Hurwitz's Complex Continued Fractions

A Historical Approach and Modern Perspectives.

Dissertationsschrift zur Erlangung

des naturwissenschaftlichen Doktorgrades

der Julius-Maximilians-Universität Würzburg

vorgelegt von

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Würzburg 2014

"Knowledge is in the end based on acknowledgement."

Ludwig Wittgenstein (1889 - 1951)

Ich freue mich, an dieser Stelle die Gelegenheit zu nutzen, einigen Menschen zu danken, die mich während der letzten drei Jahre unterstützt und begleitet haben. Mein erstes großes Dankeschön gebührt natürlich Jörn Steuding, der mir diese Promotionsstelle nicht nur ermöglicht hat, sondern mir seitdem als Betreuer stets mit Rat und Tat zur Hilfe steht. Er ermöglichte mir den Besuch zahlreicher Konferenzen, meine Forschungsinteressen zu finden und selbstständig eigene Projekte zu verwirklichen. In unserer gemeinsamen Arbeitsgruppe zusammen mit Thomas Christ, sowie am Institut, genoss ich sehr die fortwährend freundliche und rücksichtsvolle Atmosphäre.

Außerdem möchte ich meinen Dank Karma Dajani aussprechen, die mir bereits zu Beginn meiner Arbeit erste Inspirationen zur Ergodentheorie gab. Während ihres Aufenthalts in Würzburg im Sommersemester 2014 hatte ich große Freude daran, dieses Interesse wieder aufzunehmen und konnte viel persönliche Motivation aus ihrer herzlichen Art schöpfen.

Im Bezug auf meine mathematik-historische Forschung möchte ich mich ausdrücklich bei meiner Mentorin Renate Tobies und Klaus Volkert bedanken. Beide halfen mir sehr bei meinem Einstieg in diesen neuen Fachbereich und ich bin ihnen für viele gute Ratschläge und Ermutigungen dankbar.

In dieser Hinsicht sind auch das Archiv der ETH Zürich, insbesondere die hervorragende Betreuung von Evelyn Boesch, die Staats- und Universitätsbibliothek Göttingen, das Staatsarchiv Basel und das Universitätsarchiv Halle-Wittenberg zu erwähnen. Ich bedanke mich herzlich für die Möglichkeit, so viele spannende Dokumente einsehen zu dürfen.

Zu guter Letzt möchte ich mich von ganzem Herzen bei Ingrid und Wolfgang, Moni und Nelly, Eva, Daria, Aniela, Lisa und natürlich Andi bedanken, die stets dafür gesorgt haben, dass meine Nerven wieder an ihren angedachten Ort wandern.

Contents

Introduction

This thesis deals with two branches of mathematics: Number Theory and History of Mathematics. On the first glimpse this might be unexpected, however, on the second view this is a very fruitful combination. Doing research in mathematics, it turns out to be very helpful to be aware of the beginnings and development of the corresponding subject.

In the case of Complex Continued Fractions the origins can easily be traced back to the end of the 19th century (see [Perron, 1954, vl. 1, Ch. 46]). One of their godfathers had been the famous mathematician Adolf Hurwitz. During the study of his transformation from real to complex continued fraction theory [Hurwitz, 1888], our attention was arrested by the article *'Ueber eine besondere Art der Kettenbruch-Entwicklung complexer Grössen'* [Hurwitz, 1895] from 1895 of an author called J. Hurwitz. We were not only surprised when we found out that he was the elder unknown brother Julius, furthermore, Julius Hurwitz introduced a complex continued fraction that also appeared (unmentioned) in an ergodic theoretical work from 1985 [Tanaka, 1985]. Those observations formed the basis of our main research questions:

What is the historical background of Adolf and Julius Hurwitz and their mathematical studies?

and

What modern perspectives are provided by their complex continued fraction expansions?

In this work we examine complex continued fractions from various viewpoints. After a brief introduction on real continued fractions, we firstly devote ourselves to the lives of the brothers Adolf and Julius Hurwitz. Two excursions on selected historical aspects in respect to their work complete this historical chapter. In the sequel we shed light on Hurwitz's, Adolf's as well as Julius', approaches to complex continued fraction expansions. Correspondingly, in the following chapter we take a more modern perspective. Highlights are an ergodic theoretical result, namely a variation on the Döblin-Lenstra Conjecture

[Bosma et al., 1983], as well as a result on transcendental numbers in tradition of Roth's theorem [Roth, 1955]. In two subsequent chapters we are concernced with arithmetical properties of complex continued fractions. Firstly, an analogue to Marshall Hall's theorem from 1947 [Hall, 1947] on sums of continued fractions is derived. Secondly, a general approach on new types of continued fractions is presented building on the structural properties of lattices. Finally, in the last chapter we take up this approach and obtain an upper bound for the approximation quality of diophantine approximations by quotients of lattice points in the complex plane generalizing a method of Hermann Minkowski, improved by Hilde Gintner [Gintner, 1936], based on ideas from geometry of numbers.

In particular in Chapter 2 the reader can find a great number of quotations. Notice that all translations from German to English have been made by the author to the best of her knowledge. The original versions can be found in footnotes.

Whenever a result of us or a related version has already been published, the corresponding reference is given in each chapter's preamble.

At this point, we align ourself to an opinion of the authors of the inspiring book *'Neverending Fractions'* [Borwein et al., 2014, p. i]:

> "[D]espite their classical nature, continued fractions are a neverending research area [...]".

1 Real Continued Fractions

This chapter shall provide a brief mathematical as well as historical introduction to the theory of real continued fractions. Since there is a great variety of excellent standard literature, here we avoid to repeat comprehensive proofs. Instead, our aim is to anticipate some results which will occur in subsequent chapters for the complex relatives of real continued fractions. To achieve a deeper understanding and an extensive overview of real continued fraction theory, as well as for references to the original papers, among others we refer to [Perron, 1913], [Opolka and Scharlau, 1980], [Kraaikamp and Iosifescu, 2002], [Steuding, 2005], and [Borwein et al., 2014].

1.1 A Brief Mathematical and Historical Introduction

Every real number x can be uniquely expanded as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \, + \cfrac{1}{a_n + \ddots}}},$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$; if x is rational, the expansion is finite and uniqueness follows from assuming the last a_n to be strictly larger than one. This often with a twinkling eye called typographical nightmare represents the *regular continued fraction expansion* of x. It is developed by the *continued fraction algorithm*, which for rational numbers is derived from the well known Euclidean Algorithm[1]. Here in each iteration the input is a real number, which is divided into an integer and a fractional part, starting with

$$x = \lfloor x \rfloor + \{x\},$$

where $\lfloor \cdot \rfloor$ denotes the Gaussian floor function. Defining $a_0 := \lfloor x \rfloor$ leads to the first iteration, the algorithm's output,

$$x = a_0 + \frac{1}{\frac{1}{\{x\}}},$$

provided $x \notin \mathbb{Z}$. Since then $\{x\} \in (0,1)$, a new real number $\alpha_1 := \frac{1}{\{x\}} > 1$ arises, which serves as input for the subsequent iteration. The algorithm's recursion starts from the beginning, producing a positive integer a_1 and a further remainder α_2 and so on.

We mainly distinguish between *finite* and *inifinite* outputs producing a finite or infinite sequence $(a_n)_n$ of so-called *partial quotients*[2]. Since in regular continued fractions each numerator is equal to 1, all relevant information is preserved in this sequence. Hence, it is sufficient to use the shortened continued fraction notation

$$x := [a_0; a_1, a_2, \ldots a_n], \quad \text{respectively} \quad x := [a_0; a_1, a_2, \ldots a_n, \ldots].$$

Firstly, we give a classical quotation to finite continued fractions refering to one of the probably first scientific publications concerning continued fractions. In his *'Descriptio automati planetarii'*, published in 1703, the Dutch universal scientist Christiaan Huygens (1629 - 1695)[3] explained a pioneering method to receive good approximations to rational numbers with large numerators and denominators. His aim was to construct a cogwheel

[1] see [Borwein et al., 2014, Ch. 1.1].
Remark: In the sequel letters in established expressions like 'euclidean algorithm' shall only be capitalized if they arise because of their mathematical function.

[2] in modern literature sometimes also *'digits'*, written in short form (a_n)

[3] **Remark:** In the sequel biographical data of mathematicians will only be given in the first three chapters with respect to their historical background.

model of our solar system. Here the ratio of the number of gear teeth should represent the proportion of the times the planets need for their orbits.

> "To find smaller numbers, which are as similar as possible to this ratio, I divide the greater by the smaller, and continue [to divide] the smaller by what remained from the division, and this again by the remainder. And so on continuing [...]".[4] [Lüneburg, 2008, p. 558]

Huygens knew the approximate length of Saturn's revolution around the sun, namely $\frac{77708431}{2640858} = 29.42544847\ldots$ years. With the above stated algorithm he was able to calculate the continued fraction expansion

$$\frac{77708431}{2640858} = [29; 2, 2, 1, 5, 1, 4, 1, 1, 2, 1, 6, 1, 10, 2, 2, 3].$$

According to his remark, the Dutch scientist was obviously not only aware of how to use the euclidean algorithm to expand a continued fraction, he moreover knew about its approximation properties.[5] Those play a decisive role in several parts of this work, in particular in Chapter 7. Indeed, from continued fractions one obtains so-called *convergents* $\frac{p_n}{q_n} \in \mathbb{Q}$ by 'cutting' them after the first n partial quotients:

$$\frac{p_0}{q_0} = [a_0],\ \frac{p_1}{q_1} = [a_0; a_1],\ \frac{p_2}{q_2} = [a_0; a_1, a_2],\ \ldots.$$

Since only a limited number of gear teeth can be realized, Huygens calculated in the same manner $\frac{p_0}{q_0} = 29$, $\frac{p_1}{q_1} = \frac{59}{2}$, $\frac{p_2}{q_2} = \frac{147}{5}$, $\frac{p_3}{q_3} = \frac{206}{7}$, $\frac{p_4}{q_4} = \frac{1177}{40}$. He concluded,

> "Consequently, 7 is to 206 an approximation ratio to the ratio 2640858 to 77708431. Thus, we give Saturn's wheel 206 gear teeth, however, the driving

[4] "Um nun kleinere Zahlen zu finden, die diesem Verhältnis möglichst nahe kommen, teile ich die größere durch die kleinere, und weiter die kleinere durch das, was bei der Division übrig bleibt, und dies wiederum durch den letzten Rest. Und so weiter fortfahrend [...]."

[5] Very likely, continued fractions have been used for centuries for the purpose of approximation before Huygens, however, his studies stand at the beginning of the modern development.

one 7."[6] [Lüneburg, 2008, p. 559]

Planet by planet Huygens used continued fractions to obtain appropriate convergents and, finally, succeeded to construct a model of our solar system.

One can easily prove (see e.g. [Steuding, 2005, p. 38]) that the convergents satisfy the recusion formulae

$$\begin{cases} p_{-1} = 1, \ p_0 = a_0, \quad \text{and} \qquad p_n = a_n p_{n-1} + p_{n-2}, \\ \\ q_{-1} = 0, \ q_0 = 1, \quad \text{and} \qquad q_n = a_n q_{n-1} + q_{n-2}. \end{cases}$$

Obviously, finitely many convergents arise from a finite continued fraction, whereas there are infinitely many convergents if the algorithm does not stop. Since the latter infinite case is not only much more interesting, however, also more complicated there was a certain need for systematization. "After many individual results and more or less incidentally found correlations by former mathematicians this theory is upraised to the rank of a systematic theory by Euler and especially Lagrange."[7] [Opolka and Scharlau, 1980, p. 54] In an initial, hereon related, paper *'De fractionibus continuis'*[8] from 1737, Leonhard Euler (1707 - 1783) proved the first part of the following theorem; the second result goes back to a work of Joseph-Louis Lagrange (1736 - 1813) in 1768.

Theorem 1.1.1 ***(Euler and Lagrange)***

A real number x is irrational if, and only if, its continued fraction expansion is infinite. If x is quadratic irrational, i. e. x is the irrational solution of a quadratic equation with rational coefficients, then its continued fraction expansion is periodic.[9]

[6] "Folglich ist 7 zu 206 Näherungsverhältnis zum Verhältnis 2640858 zu 77708431. Wir geben also dem Saturnrad 206 Zähne, dem treibenden aber 7."

[7] "Nach vielen Einzelergebnissen und mehr oder weniger zufällig gefundenen Zusammenhängen durch frühere Mathematiker wird diese Theorie durch Euler und vor allem Lagrange in den Rang einer systematischen Theorie erhoben"

[8] for more details we refer to [Brezinski, 1991, pp. 97]

[9] Here *periodic* means that after a certain partial quotient, the consecutive sequence of partial quotients is periodic.

Moreover, Euler gave a nice example of an infinite continued fraction. He expanded

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$$

where e denotes as usual Euler's Number[10]. In the following years many mathematicians, as Johann Heinrich Lambert (1728 - 1777), Adrien Marie Legendre (1752 - 1833), Evariste Galois (1811 - 1832), and Carl Friedrich Gauss (1777 - 1855), made significant contributions to the further development of the theory of continued fractions. Apart from e, one of their major concerns was to receive more information about the number π. The first continued fraction for π was found by the English school, namely William Brouncker (1620 - 1684) gave the *semi-regular*[11] expansion

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}$$

in response to John Wallis' (1616 - 1703) infinite product [Dutka, 1982]. In 1844, Joseph Liouville (1809 - 1882) proved that for any algebraic number x of degree $d > 1$, there exists a positive constant c, depending only on x, such that

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^d}$$

for all rationals $\frac{p}{q}$ with $q > 0$. Consequently, algebraic numbers cannot be approximated too good by rational numbers; moreover, numbers which allow better rational approximations are transcendental. Taking this observation into account, Liouville gave first

[10]Many years later, in 1896, also Adolf Hurwitz, one of the central figures of Chapters 2 and 3, published an article on the continued fraction expansion of e. [Hurwitz, 1896a]

[11]A general definition of semi-regular continued fractions is given in Section 3.1

explicit examples of transcendental numbers by the regular continued fraction expansions $x = [0; a_1, a_2, \ldots]$[12] with partial quotients a_n satisfying

$$\limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n} = \infty;$$

notice that, given $a_1, \ldots, a_n$ (and therefore q_n), the latter condition can be fulfilled by an appropriate choice of a_{n+1}. Certainly, this mathematical benchmark has paved the way and was encouraging for a great number of results on transcendental numbers.[13]

More information about the history of continued fractions can be found in Brezinski's work [Brezinski, 1991]. The first classic and yet standard reference for continued fractions is the treatise '*Die Lehre von den Kettenbrüchen*' [Perron, 1913] by Oskar Perron (1880 - 1975). For definitions, details, and fundamental facts we refer to this source. Perron's wonderful monograph covers both, arithmetic and approximation theory of continued fractions, and its third edition [Perron, 1954, vl. 1, Ch. 46] includes a brief introduction to Adolf Hurwitz's approach to extend the theory of continued fractions from real numbers to complex numbers what is also subject of Chapter 3 in this work.

1.2 Some Classical Results

With respect to Christaan Huygen's investigations and Joseph Liouville's result on transcendental numbers, we have already indicated that continued fractions are the method of choice when a rational approximation for a given (irrational) real number is needed. Since the approximation properties refer to the arising convergents $\frac{p_n}{q_n}$, we point out some results concerning their characteristics. A first simple, however, important observation is

[12]**Remark:** Notice that if $a_0 = 0$ appears, sometimes this partial quotient is omitted. In this case there are only commas, no semicolon in this way of notation.

[13]The transcendence of e and π were proved by Charles Hermite (1822 - 1901), respectively Ferdinand von Lindemann. Their original approaches were simplified several times, among others by David Hilbert and Adolf Hurwitz. These proofs do not rely on continued fractions.

the following

Lemma 1.2.1 *The sequence of denominators* $(q_n)_n$ *of convergents* $\frac{p_n}{q_n}$ *from a continued fraction expansion is strictly increasing for* $n \geq 2$.

Notice that also the sequence of numerators of the convergents increases strictly. Moreover, the convergents approximate a number alternating, those with even index from below and the others from above. These observations, together with Recursion Formulae (1.1), form the basis for deeper results.

Theorem 1.2.2 ***(Law of Best Approximation, Lagrange, 1770)***
Let x *be a real number. If* $n \geq 2$ *and* p, q *are positive integers satisfying* $0 < q < q_n$ *with* $\frac{p}{q} \neq \frac{p_n}{q_n}$, *then*

$$|q_n x - p_n| < |qx - p|.$$

In other words, Lagrange [Lagrange, 1770] showed that the convergents to an irrational number provide the best possible rational approximations. This result is rather impressive and elucidates the importance of continued fractions for approximation theory. More information on the convergent's approximation quality is contained in another result of this period, proven by Legendre in [Legendre, 1798, p. 139].

Theorem 1.2.3 ***(Legendre, 1798)***
Every reduced rational $\frac{p}{q} \in \mathbb{Q}$ *satisfying*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

is a convergent to x. *Among two consecutive convergents* $\frac{p}{q}$ *to a given real* x, *there is at least one satisfying this inequality.*

In fact the approximation quality of convergents also turns out to be extremely helpful for solving the Pell equation

$$X^2 - dY^2 = 1$$

for $x, y \in \mathbb{Z}$, where $d \in \mathbb{N}$. If d is a perfect square, there are only finitely many solutions. Thus, we consider the case $\sqrt{d} \notin \mathbb{Q}$ and assume that x, y is a solution. The left hand-side can be factored as

$$(x - \sqrt{d}y)(x + \sqrt{d}y) = x^2 - dy^2 = 1,$$

which implies

$$\left|\sqrt{d} - \frac{x}{y}\right| = \frac{1}{y^2|\sqrt{d} + \frac{x}{y}|} < \frac{1}{2y^2}.$$

This modification provides obviously the same inequality as in Theorem 1.2.3. Consequently, all solutions of the Pell equation can be found among the convergents of $\sqrt{d}$.[14] In fact, they may be precisely determined.

Theorem 1.2.4 *Assume $\sqrt{d} \notin \mathbb{Q}$. The Pell equation $X^2 - dY^2 = 1$ has infinitely many integer solutions (x_k, y_k) besides the trivial solution $(\pm 1, 0)$; up to the sign they are all given by*

$$(p_{kl-1}, q_{kl-q}) \quad \text{if } l \equiv 0 \bmod 2,$$
$$(p_{2kl-1}, q_{2kl-q}) \quad \text{if } l \equiv 1 \bmod 2.$$

Here l denotes the minimal period of the periodic continued fraction expansion of $\sqrt{d}$.

In Chapter 7 we study the complex analogue of this remarkable theorem.

We shall come back to all results stated in this section in later chapters. Of course, apart from those few classical results mentioned here, there is an enormous number of publications on real continued fractions that would be worth to discuss. However, since this work is dedicated to complex continued fractions, here we do not go further and draw the interested reader's attention to the above mentioned literature.

[14] Even all solutions of the more general equation $X^2 - dY^2 = \pm 1$ can be found among the convergents.

2 The Hurwitz Brothers

This chapter is dedicated to the lives of the brothers Julius and Adolf Hurwitz, both gifted with a smart intellect, a keen perception, and great curiosity in science. They were born in the middle of the nineteenth century in a small town not far from Göttingen. Already during their schooldays the two of them became acquainted with mathematical problems and both started studies in mathematics. So far nothing extraordinary. While the younger brother turned out to be extremely successful in his research, the elder brother and his work, however, seem to be almost forgotten. Besides their biographies, we furthermore give two excursions connected to their mathematical careers. Regarding Adolf Hurwitz, we provide an insight into his estate stored in the archive of the polytechnic *Eidgenössische Technische Hochschule* in Zurich, whereas in Julius Hurwitz's case we examine a letter exchange concerning his doctoral thesis. The majority of the biographical parts is published in [Oswald and Steuding, 2014], concerning the first excursion some results can also be found in the articles [Oswald, 2014a] and [Oswald, 2014c], both relying on Adolf Hurwitz's mathematical diaries.

2.1 The Younger: Adolf Hurwitz

Figure 2.1: Portrait of Adolf Hurwitz (1859 - 1919), taken from Riesz's register in Acta Mathematica from 1913 [Riesz, 1913].

Adolf Hurwitz was born into a Jewish family on March 26, 1859, in Hildesheim near Hanover. The common Jewish surname Hurwitz[1]is a reference to the historically portentous small town Hořovice in Central Bohemian Region of the Czech Republic. Already during his schooldays Adolf Hurwitz must have made an excellent impression in mathematics. His teacher at school was Hermann Cäsar Hannibal Schubert (1848 - 1911), a doctor of mathematics who invented the so-called Schubert calculus in enumerative geometry.

Schubert gave his pupil Adolf private lessons each Sunday afternoon. Obviously, this was a fruitful investment. Hurwitz's first paper is a joint work [Hurwitz and Schubert, 1876] with Schubert on Chasles' theorem on counting the number of curves satisfying certain algebraic conditions within a family of conics[2]; this generalizes Bézout's theorem and plays a certain role in algebraic geometry. The team play of Adolf and his teacher was indeed good luck since Schubert left Hildesheim for Hamburg in 1876 after only five years.[3]

According to George Pólya (1887 - 1985) [Pólya, 1987, p. 25], "Hurwitz had great math-

[1]smilar to Horowitz and Hurewicz

[2]The famous Apollonian problem about a circle being tangent to three further circles can be seen as forefather of those geometrical questions. A nice reading [Hellwig, 1856] in this topic was published in Halle in 1856, where H.C.H. Schubert did his doctorate in 1870, only a couple of years later.

[3]For more details on Schubert, who is nowadays better know for his mathematician influence in Hamburg, see [Burau, 1966, Burau and Renschuch, 1993].

ematical breadth, as much as was possible in his time. He had learned algebra and number theory from Kummer and Kronecker, complex variables from Klein and Weierstrass." Indeed, this great opportunity of having studied both at that time opposing perspectives on mathematics, the analytical approach of the Berlin school as well as the geometrical approach of the Göttingen school, helped Adolf Hurwitz during his entire career. At the age of 18 he started to study at the polytechnic university in Munich attending the lectures of the eminent Felix Klein (1849 - 1925). Refering to these years of study, Adolf Hurwitz's later wife Ida Samuel-Hurwitz (1864 - 1951) wrote:

> "From the very first beginnings Klein thought he could demand the highest requirements on him. [...] Also he made him execute complicated calculations for his (Klein's) work and gave him presentations for the seminar about the most difficult papers [...]. The mental exhaustion had naturally a very unfavorable effect on his state of health and the existing tendency for migraine increased immensely."[4] [Samuel-Hurwitz, 1984, p. 5]

At that time Klein was lecturing on number theory, a topic which became Adolf Hurwitz's main line of investigation. He, furthermore, supported Adolf to move to Berlin, where Leopold Kronecker (1823 - 1891) and Karl Weierstrass (1815 - 1897) were his teachers. After three semesters in the capital, in 1880, Adolf Hurwitz followed Felix Klein to Leipzig for a continuation of his studies.[5] Already one year later, in 1881, Hurwitz obtained a doctoral degree[6] for his work on modular functions [Hurwitz, 1881]; more precisely, on arithmetic properties of the Fourier coefficients of Eisenstein series and congruence subgroups. In retrospective, his supervisor, Felix Klein, wrote that when it is about

[4] "Klein glaubte von vornherein, die höchsten Anforderungen an ihn stellen zu dürfen. [...] Auch liess er ihn complicierte Rechnungen für seine (Klein's) eigene Arbeiten ausführen und übertrug ihm für das Seminar Referate über die schwierigsten Abhandlungen [...]. Die geistige Überanstrengung wirkte naturgemäss sehr ungünstig auf sein Befinden, und die schon von früh an vorhandene Neigung zu Migräne nahm bedeutend zu."

[5] An excerpt of a letter from Klein to Adolf's father Salomon Hurwitz explaining the situation and the perspective for his son can be found in [Rowe, 2007, p. 22].

[6] According to Ida Samuel-Hurwitz [Samuel-Hurwitz, 1984, p. 6], he had to borrow a tailcoat from a fellow student for his doctoral viva.

Riemann's theory of functions from his school Hurwitz and Dyck made the most important contributions.[7] In fact, Klein was at the peak of his scientific career when Adolf did his doctorate and Klein's research definitely benefited a lot from Adolf's investigations.

Doing further steps in his academic career, Adolf Hurwitz had to face certain difficulties at the University of Leipzig with respect to his school education at the *Realgymnasium Andreanum* at Hildesheim. This type of school had been introduced in Prussia in the middle of the nineteenth century for the sake of an advanced education for more than the priviliged youth. However, there were critical voices[8] and some universities did not value these institutions well. Students with such an educational background, lacking sufficient knowledge of Greek and Latin, could not obtain higher degrees (cf. [Hilbert, 1921, Rowe, 2007]).

In 1882, Adolf Hurwitz visited again the University of Berlin, where especially Weierstrass was interested in his function-theoretical efforts and gave him his post-doctoral subject. "For his habilitation Kronecker and Klein strongly recommended to him the at that time full professor in Göttingen H.A. Schwarz, whereas he himself was already in correspondence with the eldest full professor M.A. Stern."[9][Samuel-Hurwitz, 1984, p. 6] Hurwitz moved in the following year to the more liberal Göttingen University where he succeeded with his habilitation. In the German system habilition granted the 'venia legendi', i.e., the permission to lecture as a *Privatdozent* which meant to collect course fees from the students without any payment from the university.[10] It seems that both, Hermann Amandus Schwarz (1843 - 1921) as well as Moritz Abraham Stern (1807 - 1894), became supporters of Hurwitz. On June 30, 1883 Schwarz reported to Weierstrass about

[7]"Seitdem ist das Interesse für Riemanns Funktionentheorie in immer weiteren Kreisen, auch des Auslandes, erwacht. Von meinen Schülern ist wohl besonders Hurwitz in Zürich und Dyck in München zu nennen." [Klein, 1926, p. 276]

[8]In the article *'Hurwitz erteilt Albert Einstein eine Abfuhr'* of the *Hildesheimer Allgemeine Zeitung* from January 24, 2014, a reference to the school chronicle says that they were even designated as "goodless, superficial, revolutionary".

[9]"Kronecker und Klein empfahlen ihm auf's eindringlichste den damaligen Göttinger Ordinarius H.A. Schwarz zur Habilitation daselbst, während er selber schon mit dem ältesten Ordinarius M.A. Stern in Korrespondenz getreten war."

[10]For more details on exploiting workers and the academic ladder in nineteenth century Germany see [Rowe, 1986].

the situation of his younger colleague.

> "First about the affairs of Dr. H[urwitz]. In the last month he droped hints several times that lead me to suspect that he has to cope with worries connected to his economic situation. [...] Dr. H[urwitz] receives from a very remote relative in America, a friend of his father, the annual sum of 1500M and this is few for a local private lecturer, when I asked him if he gets along with this sum, without incuring debts, he was silent."[11] [Confaloniere, 2013, p. 161]

Some months later, in a letter from December 31, 1883 to Weierstrass, Schwarz wrote about a positive development. "Dr. Hurwitz stays for the time being in Göttingen and hopefully will receive the grant for private lecturers starting Easter."[12] [Confaloniere, 2013, p. 189]

Friends from this highly prolific period in Göttingen were the famous physicist Wilhelm Weber (1804 - 1891) and the mathematician Stern; while Adolf visited dance events with the first named,[13] the latter one must have played a rather important role in Adolf's life. Born in Frankfurt in 1807, Stern was, although being a protégé of Carl Friedrich Gauss, faced with racist obstacles in his scientific career. Only in 1859, Stern became an ordinary professor at Göttingen University, being the first unbaptized Jew who was appointed with an ordinary professorship in the rather anti-Semitic Prussia of the nineteenth century. Besides, Stern was the granduncle of Anne Frank, another target of racism during the Nazi-time about one hundred years later.[14] A first obvious indication that Adolf Hurwitz's career was also affected by the rigidity and anti-Semitic attitude of the Prussian system

[11]"Zunächst die Angelegenheit des Herrn Dr. H[urwitz]. Dieser hatte in dem letzten Monate zu verschiedenen Malen mir gegenüber Andeutungen fallen lassen, die mich vermuthen ließen, daß er mit Sorgen zu kämpfen habe, welche mit seiner ökonomischen Lage zusammenhängen. [...] Dr. H[urwitz] bekomme von einem sehr entfernten Verwandten in Amerika, einem Freund seines Vaters, die Summe 1500 M jährlich und dies ist für einen hiesigen Privatdocenten wenig, als ich ihn fragte, ob er mit diesem Summe auskommen könne, ohne Schulden machen zu müssen, schwieg er."

[12]"Herr Dr. Hurwitz bleibt vorläufig in Göttingen und wird, wie ich hoffe, von Ostern ab das Privatdocentenstipendium erhalten."

[13]"An der lebhaften Göttinger Geselligkeit nahm H. auch sonst eifrig teil, so schwang er das Tanzbein bei dem grossen Physiker Wilh. v. Weber, [...]" [Samuel-Hurwitz, 1984, p. 7]

[14]See [Rowe, 1986, Rowe, 2007] for further details.

is manifested in another part of Schwarz's letter to Weierstrass. Here Schwarz refered to initial difficulties that arised when another pupil of Klein, Ferdinand von Lindemann (1852 - 1939), tried to offer Hurwitz a position in Königsberg.

> "Since Dr. Hurwitz is a former graduate of a *Realschule* and furthermore an Israelite; one can not blame him that he does not want to engage himself with this business, to which also nobody, least Prof. Lindemann advises him. I talked to G.K. Althoff about it. Prof. Stern and I suggested to send Dr. Hurwitz as Professor to Königsberg [...]"[15] [Confaloniere, 2013, p. 189]

Unfortunately, it is difficult to reconstruct on what Hurwitz did not want to engage himself and what difficulties actually have arised. However, we notice that a first attempt of Lindemann to support Hurwitz's career failed. Lindemann just had proved the transcendence of π and thereby the impossibility of squaring the circle and became aware of Hurwitz because of his *new Representations of the proof of Weierstrass for the transcendence of e and* π.[16].

However, the year 1884 brought many changes. Stern retired and started to live in Bern, where his son was a professor of history, and Stern's chair at Göttingen was filled by Klein. Furthermore, in his memoirs Lindemann wrote that he still had "[...] envisaged the private lecturer Hurwitz from Göttingen. Kronecker agreed with [him], called a coach and drove with [him] to Althoff to support Hurwitz's employment. The latter was indeed called to Königsberg."[17] [von Lindemann, 2071, p. 88][18] In the same year Adolf Hurwitz moved to the University of Königsberg (nowadays Kaliningrad in Russia). There he obtained an

[15] "Nun ist Herr Dr. Hurwitz ehemaliger Realschulabiturient und außerdem Israelit; man kann es ihm daher gewiß nicht verdanken, wenn er sich auf eine so zweifelhafte Sache nicht einlassen will, wozu ihm auch Niemand, am wenigstens Herr Prof. Lindemann räth. Ich habe mit Herrn G.K. Althoff über die Sache geredet. Herr Prof. Stern und ich schlugen vor, Herrn Dr. Hurwitz als Professor nach Königsberg zu schicken [...]"

[16] *Neue Darstellung des Weierstrassschen Beweises für die Transzendenz von e und* π, enclosed in Adolf Hurwitz's mathematical diaries [Hurwitz, 1919a, No. 3]

[17] "[...] den Göttinger Privatdozenten Hurwitz ins Auge gefasst [hatte]. Kronecker gab [ihm] darin recht, liess eine Droschke kommen und fuhr mit [ihm] zu Althoff, um bei ihm die Anstellung Hurwitz's zu befürworten. Letzterer wurde in der Tat dann nach Königberg berufen."

[18] Lindemann's memoirs are written in a rather flowery language.

extraordinary professorship[19] comparable to an associate professorship in modern terms. In a letter from April 1, 1884 to his brother Julius[20], his father Salomon Hurwitz wrote,

> "It is an extraordinary event, and we cannot thank enough the destiny, that our Adolf is so gifted, so acclaimed and is already recognized by the most important mathematicians as excellent person. [...] So your youngest brother is professor by the age of 25 and after only two years of being private lecturer!"[21]

Although the working conditions at Königsberg must have been desastrous, the students were excellent, among them the young Hermann Minkowski (1864 - 1909) and David Hilbert (1862 - 1943). Adolf became not only their guide to mathematics but also a lifelong friend of both. At that time "Adolf Hurwitz was at the height of his powers and he opened up whole new mathematical vistas to Hilbert who looked up to him with admiration mixed with a tinge of envy" [Rowe, 2007, p. 25]. (In Subsection 2.2.3 we give a deeper insight into their relationship.) Besides his mathematical dedication, Adolf was rooted in the academic social and cultural life of Königsberg. Thereby he established the contact to professor's families. After years of restraints because of his changing state of health, Adolf Hurwitz married Ida Samuel[22], a daughter of the professor for pathology Simon Samuel at the University of Königsberg in summer 1892.

"[After] years of diligent work and nice results. A request from Rostock, if he was willing to be baptizised when he was appointed as professor, he refused the request. In the beginning of 1892 he was suggested to be H.A. Schwarz's successor in Göttingen, at the same time he was considered as successor of Frobenius who was appointed to

[19] in German: *Extraordinariat*

[20] to whom we refer in Section 2.3

[21] "Es ist ein ordentliches Ereignis, und wir können der Vorsehung nicht genug danken, daß unser Adolf so begabt, so beliebt und schon von den bedeutendsten Mathematikern als ein hervorragender Mensch anerkannt wird. [...] Also Dein jüngster Bruder ist Professor mit 25 Jahren und nachdem er nur 2 Jahren Privatdozent gewesen!" Letter from Salomon to Julius, April 1, 1884, in possession of the ETH library, HS 583

[22] Later Ida Samuel-Hurwitz (1864 - 1951); author of the brief and very readable account [Samuel-Hurwitz, 1984] on the Hurwitz family and her husband in particular

Berlin."[23] [Samuel-Hurwitz, 1984, p. 8] Those memories of Ida Samuel-Hurwitz reflect that in nineteenth century Prussian prejudices against Jews and racism were common; in particular, when the founding of the Reich was followed by a financial crash and the long depression afterwards the atmosphere became rather unfriendly.[24] In fact, Adolf Hurwitz was never officially considered to be Schwarz's successor. In a letter to Friedrich Engel (1861 - 1941), Friedrich Schur (1856 - 1932) described Klein's disappointment that Hurwitz was not appointed. Moreover, he wrote, "it did not really become clear to me whether H.[urwitz] indeed was proposed by the faculty [...] ."[25] [26]

Adolf Hurwitz accepted the call to Georg Frobenius' chair at the *Eidgenössische Technische Hochschule Zürich*[27] (in the sequel ETH for short), a polytechnic, and moved to Zurich, where he remained for the rest of his life. It was a fortunate coincidence that Hurwitz's paternal friend Moritz Stern had meanwhile moved from Bern to Zurich. There Stern was made a honorary member of the local Society of Natural Scientists and they could spend some more years together before the elder died in 1894.[28] Adolf Hurwitz's successor at Königsberg was Hilbert who shortly after moved to Göttingen in 1895 on promotion of Klein. Certainly, Adolf did not want to leave Zurich for Göttingen. According to the biographical notes of his wife [Samuel-Hurwitz, 1984, p. 11], it was not only the restless mathematical bustle, but also the lack of social amenities in the smaller German town, which prevented him to change universities.[29]

[23] "[Nach] Jahre[n] fleissiger Arbeit und schöner Erfolge. Eine Anfrage aus Rostock, ob er bereits wäre sich taufen zu lassen, wenn man ihn als Ordinarius an die dortige Universität beriefe, hatte er abschlägig beantwortet. Zu Beginn des Jahres 1892 wurde als Nachfolger von H.A. Schwarz vorgeschlagen, gleichzeitig reflektierte man in Zürich auf ihn als Nachfolger des nach Berlin berufenen Frobenius."

[24] An excellent reading on this 'Game of Mathematical Chairs' and the difficulties for Jewish mathematicians at that time is Rowe's article [Rowe, 2007] as well as the correspondence [Frei, 1985] between Hilbert and Klein.

[25] "es ist mir nicht ganz klar geworden, ob H.[urwitz] wirklich von der Fakultät [...] vorgeschlagen wurde."

[26] This letter from May 21, 1892 can be found as digital reproduction on http://digibib.ub.uni-giessen.de

[27] founded in 1855 under the name *Polytechnikum*, renamed in 1911

[28] Adolf Hurwitz and his colleague Ferdinand Rudio, a colleague from ETH and former friend from his study times at Berlin, published the collected letters from Eisenstein to Stern [Eisenstein, 1975]; this had been a wish of Stern and it affirms his close relation with Adolf.

[29] "Obgleich sein mangelhafter Gesundheitszustand natürlich bekannt war, trachtete man in Deutschland mehrfach, ihn wieder dorthin zu ziehen. Als gleichwertig mit seinem Züricher Lehrstuhl, den nacheinander eine Reihe der hervorragendsten Mathematiker bekleidet hatten, konnten freilich nur

Refering to this difficult situation of appointments and job placings, later Hilbert described Hurwitz's reserved behavior.

> "[He] who was so deeply inward modest and at the same time free of all outward ambitions, that he did not consider it as an affront when a mathematician who was inferior in importance, was preferred to appointments. [...] Finally, it was his luck that he stayed in Switzerland, since he would not have been able to stand the physical and mental efforts a life in Germany in wartime would have brought for him."[30] [Hilbert, 1921, p. 165][31]

Indeed, Zurich turned out to be a good choice. Firstly, in Switzerland were less resentments against Jews than in Prussia. Moreover, there was a rich academic life with a polytechnic and a university as well as a mathematical society. The ETH was founded in 1855. While in the beginning it was a stepping stone for young researchers to obtain better positions at respectable German universities (e.g., Richard Dedekind, Schwarz, Frobenius), the situation improved quickly thanks to the tight collaboration with the established University of Zurich. Of course, there was also a certain competition between the two institutions.[32] The various activities culminated in the first International Congress for Mathematicians held at Zurich in 1897. Adolf Hurwitz was not only involved in its organization but, together with Klein, Guiseppe Peano (1858 - 1932), and Henri Poincaré (1854 - 1912), Adolf was one of the distinguished invited speakers giving a talk on Georg Cantor's (1845 - 1918)

Berlin, München und Leipzig in Frage kommen. An die letzte Universität war die Berufung eines Juden ausgeschlossen; für Berlin war er bei einer Vakanz an erster Stelle vorgeschlagen. Von Göttingen war natürlich öfters die Rede, doch scheute er einerseits den enormen mathematischen Betrieb dort, während ihm andererseits die Kleinstadt unsympatisch war." [Samuel-Hurwitz, 1984, p. 11]

[30] "[Er] der so tief innerlich bescheiden und zugleich frei von allem äußeren Ehrgeiz war, daß er keine Kränkung darüber empfand, wenn ein Mathematiker, der ihm an Bedeutung nachstand, ihm bei Berufungen vorgezogen wurde. [...] Es ist ihm schließlich zum Glück ausgeschlagen, daß er in der Schweiz blieb, da er den körperlichen und seelischen Anstrengungen, die das Leben in Deutschland während des Krieges für ihn mit sich gebracht hätte, nicht gewachsen gewesen wäre."

[31] Here we refer to the commemorative speech 'Adolf Hurwitz' of David Hilbert from May 15, 1919 held in the public meeting of the Royal Society of Science in Göttingen, which was also published as [Hilbert, 1920].

[32] for example, when it was about the appointment of Minkowski; see letter 107 from Hilbert to Klein in [Frei, 1985]

at that time controversal foundation of set theory and its application to analysis.[33]

Similar to the situation in Königsberg there were excellent students in Zurich. For instance, around 1900, Albert Einstein (1879 - 1955) applied for assisting Adolf Hurwitz, however, as Einstein reported, Hurwitz must have been puzzled that a student who was never ever seen in the mathematical seminar asked for such a position; according to Einstein, for a physicist it suffices to know and apply the elementary mathematical notions.[34] About a decade later, both, Adolf and Albert performed some chamber music together[35]. Especially in duet with the youngest daughter, Lisi Hurwitz, Einstein performed a long series of violin playings of Bach and Händel, whereas Hurwitz himself enjoyed playing the piano with his oldest daughter Eva [Samuel-Hurwitz, 1984, p. 12]. The physicist Max Born (1882 - 1879) was another student of Adolf around 1902/03 whose memories give a nice picture of the teacher Adolf Hurwitz.

> "Once when I missed a point in a lecture I went to Hurwitz afterwards and asked for a private explanation. He invited me [...] to his house and gave us a series of private lectures on some chapters of the theory of functions of complex variables, in particular on Mittag-Leffler's theorem, which I still consider as one of the most impressive experiences of my student life. I carefully worked out the whole course, including these private appendices, and my notebook was used by Courant when he, many years later and after Hurwitz's death, published his well-known book. [...]" (cf. [Rowe, 2007, p. 29])[36]

Adolf's health was never good. Already during his studies in Munich he suffered from typhus, a reason for Adolf to stay and study in Hildesheim. Actually, his early paper

[33] Adolf Hurwitz supported the forerunner of the ICM at Chicago's World's Columbian Exposition in 1893 by submitting a contribution *in absentia*; see [Lehto, 1998, p. 5], resp. www.mathunion.org/ICM/.

[34] "der Herr Professor [mag] darüber ein wenig verwundert gewesen sein, war doch dieser Student niemals in den mathematischen Seminaren zu sehen gewesen, da er sich mangels an Zeit nicht beteiligen konnte. [...] dass es für einen Physiker genüge, die elementaren mathematischen Begriffe zu kennen und anzuwenden, der Rest für ihn aus 'unfruchtbaren Subtilitäten' bestehe." [Fölsing, 1993, p. 634]

[35] see [Pólya, 1987, p. 24] as well as its title page for a nice photograph

[36] The book in question is [Courant and Hurwitz, 1992].

[Hurwitz, 1882] on the class number of binary quadratic forms, in which he introduced the so-called Hurwitz zeta-function, has the footer '*Hildesheim, den 10. Oktober 1881*'; it plays an important role in the theory of L-functions associated with number fields. In 1905, one of Adolf's kidneys was removed; later also his second kidney gave up working properly. In the sequel, Hurwitz's life became more calm and remote than before. Pólya wrote, "His health was not too good so when we walked it had to be an level ground, not always easy in the hilly part of Zürich, and if we went uphill, we walked very slowly." [Pólya, 1987, p. 25] Pólya at that time was around thirty years old, whereas Adolf was in the mid-fifties. In May 1919, Adolf Hurwitz finished his last big project, his monograph [Hurwitz, 1919b] on number theory of quaternions based on his lectures at the *Königl. Gesellschaft der Wissenschaften zu Göttingen* in 1896. Still on October 28, he ran a seminar at home where his family was "[we,] listening at the door admired the control and clearness, with which he knew to talk."[37] [Samuel-Hurwitz, 1984, p. 14] He died from kidney malfunction some days later, November 18, 1919 in Zurich. Life expectancy was just around 54 years at that time.

Adolf Hurwitz's collected papers [Hurwitz, 1932] were edited by Pólya and appeared in 1932. In [Pólya, 1987, p 25] George Pólya wrote: "My connection with Hurwitz was deeper and my debt to him greater than to any other colleague." It was indeed on Adolf's invitation that Pólya was offered an appointment as *Privatdozent* at Zurich. In the next subsection we present the complete rack of 30 mathematical diaries [Hurwitz, 1919a] of Adolf Hurwitz ranging from 1882 until 1919, held by the *Eidgenössische Hochschule Zürich.* During his scientific life he supervised altogether at least 23 doctoral students[38]. Among his pupils one can find the later professors Gustave du Pasquier (1876 - 1957) at the Université de Neuchâtel, Eugène Chatelain (1885 - 1956), Alfred Kienast (1879 - 1969), Émile Marchand (1890 - 1971), and Ernst Meissner (1883 - 1939), all at ETH Zurich,

[37] "[wir,] an der Tür Lauschenden bewunderten die Beherrschtheit und Klarheit, mit der er vorzutragen vermochte."

[38] According to the Mathematics Genealogy Project http://genealogy.math, all within the period 1896-1919; however, in his collected works [Hurwitz, 1932] there are just 21 listed.

as well as Kerim Erim (1894 - 1952), who obtained his doctorate at the University of Erlangen-Nuremberg and later became a professor at the University of Istanbul. A good account on Adolf Hurwitz's life and work is given by his wife [Samuel-Hurwitz, 1984] and Frei [Frei, 1995].

In texts about Adolf Hurwitz's life and work it is often mentioned that he had greatly benefited from his teacher Felix Klein. It is well-known that Klein himself had a very high opinion on the triangle Hurwitz, Hilbert, and Minkowski at Königsberg; in his treatise on the development of mathematics in the nineteenth century [Klein, 1926] Klein attributed the description 'aphorist' to Hurwitz and considered him as a 'problem solver' writing 'complete works', whereas Minkowski is a theory builder who found new links between 'geometrical view' and 'number theoretical problems'.[39] This is the picture of a *frog* and a *bird* according to Dyson's classification of mathematicians characters [Dyson, 2009]. Actually, Minkowski had encounters with both, Hurwitz and Hilbert, after their common time at Königsberg, with the first named during his time from 1896 until 1902 at Zurich and with the latter until his untimely death in 1909. It seems that Minkowski and Hilbert were closer than the other vertices of this unequal triangle. Nevertheless, the lifelong relationship of Hilbert and Hurwitz, on which we shed light in Sebsection 2.2.3, was as well very fruitful.

[39]" Und glücklicherweise findet sich um 1885 für fast wieder ein Jahrzehnt, eben auch wieder in Königsberg, ein Dreibund junger Forscher zusammen, welche diese Tendenz in neuer Weise in die Tat umsetzen und damit denjenigen Standpunkt schaffen, von dem aus die Neuzeit, wenn sie es vermag, weiterzugehen hat. Es sind dies Hurwitz, Hilbert und Minkowski. [...] und so möchte ich über Hurwitz und Minkowski hier vorweg ein paar Worte sagen, welche deren Arbeitsweise charakterisieren sollen. Man hat Hurwitz einen Aphoristiker genannt. In voller Beherrschung der in Betracht kommenden Disziplinen sucht er sich hier und dort ein wichtiges Problem heraus, das er jeweils um ein bedeutendes Stück fördert. Jede seiner Arbeiten steht für sich und ist ein abgeschlossenes Werk. [...] Minkowskis hier in Betracht kommende Arbeiten beruhen zumeist auf der Verbindung durchsichtiger geometrischer Anschauung mit zahlentheoretischen Problemen. [...] Ich selbst habe mich seinerzeit darauf beschränkt, gewisse schon bekannte Grundlagen geometrisch klarzustellen, während Minkowski Neues zu finden unternahm. Diese Untersuchungen zeigen deutlich, daß Geometrie und Zahlentheorie keineswegs einander ausschließen, sofern man sich in der Geometrie nur entschließt, diskontinuierliche Objekte zu betrachten." [Klein, 1926, pp. 326]

2.2 Excursion: The Personal Hurwitz Estate from the Archive of the ETH Zurich

In the archive of the polytechnic *Eidgenössische Technische Hochschule*, short ETH, in Zurich the estate of Adolf Hurwitz is stored. In the directories HS 582 and HS 583 various documents concerning Adolf Hurwitz and his family can be found. Apart from a large number of manuscripts and lecture notes of Hurwitz himself and colleagues, there are also personal documents. In the previous Section 2.1 we have already stated several quotations from the biographical dossier of Ida Samuel-Hurwitz [Samuel-Hurwitz, 1984] (HS 583a:2). Together with a second dossier of Elsbeth Meyer-Neumann (HS 583a:1) those provide an overview of the family history. Furthermore, some letters of condolence (HS. 583: 15 - 50) and a part of Hurwitz's correspondence (HS 583pp) are stored.[40] In the following we mainly refer to Adolf Hurwitz's mathematical diaries (HS 582: 1 - 30).

Some Remarks on the Mathematical Diaries

> "Since his habilitation in 1882, Hurwitz took notes of everything he spent time on with uninterrupted regularity and in this way he left a series of 31 diaries, which provide a true view of his constantly progresssive development and at the same time they are a rich treasure trove for interesting and further examination appropriate thoughts and problems."[41] [Hilbert, 1921, p. 166]

[40] The main part of the Hurwitz correspondence is stored in the *'Niedersächsische Staats- und Universitätsbibliothek'* in Göttingen.

[41] "Hurwitz hat seit seiner Habilitation 1882 in ununterbrochenender Regelmäßigkeit von allem, was ihn wissenschaftlich beschäftigte, Aufzeichnungen gemacht und auf diese Weise eine Serie von 31 Tagebüchern hinterlassen, die ein getreues Bild seiner beständig fortschreitenden Entwicklung geben und zugleich eine reiche Fundgrube für interessante und zur weiteren Bearbeitung geeignete Gedanken und Probleme sind."

Figure 2.2: Front pages of diary No. 4 and diary No. 5 with mottos "Durch Nacht zum Licht." and "Rast ich, so rost ich.".[42]

With David Hilbert's description, taken from his commemorative speech, he provided a good picture of Adolf Hurwitz's mathematical diaries. Alongside a variety of publications, the zealous Hurwitz wrote in a meticulous manner mathematical notebooks from March 1882 to September 1919. After his death those were reviewed and registered in an additional notebook [Hurwitz, 1919a, No. 32] by his confidant and colleague George Pòlya, who considered Hurwitz as "colleague who he felt influenced the most." [Pólya, 1987, p. 25] Mostly with an accurate writing and an impressive precision Adolf Hurwitz worked on proofs of colleagues, made notes for future dissertation topics and developed his own approaches to various mathematical problems. However, in a few places in the collected, in fact, 30 diaries[43], we can also find quotations giving an insight into Adolf Hurwitz's personal side.

[42]freely translated "Through the night to the light." [Hurwitz, 1919a, No. 4] and "When I rest, I rust." [Hurwitz, 1919a, No. 5]

[43]Hilbert mentioned 31 diaries. However, the last notebook contains only a collection of articles of colleagues and no own ideas of Hurwitz.

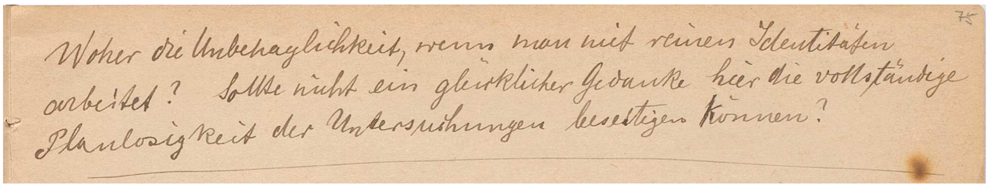

Woher die Unbehaglichkeit, wenn man mit reinen Identitäten arbeitet? Sollte nicht ein glücklicher Gedanke hier die vollständige Planlosigkeit der Untersuchungen beseitigen können?

Figure 2.3: "Where does the uneasiness come from, when you work with pure identities? Shouldn't a fortunate idea be able to eliminate the complete lack of planning in the studies?"[44]

This nice note is taken from Hurwitz's first diary [Hurwitz, 1919a, No. 1]. Accordingly, we give a second example from his last diary [Hurwitz, 1919a, No. 30], a quotation from Johann Wolfgang von Goethe.

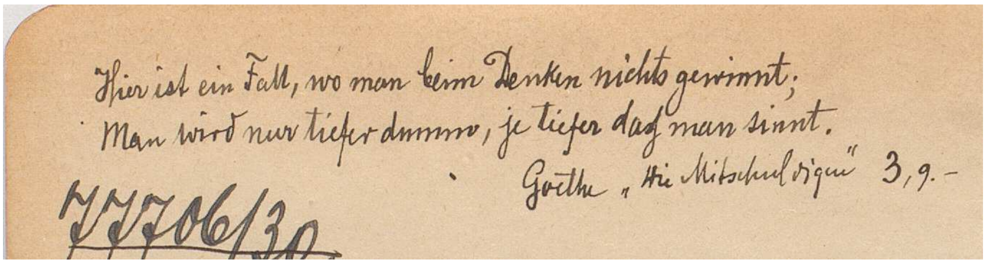

Hier ist ein Fall, wo man beim Denken nichts gewinnt;
Man wird nur tiefer dumm, je tiefer daß man sinnt.
Goethe „Die Mitschuldigen" 3,9.-

Figure 2.4: "Here is the case, where we win nothing from thinking; We will only become more stupid, the deeper we ponder. Goethe 'Die Mitschuldigen' "[45]

[44] "Woher die Unbehaglichkeit, wenn man mit reinen Identitäten arbeitet? Sollte nicht ein glücklicher Gedanke hier die vollständige Planlosigkeit der Untersuchungen beseitigen können?"

[45] "Hier ist der Fall, wo man beim Denken nichts gewinnt; Man wird nur tiefer dumm, je tiefer daß man sinnt. Goethe 'Die Mitschuldigen' "

2.2.1 Recreational Mathematics in the Mathematical Diaries

There is some evidence that Adolf Hurwitz was of rather serious nature. Colleagues described him as "carrying out his duties" [Stern, 1919, p. 859], being "quiet and considerate" even "passionless in judging" [Meissner, 1919, p. XXIV] and also to "devote himself to calm mental work" [Hilbert, 1921, p. 168]. Furthermore, his wife Ida Samuel-Hurwitz characterized him as restrained and having a "certain shyness" [Samuel-Hurwitz, 1984, p. 15] concerning acquaintances. Historically, Adolf Hurwitz belonged to a last generation of universal mathematicians: He achieved outstanding results in complex anaylsis, algebra, number theory as well as in geometry [Hurwitz, 1932]. Did this serious mathematician also spend time on recreational mathematics, a field growing in popularity at that time?

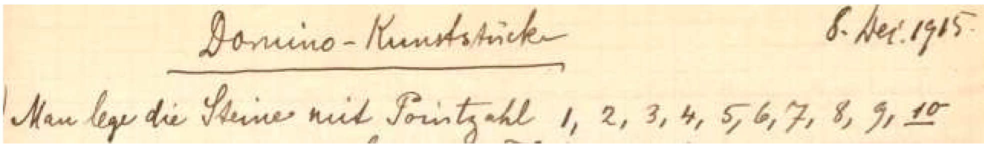
In a few places in his diaries, distributed over the years, sometimes rather unobtrusive entries with titles like "Funny riddle (from Landau)"[46] [Hurwitz, 1919a, No. 25], "Trick with dominoes"[47] [Hurwitz, 1919a, No. 28] or "Most coloured rings"[48] [Hurwitz, 1919a, No. 28] can be found from time to time. A complete list of entries being ordered in the category 'Recreational Mathematics' is given in Appendix 9.1.

First, we analyze three examples of these unusual entries with exercise character. The formulation of those problems is often misleadingly clear, whereas, the complexity of their solutions can be astonishingly advanced.

Dominoes

Under the titel "Trick with dominoes" an entry from December 08, 1915 in the 28th diary[49] on page 61 is concealed.

[46] "Scherzaufgabe(von Landau)"
[47] "Dominokunststück"
[48] "Bunteste Ringe"
[49] from 1915 II.16. to 1917 III.22.


Domino-Kunststücke 6. Dez. 1915

Man lege die Steine mit Pointzahl 1, 2, 3, 4, 5, 6, 7, 8, 9, $\underline{10}$

Figure 2.5: "One places the dominoes with numbers of points $1, 2, 3, 4, 5, 6, 7, 8, 9, \underline{10}$ one after the other face down on the table and let somebody shift a number of dominoes from the beginning to the end."[50]

The aim is to be always able to state the number of shifted dominoes. This can be realized with the trick that "the number [is] then equal to the number of points which is at the position where the 10 had been."


sind 4 Steine geschoben, so hat man: 5, 6, 7, 8, 9, $\underline{\underline{10}}$, 1, 2, 3, $\underline{4}$ erhalten

Figure 2.6: "E.g. when 4 dominoes were shifted, we receive $5, 6, 7, 8, 9, \underline{\underline{10}}, 1, 2, 3, \underline{4}$ and 4 is at the position where 10 had been before."[51]

Consequently, one only has to keep in mind where 10 was initially placed to be able to determine the number of shifted dominoes. "In this order, 10 is placed at the 5th last position."

[50] "Man lege die Steine mit Pointzahl $1, 2, 3, 4, 5, 6, 7, 8, 9, \underline{10}$ der Reihe nach zugedeckt auf den Tisch und lasse jemand eine Anzahl Steine vom Anfang ans Ende schieben."

[51] "die Anzahl [...] dann der Pointzahl gleich [ist], die sich an der Stelle befindet, wo die 10 lag. Z.B. sind 4 Steine geschoben, so hat man: $5, 6, 7, 8, 9, \underline{\underline{10}}, 1, 2, 3, \underline{4}$ erhalten und 4 liegt nun an der letzten Stelle, also an derjenigen, wo 10 früher lag."

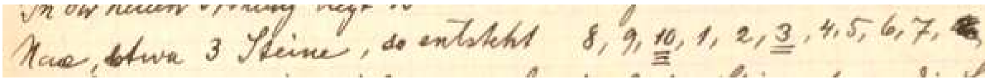

Neue, etwa 3 Steine, so entsteht 8, 9, 10, 1, 2, 3, 4, 5, 6, 7,

Figure 2.7: "Shifting again, for example 3 dominoes, leads to $8, 9, \underline{\underline{10}}, 1, 2, 3, \underline{\underline{4}}, 5, 6, 7$. The number 3 is given by the dominoe, which was shifted at the position of 10. In this way one can arbitrarily continue."[52]

This mathematical pastime certainly belongs to the easier questions in recreational mathematics. Related considerations can be designed much more difficult when the dominoe-specific characteristic of the bivalence is taken into account. In fact, this diary entry is followed by a far more complicated trick concerning N dominoes, where $N \in \mathbb{N}$ is arbitrary. Here the respective valence is expressed by an integral function. Hurwitz solved the problem on behalf of a finite series.

Spider meets Fly?

In diary No. 23[53] on the first page under the title "Exercise about the shortest line on a parallelepiped"[54] one can read about a problem, which nowadays can be considered as classic in recreational mathematics. Adolf Hurwitz wrote: "Yesterday after the lecture Dr. Du Pasquier[55] told me about a nice exercise (which he himself had received from Herrn von Mises): In a room of height 12m and of length 36m on opposite walls on the middle lines a fly F and a spider S are sitting, F is 3m from the ceiling, S is 3m from the floor. The fly says to the spider: If you come to me, without crawling through a path of 48 or more meters, so I keep sitting and you catch me; however, if it takes you 48 or more

[52] "In der Ordnung liegt 10 an der 5t-letzten Stelle. Schiebt man aufs Neue, etwa 3 Steine, so entsteht $8, 9, \underline{\underline{10}}, 1, 2, 3, \underline{\underline{4}}, 5, 6, 7$. Die Anzahl 3 wird wieder angegeben durch den Stein, der an die Stelle von 10 gerückt ist. Auf diese Weise kann man beliebig fortfahren."

[53] from 1908 I. 23. to 1910 II.18.

[54] "Aufgabe über kürzeste Linie auf einem Parallelepiped"

[55] Certainly he wrote about his former doctoral student Louis-Gustave Du Pasquier (see Subsection 2.1) and the Austrian mathematician Richard Edler von Mises (1883 - 1953).

meters, I fly away before you reach me. How should the spider crawl to reach the fly?"[56]

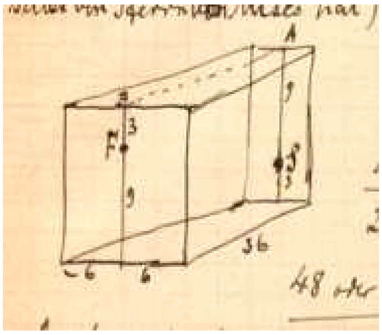


Figure 2.8: Sketch of the positions of fly F and spider S.

Obviously the alleged easiest way $SABF$ with $\overline{SABF} = 9 + 36 + 3 = 48$ would not be successful. Adolf Hurwitz reformulated the problem more generally:

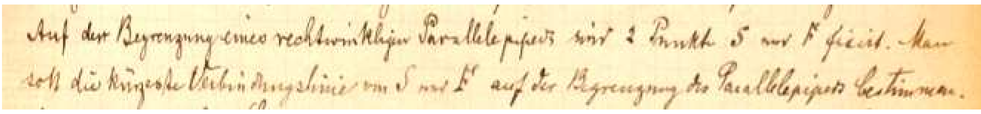

Auf der Begrenzung eines rechtwinkligen Parallelepipeds sind 2 Punkte S und F fixiert. Man soll die kürzeste Verbindungslinie von S und F auf der Begrenzung des Parallelepipeds bestimmen.

Figure 2.9: "On the surface of a rectangular parallelepiped two points F and S are fixed. The shortest connecting line along the surface [...] is to be determined."[57]

It is provided that on a planar surface the way is geometrically linear. Hurwitz's idea is to rotate each two faces f_1 and f_2 which are passed through and "which have the edge CD in common"[58], into one plane.

[56] "Dr. Du Pasquier teilte mir gestern nach dem Colleg folgende hübsche Aufgabe mit (die er selbst von Herrn von Mises hat): In einem Zimmer von der Höhe 12m der Breite 12m und der Länge 36m sitzen an gegenüberliegenden Wänden in der Mittellinie derselben eine Fliege F und eine Spinne S, F 3m von der Decke, S 3m vom Fussboden. Die Fliege sagt zur Spinne: Wenn Du zu mir kommst, ohne einen Weg von 48 oder gar mehr Metern zu durchkriechen, so bleib ich sitzen und Du fängst mich; wenn Du aber 48 oder mehr Meter gebrauchst, so fliege ich fort, ehe Du zu mir kommst. Wie muss die Spinne kriechen, um die Fliege zu erhalten?"

[57] "Auf der Begrenzung eines rechtwinkligen Parallelepipeds sind 2 Punkte S und F fixiert. Man soll die kürzeste Verbindungslinie von S und F auf der Begrenzung [...] bestimmen."

[58] "welche die Kante CD gemeinsam"

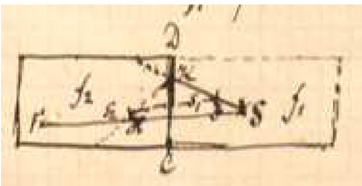


Figure 2.10: Illustration of f_1 and f_2 as well as $S_2, \mathcal{H}, \mathcal{J}$ and $\mathcal{K}$.

"So the continuation of the path on f_2 needs to be on a straight line with the path on f_1. Because if it would continue refracted on f_2 (like in the piece S_2) the path could be shortened, by e.g. letting $\mathcal{HJ}$ take the place of $\mathcal{HK} + \mathcal{KJ}$."[59]

The diarist continued his considerations to n planar faces $f_i, i = 1, \cdots, n$, which are rotated into one plane on which the path needs to "merge into one straight line"[60]. Here it is excluded that a face is entered twice. "Since was the way like $SS_1, S_1S_2, S_2S_3, \ldots, S_{k-1}S_k$ and were $S_{i-1}S_i$ and $S_{h-1}S_h$ located on the same face, the section of the path $S_{i-1}S_i \ldots S_h$ could be replaced by the shorter straightlined $S_{i-1}S_h$."[61] Denoting in our special case start and end faces with f and g and each opposite faces with f_1, f_2 and g_1, g_2 (see Fig. 2.8 and 2.11), we receive the opportunities

$$f, f_1, g; \quad f, f_2, g; \quad f, g_1, g; \quad f, g_2, g; \quad f, f_1, f_2, g \text{ "(and the analogous)"};$$

$$f, f_1, f_2, g_1, g \text{ "(and the analogous)"}; \quad f, f_1, f_2, g_1, g_2, g \text{ "(and the analogous)"}.$$

According to Hurwitz "[T]he exercise [is] here principally done." He gives an explicit calculation for the path fg_2g_1g, illustrated by a sketch (see Fig. 2.11).

[59] "Dann muss die Fortsetzung des Weges auf f_2 mit dem Weg auf f_1 in gerade Linie fallen. Denn würde er gebrochen auf f_2 weitergehen (etwa in dem Stücke S_2) so würde man den Weg verkleinern können, indem man z.B. $\mathcal{HJ}$ an die Stelle von $\mathcal{HK} + \mathcal{KJ}$ treten ließe."

[60] "in eine Gerade übergehen"

[61] "Denn wäre der Weg dieser $SS_1, S_1S_2, S_2S_3, \cdots, S_{k-1}S_k$ und würden $S_{i-1}S_i$ und $S_{h-1}S_h$ auf der selben Seitenfläche $f_i = f_h$ liegen, so würde das Wegstück $S_{i-1}S_i \cdots S_h$ durch das kürzere geradlinige $S_{i-1}S_h$ ersetzt werden können."

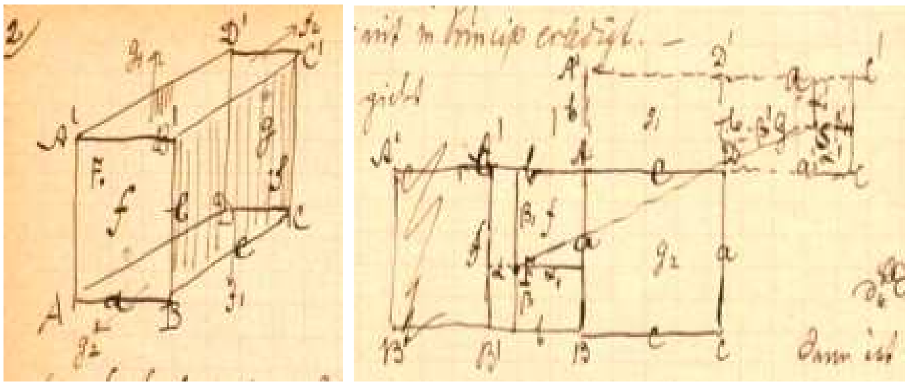


Figure 2.11: The parallelepiped, in particular the path fg_2g_1g.

"[...] F in f shall have distance α of the edge $B'A'$, β of the edge $B'B$, [...] S in g the distance α' of DC and β' of DD'. Then regarding the axes of coordinates $(B'B, B'A')$ (α, β) is the point F and $(b + c + \beta', a + \alpha')$ the point S."[62] Therewith, Hurwitz receives the straight line equation for FS

$$\frac{x - \alpha}{y - \beta} = \frac{b + c + \beta' - \alpha}{a + \alpha' - \beta},$$

with $B'B$ as x- and $B'A'$ as y-axis. The straight line intersects the line $A'D$ in the point with coordinates $y = a$ and

$$x = \alpha - (a - \beta)\frac{b + c + \beta' - \alpha}{a + \alpha' - \beta} = \frac{\alpha\alpha' + (a - \beta)(b + c + \beta')}{a + \alpha' - \beta}.$$

In our case, the x-coordinate must be located on the edge of g_2, in between b and $b + c$. Consequently, we have

$$b(a + \alpha' - \beta) < \alpha\alpha' + (a - \beta)(b + c + \beta') < (b + c)(a + \alpha' - \beta),$$

[62] "[...] F in f habe die Entfernung α von der Kante $B'A'$, β von der Kante $B'B$, [...] S in g die Entfernung α' von DC und β' von DD'. Dann ist bezüglich des Achsenkreuzes $(B'B, B'A')$ (α, β) der Punkt F und $(b + c + \beta', a + \alpha')$ der Punkt S."

which can be splitted into

$$\alpha'(b-\alpha) < (a-\beta)(c+\beta') \text{ and } (a-\beta)\beta' < \alpha'(b+c-\alpha).$$

We determine α_1, β_1 und α'_1, β'_1, such that $\alpha + \alpha_1 = \alpha' + \alpha'_1 = b$ and $\beta + \beta_1 = \beta' + \beta'_1 = a$. Therewith, we receive the constraints

$$\alpha\alpha' < \beta_1(c+\beta') \text{ and } \beta_1\beta' < \alpha'(c+\alpha_1).$$

Moreover, for the distance of the points S and F we obtain

$$\overline{SF} = \sqrt{(c+\beta'+\alpha_1)^2 + (\alpha'+\beta_1)^2}.$$

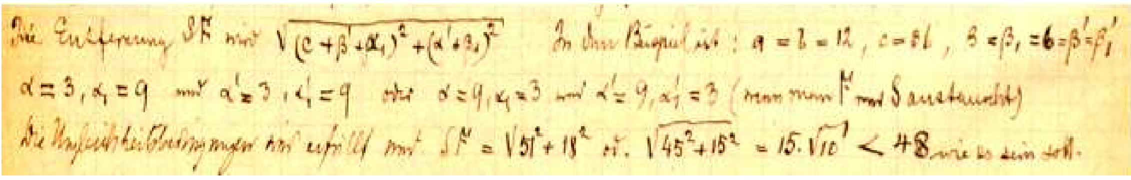

Die Entfernung SF wird $\sqrt{(c+\beta'+\alpha_1)^2+(\alpha'+\beta_1)^2}$ In dem Beispiel ist: $a = b = 12$, $c = 36$, $\beta = \beta_1 = 6 = \beta' = \beta'_1$
$\alpha = 3, \alpha_1 = 9$ und $\alpha' = 3, \alpha'_1 = 9$ oder $\alpha = 9, \alpha_1 = 3$ und $\alpha' = 9, \alpha'_1 = 3$ (wenn man F und S austauscht)
Die Ungleichheitsbedingungen sind erfüllt mit $SF = \sqrt{51^2+18^2}$ od. $\sqrt{45^2+15^2} = 15.\sqrt{10} < 48$ wie es sein soll.

Figure 2.12: "The distance of SF is $\overline{SF} = \sqrt{(c+\beta'+\alpha_1)^2 + (\alpha'+\beta_1)^2}$. In the [initial] example one has $a = b = 12, c = 36, \beta = \beta_1 = 6 = \beta' = \beta'_1, \alpha = 3, \alpha_1 = 9$ and $\alpha' = 3, \alpha'_1 = 9$ or $\alpha = 9, \alpha_1 = 3$ and $\alpha' = 9, \alpha'_1 = 3$ (if we exchange F and S)."[63]

Hurwitz wrote, "the condition of inequality is satisfied by $SF = \sqrt{51^2 + 18^2}$ resp. $\sqrt{45^2 + 15^2} = 15.\sqrt{10} < 48$ as ist should be." Indeed, we have $15.\sqrt{10} \sim 47.4341649\ldots$.

Nowadays, the fly-spider-riddle is very well know. In different variations it is mentioned in a great number of books on recreational mathematics and, in particular, students could get in touch with it on lessons concerning the Pythagorean Theorem. There are many

[63] "Die Entfernung SF wird $\overline{SF} = \sqrt{(c+\beta'+\alpha_1)^2 + (\alpha'+\beta_1)^2}$. In dem [ursprünglichen] Beispiel ist $a = b = 12, c = 36, \beta = \beta_1 = 6 = \beta' = \beta'_1, \alpha = 3, \alpha_1 = 9$ und $\alpha' = 3, \alpha'_1 = 9$ oder $\alpha = 9, \alpha_1 = 3$ und $\alpha' = 9, \alpha'_1 = 3$ (wenn man F und S austauscht). Die Ungleichheitsbedingung wird erfüllt mit $\overline{SF} = \sqrt{51^2 + 18^2}$ od. $\sqrt{45^2 + 15^2} = 15.\sqrt{10} < 48$ wie es sein soll."

similarly formulated problems concerning shortest connecting lines, so-called geodesics, on given objects. Those have names like 'Fly-Honey-' or 'Ants-Problem'. However, when Adolf Hurwitz wrote his diary entry, the Fly-Spider-Problem was brandnew. According to the article '*Henry Ernest Dudeney: Englands größter Rätselerfinder'*[64] from Martin Gardner (1914 - 2012) published in his collection [Gardner, 1968, p. 73pp], the nowadays known formulation of this brain teaser goes back to Henry Ernest Dudeney (1857 - 1930). On February 01, 1905, he had first published the exercise in the 'Daily Mail' [Dudeney, 1905a] followed by the solution [Dudeney, 1905b] one week later, on February 08, 1905 (see also [Hemme, 2013]). The British puzzle fan published a multitude of collected exercises[65] from recreational mathematics, regularly he sent in puzzles in newspapers under the pseudonym 'Sphinx'. Furthermore, he was in close contact with his probably better known American colleague Sam Loyd (1841 - 1911) and worked with him "on a series of articles about riddles for the English newspaper *'Tit-Bits'*." [Gardner, 1968, p. 70] According to Martin Gardner, Dundeley was "of both [...] the better mathematician" and "his work [was] more sophisticated (he once described a picture puzzle of which Loyd produced thousands as "youthful banality, which could only attract the interest of imbeciles")." Those harsh words indicate that Dundeley accused Loyd to have published some of his exercises without permisssion. This situation is elucidated in [Newing, 1994, p. 299], where Angela Newing described that "Henry Ernest became increasingly more infuriated as he saw his puzzles passed off in the States as Sam's own."

Conundrum

Another rather entertaining exercise with recreational character can be encountered in diary No. 25[66] on page 138. Although the exercise and its solution are certainly short, Adolf Hurwitz wrote: "An exercise given by Landau generalized and modified".

[64] *'Henry Ernest Dudeney: Englands greatest riddle inventor'*

[65] including similar problems, e.g. *'A Fly's Journey'* [Dudeney, 1967, p. 110] and *'The Fly and the Honey'* [Dudeney, 1967, p. 11].

[66] from 1912 XII. 27. to 1914 IV. 30.

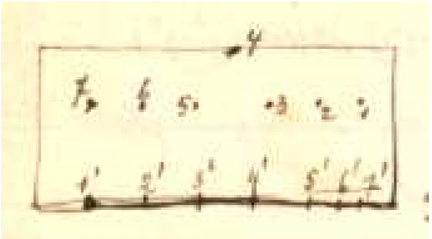


Figure 2.13: Positions of ladies and gentlemen in the hall.

"On the wall of a ballroom 7 (n) ladies are sitting $1', 2', 3', 4', 5', 6', 7'$; in the hall the 7 ($n$) gentlemen $1, 2, 3, 4, 5, 6, 7$ (4 on the opposide wall), as indicated in the drawing. Each gentleman a asks the lady a' for the dance; hereby the taken paths from the gentlemen to their ladies shall not cross."[67]

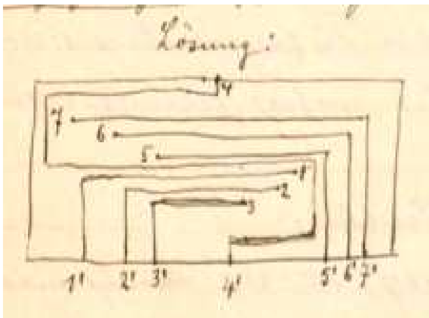


Figure 2.14: The solution.

Here the generalizations (n) were inserted subsequently by Hurwitz. The solution turns out to be conceivably simple and is solely given by a sketch (see Fig. 2.14).

We may ask if the shortness of the solution is connected to the note in the table of contents (see Fig. 2.15).

Figure 2.15: Excerpt from the handwritten table of contents of Adolf Hurwitz [Hurwitz, 1919a, Nr. 28]: "Conundrum (Analysis Situs) Landau"[68]

It should be mentioned that the number theorist Edmund Landau (1877 - 1938) was not at all famous for his humour. Moreover, he was regarded as "not easy to han-

[67] "Eine von Landau gestellte Aufgabe verallgemeinert und modifiziert: An der Wand eines Ballsaales sitzen 7 (n) Damen $1', 2', 3', 4', 5', 6', 7'$; im Saale die 7 (n) Herren $1, 2, 3, 4, 5, 6, 7$ (4 an der Wand gegenüber), wie in der Figur angedeutet. Jeder Herr a fordert die Dame a' zum Tanz auf; dabei sollen aber die von den Herren zu ihren Damen zurückgelegten Wege sich gegenseitig nicht treffen."

[68] "Scherzaufgabe (Analysis Situs) Landau"

dle" and was known for "his personal, severe and the reader completely informing style"[69][Rechenberg, 1982, S. 480]. However, not least because of their common Jewish religious affiliation, the two mathematicians had a very close friendship. Landau visited the Hurwitz family many times in Zurich and sent a very worth reading, emotional letter of condolence to Ida Samuel-Hurwitz on November 28, 1919, only ten days after Adolf Hurwitz's death. Landau wrote: "[...] Now the center of mathematical life in Zurich is gone and I have no longer a home in this beautiful city. And that this great man was a Jew, makes me particularly proud."[70]. Unfortunately, we may only assume on which occasion he could have posed this riddle to Adolf Hurwitz.

In fact, the note in the table of contents itself poses another riddle: Why did Hurwitz describe the exercise by 'Analysis situs'? This term can be translated to 'Geometrie der Lage'[71] going back to Gottfried Wilhelm Leibniz (1646 - 1716), who characterized therewith a new way of a mathematical model of space[72]. Later 'Analysis situs' has been used synonymously to the term 'topology', a section of mathematics, in which mathematical structures are examined in respect to continuous deformations. Furthermore, Henri Poincaré published in 1895, some years before Hurwitz's diary entry, the article [Poincaré, 1895] entitled "Analysis Situs". Combined with five hereon based articles, so-called *Compléments*, Poincaré's work is nowadays considered as foundation of a first algebraic approach to topological objects: the so-called 'Algebraic Topology'.

Here various speculative references could be created, however, we may assume that the exercise was indeed simply considered as humouristic, so to say as 'analysis of the ballroom-space'.

[69] "nicht leicht im Ungang", "seinen eigenen, strengen und den Leser vollständig informierenden mathematischen Stil"

[70] "[...] Nun ist der Mittelpunkt des Zürcher mathematischen Lebens fort und ich habe kein Heim mehr in dieser schönen Stadt. Und dass dieser grosse Mann ein Jude war, macht mich ganz besonders stolz.", ETH Archives, Zurich (Hs 583:33)

[71] 'geometry of location'

[72] A detailed explanation is given in [Loemker, 1969, Ch. 27]

2.2.2 Adolf Hurwitz Folding and Cutting Paper

In particular in diary No. 22[73] a special feature can be discovered. Carefully kept in a small envelope, there are original pieces of paper, partially coloured with crayons. Some of them illustrate foldable mathematical objects; others help to illustrate a proof of the Theorem of Pythagoras.

Pythagoras

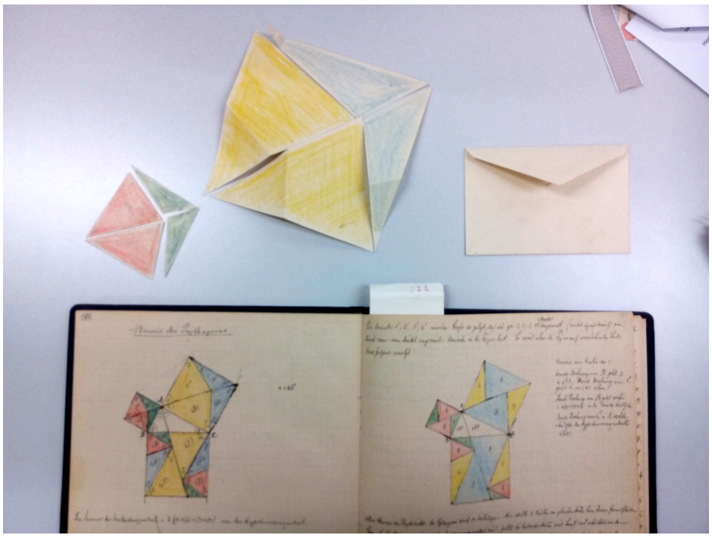


Figure 2.16: Cutted out, congruent triangles; drawings by Adolf Hurwitz illustrating a proof of the Theorem of Pythagoras.

Adolf Hurwitz wrote: "In class the theorem of Pythagoras can also be verified as follows: We manufacture 3 boxes of equal height, whose base areas resp. the squares of legs and the square of the hypotenuse, fill the boxes of legs with sand and pour them out into the box of hypotenuse. Here those will be filled completely. - We can also take 3 cyclic vessels of equal height (3 jars), whose base areas have the legs resp. the hypotenuse as diameters,

[73]from 1906 XIII.8. to 1908 I.22.

fill the jars of legs with water [...] We can also cut out the squares and verify with a balance that such a square weights as much as both other squares together."[74]

Certainly, many more similar proof ideas can be found and are widely known. However, what makes this diary entry remarkable is the didactic aspect Adolf Hurwitz pointed out here. First, it is in general very unusual that this ambitious mathematician is reflecting on school lessons. Second, he even cut out triangles by hand and coloured them. Why was he so intrigued by the Pythagorean Theroem?

Initially Hurwitz continued his thoughts by using polygons, however, then he dedicated himself to "a more difficult question [is] concerning the minimal number of pairwise congruent pieces, in which two [...] equal areas can be decomposed."[75] Three-dimensionally this recalls the third of Hilbert's Problems, which goes back to Gauss[76]: Can two given polyhedra of equal volume always be transferred into one another? This question was already negatively answered by Hilbert's student Max Dehn (1878 - 1952) in [Dehn, 1901].[77] However, for the two-dimensional version of the problem, the corresponding positive answer was known for a long time. We may only speculate whether Hurwitz could have been inspired by Hilbert's problem to deal with this issue concerning the minimal decomposition.

[74] "Man kann im Unterricht den Pythagoras auch so bestätigen: Man stellt 3 Kästen in gleicher Höhe her, deren Grundfläche bez. die Kathetenquadrate und das Hypothenusenquadrat sind, füllt die Kathetenkästen mit Sand und schüttet sie dann in den Hypothenusenkasten aus. Hier wird dieser ganz gefüllt werden. - Man kann auch 3 cyklische Gefäße von gleicher Höhe (3 Gläser) nehmen, deren Grundfläche die Katheten bez. die Hypothenuse zu Durchmessern besitzen, die Kathetengefäße mit Wasser füllen [...] Man kann auch die Quadrate ausschneiden und mit einer Waage bestätigen, daß das eine Quadrat so viel wiegt wie die beiden anderen zusammen."

[75] "einer schwierigeren Frage [ist die] nach der Minimalanzahl von paarweise congruenten Stücken, in welche zwei [...] gleiche Flächen (od. Systeme) zerlegbar sind."

[76] Gauss mentioned this problem in a letter to Christian Ludwig Gerling (1788 - 1864) (see [Kellerhals, 1999, p. 1]).

[77] Interestingly in [Piotrowski, 1985] it is explained that Hilbert's Third Problem was actually already posed by Wladyslaw Kretkowski (1840 - 1910) in 1882 and furthermore, there existed a, at least partial, solution due to Ludwik Antoni Birkenmajer (1855 - 1929) since 1883.

Paper Folding

A second unusual entry is written down in the twenty second mathematical diary on page 137, obviously inspired by Hurwitz's nine-year old son Otto. On December 24, 1907, the mathematician noted under the heading "Folding construction of the golden section and a regular pentagon"[78]

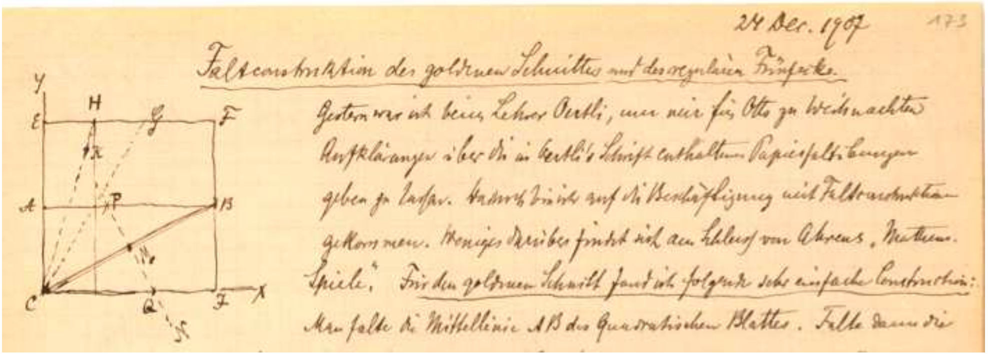


Figure 2.17: "Yesterday I visited teacher Oertli to receive for Otto on christmas enlightment about paper folding exercises included in Oertli's scripture. Here I came to the treating of folding constructions."

Hurwitz continued: "A few about this can be found at the end of Ahren's "Mathematische Spiele"[79]. For the golden section I found the following very simple construction: One folds the centerline AB of the squared paper. Then, fold the diagonal CB, and the halving line CG of the angle ECB (by placing CE on CB). Then EF is intersected in point G." (see Fig. 2.17) In the following, Hurwitz proved the golden section on behalf of elementary

[78] "Faltconstruktion des goldenen Schnittes und des regulären Fünfecks. Gestern war ich beim Lehrer Oertli, um mir für Otto zu Weihnachten Aufklärung über die in Oertli's Schrift enthaltenen Papierfaltübungen geben zu laßen. Dadurch bin ich auf die Beschäftigung mit Faltconstruktionen gekommen. Weniges darüber findet sich am Schluß von Ahrens "Mathematische Spiele". Für den goldenen Schnitt fand ich folgende sehr einfache Construction: Man falte die Mittellinie AB des quadratischen Blattes. Falte dann die Diagonale CB, und die Halbierungslinie CG des Winkels ECB (indem man CE auf CB legt). Dann wird EF im Punkte G geteilt."

[79] One might assume that [Ahrens, 1907] was meant, which is an abbreviated version of [Ahrens, 1901]. However, only [Ahrens, 1901] includes a chapter on folding constructions.

geometrical methods. He stated, "Equation of CB: $y = \frac{1}{2}x$ or $\frac{x-2y}{\sqrt{5}} = 0$"[80]. With this 'normalization' by the factor $\frac{2}{\sqrt{5}}$ he provided a straight line on CB of length 1. This equals the length of the straight line CE with the "Equation of CE: $x = 0$"[81]. By summarizing those two, he received an equation for CG[82] (see Fig. 2.18):

$$\frac{x-2y}{\sqrt{5}} + x = 0 \Leftrightarrow x - 2y + \sqrt{5}x = x(\sqrt{5}+1) - 2y = 0.$$

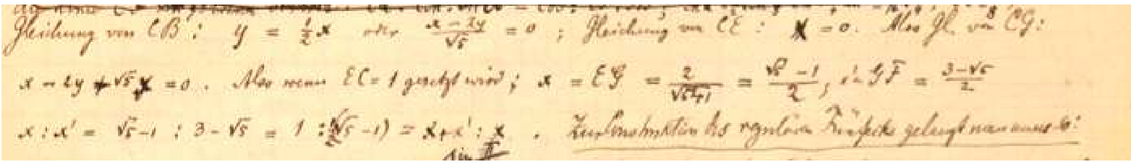


Figure 2.18: "So, when we set $EC = 1$; $x = EG = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2}$; $x' = GF = \frac{3-\sqrt{5}}{2}$ $x : x' = \sqrt{5}-1 : 3-\sqrt{5} = 1 : \frac{1}{2}(\sqrt{5}-1) = x + x' : x$. Thus, this leads to the construction of a regular pentagon:"[83]

This simple paper folding of the golden section is Hurwitz's basis for the construction of a regular pentagon. He pointed out that by halving EG in a new point H, he got $EH = \frac{1}{2}EG = \frac{\sqrt{5}-1}{4} = \sin(\frac{\pi}{10})$, where $\frac{\pi}{10}$ is obviously the central angle of a regular 20-gon.

[80] "Gleichung von CB: $y = \frac{1}{2}x$ oder $\frac{x-2y}{\sqrt{5}} = 0$"

[81] "Gleichung von CE: $x = 0$"

[82] In modern words this corresponds with the normalization of vectors and receiving of the angle bisector by summarizing those.

[82] In Theorem 11 in the second book of Euclid's Elements [Euclides, 1996, B. II] there is a very similar construction of the golden section.

[83] "Also wenn $EC = 1$ gesetzt wird; $x = EG = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2}$; $x' = GF = \frac{3-\sqrt{5}}{2}$ $x : x' = \sqrt{5}-1 : 3-\sqrt{5} = 1 : \frac{1}{2}(\sqrt{5}-1) = x + x' : x$. Zur Construktion des regulären Fünfecks gelangt man nun so:"

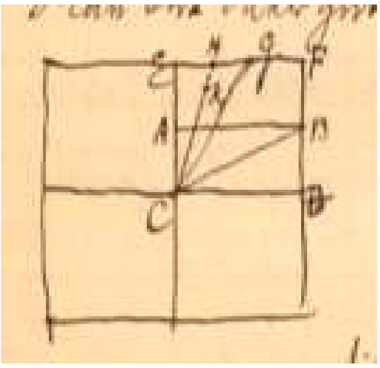


Figure 2.19: The idea behind Hurwitz's pentagon construction.

Therewith, Hurwitz received $\angle HCE = \frac{\pi}{10}$, respectively $\angle HCJ = \frac{\pi}{2} - \frac{\pi}{10} = \frac{2\pi}{5}$, which is the central angle of the regular pentagon. To fullfill the condition of equilaterality, Hurwitz folded CJ on CH and received a new point K, where the length of CJ equals the length of CK. Consequently, one can construct the pentagon using C as center and arranging around C five triangles of the type KCJ.

It seems that Hurwitz became really enthusiastic about paper folding. He continued his entry with stating a set of rules, which we want to sketch here:

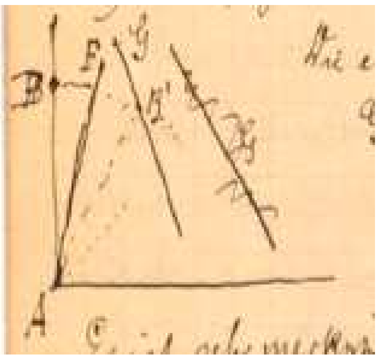


Figure 2.20: Illustration of rule 4).

"In the practical operation of foldings one will soon notice that only the following operations can safely be done: 1.) Determination of the intersection of two folding lines [...] 2) Determination of the vertical line in the middle of the line connecting two points on this connecting line. [...] 3) Halving of a given angle by placing one adjacent line on the other. 4.) Placing a boundary point B on a straight line G until the folding line AF passes through another boundary point A."[84]

[84]"Bei praktischer Ausführung von Faltungen wird man bald beobachten, daß nur folgende Operationen mit Sicherheit auszuführen sind: 1.) Bestimmung des Durchschnitts zweier Faltlinien [...] 2) Bestimmung der Senkrechten in der Mitte der Verbindungsgeraden zweier Punkte auf dieser Verbindungsgeraden. [...] 3) Halbierung eines bekannten Winkels durch Auflegen des einen Schenkels auf den anderen. 4.) Auflegen eines Randpunktes B auf eine Gerade G bis die Faltlinie AF durch einen anderen Rand-

Hurwitz continued with constructions of a regular hexagon and an equilateral triangle from a rectangular paper and the regular octagon from a squared paper (see Fig. 2.21).

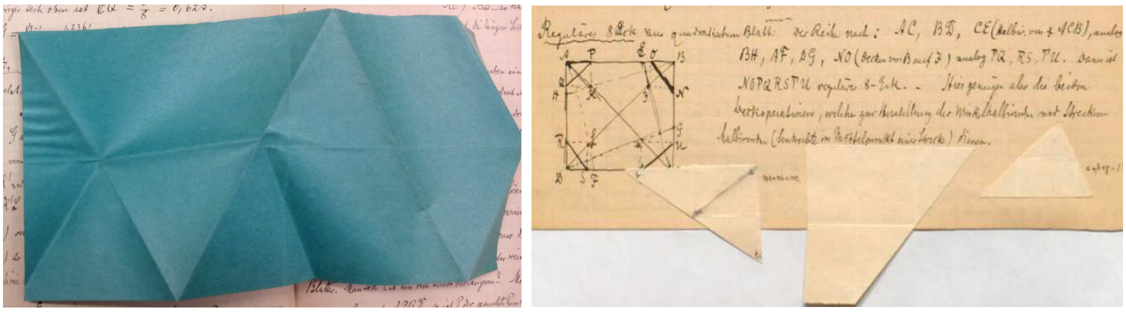


Figure 2.21: The left picture shows a piece of paper that was found loosely in the last pages of the 22nd diary. It cannot be said for sure that Hurwitz did the folding. In the picture on the right an illustration of the octagon can be seen. Here we are certain that the pieces of folded papers are originals: Hurwitz additionaly glued them on his diary page.

Furthermore, he dealt with other paper folding constructions, however, we want to concentrate on two more ideas concerning the pentagon. There are many ways of paper folding constructions of the pentagon. Hurwitz's first approach left some room for improvement. And indeed, on the next page of his mathematical diary, there is an additional entry with a corresponding title.

punkt A geht."

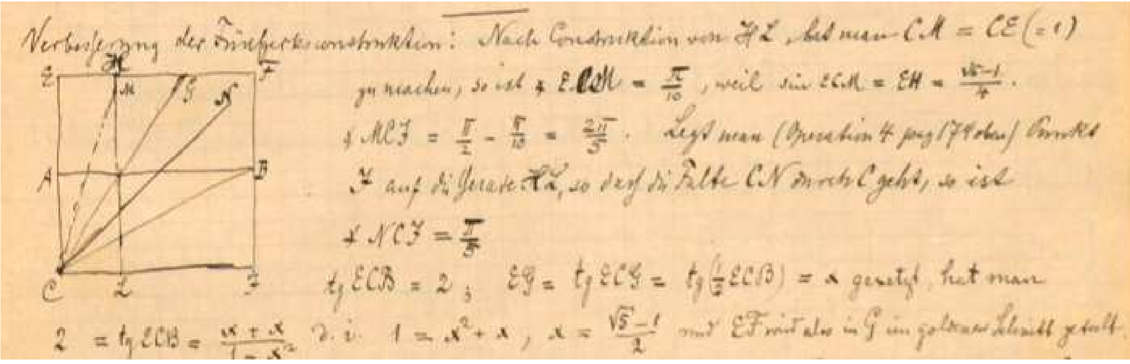


Figure 2.22: "Improvements of the construction of the pentagon: After the construction of HL, one has to set $CM = CE(= 1)$, so $\angle ECM = \frac{\pi}{10}$, because $\sin ECM = EH = \frac{\sqrt{5}-1}{4}$. $\angle MCJ = \frac{\pi}{2} - \frac{\pi}{10} = \frac{2\pi}{5}$."

"Placing (operation 4 pag 174 above) the point J on the straight line HL, so that the fold CN passes through C, then is $\angle NCJ = \frac{\pi}{5}$
Setting tgECB[85] $= 2, EG = \text{tg}ECG = \text{tg}(\frac{1}{2}ECB) = \alpha$, one has $2 = \text{tg}ECB = \frac{x+x}{1-x^2}$ that is $1 = x^2 + x$, $x = \frac{\sqrt{5}-1}{2}$ and EF will be divided in G in the golden section."[86]

In the end of his diary entry, Hurwitz finally stated as main result a simplification of the construction of a pentagon from a rectangled paper.

[85] tg is nowadays tangent

[86] "Verbesserung der Fünfecksconstruction: Nach Construktion von HL, hat man $CM = CE(= 1)$ zu machen, so ist $\angle ECM = \frac{\pi}{10}$, weil $\sin ECM = EH = \frac{\sqrt{5}-1}{4}$. $\angle MCJ = \frac{\pi}{2} - \frac{\pi}{10} = \frac{2\pi}{5}$. Legt man (Operation 4 pag 174 oben) Punkt J auf die Gerade HL, so daß die Falte CN durch C geht, so ist $\angle NCJ = \frac{\pi}{5}$
tg$ECB = 2, EG = \text{tg}ECG = \text{tg}(\frac{1}{2}ECB) = \alpha$ gesetzt, hat man $2 = \text{tg}ECB = \frac{x+x}{1-x^2}$ d. i. $1 = x^2 + x$, $x = \frac{\sqrt{5}-1}{2}$ und EF wird also in G im goldenen Schnitt geteilt."

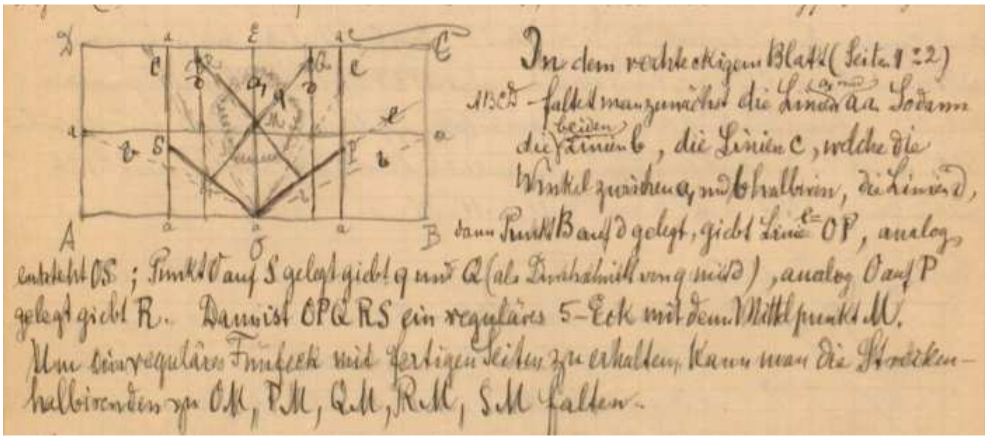


Figure 2.23: "In the rectangular paper (Page 1 : 2) $ABCD$ one firstly folds the lines a_1 and a_2. Then the two lines b, the lines c, which halve the angles between a_1 and b, the lines d, then placing point B on d, results in $e = OP$, analogously occurs OS; placing point O on S results in q and Q (as intersection of q and d), analogously placed O on R provides R. Then $OPQRS$ is a regular pentagon with center M. To receive a regular pentagon with completed sides, one can fold the halving lines of OM, PM, QM, RM, SM."[87]

It is remarkable, that Hurwitz obviously took the idea of paper folding and its possibilities very seriously and wanted to give exact instructions for the construction of the pentagon. This is underlined by the fact that it looks as if Hurwitz had first written with a pencil and then overwritten his construction with ink. The mathematician also gave a proof of his final construction: "As proof of this construction one considers a regular 5-gon $OPQRS$ and set the diagonal $SP = 1$. Then $\angle POB = \frac{\pi}{5} = \angle POQ = \angle QOR = \angle ROS$. Therewith $\angle QOM = \frac{\pi}{10}$ and Q lies on the straight line d, which cut off $\frac{\sqrt{5}-1}{4} = \sin\frac{\pi}{10}$ from EC. Since furthermore Q is vertical on the middle of OS, the correctness of the

[87]"In dem rechtwinkligen Blatt (Seite 1 : 2) $ABCD$ faltet man zunächst die Linien a_1 und aa. Sodann die beiden Linien b, die Linien c, welche die Winkel zwischen a_1 und b halbieren, die Linien d, dann Punkt B auf d gelegt, giebt Linie $e = OP$, analog entsteht OS; Punkt O auf S gelegt giebt q und Q (als Durchschnitt von q und d), analog O auf P gelegt giebt R. Dann ist $OPQRS$ ein reguläres 5-Eck mit dem Mittelpunkt M. Um ein reguläres Fünfeck mit fertigen Seiten zu erhalten, kann man die Streckenhalbierenden zu OM, PM, QM, RM, SM falten."

construction is evident."[88]

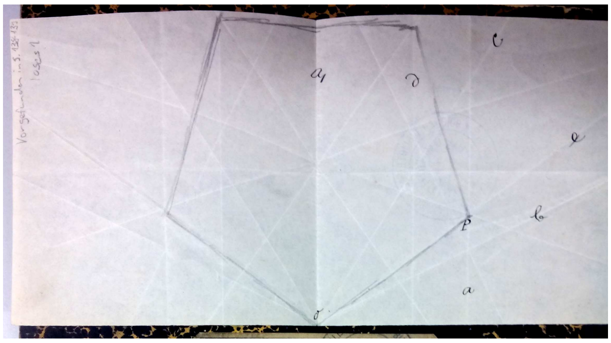


Figure 2.24: Find from diary No. 24.

That this construction was absolutely not trivial, illustrates a loose sheet, which we could find two diaries later. The handwriting of the small letters indicate that it was indeed Hurwitz who folded this pentagon.

Some remarks on paper folding in the beginning of the 20th century.

Hurwitz's remark about Otto's teacher from the beginning already indicates that paper folding played a certain role in mathematical education of children at that time. In fact, the pedagogical method of paper folding goes back to the German Friedrich Fröbel (1782 - 1852), founder of the 'kindergarten' movement and creater of so-called 'Kindergarten Gifts'[89]. He "[...] recognized very early [..] among other things the great utility of children's occupation with paper folding and braiding."[90] [Flachsmeyer, 2008, p. 8][91] His far-reaching influence on using paper folding in mathematical education is manifested in [Sundara Row, 1893], an Indian textbook. Interestingly, in its introduction, Sundara

[88]"Zum Beweis dieser Construction betrachte man ein reguläres 5-Eck $OPQRS$ und setze die Diagonale $SP = 1$. Damit ist $\angle POB = \frac{\pi}{5} = \angle POQ = \angle QOR = \angle ROS$. Daher $\angle QOM = \frac{\pi}{10}$ und Q liegt auf der Geraden d, welche das Stück $\frac{\sqrt{5}-1}{4} = \sin\frac{\pi}{10}$ von EC abschneidet. Da außerdem Q senkrecht über der Mitte von OS liegt, so leuchtet die Richtigkeit der Construction ein."

[89]a collection of educational 'toys'; a special one will be explained in the following

[90]" [...] erkannte schon frühzeitig [...] unter anderem die große Nützlichkeit der kindlichen Beschäftigung mit dem Papierfalten und Flechten."

[91]An indication, that Fröbel's ideas were sustainable can be found in [Timerding, 1914, p. A 135]: "Of course, who has the preconceived idea of mathematics [...] will find it disconcerting that it already takes place in games of small children like placing sticks, paper folding and cutting [...]." ("Wer freilich von der vorgefaßten Meinung über die Mathematik ausgeht [...] wird es befremdlich finden, daß sie schon in Spielen des kleinen Kindes wie Stäbchenlegen, Papierfalten und Ausschneiden [...] ihren Ausdruck findet.")

Row[92] wrote: "The idea of this book was suggested to me by Kindergarten Gift No. VIII - Paper-folding. The gift consists of 200 variously coloured squares of paper, a folder, and diagrams and instructions for folding."[93] However, the German-Indian exchange even went on: Row's work was well known to the German mathematician and expert for recreational mathematics Wilhelm Ahrens (1872 - 1927). In his two-volume collection *'Mathematische Unterhaltungen und Spiele'* [Ahrens, 1901, Ch. XXIII] from 1901, he dedicated one chapter to paper folding and remarked: "In this chapter we intend to give some extracts of a by the Indian mathematician Mathematiker Sundara Row 1893 published book, in which is shown how to proceed, when one wants to realize the geometric construction with nothing else then with folding of paper [...]"[94]. In his chapter Ahrens continued Row's purpose "not only to aid the teaching of Geometry in school and colleges, but also to afford mathematical recreation to young and old [...]" [Sundara Row, 1893, p. vi], which was certainly successfull in view of Adolf Hurwitz.

2.2.3 Relation to his Student David Hilbert

Examination of Adolf Hurwitz's estate furthermore highlights that his lifelong friendship to his former student and later colleague David Hilbert was not only exceptionally fruitful, but also rather interesting. It seems that their relation had undergone a certain change between 1884 and 1919. In this section we try to investigate when the famous Hilbert became completely emancipated from his teacher Hurwitz.

Hereby, it is important to emphasize that the following analysis is mainly based on the mentioned Hurwitz estate and its comparison to additional biographical informations and mathematical details extracted from the *'Gesammelte Abhandlungen'* [Hilbert, 1935] and

[92] sometimes also refered to as Rao instead of Row

[93] Probably, Row could know the kindergarten gift because of Pandita Ramabai Sarawati, an Indian activist for the emancipation of women and pioneer in education. During her time in the USA from 1886 to 1889, she became a follower of Fröbel's method of education (see [Ramabai, 2003, p. 145]).

[94] "In diesem Kapitel beabsichtigen wir, einige Proben aus einem von dem indischen Mathematiker Sundara Row 1893 herausgegebenen Buche zu geben, in dem gezeigt wird, wie man zu verfahren hat, wenn man die geometrische Konstruktion lediglich durch Falten von Papier ausführen will [...]"

from the fourth and fifth supplements of the *'Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen.'* [Hilbert, 1904, Hilbert, 1906] of David Hilbert. This will not provide a complete overview of the lifelong friendship between Hilbert and Hurwitz or their extensive interdisciplinary exchange of mathematical ideas. However, since most of the documents stored in Zurich have not been published yet, their perception should nevertheless extend the well known facts of their relationship. We will concentrate on the teacher-student-aspect and tackle the following question. Who was influenced by whom at which time?

The Beginning of a Friendship.

In 1884, when Hurwitz received his first full professorship in Königsberg, he and David Hilbert met for the first time. The younger wrote later: "His friendly and open nature won him, when he came to Königsberg, quickly the hearts of all who got to know him there [...]"[95] [Hilbert, 1921, p. 167]. Hilbert - born, grown up and studying in Königberg[96] - was an extraordinarily inquisitive young mathematician, who craved for progressive mathematical knowledge. For him and his two-years younger friend Hermann Minkowski, who was said to be an exceptional talent, it was definitely a very fortunate coincidence that Adolf Hurwitz became their teacher. Since Hurwitz was not only familiar with the mathematical school created by Alfred Clebsch (1833 - 1872) and Felix Klein, but also had learned in Berlin from Leopold Kronecker and especially Karl Weierstrass (see Section 2.1), the contribution of his knowledge was enormous[97]. In fact, Hilbert himself mentioned in his obituary about Adolf Hurwitz [Hilbert, 1921, p. 162],

> "Here I was, at that time still a student, soon asked for scientific exchange and had the luck by being together with him to get to know in the easiest and most interesting way the directions of thinking of the at time opposite however

[95] "Sein freundliches und offenes Wesen gewann ihm, als er nach Königsberg kam, rasch die Herzen aller, die ihn dort kennenlernten [...]"

[96] with the exception of one year, 1881, at the University of Heidelberg (see [Reid, 1970])

[97] in particular concerning both faces of complex analysis [Blumenthal, 1932, p. 390]

> each other excellently complementing schools, the geometrical school of Klein and the algebraic-analytical school of Berlin. [...] On numerous, sometimes day by day undertaken walks at the time for eight years we have browsed through probably all corners of mathematical knowledge, and Hurwitz with his as well wide and multifaceted as also established and well-ordered knowledge was always our leader."[98]

Furthermore, in his colourful biography of David Hilbert, Otto Blumenthal[99] (1876 - 1944) quoted, "We, Minkowski and me, were quite overwhelmed with his knowledge and did not believe that we could ever get that far."[100] [Blumenthal, 1932, p. 390]. However, Adolf Hurwitz also cared a lot about his students. "In the lessons he always took great care by interesting exercises to motivate for participation, and it was characteristic, how often one could find him in his thoughts searching for appropriate exercises and problems."[101][Hilbert, 1921, p. 166], remembered the former student Hilbert as well as, "Inspirations were given by the mathematical Colloquium [...] in particular, however, by the walks with Hurwitz "in the afternoon precisely at 5 o'clock next to the apple tree" "[102][Blumenthal, 1932, p. 393]. This tradition of mathematical group walks with students had been continued by Hilbert for all of his academic life. We can conclude that in the beginning it was naturally Hilbert who benefited a lot from his teacher Hurwitz. In 1892, he received his first professorship as successor of Hurwitz in Königsberg.

[98] "Hier wurde ich, damals noch Student, bald von Hurwitz zu wissenschaftlichem Verkehr herangezogen und hatte das Glück, durch das Zusammensein mit ihm in der mühelosesten und interessantesten Art die Gedankenrichtungen der beiden sich damals gegenüberstehenden und doch einander so vortrefflich ergänzenden Schulen, der geometrischen Schule von Klein und der algebraisch-analytischen Berliner Schule kennenzulernen. [...] Auf zahllosen, zeitweise Tag für Tag unternommenen Spaziergängen haben wir damals während acht Jahren wohl alle Winkel mathematischen Wissens durchstöbert, und Hurwitz mit seinen ebenso ausgedehnten und vielseitigen wie festbegründeten und wohlgeordneten Kenntnissen war uns dabei immer der Führer."

[99] Hilbert's first doctoral student (in 1898)

[100] "Wir, Minkowski und ich, waren ganz erschlagen von seinem Wissen und glaubten nicht, dass wir es jemals so weit bringen würden."

[101] "In den Übungen war er ständig darauf bedacht, durch anregende Aufgaben zur Mitarbeit heranzuziehen, und es war charakteristisch, wie oft man ihn in seinen Gedanken auf der Suche nach geeigneten Aufgaben und Problemstellungen für Schüler antraf."

[102] "Anregungen vermittelten das Mathematische Kolloquium [...], vor allem aber die Spaziergänge mit Hurwitz "nachmittags präzise 5 Uhr nach dem Apfelbaum" "

Two Productive Universal Mathematicians

Both mathematicians, Adolf Hurwitz as well as David Hilbert, belonged to a dying species in their profession: They can be considered as universal mathematicians having comprehensive knowledge and scientific results in various mathematical disciplines.[103] Furthermore, both were extremely productive. We can get an impression of their work by noticing that the *'Gesammelten Abhandlungen I - III'* of Hilbert [Hilbert, 1935] consist of more than 1350 pages of mostly influential mathematics similar to the *'Mathematisches Werk I + II'* of Adolf Hurwitz [Hurwitz, 1932] having more than 1400 pages.

Without claiming to be exhaustive, we want to give a strongly shortened overview of the mathematical work of David Hilbert following the biographical essay 'Lebensgeschichte' [Blumenthal, 1932] written by his former doctoral student. The subsequent list sketches Hilbert's wide scientific spectrum concerning all main mathematical disciplines, ordered by modern terms, highlighting some results, publications or speeches, which we want to analyze in respect to their connection to Hurwitz's work in the following subsection.

- 1885 - 1892 **Algebra**: Theory of Invariants
- 1890 *'Ueber die Theorie der algebraischen Formen'* [Hilbert, 1890]
- 1892 *'Ueber die Irrationalität ganzer rationaler Funktionen mit ganzzahligen Koeffizienten'* [Hilbert, 1935, vl. II, No. 18] with 'Irreduzibilitätssatz' [Blumenthal, 1932, p. 393]
- 1892 - 1899 **Number Theory**: Theory of Number Fields
- 1893 Simplification of the Hermite-Lindemann proof of the transcendence of e and π [Hilbert, 1935, vl. I, No. 1]

[103] In view of the growth of the mathematical community and its insights around the turn to the twentieth century, Hilbert and Henri Poincaré are said to be the last knowing almost everything about the whole developments in mathematics.

- 1894 *'Zwei neue Beweise für die Zerlegbarkeit der Zahlen eines Körpers in Primideale'* [Hilbert, 1935, vl. I, No. 2]
- 1896 *'Die Theorie der algebraischen Zahlkörper'* [Hilbert, 1935, vl. I, No. 7] , also called 'Zahlbericht' including ideal theory
- 1891 - 1902 **Geometry**: Axiomatization of Geometry
- 1895 - 1903 *'Grundlagen der Geometrie'* including complements [Hilbert, 1930]
- 1895 *'Über die gerade Linie als kürzeste Verbindung zweier Punkte'* [Hilbert, 1930, compl. I]
- 1900 *'Über den Zahlbegriff'* [Hilbert, 1900b]: Axiomatization of Arithmetic
- 1900 Hilbert stated his 23 Mathematical Problems at the International Congress of Mathematicians in Paris [Hilbert, 1900a]
- 1902 - 1910 **Complex Analysis**: variation problems, Independence Theorem
- 1904 - 1910 **Linear Algebra, Functional Analysis**: *'Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen'* with supplements [Hilbert, 1904, Hilbert, 1906] including new terminology
- 1907 (published 1910) **Analysis meets Geometry**: Analytical refounding of Minkowski's theory of volumes and surfaces of convex bodies in [Hilbert, 1910]
- 1907 **Analysis meets Number Theory**: *'Beweis der Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl nter Potenzen (Waringsches Problem).'* [Hilbert, 1909]
- 1902 - 1918 Axiomatization of Physics and Mechanics: Theory of Relativity
- 1904 - 1934 **Mathematical foundation**

- 1904 A first talk in Heidelberg about *'Axiomatisierung der Zahlenlehre'* [Blumenthal, 1932, p. 421]
- 1922 - 1934 **Hilbert Program**: Formalism and Proof Theory
- 1931 *'Die Grundlegung der elementaren Zahlenlehre.'* [Hilbert, 1931]

Although David Hilbert and Adolf Hurwitz are very similar with respect to their extraordinary productivity, their different characters can be recognized in their academic behavior. Hilbert was noticed as extroverted and "became used to be a famous man"[104] [Blumenthal, 1932, p. 407], whereas Hurwitz "avoided any personal being apparent in academic and public life"[105] [Hilbert, 1921, p. 167] and preferred to work contiuously, however, silently. This is particularly visible in his consequent, nearly peerless way of taking notes of mathematical ideas in his diaries. In the following we take a glance at those diaries in view of parallels to the above listed fields of research of David Hilbert.

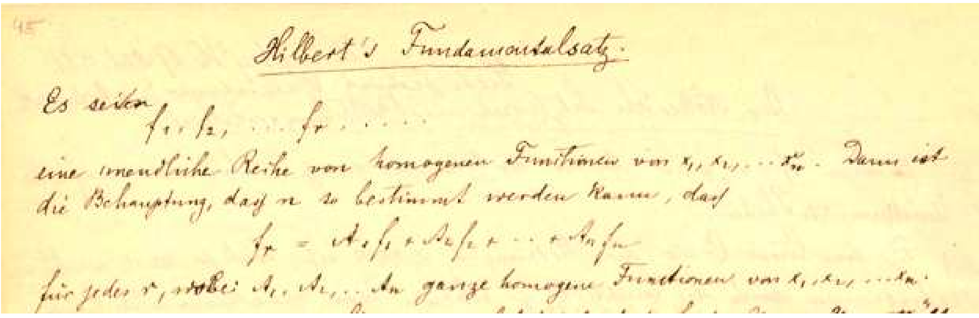
Mathematical Exchange.

"Already here, anticipating, it shall be reported about the seldom harmonic and fruitful cooperation of those three mathematicians."[106] [Blumenthal, 1932, p. 390], noted Otto Blumenthal and refered to the active mathematical exchange between Minkowski, Hilbert and Hurwitz. In the mathematical diaries [Hurwitz, 1919a] various entries, directly or indirectly related to publications of Hilbert, can be found - those are listed in Appendix 9.2. Furthermore, some ideas in selected diaries suggest to be inspired by Hilbert.

We already pointed out, that without doubt, in their first years it was essentially David Hilbert who benefited from his teacher. However, his teacher Hurwitz became very soon aware of his talented student. In diary No. 6[107] [Hurwitz, 1919a, No. 6] on page 44 is a

[104] "gewöhnte sich daran, ein berühmter Mann zu sein"
[105] "mied jedes persönliche Hervortreten im akademischen und öffentlichen Leben"
[106] "Es soll schon hier vorgreifend über das selten harmonische und fruchtbare Zusammenarbeiten dieser drei Mathematiker berichtet werden."
[107] from 1888 IV. to 1889 XI.

first entry related to Hilbert, entitled "On Noether's Theorem (concerning a message of Hilbert)"[108]. Here Hurwitz familarized himself with the nowadays called Residual Intersection Theorem, sometimes also Fundamental Theorem, of Max Noether (1844 - 1921) dealing with a linear form associated with two algebraic curves. Interestingly one page later follows the entry "Hilbert's Fundamental Theorem"[109] dealing with a linear form of homogeneous functions.


45

Hilbert's Fundamentalsatz.

Es seien $f_1, f_2, \ldots f_r \ldots$

eine unendliche Reihe von homogenen Functionen von $x_1, x_2, \ldots x_n$. Dann ist die Behauptung, daß n so bestimmt werden kann, daß

$$f_r = A_1 f_1 + A_2 f_2 + \cdots + A_n f_n$$

für jedes r, wobei $A_1, A_2, \ldots A_n$ ganze homogene Functionen von $x_1, x_2, \ldots x_n$.

Figure 2.25: "Hilbert's Fundamental Theorem. Let $f_1, f_2, \ldots, f_r, \ldots$ be an infinite series of homogeneous functions of $x_1, x_2, \ldots x_n$. We claim that n can be determined in such a way that

$$f_r = A_1 f_1 + A_2 f_2 + \cdots A_n f_n$$

for any r, where $A_1, A_2, \ldots, A_n$ are entire homogeneous functions of $x_1, x_2, \ldots, x_n$."[110]

On the one hand, this can be considered as continuation and extension of Noether's theorem, on the other hand this is a previous version of Hilbert's Basic Theorem. Although Hurwitz was obviously interested perhaps even inspired from Hilbert's Form and Invariant Theory, in his work *'Über die Erzeugung der Invarianten durch Integration'*[111] Hurwitz discovered a "new generating principle for algebraic invariants which allowed him to apply

[108] "Der Nöther'sche Satz (nach einer Mitteilung von Hilbert)"

[109] "Hilberts Fundamentalsatz"

[110] "Hilbert's Fundamentalsatz. Es seien $f_1, f_2, \ldots, f_r, \ldots$ eine unendliche Reihe von homogenen Funktionen von $x_1, x_2, \ldots x_n$. Dann ist die Behauptung, daß n so bestimmt werden kann, daß $f_r = A_1 f_1 + A_2 f_2 + \cdots A_n f_n$ für jedes r, wobei $A_1, A_2, \ldots, A_n$ ganze homogene Functionen von $x_1, x_2, \ldots, x_n$."

[111] 'On generating invariants by integration.'

an [by Hilbert] introduced method [...] ."[112] [Hilbert, 1921, p. 164]. This nice formulation comes from Hilbert himself and can be interpreted that Hurwitz kind of improved the use of one of Hilbert's methods. At least three more entries in the mathematical diaries (see Appendix II: No. 8, p. 207; No. 14, p. 204; No. 25, p. 77) are directly dedicated to Hilbert's "Formensatz", nowadays called Basic Theorem. In the entry of the fourteenth diary[113] [Hurwitz, 1919a, No. 14], Hurwitz's comments on Hilbert's theorems have the tendency to sound like amendments. In the beginning, he wrote

Figure 2.26: "The proof of Hilbert's Theorem (Ann 36. p. 485) seems to be the most easiest understandable in such a way: [...]"[114]

Here Hurwitz sounds as if Hilbert's ideas respectively the way Hilbert had put the proof to language could be simplified. Another example is on the next page:

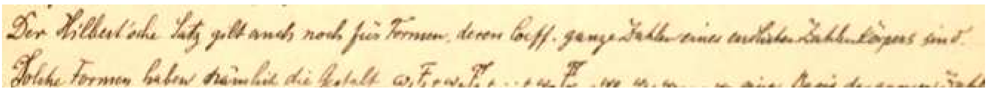


Figure 2.27: "Hilbert's Theorem holds also for forms whose coeff. are integers of a finite number field."[115]

This generalization of the theorem of Hilbert is remarkable, because it obviously paved the way for one of the later known versions of Hilbert's Basic Theorem: *The ring of*

[112]"neues Erzeugungsprinzip für algebraische Invarianten, das es ihm ermöglicht, ein [von Hilbert] eingeschlagenes Verfahren [...] anzuwenden."
[113]from 1896 I.1. to 1897 II.1.
[114]"Der Beweis des Hilbert'schen Theorems II (Ann 36. p. 485) ist wohl am leichtesten so aufzufassen: [...]"
[115]"Der Hilbert'sche Satz gilt auch noch für Formen, deren Coeff. ganze Zahlen eines endlichen Zahlkörpers sind."

polynomials $K[X_1, \ldots, X_n]$ *over a field* K *is Noetherian.*[116]

In 1891, Hurwitz and Hilbert published a first joint note [Hilbert, 1935, vl. II, Nr. 17] in which they used "that a certain, a number of parameters including irreducible ternary form still is irreducible for general integral values of those parameters."[117] [Blumenthal, 1932, p. 393] Otto Blumenthal considered this as foundation of Hilbert's famous 'Irreduzibilitätssatz' [Hilbert, 1935, vl. II, Nr. 18] from 1892. Apparently, there was still a great influence from Hurwitz on Hilbert.

In his first years being *Privatdozent*[118] in Königsberg, when Hilbert adressed himself to the theory of number fields, he reported from his and Adolf Hurwitz's common walks, discussing theories of Dedekind and Kronecker. "One considered Kronecker's proof for a unique decomposition of prime ideals, the other the one of Dedekind, and we thought both were awful."[119][Blumenthal, 1932, p. 397]. However, their cooperation turned out to be successfull: Firstly, Hilbert began a publication from 1894 building on his talk *'Zwei neue Beweise für die Zerlegbarkeit der Zahlen eines Körpers in Primideale'*[120] from 1893. Secondly, Hurwitz, also working on algebraic number fields, published another proof one year later in his paper '*Der Euklidische Divisionssatz in einem endlichen algebraischen Zahlkörper*'[121]. Hilbert considered this work as "remarkable in view of the analogy with the Euclidean Algorithm in number theory"[122] [Hilbert, 1921, p. 165] and even preferred Hurwitz's proof in his famous *'Zahlbericht'*[123]. It seems that even after Hurwitz's moving to Zurich respectively Hilbert's full professorship in Königsberg starting from 1892, Hilbert was still slightly influenced by his former teacher. We investigate this also on behalf of

[116] We may assume that Hurwitz had meant 'algebraic' instead of 'finite'.

[117] "daß eine gewisse, eine Anzahl Parameter enthaltende irreduzible ternäre Form auch für allgemeine ganzzahlige Werte dieser Parameter irreduzibel bleibt."

[118] we explained this old German term in Section 2.1

[119] "Einer nahm den Kroneckersches Beweis für die eindeutige Zerlegung in Primideale vor, der andere den Dedekindschen, und beide fanden wir scheußlich."

[120] *'Two new proofs of the decomposability of numbers of a number field in prime ideals'*, spoken in September 1893 at the meeting of the 'Deutsche Mathematiker-Vereinigung in Munich

[121] *'The Euclidean Division Theorem in a finite algebraic number field.'*

[122] "bemerkenswert durch die Analogie mit dem Euklidischen Algorithmus in der elementaren Zahlentheorie"

[123] actually *'Die Theorie der algebraischen Zahlkörper'*, report of algebraic number theory

two entries from 1898 and 1899 in Hurwitz's diaries No. 15[124] [Hurwitz, 1919a, No. 15] and No. 16[125] [Hurwitz, 1919a, No. 16] directly refering to Hilbert's *'Zahlbericht'*. The first is listed as "Concerning Hilbert's "Report on Number Fields" "[126].

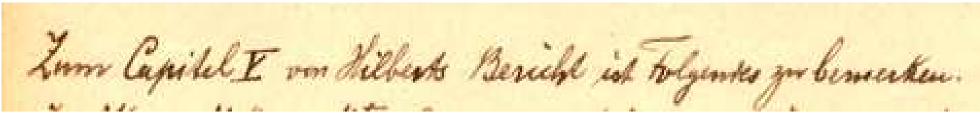

Zum Capitel V von Hilberts Bericht ist Folgendes zu bemerken.

Figure 2.28: "Concerning Chapter V of Hilbert's report the following is to remark."[127]

The subsequent entry gives the impression that Hurwitz continued Hilbert's work using a result of Hilbert about composing ideals of subfields of number fields. Hereafter, number fields will be defined by K respectively k_i and ideals by ν respectively ν_i, $i = 1, 2, 12$.

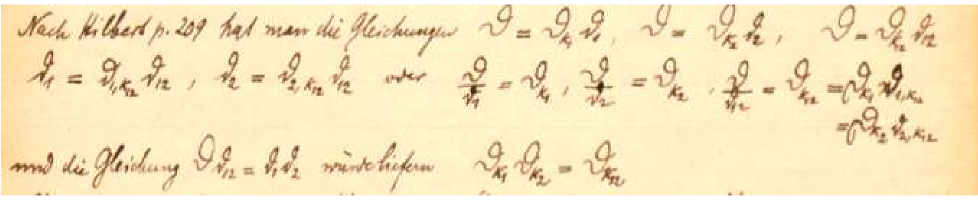

Nach Hilbert p. 209 hat man die Gleichungen $\vartheta = \vartheta_{k_1}\vartheta_1$, $\vartheta = \vartheta_{k_2}\vartheta_2$, $\vartheta = \vartheta_{k_{12}}\vartheta_{12}$
$\vartheta_1 = \vartheta_{1,k_2}\vartheta_{12}$, $\vartheta_2 = \vartheta_{2,k_{12}}\vartheta_{12}$ oder $\frac{\vartheta}{\vartheta_1} = \vartheta_{k_1}$, $\frac{\vartheta}{\vartheta_2} = \vartheta_{k_2}$, $\frac{\vartheta}{\vartheta_{12}} = \vartheta_{k_{12}} = \vartheta_{k_1}\vartheta_{1,k_2} = \vartheta_{k_2}\vartheta_{2,k_{12}}$
und die Gleichung $\vartheta\vartheta_{12} = \vartheta_1\vartheta_2$ würde liefern $\vartheta_{k_1}\vartheta_{k_2} = \vartheta_{k_{12}}$

Figure 2.29: "According to Hilbert p. 209 we have [...] and the equation $\nu\nu_{12} = \nu_1\nu_2$ would lead to $\nu_{k_1}\nu_{k_2} = \nu_{k_{12}}$"[128]

Then Hurwitz first verified this consequence $\nu_{k_1}\nu_{k_2} = \nu_{k_{12}}$ of Hilbert's formula, before he compared it with his conjecture.

124 from 1897 II.1. to 1898 III.19.
125 from 1898 III.20. to 1899 II.23.
126 "Zu Hilbert's "Körperbericht" "
127 "Zum Capitel V von Hilberts Bericht ist Folgendes zu bemerken."
128 "Nach Hilbert p. 209 hat man [...] und die Gleichung $\nu\nu_{12} = \nu_1\nu_2$ würde liefern $\nu_{k_1}\nu_{k_2} = \nu_{k_{12}}$"

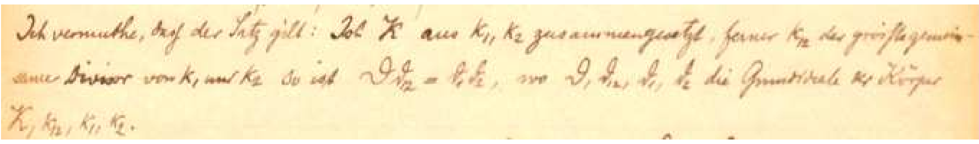

Ich vermuthe, daß der Satz gilt: Ist K aus k_1, k_2 zusammengesetzt, ferner k_{12} der größte gemeinsame Divisor von k_1 und k_2 so ist $\vartheta\vartheta_{k_{12}} = \vartheta_{k_1}\vartheta_{k_2}$, wo $\vartheta, \vartheta_{12}, \vartheta_1, \vartheta_2$ die Grundideale der Körper K, k_{12}, k_1, k_2.

Figure 2.30: "I assume that the theorem holds: Is K a composition of k_1, k_2, moreover k_{12} the greatest common divisor of k_1 and k_2 then $\nu\nu_{k_{12}} = \nu_{k_1}\nu_{k_2}$, where $\nu, \nu_{12}, \nu_1, \nu_2$ are Grundideale of the number fields K, k_{12}, k_1, k_2."[129]

Consequently, Hurwitz deduced a general new result on algebraic number fields.

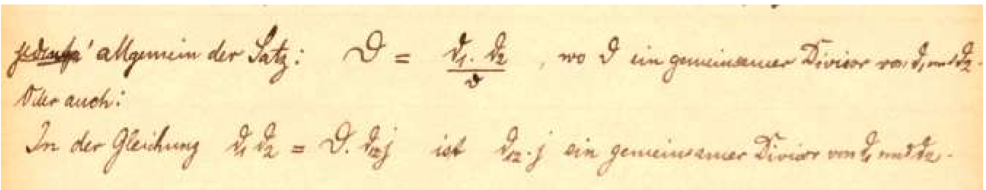
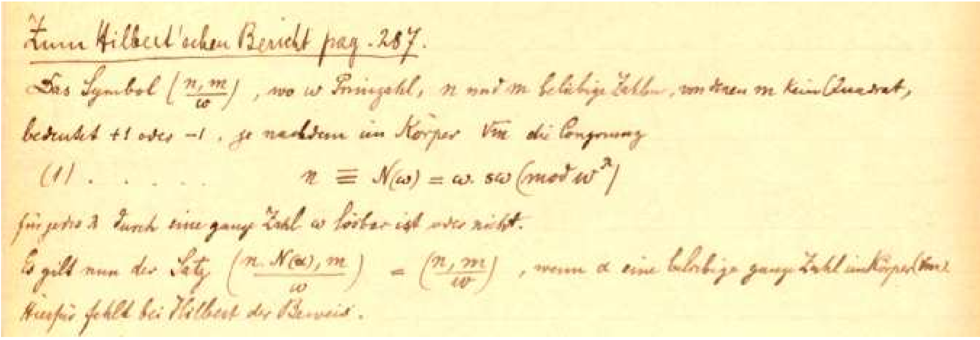

allgemein der Satz: $\vartheta = \frac{\vartheta_1 \cdot \vartheta_2}{\vartheta}$, wo ϑ ein gemeinsamer Divisor von ϑ_1 und ϑ_2.
Oder auch:
In der Gleichung $\vartheta_1\vartheta_2 = \vartheta \cdot \vartheta_{12} j$ ist $\vartheta_{12} \cdot j$ ein gemeinsamer Divisor von ϑ_1 und ϑ_2.

Figure 2.31: "[...] consequently the general theorem holds: $\nu = \frac{\nu_1 \cdot \nu_2}{\nu}$, with ν a common divisor of ν_1 und ν_2.[130]Or also: In the equation $\nu_1\nu_2 = \nu\nu_{12} \cdot j$ is $\nu_{12} \cdot j$ a common divisor of ν_1 and ν_2."

Here we may interpret that Hurwitz used the *'Zahlbericht'* as a textbook, however, the entry in No. 16 [Hurwitz, 1919a, No. 16] emphasizes that the teacher Hurwitz is still one step ahead.

[129] The use of the notation 'Grundideal' is here a bit confusing. According to [Renschuch, 1973] it goes back to a work of Emmy Noether from 1923. However, this comment is misleading. According to [Hilbert, 1935, vl. I, p. 90] the term 'Grundideal' was already used by Dedekind and Hilbert himself invented the new term 'Differente' which is still used today. Interestingly, Hurwitz refering to Hilbert kept Dedekind's notation.
"Ich vermute, daß der Satz gilt: Ist K aus k_1, k_2 zusammengesetzt, ferner k_{12} der größte gemeinsame Divisor von k_1 und k_2 so ist $\nu\nu_{k_{12}} = \nu_{k_1}\nu_{k_2}$, wo $\nu, \nu_{12}, \nu_1, \nu_2$ die Grundideale der Körper K, k_{12}, k_1, k_2."

[130] probably ν_{12} and ν were interchanged
"[...] folglich besteht allgemein der Satz: $\nu = \frac{\nu_1 \cdot \nu_2}{\nu}$, wo ν ein gemeinsamer Divisor von ν_1 und ν_2. Oder auch: In der Gleichung $\nu_1\nu_2 = \nu\nu_{12} \cdot j$ ist $\nu_{12} \cdot j$ ein gemeinsamer Divisor von ν_1 und ν_2."


Zum Hilbert'schen Bericht pag. 287.

Das Symbol $\left(\frac{n,m}{\omega}\right)$, wo ω Primzahl, n und m beliebige Zahlen, von denen m kein Quadrat, bedeutet $+1$ oder -1, je nachdem im Körper $k(\sqrt{m})$ die Congruenz

(1) . . . . . $n \equiv N(\omega) = \omega \cdot s\omega \pmod{\omega^\lambda}$

für jedes λ durch eine ganze Zahl ω lösbar ist oder nicht.

Es gilt nun der Satz $\left(\frac{n \cdot N(\alpha), m}{\omega}\right) = \left(\frac{n,m}{\omega}\right)$, wenn α eine beliebige ganze Zahl im Körper $k(\sqrt{m})$

Hierfür fehlt bei Hilbert der Beweis.

Figure 2.32: Under the heading "Concerning Hilbert's report pag. 287" the symbol $\left(\frac{n,m}{\omega}\right)$, where ω is a prime number, n and m are abitrary numbers and m is not a square number is treated.[131]

It is defined to be equal to $+1$ or -1, according to the property whether in the field $\mathbb{Q}(\sqrt{m})$ the congruence

$$n \equiv N(\omega) = \omega \cdot s\omega (\mathrm{mod} \omega^\lambda)$$

has for any λ a solution for an integer number ω or not (here s and λ are not defined precisely)[132]. Hurwitz stated, "So we have the theorem

$$\left(\frac{n \cdot N(\alpha), m}{\omega}\right) = \left(\frac{n,m}{\omega}\right),$$

if α is an arbitrary entire number in the number field $(\sqrt{m})$."[133][134] and continued "Therefore Hilbert lacks a proof."[135] Indeed, we can find the stated equation some pages later in Hilbert's work without a sound verification:

[131] "Zum Hilbert'schen Bericht pag. 287"

[132] Hilbert characterized with his sympol so-called 'Normenreste' respectively 'Normennichtreste' [Hilbert, 1935, vl. I, p. 164] of a number field. Today the symbol is known as 'Hilbert symbol'.

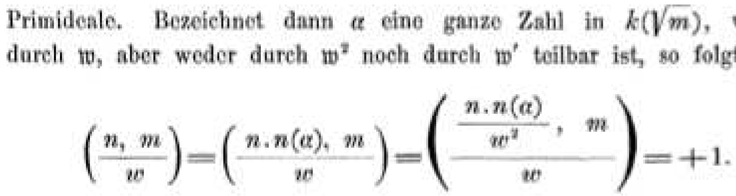
[133] "Es gilt nun der Satz

$$\left(\frac{n \cdot N(\alpha), m}{\omega}\right) = \left(\frac{n,m}{\omega}\right),$$

wenn α eine beliebige ganze Zahl im Körper $(\sqrt{m})$."

[134] We may assume that an algebraic number field $K(\sqrt{m})$ was meant here.

[135] "Hierfür fehlt Hilbert der Beweis."


Primideale. Bezeichnet dann α eine ganze Zahl in $k(\sqrt{m})$, welche durch $\mathfrak{w}$, aber weder durch $\mathfrak{w}^2$ noch durch $\mathfrak{w}'$ teilbar ist, so folgt:

$$\left(\frac{n,\ m}{w}\right)=\left(\frac{n.n(\alpha),\ m}{w}\right)=\left(\frac{\frac{n.n(\alpha)}{w^2},\ m}{w}\right)=+1.$$

Figure 2.33: Excerpt of Hilbert's *'Zahlbericht'*, page 289 respectively [Hilbert, 1935, vl. I, p. 164].

Within one page Hurwitz filled Hilbert's gap proving this equation by use of a clever case distinction.

Figure 2.34: "Thus $nn \cdot sn \equiv xn \cdot sx \cdot sn(\omega^{\lambda-1})$ and $n \equiv x \cdot sx(\omega^{\lambda-2})$ q.e.d."[136]

Obviously, Hurwitz worked not only with Hilbert's *'Zahlbericht'*, however, he was furthermore able to obtain some improvements.

Another diary entry has to be mentioned because of completeness, any conclusions are rather speculative: In No. 23 [Hurwitz, 1919a, No. 23] from 1908 Hurwitz dealed with a short exercise entitled *'Über die kürzeste Linie auf einem Parallelepiped'* (which is explained in detail in Section 2.2.1). Remarkably one of Hilbert's papers from 1895 has the very similar title *'Über die gerade Linie als kürzeste Verbindung zweier Punkte'*. However, since these two topics were examined with a distance of 13 years, we will not take this coincidence into consideration.

Instead, we go on chronologically and take a look at diary No. 19[137] [Hurwitz, 1919a, No. 19] from 1902.

[136] "Also wird $nn \cdot sn \equiv xn \cdot sx \cdot sn(\omega^{\lambda-1})$ und $n \equiv x \cdot sx(\omega^{\lambda-2})$ q.e.d."
[137] from 1901 XI.1. to 1904 III.16.

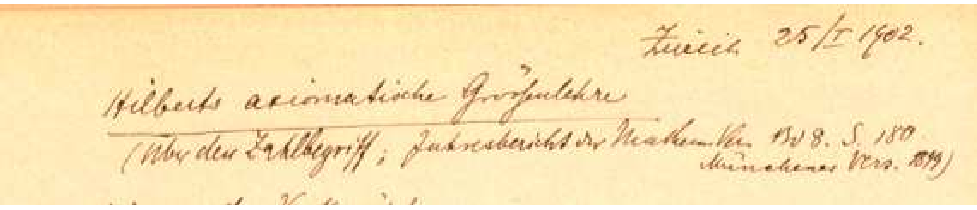
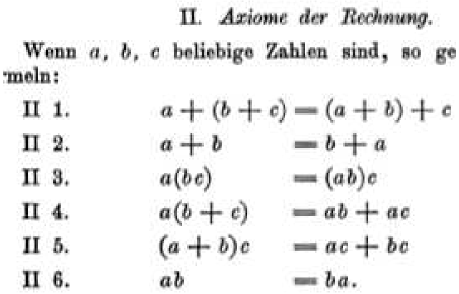


Figure 2.35: The entry is entitled "Hilbert's axiomatic Theory of Quantities"[138], where Hurwitz refered to Hilbert's talk *'Über den Zahlbegriff'* [Hilbert, 1900b].

After his axiomatization of geometry, Hilbert continued working on developing an axiomatic system for arithmetic. He was demanding for complete freedom from contradictions in mathematics. Therewith, Hilbert became one of the founders of a new philosophical movement in mathematics, the complete establishment of mathematics on an axiomatic system, being the first advocate of the so-called Formalism.[139] His above mentioned talk and subsequent publication [Hilbert, 1900b] can be considered as milestone. What makes Hurwitz's entry so interesting, is that it contains nothing but a nearly exact copy of Hilbert's ideas.


II. *Axiome der Rechnung.*

Wenn a, b, c beliebige Zahlen sind, so ge
·meln:

II 1.	$a+(b+c)$	$=(a+b)+c$
II 2.	$a+b$	$=b+a$
II 3.	$a(bc)$	$=(ab)c$
II 4.	$a(b+c)$	$=ab+ac$
II 5.	$(a+b)c$	$=ac+bc$
II 6.	ab	$=ba.$

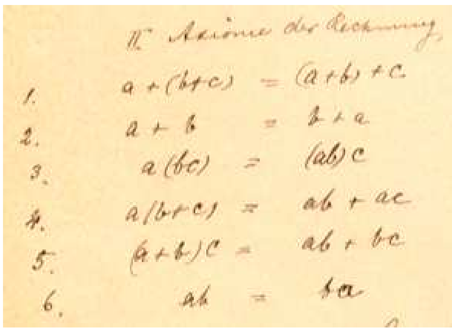


Figure 2.36: Excerpts of Hilbert's publication [Hilbert, 1900b] and Hurwitz's diary entry, "II. Axioms of Calculation. [...]".

[138]"Hilbert's axiomatische Größenlehre"

[139]Due to inspirations of various mathematicians his attitude was in a steadily evolvement. For more information we refer to [Tapp, 2013].

Here Hurwitz followed step by step, axiom by axiom, Hilbert's rules for operations, calculations, order and continuity and even his consequences,


Einige Bemerkungen über die Abhängigkeit der Axiome hat Hilbert hinzugefügt:

Figure 2.37: "Some remarks on the dependence of axioms were added by Hilbert:"[140]

as well as Hilbert's new terminology,

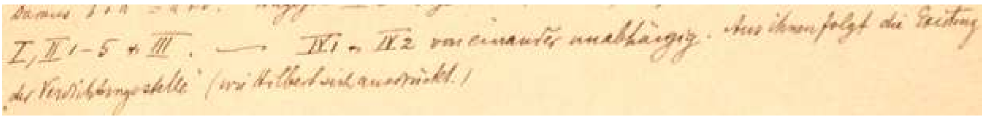

I, II 1-5 u. III. → IV1 u. IV2 von einander unabhängig. Aus ihnen folgt die Existenz der Verdichtungsstelle" (wie Hilbert sich ausdrückt.)

Figure 2.38: "From them the existence of a "Verdichtungsstelle" follows (as Hilbert expresses himself.)".[141]

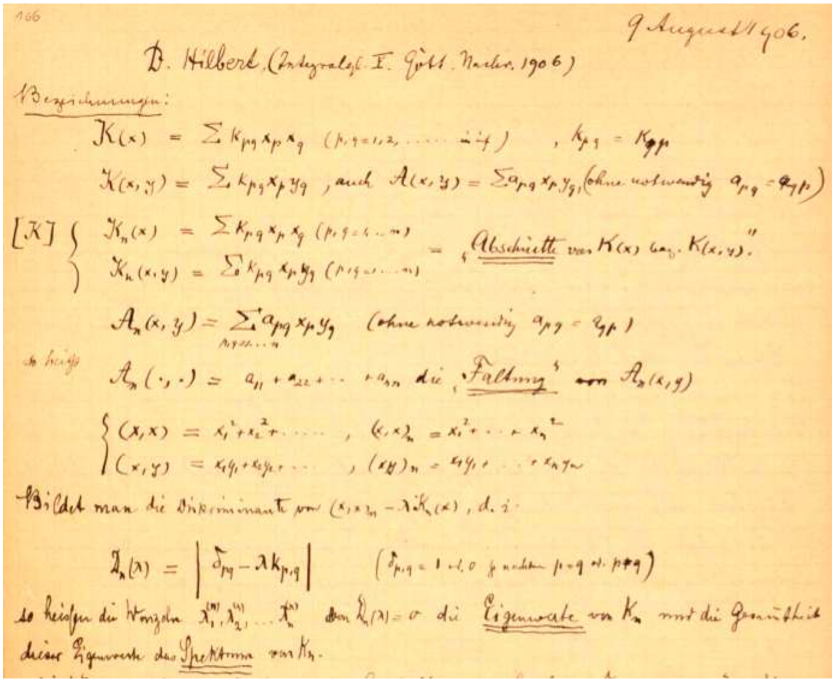
It seems that this concept was completely new for Hurwitz and, furthermore, that he was willing to understand Hilbert's axiomatization. Here we get a first idea that the status of their relation was about to change. In any case, David Hilbert and Adolf Hurwitz are on an equal footing at the turn of the century.

The year 1900 was also the year when Hilbert introduced his famous 23 problems on the International Conference of Mathematicians in Paris [Hilbert, 1900a]. In the previous section we already pointed out that at least one entry in diary No. 22 [Hurwitz, 1919a, No. 22] concerning the proof of the Theorem of Pythagoras could have been inspired by the third problem.

More obvious, however, is Hilbert's influence on an entry in diary No. 21[142] [Hurwitz, 1919a, No. 21] from August 09, 1906, which is entitled "D. Hilbert (Integralequ.

[140] "Einige Bemerkungen über die Abhängigkeit der Axiome hat Hilbert hinzugefügt:"
[141] "Aus ihnen folgt die Existenz der "Verdichtungsstelle" (wie Hilbert sich ausdrückt.)"
[142] from 1906 II.1. to 1906 XII.8.

V. Gött, Nachr. 1906)"[143] and refers to the fifth supplement of Hilbert's article *'Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen'* [Hilbert, 1906] from 1906. In this work Hilbert defined a new terminology and presented an innovative concept of handling linear algebra problems by applying integral equations. His method is based on the symmetry of the coefficients which is equivalent to the symmetry of the kernel[144] of the integral equation (see [Blumenthal, 1932, p. 411]). The fourth [Hilbert, 1904] and fifth supplement, to which Hurwitz refered, extend Hilbert's previous results on bilinear forms with infinitely many variables. The diary entry begins with the section "Bezeichnungen", which means 'notations'. One after the other Hurwitz reproduced Hilbert's definitions of the terms "Abschnitte" and "Faltung" as well as "Eigenwerte", "Spektrum" and "Resolvente".[145]


166 9 August 1906.

D. Hilbert (Integralgl. I. Gött. Nachr. 1906)

1 Bezeichnungen:

$K(x) = \sum k_{pq} x_p x_q \quad (p,q = 1,2,\dots\infty), \quad k_{pq} = k_{qp}$

$K(x,y) = \sum k_{pq} x_p y_q$, auch $A(x,y) = \sum a_{pq} x_p y_q$ (ohne notwendig $a_{pq} = a_{qp}$)

[K] $K_n(x) = \sum k_{pq} x_p x_q \; (p,q = 1 \dots n)$, $K_n(x,y) = \sum k_{pq} x_p y_q \; (p,q = 1 \dots n)$ = „Abschnitte" von $K(x)$ bez. $K(x,y)$."

$A_n(x,y) = \sum_{p,q=1\dots n} a_{pq} x_p y_q$ (ohne notwendig $a_{pq} = a_{qp}$)

so heisst $A_n(\cdot,\cdot) = a_{11} + a_{22} + \dots + a_{nn}$ die „Faltung" von $A_n(x,y)$

$(x,x) = x_1^2 + x_2^2 + \dots, \quad (x,x)_n = x_1^2 + \dots + x_n^2$

$(x,y) = x_1y_1 + x_2y_2 + \dots, \quad (xy)_n = x_1y_1 + \dots + x_ny_n$

Bildet man die Diskriminante von $(x,x)_n - \lambda K_n(x)$, d. i.

$\Delta_n(\lambda) = |\delta_{pq} - \lambda k_{pq}|$ ($\delta_{pq} = 1$ od. 0 je nachdem $p=q$ od. $p \neq q$)

so heissen die Wurzeln $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots \lambda_n^{(n)}$ von $\Delta_n(\lambda) = 0$ die Eigenwerte von K_n und die Gesamtheit dieser Eigenwerte das Spektrum von K_n.

Figure 2.39: Excerpt of Hurwitz's diary entry. The important terms are double underlined.

143"D. Hilbert (Integralgl. V. Gött. Nachr. 1906)"

144Hilbert's term 'Kern' became internationally used, in English it was transformed to 'kernel'.

145"segments", "convolution", "eigenvalue", "spectrum", "resolvent"

In fact, those terms were introduced by Hilbert himself. On page 459 of the fifth supplement he wrote: "Values are significantly determined by the kernel(s,t); I named them *Eigenwerte* resp. *Eigenfunktionen* [...]."[146] It seems that Hurwitz was not yet familiar with Hilbert's concept. One indication is the special highlighting of all those terms by double underlining, another indication is the placing of certain terms in quotation marks on the second page, where Hurwitz became acquainted with resulting equations.

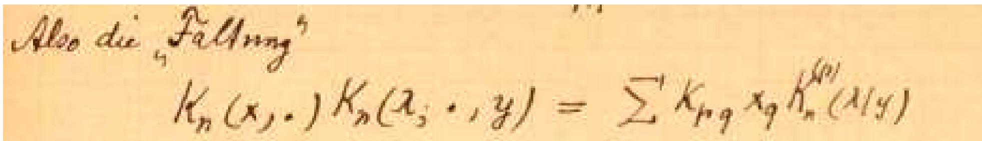

Also die „Faltung"
$K_n(x,\cdot)K_n(\lambda_j\cdot,y) = \sum K_{pq}x_qK_n^{(p)}(\lambda/y)$

Figure 2.40: "Thus the "convolution" $K_n(x,\cdot)K_p(\lambda_j\cdot,y) = \sum K_{pq}x_qK_n^{(p)}(\lambda/y)$" [147]

This shows the unfamiliar use of this term. Some days later, in an entry from September 02, 1906, Hurwitz continued his analysis. He wrote,

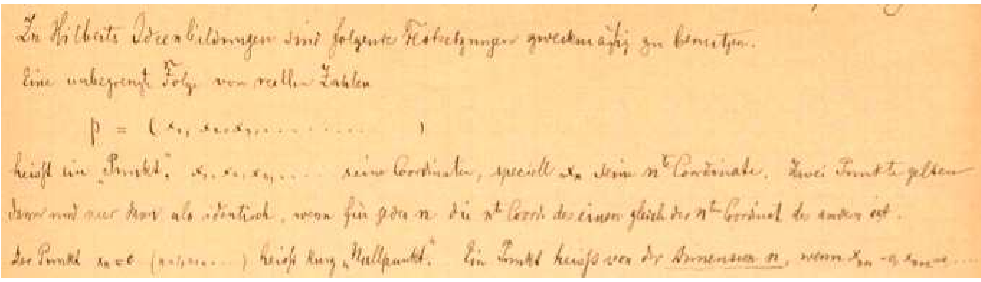

In Hilberts Ideenbildungen sind folgende Fortsetzungen zweckmäßig zu benutzen.
Eine unbegrenzte Folge von reellen Zahlen

Figure 2.41: "In Hilbert's concept the following continuations are expediently to be used [...]" [148]

Then Hurwitz defined again a variety of new terms. This is followed up through an entry from September 16, where Hurwitz stated a theorem on quadratic forms.

[146] "Die Werte [...] sind wesentlich durch den Kern(s,t) bestimmt; ich habe sie *Eigenwerte* bez. *Eigenfunktionen* [...] genannt."

[147] "Also die "Faltung" $K_n(x,\cdot)K_p(\lambda_j\cdot,y) = \sum K_{pq}x_qK_n^{(p)}(\lambda/y)$"

[148] "In Hilberts Ideenbildung sind folgende Fortsetzungen zweckmäßig zu benutzen [...]"

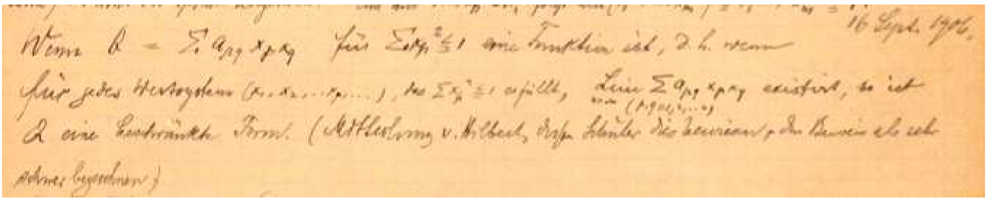


Figure 2.42: "If $Q = \sum a_{pq} x_p x_q$ for $\sum x_p^2 \leq 1$ is a function, i.e. if for any system of values $(x_1, x_2, \cdots, x_p, \cdots)$, which satisfies $\sum x_p^2 \leq 1$, $Lim_{n=\infty(p,q=1,2,\dots n)} \sum a_{pq} x_p x_q$ exists, then Q is a limited form. (notification of Hilbert, that students proved this, the proof is to be considered as very difficult.)"[149]

Hurwitz examined the theorem as well as the proof and found some reformulations, noted on diary pages 170 and 171. Two pages later he stated a further theorem.

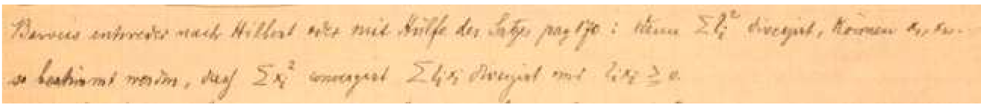


Figure 2.43: "Proof either according to Hilbert or with the theorem on pag 170 [...]" [150]

The next section continues with the consideration of linear and quadratic forms, which we do not want to deepen here. However, what we remark is that Hurwitz is not only dealing with Hilbert's ideas, he is even adopting a completely new theory from his colleague. Certainly, Hilbert was on the fast lane.

One year later this overtaking became even more apparent. In 1907, David Hilbert solved the task of developing an analytical refounding of Minkowski's theory of volumes and surfaces of convex bodies in his sixth supplement[151] [Hilbert, 1910]. This task had already

[149] "Wenn $Q = \sum a_{pq} x_p x_q$ für $\sum x_p^2 \leq 1$ eine Funktion ist, d.h. wenn für jedes Wertsystem $(x_1, x_2, \cdots, x_p, \cdots)$, das $\sum x_p^2 \leq 1$ erfüllt, $Lim_{n=\infty(p,q=1,2,\dots n)} \sum a_{pq} x_p x_q$ existiert, so ist Q eine beschränkte Form. (Mitteilung v. Hilbert, daß Schüler dies bewiesen, den Beweis als sehr schwer bezeichnen.)"

[150] "Beweis entweder nach Hilbert oder mit Hilfe des Satzes pag 170 [...]".

[151] which was published three years later in 1910

been tackled by Hurwitz in 1901 and 1902. An entry in diary No. 18[152] [Hurwitz, 1919a, No. 18] about his colloquium talk from January 21, 1901 is entitled:

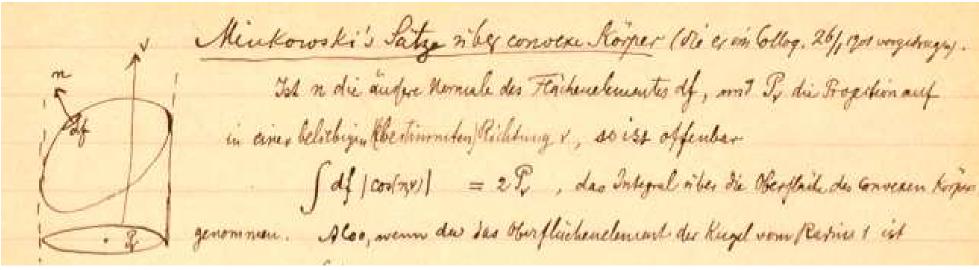

Minkowski's Sätze über convexe Körper

Figure 2.44: "Minkowski's theorems on convex bodies [...]". [153]

Here Hurwitz discussed several questions on convex bodies. However, four pages of calculations later, he stated,


Es bleibt fraglich, ob man hier zu einfachen Resultaten durchdringen kann.

Figure 2.45: "It remains doubtful if simple results can be discovered here."[154]

Obviously, he was not content with his considerations. One year later, Adolf Hurwitz published the article [Hurwitz, 1902a] in which he tried a first attempt of an appropriate refoundation "using his theory of spherical functions [...], however, he only had a partial success. Hilbert, with his powerful tool on integral equations, replaces the spherical function by more generalized ones and passes through."[155] [Blumenthal, 1932, p. 414][156].

While Hilbert developed as guidepost and unique mathematician, his scientific supe-

[152]from 1900 XII. to 1901 X.

[153]"Minkowski's Sätze über konvexe Körper [...]"

[154]"Es bleibt fraglich, ob man hier zu einfachen Resultaten durchdringen kann."

[155]"mit seiner Theorie der Kugelfunktionen [...], hatte aber nur einen Teilerfolg erzielt. Hilbert, im Besitze der mächtigen Hilfsmittel der Integralgleichungen, ersetzt die Kugelfunktionen durch allgemeinere, und kommt durch."

[156]Firstly, it is remarkable that Hurwitz, studying Hilbert's supplements, did not apply the integral equation method. Secondly, notice that Hilbert's dissertation thesis was about spherical functions. Interestingly, the thesis is dedicated to Hurwitz.

riority over Hurwitz became more and more clear. At the latest, the year 1907 can be considered as final turnaround of their teacher-student-relation. A therefore remarkable situaton is explained in detail by Otto Blumenthal: In a short note [Hurwitz, 1908] from November 20, 1907, Adolf Hurwitz proved a variation of the so-called Waring problem. This number theoretical question, named after the English mathematician Edward Waring (1736 - 1798) and published in his work *'Meditationes algebraicae'* from 1770, claimed that for every exponent $k \in \mathbb{N}$ a natural number n exists such that every natural number can be expressed as a sum of at least n many k-th powers. Hurwitz showed, " Is the nth power of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ equal to a sum of $2n$th powers of a linear rational form of x_1, x_2, x_3, x_4, and does the Waring Conjecture hold for n, it is also valid for $2n$."[157] [Blumenthal, 1932, p. 415] According to Otto Blumenthal, "This theorem gave Hilbert the inspiration and direction for his examinations. He found an unexpected way to state an identity of the by Hurwitz demanded kind for arbitrary n."[158] Moreover, Hilbert deduced "from a general principle which was used by Hurwitz in the theory of invariants in 1897, a formula [...]"[159] and he managed "to transfer the by the integration demanded taking the limits in the coefficients of the sum and finally, on behalf of another trick, to replace those coefficients by positive rationals. Therewith the foundation for the proof of Waring's theorem is laid."[160] [Blumenthal, 1932, p. 415] Finally, Hilbert solved Waring's Problem [Hilbert, 1909] and presented a wonderful example for his definite mathematical emancipation: "Because he fought together with a master of Hurwitz's high level and won with the weapons from Hurwitz's armor chamber on a point, when [Hurwitz] had no prospect of success."[161] [Blumenthal, 1932, p. 416]

[157] "Ist die n-te Potenz von $x_1^2 + x_2^2 + x_3^2 + x_4^2$ identisch gleich einer Summe $(2n)$-ter Potenzen linearer rationaler Formen der x_1, x_2, x_3, x_4, und gilt die Waringsche Behauptung für n, so gilt sie auch für $2n$."

[158] "Dieser Satz gab Hilbert die Anregung und Richtung zu seinen Untersuchungen. Er fand nämlich einen ungeahnten Weg, um für beliebige n eine Identität der von Hurwitz geforderten Art aufzustellen."

[159] "aus einem allgemeinen Prinzip, das Hurwitz 1897 in der Invariantentheorie benutzt hatte, eine Formel [...]"

[160] "den durch die Integration geforderten Grenzübergang in die Koeffizienten der Summe zu verlegen und schließlich durch einen weiteren Kunstgriff diese Koeffizienten durch positive rationale zu ersetzen. Damit ist die Grundlage für den Beweis des Waringschen Satzes gelegt."

[161] "Denn er kämpfte zusammen mit einem Meister von dem hohem Range Hurwitz's und siegte mit den

With this meaningful characterization we close the analysis of the diary entries related to the mathematical exchange between those two great mathematicians with some last significant words from Hilbert, expressing that Hurwitz was "[...] more than willing to appreaciate the achievements of others and he was genuinely pleased about any scientific progress: an idealist in the good old-fashioned meaning of the word."[162] [Hilbert, 1921, p. 164]

Personal Relation: Lifelong and even Longer.

Besides the mathematical diaries there are some more hints on the multifaceted relationship between David Hilbert and Adolf Hurwitz to be discovered in the ETH estate in Zurich. Hurwitz, who suffered during all of his life from an unstable health, was a very rare guest at conferences outside Zurich. Accordingly several greeting cards[163] sent from mathematical events can be found, from 'Lutetia Parisiorum, le 12 aout 1900', the 'Landau-Kommers 18. Jan. 1913' and from the 'Dirichletkommers am 13. Februar 1905'.

Figure 2.46: Greeting cards from the 'Lutetia Parisiorum' and the 'Landau-Kommers'.

All of those were signed by a great number of mathematicians, the first two show the handwriting of David Hilbert. On the card from Paris (see left picture in Fig. 2.46) he

Waffen aus Hurwitz's Rüstkammer an einem Punkte, wo dieser keine Aussicht auf Erfolg gesehen hatte."

[162] "[...] gern bereit zur Anerkennung der Leistungen anderer und von aufrichtiger Freude erfüllt über jeden wissenschaftlichen Fortschritt an sich: ein Idealist im guten altmodischen Sinne des Wortes."

[163] under the directory HS 583: 52,53 and 57

wrote, "Sending warm greetings, wishing good recovery and hoping for a soon reunion longer than the last time. Hilbert"[164]

Furthermore, some documents testify that there had even been a close relationship between the families Hurwitz and Hilbert. Adolf Hurwitz's elder brother Julius, who also studied mathematics in Königsberg, edited several of Hilbert's lectures. On some lecture notes[165] comments of Hilbert himself can be found and in the extensive collected correspondence of Adolf Hurwitz[166] a letter exchange between Julius Hurwitz and Hilbert has been discovered[167].

In the additional book No. 32[168] Georg Pólya completed the mathematical diaries with a register. Next to this list the notations "the first 9 volumes and table of contents are for the purpose of editing temporarily at Prof. Hilbert in Göttingen"[169] and

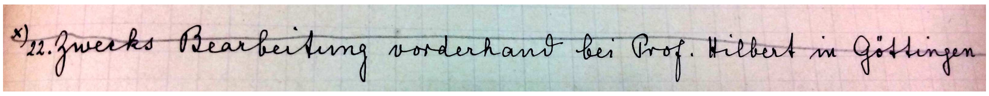

x) 22. zwecks Bearbeitung vorderhand bei Prof. Hilbert in Göttingen

Figure 2.47: "22. for editing temporarily at Prof. Hilbert in Göttingen"[170]

are written and later crossed out with a pencil. Obviously, Hilbert had lent the first nine as well as the twentysecond diary and had returned them after a while. It is difficult to reconstruct when exactly he borrowed the diaries, however, there are two hints. In the beginning of this section we already stated Hilbert's remarks about Hurwitz's diaries. He wrote that they "provide a true view of his constantly progresssive development and at the same time they are a rich treasure trove for interesting and for further examination

[164] "Herzliche Grüße sendend, gute Erholung wünschend und baldiges Wiedersehen auf länger, wie das letzte Mal erhoffend. Hilbert".

[165] under the directory HS 582: 154

[166] which is stored in the archive of Göttingen

[167] We will leave further details to Section 2.4.

[168] in HS 582: 32

[169] "die ersten 9 Bände und Inhaltsverzeichnis sind zwecks Bearbeitung vorderhand bei Prof. Hilbert in Göttingen"

[170] "22. zwecks Bearbeitung vorderhand bei Prof. Hilbert in Göttingen"

appropriate thoughts and problems." [Hilbert, 1921, p. 166] Since this quotation is taken from his commemorative speech, Hilbert had viewed the diaries before 1920 and considered them as a rich source of new mathematical inspirations.

In a letter of condolence to Ida Samuel-Hurwitz[171] from December 15, 1919 - only four weeks after Adolf Hurwitz's death - Hilbert wrote that for Pólya and him the "matter of publishing the Hurwitz's treatises [is] of utmost concern".[172] [173] He offered, "The negotiations could be done verbally with Springer by a local, very skillful, math. colleague."[174][175] It took another few years before Hilbert's and Pólya's support of their mathematical and personal friend finally turned out to be successfull. Adolf Hurwitz's 'Mathematische Werke' [Hurwitz, 1932] were published in 1932.

[171] under the directory HS 583: 28
[172] "Angelegenheit der Herausgabe der Abhandlungen von Hurwitz [ist] unsere wichtigste Sorge"
[173] Pólya himself remembered, "I played a large role in editing his collected works." [Pólya, 1987, p. 25]
[174] "Die Verhandlungen könnte ich durch einen hiesigen sehr gewandten math. Kollegen mündlich mit Springer führen lassen."
[175] Probably, Hilbert meant Richard Courant, his former student and at that time professor in Göttingen, with whom he had created the 'Gelbe Buchreihe' with publisher Springer.

2.3 The Elder Brother: Julius Hurwitz

Figure 2.48: Portrait of Julius Hurwitz (1857 - 1919), taken from Riesz's register in Acta Mathematica from 1913. [Riesz, 1913]

Compared to his younger brother Adolf, much less is known about Julius [Jakob] Hurwitz. He was also born in Hildesheim, two years previous to Adolf, more precisely on July 14, 1857. There had been two more children in the Hurwitz family, namely sister Jenny who died already in 1855 at age one, and the elder brother Max [Mosche], born August 22, 1855 who died July 17, 1910 in Zurich. In the beginning Julius' education went pretty parallel to that of his younger brother Adolf Hurwitz.

Both benefited from their ambitious father Salomon Hurwitz and were sent to the Realgymnasium Adreanum. According to their school certificates[176] both were rather good pupils with quite good marks, in particular in music and math, and both received extra lessons by their teacher Hermann Caesar Hannibal Schubert. However, it seems that Julius was a little more open for distraction. In one of his school certificates the note "Julius visited a tavern without permission"[177] can be found. Nevertheless, also Julius spent Sunday afternoon at Schubert's home learning about geometry and he could have been an aspirant for an academic career as well. Unfortunately, the father of Adolf and Julius was sceptical about those plans, and moreover he was not very well off. He, Salomon

[176] Those certificates can be found in municipal archives of Hildesheim (cf. [Rasche, 2011]).
[177] "H. hat einmal (unerlaubter Weise ein) Wirtshaus besucht."

Eduard Hurwitz was a merchant, and widower after the boys' mother, Elise Wertheimer, died[178] in 1862 when Julius was five and Adolf three years old. Although the partnership of Salomon and Elise might have been not the best, the relation of the three Hurwitz boys with their father must have been very close; Adolf's wife Ida described the father as follows, "Moreover he set a high value that the young boys started smoking since he could not imagine a proper man without cigar or better a pipe."[179] [Samuel-Hurwitz, 1984, p. 2] When the teacher Schubert came to Salomon Hurwitz "to convince him to let both sons choose the studies of mathematics"[180] [Samuel-Hurwitz, 1984, p. 4], Salomon consulted a prosperous childless friend E. Edward[181] who offered to finance the studies of one of the sons. After questioning the teacher Schubert, finally Adolf was elected and Julius had to follow the profession of his elder brother Max. He did an apprenticeship at Nordhausen and became a bank clerk. In this or similar professions he worked for many years, probably first in Hamburg. There are postcards from June 17, 1881 and September 04, 1881 from Salomon to Julius in Hamburg, in the archive of the ETH Zürich and in a letter to his Italian colleague and friend Luigi Bianchi (1856 - 1928), Adolf Hurwitz reported: "My brother Julius is working as an exporter in Hamburg" [Bianchi, 1959]. Later he went to Hanover, where he and his brother Max took over the banking business of their deceased uncle Adolf Wertheimer. Despite this disappointing working life Julius must have studied besides all the time and he never lost his love for mathematics. Thus, in retrospect, his sister-in-law Ida Samuel-Hurwitz wrote in her biographical scetch,

> "Since his uncle Adolph's death Max and Julius were owning the bank 'Adolph M. Wertheimer's Nachf.' in Hanover, however, they felt uncomfortable in this business. Hence, first Julius quitted in order to return to

[178] on liver malfunction according to Adolf's wife Ida [Samuel-Hurwitz, 1984], resp. kidney malfunction according to Frei [Frei, 1995]

[179] " Ebenso legte er auch Wert darauf, dass die Knaben schon sehr frühzeitig zu rauchen begannen, da ihm ein rechter Mann ohne Cigarre oder besser noch Pfeife kaum denkbar war."

[180] "Schubert suchte sogar den Vater auf, um ihn zu bestimmen, beide Söhne das Studium der Mathematik ergreifen zu lassen."

[181] probably the American distant relative of whom Schwarz reported in his letter to Weierstrass, see Section 2.1

> school at the age of 33 years and finish with the school examination for studying Mathematics afterwards under supervision of his younger brother."[182]
> [Samuel-Hurwitz, 1984, p. 8]

Julius Hurwitz visited a school in Quakenbrück, a small town in northern Germany (not far from Bremen). One can still find his school leaving certificate from September 9, 1890 in the archive of the University of Halle[183]. It is rather impressive to read: He not only had to complete an exam in mathematics, but also in various subjects like geography, French and even gymnastics and drawing. Furthermore he was assessed in the category of 'moral behavior', which was considered to be excellent in his case. A big motivation for Julius to take the risk to change his career so drastically could have been the professorship of his brother in Königsberg in 1884. The thereof informing letter of Salomon Hurwitz to Julius (see Section 2.1) shows again the very close familial bonds. In another letter of congratulations to Adolf Hurwitz from April 7, 1884, written in Hanover, the mathematician Hans von Mangoldt (1854 - 1925) mentioned "[...] your brother [told me] about the news of your calling" and "[he] will have informed you about my well-being" [184][von Mangoldt, 1884]. Obviously Julius Hurwitz maintained active contact with mathematicians from his home city. Moreover, several remarks in the correspondence between David Hilbert and Hermann Minkowski, studying under Adolf Hurwitz in Königsberg, indicate that Julius visited his brother frequently. On December 22, 1890 Minkowski wrote for example "Best greetings to Hurwitz the older and Hurwitz the younger [...]" [Minkowski, 1910, p. 42]. Julius had finally moved to his brother and pursued his interest in mathematics. It is known that he edited a variety of lecture notes not only from his brother, but also, among others, from

[182] "Seit dem Tode Onkel Adolphs waren Max u. Julius Inhaber des Bankgeschäfts 'Adolph M. Wertheimer's Nachf.' in Hannover, fühlten sich aber im Kaufmannstande nicht glücklich. So trat zuerst Julius aus, setzte sich mit 33 Jahren nochmals auf die Schulbank, um das Abiturientenexamen nach zumachen und dann bei seinem jüngeren Bruder Mathematik zu studieren."

[183] Archive of Halle University, Rep. 21 Nr. 162

[184] "[...] durch Ihren Herrn Bruder die Nachricht von Ihrer Berufung [...] erhalten. [...] Von meinem Wohlergehen wird Ihnen Ihr Herr Bruder berichtet haben."

Viktor Eberhard (1861 - 1927) in 1890/91[185] and David Hilbert in 1892[186]. On the first page of the second one can find Julius' handwriting: "reviewed by Dr. Hilbert a. equipped with his personal sidenotes"[187]. Furthermore, there is a letter from Hilbert to Klein with date March 04, 1891 ending with the sentence: "(Adolf) Hurwitz send you his regards. At the moment his eldest brother is staying here for a visit."[188] However, the word 'visit' does not explain the situation as it were considered by Adolf's wife Ida. As a matter of fact, Julius edited some course notes of his brother Adolf's lectures at Königsberg University and, when in 1892 his brother Adolf followed the call of the ETH Zurich, he accompanied him; Adolf's wife Ida wrote about that: "[Also] his brother Julius followed him soon to Zürich, where he wrote his doctoral thesis for which he had received the subject from his brother."[189] [Samuel-Hurwitz, 1984] However, in Section 2.4 we explain that for his final doctoral viva Julius Hurwitz actually returned to Germany, more precisely, to the University of Halle-Wittenberg and provide more details about Julius Hurwitz's time in Halle.

Here we shall give an impression of the University of Halle in the second half of the nineteenth century.[190] Of course, Georg Cantor was playing an important role at that time, succeeding Eduard Heine (1821 - 1881) and H.A. Schwarz. Born in St. Petersburg in 1845, after studies in Berlin and Göttingen, Cantor wrote both theses, his dissertation and his habilitation on number theory and was appointed to Halle in 1869 on promotion of Schwarz; there he obtained an extraordinary professorship in 1872 and became ordinary

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[185] The lecture notes can be found in the archive of the ETH Zurich, HS 582: 157, under the reference 'Eberhard, Victor; Determinantentheorie nach Vorlesungen in Königsberg 1890/1891, ausgearbeitet von Julius Hurwitz.'

[186] The lecture notes can be found in the archive of the ETH Zurich, HS 582: 154, under the reference 'Hilbert, David; Die eindeutigen Funktionen mit linearen Transformationen in sich, nach Vorlesungen in Königsberg 1892, ausgearbeitet von Julius Hurwitz.'

[187] "von Dr. Hilbert durchgesehen u. mit eigenhändigen Randbemerkungen versehen.", archive ETH, HS 582: 154

[188] "(Adolf) Hurwitz lässt Sie bestens grüssen. Augenblicklich ist auch sein ältester Bruder hier bei ihm zum Besuch.", letter 63 in [Frei, 1985]; actually, Max is the eldest, and Julius is second.

[189] "[Auch] sein Bruder Julius folgte ihm bald nach Zürich, wo er an der Doktorarbeit schrieb, deren Thema er von seinem Bruder erhalten hatte."

[190] In Section 2.4 we state some considerations about easrlier years.

professor in 1879 on Heine's recommendation. Around these years Cantor published a series of papers which are nowadays considered as the foundation of set theory; however, the reception of his arithmetic of the infinite was rather controversal ranging from Kronecker's offensive opposition to the tonic support from renown mathematicians as Magnus Gösta Mittag-Leffler (1846 - 1927), Klein, and, later, Hilbert, Jacques Hadamard (1865 - 1963), and Adolf Hurwitz. It was not least Cantor who was responsible for the foundation of the *Deutsche Mathematiker Vereinigung* (DMV) and their first meeting in 1890; it should be noticed that as well Klein's school was taking part in erecting these structures in scientific mathematics (with the exception of the Hurwitz brothers). Cantor as well as Klein were also involved in organizing the first International Congress for Mathematicians in Zurich in 1897 which also brought some kind of approval for his foundations of set theory: "... [Adolf] Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments." [Johnson, 1972, p. 17] Adolf Hurwitz had his first encounter with Cantor in summer 1888 when both spent some time in a group of mathematicians around Weierstrass in the Harz Mountains.[191] Georg Cantor died in Halle in 1918. In some periods of his life he suffered from depressions, and lived in a sanatorium; his theory about Francis Bacon being the author of Shakespeare's plays had been considered as odd by his contemporaries, nevertheless this rumour continues until today. There is a letter from Minkowski to Hilbert from 1897 saying "Hurwitz's brother [Julius] writes that Cantor of Halle has been offered a chair in Munich. Seems peculiar. The chair of Shakespearology?"[192] This gives a certain insight how Cantor was recognised at that time (among people who considered his work as outstanding) and it indicates that there had been more communication among the

[191] "Im Sommer 1888 verbrachte er [Adolf Hurwitz] auf Anregung von Prof. Mittag-Leffler in Stockholm, einige Tage in Wernigerode i/H. in interessantem mathematischen Kreisen der sich um den Altmeister Prof. Weierstrass aus Berlin geschart hatte. Dort lernte er Georg Cantor und Sonja Kowalewski näher kennen." [Samuel-Hurwitz, 1984]

[192] "Hurwitz's Bruder [Julius] schreibt, dass Cantor aus Halle nach München berufen sei. Die Nachricht klingt sehr seltsam. Etwa auf einen Lehrstuhl für Shakespearology?", cf. [Thiele, 2005]

Königsberg clique than could be found in archives. (For more information about Cantor's life and work we refer to Thiele [Thiele, 2005].)

As a professor of the department at Halle, Cantor had to review Julius' dissertation and his signature can be found on the relevant document. In 1895, Julius Hurwitz finished his thesis. Besides Cantor's signature it bears the dedication "My dear brother and distinguished teacher Prof. Dr. A. Hurwitz."[193] Moreover, he expressed his gratitude to his younger brother for "many advises with which he had supported this work."[194] Remarkably, the topic of Julius' dissertation thesis is not only pretty close to his brother's treatment of complex continued fractions, according to the official description of Albert Wangerin (1844 - 1933), it is even based on a published article of Adolf Hurwitz. He wrote,

> "Mister J. Hurwitz examines, following a in Acta Mathematica, volume XI, published work by his brother, Prof. A. Hurwitz in Zürich, a certain kind of continued fraction expansion of complex numbers."[195] [Universitätsarchiv, 1895]

In his *Jahrbuch über die Fortschritte der Mathematik* review [Hurwitz, 1894a], Adolf Hurwitz himself wrote about his brother Julius' dissertation:

> "The author examines in the present paper a certain kind of a continued fraction expansion of complex numbers from similar points of view as the referee took them as a basis to handle certain other continued fraction expansions of real and complex numbers."[196]

[193] "Meinem lieben Bruder und verehrten Lehrer Herrn Prof. Dr. A. Hurwitz."

[194] "Es sei mir gestattet, meinem Lehrer, Herrn Professor A. Hurwitz, für die mannigfachen Ratschläge, mit welchen er mich bei dieser Arbeit unterstützt hat, auch an dieser Stelle meinen herzlichsten Dank auszusprechen."

[195] "Herr J. Hurwitz untersucht im Anschluß an eine in den Acta mathematica, Band XI, veröffentlichte Arbeit seines Bruders, des Prof. A. Hurwitz in Zürich, eine besondere Art der Kettenbruchentwickelung complexer Größen."

[196] "Der Verfasser untersucht in der vorliegenden Arbeit eine besondere Art der Kettenbruchentwickelung complexer Grössen nach ähnlichen Gesichtspunkten, wie sie der Referent der Behandlung gewisser anderer Kettenbruchentwickelungen reeller und complexer Grössen [...] zu Grunde gelegt hat."

This summary may be seen in the light of a rough quotation by Adolf Hurwitz saying that "A PhD dissertation is a paper of the professor written under aggravating circumstances." (cf. [Krantz, 2005, p. 24]). However, Julius' thesis must have made a good impression. In the further course of description, Wangerin emphasized that Julius Hurwitz worked diligently, efficiently and in particular independently.[197] Besides Wangerin, the examination committee consisted of the dean Prof. Haym, Prof. Dorn (physics), and Prof. Vaihinger (philosophy). In the minutes of the examination[198] can be read that Julius was not only questioned about mathematics, but also about Carnot's theory of heat, the philosophy of Leibniz and related subjects. In the end all examiners agreed in the grading *magna cum laude*.

Shortly after his defence, Julius returned to Switzerland. Already in 1896, Julius became a member of the 'Naturforschende Gesellschaft' in Basel[199] and in the same year he was engaged as *Privatdocent*[200] at the University of Basel, the first university in Switzerland, founded in 1460. One of the contacts the Hurwitz brothers could have had in Basel was Karl von der Mühll (1841 - 1912) who worked in Leipzig until 1889 before he was appointed a professorship in Basel. It might be astonishing how fast Julius submitted the necessary habilitation thesis *'Über die Reduction der binären quadratischen Formen mit complexen Coeffizienten und Variablen'*; it must have been written in the same year 1896 although it was published only in 1901 as [Hurwitz, 1902b].

In a meeting of the faculty council on June 02, 1896 the examiners decided that "both works are properly and well conducted. In the habilitation thesis, new methods were applied successfully. [...] The petitioner has zealously and skillfully become acquainted with the field of mathematics in a short time."[201] [Staatsarchiv, 1896, 02.06.1896] Moreover, the

[197] "Die Durchführung der Untersuchung zeugt nicht nur von Fleiß und tüchtiger mathematischer Schulung, sondern auch von selbständigem Nachdenken." Certificate of the dissertation of A. Wangerin, Rep. 21 Nr. 162, University archive Halle-Wittenberg

[198] Certificate of the dissertation, Rep. 21 Nr. 162, University archive Halle-Wittenberg

[199] University Library Basel

[200] Personal file, Dozentenkartei, University archive F 6.2.1, State archive Basel

[201] "Beide Arbeiten sind sauber und gut durchgeführt. In der Habilitationsschrift werden neue Methoden mit Erfolg angewendet. [...] Der Petent hat sich in kurzer zeit mit Eifer und Geschick in das Gebiet

application procedure included a colloquium, which took place on June 17, 1896. We want to briefly look at the protocols of the examination board: First Julius Hurwitz spoke freely about the 'Principles and Developments of modern Number Theory' for twenty minutes and was questioned afterwards. According to his examiners[202], "The talk left something [...] to be desired as far as formality is concerned. It lacked clarity and wasn't very well presented. However, the academic requirements seem to have been fulfilled completely. It was decided to submit the proposal to the Senate: The venia docendi[203] for the subject of mathematics is to be given to Dr. J. Hurwitz."[204][Staatsarchiv, 1896, 17.06.1896] Consequently, Julius Hurwitz became a lecturer at the University of Basel and, by the age of 39, for the first time academically independent of his younger brother.

Curiously, it was only in July 1899 that Julius finished the summary of dissertation and habilitation, and it took even another two years that it was printed as [Hurwitz, 1902b]. In any case, it is remarkable that both, Adolf and Julius realized their habilitation in very short time.


UNIVERSITÄT BASEL

Personalkarte für Dozenten.

1. Name: Hurwitz, Julius
2. Heimatort: Hildesheim Staat: Preussen
3. Geburtsdatum: 14. Juli 1857 Geburtsort: Hildesheim
4. Ort und Datum des Doktorexamens: 1895 Dr. phil. Halle
5. Assistent, wo:
6. Datum der Habilitation: 29. Juni 1896
   a) an welcher Universität: Basel
   b) Lehrfach: Mathematik

Figure 2.49: Excerpt from Julius Hurwitz's personnel file at the University of Basel.[205]

der Mathematik eingearbeitet."

[202] Hermann Kinkelin, Karl von der Mühll and probably Eduard Hagenbach

[203] the Swiss version of the 'venia legendi'

[204] "Der Vortrag liess [...] formell manches zu wünschen übrig. Es fehlte an Uebersichtlichkeit und klarer Beherrschung der Darstellung. Doch scheinen die wissenschaftlichen Anforderungen ganz erfüllt zu sein. Es wird beschlossen der Regenz den Antrag zu unterbreiten: Es sei dem Herrn Dr. J. Hurwitz die Venia docendi für das Fach der Mathematik zu erteilen."

[205] taken from [Staatsarchiv, 1896]

Julius Hurwitz did attend the first International Congress of Mathematicians at Zurich in 1897, but he did not attend the next one in Paris nor any other International Congress (as well as Adolf did not). No further scientific work of Julius Hurwitz is known and we can assume that he did not lecture a lot at the university. In 1898, he already submitted a request for leave, "because he had to spend the winter in Italy on grounds of illness."[206] [Staatsarchiv, 1896, 04.05.1899] In a letter from August 25, 1898 to Adolf Hurwitz, Hermann Minkowski wrote: "I was very sorry, that your brother Julius is not feeling quite well and I wish him speedy recovery."[207] Three years later, in 1901, Julius Hurwitz stopped lecturing completely and moved to Lucerne.[208] He stayed there until 1916 when he accompanied his companion[209] Franz Sieckmeyer to Germany. The brief note "Deutschland, Krieg" in the file at the municipal archives of Lucerne suggests that this return was not of his own free will. Refering to their last meeting, Ida Samuel-Hurwitz wrote,

> "[..] Julius was visting us after a long break (he had followed his conversionalist Franz Sieckmeyer to Freiburg i/Br., where he rendered service at a military hospital). Also Julius was suffering for many years (heart disease and arteriosclerosis), but at this last being-together, which was rather unexpected for both brothers, Adolf made a far more sickly impression."[210] [Samuel-Hurwitz, 1984, p. 13]

For June 2/3, 1919 Julius' reentry in Lucerne was registered, where he checked in at the 'Hotel des Alpes'. Already some days later he wanted to leave for Lugano.[211] Although

[206] "da er den Winter krankheitsbedingt in Italien verbringen muss."

[207] "Daß Ihr Bruder Julius nicht ganz wohl ist, that mir herzlich leid und ich wünsche ihm baldige Genesung", HS 583: 51 ETH Archive

[208] Municipal archive of Lucerne, Alphabetisches Einwohnerverzeichnis 1904-1907, Bd. H-Q, B3.22 B09: 011; Journal P zu den alphabetischen Einwohnerverzeichnissen, 1905, B 3.22 B10: 016; Häuserverzeichnis V552 S 11 R-S

[209] in German: Gesellschafter

[210] "[...] war Julius nach langer Pause wieder einmal bei uns zu Besuch (er war seinem Gesellschafter Franz Sieckmeyer nach Freiburg i/Br. gefolgt, wo dieser Lazarettdienst leistete). Auch Julius war seit Jahren sehr leidend (Herzleiden und Arterienverkalkung), auch machte bei diesem letzten Zusammensein, auf welches beide Brüder wohl kaum mehr gerechnet hatten, Adolf den weitaus kränklicheren Eindruck."

[211] Municipal archive of Lucerne, Alphabetische Ausländerkontrolle, F 8/7: 10

his notice of departure can be found in the city's records, it never came that far.

> "The 15 June during one of the frequent heart attacks he endured, in Lucerne Julius closed all of a sudden his eyes forever. [Adolf] H. accepted this news, which was delivered to him with the greatest caution, with the fullest conviction, praising the destiny of his beloved brother, who had overcome everything now, and longing for the same for himself."[212] [Samuel-Hurwitz, 1984, p. 13]

2.4 Excursion: A Letter Exchange Concerning Julius Hurwitz's PhD Thesis

We may assume that Julius Hurwitz was nicely integrated into the academic circle of mathematicians in Königsberg, even though there had always been a strong dependence on his successful brother. As we stated in Section 2.3, Ida Samuel-Hurwitz even reported that Julius was with them "for studying mathematics under the supervision of his younger brother."[Samuel-Hurwitz, 1984, p. 8] Consequently it is not surprising that when Adolf Hurwitz was called to the ETH Zurich in 1892, she noted "[Also] his brother Julius soon followed him to Zurich [...]", whereas the continuation of her explanation might be slightly confusing: "[to Zurich,] where he wrote his doctoral thesis, for which he had received the subject from his brother."[Samuel-Hurwitz, 1984, p. 9] This interesting remark leads to some considerations: *Was it possible for Julius Hurwitz to do his doctorate with his brother as supervisor in Zurich?* Here a definite answer can be given: At that time, professors of the ETH did not have the permission to award doctorates. Consequently there was not even an option for Julius Hurwitz to proceed at the polytechnic.[213] In a letter from July 22, 1893 to David Hilbert, Adolf Hurwitz clarified that Julius did not

[212] "Am 15. Juni schloss Julius in Luzern bei einem der häufigen Herzanfälle, die er erlitt, ganz plötzlich seine Augen für immer. [Adolf] H. nahm die Nachricht, die ihm mit größter Vorsicht beigebracht wurde, voll Ergebung auf, das Geschick des Bruders preisend, der nun alles überstanden habe, und für sich selber das Gleiche ersehnend."

[213] The University of Zurich would have been an opportunity, Adolf used to supervise his doctral students at this institution until 1909; however, the relation between Julius and Adolf might have been to close.

stay for a long time in Switzerland. "My brother, warmly greeting you, leaves Zurich now, to go to Halle and gradually prepare there his doctorate."[214] [Hurwitz, 1895, let. 13] Here another question arises: Why Halle? Some years earlier, such a decision of an ambitious young mathematician would have been very understandable, since Halle had the great advantage of a neutral position in the turmoils of the Prussian education policy. The mathematical institute managed to combine influences of the school of Göttingen, the school of Berlin as well as the school of Königsberg. Because of "[...] the excessive number of lecturers preventing the fruitful education of a beginner [lecturer][215][Göpfert, 2002, p. 22], the Göttingen mathematician Carl Thomae (1840 - 1921) had transferred to Halle in 1867 and stayed there for seven years. Besides the powerful Eduard Heine, who had been in Königsberg previously, the nowadays famous Georg Cantor was in Halle at the same time. During his studies in Berlin, Cantor was strongly influenced by his advisors Ernst Eduard Kummer (1810 - 1893) and Weierstrass. It was, however, in Halle, where he developed his infinite set theory and stayed until the end of his life. Cantor's colleague Albert Wangerin wrote,

> "[...] a work demanding the greatest ingenuity, high mathematical creativity and a vivid imagination; Cantor possessed all those characteristics. Thus he became the creator of a completely new branch of mathematics." [216] [Wangerin, 1918]

In consequence there had been a wide range of courses and a good possibility to learn about progressive mathematics in the seventies and eighties of the 19th century.[217] However, when Julius Hurwitz went to the University of Halle a couple of years later, the situation

[214]"Mein Bruder, der Sie herzlich grüßt, verlässt jetzt Zürich, um nach Halle zu gehen und dort allmählich den Doktor vorzubereiten."

[215]"[...] die übergroße Anzahl an Docenten desselben Faches an diesem Platze einer gedeihlichen Ausbildung eines Anfängers entgegenwirkt [...]"

[216]"[...] eine Arbeit die den größten Scharfsinn, hohe mathematische Schöpfergabe und rege Phantasie erforderte; alle diese Eigenschaften besaß Cantor. Er wurde so der Schöpfer eines ganz neuen Zweiges der Mathematik."

[217]For a detailed listing of lectures we refer to [Göpfert, 1999] and [Richter and Richter, 2002]

had changed drastically. At least because of the instable mental condition of Cantor, who suffered greatly from a lack of recognition of his discoveries.
So the question remains: *Why did Julius Hurwitz choose Halle?* It should be noticed that also the teacher of the Hurwitz's brothers, Hermann Schubert, received his doctorate in 1870 from the University of Halle. He wrote his dissertation on enumerative geometry during his studies in Berlin, however, after the death of his teacher Gustav Magnus, he decided to finish his doctorate at the University of Halle without official supervisor (see [Burau and Renschuch, 1993] for further details). Therefore it is not clear whether Schubert's experience could have been a reason for Julius to choose Halle for his doctorate. Another link could have been Viktor Eberhard who was appointed extraordinary professor at Königsberg in 1894 and became ordinary professor at the University in Halle in the following year. Eberhard was blind; Hilbert and Klein tried to support his career during his time at Königsberg. Adolf and Eberhard knew each other[218] and, probably, the same holds true for Julius and Eberhard. However, those considerations are only speculative.

Even for David Hilbert the choice of Halle was astonishing. In a letter of September 05, 1893 to Adolf Hurwitz he wrote:

> "I received your brother's congratulations [...]. Why did he choose Halle as his place of study? Where, as I imagine, not much is going on right now, since for Cantor, as he himself declares, mathematics is an atrocity."[219] [Hilbert, 1895, let. 330]

One can see that Hilbert is not only interested in his friend's brother, but also uses the opportunity to drop a remark on Georg Cantor's behavior. This remark is being followed

[218] as follows from letter 53 from Hilbert to Klein in [Frei, 1985] with date February 15, 1890

[219] "Ihres Bruders Gückwünsche habe ich erhalten [...]. Warum hat derselbe gerade Halle zu seinem Studienort gewählt? Wo wie ich denke jetzt sehr wenig los ist, da Cantor wie er selbst eingesteht die Mathematik ein Gräuel ist."
The congratulations of Julius probably referred to the birth of his son Franz Hilbert on August 11, 1893. Tragically, Franz suffered all of his life under mental disorder and never had a good relation to his father, who could hardly accept his lack of cleverness. In 1914, when he was taken to a psychiatry, Hilbert declared "From now on, I must consider myself as not having a son."[Reid, 1970, p. 139]

by the even more ironical answer of Adolf Hurwitz in a letter from October 10, 1893:

> "My brother [...] is feeling very well in Halle. He chose Halle because of the comfortable location and because of the inspirations that are still to be expected. Wangerin is a very good lecturer, [Hermann] Wiener at least quite active [...]. Cantor is lecturing on number theory, is however said to be preferably working on Shakespearology."[220] [Hurwitz, 1895, let. 15]

This remark refers to the fact we already stated in the previous section that Cantor was a great believer in the theory of Francis Bacon being Shakespeare and it shows again the amusement of contemporary mathematicians about it. Adolf Hurwitz explained the decision for Halle with the "inspirations that are still to be expected" and with "the comfortable location". The latter already hints at the fact that Julius was not only focused on studying the most progressive mathematics, but that there were various aspects of Halle appealing to him.

Another open question arises from Ida's memories: *What topic did he receive?* This is also asked in a later letter of Hilbert from April 20, 1894. He wrote, "Your brother is said to approach his doctorate, what topic has he actually?[221] [Hilbert, 1895, let. 253] In a surprising answer from April 26, 1894, Adolf Hurwitz explained,

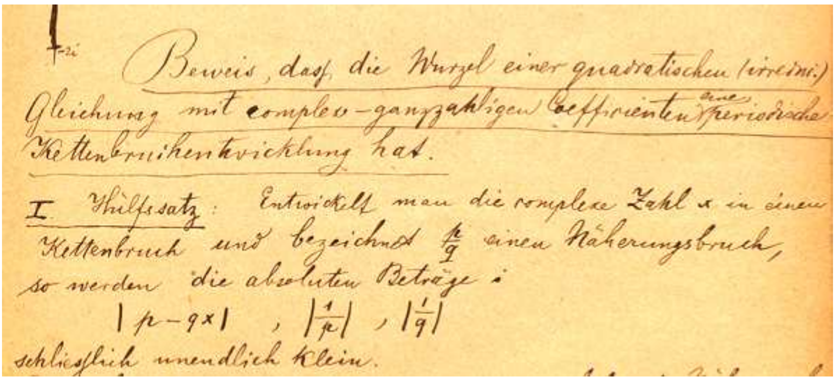
> "Certainly my brother has the intention to do his doctorate in the course of the next year. However, he is not yet sure, what topic he wants to work on. Perhaps he continues on my work about the expansion of complex numbers into continued fractions. But I initially proposed him to ask G. Cantor for a topic."[222] [Hurwitz, 1895, let. 18]

[220] "Mein Bruder [...] fühlt sich einstweilen in Halle sehr wohl. Er hat Halle namentlich der angenehmen Lage und der immerhin zu erwartenden Anregungen wegen gewählt. Wangerin ist ein sehr guter Docent, Wiener immerhin recht rege [...]. Cantor liest Zahlentheorie, soll allerdings vorzugsweise mit Shakespeare-Forschung beschäftigt sein."

[221] "Ihr Bruder soll sich ja schon zum Doktor vorarbeiten, Was für ein Thema hat er denn?"

[222] "Mein Bruder hat freilich die Absicht im Laufe des nächsten Jahres zu promovieren. Er ist aber noch nicht sicher, welches Thema er bearbeiten will. Vielleicht knüpft er an meine Arbeit über die Entwicklung complexer Zahlen in Kettenbrüchen an. Ich habe ihm aber zunächst vorgeschlagen, G. Cantor um ein Thema anzugehen."

Obviously, despite his amusement, Adolf Hurwitz still believed in Cantor's mathematical expertise. Nevertheless, observing Julius Hurwitz's final doctoral thesis [Hurwitz, 1895] shows immediately that he chose the first option. Comparing his work with some notes in Adolf Hurwitz's mathematical diaries (Fig. 2.50), it is a justifiable conclusion that the topic included also quadratic forms, a central theme in mathematical research in the second half of the 19th century. One possible task could have been to use a new kind of complex continued fraction expansion for the reduction theory of quadratic forms with complex coefficients.


Beweis, daß die Wurzel einer quadratischen (irreduc.) Gleichung mit complex-ganzzahligen Coefficienten eine periodische Kettenbruchentwicklung hat.

I. Hülfssatz: Entwickelt man die complexe Zahl x in einen Kettenbruch und bezeichnet $\frac{p}{q}$ einen Näherungsbruch, so werden die absoluten Beträge:

$$|p - qx| \quad , \left|\frac{1}{p}\right| , \left|\frac{1}{q}\right|$$

schliesslich unendlich klein.

Figure 2.50: Excerpt from the mathematical diaries of Adolf Hurwitz: Proof that the root of a quadratic (irreducible) equation with complex integer coefficients has a periodic continued fraction expansion [Hurwitz, 1919a, No. 5, p. 52].

For the understanding of the following it is not necessary to go deeper into details of the topic but we want to keep in mind this task, which would have been a continuation of investigations of his younger brother (see Fig. 2.50).[223] In view of the choice of Halle, a letter from Julius Hurwitz himself to David Hilbert from June 09, 1894, which was recovered by coincidence in the collected correspondence of Adolf Hurwitz in the archive

[223] A good overview of how quadratic forms and continued fractions are connected can be found in [Halter-Koch, 2013].

of the University of Göttingen, might reinforce this impression. Julius Hurwitz described the situation of studying in Halle as follows:

> "The number of students [...] increased, including two refugees from Göttingen, for whom Prof. Klein was too demanding and Prof. Weber was too complicated. Halle is, concerning mathematical lectures, far behind Göttingen, but it has the great advantage that lectures are presented clearly and there are no high demands on students."[224] [Hurwitz, 1894, let. 22]

With his choice of words Julius Hurwitz presented the situation in Göttingen in an interesting light. One gets the impression that the sentiment of students towards the famous university is rather sceptical. Furthermore, it is amusing and perhaps even a bit revealing that Julius wrote he prefers "no high demands on students". By contrast, he is obviously an optimist and a zealous student. In a letter to a friend from July 07, 1894, Georg Cantor thanked his colleague Émile Lemoine (1840 - 1912) "for the friendly acceptance of the notes of [my] pupil Hurwitz."[Décaillot, 2008, p. 152] Based on our current state of knowledge, we may assume, that Cantor was not talking about Adolf Hurwitz, but his elder brother being an attentive student.

In a further letter to Adolf Hurwitz from June 13, 1894 David Hilbert refered to Julius' letter, however, avoiding a direct response on his description of Göttingen and Halle. Hilbert wrote, "Some days ago I received a letter of your brother in which he comprehensively reportet about Halle. I was very pleased to once again hear from him personally."[225] [Hilbert, 1895, let. 255] In a second letter to Hilbert from October 4, 1894 Julius Hurwitz wrote,

> "In these holidays I was working on my dissertation (continued fraction

[224] "Die Zahl der Studierenden der Mathematik hat sich [...] vermehrt, darunter 2 Flüchtlinge aus Göttingen, denen Prof. Klein zu hoch und Prof. Weber zu umständlich waren. Halle steht zwar, was die mathematischen Vorlesungen betrifft weit hinter Göttingen zurück, aber es hat den großen Vorzug, dass das, was gebracht wird, klar vorgetragen wird und dass nicht übergroße Anforderungen an die Studierenden gestellt werden."

[225] "Vor einigen Tagen erhielt ich von Ihrem Bruder einen Brief in welchem er mir ausführlich über Halle erzählt. Ich habe mich sehr darüber gefreut, wieder einmal etwas von ihm selbst zu hören."

expansions in complex areas and applications on the reduction of quadratic forms) [...]. In the mentioned dissertation there is still a deficiency; once it is resolved, I will register for the graduation in winter."[226] [Hurwitz, 1894, let. 25]

Later we will come back to the mentioned title of the thesis (in brackets) and we should keep in mind Julius' aim to finish his work in the winter of 1894. Concerning the "deficiency", we examine an entry in Adolf Hurwitz's mathematical diary [Hurwitz, 1919a, No. 9, pp. 94] from November 04, 1894, exactly one month after Julius' letter, authored under the header "Concerning Julius' work"[227]. We find several pages filled with calculations on partial quotients from complex continued fractions and a nice result concerning possible (admissible) sequences of partial quotients (see Fig. 2.51).

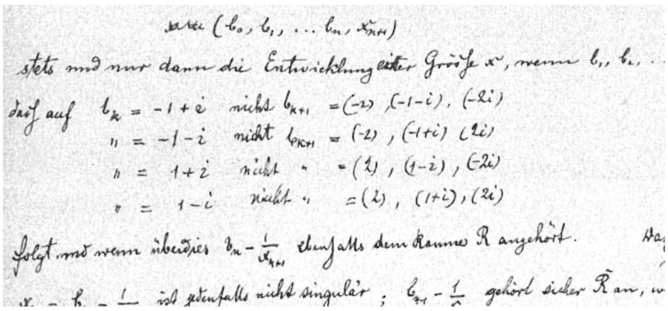
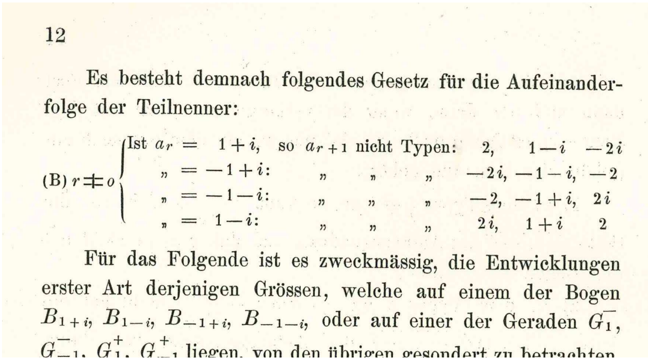

$(b_0, b_1, \ldots b_n, x_{n+1})$
stets und nur dann die Entwicklung einer Größe x, wenn $b_1, b_2 \ldots$
daß auf $b_k = -1+i$ nicht $b_{k+1} = (-2), (-1-i), (-2i)$
" $= -1-i$ nicht $b_{k+1} = (-2), (-1+i), (2i)$
" $= 1+i$ nicht " $= (2), (1-i), (-2i)$
" $= 1-i$ nicht " $= (2), (1+i), (2i)$
folgt, und wenn überdies $b_n - \frac{1}{x_{n+1}}$ ebenfalls dem Raume R angehört.

Figure 2.51: Excerpt from Adolf Hurwitz's mathematical diary entry concerning Julius' work. [Hurwitz, 1919a, No. 9, pp. 94]

When compared to a certain paragraph of Julius Hurwitz's doctoral thesis (Fig. 2.52), we recognize a definite similarity which can be taken as an indication that he received direct help from his brother.

[226] "Ich habe in diesen Ferien an meiner Dissertation (Kettenbruch-Entwicklung im complexen Gebiete und Anwendung zur Reduction der quadr. Formen) gearbeitet [...]. Mit der erwähnten Dissertation hapert es noch an einer Stelle; ist diese überwunden, so werde ich mich im Winter zur Promotion anmelden."
[227] "Zu Julius' Arbeit."



12

Es besteht demnach folgendes Gesetz für die Aufeinanderfolge der Teilnenner:

$$(B)\ r \neq o \begin{cases} \text{Ist } a_r = 1+i, \text{ so } a_{r+1} \text{ nicht Typen: } 2,\ 1-i\ -2i \\ \quad "\ = -1+i: \quad " \quad " \quad " \quad -2i,\ -1-i,\ -2 \\ \quad "\ = -1-i: \quad " \quad " \quad " \quad -2,\ -1+i,\ 2i \\ \quad "\ = 1-i: \quad " \quad " \quad " \quad 2i,\ 1+i\ \ 2 \end{cases}$$

Für das Folgende ist es zweckmässig, die Entwicklungen erster Art derjenigen Grössen, welche auf einem der Bogen B_{1+i}, B_{1-i}, B_{-1+i}, B_{-1-i}, oder auf einer der Geraden G_1^-, G_{-1}^-, G_1^+, G_{-1}^+ liegen, von den übrigen gesondert zu betrachten.

Figure 2.52: Excerpt from Julius Hurwitz's doctoral thesis. [Hurwitz, 1895, p. 12]

This is even underlined by another striking statement from Adolf Hurwitz himself in a letter to David Hilbert from June 19, 1895:

> "Eight days ago the news from my brother came that he passed the doctoral exam with very good results. The dissertation concerns continued fraction expansions of complex numbers. I proposed the topic."[228] [Hurwitz, 1895, let. 29]

What is so remarkable about it? First of all, winter time became summer time, which means Julius needed some more months than he had expected; secondly, the announced title was shortened since Julius' letter to Hilbert. There is no mention of "quadratic forms" anymore. What is most curious, however, Adolf Hurwitz again wrote that he himself proposed the subject. Although we are aware of his support over the years, this is rather confusing and leads indirectly to another possible reason for Julius' decision in favor of Halle: his official thesis advisor Albert Wangerin. Although his advisor is not widely known today, in the second half of the 18th century he must have been a rather influential mathematician. Born 1844 in Greifenberg, Wangerin worked after his studies

[228] "Von meinem Bruder traf vor acht Tagen die Nachricht ein, daß er in Halle das Doktor-Examen sehr gut bestanden hat. Die Dissertation betrifft Kettenbruch-Entw. complexer Größen. Das Thema habe ich gestellt."

at Halle under Heine first as a teacher before he became a professor at the University of Berlin in 1876; there he was responsible for teaching beginners, duties which more prominent mathematicians as Kronecker, Weierstrass, and Kummer tried to avoid. In 1882 Wangerin became professor at the University of Halle and remained there until his retirement in 1919. During his life Wangerin advised the remarkable number of 53 students to their dissertation, Julius Hurwitz being number 28 of them. The supervised topics range from calculus, in particular, differential equations, via analytic and differential geometry to topics from mathematical physics; there are only two theses from number theory.[229] Wangerin died 1933 in Halle. In his report on Julius' dissertation Wangerin declared "J. Hurwitz examines a certain kind of continued fraction expansion of complex numbers following work by his brother, Prof. A. Hurwitz in Zurich, published in Acta Mathematica, volume XI."[Wangerin, 1895] Obviously, Julius' thesis advisor was very well aware of Adolf Hurwitz's position with regard to Julius' work and one can assume that it was considered a positive constellation for the three of them.

Concerning Julius Hurwitz's work, there is one more remark from David Hilbert in a letter from June 25, 1895 to Adolf Hurwitz. Noticing that Hilbert just followed a call to Göttingen and having in mind the description of Göttingen that Julius gave in his first letter, it sounds a bit ironic: "I am very pleased that your brother did his doctorate, I am very curious about his thesis. Perhaps he comes to Göttingen in the near future? One can learn a lot here."[230] [Hilbert, 1895, let. 260] There is no indication that Julius ever visited Göttingen. From another letter between the families Hurwitz and Hilbert we receive an insight into Julius' first activities after his doctoral graduation. This time the women have their say: Ida Samuel-Hurwitz wrote to Käthe Hilbert (1864 - 1945) on April 07, 1896,

> "[...] Julius just visited us for a couple of weeks, from here he traveled to Upper Italy (right now he is at the Riviera), we expect him back at the end

[229] see www.mathematikuni-halle.de/history for a list

[230] "Dass ihr Bruder den Doktor gemacht hat freut mich sehr, ich bin auf seine Dissertation sehr gespannt. Vielleicht kommt derselbe nächstens nach Göttingen? Man kann hier viel lernen."

> of April. He has not yet decided about his future."[231] [Samuel-Hurwitz, 1895, let. 32]

From Section 2.3 we know about Julius Hurwitz later position at the University of Basel. In addition to his doctoral thesis, Julius Hurwitz handed in his habilitation thesis titled *'On the reduction of binary quadratic forms with complex coefficients and variables.'* [Hurwitz, 1902b] In the introduction he described that the type of continued fraction expansion appearing in his doctoral thesis is used in a similar way "as the continued fraction expansion of real numbers for the reduction of real quadratic forms [...] is used". We can conclude that Julius finally tackled his original task concerning quadratic forms. For his public habilitation lecture, Julius Hurwitz chose a subject which allowed inferences about his studies in Halle: "On Tuesday, the 27 October [1896], Dr. Julius Hurwitz held his public habilitation lecture [...] on 'The infinite in mathematics'."[232] The proximity to Georg Cantor's research field may show the impression his former professor at Halle made on him.

[231] "Soeben war [...] Julius einige Wochen unser Gast, von hier ist er nach Oberitalien gereist, (augenblicklich befindet er sich an der Riviera), Ende April erwarten wir ihn zurück. Ueber seine Zukunft hat er sich noch nicht schlüßig gemacht."

[232] "Dienstags, den 27. Oktober, hielt Herr Dr. Julius Hurwitz seine öffentliche Habilitationsvorlesung [...] über 'Das Unendliche in der Mathematik'." [Staatsarchiv, 1896]

3 Hurwitz's Approach to Complex Continued Fractions

This chapter provides an introduction to the expansion of complex numbers into continued fractions which forms the basis for our subsequent studies. The main characters of the previous chapter, Adolf and Julius Hurwitz, can be considered as co-founders of the arithmetical theory of those complex continued fractions. Here we highlight their different approaches and we outline what arithmetical properties they were interested in. A less comprehensive version of the following considerations can be found in [Oswald and Steuding, 2014].

3.1 Continued Fractions According to Adolf

Already in Adolf Hurwitz's first mathematical diary[1] [Hurwitz, 1919a, No. 1] an entry entitled "Expansion of $\sqrt[3]{A} = x$ into a continued fraction"[2] is noted. An indication that his interest in continued fractions outlasted the time of his research life is provided by taking a glance at the unfortunately incomplete list of related entries made by Georg Pólya.

[1] from April 25, 1882 to December 1882
[2] "Entwicklung von $\sqrt[3]{A} = x$ in einem Kettenbruch"

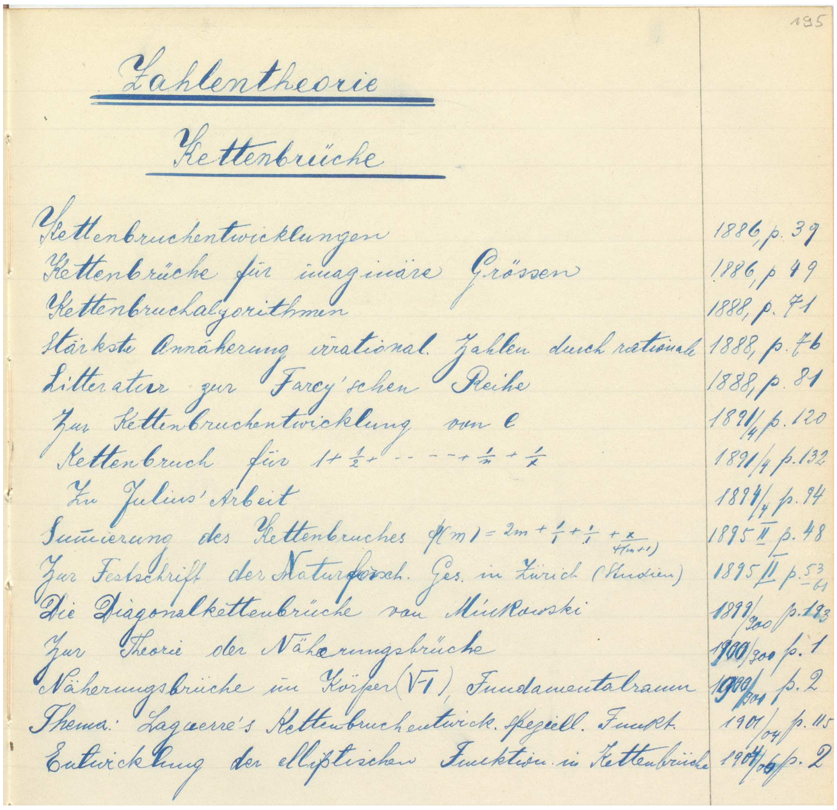


Figure 3.1: Georg Pólya's list of diary entries related to continued fractions [Hurwitz, 1919a, No. 33].

In the sequel our main focus is basically on Adolf Hurwitz's entry "Continued Fractions for imaginary Numbers"[3] [Hurwitz, 1919a, No. 5] from 1886 and on the entry "Concerning Julius' work"[4] [Hurwitz, 1919a, No. 9] from 1894 related to his brother's dissertation.

Of course, there is much more to say about Adolf Hurwitz's research interests. For instance, the detailed obituary [W.H.Y., 1922] published by the London Mathematical Society makes a good reading. Adolf Hurwitz was an honorary member of this society since 1913 and it might be worth to notice that he was also an honorary member of the mathematical societies of Hamburg and Charkov, and a corresponding member of the

[3] "Kettenbrüche für imaginäre Grössen"
[4] "Zu Julius' Arbeit"

Academia di Lincei at Rome (which is rather different from the image of a couch potato one could have in view of his absence from International Congresses outside Zurich). Curiously, the author of the obituary [W.H.Y., 1922] signed with his intitials W.H.Y., so we may only guess that it was William Henry Young (1863 - 1942), the president of the London Mathematical Society from 1922 to 1924. However, we shall concentrate on Adolf Hurwitz's work on continued fractions. About this topic W.H.Y. wrote,

> "His papers on continued fractions and on the approximate representation of irrational numbers are also very original, as well as curious."

Yet it is not further explained what is meant by *curious*. Another evidence highlighting Adolf Hurwitz's contribution to the development of continued fraction theory can be found in David Hilbert's commemorative speech which we already mentioned in Section 2.2.3. Here Hilbert remembered,

> "A with preference treated subject of [Adolf Hurwitz] was the theory of arithmetical continued fractions. In his work *Über die Entwicklung komplexer Größen in Kettenbrüche* he went beyond the to that point only considered area of real numbers [...] ."[Hilbert, 1921, p. 163][5]

In this quotation Hurwitz's publication [Hurwitz, 1888] is mentioned which can be considered as first approach to the complex case. We shall return to this work later, however, first we give a short preparatory account of Adolf Hurwitz's work on continued fractions in general.

Given a real number $x \in [-\frac{1}{2}, \frac{1}{2})$, its *continued fraction to the nearest integer* is of the

[5]"Ein mit Vorliebe von [Adolf] Hurwitz behandeltes Thema war die Theorie der arithmetischen Kettenbrüche. In seiner Arbeit *Über die Entwicklung komplexer Größen in Kettenbrüche* ging er dabei über den bisher allein berücksichtigten Bereich der reellen Zahlen hinaus [...]."

form

$$x = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{a_3 + \ddots}}},$$

where $\epsilon_n = \pm 1, a_n \in \mathbb{N}$, and $\epsilon_{n+1} + a_n \geq 2$. This convergent expansion is obtained by iterating the mapping[6]

$$x \mapsto Tx := T(x) := \frac{\epsilon}{x} - \left\lfloor \frac{\epsilon}{x} \right\rfloor \qquad \text{for} \quad x \neq 0$$

and $T0 = 0$, where $\lfloor z \rfloor$ denotes the integral part of z. Here the subsequent partial quotients are given by $a_n := \lfloor \frac{\epsilon_n}{T^{n-1}x} + \frac{1}{2} \rfloor$ and the sign $\epsilon_n = \epsilon$ equals the sign of $T^{n-1}x$. The first iteration leads to

$$x = \frac{\epsilon_1}{\lfloor \frac{\epsilon_1}{x} \rfloor + Tx} = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + T^2 x}},$$

and so forth. Obviously, a real number x has a finite nearest integer continued fraction if, and only if, x is rational. It seems that Bernhard Minnigerode (1837 - 1896) [Minnigerode, 1873] was the first who considered continued fractions to the nearest rational integer and used them as alternative approach to solve the Pell equation (although his notation differs from ours). In [Roberts, 1884] a related approach was developed independently about a decade later.

Already in an entry in his first diary from 1882[7] Adolf Hurwitz wrote down his idea to develop complex numbers into continued fractions.

> "In the case of numbers $a + bi$ and $a + b\rho$ ($i = \sqrt{-1}, \rho = \frac{-1+i\sqrt{3}}{2}$) there is in principle no difficulty in developing a corresponding theory in view of the

[6] In the previous chapter we will introduce the more modern term 'transformation' for T.

[7] Although the entry bears no date, we may assume that it was before April 13, 1883, which is the date of a later entry.

possibility of Euclid's method for determining the greatest common divisor. Nevertheless, a careful and thorough foundation of such a theory appears to be of great value, for example, for solving diophantine equations of second degree for the corresponding number systems. This is a good doctoral thesis for a young and ambitious mathematician."[8] [Hurwitz, 1919a, No. 1]

In the end, it was not a doctoral student but Adolf Hurwitz himself who tackled the required foundation. In [Hurwitz, 1888] he extended Minnigerode's approach by considering complex numbers $z = x + iy$ instead of real numbers and replacing rational integers by Gaussian integers; here and in the sequel the imaginary unit $i = \sqrt{-1}$ denotes the square root of -1 in the upper half-plane. It is not difficult to see that this yields a continued fraction expansion with partial quotients in the ring $\mathbb{Z}[i]$. In analogy to the real situation a complex number has a finite continued fraction to the nearest Gaussian integer if, and only if, it is a rational Gaussian integer. The proof relies on a variation of the euclidean algorithm in $\mathbb{Z}[i]$. However, Adolf Hurwitz was considering a far more general situation.

Let S be any set of complex numbers such that i) sum, difference and product of any two elements in S belong to S, ii) any finite domain of the complex plane contains only finitely many points from S (from which already follows that besides zero there is no point from the open unit disc inside S), and, finally, iii) $1 \in S$. Starting from some complex number z, Adolf built up the following chain of equations:

$$z = a_0 + \frac{1}{z_1}, \quad z_1 = a_1 + \frac{1}{z_2}, \quad \ldots \ , \quad z_n = a_n + \frac{1}{z_{n+1}},$$

where $a_n \in S$ and none of the z_j is assumed to vanish. This leads to a continued fraction

[8]"Im Falle der Zahlen $a + bi$ und $a + b\rho$ ($i = \sqrt{-1}, \rho = \frac{-1+i\sqrt{3}}{2}$) ist, wegen der Möglichkeit des Euklid. Verfahrens zur Bestimmung des größten gemeins. Theilers, die Entwickl. der betreffenden Theorie ohne prinzipielle Schwierigkeiten. Nichts desto weniger scheint eine sorgfältige und gründliche Durchführung derselben von großem Werte, z.B. für die Lösung Diophant. Gleichungen des zweiten Grades für die betr. Zahlengebiete. Dieses ist eine gute Doctor-Arbeit für einen jüngeren strebsamen Mathematiker."

expansion

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{a_n + \cfrac{1}{z_{n+1}}}}},$$

which one can continue ad infinitum if all $z_j \neq 0$. Supposing further that the nth convergent $\frac{p_n}{q_n} := [a_0; a_1, a_2, \ldots, a_n]$ is distant to z by a quantity less than a fixed constant multiple of $\frac{1}{q_n^2}$, Adolf Hurwitz obtained the following

Theorem 3.1.1 *If all z_j are non-zero, both, the infinite continued fraction*

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_n + \ddots}}}}$$

as well as the sequence of convergents $\frac{p_n}{q_n}$ converge with limit z (which cannot be an element of S). Moreover, if z is the solution of a quadratic equation with coefficients from S, then the sequence of the z_n takes only finitely many values.

If the numbers a_n are taken according a certain rule, e.g., as nearest Gaussian integer, then the sequence of z_n and, henceforth, the sequence of partial quotients a_n is eventually periodic. For regular continued fractions of real numbers this is the celebrated Theorem 1.1.1 of Euler and Lagrange [Lagrange, 1770]; the same reasoning holds for continued fractions to the nearest integer, and even in the case of complex numbers z when the partial quotients are taken to be the nearest Gaussian integers.

Concerning Adolf Hurwitz's assumptions on the 'system' S (that is how he called a set of numbers satisfying conditions i)-iii)), it should be mentioned that S is neither a ring nor a module nor a lattice but shares similar properties. The notion of a ring came only in the 1890s with Hilbert's work on his *'Zahlbericht'* although the concepts of ideal and module have been used since Dedekind's pathbreaking work on the arithmetic of number fields (cf. [Dieudonné and Guérindon, 1985]). Already in 1894, Adolf Hurwitz [Hurwitz, 1894c] published a note on principle ideals. In Section 2.2.3 we pointed out that Hurwitz used Hilbert's *'Zahlbericht'* as a textbook and here we may assume that this influence became visible. Outside algebraic number theory rings and their substructures became widely accepted only with Emmy Noether's (1882 - 1935) modern algebra. The notion of a lattice, however, has been accomplished by the important work of Hermann Minkowski in the 1890s. His *Geometry of Numbers* is based on lattices; there had previously been relevant work on lattices by Gotthold Eisenstein (1823 - 1852), Liouville, Carl Gustav Jacob Jacobi (1804 - 1851), and Weierstrass dealing with elliptic functions. In Chapter 7 we follow up Minkowski's work and, in particular, we give an application concerning an approximation quality result of diophantine approximations formed from lattice points in the complex plane.

As already mentioned, in [Hurwitz, 1888] Adolf Hurwitz applied his general theorem to the case of S being the ring of Gaussian integers $\mathbb{Z}[i]$. He furthermore wrote,

> "Apart from the here considered continued fraction expansion in the range of complex numbers $m+ni$ there exist more for which the above theorem holds, however, I do not want to go on with this here."[9]

Another continued fraction expansion with partial quotients from the set of Gaussian integers was indeed studied by Adolf's brother Julius Hurwitz as main topic of his dissertation; we shall investigate this in detail in the following section. Moreover, Adolf Hurwitz mentioned that his approach could be used to build up a *complex* theory of the Pell equation

[9] "Ausser der hier betrachteten giebt es übrigens noch andere Kettenbruchentwicklungen im Gebiete der complexen Zahlen $m+ni$, für welche der obige Satz ebenfalls gilt, worauf ich indessen an dieser Stelle nicht eingehen will."

$t^2 - Du^2 = 1$ where D is a given number and solutions t and u shall be complex integers. This topic as well had later been considered by his elder brother Julius. Furthermore, in Section 7.4 we transfer a classical theorem concerning Pell equations in the real case to the complex case. Finally, in [Hurwitz, 1888] Adolf Hurwitz studied the ring of Eisenstein integers $\mathbb{Z}[\rho]$ associated with a primitive third root of unity $\rho = \frac{1}{2}(-1 + i\sqrt{3})$. He pointed out that his continued fraction expansion to the nearest Eisenstein integer is different from that what one would obtain from Paul Gustav Heinrich Bachmann's (1837 - 1920) euclidean algorithm for $\mathbb{Q}(\rho)$ in [Bachmann, 1872]. Further applications are possible and were considered by his contemporaries. We indicate some of them in Section 3.3. Generalizing from Hurwitz's system S, in Chapter 6 we imagine to start with a discrete infinite set of complex numbers and consider their Voronoï cells as suitable tiling of the complex plane in order to obtain further complex continued fraction expansions.

Klein's influence is apparent in much of Adolf Hurwitz's work, however, the research on continued fractions and related topics from diophantine approximation seems to have independent roots. Both of Hurwitz's teachers, Schubert and Klein have included continued fractions in some of their writings, namely the textbooks [Schubert, 1902] and [Klein, 1932], but these sources are addressed to beginners in mathematics and do not indicate any deeper relation to this topic. We may ask *what might have been the motivation for Adolf Hurwitz to start a new direction of research and investigate continued fractions?* A possible answer could lead to his friend from Göttingen, Moritz Stern, who had received a doctorate on the theory of continued fractions [Stern, 1829]. He was the first candidate examined by Gauss (cf. [Rowe, 1986]). For some period, the time after the work of Euler and Lambert and the investigations of the French school around Lagrange and Legendre, Stern had been the leading expert in this subject; he published quite a few papers on continued fractions in Crelle's journal and his *Lehrbuch der Algebraischen Analysis* [Stern, 1860] contains a whole chapter on this topic.

Actually, continued fractions were a major line of investigation in the eighteenth and nineteenth century. The well-known contributions of Euler, Lagrange and others had a focus on arithmetical questions as, for example, solving the Pell equation or periodic expansions, and Gauss' research marks the beginning of the metrical theory of continued fractions. First results in the nineteenth century highlight analytic questions about convergence and divergence, e.g. the Seidel-Stern convergence theorem due to Philipp Ludwig von Seidel (1821 - 1896) [Seidel, 1847] and (independently) Stern [Stern, 1848]. Building on previous work of Stern [Stern, 1860], Otto Stolz (1842 - 1905) [Stolz, 1885] investigated periodic continued fractions with complex entries with respect to convergence; this is now known as the Stern-Stolz divergence theorem (for details we refer to [Lorentzen and Waadeland, 1992]). The very first complex continued fractions can be found implicitly in the general approach of Jacobi [Jacobi, 1868] from 1868, published seventeen years after his death.

Starting in Königsberg with his research on continued fractions around 1886/87,[10] Adolf Hurwitz was breaking into a new market, independent of Klein's business. Hurwitz's contribution [Hurwitz, 1888] on complex continued fractions was submitted 29 November 1887 to the renown journal *Acta Mathematica* and published there in March 1888. It seems that this paper is the first to study complex continued fractions in a systematic way. At this time there was another study of this topic by the Italian mathematician Michelangeli [Michelangeli, 1887], and in a letter [Bianchi, 1959, p. 99] to his friend Bianchi from January 12, 1891 Adolf Hurwitz asked about Michelangeli's work [Michelangeli, 1887] on continued fractions. However, Hurwitz's results seem to go far beyond Michelangeli.[11] We cannot be sure whether he was aware of previous results as those of Jacobi or Stolz mentioned above. We may only speculate that he could have gotten a severe introduction to *real* continued fractions by his friend Stern; any research on *complex* continued fractions might have been unknown to him. During his studies in Munich, Adolf attended courses

[10] according to his [Hurwitz, 1919a, No. 5] from February 1886 to March 1888

[11] At least if the summery of Vivanti in the *Jahrbuch über die Fortschritte der Mathematik* [Vivanti, 2005] provides an appropriate picture of [Michelangeli, 1887]. Unfortunately, the author was not able to find a copy of Michelangeli's work.

by Seydel and Alfred Pringsheim (1859 - 1941), both well-known for their contributions to the theory of continued fractions; however, their research went in a different direction at that time, which does not exclude the possibility that continued fractions were a topic of their courses and interests.[12]

Summing up, Hurwitz's point of view is rather different from Jacobi or Stolz – namely, arithmetical, not analytical –, and it led to a revival of the arithmetical theory of continued fractions. His approach generalizes Minnigerode's approach from real to complex numbers and it indicates certain phenomena occuring with complex continued fractions that do not appear in the real case.

We shall briefly mention further work of Adolf Hurwitz on continued fractions. In [Hurwitz, 1889] he introduced a new type of semi-regular continued fractions and studied those with respect to equivalent numbers and quadratic forms. These so-called singular continued fractions are a mixture of the regular continued fraction and the continued fraction to the nearest integer. To explain that we recall Hitoshi Nakada's α-continued fraction from [Nakada, 1981]. Given a fixed real number $\alpha \in [\frac{1}{2}, 1]$, the α-continued fraction of a real number $x \in I_\alpha := [\alpha - 1, \alpha)$ is a convergent finite or infinite semi-regular continued fraction of the form

$$x = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \ddots + \cfrac{\epsilon_n}{a_n + \ddots}}},$$

where the partial quotients a_n are positive integers and the $\epsilon_n = \pm 1$ are signs determined

[12]"*Neben Klein, dessen Vorlesungen über seine Forschungen im Gebiet der Modulfunktionen ihn in hohem Masse fesselten, hörte er bei Gustav Bauer, Seydel, Pringsheim, Brill und Beetz. Bauer u. Pringsheim trat er persönlich näher ...*" [Samuel-Hurwitz, 1984, p. 6] We assume that *Seydel* is misspelled here. Indeed, Philipp Ludwig von Seidel (1821 - 1896), at that time professor in Munich, was working on continued fractions.

by iterations of the mapping T_α on $[\alpha-1,\alpha)$ given by $T_\alpha(0)=0$ and

$$T_\alpha(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$$

otherwise. For $\alpha = 1$ Nakada's α-continued fraction expansion is nothing but the regular continued fraction, for $\alpha = \frac{1}{2}$ one obtains the continued fraction to the nearest integer, and for $\alpha = \frac{\sqrt{5}-1}{2}$ it is the singular continued fraction due to Adolf Hurwitz [Hurwitz, 1889].

In [Hurwitz, 1891], Adolf showed that for any irrational real number x there exists an infinite sequence of rational numbers $\frac{p}{q}$ such that their distance to x is strictly less than $\frac{1}{\sqrt{5}q^2}$ and that this bound is best possible; this improves upon a previous result due to Charles Hermite in [Hermite, 1885] and is the starting point for all investigations on the Markov spectrum. Hurwitz's method of proof is based on the regular continued fraction expansion and the result can be found in diophantine textbooks under the keyword 'Hurwitz's Approximation Theorem' (although often with a proof using the Farey sequence, avoiding continued fractions). There are some refinements of this result, e.g., Borel's work [Borel, 1903]. Adolf Hurwitz's paper [Hurwitz, 1894b] provides a link between the Farey sequence and continued fractions of irrationals with an application to the Pell equation. In Section 7.3 we give an application of this flavor. Finally, in [Hurwitz, 1896b], Hurwitz generalized some classical results due to Euler and Lambert on e and related values to more general continued fractions with partial quotients which form an arithmetic progression.

This impressive list of publications may be seen as proof of our assumption from the beginning that continued fractions played a central role in Adolf Hurwitz's investigations.

3.2 A Complex Continued Fraction According to Julius

In Section 2.4 we have already stated some quotations of Ida Samuel-Hurwitz which indicate how closely the mathematical work of the elder Julius Hurwitz was dependening on of his younger brother. In fact, the support from Adolf is emphasized on the very first page of Julius' dissertation where he wrote that "*the thesis follows in aim and method two publications due to Mr. A. Hurwitz to whom I owe the encouragement for this investigation.*"[13] The two mentioned publications are [Hurwitz, 1888, Hurwitz, 1889]. Interestingly, it is his brother Adolf who wrote a review in Zentralblatt [Hurwitz, 1894a] about his brother's doctorate starting as follows:

> "The complex plane may be tiled by straight lines $x+y=v,\ x-y=v$, where v ranges through all positive and negative odd integers, into infinitely many squares. The centers of the squares are complex integers divisible by $1+i$. For an arbitrary complex number x, one may develop the chain of equations
>
> $$(1) \qquad x = a - \frac{1}{x_1}, \qquad x_1 = a_1 - \frac{1}{x_2}, \quad \ldots, \quad x_n = a_n - \frac{1}{x_{n+1}}, \quad \ldots$$
>
> following the rule that in general a_i is the center of the square which contains x_i. In the case when x_i is lying on the boundary of a square, some further rule has to be applied which we ignore here for the sake of brevity. For x the chain of equations (1) leads to a continued fraction expansion $x = (a, a_1, \ldots, a_n, x_{n+1})$ which further investigation is the topic of this work."[14]

[13] "*Die Arbeit schliesst sich, nach Ziel und Methode, eng an die nachstehend genannten zwei Abhandlungen des Herrn A. Hurwitz an, dem ich auch die Anregung zu dieser Untersuchung verdanke.*"

[14] "Die complexe Zahlenebene werde durch die Geraden $x+y=v, x-y=v$, wo alle positiven und negativen ungeraden ganzen Zahlen durchläuft, in unendlich viele Quadrate eingeteilt. Die Mittelpunkte dieser Quadrate werden durch die durch 1+i teilbaren ganzen complexen Zahlen besetzt. Wenn nun x eine beliebige complexe Zahl ist, so bilde man die Gleichungskette:

$$(1) \qquad x = a - \frac{1}{x_1}, \qquad x_1 = a_1 - \frac{1}{x_2}, \quad \ldots, \quad x_n = a_n - \frac{1}{x_{n+1}}, \quad \ldots$$

nach der Massgabe, dass allgemein a_i den Mittelpunkt desjenigen Quadrates bezeichnet, in welches der Punkt x_i hineinfällt. Dabei sind noch bezüglich des Falles, wo x_i auf den Rand eines Quadrates fällt, On the first view Julius' expansion could be mistaken as a specification of the general complex continued fraction investigated by his younger brother. The key difference is a modification of the set of possible partial quotients. While Adolf Hurwitz allowed all Gaussian integers, Julius restricted this set to the ideal generated by $\alpha := 1+i$. It shall be noticed that 1 is not an element of this ideal, hence condition iii) of Adolf's setting for his 'system' S of partial quotients is not fulfilled. Julius' approach results in a thinner lattice in $\mathbb{C}$. This leads to a tiling of the complex plane and enables consequently the definition of a 'nearest' partial quotient $a_n \in (\alpha) = (1+i)\mathbb{Z}[i]$ to each complex number $z \in \mathbb{C}$. What turns up is an expansion of a complex continued fraction

$$z = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 + T^3 z}}},$$

where each iteration is determined by a mapping T. In Chapter 4 we specify the definition of T in modern language, slightly different from what can be found in Julius' dissertation [Hurwitz, 1895].

A natural question is *which complex numbers have a finite continued fraction expansion?* Of course, the answer may depend on the type of continued fraction expansion, and, indeed, there are differences between the continued fraction proposed by Adolf Hurwitz and the one investigated by his brother Julius. For Adolf's continued fractions the expansion for a complex number z terminates if, and only if, $z \in \mathbb{Q}(i)$, as follows from the analogue of the euclidean algorithm for the ring of integers $\mathbb{Z}[i]$. However, the situation in the case of

besondere Festsetzungen getroffen, die wir der Kürze halber übergehen. Durch die Gleichungskette (1) wird nun für x eine bestimmte Kettenbruchentwickelung $x = (a, a_1, \ldots, a_n, x_{n+1})$ gegeben, deren nähere Untersuchung der Gegenstand der Arbeit ist."

Julius Hurwitz-continued fractions is different:

$$\begin{aligned} \frac{1+8i}{5+7i} &= [1-i, -2i, -4], \\ \frac{18}{95} &= [6, -2, 2, -2, -2, 2], \end{aligned}$$

whereas

$$\begin{aligned} \frac{1+7i}{5+7i} &= [1-i, -2i, 3+i, 0, \ldots] \neq [1-i, -2i, 3+i], \\ \frac{17}{95} &= [6, -2, -2, -2, 0, \ldots] \neq [6, -2, -2, -2]. \end{aligned}$$

The latter examples are related to the case of iterates of z coming from the complementary set $1 + (1+i)\mathbb{Z}[i]$ of partial quotients; we observe

$$i = \frac{1}{-i} = \frac{1}{1-i-1} = \frac{1}{1-i+\frac{1}{-i}} = \ldots = [1-i, 0, 0, \ldots] \neq [1-i] = \frac{1}{1-i} = \frac{1+i}{2},$$

here $[1-i, 0, 0, \ldots]$ is not convergent, which shows that there is no convergence in such cases. Generalizing from these examples one can find the following

Theorem 3.2.1 *The Julius Hurwitz-continued fraction for a complex number z is finite if, and only if, $z = \frac{a}{b}$ with coprime $a, b \in \mathbb{Z}[i]$ satisfying either $a \equiv 1, b \equiv 0 \mod \alpha$ or $a \equiv 0, b \equiv 1 \mod \alpha$.*

This result is due to Julius Hurwitz's thesis [Hurwitz, 1895][15]; we give a short algebraic proof in place of Julius' lengthy geometric reasoning.

Proof. First of all, recall that any partial quotient a_j is a multiple of $\alpha = 1+i$. Thus, each partial quotient is an element of the ideal generated by α, that is the set $(\alpha) = (1+i)\mathbb{Z}[i]$. The set of Gaussian integers is a disjoint union of (α) and $1 + (1+i)\mathbb{Z}[i]$; this follows

[15] "*Die Kettenbruch-Entwicklung erster Art einer complexen rationalen Zahl endigt dann und nur dann mit einem nicht durch $1+i$ teilbaren Teilnenner, wenn, nach Forthebung gemeinsamer Faktoren, weder der Zähler noch der Nenner der Zahl durch $1+i$ teilbar sind.*"

immediately from the fact that $x + iy \in (\alpha)$ if, and only if, $x \equiv y \bmod 2$ (as explained in Subsection 4.1.1). Denoting the Julius Hurwitz-continued fraction of a complex number by $z = [a_0; a_1, \ldots, a_n, \ldots]$, the numerators p_n and denominators q_n of its convergents $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ satisfy the following recursion formulae:

$$\begin{cases} p_{-1} = \alpha,\ p_0 = a_0, \quad \text{and} \qquad p_n = a_n p_{n-1} + p_{n-2}, \\ \\ q_{-1} = 0,\ q_0 = \alpha, \quad \text{and} \qquad q_n = a_n q_{n-1} + q_{n-2}. \end{cases}$$

The proof is analogous to the one for regular continued fractions, only the initial values differ. Without loss of generality we may assume that $a_0 = 0$. As for real continued fractions we may rewrite the recursion formulae in terms of 2×2-matrices as

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad (n \in \mathbb{N}_0).$$

In view of

$$\frac{p_1}{q_1} = \frac{\alpha}{a_1 \alpha} = \frac{1}{a_1}, \qquad \frac{p_2}{q_2} = \frac{a_2 \alpha}{a_2 a_1 \alpha + \alpha} = \frac{a_2 \cdot \alpha}{(a_2 a_1 + 1) \cdot \alpha} = \frac{a_2}{a_2 a_1 + 1},$$

we find, after reducing the convergents $\frac{p_n}{q_n}$ with respect to common powers of α in the numerator and denominator, that

$$\begin{pmatrix} p_2 & p_1 \\ q_2 & q_1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod \alpha,$$

where the congruence is with respect to each entry. It thus follows from the recurrence formulae by a simple induction on n that

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \mod \alpha,$$

Hence, comparing the first columns on both sides, we have shown that each finite Julius Hurwitz-continued fraction is of the form predicted by the theorem.

The converse can be verified by use of Fermat's descend method (see proof of Lemma 4.1.4). q.e.d.

As often the complex viewpoint provides new insights about the real case. Julius Hurwitz's continued fraction applied to real numbers leads to partial quotients from $2\mathbb{Z}$, and indeed, these continued fraction expansions coincide with the continued fraction with even partial quotients which has been studied first by Schweiger [Schweiger, 1982].[16]

In order to have always a convergent expansion one may allow a last partial quotient from $\mathbb{Z}[i]$; then all rational Gaussian integers have a finite Julius Hurwitz-continued fraction. A related question is what kind of partial quotients can occur. For instance, if a partial quotient equals $a_n = 1 + i$, then a simple geometric analysis shows that the next partial quotient a_{n+1} is different from $2, 1 - i, -2i$. A similar problem was already considered by Adolf Hurwitz for his continued fractions to the nearest Gaussian integers. Julius' doctorate as well contains results about those *admissible* sequences. Therefore, he separated partial quotients into those of type $\pm 1 \pm i$ and those of type ± 2 respectively $\pm i2$. In the following course of his studies, Julius Hurwitz stated a table listing all *impossible*, not admissible, sequences. To explain this table, we first need to analyze his notation. Partial quotients are said to be of *type* $1 + i$ if they are of the form $k(1 + i); k = +1, +2, \ldots$, they

[16] In [Perron, 1954, vl. 1, p. 186] Perron wrote, "Eine andere Vorschrift für die Wahl der b_ν [Teilnenner] stammt von J. Hurwitz [1]; sie bildet das Analogon zu den halbregelmäßigen Kettenbrüchen mit geraden Teilnennern." Actually, Julius Hurwitz-continued fractions of real numbers are exactly continued fractions with even partial quotients.

are of *type* $1-i$ if they are of the form $k(1-i); k = +1, +2, \ldots$, they are of *type* $-1+i$ if they are of the form $k(1-i); k = -1, -2, \ldots$ and the are of *type* $-1-i$ if they are of the form $k(1+i); k = -1, -2, \ldots$. Furthermore, partial quotients are of *type* 2 when they are located in between the angle bisector of the first and the fourth quadrant, -2 when they are located between second and third quandrant and so on.

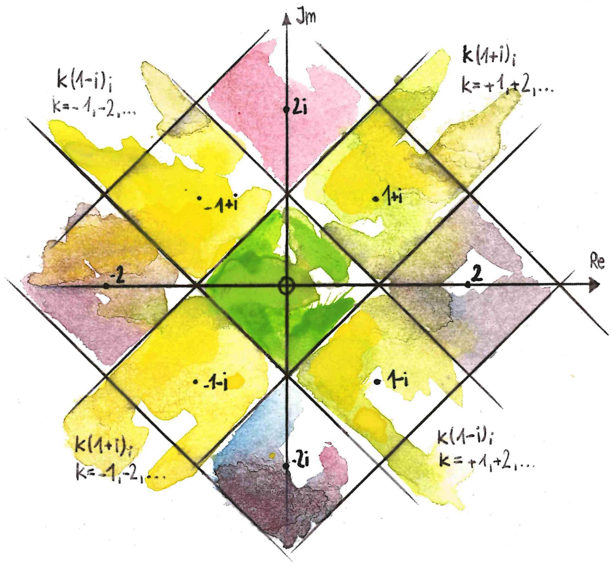


Figure 3.2: Illustration of the types of partial quotients and the associated complete tiling of the complex plane.

Notice that this typification provides a complete tiling of the complex plane (see Fig 3.2). On page 12 in [Hurwitz, 1895] Julius Hurwitz proved

Lemma 3.2.2 ***(J. Hurwitz, 1895)***

The following rules for consecutive partial quotients hold:

If a_r is of the type $1+i$,then a_{r+1} is not of type $2, 1-i$ or $-2i$.

If a_r is of the type $-1+i$,then a_{r+1} is not of type $-2i, -1-i$ or -2.

If a_r is of the type $-1-i$,then a_{r+1} is not of type $-2, -1+i$ or $2i$.

If a_r is of the type $1-i$,then a_{r+1} is not type $2i, 1+i$ or 2.

Remark: This is exactly the part in Adolf Hurwitz's diary entry entitled "Concerning Julius' work" (see Fig. 2.51) to which we already refered in Section 2.4.

Adolf described in his *Jahrbuch über die Fortschritte der Mathematik* review [Hurwitz, 1894a] that if the partial quotients

> "[...] $a_0, a_1, \ldots, a_n, x_{n+1}$ fulfill certain constraints. [...] Then it is proved that the expansion of any complex quantity converges, that it is finite, resp. periodic, if the quantity is a rational complex number, resp. satisfies a quadratic equation with complex integer coefficients. There is a relation of the investigated continued fraction expansion with another in which the tiling of the complex plane in the above mentioned squares is replaced by domains bounded by circular arcs. For this second expansion analoguous results are proved as for the first."[17]

These continued fractions of the second kind are the singular continued fractions developed by Adolf Hurwitz in [Hurwitz, 1889].

Building on his continued fraction expansion to the nearest Gaussian integer in $(1+i)\mathbb{Z}[i]$ from his dissertation, in his habilitation thesis Julius Hurwitz developed a method for reduction of quadratic forms with complex coefficients and variables; this was published as [Hurwitz, 1902b] in the same renown journal *Acta Mathematica* as his younger brother's initial paper [Hurwitz, 1888]. Already during the preparation of his dissertation Julius had this very application in mind. Actually, Wangerin wrote in his report about Julius Hurwitz's dissertation that

> "The author believes that the investigated continued fraction expansion may

[17] "[...] $a, a_1, \ldots, a_n, x_{n+1}$ gewisse Bedingungen erfüllen. [...] Sodann wird der Nachweis geführt, dass die Entwickelung jeder complexen Grösse convergirt, dass sie abbricht, resp. periodisch wird, wenn die Grösse eine complexe rationale Zahl ist, resp. einer quadratischen Gleichung mit ganzzahlig complexen Coefficienten genügt. Mit der untersuchten Kettenbruchentwickelung steht nun ferner eine andere in genauestem Zusammenhange, bei welcher an Stelle der Einteilung der complexen Zahlenebene in die oben genannten Quadrate eine solche in Gebiete tritt, die von Kreisstücken begrenzt sind. Für diese zweite Entwickelung werden die analogen Sätze wie für die erste bewiesen."

> serve as basis for a theory of quadratic forms with complex variables and complex coefficients."[18] [Universitätsarchiv, 1895]

Julius' approach is quite similar to the one for real quadratic forms with positive discriminant. He solved the problem of determining whether two given forms with equal discriminant are equivalent, and to find all substitutions which transform one form to any of its equivalent forms. Both questions were already solved by Peter Gustav Lejeune Dirichlet (1805 - 1859) [Dirichlet, 1842] by a different method.

3.3 Some Historical Notes

Continued fractions are always linked with the name *Perron*. Oskar Perron was born May 7, 1880 in Frankenthal, and died in Munich February 22, 1975. Perron studied in Munich and obtained there a doctorate in 1902; his dissertation was about the rotation of a rigid body and was supervised by Lindemann (who was already responsible for Adolf Hurwitz's appointment to Königsberg). In his post-doctoral research, however, Perron became interested in the work of Pringsheim, another professor at Munich with an expertise in complex analysis (and father of Katharina Pringsheim the later Katia Mann, wife of Thomas Mann). At that time Pringsheim [Pringsheim, 1900] investigated the Stern-Stolz criterion [Stolz, 1885] for convergence of periodic complex continued fractions. Perron did his habilitation [Perron, 1907] in 1907 on a related question, namely Jacobi's general continued fraction algorithm [Jacobi, 1868]; this topic led Perron and later Frobenius to the discovery of the famous Perron-Frobenius theorem for matrices with non-negative entries (see [Hawkins, 2008]).

The standard reference in the theory of continued fractions is Perron's monograph [Perron, 1913] but neither the first edition from 1913 nor the second edition from 1929 mention Adolf or Julius Hurwitz's work on complex continued fractions. However, the

[18] "Der Verfasser glaubt, dass die von ihm genauer erforschte Art der Kettenbruchentwicklung als Grundlage für die Theorie der quadratischen Formen mit complexen Variablen und complexen Koeffizienten dienen könne."

third edition [Perron, 1954, vl. 1] from 1954 contains a whole section on continued fractions in imaginary quadratic number fields (§46). Here Perron gives a brief introduction to Adolf Hurwitz's work [Hurwitz, 1888]. A proof is given that this expansion is finite if, and only if, the number in question is a Gaussian rational. Much of attention is also paid to approximation properties as well as to admissible sequences, and periodic complex continued fractions. It is a natural question to ask for what other fields than $\mathbb{Q}(i)$ one can obtain similar results. It seems that Leonard Eugene Dickson (1874 - 1954) was the first to investigate in which quadratic fields $\mathbb{Q}(\sqrt{D})$ an analogue of the euclidean algorithm is possible [Dickson, 1927]. He proved that for imaginary quadratic fields there exists a euclidean algorithm if, and only if, $D = -1, -2, -3, -7, -11$; however, his proof for real quadratic fields turned out to be false, and was corrected by Perron [Perron, 1932]. With regard to this, Paul Lunz (1909 - ?) [Lunz, 1937] considered in his dissertation (supervised by Perron) the field $\mathbb{Q}(\sqrt{-2})$. Already in this case fundamental questions as, e.g., the growth of the denominators of the convergents in absolute value, seem to be more difficult to answer than in the Gaussian number field. Further studies were made by Axel Arwin (1879 - 1935) [Arwin, 1926, Arwin, 1928] for several other imaginary quadratic fields. Hilde Gintner proved in her PhD thesis [Gintner, 1936] at the University of Vienna in 1936 that in non-euclidean imaginary quadratic number fields one can find examples where the corresponding continued fraction expansion does not converge, e.g.,

$$z = \tfrac{1}{2}\sqrt{-d} \quad \text{if } d \not\equiv 3 \bmod 4, \qquad z = \tfrac{2d+1}{2d}\sqrt{-d} \quad \text{if } d \equiv 3 \bmod 4.$$

Summing up, it follows that a continued fraction expansion to the nearest integer is possible if, and only if, the order of the imaginary quadratic field is euclidean. Moreover, she studied diophantine approximation in imaginary quadratic fields not only with continued fractions but using Minkowski's geometry of numbers in a rather general setting. Further results along these lines were found by her thesis advisor Nikolaus Hofreiter (1904 - 1990) [Hofreiter, 1938] and several more mathematicians. Hereon, an overview will be outlined in

Chapter 7 where we state some new results arising from application of Gintner's approach.

It seems that Julius Hurwitz's work had not found many readers. In 1918, Ford [Ford, 1918] wrote in a footnote: "The continued fractions involving complex integers have been little studied. Only one of such fraction has, so far as I know, appeared in the literature. See [Adolf] Hurwitz, Acta Mathematica, vol. 11 (1887), pp. 187-220; Auric, Journal de mathématiques, 5th ser., vol. 8 (1902), pp. 387-431." Ford is mostly interested in extending Hermite's approach for rational approximations to complex numbers rather than in Adolf Hurwitz's treatise of complex continued fractions. Auric [Auric, 1902] gave further applications of Adolf Hurwitz's continued fractions.

In 1912, George Ballard Mathews (1861 - 1922) considered binary quadratic forms with complex coefficients in [Mathews, 1912]; he stressed that his approach differs from Julius Hurwitz's method in his habilitation thesis, published as [Hurwitz, 1902b]. Mathews avoided to consider points of condensation; his reasoning showed immediately that the number of reduced froms is finite. Moreover, "the roots of a reduced form are expressible as pure recurrent chain-fractions[19] appears as a corollary, instead of being a definition" (see [Mathews, 1912]).

In 1927, Anna Stein (1894 - ?) [Stein, 1927] used in her dissertation Julius Hurwitz-continued fractions in order to compute units in quadratic extensions of number fields. This line of investigation was proposed by her supervisor Helmut Hasse (1898 - 1979). The divisibility of the denominators of the convergents by $1 + i$ is here an essential tool.

[19] Here 'chain-fraction' is the word-for-word translation of the German term for continued fraction: 'Kettenbruch'.

4 Modern Developments of Complex Continued Fractions

In the twentieth century continued fractions were studied for many different reasons. Whereas in the nineteenth century much attention was given to convergence criteria and diophantine approximation new lines of investigation were the ergodic theory of continued fractions and the consideration of a continued fraction expansion as a product of linear fractional transformations. The latter approach is related to the modular group while the former has roots in an old problem on the statistics of the partial quotients of the regular continued fraction expansion posed by Gauss.

The first sentence of [Ito and Tanaka, 1981] illustrates the different approaches for research on continued fractions: "The simple continued-fraction expansion of real numbers is an important concept in the theory of numbers. And the continued-fraction expansion defined by Hurwitz is also important because it is the expansion by the nearest integers. These two continued-fraction expansions give rise to many interesting problems not only in the theory of numbers but also in ergodic theory." Here Adolf Hurwitz's continued fraction is meant. Later we stress that there is previous work by Kaneiwa, Shiokawa and Tamura [Kaneiwa and Shiokawa, 1975, Kaneiwa and Shiokawa, 1976] which already contains a certain similarity to Julius' continued fraction. Their intention was likewise: admissible sequences are discussed and diophantine properties investigated, and a Lagrange-type theorem for general quadratic extensions is included in [Kaneiwa and Shiokawa, 1976]. In a subsequent paper [Shiokawa, 1976], Shiokawa introduced tools from ergodic theory in order to deduce several metrical results about this complex continued fraction expansion. In none of these papers Julius' work is mentioned, it seemed that his insights were fallen into oblivion.

In 1985, about one century after Julius' doctorate, Shigeru Tanaka published the identical continued fraction transformation a second time [Tanaka, 1985]. Although he refers to the work of Adolf Hurwitz, it is rather unlikely that he was familiar with the thesis of Julius which was only published in German and is hardly accessible (if one does not know that a short version is contained as first part in [Hurwitz, 1902b]).[1] Furthermore, his approach is very different. While Julius' point of view is quite geometrical, Tanaka's motivation was to investigate ergodic properties of the continued fraction. In the following we outline how he succeeded to determine a natural extension of the transformation T and therewith to construct an invariant measure with respect to whom the continued fraction map T is ergodic. With this property Tanaka was able to transfer results from Shiokawa's ergodic theoretical approach to Adolf's continued fraction to the theory of Julius' complex continued fraction. In that sense Tanaka's work is building on Shiokawa's in a similar way as Julius was continuing Adolf's approach.

4.1 J. Hurwitz's Algorithm = Tanaka's Algorithm

On the first glance, with its modern terminology, Shigeru Tanaka's algorithm seems to provide a typical recent, innovative approach to complex continued fractions. However, on the second glance, it turns out that Tanaka's continued fraction expansion equals Julius Hurwitz's complex continued fractions. In this section we adopt Tanaka's approach to receive new tools for handling characteristics of this complex continued fraction.

[1] Perron's monograph [Perron, 1913] is not yet translated into English!

4.1.1 Basics

Tanaka's approach is based on the following representation of complex numbers z,

$$z = x\alpha + y\overline{\alpha},$$

with $x, y \in \mathbb{R}$ and $\alpha = 1 + i$ respectively $\overline{\alpha} = 1 - i$. Obviously, all complex numbers z have therewith a representation

$$z = x(1+i) + y(1-i) = (x+y) + i(x-y), \tag{4.1}$$

where x and y are uniquely determined by solving the system of linear equations $\operatorname{Re} z = x + y$ and $\operatorname{Im} z = x - y$.

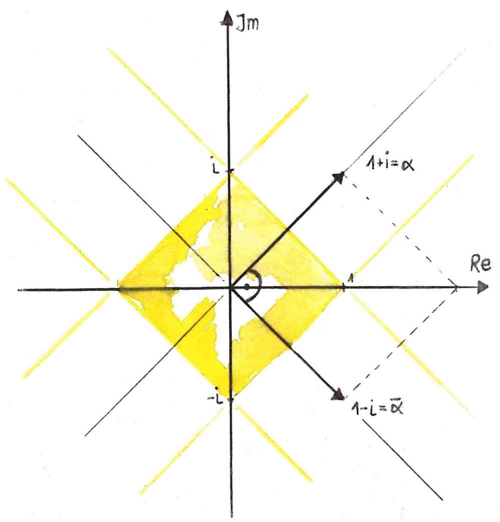


Figure 4.1: Illustration of Tanaka's change of coordinates $\{1, i\} \to \{\alpha, \overline{\alpha}\}$ along the angle bisectors of the first and third quadrant.

To facilitate the following calculations, we illustrate this representation of complex numbers. We have

$$z = a + ib = \left(\frac{1}{2}(a+b)\right)\alpha + \left(\frac{1}{2}(a-b)\right)\overline{\alpha},$$

which leads to

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Accordingly, the sets

$$I_0 := \{n\alpha + m\overline{\alpha} \ : \ n, m \in \mathbb{Z}\},$$

respectively

$$I := I_0 \setminus \{0\}$$

describe subsets of the set of Gaussian integers $\mathbb{Z}[i]$. Since

$$n + m \equiv n - m \bmod 2 \tag{4.2}$$

with $n, m \in Z$, I_0 contains exactly those numbers z, whose real part and imaginary part have even distance. Writing I_0 in the form

$$I_0 = \{a + ib \in \mathbb{Z}[i] \ : \ a \equiv b \bmod 2\} = (1+i)\mathbb{Z}[i] = (\alpha),$$

it occurs that I_0 is the principal ideal in the ring of integers $\mathbb{Z}[i]$ generated by α. In fact, this ideal is identical to the set of possible partial quotients in Julius Hurwitz's case.

To construct a corresponding continued fraction expansion, firstly a complex analogue to the Gaussian brackets is required. To identify the nearest integer to z from I_0, we define

$$[\,.\,]_T : \mathbb{C} \to I_0,$$

$$[z]_T = \left\lfloor x + \frac{1}{2} \right\rfloor \alpha + \left\lfloor y + \frac{1}{2} \right\rfloor \overline{\alpha}.$$

Furthermore, we specify a fundamental domain

$$X = \left\{ z = x\alpha + y\overline{\alpha} \ : \ \frac{-1}{2} \leq x, y < \frac{1}{2} \right\},$$

and define a map $T : X \to X$ by $T0 = 0$ and

$$Tz = \frac{1}{z} - \left[\frac{1}{z}\right]_T \quad \text{for } z \neq 0.$$

By iteration, we obtain the expansion

$$z = \cfrac{1}{a_1 + Tz} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2 z}} = \cdots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + T^n z}}},$$

with partial quotients $a_n := a_n(z) = \left[\frac{1}{T^{n-1}z}\right]_T$. As in the real case, convergents $\frac{p_n}{q_n}$ to z are defined by 'cutting' the continued fraction after the nth partial quotient. Setting

$$p_{-1} = \alpha,\ p_0 = 0, \ \text{ and } p_{n+1} = a_{n+1}p_n + p_{n-1} \text{ for } n \geq 1, \tag{4.3}$$

as well as

$$q_{-1} = 0,\ q_0 = \alpha, \ \text{ and } q_{n+1} = a_{n+1}q_n + q_{n-1} \text{ for } n \geq 1, \tag{4.4}$$

by the same reasoning as for regular continued fractions for real numbers, z can be expressed as

$$z = \frac{p_n + T^n z p_{n-1}}{q_n + T^n z q_{n-1}}. \tag{4.5}$$

Remark: In the beginning of this chapter we mentioned that there is a certain similarity between Julius Hurwitz's continued fraction and work of Kaneiwa, Shiokawa and

Tamura in 1975/6 [Kaneiwa and Shiokawa, 1975, Kaneiwa and Shiokawa, 1976]. In fact, if ρ denotes the third root of unity in the upper half-plane, they defined a bracket by

$$[z] := \lfloor u \rfloor \rho + \lfloor v \rfloor \overline{\rho} \qquad \text{for} \quad z = u\rho + v\overline{\rho}$$

with real u and v. Given a complex number z, iterations of the transform $z \mapsto \frac{1}{z} - [\frac{1}{z}]$ yield a continued fraction in a similar way as for Julius' expansion.

4.1.2 Example

As illustrating example we choose $x = \frac{1}{8}, y = \frac{3}{8}$ and receive

$$z = \frac{1}{2}(1+i) - \frac{1}{4}(1-i) = \frac{1}{8} + i\frac{3}{8} = \frac{1}{8}(1-3i).$$

For the first transformation Tz, we calculate $\frac{1}{z}$ and $\left[\frac{1}{z}\right]_T$:

$$\frac{1}{z} = \frac{8}{1-3i} = \frac{8(1+3i)}{(1-3i)(1+3i)} = \frac{8}{10}(1+3i) = \frac{4}{5}(1+3i).$$

Following Tanaka, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{4}{5} \\ \frac{12}{5} \end{pmatrix}$$

respectively $x = \frac{8}{5}, y = -\frac{4}{5}$. Thus, we have

$$a_1 = \left[\frac{1}{z}\right]_T = \left\lfloor \frac{8}{5} + \frac{1}{2} \right\rfloor \alpha + \left\lfloor -\frac{4}{5} + \frac{1}{2} \right\rfloor \overline{\alpha} = 2\alpha - \overline{\alpha} = 1 + 3i.$$

Consequently, we receive the continued fraction expansion with first partial quotient

$$z = \frac{1}{8}(1-3i) = \frac{1}{\frac{1}{z}} = \frac{1}{\frac{4}{5}(2\alpha - \overline{\alpha})} = \frac{1}{2\alpha - \overline{\alpha} + \frac{4}{5}(2\alpha - \overline{\alpha}) - (2\alpha - \overline{\alpha})} = \frac{1}{a_1 + Tz}.$$

For the second partial quotient, we calculate

$$Tz = \frac{4}{5}(2\alpha - \overline{\alpha}) - (2\alpha - \overline{\alpha}) = -\frac{2}{5}\alpha + \frac{1}{5}\overline{\alpha} = -\frac{1}{5} - \frac{3}{5}i = -\frac{1}{5}(1+3i),$$

$$\frac{1}{Tz} = \frac{-5(1-3i)}{(1+3i)(1-3i)} = \frac{-5(1-3i)}{10} = -\frac{1}{2}(1-3i) = \frac{1}{2}(\alpha - 2\overline{\alpha})$$

and

$$a_2 = \left[\frac{1}{Tz}\right]_T = \left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor \alpha + \left\lfloor -1 + \frac{1}{2} \right\rfloor \overline{\alpha} = \alpha - \overline{\alpha} = 2i.$$

For the third partial quotient, we calculate

$$T^2 z = -\frac{1}{2}(1+i),$$

$$\frac{1}{T^2 z} = \frac{-2(1-i)}{(1+i)(1-i)} = -1 + i = -\overline{\alpha}$$

and

$$a_3 = \left[\frac{1}{T^2 z}\right]_T = \left\lfloor -1 + \frac{1}{2} \right\rfloor \overline{\alpha} = -1 + i.$$

Since

$$T^3 z = \frac{1}{T^2 z} - \left[\frac{1}{T^2 z}\right]_T = 0,$$

it follows that the algorithm terminates and that the finite continued fraction is given by

$$\frac{1-3i}{8} = \cfrac{1}{1+3i+\cfrac{1}{2i+\cfrac{1}{-1+i}}}.$$

4.1.3 Some Considerations and Characteristics

To deepen the understanding of the behaviour of T we examine some of its characteristcs generating the complex continued fraction expansion. First of all, we verify that T indeed maps X to X and determine the region in which $\frac{1}{z}$ is located.

Lemma 4.1.1 *For $z \in X$ we have $Tz \in X$.*

Proof. Considering $0 \neq z = a + ib = \frac{1}{2}(a+b)\alpha + \frac{1}{2}(a-b)\overline{\alpha} \in X$, it follows

$$\frac{1}{z} = \frac{(a-b)}{2(a^2+b^2)}\alpha + \frac{(a+b)}{2(a^2+b^2)}\overline{\alpha}$$

and

$$\left[\frac{1}{z}\right]_T = \left\lfloor \frac{a^2+a-b+b^2}{2(a^2+b^2)} \right\rfloor \alpha + \left\lfloor \frac{a^2+a+b+b^2}{2(a^2+b^2)} \right\rfloor \overline{\alpha}.$$

To show that $Tz = \frac{1}{z} - \left[\frac{1}{z}\right]_T$ lies in X one takes the difference of the respective x or y-values. We receive

$$-\frac{1}{2} \leq \frac{(a \pm b)}{2(a^2+b^2)} - \left\lfloor \frac{a^2+a\pm b+b^2}{2(a^2+b^2)} \right\rfloor \leq \frac{1}{2},$$

because leaving off the Gaussian brackets in the middle leads to $-\frac{1}{2}$. q.e.d.

In view of Lemma 4.1.1 the continued fraction expansion follows from iterating T over and over again. Since $Tz \in X$, we can localize $\frac{1}{z}$ by inverting the minimal and maximal possible value of $|z|$.

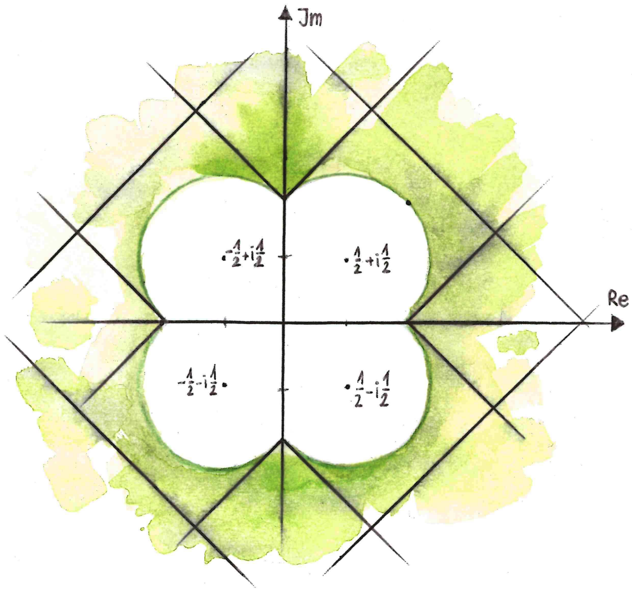


Figure 4.2: Taking the geometry of X into account, an easy computation shows that $\frac{1}{z}$ is always located in the infinite region outside the four semicircles centered at $\pm\frac{1}{2} \pm i\frac{1}{2}$ with radius $\frac{1}{\sqrt{2}}$ (see Subsection 4.1.4 for more details).

A natural question we have already answered in Section 3.2 by Theorem 3.2.1 is: *When does the algorithm terminate?* In this section we give a more detailed proof of this theorem without using of matrices, that provides slightly more information. Therefore, we introduce a certain tiling of the set of Gaussian integers on behalf of

$$J := 1 + (\alpha) = \{c + id \in \mathbb{Z}[i] \; : \; c \not\equiv d \bmod 2\},$$

which leads to the disjoint decomposition

$$\mathbb{Z}[i] = I_0 \,\dot\cup\, J$$

with I_0 as defined in Subsection 4.1.1.

Theorem 4.1.2 *The algorithm terminates if, and only if,*

$$z \in Q = \left\{ z = \frac{u}{v} \ : \ \textit{either } u \in I, v \in J \textit{ or } v \in I, u \in J \right\}.$$

In the sequel the set

$$M := \{z \in \mathbb{C} \ : \ T^n z = 0 \text{ for some } n \in \mathbb{N}\} \subset \mathbb{Q}[i]$$

will be examined. Since the algorithm terminates if, and only if, z is equal to a convergent $\frac{p_n}{q_n}$, M can also be written as

$$M := \left\{ z = \frac{p_n}{q_n} \text{ for some } n \in \mathbb{N} \right\}.$$

The proof of the theorem will be seperated into two parts. After showing $M \subset Q$, we verify the inverse inclusion $Q \subset M$. However, first some preliminary work needs to be done.

By definition $I_0 = \{n\alpha + m\overline{\alpha} \ : \ m, n \in \mathbb{Z}\} = (1+i)\mathbb{Z}[i] = (\alpha)$. Hence, for $z \in M$, we have

$$a_n = \left[\frac{1}{T^{n-1}z} \right]_T = \alpha b_n \ \text{ with } \ b_n \in \mathbb{Z}[i].$$

In accordance with (4.3) and (4.4) the first convergents are of the form

$$\frac{p_1}{q_1} = \frac{\alpha}{a_1\alpha} = \frac{\alpha}{\alpha^2 b_1} \ , \ \frac{p_2}{q_2} = \frac{a_2\alpha}{(a_2a_1+1)\alpha} = \frac{\alpha^2 b_2}{\alpha b_2'} \ , \ \cdots$$

with $b_2' \in \mathbb{Z}[i]$ and so on. We observe that the parity of the α-parts of numerator and denominator alternate in their exponents.

Lemma 4.1.3 *We have*

$$\frac{p_n}{\alpha} \in (\alpha) \ \Leftrightarrow \ 2|n \ \Leftrightarrow \ \frac{q_n}{\alpha} \notin (\alpha),$$

respectively

$$\frac{p_n}{\alpha} \notin (\alpha) \quad \Leftrightarrow \quad 2 \nmid n \quad \Leftrightarrow \quad \frac{q_n}{\alpha} \in (\alpha).$$

Proof. It is sufficient to consider the sequence of nominators (p_n), since (q_n) can be treated analogously. According to the recursion formula (4.3) we may write

$$\frac{p_n}{\alpha} = a_n \frac{p_{n-1}}{\alpha} + \frac{p_{n-2}}{\alpha},$$

what leads to a simple proof by induction. We begin with

- $n = -1 : \frac{p_{-1}}{\alpha} = \frac{\alpha}{\alpha} = 1 \notin (\alpha)$, and
- $n = 0 : \frac{p_0}{\alpha} = 0 \in (\alpha)$,

which satisfy the assertion. In the induction step we distinguish the two cases of n being even or odd.

1. case: $2|n$

   Since $n-2$ is even as well, the induction hypothesis provides $\frac{p_{n-2}}{\alpha} \in (\alpha)$. Moreover, we know $a_n \in (\alpha)$ and respectively $a_n \frac{p_{n-1}}{\alpha} \in (\alpha)$. On behalf of the recursion formula follows $\frac{p_n}{\alpha} \in (\alpha)$. Here we have used that (α) is an ideal.

2. case: $2 \nmid n$

   This case can be treated similarly: $\frac{p_{n-2}}{\alpha} \notin (\alpha), a_n \in (\alpha)$ implies $\frac{p_n}{\alpha} \notin (\alpha)$.

q.e.d.

Thus, we have verified $M \subset Q$ and for the proof of Theorem 4.1.2 it remains to prove the inverse inclusion.

Lemma 4.1.4 *We have*

$$Q \subset M.$$

In particular, all $z = \frac{u}{v} \in \mathbb{Q}(i)$ with either $u \in I, v \in J$ or $v \in I, u \in J$ lead to a terminating algorithm.

Proof. We assume that $O := Q \setminus M$ is not an empty set. Then there exists a number $z \in O$ of the form $z = \frac{u}{v} \in \frac{I}{J}$ (respectively $\in \frac{J}{I}$) with $T^n \frac{u}{v} \neq 0$, for which $|u|^2 + |v|^2$ is minimal. Without loss of generality we may assume $u \in I$ and $v \in J$.

We have

$$z = \frac{u}{v} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n \frac{u}{v}}}}}.$$

Furthermore, we put

$$z' = \frac{x}{y} = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n \frac{u}{v}}}}}.$$

Because of $T^n \frac{u}{v} = T^{n-1} \frac{x}{y}$, it follows that z' is as well an element of the set Q and satisfies the condition

$$T^n \frac{x}{y} \neq 0.$$

Moreover, we have

$$\frac{u}{v} = \frac{1}{a_1 + \frac{x}{y}} = \frac{y}{a_1 y + x} \tag{4.6}$$

and therewith $u = y \in I$ (respectively $\in J$) and $v = a_1 y + x \in J$ (respectively $\in I$). To go on, we observe the multiplicative and additive structure of I and J by noting

$$J \cdot I = I \cdot J = I \ , \ J \cdot J = J \ , \ I \cdot I = I,$$

and

$$J + I = I + J = J \ , \ J + J = I \ , \ J + J = J.$$

Of course, this is meant in the sense of Minkowski's multiplication and addition of sets, respectively, and can easily be shown on behalf of complex calculation.

With the constraint $a_1 \in I$, the structure of $\frac{x}{y}$ occurs:

1. For $u = y \in I$ and $v = a_1 y + x \in J$ we have that $a_1 y + x$ is an element of $I \cdot I + x$, which is a subset of J, if, and only if, $x \in J$.

2. For $u = y \in J$ and $v = a_1 y + x \in I$ we have that $a_1 y + x$ is an element of $I \cdot J + x$, which is a subset of I, if, and only if, $x \in I$.

Obviously, the structure of $z = \frac{u}{v}$ changes for $z' = \frac{x}{y}$ in an alternating way. The property of non-terminating as well as the observed structure above do also hold if $z = \frac{x}{y}$ is replaced by $\frac{x}{-y}$. With (4.6) we additionally obtain

$$|u| = |y| \text{ and } |v| = |a_1 y + x|$$

what leads to

$$|u|^2 + |v|^2 = |y|^2 + |a_1 y + x|^2.$$

Since $|u|^2 + |v|^2$ was assumed to be minimal, the inequality

$$|a_1 y + x|^2 \leq |x|^2 \tag{4.7}$$

follows. Since setting $z' = \frac{x}{y}$ or $z' = \frac{x}{-y}$ will not change anything of our previous observations, it is guaranteed that the case of real part and imaginary part of $a_1 y$ being positive can always be achieved. With $x = a + ib$, the inequality

$$|a_1 y + x|^2 = (\mathrm{Re}\,(a_1 y) + a)^2 + (\mathrm{Im}\,(a_1 y) + b)^2 > a^2 + b^2 = |x|^2$$

holds. This is a contradiction to (4.7), i.e., to the minimality of $z = \frac{u}{v}$, which proves that there is no number satisfying the assumption. Consequently, O is an empty set. q.e.d

Concluding Lemma 4.1.4 and Lemma 4.1.3, it follows that complex numbers $z \in \mathbb{C}$, having a finite continued fraction, are either of the form $z \in \frac{I}{J}$ or $z \in \frac{J}{I}$.

Remark: In Julius Hurwitz's doctoral thesis a whole chapter is dedicated to the question which complex numbers have a finite continued fraction expansion. There it is described that the continued fraction is finite in any case of rational complex numbers. This is because Hurwitz allowed a last partial quotient from $\mathbb{Z}[i]$ in those cases when irregularities occur from rational complex numbers with numerator and denominator divisible by $(1+i)$. That Shigeru Tanaka did not refer to those difficulties is certainly due to the fact that under an ergodic theoretical point of view those few irregularities are insignificant. The set $\mathbb{Q}(i)$ is countable and thus negligible with respect to applications from ergodic theory. However, this also indicates that Julius Hurwitz's thesis was probably unknown to Tanaka.

Next we examine some characteristics of the algorithm concerning its approximation property.

Lemma 4.1.5 *Let $a_{n+1} \neq 0$. For $k_n := \frac{q_{n+1}}{q_n}$ we have $|k_n| > 1$.*

Notice that for $a_{n+1} = 0$ the recursion formula leads to $q_{n+1} = q_{n-1}$ and that the continued fraction is finite.

Proof. We suppose that all previous $k_1, \ldots, k_{n-1}$ are of absolute value > 1, here certainly $|k_1| \geq \sqrt{2}$ (because $|a_j| \geq \sqrt{2}$ for every $a_j \in I$). Thus,

$$k_n = a_{n+1} + \frac{1}{k_{n-1}} \in \{z \in \mathbb{C} \,:\, |z - a_n| < 1\}.$$

We assume $|k_n| < 1$, hence,

$$|a_{n+1}| = \left| k_n - \frac{1}{k_{n-1}} \right| \leq |k_n| + \frac{1}{|k_{n-1}|} < 2$$

and consequently $a_{n+1} = \pm 1 \pm i$. Without loss of generality we consider $a_{n+1} = 1 + i$. By a backwards calculation we determine

$$a_{n-2k} = -2 + 2i \text{ and } a_{n-2k-1} = 2 + 2i$$

for all $k \in \mathbb{N}_0$ with $2k < n$. The following considerations are by induction:
For $n = 1$ we have $k_1 = a_2 = 1 + i$ which leads to $|k_1| = \sqrt{2} > 1$.
For $n = 2$ we have $k_2 = a_3 + \frac{1}{-2+2i} = \frac{3}{4}(1 + i)$ which leads to $|k_1| = \frac{3}{4}\sqrt{2} > 1$.
For $n \geq 3$ we have $k_1 = \pm 2 + 2i$ which leads to $|k_1| = \sqrt{8}$ and $k_2 = \pm 2 + 2i + \frac{1}{k_1}$ which leads to $|k_2| \geq \sqrt{8} - \frac{1}{|k_1|}$.
Therefore, we observe $k_j = \pm 2 + 2i + \frac{1}{k_{j-i}}$ as well as $|k_j| \geq \sqrt{8} - \frac{1}{|k_{j-1}|} =: y_j \in \mathbb{R}$. We consider the recursion $y_j = \sqrt{8} - \frac{1}{y_{j-1}}$ with

$$y_1 = \sqrt{8}, y_2 = \sqrt{8} - \frac{1}{\sqrt{8}} = \frac{7}{4}\sqrt{2} < \sqrt{2} + 1.$$

Under the assumption $y_j < y_{j-1} < \ldots$ it follows by induction that

$$y_{j+1} = \sqrt{8} - \frac{1}{y_j} < \sqrt{8} - \frac{1}{y_{j-1}} = y_j$$

as well as

$$y_{j+1} = \sqrt{8} - \frac{1}{y_j} > \sqrt{8} - \frac{1}{\sqrt{2} + 1} = \sqrt{2} + 1.$$

Hence, for all $j < n$ the inequality

$$k_j \geq y_j > \sqrt{2} + 1$$

holds. Consequently, with $k_n = a_n + \frac{1}{k_{n-1}}$ this leads to

$$|k_n| = \left|1 + i + \frac{1}{k_{n-1}}\right| > \sqrt{2} - \frac{1}{\sqrt{2} + 1} = 1.$$

q.e.d.

In the following, we shall use the previous lemma in order to show that the continued fraction expansion actually converges.

4.1.4 Geometrical Approach to the Approximation Behaviour

We have already indicated that, following Tanaka's line of argumentation, the approximation properties of his algorithm can be illustrated geometrically. This happens essentially on behalf of two tilings.

Firstly, the fundamental domain X is split into disjoint so-called *T-cells*. This is done with respect to the first n partial quotients as follows. We define the set $A(n)$ of sequences of partial quotients by

$$A(n) = \{a_1(z), a_2(z), \ldots, a_n(z) : z \in X\}.$$

Such sequences are called *T-admissible*. One should notice that certain sequences of numbers from $(1+i)\mathbb{Z}[i]$ cannot appear as sequence of partial quotients, which was examined by Julius Hurwitz (see Lemma 3.2.2).

Corresponding to each admissible sequence $a_1, a_2, \ldots, a_n \in A(n)$ the subset $X(a_1, a_2 \ldots, a_n)$ of X arises as

$$X(a_1, a_2 \ldots, a_n) = \{z \in X : a_k(z) = a_k \text{ for } 1 \leq k \leq n\};$$

each such set is called a T-cell. We have

$$X = \bigcup_{a_1, a_2 \ldots, a_n \in A(n)} X(a_1, a_2 \ldots, a_n).$$

In other words, T-cells describe a 'close' neighborhood of a certain $z \in X$.

The second tiling is corresponding to the set of reciprocals

$$X^{-1} = \left\{ \frac{1}{z} : z \in X, z \neq 0 \right\}.$$

In each iteration of the continued fraction algorithm a complex number

$$T^n z = \frac{1}{T^{n-1}z} - \left[\frac{1}{T^{n-1}t} \right]_T$$

arises, which is split into an integral partial quotient and a remainder. The latter is going to be iterated again. We thus receive a sequence of elements of X^{-1}. By taking the reciprocals, the edges of X are transformed to arcs of discs. We define

$$U_1 := \left\{ z \in X : \left| z + \frac{\alpha}{2} \right| \geq \frac{1}{\sqrt{2}} \right\}.$$

Writing $z = x\alpha + y\overline{\alpha}$, we have

$$\left| z + \frac{\alpha}{2} \right| = \left| \left(x + y + \frac{1}{2} \right) + i \left(x - y + \frac{1}{2} \right) \right| = \sqrt{ \left(\mathrm{Re}\,(z) + \frac{1}{2} \right)^2 + \left(\mathrm{Im}\,(z) + \frac{1}{2} \right)^2 } \geq \frac{1}{\sqrt{2}}$$

including elements $z \in X$, while excluding numbers which lie inside the disc of radius $\frac{1}{\sqrt{2}}$ and center $-\frac{1}{2}(1+i) = -\frac{1}{2}\alpha$ (see Fig. 4.2). Analogously, we definde

$$U_2 := -i \times U_1, \;\; U_3 := -i \times U_2 \text{ and } U_4 := -i \times U_3.$$

Setting

$$U(\alpha) := U_1, \;\; U(\overline{\alpha}) := U_2, \;\; U(-\alpha) := U_3, \;\; U(-\overline{\alpha}) := U_4$$

and

$$U(a) := X, \text{ if } a \neq \alpha, \overline{\alpha}, -\alpha, -\overline{\alpha},$$

we attach to each Gaussian integer $a \in I$ an area. With (4.2) there is always an even integer distance between real part $\operatorname{Re} a$ and imaginary part $\operatorname{Im} a$.

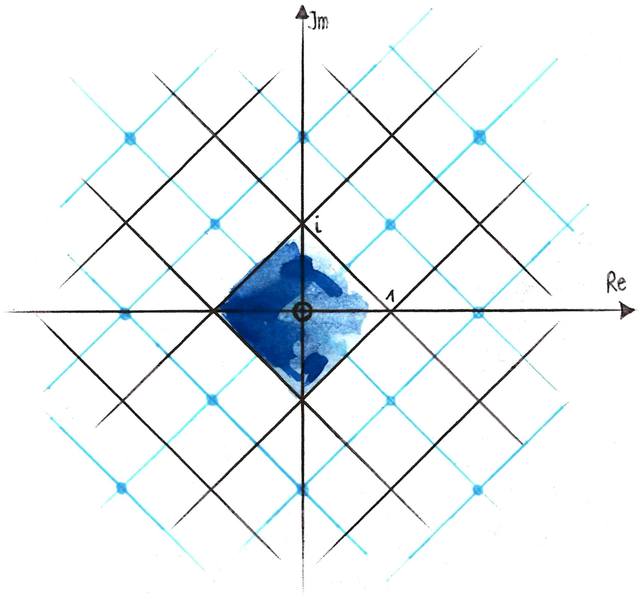


Figure 4.3: Thus, the numbers $a \in I$ are located on the angle bisectors of the quadrants or on parallel lines shifted by a multiple of 2 excluding the origin.

Thus, the set of reciprocals X^{-1} can be composed from translates of the sets $U(a)$ shifted by a, i.e.,

$$X^{-1} = \bigcup_{a \in I} (a + U(a)).$$

These geometrical observations are related to one another on behalf of the defined transformation T. We have

$$T^n X(a_1, \dots, a_n) = U(a_n).$$

This implies that the nth iteration of the transformation applied to the remainder of the n first partial quotients maps to the domain U, which is located around the nth partial quotient. This is interesting in view of Equation (4.5), from which the uniqueness of the inverse map T^n follows. We define $\phi_{a_1\cdots a_n} := (T^n)^{-1}$ by

$$\phi_{a_1\cdots a_n} : U(a_n) \to X(a_1, \ldots, a_n)$$

with

$$\phi_{a_1\cdots a_n}(z) = \frac{p_n + zp_{n-1}}{q_n + zq_{n-1}}.$$

The 'forward' mapping to the non-integer remainder of $\frac{1}{T^{n-1}z}$ becomes a unique inverse mapping 'backwards' in the algorithm. Since in each set $U(a_n) = X, U_1, \ldots, U_4$ the origin is included, for all admissible sequences of partial quotients $a_1, \ldots, a_n \in A(n)$ the nth convergent is located in the corresponding T-cell, that is

$$\frac{p_n}{q_n} = \phi_{a_1\cdots a_n}(0) \in X(a_1 \cdots a_n).$$

Consequently, the algorithm indeed produces convergents approximating the initial values better and better. The more partial quotients are chosen as fixed, the smaller the T-cell becomes in X and therewith the closer the corresponding convergent is located.

4.1.5 Approximation Quality

In view of Lemma 4.1.5 we have $|q_{n-1}| < |q_n|$, hence, the factor $\frac{1}{|q_n|}$ becomes smaller with increasing $n \in \mathbb{N}$. To consider further questions concerning the approximation behaviour, we examine

$$\theta'_n := |q_n|^2 \left| z - \frac{p_n}{q_n} \right|,$$

an analogue of the so-called *approximation coefficients* from real theory.

In [Tanaka, 1985] the equation

$$\left| z - \frac{p_n}{q_n} \right| = \frac{|2i(-1)^n T^n z|}{|q_n(q_n + q_{n-1}T^n z)|} \tag{4.8}$$

is given. The proof is by induction based on recursion formulae (4.3) and (4.4). Hence,

$$\theta'_n = \frac{2|T^n z|}{\left|1 + \frac{q_{n-1}}{q_n} T^n z\right|} \leq \frac{2}{\left(1 - \left|\frac{q_{n-1}}{q_n} T^n z\right|\right)}. \tag{4.9}$$

This upper bound implies that there exists a constant κ such that

$$\left| z - \frac{p_n}{q_n} \right| < \frac{\kappa}{|q_n|^2}$$

for all z and all its convergents $\frac{p_n}{q_n}$. To specify this constant further we perform an approach to bound it from below. Therefore, we first prove

Lemma 4.1.6 *We have*

$$\min\{|p^2 + pq + q^2| : p, q \in (1+i)\mathbb{Z}[i] \setminus \{0\}\} = 2.$$

Proof. The set of values taken by the quadratic form

$$p^2 + pq + q^2 \text{ for } (p, q) \in ((1+i)\mathbb{Z}[i])^2 \setminus \{(0,0)\}$$

is contained in $(1+i)\mathbb{Z}[i] \setminus \{0\}$. Hence, the smallest possible absolute values of p^2+pq+q^2 are $\sqrt{2}$ related to possible solutions of $p^2+pq+q^2 = \pm 1 \pm i$, and 2 related to $p^2+pq+q^2 = \pm 2i, \pm 2$. We assume

$$p^2 + pq + q^2 = \pm 1 \pm i. \tag{4.10}$$

Setting $p = a + ib, q = c + id$ with $a \equiv b, c \equiv d \bmod 2$ leads to

$$p^2 = a^2 - b^2 + 2iab \text{ and } q^2 = c^2 - d^2 + 2icd.$$

This transforms (4.10) to

$$(a^2 - b^2) + (c^2 - d^2) + (ac - bd) + i\,(2ab + 2cd + (ad + bc)) = \pm 1 \pm i.$$

Separating real and imaginary part, we end up with

$$\text{(I)} \quad a^2 - b^2 + c^2 - d^2 + ac - bd = \pm 1$$

and

$$\text{(II)} \quad 2ab + 2cd + (ad + bc) = \pm 1.$$

Since

$$ad + bc \equiv bd + bd = 2bd \equiv 0 \bmod 2,$$

we have $2ab + 2cd + (ad + bc) \equiv 0 \bmod 2$ showing that (II) is not solvable. Consequently,

$$\min\{|p^2 + pq + q^2| : p, q \in (1+i)\mathbb{Z}[i] \setminus \{0\}\} > |1+i| = \sqrt{2},$$

respectively

$$|p^2 + pq + q^2| \geq 2,$$

since the next 'larger' elements of $(1+i)\mathbb{Z}[i] \setminus \{0\}$ are $\pm 2, \pm 2i$. Because $p = 1+i, q = 0$ leads to $p^2 + pq + q^2 = (1+i)^2 = 2i$ the proposition is proven. q.e.d

The previous lemma allows to estimate a lower bound $c \leq \kappa$ as follows. For $z \notin \mathbb{Q}(i)$ we assume

$$z = \frac{1}{2}(-1 + i\sqrt{3}) = \frac{p}{q} + \frac{c'}{q^2}$$

with $p, q \in (1+i)\mathbb{Z}[i]$ and $q \neq 0$ as well as $c' \neq 0$, since $\frac{p}{q} + \frac{c'}{q^2} \notin \mathbb{Q}(i)$. Multiplying with q

$$\frac{i\sqrt{3}}{2}q - \frac{c'}{q} = p + \frac{q}{2},$$

and squaring,

$$-\frac{3}{4}q^2 - i\sqrt{3}c' + \frac{c'^2}{q^2} = p^2 + pq + \frac{q^2}{4}$$

provides

$$\frac{c'^2}{q^2} - i\sqrt{3}c' = p^2 + pq + q^2.$$

From Lemma 4.1.6 we have $|p^2 + pq + q^2| \geq 2$ and thus, we can estimate

$$\frac{|c'|^2}{|q|^2} + \sqrt{3}|c'| \geq 2.$$

For $|c'| < c$, with a constant $c < \frac{\sqrt{3}}{2}$, this would lead to

$$\frac{c^2}{|q|^2} + c\sqrt{3} > 2,$$

respectively

$$|q|^2 < \frac{c^2}{2 - c\sqrt{3}},$$

which can be satisfied by only finitely many numbers q.

Consequently, we have

Lemma 4.1.7 *The best possible constant κ is greater than or equal to $c = \frac{\sqrt{3}}{2}$.*

Remark: In fact, we expect c from the reasoning above to be best possible, which is equivalent to $\kappa = \frac{\sqrt{3}}{2}$. From a geometrical point of view, this assumption is rather natural. In [Ford, 1925] Leister R. Ford proved that for the Adolf Hurwitz continued fraction, where $p, q \in \mathbb{Z}[i]$, the best possible constant for the approximation bound is $\sqrt{3}$. In Subsection 7.2.4 we examine that the upper bounds for the approximation quality obtained by a geometrical method depend on the area of the fundamental domains. In our case, these areas differ by a factor 2, which matches exactly to the corresponding difference of Ford's approximation bound $\frac{1}{\sqrt{3}}$ and the here expected bound $\frac{2}{\sqrt{3}}$ (for more details see Subsection 7.2.4). Following [Ford, 1925] would shed light on this question, however, here we do not study this topic any further.

4.2 Ergodic Theory

In ergodic theory one examines so-called *measure preserving dynamical systems*. In general such a dynamical system describes a mathematical concept which models a certain time-process in a certain space on behalf of fixed mathematical regularities.

4.2.1 Transformations

We consider a probablility space (X, Σ, μ) with non-empty set X, a σ-algebra Σ on the set X, a probability measure μ on (X, Σ) and a measure preserving *transformation*

$$T : X \to X.$$

The above mentioned time-process is explained by assuming T as 'shift into the future', whereas its inverse T^{-1} can be considered as 'shift into the past'. Here T is said to be *measure preserving*, if, and only if, for $E \in \Sigma$,

$$\mu(T^{-1}E) = \mu(E),$$

which means that the measure of E is preserved under T. Furthermore, a measurable set E is called T*-invariant* when

$$T^{-1}E = E.$$

The corresponding dynamical system is written as the quadrupel (X, Σ, μ, T).

In addition, T is called *ergodic* when for each μ-measurable, T-invariant set E either

$$\mu(E) = 0 \text{ or } \mu(E) = 1.$$

A very powerful result dealing with ergodic transformations was realized by Georg David Birkhoff [Birkhoff, 1931] and in the general form below by Aleksandr Jakovlevich Khinchine [Khintchine, 1933][2].

Theorem 4.2.1 ***(Pointwise Ergodic Theorem, Birkhoff, 1931)***
Let T be a measure preserving, ergodic transformation on a probablitiy space (X, Σ, μ). If f is integrable, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n\le N} f(T^n x) = \int_X f d\mu$$

for almost all $x \in X$.

The nth transformation T^n is defined recursively as follows

$$T^0 = id, \;\; T^1 = T \text{ and } T^n = T^0 \circ T^{n-1}.$$

4.2.2 Continued Fraction Transformation

In this subsection we shortly introduce an ergodic approach to continued fractions. Hereby, we follow a classical method[3] firstly explained for the real case.

We define a transformation $T : [0, 1) \to [0, 1)$ which serves as operator in the well known regular continued fraction algorithm (similar to the map T introduced in Tanaka's complex approach in Subsection 4.1.1). Choosing the unit interval as fundamental set X, we define $T0 = 0$ and, for $x \in X$,

$$Tx = T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ if } x \neq 0.$$

For an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ and $a_0 := \lfloor x \rfloor \in \mathbb{Z}$ we obviously have $x - a_0 \in [0, 1)$. Setting

$$T^0x := x - a_0, \quad T^1x := T(x - a_0), \quad T^2x := T(T^1x), \ldots,$$

[2] for a comprehensive version see [Dajani and Kraaikamp, 2002]
[3] once more we refer to [Dajani and Kraaikamp, 2002, p. 20]

the definition above provides

$$T^n x \in [0,1) \setminus \mathbb{Q} \text{ for all } n \geq 0.$$

With

$$a_n = a_n(x) := \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor \text{ for } n \geq 1$$

one receives the well-known regular continued fraction expansion

$$\begin{aligned} x &= a_0 + \cfrac{1}{a_1 + T^1 x} = a_0 \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2 x}} = \cdots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + T^n x}}} \\ &= [a_0; a_1, a_2, \cdots, a_n + T^n x]. \end{aligned}$$

The existence of the limit

$$x = [a_0; a_1, a_2 \cdots, a_n, \cdots] = \lim_{n \to \infty} [a_0; a_1, a_2 \cdots, a_n + T^n x]$$

follows on behalf of the representation related to the nth convergent $\frac{p_n}{q_n} \in \mathbb{Q}$ to x. In fact, we have

$$x = \frac{p_n + T^n x p_{n-1}}{q_n + T^n x q_{n-1}}$$

and $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ for $n \geq 1$. In view of $T^n x \in [0,1)$ the inequality

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}$$

can be derived. Here $(q_n)_{n \geq 0}$ is a strictly increasing sequence (see Chapter 1 respectively Lemma 4.1.5 for the complex case) of positive integers.

In the complex case, the fundamental set is naturally two-dimensional corresponding to real and imaginary parts of the expanded complex number. Hence, in Shigeru Tanaka's, respectively Julius Hurwitz's algorithm the transformation $T : X \to X$ is defined on the fundamental domain $X = \{z = x\alpha + y\overline{\alpha} \ : \ \frac{-1}{2} \leq x, y < \frac{1}{2}\}$ with $\alpha = 1 + i$ (and $\overline{\alpha} = 1 - i$) by

$$Tz := \frac{1}{z} - \left[\frac{1}{z}\right]_T \quad \text{for } z \neq 0 \quad \text{and} \quad T0 = 0,$$

where

$$[z]_T := \left\lfloor x + \frac{1}{2} \right\rfloor (1+i) + \left\lfloor y + \frac{1}{2} \right\rfloor (1-i).$$

The analogy between the real and complex approach is obvious. However, for the complex continued fractions some adjustments need to be done. In order to apply ergodic methods a main challenge is to define an invariant measure for a given transformation. In the real case, there is the so-called *Gauss-measure*[4], defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx,$$

for all Lebesgue sets $A \subset [0, 1)$. For complex algorithms some difficulties may appear. Here a so-called natural extension of the underlying transformation can be helpful.

4.2.3 Dual Transformation and Natural Extension for Tanaka's Algorithm

In Section 4.1 it was shown that Tanaka's transformation T provides a complex continued fraction expansion. Here we sketch another related transformation $S : Y \to Y$, introduced by Tanaka, where

$$Y = \{w \in \mathbb{C} : |w| \leq 1\}$$

is the unit disc centered at the origin in the complex plane. We define subsets V_j of Y by

$$V_1 := \{w \in Y : |w + \alpha| \geq 1\},$$

[4] discovered by Carl Friedrich Gauß

$$V_2 = -i \times V_1, \;\; V_3 = -i \times V_2, \;\; V_4 = -i \times V_3, \;\; V_5 = V_1 \cap V_2,$$

$$V_6 = -i \times V_5, \;\; V_7 = -i \times V_6, \;\; V_8 = -i \times V_7$$

and a partition of $I_0 = \{n\alpha + m\overline{\alpha} \;:\; n, m \in \mathbb{Z}\}$, respectively $I = I_0 \setminus \{0\}$ by

$$J_1 = \{n\alpha : n > 0\}, J_2 = -i \times J_1, \;\; J_3 = -i \times J_2, \;\; J_4 = -i \times J_3,$$

$$J_5 = \{n\alpha + m\overline{\alpha} : m > 0\}, J_6 = -i \times J_5, \;\; J_7 = -i \times J_6, \;\; J_8 = -i \times J_7.$$

Setting

$$V(a) := \begin{cases} Y, & \text{if } a = 0, \\ V_j, & \text{if } a \in J_j, \end{cases}$$

for $1 \leq j \leq 8$, we obtain a complete tiling of the complex plane

$$\mathbb{C} = \bigcup_{a \in I_0} (a + V(a)).$$

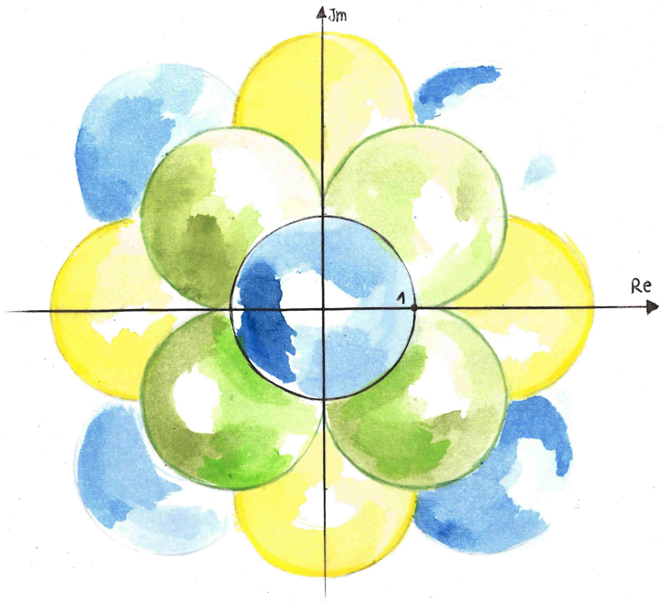


Figure 4.4: Illustration of the tiling of the complex plane with respect to S.

Furthermore, we define $S0 = 0$ and

$$Sw = \frac{1}{w} - \left[\frac{1}{w}\right]_S \text{ for } w \neq 0$$

with $[w]_S = a$ if $w \in a + V(a)$. As above partial quotients arise through

$$b_n = b_n(w) = \left[\frac{1}{S^{n-1}w}\right]_S \in I_0.$$

This leads to an expansion of $w \in Y$ as

$$w = [b_1, b_2, \cdots, b_n + S^n w]$$

with convergents $V_n =: [b_1, b_2, \cdots, b_n]$. Tanaka proved that the transfomation S satisfies a certain duality.

Lemma 4.2.2 ***(Duality)***

Let $a_1, \ldots, a_n$ be a sequence of numbers in I. Then $a_1, \ldots, a_n$ is T-admissible if, and only if, the inverse sequence of partial quotients $a_n, \ldots, a_1$ is S-admissible.

The proof is based on geometrical constraints concerning a finely tiling of Y respectively Y^{-1} and can be found in [Tanaka, 1985, p. 200].

By Lemma 4.2.2, we know that for each T-admissible sequence there exists an associated sequence of convergents of a given $w \in Y$, namely

$$\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \ldots, a_1] = V_n \in Y.$$

This provides another proof of Lemma 4.1.5 that the sequence $(q_n)_{n\in\mathbb{N}}$ increases monotonously in absolute value. Given this *dual transformation* S, one can construct the so-called *natural extension* $\mathbb{T}$ containing information of T as well as information of S, which are considered as 'future' and 'past' of the sequence of partial quotients. On behalf of Lemma 4.2.2, we define $\mathbb{T} : X \times Y \to X \times Y$ by[5]

$$\mathbb{T}(z, w) = \left(Tz, \frac{1}{a_1(z) + w} \right), \tag{4.11}$$

where $a_1 = \left[\frac{1}{z}\right]_T$. On pages 202 and 210 in [Tanaka, 1985] Tanaka proved

Theorem 4.2.3 ***(Natural Extension)***

The transformation $\mathbb{T}$ is a natural extension; in particular T and S are ergodic and the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(z, w) = \frac{1}{|1 + zw|^4}$$

is the density function of an absolutely continuous $\mathbb{T}$-invariant measure.

[5]In [Tanaka, 1985] a more exact, more finely, tiling of X resp. Y was given on behalf of sets U and V. For our needs, it is sufficient to consider the whole sets X and Y.

4.2.4 Variation on the Döblin-Lenstra Conjecture

In this section we give a proof of an analogue of the so-called Döblin-Lenstra Conjecture for the complex case of Tanaka's continued fraction algorithm. Therefore, we first state the original conjecture for the regular continued fraction algorithm for real numbers.

Theorem 4.2.4 ***(Döblin-Lenstra Conjecture)***
Let x be any irrational number with continued fraction convergents $\frac{p_n}{q_n}$. Define the approximation coefficients by $\theta_n := q_n|q_n x - p_n|$. Then for almost all x we have

$$\lim_{n\to\infty} \frac{1}{n} |\{j \ : \ j \leq n, \theta_j(x) \leq c\}| = \begin{cases} \frac{c}{\log 2}, & \text{for } 0 \leq c \leq \frac{1}{2}, \\ \frac{-c+\log 2c+1}{\log 2}, & \text{for } \frac{1}{2} \leq c \leq 1. \end{cases} \tag{4.12}$$

Notice that $\theta_j(x) \leq 1$ for all j and all x.

Indeed, the conjecture has been proven in [Bosma et al., 1983] by ergodic methods, in particular by using the natural extension $\mathbb{T}$ of the continued fraction transformation T as main tool. To transfer their approach to Tanaka's and, respectively, Julius Hurwitz's complex continued fraction, some preparatory work needs to be done. Our first aim is to find an analogous expression for θ_n.

Subsequently to Lemma **??** in Subsection 4.1.5, we define normalized approximation coefficients $\varphi_n := \frac{\theta'_n}{2}$ satisfying

$$\varphi_n \leq 2 + \sqrt{2} \text{ for } n \in \mathbb{N}.$$

Analogously to the real case, we point out that for almost all z, in the Lebesgue sense, the arithmetical function $n \mapsto \varphi_n(z)$ has a limiting distribution function. Since this function will be constant for values greater than a certain upper bound $\frac{\kappa}{2}$ (see also Subsection 4.1.5, where we have shown $\frac{\sqrt{3}}{2} \leq \kappa \leq 4 + 2\sqrt{2}$), we consider only the corresponding interval.

Following the line of argumentation of [Bosma et al., 1983], we define, for $0 \leq g \leq \frac{\kappa}{2}$,

$$l(g) := \lim_{n\to\infty} \frac{1}{n} \left|\{j \; : \; j \leq n, \varphi_j(z) \leq g\}\right|.$$

From the dual transformation we know $V_n = \frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_2, a_1]$ and thus from (4.9) we deduce the equivalence

$$\varphi_j(z) \leq g \quad \Leftrightarrow \quad \left|\frac{T^j z}{1 + V_j T^j z}\right| \leq g, \tag{4.13}$$

which will play a decisive role in the following steps. Recall definition (4.11) of the natural extension $\mathbb{T} : X \times Y \to X \times Y$,

$$\mathbb{T}(z, y) = \left(Tz, \frac{1}{a_1 + y}\right),$$

where $a_1 := [\frac{1}{z}]_T$ is the first partial quotient of z. We have

$$\mathbb{T}^i(z, y) = \left(T^i z, [0; a_i, a_{i-1}, \ldots, a_2, a_1 + y]\right),$$

and in particular

$$\mathbb{T}^i(z, 0) = \left(T^i z, V_i\right).$$

Because of (4.13), this show that $\varphi_i(z) \leq g$ if, and only if,

$$\mathbb{T}^i(z, 0) \in A_g := \{(u, v) \in X \times Y : \left|\frac{u}{1 + uv}\right| \leq g\},$$

respectively

$$\mathbb{T}^i(z, 0) \in A_g = \{(u, v) \in X \times Y : \left|\frac{1}{u} + v\right| \geq \frac{1}{g}\}.$$

Comparing

$$\mathbb{T}^n(z, y) = (T^n z, [0; a_n, a_{n-1}, \ldots, a_1 + y])$$

and

$$\mathbb{T}^n(z,0) = (T^n z, [0; a_n, a_{n-1}, \ldots, a_1]),$$

it follows that for every $\epsilon > 0$ there exists $n_0(\epsilon)$ such that, for all $n \geq n_0(\epsilon)$ and all $y \in Y$, we have

$$\mathbb{T}^n(z,y) \in A_{g+\epsilon} \Rightarrow \mathbb{T}^n(z,0) \in A_g,$$

as well as

$$\mathbb{T}^n(z,0) \in A_g \Rightarrow \mathbb{T}^n(z,y) \in A_{g-\epsilon}.$$

We define $A_g^i := \{i \ : \ i \leq n, \mathbb{T}^i(z,y) \in A_g\}$. Since $\mathbb{T}$ is ergodic, the following limits exist and the inequalities in between hold:

$$\lim_{n\to\infty} \frac{1}{n}|A^i_{g+\epsilon}| \leq \liminf_{n\to\infty} \frac{1}{n}|A^i_g| \leq \limsup_{n\to\infty} \frac{1}{n}|A^i_g| \leq \lim_{n\to\infty} \frac{1}{n}|A^i_{g-\epsilon}|.$$

As shown by Tanaka, there exists a probability measure μ with density function $h(z,y) = \frac{1}{|1+zy|^4}$, such that the quadrupel $(X \times Y, \Sigma, \mu, \mathbb{T})$ forms an ergodic system with $A^i_g \in \Sigma$. Applying the Ergodic Theorem 4.2.1 of Birkhoff [Birkhoff, 1931] leads to

Theorem 4.2.5 *For almost all z the distribution function exists and is given by*

$$l(g) = \lim_{n\to\infty} \frac{1}{n} \sum_{i\leq n} \mathbb{1}_{A_g}(\mathbb{T}^i(z,0)) = \mu(A_g) = \frac{1}{G} \int\int_{A_g} \frac{d\lambda(u,v)}{|1+uv|^4}, \tag{4.14}$$

where $G := \int\int_{X\times Y} \frac{d\lambda(u,v)}{|1+uv|^4}$.

We notice that A_g describes that part of $X \times Y$ which lies above $\left|\frac{1}{u} + v\right| = \frac{1}{g}$. This leads to

$$l(g) = \frac{1}{G} \int\int_{A_g} \frac{d\lambda(u,v)}{|1+uv|^4} = \left(\int\int_{X\times Y} \frac{d\lambda(u,v)}{|1+uv|^4} \right)^{-1} \int\int_{|\frac{1}{u}+v| \geq \frac{1}{g}} \frac{d\lambda(u,v)}{|1+uv|^4}.$$

For tackling the explicit calculation of the normalizing factor G, we keep in mind the formula

$$\frac{1}{(1-x)^2} = \sum_{m\geq 1} m x^{m-1}, \tag{4.15}$$

valid for $|x| < 1$. To prevent difficulties that could arise from singularitiesof h on the boundaries of X and Y, for $0 < r < 1$ we define subdomains

$$X_r := \{z = x + iy : \frac{-r}{2} \leq x, y \leq \frac{r}{2}\}$$

and

$$Y_r := \{w \in \mathbb{C} : |w| \leq r\},$$

which inherit the symmetrical properties of X and Y. In view of $|1+uv|^2 = (1+uv)(1+\overline{uv})$ and applying (4.15), which converges for $x = uv \in X_r \times Y_r$ uniformly, we have

$$G_r = \int\int_{X_r\times Y_r} \frac{d\lambda(u,v)}{|1+uv|^4} = \sum_{m,n\geq 1} mn(-1)^{m+n} \int_{X_r} u^{m-1}\overline{u}^{n-1} d\lambda(u) \int_{Y_r} v^{m-1}\overline{v}^{n-1} d\lambda(v).$$

Without loss of generality we assume $m \geq n$ and receive

$$\begin{aligned} \int_{Y_r} v^{m-1}\overline{v}^{n-1} d\lambda(v) &= \int_{Y_r} |v|^{n-1} v^{m-n} d\lambda(v) = \int_0^{2\pi}\int_0^r a^m a e^{i\varphi(m-n)} da d\varphi \\ &= \int_0^{2\pi} e^{i\varphi(m-n)} d\varphi \int_0^r a^m da = \begin{cases} \pi\frac{1}{n}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Applying (4.15) once more, leads to

$$G_r = \pi \sum_{n\geq 1} n \int_{X_r} |u|^{2(n-1)} d\lambda(u) = \pi \int_{X_r} \sum_{n\geq 1} n|u|^{2(n-1)} d\lambda(u) = \pi \int_{X_r} \frac{d\lambda(u)}{1-|u|^2}.$$

Now we let r tend to 1, respectively $G_r \to G_1 =: G$, which is possible by Lebesgue's theorem on monotone convergence applied to

$$f(r) := \pi \int_{X_r} \frac{d\lambda(u)}{1 - |u|^2}.$$

Furthermore, we use the symmetry of the fundamental domain and split it along the axes into four parts of equal sizes. We consider the part located in the first quadrant

$$\triangle := \{a + ib : 0 \leq a, b \leq 1; b - 1 < a\}$$

and receive

$$G = 4\pi \int \int_{\triangle} \frac{dadb}{1 - |a + ib|^2} = 4\pi \int_0^1 \left(\int_0^{1-a} \frac{db}{1 - (a^2 + b^2)} \right) da,$$

where

$$\begin{aligned} \int_0^{1-a} \frac{db}{1 - a^2 - b^2} &= \int_0^{1-a} \left(\frac{1}{\sqrt{1 - a^2} - b} + \frac{1}{\sqrt{1 - a^2} + b} \right) db \frac{1}{2\sqrt{1 - a^2}} \\ &= \left[-\log |b - \sqrt{1 - a^2}| + \log(b + \sqrt{1 - a^2}) \right]_{b=0}^{1-a} \cdot \frac{1}{2\sqrt{1 - a^2}} \\ &= \frac{1}{2\sqrt{1 - a^2}} \log \frac{\sqrt{1 - a^2} + 1 - a}{\sqrt{1 - a^2} - 1 + a}. \end{aligned}$$

Altogether this gives

$$\begin{aligned} G &= 2\pi \int_0^1 \log \frac{\sqrt{1 - a^2} + 1 - a}{\sqrt{1 - a^2} - 1 + a} \frac{da}{\sqrt{1 - a^2}} \\ &= 2\pi \left[A - B + i \sum_{k=1}^{\infty} \frac{(-\exp(i \arcsin(a)))^k}{k^2} - i \sum_{k=1}^{\infty} \frac{(\exp(i \arcsin(a)))^k}{k^2} \right]_{a=0}^1 \end{aligned}$$

with $A = \arcsin(a)\log(1 - \exp(i \arcsin(a)))$ and $B = \arcsin(a)\log(1 + \exp(i \arcsin(a)))$. This leads to

$$G = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

Thus, we can state

Theorem 4.2.6 *The normalizing constant G is given by*

$$G = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Notice that the appearing infinite series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is *Catalan's constant.* Interestingly, Catalan's constant appears in various ergodic theory results, see for example [Nakada, 1990], [Kasteleyn, 1961, p. 1217] or [Felker and Russell, 2003, p. 2].

Remark: In view of this representation of the normalization constant G as infinite series we observe a certain phenomenon that appears in applications of ergodic theory to complex continued fractions frequently. In contrast to the real case, analogous expressions seem to be far more complicated and usually cannot easily be made explicit. A similar example is given in [Tanaka, 1985, p. 212]. Here Shigeru Tanaka gave a non-explicit respresentation of the entropy of T and S. Therefore, we do not attempt further to find an explicit expression for the distribution function l.

4.3 Transcendental Numbers

In this section we transfer a method to construct transcendental numbers from real theory to the complex case.

4.3.1 Roth's Theorem

We shall use an extension of Roth's theorem [Schmidt, 1980, p. 278][6] on algebraic numbers respectively a number field version thereof.

Theorem 4.3.1 *For any algebraic number $\alpha \in \mathbb{C}$, integer $d \geq 1$ and $\epsilon > 0$, there exist only finitely many algebraic numbers $\xi \in \mathbb{C}$ of degree d, such that the inequality*

$$|\alpha - \xi| < \frac{1}{H(\xi)^{d+1+\epsilon}} \tag{4.16}$$

holds. Here, $H(\xi)$ is the so-called height of ξ, the maximum of the absolut values of the coefficients of the minimal polynomial of ξ.

In other words: For any algebraic α there exists a positive constant $c(\alpha) \in \mathbb{C}$ with

$$|\alpha - \xi| \geq \frac{c(\alpha)}{H(\xi)^{d+1}}.$$

Thus, in order to show the transcendence of a number α, infinitely many ξ need to be found for which (4.16) holds.

In real theory we can perform an approach on behalf of continued fraction theory (see Section 1.2). The convergents $\frac{p_n}{q_n}$ to an irrational $\alpha = [a_0; a_1, a_2 \ldots]$ provide an estimate from above

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1} q_n^2}.$$

If we suppose α to be algebraic we may conclude the inequality

$$\frac{c(\alpha)}{q_n^{2+\epsilon}} < \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1} q_n^2} \tag{4.17}$$

which leads to the condition

$$c(\alpha) a_{n+1} < q_n^{\epsilon}.$$

[6] which can be considered as successor of Liousville's result on transcendental numbers, see Section 1.1

This implies that if we find a continued fraction in which the sequence of partial quotients increases sufficiently fast with respect to the sequence of the convergent's denominators it is a transcendental number. In other words, if

$$\limsup_{n\to\infty} \frac{a_{n+1}}{q_n^{\epsilon}} = \infty,$$

then, for any positive ϵ, there exist infinitely many convergents to α satisfying $a_{n+1} \geq q_n^{\epsilon}$, which implies the transcendence of α.

4.3.2 Transfer to Tanaka's Continued Fraction

Concerning the complex continued fraction algorithm of Tanaka, a corresponding inequality to (4.17) shall be stated. First, we consider an upper bound. We have

$$\begin{aligned} z - \frac{p_n}{q_n} &= \frac{p_n + T^n z p_{n-1}}{q_n + T^n z q_{n+1}} - \frac{p_n}{q_n} = \frac{(p_n + T^n z p_{n-1})q_n - (q_n + T^n z q_{n+1})p_n}{q_n(q_n + T^n z q_{n+1})} \\ &= \frac{T^n z(p_{n-1}q_n - p_n q_{n-1})}{q_n(q_n + T^n z q_{n+1})} = \frac{2i(-1)^n}{q_n^2 (T^n z)^{-1} + q_n q_{n-1}}, \end{aligned}$$

what also proves Formula (4.8) from Subsection 4.1.5. Following the definition $Tz := \frac{1}{z} - \left[\frac{1}{z}\right]_T$, we receive

$$(T^n z)^{-1} = a_{n+1} + T^{n+1} z,$$

with 'small' $T^{n+1}z \in X$ and partial quotients $a_{n+1} = a_{n+1}(z) = \left[\frac{1}{T^n z}\right]_T \in (1-i)\mathbb{Z}[i]$. This leads to

$$z - \frac{p_n}{q_n} = \frac{2i(-1)^n}{q_n^2(a_{n+1} + T^{n+1}z) + q_n q_{n-1}}.$$

It follows from Lemma 4.1.5 that $|a_{n+1}| \geq \sqrt{2}$ is bigger than $|\frac{q_{n-1}}{q_n}| < 1$ and furthermore, $T^{n+1}z \in X$. Consequently, the estimate

$$\left| z - \frac{p_n}{q_n} \right| = \frac{2}{|q_n|^2 |a_{n+1} + T^{n+1}z + \frac{q_{n-1}}{q_n}|} < \frac{c_1(z)}{|q_n|^2 |a_{n+1}|}$$

holds with a constant $c_1(z)$ dependending only on z. Concerning an estimation from below, we follow Roth and consider the inequality

$$|z-\xi| \geq \frac{c(z)}{H(\xi)^{d+1}}$$

with another positive constant $c(z)$ depending only on z.

The general case $\xi = \frac{a+ib}{n}$ with $n \in \mathbb{N}, a+ib \in \mathbb{Z}[i]$ shall be observed first. The minimal polynomial of ξ is given by

$$P := n^2X^2 - 2anX + a^2 + b^2.$$

On behalf of its coefficients the general height

$$H\left(\frac{a+ib}{n}\right) = \max\{n^2, 2|a|n, |a+ib|^2\}$$

arises. Next we consider the specific case of Gaussian rationals, that is,

$$\xi = \frac{p_n}{q_n} = \frac{p_n\overline{q_n}}{|q_n|^2}$$

with elements $p_n, q_n \in \mathbb{Z}[i]$ from the sequences of numerators and denominators of the complex continued fraction expansion of z. Following our approach from above, we compute their height as

$$\begin{aligned} H(\xi) &= \max\{|q_n|^4, 2|Re\{p_n\overline{q_n}\}||q_n|^2, |p_n\overline{q_n}|^2\} \\ &= |q_n|^2 \cdot \max\{|q_n|^2, 2|Re\{p_n\overline{q_n}\}|, |p_n|^2\} \\ &\leq |q_n|^2 \cdot (|q_n| + |p_n|)^2 . \end{aligned}$$

Taking into account $|z| \leq 1$, we have $|q_n| \leq |p_n|$ and deduce

$$\frac{c'(z)}{|q_n^4|^3} \leq \left| z - \frac{p_n}{q_n} \right|$$

with another positive constant $c'(z)$ only depending on z. Hence,

$$\frac{c'(z)}{|q_n|^{12}} \leq \left| z - \frac{p_n}{q_n} \right| < \frac{c_1(z)}{|a_{n+1}||q_n|^2}.$$

Consequently, if $\frac{|a_{n+1}|}{|q|^{10}}$ exceeds any positive quantity, the underlying complex number z is transcendental.

Theorem 4.3.2 *If*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{q_n^{10}} \right| = \infty,$$

then the complex number $z = [a_0, a_1, \ldots]$ is transcendental.

4.3.3 Example of a Transcendental Number

An example shall illustrate the previous result. We are looking for a number with partial quotients and denominators of convergents corresponding satisfying the consition of Theorem 4.3.2. Considering

$$z = [0; i10^{1!}, i^2 10^{2!}, i^3 10^{3!}, \ldots],$$

with partial quotients $a_n = i^n 10^{n!}$, we first need to verify

Lemma 4.3.3 *The sequence of partial quotients $a_n = (i^n 10^{n!})_{n\in\mathbb{N}}$ is T-admissible.*

Remark: In Julius Hurwitz's work [Hurwitz, 1895, p. 12] a list of impossible consecutive partial quotients is given, see Lemma 3.2.2. According to [Hensley, 2006, p. 74], here "it was apparently possible to ferret out the details of which sequences of (a_n) can occur", and we may deduce this list to be complete.

Proof. According to Julius only for partial quotients of the form $\pm 1 \pm i$ certain restrictions need to be taken into account. In our sequence (a_n) only numbers of the form

$$a_n = 10^{n!} \text{ if } n \equiv 0 \bmod 4,$$

$$a_n = i10^{n!} \text{ if } n \equiv 1 \bmod 4,$$

$$a_n = -10^{n!} \text{ if } n \equiv 2 \bmod 4,$$

and

$$a_n = -i10^{n!} \text{ if } n \equiv 3 \bmod 4$$

occur. Those are all of type ± 2 respectively $\pm i2$ according to Julius Hurwitz's notation explained in Section 3.2 and, consequently, they form an admissible sequence. q.e.d.

In view of Recursion Formula (4.4) the denominators of the sequence of convergents need to be determined. We have

$$\begin{aligned} q_1 &= a_1\alpha + 0 = 10i(1+i) = -10 + 10i \sim 10^1, \\ q_2 &= -10^2 \cdot (-10 + 10i) + 1 + i = 1001 - 999i \sim 10^3, \\ q_3 &= -i10^6(1001 - 999i) - 10 + 10i \sim 10^9, \end{aligned}$$

and, by induction,

$$q_n \sim 10^{\sum_{m=1}^{n} m!}.$$

Concerning Theorem 4.3.2, we observe

$$\left| \frac{a_{n+1}}{q_n^{10}} \right| \sim 10^{(n+1)! - 10\sum_{m=1}^{n} m!}.$$

Since

$$\begin{aligned}(n+1)! - 10\sum_{m=1}^{n} m! &= (n+1-10)n! - 10\sum_{m=1}^{n-1} m! \\ &> (n-9)n! - 10(n-1)(n-1)! \\ &= \left(n(n-9) - 10(n-1)\right)(n-1)! \\ &= (n^2 - 19n + 10)(n-1)!,\end{aligned}$$

we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{q_n^{10}}\right| = \infty$$

which proves that the number in question is transcendental.

5 Representation of Real Numbers as Sum of Continued Fractions

In this chapter we outline a result that has already been published in [Oswald and Steuding, 2013] and our presentation follows this article closely. Firstly, we prove that every real number can be written as a sum of an integer and at most $\lfloor \frac{b+1}{2} \rfloor$ continued fractions to the nearest integer each of which having partial quotients at least b, where b is a positive integer. Secondly, we give an application to complex numbers and their representation by complex continued fractions.

5.1 Introduction and Main Result

In 1947, Marshall Hall [Hall, 1947] showed that every real number can be written as a sum of an integer and two regular continued fractions each of which having partial quotients less than or equal to four. We denote by $\mathsf{F}(b)$ the set of those real numbers x having a regular continued fraction expansion $x = [a_0; a_1, a_2, \ldots, a_n, \ldots]$ with arbitrary $a_0 \in \mathbb{Z}$ and partial quotients $a_n \leq b$ for $n \in \mathbb{N}$, where b is a positive integer. Hall's theorem can be stated as

$$\mathsf{F}(4) + \mathsf{F}(4) = \mathbb{R}.$$

Here the sumset $A + B$ is defined as the set of all pairwise sums $a + b$ with $a \in A$ and $b \in B$ (we already used this 'Minkowski Sum' in Section 4.1.3). There have been several generalizations of Hall's remarkable result. For example, Cusick [Cusick, 1973] and Diviš

[Diviš, 1973] showed independently that $\mathsf{F}(3) + \mathsf{F}(3) \neq \mathbb{R}$; Hlavka [Hlavka, 1975] obtained $\mathsf{F}(3) + \mathsf{F}(4) = \mathbb{R}$ as well as $\mathsf{F}(2) + \mathsf{F}(4) \neq \mathbb{R}$; Astels [Astels, 2001] proved among other things that $\mathsf{F}(5) \pm \mathsf{F}(2) = \mathbb{R}$ and, quite surprisingly, $\mathsf{F}(3) - \mathsf{F}(3) = \mathbb{R}$.

On the contrary, one may ask what one can get by adding continued fractions where all partial quotients are larger than a given quantity. For this purpose Cusick [Cusick, 1971] defined for $b \geq 2$ the set $\mathsf{S}(b)$ consisting of all $x = [0; a_1, a_2, \ldots, a_n, \ldots] \leq b^{-1}$ containing no partial quotient less than b and proved

$$\mathsf{S}(2) + \mathsf{S}(2) = [0, 1].$$

In [Cusick and Lee, 1971], Cusick and Lee extended this result by proving

$$b\mathsf{S}(b) = [0, 1] \qquad \text{for any integer } b \geq 2, \tag{5.1}$$

where the left hand-side is defined as the sumset of b copies of $\mathsf{S}(b)$. The result of Cusick and Lee is best possible as the following example illustrates:

$$(\tfrac{7}{12}, \tfrac{3}{5}) \not\subset 2\mathsf{S}(3) \subset [0, \tfrac{2}{3}].$$

Here we are concerned about an analogue of this result for continued fractions to the nearest integer which has been explained in Section 3.1.

Notice that for all those continued fractions

$$a_n + \epsilon_{n+1} \geq 2 \tag{5.2}$$

for $n \in \mathbb{N}$. For further details we refer to Perron's monograph [Perron, 1913].

We denote by $\mathcal{L}(b)$ the set of all real numbers $x \in [-\frac{1}{2}, \frac{1}{2})$ having a continued fraction to the nearest integer with all partial quotients a_n being larger than or equal to b, where b is a positive integer. Following Cusick [Cusick, 1971] it is not difficult to show that $\mathcal{L}(b)$

is a Cantor set and, in particular, of Lebesgue measure zero (see also Rockett and Szüsz [Rockett and Szüsz, 1992, Ch. V]). The following theorem extends the theorem of Cusick and Lee (5.1) to continued fractions to the nearest integer:

Theorem 5.1.1 *Let b be a positive integer. Every real number can be written as sum of an integer and at most $\lfloor \frac{b+1}{2} \rfloor$ continued fractions to the nearest integer each of which having partial quotients at least b, more precisely,*

$$\left\lfloor \frac{b+1}{2} \right\rfloor \mathcal{L}(b) = \left[- \left\lfloor \frac{b+1}{2} \right\rfloor \beta, \left\lfloor \frac{b+1}{2} \right\rfloor \beta \right],$$

with $\beta = \frac{1}{2}(b - \sqrt{b^2-4})$, and the interval on the right hand-side has length larger than one. The result is best possible in the following sense: if $m < \lfloor \frac{b+1}{2} \rfloor$, then $m\mathcal{L}(b) = [-m\beta, m\beta]$ and the interval on the right has length less than one.

5.2 Some Preliminaries

In the sequel we sometimes denote the nth partial quotient and the nth sign in the continued fraction expansion to the nearest integer of x by $a_n(x)$ and $\epsilon_n(x)$, respectively.

Lemma 5.2.1 *Given $j, n \in \mathbb{N}$, we have $\epsilon_n(x)a_n(x) = \pm j$ if, and only if,*

$$T^{n-1}(x) \in \left\{ \left[\frac{-1}{j-\frac{1}{2}}, \frac{-1}{j+\frac{1}{2}} \right) \cup \left(\frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}} \right] \right\} \cap [-\tfrac{1}{2}, \tfrac{1}{2}).$$

More precisely, for positive $T^{n-1}(x)$, we have $\epsilon_n(x) = +1$ and

$$\begin{aligned} a_n(x) = j \geq 3 \quad &\Longleftrightarrow \quad T^{n-1}(x) \in \left(\frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}} \right], \\ a_n(x) = 2 \quad &\Longleftrightarrow \quad T^{n-1}(x) \in \left(\tfrac{2}{5}, \tfrac{1}{2} \right). \end{aligned}$$

while, for negative $T^{n-1}(x)$, we have $\epsilon_n(x) = -1$ and

$$a_n(x) = j \geq 3 \quad \Longleftrightarrow \quad T^{n-1}(x) \in \left(\frac{-1}{j-\frac{1}{2}}, \frac{-1}{j+\frac{1}{2}}\right],$$
$$a_n(x) = 2 \quad \Longleftrightarrow \quad T^{n-1}(x) \in \left[-\tfrac{1}{2}, -\tfrac{2}{5}\right).$$

A partial quotient equal to 1 is impossible.

This lemma indicates a symmetry in the distribution of partial quotients with respect to zero for the interior of the intervals. Furthermore, the lemma implies Condition (5.2). Another trivial consequence is $\mathcal{L}(2) = [-\frac{1}{2}, \frac{1}{2})$; hence, *every real number has a continued fraction expansion to the nearest integer with all partial quotients being larger than or equal to two* which is an assertion of the theorem for $b = 2$.

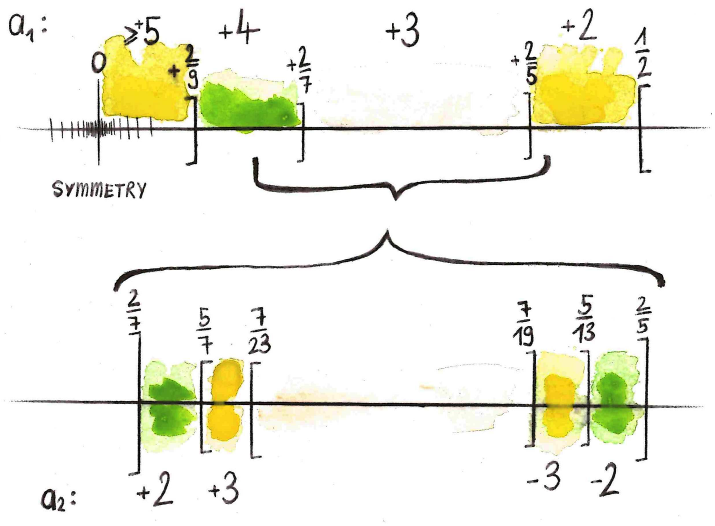


Figure 5.1: Illustration of all partial quotients being larger than or equal to two.

Proof. Writing

$$x = \frac{\epsilon_1(x)}{\frac{1}{|x|}} = \frac{\epsilon_1(x)}{\lfloor \frac{1}{|x|} + \frac{1}{2} \rfloor + \frac{1}{|x|} - \lfloor \frac{1}{|x|} + \frac{1}{2} \rfloor} = \frac{\epsilon_1(x)}{a_1(x) + T(x)},$$

we find $a_1(x) = j$ if, and only if,

$$|x| \in \left(\frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}}\right] \cap [-\tfrac{1}{2}, \tfrac{1}{2}),$$

where the intersection on the right is with respect to the condition $x \in [-\frac{1}{2}, \frac{1}{2})$. The corresponding intervals may or may not lie completely inside $[-\frac{1}{2}, \frac{1}{2})$. In order to obtain precise intervals for the partial quotients we observe that on the positive real axis

$$\left(\frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}}\right] \subset [-\tfrac{1}{2}, \tfrac{1}{2}),$$

provided $j \geq 3$; the partial quotient 2 is assigned to the interval $(\frac{2}{5}, \frac{1}{2})$, and a partial quotient 1 is impossible. The case of negative x follows from symmetry by switching the sign ϵ_1. Replacing x in the previous lemma by some iterate $T^{n-1}(x)$, the formulae of the lemma follow. q.e.d.

The following lemma is about a certain continued fraction to the nearest integer which is involved in the statement of Theorem 5.1.1 and in many estimates needed for its proof.

Lemma 5.2.2 *For* $3 \leq b \in \mathbb{N}$, *denote by*

$$\beta := [0; +1/b, \overline{-1/b}] := [0; +1/b, -1/b, -1/b, \ldots]$$

the infinite eventually periodic continued fraction to the nearest integer with all partial quotients $a_n = b$ *and signs* $\epsilon_1 = +1 = -\epsilon_{n+1}$ *for* $n \in \mathbb{N}$. *Then,*

$$\beta = \tfrac{1}{2}(b - \sqrt{b^2 - 4}) \sim \frac{1}{b}.$$

For $b = 2$ the formula yields $\beta = 1$, however, the expansion is not the continued fraction expansion for 1 since Condition (5.2) is not fulfilled; fortunately, this case of the theorem is already proved by the previous lemma. For $b \geq 3$, however, Condition (5.2) is satisfied and β is represented by the above continued fraction expansion to the nearest integer; in all these cases β is an irrational number inside $[-\frac{1}{2}, \frac{1}{2})$.

Proof. In view of the definition of β,

$$\beta = [0; +1/b, -1/b, \overline{-1/b}] = \frac{1}{b - \beta};$$

hence, β is the positive root of the quadratic equation $\beta^2 - b\beta + 1 = 0$. The asymptotic formula for β follows easily from the Taylor expansion

$$\beta = \frac{b}{2}\left(1 - \sqrt{1 - \frac{4}{b^2}}\right) = \frac{1}{b} + \frac{1}{b^3} + \frac{2}{b^5} + O\left(\frac{1}{b^7}\right).$$

The lemma is proved. q.e.d.

The next and final preparatory lemma is due to Cusick and Lee [Cusick and Lee, 1971]. It is a generalization of Hall's interval arithmetic for the addition of Cantor sets which is the core of his method. We denote the length of an interval I by $|I|$.

Lemma 5.2.3 *Let $I_0, I_1, \ldots, I_n$ be disjoint bounded closed intervals of real numbers. Suppose that an open interval G is removed from the middle of I_0, leaving two closed intervals L and R on the left and right, respectively. If*

$$|G| \leq (m-1) \min\{|L|, |R|, |I_1|, \ldots, |I_n|\} \tag{5.3}$$

for some positive integer m, then

$$m\left(L \cup R \cup \bigcup_{j=1}^{n} I_j\right) = m \bigcup_{j=0}^{n} I_j.$$

Hence, if a sufficiently small interval is removed from the middle of some interval in a certain disjoint union, still the m-folded sum of the shrinked union adds up to the m-folded sum of the complete union. For the straightforward proof we refer to Cusick and

Lee [Cusick and Lee, 1971].

5.3 A Cusick and Lee-type theorem

The method of proof is along the lines of Hall's original paper [Hall, 1947] and Cusick and Lee [Cusick and Lee, 1971] as well. Since the case $b = 2$ has already been solved in the previous section, we may suppose $b \geq 3$.

Assume $x = [0; \epsilon_1/a_1, \epsilon_2/a_2, \ldots, \epsilon_n/a_n, \ldots] \in \mathcal{L}(b)$, then, by Lemma 5.2.1, the condition $a_1 \geq b$ on the first partial quotient implies

$$-\frac{1}{b - \frac{1}{2}} \leq x \leq \frac{1}{b - \frac{1}{2}}.$$

In view of the second partial quotient $a_2 \geq b$ we further find by a simple calculation

$$[0; -1/b, -1/b, -1/2] \leq x \leq [0; +1/b, -1/b, -1/2].$$

Going on, we find via $a_n \geq b$ the inequality

$$-\beta \leq x \leq \beta \tag{5.4}$$

with $\beta = [0; +1/b, -1/b, -1/b, \ldots] = \frac{1}{2}(b - \sqrt{b^2 - 4})$ as in Lemma 5.2.2. Hence, $\mathcal{L}(b) \subset [-\beta, \beta]$ and a necessary condition to find a representation of an arbitrary real number as a sum of an integer and m continued fractions to the nearest integer each of which having no partial quotient less than b is that $m\mathcal{L}(b)$ covers an interval of length at least one. We thus obtain the necessary inequality

$$\lambda\left([-m\beta, m\beta]\right) = 2m\beta \geq 1$$

where λ is the Lebesgue measure. In view of $\beta \sim b^{-1}$ by Lemma 5.2.2 we thus may expect

m to be about $\frac{b}{2}$. However, $m = \lfloor \frac{b-1}{2} \rfloor$ will not suffice since

$$m = \left\lfloor \frac{b-1}{2} \right\rfloor < \frac{1}{2\beta} = \frac{b + \sqrt{b^2-4}}{4} < \frac{b}{2},$$

as a simple computation shows.

For a start we remove from the complete interval $[-\frac{1}{2}, \frac{1}{2})$ the intervals $[-\frac{1}{2}, -\beta)$ and $(\beta, \frac{1}{2})$ according to Condition (5.4); obviously, the *two* signs $\epsilon_1 = \pm 1$ are responsible for removing intervals on *both* sides. Notice that $0 < \beta \leq \frac{1}{2}(3 - \sqrt{5}) < \frac{1}{2}$ for any $b \geq 3$. In the remaining closed interval $J_0 := [-\beta, \beta]$ all real numbers $x = [0; \epsilon_1/a_1, \epsilon_2/a_2, \ldots, \epsilon_n/a_n, \ldots]$ have a first partial quotient $a_1 \geq b$ as already explained above. Now consider all such x having sign $\epsilon_1 = \epsilon$ for some fixed $\epsilon \in \{\pm 1\}$ and partial quotient $a_1 = a$ for some $a \geq b$. Clearly, the set of those x forms an interval $I_1(\epsilon/a)$, say. Since each element of $I_1(\epsilon/a)$ is of the form

$$x = [0; \epsilon/a + T(x)] = \frac{\epsilon}{a + T(x)}$$

with $T(x) \in [-\frac{1}{2}, \frac{1}{2})$, we have either

$$I_1(-1/a) = \{x = [0; -1/a + t] \,:\, t \in [-\tfrac{1}{2}, \tfrac{1}{2})\} = \big[[0; -1/a - \tfrac{1}{2}], [0; -1/a + \tfrac{1}{2}]\big)$$

or

$$I_1(+1/a) = \{x = [0; +1/a + t] \,:\, t \in [-\tfrac{1}{2}, \tfrac{1}{2})\} = \big([0; +1/a + \tfrac{1}{2}], [0; +1/a - \tfrac{1}{2}]\big]$$

according to the sign $\epsilon = \pm 1$. In view of the condition $a_2 \geq b$ we remove from any such $I_1(\epsilon/a)$ in the next step two semi-open intervals with boundary points $[0, \pm 1/a \pm \frac{1}{2}]$ and $[0, \pm 1/a \pm \beta]$ on both sides. Consequently, the remaining intervals are

$$J_1(-1/a) := [[0; -1/a - \beta], [0; -1/a + \beta]]$$

and

$$J_1(+1/a) := [[0;+1/a+\beta],[0;+1/a-\beta]]\,.$$

In general, we consider an interval $J_n(\mathfrak{a})$ consisting of those real numbers x having a prescribed continued fraction expansion to the nearest integer. Denoting by $\mathfrak{a}$ an arbitrary admissible sequence of signs and partial quotients $\epsilon_1/a_1,\ldots,\epsilon_n/a_n$, namely positive integers $a_j \geq b$ and $\epsilon_j \in \{\pm 1\}$, then $J_n(\mathfrak{a})$ is the closed interval

$$J_n(\mathfrak{a}) := [[0;\epsilon_1/a_1,\ldots,\epsilon_n/a_n-\beta],[0;\epsilon_1/a_1,\ldots,\epsilon_n/a_n+\beta]]\,.$$

Here and in the sequel it may happen that in an interval $[A,B]$ or (A,B) we have the relation $A > B$ for the boundary points in which case the interval is meant to be equal to $[B,A]$, resp. (B,A). From such an interval $J_n(\mathfrak{a})$ we remove the open intervals of the form

$$G_{n+1}(\mathfrak{a}') := ([0;\epsilon_1/a_1,\ldots,\epsilon_n/a_n,\epsilon/a+1-\beta],[0;\epsilon_1/a_1,\ldots,\epsilon_n/a_n,\epsilon/a+\beta])$$

for any $a \geq b$ and $\epsilon = \pm 1$, where

$$\mathfrak{a}' := \mathfrak{a},\epsilon/a := \epsilon_1/a_1,\ldots,\epsilon_n/a_n,\epsilon/a$$

(by adding ϵ/a to $\mathfrak{a}$ at the end). This leads to further intervals of the form $J_{n+1}(\mathfrak{a}')$. Following Cusick and Lee [Cusick and Lee, 1971], we call this the Cantor dissection process.

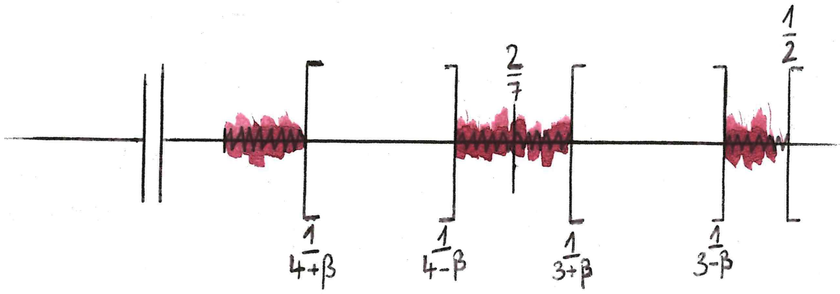


Figure 5.2: Illustration of the further intervalls.

The main idea now is applying Lemma 5.2.3 to this dissection process over and over again. In the beginning (when $n = 0$) we have $J_0 = [-\beta, \beta]$ and we remove step by step all open intervals of the form $G_1(+1/a) = ([0; +1/a + 1 - \beta], (0; +1/a + \beta])$ for all $a \geq b$ and their counterparts on the negative real axis. In fact, these are two closed intervals $L := [-\beta, [0; +1/a + \beta]]$ and $R := [[0; +1/a - \beta], \beta]$ on the left and right of $G_1(+1/a)$. Lemma 5.2.3 implies

$$m(L \cup R) = m(L \cup G_1(+1/a) \cup R) = mJ_0,$$

provided Condition (5.3) for the lengths of the intervals of type G_1 and L, R is fulfilled. It is an easy computation to prove that the start of the Cantor dissection process gives no obstruction to the general case which we shall consider below. In view of the symmetry the situation on the left is similar.

In the general case, we have to find the least positive integer m satisfying

$$|G_{n+1}(\mathfrak{a}, \pm 1/a_{n+1})| \leq (m-1) \min_{a_{n+1} \geq b} |J_{n+1}(\mathfrak{a}, \pm 1/a_{n+1})| \tag{5.5}$$

with arbitrary $a_{n+1} \geq b$. If this quantity m is found, then it follows from Lemma 5.2.3

in combination with $J_0 = [-\beta, \beta]$ that $m\mathcal{L}(b) = [-m\beta, m\beta]$ and we are done, provided $2m\beta \geq 1$ in order to cover an interval of length at least one.

For this aim we compute the lengths of the corresponding intervals by the standard continued fraction machinery as follows. Firstly, any continued fraction to the nearest integer can be written as a convergent

$$x = [0; \epsilon_1/a_1, \ldots, \epsilon_n/a_n] = \frac{p_n}{q_n}$$

with coprime p_n and $q_n > 0$. The numerators and denominators p_n, q_n satisfy certain recursion formulae (as in the case of regular continued fractions, see the Formulae (4.3) and (4.4)), which leads to

$$[0; \epsilon_1/a_1, \ldots, \epsilon_n/a_n \pm \beta] = [0; \epsilon_1/a_1, \ldots, \epsilon_n/a_n, \pm 1/b - \beta] = \frac{(b-\beta)p_n \pm p_{n-1}}{(b-\beta)q_n \pm q_{n-1}},$$

as well as

$$[0; \epsilon_1/a_1, \ldots, \epsilon_n/a_n, \pm 1/a_{n+1} + 1 - \beta] = \frac{(a_{n+1} + 1 - \beta)p_n \pm p_{n-1}}{(a_{n+1} + 1 - \beta)q_n \pm q_{n-1}},$$

and

$$[0; \epsilon_1/a_1, \ldots, \epsilon_n/a_n, \pm 1/a_{n+1} + \beta] = \frac{(a_{n+1} + \beta)p_n \pm p_{n-1}}{(a_{n+1} + \beta)q_n \pm q_{n-1}}.$$

Using this in combination with

$$p_{n+1}q_n - p_nq_{n+1} = (-1)^n \prod_{j=1}^{n+1} \epsilon_j = \pm 1,$$

yields

$$|G_{n+1}(\mathfrak{a}')| = \frac{1 - 2\beta}{((a_{n+1} + \beta)q_n \pm q_{n-1})((a_{n+1} + 1 - \beta)q_n \pm q_{n-1})},$$

as well as

$$|J_{n+1}(\mathfrak{a}')| = \frac{a_{n+1}+2\beta-b}{((b-\beta)q_n \pm q_{n-1})((a_{n+1}+\beta)q_n \pm q_{n-1})},$$

and

$$|J_{n+1}(\mathfrak{a}')| = \frac{a_{n+1}+1-b}{((a_{n+1}+1-\beta)q_n \pm q_{n-1})((b-\beta)q_n \pm q_{n-1})},$$

depending on $J_{n+1}(\mathfrak{a}')$ lying on the left or on the right of $G_{n+1}(\mathfrak{a}')$. Plugging this into (5.5), leads to

$$\begin{aligned} m-1 \quad \geq \quad \max\Bigg\{ & \frac{1-2\beta}{a_{n+1}+2\beta-b} \cdot \frac{(b-\beta)q_n \pm q_{n-1}}{(a_{n+1}+1-\beta)q_n \pm q_{n-1}}, \\ & \frac{1-2\beta}{a_{n+1}+1-b} \cdot \frac{(b-\beta)q_n \pm q_{n-1}}{(a_{n+1}+\beta)q_n \pm q_{n-1}} \Bigg\}. \end{aligned}$$

In view of $a_{n+1} \geq b$ we deduce the condition

$$m \geq 1 + \frac{1-2\beta}{2\beta} = \frac{1}{2\beta} = \frac{b+\sqrt{b^2-4}}{4}$$

Hence, we may choose $m = \lfloor \frac{b+1}{2} \rfloor$ as another short computation shows. This proves Theorem 5.1.1.

5.4 Transfer to the Complex Case

We conclude with some observations for complex continued fractions. Given a complex number $z = x + iy$, with $i = \sqrt{-1}$, we may apply results for real continued fractions to both, the real- and the imaginary part of z seperately. A short computation shows

$$i\left(a_0 + \frac{1}{a_1} + \frac{1}{\alpha_2}\right) = ia_0 + \frac{1}{-ia_1} + \frac{1}{i\alpha_2},$$

where a_0, a_1 are integers and α_2 is real (such that the expression on the left makes sense). Hence, the theorem of Cusick and Lee (5.1) immediately implies that *every complex number*

z can be written as the sum of a Gaussian integer and $2b$ regular continued fractions, where b of them have real partial quotients $a_n \geq b$ while the others have partial quotients of the form $\pm ia_n$ with integral $a_n \geq b$. Here the set of partial quotients $\mathbb{Z}$ is replaced by the set of Gaussian integers $\mathbb{Z}[i]$. Using Theorem 5.1.1 we may deduce in the same way a comparable result for continued fractions to the nearest integer in the complex case:

Corollary 5.4.1 *Every complex number z can be written as the sum of a Gaussian integer and $2\lfloor \frac{b+1}{2} \rfloor$ continued fractions to the nearest Gaussian integer, where half of them have real partial quotients $a_n \geq b$ while the other half have partial quotients of the form $\pm ia_n$ with integral $a_n \geq b$.*

Remark: It might be interesting to apply complex methods for continued fractions to the nearest Gaussian integer as introduced by Adolf Hurwitz [Hurwitz, 1888]. We expect that a careful analysis of the complex case will allow representations with less complex continued fractions. Of course, the partial quotients will not carry as much structure as in the above application; they just will be 'random' Gaussian integers of absolute value at least b.

6 A New Type of Continued Fractions with Partial Quotients from a Lattice

The contents of this chapter were established in the course of the *'Fifth International Conference on Analytic Number Theory and Spatial Tessellations'*[1] and will appear in the conference proceedings [Oswald et al., 2013]. Although Hurwitz's approach to a continued fraction expansion for complex numbers cannot be applied directly to the ring of integers of a non-quadratic cyclotomic field, we show that with a certain modification in the explicit example $\mathbb{Q}(\exp(\frac{2\pi i}{8}))$ an analogue of such a continued fraction expansion is derived following an idea of Hans Höngesberg from his Bachelor thesis. Moreover, using the geometry of Voronoï diagrams, we give far-reaching further generalizations of complex continued fractions associated with lattices.

6.1 Cyclotomic Fields: Union of Lattices

Let $n \geq 3$ be an integer. Given a primitive nth root of unity ζ_n (e.g., $\zeta_n = \exp(\frac{2\pi i}{n})$), the associated cyclotomic field $\mathbb{Q}(\zeta_n)$ is an algebraic extension of $\mathbb{Q}$ of degree $\varphi(n)$, where $\varphi(n)$ is Euler's totient, i.e., the number of prime residue classes modulo n, and its ring of integers is given by $\mathbb{Z}[\zeta_n]$[2]. In Section 3.1 we stated Hurwitz's restriction ii) that his system S shall be discrete, respectively that there shall be only finitely many elements in

[1] which took place in the Institute of Physics and Mathematics of the National Pedagogical Dragomanov University in Kiev, Ukraine, on September 16 - 20, 2013

[2] see [Neukirch, 1992, Ch. 1] for this and other details about cyclotomic fields

any finite region of the complex plane. This is not valid until $n = 3, 4, 6$, which are exactly the values for which $\varphi(n) = 2$ and $\mathbb{Q}(\zeta_n)$ is an imaginary quadratic number field. In fact, for all other values $n \geq 3$, there exist algebraic integers inside the unit circle: if $n \geq 7$, then

$$0 \neq |1 - \zeta_n|^2 = 2 - 2\cos\tfrac{2\pi}{n} < 1,$$

giving a contradiction to ii) by taking powers of $1-\zeta_n$; for $n = 5$ one finds, by the geometry of the regular pentagon,

$$0 \neq |1 + \zeta_5^3|^2 = \tfrac{1}{2}(\sqrt{5} - 1) < 1.$$

It should be mentioned that $\mathbb{Z}[\zeta_8]$ is norm-euclidean as already shown by Eisenstein [Eisenstein, 1975, v. II, pp. 585]. Here the notion 'norm-euclidean' means that the ring in question is euclidean with the canonical norm. Lenstra [Lenstra, 1975] proved that $\mathbb{Z}[\zeta_n]$ is norm-euclidean if $n \neq 16, 24$ is a positive integer with $\varphi(n) \leq 10$.

Although Hurwitz's approach does not apply to cyclotomic fields of degree strictly larger than two we shall introduce a *modified* continued fraction expansion. For the sake of simplicity we consider the explicit example of $\mathbb{Q}(\zeta_8)$ with the primitive eighth root of unity $\zeta_8 = \exp(\frac{2\pi i}{8})$ having degree four over the rationals.

Recall that a two-dimensional lattice Ω in $\mathbb{C}$ is a discrete additive subgroup. Any such lattice has a representation as $\Omega = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with complex numbers ω_1 and ω_2 being linearly independent over $\mathbb{R}$; this representation is not unique. Defining a fundamental parallelogram by $\mathcal{F}_\Omega = \{0 \leq \lambda_1, \lambda_2 < 1 : \lambda_1\omega_1 + \lambda_2\omega_2\}$, the set of its translates

$$\mathcal{F}_\Omega(\omega) := \omega + \tfrac{1}{2}(\omega_1 + \omega_2) + \mathcal{F}_\Omega$$

with lattice points ω yields a tiling of the complex plane by parallelograms of equal size each of which having exactly one lattice point in the interior which appears to be at its center. We shall call this the *lattice tiling of* Ω (with respect to the representation $\Omega = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$) consisting of lattice parallelograms $\mathcal{F}_\Omega(\omega)$.

The numbers ζ_8^j with $0 \leq j < 4 = \varphi(8)$ form an integral basis for $\mathbb{Z}[\zeta_8]$; obviously, we may also choose $\{1, i, \zeta_8, \overline{\zeta_8}\}$ as integral basis. Note that $\mathbb{Q}(\zeta_8 + \overline{\zeta_8})$ is the maximal real subfield of $\mathbb{Q}(\zeta_8)$. We shall associate two lattices. The first lattice is given by

$$\Lambda_1 := \mathbb{Z} + \mathbb{Z}i \quad (= \mathbb{Z}[i]).$$

For a complex number z we have $z \in \mathcal{F}_{\Lambda_1}(a + ib)$ with some lattice point $a + ib \in \Lambda_1$ by construction, and we write $[z]_1 = a + ib$ for the lattice point associated with z in this way. Notice that $[z]_1$ is the closest lattice point to z, however, for general lattices this is not true. In fact, *for any element from a parallelogram* $\mathcal{F}_\Omega(\omega)$ *the interior lattice point is the nearest lattice point* (in euclidean distance) *if, and only if, the diagonals of the parallelogram are of equal length, i.e.,* $\mathcal{F}_\Omega(\omega)$ *is rectangular.* This holds true for Λ_1 as well as for the second lattice we shall consider, namely the one defined by

$$\Lambda_2 := \mathbb{Z}\zeta_8 + \mathbb{Z}\overline{\zeta_8}.$$

Here we shall write $[z]_2 = c\zeta_8 + d\overline{\zeta_8}$ for the lattice point $c\zeta_8 + d\overline{\zeta_8}$ such that $z \in \mathcal{F}_{\Lambda_2}(c\zeta_8 + d\overline{\zeta_8})$. Notice that also Λ_2 is rectangular; actually both, Λ_1 and Λ_2 are even quadratic as follows from the geometry of the eighth roots of unity. In order to have a unique assignment on the boundary of our lattices we may assume that in such cases the larger coefficient shall be chosen. Finally, let

$$[z] := \tfrac{1}{2}([z]_1 + [z]_2) = \tfrac{1}{2}(a + bi + c\zeta_8 + d\overline{\zeta_8}) =: (a, b, c, d)_{1,2} \tag{6.1}$$

denote the arithmetical mean of the associated lattice points. It follows that $[z]$ is half an algebraic integer, i.e., an element of $\frac{1}{2}\mathbb{Z}[\zeta_8]$. The union of the lattices, $\Lambda_1 \cup \Lambda_2$, is again a discrete set of complex numbers but is neither a lattice nor a system S in the sense of Hurwitz [Hurwitz, 1888]. The lattice tilings of Λ_1 and Λ_2 provide a tiling of the complex plane in polygons by subdividing the parallelograms of the respective lattices into smaller polygons which we shall denoted by $Z((a,b,c,d)_{1,2})$ according to the unique assignment of the half algebraic integer $(a,b,c,d)_{1,2} = \frac{1}{2}(a + bi + c\zeta_8 + d\overline{\zeta_8})$.

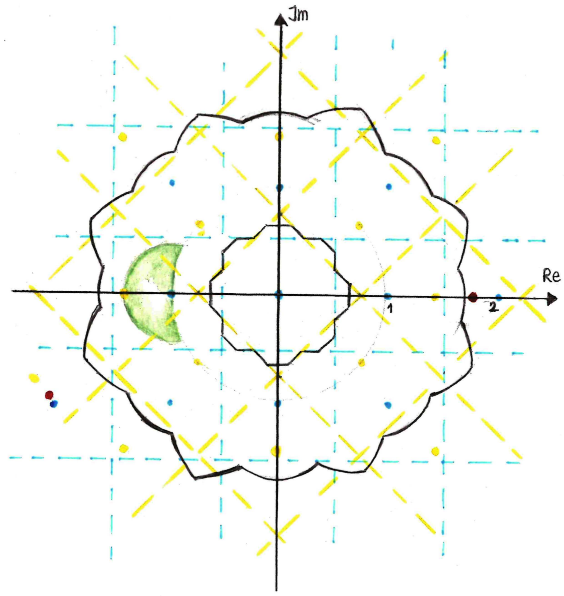


Figure 6.1: Illustration of the union of the lattices Λ_1 and Λ_2.

Following Adolf Hurwitz we consider the sequence of equations

$$z = a_0 + \frac{1}{z_1}, \quad z_1 = a_1 + \frac{1}{z_2}, \quad \ldots, \quad z_n = a_n + \frac{1}{z_{n+1}} \tag{6.2}$$

with $a_n = [z_n] = (a,b,c,d)_{1,2}$; here the z_n are assumed not to vanish. This leads to a continued fraction expansion

$$z =: [a_0, a_1, \ldots, a_n, z_{n+1}] \tag{6.3}$$

having partial quotients in the set $\frac{1}{2}\mathbb{Z}[\zeta_8]$. Similarly to Hurwitz's continued fraction this expansion can be described by $z \mapsto T(z) = \frac{1}{z} - [\frac{1}{z}]$, where the Gauss bracket $\lfloor \cdot \rfloor$ is replaced by $[\cdot]$ defined in (6.1). Obviously, a vanishing z_n would imply a finite expansion going along with $z \in \mathbb{Q}(\zeta_8)$. In the sequel we shall assume $z \notin \mathbb{Q}(\zeta_8)$ in order to have an infinite continued fraction.

As in the case of Hurwitz's complex continued fraction certain sequences of partial quotients are impossible. By construction, $z_n - a_n$ lies inside a 24-sided polygon P with center at the origin, which is determined by the straight lines $x = \pm\frac{1}{2}$, $y = \pm\frac{1}{2}$, $x = \pm(\frac{1}{4} + \frac{1}{4}\sqrt{2})$, $y = \pm(\frac{1}{4} + \frac{1}{4}\sqrt{2})$, $y \pm x = \pm(\frac{1}{2} + \frac{1}{4}\sqrt{2})$ and $x \pm y = \pm\frac{1}{2}\sqrt{2}$ defining the boundary in the $x + iy$-plane (see Fig 6.1). Hence,

$$z_{n+1} = \frac{1}{z_n - a_n} \in R := P^{-1};$$

here we have used the notation $\mathcal{M}^{-1} := \{m^{-1} : m \in \mathcal{M}\}$ for any set $\mathcal{M}$ not containing zero. In the sequel we shall also use the notation $\mathsf{D}_r(m)$ (resp. $\overline{\mathsf{D}}_r(m)$) for the open (closed) disc of radius r with center m. Therefore, the following half algebraic integers cannot occur as partial quotients:

$$\begin{aligned}
&(0,0,0,0)_{1,2},\\
&(\pm1,0,0,0)_{1,2}, (0,\pm1,0,0)_{1,2}, (0,0,\pm1,0)_{1,2}, (0,0,0,\pm1)_{1,2},\\
&(\pm1,0,0,\pm1)_{1,2}, (\pm1,0,\pm1,0)_{1,2}, (0,\pm1,\pm1,0)_{1,2}, (0,\pm1,0,\mp1)_{1,2},\\
&(\pm1,0,\pm1,\pm1)_{1,2}, (\pm1,\pm1,\pm1,0)_{1,2}, (0,\pm1,\pm1,\mp1)_{1,2}, (\pm1,\mp1,\pm0,\pm1)_{1,2},\\
&(\pm1,\pm1,\pm1,\pm1)_{1,2}, (\pm1,\pm1,\pm1,\mp1)_{1,2}, (\mp1,\pm1,\pm1,\mp1)_{1,2}, (\pm1,\mp1,\pm1,\pm1)_{1,2},\\
&(\pm2,\pm1,\pm1,\pm1)_{1,2}, (\pm1,\pm1,\pm2,\pm1)_{1,2}, (\pm1,\pm1,\pm2,\mp1)_{1,2}, (\pm1,\pm2,\pm1,\mp1)_{1,2},\\
&(\mp1,\pm2,\pm1,\mp1)_{1,2}, (\mp1,\pm1,\pm1,\mp2)_{1,2}, (\mp1,\pm1,\mp1,\mp2)_{1,2}, (\mp2,\pm1,\mp1,\mp1)_{1,2}.
\end{aligned}$$

Next we investigate the sequence of partial quotients a_n with respect to convergence. Suppose that $a_n = (2,0,1,1)_{1,2}$, then $z_{n+1} \in Z((2,0,1,1)_{1,2})$. In view of (6.2) we have $z_n - a_n \in Z^{-1}((2,0,1,1)_{1,2})$. The latter set is bounded by $\mathsf{D}_{\frac{2}{6}}(\frac{1}{6}\sqrt{2} + \frac{1}{6}\sqrt{2}i)$,

$\mathsf{D}_{\frac{2}{6}}(\frac{1}{6}\sqrt{2}-\frac{1}{6}\sqrt{2}i)$, $x \pm y = \frac{1}{2}\sqrt{2}$, and $x = \frac{1}{4}+\frac{1}{4}\sqrt{2}$. Hence, the set $Z^{-1}((2,0,1,1)_{1,2})$ intersects with the real axis at $x = \frac{1}{3}\sqrt{2}$. However, the polygons $Z((-2,0,-1,-1)_{1,2})$, $Z((-2,1,0,-2)_{1,2})$, $Z((-2,-1,-2,0)_{1,2})$, $Z((-1,1,0,-2)_{1,2})$, and $Z((-1,-1,-2,0)_{1,2})$ have all in common that their respective lattice points have distance at least $\frac{1}{3}\sqrt{2}$ in x-direction from the boundary. A similar reasoning provides restrictions for their predecessors of $a_n = (-2,-1,-2,-1)_{1,2}$. This leads to a list of pairs which do not occur as consecutive partial quotients. In order to prove the convergence of this continued fraction

a_n	a_{n+1}
$(-2,0,-1,-1)_{1,2}$, $(-2,1,0,-2)_{1,2}$, $(-2,-1,-2,0)_{1,2}$, $(-1,1,0,-2)_{1,2}$, $(-1,-1,-2,0)_{1,2}$	$(2,0,1,1)_{1,2}$
$(2,0,1,1)_{1,2}$, $(2,-1,0,2)_{1,2}$, $(1,-1,0,2)_{1,2}$	$(-2,-1,-2,-1)_{1,2}$

Table 6.1: Impossible pairs of consecutive partial quotients

expansion we shall show

$$|k_n| > 1 \qquad \text{with} \quad k_n := \frac{q_n}{q_{n-1}} \tag{6.4}$$

by induction on n, which is an analogue of Lemma 4.1.5. This implies convergence by the standard reasoning. One has

$$z - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2(z_{n+1}+k_n^{-1})} \qquad \text{and} \qquad z - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}^2(z_{n+1}^{-1}+k_n)}.$$

Here p_j and q_j denote the numerator and denominator to the convergents of the continued fraction expansion defined in the same way as in previous sections. Moreover, as in Subsection 4.1.3, we shall use the recursive formula $k_n = a_n + \frac{1}{k_{n-1}}$.

For $k_1 = a_1$ assertion (6.4) obviously holds since $a_n \in R$. Now assume $|k_j| > 1$ for $1 \le j < n$ and $|k_n| \le 1$ with some positive integer n. Since $k_n = a_n + \frac{1}{k_{n-1}} \in \mathsf{D}_1(a_n)$ and, by assumption, $|k_n| \leq 1$, it follows that a_n has to be one of the following numbers:

$$(\pm 2, 0, \pm 1, \pm 1)_{1,2}, (\pm 1, \pm 1, \pm 2, 0)_{1,2}, (0, \pm 2, \pm 1, \mp 1)_{1,2},$$
$$(\mp 1, \pm 1, 0, \mp 2)_{1,2}.$$

By symmetry, we may assume without loss of generality that

$$a_n = (2, 0, 1, 1)_{1,2} = 1 + \tfrac{1}{2}\sqrt{2}.$$

Hence, $k_n = a_n + \frac{1}{k_{n-1}}$ is located in the intersection of the unit disc and $\mathsf{D}_1((2,0,1,1)_{1,2})$. Consequently, $\frac{1}{k_{n-1}} = k_n - a_n$ lies in the intersection of the unit disc and $\mathsf{D}_1((-2,0,-1,-1)_{1,2})$. Hence,

$$k_{n-1} = \frac{1}{k_n - a_n} = a_{n-1} + \frac{1}{k_{n-2}}$$

is located outside the unit disc but in the interior of $\mathsf{D}_{\frac{1}{7}(-2+4\sqrt{2})}(\frac{1}{7}(-2-3\sqrt{2}))$ (the set in 6.1 coloured in green). Since $|k_{n-2}| > 1$, it follows that k_{n-1} lies as well in $\mathsf{D}_1(a_{n-1})$. Hence, a_{n-1} can take only one of the following values:

$$(-2, 0, -1, -1)_{1,2}, (-2, -1, -2, 0)_{1,2}, (-2, 1, 0, -2)_{1,2}, (-1, 1, 0, -2)_{1,2},$$
$$(-1, -1, -2, 0)_{1,2}, (-2, -1, -2, -1)_{1,2}, (-2, 1, -1, -2)_{1,2}.$$

In view of our list of non-admissible partial quotients (see the table above) the value for a_{n-1} can be found amongst

$$(-2, -1, -2, -1)_{1,2}, (-2, 1, -1, -2)_{1,2}.$$

Again, by symmetry, we may suppose without loss of generality that $a_{n-1} = (-2, -1, -2, -1)_{1,2}$. It follows that $k_{n-1} = a_{n-1} + \frac{1}{k_{n-2}}$ lies in the intersection of the discs $\overline{\mathsf{D}}_{\frac{1}{7}(-2+4\sqrt{2})}(\frac{1}{7}(-2-3\sqrt{2}))$ and $\mathsf{D}_1((-2,-1,-2,-1)_{1,2})$. Hence, $\frac{1}{k_{n-2}} = k_{n-1} - a_{n-1}$ is in the intersection of the unit disc and

$$\overline{\mathsf{D}}_{\frac{1}{7}(-2+4\sqrt{2})}(\tfrac{1}{7}(5+\tfrac{9}{4}\sqrt{2}) + \tfrac{1}{2}(1+\tfrac{1}{2}\sqrt{2})i) \subseteq \mathsf{D}_{0.53}(1.17+0.85i).$$

Thus, we find k_{n-2} outside the unit disc and inside $\mathsf{D}_{0.3}(0.65+0.47i)$. Since $k_{n-2} = a_{n-2} + \frac{1}{k_{n-3}}$ lies inside $\mathsf{D}_1(b_{n-2})$, we conclude that a_{n-2} has to be one of the following numbers:

$$(1,-1,0,2)_{1,2}, (2,-1,0,2)_{1,2}, (2,0,1,1)_{1,2}.$$

However, all these values appear in the list of impossible partial quotients (see the table on the previous page), giving the desired contradiction. Thus we have proved

Theorem 6.1.1 *The continued fraction expansion (6.3) with partial quotients (6.1) from* $\frac{1}{2}\mathbb{Z}[\zeta_8]$ *converges.*

Remark: Our line of argumentation runs similar to Julius', as well as Adolf Hurwitz's, original proofs for $|k_n| < 1$, see [Hurwitz, 1895, p. 31] and [Hurwitz, 1888, p. 195].

To overcome the minor flaw that the partial quotients might be not algebraic integers one may exchange Λ_1 and Λ_2 by taking their sublattices $2\Lambda_1 = 2\mathbb{Z}+2i\mathbb{Z}$ and $2\Lambda_2 = 2\zeta_8\mathbb{Z}+2\overline{\zeta_8}\mathbb{Z}$ and follow the above analysis of the corresponding continued fraction expansion.

There are several aspects which could be studied further. Firstly, what are the arithmetical properties of this new continued fraction expansion? Can one prove a similar result on bounded expansions and quadratic equations as Hurwitz did for his complex continued fractions? Moreover, what are the limits of the construction for $\mathbb{Q}(\zeta_8)$ sketched above? Does this lead to continued fraction expansions for other cyclotomic fields as well? We do not answer these questions here but provide another generalization of Adolf Hurwitz's approach to complex continued fractions.

6.2 Generalization: Voronoï Diagrams

There is a lot of literature about Voronoï diagrams and Voronoï cells; the monographies of Gruber [Gruber, 2007] and Matousek [Matoušek, 2002] provide excellent readings on this topic. In the sequel we shall concentrate on the two-dimensional situation.

Given a discrete set S of points in the complex plane, the *Voronoï cell* for a point $p \in S$ is defined by

$$\mathcal{V}_S(p) = \{z \in \mathbb{C} \,:\, |z - p| \leq |z - q| \ \forall q \in S\},$$

i.e., the set of all z that are closer to p than to any other element of S (in euclidean distance). Any Voronoï cell $\mathcal{V}_S(p)$ is a convex polygon and their union over all $p \in S$ is called *Voronoï diagram* and yields a tiling of the complex plane. The earliest appearance of Voronoï cells is in a picture in Descartes' solar system in his *Principia Philosophiae* from 1644 (cf. [Matoušek, 2002], p. 120). A rigour mathematical definition was first given by Dirichlet [Dirichlet, 1850] and Voronoï [Voronoï, 9089] in the setting of quadratic forms.

The Voronoï diagram of the lattice $\mathbb{Z}[i]$ of Gaussian integers coincides with the lattice tiling by squares $\mathcal{F}_{\mathbb{Z}[i]}(a + ib)$ introduced in the previous section.

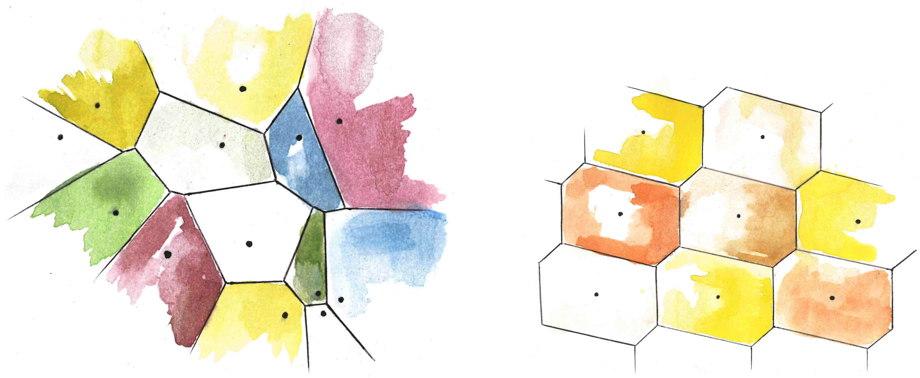


Figure 6.2: On the left a *random* Voronoï diagram. On the right the one for the lattice generated by 1 and $\frac{1}{4}(1 + 3i)$; here the cells are of honeycomb shape.

We have already noticed there, although in different language, that this is a rare event, namely, that *a lattice tiling is a Voronoï diagram if, and only if, the lattice is rectangular.* Otherwise the Voronoï cells are hexagonal.[3] In the sequel we shall consider lattices of the form $\Lambda = \delta\mathbb{Z} + \tau\mathbb{Z}$ with a real number $\delta > 0$ and $\tau = x + iy \in \mathbb{C}$ from the upper half-plane (i.e., $y > 0$). This is not a severe restriction since we are concerned with approximations by fractions $\frac{p}{q}$ built from our lattice, $p, q \in \Lambda$, and

$$\frac{p}{q} = \frac{\frac{\omega_1}{\delta}p}{\frac{\omega_1}{\delta}q} = \frac{P}{Q} \tag{6.5}$$

with $P = \frac{\omega_1}{\delta}p, Q = \frac{\omega_1}{\delta}q \in \Omega$, where

$$\Omega := \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \frac{\omega_1}{\delta}(\delta\mathbb{Z} + \tau\mathbb{Z}) = \frac{\omega_1}{\delta}\Lambda$$

by setting $\tau = \delta\frac{\omega_2}{\omega_1}$ (which is not real by the linear independence of ω_1 and ω_2 over $\mathbb{R}$ and, hence, can be chosen as an element from the upper half-plane). Therefore, any approximation by a quotient from Λ corresponds to an approximation by a quotient of the equivalent lattice Ω and vice versa. Lattices Ω_1 and Ω_2 are said to be *equivalent* if there exists a complex number $\omega \neq 0$ such that $\Omega_1 = \omega\Omega_2$.

Similarly to Adolf Hurwitz, (6.2) and (6.3), respectively, we consider a sequence of equations,

$$z = a_0 + \frac{1}{z_1}, \quad z_1 = a_1 + \frac{1}{z_2}, \quad \ldots, \quad z_n = a_n + \frac{1}{z_{n+1}}$$

with $a_n \in \Lambda$ chosen such that z_n is from the Voronoï cell $\mathcal{V}_\Lambda(a_n)$ around a_n; of course, the appearing z_j are assumed not to vanish. This leads to a continued fraction expansion

$$z = [a_0; a_1, \ldots, a_n, z_{n+1}] \tag{6.6}$$

with partial quotients in the lattice Λ. Obviously, a vanishing z_n would imply that z has a finite expansion and, thus, z would have a representation as a quotient of two lattice

[3] see also [Gruber, 2007]

points. In the sequel we shall assume that z is not of this type, that is $z \in \mathbb{C} \setminus \mathbb{Q}(\Lambda)$, where $\mathbb{Q}(\Lambda) := \{\frac{p}{q} : p, q \in \Lambda\}$, and the continued fraction expansion for z is infinite.

In order to prove the convergence of this continued fraction expansion we define once again $k_n = \frac{q_n}{q_{n-1}}$ and show the analogue of (6.4), i.e., $|k_n| > 1$; here q_n again denotes the denominator of the nth convergent to the just defined new continued fraction for z. It is not too difficult to deduce the desired convergence in just the same way as for (6.4) in the previous section.

In this general setting our reasoning, however, can be less detailed than in the explicit example of the previous section. By definition, we find for the Voronoï cell $\mathcal{V}_\Lambda(0) \subset \overline{\mathsf{D}}_\rho(0)$ with

$$\rho := \tfrac{1}{2} \max\{\delta, |\tau|, |\tau \pm \delta|\}; \tag{6.7}$$

this follows by considering the neighbouring lattice points $\pm\delta, \pm\tau, \pm\tau \pm \delta$ of the origin. Of course, here one could be more precise by exploiting the geometry and using the knowledge that the volume of each cell equals the determinant of the lattice. Since $a_n - z_n \in \mathcal{V}_\Lambda(0)$ it follows that

$$z_{n+1} \in \overline{\mathsf{D}}_\rho(0)^{-1} = \mathbb{C} \setminus \mathsf{D}_{\rho^{-1}}(0).$$

Hence, z_{n+1} lies outside the disc of radius ρ^{-1} with center at the origin. In order to prevent that z_{n+1} is located inside the Voronoï cell $\mathcal{V}_\Lambda(0)$ (which would cause difficulties for convergence) we need to put a restriction on ρ. In view of $z_{n+1} \in \mathcal{V}_\Lambda(a_{n+1})$ and $|a_{n+1} - z_{n+1}| \leq \rho$ we obtain $|a_{n+1}| \geq \rho^{-1} - \rho$. To conclude with the proof by induction we assume $|k_n| > 1$ and deduce via $k_{n+1} = a_{n+1} + \frac{1}{k_n}$ the inequality

$$|k_{n+1}| = |a_{n+1}| - \frac{1}{|k_n|} > \rho^{-1} - \rho - 1$$

which is greater than or equal to one for $\rho \leq \sqrt{2} - 1$. Hence,

Theorem 6.2.1 *The continued fraction expansion (6.6) with partial quotients in the lat-*

tice $\Lambda = \delta\mathbb{Z} + \tau\mathbb{Z}$ *converges provided* $\rho \le \sqrt{2} - 1$*, where* ρ *is given by (6.7).*

The bound on ρ is not completely satisfying. Indeed, the statement of the theorem does not imply the cases of the Gaussian lattice $\mathbb{Z}[i]$ and the Eisenstein lattice $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$ considered by Hurwitz [Hurwitz, 1888]. A more sophisticated analysis would lead to an extension of the above theorem covering these cases. Another, more simple solution, relies on the observation (6.5) that for approximation by quotients from a lattice one may exchange the lattice in question by an equivalent lattice. Hence, by using an appropriate scaling, *one can obtain a continued fraction expansion with partial quotients from any given lattice.*

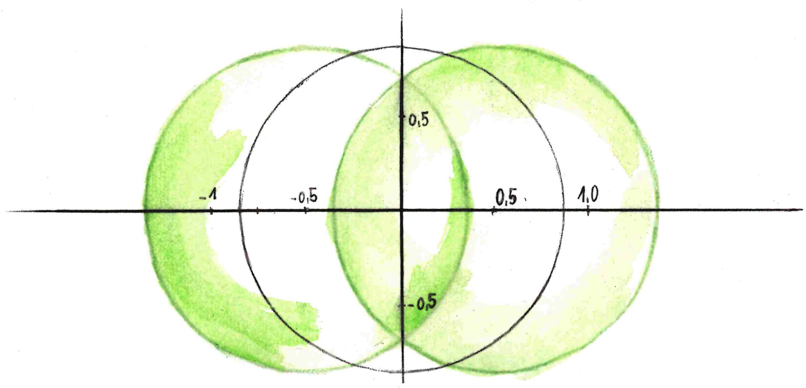


Figure 6.3: The restrictions for ρ are indicated by the black circle ($|\tau| \le 2(\sqrt{2}-1)$) and the green circles ($|\tau \pm \delta| \le 2(\sqrt{2}-1)$) with the special value $\delta = \frac{1}{2}$. The set of admissible τ is given by the non-empty intersection of all three circles.

Remark: In a similar way one could also consider arbitrary discrete point sets S in the complex plane in place of a lattice provided the corresponding Voronoï diagram would share the essential property of having sufficiently small Voronoï cells. This would lead to another continued fraction expansion with partial quotients from S, however, for practical purposes discrete sets S with structure seem to be more useful than others.

7 Approximation Quality

Since the continued fraction expansion of a real or complex number x provides a sequence of convergents, it is a natural question to ask about the approximation quality of those. In the real case, we know some very strong results, starting with the work of Legendre. Already in Chapter 1 we mentioned that every fraction $\frac{p}{q} \in \mathbb{Q}$ satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

can be found in the sequence of convergents; furthermore, among two consecutive convergents at least one satisfies this inequality. This result was improved, inter alia, by Emil Borel [Borel, 1903] and Adolf Hurwitz [Hurwitz, 1891].

In this chapter, building on [Oswald, 2014b], we prove upper bounds for the approximation quality of diophantine approximations formed from lattice points in the complex plane. These upper bounds depend on a certain lattice invariant. In particular, we generalize a method based on geometrical ideas of Hermann Minkowski and improved by Hilde Gintner. Subsequently, we examine the spectrum arising from the infimum of the constants occuring in the upper bound and give a proof of the existence of infinitely many solutions of generalized Pell equations in the complex case.

7.1 Previous Results

In the real case, the quality of diophantine approximation is ruled by A. Hurwitz's famous result [Hurwitz, 1891] that for every irrational number z there are infinitely many rationals $\frac{p}{q} \in \mathbb{Q}$ such that

$$\left| z - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}, \tag{7.1}$$

and if the constant $\frac{1}{\sqrt{5}}$ is replaced by any smaller quantity there are only finitely many $\frac{p}{q} \in \mathbb{Q}$ approximating $\frac{1}{2}(\sqrt{5}+1)$ and all equivalent numbers[1] z with the corresponding quality. Referring to this best possible result, there were various attempts in the complex case. A first approach by Hermann Minkowski [Minkowski, 1910] (cf. [Hofreiter, 1935]), using geometry of numbers, led to the upper bound $\frac{\sqrt{6}}{\pi|q|^2}$ in the case of $\frac{p}{q} \in \mathbb{Q}(i)$.

In 1925, Lester R. Ford finally found the sharp complex analogue [Ford, 1925]: *If $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ is any complex irrational number, then there exist infinitely many $p, q \in \mathbb{Z}[i]$ such that*

$$\left| z - \frac{p}{q} \right| < \frac{1}{\sqrt{3}|q|^2}.$$

This upper bound is as well best possible.[2] Around fifty years later, Richard B. Lakein [Lakein, 1975] gave a constructive proof using continued fractions in the tradition of A. Hurwitz. In 1933, Oskar Perron [Perron, 1933] extended Ford's examinations to other imaginary quadratic number rings. *If $z \in \mathbb{C} \setminus \mathbb{Q}(i\sqrt{2})$ is any complex irrational number, then there exist infinitely many $p, q \in \mathbb{Z}[i\sqrt{2}]$ such that*

$$\left| z - \frac{p}{q} \right| < \frac{1}{\sqrt{2}|q|^2};$$

if the constant $\frac{1}{\sqrt{2}}$ is replaced by any smaller quantity in general there are only finitely

[1] Two irrational numbers β, γ are called *equivalent* if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ and $\gamma = \frac{a\beta+b}{c\beta+d}$.

[2] In 1930, Oskar Perron published the same result, since he had forgotten that he even refereed Ford's work. Enbedded in a 'historical correction', Perron expressly apologized for his failure in a second version of 1931 [Perron, 1931].

many $\frac{p}{q}$*, where* $p, q \in \mathbb{Z}[i\sqrt{2}]$. Interestingly, Perron used, among other tools, again geometrical methods of Minkowski, in particular his lattice point theorem. In 1936, Hilde Gintner[3] [Gintner, 1936], generalizing results of her thesis advisor Nikolaus Hofreiter [Hofreiter, 1935], continued this approach, applying additionally Minkowski's theorem on linear forms [Minkowski, 1910]. She proved that *in any ring* $O_{\sqrt{-D}}$ *of integers associated with an imaginary quadratic number field* $\mathbb{Q}(\sqrt{-D})$ *there exist infinitely many integers* p, q *such that for any arbitrary complex number* $z \in \mathbb{C}$*, one has*

$$\left| z - \frac{p}{q} \right| < \frac{\sqrt{6D}}{\pi |q|^2} \qquad \text{if } D \not\equiv 3 \bmod 4,$$

respectively

$$\left| z - \frac{p}{q} \right| < \frac{\sqrt{6D}}{2\pi |q|^2} \qquad \text{if } D \equiv 3 \bmod 4.$$

Notice that only in the few examples of euclidean rings of integers of imaginary quadratic fields (for $D = 1, 2, 3, 7, 11$) an analogue of A. Hurwitz's continued fraction can be realized in a straightforward manner (see Section 6). Although Gintner's result provides better, probably not best possible, upper bounds for arbitrary imaginary quadratic number fields than known before, all considerations are still restricted to certain imaginary quadratic number rings.

7.2 Minkowski and Gintner

In the following, we generalize the method of Hilde Gintner. First, we apply her approach to the generalized Julius Hurwitz-lattice $(1 + i\sqrt{m})\mathbb{Z}[\sqrt{m}]$ where $m \in \mathbb{N}$ is squarefree and subsequently to arbitrary lattices.

[3] In Section 3.3 we have already stated results of her.

7.2.1 Preliminaries

Main tools in [Gintner, 1936] are two theorems arising from Minkowksi's famous *'Geometry of Numbers'* [Minkowski, 1910]. Those theorems are also stated in [Cassels, 1959], where John William Scott Cassels so to say translated them to a more modern language of lattices. He considered these structures as "the most important concept in geometry of numbers" [Cassels, 1959, p. 9]. Here we stay close to the original version of Minkowski.

Theorem 7.2.1 ***(Lattice Point Theorem, Minkowski, 1889/91)***
Let Λ be a full lattice in $\mathbb{R}^n$ and $C \in \mathbb{R}^n$ be a convex and symmetric body with bounded volume $vol(C) \geq 2^n \det(\Lambda)$. Then C contains at least one lattice point of Λ different from the origin.

Theorem 7.2.2 ***(Linear Forms Theorem, Minkowski, 1896)***
Let $Y_1, Y_2, \cdots, Y_{2s-1}, Y_{2s}$ be s pairs of linear forms with complex conjugated coefficients z_{jk} and another $r = n - 2s$ linear forms $Y_{2s+1}, \ldots Y_n$ with real coefficients z_{jk}. Then there exist integers $x_1, \ldots, x_n$, not all zero, such that

$$|Y_j(x_1, \cdots, x_n)| \leq \left(\frac{2}{\pi}\right)^{\frac{s}{n}} |\det(z_{jk})|^{\frac{1}{n}}$$

for $j = 1, \cdots n$.

Gintner considered integers of an arbitrary imaginary quadratic number ring $O_{\sqrt{-D}}, D \in \mathbb{N}$, as lattice points in the complex plane. Using Theorem 7.2.2, she showed that there exist infinitely many numbers $p, q \in O_{\sqrt{-D}}$ such that $\frac{p}{q}$ approximates a given arbitrary complex number $z \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-D})$ with a certain accuracy. This upper bound was subsequently improved by Theorem 7.2.1.

Remark: The restriction to integer rings in former studies providing a unique factorisation of the approximants $\frac{p}{q}$ can generally be overcome. Since for algebraic u_j their norm satisfies $|N(u_j)| \in \mathbb{N}_0$, there can only be finitely many factorisations $q = u_1 \cdot \ldots \cdot u_r$ for $q \in O_{\sqrt{-D}}$.

This eventual finite ambiguity of $|q|$ does not change the fact that there exist infinitely many approximations. Of course, this remark also applies to quadratic number fields with slightly different rings of integers.

One significant advantage of the method of Minkowski and Gintner is that they are not limited to quadratic number fields with euclidean algorithms.

7.2.2 Application of Linear Forms

We define pairs of complex conjugated linear forms in variables $X_1, \dots, X_4$ by

$$Y_1 = \frac{(1+i\sqrt{m})}{\sqrt{2}} \left((X_1 + i\sqrt{m}X_2) - (\beta + i\gamma)(X_3 - i\sqrt{m}X_4) \right), Y_2 = \overline{Y_1}$$

and

$$Y_3 = \frac{(1+i\sqrt{m})}{\sqrt{2}t^2}(X_3 + i\sqrt{m}X_4), Y_4 = \overline{Y_3},$$

where $m \in \mathbb{N}$ is squarefree and $t \in \mathbb{R}$. We compute the determinant of the coefficient matrix as

$$\begin{aligned}
\det(z_{jk}) &= \frac{(1+m)^2}{(\sqrt{2})^4} \det \begin{pmatrix} 1 & i\sqrt{m} & -(\beta+i\gamma) & (\beta+i\gamma)i\sqrt{m} \\ 1 & -i\sqrt{m} & -(\beta-i\gamma) & -(\beta-i\gamma)i\sqrt{m} \\ 0 & 0 & t^{-2} & t^{-2}i\sqrt{m} \\ 0 & 0 & t^{-2} & -t^{-2}i\sqrt{m} \end{pmatrix} \\
&= \left(\frac{1+m}{2t^2}\right)^2 \det \begin{pmatrix} 1 & i\sqrt{m} \\ 1 & -i\sqrt{m} \end{pmatrix}^2 = (1+m)^2 \frac{(-2i\sqrt{m})^2}{4t^4} \\
&= (1+m)^2 \frac{-m}{t^4}.
\end{aligned}$$

Now the direct application of the theorem above leads to the existence of integers

x_1, x_2, x_3, x_4, not all equal zero, satisfying

$$|Y_j(x_1, ..., x_4)| < \sqrt{\frac{2(1+m)}{\pi} \frac{\sqrt[4]{m}}{t}} \tag{7.2}$$

for $1 \leq j \leq 4$. In particular, we have

$$\begin{aligned} 0 \quad &\leq \quad |Y_1| \cdot |Y_3| = \frac{|1+i\sqrt{m}|^2}{2t^2} |(x_1 + i\sqrt{m}x_2) - (\beta + i\gamma)(x_3 - i\sqrt{m}x_4)| \cdot |x_3 + i\sqrt{m}x_4| \\ &< \quad \frac{2(1+m)}{\pi} \frac{\sqrt{m}}{t^2}. \end{aligned} \tag{7.3}$$

Remark: If $Y_3 = 0$, in view of (7.2), then $x_3 = x_4 = 0$ and $|Y_1| = \frac{(1+i\sqrt{m})}{\sqrt{2}} |x_1 + i\sqrt{m}x_2|$ tends to zero as $t \to \infty$. Since the case $x_1 = x_2 = 0 = x_3 = x_4$ is excluded, it follows that $Y_3 \neq 0$. On the other hand, if we have $Y_1 = 0$, then $(1 + i\sqrt{m})(x_1 + i\sqrt{m}x_2) = (\beta + i\gamma)(1 + i\sqrt{m})(x_3 + i\sqrt{m}x_4)$ and it follows that $(\beta + i\gamma) \in (1 + i\sqrt{m})\mathbb{Q}(i\sqrt{m})$ is a rational complex number. Consequently, we may suppose $Y_1, Y_3 \neq 0$. Thus, we can divide (7.3) by $|1 + i\sqrt{m}||x_3 + i\sqrt{m}x_4|$ and receive

$$0 < |1 + i\sqrt{m}||(x_1 + i\sqrt{m}x_2) - (\beta + i\gamma)(x_3 - i\sqrt{m}x_4)| < \frac{4(1+m)\sqrt{m}}{\pi |x_3 + i\sqrt{m}x_4||1 + i\sqrt{m}|}.$$

Setting $p = (1 + i\sqrt{m})(x_1 + i\sqrt{m}x_2), q = (1 + i\sqrt{m})(x_3 + i\sqrt{m}x_4)$ and $z = \beta + i\gamma$, then for each arbitrary $z \in \mathbb{C}$, there exist complex numbers $p, q \in (1 + i\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ with

$$\left| z - \frac{p}{q} \right| < \frac{4(1+m)\sqrt{m}}{\pi |q|^2}.$$

Furthermore, we may conclude that there are infinitely many $p, q \in (1 + i\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ fulfilling this inequality. Since, for arbitrary $t > 0$, (7.2) holds for certain $x_1, ..., x_4 \in \mathbb{Z}$, not all equal to zero, increasing the parameter t leads to an infinitude of such integer solutions. Therewith, we receive infinitely many $\frac{p}{q}$ satisfying the corresponding approximation quality.

Given $m \in \mathbb{N}$ and $z \in \mathbb{C}$, there exist infinitely many $p, q \in (1 + i\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ with

$$\left| z - \frac{p}{q} \right| < \frac{4(1+m)\sqrt{m}}{\pi |q|^2}.$$

Example:

For $m = 1$, the Julius Hurwitz case, we have infinitely many $p, q \in (1+i)\mathbb{Z}[i]$ satisfying

$$\left| z - \frac{p}{q} \right| < \frac{8}{\pi |q|^2}.$$

7.2.3 Improvement by the Lattice Point Theorem

However, those approximations are certainly not best possible. By using the Lattice Point Theorem the result can be improved significantly. Based on the same definition of linear forms $Y_1, ..., Y_4$ from above, we define a symmetric convex body by

$$C := \{x \in \mathbb{R}^4 : |Y_1| + |Y_3| \leq M, |Y_2| + |Y_4| \leq M\}.$$

For calculating the volume

$$vol(C) = \int\int\int\int \mathbb{1}_C dY_1 dY_2 dY_3 dY_4,$$

where $\mathbb{1}_C$ is the indicator function associated with C, we divide those linear forms into real and imaginary parts as

$$Y_1 = \overline{Y_2} = \frac{1}{\sqrt{2}}(\varphi_1 + i\psi_1) \text{ and } Y_3 = \overline{Y_4} = \frac{1}{\sqrt{2}}(\varphi_2 + i\psi_2).$$

The functional determinant of this transformation is

$$\det(Df(\varphi_1, \psi_1) \cdot Df(\varphi_2, \psi_2)) = \left(\frac{1}{\sqrt{2}}\right)^4 \det \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^2 = \frac{1}{4}(-2i)^2 = -1.$$

In view of the transformation formula we deduce

$$vol(C) = |\det(Df)| \int\int\int\int \mathbb{1}_{\sqrt{\phi_1^2+\psi_1^2}+\sqrt{\varphi_2^2+\psi_2^2}\leq\sqrt{2}M} d\varphi_1 d\varphi_2 d\psi_1 d\psi_2.$$

Using polar coordinates $\varphi_j = Mr_j \cos\alpha_j, \psi_j = Mr_j \sin\alpha_j$, we find

$$vol(C) = M^4 \iint_{[0,2\pi)^2} d\alpha_1 d\alpha_2 \iint_{r_1+r_2\leq\sqrt{2}, r_j\geq 0} r_1 r_2 dr_1 dr_2 = \frac{2\pi^2}{3} M^4.$$

On the other hand, the Lattice Point Theorem implies that there exists a point $x = (x_1, \cdots, x_4) \in \mathbb{Z}^4 \setminus \{0\}$ in C whenever

$$vol(C) \geq 2^4 |\det(\alpha_{jk})| = 2^4 (1+m)^2 \frac{|m|}{t^4}.$$

Therewith, we get the estimate

$$M^4 \geq 3 \cdot 2^3 \cdot (1+m)^2 \frac{|m|}{t^4 \pi^2}, \text{ respectively } M \geq \sqrt{2(1+m)} \frac{\sqrt[4]{6|m|}}{\sqrt{\pi} t}.$$

Choosing M such that we receive an equality, leads to the existence of $x \in \mathbb{Z}^4 \setminus \{0\}$ with

$$|Y_1| + |Y_3| \leq \sqrt{2(1+m)} \frac{\sqrt[4]{6|m|}}{\sqrt{\pi} t},$$

respectively

$$|Y_1| \cdot |Y_3| \leq (\frac{1}{2}(|Y_1| + |Y_3|))^2 \leq \frac{1+m}{2} \cdot \frac{\sqrt{6|m|}}{\pi t^2}.$$

Since $|Y_1|, |Y_3| \neq 0$ we continue as before and have

$$0 < \frac{|1+i\sqrt{m}|^2}{2t^2} |(x_1 + i\sqrt{m}x_2) - (\beta + i\gamma)(x_3 - i\sqrt{m}x_4)| \cdot |x_3 + i\sqrt{m}x_4| < \frac{1+m}{2} \cdot \frac{\sqrt{6|m|}}{\pi t^2},$$

respectively

$$0 < \left| \frac{(x_1 + i\sqrt{m}x_2)(1 + i\sqrt{m})}{(x_3 + i\sqrt{m}x_4)(1 + i\sqrt{m})} - (\beta + i\gamma) \right| < (1 + m) \frac{\sqrt{6|m|}}{\pi |x_3 + i\sqrt{m}x_4|^2 |1 + i|^2}.$$

Setting $p = (x_1 + i\sqrt{m}x_2)(1 + i\sqrt{m}), q = (x_3 + i\sqrt{m}x_4)(1 + i\sqrt{m})$ and $z = \beta + i\gamma$, we obtain

$$0 < \left| \frac{p}{q} - z \right| < (1 + m) \frac{\sqrt{6|m|}}{\pi |q|^2},$$

and by the same argument (7.2) as before concerning an unlimited possible increase of t, there arise infinitely many such numbers p and q.

Theorem 7.2.3 *Let $m \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Q}(i\sqrt{m})$ be arbitrarily, then there exist infinitely many $p, q \in (1 + i\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ with*

$$\left| z - \frac{p}{q} \right| < (1 + m) \frac{\sqrt{6|m|}}{\pi |q|^2}.$$

Examples:

For $m = 1$, the Julius Hurwitz case, we have infinitely many $p, q \in (1 + i)\mathbb{Z}[i]$ satisfying

$$\left| z - \frac{p}{q} \right| < \frac{2\sqrt{6}}{\pi |q|^2}.$$

For $m = 2$ we have infinitely many $p, q \in (1 + i\sqrt{2})\mathbb{Z}[i\sqrt{2}]$ satisfying

$$\left| z - \frac{p}{q} \right| < \frac{3\sqrt{12}}{\pi |q|^2} = \frac{6\sqrt{3}}{\pi |q|^2}.$$

7.2.4 Geometrical Interpretation

Considering the set of partial quotients as lattice, we define so-called *fundamental domains* in a similar manner as, for example, in Sections 3.2 or 4.2.2. Those domains shall be located around the origin and they provide a complete tiling of the Gaussian Complex Plane by shifting them along the lattice. It is natural to expect that the above stated results on approximation quality are connected with the size of those domains: An increase of the area should lower the approximations quality and vice versa.

For a generalized lattice in tradition of Julius Hurwitz we define its fundamental domain by

$$X(m) := \left\{ (1+i\sqrt{m})x + \overline{(1+i\sqrt{m})}y \ : \ -\frac{1}{2} < y, x \leq \frac{1}{2} \right\}.$$

This implies that the domain is generated by $1+i\sqrt{m}$ and its complex conjugated. Notice that $\lambda := (1+i\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ is a sublattice of $\Lambda := \mathbb{Z}[i\sqrt{m}] = \mathbb{Z} + i\sqrt{m}\mathbb{Z}$. It is easy to calculate the corresponding areas of the fundamental domains

$$\det(\Lambda) = \det \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{m} \end{pmatrix} = \sqrt{m}$$

and

$$\det(\lambda) = \det((1+i\sqrt{m})\mathbb{Z} + (-m+i\sqrt{m})\mathbb{Z}) = \det \begin{pmatrix} 1 & -m \\ \sqrt{m} & \sqrt{m} \end{pmatrix} = \sqrt{m}(1+m).$$

The quanitity $1+m$, appearing in Theorem 7.2.3, is exactly the stretching factor of the corresponding fundamental domains of λ and Λ.

Remark: The fundamental domains loose there symmetry with respect to the real and imaginary axes, however, they are still parallelograms.

7.2.5 Generalization of the Result

Considering the more general lattice $(n+ik\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ with $n, k \in \mathbb{Q}$ and $m \in \mathbb{N}$ square-free, provides a respective straightforward generalization of the result. Let $m \in \mathbb{N}$ be and natural number and $z \in \mathbb{C} \setminus \mathbb{Q}(i\sqrt{m})$ be an arbitrary not-lattice-point complex number, then there exist infinetly many $p, q \in (n + ik\sqrt{m})\mathbb{Z}[i\sqrt{m}]$ with

$$\left| z - \frac{p}{q} \right| < (n^2 + k^2 m) \frac{\sqrt{6|m|}}{\pi |q|^2}.$$

However, more interesting is that we can generalize our result to arbitrary lattices in the complex plane as follows. Considering a lattice λ generated by two arbitrary $\mathbb{R}$-linearly independent vectors $\omega_1 = a + ib, \omega_2 = c + id$, where $a, b, c, d \in \mathbb{R}$, we compute for

$$\lambda = (a + ib)\mathbb{Z} + (c + id)\mathbb{Z} = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$$

the lattice determinant by

$$\Delta(\lambda) = \left| \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right| = |ad - bc| > 0.$$

We define pairs of linear forms

$$Y_1 = \frac{1}{\sqrt{2}}((\omega_1 X_1 + \omega_2 X_2) - (\beta + i\gamma)(\omega_1 X_3 + \omega_2 X_4)), Y_2 = \overline{Y_1}$$

and

$$Y_3 = \frac{1}{\sqrt{2} t^2}(\omega_1 X_3 + \omega_2 X_4), Y_4 = \overline{Y_3},$$

where t is a positive real parameter. The determinant of the coefficient matrix is computed as follows

$$\begin{aligned}\det(z_{ij}) &= \frac{1}{4t^4}\det\begin{pmatrix}\omega_1 & \omega_2 & \star & \star \\ \overline{\omega_1} & \overline{\omega_2} & \star & \star \\ 0 & 0 & \omega_1 & \omega_2 \\ 0 & 0 & \overline{\omega_1} & \overline{\omega_2}\end{pmatrix} = \frac{1}{4t^4}\det\begin{pmatrix}\omega_1 & \omega_2 \\ \overline{\omega_1} & \overline{\omega_2}\end{pmatrix}^2 \\ &= \frac{1}{4t^4}\left((a+ib)(c-id)-(c+id)(a-ib)\right)^2 \\ &= \frac{1}{4t^4}(2i(bc-ad))^2 = -\frac{1}{t^4}(ad-bc)^2.\end{aligned}$$

Application of Theorem 7.2.2 provides the simultaneous estimate

$$|Y_j(x_1,\cdots,x_4)| \leq \sqrt{\frac{2}{\pi}}\sqrt[4]{|\frac{1}{t^4}(ad-bc)^2|} = \frac{\sqrt{2}\sqrt{|ad-bc|}}{\sqrt{\pi}t}$$

with a certain lattice point $(x_1,...,x_4) \in \mathbb{Z}^4 \setminus \{0\}$. Therefore, for the product of two linear forms we get

$$|Y_1|\cdot|Y_3| = \frac{1}{2t^2}|(\omega_1x_1+\omega_2x_2)-(\beta+i\gamma)(\omega_1x_3+\omega_2x_4)||\omega_1x_3+\omega_2x_4| \leq \frac{2|ad-bc|}{\pi t^2}.$$

Setting $p = (\omega_1x_1+\omega_2x_2), q = (\omega_1x_3+\omega_2x_4)$ and $z = \beta+i\gamma$, there exist complex numbers $p, q \in \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ such that

$$\left|z-\frac{p}{q}\right| \leq \frac{4|ad-bc|}{\pi|q|^2}. \qquad (\star\,\star)$$

Since $t > 0$ can increase arbitrarily in Y_3, we may conclude once more: If $z \in \mathbb{C}\setminus\mathbb{Q}(\lambda)$ is an arbitrary non-lattice-point complex number, there exist infinitely many $p, q \in \lambda$ satisfying

$$\left|z-\frac{p}{q}\right| \leq \frac{4\Delta(\lambda)}{\pi|q|^2}.$$

Here $\mathbb{Q}(\lambda)$ is defined as the set of rational numbers with numerator and denominator being lattice points, as in Chapter 6.

On the one hand, this result shows again the influence of the lattice structure on the bound of the approximation quality by this method; on the other hand we notice that the obtained bound on the right hand-side is invariant under any basis change of the lattice.

To improve the approximation quality, we shall apply Theorem 7.2.1. Hence, we define the symmetric convex body

$$C := \{x = (x_1, \cdots, x_4) \in \mathbb{R}^4 \ : \ |Y_1| + |Y_3| \leq M, |Y_2| + |Y_4| \leq M\},$$

with the same linear forms Y_j from above. We calculate the volume as in Subsection 7.2.3 and obtain once again

$$vol(C) = M^4 \iint_{[0,2\pi)^2} dz_1 dz_2 \iint_{r_1+r_2 \leq \sqrt{2}, r_j \geq 0} r_1 r_2 dr_1 dr_2 = \frac{2\pi^2}{3} M^4.$$

Following Theorem 7.2.2, there exists a lattice point $x = (x_1, ..., x_4) \in \mathbb{Z}^4 \setminus \{0\}$ in C whenever

$$vol(C) \geq 2^4 \left| \frac{1}{t^4}(ad - bc)^2 \right|.$$

This leads to the restriction

$$M \geq \frac{\sqrt{2}}{t\sqrt{\pi}} \sqrt[4]{6|ad - bc|^2},$$

giving

$$|Y_1| + |Y_3| \leq \frac{\sqrt{2}}{t\sqrt{\pi}} \sqrt[4]{6|ad - bc|^2}$$

and

$$|Y_1| \cdot |Y_3| \leq (\frac{1}{2}(|Y_1| + |Y_3|))^2 \leq \frac{\sqrt{6}|ad - bc|}{2t^2\pi}.$$

Applying the same argument as above this proves

Theorem 7.2.4 *Let λ be a lattice in $\mathbb{C}$ of full rank. For any $z \in \mathbb{C} \setminus \mathbb{Q}(\lambda)$ there exist infinitely many lattice points $p, q \in \lambda$ such that*

$$\left| z - \frac{p}{q} \right| < \frac{\sqrt{6} \cdot \Delta(\lambda)}{\pi |q|^2},$$

where $\Delta(\lambda)$ denotes the determinant of the lattice λ.

Remarks: Whereas in Hilde Gintner's arithmetical approach it was necessary to distinguish the cases $D \not\equiv 3 \bmod 4$ and $D \equiv 3 \bmod 4$, in our geometrical approach those are already combined.

7.3 Modular Group

We consider the arising modular form classifying the upper bound depending on the lattice. It is a natural question to ask how good an approximation of a complex number z by quotients of lattice points in the sense of Theorem 7.2.4 can be. In the following we prove some characteristics of the subsequently defined quantity.

Corollary 7.3.1 *The function*

$$\mu(\tau) := \mu(\lambda) := \sup L(\tau) = \inf\{c > 0 \ : \ \forall z \in \mathbb{C} \ \exists^\infty p, q \in \lambda : \left| z - \frac{p}{q} \right| < \frac{c}{|q|^2}\},$$

where $\lambda := \mathbb{Z} + \tau\mathbb{Z}$, is

1. *automorphic,*
2. *continuous,*
3. *non-constant and always satisfies $\mu(\tau) \geq \frac{1}{\sqrt{5}}$.*

Here we call a function *automorphic* when it is invariant under the action of $SL_2(\mathbb{Z})$.

Proof.

1. The existence of the automorphic function is a direct consequence of Theorem 7.2.4 where we found an explicit, probably not optimal, upper bound. For

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

we have

$$M\tau = \frac{a\tau + b}{c\tau + d} = \frac{(a\tau + b)(c\overline{\tau} + d)}{|c\tau + d|^2}.$$

Since $\tau = x + iy, y > 0$ and $\det M = ad - bc = +1$, it follows that

$$M\tau = \frac{x(ad + bc) + iy(ad - bc)}{|c\tau + d|^2} \in \mathbb{H},$$

where $\mathbb{H}$ is the upper half plane. A short computation shows

$$\frac{p}{q} = \frac{p_1\tau + p_2}{q_1\tau + q_2} = \frac{P_1 M\tau + P_2}{Q_1 M\tau + Q_2} = \frac{P}{Q}$$

with

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consequently, the best approximations to a given $z \in \mathbb{C}$ by lattice points from $\lambda = \mathbb{Z} + \tau\mathbb{Z}$ and from $\mathbb{Z} + M\tau\mathbb{Z}$ equal one another. Hence,

$$\mu(\tau) = \mu(M\tau)$$

for any $M \in SL_2(\mathbb{Z})$, which proves that μ is an automorphic function.

2. In order to prove continuity we consider a second lattice $\lambda' := \mathbb{Z} + \tau'\mathbb{Z}$ for which we

have
$$|\tau - \tau'| < \delta$$
with an arbitrary small $\delta > 0$. According to $p = a + b\tau$ and $q = c + d\tau$, we define lattice points
$$p' = a + b\tau' = a + b\tau + b(\tau' - \tau) = p + b\delta$$
and
$$q' = c + d\tau' = c + d\tau + d(\tau' - \tau) = q + d\delta$$
with $\delta = \tau - \tau'$. It follows that
$$\begin{aligned} \left|z - \frac{p'}{q'}\right| &= \left|z - \frac{p}{q} + \frac{p}{q} - \frac{p'}{q'}\right| \\ &\leq \left|z - \frac{p}{q}\right| + \frac{|pq' - p'q|}{|qq'|} < \frac{c+\epsilon}{|q|^2} + \frac{|p(q + d\delta) - (p + b\delta)q|}{|q(q + d\delta)|} \\ &= \frac{c+\epsilon}{|q|^2} + \frac{|(a + b\tau)d - (c + d\tau)b||\delta|}{|q|^2|1 + \frac{d\delta}{q}|}. \end{aligned}$$
We simplify the numerator by
$$|ad + bd\tau - bc - bd\tau| = |(ad - bc)||\delta| = |\delta|,$$
since we may assume p and q to be coprime (respectively $ad - bc = \pm 1$). For the denominator we get, by geometrical series expanion,
$$\frac{1}{|q|^2|1 + \frac{d\delta}{q}|} = \frac{1}{|q|^2}\left(1 + O\left(\frac{|\delta|}{|q|}\right)\right).$$
Hence, we obtain
$$\left|z - \frac{p'}{q'}\right| < \frac{c + \epsilon^* + |\delta|}{|q|^2},$$
where ϵ^* can be made arbitrarily small depending only on ϵ. If δ is sufficiently small,

this leads to

$$\left|z - \frac{p'}{q'}\right| < \frac{c + \epsilon'}{|q|^2}$$

with ϵ' as small as we please, which implies the continuity.

3. That μ is non-constant is a direct consequence of existing results (see Section 7.1). We know for example that for $\tau = i$, the bounding constant is $\mu(\lambda) = \frac{1}{\sqrt{3}}$, whereas for $\tau = i\sqrt{2}$, we have $\mu(\lambda) = \frac{1}{\sqrt{2}}$. Furthermore, Hurwitz's approximation result (7.1) provides a lower bound for c. We have

$$\left|z - \frac{p}{q}\right| < \frac{c + \epsilon}{|q|^2}$$

and since

$$\left|z - \frac{p}{q}\right| \geq \left|\operatorname{Re} z - \operatorname{Re} \frac{p}{q}\right|,$$

it follows that $c \geq \frac{1}{\sqrt{5}}$.

Remark: Unfortunately, μ is not analytic since $\mu(\tau) \in \mathbb{R}$ for all $\tau \in \mathbb{H}$.

7.4 Arithmetical Application

A nice arithmetical application of Hilde Gintner's result and Theorem 7.2.4 is given by considering the complex analogue of the classical Pell equation (see Theorem 1.2.4 in Section 1.2).

Corollary 7.4.1 *Let $d \in \mathbb{Z}[i\sqrt{D}]$ with squarefree $D \in \mathbb{N}$ and $\sqrt{d} \notin \mathbb{Q}(i\sqrt{D})$. Then the Pell equation*

$$X^2 - dY^2 = 1$$

has infinitely many non-trivial solutions in numbers $x, y \in \mathbb{Z}[i\sqrt{D}]$.

This corollary can be proved easily by the classical application of complex continued fractions whenever $\mathbb{Z}[i\sqrt{D}]$ forms a euclidean ring. However, this constraint is not necessary.

According to Theorem 7.2.4 there are infinitely many $x, y \in \mathbb{Z}[i\sqrt{D}]$ satisfying

$$\left|\sqrt{d} - \frac{x}{y}\right| < \frac{c(D)}{|y|^2}$$

with a certain positive constant $c(D) := \mu(i\sqrt{D})$, depending on $\mathbb{Z}[i\sqrt{D}]$. In view of Ford's result [Ford, 1918], we have

$$c(D) = \frac{\sqrt{6D}}{\pi} \geq c(1) = \frac{1}{\sqrt{3}}.$$

Due to

$$\left|\sqrt{d} + \frac{x}{y}\right| = \left|\frac{x}{y} - \sqrt{d} + 2\sqrt{d}\right| < \frac{c(D)}{|y|^2} + 2\sqrt{|d|},$$

it follows that

$$\begin{aligned} |x^2 - y^2 d| &= |x - y\sqrt{d}| \cdot |x + y\sqrt{d}| \\ &< \frac{c(D)}{|y|}\left(\frac{c(D)}{|y|} + 2|y|\sqrt{|d|}\right) = \left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}. \end{aligned}$$

Consequently, for all forms $X^2 - dY^2$ there exists a $k \in \mathbb{Z}[i\sqrt{D}]$ satisfying

$$-\left(\left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}\right) < k < \left(\left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}\right)$$

such that, for infinitely many $x, y \in \mathbb{Z}[i\sqrt{D}]$,

$$x^2 - dy^2 = k.$$

Since $\sqrt{d} \notin \mathbb{Q}(i\sqrt{D})$ this yields $k \neq 0$. If $k = 1$ we are finished.

Otherwise, if $k \neq 1$, we define equivalence classes in the set of such pairs x, y as follows. Let $N(k)$ denote the norm of the ideal generated by k in the ring $\mathbb{Z}[i\sqrt{D}]/k\mathbb{Z}[i\sqrt{D}]$. Two pairs x_1, y_1 and x_2, y_2 belong to the same class if, and only if,

$$x_1 \equiv x_2 \bmod k \quad \text{and} \quad y_1 \equiv y_2 \bmod k.$$

Since there are only $N(k)^2 < \infty$ classes but infinitely many pairs, some class contains at least two of such pairs where $x_1 \neq \pm x_2$ and $y_1 \neq \pm y_2 \neq 0$. Furthermore, we define

$$x_0 = \frac{x_1 x_2 - y_1 y_2 d}{k} \quad \text{and} \quad y_0 = \frac{x_1 y_2 - x_2 y_1}{k}.$$

Obviously, the numerators of x_0 and y_0 are $\equiv 0 \bmod k$ and $x_0, y_0 \in \mathbb{Z}[i\sqrt{D}]$. Moreover, we observe

$$x_0^2 - y_0^2 d = \frac{1}{k^2}(x_1^2 - y_1^2 d)(x_2^2 - y_2^2 d) = 1,$$

which shows that there is always a solution $x_0, y_0 \in \mathbb{Z}[i\sqrt{D}]$ of the Pell equation. Next we shall exclude the trivial case $y_0 = 0$. In view of

$$k = x_1^2 - y_1^2 d = \left(\frac{x_2 y_1}{y_2}\right)^2 - y_1^2 d = \left(\frac{y_1^2}{y_2^2}\right)(x_2^2 - y_2^2 d) = \left(\frac{y_1^2}{y_2^2}\right) k,$$

it follows that $y_1^2 = y_2^2$, however, since $y_1 \neq \pm y_2$ this is a contradiction to the assumption of the chosen pairs.

Additionally, we can show that there are even infinitely many solutions. Starting with two (not necessarily distinct) non-trivial solutions $x_j + y_j\sqrt{d}$ to $X^2 - dY^2 = 1$, we find further solutions by

$$(x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d}) = (a + b\sqrt{d}).$$

Since $a = x_1 x_2 + y_1 d y_2$ and $b = x_1 y_2 + x_2 y_1$, we have $a, b \in \mathbb{Z}[i\sqrt{D}]$ and

$$a^2 - b^2 d = (x_1^2 - y_1^2 d)(x_2^2 - y_2^2 d) = 1.$$

This proves that the product of two solutions provides again a solution and furthermore, we obtain infinitely many solutions by raising them to the nth power of a non-trivial one:

$$(x_0 + y_0\sqrt{d})^n = a_n + b_n\sqrt{d}.$$

Corollary 7.4.1 can be illustrated by a concrete example. Therefore, we apply a real case method of Arturas Dubickas and Jörn Steuding [Dubickas and Steuding, 2004] giving a generalization of Melvyn B. Nathanson's result [Nathanson, 1976] on families of polynomial solutions for Pell equations. They showed that for the polynomial equation

$$P(X)^2 - (X^2 + 1)Q(X)^2 = 1$$

a family of solutions can be generated by the sequence of polynomials

$$P_n(X) := (2X^2 + 1)P_{n-1}(X) + 2X(X^2 + 1)Q_{n-1}(X)$$

and

$$Q_n(X) := 2XP_{n-1}(X) + (2X^2 + 1)Q_{n-1}(X),$$

where $P_0(X) = 1$ and $Q_0(X) = 0$. We easily calculate $P_1 = 2X^2 + 1, Q_1 = 2X$ and receive the (since Euclid known) equation

$$(2X^2 + 1)^2 - (X^2 + 1)(2X)^2 = 1.$$

Now we can chose an arbitrary X. For $x = 1 + i\sqrt{5}$ respectively $x^2 = -4 + 2i\sqrt{5}$, we get

$$(4i\sqrt{5} - 7)^2 - (2i\sqrt{5} - 3)(2 + 2i\sqrt{5})^2 = 1,$$

which provides an example for $d = 2i\sqrt{5} - 3$.

7.5 Law of Best Approximation

A fundamental problem of diophantine approximation is to find 'good' rational approximations $\frac{p}{q}$ to a given z ($\in \mathbb{R}$ or $\mathbb{C}$) - finitely or infinitely many. In the real case the situation was clarified by Lagrange's Law of Best Approximation (see Section 1.2). He stated that *for any $z \in \mathbb{R}$ there are no better rational approximantions than the convergents to z arising from the regular continued fraction expansion.*

In 1991, Cor Kraaikamp proved the following equivalence: *The sequence of convergents of a semi-regular continued fraction forms a subsequence of the sequence of convergents arising from the regular continued fraction if, and only if, $\epsilon_{n+1} + a_n > 2 - a_n - \epsilon_n$ for $n \geq 1$.* [Kraaikamp, 1991, p. 11] (Here we use the same notation as in Section 3.1.) In Chapter 5 we stated for continued fractions to the nearest integer[4] Condition (5.2), namely $\epsilon_{n+1} + a_n \geq 2$. Furthermore, since their partial quotients satisfy $a_n \geq 2$, the inequality $2 - a_n - \epsilon_n \leq 1$ holds. Noticing that continued fractions to the nearest integer are a special type of semi-regular continued fractions, the Law of Best Approximation consequently also holds for continued fractions to the nearest integer. Since those can be considered as precursors to Adolf Hurwitz' complex continued fractions, this might indicate that the latter also provide best convergents. However, in fact, *the law is no longer true for complex numbers.*

Following [Lakein, 1973, p. 400] we consider the example

$$z = [0; -2+2i, 1+i] \left(= \frac{-1}{3}(1+i) \right).$$

Notice that this continued fraction could arise from Adolf Hurwitz's algorithm as well as from Julius Hurwitz's algorithm. Here the convergent

$$\frac{p_1}{q_1} = [0; -2+2i] = \frac{1}{-2+2i} \left(= \frac{-1}{4}(1+i) \right)$$

[4] which were also introduced in Section 3.1

is *not a best approximation* to z, since $\frac{p}{q} = \frac{1}{-1+i}\left(= \frac{-1}{2}(1+i)\right)$ satisfies

$$|q| = |-1+i| = \sqrt{2} < \sqrt{8} = |-2+2i| = |q_1|$$

and

$$|qz - p| = |q|\left|z - \frac{p}{q}\right| = \sqrt{2}|1+i|\left|\frac{-1}{3} + \frac{1}{2}\right| = \frac{1}{3} = |q_1|\left|z - \frac{p_1}{q_1}\right| = |q_1 z - p_1|.$$

However, in view of

$$\left|z - \frac{p}{q}\right| = \frac{1}{3\sqrt{2}} > \frac{1}{3\sqrt{8}} = \left|z - \frac{p_1}{q_1}\right|,$$

the notation of 'best approximation' seems to loose its meaning by stepping from $\mathbb{R}$ to $\mathbb{C}$.

Remark: In the above mentioned work [Lakein, 1973] Richard B. Lakein proved that nevertheless *for almost all* z the convergents are best approximations in Adolf Hurwitz's continued fraction.

8 Last words

Recently, complex continued fractions have been studied in a rather different context. Nearly nothing is known about the regular continued fraction expansion of real algebraic irrationals of degree strictly larger than two. For instance, it is an open question whether the sequence of partial quotients of such a real algebraic irrational is bounded or not; the same problem is also unanswered for other real continued fractions. As follows from Adolf Hurwitz's work already the situation for complex algebraic irrationals is pretty different: complex irrationals satisfying an irreducible quadratic equation with coefficients from $\mathbb{Z}[i]$ have a periodic, henceforth bounded sequence of partial quotients (extending Lagrange's celebrated theorem). However, Hensley [Hensley, 2006] discovered a far more surprising phenomenon: there exist complex algebraic irrationals having a bounded but not eventually periodic sequence of partial quotients in Adolf's continued fraction expansion; an example is $z = \sqrt{2} - 1 + i(\sqrt{5} - 2)$ which is a solution of the irreducible biquadratic equation

$$Z^4 + (4 + 8i)Z^3 - (12 - 24i)Z^2 - (32 - 16i)Z + 24 = 0.$$

Bosma and Gruenewald [Bosma and Gruenewald, 2011] proved the existence of complex algebraic numbers of arbitrary even degree having a continued fraction expansion with bounded partial quotients (being non-periodic for degree larger than two over the Gaussian number field).

What makes continued fraction algorithms for complex numbers interesting for current research in number theory is that there are quite many open questions concerning algebraic and ergodic features of complex continued fraction expansions. The same could be applied to another approach to diophantine approximation of complex numbers via continued fractions due to A.L. Schmidt [Schmidt, 1975, Schmidt, 1982]; his papers include references to the work of both, Adolf and Julius Hurwitz. Schmidt's type of continued fraction is superior if the quality of approximation is paramount, and it allows the use of tools from ergodic theory too at the expense that his continued fraction lacks the simplicity of the continued fraction expansions found by the Hurwitz brothers.

In view of our historical investigations, we agree with David Hilbert's opinion (see Subsection 2.2.3): The mathematical diaries of Adolf Hurwitz can be considered as treasure trove of mathematical ideas. We believe that they, furthermore, provide a good picture of the mathematical community at that time. The turn of the 19th to the 20th century was certainly also a turn in mathematics. In [Gray, 2008] Jeremy Gray describes this as mathematical modernism. In the second half of the 19th century the community of mathematicians was very manageable and we believe it is worth to investigate the role of Adolf Hurwitz being considered as focal point of mathematical exchange (see also Section 2.1). As Jewish mathematician he certainly had not had the best prospects, see [Rowe, 2007]. Nevertheless, he managed with his unobtrusive professional attitude to reach a highly regarded expert's position. In subsequent investigations we plan to continue to examine his exchange with David Hilbert as well as with other mathematicians; here the focus shall be on the development and exchange of mathematical ideas. The diaries promise to provide a basis for this examination of Adolf Hurwitz's position in the scientific community.

9 Appendix

The appendix contains three parts: a list of entries related to recreational mathematics in the mathematical diaries [Hurwitz, 1919a] of Adolf Hurwitz, a list of direct or indirect references to David Hilbert in Hurwitz's ETH estate in Zurich and a table of figures of this work.

9.1 Appendix I: Recreational Mathematics in the Mathematical Diaries

In the directory HS 582: 1 - 30 at the archive of the ETH Zurich the thirty diaries [Hurwitz, 1919a] of Adolf Hurwitz are stored. Here we give a list of diary entries dealing with recreational mathematics (see Subsection 2.2.1) in addition of one philosophical entry in [Hurwitz, 1919a, No. 26].

- No. 4: 1885 I. - ,

  p. 140 "Kartenkunststück"

- No. 22: 1906 XII.18. - 1908 I.22.,

  p. 173 "Konstruktionen u. Beweise durch Papierfalten"

  p. 184 "Beweis des Pythagoras"

- No. 23 1908 I.23. - 1910 II.18.,

  p. 1 "Aufgabe über kürzeste Linie auf einem Parallelepiped"

- No. 24: 1910 II.19. - 1911 X.26.,

  loose sheet of paper, a folded pentagon

- No. 25: 1911 X.27. - 1912 XII.27.,

  p. 138 ”Scherzaufgabe (Analysis situs) von Landau”

- No. 26: 1912 XII.27. - 1914 IV.30.,

  p. 138 ”Über den mathematischen Beweis”

- No. 28: 1915 II.16. - 1917 III.22.,

  p. 61 ”Domino-Kunststücke”

  p. 188 ”Bunteste Ringe”

9.2 Appendix II: Links to Hilbert in the ETH Estate of Hurwitz

In Subsection 2.2.3 we consider the teacher-student-relation of Adolf Hurwitz and David Hilbert. Here we give a list of documents of Adolf Hurwitz's ETH estate (in the directories HS 582 and HS 583) directly or indirectly connected to David Hilbert including a great number of diary entries [Hurwitz, 1919a] (HS 582 : 1 - 30) with remarks related to Hilbert.

- Lectures of David Hilbert edited by Julius Hurwitz

  HS 582: 154, 'Die eindeutigen Funktionen mit linearen Transformationen in sich' (Königsberg 1892 SS) with handwritten remarks of Hilbert (e.g. on pages -1, 34, 69, 80, 81, ...)

  HS 582: 158, 'Geometrie der Lage' (Königsberg 1891 SS)

- No. 6: 1888 IV. - 1889 XI.,

  p. 44 ”Der Nöther'sche Satz (nach einer Mitteilung von Hilbert)”

  p. 45 ”Hilberts Fundamentalsatz”

  p. 93 ”Hilbert beweist die obigen Sätze so” (study on convergent series)

- No. 7: 1890 IV.9. - 1891 XI.,

  p. 94 "[...] die Hilbert'schen Figuren" ('Lines on square'-figures)

- No. 8: 1891 XI.3. - 1894 III.,

  p. 207 "Zweiter Hilbert'scher Formensatz"(Hilbert's Basic Theorem)

- No. 9: 1894 IV.4. - 1895 I.6.,

  loose sheet concerning "[...] von Hilbert, betreffend die Anzahl von Covarianten"

- No. 13: 1895 VI.19. - XII.31.,

  p. 19 letter to Hilbert in stenography

- No. 14: 1896 I.1. - 1897 II.1.,

  p. 204 "Hilberts 2tes Theorem" (related to [Hilbert, 1890, p. 485])

- No. 15: 1897 II.1. - 1898 III.19.,

  p. 175 "Zu Hilberts "Körperbericht"" (related to Hilbert's *'Zahlbericht'*)

- No. 16: 1898 III.20. - 1899 II.23.,

  p. 129 "Zum Hilbertschen Bericht pag. 287" (related to Hilbert's *'Zahlbericht'*)

- No. 19: 1901 XI.1. - 1904 III.16.,

  p. 29 "Hilberts axiomatische Größenlehre" (related to [Hilbert, 1900b])

  p. 114 "Abbildung einer Strecke auf ein Quadrat", "[..] die Hilbert'schen geometrisch erklärten Funktionen sollen arithmetisch charakterisiert werden." ('Lines on square'-figures)

- No. 20: 1904 III.16. - 1906 II.1.,

  p. 163 "Hilberts Beweis von Hadamards Determinantensatz"

- No. 21: 1906 II.1. - 1906 XII.8.,

  p. 166 "Hilberts Vte Mitteilung über Integralgleichungen"

  (related to [Hilbert, 1906])

- No. 22: 1906 XII.18. - 1908 I.22.,

  p. 36 "Convergenzsätze von Landau + Hilbert"

- No. 25: 1911 X.27. - 1912 XII.27.,

  p. 77 "Zu Hilberts Formenarbeit (Charakt. Funktion eines Moduls)"

- In the directory HS 582: 32 done by Georg Pòlya on page 4 and 6 there are remarks "die ersten 9 Bände und Inhaltsverzeichnis sind zwecks Bearbeitung verderhand bei Prof. Hilbert in Göttingen" and "22. zwecks Bearbeitung vorderhand bei Prof. Hilbert in Göttingen", crossed out with pencil

- HS 582: 28, Letter of condolences from David Hilbert to Ida Samuel-Hurwitz (December 15, 1919)

- HS 583: 52, Greeting cards from conferences with Hilbert's handwritting: 'Lutetia Parisiorum, le 12 aout 1900' and the 'Landau-Kommers 18. Jan. 1913'

- Remarks in the biographical dossier written by Ida Samuel-Hurwitz (HS 583a: 2)

9.3 Appendix III: Table of Figures

- Fig. 2.3: Excerpt of [Hurwitz, 1919a, No. 1, p. 75], ETH Zurich University Archives, Hs 582:1, DOI: 10.7891/e-manuscripta-12817.
- Fig. 2.4: Excerpt of the back of the front page in [Hurwitz, 1919a, No. 30], ETH Zurich University Archives, Hs 582:30, DOI: 10.7891/e-manuscripta-12830.
- Fig. 2.5, 2.6 and 2.7: Excerpts of [Hurwitz, 1919a, No. 28, p. 61], ETH Zurich University Archives, Hs 582:28, DOI: 10.7891/e-manuscripta-12837.
- Fig. 2.8, 2.9 and 2.10: Excerpts of [Hurwitz, 1919a, No. 23, p. 1], ETH Zurich University Archives, Hs 582:23, DOI: 10.7891/e-manuscripta-12818.
- Fig. 2.11 and 2.12: Excerpts of [Hurwitz, 1919a, No. 23, p. 2], ETH Zurich University Archives, Hs 582:23, DOI: 10.7891/e-manuscripta-12818.
- Fig. 2.13 and 2.14: Excerpts of [Hurwitz, 1919a, No. 25, p. 138], ETH Zurich University Archives, Hs 582:25, DOI: 10.7891/e-manuscripta-12820.
- Fig. 2.15: Excerpt of the index in [Hurwitz, 1919a, No. 25], ETH Zurich University Archives, Hs 582:25, DOI: 10.7891/e-manuscripta-12820.
- Fig. 2.16: Excerpt of [Hurwitz, 1919a, No. 22, p. 184 and p. 185] and a loose envelope with colored cutted out triangles also in [Hurwitz, 1919a, No. 22], ETH Zurich University Archives, Hs 582:22, made by the author.
- Fig. 2.17, 2.18 and 2.19: Excerpts of [Hurwitz, 1919a, No. 22, p. 173], ETH Zurich University Archives, Hs 582:22, DOI: 10.7891/e-manuscripta-12811.
- Fig. 2.20 and 2.21: Excerpts of [Hurwitz, 1919a, No. 22, p. 174] and picture of a loose sheet found in the same diary, ETH Zurich University Archives, Hs 582:22, made by the author.
- Fig. 2.22: Excerpt of [Hurwitz, 1919a, No. 22, p. 175], ETH Zurich University Archives, Hs 582:22, DOI: 10.7891/e-manuscripta-12811.

- Fig. 2.23: Excerpt of [Hurwitz, 1919a, No. 22, p. 176], ETH Zurich University Archives, Hs 582:22, DOI: 10.7891/e-manuscripta-12811.
- Fig. 2.24: Loose folded sheet in [Hurwitz, 1919a, No. 24], ETH Zurich University Archives, Hs 582:24, made by the author.
- Fig. 2.25: Excerpt of [Hurwitz, 1919a, No. 6, p. 45], ETH Zurich University Archives, Hs 582:6, DOI: 10.7891/e-manuscripta-12821.
- Fig. 2.26 and 2.27: Excerpts of [Hurwitz, 1919a, No. 14] on pages 204 and 205, ETH Zurich University Archives, Hs 582:14, DOI: 10.7891/e-manuscripta-12840.
- Fig. 2.28: Excerpt of [Hurwitz, 1919a, No. 15, p. 175], ETH Zurich University Archives, Hs 582:15, DOI: 10.7891/e-manuscripta-12831.
- Fig. 2.29 and 2.30: Excerpts of [Hurwitz, 1919a, No. 15, p. 177], ETH Zurich University Archives, Hs 582:15, DOI: 10.7891/e-manuscripta-12831.
- Fig. 2.31: Excerpt of [Hurwitz, 1919a, No. 15, p. 178], ETH Zurich University Archives, Hs 582:15, DOI: 10.7891/e-manuscripta-12831.
- Fig. 2.32 and 2.34: Excerpts of [Hurwitz, 1919a, No. 16] on pages 129 and 130, ETH Zurich University Archives, Hs 582:16, DOI: 10.7891/e-manuscripta-12830.
- Fig. 2.33: Excerpt of Hilbert's *'Zahlbericht'*, page 289 respectively [Hilbert, 1935, vl. I, p. 164].
- Fig. 2.35: Excerpt of [Hurwitz, 1919a, No. 19, p. 29], ETH Zurich University Archives, Hs 582:19, DOI: 10.7891/e-manuscripta-12819.
- Fig. 2.36: Excerpt of [Hilbert, 1900b, p. 182] and [Hurwitz, 1919a, No. 19, p. 29], ETH Zurich University Archives, Hs 582:19, DOI: 10.7891/e-manuscripta-12819.
- Fig. 2.37 and 2.38: Excerpts of [Hurwitz, 1919a, No. 19, p. 30], ETH Zurich University Archives, Hs 582:19, DOI: 10.7891/e-manuscripta-12819.

- Fig. 2.39, 2.40, 2.41, 2.42 and 2.43: Excerpts from [Hurwitz, 1919a, No. 21] on pages 166, 167, 168, 169 and 172, ETH Zurich University Archives, Hs 582:21, DOI: 10.7891/e-manuscripta-12836.
- Fig. 2.44 and 2.45: Excerpts of [Hurwitz, 1919a, No. 18, p. 75, p. 81], ETH Zurich University Archives, Hs 582:18, DOI: 10.7891/e-manuscripta-12810.
- Fig. 2.46: Greeting cards from 'Lutetia Parisiorum, le 12 aout 1900' and the 'Landau-Kommers 18. Jan. 1913', in Hs 583:53 and 57, ETH Zurich University Archives.
- Fig. 2.47: Excerpt of Georg Pòlya's list [Hurwitz, 1919a, No. 32, p. 6], ETH Zurich University Archives, Hs 582:32, DOI: 10.7891/e-manuscripta-16074.
- Fig. 2.48: Portrait of Julius Hurwitz, taken from Riesz's register in *Acta Mathematica* from 1913 [Riesz, 1913].
- Fig. 2.49: Excerpt of Julius Hurwitz's personnel file at the University of Basel [Staatsarchiv, 1896], State Archive Basel.
- Fig. 2.50: Excerpt of [Hurwitz, 1919a, No. 5, p. 52], ETH Zurich University Archives, Hs 582:5, DOI: 10.7891/e-manuscripta-12823.
- Fig. 2.51: Excerpt of [Hurwitz, 1919a, No. 9, p. 100], ETH Zurich University Archives, Hs 582:9, DOI: 10.7891/e-manuscripta-12816.
- Fig. 2.52: Excerpt of [Hurwitz, 1895, p. 12].
- Fig. 3.1: Excerpt of [Hurwitz, 1919a, No. 33, p. 195], ETH Zurich University Archives, Hs 582:32, DOI: 10.7891/e-manuscripta-16074.
- Fig. 3.2: Illustration of Julius Hurwitz's types of partial quotients, made by the author.

- Fig. 4.1: Illustration of Tanaka's change of coordinates $\{1, i\} \to \{\alpha, \overline{\alpha}\}$, made by the author.
- Fig. 4.2: Illustration of the set of reciprocals X^{-1}, made by the author.
- Fig. 4.3: Illustration of the numbers $a \in I$, made by the author.
- Fig. 4.4: Illustration of the tiling of the complex plane in respect to the dual transformation, made by the author.
- Fig. 5.1: Illustration of all partial quotients being larger than or equal to two, made by the author.
- Fig. 5.2: Illustration of certain further intervalls, made by the author.
- Fig. 6.1: Illustration of a union of lattices, made by the author.
- Fig. 6.2: Illustration of a random Voronoï diagram and a honeycomb shaped lattice, made by the author.
- Fig. 6.3: Illustration of the set of admissible lattice constants τ, given by the non-empty intersection of three circles, made by the author.

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