Stability and Stabilization of Large-Scale Digital Networks

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To my parents, without whom this
would never have been possible.

*Carefully directed ignorance
is the key to all knowledge.*
Terry Pratchett

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Introduction

Since the beginning of control theory two main questions arose frequently: Does a system behave "nicely"?
If not, can we make the system do what we want in a "nice" way?
Although these questions essentially remained the same since [Max67], the class of systems under consideration changed in the course of time. It became apparent that a simple feedback loop as depicted in Figure 1 is not well suited for modern challenges in control theory. For example, a modern car can

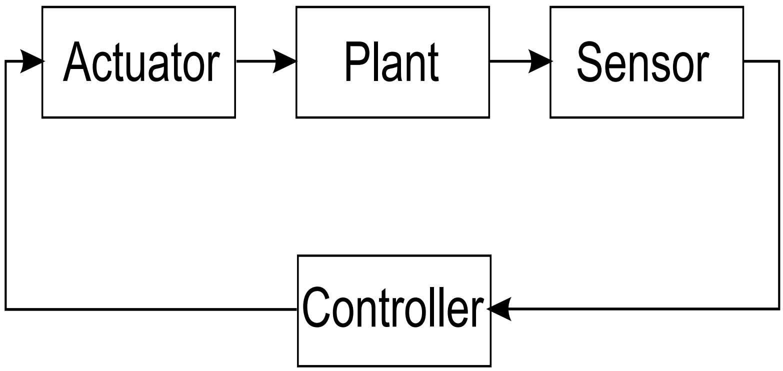

Figure 1: A simple feedback system

easily consist of hundreds of sensors, communicating over thousands of meters of cable with several controllers, which in turn coordinate many actuators. The sheer number of different devices of such a system already demands for new tools for analysis, design, and modeling. To this end Figure 2 is more appropriate to display the structure of a modern car than Figure 1. A system of the form depicted in Figure 2 is often referred to as a large-scale system. Although no precise definition can be found in literature, according to [MH05] a system is considered to be large-scale, if it has at least one of the following properties:

Decomposition: The system can be decomposed into smaller systems.

Centrality: There is no central controller.

Complexity: The system is too complex for traditional methods.

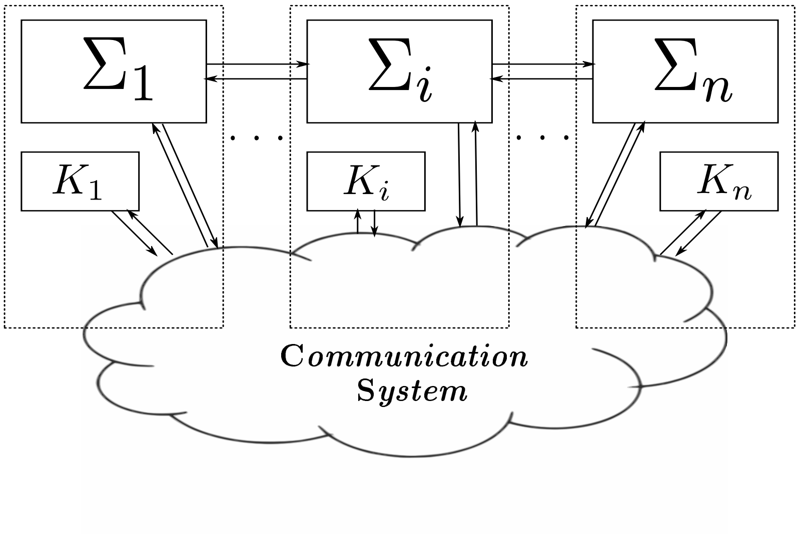

Figure 2: A large-scale system communicating over a digital channel

The somewhat vague definition itself points out that many open problems exist in the analysis, design and modeling of large-scale systems.
The complexity as well as the centrality aspect of large-scale systems leads to failure of a holistic approach of modeling, design and analysis. In this regard, it is often only possible to treat the influence of one subsystem to another as a disturbance. This approach was followed by e.g., [Vid81, Šil91], in which this influence is modeled by linear functions.
Sontag introduced in [Son89] a certain class of nonlinear systems that carry particular "nice" features of linear systems. Basically, these systems share the property that if a disturbance of the system is bounded, the state of the system will also be bounded. This "nice" behavior is termed *input to state stability* (ISS). Within the ISS framework, which plays a prominent role throughout this thesis, it was possible to generalize the ideas of [Vid81, Šil91]

to a larger class of systems in [DRW07] by allowing nonlinear functions to model the influence of one subsystem to another.

Let us consider the previous example of a modern car again. If there is a great number of parts within the system which share the same communication medium, new approaches are needed. To date, control theory often neglected these communication aspects during the analysis and design process of control systems. However, this cannot be done anymore, if parts of the system are far from each other, or if there is a large number of subsystems. By taking the communication aspect explicitly into account in the analysis and design, ideas from signal and information theory become more and more important in control theory. See, e.g., [NFZ07].
As the title of this thesis suggests, we develop tools for the stabilization and stability analysis of large scale systems communicating over digital communication channels. Before we describe our contribution in detail, we briefly discuss the impact of digital communication on control systems.

Digital Communication Channels and Control

Since the fifties of the last century the introduction of digital (digitus (lat): *finger*) communication channels, respectively, digital devices themselves, revolutionized a vast field of technical disciplines.
A control system that communicates over a digital channel is called a networked control system. The system itself may be large-scale or not. For an overview of networked control systems up to the year 2007 see [HNX07].
In this work we are considering large-scale systems that communicate over digital communication channels. The key property of digital communication channels is that they only transmit two different symbols (usually named 0 and 1). Besides the mathematically interesting problems raised by the introduction of digital channels, from a engineering point of view the merit of digital communication lies in its flexibility, robustness, and affordability. As every coin has two sides, the introduction of digital communication within the control community gave rise to new problems and challenges. In particular, the introduction of digital communication channels poses problems that are typical for networked control systems, such as:

Delay Information sent at some point in time is received at a later point.

Packet Loss Data gets lost.

Quantization Rounding errors occur due to finiteness of the channel.

Arbitration Not every sender gets access to the channel at every time.

Bandwidth The capacity of the channel is limited.

Note that delay and packet loss are closely related as well as quantization and bandwidth. The former, because packet loss can be modeled as an infinite delay. The latter, because there is a trade-off between the available bandwidth of the channel and the size of the rounding error.
The problem of arbitration demands for more sophisticated methods to coordinate the communication among several senders and receivers. Usually, this coordination is done by a so called protocol, which prepares the information and decides, when to send data and how to handle collisions and congestions. A widely used protocol today is called TCP (Transmission Control Protocol, see [Ste93] for an introduction). TCP is a packet based transmission protocol, i.e., information is gathered in packets of equal size and sent from the sender to the receiver. A network of senders and receivers that uses TCP or similar protocols is itself a large-scale system, whose analysis is far away from being trivial. Early accounts of modeling and analysis of such a system can be found in [HMTG01, Sri04, WSSL06, Bia00]. Despite the complexity of the dynamics inherent to such protocols these models deliver astonishing accuracy. However, the interaction between the dynamics of the communication channel with another dynamical system using the channel is not understood sufficiently.
One particular problem of the mentioned models is that they are mostly interested in the steady state behavior of the communication channel, while for control purposes the transient is also of vital importance. To the best of the author's knowledge, up to now, there is no model well suited for control applications.
All of the above mentioned effects, i.e., delay, packet loss, quantization, and bandwidth limitations can severely deteriorate the stability and performance of a system. To counteract these effects three different approaches are used frequently. The first is to design a controller for the nominal system, i.e., the system without a digital channel and thus without the negative effects of communication. And then to give conditions for the communication channel to ensure stability of the system despite the effects of delay, loss, and quantization.
The second approach is the introduction of two new devices into the feedback loop often called encoder and decoder. The encoder prepares the measured state in a certain way, better suited for the digital communication channel and control applications. The decoder on the other side inverts this procedure. Note that the concept of encoder/decoder is also known in information theory. To avoid confusion, the encoder and decoder are sometimes called smart sensor and smart actuator in control literature. The first and the sec-

ond method are referred to as model based or emulation approach, as it uses the system without communication as a model to design a controller.
The third approach is explicitly incorporating knowledge of the communication channel into the design of the controller. For instance, in [WCdWGA07] a method is presented to stabilize the system despite the presence of delay, provided that a model of the communication channel is given, i.e., there is a model of the communication which gives a forecast of the expected delay with sufficient accuracy.
In general, the third approach has the potential to yield better results than the other two, but as mentioned before, up to now no model of the communication channel exists that is well suited for control purposes. The first approach is followed by, e.g., [HTvdWN10], where ideas are given to handle the effect of delay, loss, and quantization.
In a similar spirit, [NL09] casts the problem into the framework of hybrid systems, where it is possible to consider the problem of arbitration. [NL09] is of interest in this thesis, as we expect that the results of Chapter 2 and 4 can be combined in a similar manner as in [NL09].
We decided to follow approach number two and to a lesser extent number one in this thesis. The rest of the introduction is devoted to the explanation of the details of our methods.

Existing Results and Contribution of the Thesis

In this work we want to derive tools for the stability analysis and stabilization problem of large-scale systems communicating over digital communication channels. We have already mentioned typical problems introduced by digital communication, such as delay, packet loss, quantization, and bandwidth limitations.
In Chapter 2 we introduce an approach which stabilizes a system despite the effects of quantization, loss, and delay. As the presented method might be computationally too complex for high dimensional systems, it is not suited for large-scale systems. Thus, additional methods to analyze interconnected systems are needed. In this regard, we present small-gain based ideas for the analysis of large-scale systems in Chapter 3, which are of independent interest.
The problem of bandwidth limitation is addressed in Chapter 4, where we present methods to lessen the amount of information we have to send over the channel, based on event triggering. In Chapter 5 we conclude the thesis with numerical simulations to show the feasibility of the presented methods. Before we go into detail on the corresponding contributions, we give a small overview on the existing literature that inspired this thesis in the particular

topics. In the note and reference section of each chapter additional information are given.

Control with Limited Information

The term control with limited information describes the situation in which the information available to the controller is in some way limited. In this work, these limitations stem from digital communication.

A subclass of control with limited information is control with encoded feedback, in which the signal that is fed back to the controller is quantized. One of the first contributions that studies the effect of quantization on a control system was made in [Del90]. In the seminal paper by Delchamps a discrete time system of the form

$$x(k+1) = Ax(k) + Bu(k)$$

where A, B are suitable real matrices with (A, B) controllable and A unstable is considered. The control action is of the form

$u(k) = f_k(q(x(0)), q(x(1)), \dots, q(x(k)))$, where q is a static quantizer.

Usually, static quantizers are of the form

$$q(x) = \Delta \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor ,$$

with $\Delta > 0$ the so called resolution. Although the author allows the controller f_k to depend on all past measurements, the set of initial states that converge to zero is thin, according to [Del90].

One key assumption for this result to hold is the restriction to static quantizers. In a static quantizer the upper bound on the quantization error $x - q(x)$ is fixed i.e., does not depend on time or x.

In [BL00] it is shown that considering non static quantizers leads to asymptotic stability of the controlled system, despite quantization. This result was generalized to nonlinear systems in [LH05]. In particular, Liberzon studies systems of the form

$$\dot{x} = f(x, u) . \tag{1}$$

It is assumed that a sensor measures the state x and transmits a quantized version of this data over a digital communication channel to a stabilizing controller g at time instances t_k. To overcome the limitations posed by quantization, two dynamical systems called encoder and decoder are introduced in [LH05], respectively, [BL00]. The quantization is done by the encoder and its dynamics are given by

$$\begin{aligned}
\dot{x}_e(t) &= f(x_e(t), g(x_e(t))) && t \neq t_k && (2)\\
\dot{\ell}_e(t) &= 0 && t \neq t_k && (3)\\
\ell_e(t) &= \Lambda \ell_e(t^-) && t = t_k && (4)\\
y(t) &= q(x(t) - x_e(t^-), \ell_e) && t = t_k && (5)\\
x_e(t) &= x_e(t^-) + y(t) && t = t_k\,. && (6)
\end{aligned}$$

The decoder on the other side of the communication channel also has a model of the plant and follows the same dynamics as the encoder. As it is assumed that encoder and decoder are initialized to the same values, the states of the encoder and decoder match for all positive times.

A brief explanation of the equations (2)–(6) is in order. The function q in (5) is a quantizer, which encodes the distance between the state x and the encoder x_e with a time varying resolution ℓ_e. The variable y caries this quantized information. At time t_k the information y is sent over the communication channel to the decoder and used to update the encoder, respectively, decoder state. By combining (5) with (6) we see that if the resolution ℓ_e converges to zero, the encoder x_e converges to the state x. Provided that $\Lambda < 1$, the resolution ℓ_e gets smaller at each time instances $t = t_k$ as can be seen in (4). One problem is that the values of q could get arbitrarily large, if the difference $x - x_e$ gets large in between t_k and t_{k+1}. When considering digital communication channels with limited bandwidth, large data transmission(s) can not be allowed. This issue is addressed by bounding the difference $x - x_e$ with the help of the Gronwall inequality.

If the difference of x and x_e converges to zero, the state of the decoder x_d also converges to the state x, because encoder and decoder agree on their states. The decoder trajectory is then used to close the loop. Due to the fact that encoder and decoder are dynamical systems, this approach is known as dynamic quantization.

Evidently for this dynamic quantization to work, encoder and decoder have to agree on their states. Moreover, the resolution has to become smaller and the evolution of the difference $x - x_e$ has to be bound.

When there is a channel delay, the encoder and decoder states will generally not match any longer. As we want to study systems that communicate over digital channels, yet any real world communication medium introduces a delay, we cannot use the approach of [LH05] as it is.

If data is sent from encoder to decoder at time t_k and there is a delay $\theta > 0$ present in the channel, the information is received by the decoder at $\theta_k := t_k + \theta$. To account for delay, De Persis uses dynamic quantization in [DP10] to achieve equality of encoder and decoder states by changing the equations for the encoder. To understand the basic idea, consider the following set of equations:

$$\begin{aligned}
\dot{x}_e(t) &= f(x_e(t), g(x_e(t-\theta))) && t \neq \theta_k \\
\dot{\bar{x}}_e(t) &= f(\bar{x}_e(t), g(\bar{x}_e(t))) && t \neq t_k \\
\dot{\ell}_e(t) &= 0 && t \neq t_k \\
x_e(t^+) &= x_e(t) + q_e(x(t-\theta) - \bar{x}_e(t-\theta), \ell_e(t-\theta)) && t = \theta_k \\
\bar{x}_e(t^+) &= \bar{x}_e(t) + q_e(x(t), \bar{x}_e(t), \ell_e(t)) && t = t_k \\
\ell_e(t^+) &= \Lambda \ell_e(t) && t = t_k \\
y(t) &= q(x(t) - \bar{x}_e(t), \ell_e) && t = t_k \,.
\end{aligned}$$

Note that the equations follow the same reasoning as the according equations from [LH05]. De Persis introduces new equations to account for a change in the control action between t_k and t_{k+1} due to θ. One can think of x_e as the "old" trajectory before y is received by the decoder and $\bar{x}_e$ as the reference trajectory, to which the distance to x is measured, as soon as y has arrived. For this approach to work, the encoder has to know the instant of time at which the data is received by the decoder. De Persis assumes a constant delay to achieve this. The assumption of constant delay θ is too restrictive for our problem of stabilizing a control system over a digital communication channel, as usually the delay is time-varying. To address this, we generalize the ideas from [DP10] to the case of an arbitrary (i.e., time varying and arbitrarily large) delay and apply them to the setup of [LH05] in Chapter 2. Furthermore, we present a mechanism to handle the effect of packet loss. To be more precise, we study systems of the form (1) which are assumed to be Lipschitz and ISS. We identify the properties needed for the dynamic quantization to be able to stabilize the system. Namely, encoder and decoder agree on certain states (information consistency) and the level of uncertainty. This uncertainty gets exponentially smaller with each transmission of data (N-contracting and L-expanding), provided that the communication channel fulfills certain bandwidth conditions. In particular, if the channel can transmit a symbol from a set of N^n different symbols where $N \in \mathbb{N}$ and n is the dimension of the system, then for the average delay τ^* it should hold that

$$e^{L\tau^*} < N \,,$$

with L the Lipschitz constant of the system. The quantity on the left hand describes the average gain of information the system exhibits and the right hand side correlates with the bandwidth of the communication channel. In

this regard the condition relates properties of the system to the bandwidth of the communication channel.
So far we introduced tools to stabilize a system despite the effects of digital communication. As already mentioned, these tools are not applicable for large-scale systems. If we want to use this approach to stabilize a large-scale system, we need conditions to ensure that the interconnection of several systems are stable. This is often done with the help of so called small-gain conditions.

Small-gain conditions for the Stability Analysis of Large-Scale Systems

Early accounts of small-gain conditions for large scale-systems are given by Siljak [Šil78]. Siljak considers systems of the form

$$\dot{x}_i = g_i(t, x_i) + h_i(t, x)\,, \quad i = 1, \dots, n\,.$$

Here, x_i is the state of the ith subsystem, g_i describes the dynamics of the subsystem, and h_i represents the interaction between other subsystems and system i. It is assumed that each subsystem without interconnection is stable. A matrix $W \in \mathbb{R}^{n \times n}$ is derived whose entries w_{ij} describe the effect of the ith subsystem onto the jth subsystem.
Stability of the interconnected system is concluded under the assumption that W is a M-matrix or equivalently $-M$ is a Metzler matrix. In [BP94] a list of over 40 properties that are equivalent to the property that W is a M-matrix can be found. The most lucid property for the presented situation is that W is a quasidominant diagonal matrix. Without going into details, this characterization states that the influence of the ith state on its own dynamics has to be much stronger than the influence of the rest of the subsystems. Or in other words, the coupling of the subsystems is weak. For our problem of stabilizing a large-scale system, the modeling of the effect of interconnection by a linear function, might be too restrictive.
More in the spirit of this work, since it also takes advantage of the ISS framework is [JTP94], although it only considers the interconnection of two systems. In [JTP94] stability of the interconnection of

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u_1) \quad \text{and} \\ \dot{x}_2 &= f_2(x_2, x_1, u_2) \end{aligned}$$

is guaranteed under the hypothesis that each subsystem is ISS and a small gain condition holds, i.e.,

$$|x_1(t)| \leq \beta(|x_1(0)|, t) + \gamma_1(\|x_2\|) + \gamma_{u1}(\|u_1\|)$$
$$|x_2(t)| \leq \beta(|x_2(0)|, t) + \gamma_2(\|x_1\|) + \gamma_{u2}(\|u_2\|)$$

and

$$\rho_1 \circ \gamma_1 \circ \rho_2 \circ \gamma_2 < \mathrm{id}\,,$$

with ρ_1, ρ_2 suitable functions. The ISS assumption allows to quantify the effect of the interconnection while the small gain condition ensures that the coupling is weak i.e., the effect one subsystem has on the other is not too strong.
As the interconnected system is ISS with u_1, u_2 as inputs, the presented approach could be used to analyze large-scale systems by starting to conclude ISS of two subsystems and adding one subsystem after another subsequently. The problem though is that the outcome of the stability analysis depends on the particular order the subsystems are chosen in. Clearly, stability should be permutation invariant, i.e., relabeling of the subsystems should not change the stability property.
To address this issue the latter result was generalized to the case of an arbitrary number of subsystems in [DRW07]. The ISS assumption for each subsystem becomes

$$|x_i(t)| \leq \beta(x_i(0), t) + \sum_{j \neq i}^{n} \gamma_{ij}(\|x_j\|) + \gamma_{iu}(\|u\|)\,.$$

The gains γ_{ij}, which model the effect of the jth subsystem onto system i, are used to define a nonlinear operator

$$\Gamma(s) = \begin{pmatrix} \sum_{j \neq 1}^{n} \gamma_{1j}(s_j) \\ \vdots \\ \sum_{j \neq n}^{n} \gamma_{nj}(s_j) \end{pmatrix}$$

mimicking matrix vector multiplication. Stability of the interconnection is inferred by the *no joint increase condition*

$$\Gamma \not\geq \mathrm{id}\ .$$

The no joint increase condition states that if Γ is applied to a nonnegative vector s, $\Gamma(s)$ must be strictly smaller than s in at least one component.

To summarize, the no joint increase condition is used to infer stability of an arbitrary number of ISS systems.

As it turns out, there is a class of systems better suited for our purpose of analyzing the stability properties of large-scale systems communicating over digital channels. This class of systems is called multichannel input to output practically stable time delay systems and is introduced in [PMTL09]. The change to infinite dimensional systems allows to account for the delay present in the communication channel over which the systems communicate.
The term multichannel describes a system/subsystem in which the inputs and outputs are divided into separate channels.
Hence, we can assign several functions to each system to model the effect of interconnection instead of having just one per system. The advantage of the multichannel approach lies in its additional flexibility to assign the gains. In [PMTL09] an example is given, in which a certain small-gain condition holds, if the multichannel approach is used, yet it does not hold if only one gain for each subsystem is assigned.
The downside of the results of [PMTL09] are that they are again only applicable for the case of two subsystems and that the presented small-gain condition is too conservative.
Moreover, in the framework of multichannel time delay systems we were not able to use the results of [DRW07] directly. The concept of an Ω-path, introduced in [DRW10] plays an important role in applying the results of [DRW07] to our problem. For an Ω-path σ it holds that

$$\Gamma(\sigma(r)) < \sigma(r)\,. \tag{7}$$

The no joint increase condition states, that Γ should decrease in at least one component. The path σ may be loosely interpreted as a nonlinear change of coordinates or to be more precise it is a path through the domain of Γ. This path allows to conclude a descent in every component of Γ. In order to benefit from an Ω-path we cast the notion into our framework in Section 3.3. Note that the introduction of the concept of an Ω-path into our framework has also importance to the results of Chapter 4.
All the mentioned small-gain conditions bound the state of the interconnected system, using bounds for the individual systems. In particular, they all end up with an inequality of the form

$$|x(t)|_{vec} \leq \beta(|x(0)|) + \Gamma(|x|_{vec}) + \gamma(\|u\|)\,,$$

with Γ modeling the effect of interconnection. To have a useful bound on the state, the inequality must be "solved" for $|x|_{vec}$. While the general idea is the same for all the presented small-gain approaches, the condition on Γ to "invert" the inequality changes. In this regard, the question arises, how to relate these different conditions and how to introduce new ones. For instance, in [Rüf10b] it was shown that the no joint increase condition is equivalent to the property that

$$s(k+1) = \Gamma(s(k)) + \gamma(w(k))$$

is ISS with respect to w. Or in other words, the ISS property of an interconnected system is concluded, provided that a comparison system, induced by the functions modeling the interconnection, is itself ISS. The latter equivalence inspired the research into more small-gain conditions and their relations in Chapter 3.

In total, the contribution of Chapter 3 may be summarized as follows. First we give a catalog of properties that are all equivalent to the no joint increase condition. In this regard we do not only add new conditions to the list of existing small-gain conditions, but show that most of the known conditions are equivalent under suitable additional assumptions. For instance, we prove that the above mentioned inversion property is already equivalent to the no joint increase condition.

The presented small-gain approach is then used to generalize [DRW07] to the case of large-scale systems that are interconnected through a digital communication channel. Or in other words, we use the derived small-gain condition to apply them to the setup of [PMTL09].

So far, we handled the effects of delay, quantization, and packet loss within a large-scale setup. In order to address bandwidth limitations, we use tools from event based control.

Event-Based Control

Usually, in control engineering, a feedback F is designed, which stabilizes the system, if it has full access to the information of the state. For instance, consider

$$\dot{x}(t) = Ax(t) + BFx(t)\,,$$

with $A + BF$ Hurwitz. Consider a sampled data version of $x(t)$ i.e.,

$$\begin{aligned} \dot{\bar{x}}(t) &= 0\,, \quad t \neq t_k \\ \bar{x}(t) &= x(t)\,, \quad t = t_k\,. \end{aligned}$$

The to be designed instances of time t_k are called sampling times. The sample and hold error is $e(t) := \bar{x}(t) - x(t)$. Clearly, although $\dot{x}(t) = (A + BF)x(t)$ is asymptotically stable, $\dot{x}(t) = Ax(t) + BF\bar{x}(t)$ does not need to have this property, if the duration between the sampling periods $t_{k+1} - t_k$ is too large.

Thus conditions for the sampling times are required. Frequently, control engineers sample the state with a fixed sampling rate, i.e., $t_{k+1} = t_k + \tau$, with $\tau > 0$ chosen heuristically. In general the approach of periodic sampling might be too conservative. Intuitively, if the error e grows fast, we have to sample more often than when the error grows slowly. Hence, it would be more effective to adapt the sampling periods accordingly. Ideas of non periodic sampling times were around since the fifties of the last century (e.g., [Ell59, TB66]). However, it was not until the introduction of the ISS framework that ideas to stabilize a system using non periodic sampling drew again more attention.
For instance, in [Tab07] a system

$$\dot{x} = f(x, k(\bar{x})) = f(x, k(x + e))$$

is considered. The existence of an ISS-Lyapunov function is used to derive a triggering condition

$$\gamma(e) \geq \alpha(x)$$

with α, γ from the ISS assumption. In essence, if the error becomes too large compared to the state, a sample has to be taken, thus resetting the error to zero.
In [Tab07] it is shown that the closed-loop system with sampling times implicitly given by the triggering condition is stable.
As [Tab07] considers a single controller for the system. The presented approach is not directly suited for the large-scale case.
The case of a distributed event-triggered approach is presented in [WL11]. The authors consider a network of systems given by

$$\begin{aligned} \dot{x}_i &= f_i(x, u_i) \\ u_i &= g_i(\bar{x}) \end{aligned}$$

where $\bar{x}$ is a sampled data version of x. It is shown that under a finite $\mathcal{L}_p$ gain assumption on the subsystems, the interconnection is stable, provided that the matrix describing the effect of the interconnection is diagonally dominant. The analysis and design of the interconnections are similar to [Šil78] as described in the last section. In particular, the effects of the interconnection is modeled by linear functions. As diagonal dominance is stronger than the quasi dominance property, the small-gain condition of [WL11] is too demanding. The basic idea of Chapter 4 stems from the observation that the simple and possibly too restrictive small-gain condition used in [WL11] can be replaced by ideas presented in [DRW10]. In particular, in [DRW10] functions σ, ρ are constructed for which

$$\bar{\Gamma}(\sigma(r), \rho(r)) < \sigma(r)$$

holds for all $r > 0$.
$\bar{\Gamma}$ is an augmented version of (7) used to ensure stability of the interconnection. The second argument models the effect of the imperfect knowledge of the states to the local controllers due to the sample and hold error e.
The Ω-path σ is used to derive a Lyapunov function V for the overall system, given the Lyapunov functions of the subsystems V_i. By the results of Chapter 3 we know that the interconnected system is stable if $e \equiv 0$. In Chapter 4 we derive triggering functions depending on ρ to ensure that the effect of imperfect knowledge of the state due to the error e does not interfere with the stability property of the interconnected system. In essence, the triggering functions are of the form

$$V_i(x_i) \leq \chi_i(e_i)$$

where χ_i is a scaling function depending on the gains of the effect of the imperfect knowledge of the states and the path σ.
The event-triggering approaches mentioned so far share one property: the corresponding event-triggered closed-loop systems are hybrid system. In hybrid systems the so-called Zeno effect may occur, i.e., the triggering and hence information transmission can happen infinitely often in finite time. To avoid this unwanted effect, we show that altering the ISS assumption to a practical notion of ISS rules out Zeno phenomena for the price of a weaker stability property. It is also shown that triggering conditions of the form

$$V(x) \leq \chi_i(e_i)$$

would ensure stability, while ruling out Zeno phenomena. A triggering function of the latter form demands knowledge of the Lyapunov function for the overall system to all subsystem. As we are heading for a purely decentralized approach, we need an approximation for the Lyapunov function V using only local information. We show that the velocity of the subsystems can be used to approximate V and thus we can derive a triggering condition called *parsimonious triggering*, which again asymptotically stabilizes the system while retaining our decentralized setup. The main tools we use to conclude stability of the event-triggered systems, besides the above mentioned, are standard Lyapunov techniques, a discrete Gronwall inequality and results from non smooth analysis.

Open Problems

Of course we will not answer the questions posed at the very beginning in their entirety. But we present tools that we hope to prove helpful in the stabilization and stability analysis of large-scale systems communicating over digital channels. In detail, we present methods to overcome limitations posed by quantization, delay and packet loss in Chapter 2 by using concepts from information theory (encoding/decoding, entropy like estimates) together with tools from signal theory (dynamic quantization).

In Chapter 3 we develop novel small-gain conditions, relate them to the literature, and apply them to analyze the stability properties of a large scale system, communicating over digital channels.

We address bandwidth limitations in Chapter 4 by lowering the amount of information we have to send over the channel. To this end we use the newly developed tools from Chapter 3 and combine them with known ideas. The basic idea from Chapter 4 stems also from signal theory and is known as e.g., event based sampling.

Besides the countless open, yet interesting, problems within the framework of large-scale systems communicating over digital channels, we should mention some that are of particular interest for this thesis.

As already sketched, to apply the tools from Chapter 3 or 4 the subsystems have to be ISS. As the stabilization method of Chapter 2 yields asymptotic stability instead of ISS, further research is required. In Chapter 3 we discuss the equivalence of small-gain conditions. We expect that the notion of an Ω-path is also equivalent to the presented ones under additional assumptions, but we were not able to identify these conditions in a satisfactorily manner. It should also be not unmentioned, that although we introduce methods to lessen the information we have to sent over the channel, the problem of arbitration is not considered.

Furthermore, we demand the controllers to render the closed-loop systems ISS, which is in general hard to achieve. In this regard especially in Chapter 3 the ISS assumption might be too demanding. We also expect that the introduction of a suitable model for the communication channel, regarding control purposes, would lead to better results by allowing to incorporate the effects of the communication directly into design, modeling, and analysis of control systems. But we hope that the tools and methods introduced in this thesis are still a considerably contribution to the field of large-scale digital networks.

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Chapter 1

Preliminaries

In this chapter we introduce the notation we use together with the basic definitions and concepts we want to consider. In particular, in Section 1.1 we present the basic notation, while the class of systems as well as preliminary stability definitions and results are given in Section 1.2. Finally, in Section 1.3 we give an introduction to the ISS framework, which is a robust notion of stability.

Of course, this can only be an excerpt of the corresponding topics. Moreover, the sketchy nature of the presentation does not reflect the importance of the mentioned results. However, a thorough discussion would go beyond the scope of this thesis.

1.1 Notations and Definitions

Here we want to introduce the basic definitions and notations that hold throughout the thesis. Sometimes we use slight modifications of the concepts introduced here. We will mention the exceptions explicitly.

Let $\mathbb{R}$ denotes the field of real numbers. By $\mathbb{R}_+$ we denote the set of non-negative real numbers. The set $\mathbb{Z}$ represents the integers and $\mathbb{N} = \{0, 1, \dots\}$ the natural numbers. We denote by $e_n \in \mathbb{R}^n$ the vector consisting of ones. If it is clear from the context, we usually omit the subscript n.

1.1.1 Monotonicity

Monotonicity or to be more precise monotone operators play an important role in this thesis. Before we can define what we mean by monotone operators, we have to state which order relation we use. Throughout the thesis we use the order relation "$\leq_n$" induced on $\mathbb{R}^n$ by the positive orthant.

Let $x, y \in \mathbb{R}^n$. We define a partial order relation on $\mathbb{R}^n$ by

1. $x \leq_n y$ if $x_i \leq y_i$ for $i = 1, \dots, n$
2. $x <_n y$ if $x_i < y_i$ for $i = 1, \dots, n$,

with $x \leq y$ for $x, y \in \mathbb{R}$ the usual total order relation. Clearly, for $n > 1$ the order relation $\leq_n$ is not a total order relation, i.e we cannot say that either $x \leq_n y$ or $y <_n x$ and hence we need the negation as well i.e.,

1. $x \not\leq_n y$ if $x_i > y_i$ for at least one $i \in \{1, \dots, n\}$
2. $x \not<_n y$ if $x_i \geq y_i$ for at least one $i \in \{1, \dots, n\}$.

We say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is monotone, if $x \leq_n y$ implies $f(x) \leq_m f(y)$ for all $x, y \in \mathbb{R}^n$.
If $x \leq_n y$ and $x \neq y$ implies $f(x) <_m f(y)$, we say that f is strictly monotone. A monotone function is also called nondecreasing, while a strictly monotone function is referred to as strictly increasing. Furthermore, f is called decreasing, if $x \leq y$ implies $f(y) \leq f(x)$.
If it is clear from the context, we will omit the subscript of the order relation. As monotonicity plays an important role in the thesis, we restrict the concept of a norm to monotone norms. Let $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ be a norm and $x = (x_1, \dots, x_n)^\top, y = (y_1, \dots, y_n)^\top \in \mathbb{R}^n$. We say that $|\cdot|$ is a monotone norm, if

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

implies $|x| \leq |y|$. Note that if we restrict $|\cdot|$ to the positive orthant, a monotone norm is a monotone function.
Let $a = (a_1, \dots, a_n)^\top, b = (b_1, \dots, b_n)^\top \in \mathbb{R}^n$. We define the maximum and supremum of vectors component-wise i.e.,

$$\max\{a, b\} := \sup\{a, b\} := \begin{pmatrix} \sup\{a_1, b_1\} \\ \vdots \\ \sup\{a_n, b_n\} \end{pmatrix}.$$

Consider a sequence $s : \mathbb{N} \to \mathbb{R}^n$. We define the lim sup also component-wise i.e.,

$$\limsup_{k \to \infty} s(k) = \lim_{k \to \infty} \sup_{l \geq k} s(l) = \begin{pmatrix} \limsup_{k \to \infty} s_1(k) \\ \vdots \\ \limsup_{k \to \infty} s_n(k) \end{pmatrix} \in (\mathbb{R} \cup \infty)^n.$$

Note that some of the component sequences $s_i(k)$ may be finite and others infinite.
We often need the following fact that we find convenient to state here. For a monotone function $f : \mathbb{R}^n \to \mathbb{R}^n$ it holds that

$$f(\sup\{a, b\}) \geq \sup\{f(a), f(b)\} \tag{1.1}$$

for all $a, b \in \mathbb{R}^n$.

1.1.2 Gain Functions and their Multidimensional Extensions

As we will see in Section 1.3 important generalizations of linear control theory utilize the notion of gain functions respectively comparison functions. Basically, gain functions, or gains for short, are used to describe the influence of a disturbance on a dynamical system.
A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$, if γ is continuous, increasing and satisfies $\gamma(0) = 0$.
If in addition γ is unbounded, we say that $\gamma \in \mathcal{K}_\infty$. We refer to a function of class $\mathcal{K}$ or $\mathcal{K}_\infty$ as gain or gain operator.
A particular nice feature of gain functions is the following lemma.

Lemma 1.1.1. *Let $\gamma \in \mathcal{K}_\infty$, then its inverse $\gamma^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists and is also of class $\mathcal{K}_\infty$.*

A proof can be found e.g., in [Rüf07].
Note that the class $\mathcal{K}$ has a semigroup structure with respect to composition and the identity (id) as the neutral element. Similarly, $\mathcal{K}_\infty$ has a group structure.
Sometimes we have to compare gains. In this regard we say that $\gamma < \alpha$ for $\gamma, \alpha \in \mathcal{K}_\infty$, if $\gamma(r) < \alpha(r)$ for all $r > 0$ holds.

Lemma 1.1.2. *Let $\gamma, \rho \in \mathcal{K}_\infty$, then $\max\{\gamma, \rho\} \in \mathcal{K}_\infty$ and $\min\{\gamma, \rho\} \in \mathcal{K}_\infty$. Furthermore, the same holds true with $\mathcal{K}_\infty$ replaced by $\mathcal{K}$.*

Proof. Let $\gamma, \rho \in \mathcal{K}_\infty$. As the maximum as well as the minimum are continuous functions over $\mathbb{R}_+$, $\max\{\gamma, \rho\}$ and $\min\{\gamma, \rho\}$ are continuous.
Let $r_1 < r_2$. As $\gamma, \rho \in \mathcal{K}_\infty$, we have $\gamma(r_1) < \gamma(r_2)$ and $\rho(r_1) < \rho(r_2)$ and thus $\max\{\gamma(r_1), \rho(r_1)\} < \max\{\gamma(r_2), \rho(r_2)\}$, which shows strict monotonicity.
Unboundedness and the property that $\max\{\gamma, \rho\}(0) = 0$ respectively $\min\{\gamma, \rho\}(0) = 0$ are obvious and the proof is complete. □

We say that $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$, if β is decreasing and $\lim_{t\to\infty} \beta(t) = 0$. A function $\beta : \mathbb{R}_+^2 \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ of class $\mathcal{L}$ for each fixed $s \in \mathbb{R}_+$.
The next lemma considers the so called weak triangle inequality.

Lemma 1.1.3. *Let $\gamma \in \mathcal{K}$ and $\rho \in \mathcal{K}_\infty$. Then for all $a, b \in \mathbb{R}_+$ we have*

$$\gamma(a+b) \leq \gamma \circ (\mathrm{id} + \rho)(a) + \gamma \circ (\mathrm{id} + \rho^{-1})(b) .$$

A proof is given in [JTP94].
We say that $\gamma : \mathbb{R}_+ \to \mathbb{R}^n_+$ with $\gamma = (\gamma_1, \ldots, \gamma_n)^\top$ is of class $\mathcal{K}^n$, if $\gamma_i \in \mathcal{K}$ for each $i = 1, \ldots, n$. A function $\gamma \in \mathcal{K}^n_\infty$ is defined analogously. The map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^{n \times m}_+$ is said to be of class $\mathcal{K}^{n \times m}_\infty$ if

$$\Gamma(s) = \begin{pmatrix} \gamma_{11}(s_1) & \ldots & \gamma_{1n}(s_n) \\ \vdots & & \vdots \\ \gamma_{m1}(s_1) & \ldots & \gamma_{mn}(s_n) \end{pmatrix}$$

with $\gamma_{ij} \in \mathcal{K}_\infty$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. If we allow some of the $\gamma_{ij} = 0$, we say that $\Gamma \in \mathcal{G} := (\mathcal{K}_\infty \cup 0)^{n \times m}$. We will refer to Γ as a gain matrix.
Similarly to the linear case, we want to use Γ to define a map from $\mathbb{R}^n_+$ to $\mathbb{R}^m_+$. To do so, we need a way to "aggregate" the gains within one row of Γ. This is done with the help of the so called monotone aggregation functions.

1.1.3 Monotone Aggregation Functions

Let $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ denote a monotone norm. To emphasize that a norm is defined on an infinite dimensional space, we use the symbol $\|\cdot\|$ for this case.

Definition 1.1.4. A continuous function $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a *monotone aggregation function* if μ is:

(Positive definite:) $\mu(v) \geq 0$ for all $v \in \mathbb{R}^n_+$ and $\mu(v) = 0$ iff $v = 0$;

(Increasing:) $\mu(v) > \mu(z)$ if $v \geq z$, $v \neq z$;

(Unbounded:) If $|v| \to \infty$ then $\mu(v) \to \infty$.

The space of monotone aggregation functions (MAFs for short) with domain $\mathbb{R}^n_+$ is denoted by MAF_n. Moreover, we say that $\mu \in MAF^m_n$ if $\mu : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ and for each $i = 1, 2, \ldots, m$, $\mu_i \in MAF_n$.
Furthermore, if additionally μ is

(Subadditive:) $\mu(v+z) \leq \mu(v) + \mu(z)$ for all $v, z \in \mathbb{R}^n_+$,

we say μ is subadditive.

A popular example of a monotone aggregation function is a monotone norm. In fact, all monotone norms are subadditive monotone aggregation functions. Of course there exists monotone aggregation functions that are not norms. For instance, consider $\mu(v) = \log(|v| + 1)$ with $|\cdot|$ an arbitrary monotone norm.
Now we want to state some easy to verify facts about monotone aggregation functions that we need throughout the thesis. There is a strong relation between monotone aggregation functions and gains, as we will see in the rest of this section.
The next lemma is a direct consequence of Definition 1.1.4, but we find it convenient to state it as a lemma nevertheless.

Lemma 1.1.5. *If $\mu : \mathbb{R}_+^n \to \mathbb{R}_+$ is a monotone aggregation function, then $\gamma(r) := \mu(\alpha(r))$ is of class $\mathcal{K}_\infty$ for any $\alpha \in \mathcal{G}^n \setminus \{0\}$.*

The next two assertions are devoted to the fact that we can always bound a particular MAF by another MAF by changing the argument.

Lemma 1.1.6. *For any $\mu \in \mathrm{MAF}_n$ there exists $\gamma_1, \ldots, \gamma_n \in \mathcal{K}_\infty$ such that for all $a_1, \ldots, a_n \in \mathbb{R}_+$ it holds*

$$\max\{a_1, \ldots, a_n\} \leq \mu(\gamma_1(a_1), \ldots, \gamma_n(a_n)) \,.$$

Proof. Define $\gamma_1^{-1}(s) := \mu(s, 0, \ldots, 0)$, $\gamma_2^{-1}(s) := \mu(0, s, 0, \ldots, 0)\,, \ldots\,,$ $\gamma_n^{-1}(s) := \mu(0, \ldots, 0, s)$. By Lemma 1.1.5 it holds that $\gamma_1^{-1}, \ldots, \gamma_n^{-1} \in \mathcal{K}_\infty$. Hence we can write

$$\begin{aligned}\max\{a_1, \ldots, a_n\} = \max\{\gamma_1^{-1} \circ \gamma_1(a_1), \ldots, \gamma_n^{-1} \circ \gamma_n(a_n)\} = \\ \max\{\mu(\gamma_1(a_1), 0, \ldots, 0), \ldots, \mu(0, \ldots, 0, \gamma_n(a_n))\} \leq \\ \max\{\mu(\gamma_1(a_1), \ldots, \gamma_n(a_n)), \ldots, \mu(\gamma_1(a_1), \ldots, \gamma_n(a_n))\} = \\ \mu(\gamma_1(a_1), \ldots, \gamma_n(a_n))\,,\end{aligned}$$

where the inequality follows from the monotonicity of μ and the proof is complete. □

Corollary 1.1.7. *Given a $\mu_1 \in \mathrm{MAF}_n$ and $a_1, \ldots, a_n \in \mathbb{R}_+$. Then we can find for any $\mu_2 \in \mathrm{MAF}_n$ functions $\gamma_1, \ldots, \gamma_n \in \mathcal{K}_\infty$ such that*

$$\mu_1(a_1, \ldots, a_n) \leq \mu_2(\gamma_1(a_1), \ldots, \gamma_n(a_n))\,.$$

Proof. By monotonicity of μ_1 we have

$$\mu_1(a_1, \ldots, a_n) \leq \max\{\mu_1(a_1, \ldots, a_1), \ldots, \mu_1(a_n, \ldots, a_n)\}\,.$$

By Lemma 1.1.5 we see that

$$\mu_1(a_i, \dots, a_i) =: \tilde{\gamma}_i(a_i) \in \mathcal{K}_\infty$$

for all $i = 1, \dots, n$. And therefore

$$\mu_1(a_1, \dots, a_n) \leq \max\{\tilde{\gamma}_1(a_1), \dots, \tilde{\gamma}_n(a_n)\} .$$

By Lemma 1.1.6 we find for any $\mu_2 \in \mathrm{MAF}_n$ functions $\bar{\gamma}_1, \dots, \bar{\gamma}_n \in \mathcal{K}_\infty$ such that

$$\max\{\tilde{\gamma}_1(a_1), \dots, \tilde{\gamma}_n(a_n)\} \leq \mu_2(\bar{\gamma}_1 \circ \tilde{\gamma}_1(a_1), \dots, \bar{\gamma}_n \circ \tilde{\gamma}_n(a_n))$$

Realizing that $\bar{\gamma}_i \circ \tilde{\gamma}_i =: \gamma_i \in \mathcal{K}_\infty$ finishes the proof. □

Sometimes we need a weaker notion than that of a MAF. To this end we summarize some concepts here that we need in order to compare values from different spaces.

Definition 1.1.8. A continuous function $\xi : \mathbb{R}^n \to \mathbb{R}^m_+$ is called positive definite if $\xi(0) = 0$ and $\xi(s) = 0$ implies $s = 0$.

Definition 1.1.9. We will call a continuous monotone function $\zeta : \mathbb{R}^n \to \mathbb{R}^m_+$ *proper* if there exists a function $\tilde{\alpha} \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}^n_+$,

$$\tilde{\alpha}(|s|)e \leq \zeta(s). \tag{1.2}$$

A way to ensure positive definiteness of a continuous and proper function ξ is to assume $\xi(0) = 0$. As we need (1.3) frequently in the thesis, we state the next Lemma in its present form.

Lemma 1.1.10. *A proper function $\zeta : \mathbb{R}^n \to \mathbb{R}^m_+$ is positive definite if and only if there exists an $\hat{\alpha} \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}^n_+$,*

$$|\zeta(s)| \leq \hat{\alpha}(|s|). \tag{1.3}$$

Proof. Let ζ be proper and so in particular monotone. If ζ is positive definite, then $|\zeta(s)|$ is monotone, continuous, and positive definite, because of the properness of ζ and the restriction to monotone norms. By the equivalence of norms on finite-dimensional spaces, there exists a $v \in \mathbb{R}_+$ such that $\hat{\alpha}(r) = |\zeta(rve)|$ for $r \geq 0$ can be chosen as the desired class $\mathcal{K}_\infty$ function.
For the other direction observe that by choosing $s = 0$ we get

$$|\zeta(0)| \leq \hat{\alpha}(0) = 0\,, \tag{1.4}$$

because $\hat{\alpha} \in \mathcal{K}_\infty$. Considering the properness of ζ for $s \neq 0$ yields

$$0 < \tilde{\alpha}(|s|)e \leq \zeta(s)\,.$$

By applying norms we arrive at

$$0 < |\zeta(s)|\,, \tag{1.5}$$

for $s \neq 0$. Combining (1.4) and (1.5) yields the positive definiteness and the proof is complete. □

The next lemma relates the concept of a MAF to proper and positive definite functions.

Lemma 1.1.11. *If $\xi : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ is proper and positive definite, then there exists a $\mu \in \mathrm{MAF}^{\mathrm{m}}_{\mathrm{n}}$ such that*

$$\xi(s) \leq \mu(s)$$

for all $s \in \mathbb{R}^n_+$.

Proof. Let $\rho \in \mathrm{MAF}_n$ and define $\mu_j = \xi_j + \rho$. Fix $s, v \in \mathbb{R}^n_+$ with $s \leq v$ and $s \neq v$. We have

$$\mu_j(s) = \xi_j(s) + \rho(s) < \xi_j(v) + \rho(v) = \mu_j(v)\,,$$

because ξ_j is monotone and $\rho \in \mathrm{MAF}_n$ and thus μ_j has the increasing property of a MAF.
It is obvious, that μ_j is positive definite, because the sum of two positive definite functions is again positive definite. By properness of ξ there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\alpha(|s|) \leq \xi(s)_j \leq \xi(s)_j + \rho(s) = \mu_j(s)$$

for all $j = 1, \ldots, m$. Clearly, $\mu_j(s)$ tends to infinity as $|s|$ tends to infinity, because $\alpha_j \in \mathcal{K}_\infty$. Hence, $\mu(s) := (\mu_1(s), \ldots, \mu_m(s))^\top$ is in $\mathrm{MAF}^{\mathrm{m}}_{\mathrm{n}}$ and the proof is complete. □

1.2 Dynamical Systems

If not explicitly mentioned otherwise, the presented material in this section is borrowed from [HP05]. For the sake of completeness we cite the needed concepts and results here.

Dynamical systems are often defined by solutions of ordinary differential equations of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1.6}$$

with $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Before we can define what a dynamical system is, we have to define the concept of a solution of an ordinary differential equation. As we want to study differential equations with non continuous right hand sides, we need also the following definition.

Definition 1.2.1. We say a function $x : [t_0, T] \to \mathbb{R}^n$ is absolutely continuous, if its derivative $\dot{x}(t)$ exists almost everywhere (i.e., except on a set of Lebesgue measure zero) for $t \in [t_0, T]$ and

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s) ds \quad t \in [t_0, T].$$

Definition 1.2.2. A function $x(\cdot) : I \to X$ is called a solution of (1.6) on an interval $I \subset T$ if it is absolutely continuous and satisfies (1.6) almost everywhere on I.

Note that contrary to the usual solution theory (i.e., in the sense of Peano [HC08, Theorem 2.24]), the trajectory $x(t)$ is only differentiable almost everywhere on I.

According to [HP05] a differentiable dynamical system can be described by the following definition.

Definition 1.2.3. A septuple $\Sigma = (T, U, \mathcal{U}, X, Y, \varphi, \eta)$ is said to be a differentiable dynamical system with time domain T, input value space U, state space X, output space Y, state transition map φ and output map η, if the following conditions are satisfied.

- $T, U, \mathcal{U}, X, Y$ are non void sets.
- U, Y are subsets of $\mathbb{R}^m$ and $\mathbb{R}^p$, and X is an open subset of $\mathbb{R}^n$, and $\mathcal{U} \subset \mathcal{U}^T$.
- There exists a function $f : T \times X \times U \to \mathbb{R}^n$ such that for all $t_0 \in T$, $x_0 \in X$, $u(\cdot) \in \mathcal{U}$ the initial value problem

  $$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), t \geq t_0 \;\; t \in T \\ x(t_0) &= x_0 \end{aligned}$$

  has a unique solution $x(\cdot)$ on a maximal open time interval I satisfying $I = T_{t_0, x_0, u(\cdot)}$. Furthermore, for the state transition map $\varphi : T^2 \times X \times \mathcal{U}$ it holds that $x(t) = \varphi(t; t_0, x_0, u(\cdot))$ for all $t \in I$.

- $\eta : T \times X \times U \to Y$ is continuous

As we frequently deal with differentiable systems, we call a differentiable system often just a system. The definition from [HP05] allows for arbitrary finite dimensional fields for state, input, and output space. Because we consider only real dynamical systems, we stated the definition in its present form. We refer to x_0 and t_0 as the initial state and initial time, respectively. By abusing notation, we sometimes denote the trajectory by $x(t)$ and sometimes we identify $x(t)$ with a single point in the state space for a fixed t. If it is clear from the context we will omit the argument of $x(t)$. On the other hand, if we want to emphasize the role of the input, we write for a solution to (1.6) $x(t; u)$.
Next we cite a remarkable result concerning the existence of solutions. To this end consider

$$\dot{x} = f(t, x) \tag{1.7}$$

where $f : T \times X \to \mathbb{R}^n$, $T \subset \mathbb{R}$ is an interval and X an open subset of $\mathbb{R}^n$. We say that f satisfies the Carathéodory conditions if

Car 1 $f(\cdot, x) : T \to \mathbb{R}^n$ is measurable for each fixed $x \in X$;

Car 2 $f(t, \cdot) : X \to \mathbb{R}^n$ is continuous for each fixed $t \in T$;

Car 3 $|f(\cdot, \tilde{x})|$ is locally integrable on T for some $\tilde{x} \in X$;

Car 4 for each compact set $C = I \times K \subset T \times X$ there exists an integrable function $L_C(\cdot) : I \times \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq L_C(t)|x - y| , \quad (t, x), (t, y) \in C .$$

Theorem 1.2.4. *If T is an open interval, X is an open subset of $\mathbb{R}^n$ and $f : T \times \mathbb{R}^n$ satisfies the Carathéodory conditions on $T \times X$, then for any $(t_0, x_0) \in T \times X$ there exists a unique solution $x(\cdot) = \psi(\cdot; t_0, x_0)$ of (1.7).*

A proof can be found e.g., in [CL55]. If uniqueness does not play a role, condition Car 4 can be dropped.
To ensure that a solution of (1.6) exists, one must verify that $g(t, x) := f(t, x, u(t))$ satisfies the Carathéodory conditions for all $u(\cdot) \in \mathcal{U}$. The following corollary gives a sufficient condition.

Corollary 1.2.5. *Suppose $T, U, \mathcal{U}, X, Y$ are sets as in Definition 1.2.3, $\eta : T \times X \times U \to Y$ is continuous and $f : T \times X \times U \to \mathbb{R}^n$ is jointly measurable in $(t, u) \in T \times U$ for every $x \in X$ and continuous in $x \in X$ for each fixed $(t, u) \in T \times U$. If $\mathcal{U} \subset U^T$ consists of locally L^p-integrable functions $(1 \leq p <$*

∞) on T and for each compact set $C = I \times K \subset T \times X$ there exists constants m_C, l_C such that

$$|f(t,x,u)| \leq m_C(\|u\|_p + 1)\,, \quad t \in I,\ u \in U \text{ for some } x \in X\,,$$
$$|f(t,x,u) - f(t,y,u)| \leq l_C(\|u\|_p + 1)|x-y|\,, \quad (t,x),(t,y) \in C\,,\ u \in U\,,$$

then (1.6) *has a unique solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ on a maximal interval of existence $T_{t_0,x_0,u(\cdot)}$ for all $(t_0, x_0, u(\cdot)) \in T \times X \times \mathcal{U}$.*

Another concept we will use frequently in this work is the following.

Definition 1.2.6. A quintuple $\Sigma = (U, X, Y, \psi, \eta)$ where U, X, Y are non void sets and $\psi : X \times U \to X$, $\eta : X \times U$ are maps, is called a discrete time system with input space U, state space X, output space Y, next state function ψ, and output function η.

The dynamics of a discrete time system are described by the following state and output equations

$$\begin{aligned} x(k+1) &= \psi(x(k), u(k))\,, \quad k \in \mathbb{N} \\ y(k) &= \eta(x(k), u(k))\,. \end{aligned} \tag{1.8}$$

Often, we are interested in stationary points of the state space.

Definition 1.2.7. Consider (1.6) respectively (1.8). We say a pair $(x^*, u^*) \in X \times U$ is an equilibrium pair, if

$$0 = f(t, x^*, u^*)$$

for all $t \geq 0$ or

$$x^* = \psi(x^*, u^*)\,.$$

Furthermore x^* is called an equilibrium state or equilibrium.

Equilibria may be stable or not in the sense that if we start sufficiently close to the equilibrium the trajectory will stay close to the equilibrium. The next definitions describe these properties.

Definition 1.2.8. An equilibrium pair (x^*, u^*) of (1.6) is called stable at time t_0, if for each $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that

$$|x^* - x(t_0)| \leq \delta(t_0, \varepsilon) \Rightarrow |x^* - x(t)| \leq \varepsilon$$

for all $t \geq t_0$.

Definition 1.2.9. An equilibrium pair (x^*, u^*) of (1.6) is called attractive at time t_0, if there exists a $\delta(t_0) > 0$ such that

$$|x^* - x(t_0)| \leq \delta(t_0) \Rightarrow \lim_{t\to\infty} |x^* - x(t)| = 0\,.$$

Definition 1.2.10. An equilibrium pair (x^*, u^*) of (1.6) is called asymptotically stable at t_0, if it is stable and attractive at t_0.

If an equilibrium point x^* is stable and attractive at t_0 for all $x(t_0) \in \mathbb{R}^n$ we say that x^* is globally asymptotically stable (or GAS for short).
If δ in the above definitions does not depend on t_0 and the convergence in Definition 1.2.9 is uniform in t_0, we say that x^* is uniformly asymptotically stable.
One concept, which proved helpful in the analysis of the stability properties of

$$\dot{x} = f(x) \tag{1.9}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, is a Lyapunov function. Informally, a Lyapunov function is a function from the state space into the positive reals. It is positive everywhere except at the equilibrium and decreases along every trajectory.

Definition 1.2.11. A differentiable, positive definite and proper function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a Lyapunov function for system (1.9) if there exist a positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\nabla V(x) f(x) \leq -\alpha(|x|) \tag{1.10}$$

for all $x \in \mathbb{R}^n$.

Intuitively, the decay condition (1.10) ensures that the "energy" function V decays along solutions. As the properness and positive definiteness ensures that $x \to 0$ as $V(x) \to 0$, zero must be attractive and stable. The success of Lyapunov functions lies in the next theorem.

Theorem 1.2.12. *If and only if there exist a Lyapunov function for system* (1.9)*, then* (1.9) *is globally asymptotically stable.*

The "if" part of the last theorem is known as Lyapunov's direct method (see e.g., [BR05, Theorem 2.2]). Direct in the sense that in order to proof stability of a system, no solution has to be calculated. The "only if" part, on the other hand, is known as a converse Lyapunov result (see e.g., [BR05, Theorem 2.4]). Pursuing the rich field of Lyapunov theory further goes beyond the scope of this thesis. The interested reader is referred to e.g., [BR05] for more details.

So far, we considered what can be regarded as classical systems theory. The title of this work suggests that we are interested in the stability properties of large-scale systems. As stated in the introduction, one aspect of large-scale systems is the decomposability. To be able to distinguish between a system and one of the smaller parts consider

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u_1)\\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, u_2)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u_n)\end{aligned}$$

with $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$. Let $x = (x_1^\top, \dots, x_n^\top)^\top$, $u = (u_1^\top, \dots, u_n^\top)^\top$ and

$$f(x,u) := \begin{pmatrix} f_1(x_1, x_2, \dots, x_n, u_1) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n, u_n) \end{pmatrix}.$$

We call $\dot{x} = f(x,u)$ an interconnected system and we will refer to $\dot{x}_i = f_i(x_1, x_2, \dots, x_n, u_i)$ for each $i = 1, \dots, n$ as the ith subsystem of the interconnected system.

In Chapter 3 we are dealing with a different class of systems. This type of system is called a *functional differential equation* (FDE). To be more precise we are dealing with a subclass of FDE's namely time delay systems. A time delay system is of the form

$$\begin{aligned}\dot{x}(t) &= f(x_{\mathtt{d}}, u^1_{\mathtt{d}}, \dots, u^l_{\mathtt{d}}, t)\\ y^1(t) &= h^1(x_{\mathtt{d}}, u^1_{\mathtt{d}}, \dots, u^l_{\mathtt{d}}, t)\\ &\vdots\\ y^r(t) &= h^r(x_{\mathtt{d}}, u^1_{\mathtt{d}}, \dots, u^l_{\mathtt{d}}, t)\,.\end{aligned} \tag{1.11}$$

In a functional differential equation the right-hand-side rather depends on a piece of trajectory than a single point in the state space, as it is the case for ordinary differential equations.

Denote by $\mathcal{X}^n = \mathcal{C}([-t_d, 0], \mathbb{R}^n)$ the set of continuous functions with domain $[-t_d, 0]$, $t_d \in \mathbb{R}_+$ and image $\mathbb{R}^n$. The operators

$$f : \mathcal{X}^n \times \mathcal{X}^{m_1} \times \cdots \times \mathcal{X}^{m_l} \times \mathbb{R}_+ \to \mathbb{R}^n$$

and

$$h^i : \mathcal{X}^n \times \mathcal{X}^{m_1} \times \cdots \times \mathcal{X}^{m_l} \times \mathbb{R}_+ \to \mathbb{R}^{v_i}$$

for $i = 1, \ldots, r$ are supposed to be Lipschitz in x_d, uniformly continuous in u_d and Lebesgue measurable in t. The state $x_{\mathtt{d}}$ lives in $\mathcal{X}^n$ and the inputs are $u^j_{\mathtt{d}} \in \mathcal{X}^{m_j}$ for $j = 1, \ldots, l$. The subscript d describes a retarded version of its variable in the following way: Let $t_{\mathtt{d}} \in \mathbb{R}_+$. If a function is defined as $g : [-t_{\mathtt{d}}, \infty) \to \mathbb{R}^k$ and $t \in [0, t_{\mathtt{d}}]$, then $g_{\mathtt{d}}(t; \cdot)$ represents a function from $[0, t_{\mathtt{d}}]$ to $\mathbb{R}^k$ by

$$g_{\mathtt{d}}(t; \tau) := g(t - \tau)\,.$$

Therefore x_d is a piece of trajectory starting at $s = t - t_d(t)$ and ending at $s = t$. The aforementioned notation is borrowed from [Tee98]. Note that it differs slightly from the classical notation used e.g. in [Hal77] by interchanging the argument of delay and time.
For a retarded function $g_{\mathtt{d}} : [0, t_{\mathtt{d}}] \times \mathbb{R} \to \mathbb{R}^k$, define
$\|g_{\mathtt{d}}(t)\| = \sup_{\tau \in [0, t_{\mathtt{d}}]} |g_{\mathtt{d}}(t; \tau)| = \sup_{s \in [t - t_{\mathtt{d}}, t]} |g(s)|$ as the supremum norm.
Often we describe the influence of one subsystem on another either qualitatively or quantitatively. To this end, we want to assign one real number to each subsystem, despite the fact that each subsystem lives in a higher Euclidean space. Here we are heading for a finer granularity, by assigning one real number to each of the "channels" given by the $u^i_{\mathtt{d}}$ respectively y^i. To this end, we introduce the following notation.
Given $x = (x_1^T, \ldots, x_k^T)^T$ with $x_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, k$, the vector of norms is given by $|x|_{\mathbf{vec}} = (|x_1|, \ldots, |x_k|)^T \in \mathbb{R}^k_+$, in a similar manner define $\|x(\cdot)\|_{\mathbf{vec}} = (\|x_1(\cdot)\|, \ldots, \|x_k(\cdot)\|)^T \in \mathbb{R}^k_+$. We need another symbol to indicate the stacking of several $\|x_i\|_{\mathbf{vec}}$. To be able to see at first glance, of which elements a symbol consists, we decided to use $\|x(\cdot)\|_{\mathbf{stc}} = (\|x_1(\cdot)\|^T_{\mathbf{vec}}, \ldots, \|x_n(\cdot)\|^T_{\mathbf{vec}})^T$ to indicate a vector which itself consists of vectors of norms ($|x|_{\mathbf{stc}}$ is defined in an obvious manner).

1.3 Input to State Stability and Related Notions

Linear systems share particular nice features. For instance if a linear system is internally stable, then all trajectories are bounded, provided that the input is bounded. Of course, this does not hold, in general, for nonlinear systems. Sontag developed a notion of stability, which tries to capture these nice properties of linear systems. For an introduction in a tutorial fashion see [Son08]. If not otherwise mentioned, the definitions found in this section are taken from [SW96]. Let $\mathcal{U} \subset \mathcal{L}^{loc}_\infty(\mathbb{R}_+, \mathbb{R}^m)$ with $\mathcal{L}^{loc}_\infty(\mathbb{R}_+, \mathbb{R}^m)$ the set of all locally essentially bounded functions with domain $\mathbb{R}_+$ and image $\mathbb{R}^m$. If $u \in \mathcal{U}$, we denote with $\|u\| := \operatorname{ess\,sup}_{t \geq 0} |u(t)|$ the essential supremum of u. Consider the system

$$\dot{x}(t) = f(x, u) \tag{1.12}$$

with $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$. We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and $f(0,0) = 0$.
It is also of interest to study discrete time systems

$$s(k+1) = g(s(k), w(k)) \tag{1.13}$$

with $s \in \mathbb{R}^n$ and $w(k) \in \mathbb{R}^m$ for all k. We assume that $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and that $g(0,0) = 0$.

Definition 1.3.1. We say system (1.12) is input-to-state stable (ISS) for $\mu \in \mathrm{MAF}_2$, if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t;u)| \le \mu(\beta(|x(0)|, t), \gamma(\|u\|))$$

for all $t \ge 0$, all $x(0) \in \mathbb{R}^n$ and all $u \in \mathcal{U}$.

Definition 1.3.2. We say system (1.13) is input-to-state stable (ISS) for $\mu \in \mathrm{MAF}_2$, if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|s(k;w)| \le \mu(\beta(|s(0)|, k), \gamma(\|w\|))$$

for all $k \in \mathbb{N}$, all $s(0) \in \mathbb{R}^n$ and all $w : \mathbb{N} \to \mathbb{R}^m$.

Usually, the definition for ISS is given with $\mu = |\cdot|_1$ or $\mu = |\cdot|_\infty$.
We will see later in this section that qualitatively the choice of μ does not matter i.e., if a certain ISS property holds for a given $\mu \in \mathrm{MAF}$ then it holds also for any other MAF, albeit with different gains.
The definition of ISS states that a trajectory can be bounded by two terms. The first corresponds to the initial condition and the other to the energy that is put into the system. Moreover, the effect of the initial condition vanishes with time. Thus it describes the transient behavior as well as the asymptotic characteristics of the trajectory. These two behaviors can also be examined separately, which we will see next.

Definition 1.3.3. System (1.12) is said to have the Asymptotic Gain (AG) property, if there exist $\gamma \in \mathcal{K}$ such that

$$\limsup_{t\to\infty} |x(t;u)| \le \gamma(\|u\|)$$

for all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{U}$.

Definition 1.3.4. System (1.12) is said to have the Global Stability (GS) property for $\mu \in \mathrm{MAF}_2$, if there exist $\beta, \gamma \in \mathcal{K}$ such that

$$\sup_{t\ge 0} |x(t;u)| \le \mu(\beta(|x(0)|), \gamma(\|u\|))$$

for all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{U}$.

Of course, there exist the corresponding discrete counterparts, which we present for reasons of convenience.

Definition 1.3.5. System (1.13) is said to have the Asymptotic Gain (AG) property, if there exist $\gamma \in \mathcal{K}$ such that

$$\limsup_{k\to\infty} |s(k;w)| \leq \gamma(\|w\|)$$

for all $s(0) \in \mathbb{R}^n$ and $w : \mathbb{N} \to \mathbb{R}^m$.

Definition 1.3.6. System (1.13) is said to have the Global Stability (GS) property for $\mu \in \mathrm{MAF}_2$, if there exist $\beta, \gamma \in \mathcal{K}$ such that

$$\sup_{k\geq 0} |s(k;w)| \leq \mu(\beta(|s(0)|), \gamma(\|w\|)$$

for all $s(0) \in \mathbb{R}^n$ and $w : \mathbb{N} \to \mathbb{R}^m$.

There is a stronger notion of AG where it is assumed that the lim sup in Definition 1.3.5 is attained uniformly with respect to initial states and all w. Before we cite a result that this stronger notion is equivalent to ISS we give a precise definition.

Definition 1.3.7. System (1.13) is said to have the Uniform Asymptotic Gain (UAG) property, if there exists a $\gamma \in \mathcal{K}$ such that for each $\varepsilon, \nu > 0$ there is a $T = T(\varepsilon, \nu) \in \mathbb{N}$ such that

$$\sup_{k\geq T} |s(k;w)| \leq \gamma(\|w\|) + \varepsilon$$

for all $|s(0)| \leq \nu$, all $w : \mathbb{N} \to \mathbb{R}^m$, and all $k \geq T$.

The next lemma relates the UAG property to the ISS property. It is taken from [GL00, Theorem 2] where a proof can be found.

Lemma 1.3.8. *System* (1.13) *is ISS if and only if it has the UAG property.*

As we can always upper bound a class $\mathcal{K}$ function by a class $\mathcal{K}_\infty$ function, we can replace $\gamma \in \mathcal{K}$ with $\gamma \in \mathcal{K}_\infty$ in all of the above definitions.
Note that the definitions from [SW95] are formulated for $\mu = |\cdot|_1$. The next lemma relates this to our definitions.

Lemma 1.3.9. *If a system or a discrete time system is ISS or has the AG or GS property for* $\mu_1 \in \mathrm{MAF}$*, then it shares the same property for any* $\mu_2 \in \mathrm{MAF}$.

Proof. This is a direct consequence of Corollary 1.1.7. □

In general, the gains change by going from one MAF to another. By Lemma 1.3.9 we see that, qualitatively the choice of $\mu \in$ MAF does not play a role. Hence we will say a system is ISS and omit the dependency on $\mu \in$ MAF from now on.

Lemma 1.3.10. *A system or a discrete time system is ISS if and only if it is AG and GS.*

A proof for continuous time can be found in [SW96] and for the discrete counter-part in [JW01].
Similar to the case of an uncontrolled system, the notion of Lyapunov functions play an important role within the ISS framework as well.

Definition 1.3.11. A differentiable, positive definite, and proper function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an ISS-Lyapunov function for system (1.12) if there exists $\gamma \in \mathcal{K}_\infty$ and positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V(x) \geq \gamma(|u|)$$

implies

$$\nabla V(x) f(x,u) \leq \alpha(|x|)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

The corresponding version for discrete time systems reads as follows.

Definition 1.3.12. A continuous, positive definite, and proper function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an ISS-Lyapunov function for system (1.13) if there exists $\gamma \in \mathcal{K}_\infty$ and positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V(x) \geq \gamma(|u|)$$

implies

$$V(g(x,u)) - V(x) \leq -\alpha(|x|)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

As in the last section there is a strong relation between Lyapunov characterization and stability.

Theorem 1.3.13. *There exists an ISS-Lyapunov function for* (1.12) *or* (1.13), *if and only if* (1.12) *respectively* (1.13) *is ISS.*

A proof can be found in e.g., [SW95] for continuous time and in [JW01] for the discrete case.
In Chapter 3 we deal with delay systems. Hence we need another stability notion, which is taken from [PMTL09].

Definition 1.3.14. A system of the form (1.11) is input-to-output-practically-stable (IOpS) at $t = t_0$ with $t_d(t) \geq 0$, $\beta \in \mathcal{K}_\infty^{r\times 1}$, IOpS gains $\Gamma \in \mathcal{G}^{r\times l}$, restrictions $\Delta_x \in \mathbb{R}_+$, $\Delta_u \in \mathbb{R}_+^l$ and offset $\delta \in \mathbb{R}_+^r$ if the conditions $\|x_d(t_0)\| \leq \Delta_x$ and $\sup_{t\geq t_0} u_d^+ \leq \Delta_u$, imply that solutions of (1.11) are well-defined for $t \geq t_0$ and the following inequalities hold:

$$\sup_{t\geq t_0} |y(t)|_{\mathbf{stc}} \leq \mu\left(\beta(\|x_{\mathbf{d}}(t_0)\|), \Gamma(\sup_{t\geq t_0} \|u_{\mathbf{d}}\|_{\mathbf{stc}}), \delta\right)$$

and

$$\limsup_{t\to\infty} |y(t)|_{\mathbf{stc}} \leq \mu\left(\Gamma(\limsup_{t\to\infty} \|u_{\mathbf{d}}\|_{\mathbf{stc}}), \delta\right) .$$

The first inequality resembles the GS property, while the second can be regarded as an AG type estimate. Note that in the context of infinite dimensional systems it is unknown yet whether AG together with GS is equivalent to ISS. So strictly speaking, the terminology IOpS for Definition 1.3.14 is not correct. For the sake of simplicity we decided to stick to the name.

1.4 Notes and References

Most of the notations and definitions used here are standard with a few exceptions. In particular the notion of a monotone aggregation function was introduced in [Rüf07].
We also use the order relation in a slightly different manner. In lattice theory the symbol $x \ll y$ is used to denote that $x_i < y_i$ in every component. And $x < y$ is used to state that $x_i \leq y_i$ in every component and $x_j < y_j$ for at least one component. As we do not need these difference more than once, we decided to use the notation that we believe is nearer to the dynamical system community.
The notation $\mathcal{K}$, $\mathcal{K}_\infty$ respectively $\mathcal{KL}$ was introduced by Hahn in [Hah59, Hah67]. Hahn did not explain why he chose the particular naming, but according to rumors it is in honor of Kamke (see [Grü02]).
Our definition of dynamical systems is taken from [HP05]. We changed the naming to fit more our needs though.
We decided to solely consider Carathèodory solutions, because any usual solution (in the sense of Peano) is a Carathèodory solution and we will deal

with discontinuous right hand sides in some of the chapters, for which Peano's Theorem is not applicable.
The definitions from Section 1.3 are taken from [SW96] with one exception. Definition 1.3.14 is taken from [PMTL09].
Eduardo Sontag introduced the ISS notion in [Son89]. For an excellent starting point into the ISS framework see [Son08].
The definition and notation for time delay systems is somewhat a mixture of [PMTL09, Hal77, Tee98]. The interested reader is referred to [Hal77] for a detailed introduction to time delay systems.

Chapter 2

Stabilizing a Single System over a Communication Channel

Before we start considering large-scale systems, it is of interest how to stabilize a single system over communication channels. In this chapter we focus on digital communication channels. In a digital channel the information we want to transmit is translated, respectively, encoded into a binary representation.
We are dealing with communication channels of finite capacity, i.e. we cannot transmit an infinite amount of information in a finite amount of time.
In general, transmitting a single real number demands for an infinite amount of information respectively binary digits. Therefore limiting the capacity of the communication channel poses some problems due to possible rounding errors. In the context of networked control systems the effect of the presence of rounding errors is known as quantization.
Moreover, in any physical implementation of a communication channel a delay is present. Delay describes the effect that information that was sent at some time is received some time later on the other side of the communication channel.
Information can even get lost, which means that information sent by one side is never received by the other side of the communication channel. This effect is known as packet loss.
In this chapter we discuss certain ways to deal with the limitations posed by the communication. In detail, we introduce an approach to deal with quantization, delay, and packet loss.
One of the key tools used in this chapter is called *dynamic quantization*. See Section 2.2 respectively Remark 2.3.6 for a discussion.
The approach presented in this chapter requires more specific assumptions

on the communication channel than those discussed in the introduction.
We start in the next section by introducing the system setup under consideration.

2.1 System Description and Further Assumptions

This chapter is in two aspects different from the rest of the thesis. First we only deal with a single system and secondly we are dealing with a more specific situation as far as the communication channel is concerned.

Problem Setup

In Figure 2.1 the setup under consideration is depicted. The devices behind these boxes will be introduced here. The communication channel, depicted as a cloud, is introduced later in this section.

The Plant

We consider systems of the form

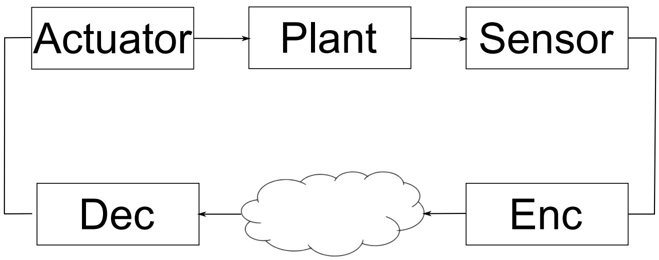

Figure 2.1: Setup of the closed-loop system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m \,, \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz in the first component uniformly with respect to u, i.e.,

$$|f(x,u) - f(y,u)| \leq L|x-y|, \; \forall x, y \in \mathbb{R}^n \,, \forall u \in U \,, \tag{2.2}$$

where $|\cdot|$ is an arbitrary norm.
Throughout this thesis the ISS framework is the key tool to achieve certain stability properties. In this regard, we assume the following.

Assumption 2.1.1. There exists a smooth $k : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto k(x)$ with $k(0) = 0$ such that

$$\dot{x}(t) = f(x(t), k(x(t) + e_d(t))) \tag{2.3}$$

is ISS with respect to a measurement error e_d. Note that this is equivalent (see Definition 1.3.1) to the existence of functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ so that the solutions of (2.3) satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma\left(\sup_{s \in [t_0, t]} |e_d(s)|\right), \quad \forall t \geq t_0\,. \tag{2.4}$$

Note that Assumption 2.1.1 states that if $e_d \equiv 0$, the controller k globally asymptotically stabilizes the unique fixed point $x^* = 0$.

Sensor

The sensor is a measuring device, capable of measuring the state at arbitrary time instances. Furthermore, we assume there are no measurement errors. Although Assumption 2.1.1 allows modeling sensor errors by e.g., $e_d = e_m + e_q$, where e_m is the measurement error due to the faulty sensor and e_q is the quantization error, we decided to neglect sensor errors to ease the presentation. In general we would not be able to conclude asymptotic stability of the presented approach by considering non-vanishing sensor error. For an explanation see Section 2.2.

Encoder

The encoder is the device that prepares the information in a suitable way such that it can be transmitted over the communication channel. To do so it uses an approximation of the state $\hat{x}_e \in \mathbb{R}^n$. As the encoder and the decoder are the heart of this chapter, they are introduced in more detail in Section 2.3.

Decoder

The decoder sits on the other end of the communication channel and translates respectively decodes the information that was encoded by the encoder and transmitted over the communication channel. In this regard, the decoder inverts the encoding procedure done by the encoder. It has a model of the plant and the equation is given by

$$\dot{\hat{x}}_d = f(\hat{x}_d, u_d)\,, \tag{2.5}$$

with $\hat{x}_d \in \mathbb{R}^n$ and $u_d = k(\hat{x}_d)$. The decoder is initialized to $\hat{x}_d(0) = 0$. Note that we assume throughout this chapter that the clocks of encoder and decoder are synchronized.

Actuator

The actuator takes the signal generated by the decoder and closes the loop with the help of this signal.

Further Assumptions on the Communication Channel

Here we give the specific assumptions on the communication channel. In particular, we consider TCP like packet based transmissions over a noiseless, error free channel with delay and packet loss. For an introduction on TCP and similar protocols see [Ste93]. Basically, information will be gathered into packets and sent through the channel. If a packet is received on the other side of the channel, the receiving side sends an acknowledgment back to the sender to signal that the information was received correctly.
The encoder encodes the state and sends a symbol from a finite alphabet (here, a finite alphabet is a finite set given by $\mathcal{S} := \{s_1, s_2, \ldots, s_m\}$, $s_i \in \mathbb{Z}$ for $i = 1, \ldots, m$) to the decoder together with the time when the state was encoded (time stamping). As soon as a packet arrives at the decoder, it reconstructs the encoded state and sends an acknowledgment (ack) back to the encoder. This ack is also time stamped. If an ack arrives at the encoder or a predefined time (called τ_{max}) elapses without receiving one, it repeats the encoding with the actual state. Let t_k, t_k^* for $k \in \mathbb{N}$ be series of time instances. We say that t_k, t_k^* are encoder-decoder time sequences, if t_k is the kth time instance the encoder received an ack (with one important exception: The first time the encoder sends an information is without receiving an ack at time $t_1 = 0$) and t_k^* is the time when the kth information sent by the encoder is received by the decoder. Note that we assume that there is no time delay between the arrival of an information and the sending of the next packet i.e. t_k and t_k^* are also the time instances when the encoder sends information and the decoder sends an ack respectively, provided that no loss occurred.
The aforementioned procedure is summarized in Figure 2.2. The sending of packets is depicted as solid lines whereas the ack's are given as dotted lines. Note that the packet sent at t_3 is lost and a new packet is sent at $t_3 + \tau_{max}$.
To be able to stabilize the system, information sent by the encoder should be received by the decoder at least sometimes. The next assumption quantifies this precisely.

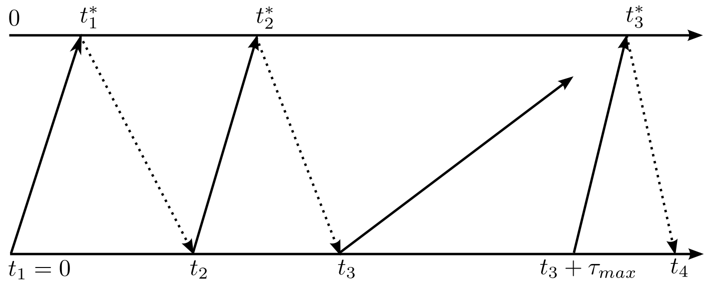

Figure 2.2: Time instances of encoder and decoder

Assumption 2.1.2. There exists a long time average of the difference $t_k - t_{k-1}$. This average is given by

$$\tau^* = \limsup_{k\to\infty} \frac{1}{k} \sum_{j=2}^{k} (t_j - t_{j-1}) = \limsup_{k\to\infty} \frac{1}{k} t_k. \tag{2.6}$$

Assumption 2.1.2 states that on average on an infinite time horizon, every τ^* units of time a packet will be successfully acknowledged. The last equation holds, because we conventionally set $t_1 = 0$, as described before. Note that the existence of such a τ^* does not hold in general despite τ_{max}, as t_k are the time instances when transmission is successful. In other words: $t_k - t_{k-1} > \tau_{max}$ for some k because of possible packet loss.
The next statement shows that also every τ^* units of time the decoder receives a packet on average.

Lemma 2.1.3. *If Assumption 2.1.2 holds, we also have*

$$\tau^* = \limsup_{k\to\infty} \frac{1}{k} t_k^* \,.$$

Proof. Note that by definition we have $t_k \le t_k^* \le t_{k+1}$. Hence by using (2.6) we get

$$\tau^* = \limsup_{k\to\infty} \frac{1}{k} t_k \le \limsup_{k\to\infty} \frac{1}{k} t_k^* \le \limsup_{k\to\infty} \frac{1}{k} t_{k+1} \,.$$

It remains to show that the right hand side of the inequality also converges to τ^*. To this end consider

$$\limsup_{k\to\infty} \frac{1}{k} t_{k+1} = \lim_{k\to\infty} \sup_{l\ge k} \frac{1}{l} t_{l+1} = \lim_{k\to\infty} \frac{k}{k+1} \sup_{l\ge k} \frac{1}{l} t_{l+1} \le$$

$$\lim_{k\to\infty} \sup_{l\ge k} \frac{l}{l+1} \frac{1}{l} t_{l+1} = \tau^* \,.$$

And we have shown the claim. □

Next we give a summary of the properties of the communication channel.

Summary 2.1.4. For the communication channel the following should hold:

1. All packets are time stamped with the current time they are sent.
2. Only packets sent from encoder to decoder are lost.
3. There exists a minimal delay from encoder to decoder, given by τ_{min}, i.e., $t_k^* - t_k \geq \tau_{min}$ and $t_k - t_{k-1}^* \geq \tau_{min}$.
4. The channel is able to transmit packets containing a value from a set of N^n discrete values within τ_{min} units of time.
5. If τ_{max} time elapses without receiving an ack, the packet sent last time is considered lost and a new packet will be sent.

By Summary 2.1.4 (1) we have to send the actual time together with the encoded state information. It is not reasonable to transmit the state information quantized, while the time information is transmitted with arbitrary accuracy. For the sake of simplicity we omit details on time quantization, see [SW09] for a discussion.
Summary 2.1.4 (2) is a major restriction on the channel used. But because the ack's are much smaller than the state information the decoder could send many ack's to ensure that at least one arrives at the encoder. Without this assumption we could not guarantee that the encoder and the decoder agree on their states by means of a simple time stamping mechanism.
Summary 2.1.4 (3) is in general not a restrictive one. In every real communication channel such a minimal delay exists.
Summary 2.1.4 (4) states that the bandwidth of the channel B must be large enough to transmit the state information within τ_{min} units of time. For instance, if binary encoding is used we require

$$B \geq \frac{n \log_2 N}{\tau_{min}} \,. \tag{2.7}$$

If this condition is not met, the decoder could introduce an artificial delay by waiting to ensure that τ_{min} is large enough to fulfill the bandwidth constraint.
Because we do not have a mechanism to detect packet loss, we introduce the design parameter τ_{max} in 2.1.4 (5) to be able to handle the effect of packet loss.

The design parameters τ_{min} and τ_{max} steer the trade-off between the performance of the system and the requirement of bandwidth. For instance, choosing τ_{max} to large can deteriorate the performance of the closed loop system, while choosing it too small can lead to too many unneeded retransmission of data.
We say that the encoder-decoder time sequences t_k, t_k^* are admissible for the encoder-decoder pair, if Assumption 2.1.2 and Summary 2.1.4.3 holds.
After describing the communication channel in detail, we are able to introduce the basic concept that will be used to overcome the limitation posed by quantization in the next section.

2.2 Quantization

The basic tools used in this chapter are input to state stability and dynamic quantization. Before we give a definition of a quantizer we want to start with briefly discussing static quantizers.

Definition 2.2.1. A map $q : \mathbb{R}^n \to \mathbb{R}^n$ is called a *static quantizer*, if there exists $M, m \in \mathbb{R}_+$ with $m < M$ such that

$$|q(x) - x| \leq m$$

whenever $|x| \leq M$, $x \in \mathbb{R}^n$.

The quantity M is often referred to as range of the quantizer, whereas m is termed the resolution. If the state x is in a certain region (i.e. the quantization region), we say the state lies within the range of the quantizer. The definition above states that whenever the state lies within the quantization region, the quantization error $q(x) - x$ is bounded by the resolution of the quantizer.
In Figure 2.3 an example of a static quantizer is given. Here, the quantization region is a hypercube of length $2M$. It is partitioned into smaller hypercubes of length $2m$ each. We will refer to these smaller hypercubes as subregions. In this regard, the range of the quantizer is M, whereas the resolution is m. Whenever the state x lies within the range of the quantizer, the quantizer determines the subregion in which the state lies and gives as a result the center of this smaller hypercube.
A static quantizer has a fixed resolution and hence a fixed guaranteed error. As we want to use the quantized value to close the loop, we cannot hope to achieve asymptotic stability, in general, by using a static quantizer (see [Del90] for a corresponding statement). Therefore we generalize the concept to quantizers in which the parameters of the quantizer can change over time, as can be seen in the next definition.

Definition 2.2.2. A map $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is called a *quantizer*, if there exist $M, m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) < M(t)$ for all t such that

$$|q(x, \hat{x}, t) - x| \leq m(t)$$

whenever $|x - \hat{x}| \leq M(t)$.
Here x is the value to be quantized, $\hat{x}$ is the center of the quantization region, M is the size of the quantization region, and m is the resolution.

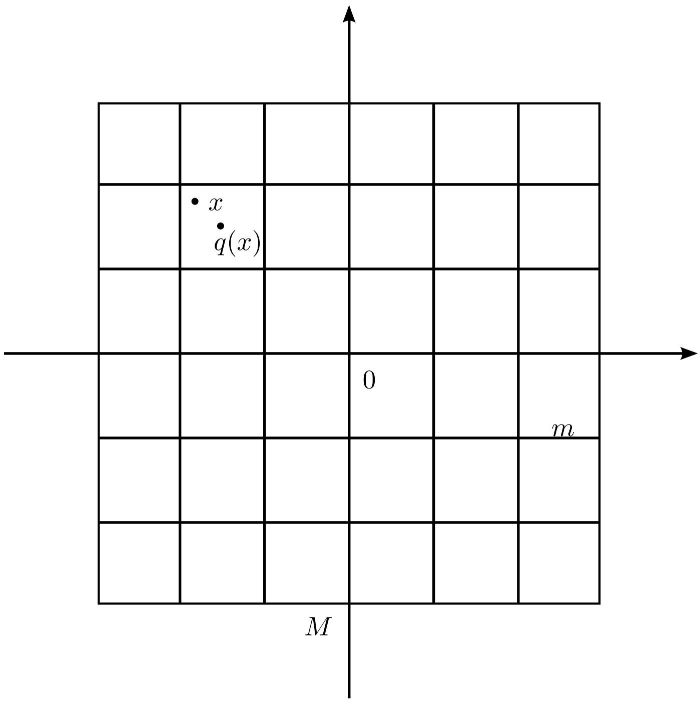

Figure 2.3: Example of a finite quantizer with range M and resolution m

The definition above states that whenever the state lies within the range of the quantizer ($|x - \hat{x}| \leq M(t)$) the quantized value differs at most $m(t)$ from the state x. In contrast to the finite quantizer the center, range, and the resolution can change over time.
For an example see Figure 2.4. In this particular example the effect of the quantizer is depicted for the parameters of the quantizer at two different times t_1, t_2. In both cases the quantization region is given by a box centered

at $\hat{x}$ with size $2M$. This box is divided into subregions, each of size $2m$. The quantizer determines the subregion in which the state x lies and gives the center of this subregion as the result of the quantization. Between t_1 and t_2 the center $\hat{x}$ and the range M changes. In this example the state at time t_2 still lies within the range of the quantizer. Of course, this does not hold automatically. We will see in the next section how we can ensure that the state always lies within the range of the quantizer.

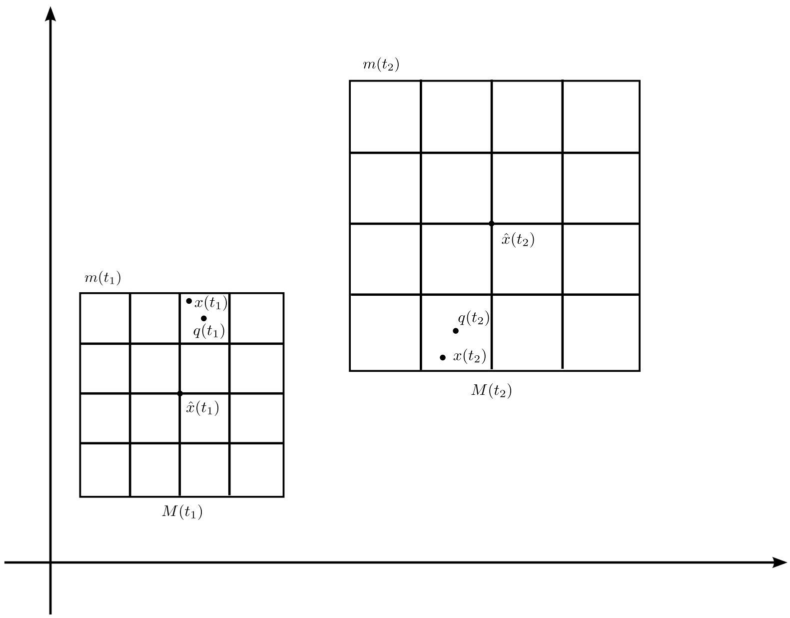

Figure 2.4: Schematic of a dynamic quantizer at two different time instances

2.3 Encoder-Decoder Pair and Description of the Closed-Loop System

We have learned in the last section that a quantizer is a map that gives an approximation of the state with a guaranteed bound on the error. However, the quantized values are still real numbers and hence they cannot be easily transmitted via a communication channel with finite capacity.
In Figure 2.3 and 2.4 we have chosen a special quantizer, in which the subregions partition the quantization region. This suggests that we can choose

a quantizer in such a way that we can assign to each subregion a certain symbol from our alphabet.
This would require some device that translates the real valued number to a symbol from our alphabet. If the capacity of the communication channel is large enough this symbol can be transmitted. On the other side of the communication channel we need also a new device, which inverts the translation into the finite alphabet.
The next definition, which labels these two devices, is borrowed from [TM04].

Definition 2.3.1. An *encoder* is a dynamical system and its output map at time t is given by

$$Enc\ :\ \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathcal{S} \times \mathbb{R}_+\ :\ Enc(x, \mathrm{ack}, \Xi_e(t), t) \mapsto (s, t_k)\,,$$

where $\mathcal{S}$ is a finite alphabet. A *decoder* is a dynamical system and its output map at time t is given by

$$Dec\ :\ \mathcal{S} \times \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n\ :\ Dec(s, t_k, \Xi_d(t), t) \mapsto \hat{x}_d(t)\,.$$

The values Ξ_e and Ξ_d are the internal states of the encoder respectively decoder. The variable ack is the time stamp of the ack.
The values x, ack are the inputs and s, t_k the outputs. For the decoder, on the other hand, s, t_k act as an input and $\hat{x}_d$ is the output.
Encoder and decoder together are called an encoder-decoder pair.
An encoder-decoder pair should be able to encode the state at time t_k in such a way that the decoder is able to reconstruct at least an approximation of the state. We will see that to achieve this goal it is important that encoder and decoder agree on their internal states.

Definition 2.3.2. If for an encoder-decoder pair

$$\Xi_e(t_k) = \Xi_d(t_k^*)$$

holds for all $k \geq 0$, the encoder-decoder pair is called information consistent.

An information consistent encoder-decoder pair has the property that at the encoding time t_k the internal state of the encoder is the same as the decoder internal state at time t_k^*. If there is no delay in the communication channel, information consistency is easily achieved by initializing encoder and decoder to the same values and let them follow the same dynamics (see [LH05]). However, if there is delay in the communication channel, information consistency does not hold automatically. An example how this consistency property can be achieved despite the presence of delay is given

in Section 2.5.
Now we are able to give the equations for the closed-loop system:

$$\dot{x}(t) = f(x(t), k(\hat{x}_d(t))), \qquad t \neq t_k^* \tag{2.8}$$

$$\dot{\hat{x}}_d(t) = f(\hat{x}_d(t), k(\hat{x}_d(t))), \qquad t \neq t_k^* \tag{2.9}$$

$$\hat{x}_e(t) = Dec\Big(Enc\big(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k\big), \Xi_e(t_k), t_k \Big), \quad t = t_k \tag{2.10}$$

$$\hat{x}_d(t) = Dec\Big(Enc\big(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k\big), \Xi_d(t_k^*), t_k^* \Big), \quad t = t_k^*. \tag{2.11}$$

The initial values are $x(0) = x_0$ and $\hat{x}_d(0) = \hat{x}_e(0) = 0$.
The first equation describes the dynamics of the plant. The control action is calculated using the decoder trajectory, which is given by (2.9). The last two equations describe the jumps in the encoder respectively decoder trajectories. To be more precise, at time t_k the encoder uses the actual state x to generate an approximation of it. This approximation is translated to a finite alphabet and transmitted over the channel. At time t_k^* this information is received by the decoder and used to approximate the state at that time.
The closed-loop system consists of continuous dynamics and jumps in the state $\hat{x}_e$ and $\hat{x}_d$ at time instances t_k respectively t_k^*. A system consisting of discrete and continuous dynamics is called a hybrid system. For an introduction on hybrid systems see [vdSS00, GST09] and the references therein.
Before we formulate the main result of this chapter we need further assumptions for encoder and decoder.

Assumption 2.3.3. Encoder and decoder have *a priori* knowledge of the initial state $x(0)$ of the system i.e., there exist $m_0 \in \mathbb{R}_+$, known to encoder and decoder such that

$$|x(0)| \leq m_0 \,. \tag{2.12}$$

The assumption above states that encoder and decoder agree on a certain region in which the initial state lies. The size of this region can be interpreted as the level of uncertainty of the encoder where the state lies.
We want the state x to be confined within a certain region for all positive times. Moreover, we want this region to become smaller with each transmission. This property is given in detail in the next definition.

Definition 2.3.4. Let L be the Lipschitz constant of f and $N \in \mathbb{N}$ with $N > 1$. Consider an arbitrary solution $x, \hat{x}_e$ of (2.8)-(2.11). Define $m(0) := m_0$ and let $m(t_k) := m(t_{k-1}) e^{L(t_k - t_{k-1})}/N$. If

$$|\hat{x}_e(t_{k-1}) - x(t_{k-1})| \leq m(t_{k-1})$$

implies

$$|\hat{x}_e(t_k) - x(t_k)| \leq m(t_k)$$

for all $k \in \mathbb{N}$, the encoder is called N-contracting. Furthermore $m(t_k)$ is called the range of the encoder.

Similar to quantizers, $m(t)$ is the range and $\hat{x}_e$ may be regarded as the center of the quantization region.
Definition 2.3.4 states that if the state lies within the range of the encoder at time t_{k-1} it will stay within the range for all t_j for $j > k-1$. Moreover, the range is divided by N at each time instance t_k and growth by $e^{L(t_k - t_{k-1})}$.
The difficulty in achieving this is that the actuator signal for the system is calculated using the decoder trajectory. As the only information the encoder receives from the decoder is a time stamp, the N-contracting property does not hold automatically.
Now that we know that the state lies within range for all positive times and this range is divided by N with each encoding step, we need a way to describe how the decoder approximation behaves.

Definition 2.3.5. Let L be the Lipschitz constant of f and t_k, t_k^* admissible encoder-decoder time sequences. If for any solution $x, \hat{x}_e, \hat{x}_d$ of (2.8)-(2.11)

$$|Dec(Enc(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k), \Xi_e(t_k), t_k) - x(t_k)| \leq \\ |\hat{x}_e(t_{k-1}) - x(t_{k-1})| e^{L(t_k - t_{k-1})} \quad (2.13)$$

and

$$|Dec(Enc(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k), \Xi_e(t_k), t_k^*) - x(t_k^*)| \leq \\ |Dec(Enc(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k), \Xi_e(t_k), t_k) - x(t_k)| e^{L(t_k^* - t_k)}, \quad (2.14)$$

for all $k \in \mathbb{N}$ hold, the encoder-decoder pair is called L-expanding.

Equation (2.13) states that the encoder approximation can at most grow by $e^{L(t_k - t_{k-1})}$ in between transmission times.
The second assertion says that if the decoding is done with the same internal state as the encoding, the decoder approximation can only deviate from the encoder approximation by a factor of $e^{L(t_k^* - t_k)}$.
For an intuition see Figure 2.5. If the state at time t_k lies within the range of the encoder (i.e. $|\hat{x}_e(t_k) - x(t_k)| \leq m(t_k)$), the encoder determines the subregion in which the state lies and transmits the corresponding symbol from the finite alphabet (here s_3) together with the time t_k to the decoder. By the jump from $\hat{x}_e(t_k)$ to the center of the subregion the error gets divided

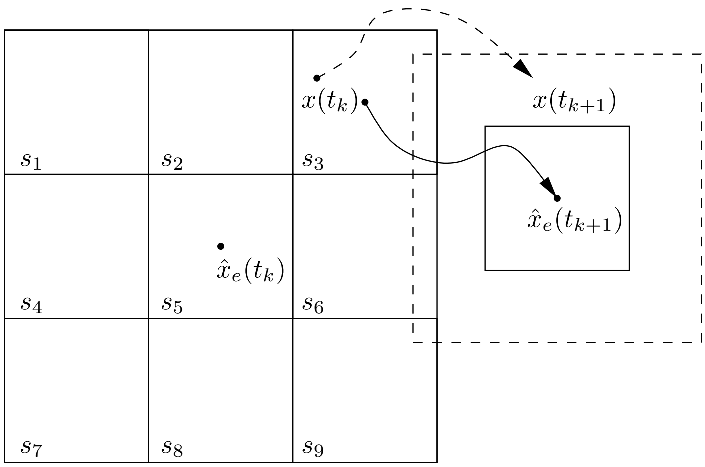

Figure 2.5: Example of a N-contracting encoder

by N (in this particular example $N = 3$). The N-contracting property states that if we enlarge the subregion by $e^{L(t_k - t_{k-1})}$ the state at time t_{k+1} still lies in a quantization region with $\hat{x}_e(t_{k+1})$ as the new center.

Remark 2.3.6. Inspecting Definition 2.3.5 reveals that the concatenation of the output maps of encoder and decoder are closely related to the definition of a quantizer. While a quantizer is time varying, encoder as well as decoder are dynamical systems. To be more precise, the center and range of a quantizer can change over time, while the evolution of center and range of encoder resp. decoder are governed by a dynamical system.
In this regard, an encoder-decoder pair is often referred to as a dynamic quantizer in the literature.

Before we can state the main result of this chapter, we want to define the particular notion of stability we are interested in. As our system is a hybrid system consisting of states that evolves only discrete in time and states that are governed by ordinary differential equations, we need another notion of stability as the one defined in Section 1.

Definition 2.3.7. Let $e = \hat{x}_d - x$ and $e_e = \hat{x}_e - x$. We say that a system of the form (2.8)-(2.11) is semi globally asymptotically stable at $t_1 = 0$ if $(x, e, e_e) = 0$ is an asymptotically stable equilibrium at $t_1 = 0$ for (2.8)-(2.11), provided that $\hat{x}_d(0) = \hat{x}_e(0) = 0$ and $|x(0)| \leq m_0$.

The term semi globally refers to the fact that the initial conditions has to lie in some compact set (see Assumption 2.3.3). In this regard we have "more" than local stability but less than global stability. At first glance system (2.8)-(2.11) seems to be time invariant. Because in general the evolution of the t_k and t_k^* are time variant, the evolution of (2.8)-(2.11) depends also on t_1.

Remark 2.3.8. By the change of coordinates

$$\begin{pmatrix} x \\ e \\ e_e \end{pmatrix} = \begin{pmatrix} I_n & 0 & 0 \\ -I_n & I_n & 0 \\ -I_n & 0 & I_n \end{pmatrix} \begin{pmatrix} x \\ \hat{x}_d \\ \hat{x}_e \end{pmatrix},$$

where I_n is the identity matrix of dimension n, we see that Definition 2.3.7 is equivalent to saying that $x, \hat{x}_e, \hat{x}_d$ are asymptotically stable at $t_1 = 0$, again provided that $x_d(0) = x_e(0) = 0$ and $|x(0)| \leq m_0$.

Now we are able to state the main result of this chapter.

2.4 Main Result: Stabilization of a Single System over a Digital Channel

Here, we will see that as long as encoder and decoder agree on certain values (i.e, information consistency) and the state always lies within the range of the encoder-decoder pair, the approximation of the state $\hat{x}_d$ becomes better by a factor of N with each information that arrives at the decoder. The next theorem gives a sufficient condition on the size of the finite alphabet needed to ensure that the range will converge to zero. In this regard, the next theorem relates a system property (the Lipschitz constant) to the capacity of the communication channel (the size of the alphabet). Basically, the input consistency together with L-expanding property allows us to bound the error between the encoder and the state with the help of the Gronwall inequality and the N-contracting property ensures that the error is divided by N with each successful transmission.

Theorem 2.4.1. *Consider system* (2.8)-(2.11) *with initial condition as before, which communicates over a digital channel with the properties given in Summary 2.1.4. Let* t_k, t_k^* *be admissible encoder-decoder time sequences as described in Section 2.1. Let Assumptions 2.1.1, 2.1.2, and 2.3.3 hold. If for an information consistent, L-expanding, and N-contracting encoder-decoder pair it holds that* $N > e^{L\tau^*}$, *where* L *is the Lipschitz constant for system* (2.3), *then the closed-loop system* (2.8)-(2.11) *is semi globally asymptotically stable at* $t_1 = 0$.

Proof. First we show that the range converges to zero. To this end consider $\tau^* = \limsup_{k\to\infty} \frac{1}{k} t_k$. Choose an $\varepsilon > 0$ such that $e^{L(\tau^*+\varepsilon)} < N$, which is possible because $e^{L\tau^*} < N$. This choice yields a $K \in \mathbb{N}$ such that $e^{L\frac{1}{k}t_k} < e^{L(\tau^*+\varepsilon)} < N$ for all $k > K$ and hence

$$\lim_{k\to\infty} m_0 e^{L(t_k - t_1)}/N^k = \lim_{k\to\infty} m_0 \left(\frac{e^{L\frac{1}{k}t_k}}{N} \right)^k = 0\,. \tag{2.15}$$

By Assumption 2.3.3, the initialization of the encoder, and $t_1 = 0$ we have

$$|\hat{x}_e(t_1) - x(t_1)| = |0 - x(t_1)| \leq m_0 = m(t_1)\,.$$

Hence by (2.13) together with (2.10) we get

$$|\hat{x}_e(t_k) - x(t_k)| \leq m(t_k) \tag{2.16}$$

for all $k \in \mathbb{N}$.
By information consistency we have $\Xi_e(t_k) = \Xi_e(t_k^*)$. Hence we can combine (2.14) with (2.11) to arrive at

$$|e(t_k^*)| = |\hat{x}_d(t_k^*) - x(t_k^*)| \leq |\hat{x}_e(t_k) - x(t_k)| e^{L(t_k^* - t_k)} \leq m(t_k) e^{L(t_k^* - t_k)}\,. \tag{2.17}$$

The evolution of $|e(t)|$ for $t \in [t_{k-1}^*, t_k^*)$ can be bounded with the help of the Lipschitz property of f and the triangle inequality by

$$\begin{aligned} &|e(t)| = |\hat{x}_d(t) - x(t)| = \\ &\left| \hat{x}_d(t_{k-1}^*) - x(t_{k-1}^*) + \int_{t_{k-1}^*}^{t} f(\hat{x}_d(s), k(\hat{x}_d(s))) - f(x(s), k(\hat{x}_d(s))) ds \right| \leq \\ &\qquad\qquad |e(t_{k-1}^*)| + L \int_{t_{k-1}*}^{t} |e(s)| ds\,. \end{aligned}$$

Application of the Gronwall Lemma yields

$$|e(t)| \leq |e(t_{k-1}^*)| e^{L(t - t_{k-1}^*)}\,, \tag{2.18}$$

for $t \in [t_{k-1}^*, t_k^*)$. Because of (2.17) the latter results for all $t \in [t_{k-1}^*, t_k^*)$ in

$$|e(t)| \leq m(t_{k-1}) e^{L(t_{k-1}^* - t_{k-1})} e^{L(t - t_{k-1}^*)} = m(t_{k-1}) e^{L(t - t_{k-1})} = m_0 e^{Lt}/N^k.$$

where we used the definition of $m(t_k)$ to get the last equality.
By monotonicity of the exponential we have

$$|e(t)| \leq m_0 e^{Lt_k^*}/N^k \tag{2.19}$$

for all $t \in [t^*_{k-1}, t^*_k)$. And hence

$$\sup_{t\geq 0} |e(t)| \leq \sup_{k\in\mathbb{N}} m_0 e^{Lt^*_k}/N^k \,.$$

By Lemma 2.1.3 we have $\tau^* = \limsup_{k\to\infty} \frac{1}{k} t^*_k$. Hence, as before, we have

$$e^{Lt^*_k} < N^k$$

for all $k > K$. This together with (2.19) allow us to conclude

$$\lim_{t\to\infty} |e(t)| = 0 \,. \tag{2.20}$$

Moreover, we have

$$\sup_{t\geq 0} |e(t)| \leq \max_{k\leq K} m_0 e^{Lt^*_k}/N^k =: W \,.$$

Hence by (2.4) we get

$$|x(t)| \leq \beta(|x(0)|, 0) + \gamma(W) =: D$$

for all $t \geq 0$. Looking at trajectories starting at t_0 gives

$$|x(t)| \leq \beta(D, t - t_0) + \gamma(\sup_{s\in[t_0,t]} |e(s)|)$$

for all $t \geq t_0$. If we let t, t_0 tend to infinity in such a way that $t - t_0$ tends to infinity, we get

$$\lim_{t\to\infty} |x(t)| \leq \lim_{t-t_0\to\infty} \beta(D, t - t_0) + \gamma(\limsup_{t_0\to\infty} |e(t_0)|) \,.$$

As $\beta \in \mathcal{KL}$ and $e(t) \to 0$ by (2.20) we have $x(t) \to 0$. Considering (2.16) gives

$$\lim_{k\to\infty} |e_e(t_k)| = \lim_{k\to\infty} |\hat{x}_e(t_k) - x(t_k)| \leq \lim_{k\to\infty} m(t_k) = 0 \,.$$

And we have shown attractivity of $(x, e, e_e) = 0$. To conclude stability let $\varepsilon > 0$ be arbitrary and define $\varepsilon' := \gamma^{-1}(\frac{\varepsilon}{2})$. Choose a $k' \in \mathbb{N}$ such that for all $k \geq k'$

$$\frac{e^{Lt^*_k} m_0}{N^k} \leq \varepsilon' \,, \tag{2.21}$$

which is possible because of (2.15).
Choose an initial condition such that

$$|x(0)| \leq \frac{\varepsilon'}{e^{Lt^*_{k'}}} \,. \tag{2.22}$$

Note that combining (2.22) and (2.21) yields

$$|x(0)| \leq \frac{m_0}{N^{k'}}\,. \tag{2.23}$$

First we consider the case $k \geq k'$. By (2.19) and (2.21) we get

$$|e(t)| \leq m_0 e^{Lt_k^*}/N^k \leq \varepsilon' \tag{2.24}$$

for all $t \in [t_{k-1}^*, t_k^*)$. And thus

$$\sup_{t \geq t_{k'}^*} |e(t)| \leq \varepsilon'\,. \tag{2.25}$$

Now let $k < k'$. Combining (2.13) and (2.14) yields

$$|e(t_k^*)| = |\hat{x}_d(t_k^*) - x(t_k^*)| \leq$$
$$|\hat{x}_e(t_k) - x(t_k)|e^{L(t_k^* - t_k)} \leq |\hat{x}_e(t_{k-1}) - x(t_{k-1})|e^{L(t_k^* - t_{k-1})}\,.$$

Repetitively applying (2.13) gives

$$|e(t_k^*)| \leq |x_e(t_1) - x(t_1)|e^{Lt_k^*} = |e(0)|e^{Lt_k^*}$$

for all $k \in \mathbb{N}$. By (2.18) we get

$$|e(t)| \leq |e(0)|e^{Lt_k^*}e^{L(t-t_k^*)} = |e(0)|e^{Lt}\,.$$

for all $t > 0$. Using (2.22) we conclude

$$\sup_{0 \leq t < t_{k'}^*} |e(t)| \leq \sup_{0 \leq t < t_{k'}^*} |e(0)|e^{Lt} = \sup_{0 \leq t < t_{k'}^*} |x(0)|e^{Lt} \leq \varepsilon'\,. \tag{2.26}$$

Combining (2.26) with (2.25) yields

$$\sup_{t \geq 0} |e(t)| \leq \varepsilon'\,, \tag{2.27}$$

as long as $|e(0)| = |x(0)| \leq \frac{\varepsilon'}{e^{Lt_{k'}^*}}$. And we have shown stability of $e(t)$. Consider again (2.4). Because $\beta \in \mathcal{KL}$ we can choose $x(0)$ such that $\beta(|x(0)|, 0) \leq \varepsilon/2$ and $|x(0)| \leq \frac{\varepsilon'}{e^{Lt_{k'}^*}}$. And hence by (2.4)

$$|x(t)| \leq \beta(|x(0)|, 0) + \gamma(\sup_{t \geq 0} |e(t)|) \leq \frac{\varepsilon}{2} + \gamma(\varepsilon') \leq \varepsilon\,,$$

where we used (2.27) and the definition of ε'. And we can conclude stability of $x(t)$.

It remains to show that e_e is stable. Combining (2.10) with (2.13) repetitively yields

$$|e_e(t_k)| = |\hat{x}_e(t_k) - x(t_k)| \leq |\hat{x}_e(t_1) - x(t_1)|e^{Lt_k} = |e_e(0)|e^{Lt_k}$$

for all $k \in \mathbb{N}$. Combining this with the N-contracting property gives

$$|e_e(t_k)| \leq \min\{|e_e(0)|e^{Lt_k}, m(t_k)\}\,.$$

Because $m(t_k) \to 0$ there exists an index k' where the minimum changes. And thus we can make the right hand side arbitrarily small by choosing $e_e(0)$ small. This shows the stability of e_e. And in summary we have shown that $(x, e, e_e) = 0$ is an attractive and stable equilibrium in the sense of Definition 2.3.7 and the proof is complete. □

We have seen that we can infer stability of the closed-loop system, if the bandwidth of the communication channel is large enough despite the presence of delay and packet loss. The main technical assumptions that allow us to achieve this are N-contracting, L-expanding and information consistency. Information consistency states that the internal states of encoder and decoder coincide at certain time instances. The N-contracting property states that if the state lies in the range of the encoder at time t_1, it will remain within range for all t_k and this range converges to zero, if the bandwidth is large enough. The L-expanding property on the other hand bounds the evolution of the errors of the encoder and state, respectively, decoder and state, provided that information consistency holds. While the N-contacting property could possibly be achieved by an observer, it is not clear how the L-expanding property should be established for this case, as it relates the error on the encoder side to the error on the decoder side. Similarly, it is not clear, how to construct an encoder decoder pair, which is information consistent. In this regard, it is not clear at this stage whether such encoder-decoder pairs do exist. The next example gives a positive answer.

2.5 Example of an Encoder-Decoder Pair

Here we give an example of an encoder-decoder pair, which is information consistent, L-expanding, and N-contracting. The internal states of encoder respectively decoder are defined as

$$\Xi_e(t_k) = (\ell_e, x_e, \bar{x}_e, N, t_{k-1}, t^*_{k-1})\,,$$
$$\Xi_d(t^*_k) = (\ell_d, x_d, \bar{x}_d, N, t_{k-1}, t^*_{k-1})$$

with $\Xi_e, \Xi_d \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+$. The subscript e always indicates an encoder variable. The subscript d is defined in an obvious manner. As the encoder variables are a direct counter part of the decoder variables, we only explain the first.
The variable ℓ_e is the size of the quantization region on the encoder side. To emphasize that we use two different variables for the size of the region on encoder resp. decoder side, we decided to use ℓ instead of the more obvious m as we used in the rest of this chapter.
The variables x_e is the center of the quantization region whereas $\bar{x}_e$ is an auxiliary variable. The value N is a natural number known to encoder an decoder. The last two are the last time instances the encoder sent data respectively when the decoder sent data the last time.
The initial states for the encoder and the decoder are:

$$\begin{aligned} &k = 1,\ t_0 = 0,\ t_0^* = 0,\ t_1 = 0\ t_1^* = 0 \text{ and } \bar{x}_d(0) = \bar{x}_e(0) = 0 \\ &x_e(0^-) = x_d(0^-) = 0 \text{ and } \ell_e(0^-) = \ell_d(0^-) = 2m_0\,. \end{aligned}$$

Following Definition 2.3.1, an encoder is a dynamical system with Ξ_e as the state, x, t_{k-1}^* as inputs and s, t_s as outputs. Its output map at time t is defined by

$$(s, t_s) = Enc(x(t), t_{k-1}^*, \Xi_e(t), t)\,.$$

The variable s carries the encoded state information and t_s the corresponding time stamp.
The state of the encoder Ξ_e evolves according to the following equations.
The encoder equations read:
(i) Every time an ack arrives at the encoder ($t = t_k$)

$$t_s = t \tag{2.28}$$

$$\ell_e(t_k) = \ell_e(t_{k-1})e^{L(t_k - t_{k-1})}/N \tag{2.29}$$

$$\begin{aligned} x_e(t_k^-) = x_e(t_{k-1}) &+ \\ &\int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds + \\ &\int_{t_{k-1}^*}^{t_k} f(x_e(s), k(x_e(s)))ds \end{aligned} \tag{2.30}$$

$$s(t_s) = q(x_e(t_k^-), x(t_k), \ell_e(t_k)) \tag{2.31}$$

$$x_e(t_k) = x_e(t_k^-) + s(t_s)\frac{\ell_e(t_k)}{N} \tag{2.32}$$

$$\bar{x}_e(t_k) = x_e(t_k^-)\,. \tag{2.33}$$

Every time the encoder receives an ack ($t = t_k$) it updates the length of the quantization region according to the growth of the error on the last interval (2.29). The center of the quantization region is updated via (2.30). Both integrals are needed to account for the change in the control action on the decoder side. The subregion in which the state lies is calculated by (2.31) with q defined in (2.51). This information will be sent to the decoder together with the actual time (2.28). The jump from the center to the subregion is done by equation (2.32). The value of the old quantization region is copied by (2.33).
To be able to cover the case of packet loss we also need:
(ii) If τ_{max} units of time without receipt of an ack elapse ($t = t_k + \tau_{max}$)

$$t_s = t \tag{2.34}$$

$$\ell_e(t) = \ell_e(t_k)e^{L(t-t_k)} \tag{2.35}$$

$$x_e(t^-) = \bar{x}_e(t) \tag{2.36}$$

$$s(t_s) = q(x_e(t^-), x(t), \ell_e(t)) \tag{2.37}$$

$$x_e(t) = x_e(t^-) + s(t_s)\frac{\ell_e(t)}{N}\,. \tag{2.38}$$

If τ_{max} units of time elapse without receiving an ack, the packet sent last time is considered lost and a new one will be sent. Similar to the case of no loss, the encoder updates the length of the quantization region (2.35). Note that there is no division by N. Equation (2.36) cancels the jump from the center to the subregion made in the last encoding step. The equations (2.37) and (2.38) follow the same reason as in the case of no loss. In both cases ((i) and (ii)) t_s and $s(t_s)$ are the output of the encoder, which will be sent from encoder to decoder.
(iii) Otherwise:

$$\dot{\bar{x}}_e(t) = f(\bar{x}_e(t), k(\bar{x}_e(t))) \tag{2.39}$$

We need (2.39) to know the trajectory which will be used to close the loop on the decoder side as can be seen from Lemma 2.5.6 and Theorem 2.5.7. It is also needed to treat the case of packet loss (2.36).
The decoder is also a dynamical system. Its internal state is Ξ_d, the inputs are s, t_s and $\bar{x}_d$ the output. Its output map at time t is given by

$$\bar{x}_d(t) = Dec(s, t_s, \Xi_d(t), t)\,.$$

The evolution of the states are governed by the following equations.

The decoder equations read:
(i) Every time a packet arrives at the decoder ($t = t_k^*$)

$$\ell_d(t_s) = \ell_d(t_{k-1})e^{L(t_s - t_{k-1})}/N \tag{2.40}$$

$$x_d(t_s^-) = \bar{x}_d(t_{k-1}^*) + \int_{t_{k-1}^*}^{t_s} f(x_d(s), k(x_d(s)))ds \tag{2.41}$$

$$x_d(t_s) = x_d(t_s^-) + s(t_s)\frac{\ell_d(t_s)}{N} \tag{2.42}$$

$$\bar{x}_d(t_k^*) = x_d(t_s) + \int_{t_s}^{t_k^*} f(x_d(s), k(\bar{x}_d(s)))ds \tag{2.43}$$

(ii) Otherwise

$$\dot{\bar{x}}_d(t) = f(\bar{x}_d(t), k(\bar{x}_d(t)))\,. \tag{2.44}$$

The decoder equation tries to mimic the corresponding equations of the encoder. We will see in Lemma 2.5.6 and Theorem 2.5.7 that decoder indeed succeeds with this goal.
The state x evolves according to

$$\dot{x} = f(x, k(\bar{x}_d))\,. \tag{2.45}$$

To see that this system is indeed of the form (2.8)-(2.11) consider the following equations.

$$\dot{x} = f(x, k(\hat{x}_d)) \tag{2.46}$$

$$\dot{\hat{x}}_d = f(\hat{x}_d, k(\hat{x}_d)) \tag{2.47}$$

$$\hat{x}_e(t_k) = x_e(t_k) \tag{2.48}$$

$$\hat{x}_d(t_k^*) = Dec\Big(Enc\big(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k\big), \Xi_d(t_k^*), t_k^*\Big)\,. \tag{2.49}$$

In contrast to system (2.8)-(2.11) we see that in the equation for $\hat{x}_e(t_k)$ we do not need knowledge of the decoder output map on the encoder side. In general, it would also be sufficient in (2.10) to use the same quantizer as on the decoder side. For the sake of simplicity, we decided to use the concatenation of encoder and decoder in (2.10) for $\hat{x}_e$ nevertheless.
It remains to show that (2.48) is of the form (2.10).

Lemma 2.5.1. *Consider encoder and decoder as above. Let $t = t_k$ be the encoding times and $t = t_k^*$ the time instances the decoder receives information. Then for the concatenation of the output maps at time $t = t_k$ it holds that*

$$x_e(t_k) = Dec\Big(Enc\big(x(t_k), t_{k-1}^*, \Xi_e(t_k), t_k\big), \Xi_e(t_k), t_k\Big)\,. \tag{2.50}$$

Proof. Let s, t_s be the output of the encoder at time $t = t_k$. Hence we have by (2.28) that $t_s = t_k$. Note that in (2.50) the output map of the decoder is envoked with the internal state of the encoder at time $t = t_k$. Thus we have $\ell_d = \ell_e$, $x_d(t_s^-) = x_e(t_s^-)$. Following (2.43), the output of the decoder is

$$\bar{x}_d(t_k) = x_d(t_k) + \int_{t_k}^{t_k} f(x_d(s), k(\bar{x}_d(s)))ds = x_d(t_k)\,.$$

Considering (2.42) yields

$$\bar{x}_d(t_k) = x_d(t_k^-) + s(t_k)\frac{\ell_d(t_s)}{N} = x_e(t_s^-) + s(t_k)\frac{\ell_e(t_s)}{N} = x_e(t_k)\,,$$

Realizing that $\bar{x}_d(t_k)$ is the output of the decoder at time $t = t_k$ shows the claim. □

The quantizer used in (2.31) and (2.37) takes the form

$$q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{Z}^n \; :$$
$$q(x_e(t^-), x(t), \ell_e(t)) = \left\lfloor \frac{N}{\ell_e(t)}(x(t) - x_e(t^-)) + \frac{1}{2}e \right\rfloor . \quad (2.51)$$

The floor function has to be understood component wise. Please note that $e \in \mathbb{R}^n$ above is the vector consisting of ones, not to be confused with the exponential.
An example of a two dimensional q can be seen in Figure 2.6. The center of the quantization region is x_e and its length ℓ_e. A vector of integers is assigned to each subregion. For instance, in Figure 2.6 it holds for the quantizer that $q(x_e, x, \ell_e) = (0, 1)^\top$. Although in the rest of this chapter the norms can be arbitrary, we want to use the max norm in this section. By utilizing the equivalence of norms on finite dimensional spaces, the consideration would still hold true for other norms. To ease the presentation and stick to the idea of quantization regions as boxes as in e.g. Figure 2.6, we decided to use the maximum norm.

Lemma 2.5.2. *Consider a quantizer q as defined in* (2.51)*. Whenever* $|x(t_k) - x_e(t_k^-)| \le \frac{\ell_e(t_k)}{2}$ *we have* $|x(t_k) - x_e(t_k)| \le \frac{\ell_e(t_k)}{2N}$.

Proof. Considering (2.31), (2.32) and (2.51) for the ith component yields

$$\begin{aligned} & x(t_k)_i - x_e(t_k)_i = \\ & \quad x(t_k)_i - x_e(t_k^-)_i - \left\lfloor \frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k^-)_i) + \frac{1}{2} \right\rfloor \frac{\ell_e(t_k)}{N} \ge \\ & x(t_k)_i - x_e(t_k^-)_i - \left(\frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k^-)_i) + \frac{1}{2} \right) \frac{\ell_e(t_k)}{N} = -\frac{\ell_e(t_k)}{2N}\,. \end{aligned}$$

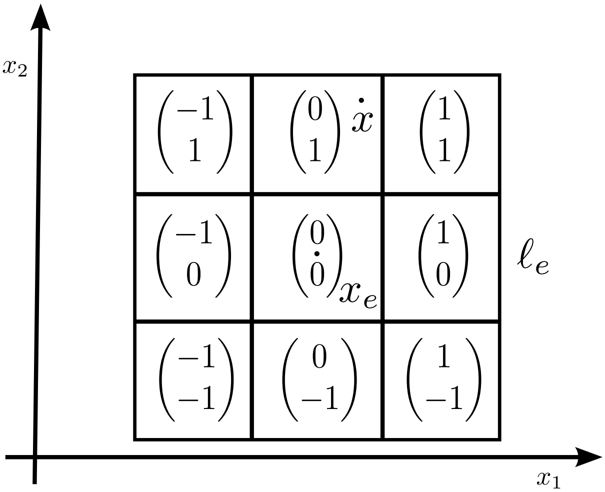

Figure 2.6: Two dimensional example of φ

Using the properties of the floor function again leads to

$$\begin{aligned}
&x(t_k)_i - x_e(t_k)_i = \\
&\qquad x(t_k)_i - x_e(t_k^-)_i - \left\lfloor \frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k^-)_i) + \frac{1}{2} \right\rfloor \frac{\ell_e(t_k)}{N} \leq \\
&x(t_k)_i - x_e(t_k^-)_i - \left(\frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k^-)_i) + \frac{1}{2} - 1 \right) \frac{\ell_e(t_k)}{N} = \frac{\ell_e(t_k)}{2N} .
\end{aligned}$$

Using the maximum norm yields the assertion. □

Lemma 2.5.2 states that whenever the state lies within the quantization region, the size of the quantization region is divided by N at each encoding step.
We assumed that the communication channel is able to transmit a symbol from a set of N^n different symbols (cf., Summary 2.1.4 (4)). The question remains whether the encoded state stays within this bound. The next lemma gives a positive answer.

Lemma 2.5.3. *Consider the encoder-decoder pair given by* (2.28)-(2.44). *Assume that* $|x(t_k) - x_e(t_k^-)| \leq \frac{\ell_e(t_k)}{2}$. *Then we have for each* $i = 1, \ldots, n$

$$\frac{-N+1}{2} \leq s(t_k)_i \leq \frac{N+1}{2} ,$$

if N is odd and

$$\frac{-N}{2} \le s(t_k)_i \le \frac{N}{2},$$

if N is an even number.

Proof. Because $|x(t_k) - x_e(t_k^-)| \le \frac{\ell_e(t_k)}{2}$ we have for each $i = 1, \dots n$ that $\frac{-\ell_e(t_k)}{2} \le x(t_k)_i - x_e(t_k^-)_i \le \frac{\ell_e(t_k)}{2}$ and hence

$$\left\lfloor \frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k)_i) + \frac{1}{2} \right\rfloor \le \left\lfloor \frac{N}{\ell_e(t_k)} \frac{\ell_e(t_k)}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{N+1}{2} \right\rfloor . \quad (2.52)$$

Similarly we get

$$\left\lfloor \frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k)_i) + \frac{1}{2} \right\rfloor \ge \left\lfloor \frac{N}{\ell_e(t_k)} \frac{-\ell_e(t_k)}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{-N+1}{2} \right\rfloor . \quad (2.53)$$

Now assume that N is odd. Hence we get

$$\left\lfloor \frac{N+1}{2} \right\rfloor = \frac{N+1}{2} \qquad \text{and} \qquad \left\lfloor \frac{-N+1}{2} \right\rfloor = \frac{-N+1}{2} .$$

Similarly, we get for N even

$$\left\lfloor \frac{N+1}{2} \right\rfloor = \frac{N}{2} \qquad \text{and} \qquad \left\lfloor \frac{-N+1}{2} \right\rfloor = \frac{-N}{2} .$$

Combining the latter with (2.52) resp. (2.53) yields the claim. □

The lemma states that $q(x_e, x, \ell_e) = s$ is in every dimension an integer between $\frac{-N+1}{2}$ and $\frac{N+1}{2}$ if N is odd respectively between $\frac{-N}{2}$ and $\frac{N}{2}$ if N is even. In both cases we can encode s from (2.31) in such a way that we need at most N^n different symbols, provided that the state always stays in range of the encoder.

Remark 2.5.4. Although our approach does not distinguish between N even or odd, the intuition that during encoding the encoder jumps from the center of the quantization region to the center of a subregion is only valid, if N is an odd number.

Before we show that the equations for encoder and decoder are information consistent an important observation is in order.

Remark 2.5.5. Although the equations for the decoder depend on the time t_k, which it is aware of because of the time stamping mechanism, we regard these as the internal state of the decoder at time t_k^*. One has also to distinguish between "real" time t and the fictional time for the encoder and decoder. As the states of encoder and decoder are most likely calculated by some digital device, they are no real trajectories. Despite this fact we will treat them in the proofs as if they actually were given by a "real" dynamical system.

In order to use the results from the last section we have to show that the encoder-decoder pair given above are information consistent, L-expanding, and N-contracting.
Before we prove that the encoder and decoder described in this section are indeed information consistent, we give a preliminary lemma, which shows that certain signals coincide on certain time intervals. This observation will be crucial for the rest of this section.

Lemma 2.5.6. *For the encoder-decoder pair described in* (2.28)-(2.44) *it holds that* $\hat{x}_d(t) = \bar{x}_d(t) = \bar{x}_e(t)$ *for all* $t \in [t_k, t_k^*)$ *and for all* $t \in [t_k^*, t_{k+1})$ *we have* $\hat{x}_d(t) = \bar{x}_d(t) = x_e(t)$.

Proof. The output of the decoder at time $t = t_k^*$ is $\bar{x}_d(t_k^*)$. And hence by (2.49) we have $\hat{x}_d(t_k^*) = \bar{x}_d(t_k^*)$. As both follow the same closed-loop dynamics ((2.47) resp. (2.44)) for $t \in [t_{k-1}^*, t_k^*)$ we have $\bar{x}_d(t) = \hat{x}_d(t)$ for all $t \geq 0$.
As soon as a packet arrives at the decoder, it knows the time when the state was encoded due to the time stamping. Hence the decoder can use (2.40) to reconstruct the length of the quantization region used to encode the state (2.29) respectively (2.35). Hence it holds

$$\ell_e(t_s) = \ell_d(t_s). \tag{2.54}$$

Because of the initial condition of the encoder and the decoder and (2.33) it holds that $\bar{x}_d(0) = \bar{x}_e(0) = 0$. Using $t_1 = 0$ and equations (2.39) and (2.44) we obtain

$$\bar{x}_d(t) = \bar{x}_e(t) \quad \forall t \in [t_1, t_1^*). \tag{2.55}$$

At time t_1^* the value $s(t_s)$ as well as the time t_s becomes available to the decoder. Because of the initialization of encoder and decoder and (2.32) respectively (2.42) and (2.54) it holds that $x_e(t_1) = x_d(t_1)$. By (2.43), (2.30) and (2.55) we have $\bar{x}_d(t_1^*) = x_e(t_1^*)$.
Since both trajectories follow the same dynamics on the interval $[t_k^*, t_{k+1})$ by (2.44) and (2.30) we get $\bar{x}_d(t) = x_e(t)\,, \quad \forall t \in [t_1^*, t_2)$.
Due to the continuity of $\bar{x}_d$ at t_k and (2.33) $\bar{x}_d(t_2) = \bar{x}_e(t_2)$ holds. From (2.30), (2.41) and (2.43) as well as (2.55) we can deduce

$$x_d(t_2^-) = x_e(t_2^-).$$

Now we can use (2.32) respectively (2.42) and (2.54) to get $x_d(t_2) = x_e(t_2)$. To conclude the proof, repeat the arguments inductively. □

Now we use the last lemma to show that not only certain signals are the same, but also that all internal states are the same at the encoding respectively decoding times.

Theorem 2.5.7. *Equations* (2.28)-(2.44) *describe an information consistent encoder-decoder pair i.e.,* $\Xi_e(t_k) = \Xi_d(t_k^*)$ *for all* $k \in \mathbb{N}$.

Proof. First we treat the case of no packet loss. In the case of packet loss the same considerations hold with just minor modifications in the equations used.
The variable t_s is an auxiliary variable, which holds the value of the time stamp of the packet sent from encoder to decoder. Thus we have in the corresponding time interval $t_s = t_k$. From Lemma 2.5.6 we know already that $\ell_e(t_k) = \ell_d(t_s) = \ell_d(t_k)$ for the same reasoning we have $\bar{x}_e(t_k) = \bar{x}_d(t_k)$. The parameter N does not change over time. Hence it is sufficient to initialize encoder and decoder to the same value of N. The last time instances an packet was sent respectively received must be stored by encoder and decoder. It remains to show that $x_e(t_k) = x_d(t_k)$ for all $k \in \mathbb{N}$.
Because of the initialization of encoder and decoder we have $x_e(t_1) = x_d(t_1)$. Now let us assume

$$x_e(t_{k-1}) = x_d(t_{k-1})\,. \tag{2.56}$$

Starting from equation (2.43) we get

$$\bar{x}_d(t_{k-1}^*) = x_d(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_d(s), k(\bar{x}_d(s)))ds$$

Using (2.56) and Lemma 2.5.6 yield

$$\bar{x}_d(t_{k-1}^*) = x_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds \tag{2.57}$$

Substituting the latter in (2.41) gives

$$x_d(t_k^-) = x_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds + \int_{t_{k-1}^*}^{t_k} f(x_e(s), k(x_d(s)))ds$$

By using Lemma 2.5.6 again we arrive at

$$x_d(t_k^-) = x_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds + \int_{t_{k-1}^*}^{t_k} f(x_e(s), k(x_e(s)))ds$$

By inspecting (2.30) we see that

$$x_d(t_k^-) = x_e(t_k^-)\,.$$

Considering equation (2.32) and (2.42) together with $\ell_e(t_k) = \ell_d(t_k)$ we conclude

$$x_e(t_k) = x_d(t_k)$$

and we have shown that $\Xi(t_k) = \Xi(t_k^*)$ for all $k \in \mathbb{N}$, if no packet loss occurred. Now let the packet sent at the time t_k be lost. The encoder waits τ_{max} units of time until it sends the next packet. Hence we have $t_s = t_k + \tau_{max}$.
Note that by definition t_k is the last time an ack arrived at the encoder, hence until that time input consistency holds by the considerations above. Of course, it could happen that consecutive packets get lost. In this case $t_s = t_k + m\tau_{max}$, where m is the number of packets lost in a row. To simplify the presentation we treat the case $m = 1$, although the considerations still hold true for larger m.
By inspecting (2.35) and (2.29) we have

$$\ell_e(t_s) = \ell_e(t_k)e^{L(t_s-t_k)} = \ell_e(t_{k-1})e^{L(t_k-t_{k-1})}/Ne^{L(t_s-t_k)} = \ell_d(t_s)\,,$$

where the last equality comes from (2.40). Here, we see that the quantization region gets only smaller by N if an ack arrives at the encoder.
Now we want to show that $x_e(t_k + \tau_{max}) = x_d(t_k + \tau_{max})$. To this end take (2.41) and $\bar{x}_e(t_{k-1}) = \bar{x}_d(t_{k-1})$ to get

$$\begin{aligned} x_d(t_s^-) &= \bar{x}_d(t_{k-1}^*) + \int_{t_{k-1}^*}^{t_s} f(x_d(s), k(x_d(s)))ds = \bar{x}_d(t_{k-1}) \\ &+ \int_{t_{k-1}}^{t_{k-1}^*} f(x_d(s), k(x_d(s)))ds + \int_{t_{k-1}^*}^{t_s} f(x_d(s), k(x_d(s)))ds = \\ &\bar{x}_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_d(s), k(x_d(s)))ds + \int_{t_{k-1}^*}^{t_s} f(x_d(s), k(x_d(s)))ds \\ &= \bar{x}_e(t_s). \end{aligned}$$

With (2.36) we conclude

$$x_e(t_s^-) = x_d(t_s^-)\,.$$

Similar as in the case of no loss we infer $x_e(t_s) = x_d(t_s)$.
It remains to show that $\bar{x}_e(t_s) = \bar{x}_d(t_s)$. As in the case of no loss we already know that $\bar{x}_e(t_k) = \bar{x}_d(t_k)$. As $\bar{x}_e$ and $\bar{x}_d$ follow the same dynamics on $[t_k, t_k^*)$ we have $\bar{x}_e(t_s) = \bar{x}_d(t_s)$ and the proof is finished. □

So far we have shown that the internal states of encoder and decoder are the same and that encoder, decoder, and the actual system close their loops with the same signal. It remains to show that the encoder-decoder pair is L-expanding.

Theorem 2.5.8. *Equations* (2.28)-(2.44) *describe an L-expanding encoder-decoder pair.*

Proof. Identify $\frac{\ell_e(t_k)}{2} = m(t_{k-1})e^{L(t_k - t_{k-1})}$ for all $k \in \mathbb{N}$. By the initialization we have

$$|\hat{x}_e(t_1) - x(t_1)| \leq m_0 = m(t_1)\,.$$

Lets assume that

$$|\hat{x}_e(t_{k-1}) - x(t_{k-1})| \leq m(t_{k-1})\,. \tag{2.58}$$

Similar as before we can use the Gronwall inequality to bound the error $|x_e(t) - x(t)|$ for $t \in [t_{k-1}, t_k)$ with the help of (2.30)

$$|x_e(t) - x(t)| = |x_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds + \int_{t_{k-1}^*}^{t} f(x_e(s), k(x_e(s)))ds - x(t_{k-1}) - \int_{t_{k-1}}^{t} f(x(s), k(\hat{x}_d(s)))ds|\,.$$

By Lemma 2.5.6 we know $\bar{x}_e(t) = \bar{x}_d(t) = \hat{x}_d(t)$ for $t \in [t_k, t_k^*)$ and $x_e(t) = \bar{x}_d(t) = \hat{x}_d(t)$ for $t \in [t_k^*, t_{k+1})$ and hence

$$|x_e(t) - x(t)| = |x_e(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}^*} f(x_e(s), k(\bar{x}_e(s)))ds + \int_{t_{k-1}^*}^{t} f(x_e(s), k(x_e(s)))ds - x(t_{k-1}) - \int_{t_{k-1}}^{t_{k-1}^*} f(x(s), k(\bar{x}_e(s)))ds - \int_{t_{k-1}^*}^{t} f(x_e(s), k(x_e(s)))ds|\,.$$

And thus using Lipschitz continuity together with the Gronwall inequality yields

$$|x_e(t_k^-) - x(t_k)| \leq |x_e(t_{k-1}) - x(t_{k-1})|e^{L(t_k - t_{k-1})} \leq m(t_{k-1})e^{L(t_k - t_{k-1})} = \frac{\ell_e(t_k)}{2}\,. \tag{2.59}$$

By Lemma 2.5.2 we have

$$|x_e(t_k) - x(t_k)| \leq \frac{\ell_e(t_k)}{2N} = m(t_k)\,. \tag{2.60}$$

To prove (2.13) we distinguish two cases. First lets assume that $|x_e(t_k^-) - x(t_k)| \geq \frac{\ell_e(t_k)}{2N}$. Using (2.60) we conclude

$$|x_e(t_k) - x(t_k)| \leq |x_e(t_k^-) - x(t_k)|\,,$$

which together with (2.59) yields

$$|x_e(t_k) - x(t_k)| \leq |x_e(t_{k-1}) - x(t_{k-1})|e^{L(t_k - t_{k-1})}$$

by Lemma 2.5.1 we have established (2.13).
Now let $|x_e(t_k^-) - x(t_k)| < \frac{\ell_e(t_k)}{2N}$ and consider (2.51) component wise. Similar as in Lemma 2.5.2 we get for each $i = 1 \ldots n$

$$0 = \frac{N}{\ell_e(t_k)}\frac{-\ell_e(t_k)}{2N} + \frac{1}{2} < \frac{N}{\ell_e(t_k)}(x(t_k)_i - x_e(t_k^-)_i) + \frac{1}{2} < \frac{N}{\ell_e(t_k)}\frac{\ell_e(t_k)}{2N} + \frac{1}{2} = 1\,.$$

Applying the floor function and using (2.51) we have $q(x_e(t_k), x(t_k), \ell_e(t_k)) = 0$ and thus $s(t_k) = 0$. Considering (2.32) yields $x_e(t_k) = x_e(t_k^-)$, which together with (2.59) readily gives (2.13). Now consider (2.43).

$$|\bar{x}(t_k^*) - x(t_k^*)| = |x_d(t_k) - x(t_k) + \int_{t_k}^{t_k^*} f(x_d(s), k(\bar{x}_d(s)))ds - \int_{t_k}^{t_k^*} f(x(s), k(\bar{x}_d(s)))ds|\,.$$

As before, using Gronwall's inequality together with Lipschitz continuity gives

$$|\bar{x}(t_k^*) - x(t_k^*)| \leq |x_d(t_k) - x(t_k)|e^{L(t_k^* - t_k)}\,.$$

By information consistency this is equivalent to

$$|\bar{x}(t_k^*) - x(t_k^*)| \leq |x_e(t_k) - x(t_k)|e^{L(t_k^* - t_k)}\,.$$

As $\bar{x}(t_k^*)$ is the output of the decoder at time t_k^* this is (2.14) and the proof is complete. □

Theorem 2.5.9. *Equations* (2.28)-(2.44) *describe an N-contracting encoder-decoder pair.*

Proof. This statement was already proven in Theorem 2.5.8. To be more precise Combine the initialization of encoder together with (2.58) and (2.60). □

Now we are able to use the main result from Section 2.4 to conclude.

Corollary 2.5.10. *Equations* (2.28)-(2.43) *describe an encoder-decoder pair, which semi globally asymptotically stabilizes system* (2.45) *at* $t_1 = 0$*, provided that* $N > e^{L\tau^*}$ *and Assumptions 2.1.1, 2.1.2, 2.1.4, and 2.3.3 hold.*

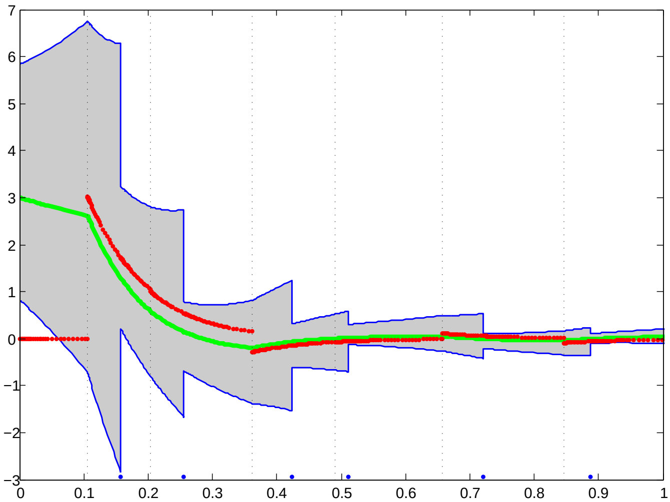

Figure 2.7: Trajectory of the closed-loop system

In Figure 2.7 a one dimensional example using the approach given in this section is depicted. This example is taken from numerical Section 5.2.
The system x is given in green and the approximation $\bar{x}_d$ in red. The shaded region is the range of the encoder-decoder pair. The vertical dotted lines are the time instances t_k^* when the decoder receives a packet. The small blue dots at the bottom of the figure are the time instances t_k when the encoder sends a packet. The encoding starts at $t = 0$. Until the first packet arrives at the decoder at $t_1^* = 0.1$ the system runs in open loop. From this time on the decoder has a better approximation of the state and uses this signal to close the loop.
The range of the encoder-decoder pair growths until $t \approx 0.15$. At this time an ack arrives at the encoder and the range can be made smaller by a factor of

N because the encoder knows that the last packet arrived successfully. Note that this procedures ensures that the system trajectory always stays within the shaded region. As can be seen at $t = 0$ this does not have to be true for the approximation on the decoder side.
The range of the encoder-decoder pair tries to follow the system trajectory. Although the system trajectory is continuous, the dynamics may have discontinuities leading to possible non differentiable states, explaining the kink of the range at time $t = 0.1$.

2.6 Notes and References

Although the idea of incorporating the effects of delay and packet-loss to the controller design is rather new, the effects of quantization was already addressed by Delchamps [Del90]. However, these early considerations only dealt with static quantizers.
The term quantizer stems from signal theory. For an introduction of quantizers see [GG92].
The idea of a non static quantizer was introduced within the signal theory community. There, it is known as adaptive quantization [GS67, GG74]. These ideas were adopted to the control theory by Tatikonda in his PhD thesis [Tat00].
More in the spirit of this chapter is [BL00] for the linear case and [LH05] for the nonlinear case, which uses also the ISS framework. This chapter is based on work previously submitted i.e. [SW09] and [SW10c]. Both papers are a generalization of [LH05] to the case of delay and packet-loss. Ideas how to handle the effects of delay in the framework of dynamic quantization was given in [DP10]. Although [BL00, LH05] uses the approach we refer to as dynamic quantization, the term itself cannot be found in these works. The name is used in later work by Liberzon, though. We could not find out, who was the first to introduce the term dynamic quantization to the control community. It should be noted that in quantum mechanics the term dynamic quantization also exists and should not be confused with our approach.
Interestingly, Assumption 2.1.2 resembles a property of Markov chains. As the communication channel is also often modeled as a Markov chain ([Bia00, WSSL06]) it seems natural to combine these ideas. Early accounts addressing this can be found in [SW10c].
As the closed-loop system considered in this chapter is a hybrid system, we had to define another notion of stability, different from the classical one. This definition is borrowed from [YMH98]. Hybrid systems consists of states that change continuously in time and states that evolve discrete in time or both.

In general, there are time dependent as well as state dependent criteria when a state evolves continuously or discrete. Of course, the analysis of such a system is more demanding than of an ordinary differential equation.
The encoder and decoder from the example are impulsive systems. Impulsive systems are a subclass of hybrid system, where the state dynamics follow an ordinary differential equation, but is allowed to "jump" during discrete time instances. For an introduction on impulsive systems see [Yan01]. The closed-loop system considered in this chapter is somewhat between impulsive and hybrid systems, as the time instances when the discrete "jumps" happen depend only on time and not on the state.
Sometimes hybrid systems exhibits an effect called Zeno-effect. Basically, Zeno means that the discrete evolution happens infinitely often in finite time. For a more thorough discussion see Chapter 4. In our case, the discrete evolution happens at the time instances t_k. Because of the design parameter τ_{min} we know that $t_k - t_{k-1} \geq \tau_{min}$ and thus Zeno cannot happen for system (2.8)-(2.11).
Assumption 2.3.3 is made just to simplify the presentation, as can be seen in [LH05]. There, the authors start with a fixed range of the quantizer. Then this range growth fast enough to eventually capture the state. As the Lipschitz constant is assumed to be known, the growth rate to achieve this is easily calculated. In [LH05] this is referred to as *zooming-out*.
Also the ISS assumption is not needed in general. For the price of a higher bandwidth it can be neglected (see [DPI04]). As the rest of the thesis is based on the ISS framework, we decided to use it in this chapter nevertheless.
Here, we gave an example how to choose an encoder respectively decoder to be able to counter the effects of a communication channel of finite capacity. It is of ongoing research, which information is really needed for an encoder to achieve this. The information available to the encoder is often termed *information pattern* ([TM04]).
Similar to information theory, a measure of information is introduced called entropy. Research in this very interesting, but more theoretical topic can be found in [NEMM04, CK09].
As can be seen in Section 2.5, encoder and decoder have both a model of the plant. Usually, this model does not capture the dynamics of a real plant precisely. This issue was addressed in [DPN08]. There, it was shown that beside the modeling error, the approach of dynamic quantization is able to achieve practical stability.
Looking at the example in Section 2.5, it becomes apparent that the property of input consistency is of importance for the presented approach to work. The N-contracting property can only be guaranteed, if the state always stays within the quantization region. The input consistency ensures that encoder

and decoder agree on their approximation of the state. Because of the exponential convergence, small errors in the calculations could result in states that are outside the quantization region. Hence, it is of interest to have a robust version, which possibly could deal with small errors between the states of encoder and decoder. First research in this direction can be found in [KN09, GN09].

We also assumed implicitly that the clocks of encoder and decoder are perfectly synchronized. In general, this is hard to achieve. For an introduction respectively an overview on clock synchronization see [Cri89, SBK05, SY04].

Here, we were able to conclude asymptotic stability by assuming an ISS property of the system. As the rest of the thesis demands for systems that have the ISS property, it would be interesting to be able to conclude ISS instead of asymptotic stability. For the case of no delay and no packet loss, ideas how to achieve this are presented in [SL12].

As the calculation errors can be modeled as disturbances, ISS results could also be used to compensate for small errors between encoder and decoder.

Another advantage of having ISS of the closed-loop system lies in the possibility to relax the computational burden employed by the encoding resp. decoding. To understand this, consider a system of high dimension. We can break down this high dimensional system in smaller parts, apply the approach from this chapter, and conclude stability of the interconnected system with the help of small-gain conditions as discussed in the next chapter, provided that the subsystems were ISS.

As Theorem 2.4.1 demands for $N > e^{L\tau^*}$, N must be large for reasonable L and τ^*. The alphabet has to be the size of N^n. At first glance it seems that this approach might be discouraging. Fortunately, a typical TCP-packet is of the size $1000 - 1500$ bits, allowing a very large alphabet.

We have seen that for the presented approach to work it is important that the state is confined within a certain region for all times. There is a strong relation to a particular class of observers called interval observers. See e.g., [ARSJ13] and the references therein.

Chapter 3

Stability Analysis of Large-Scale Systems via Small-Gain Theorems

In the last chapter we presented a method to stabilize a system communicating over a digital channel. Here we want to present conditions under which a large-scale system is stable. Theoretically, the approach from Chapter 2 can be applied to a large-scale system as well by interpreting the interconnection as a single system. The problem is that the methods presented in Chapter 2 become computationally unfeasible in larger dimensions. Hence we need different tools, which are well suited for large-scale systems.
We start with a motivating example to familiarize the reader with the problems that typically arise.
We will see that similar to Chapter 2 the ISS framework will play an important role. We regard the influence of the subsystems on each other as a disturbance and thus the stability is concluded following small-gain ideas. These small-gain conditions are a way to ensure that the coupling among the subsystems is weak. We present generalizations of known small-gain theorems and show that most of the small-gain conditions known from the literature are equivalent, if looked upon the "right" way.
After the small excursus concerning different small-gain conditions, we present a way to ensure that the interconnection of several systems is stable.

3.1 Motivating Example

The ensuing example shows how the small-gain idea comes into play and which typical problems arise by studying the stability properties of large-scale systems within the ISS framework.

To this end consider a system consisting of n subsystems:

$$\left.\begin{array}{rcl}\dot{x}_1 & = & f_1(x_1,\dots,x_n,u)\\ \dot{x}_2 & = & f_2(x_1,\dots,x_n,u)\\ & \vdots & \\ \dot{x}_n & = & f_n(x_1,\dots,x_n,u)\end{array}\right\} \tag{3.1}$$

with $x_i \in \mathbb{R}^{n_i}$ being the state of the ith subsystem and $u \in \mathbb{R}^m$ a external disturbance.
As already discussed in [DRW07] it is no loss of generality to consider a single u for all subsystems.
Assume that all subsystems are ISS as in Definition 1.3.1 with $\mu = \sum$ and u as well as the states of the other subsystems as inputs. To be more precise, let the next assumption hold.

Assumption 3.1.1. For each subsystem $i = 1,\dots,n$ denote $x_i(t)$ the solution to the corresponding subsystem of (3.1). Furthermore, there exists $\beta_i \in \mathcal{K}_\infty$, $\gamma_{ij} \in \mathcal{K}_\infty$, $j = 1,\dots,n$, $i \neq j$ and $\gamma_i \in \mathcal{K}_\infty$ for all $i = 1,\dots,n$ such that

$$|x_i(t)| \le \beta_i(|x_i(0)|) + \sum_{i\neq j}^{n} \gamma_{ij}(\|x_{j[0,t]}\|) + \gamma_i(\|u\|), \quad \forall t \ge 0 \tag{3.2}$$

and

$$\limsup_{t\to\infty} |x_i(t)| \le \sum_{i\neq j}^{n} \gamma_{ij}(\limsup_{t\to\infty} |x_j(t)|) + \gamma_i(\limsup_{t\to\infty} |u(t)|)\ , \tag{3.3}$$

for all initial conditions $x_i(0)$ and all essentially bounded inputs $u \in \mathbb{R}^m$.

To ease notation let $x := (x_1^\top,\dots,x_n^\top)^\top$, $\beta := \operatorname{diag}(\beta_i)$, $\gamma = (\gamma_1,\dots,\gamma_n)^\top \in \mathcal{K}_\infty^n$, and define $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Gamma(s) := \begin{pmatrix} \sum_{j\neq 1}^{n} \gamma_{1j}(s_j) \\ \vdots \\ \sum_{j\neq n}^{n} \gamma_{nj}(s_j) \end{pmatrix}.$$

Now stack all the inequalities from Assumption 3.1.1 in a single inequality to get

$$|x(t)|_{\mathbf{vec}} \le \beta(|x(0)|_{\mathbf{vec}}) + \Gamma(\left\|x_{[0,t]}\right\|_{\mathbf{vec}}) + \gamma(\|u\|), \tag{3.4}$$

for all $t \ge 0$, respectively

$$\limsup_{t\to\infty} |x(t)|_{\mathbf{vec}} \le \Gamma(\limsup_{t\to\infty} |x(t)|_{\mathbf{vec}}) + \gamma(\limsup_{t\to\infty} |u(t)|)\ . \tag{3.5}$$

The main problem to conclude the ISS property of the interconnected system is that in inequality (3.4) and (3.5) the quantity on the left hand side appears also in the right hand side of the corresponding inequality. Hence we need some conditions which allows us to "invert" these inequalities.
Before we do so, we want to introduce the map

$$g \, : \, \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n \, , \; g(s,w) = \Gamma(s) + w \, . \tag{3.6}$$

As Γ consists only of $\mathcal{K}_\infty$ functions or 0, the map g is continuous and monotone in both of its arguments.
Before we can benefit from the map g, we have to manipulate (3.4) by applying the supremum over $[0,t]$ on both sides. Because neither the β-term nor the γ-term depends on t we get

$$\left\|x_{[0,t]}\right\|_{\mathbf{vec}} \leq \beta(|x(0)|_{\mathbf{vec}}) + \Gamma(\left\|x_{[0,t]}\right\|_{\mathbf{vec}}) + \gamma(\|u\|) \tag{3.7}$$

for all $t \geq 0$. With the help of g this reads as

$$\left\|x_{[0,t]}\right\|_{\mathbf{vec}} \leq g\left(\left\|x_{[0,t]}\right\|_{\mathbf{vec}}, \beta(|x(0)|_{\mathbf{vec}}) + \gamma(\|u\|)\right) \, . \tag{3.8}$$

Similarly, (3.5) yields

$$\limsup_{t\to\infty} |x(t)|_{\mathbf{vec}} \leq g(\limsup_{t\to\infty} |x(t)|_{\mathbf{vec}}, \gamma(\limsup_{t\to\infty} |u(t)|)) \, . \tag{3.9}$$

In order to get rid of the dependency of x on both sides of the inequalities, assume the following.

Assumption 3.1.2. Consider $g \, : \, \mathbb{R}_+^n \times \mathbb{R}_+^m \to \mathbb{R}_+^n$. Assume there exists $\rho \in \mathcal{K}_\infty$ such that whenever

$$s \leq g(s,w) \;\text{ for } s \in \mathbb{R}_+^n \text{ and } w \in \mathbb{R}_+^m \, ,$$

we have

$$|s| \leq \rho(|w|) \, .$$

If the latter assumption holds true, we can rewrite (3.8) and (3.9) to

$$|x(t)| \leq |\left\|x_{[0,t]}\right\|_{\mathbf{vec}}| \leq \rho\Big(|\beta(|x(0)|_{\mathbf{vec}}) + \gamma(\|u\|)|\Big) \tag{3.10}$$

for all $t \geq 0$, respectively

$$\limsup_{t\to\infty} |x(t)| \leq |\limsup_{t\to\infty} |x(t)|_{\mathbf{vec}}| \leq \rho(|\gamma(\limsup_{t\to\infty} |u(t)|)|) \, . \tag{3.11}$$

Using the triangle inequality, monotonicity of ρ, and (3.10) yields for all $t \geq 0$

$$|x(t)| \leq \rho(2|\beta(|x(0)|_{\mathbf{vec}})|) + \rho(2|\gamma(\|u\|)|) .$$

As we are only considering monotone norms and because the class of $\mathcal{K}_\infty$ functions forms a group the latter inequality is a GS type estimate for the interconnected system.
The same arguments let us conclude that (3.11) is a AG type estimate, which together with GS is ISS of the interconnected system by Lemma 1.3.10.
To summarize, starting from an ISS assumption on each subsystem, we can conclude ISS of the interconnected system, as long as Assumption 3.1.2 holds true.
In the next section we will see, which conditions are important for the map g and which conditions ensure that Assumption 3.1.2 holds. At first glance, the introduction of the map g is not of great benefit, but it will prove helpful in generalizing the properties that are needed, let us simplify the notations, and let the results from the next section be applicable for a larger class of systems.

3.2 Equivalent Small-Gain Conditions

In the last section we have seen that in the ISS framework the analysis of the interconnection of several subsystems leads in a natural way to a certain map g. We have also seen that certain properties of the map g help us to conclude stability of the interconnection. The question remains how we can check if this inversion property from the motivating example holds. To answer this question we want to give a catalog of properties that are equivalent to this "inversion" property.
We will see that properties of an induced system

$$s(k+1) = g(s(k), w(k)) , \quad k \in \mathbb{N} \tag{3.12}$$

will also play an important role. We assume that $g : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ is continuous and monotone and satisfies $g(0,0) = 0$. Note that (3.6) from the last section fulfills these assumptions.
If it is clear from the context, we identify the map g with the induced system (3.12).
Before we can give the afore mentioned catalog in detail, we have to introduce some notations and technical statements. Throughout this chapter we denote by $|\cdot|$ the maximum norm. Often we are only interested in bounds for the map g resp. a trajectory of (3.12). As g is monotone it is often sufficient to

consider constant inputs. For this case we introduce the following recursively defined notation:

$$g_w^0(s) := s\,, \qquad g_w^k(s) := g(g_w^{k-1}(s), w)\,, \quad k \in \mathbb{N}\,, \quad w \in \mathbb{R}_+^m\,.$$

Before we formulate the main theorem of this section, we specify the technical conditions for the map g.

Definition 3.2.1. A continuous, monotone function $g : \mathbb{R}_+^n \times \mathbb{R}_+^m \to \mathbb{R}_+^n$ with $g(0,0) = 0$ is called *weakly increasing* if for all $s \in \mathbb{R}_+^n$ there exists a $k \geq 1$ and $w \in \mathbb{R}_+^m$ such that

$$s \leq g_w^k(s). \tag{3.13}$$

Theorem 3.2.2. *If g is weakly increasing then the following statements are equivalent:*

(ISS-LF) *there exists a monotone ISS Lyapunov-function for* (3.12)*;*

(FDBK) *there exists a proper and positive definite map $\zeta : \mathbb{R}_+^n \to \mathbb{R}_+^m$ so that the origin is globally asymptotically stable with respect to*

$$s(k+1) = f(s(k)) := g(s(k), \zeta(s(k)))\,, \quad k \in \mathbb{N}; \tag{3.14}$$

(ISS) *system* (3.12) *is input-to-state stable;*

(AG) *system* (3.12) *has the asymptotic gain property;*

(UOC) *(uniform order condition) there exists a proper and positive definite $\zeta : \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that*

$$g(s,w) \not\geq s \text{ for all } s \not\leq \zeta(w); \tag{3.15}$$

(NP) *(Neumann property) there exists a proper and positive definite $\zeta : \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that for all $s \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^m$,*

$$s \leq g(s,w) \implies s \leq \zeta(w); \tag{3.16}$$

We postpone the proof to the end of this section. The corresponding assertions are proved in the lemmas according to Figure 3.1. The star marks the only point were we need the additional assumption that g is weakly increasing.
Is is easy to see that (3.6) from the last section is weakly increasing. The next proposition gives three important examples of function classes that are also weakly increasing.

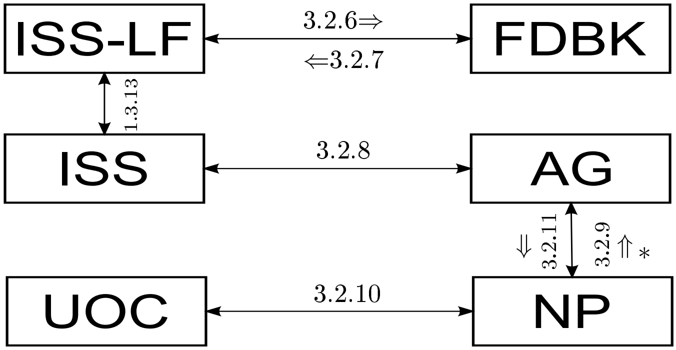

Figure 3.1: Guideline to the proofs of Theorem 3.2.2

Proposition 3.2.3. *A continuous, monotone function* $g : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ *with* $g(0,0) = 0$ *is weakly increasing if one of the following conditions is satisfied:*

1. *the map* $g(s,w)$ *can be decomposed into* $g(s,w) = g_1(s) + g_2(s)w$ *with a continuous, monotone, and positive definite* $g_2 : \mathbb{R}^n_+ \to \mathbb{R}^{n\times m}_+$*;*
2. *(increasing) for all proper* $\zeta : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ *the map* $f(s) := g(s, \zeta(s))$ *is proper.*
3. *the map* g *is in* $\mathrm{MAF}^{\mathrm{n}}_{\mathrm{n+m}}$.

Condition 2 is inspired by the concept of irreducibility for nonnegative matrices.
In the last section we used Assumption 3.1.2 to conclude ISS of the closed-loop system. The next remark relates this assumption to the property NP.

Remark 3.2.4. Property NP can equivalently be stated as follows: There exists a $\gamma \in \mathcal{K}_\infty$ such that

$$\text{for all } s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+, \quad s \le g(s,w) \implies |s| \le \gamma(|w|). \tag{3.17}$$

To see that (3.16) implies (3.17) we apply norms to (3.16) in order to get

$$|s| \le |\zeta(w)| \le |\zeta(|w|e)| \le \max_{1\le i\le m} \zeta_i(|w|e) =: \gamma(|w|),$$

Because ζ is monotone, continuous, positive definite and proper, we have $\zeta_i : \mathbb{R}_+ \to \mathbb{R}_+$ in class $\mathcal{K}_\infty$. By Lemma 1.1.2 it follows that $\gamma \in \mathcal{K}_\infty$. For the other direction define $\zeta(w) := \gamma(|w|)e$. Then $|s| \le \gamma(|w|)$ implies $s \le |s|e \le \gamma(|w|)e = \zeta(w)$.

Remark 3.2.5. A Lyapunov function for a monotone system of the form

$$s(k+1) = f(s(k)), \quad s \in \mathbb{R}^n_+, \quad k \in \mathbb{N}$$

can always be assumed to be itself a monotone function $V : \mathbb{R}^n_+ \to \mathbb{R}_+$. This follows from converse Lyapunov results like the converse ISS Lyapunov result [JW01] or the converse Lyapunov result [TP00] for autonomous systems. Indeed, constructions like [TP00] utilize Sontag's Lemma on $\mathcal{KL}$ functions to define a Lyapunov function $U : \mathbb{R}^n_+ \to \mathbb{R}_+$ via

$$U(s) := \sup_{k \geq 0} \alpha(|f^k(s)|)e^k$$

with a locally Lipschitz $\alpha \in \mathcal{K}_\infty$. From this definition it is immediate that U must be monotone and locally Lipschitz in s. In the literature on converse Lyapunov theorems the candidate function U usually undergoes additional smoothing steps to obtain a continuously differentiable Lyapunov function, which we do not need here.

Some of the properties in Theorem 3.2.2 use negations of order relations. Clearly, these negations are not transitive. Thus it is sometimes easier to work with the corresponding logical negation in proofs. To this end we summarize the needed statements here.

¬AG: For all $\gamma \in \mathcal{K}_\infty$ there exists $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ such that

$$\limsup_{k \to \infty} |g^k_w(s)| > \gamma(|w|). \tag{3.18}$$

¬NP: For all proper and positive definite $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ there exists $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ such that

$$s \leq g(s, w) \tag{3.19}$$

and

$$s \not\leq \zeta(w). \tag{3.20}$$

To enhance readability of the proof of Theorem 3.2.2 we split the proof into the following lemmas.

Lemma 3.2.6. *ISS-LF implies FDBK.*

Proof. We start with an ISS Lyapunov function $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ which satisfies

$$V(s) \geq \gamma(|w|) \Longrightarrow V(g(s, w)) - V(s) \leq -\alpha(V(s)) \tag{3.21}$$

for some $\gamma \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$. Define a proper and positive definite map $\zeta : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $\zeta(s) := \gamma^{-1}(V(s))e$ and consider the input $w = \zeta(s)$, i.e.,

$$s(k+1) = g(s(k), \zeta(s(k))) , \quad k \in \mathbb{N} . \tag{3.22}$$

The choice of ζ ensures that the decay condition in (3.21) is always satisfied, because

$$V(s) = \gamma(\gamma^{-1}(V(s))) = \gamma(|\gamma^{-1}(V(s))e|) = \gamma(|\zeta(s)|) = \gamma(|w|).$$

Now that we established that the left-hand side of the implication in (3.21) is always true, we can conclude by standard Lyapunov arguments (see e.g., [Theorem 5.9.2][Aga00]) that (3.22) is globally asymptotically stable and we have established FDBK. □

Clearly, the last lemma does also hold in a more general setting, e.g., where g is not monotone and hence it is of interest of its own. For sake of completeness we state the result here.

Lemma 3.2.7. *FDBK implies ISS-LF.*

Proof. As (3.14) is globally asymptotically stable there exists a locally Lipschitz, proper, positive definite, and monotone Lyapunov function V : $\mathbb{R}^n_+ \to \mathbb{R}_+$, cf. Remark 3.2.5 and Lemma 1.2.12, such that for some $\overline{\alpha}, \underline{\alpha} \in \mathcal{K}_\infty$ we have

$$\underline{\alpha}(|s|) \leq V(s) \leq \overline{\alpha}(|s|) . \tag{3.23}$$

By FDBK there exists a map $\zeta : \mathbb{R}^n \to \mathbb{R}^m$, which is proper and positive definite, hence by Definition 1.1.9 there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that

$$\tilde{\alpha}(|s|)e \leq \zeta(s). \tag{3.24}$$

Define $\gamma := \overline{\alpha} \circ \tilde{\alpha}^{-1} \in \mathcal{K}_\infty$. Now consider the case that $V(s) \geq \gamma(|w|)$. This implies by (3.23) that

$$\overline{\alpha}(|s|) \geq \gamma(|w|) = \overline{\alpha}(\tilde{\alpha}^{-1}(|w|))$$

and thus

$$\tilde{\alpha}(|s|)e \geq |w|e \geq w.$$

Application of (3.24) yields

$$\zeta(s) \geq \tilde{\alpha}(|s|)e \geq w . \tag{3.25}$$

By monotonicity of V and using (3.25) we get

$$V(g(s,w)) \leq V(g(s,\zeta(s))) < V(s),$$

proving that $V(s) \geq \gamma(|w|)$ implies $V(g(s,w)) < V(s)$. This shows that V is an ISS-Lyapunov function for system (3.12). □

For monotone systems attractivity already implies stability (see e.g., [HS06]). Interestingly, this observation carries over to the ISS framework.

Lemma 3.2.8. *The properties ISS and AG are equivalent.*

Proof. It is obvious that input-to-state stability implies the asymptotic gain property. For the other direction consider the AG property (1.3.5). Fix $\varepsilon > 0$ and $w \in \mathbb{R}^m$. Then for any $s_0 \in \mathbb{R}^n_+$ we can find a $T = T(s_0, \varepsilon)$ such that

$$\sup_{k \geq T} |g_w^k(s_0)| \leq \gamma(|w|) + \varepsilon.$$

By monotonicity of g we obtain

$$\sup_{k \geq T} |g_w^k(s)| \leq \sup_{k \geq T} |g_w^k(s_0)| \leq \gamma(|w|) + \varepsilon$$

for all $s \leq s_0$. One can verify that this property thus coincides with the (in general stronger) *uniform* asymptotic gain (UAG) property (see Definition 1.3.7). Considering Lemma 1.3.8 finishes the proof. □

We stress that for the last lemma to hold, the monotonicity of g is instrumental.

Lemma 3.2.9. *NP implies AG.*

Proof. We will show ¬AG implies ¬NP. To this end let $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ be proper and positive definite. Choose $\gamma \in \mathcal{K}_\infty$ such that

$$\gamma(r)e \geq \zeta(re) \tag{3.26}$$

for all $r \in \mathbb{R}_+$. By ¬AG, see (3.18), there exist $s^* \in \mathbb{R}^n_+$ and $w^* \in \mathbb{R}^m_+$ so that

$$\limsup_{k \to \infty} |g_{w^*}^k(s^*)| > \gamma(|w^*|)\,. \tag{3.27}$$

Define $\overline{s} := \limsup_{k \to \infty} g_{w^*}^k(s^*) \in (\mathbb{R}_+ \cup \infty)^n$.
First we assume that $\overline{s}$ is finite. Hence by (3.27) and using monotonicity of g and the norm we have

$$|\overline{s}| = |\limsup_{k \to \infty} g_{w^*}^k(s^*)| \geq \limsup_{k \to \infty} |g_{w^*}^k(s^*)| > \gamma(|w^*|). \tag{3.28}$$

Similarly, we deduce

$$g(\overline{s}, w^*) = g(\limsup_{k \to \infty} g_{w^*}^k(s^*), w^*) \geq \limsup_{k \to \infty} g_{w^*}^{k+1}(s^*) = \overline{s},$$

which is (3.19), respectively the first part of ¬NP. Because of the restriction to the maximum norm, (3.28), and (3.26) there exists an index i with

$$\overline{s}_i = |\overline{s}| > \gamma(|w^*|) \geq \zeta_i(|w^*|e) \geq \zeta_i(w^*),$$

which is equivalent to (3.20). Equation (3.19) together with (3.20) is ¬NP. Now assume that at least one of the components of $\overline{s}$ is infinite. Thus

$$\limsup_{k\to\infty} g_{w^*}^k(s^*) \not< \infty\,. \tag{3.29}$$

It is easy to see that because g is weakly increasing, there exist $\bar{k} \geq 1$ and $\overline{w} \geq w^*$ such that

$$s^* \leq g_{\overline{w}}^{\bar{k}}(s^*)\,.$$

Applying $g_{\overline{w}}$ on both sides repetitively yields, by monotonicity of $g_{\overline{w}}$,

$$g_{\overline{w}}^n(s^*) \leq g_{\overline{w}}^{n+\bar{k}}(s^*) \tag{3.30}$$

for all $n \in \mathbb{N}$. Observe that by monotonicity (3.29) still holds if w^* is replaced by $\overline{w}$ and hence for all monotone, proper, and positive definite $\zeta : \mathbb{R}_+^m \to \mathbb{R}_+^n$ there exists a $K \in \mathbb{N}$ such that

$$s^\# := \sup_{K\leq l\leq K+\bar{k}-1} g_{\overline{w}}^l(s^*) \not\leq \zeta(\overline{w}),$$

establishing (3.20). Using monotonicity of g gives

$$\begin{aligned} g(s^\#, \overline{w}) = g\Big(\sup_{K\leq l\leq K+\bar{k}-1} g_{\overline{w}}^l(s^*), \overline{w}\Big) \geq \sup_{K\leq l\leq K+\bar{k}-1} g_{\overline{w}}^{l+1}(s^*) = \\ \sup\{g_{\overline{w}}^{K+1}(s^*), \dots, g_{\overline{w}}^{K+\bar{k}-1}(s^*), g_{\overline{w}}^{K+\bar{k}}(s^*)\} \geq \\ \sup\{g_{\overline{w}}^{K+1}(s^*), \dots, g_{\overline{w}}^{K+\bar{k}-1}(s^*), g_{\overline{w}}^{K}(s^*)\} = s^\#, \end{aligned}$$

where in the last inequality we have used (3.30) for $n = K$. This establishes (3.19) and thus completes the proof. □

Lemma 3.2.10. *The properties NP and UOC are equivalent.*

Proof. First note that UOC can be equivalently rephrased as:

> There exists a proper and positive definite $\zeta : \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that for all $s \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^m$, $s \not\leq \zeta(s)$ implies $g(s, w) \not\geq s$.

This in turn can be rephrased to: for all $s \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^m$, $s \leq g(s, w)$ implies $s \leq \zeta(s)$ with ζ proper and positive definite. From here it is obvious that NP and UOC are equivalent. □

Lemma 3.2.11. *AG implies NP.*

Proof. We will show ¬NP implies ¬AG. Let $\gamma \in \mathcal{K}_\infty$. Define $\zeta(s) := \gamma(|s|)e$. It is easy to see that ζ is proper and positive definite. By ¬NP for this choice of γ and ζ there exist $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ satisfying (3.19) and (3.20). From the monotonicity of g it follows that

$$s \leq g(s,w) \leq \cdots \leq g_w^k(s)$$

for all $k \in \mathbb{N}$. And thus by taking norms

$$|s| \leq \limsup_{k\to\infty} |g_w^k(s)|. \tag{3.31}$$

Due to (3.20) there exists an index i such that $\zeta_i(w) < s_i$. Combining the latter with (3.31) results in

$$\gamma(|w|) = \zeta_i(w) < s_i \leq |s| \leq \limsup_{k\to\infty} |g_w^k(s)|$$

where the equality follows from the definition of ζ. As γ was arbitrary ¬AG is established, cf. (3.18). □

Now that we established the technical details, the proof of the main Theorem is easily achieved.

Proof of Theorem 3.2.2. The equivalence of ISS-LF and ISS is standard and can be found for discrete time systems e.g. in [JW01].
Property ISS-LF implies FDBK is established in Lemma 3.2.6, whereas the other direction is given in Lemma 3.2.7.
That ISS holds if and only if AG holds is stated in Lemma 3.2.8.
Lemma 3.2.10 accomplishes the equivalence of NP and UOC. By Lemma 3.2.9 NP implies AG, which together with Lemma 3.2.11 shows the equivalence and the proof is complete. □

The only instance where we used the condition weakly increasing is in Lemma 3.2.9. The next example shows that we indeed need a condition in the spirit of the weak increase property for Lemma 3.2.9 to hold.

Example 3.2.12. Let

$$g(s) := \begin{pmatrix} \frac{1}{2}s_1 \\ 2s_2 \end{pmatrix}.$$

Because g does not depend on w it is easy to see that g is not weakly increasing. Clearly, $\tilde{g}(s,w) = g(s)$ is monotone, continuous and $\tilde{g}(0,0) = 0$. The map $\tilde{g}$ fulfills the uniform order condition, because $\tilde{g}(s,w) = g(s)$ decays in the first component for all $s \neq 0$ and all w. As the second argument always increases, the system induced by $\tilde{g}$ cannot be ISS.

It was already pointed out in [Rüf07] that $g(s,0) \not\geq s$ for all $s \neq 0$ is not equivalent to the property that $s(k+1) = g(s(k),0)$ is GAS. Although the assertion is true locally. Hence we need a condition that ensures that at least $g(s,w)$ does explicitly depend on w. We decided to use condition weakly increasing to ensure this, although we do not know yet whether weakly increasing is too strong or not.
In [Rüf07] the unboundedness of some region Ω plays a crucial role in altering the local property to a global one. Furthermore, certain paths $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ through the region Ω are important for the construction of Lyapunov functions. The next section is devoted to these paths.
The unboundedness of Ω is ensured in [Rüf07] by an irreducibility assumption on a matrix modeling the effects of the interconnection. See Remark 3.3.6 for details. This irreducibility rules out examples of the form of Example 3.2.12. We hope that a condition in the spirit of weakly increasing together with the fact that UOC is stronger than $g(s,0) \not\geq s$ is sufficient to also ensure the unboundedness of Ω, but we were not able to prove this yet.

3.3 On ISS-Ω Paths

In [DRW10, Rüf10a], among others, the existence of an so called Ω-path proved helpful in constructing Lyapunov functions.
Loosely speaking, an Ω-path is an "inverse" Lyapunov function. Basically, a Lyapunov function decays along trajectories, whereas the trajectory itself decays "along" the Ω-path. In this regard we want to generalize the concept of an Ω-path to the following definition.

Definition 3.3.1. If there exists a $\sigma \in \mathcal{K}^n_\infty$ and a $\gamma \in \mathcal{K}_\infty$ such that

- for each i, the function σ_i^{-1} is locally Lipschitz on $(0,\infty)$;
- for every compact set $K \subset (0,\infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1}

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C\,, \quad \forall r \in K\,, i = 1,\ldots,n\,;$$

- For all $r > 0$

$$|\sigma(r)| \geq \gamma(|w|) \Rightarrow g(\sigma(r),w) < \sigma(r)\,,$$

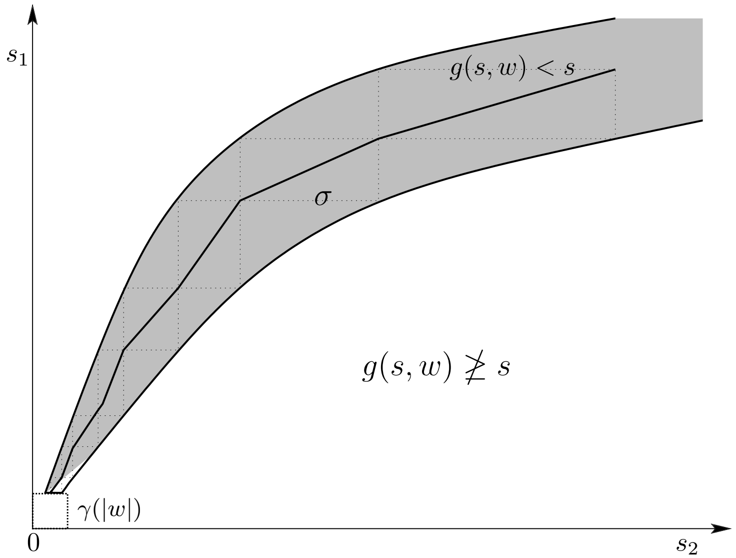

Figure 3.2: Example of an ISS-Ω-path.

we call σ an ISS-Ω-path.

If we set $w = 0$ the notion of an Ω-path from [Rüf10a] is recovered. Hence we will not distinct between Ω-path and ISS-Ω-path for systems without inputs.
In [Rüf10a] it is shown that the existence of an Ω-path implies that g is GAS. In this regard, Definition 3.3.1 tries to carry these properties over to the ISS framework. An example of an ISS-Ω-path is depicted in Figure 3.2. The dotted box close to zero has a size of $\gamma(|w|)$. Definition 3.3.1 states that as long we are outside this small box, g decays along the path σ in every dimension. As g is continuous, this does also hold at least in a small neighborhood around σ. This region is shaded in grey. The next theorem states that the existence of an ISS-Ω-path implies ISS.

Theorem 3.3.2. *If there exists an ISS-Ω-path for* (3.12)*, then there exists an ISS-LF for* (3.12)*.*

Proof. Let $\sigma \in \mathcal{K}_\infty^n$ be an ISS-Ω-path for (3.12).
Define $V(s) := \max_{i=1,\dots,n} \sigma_i^{-1}(s_i)$. We will show that this function is an ISS Lyapunov function for (3.12). As $\sigma \in \mathcal{K}_\infty^n$ we have $\sigma_i^{-1} \in \mathcal{K}_\infty$ for all $i = 1,\dots,n$. Define $\overline{\alpha} := \max_{i=1,\dots,n} \sigma_i^{-1}$ and $\underline{\alpha} := \sigma_1^{-1}$. By Lemma 1.1.2

the maximum of $\mathcal{K}_\infty$ functions is again in $\mathcal{K}_\infty$ and we have $\overline{\alpha} \in \mathcal{K}_\infty$. It follows that

$$\underline{\alpha}(|s|) = \sigma_1^{-1}(|s|) = \sigma_1^{-1}(\max_{i=1,\dots,n} s_i) \leq \max_{i=1,\dots,n} \sigma_i^{-1}(s_i) = V(s) \leq \overline{\alpha}|s|$$

and properness and positive definiteness of V is evident.
Fix $s \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$ with $s \neq 0$. Let $r := V(s)$ and $\overline{\gamma} := \overline{\alpha} \circ \gamma$. Note that by definition of V we have

$$\sigma(r) = \sigma(V(s)) = \sigma(\max_{i=1,\dots,n} \sigma_i^{-1}(s_i)) \geq s\,. \tag{3.32}$$

Moreover, we can deduce

$$V(\sigma(r)) = \max_{i=1,\dots,n} \sigma_i^{-1}(\sigma_i(r)) = r = V(s)\,. \tag{3.33}$$

Now assume $V(s) \geq \overline{\gamma}(|w|)$. We can rewrite the last statement to

$$\overline{\alpha}(|s|) \geq V(s) \geq \overline{\gamma}(|w|) = \overline{\alpha} \circ \gamma(|w|)\,.$$

This implies

$$|s| \geq \gamma(|w|)\,.$$

By (3.32) we get

$$|\sigma(r)| \geq \gamma(|w|)\,.$$

Hence the antecedence of the implication of the ISS-Ω-path holds and we have

$$g(\sigma(r), w) < \sigma(r)\,. \tag{3.34}$$

Considering (3.32) together with the monotonicity of g and (3.34) yields

$$g(s, w) \leq g(\sigma(r), w) < \sigma(r)\,.$$

Clearly, V is a monotone function and thus

$$V(g(s, w)) < V(\sigma(r)) = V(s)\,,$$

where we have used (3.33) to get the last equality. Hence we have shown that V is an ISS Lyapunov function for (3.12) and the proof is complete. □

In the last section we have seen that ISS-LF is equivalent to UOC for weakly increasing g. Thus by combining Theorem 3.3.2 with Theorem 3.2.2 we know that ISS-Ω-path implies UOC, as long as g is weakly increasing. This justifies $g(s, w) \not\geq s$ in Figure 3.2.

In [DRW10] conditions for the existence of $\sigma \in \mathcal{K}_\infty^n$ and $\rho \in \mathcal{K}_\infty^m$ are given such that

$$g(\sigma(r), \rho(r)) < \sigma(r)$$

for all $r > 0$ holds, albeit under different assumptions on g (see Remark 3.3.6 for details). The next two lemmas show that this is equivalent to the notion of an ISS-Ω-path, if we neglect the difference of the assumptions on g.

Lemma 3.3.3. *Let σ be an ISS-Ω-path for g, then there exists $\rho \in \mathcal{K}_\infty^m$ such that*

$$g(\sigma(r), \rho(r)) < \sigma(r) \tag{3.35}$$

for all $r > 0$.

Proof. Let

$$\rho(r) = \begin{pmatrix} \gamma^{-1} \circ \sigma_1(r) \\ \vdots \\ \gamma^{-1} \circ \sigma_m(r) \end{pmatrix} .$$

Clearly, $\rho \in \mathcal{K}_\infty^m$.
Fix an $r > 0$ and let $w = \rho(r)$. By definition of ρ we have

$$\gamma(|w|) = \gamma(|\rho(r)|) = |\sigma(r)| .$$

Hence the antecedence of the implication of Definition 3.3.1 holds and we get

$$\sigma(r) > g(\sigma(r), w) = g(\sigma(r), \rho(r)) .$$

And the claim follows. □

The other direction is also true, as can be seen in the next lemma.

Lemma 3.3.4. *If there exists $\sigma \in \mathcal{K}_\infty^n$ and $\rho \in \mathcal{K}_\infty^m$ such that* (3.35) *holds for all $r > 0$, then there exists an ISS-Ω-path for g.*

Proof. Define $\gamma := \max_{i=1,\dots,n} \sigma_i \circ \max_{j=1,\dots,m} \rho_j^{-1}$, fix a $r > 0$, and assume $|\sigma(r)| \geq \gamma(|w|)$. Let i be the index where the maximum of $\sigma(r)$ is attained. Hence

$$\sigma_i(r) \geq \gamma(|w|) \geq \sigma_i(\max_{j=1,\dots,m} \rho_j^{-1}(|w|)) ,$$

where the last inequality follows from the definition of γ. By applying ρ on both sides we get

$$\rho(r) \geq \rho(\max_{j=1,\dots,m} \rho_j^{-1}(|w|)) \geq w .$$

Using the latter together with the monotonicity of g yields

$$g(\sigma(r), w) \leq g(\sigma(r), \rho(r)) < \sigma(r) ,$$

provided that $|\sigma(r)| \geq \gamma(|w|)$ and the proof is complete. □

The last two lemmas justify the small abuse of notation to name the pair σ, ρ also an ISS-Ω-path.
In the spirit of the last section the question arises, if the existence of an ISS Lyapunov function implies the existence of an ISS-Ω-path.
For special cases of the map g this question has already been studied in [DRW10, Theorem 5.2]. Although with a particular application in mind. As we are heading for a similar application in Chapter 4 we cite the corresponding results. The differences are discussed in Remark 3.3.6.

Theorem 3.3.5. *Consider a $g \in \mathrm{MAF}_{n+m}^n$. Assume that one of the following conditions is satisfied*

1. *$g(\cdot, 0)$ is linear and the spectral radius of $g(\cdot, 0)$ is less than one;*
2. *$g(s, 0) \not\geq s$ for all $s \neq 0$;*

Then there exists an Ω-path σ for g.

Condition 2 resembles the uniform order condition (UOC) from the last section for $w = 0$.

Remark 3.3.6. Note that the original theorem is formulated in a different manner. In particular, another form of g is considered. To be more precise let $\overline{\Gamma} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ given by

$$\overline{\Gamma}(s) := \begin{pmatrix} \gamma_{11}(s_1) & \dots & \gamma_{1n}(s_n) \\ \vdots & & \vdots \\ \gamma_{n1}(s_1) & \dots & \gamma_{nn}(s_n) \end{pmatrix}, \tag{3.36}$$

with $\gamma_{ij} \in \mathcal{K}_\infty$. We can associate a graph in a natural way to the gain matrix $\overline{\Gamma}$. Consider a graph with n vertices. Whenever $\gamma_{ij} \neq 0$ there is an edge from vertex i to vertex j and no edge, if $\gamma_{ij} = 0$. If this graph is strongly connected, we say that $\overline{\Gamma}$ is irreducible. For an introduction on graph theory resp. strongly connected graphs see [Die05].
Now let $\mu \in \mathrm{MAF}_n^n$ and consider $\overline{\Gamma}_\mu : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g(s, 0) := \overline{\Gamma}_\mu(s) := \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \dots, \gamma_{1n}(s_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \dots, \gamma_{nn}(s_n)) \end{pmatrix}. \tag{3.37}$$

We say that g respectively $\overline{\Gamma}_\mu$ is irreducible if $\overline{\Gamma}$ is.
The original version of Theorem 3.3.5 from [DRW10, Theorem 5.2] was not formulated for $g \in \mathrm{MAF}$, but for an operator of the form (3.37). Note that

if we let $\gamma_{ij} = id$ in (3.37) for all $i, j = 1, \ldots, n$, we have $\Gamma_\mu \in \mathrm{MAF}_n^n$ respectively $g \in \mathrm{MAF}_{n+m}^n$. Clearly, this operator is irreducible. Or in general, interpreted in this way, any $g \in \mathrm{MAF}$ is irreducible and of the form (3.37) with $\gamma_{ij} = id$ for all $i, j = 1, \ldots, n$. This observation is crucial, because the cited results are all formulated for the case of an irreducible operator of the form (3.37).

Proof of Theorem 3.3.5. A proof of the first part can be found in [DRW10, Theorem 5.2 (i)].
By Remark 3.3.6 $g(\cdot, 0) \in \mathrm{MAF}_n^n$ can be written as

$$g(s,0) = \overline{\Gamma}_\mu(s) = \begin{pmatrix} \mu_1(s_1, \ldots, s_n) \\ \vdots \\ \mu_n(s_1, \ldots, s_n) \end{pmatrix} .$$

Clearly, $g(\cdot, 0)$ is irreducible and [DRW10, Theorem 5.2 (ii)] applies. Realizing that this shows the second assertion finishes the proof. □

Now that we know some special cases when an Ω-path for $g(s,0)$ exists, we devote the rest of this section to construct a function $\rho \in \mathcal{K}_\infty^m$ such that (3.35) holds. This problem was also already discussed in [DRW10], again for very special cases of g. It is worth mentioning that one of the cases considered in [DRW10] cannot be expressed in terms of a monotone aggregation function. To be more specific, an ISS-Ω-path is constructed for $g(s,w) = g_1(s)g_2(w)$ (see [DRW10, Corollary 5.7]). Clearly, because of the multiplicative structure g can not be expressed as a MAF. As this case does not play a role in this thesis, we neglect it.

Lemma 3.3.7. *Let $g \in \mathrm{MAF}_{n+m}^n$ be subadditive. If there exists $D := \mathrm{diag}(id + \gamma)$ with $\gamma \in \mathcal{K}_\infty$ such that*

$$D \circ g(s,0) \not\geq s$$

for all $s \in \mathbb{R}_+^n$ holds, then there exist $\sigma \in \mathcal{K}_\infty^n$ and $\rho \in \mathcal{K}_\infty^m$ such that

$$g(\sigma(r), \rho(r)) < \sigma(r) \tag{3.38}$$

for all $r > 0$.

Proof. Let $f(s,0) := \tilde{g}(s) := D \circ g(s,0)$. Clearly, $f \in \mathrm{MAF}_{n+m}^n$ and thus Theorem 3.3.5 applies and we have for all $r > 0$

$$\tilde{g}(\sigma(r)) = D \circ g(\sigma(r), 0) < \sigma(r) . \tag{3.39}$$

We will use the extra space provided by D to construct an $\rho \in \mathcal{K}_\infty^m$ such that (3.38) holds.
To this end consider for all $i = 1, \dots, n$

$$\alpha_i(r) := g(0, re)_i \,.$$

Because $g \in \mathrm{MAF}_{n+m}^n$, we know that $\alpha_i \in \mathcal{K}_\infty$ for all $i = 1, \dots, n$ by Lemma 1.1.5. Now define

$$\rho(r) := \min_{i=1,\dots,n} \alpha_i^{-1} \circ \gamma(g(\sigma(r), 0)_i) e \,.$$

Again using Lemma 1.1.5 together with Lemma 1.1.2 yields $\rho \in \mathcal{K}_\infty^m$.
For $j = 1, \dots n$ we get by subadditivity of g

$$\begin{aligned} g(\sigma(r), \rho(r))_j \leq g(\sigma(r), 0)_j + g(0, \rho(r))_j = \\ g(\sigma(r), 0)_j + g(0, \min_{i=1,\dots,n} \alpha_i^{-1} \circ \gamma(g(\sigma(r), 0)_i) e)_j \leq \\ g(\sigma(r), 0)_j + g(0, \alpha_j^{-1} \circ \gamma(g(\sigma(r), 0)_j) e)_j = \\ g(\sigma(r), 0)_j + \gamma(g(\sigma(r), 0)_j) = D \circ g(\sigma(r), 0)_j = \tilde{g}(s)_j < \sigma_j(r) \,, \end{aligned}$$

which is (3.38) and the proof is complete. □

After the submission of the thesis the author discovered a slightly more elegant way, which does not need the slack D as well as the subadditivity property to ensure the existence of an ISS-Ω-path. It can be found in Appendix D.
We want to stress that we are only interested in the existence of a function ρ. Having a particular application in mind, different constructions of ρ could lead to "better" results, where "better" depends on the particular application.
Next we give an example, which will play a prominent role in Chapter 4 as well as in Section 3.4. To this end consider an operator $\overline{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^{n \times n}$

$$\overline{\Gamma}(s) := \begin{pmatrix} \gamma_{11}(s_1) & \dots & \gamma_{1n}(s_n) \\ \vdots & & \vdots \\ \gamma_{n1}(s_1) & \dots & \gamma_{nn}(s_n) \end{pmatrix} , \tag{3.40}$$

with $\gamma_{ij} \in \mathcal{G}$ for each $i, j = 1, \dots, n$.
Now we augment $\overline{\Gamma}$ to $\Gamma : \mathbb{R}_+^n \times \mathbb{R}_+^{n \times m} \to \mathbb{R}_+^{n \times n+m}$ given by

$$\Gamma(s, w) = \begin{pmatrix} \gamma_{11}(s_1) & \dots & \gamma_{1n}(s_n) & w_{11} & \dots & w_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{n1}(s_1) & \dots & \gamma_{nn}(s_n) & w_{n1} & \dots & w_{nm} \end{pmatrix} ,$$

with w_{ij} as an input for $i = 1, \ldots, n\,, j = 1, \ldots, m$.
Let $\mu \in \mathrm{MAF}^n_{n+m}$ and consider

$$\Gamma_\mu(s, w) := \mu \circ \Gamma(s, w) = \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \ldots, \gamma_{1n}(s_n), w_{11}, \ldots, w_{1m}) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \ldots, \gamma_{nn}(s_n), w_{n1}, \ldots, w_{nm}) \end{pmatrix}. \quad (3.41)$$

Because some of the γ_{ij} might be zero, $\mu \circ \Gamma$ have not to be in MAF^n_{n+m} and thus Theorem 3.3.5 respectively Lemma 3.3.7 is not applicable.
Fortunately, this case was also already discussed in [DRW10].

Lemma 3.3.8. *Consider an irreducible $\Gamma_\mu(s, w)$ as in (3.41) with $\mu \in \mathrm{MAF}^n_{n+m}$ subadditive. If there exists $D := \mathrm{diag}(id + \gamma)$ with $\gamma \in \mathcal{K}_\infty$ such that $D \circ \Gamma_\mu(s, 0) \ngeq s$ for all $s \neq 0$, then there exists $\sigma \in \mathcal{K}^n_\infty$ and $\rho \in \mathcal{K}^m_\infty$ such that*

$$\Gamma_\mu(\sigma(r), \rho(r)) < \sigma(r)$$

for all $r > 0$ holds.

Proof. Let $\tilde{\Gamma}_\mu(s) := D \circ \Gamma_\mu(s, 0)$. By [DRW10, Theorem 8.11] we know that there exists $\sigma \in \mathcal{K}^n_\infty$ such that

$$D \circ \Gamma_\mu(\sigma(r), 0) = \tilde{\Gamma}_\mu(\sigma(r)) < \sigma(r)$$

for all $r > 0$. From here we can copy the proof of Lemma 3.3.7 after (3.39) word by word and the proof is complete. □

After this small excursus about small-gain conditions and ISS-Ω-paths, we collected the necessary tools to come back to the original problem as sketched in Section 3.1, namely the stability analysis of interconnected systems.

3.4 Stability Analysis of Interconnected Systems Communicating over Digital Channels

Here we want to use the ideas from the last sections to infer stability of a multichannel large-scale system as defined in Section 1.2. To this end, consider n systems of FDEs as in (1.11) Σ_i, $i = 1, 2, \ldots, n$, $\quad n \in \mathbb{N}$ of the form

$$
\begin{aligned}
\dot{x}_i &= f_i(x_{i\mathtt{d}}, u^1_{i\mathtt{d}}, \dots, u^{l_i}_{i\mathtt{d}}, w^1_{i\mathtt{d}}, \dots, w^{v_i}_{i\mathtt{d}}, t) \\
y^1_i &= h^1_i(x_{i\mathtt{d}}, u^1_{i\mathtt{d}}, \dots, u^{l_i}_{i\mathtt{d}}, w^1_{i\mathtt{d}}, \dots, w^{v_i}_{i\mathtt{d}}, t) \\
&\vdots \\
y^{r_i}_i &= h^{r_i}_i(x_{i\mathtt{d}}, u^1_{i\mathtt{d}}, \dots, u^{l_i}_{i\mathtt{d}}, w^1_{i\mathtt{d}}, \dots, w^{v_i}_{i\mathtt{d}}, t).
\end{aligned}
\tag{3.42}
$$

Here we distinguish between the controlled inputs u and the disturbances w. The dimensions of the state spaces and the input spaces are as follows $x_i \in \mathbb{R}^{n_i}$, $u^j_i \in \mathbb{R}^{p_{ij}}$, $j = 1, .., l_i$ and $w^j_i \in \mathbb{R}^{q_{ij}}$, $j = 1, .., v_i$.
Define $u_{i\mathtt{d}} := (u^{1\top}_{i\mathtt{d}}, \dots, u^{l_i\top}_{i\mathtt{d}})^\top$, $w_{i\mathtt{d}} := (w^{1\top}_{i\mathtt{d}}, \dots, w^{v_i\top}_{i\mathtt{d}})^\top$ and $y_{i\mathtt{d}} := (y^{1\top}_{i\mathtt{d}}, \dots, y^{r_i\top}_{i\mathtt{d}})^\top$. The reason for this multichannel approach lies in its greater flexibility of modeling the influence between subsystems. For instance, in [PMTL09] an example is given in which a particular small-gain condition is not satisfied if a traditional approach is used (i.e., one input and one output per subsystem), but it is satisfied with the help of the multichannel approach. In order not to lose this flexibility and still use the small-gain ideas, we need a way to build vectors consisting of norms rather than using a norm. To this end we use the $|\cdot|_{\mathbf{vec}}$ and $|\cdot|_{\mathbf{stc}}$ notation introduced in Section 1.2. For instance, $|y_i|_{\mathbf{vec}} = (|y^1_i|, \dots, |y^{r_i}_i|)$. As we are considering a different system class as in Section 3.1 we have to use another stability notion. To be more specific the next assumption should hold.

Assumption 3.4.1. The systems Σ_i, $i = 1, 2, \dots, n$ are IOpS as in Definition 1.3.14 at $t = t_0$ with $t_{i\mathtt{d}}(t_0) \geq 0$, restrictions $\Delta_{xi} \in \mathbb{R}$, $\Delta_{ui} \in \mathbb{R}^{l_i}$, $\Delta_{wi} \in \mathbb{R}^{v_i}$ and offsets $\delta_i \in \mathbb{R}^{r_i}$. More precisely, there exist $\beta_i \in \mathcal{K}^{r_i \times 1}$, $\Gamma_{iu} \in \mathcal{G}^{r_i \times l_i}$ and $\Gamma_{iw} \in \mathcal{G}^{r_i \times v_i}$, such that for each $i = 1, 2, \dots, n$ and each $t_0 \in \mathbb{R}$ the condition $\|x_{id}(t_0)\| \leq \Delta_{xi}$, $\sup_{t \geq t_0} \|u_{i\mathtt{d}}(t)\|_{\mathbf{vec}} \leq \Delta_{ui}$ and $\sup_{t \geq t_0} \|w_{i\mathtt{d}}(t)\|_{\mathbf{vec}} \leq \Delta_{iw}$ imply that the corresponding solution of Σ_i is well-defined for all $t \geq t_0$ and the following inequalities hold

$$
\sup_{t \geq t_0} |y_i(t)|_{\mathbf{vec}} \leq \mu_i \left(\beta(\|x_{id}(t_0)\|), \Gamma_{iu}(\sup_{t \geq t_0} \|u_{i\mathtt{d}}(t)\|_{\mathbf{vec}}), \Gamma_{iw}(\sup_{t \geq t_0} \|w_{i\mathtt{d}}(t)\|_{\mathbf{vec}}), \delta_i \right) \tag{3.43}
$$

and

$$
\limsup_{t \to \infty} |y_i(t)|_{\mathbf{vec}} \leq \mu_i \left(0, \Gamma_{iu}(\limsup_{t \to \infty} \|u_{i\mathtt{d}}(t)\|_{\mathbf{vec}}), \Gamma_{iw}(\limsup_{t \to \infty} \|w_{i\mathtt{d}}(t)\|_{\mathbf{vec}}), \delta_i \right), \tag{3.44}
$$

with $\mu_i \in \mathrm{MAF}^{r_i}_{l_i+v_i+2}$.

Following the approach from Section 3.1 we stack the inequalities from Assumption 3.4.1 into a single inequality. To ease notation, define $m := \sum_{i=1}^n r_i$, $l := \sum_{i=1}^n l_i$ and $v := \sum_{i=1}^n v_i$. Furthermore, let

$$B(\|x_{\mathtt{d}}(t_0)\|_{\mathbf{vec}}) = (\beta_1(\|x_{1\mathtt{d}}(t_0)\|)^\top, \ldots, \beta_n(\|x_{n\mathtt{d}}(t_0)\|)^\top)^\top,$$

$$\Gamma_U = \mathrm{diag}(\Gamma_{1u}, \ldots, \Gamma_{nu}), \quad \Gamma_W = \mathrm{diag}(\Gamma_{1w}, \ldots, \Gamma_{nw}), \text{and}$$

$$\delta = \left(\delta_1^T, \ldots, \delta_n^T\right)^T .$$

Moreover, we need a notation, which allows us to lump several $|\cdot|_{\mathbf{vec}}$ together. To this end, we use the $|\cdot|_{\mathbf{stc}}$ notation introduced in Section 1.2. Recall that $\|x_{\mathtt{d}}(t_0)\|_{\mathbf{vec}} \in \mathbb{R}^n$, $\|u_{\mathtt{d}}(t_0)\|_{\mathbf{stc}} \in \mathbb{R}^l$, and $\|w_{\mathtt{d}}(t_0)\|_{\mathbf{stc}} \in \mathbb{R}^v$.
Using this notation, we can follow the path of Section 3.1 and equivalently rewrite (3.43) and (3.44) to get a single inequality given by

$$|y(t)|_{\mathbf{stc}} \leq \mu\left(B(\|x_{\mathtt{d}}(t_0)\|_{\mathbf{vec}}), \Gamma_U(\sup_{t\geq t_0} \|u_{\mathtt{d}}(t)\|_{\mathbf{stc}}), \Gamma_W(\sup_{t\geq t_0} \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}), \delta\right) \quad (3.45)$$

for all $t \geq t_0$, respectively

$$\limsup_{t\to\infty} |y(t)|_{\mathbf{stc}} \leq \mu\left(0, \Gamma_U(\limsup_{t\to\infty} \|u_{\mathtt{d}}(t)\|_{\mathbf{stc}}), \Gamma_W(\limsup_{t\to\infty} \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}), \delta\right) \quad (3.46)$$

with $\mu \in \mathrm{MAF}^m_{l+v+2}$.
In Section 3.1 we have seen that certain properties of a map g was helpful in proving the ISS property of the overall system. Next we want to derive this map. To this end let

$$\tilde{g}(s, w_1, w_2, \delta) := \mu\left(B(w_1), \Gamma_U(s), \Gamma_W(w_2), \delta\right) , \quad (3.47)$$

where $\tilde{g} : \mathbb{R}^l_+ \times \mathbb{R}^n_+ \times \mathbb{R}^v_+ \times \mathbb{R}^n_+ \to \mathbb{R}^m_+$. Observe that we change the sequence of the arguments to be compatible with the convention that the first argument is the state of the corresponding comparison system. Note that the introduction of $\tilde{g}$ eases the notation and brings us into the opportunity to benefit from the results of the last sections.
This can now be used to rewrite (3.45) and (3.46) equivalently to

$$|y(t)|_{\mathbf{stc}} \leq \tilde{g}(\|u_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \|x_{\mathtt{d}}(t_0)\|_{\mathbf{vec}}, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \delta) \quad (3.48)$$

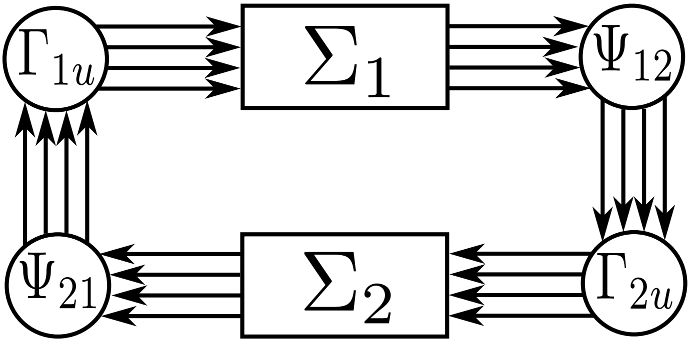

Figure 3.3: Schematic sketch of the interconnection of two systems communicating over multiple channels

and

$$\limsup_{t\to\infty} |y(t)|_{\mathbf{stc}} \leq \tilde{g}(\limsup_{t\to\infty} \|u_{\mathtt{d}}(t)\|_{\mathbf{stc}}, 0, \limsup_{t\to\infty} \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \delta)\,. \qquad (3.49)$$

So far, we equivalently rewrote the inequalities from Assumption 3.4.1 using the introduced notation. As we want to study the interconnection of the subsystem, we need a relation between the outputs y and the inputs $u_{\mathtt{d}}$. Different from Section 3.1 we want to consider systems that communicate over digital channels.

Hence we need some assumptions, that describe the effect of the communication. Before we can do so, we have to introduce the delayed version of the output. Thus, define

$$|\hat{y}_i(t)|_{\mathbf{vec}} := \left(\left|y_i^1(t-\tau_i^1)\right|, \ldots, |y_i^{r_i}(t-\tau_i^{r_i})|\right)$$

with $\tau_i^j : \mathbb{R} \to \mathbb{R}_+, i = 1, \ldots, n, j = 1, \ldots, r_i$ Lebesgue measurable functions. They describe the delay of the j-th component ot the output of the i-th subsystem.

As before, we stack all the outputs of the different subsystems into one vector given by

$$|\hat{y}(t)|_{\mathbf{stc}} := \left(|\hat{y}_1(t)|_{\mathbf{vec}}^\top, \ldots |\hat{y}_n|_{\mathbf{vec}}^\top\right)^\top$$

This can now be used to specify the conditions on the interconnection.

Assumption 3.4.2. The interconnection of the n subsystems is described by

$$\|u_{\mathtt{d}}(t)\|_{\mathbf{stc}} \equiv 0,\ \forall t < T_0 \qquad (3.50)$$

$$\|u_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \Psi_\mu(|\hat{y}(t)|_{\mathbf{stc}}),\ \forall t \geq T_0 \tag{3.51}$$

where the operator Ψ_μ is of the form

$$\Psi_\mu(|s|_{\mathbf{stc}}) := \begin{pmatrix} \mu_1(0, \Psi_{12}(s_1), \ldots, \Psi_{1n}(s_n)) \\ \vdots \\ \mu_n(\Psi_{n1}(s_1), \ldots, \Psi_{nn-1}(s_{n-1}), 0) \end{pmatrix}$$

with $\Psi_{ij} \in \mathcal{G}^{l_i \times r_j}$ for all $i, j = 1, \ldots, n$ and $\mu_i \in \mathrm{MAF}_l^{r_i}$.

Remark 3.4.3. Assumption 3.4.2 states that there exists a $T_0 \in \mathbb{R}$ which is the first time instance a connection has been established. Before that time the input is constant 0. After T_0 the operator Ψ_{ij} describes how the output of the j-th subsystem influences the input of the i-th subsystem. Hence the overall influence of the output of the i-th system to the output of the j-th system is given by $\Gamma_{ju} \circ \Psi_{ji}$.
For a schematic figure of the interconnection of two multichannel systems see Figure 3.3.
Note that by definition we have

$$|\hat{y}|_{\mathbf{stc}} \leq \|y_{\mathtt{d}}\|_{\mathbf{stc}}\ . \tag{3.52}$$

To ensure that communication between the subsystems happens at least sometimes, we have to make the following assumption on the delays.

Assumption 3.4.4. There exists $\tau_* > 0$ and a piecewise continuous function $\tau^* : \mathbb{R} \mapsto \mathbb{R}_+$ with $\tau^*(t_2) - \tau^*(t_1) \leq t_2 - t_1$ for all $t_2 \geq t_1$ such that

$$\tau_* \leq \min_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \leq \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \leq \tau^*(t), \tag{3.53}$$

and

$$t - \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,r_i}} \{\tau_i^j(t)\} \to \infty \text{ as } t \to \infty \tag{3.54}$$

for all $t \geq 0$.

Remark 3.4.5. The inequalities (3.53) say that the delays should be bounded from above by $\tau^*(t)$ and from below by $\tau_* > 0$. Because of the propagation delay of any physical system the existence of a lower bound τ_* is not restrictive.
Basically, (3.54) states that the delay should not grow faster than the time itself. In the literature an assumption of the form $\dot{\tau}^*(t) < 1$ can be found to

ensure property (3.54). To account for possible information losses we have to adopt the more general Assumption 3.4.4.
In [PMTL09] a methodology to satisfy Assumption 3.4.4 either by time-stamping or by sequence numbering can be found. Time-stamping refers to an approach where in a packet based transmission (e.g., TCP) every packet is marked with the current time (see Chapter 2 for more details), while sequence numbering maps an uniquely defined number to every packet.

Now we have collected all the necessary steps to define a discrete comparison system, which will be used to infer stability of the interconnection. To this end consider

$$g(s, w_1, w_2, \delta) := \tilde{g}(\Psi_\mu(s), w_1, w_2, \delta) = \mu\left(B(w_1), \Gamma_U \circ \Psi_\mu(s), \Gamma_W(w_2), \delta\right) ,$$

with $g : \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}_+^v \times \mathbb{R}_+^n \to \mathbb{R}_+^m$. The first argument $s \in \mathbb{R}_+^n$ describes the effect of the interconnection by modeling the influence from the multichannel outputs of all subsystems to the multichannel inputs of all other subsystems via a communication channel. The influence of the initial state is given by $w_1 \in \mathbb{R}_+^n$, while $w_2 \in \mathbb{R}_+$ models the influence of external disturbances. The offset is given by $\delta \in \mathbb{R}_+^n$.
Finally, this leads to

$$s(k+1) = g(s(k), w_1(k), w_2(k), \delta) . \tag{3.55}$$

Obviously, g is monotone and continuous in all of its arguments. Furthermore, it holds that $g(0,0,0,0) = 0$.
Let $\Delta_x := (\Delta_{1x}, \dots, \Delta_{nx})^\top$, $\Delta_u := \left(\Delta_{1u}^\top, \dots, \Delta_{nu}^\top\right)^\top$ and
$\Delta_w := \left(\Delta_{1w}^\top, \dots, \Delta_{nw}^\top\right)^\top$. The ensuing theorem is the main contribution of this section. It shows that an interconnection of IOpS subsystem is stable in the IOpS sense, provided that a discrete comparison system is ISS.

Theorem 3.4.6. *Suppose system* (3.42) *satisfies Assumptions 3.4.1,3.4.2 and 3.4.4. Let the discrete comparison system* (3.55) *be ISS. Furthermore, assumes that $g(s, 0, 0, 0)$ is irreducible.*
Let $\Delta^* \in \mathbb{R}^m$ such that $\Delta^* \geq g(\Delta^*, \Delta_x, \Delta_w, \delta)$ and

$$\Psi_\mu(\Delta^*) \leq \Delta_u . \tag{3.56}$$

Then system (3.42) *is IOpS as in Definition 1.3.14 at $t = T_0$ with*

$$t_d(T_0) = \max_{i=1,\dots,n} \{t_{id}(T_0)\} + \tau^*(T_0) + \tau^*(T_0 - \tau^*(T_0)) . \tag{3.57}$$

and restrictions Δ_x, Δ_w and offset δ. More precisely, the conditions $\|x_{\mathsf{d}}(T_0)\|_{\mathbf{vec}} \leq \Delta_x$ and $\sup_{t \geq t_0} \|w_{\mathsf{d}}(t)\|_{\mathbf{vec}} \leq \Delta_w$ imply that there exist $\mu_1 \in$

MAF_{v+2}^{m} *such that*

$$\sup_{t \geq T_0} |y(t)|_{\mathbf{stc}} \leq \mu_1(\|x_{\mathtt{d}}(T_0)\|_{\mathbf{vec}}, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \delta) \tag{3.58}$$

and there exist $\mu_2 \in \mathrm{MAF}_{v+1}^{m}$ *such that*

$$\limsup_{t \geq T_0} |y(t)|_{\mathbf{vec}} \leq \mu_2(\|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \delta) \,. \tag{3.59}$$

Proof. Before we show that the interconnection is IOpS we prove that Δ^* exists. Because g is irreducible, we know by Lemma D.0.18 that there exists a $\sigma \in \mathcal{K}_\infty^m$ and $\rho \in \mathcal{K}_\infty^{n+v+n}$ such that

$$g(\sigma(r), \rho(r)) < \sigma(r)$$

for all $r > 0$. Now choose r^* such that

$$\rho(r^*) \geq \begin{pmatrix} \Delta_x \\ \Delta_w \\ \delta \end{pmatrix} ,$$

which is possible because $\rho \in \mathcal{K}_\infty^{n+v+n}$.
This yields

$$\sigma(r^*) > g(\sigma(r^*), \rho(r^*)) \geq g(\sigma(r^*), \Delta_x, \Delta_w, \delta) \,.$$

Defining $\Delta^* := \sigma(r^*)$ yields the existence of a Δ^* for which
$\Delta^* \geq g(\Delta^*, \Delta_x, \Delta_w, \delta)$ holds.
In order to achieve the IOpS property of the overall system, we have to show that the restrictions are not violated. Because of the relation between the output and the input of the system (3.51) we have to bound the output by Δ^* for all positive times. Now assume that

$$\|x_{\mathtt{d}}(T_0)\|_{\mathbf{vec}} \leq \Delta_x \text{ and } \sup_{t \geq T_0} \|w_{\mathtt{d}}(t)\|_{\mathbf{vec}} \leq \Delta_w \,. \tag{3.60}$$

Assumption 3.4.1 respectively (3.48) and (3.50) together with causality arguments imply that

$$|y(T_0)|_{\mathbf{stc}} \leq g(0, \Delta_x, \Delta_w, \delta) \leq \Delta^* ,$$

because of the monotonicity of g and the definition of Δ^*.
By using (3.50), (3.51) and Assumption 3.4.4 we get

$$\sup_{t \in [T_0 - t_{\mathbf{d}}(T_0), T_0 + \tau_*]} |u(t)|_{\mathbf{stc}} \leq \Psi_\mu(\Delta^*) \leq \Delta_u ,$$

where the last inequality follows from (3.56). The latter chain of inequalities states that the restriction on the input Δ_u is not violated up to time T_0. Because of the minimal delay τ_*, the restrictions are still not violated on the interval $t \in [t_0 - t_{\mathtt{d}}(T_0), T_0 + \tau^*]$.
Hence there exists a $T_{max} > T_0 + \tau^*$ such that the solutions of (3.42) are well-defined for all $t \in [T_0, T_{max})$. Now we want to show that the bound on the output Δ^* is not violated on the interval $t \in [T_0, T_{max})$. To be precise, we want to show that

$$\sup_{t\in[T_0,T_{max})} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \Delta^* . \tag{3.61}$$

To prove this by contradiction, assume that there exists a $T_1 \in [T_0, T_{max} - \tau_*)$ such that

$$\sup_{t\in[T_0,T_1]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \Delta^* \text{ and } \sup_{t\in[T_0,T_1+\tau_*]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \not\leq \Delta^* . \tag{3.62}$$

Again by using (3.48) we get

$$\sup_{t\in[T_0,T_1+\tau_*]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq g(\sup_{t\in[T_0,T_1]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}}, \Delta_x, \Delta_w, \delta) .$$

By using the first part of (3.62) and the monotonicity of g we arrive at

$$\sup_{t\in[T_0,T_1+\tau_*]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq g(\Delta^*, \Delta_x, \Delta_w, \delta) \leq \Delta^* ,$$

again by monotonicity of g and the definition of Δ^*. This contradicts the second inequality in (3.62). Hence we have

$$\sup_{t\in[T_0,T_{max})} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \Delta^* . \tag{3.63}$$

Next we want to show that $T_{max} = \infty$.
Again we will prove this by contradiction. Due to Assumption 3.4.1 the inequality $T_{max} < \infty$ implies that at least one of the restrictions must be violated. By (3.60) we see that

$$\sup_{t\in[T_0,T_{max})} |u(t)|_{\mathbf{stc}} \not\leq \Delta_u .$$

This in turn implies together with (3.51) and (3.56)

$$\Psi_\mu(\sup_{t\in[T_0,T_{max})} |\hat{y}(t)|_{\mathbf{stc}}) \not\leq \Psi_\mu(\Delta^*) .$$

By monotonicity of Ψ_μ and the definition of $y_{\mathtt{d}}$ (see (3.52)) we arrive at

$$\sup_{t\in[T_0,T_{max}]} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \not\leq \Delta^*\,,$$

which contradicts (3.63), and therefore $T_{\max} = \infty$.
Now we have established that the restrictions hold on $t \in [T_0, \infty)$. Hence we can use (3.48) together with (3.51) and the monotonicity of g to get

$$\sup_{t\geq T_0} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq g\left(\sup_{t\geq T_0} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \|x_{\mathtt{d}}(T_0)\|_{\mathbf{vec}}\,, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta\right).$$

Because (3.55) is ISS, we know by Theorem 3.2.2 that there exists a positive definite and proper $\xi : \mathbb{R}^{v+2} \to \mathbb{R}^m_+$ such that

$$\sup_{t\geq T_0} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \xi(\|x_{\mathtt{d}}(T_0)\|_{\mathbf{vec}}\,, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta)\,.$$

By Lemma 1.1.11 there exists an $\mu_1 \in \mathrm{MAF}^m_{v+2}$ such that

$$\sup_{t\geq T_0} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \mu_1(\|x_{\mathtt{d}}(T_0)\|_{\mathbf{vec}}\,, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta)\,.$$

Of course, $\sup_{t\geq T_0} |y(t)|_{\mathbf{stc}} \leq \sup_{t\geq T_0} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}}$ and we have shown (3.58). Similarly we get

$$\limsup_{t\to\infty} \|y_{\mathtt{d}}(t)\|_{\mathbf{stc}} \leq \xi(0, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta) \leq$$
$$\mu_1(0, \|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta) =: \mu_2(\|w_{\mathtt{d}}(t)\|_{\mathbf{stc}}\,, \delta)\,,$$

where we used Lemma 1.1.11 again. Realizing that (3.59) is easily deduced finishes the proof. □

Remark 3.4.7. In the premise of Theorem 3.4.6 we demand $g(s, 0, 0, 0) = \mu\,(0, \Gamma_U \circ \Psi_\mu(s), 0, 0)$, to be irreducible. Recall that we can interpret

$$\Gamma_U \circ \Psi = \begin{pmatrix} 0 & \Gamma_{1u}\circ\Psi_{12} & \cdots & \Gamma_{1u}\circ\Psi_{1n} \\ \Gamma_{2u}\circ\Psi_{21} & 0 & \cdots & \Gamma_{2u}\circ\Psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{nu}\circ\Psi_{n1} & \Gamma_{nu}\circ\Psi_{n2} & \cdots & 0 \end{pmatrix}.$$

And we say that $\Gamma_U \circ \Psi_\mu$ is irreducible, if the graph associated to $\Gamma_U \circ \Psi$ is. The irreducibility states that eventually every input of every subsystem effects every output. This assumption seems too strong as we know that cascades of ISS systems are ISS (see [ST95]). A way how to relax the irreducibility assumption can be found in [DRW10]. We expect that similar techniques can be used here. Although we lose the multichannel structure inherent to $\Gamma_U \circ \Psi$, i.e., each block $\Gamma_{iu} \circ \Psi_{ij}$ models the effect of the different outputs of the jth subsystem to the inputs of the ith subsystem. Because the presented technique of [DRW10] uses a transformation of the matrix into upper block diagonal form by permuting rows and columns, the multichannel structure may be lost.

3.5 Notes and References

Since the sixties of the last century, small-gain type conditions [Zam66] have proved to be a valuable tool for analyzing the stability properties of the interconnection of two systems, where the influence from one to the other system is described by a linear function.
In this context the small-gain theorem was extended to the interconnection of several $\mathcal{L}_p$-stable subsystems. Early accounts of this approach are [Vid81] (see also [Šil78]) and references therein. For instance, in [Vid81], Theorem 6.12, the influence of each subsystem on the others is measured via an $\mathcal{L}_p$-gain, $p \in [1, \infty]$ and the $\mathcal{L}_p$-stability of the interconnected system holds provided that the spectral radius of the matrix of the gains is strictly less than unity. In other words, the stability of interconnected $\mathcal{L}_p$-stable systems holds under a condition of weak coupling.
In the nonlinear case a notion of robustness with respect to exogenous inputs is input-to-state stability (ISS) (see Section 1.3). If in a large-scale system each subsystem is ISS (or related notions as discussed in Section 1.3), then the influence between the subsystems is typically modeled via nonlinear gain functions. Small-gain theorems have been developed for ISS systems as well ([JTP94, JMW96, Tee96]) and more recently they have been extended to the interconnection of several ISS subsystems ([DRW07, DRW10]). For a recent comprehensive discussion about the literature on ISS small-gain results see [LJH10].
The notion of weak coupling in the nonlinear case is ensured for instance by the no joint increase condition as in e.g., [DRW07].
Another approach is followed by [JW08]. The corresponding condition is called *cycle condition*, which is known to be equivalent to the no joint increase condition for the case $\mu = |\cdot|_\infty$.
Here we are following another approach. Instead of checking some topological conditions of the gain operator, we demand an ISS property of a comparison system which is induced by the gain operator.
First steps in this direction can be found in [Rüf07]. There the connection between the no joint increase condition and the stability property of the

induced dynamical system was examined.
For the special case of $\mu = |\cdot|_\infty$ and $\mu = |\cdot|_1$ the stability condition as well as the equivalence between the ISS property and the no joint increase condition can be found in [Rüf10b].
Our approach has several advantages compared to the above mentioned. It allows us to use a more general class of comparison functions between the gains. Which means that we are not limited to $|\cdot|_\infty$, $|\cdot|_1$ or even general MAF.
Also in the literature it is not clear how the effects of disturbances of the gains should be handled. In the ISS context we are following this question has a natural answer, as we will see in Chapter 4. The motivating example from Section 3.1 is borrowed from [DRW07]. However, validity of Assumption 3.1.2 is guaranteed in a different way.
To be more precise, in [DRW07] conditions of the type

$$g(s,0) \not\geq s \text{ for all } s \neq 0$$

where studied. This condition resembles the UOC property from Section 3.2 for the case $w = 0$. The property NP was already discussed in [KJ11], again for the case $w = 0$.
In Section 3.3 we discussed ISS-Ω-path, but we only give a way to construct ρ. An approach to construct the corresponding σ numerically is given in [GW12]. Remarkably, the presented approach can handle systems of the dimension of several hundreds on a typical computer.
The main theorem from Section 3.4 is based on [SW10b, SW10a]. But the theorem given here is slightly more general. We want to stress that in order to prove Theorem 3.4.6, we had to use the concept of an ISS-Ω-path to conclude the existence of a "decay" point Δ^*. The proofs of the corresponding results of [SW10b, SW10a] does not need this, as this existence is trivially fulfilled in [SW10b, SW10a].
The stability notion which we named IOpS was first introduced in [PMTL09]. We should mention that it is unknown yet, whether the AG and the GS property are equivalent to ISS for the case of infinite dimensional systems. Hence it is a abuse of notation to name the stability notion IOpS, but we find it convenient to do so nevertheless.
Also the multichannel system stems from [PMTL09], but only for 2 subsystems.
An extension to the case of an arbitrary number of subsystems is given in [SW10b, SW10a]. However, the material presented in Section 3.4 is still more general than the results from [SW10a].
A rather strong small-gain type condition is used in [PMTL09] to conclude stability. This has been relaxed in [RSW10].

Some of the relations discussed in Section 3.2 and 3.3 may appear more familiar for functions g of a very special form. To this end let $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, and consider $g(s,w) := As + Bw$. It is clear that the stability of the time-discrete systems (3.12) is intimately related to the condition that the spectral radius $\rho(A)$ is less than unity. This in turn allows to compute $s = (I - A)^{-1}Bw$, the unique solution to

$$s \leq As + Bw\,.$$

Perhaps not so well known is that $\rho(A) < 1$ if and only if $As \not\geq s$ holds. Finally, the paths σ and ρ are closely connected to an eigenvector of A via the celebrated Perron-Frobenius Theorem, cf. [Rüf10a].

For functions g of a special form, some of the above conditions and their relations have been investigated in previous works, in particular [Rüf10a] and [Rüf10b].

It should be stressed that the equivalences here are strongly based on the monotonicity of g and the resulting forward invariance of $\mathbb{R}_+^n$ with respect to (3.12). In this regard our results are not a mere extension of the impressive list of equivalent notions of input-to-state stability given in [GL00] and [JW01], as the equivalences in these works also hold without the monotonicity requirement. Of course, we could use any of these equivalent notions under the additional assumptions of Theorem 3.2.2 to use them as a small-gain condition.

Chapter 4

Event-Triggered Control

In Chapter 2 a way to overcome certain limitations posed by a communication channel was discussed. In particular, the effects of quantization and delay on a single subsystem was considered. If the number of subsystems grows, bandwidth limitations may become more severe. In this chapter we will use ideas from Chapter 3 to deal with this issue. To this end we try to lower the amount of information transmissions. This is achieved by only transmitting information whenever certain events happen. In the literature this approach is known as *event-triggered control*. An event could be that a certain amount of time passes or that some state crosses some boundaries. Here the events will be that the state crosses some error threshold.

In Figure 4.1 a situation known as periodic sampling is depicted. In periodic sampling the current state is sampled every τ units of time, a control action is calculated, and this value is held constant until the next sample is taken. The approach of keeping the control action constant in between sampling times is named zero order hold in the control engineering community.

In Figure 4.2 an example of event-triggered control, termed δ-sampling (or deadband sampling resp. send-on-delta) can be seen. Whenever the last sampled value deviates from the current state by a number of delta, an event is triggered. In regions where the trajectory does not change much, the approach of δ-sampling can lead to fewer events than periodic sampling.

In both cases it is not trivial to decide how to choose the sampling period ($\tau = t_k - t_{k-1}$) respectively δ to ensure stability of the closed loop system. For a discussion on the subject of choosing the sampling period τ for a given system, see [ÅW84, ZOB90, NTS99].

In this chapter we try to convince the reader that an event based sampling scheme, which compares the error between the sampled value and the actual state to a Lyapunov function will result in event times that stabilizes the

system.
As our general setup is large-scale, we are heading for a purely decentralized approach. Hence the subsystems should be able to compute the necessary values by only using local information.

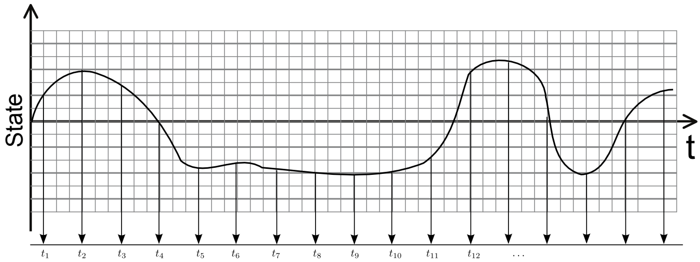

Figure 4.1: Periodic sampling

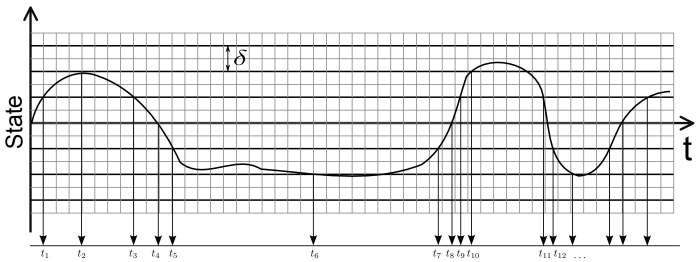

Figure 4.2: Example of event based sampling

4.1 Triggering Functions

Consider the interconnection of N systems described by equations of the form:

$$\dot{x}_i = f_i(x, u_i)\,, \quad u_i = g_i(x + e)\,, \tag{4.1}$$

where $i \in \mathcal{N} := \{1, 2, \ldots, N\}$, $x = (x_1^\top \ldots x_N^\top)^\top$, with $x_i \in \mathbb{R}^{n_i}$, is the state vector and $u_i \in \mathbb{R}^{m_i}$ is the ith control input. The g_i is a local controller of system i.
The vector e, with $e = (e_1^\top \ldots e_N^\top)^\top$ and $e_i \in \mathbb{R}^{n_i}$ is an error affecting the state.

We shall assume that the maps f_i satisfy appropriate conditions which guarantee existence and uniqueness of solutions for $\mathcal{L}_\infty$ inputs e. In particular, the f_i are locally Lipschitz in the first argument and continuous in the second. Also we assume that the g_i are locally bounded, i.e. for each compact set $K \subset \mathbb{R}^n$ ($n := \sum_{i=1}^N n_i$) there exists a constant C_K with $|g_i(x)| \leq C_K$ for each $x \in K$. Moreover, we assume that $0 = f_i(0,0)$ and $0 = g_i(0)$ for all $i \in \mathcal{N}$.
For future use we denote the set of states entering the dynamics of system i by

$$\Sigma(i) = \{j \in \mathcal{N} \ : \ f_i \text{ explicitly depends on } x_j\}\,.$$

We say that f_i does *not* depend on x_j, if $\partial f_i/\partial x_j \equiv 0$. Similarly for the controllers we denote

$$C(i) = \{j \in \mathcal{N} \ : \ g_i \text{ explicitly depends on } x_j\}\ .$$

It is also convenient to define the set of the controllers to which the state of system i is broadcast:

$$Z(i) = \{j \in \mathcal{N} \ : \ g_j \text{ depends explicitly on } x_i\}\ .$$

As an easy consequence of the above definition we have

Lemma 4.1.1. *An index i belongs to the set $C(j)$, if and only if j belongs to $Z(i)$.*

The next definition may be regarded as the heart of the event-triggering approach, as it introduces the condition each subsystem has to check to decide when an event happens.

Definition 4.1.2. A map $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a *triggering function*, if T is jointly continuous and

$$T(\cdot, 0, \cdot) < 0\,. \tag{4.2}$$

We sometimes use triggering functions, which only depend on two arguments. In this case we write $T(\cdot,\cdot)$.
Consider the interconnection of N subsystems as before

$$\begin{aligned} \dot{x}_i &= f_i(x, u_i) \quad & u_i &= g_i(x+e) \\ \dot{\hat{x}} &= 0 & e &= \hat{x} - x \end{aligned} \tag{4.3}$$

with condition on the triggering function

$$T_i(x_i, e_i, \dot{x}_i) \geq 0\ . \tag{4.4}$$

Now we are able to describe the evolution of the system (4.3) together with a triggering condition. For simplicity we set the initial controller error to $e_0 = 0$ and $t_0 = 0$. Given an initial condition x_0 we define

If $\exists i \in \mathcal{N}$ such that $T_i(x_i(t), e_i(t), \dot{x}_i(t)) \geq 0$:

$$t_k = t \tag{4.5}$$

$$\hat{x}_i(t) = x_i(t) \quad \forall i \in \mathcal{N}_{t_k} := \{j \in \mathcal{N} \ : \ T_j(x_i(t), e_i(t), \dot{x}_i(t)) \geq 0\} \tag{4.6}$$

Otherwise:

$$\dot{x}_i(t) = f_i(x(t), g_i(x(t) + e(t))) \,, \quad \forall i \in \mathcal{N} \tag{4.7}$$

$$\dot{\hat{x}}_i(t) = 0 \,, \qquad \forall i \in \mathcal{N} \tag{4.8}$$

$$e_i(t) = \hat{x}_i(t) - x_i(t) \,, \qquad \forall i \in \mathcal{N} \tag{4.9}$$

The set $\mathcal{N}_{t_k}$ describes the set of all subsystems i that trigger an event at time t_k.
Solutions to such a triggered feedback are defined as follows. By (4.5) together with the condition for (4.5) and (4.6) we have

$$t_k := \inf\{t > t_{k-1} \ : \ \exists i \in \mathcal{N} \text{ s.t. } T_i(x_i(t), e_i(t), \dot{x}_i(t)) \geq 0\} \,,$$

for all $k \geq 1$. Equation (4.6) states that at time instant t_k the systems i for which $T_i(x_i, e_i, \dot{x}_i) \geq 0$ broadcast their respective state x_i to all controllers g_j with $i \in C(j)$. The systems j which use the state x_i in the control law $g_j(x)$ update only the state x_i while all the other variables are kept equal to the previously set values.
For obvious reasons the time instances t_k are called event times or triggering times.

Remark 4.1.3. The condition (4.2) will be used to ensure that no event can be triggered as long as the error e is zero, but in many useful triggering conditions we have that $T_i(0, 0, \dot{x}_i) = 0$. If the system were to remain at $x = 0$ this would lead to a continuum of triggering events, which do not provide information. To avoid this (academic) problem we propose to add the condition that information is broadcast once x_i reaches the state zero, but no further transmission by system i occurs as long as it stays at zero.

The next proposition follows directly from the definition of t_k and continuity of T.

Proposition 4.1.4. *The event times are a strictly monotonically increasing sequence, i.e.*

$$t_{k+1} > t_k$$

for all $k \in \mathbb{N}$.

Proof. By convention we have $e(0) = 0$ and thus by Definition 4.1.2 we have $T_i(x_i(0), 0, \dot{x}_i(0)) < 0$ for all $i \in \mathcal{N}$. Because of the continuity of T_i we conclude $t_1 > 0$. By combining (4.6) and (4.9) we get $e_i(t_1^+) = 0$ for all $i \in \mathcal{N}_{t_1}$. Note that we have for all $i \notin \mathcal{N}_{t_1}$ that $e_i(t_1^+) = e_i(t_1)$ and thus for all $i \in \mathcal{N}$ we have $T_i(x_i(t_1^+), e_i(t_1^+), \dot{x}_i(t_1^+)) < 0$. Repeating this arguments yields the claim. □

System (4.3) consists of a continuous part and a discrete part. Such systems are often called hybrid systems. For a definition of respectively an introduction to hybrid systems see [vdSS00]. In contrast to ordinary differential equations it might happen that the event times accumulate in finite time.

Definition 4.1.5. The accumulation of triggering times in finite time is called *Zeno effect* i.e.

$$\lim_{k\to\infty} t_k = t^* < \infty\,.$$

The time instance t^* is called *Zeno time*.

Definition 4.1.6. If for an initial value x_0 it holds for the solution of (4.3) that the Zeno effect occurs i.e.,

$$\lim_{k\to\infty} t_k < \infty\,,$$

we say that the triggering function causes Zeno.

For a more thorough discussion on Zeno see [vdSS00]. We will use the terminology Zeno solutions or Zeno behavior synonym for Definition 4.1.6. As we heading for reducing the amount of information, Zeno is an unwanted effect.
The Zeno case will be dealt with explicitly in Section 4.4 and 4.5.
The basic idea behind our Lyapunov based event-triggering will be explained in Section 4.3. Before we do so, we have to describe how the effects of the interconnection and the imperfect knowledge of the states to the controllers are modeled.

4.2 Gain Operators

In order to use the ideas from Chapter 3 we have to assume that each subsystem from (4.3) is ISS. To be precise, we assume the following.

Assumption 4.2.1. For $i = 1, 2, \ldots, N$, there exist a differentiable function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, and class-$\mathcal{K}_\infty$ functions α_{i1}, α_{i2} such that

$$\alpha_{i1}(|x_i|) \leq V_i(x_i) \leq \alpha_{i2}(|x_i|) \ . \tag{4.10}$$

Moreover there exist functions $\mu_i \in \mathrm{MAF}_{2N}$, $\gamma_{ij}, \eta_{ij} \in \mathcal{G}$, $j = 1, \ldots, N$, and α_i positive definite such that

$$\begin{aligned} V_i(x_i) \geq \mu_i(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{iN}(V_N(x_N)), \eta_{i1}(|e_1|), \ldots, \eta_{iN}(|e_N|)) \\ \implies \nabla V_i(x_i) f_i(x, g_i(x+e)) \leq -\alpha_i(|x_i|) \ . \end{aligned} \tag{4.11}$$

Loosely speaking, the function γ_{ij} describes the overall influence of system j on the dynamics of system i, while the function η_{ij} describes the influence of the system j on the system i via the controller g_i. In particular, $\eta_{ij} \neq 0$ if and only if the controller g_i is using information from system j. In this regard η_{ij} describes the influence of the imperfect knowledge of the state of system j on system i caused by e.g., measurement noise. On the other hand, if $i \neq j$ and $\gamma_{ij} \neq 0$, then system j influences the system i either explicitly or implicitly. Here, explicit influence means that f_i depends on x_j, whereas implicit influence means that g_i depends on x_j.
We assume that $\gamma_{ii} = 0$ for any i. In Figure 4.3 the interconnection with the corresponding gains is depicted. The subsystems are given as Σ_i and the controllers as g_i. There can either be a direct (physical) connection between systems or the systems interchange information via the controllers. In the first case the effects are modeled by the gains γ_{ij} whereas in the second case the effect of the interconnection is modeled by γ_{ij} and η_{ij}.
Observe that if the system i is not influenced by any other system $j \neq i$, and there is no error e_i on the state information x_i used by the controller g_i, then the assumption amounts to saying that the system i is globally asymptotically stable.
In the spirit of Chapter 3 we use the gains γ from Assumption 4.2.1 to define a gain operator $\Gamma_\mu \ : \ \mathbb{R}_+^N \times \mathbb{R}_+^{N \times N} \to \mathbb{R}_+^N$

$$\Gamma_\mu(s, w) = \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \ldots, \gamma_{1N}(s_N), w_{11}, \ldots, w_{1N}) \\ \vdots \\ \mu_n(\gamma_{N1}(s_1), \ldots, \gamma_{NN}(s_N), w_{N1}, \ldots, w_{NN}) \end{pmatrix},$$

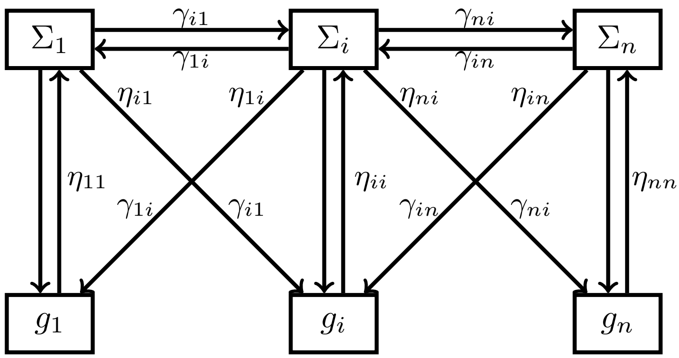

Figure 4.3: Schematic of the interconnection

with w as an arbitrary input. It will become clear later how to relate w to the gains η from Assumption 4.2.1.
Again, in the framework of the last chapter we assume the following.

Assumption 4.2.2. There exists an ISS-Ω-path for Γ_μ, i.e. there exist $\sigma \in \mathcal{K}_\infty^N$ and $\varphi \in \mathcal{G}^{N\times N}$ such that

$$\Gamma_\mu(\sigma(r), \varphi(r)) < \sigma(r)$$

for all $r > 0$.

In Chapter 3 we have seen under which conditions such an ISS-Ω-path exists. We also know from the last chapter that if $w_{ij} \le \varphi_{ij}(r)$ the disturbance w does not interfere with the stability properties.
As we want to derive triggering conditions comparing φ and the error e there is no loss in generality if we set $\varphi_{ij} = 0$ whenever $\eta_{ij} = 0$.

Proposition 4.2.3. *For $i \neq j$ the gains γ_{ij} may be chosen to be 0 if and only if $j \notin \Sigma(i)$ and $j \notin C(i)$. Analogously, $\eta_{ij} = 0$ if and only if $j \notin C(i)$ and finally $\varphi_{ij} = 0$ if and only if $j \notin C(i)$.*

Proof. Let $i \neq j$. By the definition of $\Sigma(i)$ respectively $C(i)$ we know that $j \notin \Sigma(i) \cap C(i)$, if and only if f_i depends not on x_j and hence the right hand side $\nabla V_i(x_i) f_i(x, g_i(x+e)) \le -\alpha(|x_i|)$ from condition (4.11) does not depend on x_j. And thus by monotonicity of a MAF we can equivalently rewrite (4.11) to a condition with $\gamma_{ij} = 0$.
Index $j \in C(i)$ means that the controller g_i depends on x_j. As all controllers

use zero order hold, there is an error affecting the ith state by imperfect knowledge of the jth state and hence $\eta_{ij} \neq 0$.
The statement for φ_{ij} follows from the convention that $\varphi_{ij} = 0$ whenever $\eta_{ij} = 0$. □

In Chapter 3 we have seen that the ISS-Ω-path helps us to conclude that the interconnection is still ISS. In the rest of this chapter we will show how we have to relate η from Assumption 4.2.1 to φ from Assumption 4.2.2 to conclude stability of the interconnected system despite the error e. In particular, we will see in the next section how the gains will lead to scaling functions, which allows us to compare the error to the Lyapunov function in such a way that we are able to conclude stability of the event-triggered closed-loop system.

4.3 A Triggering Condition Using Small-Gain Ideas

Throughout this chapter we use as a candidate for a Lyapunov function a scaled version of the maximum of the Lyapunov functions of the subsystems. Before we show that this function decreases along trajectories, we have the following proposition.

Proposition 4.3.1. *The Lyapunov function candidate* $V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$ *is positive definite and proper. Furthermore, V is locally Lipschitz.*

Proof. Observe that the composition of positive definite functions is again positive definite. As the maximum of positive definite functions is also positive definite, we have the first assertion. For the properness, define $\alpha_1(|x|) := \min_{i\in\mathcal{N}} \sigma_i^{-1} \circ \alpha_{i1}(|x_i|)$ and $\alpha_2(|x|) := \max_{i\in\mathcal{N}} \sigma_i^{-1} \circ \alpha_{i2}(|x_i|)$. As all $\sigma_i, \alpha_{i1}, \alpha_{i2} \in \mathcal{K}_\infty$ the functions α_1, α_2 belong also to $\mathcal{K}_\infty$ by Lemma 1.1.2. A straightforward calculation shows by using (4.10) $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and the properness follows.
Clearly, the concatenation of locally Lipschitz functions is still locally Lipschitz. The statement that the maximum of finitely many locally Lipschitz functions is again locally Lipschitz can be found in the Appendix (B.0.11) and we have shown the claim. □

The next theorem shows that there exist a set of decentralized conditions for the error e, which guarantee that V, as defined in the last proposition, decreases along trajectories of the interconnected system (4.5)-(4.9).

Theorem 4.3.2. *Let Assumptions 4.2.1 and 4.2.2 hold. Let* $V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$ *and for each* $j \in \mathcal{N}$*, define:*

$$\chi_j = \sigma_j \circ \hat{\eta}_j \ , \ \textit{with } \hat{\eta}_j = \max_{i\in Z(j)} \varphi_{ij}^{-1} \circ \eta_{ij} \ . \tag{4.12}$$

Then there exists a positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the condition*

$$V_i(x_i) \geq \chi_i(|e_i|), \quad \forall\, i \in \mathcal{N} \tag{4.13}$$

implies

$$\langle p, f(x, g(x+e))\rangle \leq -\alpha(|x|),\ \forall p \in \partial V(x) \ ,$$

where ∂V *denotes the Clarke generalized gradient (see Appendix B for details) and*

$$f(x, g(x+e)) = \begin{pmatrix} f_1(x, g_1(x+e)) \\ \vdots \\ f_n(x, g_n(x+e)) \end{pmatrix} .$$

Proof. If $x = 0$, we have $e = 0$ by (4.13) and hence $0 = f(x, g(x+e))$. In particular, the consequent of the theorem is true. Now assume $x \neq 0$.
Define $\mathcal{N}(x) \subseteq \mathcal{N}$ as the set of indices i for which $V(x) = \sigma_i^{-1}(V_i(x_i))$. Let $i \in \mathcal{N}(x)$ and set $r = V(x)$. Then

$$\begin{aligned} V_i(x_i) = \sigma_i(r) > \Gamma_{\mu,i}(\sigma(r), \varphi(r)) = \\ \mu_i(\gamma_{i1}(\sigma_1(r)), .., \gamma_{iN}(\sigma_N(r)), \varphi_{i1}(r), .., \varphi_{iN}(r)). \end{aligned} \tag{4.14}$$

Observe that by definition of $V(x)$, for any $i \in \mathcal{N}(x)$ and any $j \in \mathcal{N}$,

$$\gamma_{ij}(\sigma_j(r)) = \gamma_{ij}(\sigma_j(V(x))) \geq \gamma_{ij}(\sigma_j(\sigma_j^{-1}(V_j(x_j)))) = \gamma_{ij}(V_j(x_j)) \ . \tag{4.15}$$

Note that for $j \notin C(i)$ we have by Proposition 4.2.3 $\varphi_{ij} = 0$ and $\eta_{ij} = 0$. Hence for $j \notin C(i)$ it holds trivially that

$$\varphi_{ij}(r) \geq \eta_{ij}(|e_j|). \tag{4.16}$$

We claim this is also true if $j \in C(i)$ (or equivalently $i \in Z(j)$). To this end assume that (4.13) holds:

$$V_j(x_j) \geq \chi_j(|e_j|) \ , \quad \chi_j = \sigma_j \circ \hat{\eta}_j \ .$$

Hence we have, using the definition of V, (4.13) and (4.12), that

$$\begin{aligned} \varphi_{ij}(r) = \varphi_{ij}(V(x)) \geq \varphi_{ij}(\sigma_j^{-1}(V_j(x_j))) \geq \varphi_{ij}(\sigma_j^{-1} \circ \sigma_j(\hat{\eta}_j(|e_j|))) \\ \geq \varphi_{ij}(\sigma_j^{-1} \circ \sigma_j(\varphi_{ij}^{-1} \circ \eta_{ij}(|e_j|))) = \eta_{ij}(|e_j|) \ . \end{aligned} \tag{4.17}$$

Observe that $\mu_i(v) \geq \mu_i(z)$ for all $v \geq z \in \mathbb{R}_+^{2N}$ since $\mu_i \in MAF_{2N}$ and as a consequence of Definition 1.1.4 (ii). Since $r = V(x) \geq \sigma_i^{-1}(V_i(x_i))$ for all $i \in \mathcal{N}$, by (4.15), (4.16) and (4.17),

$$\begin{aligned}\mu_i(\gamma_{i1}(\sigma_1(r)), \ldots, \gamma_{iN}(\sigma_N(r)), \varphi_{i1}(r), \ldots, \varphi_{iN}(r)) \geq \\ \mu_i(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{iN}(V_N(x_N)), \eta_{i1}(|e_1|), \ldots, \eta_{iN}(|e_N|)) \, .\end{aligned}$$

The inequality above and (4.14) yield that for each $i \in \mathcal{N}(x)$

$$\begin{aligned}V_i(x_i) &> \mu_i(\gamma_{i1}(\sigma_1(r)), .., \gamma_{iN}(\sigma_N(r)), \varphi_{i1}(r), .., \varphi_{iN}(r)) \\ &\geq \mu_i(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{iN}(V_N(x_N)), \eta_{i1}(|e_1|), \ldots, \eta_{iN}(|e_N|)) \, .\end{aligned}$$

Hence, by (4.11),

$$\nabla V_i(x_i) f_i(x, g_i(x+e)) \leq -\alpha_i(|x_i|) \tag{4.18}$$

for all $i \in \mathcal{N}(x)$.
We now provide a bound to $\langle p, f_i(x, g_i(x+e))\rangle$ for each $p \in \partial[\sigma_i^{-1} \circ V_i(x_i)]$ and $i \in \mathcal{N}(x)$. Observe that σ^{-1} is only locally Lipschitz and the Clarke generalized gradient must be used for $\sigma_i^{-1} \circ V_i$. By the chain rule for Lipschitz continuous functions (see Appendix B) we have

$$\partial[\sigma_i^{-1} \circ V_i](x_i) \subset \{c\xi \ : \ c \in \partial\sigma_i^{-1}(y),\, y = V_i(x_i),\, \xi \in \partial V_i(x_i)\} \, . \tag{4.19}$$

By $x \neq 0$ we have for all $i \in \mathcal{N}(x)$ that $|x_i| > 0$, because $0 < V(x) = \sigma_i^{-1}(V_i(x_i))$. For each $i \in \mathcal{N}(x)$ let $\rho_i > 0$ be such that $|x_i| = \rho_i$. Define the compact set $K_\rho = \{V_i(x_i) \in \mathbb{R}_+ : \rho_i/2 \leq |x_i| \leq 2\rho_i\}$ and let

$$c_{\rho,i} = \min_{r \in K_\rho} (\sigma_i^{-1})'(r) \, .$$

Note that by the definition of an ISS Ω-path $c_{\rho,i}$ is positive.
By (4.19) and (4.18) we get

$$\langle p, f_i(x, g_i(x+e))\rangle \leq -c_{\rho,i}\alpha_i(\rho_i) \tag{4.20}$$

for all $p \in \partial[\sigma_i^{-1} \circ V_i(x_i)]$ and $i \in \mathcal{N}(x)$.
Set $\tilde{\alpha}_i(\rho) := c_{\rho,i}\alpha_i(\rho)$ and

$$\alpha(r) := \min\{\tilde{\alpha}_i(|x_i|) \ : \ r = |x| \ , \ i \in \mathcal{N}(x)\} > 0 \, .$$

The function α is positive definite because the norm is positive definite, $\tilde{\alpha}$ is positive definite, and the minimum of positive definite functions is again positive definite. Hence,

$$\langle p_i, f_i(x, g_i(x+e))\rangle \leq -\tilde{\alpha}_i(|x_i|) \leq -\alpha(|x|) \tag{4.21}$$

for each $p_i \in \partial[\sigma_i^{-1} \circ V_i(x_i)]$ and $i \in \mathcal{N}(x)$. In particular, the right hand side depends on x rather than x_i.
Because of the maximization structure of V we have (see gain Appendix B)

$$\partial V(x) \subset \operatorname{conv}\left\{ \bigcup_{i\in\mathcal{N}(x)} \partial[\sigma_i^{-1} \circ V_i \circ \pi_i](x) \right\}, \tag{4.22}$$

where π_i is the projection on the ith component i.e., if $x = (x_1^\top, \dots x_n^\top)^\top$, then $\pi_i(x) = x_i$.
Using (4.19) and (4.22) there exist for each $p \in \partial V(x)$ suitable $\lambda_i \geq 0$ with $\sum_{i\in\mathcal{N}(x)} \lambda_i = 1$, $\xi_i \in \partial[V_i \circ \pi_i](x)$, and $c_i \in \partial\sigma_i^{-1}(V_i(x_i))$ such that

$$p = \sum_{i\in\mathcal{N}(x)} \lambda_i c_i \xi_i \,.$$

Note that because $\xi_i \in \partial[V_i \circ \pi_i](x)$ we have $\langle \xi_i, a\rangle = \langle \pi_i(\xi_i), \pi_i(a)\rangle$ for all $a \in \mathbb{R}^n$ or in other words, we get ξ_i from $\pi_i(\xi_i)$ by padding zeros to the corresponding components, because in the Clarke gradient of $V_i \circ \pi_i$ the projection on the ith component is used and hence no change in the other variables is present. By similar arguments, we have $c_i\pi_i(\xi_i) \in \partial[\sigma_i^{-1} \circ V_i](x_i)$. Now we can use (4.21) to bound

$$\langle p, f(x, g(x+e))\rangle = \sum_{i\in\mathcal{N}(x)} \lambda_i \langle c_i\xi_i, f(x, g(x+e))\rangle =$$

$$\sum_{i\in\mathcal{N}(x)} \lambda_i \langle c_i\pi_i(\xi_i), f_i(x, g_i(x+e))\rangle \leq - \sum_{i\in\mathcal{N}(x)} \lambda_i \alpha(|x|) = -\alpha(|x|)$$

for all $p \in \partial V(x)$ and the proof is finished. □

Now that we have established the existence of a Lyapunov function for the overall system, we have to state the triggering condition that ensures that (4.13) holds to conclude stability.

Theorem 4.3.3. *Let Assumptions 4.2.1 and 4.2.2 hold. Consider the interconnected system*

$$\dot{x}_i(t) = f_i(x(t), g_i(\hat{x}(t))) \,, \quad i \in \mathcal{N} \,, \tag{4.23}$$

as in (4.3) *with triggering conditions given by*

$$T_i(x_i, e_i) = \chi_i(|e_i|) - V_i(x_i) \,,$$

with χ_i defined in (4.12) for all $i \in \mathcal{N}$. Assume that no Zeno behavior is induced. Then the origin is a globally uniformly asymptotically stable equilibrium for (4.23).

Proof. To analyze the event-based control scheme introduced in (4.5)-(4.9), we define the time-varying map $\tilde{f}(t,x) = f(x, g(x + e(t)))$. Clearly, the solution starting at $x(0) = x_0$ and $e(0) = 0$ for

$$\dot{x}(t) = \tilde{f}(t,x)$$

with $e(t_k) = 0$ for all $k \in \mathbb{N}$, where t_k comes from (4.5) is the same as the solution starting from the same initial condition of system (4.5)-(4.9). The map $\tilde{f}(t,x)$ satisfies the Carathéodory conditions for the existence of solutions (see Section 1.2) and thus because of the conditions on f (see Section 4.1), the solution exists and is unique. Along the solutions of $\dot{x} = \tilde{f}(t,x)$, the locally Lipschitz positive definite and proper Lyapunov function $V(x)$ (by Proposition 4.3.1) introduced in Theorem 4.3.2 satisfies

$$V(x(t'')) - V(x(t')) = \int_{t'}^{t''} \frac{d}{dt} V(x(t))dt$$

for each pair of times $t'' \geq t'$ belonging to the interval of existence of the solution. Moreover, by a property of the Clarke generalized gradient (see Appendix, Lemma B.0.10) for almost all $t \in \mathbb{R}_+$, there exists $p \in \partial V(x(t))$ such that:

$$\frac{d}{dt} V(x(t)) = \langle p, \tilde{f}(t, x(t)) \rangle \ .$$

Note that the triggering conditions $T_i(x_i, e_i) = \chi_i(|e_i|) - V_i(x_i) \geq 0$ ensures that $V_i(x_i) \geq \chi_i(|e_i|)$ for all positive times. Hence we can use Theorem 4.3.2 together with the definition of $\tilde{f}(t,x)$, to infer (see [SGT07], Section IV.B, for similar arguments)

$$V(x(t'')) - V(x(t')) \leq - \int_{t'}^{t''} \alpha(|x(t)|)dt \ .$$

We can now apply [BR05], Theorem 3.2, to conclude that the origin of $\dot{x} = \tilde{f}(t,x)$, and therefore of $\dot{x} = f(x, g(\hat{x}))$, is uniformly globally asymptotically stable. □

Theorem 4.3.3 gives a first triggering condition, which allows us to conclude stability of the closed-loop system. For a typical behavior of the triggering function see Figure 4.4. It is taken from a numerical example, which

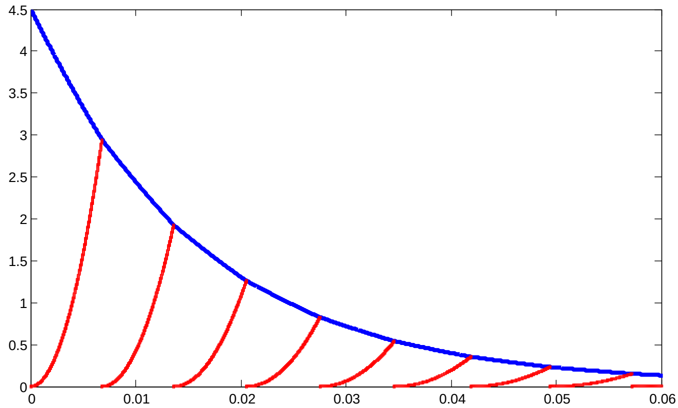

Figure 4.4: Evolution of the Lyapunov function and the error

will be discussed in detail in Section 5.2. Figure 4.4 shows the evolution of the Lyapunov function of a single subsystem (blue) together with the error e (red). Whenever the red curve hits the blue line, an event is triggered.
Which means, as long as the red curve stays below the blue line condition (4.13) holds.
To summarize, the small-gain condition (Assumption 4.2.2) ensures that the interconnection with $e \equiv 0$ does not interfere with the stability properties of the subsystems. Theorem 4.3.2 tells us under which triggering condition involving the error e and the state allows us to conclude the existence of a Lyapunov function for the overall system. In the absence of Zeno phenomena the stability of the interconnected system with event-triggered feedback is then concluded by fairly standard arguments.
As the assumption on the Zeno effects might be too demanding, the rest of this chapter is devoted to show how to handle Zeno in a different manner.

4.4 A Practical Way to Overcome Zeno

In Theorem 4.3.2 we had to rule out the occurrence of Zeno phenomena explicitly. Here we will see that altering Assumption 4.2.1 to a notion of practical stability will help to ensure that Zeno solutions cannot occur.

Assumption 4.4.1. For $i \in \mathcal{N}$, there exist a differentiable function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, and class-$\mathcal{K}_\infty$ functions α_{i1}, α_{i2} such that

$$\alpha_{i1}(|x_i|) \leq V_i(x_i) \leq \alpha_{i2}(|x_i|) \; .$$

Moreover there exist functions $\mu_i \in \mathrm{MAF}_{2N}$, $\gamma_{ij}, \eta_{ij} \in \mathcal{K}_\infty$, for $j \in \mathcal{N}$, positive definite functions α_i and positive constants c_i, for $i \in \mathcal{N}$, such that

$$V_i(x_i) \geq \max\{\mu_i\big(\gamma_{i1}(V_1(x_1)), .., \gamma_{iN}(V_N(x_N)), \eta_{i1}(|e_1|), .., \eta_{iN}(|e_N|)\big), c_i\} \\ \Longrightarrow \nabla V_i(x_i) f_i(x, g_i(x+e)) \leq -\alpha_i(|x_i|) \,. \quad (4.24)$$

Note that the only difference to Assumption 4.2.1 are the offsets c_i in the left hand side of the implication (4.24). Assumption 4.2.1 amounts to saying that each subsystem is ISS whereas Assumption 4.4.1 states that all subsystems are practically ISS with offset c_i.
In the same spirit we alter Assumption 4.2.2 to the following.

Definition 4.4.2. If there exists a $\sigma, \rho \in \mathcal{K}_\infty^n$ and a $\delta > 0$ such that

- for each i, the function σ_i^{-1} is locally Lipschitz on (δ, ∞);
- for every compact set $K \subset (\delta, \infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1}

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C \,, \quad \forall r \in K \,, i = 1, \ldots, n \,;$$

- For all $r > 0$

$$g(\sigma(r), \rho(r)) < \sigma(r) \,,$$

we call σ an ISpS-Ω-path.

In the last section we used Assumption 4.2.2 to conclude stability of the unperturbed system. As we use here practical stability we can weaken Assumption 4.2.2 to

Assumption 4.4.3. There exists an ISpS Ω-path as in Definition 4.4.2 for the discrete comparison system induced by Γ_μ, i.e. there exist $\sigma \in \mathcal{K}_\infty^N$ and $\varphi \in \mathcal{G}^N$ such that

$$\Gamma_\mu(\sigma(r), \varphi(r)) < \sigma(r)$$

for all $r \geq \tilde{c} > 0$.

Without loss of generality we may assume that $\tilde{c} \leq \max\{c_i\}$, $i \in \mathcal{N}$. The next theorem is the analogue of Theorem 4.3.2 for the case of practical ISS.

Theorem 4.4.4. *Let Assumptions 4.4.1 and 4.4.3 hold. Let* $V(x) = \max_{i \in \mathcal{N}} \sigma_i^{-1}(V_i(x_i))$. *Assume that for each* $j \in \mathcal{N}$,

$$\max\{\sigma_j^{-1}(V_j(x_j)), c_j\} \geq \hat{\eta}_j(|e_j|) \,, \quad \textit{with } \hat{\eta}_j = \max_{i \in Z(j)} \varphi_{ij}^{-1} \circ \eta_{ij} \,. \quad (4.25)$$

Then there exists a positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\langle p, f(x, g(x+e))\rangle \le -\alpha(|x|),\ \forall p \in \partial V(x)\ ,$$

for all $x = (x_1^\top\, x_2^\top\, \dots\, x_N^\top)^\top \in \{x : V(x) \ge \hat{c} := \max_i\{c_i, \sigma_i^{-1}(c_i)\}\}$, *where*

$$f(x, g(x+e)) = \begin{pmatrix} f_1(x, g_1(x+e)) \\ \dots \\ f_N(x, g_N(x+e)) \end{pmatrix} .$$

Proof. Let $\mathcal{N}(x) \subseteq \mathcal{N}$ be the set of indices i such that $V(x) = \sigma_i^{-1}(V_i(x_i))$. Take any pair of indices $i, j \in \mathcal{N}$. By definition, $V(x) \ge \sigma_j^{-1}(V_j(x_j))$ and

$$\gamma_{ij}(\sigma_j(V(x))) \ge \gamma_{ij}(V_j(x_j)) \tag{4.26}$$

by monotonicity of γ_{ij}. Let $i \in \mathcal{N}(x)$ and assume $V(x) \ge \hat{c}$. Because $\hat{c} \ge \tilde{c}$ we have by Assumption 4.4.3

$$\begin{aligned} &V_i(x_i) = \sigma_i(V(x)) > \\ &\mu_i(\gamma_{i1}(\sigma_1(V(x))), \dots, \gamma_{iN}(\sigma_N(V(x))), \varphi_{i1}(V(x)), \dots, \varphi_{iN}(V(x)))\ . \end{aligned} \tag{4.27}$$

Bearing in mind (4.26), we also have

$$\begin{aligned} &V_i(x_i) = \sigma_i(V(x)) > \\ &\qquad \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N)), \varphi_{i1}(V(x)), \dots, \varphi_{iN}(V(x)))\ . \end{aligned} \tag{4.28}$$

Let us partition the set $\mathcal{N} := \mathcal{P} \cup \mathcal{Q}$. We say $i \in \mathcal{P} :\Leftrightarrow \sigma_i^{-1}(V_i(x_i)) \ge c_i$; also $\mathcal{Q} := \mathcal{N} \setminus \mathcal{P}$. For all $j \in \mathcal{P}$ we have by (4.25) $\sigma_j^{-1}(V_j(x_j)) \ge \hat{\eta}_j(|e_j|)$ and hence using (4.25) (the case $j \notin C(i)$ is trivial because in this case $\varphi = \eta = 0$ by Proposition 4.2.3)

$$\begin{aligned} &\varphi_{ij}(V(x)) \ge \varphi_{ij} \circ \sigma_j^{-1}(V_j(x_j)) \ge \varphi_{ij} \circ \hat{\eta}_j(|e_j|) \ge \\ &\qquad\qquad \varphi_{ij} \circ \varphi_{ij}^{-1} \circ \eta_{ij}(|e_j|) = \eta_{ij}(|e_j|)\,. \end{aligned} \tag{4.29}$$

For all $j \in \mathcal{Q}$ we have by (4.25) $c_j \ge \hat{\eta}_j(|e_j|)$ and so

$$\varphi_{ij}(V(x)) \ge \varphi_{ij}(\hat{c}) \ge \varphi_{ij}(c_j) \ge \varphi_{ij} \circ \hat{\eta}_j(|e_j|) \ge \eta_{ij}(|e_j|). \tag{4.30}$$

Combining (4.29) and (4.30) we get for all $j \in \mathcal{N}$ that $\varphi_{ij}(V(x)) \ge \eta_{ij}(|e_j|)$, provided that (4.25) holds and that $V(x) \ge \hat{c}$. Substituting $\varphi_{ij}(V(x)) \ge \eta_{ij}(|e_j|)$ in (4.28) yields

$$\begin{aligned} &V_i(x_i) = \sigma_i(V(x)) > \\ &\qquad \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{iN}(V_N(x_N)), \eta_{i1}(|e_1|), \dots, \eta_{iN}(|e_N|))\ . \end{aligned} \tag{4.31}$$

For all $i \in \mathcal{N}(x)$ we have if $V(x) \geq \hat{c} = \max_i\{c_i, \sigma_i^{-1}(c_i)\}$, that $V(x) = \sigma_i^{-1}(V_i(x_i)) \geq \hat{c} \geq \sigma_i^{-1}(c_i)$ and finally as σ_i^{-1} is monotone $V_i(x_i) \geq c_i$. The latter together with (4.31) implies the left-hand side of the implication (4.24). Hence, for all $i \in \mathcal{N}(x)$: $\nabla V_i(x_i) f_i(x, g_i(x+e)) \leq -\alpha_i(|x_i|)$. Now we can repeat nearly the same arguments of the last part of the proof of Theorem 4.4.4, and conclude that for all x such that $V(x) \geq \hat{c}$ and for all $p \in \partial V(x)$, $\langle p, f(x, g(x+e))\rangle \leq -\alpha(|x|)$. The only difference is that here we use Assumption 4.4.3 instead of Assumption 4.2.2, which we can because $V(x) \geq \hat{c}$ and the proof is complete. □

Now that we established the existence of a Lyapunov function for the overall system, we show that with a suitable triggering scheme the conditions of Theorem 4.4.4 holds. We stress that the information needed for the triggering condition is again purely local.

Theorem 4.4.5. *Let Assumptions 4.4.1 and 4.4.3 hold. Consider the interconnected system*

$$\dot{x}_i(t) = f_i(x(t), g_i(\hat{x}(t))) \,, \quad i \in \mathbb{N} \,, \tag{4.32}$$

as in (4.3) *with triggering conditions given by*

$$T_i(x_i, e_i) = \hat{\eta}_i(|e_i|) - \max\{\sigma_i^{-1} \circ V_i(x_i), \hat{c}_i\} \,, \tag{4.33}$$

with $\hat{\eta}_i$ defined in (4.25) *for all $i \in \mathcal{N}$. Then the origin is a globally uniformly practically stable equilibrium for (4.32). In particular, no Zeno behavior is induced.*

Proof. The triggering condition (4.33) ensures that (4.25) holds. Hence by Theorem 4.4.4 we have our usual Lyapunov function
$V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$ for the interconnected system. By fairly standard arguments (cf. [HP05, Proposition 3.2.32]) practical stability of the system follows similarly to the proof of Theorem 4.3.3.
It remains to show that no Zeno behavior is induced. In between triggering events $\dot{e}(t) = -\dot{x}(t)$ for all $t \in (t_k, t_{k+1})$ by (4.3).
By Theorem 4.4.4 $V(x(t))$ is decreasing along the solution $x(t)$ on its domain of definition as long as $V(x) \geq \hat{c}$. Hence, $x(t)$ is bounded on its domain of definition. Since $\max\{\sigma_i^{-1} \circ V_i(x_i(t)), \hat{c}_i(t)\} \geq \hat{\eta}_i(|e_i(t)|)$, then also $e(t)$ is bounded and so is $\hat{x}(t) = x(t)+e(t)$. Let us assume the jth subsystem induces Zeno. By the considerations above all the involved quantities are bounded, hence there exists a $C \in \mathbb{R}_+$ such that $|\dot{e}_j(t)| = |\dot{x}_j(t)| = |f_j(x, g(\hat{x}))| \leq C$ for all t. By (4.33) an event is triggered from the jth subsystem if

$$\hat{\eta}_j(|e_j|) \geq \max\{\sigma_j^{-1}(V_j(x_j)), \hat{c}_j\} \geq \hat{c}_j \,.$$

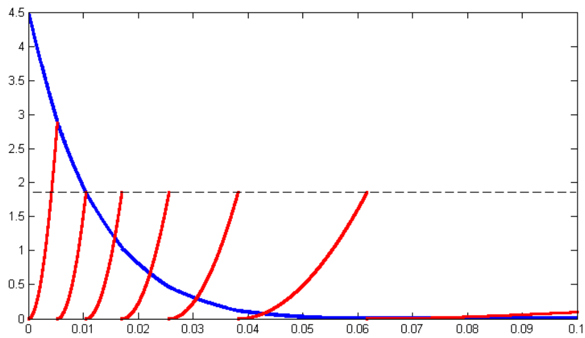

Figure 4.5: Lyapunov function together with the error and the offset

As $\hat{\eta}_j \in \mathcal{K}_\infty$ and $\hat{c}_j > 0$ we have

$$0 < \hat{\eta}_j^{-1}(\hat{c}_j) \le |e_j|\,.$$

Because $e_j(t_k^+) = 0$ we have for the evolution of $e(t)$ between t_k and the next triggering event

$$0 < \hat{\eta}_j^{-1}(\hat{c}_j) \le \left| e_j(t_k) + \int_{t_k}^{t_{k+1}} f_j(x, g(\hat{x}))ds \right| \le \int_{t_k}^{t_{k+1}} |f_j(x, g(\hat{x}))|ds \le \int_{t_k}^{t_{k+1}} Cds = C(t_{k+1} - t_k)$$

and hence

$$0 < \hat{\eta}_j^{-1}(\hat{c}_j)/C \le t_{k+1} - t_k\,. \tag{4.34}$$

Because the jth subsystem induces Zeno we have $\lim_{k\to\infty} t_k = t^* < \infty$. And thus t_k is a Cauchy sequence. This clearly contradicts (4.34) and hence no Zeno behavior can occur. □

In Figure 4.5 the quantities appearing in the triggering condition (4.25) are depicted. Compared to Figure 4.4, an event is triggered when the error hits the blue line or the dotted line, whichever is larger.
Similarly to Section 4.3, condition (4.25) holds as long as the red line stays below the blue line or the dotted line.
Note that each ISS system is also practically ISS with arbitrary offset. In this regard, the results from this section are applicable to a larger class of

systems than the results from Section 4.3 and Section 4.5.
Although the results presented here are applicable to a larger class of systems, we are only able to conclude practical stability for the overall system. Sometimes stronger stability notions than practical stability are desired. An approach which rules out the occurrence of Zeno solutions, while retaining asymptotic stability is given in the next section.

4.5 A Parsimonious Way to Overcome Zeno

The aim of this section is to show that it is possible to design distributed event-triggered control schemes for which the accumulation of the sampling times in finite time does not occur. The focus is again on the system (4.1), namely:

$$\dot{x}_i = f_i(x, g_i(x+e)) \ . \tag{4.35}$$

In this section we introduce a new triggering condition, which is termed *Parsimonious triggering*. This triggering condition not only prevents the occurrence of Zeno behavior, but it can also lead to fewer events by reducing unneeded information transmissions.
The main idea behind the new triggering scheme, which will be introduced in Theorem 4.5.7 is that if the error of the ith subsystem is bigger than its Lyapunov function but still small compared to the Lyapunov function of the overall system, no transmission of the ith subsystem is required.
For future use we need also a slight variation of Theorem 4.3.2. Here we exploit the fact that we can either compare each state to its corresponding error (as in Theorem 4.3.2) or each error to the Lyapunov function of the overall system as shown in the next theorem.

Theorem 4.5.1. *Let Assumptions 4.2.1 and 4.2.2 hold.*
Let $V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$ and, for each $j \in \mathcal{N}$, define:

$$\hat{\eta}_j = \max_{i\in Z(j)} \varphi_{ij}^{-1} \circ \eta_{ij} \ . \tag{4.36}$$

Then there exists a positive definite $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that the condition

$$V(x) \geq \hat{\eta}_j(|e_j|), \ \forall j \in \mathcal{N} \tag{4.37}$$

implies

$$\langle p, f(x, g(x+e))\rangle \leq -\alpha(|x|), \ \forall p \in \partial V(x) \ .$$

Proof. The proof follows by a slight modification of the proof of Theorem 4.3.2. For each x, let $\mathcal{N}(x) \subset \mathcal{N}$ be set of indices for which $V(x) = \sigma_i^{-1}(V_i(x_i))$.

It is sufficient to show that for all $i \in \mathcal{N}(x)$, $j \in \mathcal{N}$ we have $\varphi_{ij}(r) \geq \eta_{ij}(|e_j|)$, with $r = V(x)$.
First recall that for $j \notin C(i)$ the latter inequality trivially holds by Proposition 4.2.3. So assume that $j \in C(i)$. Using (4.36) and (4.37), we have

$$\varphi_{ij}(V(x)) \geq \varphi_{ij}(\hat{\eta}_j(|e_j|)) \geq \varphi_{ij} \circ \varphi_{ij}^{-1} \circ \eta_{ij}(|e_j|) = \eta_{ij}(|e_j|) \ .$$

i.e. (4.17) in the proof of Theorem 4.3.2. To finish the proof, we use essentially the same arguments after (4.17). □

A triggering condition for the jth subsystem which yields the validity of condition (4.37) would make the knowledge of the Lyapunov function V of the overall system to system j necessary. This would contradict our wish for a decentralized approach.
The next lemma provides a decentralized way to ensure that condition (4.37) holds. To this end, we give an approximation of the other states (termed W) which will be compared to the error instead of the Lyapunov function of the overall system. Appropriately scaled, W is a lower bound on the Lyapunov function of the overall system and hence can be used to check the validity of (4.37). The important advantage is, that this approximation can be calculated using only local information.
Before we state the next lemma, define

$$\xi^{j,x_j} := \left(\xi_1^\top, \ldots, \xi_{j-1}^\top, x_j^\top, \xi_{j+1}^\top, \ldots, \xi_N^\top\right)^\top$$

as the vector $\xi \in \mathbb{R}^n$ where the jth component is replaced by x_j.

Lemma 4.5.2. *Let Assumptions 4.2.1 and 4.2.2 hold and let* $V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$. *Assume for all* $j \in \mathcal{N}$ *there exist constants* $\tilde{\kappa}_j > 0$ *such that for all* $(x,e) \in \mathbb{R}^n \times \mathbb{R}^n$ *there is an approximation* $d_j = d_j(x_j, e_j)$ *of* $|f_j(x, g_j(x+e))|$ *with the property that* $V(x) \geq \hat{\eta}_i(|e_i|)$ *for all* $i \neq j$ *implies* $|f_j(x, g_j(x+e))| - d_j| \leq \tilde{\kappa}_j \max\{|x_j|, |e_j|\}$.
Assume furthermore that for all $j \in \mathcal{N}$ *there exist functions* $\Theta_j : \mathbb{R}^{N+1} \to \mathbb{R}$ *such that* $V(x) \geq \hat{\eta}_i(|e_i|)$ *for all* $i \neq j$ *implies*

$$\Theta_j(|x_1|, \ldots, |x_N|, |e_j|) \geq |f_j(x, g_j(x+e))| \, . \tag{4.38}$$

Define

$$W(j, x_j, e_j, d_j) = \min\{\max_{i\neq j} |\xi_i| \ : \ \xi \in \mathcal{A}(j, x_j, e_j, d_j)\}$$

with

$$\mathcal{A}(j, x_j, e_j, d_j) = \\ \{\xi^{j,x_j} \in \mathbb{R}^n \ : \ \Theta_j(|\xi_1|, \ldots, |\xi_N|, |e_j|) \geq d_j - \tilde{\kappa}_j \max\{|x_j|, |e_j|\} \ \} \, . \tag{4.39}$$

Then the condition

$$W(j, x_j, e_j, d_j) \geq \psi^{-1} \circ \hat{\eta}_j(|e_j|), \tag{4.40}$$

with $\psi = \min_{j \in \mathcal{N}} \sigma_j^{-1} \circ \alpha_{j1}$ *implies*

$$V(x) \geq \hat{\eta}_j(|e_j|) \; .$$

Proof. For later use define

$$\begin{aligned} A(j, x_j, e_j, \dot{x}_j) = \{\xi^{j,x_j} \; : \; \exists\, \epsilon \in \mathbb{R}^n \text{ s.t.} \\ f_j(\xi^{j,x_j}, g_j(\xi^{j,x_j} + \epsilon^{j,e_j})) = \dot{x}_j \text{ and } V(\xi^{j,x_j}) \geq \hat{\eta}_i(|\epsilon_i|) \; \forall i \neq j\} \; . \end{aligned} \tag{4.41}$$

The set $A(j, x_j, e_j, \dot{x}_j)$ describes the set of all ξ^{j,x_j} for which a pair $(\xi^{j,x_j}, \epsilon^{j,e_j})$ exists that satisfies the right hand side of the jth subsystem for a given $\dot{x}_j, x_j, e_j$ and for which $V(\xi^{j,x_j}) \geq \hat{\eta}_j(|\epsilon_i|)$ for all $i \neq j$ hold. As the system's state satisfies the dynamics, it holds that $x \in A$.
Before we proceed, we want to show that $A(j, x_j, e_j, \dot{x}_j) \subset \mathcal{A}(j, x_j, e_j, d_j)$. To this end take a $\xi \in A(j, x_j, e_j, \dot{x}_j)$. Hence we have $f_j(\xi^{j,x_j}, g_j(\xi^{j,x_j} + \epsilon^{j,e_j})) = \dot{x}_j$. Taking the norm and using (4.38) yields

$$\begin{aligned} \Theta_j(|\xi_1|, \dots, |\xi_n|, |e_j|) \geq |f_j(\xi^{j,x_j}, g_j(\xi^{j,x_j} + \epsilon^{j,e_j}))| = \\ |\dot{x}_j| \geq d_j - \tilde{\kappa}_j \max\{|x_j|, |e_j|\} \; , \end{aligned}$$

where the last inequality follows from the condition on the approximation for $|\dot{x}_j|$. And we can conclude $A(j, x_j, e_j, \dot{x}_j) \subset \mathcal{A}(j, x_j, e_j, d_j)$.
From condition (4.40) we deduce

$$\begin{aligned} \psi^{-1} \circ \hat{\eta}_j(|e_j|) \leq W(j, x_j, e_j, d_j) = \min\{\max_{i \neq j} |\xi_i| \; : \; \xi \in \mathcal{A}(j, x_j, e_j, d_j)\} \leq \\ \min\{\max_{i \neq j} |\xi_i| \; : \; \xi \in A(j, x_j, e_j, \dot{x}_j)\} \leq \max_{i \neq j} |x_i| \; . \end{aligned} \tag{4.42}$$

The second inequality follows from $A(j, x_j, e_j, \dot{x}_j) \subset \mathcal{A}(j, x_j, e_j, d_j)$ and the last can be deduced from $x \in A$. Now we rewrite (4.42) to get

$$\hat{\eta}_j(|e_j|) \leq \psi(\max_{i \neq j} |x_i|) \; .$$

With the help of (4.42), the definition of ψ, and Assumption 4.2.1 we arrive at

$$\begin{aligned} \hat{\eta}_j(|e_j|) \leq \min_{k \in \mathcal{N}} \sigma_k^{-1} \circ \alpha_{1k}(\max_{i \neq j} |x_i|) \leq \\ \max_{i \neq j} \sigma_i^{-1} \circ \alpha_{1i}(|x_i|) \leq \max_{i \in \mathcal{N}} \sigma_i^{-1}(V_i(x_i)) = V(x) \; , \end{aligned}$$

and the proof is complete. ☐

Before we can state another event-triggering scheme, which does not induce Zeno behavior we have to note that if Zeno behavior occurs, one of the states has to approach the equilibrium.

Lemma 4.5.3. *Consider a large scale system with triggered control of the form* (4.3) *satisfying Assumptions 4.2.1 and 4.2.2. Let χ_i, $i \in \mathcal{N}$ be given by* (4.12)*. Consider the triggering conditions*

$$T_i(x_i, e_i) = \chi_i(|e_i|) - V_i(x_i)\,, i \in \mathcal{N}\,.$$

If for a given initial condition x_0 the triggering scheme T_{i^*} for some $i^* \in \mathcal{N}$ induces Zeno behavior, then the corresponding solution

$$x_{i^*}(t_k) \to 0\,.$$

Proof. Denote $t^* = \lim_{k\to\infty} t_k$. By definition of the triggering condition we have for each k an index $i(k) \in \mathcal{N}$ such that

$$V_{i(k)}(x_{i(k)}(t_k)) = \chi_{i(k)}(|e_{i(k)}(t_k)|)\,.$$

Choose $i^* \in \mathcal{N}$ such that $i(k) = i^*$ for infinitely many k. Such a i^* exists because $\mathcal{N}$ is finite and k ranges over all of $\mathbb{N}$. Let K be the set of indices for which $i(k) = i^*$. For ease of notation let $K = \{s_1, s_2, \dots\}$. By Theorem 4.3.2 V is a Lyapunov function for the event triggered system on the interval $[0, t^*)$. Thus the trajectory $x_{|[0,t^*)}$ is bounded and $e_{|[0,t^*)}$ is bounded because $\chi_i(|e_i(t)|) \leq V_i(x_i(t))$ for all $i \in \mathcal{N}$, $t \in [0, t^*)$. It follows that $u_{i|[0,t^*)}$ is bounded and so $\dot{x}_i$ is bounded on $[0, t^*)$ for all $i \in \mathcal{N}$.
Then we have by uniform continuity of x_{i^*} on $[0, t^*)$ that the following limit exists

$$\lim_{k\to\infty} \chi_{i^*}(|e_{i^*}(s_k)|) = \lim_{k\to\infty} V_{i^*}(x_{i^*}(s_k)) = V_{i^*}(x_{i^*}(t^*)). \tag{4.43}$$

By definition $e_{i^*}(s_k^+) = 0$. Considering (4.3) we have that $\dot{e}_{i^*} = -\dot{x}_{i^*}$ almost everywhere on (s_k, s_{k+1}). Since $\dot{x}_{i^*}$ is bounded and $s_{k+1} - s_k \to 0$, then condition $e_{i^*}(s_k^+) = 0$ implies that

$$e_{i^*}(s_{k+1}) = e_{i^*}(s_k^+) + \int_{s_k}^{s_{k+1}} \dot{e}_{i^*}(\tau) d\tau = \int_{s_k}^{s_{k+1}} \dot{e}_{i^*}(\tau) d\tau\,,$$

which tends to 0 for $k \to \infty$. Hence by (4.43) we obtain that $V_{i^*}(x_{i^*}(t^*)) = 0$. This shows the assertion. □

The next lemma provides an inequality for the state and the corresponding dynamics. Besides the rather technical nature of Lemma 4.5.4 and 4.5.3 they are essential to be able to compare the ith state to the rest of the states as will be seen in Theorem 4.5.7.

Lemma 4.5.4. *Consider system*

$$\dot{x} = f(x, g(x+e)) \tag{4.44}$$

as in (4.3)*. If there are triggering instances* $t_k \to t^*$ *for* $k \to \infty$ *and an index* i *such that* $x_i(t_k) \to 0$*, then for all* $M > 0$ *there exists a* $k^* \in \mathbb{N}$ *such that for some* $k \geq k^*$

$$\frac{|x_i(t_{k+1}) - x_i(t_k)|}{t_{k+1} - t_k} > M|x_i(t_{k+1})|.$$

Proof. The proof will be by contradiction. To this end assume that for some fixed $M > 0$ and all k sufficiently large we have

$$|x_i(t_{k+1}) - x_i(t_k)| \leq M(t_{k+1} - t_k)|x_i(t_{k+1})| \ . \tag{4.45}$$

The evolution of x_i between t_l and t_k for $k > l$ can be bounded by using a telescoping sum, the triangle inequality, applying (4.45), and a judicious addition of 0:

$$\begin{aligned}
|x_i(t_k) - x_i(t_l)| &\leq \sum_{j=l+1}^{k} |x_i(t_j) - x_i(t_{j-1})| \leq \sum_{j=l+1}^{k} M(t_j - t_{j-1})|x_i(t_j)| \leq \\
&\sum_{j=l+1}^{k} M(t_j - t_{j-1})|x_i(t_j) - x_i(t_l)| + \sum_{j=l+1}^{k} M(t_j - t_{j-1})|x_i(t_l)| = \\
&\sum_{j=l+1}^{k-1} M(t_j - t_{j-1})|x_i(t_j) - x_i(t_l)| + \\
&\qquad M(t_k - t_{k-1})|x_i(t_k) - x_i(t_l)| + M(t_k - t_l)|x_i(t_l)| \ .
\end{aligned}$$

If we choose $D > 0$ and a k' such that $0 < \frac{1}{1-M(t_k - t_{k-1})} \leq D$ for all $k > k'$, we can rewrite the latter to

$$\begin{aligned}
|x_i(t_k) - x_i(t_l)| \leq \frac{1}{1 - M(t_k - t_{k-1})} &\sum_{j=l+1}^{k-1} M(t_j - t_{j-1})|x_i(t_j) - x_i(t_l)| \\
&+ \frac{M(t_k - t_l)}{1 - M(t_k - t_{k-1})}|x_i(t_l)| \ .
\end{aligned}$$

Using the discrete Gronwall inequality (see Appendix C) yields

$$|x_i(t_k) - x_i(t_l)| \le \frac{M(t_k - t_l)}{1 - M(t_k - t_{k-1})}|x_i(t_l)| + \frac{1}{1 - M(t_k - t_{k-1})} \sum_{j=l+1}^{k-1} \frac{M(t_j - t_l)}{1 - M(t_j - t_{j-1})}|x_i(t_l)|M(t_j - t_{j-1}) \times \prod_{s=j+1}^{k-1} \left(1 + \frac{M(t_s - t_{s-1})}{1 - M(t_s - t_{s-1})}\right) .$$

Exploiting that t_k is a monotone sequence, that $0 \le \frac{1}{1-M(t_k-t_{k-1})} \le D$ and that $1 + x \le e^x$ for all $x \in \mathbb{R}$ and collapsing the telescoping sum again gives

$$|x_i(t_k) - x_i(t_l)| \le \underbrace{\left(MD(t_k - t_l) + M^2D^2(t_{k-1} - t_l)^2 e^{MD(t_{k-1}-t_l)}\right)}_{:=C} |x_i(t_l)| .$$

Because of the finite accumulation point t^*, C is arbitrary small for large k and l. Hence there exist an $k^* > k'$ such that

$$|x_i(t_k) - x_i(t_l)| \le C|x_i(t_l)|$$

for all $k \ge l \ge k^*$ with $C < 1$. As this is true for all large k we can write

$$\lim_{k\to\infty} |x_i(t_k) - x_i(t_l)| \le C|x_i(t_l)|$$

and hence

$$|\lim_{k\to\infty} x_i(t_k) - x_i(t_l)| \le C|x_i(t_l)| .$$

Because $C < 1$ this contradicts $|x_i(t^*)| = 0$ and the proof is complete. □

Before we can state an immediate corollary, we have to recall a consequence of Gronwall's lemma.

Lemma 4.5.5. *If we have*

$$|\dot{x}(t)| \le L|x(t)| \tag{4.46}$$

for almost all $t \in [t_1, t_2]$, $t_2 > t_1$ *and* $L \in \mathbb{R}_+$, *then it holds*

$$|x(t_2)| \ge e^{-L(t_2-t_1)}|x(t_1)| .$$

Proof. Define $z(t) = e^{2Lt}|x(t)|^2$. For the derivative of z we have

$$\dot{z}(t) = 2Lz(t) + 2e^{2Lt}\langle x(t), \dot{x}(t)\rangle \,.$$

By using the Cauchy-Schwarz inequality and (4.46) we arrive at

$$\dot{z}(t) \geq 2L\left[z(t) - e^{2Lt}|x(t)|^2\right] = 0 \tag{4.47}$$

for almost all $t \in [t_1, t_2]$. Hence

$$z(t_2) = z(t_1) + \int_{t_1}^{t_2} \dot{z}(s)ds \geq z(t_1)\,.$$

As the square root is a monotone function we conclude

$$e^{L(t_2-t_1)}|x(t_2)| \geq |x(t_1)|\,,$$

which is the desired property. □

Corollary 4.5.6. *Under the conditions of Lemmas 4.5.3 and 4.5.4, assume that the functions Θ_j from Lemma 4.5.2 satisfying* (4.38) *may be chosen to be Lipschitz and so that $\Theta_i(0, \ldots, 0) = 0$ holds. Furthermore, assume for all $i \in \mathcal{N}$ the functions $\chi_i^{-1} \circ \alpha_{2i}$ are Lipschitz. Consider an initial condition $x(0) = x_0 \neq 0$. If there is Zeno behavior at t^*, i.e. if there are triggering instances $t_k \to t^*$, then for the overall state x of* (4.44)

$$x(t_k) \not\to 0 \qquad \text{as } k \to \infty\,. \tag{4.48}$$

Proof. We first exclude that there is a $s^* \in [0, t^*)$ such that $x(s^*) = 0$. Otherwise choose Lipschitz constants L_i for Θ_i valid on the compact set $\{x(s)\ ;\ s \in [0, s^*]\}$ and note that we have for each i almost everywhere on $[0, s^*]$

$$|\dot{x}_i(t)| = |f_i(x(t), g_i(x(t) + e(t)))| \leq \\ \Theta_i(|x_1(t)|, \ldots, |x_N(t)|, |e_i(t)|) \leq L_i|x(t)|\,, \tag{4.49}$$

where we assumed without loss of generality that L_i is larger than the Lipschitz constant from $\chi_i^{-1} \circ \alpha_{2i}$. Recall that by Assumption 4.2.1 and the triggering condition $|e_i| \leq \chi_i^{-1} \circ \alpha_{2i}(|x_i|)$.
Note that we can use Θ_i as a bound for the dynamics as in (4.38), because the validity of $V_i(x_i) \geq \chi_i(|e_i|)$ for all i trivially implies $V(x) \geq \hat{\eta}_i(|e_i|)$.
As (4.49) is true for all i, this implies $|\dot{x}(t)| \leq L|x(t)|$ for L sufficiently large and almost all $t \in [0, s^*]$. From Proposition 4.5.5 it follows that

$|x(s^*)| \geq e^{-Ls^*}|x(0)| > 0$, so that $x(s^*) \neq 0$.
If $x(t_k) \to 0$, then $x(t) \to 0$ for $t \nearrow t^*$. Hence for each i, and k sufficiently large we have that (4.49) holds almost everywhere on (t_k, t^*). As in the first part of the proof it follows from Proposition 4.5.5 that $|x(t^*)| \geq e^{-L(t^*-t_k)}|x(t_k)| > 0$, because by the first step of the proof $x(t_k) \neq 0$. This contradicts the assumption that $x(t_k) \to 0$. □

The rest of this section is devoted to constructing an event-triggered control scheme which ensures that Condition (4.37) holds.
From Lemma 4.5.3 we know that if Zeno behavior occurs, then one of the subsystems approaches the origin in finite time. Corollary 4.5.6 shows that under certain regularity assumptions, a number of subsystems do not converge to 0 as we approach the Zeno point. Hence, from a certain time on, the Lyapunov function corresponding to the subsystem which tends to the origin does not contribute to the Lyapunov function for the overall system. As a consequence no information transfer from this subsystem is necessary using parsimonious triggering. This observation is made rigorous in the rest of the section.
In the next theorem we use the triggering condition as in Theorem 4.3.3 but we add another triggering condition T_{i2}, which checks whether the ith subsystem contributes to the Lyapunov function of the overall system. It does so by comparing the local error of system i with the approximation W of the other states as described in Lemma 4.5.2. The main idea is that if the dynamics of the ith system is large compared to its own state, other states must be large. As the correct value of the dynamics is not known to system i, an approximation of $|\dot{x}_i|$ is used.
As the aim is to use only local information, we will use the difference quotients to approximate the size of the derivative at the triggering points. Furthermore, we do not wish to assume that all subsystems are aware of all triggering events. Hence in the following we will use the notation t_k^i to denote those triggering events initiated by system i. We define

$$d_i(t) = \frac{|x_i(t) - x_i(t_{k-1}^i)|}{t - t_{k-1}^i} \tag{4.50}$$

as the difference quotient approximating $|\dot{x}_i(t)|$ after the triggering event t_{k-1}^i.
Adding the new triggering condition that uses (4.50) allows us to exclude the occurrence of Zeno behavior as will be seen in the next theorem.

Theorem 4.5.7. *Consider a large scale system with triggered control of the form* (4.3) *satisfying Assumptions 4.2.1 and 4.2.2.*

Let $V(x) = \max_{i\in\mathcal{N}} \sigma_i^{-1}(V_i(x_i))$. *Define*

$$T_{i1}(x_i, e_i) = \chi_i(|e_i|) - V_i(x_i)$$

with χ_i *as in Theorem 4.3.3 and*

$$T_{i2}(x_i, e_i, d_i) = \psi^{-1} \circ \hat{\eta}_i(|e_i|) - W(i, x_i, e_i, d_i) \;,$$

where ψ, $W(i, x_i, e_i, d_i)$ *and* $\hat{\eta}_i$ *are defined as in Lemma 4.5.2. Furthermore, assume that for all* $i \in \mathcal{N}$ *the* Θ_i *from Lemma 4.5.2 and* $\psi^{-1} \circ \hat{\eta}_i$ *are Lipschitz with Lipschitz constant* L_i *respectively* K_i *and that* $\Theta_i(0, \ldots, 0) = 0$ *holds. Consider the interconnected system*

$$\dot{x}_i(t) = f_i(x(t), g_i(\hat{x}(t))) \;, \quad i \in \mathcal{N} \;, \tag{4.51}$$

as in (4.3) *with triggering conditions given by*

$$T_i(x_i, e_i, d_i) = \min\{T_{i1}(x_i, e_i), T_{i2}(x_i, e_i, d_i)\} \;, \tag{4.52}$$

for all $i \in \mathcal{N}$. *Then the origin is a globally uniformly asymptotically stable equilibrium for (4.51), if there are constants* $\kappa_j > 0$, $j \in \mathcal{N}$ *such that at the triggering times* t_k, *which are implicitly defined by* (4.51) *and* (4.52) *as described in Section 4.1, the following condition is satisfied:*

$$||\dot{x}_j(t_k^j)| - d_j(t_k^j)| \leq \kappa_j |x_j(t_k^j)| \tag{4.53}$$

where $d_j(t_{k_j}^j)$ *is defined by* (4.50). *In particular, no Zeno behavior occurs.*

Proof. Before we can use Theorem 4.3.3 respectively Theorem 4.5.1 to conclude stability, we have to exclude the occurrence of Zeno behavior. First note that condition (4.52) triggers an event if and only if $T_{i1} \geq 0$ and $T_{i2} \geq 0$ respectively condition (4.13) and (4.37) are violated. Now assume that the jth subsystem induces Zeno behavior. For simplicity, we omit the index j of the triggering times t_k^j. Hence, let t_k be the triggering times of the jth subsystem and $t^* = \lim_{k\to\infty} t_k$ the finite accumulation point. From Lemma 4.5.3 we know that the jth subsystem has to approach the equilibrium, i.e. $\lim_{t_k\to t^*} x_j(t_k) = 0$. Lemma 4.5.4 tells us that for all M there exists a k^* such that for some $k \geq k^*$

$$\frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} > M|x_j(t_k)| \;. \tag{4.54}$$

As discussed in the proof of Lemma 4.5.2, the full state $x \in A \subset \mathcal{A}$. But the knowledge of x is not available to a single subsystem. Hence, we take $\xi(t_k) \in$

$\mathcal{A}(j, x_j(t_k), e_j(t_k), d_j(t_k))$ as in the definition of W as an approximation for the states of the other subsystems. For this ξ we can deduce together with the Lipschitz continuity of Θ_j, (4.54) and the fact that $e_j(t_k) = 0$

$$L_j \max_{i \neq j}\{|\xi_i(t_k)|, |x_j(t_k)|\} \geq \Theta_j(|\xi_1|, \ldots, |x_j|, \ldots, |\xi_N|, 0) \geq \frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} - \kappa_j |x_j(t_k)| > (M - \kappa_j)|x_j(t_k)| \,. \quad (4.55)$$

And hence for the k given in (4.54)

$$\max_{i \neq j}\{|\xi_i(t_k)|, |x_j(t_k)|\} > \frac{M - \kappa_j}{L_j}|x_j(t_k)| \,. \quad (4.56)$$

Now choose

$$M > \max\{\kappa_j + L_j, \kappa_j + L_j K_j\} \,, \quad (4.57)$$

where K_j is the Lipschitz constant of $\psi^{-1} \circ \hat{\eta}_j$. From Lemma 4.5.4 we know that this choice of M yields a k^* such that we can conclude together with (4.56) $\max_{i \neq j}\{|\xi_i(t_k)|, |x_j(t_k)|\} = \max_{i \neq j} |\xi_i(t_k)|$ for some $k \geq k^*$. For this k we want to show that the corresponding t_k is not a triggering time.
To this end we use (4.55) and (4.54) to get

$$\max_{i \neq j} |\xi_i(t_k)| \geq \frac{1}{L_j}(1 - \frac{\kappa_j}{M})\frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} \,. \quad (4.58)$$

Note that for the jth subsystem (4.58) is true for all $\xi \in \mathcal{A}$ and therefore by the definition of W

$$W(j, x_j, e_j, d_j) \geq \frac{1}{L_j}(1 - \frac{\kappa_j}{M})\frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} \,.$$

Using the latter inequality and the Lipschitz constant for $\psi^{-1} \circ \hat{\eta}_j$ we can bound T_{j2} by

$$T_{j2} \leq K_j|e_j(t_k)| - \frac{1}{L_j}(1 - \frac{\kappa_j}{M})\frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} \,.$$

From the definition of $e_j(t_k) = x_j(t_{k-1}) - x_j(t_k)$ we arrive at

$$T_{j2} \leq K_j|x_j(t_k) - x_j(t_{k-1})| - \frac{1}{L_j}(1 - \frac{\kappa_j}{M})\frac{|x_j(t_k) - x_j(t_{k-1})|}{t_k - t_{k-1}} \,.$$

We may assume that k^* is sufficiently large so that $t_k - t_{k-1} < M^{-1}$ for all $k \geq k^*$. Together with (4.57) we obtain

$$K_j < \frac{1}{L_j(t_k - t_{k-1})}(1 - \frac{\kappa_j}{M})$$

and hence $T_{j2} < 0$ in contradiction to the assumption that t_k is a triggering time. Because the only further assumption on the solution of (4.51) and (4.52) is the occurrence of Zeno behavior, the aforementioned contradiction shows that Zeno behavior cannot occur.
To conclude stability define

$$I(x,e) := \{j \in \mathcal{N} \ : \ V_j(x_j) \geq \chi_j(|e_j|)\},$$

$$J(x,e) := \{j \in \mathcal{N} \ : \ V(x) \geq \hat{\eta}_j(|e_j|)\},$$

and

$$\mathcal{J}(x,e) := \{j \in \mathcal{N} \ : \ \psi(W(j,x_j,e_j,d_j)) \geq \hat{\eta}_j(|e_j|)\}.$$

Note that the triggering condition T_j ensures that $j \in I \cup \mathcal{J}$. For $j \in I$ we can use exactly the same reasoning as in Theorem 4.3.3.
Lemma 4.5.2 tells us that from $j \in \mathcal{J}$ we can deduce $j \in J$. For the case $j \in J$ we can adopt nearly the same reasoning as in Theorem 4.3.3. Only the reasoning for the existence of a Lyapunov function for the overall system changes. In Theorem 4.3.3 this can be deduced from Theorem 4.3.2 whereas here we have to use Theorem 4.5.1 to conclude the existence of a Lyapunov function. The rest of the proof can be copied word by word from Theorem 4.3.3. This ends the proof. □

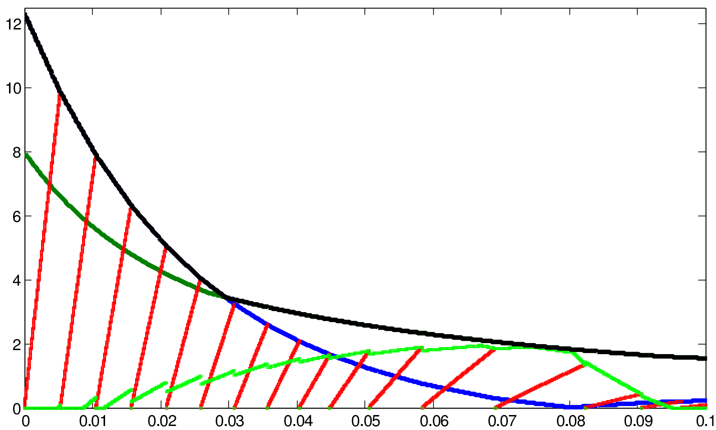

Figure 4.6: Lyapunov function together with the error and the approximation of the Lyapunov function of another subsystem

Figure 4.6 shows the Lyapunov function of two subsystems (for system one in blue and for system two in black) together with the error of the first

subsystem (in red) and the approximation of the Lyapunov function of the second subsystem calculated by the first subsystem (in green).
Similarly, as in Figure 4.4 respectively Figure 4.5 an event is triggered whenever the red curve hits the blue line or the green line, whichever is larger.
In the situation depicted in this figure, using only the triggering condition T_{i1} would lead to Zeno solutions.
Lemma 4.5.3 tells us that if an event-triggering scheme like (4.13) would lead to Zeno solutions, the corresponding state and hence the Lyapunov function has to go to zero in finite time. At around $t \approx 0.08$ in Figure 4.6 exactly this happens.
By the definition of the approximation W it can be seen that W is a lower bound on the state (or the Lyapunov function, depending on the scaling). We can interpret (4.56) together with (4.54) and Corollary 4.5.6 in such a way that as we approach the Zeno point, the approximation gets larger. In Figure 4.6 the approximation (green) grows until it nearly hits the black line, confirming the aforementioned interpretation. Interestingly, the approximation is only nonzero around the Zeno point.

4.6 Notes and References

The material from Section 4.4 appears in the proceedings of the MTNS conference 2012 in Melbourne [DPSW12]. The basic idea (Section 4.3) appeared in the proceedings of the IFAC world congress 2011 ([DPSW11]).
Section 4.5 is based on work appeared in Automatica ([DPSW13b]).
As in Chapter 2 the basic idea of event-triggering stems from signal theory. Usually, it is referred to as δ-sampling. The signal space is partitioned in equally spaced regions, δ apart. A new sample is taken whenever the signal crosses one of the boundaries. Ideas of non periodic sampling where already presented as early as in [Ell59, TB66].
An approach similar to the presented framework is [MT11, Tab07].
The approach of [MT11] builds upon [Tab07] in that it requires a *centralized* controller $u = g(x)$ to make the closed-loop system $\dot{x} = f(x, g(x + e))$ ISS with respect to the measurement errors e. Assuming that the comparison functions appearing in the ISS inequality are locally Lipschitz and that semi-global asymptotic stability is of interest, the triggering function of [MT11] takes the form $|e(t)|^2 \leq \sigma|x(t)|^2$ for some positive constant σ depending on the set of initial conditions of the system. In [MT11], this triggering condition is replaced by the triggering functions $|e_i(t)|^2 \leq \sigma|x_i(t)|^2 + \theta_i$, for $i \in \mathcal{N}$, where the θ_i's are time-varying scalar parameters satisfying $\sum_{i \in \mathcal{N}} \theta_i = 0$ and adjusted on-line via a heuristic. Whenever one of the N triggering conditions

is fulfilled, then the entire controller $u = g(x)$ is updated. This centralized update of the control law also allows the authors to avoid the Zeno phenomenon.
Compared with [MT11], the approach presented here is fully decentralized not only because the sensors sample in a decentralized fashion (and in a way very different from [MT11]) but also as far as the control design is concerned. Indeed: (i) for each subsystem a controller is designed which guarantees ISS of the subsystem (cf. Assumption 4.2.1); (ii) these controllers depend on local measurements only (cf. Assumption 4.2.1); (iii) if the measurement x_i is sampled, only the controllers which use x_i are updated (cf. Section 4.1). For truly large-scale systems, it can be much more convenient to design the decentralized controllers proposed in this paper than a single centralized controller.
An approach to event-triggered control of large-scale systems closer to the one proposed in this chapter is presented in [WL11]. A number of important differences, however, must be emphasized. First of all, we remark that the standing Assumption 4.1 in [WL11] implies but it is not implied by our Assumption 4.2.1. In fact, recall that [WL11], Assumption 4.1 requires

$$\nabla V_i(x_i) f_i(x, g_i(x+e)) \leq \\ -\alpha_i |x_i|^p + \sum_{j \in \Sigma(i) \cup C(i), j \neq i} \beta_j |x_j|^p + \sum_{j \in C(i)} \delta_j |e_j|^p \ , \quad (4.59)$$

where V_i is as before. This is the dissipative formulation of the ISS property of system i with respect to the other systems and the error variables. It is well known that dissipative ISS formulations are equivalent to the implication formulation we used in Assumption 4.2.1. In particular, (4.59) can be transformed to (4.11). On the other hand, because of the specific form of the comparison functions in (4.59), the assumption used in this chapter is more general.
This has a considerable impact in the analysis presented in [WL11], Theorem 4.3, where it is enough to assume a small gain condition involving the constant gains α_i, β_i (see [WL11], condition (6)). To be more precise, condition (6) in [WL11], which is formulated for a dissipative ISS condition, ensures that the coupling matrix is diagonally dominant and thus Hurwitz and Metzler. We note that this condition is restrictive when compared to ISS small-gain theorems for the dissipative case, cp. [DIW11, Section 3]. For a discussion on the difference of small-gain theorems see also the notes and reference section of Chapter 3.
Although in [WL11], Remark 4.5, the authors observe that the gain functions in (4.59) can be replaced by other $\mathcal{K}$ *Lipschitz* functions, for Theorem 4.3 to

continue to hold it is very likely that such functions will be much less general then the ones considered here.

In our case, the use of general nonlinear *heterogeneous* gain functions and a small gain condition expressed in terms of nonlinear functions rather than constants enlarges considerably the class of systems to which our approach applies. Furthermore, our generalization requires a different analysis than the one in [WL11].

The proof of Theorem 4.3.2 is inspired by [DRW10]. In fact, letting $e \equiv 0$ recovers the proof of [DRW10, Theorem 5.3]. With the difference that in [DRW10] a somewhat different small-gain condition was used. See notes and references of Chapter 3 for details.

One major drawback of event-triggered control is the need for constantly measuring the state. One approach trying to overcome this issue is termed self-triggered control [MJAT10, AT10].

In self-triggered control a mechanism to predict the next event-time is introduced. Although it is no longer necessary to monitor the state constantly, the robustness immanent to event-triggered control is lost.

In the hybrid system community the approach to deal with Zeno effects with the help of practical stability as in Section 4.4 is termed temporal regularization (c.f. [GST09]). For a more general discussion on solution theory for Zeno phenomena see [AZGS06].

In Section 4.5 a new way to deal with Zeno was introduced. With the help of the dynamics of a subsystem an approximation on the other states was derived. It would be of interest to pursue this idea further in such a way that the approximation is generated by different means. In particular, it could be possible to construct an observer, which gives sufficient good knowledge of the other states to follow the same ideas as in Section 4.5. This is also of interest, because the computation of the approximation involves solving a possibly non convex optimization and hence is possibly computationally unfeasible. Another direction for further research could be to investigate whether this optimization can be carried out offline instead of solving it on the fly.

It is also of interest to investigate how to handle the effects of delay and packet loss. We have seen in the last chapter that the small-gain theorems used here are also applicable for the case of delay differential equations. Hence we hope that by a more conservative triggering condition we could still use the approach from this chapter.

In [WL11] ideas how to handle packet loss are given. We expect that similar ideas can also be used here.

Another issue is collision avoidance on the communication channel. In realistic media, two subsystems cannot communicate at the same time. Ideas

how to address this problem can be found in [NT07]. In this paper a large class of medium access protocols are treated as dynamical systems and the stability analysis in the presence of communication constraints is carried out by including the protocols in the closed-loop system.
In Chapter 2 we dealt with the effects of quantization. As we already model the effects of the imperfect knowledge of the states to the controllers, we could handle the effect of quantization with just minor modifications. In particular, interpreting $e = \max\{e_1, e_2\}$, where e_1 is the zero order hold error and e_2 is the quantization error would help us to handle the effects of quantization within our event-triggering approach.
In this chapter we assumed out of convenience that $\dot{\hat{x}} \equiv 0$. It could lead to fewer events if we would use more sophisticated approaches than zero order hold. For instance, if all sensors have models of all other subsystems (similarly to Chapter 2), the evolution of the error would be less conservative, leading to fewer events at the cost of higher computational complexity.
Carefully inspecting the proofs show, that in this case all our considerations still hold true for the case $\dot{\hat{x}} \not\equiv 0$ with only minor modifications.
Although we presented triggering conditions that avoid Zeno phenomena, it would be of interest for applications to have a bound on the minimal inter event times. First ideas addressing this problem within our setup is given in [DPSW13a].

Chapter 5

Numerical Simulations

The simulations presented here already appeared in [SW14a] and [SW14b]. The corresponding Matlab code is given in Appendix A.

5.1 Dynamic Quantization

Here we give a numerical example, which shows the behavior of our approach from Chapter 2. To be more precise we implement the example given in Section 2.5. We use as a showcase the celebrated pendulum on a cart. For a derivation of the linearized model see [HP05].

The equation of the pendulum on a cart is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mg}{M} & 0 & 0 \\ 0 & -\frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

where x_2, x_4 are angle and angular velocity of the pendulum and x_1 resp. x_3 denote the position and velocity of the cart.

In the simulations the values $m = 0.329, M = 3.2, l = 0.44, g = 9.81$ have been used.

As described in Chapter 2 the encoder and decoder use a copy of the dynamics. We denote the states of the decoder with $\hat{x}_1, \ldots, \hat{x}_4$.

The control action, which stabilizes the upright position is given by

$$u(t) = Fx(t)\,,$$

where $F \in \mathbb{R}^{1\times 4}$ is calculated via standard pole placement techniques.

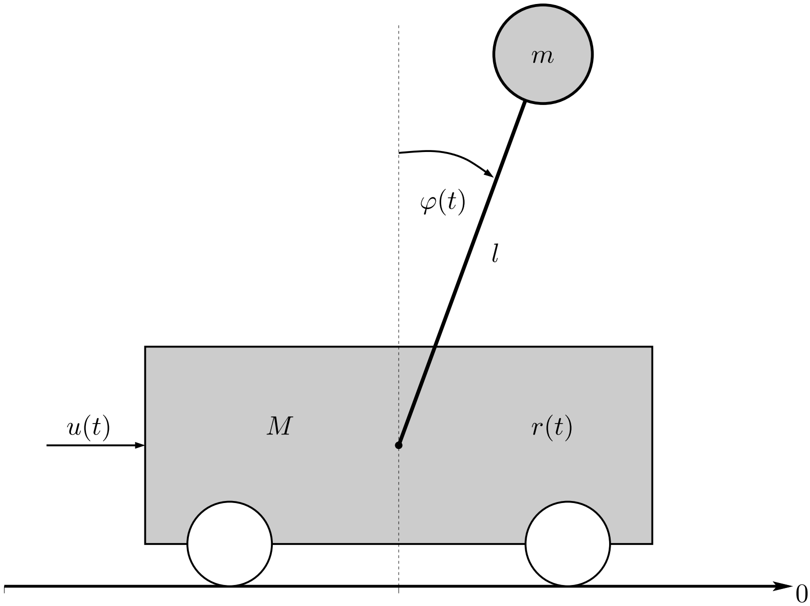

Figure 5.1: An inverted pendulum on a cart. ($x_2 = \varphi$, $x_4 = \dot{\varphi}$, $x_1 = r$, $x_3 = \dot{r}$)

Simulation results for the initial state $x(0) = (3, 7, -4, 2)^\top$ can be found in Figure 5.2 for the position of the pendulum respectively the cart and Figure 5.3 for the corresponding velocities.
In Figure 5.2 the angle of the pendulum is depicted in blue and the position of the cart in green.
The control action is calculated via the turquoise and red trajectory respectively.
The angular velocity of the pendulum is given in blue in Figure 5.3, while the velocity of the cart is the green trajectory. The control action is calculated via the red line for the cart and in turquoise for the pendulum.

The corresponding Matlab code is given in Section A. The physical parameters are taken from [fAuPRUB].
The blue dots at the bottom of the figures denote the time instances the encoder sends information while the vertical dotted lines are the time instances the decoder receives information.
In Figure 5.5 a zoom into the transient of x_1 is depicted together with the quantization region and the value $\hat{x}_1$, which is used to close the loop. In

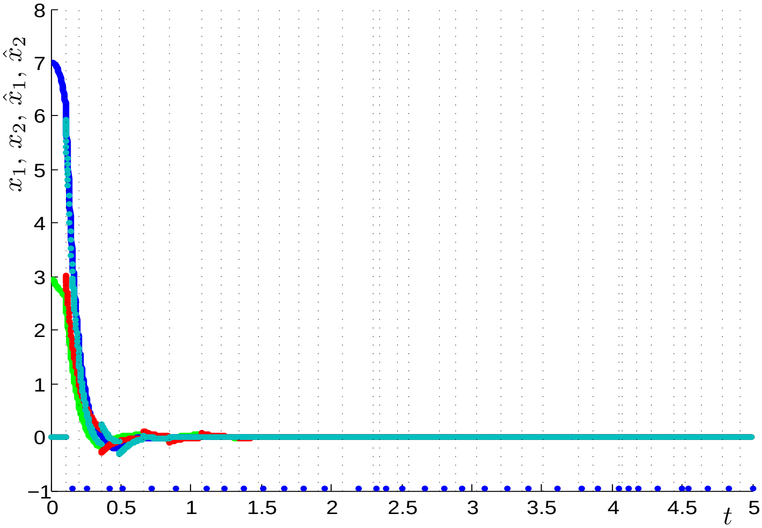

Figure 5.2: Position of cart and pendulum (x_1 green, x_2 blue, $\hat{x}_1$ red, $\hat{x}_2$ turquoise)

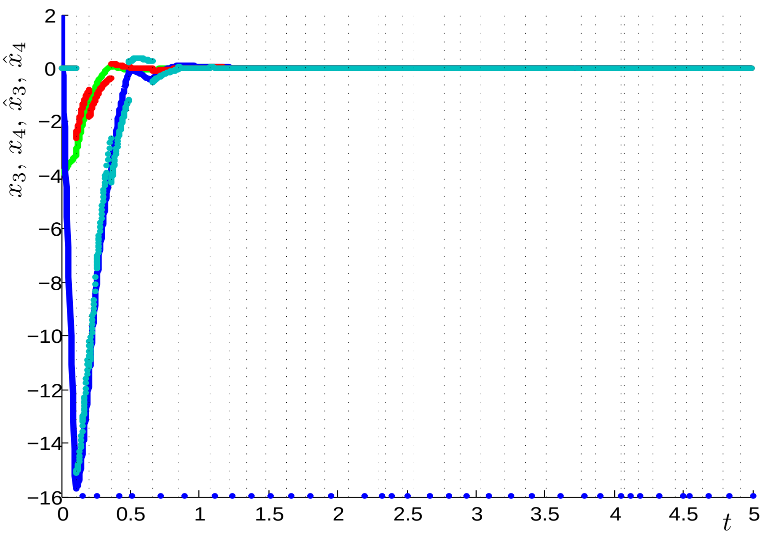

Figure 5.3: Speed of cart and pendulum (x_3 green, x_4 blue, $\hat{x}_3$ red, $\hat{x}_4$ turquoise)

Figure 5.6 the same for x_3 respectively $\hat{x}_3$ can be seen. In both figures the shaded region depicts the quantization region. Every time, the decoder receives information, the quantization region shrinks. Between these events the quantization region grows exponentially, as the theory predicts. Up to the first time the decoder receives data, the system runs in open loop. Every time the decoder receives information, the center of the quantization region jumps. This leads to a jump in the trajectory used to close the loop. At this time instances the trajectory of the pendulum are not differentiable explaining the kinks in the trajectory.

As predicted by the theory, the state always stays within the quantization region. Note that this does not hold in general for the decoder trajectory, as can be seen for instance at $t = 0$. Consistently to the theory, at the time instances the decoder receives information (e.g., $t \approx 0.1$) the decoder trajectory and the center of the quantization region coincides.

We choose a large initial size of the quantization region as well as a small N in order to enhance the visibility of the quantization region. Because of the exponential decay it would be hard to distinguish the trajectory and the quantization region otherwise.

The choice of a small N demands for rather small delays in order to satisfy condition $e^{L\tau^*} < N$. The delays are drawn randomly from a normal distribution with mean 0.15 and variance 0.002. The distribution of the delays is plotted in Figure 5.4.

The evolution of the error $e_1 = |x_1 - \hat{x}_1|$ can be found in Figure 5.7 and for $e_3 = |x_3 - \hat{x}_3|$ in Figure 5.8. Theory predicts that $e^{L\tau^*}/N < 1$ can be uses as an exponentially decaying bound on the error. The red line is this upper bound on the error and the green line denotes the bound on the error if the actual mean delay is used.

The error itself is given in black. Please note that the figures are logarithmically scaled.

We want to stress that we have chosen a large initial size of the quantization region as well as a small N in order to have a slow convergence of the quantization region. To showcase a more realistic situation, we simulated the non-linear inverse pendulum with larger delays. The size of the delays demands for a larger N. The dynamics are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \sin x_1 - u \cos x_1$$

with $u = k(x)$ a simple controller for the linearized model. The results are given in Figure 5.9. After the first time the decoder receives information around $t \approx 0.3$ the trajectory of the system and the trajectory used to close

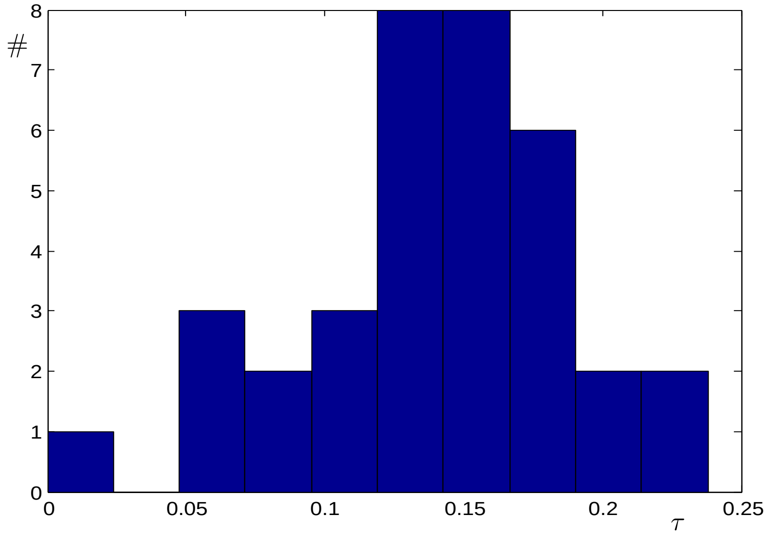

Figure 5.4: Distribution of the delays

the loop cannot be distinguished anymore. The simple controller used does not render the closed loop system ISS. In Theorem 2.4.1 we see that the error dynamics are asymptotically stable without the ISS assumption. This is demonstrated in Figure 5.10. If we choose the initial condition outside the basin of attraction of the simple controller used, the closed-loop system is unstable. But for the presented approach it still holds that the difference of the state and the trajectory used to close the loop converges. After $t \approx 1.7$ no difference between the trajectories of system and decoder can be seen.

5.2 Event Triggered Control

Here we want to give an example which shows the feasibility of our event based approach. The Matlab code for the presented simulations can be found in Appendix A.
The following interconnection of $N = 2$ subsystems

$$\dot{x}_1 = x_1 x_2 + x_1^2 u_1$$
$$\dot{x}_2 = x_1^2 + u_2$$

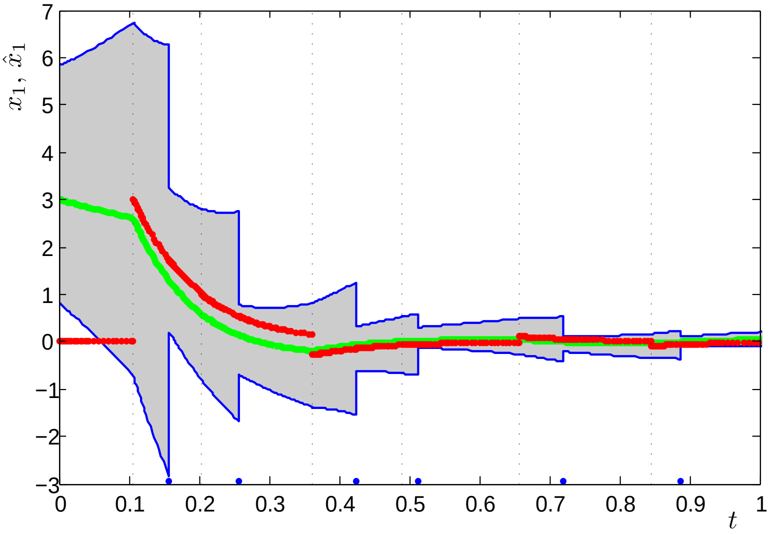

Figure 5.5: Position of the cart

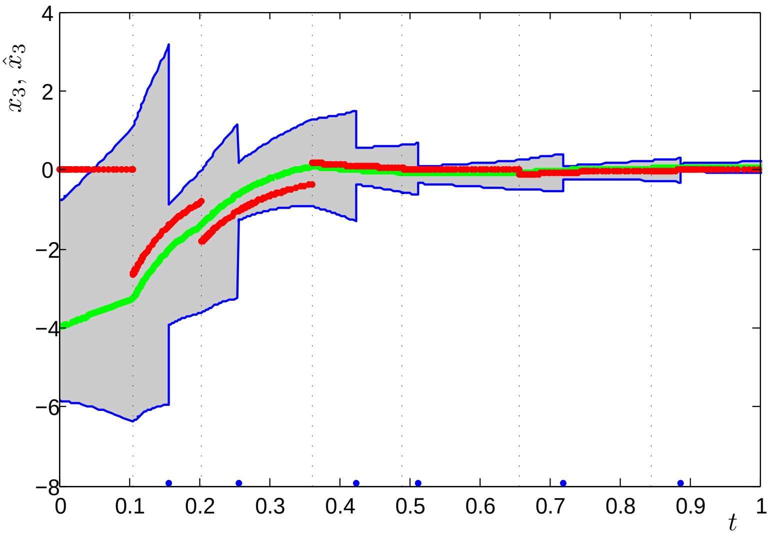

Figure 5.6: Velocity of the cart

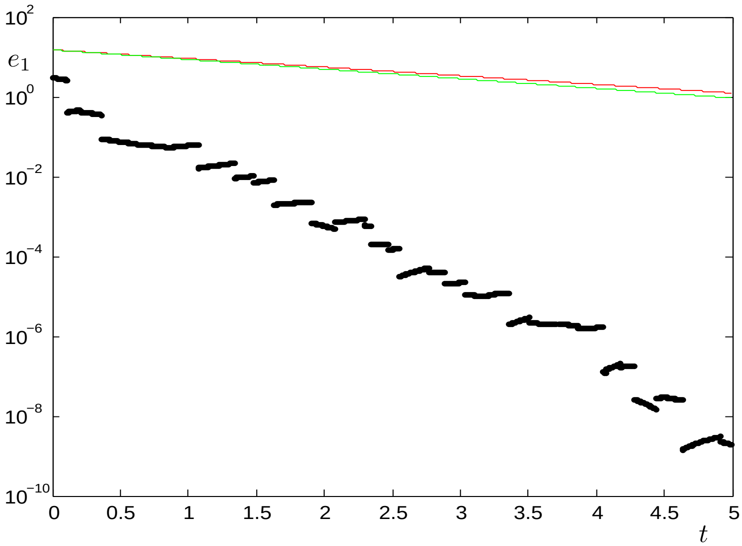

Figure 5.7: Difference between x_1 and $\hat{x}_1$

is considered under the assumption that each controller can only access the state of the system it controls. The control laws are chosen accordingly as

$$u_1 = -(x_1 + e_1)\ ,\ u_2 = -k(x_2 + e_2)\ ,\ k > 0\ .$$

Let $V_i(x_i) = \frac{1}{2}x_i^2$ for $i = 1, 2$. Then

$$\dot{V}_1(x_1) := \nabla V_1(x_1)(-x_1^3 + x_1 x_2 - x_1^2 e_1) \leq x_1^2(-\frac{1}{2}x_1^2 + |x_2| + \frac{1}{2}e_1^2) \qquad (5.1)$$

from which we can deduce

$$\frac{1}{4}x_1^2 \geq |x_2| + \frac{1}{2}e_1^2 \quad \Rightarrow \quad \dot{V}_1(x_1) \leq -\frac{1}{4}x_1^4\ .$$

Since the left-hand side of the implication is in turn implied by $V_1(x_1) \geq \max\{\sqrt{32V_2(x_2)}, 2e_1^2\}$, this shows that the first subsystem fulfills Assumption 4.2.1 with

$$\mu_1 = \max\ ,\ \gamma_{11}(r) = 0\ ,\ \gamma_{12}(r) = \sqrt{32r}\ , \eta_{11}(r) = 2r^2\ ,\ \eta_{12}(r) = 0\ . \qquad (5.2)$$

Similarly

$$\dot{V}_2(x_2) := \nabla V_2(x_2)(x_1^2 - kx_2 - ke_2) \leq |x_2|(-k|x_2| + x_1^2 + k|e_2|)$$

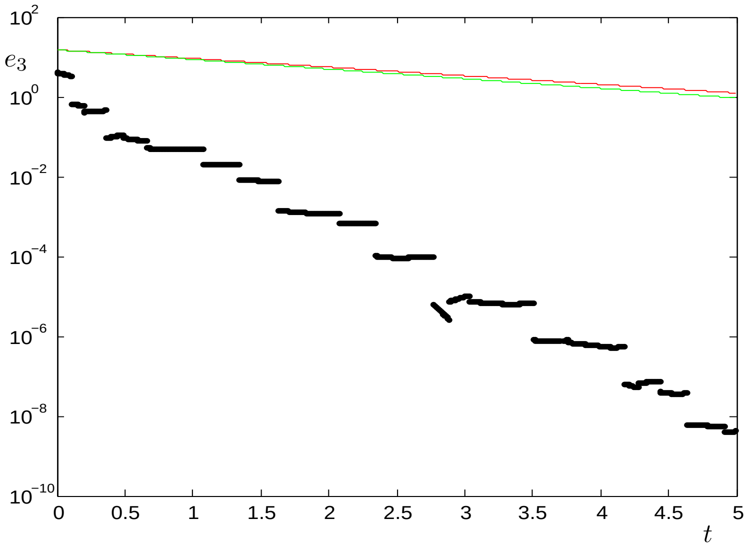

Figure 5.8: Difference between x_3 and $\hat{x}_3$

and therefore

$$V_2(x_2) \geq \max\{\frac{32}{k^2}V_1^2(x_1), 8e_2^2\} \quad \Rightarrow \quad \dot{V}_2(x_2) \leq -\frac{k}{2}x_2^2 \; ,$$

i.e. the second subsystem satisfies Assumption 4.2.1 with

$$\mu_2 = \max \; , \; \gamma_{21}(r) = \frac{32}{k^2}r^2 \; , \; \gamma_{22}(r) = 0 \; , \eta_{21}(r) = 0 \; , \; \eta_{22}(r) = 8r^2 \; . \qquad (5.3)$$

For the case $N = 2$ the conditions for an Ω-path are

$$\gamma_{12} \circ \sigma_2 < \sigma_1 \, , \qquad \gamma_{21} \circ \sigma_1 < \sigma_2 \, .$$

It is easy to see that if we choose $\sigma_1 = \mathrm{id}$, the latter is equivalent to $\gamma_{21} < \sigma_2 < \gamma_{12}^{-1}$. Provided that $k > 32$, one can set $\sigma_2(r) = \overline{\sigma}^2 r^2$, with $\overline{\sigma}^2 \in (\frac{32}{k^2}, \frac{1}{32})$. If $\varphi \in \mathcal{G}^{N\times N}$ is additionally chosen as

$$\varphi_{11}(r) = \sqrt{32}\overline{\sigma}r \; , \; \varphi_{12} \equiv \varphi_{21} \equiv 0 \; , \; \varphi_{22}(r) = \frac{32}{k^2}r^2 \; ,$$

then Assumption 4.2.2 is satisfied. In view of the choice of σ, μ and φ, the requirement $\Gamma_\mu(\sigma(r), \varphi(r)) < \sigma(r)$ boils down to the condition $\Gamma_\mu(\sigma(r)) < \sigma(r)$ which is equivalent to the small-gain condition $\gamma_{12} \circ \gamma_{21} < \mathrm{Id}$. This

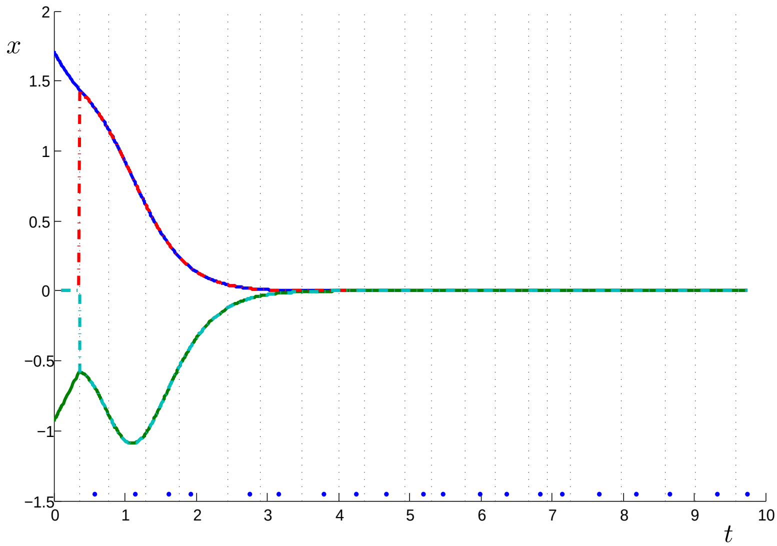

Figure 5.9: Trajectories of the pendulum

small-gain condition is fulfilled by the choice of k, since $\gamma_{12} \circ \gamma_{21}(r) = \frac{32}{k} r$. Hence Theorem 4.3.2 applies and provides an expression for the functions χ_i used in the event-triggered implementation of the control laws. The functions are given explicitly by

$$\chi_1(r) := \frac{1}{\sqrt{8\overline{\sigma}}} r^2, \quad \chi_2(r) := \frac{\overline{\sigma}^2 k^2}{4} r^2 \; .$$

Hence the triggering function introduced in Theorem 4.3.3 is explicitly given by

$$T_i(x_i, e_i) = \chi_i(|e_i|) - V_i(x_i) \, . \tag{5.4}$$

Simulation results for the initial condition $x_1(0) = -4$, $x_2(0) = 3$, $\hat{x}_1(0) = -4$ and $\hat{x}_2(0) = 3$ can be found for $t \in [0, 2]$ in Fig. 5.11 and 5.12. The trajectory of the first system is given in blue and for the second system in green. The input is calculated using the red and turquoise values accordingly. Figure 5.11 shows the event triggering scheme from Theorem 4.3.3 as presented in (5.4). Between $t = 0$ and $t = 2$, 39 events are triggered.
In Fig. 5.12 a periodic sampling scheme was used with a sampling period equal to the shortest time between events from Fig. 5.11 resulting in 286 samples. No major difference in the behavior can be seen despite of the fact that more than 7 times the amount of information was transmitted. Using a

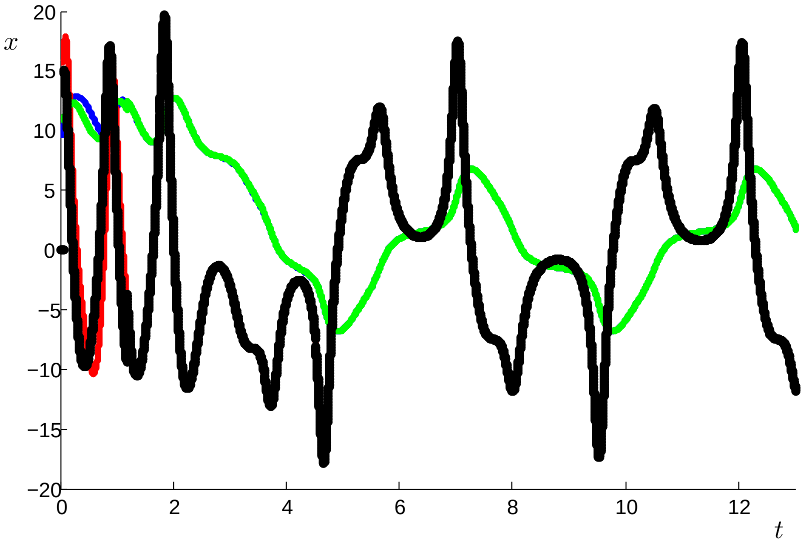

Figure 5.10: Trajectories of the pendulum

periodic sampling scheme with 39 samples between $t = 0$ and $t = 2$ results in instability of the system. Numerical simulations suggests that the smallest period, which stabilizes the system results in 66 events between $t = 0$ and $t = 2$ (see Figure 5.13). In Figs. 5.14 and 5.15 the Lyapunov function and the evolution of the error can be seen. Whenever the red curve (the error) hits the blue line (the Lyapunov function) an event is triggered.

5.2.1 Practical Stabilization

If we change the initial condition from our example to $x_1(0) = 4$, $x_2(0) = -3$ the event triggering scheme introduced in Theorem 4.3.3 exhibits Zeno behavior and hence the Theorem is not applicable.
One possible way to deal with the Zeno phenomenon is to alter the assumption of ISS-Lyapunov functions to a notion of practical stability. See Section 4.4 for details.
In particular, a new design parameter $c_i > 0$ is introduced in Assumption 4.4.1 which allows us a trade-off between the size of the region we are converging to and the minimal time between events. Of course, if we can bound the minimal time between events away from zero, Zeno cannot occur.

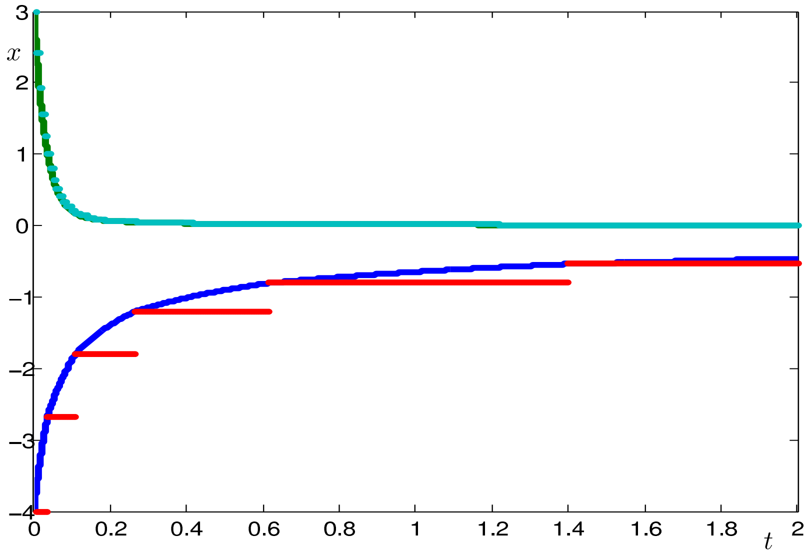

Figure 5.11: Trajectories of the system with event triggered control

In this case, the triggering condition is

$$T_i^2(x, e_i) = \hat{\eta}_i(\|e_i\|) - \max\{\sigma_i^{-1}(V_i(x_i)), c_i\} \geq 0 \,. \tag{5.5}$$

The only difference to the triggering function from Theorem 4.3.3 is that here after the error is reset to zero it must evolve at least until the norm of the error is larger than c_i before a new event is triggered.
In the triggering condition (5.4) or (5.5), each system compares its local error to its local Lyapunov function. Hence each system can decide based purely on local information when to trigger an event.
In Fig. 5.16 the trajectories of the system using (5.5) is depicted for $c_1 = 35$ and $c_2 = 1.86$ resulting in 10 events between $t = 0$ and $t = 80$. The system enters a stable limit cycle, the size of which depends on c_1 and c_2.

In Fig. 5.17 the Lyapunov function of the second subsystem together with the error e_2 is given. Whenever the red line (the error) hits the blue line (the Lyapunov function) or the threshold c_2 (the dashed line), whichever is greater, an event is triggered.
For the short time evolution of the system for $t \in [0, 0.5]$ (cf. Fig. 5.18 and for the corresponding Lyapunov function in Fig. 5.19).

An interesting question is how the threshold c_2 affects the number of triggered events. In Fig. 5.22 the threshold is plotted against the resulting

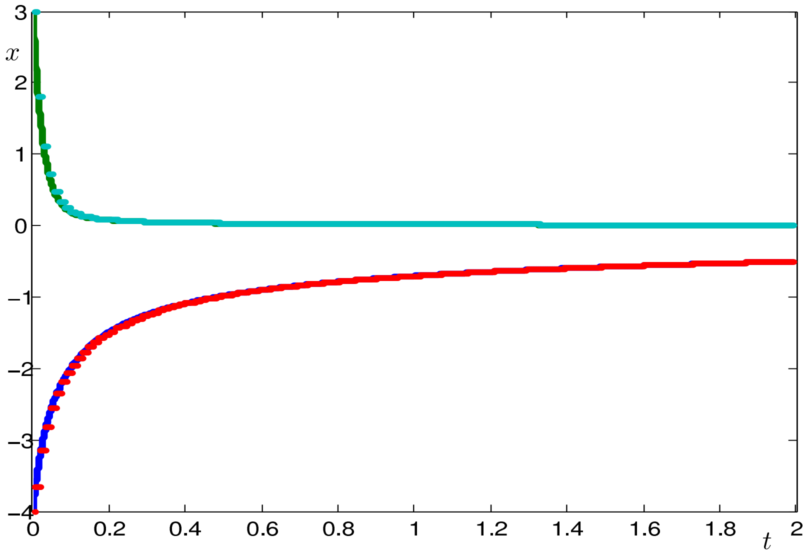

Figure 5.12: Trajectories of the system with periodic sampling

number of events triggered by the second subsystem. It can be seen that there is no heuristic that larger c_2 leads to fewer events. The reason becomes apparent by inspecting Figs. 5.20 and 5.21, where we used $c_2 = 0.16$ resulting in 339 events. Compared to Fig. 5.18 or Fig. 5.19, respectively, the trajectory oscillates faster within the limit cycle while triggering more events.

5.2.2 Parsimonious Triggering and Asymptotic Stabilization

Although in many applications a notion of practical stability is enough, sometimes it is desired to have asymptotic stability. To achieve this we introduce a further mechanism for reducing the number of triggering times. See Section 4.5 for a detailed discussion.

The approach is based on the observation that besides comparing each state to the corresponding error, it would be sufficient to compare each error to the largest Lyapunov function among the subsystems.

Intuitively, if the errors of all subsystems are smaller than the largest Lyapunov function, then these errors cannot affect the stability of the system. In particular, implication (4.11) suggests that one can either compare each error to each state as in (5.4) and (5.5) or compare all the errors to the largest Lyapunov function to ensure that the maximum in (4.11) is attained by one of the states (the γ part) instead of the error.

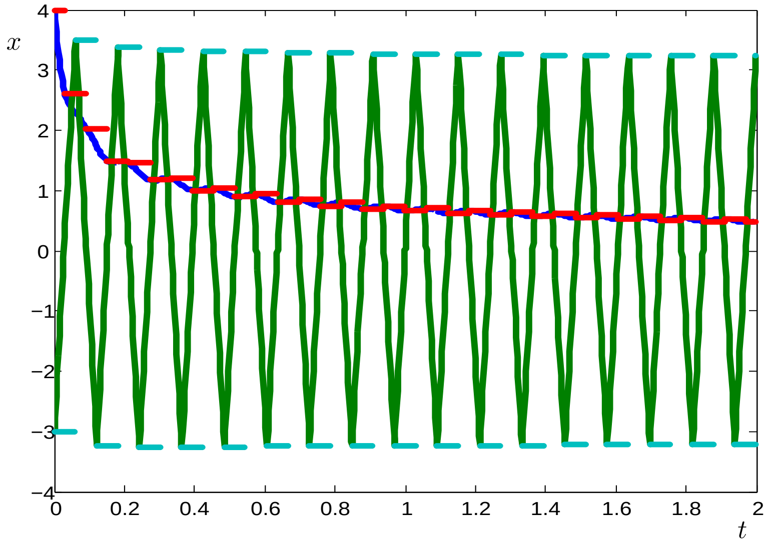

Figure 5.13: Smallest period that stabilizes the system

A triggering condition following these ideas is of the form

$$T_i(x_i, e_i) = \hat{\eta}_i(\|e_i\|) - \max_j\{\sigma_j^{-1}(V_j(x_j))\} \geq 0\,. \tag{5.6}$$

The drawback of this approach is that it requires the knowledge of all states, contradicting our wish for a decentralized setup.

However, under suitable regularity assumptions on the involved gains, respectively scaling functions, we can give an approximation on the size of the other Lyapunov functions, which can be used for a triggering condition; based only on local information. For details see Lemma 4.5.2.

The intuition behind this approximation is that a subsystem can decide based on its local dynamics (i. e., $\|\dot{x}_i\|$) whether there must be other states that are larger than the state of the i-th system. It is then reasonable to include this information in an augmented triggering condition. This can be used to obtain asymptotic stability but still rule out the occurrence of Zeno behavior, as can be seen in Theorem 4.5.7.

We know that if Zeno occurs, the corresponding state must approach zero at the Zeno time (Lemma 4.5.3). On the other hand, at least one other subsystem is bounded away from zero at the Zeno point, because the "strength" to force a Lipschitz continuous system to zero in finite time cannot result from its own state (Corollary 4.5.6). Hence we know that at some point the largest

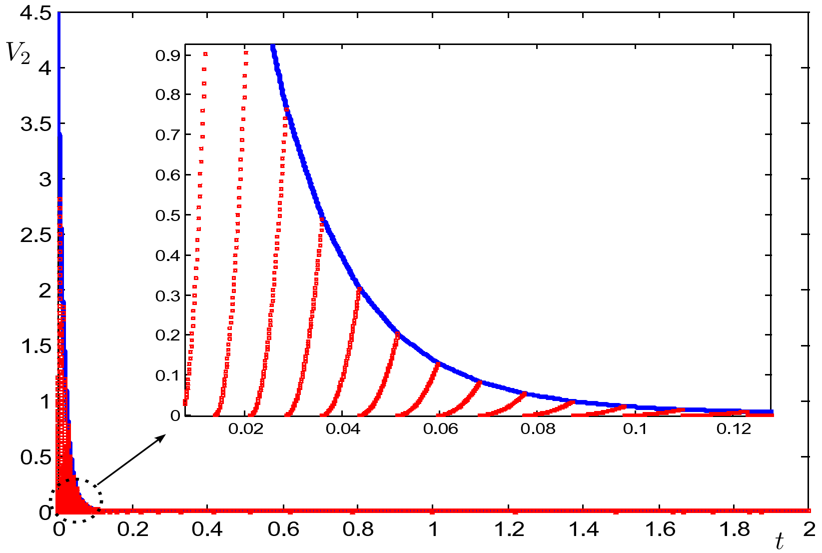

Figure 5.14: Lyapunov function of the second subsystem together with the error

Lyapunov function in (5.6) is not the Lyapunov function of the system that induces Zeno behavior. Moreover, this Lyapunov function is bounded away from zero. Therefore, no Zeno behavior can occur, because the time until the error evolves until it reaches the level of this larger Lyapunov function is bounded away from zero.

Theorem 4.3.2 shows that if the small-gain condition holds, a Lyapunov function for the interconnected system is given by the maximum of the individual Lyapunov functions (properly scaled). In Fig. 5.23 the Lyapunov function of the first subsystem is given in dark green and for the second in blue. The Lyapunov function of the interconnected system is depicted in black. At approximately $t \approx 0.03$ the maximum of the Lyapunov functions changes. Before that time the Lyapunov function of the second subsystem is equal to the overall Lyapunov function. After the maximum changed, the Lyapunov function is equal to the Lyapunov function of the first subsystem.

in Fig. 5.23 the error of the first subsystem is given in red. Again, whenever the red line hits the green line, an event is triggered.

Please note that the Lyapunov functions in Figs. 5.23 and 5.25 are the same. The only difference is that in Fig. 5.23 the error of the first subsystem is given, whereas the error of the second subsystem is depicted in Fig. 5.25.

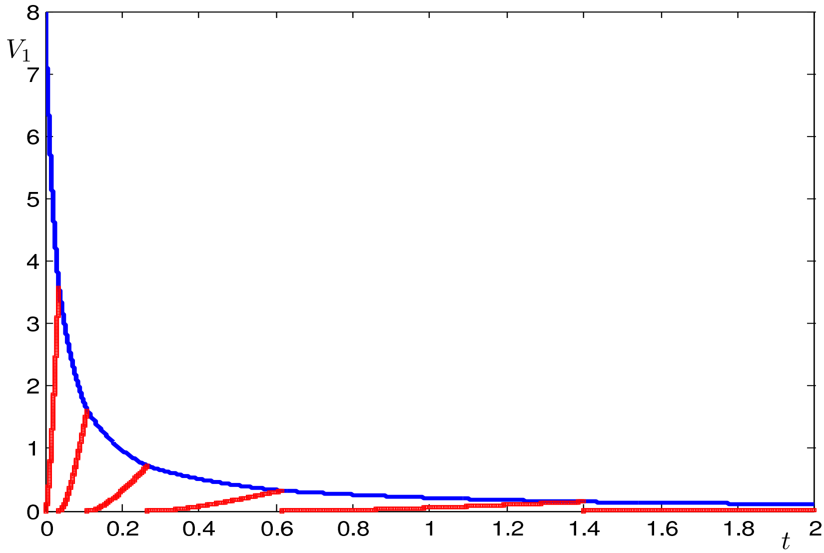

Figure 5.15: Lyapunov function of the first subsystem together with the error

We know that if Zeno occurs, the corresponding state must approach zero at the Zeno time. In Fig. 5.25 it can be seen that the blue line (the Lyapunov function of the second subsystem) hits 0 at $t \approx 0.08$. On the other hand, at least one other subsystem is bounded away from zero at the Zeno point, because the "strength" to force a Lipschitz continuous system to zero in finite time cannot result from its own state. Hence we know that at some point the largest Lyapunov function in (5.6) is not the Lyapunov function of the system that induces Zeno behavior. Moreover, this Lyapunov function is bounded away from zero.

In theory, it would be sufficient to trigger an event whenever the red line (the error) hits the black line (Lyapunov function of the overall system), ruling out the occurrence of Zeno behavior. But the knowledge of the Lyapunov function of the overall system would make the knowledge of all states to all systems necessary, contradicting our decentralized approach.

With the help of an approximation of the dynamics of the subsystems, we can give lower bounds of the Lyapunov functions of the other states (given in light green). Using this bound also rules out the occurrence of Zeno behavior. Basically, the theory predicts that the bound becomes tighter around the Zeno point ($t^* \approx 0.8$). Hence with the above considerations, we know that it is sufficient to trigger an event, if the red line hits the light green line to

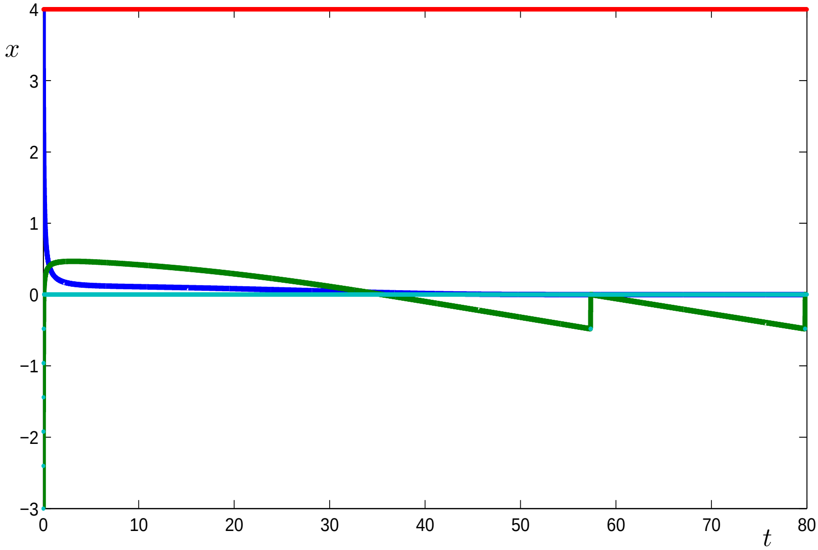

Figure 5.16: Trajectories with event triggering (5.5), $c_2 = 1.86$

infer asymptotic stability of the system and ruling out Zeno behavior.

5.2.3 Comparison of the Different Approaches

Both approaches that have been presented in the preceding sections have advantages and drawbacks, which will be discussed in this section. Both Theorem 4.4.5 and 4.5.7 can be regarded as generalizations of Theorem 4.3.3. The major drawback of Theorem 4.3.3 is that Zeno behavior may occur. The advantage of Theorem 4.4.5 lies in its simplicity. There is no difference in implementation and numerical complexity compared to Theorem 4.3.3. Note that the class of systems that are practical ISS is larger than the class of systems that are only ISS. Hence Theorem 4.4.5 is applicable to a larger number of systems than Theorems 4.3.3 and 4.5.7. On the other hand, the result is limited to practical stability of the interconnected system.
The advantage of Theorem 4.5.7 is that it rules out the Zeno phenomenon, while retaining asymptotic stability. The price to pay lies in a higher computational complexity and some regularity assumptions on the involved triggering functions. As event-triggering tries to reduce the amount of information transmissions, the question which of the approaches results in the fewest events is of interest.

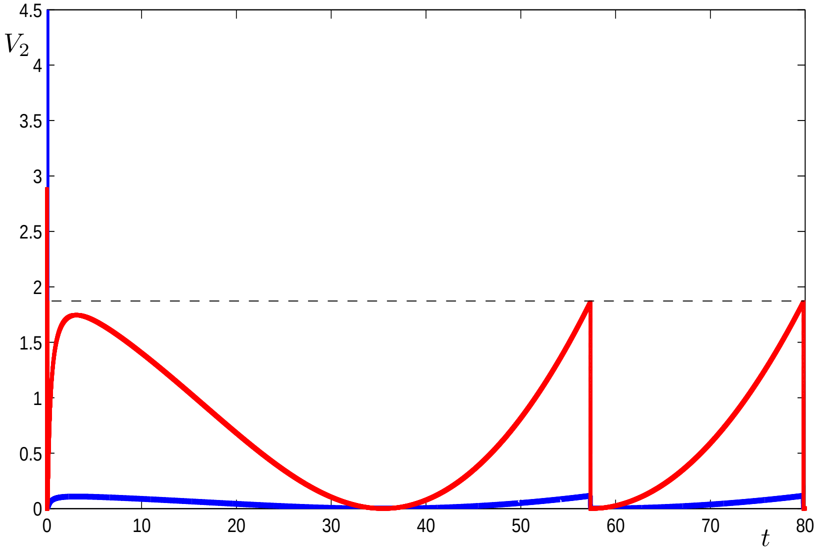

Figure 5.17: Lyapunov function of the second subsystem together with the error and the threshold c_2

In our example Theorem 4.3.3 behaves exactly like Theorem 4.5.7, if no Zeno behavior occurs. For the Zeno case, Theorem 4.4.5 can lead to fewer events, although it is not a priori clear how to choose the offset to achieve this. The wrong choice of the offset leads to an increase in the number of events beyond the number of events resulting from Theorem 4.5.7. See Figure 5.22 for a plot of the offset against the resulting number of events. Further investigations that analyze the impact of the choice of the offset are required here. The resulting number of events of the different approaches for different parameters are given in Table 5.1. The offset for the triggering condition of Theorem 4.4.5 are chosen as $c_1 = 35$ and $c_2 = 1.86, 1, 0.16$ corresponding to the first, second, and third number in the rows for Theorem 4.4.5.

End Time	$x_1(0) = -4$ and $x_2(0) = 3$			$x_1(0) = 4$ and $x_2(0) = -3$		
	T 4.3.3	T 4.4.5	T 4.5.7	T 4.3.3	T 4.4.5	T 4.5.7
t=2	35	6, 35, 40	35	Zeno	6, 37, 42	38
t=20	142	8, 303, 337	142	Zeno	6, 307, 339	150
t=40	300	8,591,667	300	Zeno	6,594,668	312

Table 5.1: Number of events for different triggering conditions

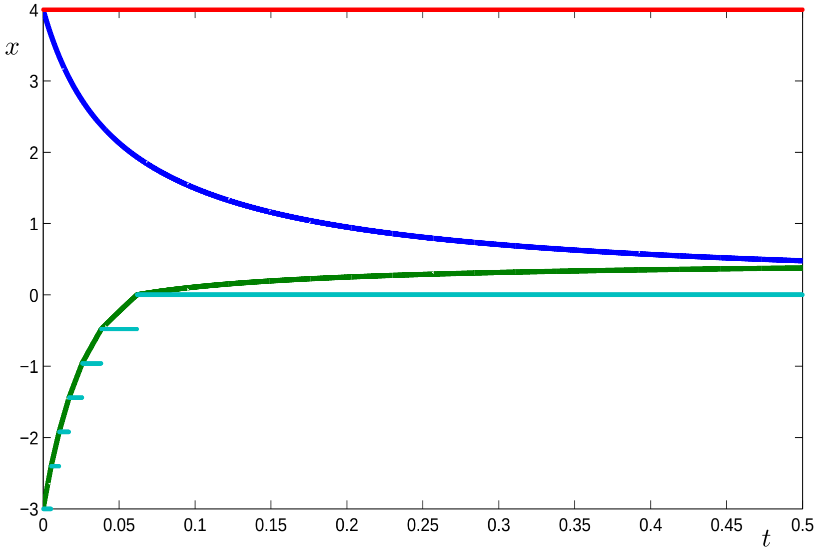

Figure 5.18: Zoom into $t \in [0, 0.5]$ of Fig. 5.16

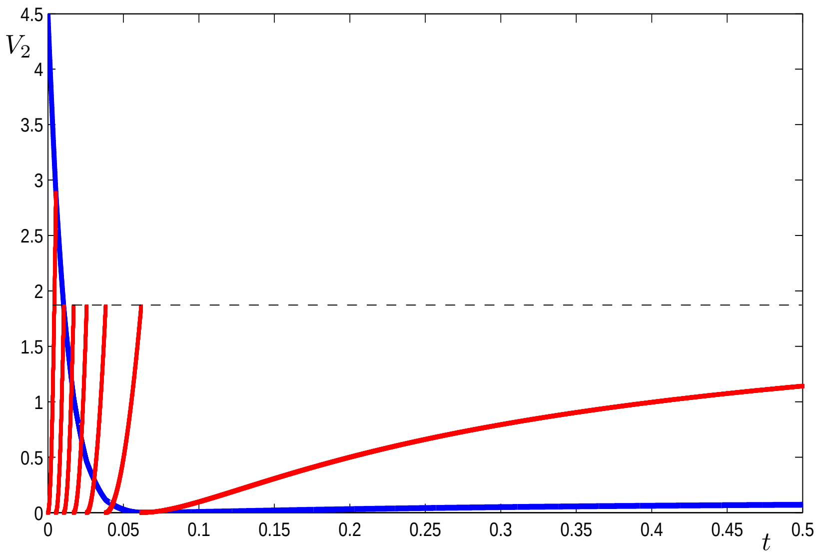

Figure 5.19: Zoom into the Lyapunov function

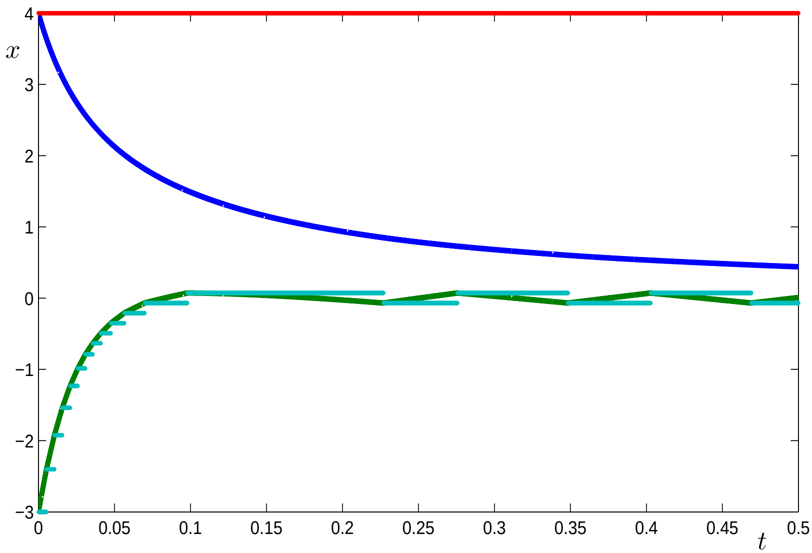

Figure 5.20: Trajectories with event triggering (5.5), $c_2 = 0.16$

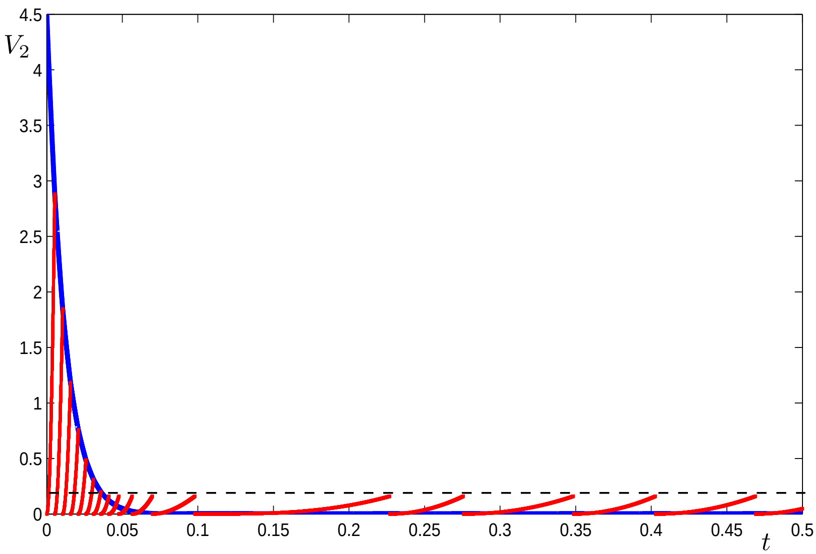

Figure 5.21: Lyapunov function for $c_2 = 0.16$

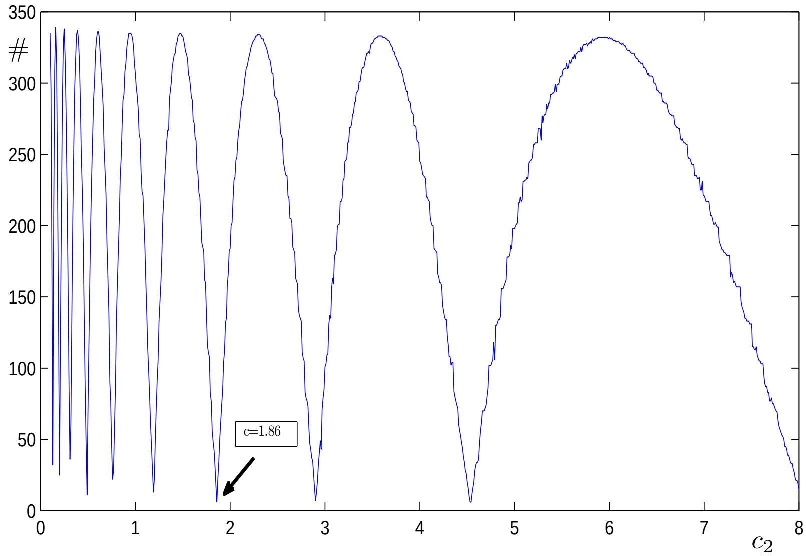

Figure 5.22: Plot of the threshold c_2 against the number of triggered events for $t \leq 20$

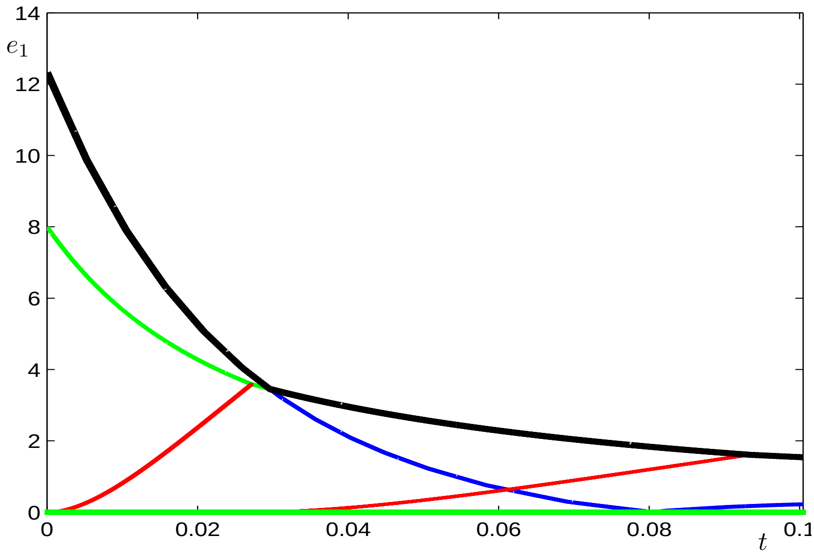

Figure 5.23: Lyapunov function of both subsystems together with e_2 and the lower bound on V_1.

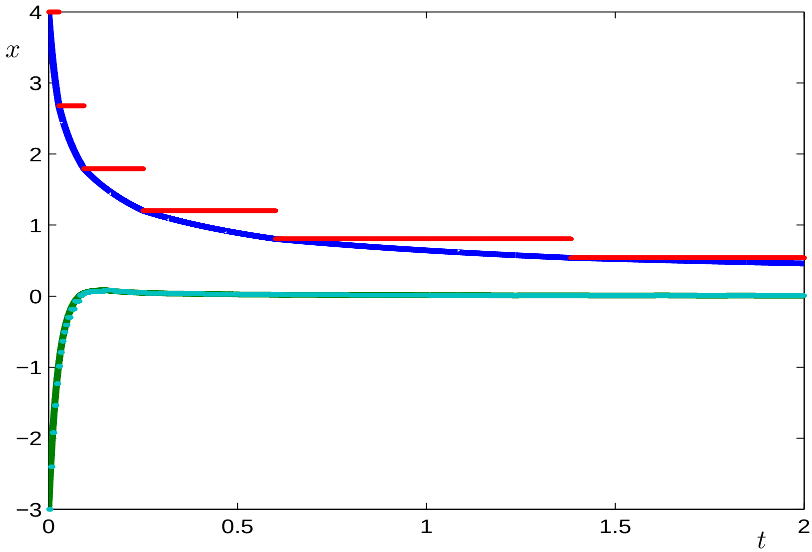

Figure 5.24: Trajectories with event triggering from (4.52)

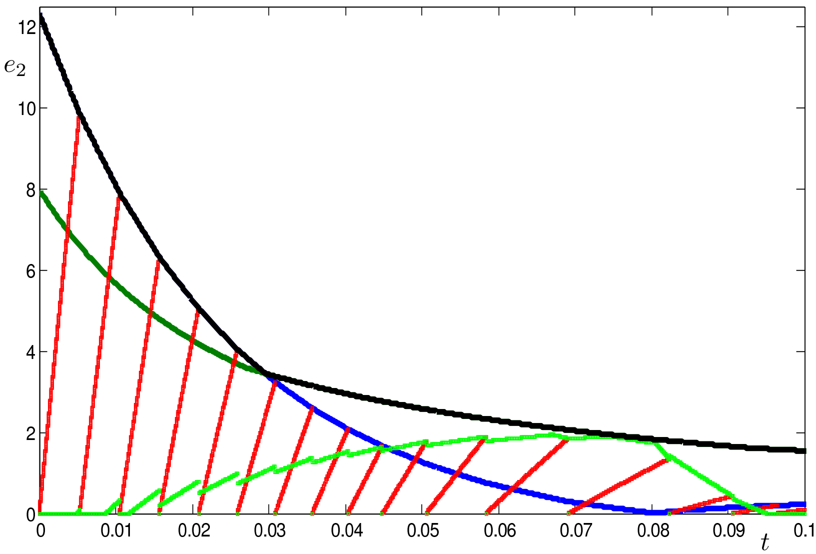

Figure 5.25: Lyapunov function of both subsystems together with e_2 and the lower bound on V_1.

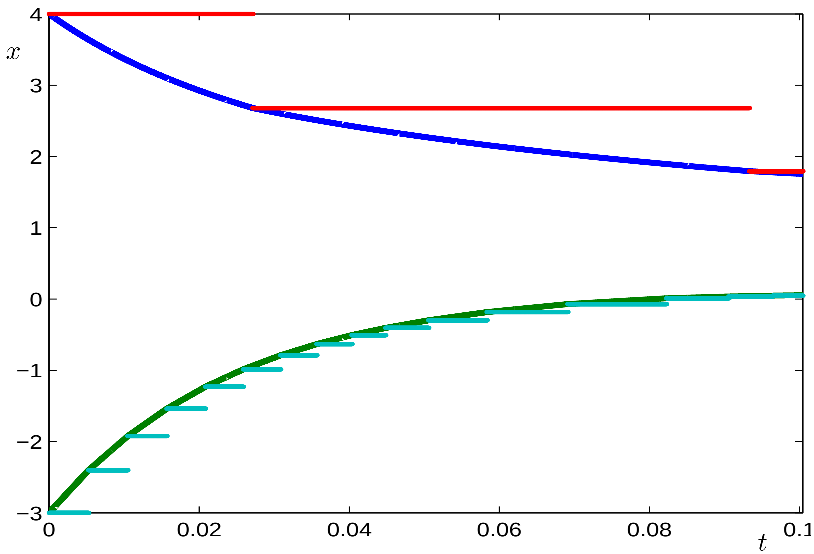

Figure 5.26: Trajectories with event triggering from (4.52)

Conclusion

In the presented work we have discussed several approaches to tackle the problem of digital control of large-scale systems.
In particular, we presented in Chapter 2 methods to use a tool known as *dynamic quantization* to asymptotically stabilize a single system despite the effects of quantization, delay, and packet loss.
As these tools are computationally expensive, they are not well-suited for large-scale systems.
Therefore we developed small-gain conditions to analyze the stability property of interconnected systems in Chapter 3.
In addition, we have shown that many of the known small-gain conditions are equivalent, if looked upon the right way.
Moreover, we have shown the feasibility of the presented methods with the help of a multichannel time delay system.
One aspect of digital control of large-scale systems is the need for more bandwidth, as the number of subsystems grows. In this regard, we presented methods to lower the amount of information needed to stabilize a system in Chapter 4.
The work is concluded by numerical examples showing the feasibility of the presented methods in Chapter 5.
As already sketched in the introduction, there are still many interesting open problems worth to pursue.
For instance, in Chapter 2 it would be wishful to conclude ISS instead of asymptotic stability, as all the other approaches presented in this work demand for an ISS property (or similar concepts).
Furthermore, we do not know yet whether the additional property of weakly increasing is the appropriate condition. Although we conjecture that it is the needed condition to prove that the existence of an ISS-Ω-path is equivalent to the ISS property of the corresponding system, but we were not able to prove this for the general case.
In Chapter 4 we presented ways to lower the amount of information needed

to stabilize a system, but we did not consider the problem of arbitration, which, of course, would be of interest in this context.
Similar to [NL09] it would be nice to cast the material from this work into a unified framework, which is part of further research.
As already stated in the introduction, we assume that a model for the communication channel, which is better suited for control purposes, could lead to more sophisticated ways to handle the negative effects of communication. As these models do not yet exist, further research is required.

Appendix A

Matlab Listings

Dynamic Quantization

Listing A.1: Main Programm

```
global L;                  %Lipschitz constant known to all subroutines
L=8;
ende=80;                   %How long should the simulation run
global h;                  %discretization size known to all subroutines
h=0.001;
global N;                  %Number of subregions per dimension
N=6;
global l0;                 %Initial size of quantisation region
l0=20;
global options;            %Options for the ode solver
options=odeset('RelTol',2.22045e-14,'AbsTol',2.22045e-14);
mindelay=10;
maxdelay=200;
xges=[];
Tges=[];
tstar=mindelay+randi(maxdelay-mindelay)+mindelay;
tk=ende/h;
tkold=0;
x=[.7125464654645764 ;-.334545774764747];%initial value
xdhat=[0; 0];                            %initializing of the decoder
s=enc(tkold,tkold,x);                            %First encoding at t=0
Talt=0;
exp(L*(maxdelay)*h)/N                    %Test if smaller than one
for t=0:ende/h
    if (t==tstar)
        [T yh]=ode113('pendel4',[0 (tstar-tkold)*h],[x; xdhat],options);
        Tges=[Tges;T+Talt];
        Talt=Tges(end);
        x=yh(end,1:2)';
        xdhat=dec(tstar,tkold,s);     %Decoding gives approx. of the state
        xges(:,length(xges)+1:length(xges)+numel(T))=yh';
        %tk=tstar+averagedelay+deviat*randn(1);
        tk=tstar+randi(maxdelay-mindelay)+mindelay;  %calculate next t_k
        tkold=tk;
        ts=tstar;
    end
     if(t==tk)
        [T yh]=ode113('pendel4',[0 (tk-tstar)*h],[x; xdhat],options);
        Tges=[Tges;T+Talt];
        Talt=Tges(end);
        x=yh(end,1:2)';
        xdhat=yh(end,3:4)';
        xges(:,length(xges)+1:length(xges)+numel(T))=yh';
       s=enc(tk,ts,x);           %calculates value to transmit
       ts=tk;
       tstar=tk+randi(maxdelay-mindelay)+mindelay;     %calculate next t^*
    end
```

```
end
plot(Tges,xges')                %trajectories
```

Listing A.2: Encoder Subroutine

```
function s=enc(tk,ts,x)
global h;
global N;
global l0;
global L;
global options;
persistent tkold;                 %Encoder needs to remember the last time
if isempty(tkold)
        tkold=0;
end
persistent xehat;                 %Also the old state
if isempty(xehat)
      xehat=zeros(2,1);
end
persistent l;
if isempty(l)
    l=l0;
end
persistent xe;                    %And its current state
if isempty(xe)
    xe=zeros(2,1);
end
l=l*exp(L*(tk-tkold)*h)/N;        %Reduce the size of the quantization region
if (tk>0)
    [t3 yh]=ode113('pendel4',[0 (ts-tkold)*h],[xe; xehat],options);
    xe=yh(end,1:2)';
    [t3 yh]=ode113('pendel4',[0 (tk-ts)*h],[xe; xe],options);
    xe=yh(end,1:2)';
end
s=floor(N/l*(x-xe)+0.5);        %Calculate in which box the state is
xehat=xe;
xe=xe+s*l/N;                    %Set the next center of the quant. region
tkold=tk;
```

Listing A.3: Decoder Subroutine

```
function y=dec(tstar,ts,s)
global h;
global N;
global l0;
global L;
global options;
persistent tstarold;              %Decoder needs to remember the last time
if isempty(tstarold)
        tstarold=0;
end
persistent xdhat;                 %Also the value which closes the loop
if isempty(xdhat)
      xdhat=zeros(2,1);
end
persistent l;                     %Lenght of quantisation region
if isempty(l)
    l=l0;
end
persistent xd;                    %And the internal state of the decoder
if isempty(xd)
    xd=zeros(2,1);
end
persistent tsold;                 %The last timestamp
if isempty(tsold)
    tsold=0;
end
l=l*exp(L*(ts-tsold)*h)/N;        %Reduce the size of the quant. region
if(ts>tstarold)                   %Test if it is the first time
    [T yh]=ode113('pendel4',[0 (ts-tstarold)*h],[xdhat; xdhat],options);
    xd=yh(end,1:2)'+s*l/N;        %Calculate the center of the box
    xdhat=yh(end,1:2)';           %Approximation of the state
else
    xd=s*l/N;
end
```

```
[T yh]=ode113('pendel4',[0 (tstar-ts)*h],[xd; xdhat],options);
xdhat=yh(end,1:2)';
y=xdhat;
tstarold=tstar;
tsold=ts;
```

Listing A.4: Dynamics of the Pendulum

```
function y=pendel4(t,x)
    u=k([x(3) x(4)]);
    y=zeros(4,1);
    y(3)=x(4);
    y(4)=sin(x(3))-cos(x(3))*u;
    y(1)=x(2);
    y(2)=sin(x(1))-cos(x(1))*u;
```

Event Triggered Control

Listing A.5: Event triggered control

```
sigma=0.172;               %   choose sigma^2 in (32/k^2,1/32)
k1=1;
k2=33;
chi1=k1/(sqrt(8)*sigma);   %   * r^2
chi2=sigma^2*k2^2/4;       %   * r^2
psi1=sqrt(2)*sigma*chi1;   %   * r^2
psi2=sqrt(k2);             %   * sqrt(r)
%
%Initializition:
x0=[4;-3;4;-3];  %Initial value that causes Zeno
x=x0;
i=0;
yges=[];          %Overall trajectory
Tges=[];          %Corrsponding time
Talt=0;
kappa1=16;  %smallest kappa1,kappa2 for conservative parameters
kappa2=6;
tkminus1=0; %Last time first system triggered
tkminus2=0; %%Last time second system triggered
p1=[];
p2=[];
% Set options for the solver
options=odeset('RelTol',2.22045e-14,'AbsTol',2.22045e-14);
while(Talt<2 && i<8000)        %Run simulatiuon until t=2 or 8000 events
    i=i+1;
    [T,y]=ode23(@fsys,[Talt,Talt+0.001],x,options); %Integration
    x=y(end,:)';     %Last value of integration will be the first for the next
    ys=y;
    V1=0.5*ys(:,1).^2;                   %Calculate V1
    chie1=chi1*(ys(:,3)-ys(:,1)).^2; %Calculate \chi_1(e_1)
    V2=0.5*ys(:,2).^2;                   %Calculate V2
    chie2=chi2*(ys(:,4)-ys(:,2)).^2; %Calculate \chi_2(e_2)
    c1= find(chie1>=V1, 1 );            %Does the first system triggers?
    c2= find(chie2>=V2, 1 );            %Does the second system triggers?
  %
  % Calculate W1 and W2:
   a1=abs(ys(1:end,1)-ys(1:end,3))./(T(1:end)-tkminus1)-
              kappa1*max(abs(ys(1:end,1)),abs(ys(1:end,1)-ys(1:end,3)));
   a2=abs(ys(1:end,2)-ys(1:end,4))./(T(1:end)-tkminus2)-
              kappa2*max(abs(ys(1:end,2)),abs(ys(1:end,2)-ys(1:end,4)));
   %W1:
   hilfe1=max(a1./abs(ys(:,1))-0.5*(ys(:,3)-ys(:,1)).^2-0.5*abs(ys(:,1)).^2,0);
   %W_2:
   hilfe2=max(a2-k2*abs(ys(:,2))-k2*abs(ys(:,4)-ys(:,2)),0);
   if (~isempty(c1) || ~isempty(c2))  % Did T1 trigger?
     %eta_1( e1 ) \geq psi( w1 ) :
     p1= find(chi1*abs((ys(c1:end,3)-ys(c1:end,1)).^2)>=
     min(0.5*hilfe1(c1:end).^2,1/(sqrt(2)*sigma)*hilfe1(c1:end)),1);
     %eta_2( e2 ) \geq psi( w2 ) :
     p2=find(1/sigma*sqrt(chi2*abs(ys(c2:end,4)-ys(c2:end,2)).^2)>=
```

```
      min(0.5*sqrt(hilfe2(c2:end)).^2,1/(sqrt(2)*sigma)*sqrt(hilfe2(c2:end))),1);
    end
    %Boolean:System 1 trigger:
    system1=(~isempty(c1) && V1(c1)>=1e-12) && ~isempty(p1);
    %Boolean:System 2 trigger:
    system2=(~isempty(c2) && V2(c2)>=1e-12) && ~isempty(p2);
    if (system1 || system2)  %Did an event happen?
        r=min([p1 p2])-1;    %When did T2 trigger?
        s=min([c1 c2])-1;    %When did T1 tri
        counter=counter+1;
        r=r+s;                   %Time when T1 and T2 trigger
        x=y(r,:)';               %Continue simulation from this time
        if(system1)              %If system 1 triggered:
            x(3)=x(1);           %Transmit the state
            tkminus1=T(r);%Save the time
        end
        if(system2)              %If system 1 triggered:
            x(4)=x(2);           %Transmit the state;
            tkminus2=T(r);%Save the time
        end
        yges=[yges y(1:r,:)'];%Concatenate the trajectory
        Tges=[Tges;T(1:r)];      %And the time
        Talt=T(r);               %Save actual time
    else                         %If no event happened continue simulation
        yges=[yges y'];
        Tges=[Tges;T];
        Talt=T(end);
    end
end
plot1=plot(Tges(1:end),yges(:,1:numel(Tges)));  %Display trajectories
```

Listing A.6: Dynamics of the system

```
function y=fsys(t,x)
    k1=1;
    k2=33;
    y=zeros(4,1);
    y(1)=x(1)*x(2)-k1*x(1)^2*x(3);
    y(2)=x(1)^2-k2*x(4);
    y(3)=0;
    y(4)=0;
```

Appendix B

Non Smooth Analysis

In Chapter 4 we deal with Lyapunov functions that are Lipschitz instead of differentiable. Hence we need some tools from the field of non-smooth analysis. The tools needed for Chapter 4 are summarized here. Moreover, to familiarize the reader with these concepts we recall a few definitions and quote some results about generalized gradients. The material is taken from [CLSW98]. Note that in Chapter 4 we combine ideas from Lyapunov constructions for large-scale systems with the concept of event based sampling. The construction of Lyapunov functions was already discussed in [Rüf07]. See notes and references of the corresponding chapter for more details. Thus the appendix here appeared already in similar form in [Rüf07].
Let X be a real Banach space. For $x \in X$ and $\epsilon > 0$ by $B(x, \epsilon)$ we denote the set $\{y \in X : \|x - y\| < \epsilon\}$. On the dual space X^* of X we use the norm

$$\|\zeta\|_* := \sup\{\langle \zeta, v\rangle : v \in V, \|v\| = 1\},$$

where $\langle \zeta, v\rangle$ is the pairing of ζ and v.

A function $f : X \to \mathbb{R}$ is *Lipschitz of rank* K near $x \in X$, if there exists an $\epsilon > 0$ and a $K \geq 0$, such that for all $y, z \in B(x, \epsilon)$ we have

$$|f(y) - f(z)| \leq K\|y - z\|.$$

The function f is *locally Lipschitz*, if for every $x \in X$ it is Lipschitz of some rank $K = K(x)$. The *generalized directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y + tv) - f(y)}{t}$$

A multivalued function $F : X \to 2^X$ is *upper semicontinuous* at x, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|x - y\| < \delta \implies F(y) \subset F(x) + B(0, \epsilon).$$

A function $g : X \to \mathbb{R}$ is *positively homogeneous* if $g(\lambda x) = \lambda g(x)$ for all $\lambda \geq 0$ and all $x \in X$, and *subadditive* if $g(v + w) \leq g(v) + g(w)$ for all $v, w \in X$.

Proposition B.0.1. *Let $f : X \to \mathbb{R}$ be Lipschitz of rank K near $x \in X$. Then*

1. *the function $v \mapsto f^\circ(x; v)$ is finite, positively homogeneous, subadditive on X, and satisfies*
$$|f^\circ(x; v)| \leq K\|v\|;$$
2. *$f^\circ(x; v)$ is upper semicontinuous as a function of (x, v) and, as a function of v alone, it is Lipschitz of rank K on X;*
3. *$f^\circ(x; -v) = (-f)^\circ(x; v)$.*

For a proof see [CLSW98, Prop. 2.1.1, p.70]. We define the *generalized gradient* ∂f of the function $f : X \to \mathbb{R}$ by

$$\partial f(x) := \{\zeta \in X^* : f^\circ(x; v) \geq \langle \zeta, v \rangle \ \forall v \in X\}.$$

We state a couple of properties of generalized gradients. The first one is stated as an exercise in [CLSW98, p.73]. For the convenience of the reader we give a proof.

Proposition B.0.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$. Then $f^\circ(x; v) = \langle f'(x), v \rangle$ and $\partial f(x) = \{f'(x)\}$.*

Proof. Since f is continuously differentiable, we have $f^\circ(x; v) = Df(x; v)$ the directional derivative of f. But $Df(x; v)$ is just equal to $\langle f'(x), v \rangle$.

We have $f'(x) = \zeta$ if and only if $\langle f'(x), v \rangle = \langle \zeta, v \rangle$ for all $v \in \mathbb{R}^n$, which is equivalent to $f^\circ(x; v) \geq \langle \zeta, v \rangle$ for all $v \in \mathbb{R}^n$, and this in turn to $\zeta \in \partial f(x)$. □

The following result is also taken from [CLSW98].

Proposition B.0.3. *Let $f : X \to \mathbb{R}$ be Lipschitz of rank K near $x \in X$. Then*

1. *$\partial f(x)$ is a nonempty, convex, weak*-compact subset of X^*, and $\|\zeta\|_* <$ K for every $\zeta \in \partial f(x)$;*

2. *for every* $v \in X$ *we have*

$$f^\circ(x; v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial f(x)\};$$

3. $\zeta \in \partial f(x)$ *if and only if* $f^\circ(x; v) \geq \langle \zeta, v\rangle$ *for all* $v \in X$*;*

4. *if* $\{x_i\}$ *and* $\{\zeta_i\}$ *are sequences in* X *and* X^* *such that* $\zeta_i \in \partial f(x_i)$ *for each* i*, and if* x_i *converges to* x *and* ζ_i *is a weak** *cluster point of the sequence* $\{\zeta_i\}$*, then we have* $\zeta \in \partial f(x)$*;*

5. *if* X *is finite dimensional, then* ∂f *is upper semicontinuous at* x*.*

This result has been proved in [CLSW98, Proposition 2.1.5, p.73]. Next we gather facts relating nonsmooth calculus to standard calculus:

Theorem B.0.4. *1. Let* $f_i : X \to \mathbb{R}$*,* $i = 1, \ldots, n$*, be Lipschitz near* $x \in X$ *and let* λ_i *be scalars,* $i = 1, \ldots, n$*. Then* $f := \sum_i \lambda_i f_i$ *is Lipschitz near* x *and*

$$\partial\left(\sum_i \lambda_i f_i\right)(x) \subset \sum_i \lambda_i \partial f_i(x).$$

2. ***(Lebourg's Mean Value Theorem)*** *Let* $x, y \in X$ *and let* $f : X \to \mathbb{R}$ *be Lipschitz on an open set containing the line segment* $[x, y]$*. Then there exists a point* u *in* (x, y) *such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x\rangle.$$

3. ***(The Chain Rule)*** *Let* $F : X \to \mathbb{R}^n$ *be Lipschitz near* x*, and let* $g : \mathbb{R}^n \to \mathbb{R}$ *be Lipschitz near* $F(x)$*. Then the function* $f(x') := g(F(x'))$ *is Lipschitz near* x*, and we have*

$$\partial f(x) \subset \overline{\mathrm{conv}}^*\{\partial\langle \gamma, F(\cdot)\rangle(x) : \ \gamma \in \partial g(F(x))\},$$

where $\overline{\mathrm{conv}}^*$ *denotes the* w^**-closed convex hull.*

Proof. From [CLSW98, Proposition 2.2.1, p. 75] we have for scalars λ and functions f that Lipschitz near x:

$$\partial(\lambda f)(x) = \lambda \partial f(x).$$

Also on [CLSW98, p. 75] it has been proved that $\partial(f+g)(x) \subset \partial f(x)+\partial g(x)$. Both results put together give the first claim. The second and third claim are [CLSW98, Proposition 2.2.4, p. 75] and [CLSW98, Proposition 2.2.5, p. 76], respectively. □

The following result relates Lipschitz continuity to differentiability almost everywhere. A property is said to hold almost everywhere, if the set where it fails to hold has Lebesgue measure zero.

Theorem B.0.5 (Rademacher's Theorem). *Let $U \subset \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable almost everywhere on U.*

See, e.g., [Eva98, Theorem 5.8.6, p.281].

Corollary B.0.6. *Let $U \subset \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}^m$ be locally Lipschitz. Then f is differentiable almost everywhere on U.*

Proof. This is a direct consequence of Theorem B.0.5, by noting that a function $f : U \to \mathbb{R}^m$ is locally Lipschitz if and only if every component function $f_i : U \to \mathbb{R}$ is locally Lipschitz. □

Theorem B.0.7 (Generalized Gradient Formula). *Let $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz at x. Let $U \subset \mathbb{R}^n$ have Lebesgue measure zero. Denote by U_f the set of points in $\mathbb{R}^n$ where f fails to be differentiable. Then we have*

$$\partial f(x) = \operatorname{conv}\left\{\lim_{i\to\infty} \nabla f(x_i) : \ x_i \to x,\, x_i \notin U,\, x_i \notin U_f\right\}.$$

For a proof see [CLSW98, Theorem 2.8.1, p.93]. The next result is an immediate consequence of Theorem B.0.7, but has not been stated explicitly in [CLSW98].

Corollary B.0.8 (Chain Rule 2). *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz, denote by U_f and U_g the sets of points where f and, respectively, g fail to be differentiable. Suppose $g^{-1}[U_f]$ has zero Lebesgue measure. Then for all $x \in \mathbb{R}^m$ we have*

$$\partial(f \circ g)(x) = \operatorname{conv}\left\{\lim_{i\to\infty} \nabla f(g(x_i)) \cdot Dg(x_i) : \ x_i \to x,\, x_i \notin U_g,\, g(x_i) \notin U_f\right\}.$$

In other words

$$\partial(f \circ g)(x) = \nabla f(g(x)) \cdot Dg(x)$$

for almost every $x \in X$.

Example B.0.9. Let $f, g \in \mathcal{K}_\infty$ be locally Lipschitz, denote by U_f and U_g the sets of points where f and, respectively, g fail to be differentiable. Note that also $g^{-1} \in \mathcal{K}_\infty$ and $(g^{-1})' = 1/(g' \circ g^{-1})$ exists almost everywhere. From

Rademacher's Theorem we know that U_f and U_g have Lebesgue measure 0. Then the Lebesgue measure of $g^{-1}[U_f]$ is given by

$$\begin{aligned}\lambda(g^{-1}[U_f]) &= \int_{\mathbb{R}_+} 1_{g^{-1}[U_f]}(y)d\lambda(y) \\ &= \int_{\mathbb{R}_+} 1_{g^{-1}[U_f]}(g^{-1}(x)) \cdot \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) \\ &= \int_{\mathbb{R}_+} 1_{U_f}(x) \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) = \int_{U_f} \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) = 0\end{aligned}$$

because U_f is a null set. Hence $(f \circ g)' = (f' \circ g) \cdot g'$ almost everywhere.

We need also the following chain rule. A proof can be found in [MV87].

Lemma B.0.10. *If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $\psi : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then for almost all $t \in \mathbb{R}$ there exists $p_0 \in \partial V(\psi(t))$ such that*

$$\frac{d}{dt} V(\psi(t)) = \langle p_0, \dot{\psi}(t) \rangle .$$

Corollary B.0.8 together with the following result on the generalized gradient of the maximum of n functions will be an essential argument in Chapter 4. The next result is taken from [CLSW98, p.83].

Proposition B.0.11. *For $i = 1, \dots, n$ let $f_i : X \to \mathbb{R}$ be Lipschitz near $x \in X$. Set $f(x) = max_{1 \le i \le n} f_i(x)$. Then f is Lipschitz near x and*

$$\partial f(x) \subset \operatorname{conv} \left\{ \bigcup_{i \in M(x)} \partial f_i(x) \right\},$$

where $M(x) = \{i \in \{1, \dots, n\} : f_i(x) = f(x)\}$.

A survey on nonsmooth analysis in control theory can be found in [Cla01]. All definitions and results in this section are stated in [CLSW98], with the exception of Corollary B.0.6 and Corollary B.0.8, Lemma B.0.10, Example B.0.9. Rademacher's Theorem has been taken from [Eva98].

Appendix C

Gronwall Type Inequalities

One of the main tools used in Chapter 2 is the Gronwall inequality or Gronwall Lemma. There exists many different versions of inequalities known as Gronwall inequality. Although the technical assumption change, all these inequalities share the same spirit: To give an explicit bound on a function, using implicit bounds involving the function itself and its derivate or integral. The original version of the Gronwall inequality was proven in [Gro19] and was formulated as follows.

Lemma C.0.12 (Gronwall: Original). *Let $\xi : [a, a+h] \to \mathbb{R}$ be a continuous function that satisfies the inequality*

$$0 \le \xi(t) \le \int_a^x A + M\xi(s)ds$$

for all $a \le x \le a+h$, where $A, M \ge 0$ are constants. Then

$$0 \le \xi(t) \le Ahe^{Mh}$$

for all $a \le t \le a+h$.

For the sake of completeness, we cite a few other versions of the latter lemma.

Lemma C.0.13 (Gronwall: Integral Form). *Suppose that T is an interval, $a \in T$, $\alpha \in \mathbb{R}$, $\beta(\cdot)$ is a locally integrable non-negative function on T and $\xi(\cdot)$ is a continuous function on T satisfying*

$$\xi(t) \le \alpha + \int_a^t \beta(r)\xi(r)dr\,, \quad t \in T\,, t \ge a\,.$$

Then

$$\xi(t) \leq \alpha \exp\left(\int_a^t \beta(s)ds\right), \quad t \in T \ t \geq a\,.$$

Lemma C.0.14 (Gronwall: Differential Form)**.** *Suppose that T is an interval, $a \in T$, $\beta(\cdot)$ is a differentiable function on T and $\xi(\cdot)$ is a continuous function on T satisfying*

$$\dot{\xi}(t) \leq \beta(t)\xi(t)\,, \quad t \in T\,, t \geq a\,.$$

Then

$$\xi(t) \leq \xi(a) \exp\left(\int_a^t \beta(s)ds\right), \quad t \in T \ t \geq a\,.$$

The proofs are standard and can be found e.g., in [Zab92].
The next may be regarded as an "inverse" version of the Gronwall inequality. It was first proven by Langenhop [Lan60]. The version we present here is taken from [TM10].

Lemma C.0.15 (Langenhop)**.** *Let $x : T \to \mathbb{R}^n$ be a solution of $\dot{x} = f(x,t)$ and let $|x(t)| > 0$ for some $t \in T$. Pick a $t_0 \in T$ with $t_0 < t$. Assume that there exists $L > 0$ such that*

$$|f(x(\tau),\tau)| \leq L|x(\tau)|$$

for almost all $\tau \in [t_0, t]$. Then

$$|x(t)| \geq |x(t_0)|e^{-L(t-t_0)}$$

One maybe not so well known version for discrete time systems is taken from [Aga00, Theorem 4.1.1]. It is of particular interest in Chapter 4.

Theorem C.0.16 (Gronwall: discrete)**.** *Consider $u, p, q, f : \mathbb{N} \to \mathbb{R}_+$. Let for all $k \in \mathbb{N}(a) := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ the following inequality be satisfied*

$$u(k) \leq p(k) + q(k) \sum_{\ell=a}^{k-1} f(\ell)u(\ell)\,. \tag{C.1}$$

Then, for all $k \in \mathbb{N}(a)$

$$u(k) \leq p(k) + q(k) \sum_{\ell=a}^{k-1} p(\ell)f(\ell) \prod_{\tau=\ell+1}^{k-1} (1 + q(\tau)f(\tau))\,. \tag{C.2}$$

Note that (C.2) is the best possible in the sense that equality in (C.1) implies equality in (C.2). Furthermore, the assertion still holds, if $p, u : \mathbb{N} \to \mathbb{R}$.

Appendix D

On ISS-Ω-path

After the submission of the thesis, we discovered a slightly more elegant way to ensure the existence of an ISS-Ω-path than the way we presented in Chapter 3. As the new results were not part of the reviewing process, we decided to present them in the appendix.

Theorem D.0.17. *Let $g \in \mathrm{MAF}^n_{n+m}$. If the induced dynamical system*

$$s(k+1) = g(s(k), w(k)) \quad k \in \mathbb{N}$$

is ISS, then there exists an ISS-Ω-path i.e., there exist $\sigma \in \mathcal{K}^n_\infty$ and $\rho \in \mathcal{K}^m_\infty$ such that

$$g(\sigma(r), \rho(r)) < \sigma(r)$$

for all $r > 0$.

Proof. By Theorem 3.2.2 we know that the ISS property implies the existence of a proper and positive definite $\zeta : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ such that

$$s(k+1) = g(s(k), \zeta(s(k)))$$

is GAS. This in turn implies that

$$g(s, \zeta(s)) \not\geq s$$

for all $s \neq 0$. To see this assume the opposite. Fix $s \neq 0$. Clearly, if $g(s, \zeta(s)) \geq s \neq 0$, the induced system cannot be GAS.
Note that $f(s) := g(s, \zeta(s)) \in \mathrm{MAF}^n_n$. Therefore we can use Theorem 3.3.5 to conclude the existence of $\sigma \in \mathcal{K}^n_\infty$ such that

$$g(\sigma(r), \zeta(\sigma(r))) < \sigma(r) \tag{D.1}$$

for all $r > 0$. By the properness of ζ, there exists an $\alpha \in \mathcal{K}_\infty$, such that

$$\alpha(|s|)e \leq \zeta(s) \tag{D.2}$$

for all $s \in \mathbb{R}^n_+$. Now define $\rho(r) := \alpha(|\sigma(r)|)e$. It is easy to see that $\rho \in \mathcal{K}^m_\infty$. Combining (D.1) and (D.2) yields

$$g(\sigma(r), \rho(r)) \leq g(\sigma(r), \zeta(\sigma(r))) < \sigma(r)\,,$$

which is the desired property and the proof is complete. □

In Theorem 3.4.6 we need the existence of an ISS-Ω-path for a different g. For the sake of completeness we state the corresponding theorem here. To this end define an operator $\overline{\Gamma} : \mathbb{R}^n_+ \to \mathbb{R}^{n\times n}_+$

$$\overline{\Gamma}(s) := \begin{pmatrix} \gamma_{11}(s_1) & \dots & \gamma_{1n}(s_n) \\ \vdots & & \vdots \\ \gamma_{n1}(s_1) & \dots & \gamma_{nn}(s_n) \end{pmatrix}, \tag{D.3}$$

with $\gamma_{ij} \in \mathcal{G}$ for each $i, j = 1, \dots, n$.
Now we augment $\overline{\Gamma}$ to $\Gamma : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^{n\times n+m}_+$ given by

$$\Gamma(s, w) = \begin{pmatrix} \gamma_{11}(s_1) & \dots & \gamma_{1n}(s_n) & w_1 & \dots & w_m \\ \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{n1}(s_1) & \dots & \gamma_{nn}(s_n) & w_1 & \dots & w_m \end{pmatrix},$$

with w_j as an input for $j = 1, \dots, m$. Let $\mu \in \mathrm{MAF}^n_{n+m}$ and consider

$$\Gamma_\mu(s, w) := \mu \circ \Gamma(s, w) = \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \dots, \gamma_{1n}(s_n), w_1, \dots, w_m) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \dots, \gamma_{nn}(s_n), w_1, \dots, w_m) \end{pmatrix}. \tag{D.4}$$

Theorem D.0.18. *Let $\overline{\Gamma} \in \mathcal{G}^{n\times n}$ be irreducible and Γ_μ as in* (D.4)*. If the induced system*

$$s(k+1) = \Gamma_\mu(s(k), w(k))$$

is ISS, then there exists an ISS-Ω-path i.e., there exist $\sigma \in \mathcal{K}^n_\infty$ and $\rho \in \mathcal{K}^m_\infty$ such that

$$\Gamma_\mu(\sigma(r), \rho(r)) < \sigma(r)$$

for all $r > 0$.

Proof. Similar as in the proof of Theorem D.0.17 we use the ISS property to conclude that there exists a proper and positive definite $\zeta : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ such that $\Gamma_\mu(s, \zeta(s)) \not\geq s$ for all $s \neq 0$. As $\overline{\Gamma}$ is irreducible we can interpret $\Gamma_\mu(s, \zeta(s))$ also as irreducible and [DRW10, Theorem 5.2 (ii)] applies. Hence there exists $\sigma \in \mathcal{K}^n_\infty$ such that

$$\Gamma_\mu(\sigma(r), \zeta(\sigma(r))) < \sigma(r)$$

for all $r > 0$. Now we can use similar arguments as in the proof of Theorem D.0.17 after (D.1) to finish the proof. $\square$

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List of Notations

$\lvert\cdot\rvert_{\mathbf{stc}}$	Stacking of several $\lvert\cdot\rvert_{\mathbf{vec}}$	32
$\lvert\cdot\rvert_{\mathbf{vec}}$	Vector consisting of norms	32
$\mathcal{K}$	continuous, increasing, zero at zero functions	21
$\mathcal{K}_\infty$	class $\mathcal{K}$ function which is unbounded	21
$\leq$	partial order induced by the positive orthant	20
$\lfloor\cdot\rfloor$	Componentwise floor function	62
$\limsup$	component-wise $\limsup$	21
$\lvert\cdot\rvert$	A monotone norm	22
MAF	Monotone aggregation function	22
$\mathcal{G}$	$\mathcal{K}_\infty \cup 0$	22
max	component-wise maximum	20
$\mathbb{N}$	Natural numbers	19
$\neg$	Logical not	84
$\not\leq$	LHS>RHS in at least one component	20
$\lVert\cdot\rVert_{\mathbf{stc}}$	Stacking of several $\lVert\cdot\rVert_{\mathbf{vec}}$	32
$\lVert\cdot\rVert_{\mathbf{vec}}$	Vector consisting of norms	32
conv	Convex hull	119
id	Identity map	21
$\mathbb{R}$	Real numbers	19
$\mathbb{R}_+$	Positive real numbers	19
sup	component-wise supremum	20
$\xi^{i,a}$	Vector ξ with ith component replaced by a	129
$\mathbb{Z}$	Integers	19
e	Vector consisting of ones	19
I_n	Identity matrix of dimension n	53
$x(t^+)$	Right limit of a function x at t	113
$x(t^-)$	Left limit of a function x at t	59
LHS/RHS	Left/right hand side	79

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