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Firm Values and Systemic Stability in Financial Networks

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1 Introduction

Published in 1974, Merton's model of asset valuation revolutionized academic finance as well as the practices of asset valuation and credit risk management. Since then, many refinements and extensions have been made (an overview may be found in a paper by Bohn [2000]), but the crucial insight that the value of a firm's equity can be regarded as a European call option on the firm's asset value with strike price equal to the firm's face value of debt, if the firm's financial structure is sufficiently simple, is still inherent to all these subsequent versions. Merton's approach provides a comparatively easily accessible framework to value a firm's debt and equity, but a great part of the models based on it are applicable to a single firm only, and hence they are unable to explain the fact that, for example, defaults of firms do not occur independently of each other, as becomes evident in the work of Lucas [1995]. As noted by Elsinger et al. [2006a] and Gouriéroux et al. [2012], among others, this correlation of defaults results from two sources. First, firms of the same industry or sector for example are exposed to common economic factors, causing dependencies in these firms' asset values and therefore between the occurrences of defaults. Further dependencies arise from financial interconnections between firms, which are "[o]ne of the most pervasive aspects of the contemporary financial environment" [Eisenberg and Noe, 2001, p. 236]. In a system of firms linked by mutual financial claims and obligations, for example in the form of bonds or shares, every firm's balance sheet contains financial assets issued by other firms in the system. This makes clear that the default of one firm – induced by a decline in its exogenous asset value, for example – might lead to the default of further firms in the system, as the triggering firm is not able to repay its debt in full and its equity value becomes zero. In addition, if a chain reaction forces an initially healthy firm to default, in the presence of bilateral or cyclical cross-holdings this event might revert to the triggering firm, causing its financial situation to deteriorate even further, "potentially a financial vicious circle" [Fischer, 2014, p. 98]. Therefore, financial interconnections between firms necessitate a simultaneous valuation of these firms and any of their liabilities, instead of applying Merton's model or one of its successors to each firm separately.

In the last 15 years, and especially in the aftermath of the recent banking crisis, this topic has received increased attention in the literature. In the spirit of Merton [1974], so-called structural models directly incorporate the network architecture of cross-holdings and the structures of the firms' balance sheets to determine the firms' values, probabilities of default and the propagation of defaults through the network. In contrast to that, "[r]educed-form models avoid these details and directly provide a stochastic model of correlated defaults" [Staum, 2013, p. 517], for example in the form of Markov chains. Hence, it is not considered *how* defaults occur, only *that* they occur, which is modelled by a stochastic process with a given default intensity that might depend on common

state variables, for example some macroeconomic factor. Overviews and some subtypes of reduced-form models are provided by Davis [2011] and Staum [2013], for example. Structural models for multiple firms can be further divided into what are called cascade models and clearing models by Staum [2013]. In cascade models, there are no feedback loops of losses, i.e. as soon as a firm defaults, its recovery value of debt is transferred to its creditors, but successive defaults of its own debtors do not reduce this residual loss retroactively. Several simulation studies on contagion and network stability using this approach have been conducted, see Furfine [2003], Nier et al. [2007] and Gai and Kapadia [2010], for example. In contrast to that, such feedback effects are taken into account in clearing models, where the statuses of all firms with respect to solvency and default are determined simultaneously.
To our knowledge, the origin of clearing models lies in the very influential work of Eisenberg and Noe [2001], who consider a system of n firms where the assets of each firm consist of an exogenous asset and endogenous assets resulting from mutual cross-holdings of debt. Each firm has a single liability, which is completely held within the system, whereas equity is completely held outside the system. For this set-up Eisenberg and Noe [2001] provide a "fictitious default algorithm" which takes the "standard rules of value division between debtors and creditors" [Eisenberg and Noe, 2001, p. 240] into account, namely the absolute priority of debt over equity, limited liability of equity and a proportionate recovery rate of debt. The resulting "clearing vector" contains the no-arbitrage prices of all firms' liabilities and therefore also reveals the financial status of all firms and the individual recovery rates of debt of the firms in default. Such a clearing vector always exists, and it is unique under rather mild conditions on the network structure. By limiting themselves to what they call "default averse clearing vectors", i.e. clearing vectors with the minimum possible number of defaults, Pokutta et al. [2011] further attenuate the conditions on the clearing vector to be unique. Later on, the model of Eisenberg and Noe [2001] was extended in several ways. For example, Cifuentes et al. [2005] include the possibility of fire sales of the illiquid part of a firm's exogenous assets, with such fire sales becoming necessary in order to maintain a regulatory solvency constraint, which might have been violated due to the default of other firms in the system. Rogers and Veraart [2013] expand the model of Eisenberg and Noe [2001] by default costs, i.e. if a firm defaults, it can sell both, its exogenous and endogenous assets at a fraction of their actual price only. Even in the presence of such default costs a clearing vector always exists, but the conditions of Eisenberg and Noe [2001] for the clearing vector to be unique are not sufficient anymore. The possible network structure of interconnections itself is considerably enlarged by Elsinger [2009], as he incorporates cross-holdings of debt of differing seniority as well as cross-holdings of equity. In doing so, he develops a new algorithm to find the no-arbitrage prices of all debt and equity. Furthermore, Elsinger [2009] allows for a part of a firm's liabilities to be held by creditors outside the system of firms. This also applies to the model of Fischer [2014], which is even more general than the model of Elsinger [2009]. In contrast to all other models mentioned, in the model of Fischer [2014] liabilities are not necessarily zero-coupon bonds. Instead, they might be derivatives of any underlying in the model. As in almost all the other models (the only exception is the work of Pokutta et al. [2011] providing an algorithm incorporating debt

of differing maturity), all liabilities have the same maturity. The main result of Fischer [2014] consists of an existence and uniqueness theorem of no-arbitrage prices of equity and liabilities of all seniorities under cross-ownership ("XOS") of possibly both, debt and equity. These no-arbitrage prices can be calculated by a fixed point iteration. The frameworks of Suzuki [2002] and Gouriéroux et al. [2012] can be seen as special cases of the models of Elsinger [2009] and Fischer [2014], since they deal with the situation of n firms linked by cross-ownership of possibly both, debt and equity, where each firm has a single, homogeneous class of debt only.

Our work directly continues the work of Suzuki [2002] and Gouriéroux et al. [2012], since we consider the same model. Each firm is assumed to have a single outstanding liability, whereas its assets consist of one system-exogenous asset, as well as system-endogenous assets comprising some fraction of other firms' liability and/or equity. In contrast to Suzuki [2002] we do not incorporate time-continuous coupon payments and dividend payments, as we almost always consider no-arbitrage prices at maturity, which are not affected by such intertemporal cash flows. Within this set-up, the aim of our work is to explore the consequences of not taking financial interconnections between firms into account properly when it comes to firm valuation at maturity, and to study the effects on the related probabilities of default. Furthermore, we aim for a better understanding of the conditions that facilitate or impede the propagation of losses and defaults throughout the system of firms. In our analysis we will mainly consider the 'firm value' of each firm, which we calculate as the sum of the no-arbitrage prices of a firm's debt and equity at maturity. By the balance sheet identity, the firm value equals the no-arbitrage price of a firm's total assets, and since a firm is in default if and only if its total assets are insufficient to repay all of its nominal debt, the firm value is directly linked to a firm's probability of default. This renders the firm value a natural object to study, with our analysis being structured as follows.

After the introduction of some notation in Section 2, Section 3 deals with the calculation of firm values under cross-ownership. It can be shown that firm values under cross-ownership are non-trivial derivatives of exogenous asset values (see Suzuki [2002] and Fischer [2014], for example), but an explicit determination of these firm values is rather tedious for the n firms case, at it requires a case differentiation with 2^n cases referring to the financial statuses (solvency or default) of the n firms. Hence, in a first step, we limit ourselves to the two firms case and employ the explicit formulae for the firm values provided by Suzuki [2002]. Recall that Merton [1974] starts from a single class of exogenous assets with values following a geometric Brownian motion, which means that asset values are lognormally distributed at maturity. Therefore, in Merton's model, firm values are lognormally distributed too, since the values of the liability and the equity add up to the exogenous asset's value, which has lognormal distribution by assumption. In a system of firms with financial interconnections, a firm's value is in general not lognormally distributed, if cross-ownership is correctly accounted for, so the question arises to what extent this distribution differs from a lognormal distribution that would emerge if we ignored that a part of the firm's assets is priced endogenously and instead treated all assets as a single, homogeneous class of assets with lognormally distributed values as in the Merton model. We address this issue in a simulation study where we examine how the discrepancy between the distribution of firm values obtained from our model and a correspondingly matched lognormal distribution depends on the realized amounts of cross-holdings of debt and equity and the ratio of nominal liabilities to expected exogenous assets value. Similarly, we compare the related probabilities of default and Values-at-Risk, where we calculate the Value-at-Risk (VaR) of a firm in the system as the negative $\alpha-$quantile of the firm value at maturity minus the firm's risk neutral price in $t = 0$, which means that we consider the $(1-\alpha)100\%$-VaR of the change in firm value.

In Section 4 we let the cross-ownership fractions, i.e. the fraction that one firm holds of another firm's debt or equity, converge to 1 (which is the supremum of the possible values that cross-ownership fractions can take), as this makes a theoretical analysis of default probabilities possible. Then we compare the resulting limiting univariate and bivariate probabilities of default arising from the actual distribution of firm values under cross-ownership and the lognormal distribution, showing that the direction of the effect strongly depends on whether the firms have established cross-ownership of debt only or cross-ownership of equity. Furthermore, we provide a formula that allows us to check for an arbitrary scenario of cross-ownership and any non-negative distribution of exogenous asset values whether the approximating lognormal model will over- or underestimate the related probability of default of a firm.

After this analysis of the univariate distribution of firm values under cross-ownership in a system of two firms with bivariate lognormally distributed exogenous asset values, Section 5 deals with the copula of these firm values as a distribution-free measure of the dependency between these firm values. Without cross-ownership, this would be the Gaussian copula. Under cross-ownership, we especially consider the behaviour of the copula of firm values in the lower left and upper right corner of the unit square, and depending on the type of cross-ownership and the considered corner, we either obtain error bounds indicating how well the copula of firm values under cross-ownership can be approximated with the Gaussian copula, or we see that the copula of firm values can be written as the copula of two linear combinations of exogenous asset values (note that these linear combinations are not lognormally distributed). These insights serve as a basis for our analysis of the tail dependence coefficient of firm values under cross-ownership. In addition, we use results of Asmussen and Rojas-Nandayapa [2008], Gao et al. [2009] and Gulisashvili and Tankov [forthcoming] on the asymptotic behaviour of the sum of lognormally distributed random variables. In general, firm values may be both, perfectly tail independent and perfectly tail dependent, depending on the parameters of the bivariate distribution of exogenous asset values and the considered type of cross-ownership.

In Section 6 we return to systems of $n \geq 2$ firms and analyze sensitivities of no-arbitrage prices of equity and the recovery claims of liabilities with respect to the cross-ownership fractions and the nominal level of liabilities. This complements the results of Liu and Staum [2010] and Gouriéroux et al. [2012], who consider such sensitivities with respect to exogenous asset values. Our insights can be used to evaluate how the no-arbitrage price of any liability or equity in the system reacts to changes within the network structure or the balance sheet of any firm at maturity, caused by a partial unwinding of cross-holdings or some debt cancellation, for example. We show that all prices are non-decreasing in any cross-ownership fraction in the model, and by use of a version of the Implicit Function Theorem we determine exact derivatives. The recovery value of debt of a firm is non-decreasing and the equity value of a firm is non-increasing in the firm's nominal level of liabilities, but the firm value is in general not monotone in the firm's level of liabilities. All these prices are in general non-monotone in the nominal level of liabilities of other firms in the system, but if we limit ourselves to one type of cross-ownership, we can derive more precise relationships. All the results can be transferred to risk-neutral prices before maturity.

As a kind of extension of the results of Liu and Staum [2010] and Gouriéroux et al. [2012] we consider in Section 7 how immediate changes in exogenous asset values of one or more firms at maturity affect the firm values and thus the financial health of a system of $n \geq 2$ initially solvent firms. In the presence of cross-ownership, a firm is affected twice by such shocks, namely by the direct effect on its exogenous asset value as well as the indirect effect consisting of the change in its endogenous asset value. We will call this change the effect of contagion, since it quantifies the additional repercussions of a system-wide shock due to financial interconnections. We explicitly calculate the effect of contagion for the two firms case, revealing that in general, the effect of contagion can have both, the same and the opposite sign as the direct effect, which means that it can mitigate and exacerbate the change in the firm value. If the effect of contagion is so large that the firm defaults, even though it could have borne the direct effect, we call this a contagious default in reference to Elsinger et al. [2006a], who define this term for firms linked by cross-ownership of debt only. As the scope of this work lies in the ramifications of financial interconnectedness between firms, we examine the occurrence of contagious defaults in more detail. For two firms with mutual cross-holdings we identify scenarios where a firm experiencing a negative shock can pass on this shock to the other firm experiencing a non-negative shock, leading to a contagious default of the latter firm. In the presence of cross-holdings of equity, it is even possible that the former firm stays solvent. We expand these considerations to the n firms case by analyzing situations where a given subset of firms receives non-positive shocks and another given subset of firms receives non-negative shocks. Extending the results of Glasserman and Young [2015] by allowing cross-ownership of debt and/or equity and by incorporating multiple shocks, we provide a necessary condition for these shocks to cause the default of all firms in the latter subset. This also yields an upper bound for the probability of such an event. After this analysis of how shocks propagate through a given system, we continue by examining to what extent the network architecture might increase the system's resilience to negative shocks. In the past, several simulation studies on this topic have been conducted (see Nier et al. [2007], Gai and Kapadia [2010], Gai et al. [2011] and Elliott et al. [2014], for example), but except for the work of Elliott et al. [2014], they are based on cascade models. Complementing the existing literature, we investigate in a simulation study how the stability of a system of firms exposed to multiple shocks depends on the model parameters within our clearing model. In doing so, we consider three network types (incomplete, core-periphery and ring networks) with simultaneous shocks on some of the

firms which wipe out a certain percentage of their exogenous assets. Then we analyze for all three types of cross-ownership how the shock intensity, the shock size and the network architecture influence several output parameters, comprising the total number of defaults, the proportion of contagious defaults and the relative loss in the sum of firm values, for example. The obtained results are compared to those of Elsinger et al. [2006a], Nier et al. [2007], Gai and Kapadia [2010] and Elliott et al. [2014]. We conclude our work with a theoretical comparison of the complete network (where each firm holds a part of any other firm) and the ring network with respect to the number of defaults caused by a shock on a single firm, as it is done by Allen and Gale [2000]. In line with the literature, we find that under cross-ownership of debt only, complete networks are "robust yet fragile" [Gai and Kapadia, 2010, p. 2403] in that a moderate shock can be completely withstood or drive only the firm directly hit by it in default, but as soon as the shock exceeds a certain size, all firms are simultaneously forced into default.
Section 8 contains some final remarks, the Appendix provides some auxiliary results and complementing analyses.
All simulations and graphs were realized with the software R 2.12.0 and R 3.0.2 (R Core Team [2013]).

2 Notation

In the following, all vectors are column vectors. For two vectors $\mathbf{a} = (a_i)_{1 \le i \le n} \in \mathbb{R}^n$ and $\mathbf{b} = (b_i)_{1 \le i \le n} \in \mathbb{R}^n$ we will write $\mathbf{a} < \mathbf{b}$ if $a_i \le b_i$ for all $i \in \{1, \ldots, n\}$ and $a_i < b_i$ for at least one $i \in \{1, \ldots, n\}$. Furthermore, we will write $\mathbf{a} \ll \mathbf{b}$ if $a_i < b_i$ for all $i \in \{1, \ldots, n\}$. The same convention will be used for matrices. The n-dimensional identity matrix will be denoted by $\mathbf{I}_n$, the zero vector and the quadratic zero matrix of dimension n will be denoted by $\mathbf{0}_n$ and $\mathbf{0}_{n,n}$, respectively. If the dimension is obvious, the index n will be left out. In the analysis of the limiting behaviour of functions $f, g : \mathbb{R} \to \mathbb{R}$, we will make use of the Landau big oh notation and small oh notation. If $g(x) \neq 0$ for all x in a neighbourhood of $x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$, f is of order $O(g(x))$ for $x \to x_0$, in symbols $f(x) = O(g(x))$, $x \to x_0$, if

$$\limsup_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < \infty. \tag{2.1}$$

f is of smaller order than g for $x \to x_0$, in symbols $f(x) = o(g(x))$, $x \to x_0$, if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{2.2}$$

Furthermore, f and g are called asymptotically equivalent for $x \to x_0$, in symbols $f(x) \sim g(x)$, $x \to x_0$, if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{2.3}$$

For a function $h : \mathbb{R}^2 \to \mathbb{R}$ in two arguments, $D_1 h$ and $D_2 h$ denote the partial derivatives of h with respect to the first and second argument, respectively, provided they exist.
For random variables X and Y let F_X, F_Y and $F_{X,Y}$ stand for the univariate distribution functions of X and Y and the bivariate distribution function of X and Y, respectively. Densities are denoted by small fs and analogous indices, survival functions are denoted by $\bar{F}$ and analogous indices. If X and Y follow the same distribution, we will write $X =^{\mathcal{D}} Y$. The distribution function and the density of the normal distribution with mean μ and variance σ^2 will be denoted by Φ_{μ,σ^2} and φ_{μ,σ^2}, respectively, and we set $\Phi := \Phi_{0,1}$ and $\varphi := \varphi_{0,1}$. Φ_ρ stands for the bivariate standard normal distribution function with correlation ρ. For a distribution function F defined on an interval I, F^{-1} denotes the generalized inverse of F, i.e.

$$F^{-1}(t) := \inf\{x \in I : F(x) \ge t\}, \tag{2.4}$$

where we follow the convention that the infimum of the empty set is ∞. If F is strictly increasing and continuous, the generalized inverse is just the ordinary inverse of F.

3 The Model of Cross-Ownership and Basic Properties

Several results of this section can be found in Karl and Fischer [2014]. In some cases, we use identical formulations.

3.1 Firm Valuation with and without Cross-Ownership

3.1.1 Merton's Model

In Merton's asset valuation model [Merton, 1974], a single firm is assumed to have one class of exogenously priced assets of value a and a certain amount of zero-coupon debt with face value d due at some future time T. In this context, "exogenously" means that the value is independent of the firm's capital structure. At maturity, debt has to be paid back, but if the asset value has fallen below the face value of debt at this time, the firm is said to be in default[1] and all assets are handed over to the creditor. Thus, the creditor receives the minimum of d and a, which we call the recovery value of debt, r. The value of equity, s, then is the value of the remaining assets, so, at maturity:

$$r = \min\{d, a\}, \tag{3.1}$$

$$s = (a - d)^+. \tag{3.2}$$

The firm's balance sheet at maturity is given in Table 3.1. This leads us to the following definition.

Definition 3.1. *Based on* (3.1)–(3.2) *we define the firm value v of a firm as the firm's total asset value:*

$$v := r + s = a. \tag{3.3}$$

Assets	Liabilities
a	s
	r

Table 3.1: Single Firm: Balance sheet at maturity.

[1] Since we have one class of perfectly liquid exogenous assets, we do not distinguish between insolvency and bankruptcy, and the terms default, insolvency and bankruptcy are used interchangeably.

A generalization of this firm value to the case of n ($n \in \mathbb{N}$) firms linked by cross-ownership is provided in the next section.

3.1.2 Suzuki's Model

For $n \geq 2$ firms linked by cross-ownership, the assets of the firms not only consist of exogenous assets of value $\mathbf{a} = (a_i)_{1 \leq i \leq n} \geq \mathbf{0}$, but also of financial assets issued by other firms, for example in the form of bonds or shares. Let $\mathbf{r} = (r_i)_{1 \leq i \leq n}$ resp. $\mathbf{s} = (s_i)_{1 \leq i \leq n}$ denote the no-arbitrage prices of liabilities resp. equity of the n firms at maturity. Then the value of firm i's assets originating from cross-ownership can be written as

$$\underbrace{\sum_{j=1}^{n} M_{ij}^{\mathrm{d}} \, r_j}_{\text{cross-owned debt}} \quad + \quad \underbrace{\sum_{j=1}^{n} M_{ij}^{\mathrm{e}} \, s_j}_{\text{cross-owned equity}}, \tag{3.4}$$

where M_{ij}^{d} and M_{ij}^{e} stand for the fraction that firm i owns of firm j's debt and equity, respectively. Note that the value of cross-owned debt is a fraction of the recovery value of debt, and not of the face value of debt.
In general, the so-called cross-ownership fractions M_{ij}^{d} and M_{ij}^{e} ($i, j \in \{1, \ldots, n\}$) can take values in the interval $[0, 1]$. We collect these cross-ownership fractions in two matrices $\mathbf{M}^{\mathrm{d}} = (M_{ij}^{\mathrm{d}})_{1 \leq i,j \leq n} \in [0, 1]^{n \times n}$ and $\mathbf{M}^{\mathrm{e}} = (M_{ij}^{\mathrm{e}})_{1 \leq i,j \leq n} \in [0, 1]^{n \times n}$ indicating the realized cross-holdings of debt and equity, respectively. We will assume that no firm's debt or equity is held completely within the system, but that some part of each firm's debt and equity is held by a firm or investor outside of the system of n firms. Hence,

$$||\mathbf{M}^{\mathrm{d}}||_1 < 1, \quad ||\mathbf{M}^{\mathrm{e}}||_1 < 1, \tag{3.5}$$

i.e. we assume $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ to be strictly left sub-stochastic. In particular, this implies M_{ij}^{d}, $M_{ij}^{\mathrm{e}} \in [0, 1)$ for all $i, j \in \{1, \ldots, n\}$. Furthermore, we suppose that no firm holds a part of its own debt or equity, i.e.

$$\operatorname{diag}(\mathbf{M}^{\mathrm{d}}) = \operatorname{diag}(\mathbf{M}^{\mathrm{e}}) = \mathbf{0}. \tag{3.6}$$

Based on the entries of $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ we define three types of cross-ownership.

Definition 3.2. *The n firms are said to be linked by*

1. *cross-ownership of debt only, if*

$$\mathbf{M}^{\mathrm{d}} > \mathbf{0}_{n,n}, \quad \mathbf{M}^{\mathrm{e}} = \mathbf{0}_{n,n}, \tag{3.7}$$

   that is at least one firm holds a part of another firm's debt, but there are no cross-holdings of equity;

2. *cross-ownership of equity only, if*

$$\mathbf{M}^{\mathrm{d}} = \mathbf{0}_{n,n}, \quad \mathbf{M}^{\mathrm{e}} > \mathbf{0}_{n,n}, \tag{3.8}$$

Firm 1

Assets	Liabilities
a_1	s_1
$M^{\mathrm{d}}_{1,2} \times r_2$	r_1
$M^{\mathrm{e}}_{1,2} \times s_2$	

Firm 2

Assets	Liabilities
a_2	s_2
$M^{\mathrm{d}}_{2,1} \times r_1$	r_2
$M^{\mathrm{e}}_{2,1} \times s_1$	

Table 3.2: Two firms linked by cross-ownership: Balance sheets at maturity.

that is at least one firm holds a part of another firm's equity, but there are no cross-holdings of debt;

3. *cross-ownership of both, debt and equity, if*

$$\mathbf{M}^{\mathrm{d}} > \mathbf{0}_{n,n}, \quad \mathbf{M}^{\mathrm{e}} > \mathbf{0}_{n,n}, \tag{3.9}$$

that is at least one firm holds a part of another firm's debt and at least one firm holds a part of another firm's equity.

Note that our definition of cross-ownership would not impose any restrictions with respect to the type of debt that is cross-owned. For example, a firm could hold a derivative on any underlying considered in the model, e.g. exogenous assets. However, following Eisenberg and Noe [2001], Elsinger [2009] and Gouriéroux et al. [2012], we will assume all liabilities to be zero-coupon bonds with identical maturity and face values $\mathbf{d} = (d_i)_{1 \leq i \leq n} \geq \mathbf{0}$. For a model with more complicated liabilities of possibly differing seniority, see Fischer [2014]. In contrast to Suzuki [2002] we assume that there are neither dividend payments on equity holdings nor coupon payments on debt before or at maturity.

It is clear from Table 3.2, which shows the balance sheets of two firms linked by cross-ownership, that the value of firm 1 also depends on the financial health of firm 2: if firm 2 defaults, this will affect both, the value of its equity and the recovery value of its debt, which will possibly be smaller than the actual outstanding amount. Therefore, the total asset value of firm 1 will decrease and thus firm 1 might also get into trouble, which again might affect firm 2 in a negative way. If we applied Merton's model of firm valuation to each firm separately in order to obtain no-arbitrage prices of debt and equity, we would ignore this circular dependence between the two firms. Of course, these considerations also hold for systems of more than two firms. The work of Suzuki [2002] shows how to overcome this problem by applying Merton's idea to all firms simultaneously.

Obviously, the total assets of firm i consist of an exogenous and an endogenous part, and we set

$$a_i^* := \underbrace{a_i}_{\substack{\text{value of}\\ \text{exogenous}\\ \text{asset}}} + \underbrace{\sum_{j=1}^{n} M_{ij}^{\mathrm{d}}\, r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}\, s_j}_{\substack{\text{value of}\\ \text{endogenous}\\ \text{assets}}}, \qquad i \in \{1,\ldots,n\}, \tag{3.10}$$

where "endogenous" means that the price is determined within the system of n firms. If we apply Merton's approach to all firms simultaneously, we obtain in analogy to (3.1)–(3.2) the following system of equations:

$$r_i = \min\{d_i,\, a_i^*\} = \min\left\{ d_i,\, a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}}\, r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}\, s_j \right\}, \tag{3.11}$$

$$s_i = (a_i^* - d_i)^+ = \left(a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}}\, r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}\, s_j - d_i \right)^+, \quad i \in \{1,\ldots,n\}, \tag{3.12}$$

or equivalently,

$$\mathbf{r} = \min\{\mathbf{d},\, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\}, \tag{3.13}$$

$$\mathbf{s} = (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^+. \tag{3.14}$$

As in Section 3.1.1, the recovery value of debt of a firm still is the minimum of the firm's liability and total asset value, but under cross-ownership this recovery value now also depends on the other firms' recovery value of debt and equity value. Similarly, the value of equity at maturity is now influenced by the other firms' recovery value of debt and equity value. Since we assume liabilities to be zero-coupon bonds, the system (3.13)–(3.14) has a unique solution by Theorem 3.8 of Fischer [2014], which can be obtained by the fixed point algorithm provided by Fischer [2014]. Hence, for given $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, $\mathbf{r}$ and $\mathbf{s}$ are deterministic functions of $\mathbf{a}$ and thus derivatives of $\mathbf{a}$, just as in the Merton model. In the Merton model for a single firm, we defined the firm value v as the sum of the recovery value of debt and the equity value (cf. Definition 3.1). Definition 3.3 transfers this definition to the case of cross-ownership.

Definition 3.3. *The firm value $\mathbf{v} = (v_i)_{1 \le i \le n}$ of n firms linked by cross-ownership is given by the firms' total asset value, i.e.*

$$v_i := a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}}\, r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}\, s_j. \tag{3.15}$$

By Lemma 3.5 of Fischer [2014], the recovery value of debt $\mathbf{r}$ (as part of a solution of

(3.13)–(3.14)) and thus the firm value $\mathbf{v}$ are always non-negative. Furthermore, (3.13)–(3.14) and Definition 3.3 imply

$$\mathbf{v} = \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} = \mathbf{r} + \mathbf{s}, \tag{3.16}$$

i.e. also firm values are derivatives of exogenous asset values, and we will sometimes write $\mathbf{v}(\mathbf{a})$ if it seems helpful. As the following lemma shows, $\mathbf{v}$ is continuous in all model parameters.

Lemma 3.4. *For a system of n firms linked by cross-ownership, let $\mathbf{r}$ and $\mathbf{s}$ be given by* (3.13)–(3.14)*, and let $\mathbf{v}$ be given as in Definition 3.3. Then $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{v}$ are continuous in each entry of $\mathbf{a}$, $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$.*

Proof. As noted by Fischer [2014], $(\mathbf{r}^T, \mathbf{s}^T)^T$ is a fixed point of the mapping

$$\mathbf{\Phi}_{\mathbf{a},\mathbf{d},\mathbf{M}^{\mathrm{d}},\mathbf{M}^{\mathrm{e}}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \mapsto \begin{pmatrix} \min\{\mathbf{d},\, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\} \\ (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^{+} \end{pmatrix}. \tag{3.17}$$

By Lemma 4.1 of Fischer [2014], $\mathbf{\Phi}_{\mathbf{a},\mathbf{d},\mathbf{M}^{\mathrm{d}},\mathbf{M}^{\mathrm{e}}}$ is a strict contraction with Lipschitz constant $L := \max\{||\mathbf{M}^{\mathrm{d}}||_1,\, ||\mathbf{M}^{\mathrm{e}}||_1\}$ with $L < 1$ because of (3.5). Furthermore, $\mathbf{\Phi}_{\mathbf{a},\mathbf{d},\mathbf{M}^{\mathrm{d}},\mathbf{M}^{\mathrm{e}}}$ is continuous in each of its $2n$ variables. Since $\mathbb{R}^{2n}$ is locally compact, Theorem 2 of Kwieciński [1992] yields the assertion. □

By definition, firm i ($i \in \{1, \ldots, n\}$) is in default if and only if its total assets (i.e. its firm value) do not suffice to pay back all of its debt, i.e. if $v_i = a_i^* < d_i$. With $\mathbf{z} = (z_i)_{1 \le i \le n} \in \{\mathrm{s}, \mathrm{d}\}^n$ indicating the conditions with respect to solvency or default of the n firms, where "s" stands for solvent and "d" for in default, we set

$$A_{\mathbf{z}} := \left(\bigcap_{\{i: z_i = \mathrm{s}\}} \left\{ \mathbf{a} \in (\mathbb{R}_0^+)^n : \, v_i(\mathbf{a}) \ge d_i \right\} \right) \cap \left(\bigcap_{\{i: z_i = \mathrm{d}\}} \left\{ \mathbf{a} \in (\mathbb{R}_0^+)^n : \, v_i(\mathbf{a}) < d_i \right\} \right), \tag{3.18}$$

i.e. on some $A_{\mathbf{z}}$, the firms with $z_i = \mathrm{s}$ have a firm value of at least d_i, whereas the firms with $z_i = \mathrm{d}$ have a firm value smaller than their outstanding liabilities, i.e. they are in default. Of course, $A_{\mathbf{z}}$ depends on $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ via $\mathbf{v}$, but we suppress them in the notation for better readability. In reference to Suzuki [2002] we will call the sets $A_{\mathbf{z}}$ Suzuki areas. Obviously, the 2^n Suzuki areas form a partition of $(\mathbb{R}_0^+)^n$.

In particular, for fixed $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, since the financial status (solvency or default) remains unchanged for all firms for all values of $\mathbf{a}$ within a certain Suzuki area, the system (3.13)–(3.14) becomes linear within a Suzuki area and can be solved by Gaussian elimination, for example. Hence, instead of determining $\mathbf{r}$ and $\mathbf{s}$ as fixed points of (3.17) by the algorithm of Fischer [2014], one could solve 2^n linear systems of equations and then check on which Suzuki area the obtained values of $\mathbf{r}$ and $\mathbf{s}$ match the imposed assumptions as to the financial status of the n firms.

For $n = 2$, the system (3.13)–(3.14) consisting of four equations and four unknowns is solved by Suzuki [2002]. The resulting no-arbitrage prices $\mathbf{r}$ and $\mathbf{s}$ at maturity are given in the following lemma.

Lemma 3.5. *For $n = 2$, the system* (3.13)–(3.14) *is solved by*

$$r_1 = \begin{cases} d_1, & (a_1, a_2) \in A_{\mathrm{ss}}, \\ d_1, & (a_1, a_2) \in A_{\mathrm{sd}}, \\ \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1}}(a_1 + M^{\mathrm{e}}_{1,2}a_2 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2), & (a_1, a_2) \in A_{\mathrm{ds}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}}(a_1 + M^{\mathrm{d}}_{1,2}a_2), & (a_1, a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.19}$$

$$r_2 = \begin{cases} d_2, & (a_1, a_2) \in A_{\mathrm{ss}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}(M^{\mathrm{e}}_{2,1}a_1 + a_2 + (M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1), & (a_1, a_2) \in A_{\mathrm{sd}}, \\ d_2, & (a_1, a_2) \in A_{\mathrm{ds}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}}(M^{\mathrm{d}}_{2,1}a_1 + a_2), & (a_1, a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.20}$$

$$s_1 = \begin{cases} \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}(a_1 + M^{\mathrm{e}}_{1,2}a_2 - (1 - M^{\mathrm{d}}_{2,1}M^{\mathrm{e}}_{1,2})d_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2, & (a_1, a_2) \in A_{\mathrm{ss}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}(a_1 + M^{\mathrm{d}}_{1,2}a_2 - (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_1), & (a_1, a_2) \in A_{\mathrm{sd}}, \\ 0, & (a_1, a_2) \in A_{\mathrm{ds}}, \\ 0, & (a_1, a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.21}$$

$$s_2 = \begin{cases} \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}(M^{\mathrm{e}}_{1,2}a_1 + a_2 + (M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1 - (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2, & (a_1, a_2) \in A_{\mathrm{ss}}, \\ 0, & (a_1, a_2) \in A_{\mathrm{sd}}, \\ \frac{1}{1-M^{\mathrm{e}}_{2,1}M^{\mathrm{d}}_{2,1}}(M^{\mathrm{d}}_{2,1}a_1 + a_2 - (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2), & (a_1, a_2) \in A_{\mathrm{ds}}, \\ 0, & (a_1, a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.22}$$

and firm values equal

$$v_1 = \begin{cases} \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}(a_1 + M^{\mathrm{e}}_{1,2}a_2 + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2), & (a_1, a_2) \in A_{\mathrm{ss}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}(a_1 + M^{\mathrm{d}}_{1,2}a_2 + M^{\mathrm{d}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1), & (a_1, a_2) \in A_{\mathrm{sd}}, \\ \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1}}(a_1 + M^{\mathrm{e}}_{1,2}a_2 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2), & (a_1, a_2) \in A_{\mathrm{ds}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}}(a_1 + M^{\mathrm{d}}_{1,2}a_2), & (a_1, a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.23}$$

$$v_2 = \begin{cases} \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}(M^{\mathrm{e}}_{2,1}a_1 + a_2 + (M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1 + M^{\mathrm{e}}_{2,1}(M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2), & (a_1,a_2) \in A_{\mathrm{ss}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}(M^{\mathrm{e}}_{2,1}a_1 + a_2 + (M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})d_1), & (a_1,a_2) \in A_{\mathrm{sd}}, \\ \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1}}(M^{\mathrm{d}}_{2,1}a_1 + a_2 + M^{\mathrm{d}}_{2,1}(M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})d_2), & (a_1,a_2) \in A_{\mathrm{ds}}, \\ \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}}(M^{\mathrm{d}}_{2,1}a_1 + a_2), & (a_1,a_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.24}$$

with

$$A_{\mathrm{ss}} = \{(a_1,a_2) \geq \mathbf{0} :\ a_1 + M^{\mathrm{e}}_{1,2}a_2 \geq (1 - M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})d_1 + (M^{\mathrm{e}}_{1,2} - M^{\mathrm{d}}_{1,2})d_2,\ M^{\mathrm{e}}_{2,1}a_1 + a_2 \geq (M^{\mathrm{e}}_{2,1} - M^{\mathrm{d}}_{2,1})d_1 + (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2\}, \tag{3.25}$$

$$A_{\mathrm{sd}} = \{(a_1,a_2) \geq \mathbf{0} :\ a_1 + M^{\mathrm{d}}_{1,2}a_2 \geq (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_1,\ M^{\mathrm{e}}_{2,1}a_1 + a_2 < (M^{\mathrm{e}}_{2,1} - M^{\mathrm{d}}_{2,1})d_1 + (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2\}, \tag{3.26}$$

$$A_{\mathrm{ds}} = \{(a_1,a_2) \geq \mathbf{0} :\ a_1 + M^{\mathrm{e}}_{1,2}a_2 < (1 - M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})d_1 + (M^{\mathrm{e}}_{1,2} - M^{\mathrm{d}}_{1,2})d_2,\ M^{\mathrm{d}}_{2,1}a_1 + a_2 \geq (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2\}, \tag{3.27}$$

$$A_{\mathrm{dd}} = \{(a_1,a_2) \geq \mathbf{0} :\ a_1 + M^{\mathrm{d}}_{1,2}a_2 < (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_1,\ M^{\mathrm{d}}_{2,1}a_1 + a_2 < (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2\}. \tag{3.28}$$

Proof. For (3.19)–(3.22) see Suzuki [2002]. Equations (3.23) and (3.24) then follow from (3.16). By (3.18), the Suzuki areas for a system of two firms are

$$A_{\mathrm{ss}} = \{(a_1,a_2) \geq \mathbf{0} :\ v_1 \geq d_1,\ v_2 \geq d_2\}, \tag{3.29}$$

$$A_{\mathrm{sd}} = \{(a_1,a_2) \geq \mathbf{0} :\ v_1 \geq d_1,\ v_2 < d_2\}, \tag{3.30}$$

$$A_{\mathrm{ds}} = \{(a_1,a_2) \geq \mathbf{0} :\ v_1 < d_1,\ v_2 \geq d_2\}, \tag{3.31}$$

$$A_{\mathrm{dd}} = \{(a_1,a_2) \geq \mathbf{0} :\ v_1 < d_1,\ v_2 < d_2\}. \tag{3.32}$$

Equivalence of (3.25)–(3.28) and (3.29)–(3.32) follows from straightforward calculations based on (3.23)–(3.24). □

An example of the Suzuki areas for $n = 2$ is provided in Figure 3.1. The boundaries between the four areas always belong to the Suzuki area above the line. Note that if $d_1 \leq M^{\mathrm{d}}_{1,2}d_2$ or $d_2 \leq M^{\mathrm{d}}_{2,1}d_1$, the area A_{ds} resp. A_{sd} vanishes.

In the following section we will compare the behaviour of firm values under cross-ownership of debt only and cross-ownership of equity only in a stochastic set-up. The restriction to one type of cross-holdings at a time not only facilitates the derivation of both, theoretical and simulation results, it also provides some valuable insights into the opposed effects of the two types of cross-ownership.

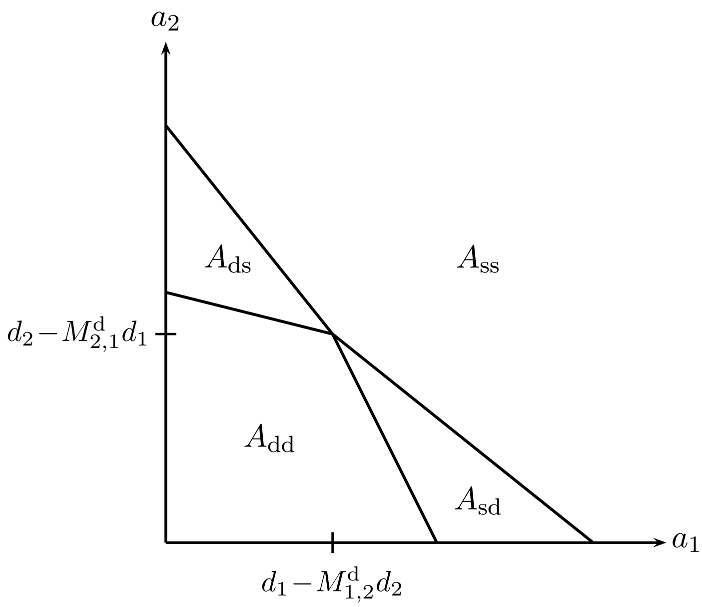


Figure 3.1: Suzuki areas if $d_1 > M^{\mathrm{d}}_{1,2} d_2$ and $d_2 > M^{\mathrm{d}}_{2,1} d_1$.

3.2 Distribution of Firm Values under Cross-Ownership for Systems of two Firms

3.2.1 Theoretical Distribution of Firm Values under XOS of Debt only and XOS of Equity only

In the following we will assume exogenous asset values to be stochastic, and since firm values are continuous functions of exogenous asset values by Lemma 3.4, this also turns the firm value v into a random variable. Henceforth, we will denote random asset values and random firm values with capital As and Vs, respectively.

Remark 3.6. In the remainder we will display several results for firm 1 only. For reasons of symmetry, the corresponding results for firm 2 can be obtained by switching the roles of firm 1 and firm 2 in the related proofs.

By (3.23) the random value of firm 1 under cross-ownership of debt only equals

$$V_1^{\mathrm{d}} := \begin{cases} A_1 + M^{\mathrm{d}}_{1,2} d_2, & (A_1, A_2) \in A_{\mathrm{ss}} \cup A_{\mathrm{ds}}, \\ A_1 + M^{\mathrm{d}}_{1,2} A_2 + M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1} d_1, & (A_1, A_2) \in A_{\mathrm{sd}}, \\ \frac{1}{1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1}} (A_1 + M^{\mathrm{d}}_{1,2} A_2), & (A_1, A_2) \in A_{\mathrm{dd}}, \end{cases} \tag{3.33}$$

and under cross-ownership of equity only, the random value of firm 1 equals

$$V_1^{\mathrm{e}} := \begin{cases} \frac{1}{1-M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}}}(A_1 + M_{1,2}^{\mathrm{e}}A_2 - M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}}d_1 - M_{1,2}^{\mathrm{e}}d_2), & (A_1, A_2) \in A_{\mathrm{ss}}, \\ A_1, & (A_1, A_2) \in A_{\mathrm{sd}} \cup A_{\mathrm{dd}}, \\ A_1 + M_{1,2}^{\mathrm{e}}A_2 - M_{1,2}^{\mathrm{e}}d_2, & (A_1, A_2) \in A_{\mathrm{ds}}. \end{cases} \tag{3.34}$$

The distribution functions of V_1^{d} and V_1^{e} are provided in the next lemma.

Lemma 3.7. *Let $(A_1, A_2) \gg \mathbf{0}$ $P-$a.s. with continuous bivariate distribution. Then, for all $v_1 \geq 0$,*

$$P(V_1^{\mathrm{d}} \leq v_1) = P(A_1 \leq \max\{v_1 - M_{1,2}^{\mathrm{d}}d_2,\, v_1 - M_{1,2}^{\mathrm{d}}M_{2,1}^{\mathrm{d}}\min\{v_1, d_1\} - M_{1,2}^{\mathrm{d}}A_2\}), \tag{3.35}$$

$$P(V_1^{\mathrm{e}} \leq v_1) = P(A_1 \leq \min\{v_1,\, v_1 + M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}}\min\{0,\, d_1 - v_1\} + M_{1,2}^{\mathrm{e}}(d_2 - A_2)\}). \tag{3.36}$$

Proof. If we let v_2 go to infinity in Lemma A.1, we obtain

$$P(V_1^{\mathrm{d}} \leq v_1) = \begin{cases} P(A_1 \leq \max\{v_1 - M_{1,2}^{\mathrm{d}}d_2,\, (1 - M_{1,2}^{\mathrm{d}}M_{2,1}^{\mathrm{d}})v_1 - M_{1,2}^{\mathrm{d}}A_2\}, & v_1 \leq d_1, \\ P(A_1 \leq \max\{v_1 - M_{1,2}^{\mathrm{d}}d_2,\, v_1 - M_{1,2}^{\mathrm{d}}M_{2,1}^{\mathrm{d}}d_1 - M_{1,2}^{\mathrm{d}}A_2\}, & v_1 > d_1. \end{cases} \tag{3.37}$$

Similarly, if we let v_2 go to infinity in Lemma A.2,

$$P(V_1^{\mathrm{e}} \leq v_1) = \begin{cases} P(A_1 \leq \min\{v_1,\, v_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2)\}), & v_1 \leq d_1, \\ P(A_1 \leq \min\{v_1,\, v_1 + M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}}(d_1 - v_1) + M_{1,2}^{\mathrm{e}}(d_2 - A_2)\}), & v_1 > d_1. \end{cases} \tag{3.38}$$

□

Lemma 3.8. *For (A_1, A_2) distributed as in Lemma 3.7, let this distribution be strictly 2-increasing*[2]*. Then $F_{V_1^{\mathrm{d}}}$ and $F_{V_1^{\mathrm{e}}}$ are strictly increasing on $\mathbb{R}_0^+$.*

Proof. Let $0 \leq v_1 < v_1'$. Then Lemma 3.7 yields $P(V_1^{\mathrm{d}} \leq v_1) = P((A_1, A_2) \in S(v_1))$ with

$$S(v_1) := \{(a_1, a_2) \geq \mathbf{0} :\, a_1 \leq \max\{v_1 - M_{1,2}^{\mathrm{d}}d_2,\, v_1 - M_{1,2}^{\mathrm{d}}M_{2,1}^{\mathrm{d}}\min\{v_1, d_1\} - M_{1,2}^{\mathrm{d}}a_2\}\}. \tag{3.39}$$

We show that $S(v_1) \subsetneq S(v_1')$ and that $P((A_1, A_2) \in S(v_1') \backslash S(v_1)) > 0$. If $v_1 < v_1' \leq d_1$

[2] Cf. Definition 5.1 and replace all three inequalities in 3. with strict inequalities.

or $d_1 < v_1 < v_1'$, then obviously $S(v_1) \subsetneq S(v_1')$. If $v_1 \leq d_1 < v_1'$,

$$\begin{aligned} S(v_1) &= \{(a_1, a_2) \geq \mathbf{0} : \, a_1 \leq \max\{v_1 - M^{\mathrm{d}}_{1,2} d_2, \, (1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1}) v_1 - M^{\mathrm{d}}_{1,2} a_2\}\} && (3.40) \\ &\subsetneq \{(a_1, a_2) \geq \mathbf{0} : \, a_1 \leq \max\{v_1' - M^{\mathrm{d}}_{1,2} d_2, \, (1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1}) v_1' - M^{\mathrm{d}}_{1,2} a_2\}\} && (3.41) \\ &\subseteq \{(a_1, a_2) \geq \mathbf{0} : \, a_1 \leq \max\{v_1' - M^{\mathrm{d}}_{1,2} d_2, \, v_1' - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1} d_1 - M^{\mathrm{d}}_{1,2} a_2\}\} && (3.42) \\ &= S(v_1'). && (3.43) \end{aligned}$$

Furthermore, for reasons of continuity, we can find a non-degenerate rectangle $R(v_1)$ such that $S(v_1) \subsetneq S(v_1) \cup R(v_1) \subsetneq S(v_1')$, and since we assume the distribution of (A_1, A_2) to be strictly 2-increasing (i.e. $P((A_1, A_2) \in R(v_1)) > 0$), we obtain $P((A_1, A_2) \in S(v_1)) < P((A_1, A_2) \in S(v_1'))$. The assertion for $F_{V_1^{\mathrm{e}}}$ can be shown similarly. □

3.2.2 Simulation Study on the Distribution of Firm Values and Probabilities of Default under Cross-Ownership

In the following we assume exogenous asset values to follow a bivariate geometric Brownian motion, similar to common extensions of the Merton model to the multivariate case, i.e. we have bivariate lognormally distributed exogenous asset values (A_1, A_2) at maturity.

Without cross-ownership, the assumption of lognormally distributed asset values would imply that firm values are also lognormally distributed because of $V_i = A_i$ in this situation (cf. Definition 3.1). But as we have seen in (3.23) and (3.24), firm values are non-trivial derivatives of exogenous asset values under cross-ownership. Consequently, the distribution of firm values is a transformation of the lognormal distribution, which is generally not lognormal anymore. However, we are not able to derive a closed-form solution of the resulting distribution function, because, alongside other problems, there is no convolution theorem for lognormal distributions, i.e. we only have formulae such as the ones given in Lemma 3.7. In this situation, one could ask to what extent the actual distribution of firm values under cross-ownership differs from the lognormal distribution. Or expressed differently: what mistake do we make if we ignore that a part of the assets is priced endogenously, and treat all assets as a single, homogeneous class of exogenous assets with values following a lognormal distribution which has the same first two moments as the actual firm value under cross-ownership? Since this approach would result in lognormally distributed firm values, this question essentially aims at the effects of applying Merton's model of firm valuation to both firms separately, despite the presence of financial interconnectedness. We address this issue in a simulation study, where we compare the actual distribution of firm values resulting from Suzuki's model to the lognormal distribution used in Merton's model. Since firm i is in default if and only if its firm value is smaller than the face value of its outstanding liability at maturity, its probability of default at maturity is given as $P(V_i < d_i)$. In our simulations we will compare this probability of default between Suzuki's model and what we will call lognormal model in the remainder, i.e. a model where firm values are assumed to follow a lognormal distribution despite the presence of cross-ownership.

3.2.2.1 Set-Up and Parameter Values

In our simulations we mainly consider two firms linked by cross-ownership of either debt or equity. For the case of cross-ownership of both, debt and equity, some exemplary analyses of the distribution of firm values are provided.
Let exogenous asset values of the two firms be independent and identically lognormally distributed at maturity T with

$$A_i \sim \mathcal{LN}(-0.5\sigma^2 + \ln(a),\, \sigma^2) \tag{3.44}$$

with some $a > 0$, implying that $E(A_i) = a$ and $\mathrm{Var}(A_i) = a^2(\exp(\sigma^2) - 1)$, $i = 1, 2$. Furthermore, we assume the liabilities of the two firms to have identical face values $d_1 = d_2 =: d$. Because of this kind of symmetry between the two firms, we only analyze the outcome of firm 1. Note that any two set-ups for which the ratio d/a is identical can be interpreted as the same set-up under a different currency at a constant exchange rate, i.e. only the relative size of d to a is important. This is why we set $a = 1$ in all our simulations and let only d take different values. In particular, we have $E(A_i) = 1$ and $\mathrm{Var}(A_i) = \exp(\sigma^2) - 1$, $i = 1, 2$. The value of the liabilities, d, runs through {0.1, 0.2, . . . , 2.9, 3}, which means that

$$\frac{\text{face value of debt}}{\text{expected ex. asset value}} = \frac{d}{a} \in \{0.1, 0.2, \ldots, 2.9, 3\}. \tag{3.45}$$

Moreover, σ^2 took values in {0.00995, 0.08618, 0.03922, 0.22314, 0.44629, 0.69315, 1, 1.17865, 1.60944, 1.98100, 2.30259, 3.25810, 4.04743, 4.61512}, which approximately resulted in coefficients of variation[3] of A_i of {0.1, 0.2, 0.3, 0.5, 0.75, 1, 1.31, 1.5, 2, 2.5, 3, 5, 7.5, 10}. The respective cross-ownership fractions $(M^{\mathrm{d}}_{1,2}, M^{\mathrm{d}}_{2,1})$ and $(M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1})$ took values in $\{0.1, 0.2, \ldots, 0.9\}^2$. Although cross-ownership fractions of equity bigger than 0.5 are not relevant from a practical point of view as such firms would be forced to create a common balance sheet, we nevertheless consider the same range of values for $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, as we hope to obtain some useful insights into the mechanics behind the two types of cross-holdings from a direct comparison.
For every combination of parameter values and both types of cross-ownership we first simulated 100,000 of values of (A_1, A_2) according to (3.44) and calculated the related empirical firm values v_1. Then we fitted a lognormal distribution $\tilde{F}_{\mathrm{LOG}}$ to the obtained empirical distribution $\hat{F}_{V_1}$ of V_1 under cross-ownership. The parameters of this lognormal distribution were determined in analogy to the Fenton–Wilkinson method [Fenton, 1960] of moment matching, where the first and second moments are chosen such that they correspond to the first and second moments of $\hat{F}_{V_1}$(cf. Section A.2.1). Note that under the risk-neutral probability, our approach of matching the first moments of firm values obtained under Suzuki's model and the lognormal model at maturity is equivalent to matching the initial firm values under the lognormal model to the initial firm values under Suzuki's model.

[3]For a random variable X with mean μ and standard deviation σ, the coefficient of variation is defined as $\frac{\sigma}{\mu}$.

Let F_{LOG} denote the lognormal distribution fitted to the theoretical distribution F_{V_1} of V_1 by the Fenton–Wilkinson method. As already mentioned, a closed formula for F_{V_1} is not available, so we approximate it with the empirical distribution function $\hat{F}_{V_1}$ based on firm values obtained from our simulation study. Of course, since the exact moments of V_1 are unknown, the obtained (theoretical) lognormal distribution $\tilde{F}_{\mathrm{LOG}}$ is only an approximation to F_{LOG}. However, the Strong Law of Large Numbers almost surely yields $\lim_{m\to\infty} \tilde{F}_{\mathrm{LOG}} = F_{\mathrm{LOG}}$, with m denoting the number of observations underlying $\hat{F}_{V_1}$. Note that V_1 is square-integrable as A_1 and A_2 are square-integrable. As a measure for the discrepancy between $\hat{F}_{V_1}$ and $\tilde{F}_{\mathrm{LOG}}$ we use the one-sample Kolmogorov–Smirnov statistic

$$\mathrm{KS} := \mathrm{KS}(\hat{F}_{V_1}, \tilde{F}_{\mathrm{LOG}}) := \sup_{x\in\mathbb{R}} \big|\hat{F}_{V_1}(x) - \tilde{F}_{\mathrm{LOG}}(x)\big|, \tag{3.46}$$

which is the supremum of the vertical distance between the two distribution functions. Furthermore, the Glivenko–Cantelli Theorem (cf. Theorem 20.6 of Billingsley [1995], for example) implies that

$$\lim_{m\to\infty} \mathrm{KS}(\hat{F}_{V_1}, \tilde{F}_{\mathrm{LOG}}) = \mathrm{KS}(F_{V_1}, F_{\mathrm{LOG}}) \quad a.s., \tag{3.47}$$

i.e. for a sufficiently large number of iterations m, the Kolmogorov–Smirnov statistic based on the empirical distribution function $\hat{F}_{V_1}$ and the matched lognormal distribution function $\tilde{F}_{\mathrm{LOG}}$ are close to the Kolmogorov–Smirnov statistic between the theoretical distribution F_{V_1} and the corresponding lognormal distribution F_{LOG}. Hence, we will interpret all results in terms of $\mathrm{KS}(F_{V_1}, F_{\mathrm{LOG}})$ instead of $\mathrm{KS}(\hat{F}_{V_1}, \tilde{F}_{\mathrm{LOG}})$.

Since firm 1 is in default if and only if $V_1 < d_1$, we estimate the probability of default under Suzuki's model by

$$\hat{p}_{\mathrm{S}} := \frac{\#\{(A_1, A_2) \in A_{\mathrm{ds}} \cup A_{\mathrm{dd}}\}}{100{,}000} \tag{3.48}$$

(cf. (3.31)–(3.32)). Since $\hat{p}_{\mathrm{S}}$ has five decimal places only, we round the probabilities of default obtained from the lognormal model, given as $\tilde{F}_{\mathrm{LOG}}(d_1)$, to five decimal places as well for better comparability. The result will be denoted by $\hat{p}_{\mathrm{LOG}}$. As a measure for the discrepancy between the two models we use the relative risk ('RR') of the two models, estimated by

$$\widehat{\mathrm{RR}} := \begin{cases} \frac{\hat{p}_{\mathrm{LOG}}}{\hat{p}_{\mathrm{S}}}, & \hat{p}_{\mathrm{S}} > 0, \\ 1, & \hat{p}_{\mathrm{S}} = 0 \text{ and } \hat{p}_{\mathrm{LOG}} = 0, \\ \infty, & \hat{p}_{\mathrm{S}} = 0 \text{ and } \hat{p}_{\mathrm{LOG}} > 0. \end{cases} \tag{3.49}$$

For cross-ownership of debt only and cross-ownership of equity only we first try to figure out which circumstances (i.e. values of cross-ownership fractions) lead to a big discrepancy between $\hat{F}_{V_1}$ and $\tilde{F}_{\mathrm{LOG}}$ and to relatively big or small values of $\widehat{\mathrm{RR}}$. After that, we carry out a more detailed simulation for scenarios with high levels of cross-ownership and maximize the discrepancy between the two distribution functions with respect to

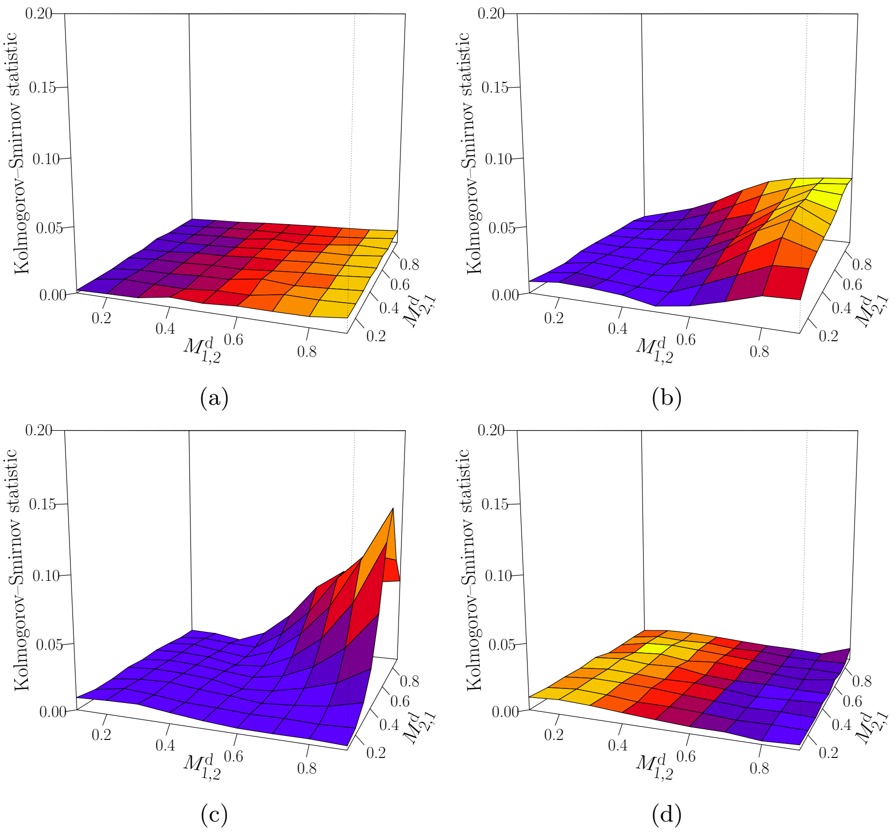


Figure 3.2: Kolmogorov–Smirnov statistics in dependence of the cross-ownership fractions under XOS of debt only, $\sigma^2 = 0.22314$; (a) $d/a = 0.1$; (b) $d/a = 2$; (c) $d/a = 4$; (d) $d/a = 20$.

the face value of liabilities d. For the constellation yielding the highest Kolmogorov–Smirnov value, we then have a look at the shape of the two distribution functions and the resulting probabilities of default. Some generic analyses for the case of simultaneous cross-ownership of debt and equity follow.

3.2.2.2 Results for XOS of Debt only and XOS of Equity only

Influence of σ^2 and d/a

For each combination of σ^2 and d/a we made surface plots of the obtained Kolmogorov–Smirnov values on the $(M^{d}_{1,2}, M^{d}_{2,1})$- resp. $(M^{e}_{1,2}, M^{e}_{2,1})$-plane. Depending on the relative size of σ^2 and d/a, we observed rather different relationships between the cross-ownership fractions and the corresponding Kolmogorov–Smirnov values (cf. Figure 3.2 and Figure 3.3), suggesting that for every considered value of σ^2, there seems to be a range of values of d/a where we observe a relationship between the realized cross-ownership fractions and the obtained Kolmogorov–Smirnov values. Outside of this range (i.e. d/a

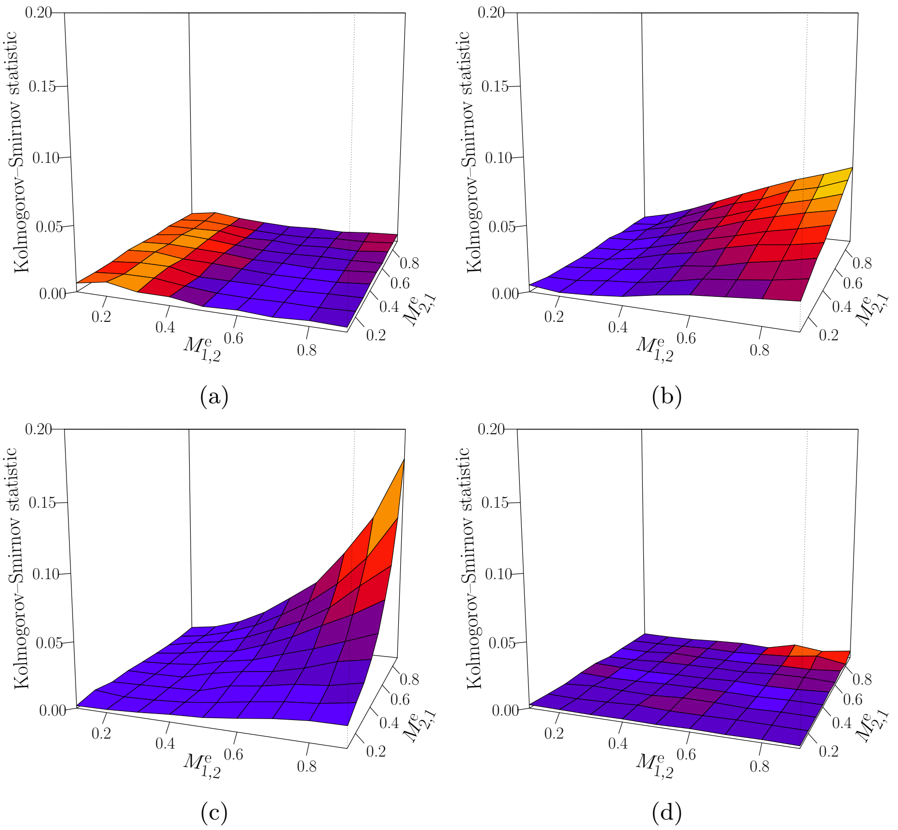
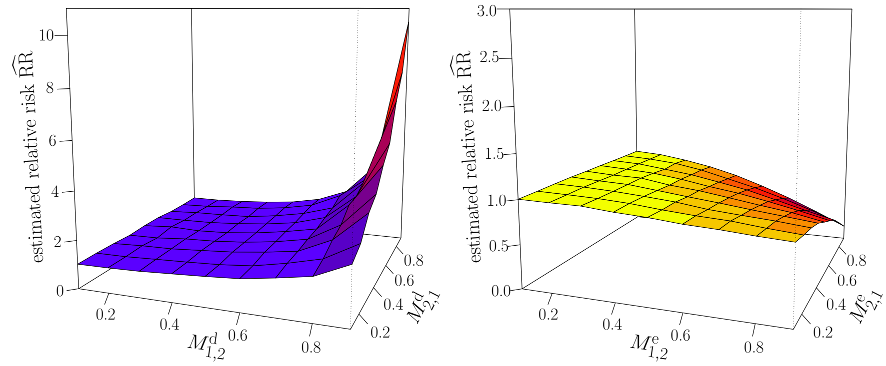


Figure 3.3: Kolmogorov–Smirnov statistics in dependence of the cross-ownership fractions under XOS of equity only, $\sigma^2 = 0.22314$; (a) $d/a = 0.1$; (b) $d/a = 0.5$; (c) $d/a = 0.8$; (d) $d/a = 3$.

chosen too small or too large), the Kolmogorov–Smirnov values appear to be essentially independent of the actual cross-ownership structure (cf. Figure 3.2(a) and (d), and Figure 3.3(a) and (d)). Note that under cross-ownership of debt only we had to extend the range of values of d/a (cf. (3.45)) by the additional values $\{4, 5, \ldots, 19, 20\}$ to observe this vanishing effect (cf. Figure 3.2(d)). These findings can be justified as follows. First, note that the Suzuki areas and their probabilities change with $d/a = d_1 = d_2$ and σ^2 (recall that $E(A_i)$ is independent of σ^2, whereas its variance increases with σ^2 increasing). If d/a is chosen relatively small compared to σ^2, $P(A_{\mathrm{ss}})$ is close to one and we have $V_1^{\mathrm{d}} \approx A_1$ (cf. (3.33)), i.e. V_1^{d} is approximately lognormally distributed, and we obtain a small Kolmogorov–Smirnov statistic, independently of the exact levels of cross-ownership. Furthermore, we have $V_1^{\mathrm{e}} \approx (A_1 + M_{1,2}^{\mathrm{e}} A_2 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_1 - M_{1,2}^{\mathrm{e}} d_2)/(1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}})$ (cf. (3.34)) for small $d/a = d_1 = d_2$, i.e. the constants can be neglected, which means we are roughly in the situation of Section A.2.2. As becomes evident in Figure A.3, $(A_1 + M_{1,2}^{\mathrm{e}} A_2)/(1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}})$ can be well approximated with a lognormal distribution by the Fenton–Wilkinson method, as long as σ^2 does not exceed a certain level. However, we could not find an intuitive explanation why the quality of the approximation does not depend on the cross-ownership fractions. If d/a is chosen large compared to σ^2, so that $P(A_{\mathrm{dd}})$ is close to one, we have $V_1^{\mathrm{d}} \approx (A_1 + M_{1,2}^{\mathrm{d}} A_2)/(1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})$ and $V_1^{\mathrm{e}} \approx A_1$, and we can reason as above.
As it was to be expected, both models yield (rounded) estimated default probabilities of 0 resp. 1 if $d/a = d_1$ is relatively small resp. large compared to σ^2, and the estimated relative risk ratios $\widehat{\mathrm{RR}}$ are 1 in such scenarios. However, note that the theoretical probabilities of default under either model can never take a value of exactly 0 or exactly 1, since we assume exogenous asset values to follow a lognormal distribution and due to $d_1 > 0$. Hence, even if d/a is chosen very small or large, the theoretical risk ratio is probably different from 1, but our short simulations cannot reveal whether we have to expect the lognormal model to over- or underestimate the actual risk in such scenarios. For very high levels of cross-ownership, the results of Section 4.1 will offer more insight.

For "medium" values of d/a the value of the Kolmogorov–Smirnov statistic seems to depend on the realized level of cross-ownership, and in most cases, this relationship was of the type as in Figure 3.3(b) and (c): the higher the levels of cross-ownership, the bigger the discrepancies between Suzuki's model and the lognormal model. Under cross-ownership of debt only we also observed scenarios where the biggest discrepancies were not obtained for the highest levels of cross-ownership, but lower ones (cf. Figure 3.2(c)). However, an intuitive explanation for this effect is not available. Similarly, the relative sizes of $\hat{p}_{\mathrm{S}}$ and $\hat{p}_{\mathrm{LOG}}$ differ most for high levels of cross-ownership, which can be explained by the fact that for low levels of cross-ownership, the endogenous part of a firm's assets can be neglected, i.e. its total asset value is approximately lognormally distributed, and the estimated probabilities of default obtained from the two models roughly coincide. Under cross-ownership of debt only, the values of $\widehat{\mathrm{RR}}$ were non-decreasing in the considered levels of cross-ownership (cf. Figure 3.4(a) for an example), i.e. for scenarios with high levels of cross-ownership (and d/a chosen appropriately in the sense explained earlier) we always obtained risk ratios greater than 1, which means that the lognormal



(a) (b)

Figure 3.4: Estimated relative risk $\widehat{\mathrm{RR}}$ in dependence of the cross-ownership fractions; (a) XOS of debt only, $\sigma^2 = 1.60944$, d/a=0.4; (b) XOS of equity only, $\sigma^2 = 0.22314$, $d/a = 0.7$.

distribution overestimates the actual probability of default in these scenarios. Under cross-ownership of equity only we observed the opposite effects, i.e. roughly speaking, the higher the cross-ownership fractions, the smaller the obtained values of $\widehat{\mathrm{RR}}$. These values tended to be bigger than 1 or about 1 if $M^{\mathrm{e}}_{1,2}$ was small. For $M^{\mathrm{e}}_{1,2}$ and $M^{\mathrm{e}}_{2,1}$ close to 1, we observed relative risks close to 0, i.e. the lognormal model then underestimates the actual probability of default. An example is given in Figure 3.4(b).

To summarize, we can state that if the quality of the lognormal approximation is poor, the biggest discrepancies between the actual distribution of firm values under cross-ownership and the matched lognormal distribution occur for high levels of cross-ownership. In our simulations, the maximal Kolmogorov–Smirnov values were 0.57 under cross-ownership of debt only and 0.34 under cross-ownership of equity only. This transfers to the estimated probabilities of default, with cross-ownership of debt only and cross-ownership of equity only having opposed effects. If the system is likely to be always solvent or to be always in default, the lognormal approximation works well, provided that σ^2 is not too big. In this case, also the estimated probabilities of default are roughly the same. Our results show that a careful estimation of the parameters of the distribution of exogenous asset values and the network structure is necessary in order to be able to assess the potential consequences of not taking the realized cross-ownership structure into account properly and treating the sum of asset values of a firm as lognormally distributed.

Qualitative Comparison of Distribution Functions

Based on the insight that in most of the considered scenarios the empirical distribution function of the firm value of firm 1 differs most from the matched lognormal distribution function if the corresponding cross-ownership fractions are close to 1, we

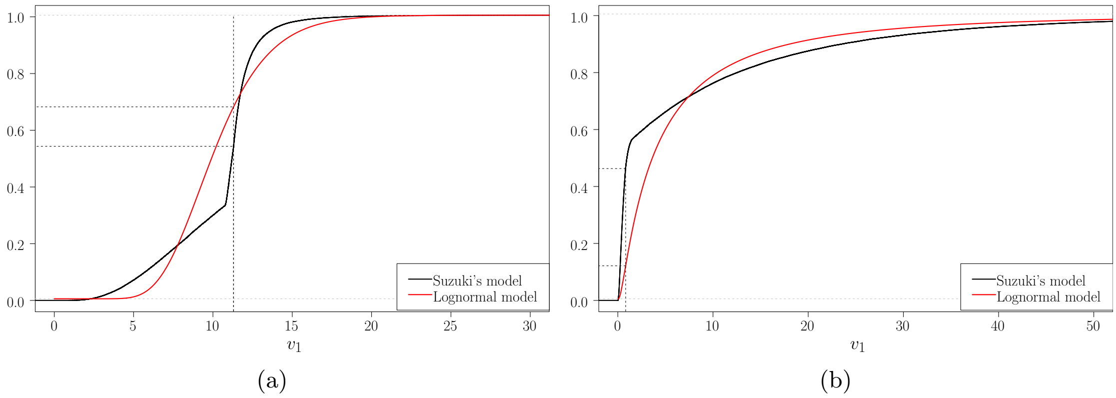


Figure 3.5: Empirical distribution function of firm values V_1 under Suzuki's model and matched lognormal distribution function with corresponding probabilities of default; the dotted vertical lines refer to $v_1 = d_1 = d/a$; $\sigma^2 = 1$; (a) XOS of debt only, $M^{\mathrm{d}}_{1,2} = M^{\mathrm{d}}_{2,1} = 0.95$, $d/a = 11.3$, KS $= 0.28$, $\hat{p}_{\mathrm{S}} = 0.53758$, $\hat{p}_{\mathrm{LOG}} = 0.67671$; (b) XOS of equity only, $M^{\mathrm{e}}_{1,2} = M^{\mathrm{e}}_{2,1} = 0.95$, $d/a = 0.8$, KS $= 0.35$, $\hat{p}_{\mathrm{S}} = 0.45733$, $\hat{p}_{\mathrm{LOG}} = 0.11582$.

fixed these fractions to 0.95 and then selected the value of d/a leading to the highest Kolmogorov–Smirnov values. Under cross-ownership of equity only we considered $d/a \in \{0.1, 0.2, \ldots, 10\}$, under cross-ownership of debt only we also considered higher levels of liabilities (up to 15) in order to be able to identify the value of d/a yielding the highest Kolmogorov–Smirnov statistic. For every level of d/a we simulated 100,000 values of (A_1, A_2) according to (3.44) with $\sigma^2 = 1$ to obtain $\hat{F}_{V_1}$ and from that a Kolmogorov–Smirnov statistic representing the discrepancy to the matched lognormal distribution.

Figure 3.5 shows the pairs of distribution functions $\hat{F}_{V_1}$ and $\tilde{F}_{\mathrm{LOG}}$ exhibiting the highest Kolmogorov–Smirnov values (XOS of debt only: 0.28, XOS of equity only: 0.35). Intuitively, one might think that the bend of the distribution functions $\hat{F}_{V_1}$ results from the section-wise definition of V_1 (cf. (3.23)). However, this is not the case, because there is no "total" order of firm values with respect to the four Suzuki areas A_{ss}, A_{sd}, A_{ds} and A_{dd}, we only have

$$V_1\big|_{A_{\mathrm{dd}} \cup A_{\mathrm{ds}}} < d_1 \le V_1\big|_{A_{\mathrm{sd}} \cup A_{\mathrm{ss}}} \tag{3.50}$$

(cf. (3.29)–(3.32)), and as becomes clear from the dotted lines referring to $v_1 = d/a = d_1$ in Figure 3.5, the bends do not occur on the boundary between $A_{\mathrm{dd}} \cup A_{\mathrm{ds}}$ and $A_{\mathrm{sd}} \cup A_{\mathrm{ss}}$. As we will see in Remark 4.2, for $M^{\mathrm{d}}_{1,2}$ and $M^{\mathrm{d}}_{2,1}$ sufficiently big, the distribution functions F_{V_1} and F_{LOG} always intersect at least twice under cross-ownership of debt only, as it is the case in Figure 3.5(a) for $\hat{F}_{V_1}$ and $\tilde{F}_{\mathrm{LOG}}$.

In consistence with Figure 3.4, the lognormal model over- resp. underestimates the actual probability of default under cross-ownership of debt only resp. cross-ownership of equity only in Figure 3.5. This was the case for all considered values of $d/a = d_1$

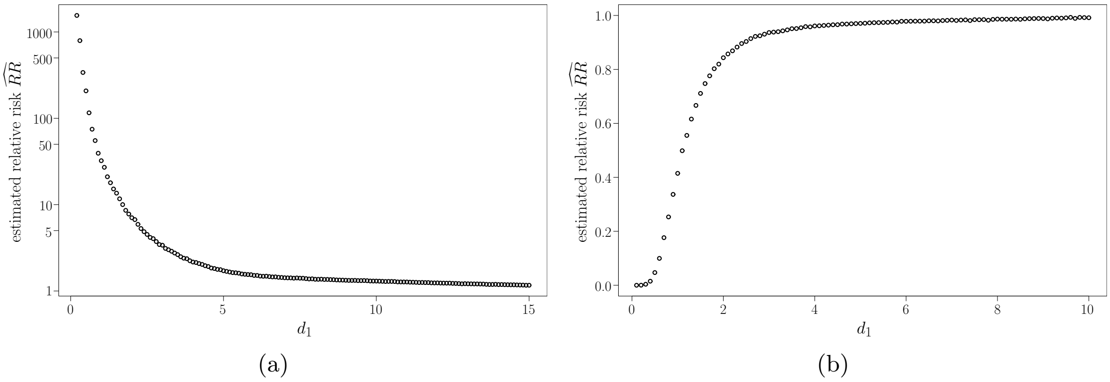


Figure 3.6: Estimated relative risk ratio $\widehat{RR}$ in dependence of d_1, $\sigma^2 = 1$; (a) XOS of debt only, $M^{\mathrm{d}}_{1,2} = M^{\mathrm{d}}_{2,1} = 0.95$; (b) XOS of equity only, $M^{\mathrm{e}}_{1,2} = M^{\mathrm{e}}_{2,1} = 0.95$.

(cf. Figure 3.6). As we will see in Section 4.1, this is no coincidence. However, we do not have an intuitive explanation why the discrepancy between the probabilities of default obtained from Suzuki's model and the lognormal model (expressed in terms of $\widehat{\mathrm{RR}}$) is biggest for small values of $d/a = d_1$ for both types of cross-ownership in Figure 3.6. For $d/a = 0.1$ the corresponding Kolmogorov-Smirnov statistic was 0.047 under cross-ownership of debt only and 0.021 under cross-ownership of equity only, showing that despite the relatively high quality of the lognormal approximation in terms of the maximum discrepancy between the two distribution functions, the relative sizes of the corresponding probabilities of default can strongly differ.

3.2.2.3 Results for XOS of both, Debt and Equity

In departure from the results of Section 3.2.2.2, we cannot state that the higher the amounts of cross-ownership, the bigger the discrepancy to the lognormal model in systems of two firms linked by cross-ownership of both, debt and equity. If for example $M^{\mathrm{d}}_{1,2} = M^{\mathrm{e}}_{1,2}$ and $M^{\mathrm{d}}_{2,1} = M^{\mathrm{e}}_{2,1}$, i.e. if firm i ($i = 1, 2$) holds identical fractions of firm j's debt and equity ($j = 1, 2,\ j \neq i$), the constants in (3.23) vanish and V_1 equals $(A_1 + M^{\mathrm{d}}_{1,2}A_2)/(1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})$. As becomes evident in Figure 3.7(a) and Figure A.3, such a sum can be well approximated by a lognormal distribution, provided that the variance σ^2 is not too big. Instead, (3.23) suggests the assumption that the discrepancy between the actual distribution of firm values and the matching lognormal distribution become largest if there is a big difference between the cross-ownership fractions of debt and equity. In this case, the constants in (3.23) in the first three Suzuki areas become relatively large (all other parameters held fixed), which means that in these areas V_1 is a weighted sum of lognormally distributed exogenous asset values shifted by some constants. Although we have no shift on A_{dd}, it is plausible that the stronger the shift in the remaining areas, the worse the fit to the (unshifted) lognormal distribution.
As it was to be expected, under cross-ownership of both, debt and equity, the shape

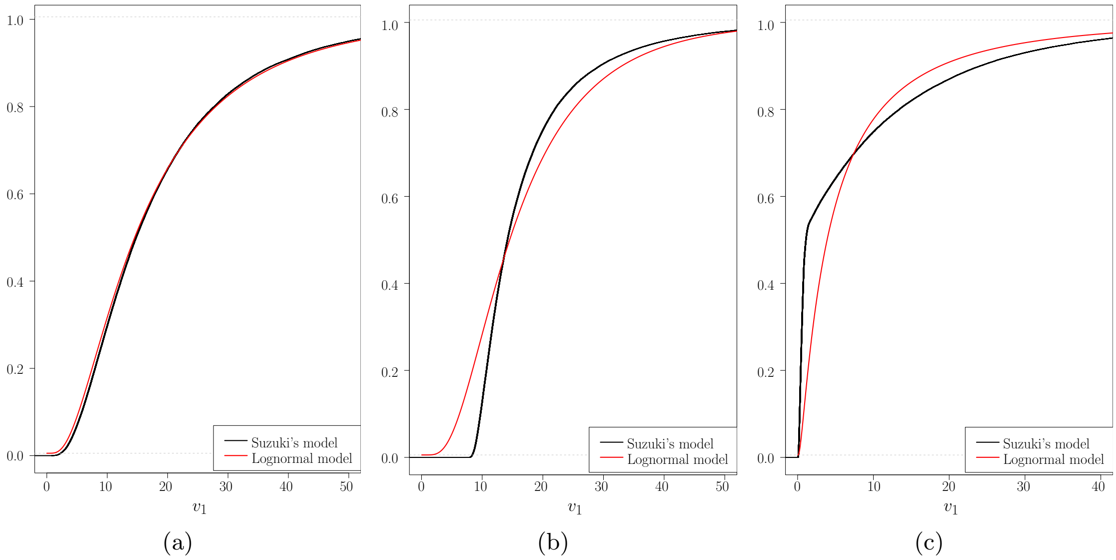


Figure 3.7: Empirical distribution function of V_1 under Suzuki's model and matched lognormal distribution function under cross-ownership of both, debt and equity, $\sigma^2 = 1$, $d/a = 0.8$; (a) balanced levels of XOS of debt and XOS of equity ($M^{\mathrm{d}}_{1,2} = M^{\mathrm{d}}_{2,1} = M^{\mathrm{e}}_{1,2} = M^{\mathrm{e}}_{2,1} = 0.95$), KS $= 0.02$; (b) high level of XOS of debt, low level of XOS of equity ($M^{\mathrm{d}}_{1,2} = M^{\mathrm{d}}_{2,1} = 0.95$, $M^{\mathrm{e}}_{1,2} = M^{\mathrm{e}}_{2,1} = 0.05$), KS $= 0.19$; (c) low level of XOS of debt, high level of XOS of equity ($M^{\mathrm{d}}_{1,2} = M^{\mathrm{d}}_{2,1} = 0.05$, $M^{\mathrm{e}}_{1,2} = M^{\mathrm{e}}_{2,1} = 0.95$), KS $= 0.35$.

of the distribution function of V_1 and the associated shortcomings of the lognormal approximation strongly depend on which type of cross-ownership is dominant in the system of firms (cf. Figure 3.7(b) and (c)). Obviously, the actual distribution of firm values under cross-ownership of both, debt and equity, can have both, a heavier and lighter head portion than the lognormal distribution. Similarly, the lognormal model can under- and overestimate the actual probability of default.

3.2.3 Simulation Study on the Value-at-Risk

In addition to our results on the distribution of firm values and the related probabilities of default, we consider the Value-at-Risk (VaR) of firm 1 linked to firm 2 by cross-ownership as well as the VaR of a portfolio (e.g. a bank portfolio) consisting of two indices representing the values of firm 1 and firm 2. According to McNeil et al. [2005], VaR "is probably the most widely used risk measure in financial institutions and has also made its way into the Basel II capital-adequacy framework" [McNeil et al., 2005, p. 37], hence we preferred it over other risk measures, although it has certain theoretical deficiencies [McNeil et al., 2005], for example it is not subadditive (cf. Axiom 6.2 of McNeil et al. [2005]). For its calculation, we proceeded as follows. In (3.44), no time horizon is specified, but if we assume exogenous asset values to follow a geometric Brownian motion with drift 0, (3.44) can be interpreted as the physical distribution of exogenous asset values with expectation a at maturity $t = T = 1$ and starting value a at time $t = 0$. If we further assume the force of interest r to be 0, (3.44) also is the risk-neutral distribution of the exogenous asset values at maturity, i.e. there is no change of measure and our σ^2 is identical to σ^2 in the Black–Scholes model. This special set-up will be used for our Value-at-Risk considerations. As the risk-neutral price of firm i's debt and equity (= firm value) at time $t = 0$ is then given by the expectation $E(V_i)$ (as there is no discounting and no change in measure), we consider the negative α-quantile of $V_i - E(V_i)$ as the $(1-\alpha)100\%$-VaR of the change in firm i's value, i.e.

$$\mathrm{VaR}_{i,1-\alpha} := -F^{-1}_{V_i - E(V_i)}(\alpha) = E(V_i) - F^{-1}_{V_i}(\alpha), \quad i = 1, 2. \tag{3.51}$$

Similarly, for the VaR of the portfolio,

$$\mathrm{VaR}_{\mathrm{P},1-\alpha} := -F^{-1}_{V_1+V_2-E(V_1+V_2)}(\alpha) = E(V_1) + E(V_2) - F^{-1}_{V_1+V_2}(\alpha). \tag{3.52}$$

For a set of m realizations $(v_{1,j}, v_{2,j})$ $(j = 1, \ldots, m)$ of (V_1, V_2) with bivariate empirical distribution function $\hat{F}_{V_1,V_2}$ we calculate the corresponding empirical VaRs as

$$\widehat{\mathrm{VaR}}_{i,1-\alpha} := \frac{1}{m}\sum_{j=1}^{m} v_{1,j} - \hat{F}^{-1}_{V_i}(\alpha), \quad i = 1, 2, \tag{3.53}$$

$$\widehat{\mathrm{VaR}}_{\mathrm{P},1-\alpha} := \frac{1}{m}\sum_{j=1}^{m}(v_{1,j} + v_{2,j}) - \hat{F}^{-1}_{V_1+V_2}(\alpha), \tag{3.54}$$

where the empirical quantiles were computed with the default method of the function 'quantile' implemented in R 2.12.0.

In our simulations we set $\alpha = 0.01$. The parameter $d/a = d_1 = d_2$ and the cross-ownership fractions were as in Section 3.2.2.1, whereas σ^2 was set to 0.09, as a somewhat realistic choice in conjunction with $T = 1$. For every scenario, 100,000 values of (V_1, V_2) were simulated. From them we calculated the estimated VaRs of firm 1 and of the portfolio as in (3.53)–(3.54). In order to obtain VaRs under the lognormal model, we fitted a bivariate lognormal distribution to $\hat{F}_{V_1,V_2}$ by the Fenton–Wilkinson method, imposed with the additional condition that the covariances of the two distributions coincide. Let (W_1, W_2) follow this bivariate lognormal distribution. In analogy to (3.53) and (3.54) the 99%-VaR of firm 1 and of the portfolio of firm 1 and firm 2 under the lognormal model were estimated from 100,000 simulated values of (W_1, W_2). Note that this approach differs from the approach of Section 3.2.2.1, where we directly used the theoretical CDF $\tilde{F}_{\mathrm{LOG}}$ instead of simulating values from it. Since the sum of lognormals is not lognormal in general, the α-quantile of $W_1 + W_2$ needed for the VaR of the portfolio under the lognormal model must be estimated from simulated values of W_1 and W_2, so we used the same rationale for the VaR under the lognormal model of a single firm for better comparability. Due to the Strong Law of Large Numbers, we will interpret all results in terms of the theoretical VaRs instead of the empirical VaRs for both, Suzuki's model and the lognormal model.

Let us first consider the results for the VaR of firm 1. Roughly speaking, for fixed cross-ownership fractions, the 99%-VaRs of firm 1 were non-decreasing in d/a under cross-ownership of debt only and non-increasing in d/a under cross-ownership of equity only. After extending the simulation under cross-ownership of debt only to values of d/a from 0.1-7, it became clear that the 99%-VaR as a function of d/a has a sigmoid shape. For small values of d/a under cross-ownership of debt only and big values of d/a under cross-ownership of equity only (cf. Figure 3.8(a) and (f)), the 99%-VaRs are independent of the established level of cross-ownership, which can be explained as follows. In the former case, firm 1 is likely to be solvent, i.e. $V_1^{\mathrm{d}} \approx A_1 + M_{1,2}^{\mathrm{d}} d_2$, where $d_2 = d/a$ is small, i.e. changes in $M_{1,2}^{\mathrm{d}}$ hardly matter. In the latter case, firm 1 is likely to be in default, and it follows that $V_1^{\mathrm{e}} \approx A_1$, which is independent of any cross-ownership fraction. Under cross-ownership of equity only, the 99%-VaR was non-decreasing in $M_{1,2}^{\mathrm{e}}$ and $M_{2,1}^{\mathrm{e}}$ for all considered values of d/a (cf. Figure 3.8(d)–(f) for some examples). This means that the higher the level of cross-ownership, the bigger the loss in value of firm 1 over time exceeded in the worst 1% scenarios. In contrast to that, very high levels of cross-ownership of debt only can reduce the 99%-VaR (cf. Figure 3.8(c)), i.e. under cross-ownership of debt only we cannot state that the tighter the cross-ownership structure the bigger the 99%-VaR.

For the comparison of the Values-at-Risk under Suzuki's model and the lognormal model we calculated the difference of the estimated VaRs under the lognormal model and Suzuki's model, i.e. positive differences indicate an overestimation of the actual VaR by the lognormal model, whereas negative differences stand for an underestimation. Some examples can be found in Figure 3.9. Under cross-ownership of equity only with a high level of d/a, we are likely to be in A_{dd}, where $V_1^{\mathrm{e}} = A_1$ by (3.33), i.e. both models yield the same VaR (cf. Figure 3.9(f)). Otherwise, any effect is possible under both, cross-ownership of debt only and cross-ownership of equity only, and it is difficult to derive general conclusions. Whether the lognormal model will over- or underestimate the actual 99%-VaR of firm 1 strongly depends on the underlying parameters, which therefore need to be analyzed carefully.
The results for the 99%-VaR of the portfolio under Suzuki's model were qualitatively the same as for the 99%-VaR of firm 1. Also over- and underestimation, expressed as the difference of VaRs between the two models, was similar.

Since the VaR is in general not subadditive, we cannot expect the VaR of the portfolio to be smaller or equal to the sum of the VaRs of firm 1 and firm 2. However, in only four of the 8100 considered scenarios, the 99%-VaR of the portfolio was greater than the sum of the 99%-VaRs of firm 1 and firm 2, i.e. in 99.95% of the scenarios, the 99%-VaR was subadditive. In the four scenarios without subadditivity, which all occurred under cross-ownership of equity only, we had $(M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}}) \in \{(0.8, 0.9), (0.9, 0.8), (0.9, 0.9)\}$, so the question arises whether the VaR of the portfolio can become arbitrarily high under a suitable cross-ownership structure. For fixed cross-ownership fractions, McNeil et al.

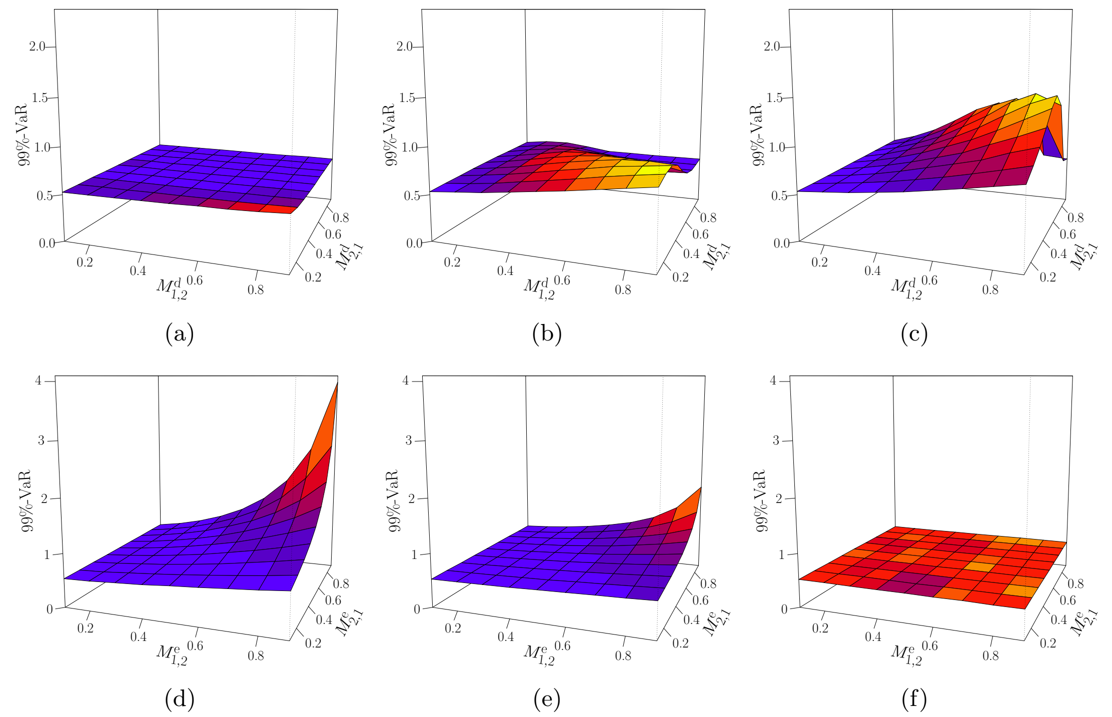


Figure 3.8: 99%-VaR of firm 1 under Suzuki's model, $\sigma^2 = 0.09$; (a) XOS of debt only, $d/a = 0.9$; (b) XOS of debt only, $d/a = 2$; (c) XOS of debt only, $d/a = 4$; (d) XOS of equity only, $d/a = 0.6$; (e) XOS of equity only, $d/a = 0.9$; (f) XOS of equity only, $d/a = 2$.

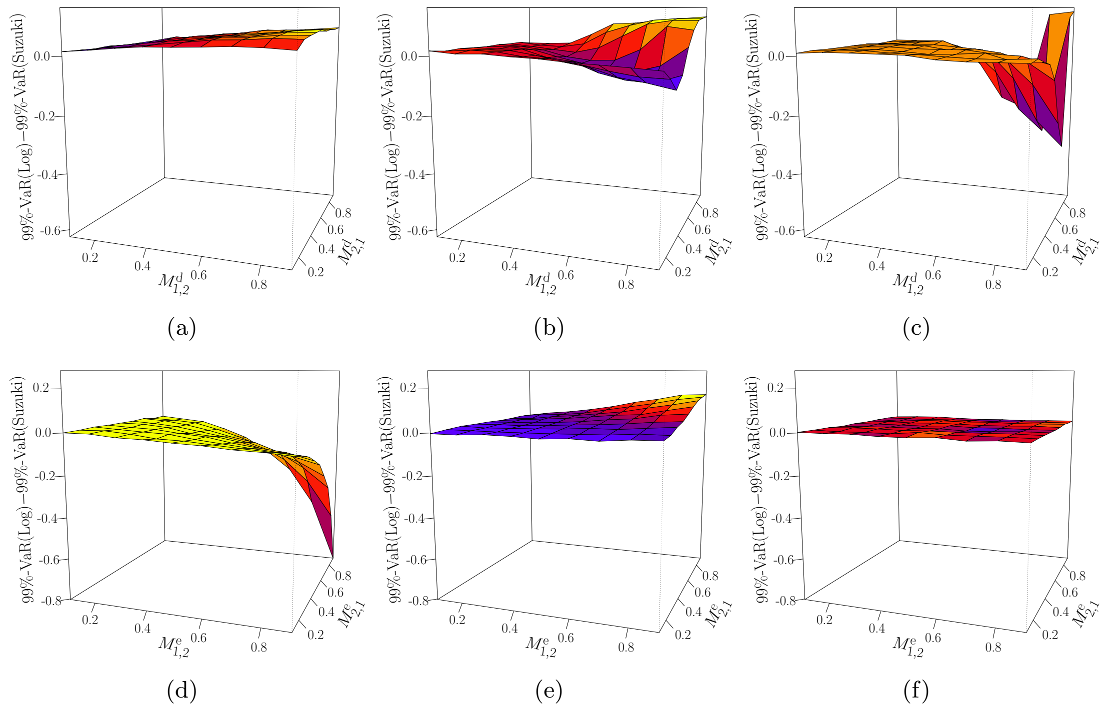


Figure 3.9: Difference of 99%-VaR of firm 1 under Suzuki's model and the lognormal model, $\sigma^2 = 0.09$; (a) XOS of debt only, $d/a = 0.9$; (b) XOS of debt only, $d/a = 2$; (c) XOS of debt only, $d/a = 4$; (d) XOS of equity only, $d/a = 0.6$; (e) XOS of equity only, $d/a = 0.9$; (f) XOS of equity only, $d/a = 2$.

[2005] provide an upper bound for the VaR of a portfolio consisting of $n \geq 2$ firms. In our set-up, the loss in value of a single firm over time equals $-(V_i - E(V_i))$ (recall that in our set-up, $E(V_i)$ is the risk-neutral price of firm 1 in $t = 0$), and the inverse distribution function of this loss evaluated at $u \in (0, 1)$ equals $E(V_i) - F_{V_i}^{-1}(1 - u)$. Hence, by equations (6.13) and (6.15) of McNeil et al. [2005] and the appendant remarks therein, an upper bound for the $(1 - \alpha)100\%$-VaR of a portfolio consisting of n firms linked by cross-ownership is

$$\inf_{\substack{(u_1,\ldots,u_n)\in[0,1]^n,\\ W_n(u_1,\ldots,u_n)=1-\alpha}} \sum_{i=1}^{n} \left(E(V_i) - F_{V_i}^{-1}(1 - u_i)\right) = \inf_{\substack{(u_1,\ldots,u_n)\in[0,\alpha]^n,\\ \sum_{i=1}^n u_i=\alpha}} \sum_{i=1}^{n} \left(E(V_i) - F_{V_i}^{-1}(u_i)\right), \tag{3.55}$$

where $W_n(u_1, \ldots, u_n) := \max\{0, \sum_{i=1}^n u_i - n + 1\}$ is the lower Fréchet-Hoeffding bound (cf. p. 47 of Nelsen [2006] for example; for $n = 2$, W_n coincides with W in (5.5)). Note that our definition of α corresponds to $1 - \alpha$ of McNeil et al. [2005]. By Theorem 6.19 of McNeil et al. [2005], the upper bound (3.55) is sharp for $n = 2$.
Since the sum of the $(1 - \alpha)100\%$-VaRs of the n firms equals $\sum_{i=1}^n (E(V_i) - F_{V_i}^{-1}(\alpha))$ by (3.51), an upper bound for the difference between the $(1 - \alpha)100\%$-VaR of the portfolio and the sum of the $(1 - \alpha)100\%$-VaRs is given by

$$\inf_{\substack{(u_1,\ldots,u_n)\in[0,\alpha]^n,\\ \sum_{i=1}^n u_i=\alpha}} \sum_{i=1}^{n} (F_{V_i}^{-1}(\alpha) - F_{V_i}^{-1}(u_i)) < \sum_{i=1}^{n} F_{V_i}^{-1}(\alpha) - \max_{1\leq i\leq n} F_{V_i}^{-1}(\alpha), \tag{3.56}$$

which reduces to $\min\{F_{V_1}^{-1}(\alpha), F_{V_2}^{-1}(\alpha)\}$ for $n = 2$.

4 More on Probabilities of Default under Cross-Ownership

Most of the results of this sections have been published in Karl and Fischer [2014], with some parts taken over unchanged.

4.1 Limiting Probability of Default

In our simulations of Section 3.2.2.2 we saw that if the two firms have established a high level of cross-ownership, the two types of cross-ownership seem to have opposite effects on the probabilities of default obtained under the lognormal model, compared to Suzuki's model. Unfortunately, we cannot calculate the exact values under either model, because we can determine neither the distribution of V_1, nor its first and second moments in closed form. Hence, we cannot justify our findings theoretically. However, the situation becomes tractable, if we let the cross-ownership fractions converge to 1. In this case, both, the Suzuki areas in (3.25)–(3.28) and the formula of V_1 simplify, which makes an analytical approach possible. In the following we consider the "limiting" probability of default of firm 1 resulting from both, Suzuki's model and the corresponding matching lognormal model. This will be done separately for cross-ownership of debt only and cross-ownership of equity only.

As in our simulations, we assume exogenous asset values to be lognormally distributed, i.e.

$$(A_1, A_2) \sim \mathcal{LN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{4.1}$$

with $\boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$ and $\boldsymbol{\Sigma} = \left(\begin{smallmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{smallmatrix}\right)$, but we do not impose any restrictions on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, except that $\sigma_1, \sigma_2 > 0$. Since we are mainly concerned with the probability of default of firm 1, we set

$$\mu := \mu_1, \quad \sigma := \sigma_1. \tag{4.2}$$

In contrast to our simulations, we do not confine ourselves to the case of $d_1 = d_2$, we only assume $d_1, d_2 > 0$ in order to exclude degenerate cases.

4.1.1 XOS of Debt only

For the comparison of the limiting probabilities of default under Suzuki's model and the lognormal model we first determine the pointwise limit of V_1^{d} for $M_{1,2}^{\mathrm{d}}$ and $M_{2,1}^{\mathrm{d}}$

converging to 1, which we will call $V_1^{\mathrm{d}^*}$. Based on (3.33) we can write

$$\begin{aligned} V_1^{\mathrm{d}} = & 1_{A_{\mathrm{ss}}}(A_1, A_2) \times (A_1 + M_{1,2}^{\mathrm{d}} d_2) \\ & + 1_{A_{\mathrm{sd}}}(A_1, A_2) \times (A_1 + M_{1,2}^{\mathrm{d}} A_2 + M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_1) \\ & + 1_{A_{\mathrm{ds}}}(A_1, A_2) \times (A_1 + M_{1,2}^{\mathrm{d}} d_2) \\ & + 1_{A_{\mathrm{dd}}}(A_1, A_2) \times (A_1 + M_{1,2}^{\mathrm{d}} A_2)/(1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}), \end{aligned} \tag{4.3}$$

where 1_A stands for the indicator function of a set A. For the determination of the pointwise limit of V_1^{d} for $M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1$, we first consider the limits of the indicator functions in (4.3). By Lemma A.3, their pointwise limits exist and we set

$$\lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} 1_{A_{ij}} =: 1_{A_{ij}^*}, \quad ij \in \{\mathrm{ss}, \mathrm{sd}, \mathrm{ds}, \mathrm{dd}\}, \tag{4.4}$$

with $A_{\mathrm{dd}}^* = \{(0,0)\}$ by Lemma A.4. Hence, $P(A_{\mathrm{dd}}^*) = 0$ and

$$\begin{aligned} V_1^{\mathrm{d}^*} := \lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} V_1^{\mathrm{d}} = & 1_{A_{\mathrm{ss}}^*}(A_1, A_2) \times (A_1 + d_2) \\ & + 1_{A_{\mathrm{sd}}^*}(A_1, A_2) \times (A_1 + A_2 + d_1) \\ & + 1_{A_{\mathrm{ds}}^*}(A_1, A_2) \times (A_1 + d_2) \qquad P\text{-a.s.} \end{aligned} \tag{4.5}$$

Since almost sure convergence implies convergence in distribution, the limiting probability of default of firm 1 under Suzuki's model equals

$$\lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} P(V_1^{\mathrm{d}} < d_1) = P(V_1^{\mathrm{d}^*} < d_1). \tag{4.6}$$

In order to determine the latter probability of default, we have to distinguish between the following two cases.

If $d_1 \leq d_2$, it follows from (4.5) that $V_1^{\mathrm{d}^*} > d_1$ P-a.s. and thus $P(V_1^{\mathrm{d}^*} < d_1) = 0$. Because of $d_1 > 0$ and $\sigma_1 > 0$ it is clear that any lognormal distribution would yield a strictly positive probability of default, i.e. the lognormal model overestimates the actual risk of a firm under cross-ownership of debt only if the cross-ownership fractions converge to 1. Recall that we had $d_1 = d_2$ in our simulations. In Figure 3.4(a) we saw that under cross-ownership of debt only, the actual risk was considerably overestimated already for cross-ownership fractions about 0.7.

If $d_1 > d_2$, the situation is somewhat trickier. Equation (4.5) and Lemma A.4 now yield $V_1^{\mathrm{d}^*} = A_1 + d_2$ P-a.s., i.e. $P(V_1^{\mathrm{d}^*} < d_1) > P(V_1^{\mathrm{d}^*} < d_2) = 0$ and the argumentation used for the case $d_1 \leq d_2$ cannot be applied. Instead, let

$$W_1^{\mathrm{d}^*} \sim \mathcal{LN}(\tilde{\mu}, \tilde{\sigma}^2), \tag{4.7}$$

where $\tilde{\mu}$ and $\tilde{\sigma}^2$ are determined such that $E(W_1^{\mathrm{d}^*}) = E(V_1^{\mathrm{d}^*}) = E(A_1) + d_2$ and

$\mathrm{Var}(W_1^{\mathrm{d}^*}) = \mathrm{Var}(V_1^{\mathrm{d}^*}) = \mathrm{Var}(A_1)$, i.e.

$$\tilde{\mu} = \frac{1}{2}\ln\left(\frac{(E(A_1)+d_2)^4}{\mathrm{Var}(A_1)+(E(A_1)+d_2)^2}\right) > \frac{1}{2}\ln\left(\frac{E(A_1)^4}{\mathrm{Var}(A_1)+E(A_1)^2}\right) = \mu, \tag{4.8}$$

$$\tilde{\sigma}^2 = \ln\left(\frac{\mathrm{Var}(A_1)}{(E(A_1)+d_2)^2}+1\right) < \ln\left(\frac{\mathrm{Var}(A_1)}{E(A_1)^2}+1\right) = \sigma^2. \tag{4.9}$$

Then we have the following limiting probabilities of default:

$$P(V_1^{\mathrm{d}^*} < d_1) = P(A_1 < d_1 - d_2) = \Phi\left(\frac{\ln(d_1-d_2)-\mu}{\sigma}\right), \tag{4.10}$$

$$P(W_1^{\mathrm{d}^*} < d_1) = \Phi\left(\frac{\ln(d_1)-\tilde{\mu}}{\tilde{\sigma}}\right), \tag{4.11}$$

i.e. for $d_1 > d_2$, in the limit the lognormal model overestimates the actual risk if and only if

$$\frac{\ln(d_1-d_2)-\mu}{\sigma} < \frac{\ln(d_1)-\tilde{\mu}}{\tilde{\sigma}} \tag{4.12}$$

$$\Leftrightarrow \quad \tilde{\sigma}\ln(d_1-d_2) - \sigma\ln(d_1) < \tilde{\sigma}\mu - \sigma\tilde{\mu} \tag{4.13}$$

$$\Leftrightarrow \quad \frac{(d_1-d_2)^{\tilde{\sigma}}}{(d_1)^{\sigma}} < \exp(\tilde{\sigma}\mu - \sigma\tilde{\mu}). \tag{4.14}$$

Straightforward calculations show that the LHS of (4.14) as a function of d_1 has a maximum value of

$$\frac{\left(\frac{\tilde{\sigma}}{\sigma-\tilde{\sigma}}d_2\right)^{\tilde{\sigma}}}{\left(\frac{\sigma}{\sigma-\tilde{\sigma}}d_2\right)^{\sigma}} =: \mathrm{LHS}_{\max} \tag{4.15}$$

taken in $d_1 = \frac{\sigma}{\sigma-\tilde{\sigma}}d_2 =: d_{1,\max} > d_2$, where the inequality holds because of $\sigma > \tilde{\sigma}$. Furthermore,

$$\lim_{d_1\to d_2^+}\frac{(d_1-d_2)^{\tilde{\sigma}}}{(d_1)^{\sigma}} = 0, \quad \lim_{d_1\to\infty}\frac{(d_1-d_2)^{\tilde{\sigma}}}{(d_1)^{\sigma}} = 0, \tag{4.16}$$

which implies that the LHS of (4.14) is a bell-shaped, continuous function of d_1 with domain (d_2,∞) and maximum value $\mathrm{LHS}_{\max}$ taken in $d_1 = d_{1,\max}$. Note that the RHS of (4.14) is independent of d_1. It can be shown (cf. Lemma A.5) that

$$\mathrm{LHS}_{\max} = \frac{\left(\frac{\tilde{\sigma}}{\sigma-\tilde{\sigma}}d_2\right)^{\tilde{\sigma}}}{\left(\frac{\sigma}{\sigma-\tilde{\sigma}}d_2\right)^{\sigma}} > \exp(\tilde{\sigma}\mu - \sigma\tilde{\mu}), \tag{4.17}$$

independently of the exact values of μ, σ and d_2, which means that the maximum value of the LHS of (4.14) as a function of d_1 is always greater than the constant $\exp(\tilde{\sigma}\mu - \sigma\tilde{\mu})$.

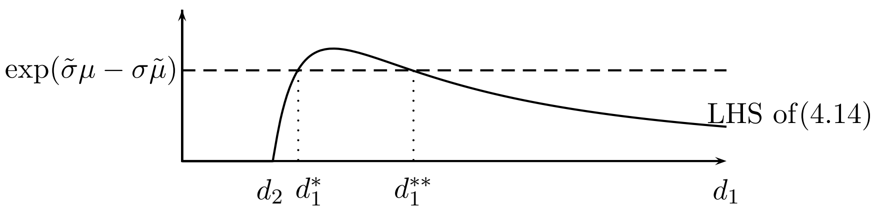


Figure 4.1: Sketch of the LHS of (4.14) as a function of d_1.

Thus (cf. Figure 4.1), there are two values d_1^* and d_1^{**}, $d_2 < d_1^* < d_{1,\max} < d_1^{**}$, such that

$$\frac{(d_1^* - d_2)^{\tilde{\sigma}}}{(d_1^*)^{\sigma}} = \frac{(d_1^{**} - d_2)^{\tilde{\sigma}}}{(d_1^{**})^{\sigma}} = \exp(\tilde{\sigma}\mu - \sigma\tilde{\mu}), \tag{4.18}$$

i.e. (4.14) holds if and only if $d_2 < d_1 < d_1^*$ or $d_1 > d_1^{**}$. In these cases, the lognormal model overestimates the actual risk, if the cross-ownership fractions of debt converge to 1.

Our case differentiation with respect to the relative sizes of d_1 and d_2 can be summarized as follows.

Proposition 4.1. *Under cross-ownership of debt only with lognormally distributed exogenous asset values, the lognormal model underestimates the actual limiting probability of default of a firm, i.e.*

$$\lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} P(V_1^{\mathrm{d}} < d_1) > P(W_1^{\mathrm{d}^*} < d_1), \tag{4.19}$$

if and only if

$$d_1^* < d_1 < d_1^{**}, \tag{4.20}$$

with d_1^* and d_1^{**} given by (4.18)*. In particular, if $d_1 \le d_2$, the actual limiting default probability of 0 is overestimated by the lognormal model.*

Remark 4.2. If we replace d_1 in (4.10)–(4.11) with an arbitrary $v_1 > d_2$, our analysis shows that in the limit of $M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1$, the distribution functions of $V_1^{\mathrm{d}^*}$ and $W_1^{\mathrm{d}^*}$ intersect twice in the interval (d_2, ∞) if $d_1 \ge d_2$ (for $d_1 < d_2$ we would not have $V_1^{\mathrm{d}} = A_1 + d_2$ $P-$a.s., which is the starting point of all considerations for the case $d_1 > d_2$). As we will see in the following, this transfers to cross-ownership fractions close to 1.
Let $(M_{1,2,n}^{\mathrm{d}})_{n\in\mathbb{N}}$ and $(M_{2,1,n}^{\mathrm{d}})_{n\in\mathbb{N}}$ be sequences in $(0,1)$ with limit 1 for $n \to \infty$, and let $V_{1,n}^{\mathrm{d}}$ be the firm value of firm 1 calculated with $M_{1,2}^{\mathrm{d}} = M_{1,2,n}^{\mathrm{d}}$ and $M_{2,1}^{\mathrm{d}} = M_{2,1,n}^{\mathrm{d}}$. Similarly to (4.7) we define

$$W_{1,n}^{\mathrm{d}} \sim \mathcal{LN}(\tilde{\mu}_n, \tilde{\sigma}_n^2), \tag{4.21}$$

with $\tilde{\mu}_n$ and $\tilde{\sigma}_n^2$ such that $E(W_{1,n}^{\mathrm{d}}) = E(V_{1,n}^{\mathrm{d}})$ and $\mathrm{Var}(W_{1,n}^{\mathrm{d}}) = \mathrm{Var}(V_{1,n}^{\mathrm{d}})$. As we will see in the following, $W_{1,n}^{\mathrm{d}}$ converges in distribution to $W_1^{\mathrm{d}^*}$ defined in (4.7). First, (3.25)–

(3.26) and (3.31)–(3.33) imply for all $M^{\mathrm{d}}_{1,2}, M^{\mathrm{d}}_{2,1} \in (0,1)$ that

$$\begin{aligned} V^{\mathrm{d}}_1 &\le 1_{A_{\mathrm{ss}}}(A_1, A_2) \times (A_1 + M^{\mathrm{d}}_{1,2} d_2) \\ &\quad + 1_{A_{\mathrm{sd}}}(A_1, A_2) \times (A_1 + M^{\mathrm{d}}_{1,2} A_2 + M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1} d_1) + 1_{A_{\mathrm{ds}} \cup A_{\mathrm{dd}}}(A_1, A_2) \times d_1 \end{aligned} \tag{4.22}$$

$$= 1_{A_{\mathrm{ss}} \cup A_{\mathrm{sd}}}(A_1, A_2) \times (A_1 + M^{\mathrm{d}}_{1,2} \min\{d_2,\, A_2 + M^{\mathrm{d}}_{2,1} d_1\}) + 1_{A_{\mathrm{ds}} \cup A_{\mathrm{dd}}}(A_1, A_2) \times d_1 \tag{4.23}$$

$$\le A_1 + A_2 + d_1, \tag{4.24}$$

which is square-integrable. Hence, application of the dominated convergence theorem to $(V^{\mathrm{d}}_{1,n})_{n\in\mathbb{N}}$ and $((V^{\mathrm{d}}_{1,n})^2)_{n\in\mathbb{N}}$ yields

$$\lim_{n\to\infty} E(W^{\mathrm{d}}_{1,n}) = \lim_{n\to\infty} E(V^{\mathrm{d}}_{1,n}) = E(V^{\mathrm{d}^*}_1) = E(W^{\mathrm{d}^*}_1), \tag{4.25}$$

$$\lim_{n\to\infty} \mathrm{Var}(W^{\mathrm{d}}_{1,n}) = \lim_{n\to\infty} \mathrm{Var}(V^{\mathrm{d}}_{1,n}) = \mathrm{Var}(V^{\mathrm{d}^*}_1) = \mathrm{Var}(W^{\mathrm{d}^*}_1). \tag{4.26}$$

Since $\tilde{\mu}_n$ and $\tilde{\sigma}_n$ are continuous functions of $E(W^{\mathrm{d}}_{1,n})$ and $\mathrm{Var}(W^{\mathrm{d}}_{1,n})$, we also have $\lim_{n\to\infty} \tilde{\mu}_n = \tilde{\mu}$ and $\lim_{n\to\infty} \tilde{\sigma}_n = \tilde{\sigma}$. Furthermore, as the normal distribution function is continuous, it follows for $x > 0$ that

$$\lim_{n\to\infty} P(W^{\mathrm{d}}_{1,n} \le x) = \lim_{n\to\infty} \Phi\left(\frac{\ln(x) - \tilde{\mu}_n}{\tilde{\sigma}_n}\right) = \Phi\left(\lim_{n\to\infty} \frac{\ln(x) - \tilde{\mu}_n}{\tilde{\sigma}_n}\right) \tag{4.27}$$

$$= \Phi\left(\frac{\ln(x) - \tilde{\mu}}{\tilde{\sigma}}\right) = P(W^{\mathrm{d}^*}_1 \le x), \tag{4.28}$$

i.e. the RHS of (4.19) is equal to $\lim_{n\to\infty} P(W^{\mathrm{d}}_{1,n} < d_1)$. Hence, since the derivation of Proposition 4.1 implies that $\lim_{n\to\infty} P(V^{\mathrm{d}}_{1,n} < x) > \lim_{n\to\infty} P(W^{\mathrm{d}}_{1,n} < x)$ if and only if $d^*_1 < x < d^{**}_1$, there are points x_1, x_2, x_3 with $d_2 < x_1 < d^*_1 < x_2 < d^{**}_1 < x_3$ and

$$P(V^{\mathrm{d}}_{1,n} < x_1) < P(W^{\mathrm{d}}_{1,n} < x_1) \quad \text{for all } n > N_1, \tag{4.29}$$

$$P(V^{\mathrm{d}}_{1,n} < x_2) > P(W^{\mathrm{d}}_{1,n} < x_2) \quad \text{for all } n > N_2, \tag{4.30}$$

$$P(V^{\mathrm{d}}_{1,n} < x_3) < P(W^{\mathrm{d}}_{1,n} < x_3) \quad \text{for all } n > N_3, \tag{4.31}$$

for some $N_1, N_2, N_3 \in \mathbb{N}$. Therefore, the CDFs of $V^{\mathrm{d}}_{1,n}$ and the approximated lognormally distributed $W^{\mathrm{d}}_{1,n}$ intersect at least twice for all $n \ge \max\{N_1, N_2, N_3\}$, i.e. generally the distribution functions of V^{d}_1 and W^{d}_1 do not have the so-called single-crossing property as defined by Diamond and Stiglitz [1974] for example.

4.1.2 XOS of Equity only

Under cross-ownership of equity only, the probability of default for $M^{\mathrm{e}}_{1,2}$, $M^{\mathrm{e}}_{2,1} \to 1$ is not very interesting from a practical point of view, since firms exceeding a certain degree of cross-holdings would be forced to create a common balance sheet, i.e. separate calculation of default probabilities is not relevant. However, the limiting probability of default is interesting in comparison to the results under cross-ownership of debt only derived in the previous section.

Due to $V_1^{\mathrm{e}}\big|_{A_{\mathrm{ss}}} = (A_1 + M_{1,2}^{\mathrm{e}}A_2 - M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}}d_1 - M_{1,2}^{\mathrm{e}}d_2)/(1 - M_{1,2}^{\mathrm{e}}M_{2,1}^{\mathrm{e}})$, we are faced with the problem that $V_1^{\mathrm{e}}\big|_{A_{\mathrm{ss}}} \to \infty$ for $M_{1,2}^{\mathrm{e}}$, $M_{2,1}^{\mathrm{e}} \to 1$ if $A_1 + A_2 > d_1 + d_2$ (by (A71), this holds for all $(A_1, A_2) \in A_{\mathrm{ss}} \setminus \{(d_1, d_2)\}$). Thus, if we want to evaluate the limiting probability of default under cross-ownership of equity only, this cannot be done by considering the pointwise limit of V_1^{e} and the resulting limiting distribution. Instead, we will first calculate the probabilities of default for $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} < 1$ under both models and then consider the limits of these probabilities if the cross-ownership fractions converge to 1.
Since firm 1 is in default if and only if its firm value is smaller than the face value of its debt at maturity, we have under Suzuki's model by (3.31)–(3.32)

$$P(V_1^{\mathrm{e}} < d_1) = P(A_{\mathrm{ds}} \cup A_{\mathrm{dd}}) = P\Big(\underbrace{\{(a_1, a_2) \geq \mathbf{0} :\ a_1 < d_1,\ a_2 < d_2 + (d_1 - a_1)/M_{1,2}^{\mathrm{e}}\}}_{=:A_{\mathrm{d.}}(M_{1,2}^{\mathrm{e}})}\Big), \tag{4.32}$$

if $M_{1,2}^{\mathrm{e}} \neq 0$, where the second equality follows from (3.27)–(3.28) with $M_{1,2}^{\mathrm{d}} = M_{2,1}^{\mathrm{d}} = 0$. With $M_{1,2}^{\mathrm{e}}$ increasing, the set $A_{\mathrm{d.}}(M_{1,2}^{\mathrm{e}})$ becomes smaller, and it follows from the continuity of a probability measure that if $M_{1,2}^{\mathrm{e}} \to 1$,

$$P(V_1^{\mathrm{e}} < d_1) \to P(\{(a_1, a_2) \geq \mathbf{0} :\ a_1 < d_1,\ a_1 + a_2 \leq d_1 + d_2\}) > 0, \tag{4.33}$$

where the strict positivity follows since we assume $d_1, d_2 > 0$ and $\sigma_1, \sigma_2 > 0$ (cf. (4.1)).

As in our simulations, let

$$W_1^{\mathrm{e}} \sim \mathcal{LN}(\tilde{\mu}, \tilde{\sigma}^2), \tag{4.34}$$

where $\tilde{\mu}$ and $\tilde{\sigma}^2$ are determined such that $E(W_1^{\mathrm{e}}) = E(V_1^{\mathrm{e}})$ and $\mathrm{Var}(W_1^{\mathrm{e}}) = \mathrm{Var}(V_1^{\mathrm{e}})$. Note that $\tilde{\sigma}$ is strictly positive since the variance of V_1^{e} is strictly positive due to $\sigma_1, \sigma_2 > 0$. Then the probability of default of firm 1 under the lognormal model equals

$$P(W_1^{\mathrm{e}} < d_1) = \Phi\left(\frac{\ln(d_1) - \tilde{\mu}}{\tilde{\sigma}}\right). \tag{4.35}$$

If the cross-ownership fractions of equity converge to 1, this affects both, expectation and variance of V_1^{e} and thus also the parameters of W_1^{e}, because

$$\tilde{\mu} = \frac{1}{2}\ln\left(\frac{E(V_1^{\mathrm{e}})^4}{\mathrm{Var}(V_1^{\mathrm{e}}) + E(V_1^{\mathrm{e}})^2}\right), \tag{4.36}$$

$$\tilde{\sigma} = \ln\left(\frac{\mathrm{Var}(V_1^{\mathrm{e}})}{E(V_1^{\mathrm{e}})^2} + 1\right)^{0.5}, \tag{4.37}$$

More specifically, we have $\tilde{\mu} \to \infty$ for $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1$, and $\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} \tilde{\sigma} < \infty$ by

Lemma A.6, i.e.

$$\frac{\ln(d_1) - \tilde{\mu}}{\tilde{\sigma}} \to -\infty, \quad M^{\mathrm{e}}_{1,2},\, M^{\mathrm{e}}_{2,1} \to 1, \tag{4.38}$$

and thus

$$P(W^{\mathrm{e}}_1 < d_1) = \Phi\left(\frac{\ln(d_1) - \tilde{\mu}}{\tilde{\sigma}}\right) \to 0, \quad M^{\mathrm{e}}_{1,2},\, M^{\mathrm{e}}_{2,1} \to 1. \tag{4.39}$$

Comparing (4.33) and (4.39), we obtain the following proposition.

Proposition 4.3. *Under cross-ownership of equity only with lognormally distributed exogenous asset values, the lognormal model underestimates the actual limiting default probability of a firm, i.e.*

$$\lim_{M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1} P(V^{\mathrm{e}}_1 < d_1) > \lim_{M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1} P(W^{\mathrm{e}}_1 < d_1) = 0. \tag{4.40}$$

In our simulation study (cf. Figure 3.4(b)), this was already evident for cross-ownership fractions about 0.7.

Remark 4.4. Note that all the results obtained for cross-ownership of equity only also hold without the assumption of lognormally distributed exogenous asset values made in (4.1), if we still approximate the distribution of V^{e}_1 with a lognormal distribution. We only have to require the distribution of exogenous asset values to be continuous, non-negative, square-integrable and to yield a strictly positive limiting probability of default $P(\{(a_1, a_2) \geq \mathbf{0} :\ a_1 < d_1,\, a_1 + a_2 \leq d_1 + d_2\})$, and both, a strictly positive expectation and variance of $A_1 + A_2 - d_1 - d_2$ on A^*_{ss}, which is needed in the proof of Lemma A.6.

4.1.3 Summary and Comparison

In general, the comparison of default probabilities obtained from Suzuki's model and the lognormal model under cross-ownership of either debt or equity can lead to the following three cases: either both firms' probabilities of default are overestimated by the lognormal model, or both firms' probabilities of default are underestimated by the lognormal model, or one probability of default is overestimated and one probability of default is underestimated. If the corresponding cross-ownership fractions converge to 1, these cases can be exactly classified. Under cross-ownership of equity only, Proposition 4.3 shows that the lognormal model will always underestimate the limiting probabilities of default of both firms. Under cross-ownership of debt only, this cannot happen. By Proposition 4.1, the limiting probability of default of firm 1 is underestimated if and only if $d^*_1 < d_1 < d^{**}_1$, where $d^*_1 > d_2$, and in this case, the limiting probability of default of firm 2 is overestimated by Proposition 4.1. If $d_1 \notin (d^*_1, d^{**}_1)$ and $d_2 \notin (d^*_2, d^{**}_2)$ (with d^*_2 and d^{**}_2 defined in analogy to d^*_1 and d^{**}_1), the limiting probabilities of default of both firms are overestimated.

4.1.4 Bivariate Probabilities of Default

In the following we consider the event that both firms default simultaneously and compare the related limiting probabilities under Suzuki's model and the lognormal model. Under the lognormal model we consider random variables

$$(W_1, W_2) \sim \mathcal{LN}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}) \tag{4.41}$$

with $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2)^{\mathrm{T}}$ and $\tilde{\boldsymbol{\Sigma}} = \left(\begin{smallmatrix} \tilde{\sigma}_1^2 & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_2^2 \end{smallmatrix}\right)$ such that $E(W_i) = E(V_i)$, $\mathrm{Var}(W_i) = \mathrm{Var}(V_i)$ and $\mathrm{Cov}(W_1, W_2) = \mathrm{Cov}(V_1, V_2)$. Then we have as in (4.36) and (4.37)

$$\tilde{\mu}_i = \frac{1}{2} \ln \left(\frac{E(V_i)^4}{\mathrm{Var}(V_i) + E(V_i)^2} \right), \tag{4.42}$$

$$\tilde{\sigma}_i = \ln \left(\frac{\mathrm{Var}(V_i)}{E(V_i)^2} + 1 \right)^{0.5} = \ln \left(\frac{E(V_i^2)}{E(V_i)^2} \right)^{0.5} > 0, \quad i = 1, 2, \tag{4.43}$$

and the work of Nalbach-Leniewska [1979] yields

$$\tilde{\sigma}_{12} = \ln \left(\frac{E(V_1 V_2)}{E(V_1) E(V_2)} \right). \tag{4.44}$$

By Sylvester's criterion (cf. Theorem 7.2.5 of Horn and Johnson [2013], for example), $\tilde{\boldsymbol{\Sigma}}$ is positive semidefinite and thus a proper covariance matrix if and only if $|\tilde{\sigma}_{12}| \leq \tilde{\sigma}_1 \tilde{\sigma}_2$, i.e. otherwise, (W_1, W_2) as in (4.41) does not exist. For $|\tilde{\sigma}_{12}| = \tilde{\sigma}_1 \tilde{\sigma}_2$, $\ln(W_1)$ and $\ln(W_2)$ are perfectly correlated. If (W_1, W_2) exists, (4.44) yields $\mathrm{Cov}(W_1, W_2) = E(W_1)E(W_2)(\exp(\tilde{\sigma}_{12}) - 1)$. For $0 \leq \tilde{\sigma}_{12} \leq \tilde{\sigma}_1 \tilde{\sigma}_2$ it follows that $\mathrm{Cov}(W_1, W_2) \leq E(W_1)E(W_2)(\exp(\tilde{\sigma}_1 \tilde{\sigma}_2) - 1)$. If $-\tilde{\sigma}_1 \tilde{\sigma}_2 \leq \tilde{\sigma}_{12} < 0$, we have $E(W_1)E(W_2)(\exp(-\tilde{\sigma}_1 \tilde{\sigma}_2) - 1) \leq \mathrm{Cov}(W_1, W_2) < 0$ and therefore

$$|\mathrm{Cov}(W_1, W_2)| \leq E(W_1)E(W_2)|\exp(-\tilde{\sigma}_1 \tilde{\sigma}_2) - 1| < E(W_1)E(W_2)(\exp(\tilde{\sigma}_1 \tilde{\sigma}_2) - 1), \tag{4.45}$$

where the second inequality can be seen as follows. For $x > 0$, $|\exp(-x) - 1| < \exp(x) - 1 \Leftrightarrow 1 - \exp(-x) < \exp(x) - 1$. Obviously, we have equality for $x = 0$, and for $x > 0$, the derivative of the LHS is smaller than the derivative of the RHS. Altogether, we have for $|\tilde{\sigma}_{12}| \leq \tilde{\sigma}_1 \tilde{\sigma}_2$,

$$|\mathrm{Cov}(W_1, W_2)| \leq E(W_1)E(W_2)(\exp(\tilde{\sigma}_1 \tilde{\sigma}_2) - 1). \tag{4.46}$$

Hence, if

$$E(V_1)E(V_2)(\exp(\tilde{\sigma}_1 \tilde{\sigma}_2) - 1) < |\mathrm{Cov}(V_1, V_2)|, \tag{4.47}$$

we cannot match a bivariate lognormally distributed random vector to (V_1, V_2). This will be important for the bivariate limiting probability of default under both, cross-ownership of debt only and cross-ownership of equity only. Note that by Lemma A.8,

$E(V_1)E(V_2)(\exp(\tilde{\sigma}_1\tilde{\sigma}_2)-1) \leq E(V_1)E(V_2)\sqrt{(\exp(\tilde{\sigma}_1^2)-1)(\exp(\tilde{\sigma}_2^2)-1)}$, with the RHS denoting the upper bound for $|\mathrm{Cov}(V_1,V_2)|$ implied by the Cauchy–Schwarz inequality, i.e. the existence of scenarios where (4.47) holds cannot be ruled out by the Cauchy–Schwarz inequality.

Keeping this in mind, we now consider the limiting bivariate probability of default for two firms linked by cross-ownership of debt only. For $M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \in (0,1)$, their bivariate probability of default equals $P(V_1^{\mathrm{d}} < d_1, V_2^{\mathrm{d}} < d_2) = P(A_{\mathrm{dd}})$, i.e. with $V_1^{\mathrm{d}^*}$ and $V_2^{\mathrm{d}^*}$ defined as and as in analogy to (4.5), respectively, we obtain

$$P(V_1^{\mathrm{d}^*} < d_1, V_2^{\mathrm{d}^*} < d_2) = \lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} P(V_1^{\mathrm{d}} < d_1, V_2^{\mathrm{d}} < d_2) \tag{4.48}$$

$$= \lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} P(A_{\mathrm{dd}}) = P(A_{\mathrm{dd}^*}) = 0, \tag{4.49}$$

where (4.48) holds because almost sure convergence implies convergence in distribution, and (4.49) follows from the fact that A_{dd} is strictly decreasing in $M_{1,2}^{\mathrm{d}}$ and $M_{2,1}^{\mathrm{d}}$ (cf. (A23)), the continuity of a probability measure and Lemma A.4. If $(V_1^{\mathrm{d}^*}, V_2^{\mathrm{d}^*})$ can be approximated with a lognormally distributed random vector $(W_1^{\mathrm{d}^*}, W_2^{\mathrm{d}^*})$ as in (4.41), the lognormal model will yield a strictly positive probability of default since we assume $d_1, d_2 > 0$ and $\sigma_1, \sigma_2 > 0$, i.e. it will overestimate the limiting bivariate probability of default. Our findings are summarized in the following proposition.

Proposition 4.5. *Under cross-ownership of debt only with bivariate lognormally distributed exogenous asset values, the lognormal model overestimates the actual limiting bivariate probability of default of firm 1 and firm 2, i.e.*

$$0 = \lim_{M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} \to 1} P(V_1^{\mathrm{d}} < d_1, V_2^{\mathrm{d}} < d_2) < P(W_1^{\mathrm{d}^*} < d_1, W_2^{\mathrm{d}^*} < d_2), \tag{4.50}$$

provided that $(W_1^{\mathrm{d}^*}, W_2^{\mathrm{d}^*})$ *exists.*

Comparing Proposition 4.5 to Proposition 4.1, we see that the relative sizes of d_1 and d_2 are irrelevant now.

Under cross-ownership of equity only the bivariate probability of default under Suzuki's model for arbitrary values of $M_{1,2}^{\mathrm{e}}$ and $M_{2,1}^{\mathrm{e}}$ is given by (cf. (3.34))

$$P(V_1^{\mathrm{e}} < d_1, V_2^{\mathrm{e}} < d_2) = P(A_{\mathrm{dd}}) = P(A_1 < d_1, A_2 < d_2), \tag{4.51}$$

which does not depend on the realized cross-ownership fractions. Hence,

$$\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} P(V_1^{\mathrm{e}} < d_1, V_2^{\mathrm{e}} < d_2) = P(A_1 < d_1, A_2 < d_2) > 0, \tag{4.52}$$

where the last inequality holds since we assume $d_1, d_2 > 0$ and $\sigma_1, \sigma_2 > 0$.

Recall that we cannot fit a bivariate lognormal distribution to $(V_1^{\mathrm{e}}, V_2^{\mathrm{e}})$ if $|\tilde{\sigma}_{12}| > \tilde{\sigma}_1\tilde{\sigma}_2$. Of course, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_{12}$ depend on $M_{1,2}^{\mathrm{e}}$ and $M_{2,1}^{\mathrm{e}}$, and we set $M := \{(M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}}) \in (0,1)^2 : |\tilde{\sigma}_{12}| \leq \tilde{\sigma}_1\tilde{\sigma}_2\}$. If there is no sequence $(M_{1,2,n}^{\mathrm{e}}, M_{2,1,n}^{\mathrm{e}})_{n\in\mathbb{N}}$ in M such that $\lim_{n\to\infty}(M_{1,2,n}^{\mathrm{e}}, M_{2,1,n}^{\mathrm{e}}) = (1,1)$, the limiting bivariate probability of default under the lognormal model is not defined, since there is no suitable sequence of lognormal distributions we could consider for that. If such a sequence $(M_{1,2,n}^{\mathrm{e}}, M_{2,1,n}^{\mathrm{e}})_{n\in\mathbb{N}}$ in M exists, let $\tilde{\mu}_{i,n}$, $\tilde{\sigma}_{i,n}$ and $W_{i,n}^{\mathrm{e}}$ denote the (well-defined) versions of $\tilde{\mu}_i$, $\tilde{\sigma}_i$ and W_i^{e} $(i = 1,2)$ calculated by use of $M_{1,2,n}^{\mathrm{e}}$ and $M_{2,1,n}^{\mathrm{e}}$. Then

$$P(W_{1,n}^{\mathrm{e}} < d_1, W_{2,n}^{\mathrm{e}} < d_2) \leq P(W_{1,n}^{\mathrm{e}} < d_1) = \Phi\left(\frac{\ln(d_1) - \tilde{\mu}_{1,n}}{\tilde{\sigma}_{1,n}}\right) \to 0, \quad n \to \infty, \quad (4.53)$$

by Lemma A.6, i.e. the limiting bivariate probability of default under the lognormal model is 0. Comparing (4.52) and (4.53), we obtain the following proposition.

Proposition 4.6. *Under cross-ownership of equity only with bivariate lognormally distributed exogenous asset values, the lognormal model underestimates the actual limiting bivariate probability of default of firm 1 and firm 2, i.e.*

$$\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} P(V_1^{\mathrm{e}} < d_1, V_2^{\mathrm{e}} < d_2) > \lim_{n\to\infty} P(W_{1,n}^{\mathrm{e}} < d_1, W_{2,n}^{\mathrm{e}} < d_2) = 0, \quad (4.54)$$

provided the existence of a sequence $(M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}})$ *in* M *with limit* $(1,1)$ *for* $n \to \infty$.

Thus, if we can approximate $(V_1^{\mathrm{e}}, V_2^{\mathrm{e}})$ with a bivariate lognormally distributed random vector for $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1$, the result is similar to Proposition 4.3 for the univariate probabilities of default under cross-ownership of equity only.

4.2 Probabilities of Default for Arbitrary Scenarios of Cross-Ownership

After our analysis of the probability of default of firm 1 if the respective cross-ownership fractions converge to 1 we now examine the probability of default of a firm if the cross-ownership fractions are strictly smaller than 1. In the previous section the assumption of lognormally distributed exogenous asset values proved to be crucial for the case of cross-ownership of debt only, whereas the results for the case of cross-ownership of equity only also hold under far less restricting conditions. In the following, we will drop any distributional assumption with respect to exogenous asset values, we only require their distribution to be square-integrable and non-degenerate in a certain sense. This will be clarified later.

Our results will not only be valid for systems of $n \geq 2$ firms and all three types of cross-ownership (hence, we do not need a case differentiation as in Section 4.1), they can even be derived under a rather general set-up that does not rely on the interpretation of V_1 as the value of a firm under cross-ownership, determined under Suzuki's model. We first derive the general formula allowing us to compare the actual CDF of V_1 to the matched lognormal CDF, then we analyze some of its properties. Finally, we apply the gained insights to our situation of interest, namely scenarios of firms linked by cross-ownership.

4.2.1 Derivation of a General Formula

In this section we consider a certain family of non-negative random variables, and to each of these random variables we fit a lognormally distributed random variable according to the Fenton–Wilkinson method of moment matching. For each pair we then compare the probabilities that the respective random variable takes a value smaller than some threshold $c > 0$. We start by constructing our family of random variables of interest. Let

$$I_1 := [0, c), \quad I_2 := [c, \infty), \tag{4.55}$$

and let X_{I_1} and X_{I_2} be two random variables with values in I_1 and I_2, respectively. We do not make any specific assumptions with respect to the distributions of $X_{I_1} : \Omega \to I_1$ and $X_{I_2} : \Omega \to I_2$, we only require them to be continuous with finite and strictly positive variances, which implies $0 < E(X_{I_i}), E(X_{I_i}^2) < \infty$, $i = 1, 2$.
Furthermore, let the family $\{X_p : p \in [0, 1]\}$ of non-negative random variables be such that $X_p \,|\, X_p \in I_1 =^{\mathcal{D}} X_{I_1}$ for all $p \in (0, 1]$ and $X_p \,|\, X_p \in I_2 =^{\mathcal{D}} X_{I_2}$ for all $p \in [0, 1)$, and with $P(X_p \in I_1) = p$. Then the distribution of X_p is uniquely determined, and for $x \geq 0$ we have $P(X_0 \leq x) = P(X_{I_2} \leq x)$, $P(X_1 \leq x) = P(X_{I_1} \leq x)$ and, for $p \in (0, 1)$,

$$\begin{aligned} P(X_p \leq x) &= P(X_p \leq x,\, I_1) + P(X_p \leq x,\, I_2) && (4.56)\\ &= p \times P(X_p \leq x \,|\, I_1) + (1-p) \times P(X_p \leq x \,|\, I_2) && (4.57)\\ &= p \times P(X_{I_1} \leq x) + (1-p) \times P(X_{I_2} \leq x). && (4.58) \end{aligned}$$

Hence, p can be interpreted as a mixing parameter of X_{I_1} and X_{I_2}. Of course, the moments of X_p, $p \in [0, 1]$, depend on p as well:

$$\begin{aligned} E(X_p) &= p \times E(X_{I_1}) + (1-p) \times E(X_{I_2}) > 0, && (4.59)\\ E(X_p^2) &= p \times E(X_{I_1}^2) + (1-p) \times E(X_{I_2}^2) > 0. && (4.60) \end{aligned}$$

Furthermore, $\mathrm{Var}(X_0) = \mathrm{Var}(X_{I_2})$, $\mathrm{Var}(X_1) = \mathrm{Var}(X_{I_1})$ and for $p \in (0, 1)$,

$$\begin{aligned} \mathrm{Var}(X_p) &= p \times \mathrm{Var}(X_p \,|\, I_1) + (1-p) \times \mathrm{Var}(X_p \,|\, I_2) \\ &\qquad + p \times (1-p) \times (E(X_p \,|\, I_1) - E(X_p \,|\, I_2))^2 && (4.61)\\ &\geq p \times \mathrm{Var}(X_p \,|\, I_1) + (1-p) \times \mathrm{Var}(X_p \,|\, I_2) && (4.62)\\ &= p \times \mathrm{Var}(X_{I_1}) + (1-p) \times \mathrm{Var}(X_{I_2}), && (4.63) \end{aligned}$$

i.e.

$$0 < \mathrm{Var}(X_p) < \infty \quad \text{for all } p \in [0, 1]. \tag{4.64}$$

We now match a lognormally distributed random variable Y_p to X_p such that $E(Y_p) =$

$E(X_p)$ and $\mathrm{Var}(Y_p) = \mathrm{Var}(X_p)$, i.e.

$$Y_p \sim \mathcal{LN}(\tilde{\mu}_p, \tilde{\sigma}_p^2) \tag{4.65}$$

with

$$\tilde{\mu}_p = \frac{1}{2} \ln\left(\frac{E(X_p)^4}{E(X_p^2)}\right), \tag{4.66}$$

$$\tilde{\sigma}_p^2 = \ln\left(\frac{E(X_p^2)}{E(X_p)^2}\right) > 0 \tag{4.67}$$

due to (4.64). In the following, we will calculate the probability $P(Y_p \in I_1)$ and compare it to the probability $P(X_p \in I_1)$, which is p by construction. For that, we set

$$E(X_{I_1}) - E(X_{I_2}) =: x_1 < 0, \tag{4.68}$$

$$E(X_{I_2}) =: x_2 \geq c, \tag{4.69}$$

$$E(X_{I_1}^2) - E(X_{I_2}^2) =: y_1 < 0, \tag{4.70}$$

$$E(X_{I_2}^2) =: y_2 \geq c^2. \tag{4.71}$$

Then

$$P(Y_p \in I_1) = P(Y_p < c) = \Phi\left(\frac{\ln(c) - \tilde{\mu}_p}{\tilde{\sigma}_p}\right) \tag{4.72}$$

$$= \Phi\left(\frac{\ln(c) - \frac{1}{2}\ln\left(\frac{E(X_p)^4}{E(X_p^2)}\right)}{\ln\left(\frac{E(X_p^2)}{E(X_p)^2}\right)^{0.5}}\right) = \Phi\left(\frac{\ln(c) - \frac{1}{2}\ln\left(\frac{(p\times x_1+x_2)^4}{p\times y_1+y_2}\right)}{\ln\left(\frac{p\times y_1+y_2}{(p\times x_1+x_2)^2}\right)^{0.5}}\right) \tag{4.73}$$

by (4.59)–(4.60) and (4.68)–(4.71). Thus, for all $p \in [0, 1]$,

$$P(Y_p \in I_1) < P(X_p \in I_1) \quad \Leftrightarrow \quad h(p) := \Phi\left(\frac{\ln(c) - \frac{1}{2}\ln\left(\frac{(p\times x_1+x_2)^4}{p\times y_1+y_2}\right)}{\ln\left(\frac{p\times y_1+y_2}{(p\times x_1+x_2)^2}\right)^{0.5}}\right) < p. \tag{4.74}$$

Hence, if we approximate an almost arbitrarily distributed non-negative random variable X_p with a lognormally distributed random variable Y_p by the method of moment matching, (4.74) yields a criterion as to which random variable has a smaller probability of falling below the threshold $c > 0$. The reason why we assume Y_p to be lognormally distributed will become clear in Section 4.2.3. In the following section we analyze the RHS of (4.74) with respect to p.

4.2.2 Comparison of Probabilities

4.2.2.1 Values of p close to or identical to 0 and 1

Let us consider the RHS of (4.74). If $p = 0$, we obtain $h(0) > 0$, because the standard normal distribution function takes values in $(0, 1)$ only. Since $h : [0, 1] \to (0, 1)$ is continuous in p, we know that there is an $\epsilon(x_1, x_2, y_1, y_2) =: \epsilon \in (0, 1)$ such that

$$h(p) > p \quad \text{for all } p \in [0, \epsilon). \tag{4.75}$$

Hence, $P(Y_p < c) > P(X_p < c)$ if $p \in [0, \epsilon)$. For $p = 1$ we obtain $h(1) < 1$, i.e. there is an $\epsilon'(x_1, x_2, y_1, y_2) =: \epsilon' \in (0, 1)$ such that

$$h(p) < p \quad \text{for all } p \in (\epsilon', 1]. \tag{4.76}$$

In this case, $P(Y_p < c) < P(X_p < c)$.

4.2.2.2 Arbitrary Values of $p \in (0, 1)$

For a given $p \in (0, 1)$, let

$$c \geq \frac{E(X_p)^2}{E(X_p^2)^{0.5}}. \tag{4.77}$$

Because of Jensen's inequality we have

$$\frac{E(X_p)^2}{E(X_p^2)^{0.5}} = E(X_p) \underbrace{\frac{E(X_p)}{E(X_p^2)^{0.5}}}_{\leq 1} \leq E(X_p) \quad \text{for all } p \in (0, 1), \tag{4.78}$$

so (4.77) is met if for example $E(X_p) \leq c$. Under assumption (4.77) the numerator of Φ in (4.74) is non-negative, which implies

$$\Phi\left(\frac{\ln(c) - \frac{1}{2}\ln\left(\frac{E(X_p)^4}{E(X_p^2)}\right)}{\ln\left(\frac{E(X_p^2)}{E(X_p)^2}\right)^{0.5}}\right) \geq 0.5, \tag{4.79}$$

i.e. $P(Y_p < c) \geq 0.5$ independently of the value of p, as long as (4.77) is met. However, the assumption of $c \geq E(X_p)^2/E(X_p^2)^{0.5}$ does not impose any restrictions on p, which can be seen as follows. Recall that $E(X_p) = p\,E(X_{I_1}) + (1 - p)\,E(X_{I_2})$ with $E(X_{I_1}) < c < E(X_{I_2})$, which means that for any conditional distribution of X_p on I_1, i.e. for any distribution of X_{I_1} on I_1, we only have to choose the distribution of X_{I_2} such that $E(X_{I_2})$ becomes small enough (i.e. close enough to c) to fulfil the sufficient condition $E(X_p) \leq c$ (cf. (4.78)). Thus, the initial condition $c \geq E(X_p)^2/E(X_p^2)^{0.5}$ can be met for any $p \in (0, 1)$, if the distributions of X_{I_1} on I_1 and X_{I_2} on I_2 are chosen suitably. Hence, the probability $P(X_p < c) = p$ can be arbitrarily small, whereas the probability

$P(Y_p < c)$ is at least 0.5, assuming that $c \geq E(X_p)^2/E(X_p^2)^{0.5}$. However, if $p \geq 0.5$, it is also possible that $P(Y_p < c) \leq P(X_p < c)$.

Let now

$$c \leq \frac{E(X_p)^2}{E(X_p^2)^{0.5}} \tag{4.80}$$

for a given $p \in (0,1)$. Then the numerator of Φ in (4.74) is non-positive, and

$$P(Y_p < c) = \Phi\left(\frac{\ln(c) - \frac{1}{2}\ln\left(\frac{E(X_p)^4}{E(X_p^2)}\right)}{\ln\left(\frac{E(X_p^2)}{E(X_p)^2}\right)^{0.5}}\right) \leq 0.5. \tag{4.81}$$

However, (4.80) does not impose any restrictions on the value of p, which can be seen as follows. By (4.78), a necessary condition for (4.80) is $E(X_p) > c$. For some $E > c$, let X_{I_1} and X_{I_2} be such that $P(X_{I_1} = 0.5\,c) = 1$ and $P(X_{I_2} = (E - 0.5\,c\,p)/(1-p)) = 1$. It is straightforward to see that $(E-0.5\,c\,p)/(1-p) > c$, i.e. indeed $(E-0.5\,c\,p)/(1-p) \in I_2$. Then, for an arbitrary $p \in (0,1)$,

$$X_p = \begin{cases} 0.5\,c, & \text{with probability } p, \\ \frac{E-0.5\,c\,p}{1-p}, & \text{with probability } 1-p. \end{cases} \tag{4.82}$$

Note that since we consider $p \in (0,1)$ instead of $p \in [0,1]$ here, it is guaranteed that $\mathrm{Var}(X_p) > 0$ for all $p \in (0,1)$, i.e. the conditions $\mathrm{Var}(X_{I_1}) > 0$ and $\mathrm{Var}(X_{I_2}) > 0$ are not necessary here. Obviously, $X_p \geq 0$ because of $E > c$, and $E(X_p) = E$, so the necessary condition for (4.80) is met. Furthermore,

$$E(X_p^2) = 0.25\,c^2\,p + \left(\frac{E - 0.5\,c\,p}{1-p}\right)^2 (1-p) \tag{4.83}$$

$$= 0.25\,c^2\,p + \frac{0.25\,c^2\,p^2 - c\,p\,E + E^2}{1-p} = \frac{0.25\,c^2\,p - c\,p\,E + E^2}{1-p} \tag{4.84}$$

and thus

$$\frac{E(X_p)^2}{E(X_p^2)^{0.5}} \geq c \quad \Leftrightarrow \quad \frac{E(X_p)^4}{E(X_p^2)} \geq c^2 \tag{4.85}$$

$$\Leftrightarrow \quad E^4(1-p) \geq 0.25\,c^4 p - c^3\,p\,E + E^2 c^2 \tag{4.86}$$

$$\Leftrightarrow \quad E^4(1-p) + c^3\,p\,E - E^2 c^2 - 0.25\,c^4 p \geq 0. \tag{4.87}$$

For given p, the inequality is always met if E is chosen large enough. For $p > 0.5$ this means that $P(Y_p < c) \leq 0.5 < P(X_p < c)$. However, if $p \leq 0.5$, it is also possible that $P(Y_p < c) \geq P(X_p < c)$.

4.2.3 Application to Probabilities of Default under Cross-Ownership

In the following we employ the insights of Section 4.2.1 and Section 4.2.2 to compare probabilities of default under Suzuki's model and the lognormal model. In detail, we are concerned with the probability of default of a firm that is part of a system of $n \geq 2$ firms linked by cross-ownership, without loss of generality we refer to this firm as firm 1. For that, we first need to see how our scenario of firms linked by cross-ownership corresponds to the set-up described in Section 4.2.1.

By definition, firm 1 is in default if and only if its total asset value is not sufficiently high to repay all of its nominal debt, i.e. if $V_1 < d_1$. Hence, we can split the non-negative real axis of possible firm values into two disjoint intervals $[0, d_1)$ and $[d_1, \infty)$, i.e. we can identify d_1 with c of (4.55), and we set $I_1 := [0, d_1)$ and $I_2 := [d_1, \infty)$, which means that I_1 and I_2 stand for sets of firm values, where firm 1 is in default, or not. Since firm values are derivatives of exogenous asset values, analogously the space $(\mathbb{R}_0^+)^n$ of exogenous asset values $\mathbf{a}$ can be divided into two subsets where firm 1 is in default or solvent, and we set (cf. (3.18))

$$A_{\text{d}} := \bigcup_{\substack{\mathbf{z} \in \{\text{s,d}\}^n, \\ z_1 = \text{d}}} A_{\mathbf{z}} = \{\mathbf{a} \in (\mathbb{R}_0^+)^n : v_1(\mathbf{a}) < d_1\}, \tag{4.88}$$

$$A_{\text{s}} := \bigcup_{\substack{\mathbf{z} \in \{\text{s,d}\}^n, \\ z_1 = \text{s}}} A_{\mathbf{z}} = \{\mathbf{a} \in (\mathbb{R}_0^+)^n : v_1(\mathbf{a}) \geq d_1\}. \tag{4.89}$$

Then the probability of default under Suzuki's model can be written as

$$P(V_1 < d_1) = P(V_1 \in I_1) = P(\mathbf{A} \in A_{\text{d}}) =: p. \tag{4.90}$$

According to the above partition of $\mathbb{R}_0^+$ we consider the distribution of V_1 as a weighted average of the two conditional distributions of V_1 on the intervals I_1 and I_2. These conditional distributions can be identified with the distribution of X_{I_1} and X_{I_2} of Section 4.2.1, i.e. V_1 corresponds to X_p with p given by (4.90). In order to be able to use the results of Section 4.2.1, we have to assume that the distribution of exogenous asset values $\mathbf{A}$ on $(\mathbb{R}_0^+)^n$ is such that the conditional distributions of V_1 on I_1 and I_2 are continuous and have strictly positive variances (cf. (4.64)). For a given cross-ownership scenario and given values of $\mathbf{d}$ we then can calculate x_1, x_2, y_1 and y_2 as in (4.68)–(4.71). Under the lognormal model, the probability of default of firm 1 is determined under the assumption that firm values are lognormally distributed, as it would be the case under Merton's model for a single firm. Hence, we fit a lognormally distributed random variable to V_1 by the Fenton–Wilkinson method, i.e. we can identify this lognormally distributed random variable with Y_p of Section 4.2.1, with p given in (4.90). If we compare the actual probability of default of firm 1, namely the probability of default derived under Suzuki's model, to the probability of default under the lognormal model, (4.74) implies that the lognormal model underestimates the actual probability of default p if and only if

$$\Phi\left(\frac{\ln(d_1) - \frac{1}{2}\ln\left(\frac{(p\times x_1+x_2)^4}{p\times y_1+y_2}\right)}{\ln\left(\frac{p\times y_1+y_2}{(p\times x_1+x_2)^2}\right)^{0.5}}\right) < p. \tag{4.91}$$

Since (4.91) cannot be solved for p, it is difficult to make a general statement for which values of p this actual probability of default will be underestimated, when all other parameters are held constant. In order to gain further insight, we assume the conditional distributions of V_1 on I_1 and I_2 to be fixed, only their mixing parameter p (i.e. the probability that firm 1 is in default) varies.

Remark 4.7. Any given distribution $\mu_{\mathbf{A}}$ of exogenous asset values $\mathbf{A}$ on $(\mathbb{R}_0^+)^n$ can be transformed into a new distribution $\mu_{\mathbf{A},\tilde{p}}$ such that the induced distribution of V_1, denoted by $\mu_{V_1,\tilde{p}}$, yields $\mu_{V_1,\tilde{p}}(I_1) = \tilde{p}$ for a desired value $\tilde{p}$, while the conditional distributions of V_1 on I_1 and I_2 remain unchanged. For that, let

$$\mu_{\mathbf{A},\tilde{p}}(A) := \tilde{p}\,\frac{\mu_{\mathbf{A}}(A\cap A_{\mathrm{d}})}{\mu_{\mathbf{A}}(A_{\mathrm{d}})} + (1-\tilde{p})\,\frac{\mu_{\mathbf{A}}(A\cap A_{\mathrm{s}})}{\mu_{\mathbf{A}}(A_{\mathrm{s}})}, \quad \tilde{p}\in[0,1],\ A\in\mathbb{B}((\mathbb{R}_0^+)^n), \tag{4.92}$$

where $\mathbb{B}(M)$ stands for the Borel-σ-algebra over a set $M \subset \mathbb{R}^m$, $m \in \mathbb{N}$. Then $\mu_{\mathbf{A},\tilde{p}}(A_{\mathrm{d}}) = \tilde{p}$ and

$$\mu_{V_1,\tilde{p}}(I_1) = \mu_{\mathbf{A},\tilde{p}}(V_1^{-1}(I_1)) = \mu_{\mathbf{A},\tilde{p}}(A_{\mathrm{d}}) = \tilde{p}, \tag{4.93}$$

with $V_1^{-1}(B) := \{\mathbf{a} \in (\mathbb{R}_0^+)^n : v_1(\mathbf{a}) \in B\}$, $B \in \mathbb{B}(\mathbb{R}_0^+)$. Straightforward calculations show that the transformation in (4.92) does not alter the conditional distributions of V_1 on I_1 and I_2, i.e. it only alters the mixing parameter p and thus the probability of default of firm 1.

Using the results of Section 4.2.2 we now analyze how the inequality in (4.91) depends on the parameter p, i.e. in the following, p stands for an arbitrarily chosen probability of default under Suzuki's model (cf. (4.90)). Note that the values of x_1, x_2, y_1 and y_2 do not vary with p since we assume the conditional distribution of V_1 on I_1 and I_2 to be fixed.
By (4.75) we know that there is an interval $[0, \epsilon)$ such that if the actual probability of default p lies in this interval, the lognormal model overestimates the actual probability of default. Recall that in Section 4.1.1 we observed a somewhat similar effect for the case of two firms linked by cross-ownership of debt only for $d_1 \leq d_2$. There, cross-ownership fractions converged to 1, which resulted in an actual limiting default probability of firm 1 of 0, whereas the lognormal model yielded a strictly positive limiting probability of default. For continuity reasons, there is a whole range of cross-ownership fractions such that the actual risk is overestimated. Hence, for two firms linked by cross-ownership of debt only with $d_1 \leq d_2$, there are at least two ways of constructing scenarios leading to an overestimation of the actual default probability of firm 1: first, as done in Section 4.1.1, we can alter the cross-ownership structure between the two firms such that the actual

probability of default converges to 0, and second, we can transform the distribution of exogenous asset values (cf. Remark 4.7) such that the actual probability of default is sufficiently close to 0, as done in this section. In both approaches, the lognormal model yields a probability of default bigger than 0. Note that the results of Section 4.1.1 for $d_1 \leq d_2$ hold without the assumption of exogenous asset values following a lognormal distribution.

Similarly, we know from (4.76) that there is an interval $(\epsilon, 1]$ such that if p, the actual probability of default of firm 1, lies in this interval, the lognormal model underestimates the actual probability of default. By Proposition 4.3 for two firms linked by cross-ownership of equity only, underestimation of default probabilities can also be constructed by either a structural approach (i.e. letting cross-ownership fractions converge to 1) or a distributional approach by weighting the distribution of exogenous asset values (cf. Remark 4.7) such that the actual probability of default is sufficiently close to 1.

Recall that the only assumption we made about the distribution of exogenous asset values was that it is continuous and yields strictly positive conditional variances of V_1 on I_1 and I_2. Any given distribution $\mu_{\mathbf{A}}$ fulfilling this weak requirement can be transformed according to (4.92) with $\tilde{p} < \epsilon(x_1, x_2, y_1, y_2) = \epsilon(\mu_{\mathbf{A}})$ or $\tilde{p} > \epsilon'(x_1, x_2, y_1, y_2) = \epsilon'(\mu_{\mathbf{A}})$, so that (4.75) resp. (4.76) follows.

Estimation of ϵ and ϵ'

In order to gain an impression of the sizes of ϵ and ϵ' we did a short simulation study for a system of two firms linked by cross-ownership of either debt or equity with the respective cross-ownership fractions taking values in $\{0.1, 0.2, \ldots, 0.9\}^2$. Furthermore, we considered lognormally distributed exogenous asset values as in (3.44) with $a = 1$ and $\sigma^2 = 1$, and $d_1 \in \{0.1, 0.5, 1, \ldots, 2.5, 3\}$. For a given scenario, we simulated 1,000,000 firm values V_1 to obtain point estimates of x_1, x_2, y_1 and y_2. Due to the high number of simulated values, the conditional variances $\mathrm{Var}(V_1 \mid I_2) = y_2 - x_2^2$ and $\mathrm{Var}(V_1 \mid I_1) = y_1 + y_2 - (x_1 + x_2)^2$ were always strictly positive (which is necessary and sufficient for (4.64)). Then we used the R package rootSolve by Soetaert [2009] to find the roots of $h(p) - p$, $p \in [0, 1]$, where the smallest and largest roots served as estimates for ϵ and ϵ', respectively. For every scenario, this procedure was repeated 500 times to obtain confidence intervals for ϵ and ϵ'. Point estimates $\hat{\epsilon}$ and $\hat{\epsilon}'$ were calculated as the median of the respective 500 simulated values.

Under both types of cross-ownership, we always had $\hat{\epsilon} = \hat{\epsilon}'$. However, we do not have a theoretical proof for that. Some of the results for moderate levels of cross-ownership are displayed in Table 4.1 and Table 4.2. Apparently, $\hat{\epsilon}$ is strongly influenced by the value of d_1, whereas the cross-ownership fractions seem to be of less importance. If d_1 is chosen relatively small, which means that the two firms tend to be healthier, the values of $\hat{\epsilon}'$ are close to 0, i.e. the lognormal model underestimates the actual probability of default p for most values of p in $(0, 1)$ (cf. Section 4.2.2.1). If d_1 is chosen relatively big, which means that the two firms are more likely to be in default, the values of $\hat{\epsilon}$ are close to 1, i.e. the lognormal model overestimates the actual probability of default p for most values of p in $(0, 1)$. Similarly, the probability of default $p_{\text{orig.}}$ associated with

		$M^{\mathrm{d}}_{1,2} = 0.1$		$M^{\mathrm{d}}_{1,2} = 0.5$	
d_1	$M^{\mathrm{d}}_{2,1}$	$\hat{\epsilon} = \hat{\epsilon}'$	95% CI	$\hat{\epsilon} = \hat{\epsilon}'$	95% CI
0.1	0.1	0.03462	0.03345 – 0.03618	0.02948	0.02838 – 0.03078
0.1	0.4	0.03465	0.03345 – 0.03611	0.02935	0.02830 – 0.03082
0.1	0.7	0.03466	0.03349 – 0.03609	0.02937	0.02826 – 0.03060
1.0	0.1	0.64907	0.64577 – 0.65260	0.53229	0.52839 – 0.53725
1.0	0.4	0.64615	0.64298 – 0.64976	0.51022	0.50634 – 0.51419
1.0	0.7	0.64254	0.63957 – 0.64663	0.48515	0.48174 – 0.49000
2.0	0.1	0.85912	0.85794 – 0.86034	0.75629	0.75501 – 0.75754
2.0	0.4	0.85629	0.85510 – 0.85758	0.72339	0.72224 – 0.72455
2.0	0.7	0.85278	0.85154 – 0.85401	0.67792	0.67673 – 0.67928
3.0	0.1	0.93391	0.93236 – 0.93510	0.87725	0.87550 – 0.87886
3.0	0.4	0.93219	0.93087 – 0.93348	0.85499	0.85337 – 0.85655
3.0	0.7	0.93012	0.92876 – 0.93131	0.82076	0.81916 – 0.82228

Table 4.1: Estimated values $\hat{\epsilon}$ and $\hat{\epsilon}'$ under cross-ownership of debt only, $\sigma^2 = 1$, 1,000,000 iterations, 500 repetitions for confidence intervals CI.

the original distribution of exogenous asset values is over- resp. underestimated by the lognormal model if $p_{\mathrm{orig.}} < \hat{\epsilon} = \hat{\epsilon}'$ resp. $p_{\mathrm{orig.}} > \hat{\epsilon} = \hat{\epsilon}'$.
In a certain sense, our empirical findings with respect to the value of d_1 are supported by the following theoretical considerations, which are based on Section 4.2.2.2.

Comparison of Default Probabilities

Let $d_1 \geq E(V_1)^2/E(V_1^2)^{0.5}$. Then, by (4.79), the lognormal model will yield a probability of default of at least 0.5, whereas the above condition does not allow any conclusions on the value of p, the probability of default under Suzuki's model. By (4.78), a sufficient condition for $d_1 \geq E(V_1)^2/E(V_1^2)^{0.5}$ is $d_1 \geq E(V_1) = p\,E(V_1 \,|\, I_1) + (1-p)\,E(V_1 \,|\, I_2)$, i.e. it is sufficient to ensure that the conditional expectation of V_1 on I_2 is close enough to d_1. Since V_1 is a continuous function of exogenous asset values $\mathbf{A}$ (cf. Lemma 3.4), this means that the conditional distribution of $\mathbf{A}$ on A_{s} has to be chosen such that much of its mass is near the 'border' to A_{d} (because we have $V_1 = d_1$ on this border). Thus, the initial condition $E(V_1)^2/E(V_1^2)^{0.5} \leq d_1$ can be met for any $p \in (0,1)$, if the distribution of exogenous asset values $\mathbf{A}$ on $(\mathbb{R}_0^+)^n$ is chosen suitably. Hence, if p, the probability of default of firm 1 under Suzuki's model, is smaller than 0.5 and if $d_1 \geq E(V_1)^2/E(V_1^2)^{0.5}$, the lognormal model will always overestimate the actual probability of default. However, if $p > 0.5$, it is also possible that the lognormal model underestimates the actual probability of default. Note that although the condition $E(V_1)^2/E(V_1^2)^{0.5} \leq d_1$ implies $E(V_1) \leq d_1$, this does not necessarily mean that a firm would have to declare bankruptcy in this circumstance. The requirement to declare bankruptcy would depend on the regulator's particular choice of the distribution of the exogenous asset values, which could be different than that of any other observer's choice.
Let now $E(V_1)^2/E(V_1^2)^{0.5} \geq d_1$. Then (4.81) implies that the probability of default under

		$M^{\mathrm{e}}_{1,2} = 0.1$		$M^{\mathrm{e}}_{1,2} = 0.5$	
d_1	$M^{\mathrm{e}}_{2,1}$	$\hat{\epsilon} = \hat{\epsilon}'$	95% CI	$\hat{\epsilon} = \hat{\epsilon}'$	95% CI
0.1	0.1	0.02142	0.02055 – 0.02248	0.00240	0.00227 – 0.00258
0.1	0.4	0.02000	0.01923 – 0.02119	0.00141	0.00133 – 0.00152
0.1	0.7	0.01871	0.01793 – 0.01966	0.00071	0.00066 – 0.00076
1.0	0.1	0.67313	0.67021 – 0.67631	0.61828	0.61499 – 0.62257
1.0	0.4	0.66912	0.66624 – 0.67238	0.58350	0.57918 – 0.58889
1.0	0.7	0.66303	0.65958 – 0.66660	0.51472	0.50744 – 0.52244
2.0	0.1	0.87887	0.87770 – 0.88006	0.86873	0.86767 – 0.86986
2.0	0.4	0.87713	0.87596 – 0.87840	0.85597	0.85487 – 0.85705
2.0	0.7	0.87429	0.87321 – 0.87544	0.83398	0.83277 – 0.83514
3.0	0.1	0.94377	0.94254 – 0.94499	0.94042	0.93922 – 0.94155
3.0	0.4	0.94302	0.94173 – 0.94404	0.93421	0.93270 – 0.93546
3.0	0.7	0.94147	0.94003 – 0.94274	0.92214	0.92046 – 0.92335

Table 4.2: Estimated values $\hat{\epsilon}$ and $\hat{\epsilon}'$ under cross-ownership of equity only, $\sigma^2 = 1$, 1,000,000 iterations, 500 repetitions for confidence intervals CI.

the lognormal model is at most 0.5, whereas Suzuki's model may yield any probability of default (cf. Section 4.2.2.2). For that, we only need to see that there is a distribution of exogenous asset values $\mathbf{A}$ on $(\mathbb{R}_0^+)^n$ such that V_1 is given as X_p in (4.82). For example, this distribution could be chosen as follows. Let $D := \{\mathbf{a} \in (\mathbb{R}_0^+)^n : v_1(\mathbf{a}) = 0.5d_1\}$ and $S := \{\mathbf{a} \in (\mathbb{R}_0^+)^n : v_1(\mathbf{a}) = (E - 0.5\,p\,d_1)/(1-p)\}$. In order to obtain the desired distribution of V_1, we thus only have to ensure that

$$P(\mathbf{A} \in D) = p, \tag{4.94}$$

$$P(\mathbf{A} \in S) = 1 - p, \tag{4.95}$$

$$P(\mathbf{A} \in (\mathbb{R}_0^+)^n \backslash (D \cup S)) = 0, \tag{4.96}$$

which can be constructed easily. In particular, the lognormal model underestimates the actual probability of default of firm 1 under Suzuki's model if this probability is bigger than 0.5.

Conclusion

The above analysis shows that we cannot arrive at definite conclusions as to whether the lognormal model over- or underestimates the actual probability of default of a firm in the general case (i.e. for arbitrary cross-ownership fractions). Although (4.74) provides an exact formula, we cannot solve this inequality for p or the conditional moments of V_1 to gain further insight. However, if p, the probability of default of firm 1 under Suzuki's model, is 0 or 1, risk is systematically over- and underestimated, respectively. Further, for given conditional distributions of V_1 on I_1 and I_2, there is a whole interval $[0, \epsilon)$ and hence a whole family of distributions $\mu_{V_1,p}$ ($p \in [0, \epsilon)$) of V_1 (cf. Remark 4.7) such that the approximating lognormal model leads to an overestimation of the actual probability of default p. Similarly, there is an interval $(\epsilon', 1]$ with corresponding distributions of V_1 such that the approximating lognormal model leads to an underestimation of the actual probability of default $p \in (\epsilon', 1]$. In particular, any given non-negative distribution of exogenous asset values fulfilling some weak requirements (cf. p. 47) can be transformed into a new, "extreme" distribution of exogenous asset values yielding such a low or high actual probability of default that the approximating lognormal model will over- and underestimate this risk, respectively.

If the expected firm value is smaller than the firm's nominal level of liabilities, the lognormal model yields a probability of default of at least 0.5, independently of the variance of the firm value. If the variance is small, the actual probability of default can be much smaller. On the other hand, there are also situations where the lognormal model grossly underestimates the actual risk of default.

5 Tail Dependence of Firm Values under Cross-Ownership

Having mainly been concerned with the analysis of the univariate distributions of firm values and probabilities of default of a firm evolving under various cross-ownership scenarios, the aim of this section is to study the bivariate behaviour of firm values in a system of two firms linked by cross-ownership. In particular, we will consider the simultaneous occurrence of extreme events, i.e. that both firm values fall below a low threshold or exceed a high threshold. For that, we will employ the tail dependence coefficient, but in order to be able to derive some of its properties and to calculate its value under cross-ownership, we first need to consider some properties of copulas and their asymptotic behaviour under cross-ownership.

5.1 Copula of Firms Values under Cross-Ownership

5.1.1 Definition and Basic Properties of Copulas

In the following we will give a short overview over the definition and the main properties of two-dimensional copulas. Most of the results can be found in Nelsen [2006]. A historical review on the subject may be found in Durante and Sempi [2010].
In the remainder we will write $\mathbf{I}$ for the unit interval $[0,1]$ and $\bar{\mathbb{R}}$ for the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$.

Definition 5.1. *A two-dimensional copula (or briefly, a copula) is a function C with the following properties:*

1. *C is a function from $\mathbf{I}^2$ to $\mathbf{I}$.*
2. *For every $u, v \in \mathbf{I}$,*

$$C(u,0) = C(0,v) = 0 \quad (\text{“groundedness”}), \tag{5.1}$$
$$C(u,1) = u, \quad C(1,v) = v. \tag{5.2}$$

3. *C is 2-increasing, that is for every u_1, u_2, v_1, v_2 such that $u_1 \le u_2$ and $v_1 \le v_2$,*

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0. \tag{5.3}$$

Copulas are 1-Lipschitz continuous with respect to the ℓ_1-norm on $\mathbf{I}^2$ and thus uniformly continuous on $\mathbf{I}^2$ (cf. Theorem 2.2.4 in Nelsen [2006]). The following bounds for copulas are an immediate consequence of (5.1)–(5.3).

Theorem 5.2. *Let C be a copula. Then for every u, v in $\mathbf{I}$,*

$$\max\{u+v-1,0\} \leq C(u,v) \leq \min\{u,v\}, \tag{5.4}$$

where

$$W(u,v) := \max\{u+v-1,0\} \text{ and } M(u,v) := \min\{u,v\} \tag{5.5}$$

are called the Fréchet-Hoeffding lower and upper bound, respectively.

In the two-dimensional case, both W and M are again copulas.
As Nelsen [2006] writes, the following theorem of Sklar "is central to the theory of copulas and is the foundation of many, if not most, of the applications of that theory to statistics. Sklar's theorem elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins" [Nelsen, 2006, p. 17].

Theorem 5.3. *Let X and Y be random variables with marginal distribution functions F_X and F_Y and bivariate distribution function $F_{X,Y}$. Then there exists a copula C such that for all x, y in $\bar{\mathbb{R}}$,*

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)). \tag{5.6}$$

If F_X and F_Y are continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran}F_X \times \text{Ran}F_Y$.

If X and Y have continuous distribution functions, we will denote the unique copula of X and Y with $C_{X,Y}$. As noted by Nelsen [2006], for U and V uniformly distributed on $\mathbf{I}$, Theorem 5.3 yields $F_{U,V}(u,v) = C_{U,V}(F_U(u), F_V(v)) = C_{U,V}(u,v)$, i.e. bivariate copulas are bivariate distribution functions with uniform margins.

Let $F^{(-1)} : \mathbf{I} \to \bar{\mathbb{R}}$ denote the quasi-inverse of a distribution function F as defined in Nelsen [2006] for example, i.e.

$$F^{(-1)}(t) \begin{cases} \text{is any number } x \in \bar{\mathbb{R}} \text{ such that } F(x) = t, & t \in \text{Ran}F, \\ = \inf\{x \in \bar{\mathbb{R}} : F(x) \geq t\} = \sup\{x \in \bar{\mathbb{R}} : F(x) \leq t\}, & t \notin \text{Ran}F. \end{cases} \tag{5.7}$$

Note that the generalized inverse defined in (2.4) is a special case of a quasi-inverse.
The following corollary to Theorem 5.3 shows how copulas can be constructed from univariate and bivariate distribution functions.

Corollary 5.4. *Let F_X, F_Y and $F_{X,Y}$ be as in Theorem 5.3, and let $F_X^{(-1)}$ and $F_Y^{(-1)}$ be quasi-inverses of F_X and F_Y, respectively. Furthermore, let F_X and F_Y be continuous. Then for any $(u,v) \in \mathbf{I}^2$,*

$$C_{X,Y}(u,v) = F_{X,Y}\left(F_X^{(-1)}(u), F_Y^{(-1)}(v)\right). \tag{5.8}$$

The next theorem deals with the stability of copulas under certain transformations.

Theorem 5.5. *Let X and Y be continuous random variables with copula $C_{X,Y}$. If α and β are strictly increasing on* Ran$X$ *and* Ran$Y$*, respectively, then $C_{\alpha(X),\beta(Y)} = C_{X,Y}$. Thus $C_{X,Y}$ is invariant under strictly increasing transformations of X and Y.*

Proofs of Theorem 5.2, Theorem 5.3, Corollary 5.4 and Theorem 5.5 can be found in Nelsen [2006].

By Theorem 5.3, copulas provide a link between bivariate distribution functions and their univariate margins, i.e. between probabilities that random variables fall below certain thresholds. In addition, one can also consider so-called survival copulas, which relate the univariate and bivariate survival functions to each other. Let

$$\bar{C}_{X,Y}(u,v) := u + v - 1 + C_{X,Y}(1-u, 1-v) \quad \text{for all } u, v \in \mathbf{I}. \tag{5.9}$$

Then straightforward calculations yield (cf. Nelsen [2006], p. 32)

$$\bar{F}_{X,Y}(x,y) = \bar{C}_{X,Y}(\bar{F}_X(x), \bar{F}_Y(y)), \tag{5.10}$$

$$\bar{C}_{X,Y}(u,v) = \bar{F}_{X,Y}\left(\bar{F}_X^{(-1)}(u), \bar{F}_Y^{(-1)}(v)\right), \tag{5.11}$$

with $\bar{F}^{(-1)}$ defined in analogy to $F^{(-1)}$, i.e. $\bar{F}^{(-1)}(t) = \inf\{x \in \bar{\mathbb{R}} : \bar{F}(x) \leq t\} = \sup\{x \in \bar{\mathbb{R}} : \bar{F}(x) \geq t\}$ for $t \notin \text{Ran}\bar{F}$. $\bar{C}_{X,Y}$ is called the survival copula of X and Y, and it can be shown that $\bar{C}_{X,Y}$ is indeed a copula.

5.1.2 Copula under Cross-Ownership

By Lemma 3.7 and Remark 3.6, the univariate distribution functions of V_1 and V_2 are continuous if we assume the bivariate distribution of exogenous asset values to be continuous. In this case, the copula of V_1 and V_2 under cross-ownership exists and is unique by Theorem 5.3. As becomes evident in Lemma 3.5, V_i in general depends on both, A_1 and A_2, i.e. the functional relationship between firm values and exogenous asset values cannot be described by strictly increasing functions f_1, f_2 such that $V_i = f_i(A_i),\ i = 1, 2$. Hence, the conditions of Theorem 5.5 are not met, and the copula of firm values is likely to differ from the copula of exogenous asset values under cross-ownership. By Corollary 5.4,

$$C_{V_1,V_2}(u,v) = F_{V_1,V_2}\left(F_{V_1}^{(-1)}(u), F_{V_2}^{(-1)}(v)\right), \tag{5.12}$$

but a detailed analysis of C_{V_1,V_2} is difficult due to the piecewise definition of V_1 and V_2 (cf. Lemma 3.5) and since we do not have explicit formulae for their univariate and bivariate distribution functions. Fortunately, we only need to know the behaviour of C_{V_1,V_2} in the lower left and upper right corner of the unit square $\mathbf{I}^2$ in order to be able to determine the tail dependence coefficient under cross-ownership of debt only and cross-ownership of equity only.

In the remainder of Section 5 we will always assume that $M^{\mathrm{d}}_{1,2}, M^{\mathrm{d}}_{2,1} > 0$ under cross-ownership of debt only and $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} > 0$ under cross-ownership of equity only. Furthermore, we will always assume exogenous asset values to be distributed as follows.

Assumption 5.6.

$$(A_1, A_2) \sim \mathcal{LN}((\mu_1, \mu_2)^T, \mathbf{\Sigma}) \tag{5.13}$$

where $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ *with*

$$0 < \sigma_1 \leq \sigma_2 \tag{5.14}$$

and with $|\rho| < 1$.

Under Assumption 5.6 the univariate distribution function of A_i $(i = 1, 2)$ is given by

$$F_{A_i}(x) = \begin{cases} \Phi\left(\frac{\ln(x)-\mu_i}{\sigma_i}\right), & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{5.15}$$

and it follows for $y \in (0,1)$ that $F_{A_i}(x) = y \Leftrightarrow x = \exp(\sigma_i \Phi^{-1}(y) + \mu_i)$, which is strictly increasing and continuous in y, i.e. the inverse of F_{A_i} on $[0, 1)$ is given by

$$F_{A_i}^{-1}(y) = \begin{cases} \exp(\sigma_i \Phi^{-1}(y) + \mu_i), & y > 0, \\ 0, & y = 0. \end{cases} \tag{5.16}$$

Furthermore,

$$f_{A_i}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_i x} \exp\left(-\frac{(\ln(x)-\mu_i)^2}{2\sigma_i^2}\right), & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{5.17}$$

The following two propositions deal with the asymptotic behaviour of C_{V_1,V_2} in the lower left and upper right corner of the unit square under Assumption 5.6 for cross-ownership of debt only and cross-ownership of equity only.

Proposition 5.7. *Let exogenous asset values* (A_1, A_2) *be distributed as in Assumption 5.6. Then*

$$C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(u,v) = C_{A_1,A_2}(u,v) + O\left(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(v))\right), \quad u,v \to 1, \tag{5.18}$$

and

$$C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,v) = \begin{cases} C_{A_1,A_2}(u,v) + o(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(v))), & u,v \to 0, \quad \text{if } \rho \leq 0, \\ C_{A_1,A_2}(u,v) + o(u+v), & u,v \to 0, \quad \text{if } \rho > 0. \end{cases} \tag{5.19}$$

Proof. In order to prove (5.18) we first show some asymptotic convergence results for the univariate and bivariate distribution functions of V_1^{d} and V_2^{d}. Let $q_1 \geq d_1$ be such that $f_{A_1}(x) \le f_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2)$ for all $x \ge q_1 - M^{\mathrm{d}}_{1,2}d_2$. Since the lognormal density is strictly decreasing for x sufficiently big, such a q_1 always exists. By Lemma 3.7,

$$\begin{aligned} P(V_1^{\mathrm{d}} \le q_1) = P(A_1 \le q_1 - M^{\mathrm{d}}_{1,2}d_2) \\ + P(q_1 - M^{\mathrm{d}}_{1,2}d_2 < A_1 \le q_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1 - M^{\mathrm{d}}_{1,2}A_2), \end{aligned} \tag{5.20}$$

where

$$P(q_1 - M^{\mathrm{d}}_{1,2}d_2 < A_1 \le q_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1 - M^{\mathrm{d}}_{1,2}A_2) \tag{5.21}$$
$$\le P(q_1 - M^{\mathrm{d}}_{1,2}d_2 < A_1 \le q_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1) \tag{5.22}$$
$$= \int_{q_1 - M^{\mathrm{d}}_{1,2}d_2}^{q_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1} f_{A_1}(x)\,\mathrm{d}x \tag{5.23}$$
$$\le M^{\mathrm{d}}_{1,2}(d_2 - M^{\mathrm{d}}_{2,1}d_1)\, f_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2) \tag{5.24}$$
$$= O(f_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2)), \quad q_1 \to \infty. \tag{5.25}$$

Hence, $P(V_1^{\mathrm{d}} \le q_1) = P(A_1 \le q_1 - M^{\mathrm{d}}_{1,2}d_2) + O(f_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2))$, $q_1 \to \infty$. Note that if $d_2 \le M^{\mathrm{d}}_{2,1}d_1$, the probability in (5.21) is zero, but nevertheless, the derived result remains valid. Analogously we obtain

$$P(V_2^{\mathrm{d}} \le q_2)$$
$$= P(A_2 \le q_2 - M^{\mathrm{d}}_{2,1}d_1) + P(q_2 - M^{\mathrm{d}}_{2,1}d_1 < A_2 \le q_2 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_2 - M^{\mathrm{d}}_{2,1}A_1) \tag{5.26}$$
$$= P(A_2 \le q_2 - M^{\mathrm{d}}_{2,1}d_1) + O(f_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1)), \quad q_2 \to \infty. \tag{5.27}$$

Furthermore, Lemma A.1 yields for q_1, q_2 sufficiently big,

$$\begin{aligned} P(V_1^{\mathrm{d}} \le q_1, V_2^{\mathrm{d}} \le q_2) = P(A_1 \le q_1 - M^{\mathrm{d}}_{1,2}d_2,\, A_2 \le q_2 - M^{\mathrm{d}}_{2,1}d_1) \\ + P(q_1 - M^{\mathrm{d}}_{1,2}d_2 < A_1 \le q_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1 - M^{\mathrm{d}}_{1,2}A_2) \\ + P(q_2 - M^{\mathrm{d}}_{2,1}d_1 < A_2 \le q_2 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_2 - M^{\mathrm{d}}_{2,1}A_1), \end{aligned} \tag{5.28}$$

where again, depending on the relative sizes of d_2 and $M^{\mathrm{d}}_{2,1}d_1$, and d_1 and $M^{\mathrm{d}}_{1,2}d_2$, the second or third probability of (5.28) might be zero. For $u, v \in (0,1)$, let $q_1, q_2 \in \mathbb{R}$ be such that $u = F_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2)$ and $v = F_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1)$, and we assume u and v to be sufficiently big so that (5.28) holds. Then

$$\begin{aligned} &|C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(u,v) - C_{A_1,A_2}(u,v)| \\ &= |C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(F_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2), F_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1)) \\ &\quad - C_{A_1,A_2}(F_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2), F_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1))| \end{aligned} \tag{5.29}$$
$$\begin{aligned} &\le |C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(F_{V_1^{\mathrm{d}}}(q_1), F_{V_2^{\mathrm{d}}}(q_2)) - C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(F_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2), F_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1))| \\ &\quad + |C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(F_{V_1^{\mathrm{d}}}(q_1), F_{V_2^{\mathrm{d}}}(q_2)) - C_{A_1,A_2}(F_{A_1}(q_1 - M^{\mathrm{d}}_{1,2}d_2), F_{A_2}(q_2 - M^{\mathrm{d}}_{2,1}d_1))| \end{aligned} \tag{5.30}$$

$$
\begin{aligned}
&\leq |F_{V_1^{\mathrm{d}}}(q_1) - F_{A_1}(q_1 - M_{1,2}^{\mathrm{d}} d_2)| + |F_{V_2^{\mathrm{d}}}(q_2) - F_{A_2}(q_2 - M_{2,1}^{\mathrm{d}} d_1)| \\
&\quad + |F_{V_1^{\mathrm{d}}, V_2^{\mathrm{d}}}(q_1, q_2) - F_{A_1, A_2}(q_1 - M_{1,2}^{\mathrm{d}} d_2, q_2 - M_{2,1}^{\mathrm{d}} d_1)| \qquad (5.31) \\
&= 2|F_{V_1^{\mathrm{d}}}(q_1) - F_{A_1}(q_1 - M_{1,2}^{\mathrm{d}} d_2)| + 2|F_{V_2^{\mathrm{d}}}(q_2) - F_{A_2}(q_2 - M_{2,1}^{\mathrm{d}} d_1)| \qquad (5.32) \\
&= O(f_{A_1}(q_1 - M_{1,2}^{\mathrm{d}} d_2)) + O(f_{A_2}(q_2 - M_{2,1}^{\mathrm{d}} d_1)), \quad q_1, q_2 \to \infty, \qquad (5.33) \\
&= O(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(v))), \quad u, v \to 1, \qquad (5.34)
\end{aligned}
$$

where (5.31) follows from the Lipschitz continuity of copulas and Theorem 5.3, (5.32) follows from (5.20), (5.26) and (5.28), and (5.33) follows from (5.25) and (5.27). Altogether, (5.18) is shown.

The proof of (5.19) has a similar structure. For $0 < q_1 < d_1$ it follows from Lemma 3.7 that

$$
P(V_1^{\mathrm{e}} \leq q_1) = P(A_1 \leq q_1) - \underbrace{P\left(q_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2) < A_1 \leq q_1\right)}_{=:p_1(q_1)} \qquad (5.35)
$$

with

$$
\begin{aligned}
p_1(q_1) &\leq P(A_1 \leq q_1, A_2 > d_2) \qquad (5.36) \\
&= \int_0^{q_1} P(A_2 > d_2 \mid A_1 = x) f_{A_1}(x) \, \mathrm{d}x \qquad (5.37) \\
&= \int_0^{q_1} \Phi\left(\frac{-1}{\sqrt{1-\rho^2}} \left(\frac{\ln(d_2) - \mu_2}{\sigma_2} - \rho \frac{\ln(x) - \mu_1}{\sigma_1}\right)\right) f_{A_1}(x) \, \mathrm{d}x \qquad (5.38) \\
&\leq q_1 \, f_{A_1}(q_1) \qquad (5.39)
\end{aligned}
$$

for q_1 sufficiently small. Note that (5.38) follows from the fact that for a bivariate standard normally distributed random vector (X, Y) with correlation ρ, we have $X \mid Y = y \sim \mathcal{N}(\rho y, 1 - \rho^2)$. Hence, by (5.39),

$$
F_{V_1^{\mathrm{e}}}(q_1) = F_{A_1}(q_1) + o(f_{A_1}(q_1)), \quad q_1 \to 0. \qquad (5.40)
$$

For $\rho > 0$, a stronger result can be derived. In this case, the Φ-term in (5.38) is strictly increasing in x, i.e.

$$
p_1(q_1) \leq \Phi\left(\frac{-1}{\sqrt{1-\rho^2}} \left(\frac{\ln(d_2) - \mu_2}{\sigma_2} - \rho \frac{\ln(q_1) - \mu_1}{\sigma_1}\right)\right) \times F_{A_1}(q_1), \qquad (5.41)
$$

and therefore

$$
F_{V_1^{\mathrm{e}}}(q_1) = F_{A_1}(q_1) + o(F_{A_1}(q_1)), \quad q_1 \to 0, \quad \text{if } \rho > 0. \qquad (5.42)
$$

Analogously, one can show that

$$F_{V_2^{\mathrm{e}}}(q_2) = F_{A_2}(q_2) + \begin{cases} o(f_{A_2}(q_2)), & q_2 \to 0, \quad \text{if } \rho \le 0, \\ o(F_{A_2}(q_2)), & q_2 \to 0, \quad \text{if } \rho > 0, \end{cases} \tag{5.43}$$

and by Lemma A.2, for $q_1 < d_1$ and $q_2 < d_2$,

$$F_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(q_1, q_2) = F_{A_1,A_2}(q_1, q_2). \tag{5.44}$$

For $0 < u < F_{A_1}(d_1)$ and $0 < v < F_{A_2}(d_2)$, let $q_1, q_2 \in \mathbb{R}$ be such that $u = F_{A_1}(q_1)$ and $v = F_{A_2}(q_2)$. Then

$$\begin{aligned} &|C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,v) - C_{A_1,A_2}(u,v)| \\ &= |C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(F_{A_1}(q_1), F_{A_2}(q_2)) - C_{A_1,A_2}(F_{A_1}(q_1), F_{A_2}(q_2))| && (5.45) \\ &= |C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(F_{A_1}(q_1), F_{A_2}(q_2)) - C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(F_{V_1^{\mathrm{e}}}(q_1), F_{V_2^{\mathrm{e}}}(q_2))| && (5.46) \\ &\le |F_{A_1}(q_1) - F_{V_1^{\mathrm{e}}}(q_1)| + |F_{A_2}(q_2) - F_{V_2^{\mathrm{e}}}(q_2)| && (5.47) \\ &= \begin{cases} o(f_{A_1}(q_1)) + o(f_{A_2}(q_2)), & q_1, q_2 \to 0, \quad \text{if } \rho \le 0, \\ o(F_{A_1}(q_1)) + o(F_{A_2}(q_2)), & q_1, q_2 \to 0, \quad \text{if } \rho > 0, \end{cases} && (5.48) \\ &= \begin{cases} o(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(v))), & u, v \to 0, \quad \text{if } \rho \le 0, \\ o(u+v), & u, v \to 0, \quad \text{if } \rho > 0, \end{cases} && (5.49) \end{aligned}$$

where (5.46) follows from Theorem 5.3 and (5.44), (5.47) follows from the Lipschitz continuity of copulas, and (5.48) follows from (5.40), (5.42) and (5.43). □

For the analysis of the behaviour of C_{V_1,V_2} in the respective "opposite" corners of $\mathbf{I}^2$ we do not need to assume that (A_1, A_2) follows a bivariate lognormal distribution.

Proposition 5.8. *Let exogenous asset values (A_1, A_2) follow a continuous bivariate distribution. Then*

$$C_{V_1^{\mathrm{d}},V_2^{\mathrm{d}}}(u,v) = C_{A_1+M_{1,2}^{\mathrm{d}}A_2,\, M_{2,1}^{\mathrm{d}}A_1+A_2}(u,v) \quad \text{if } u < \min\left\{F_{V_1^{\mathrm{d}}}(d_1),\, F_{V_1^{\mathrm{d}}}(M_{1,2}^{\mathrm{d}}d_2)\right\} \text{ and } v < \min\left\{F_{V_2^{\mathrm{d}}}(d_2),\, F_{V_2^{\mathrm{d}}}(M_{2,1}^{\mathrm{d}}d_1)\right\}, \tag{5.50}$$

and

$$C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,v) = C_{A_1+M_{1,2}^{\mathrm{e}}A_2,\, M_{2,1}^{\mathrm{e}}A_1+A_2}(u,v) \quad \text{if } u > F_{V_1^{\mathrm{e}}}\left(d_1 + \tfrac{1}{M_{2,1}^{\mathrm{e}}}d_2\right) \text{ and } v > F_{V_2^{\mathrm{e}}}\left(d_2 + \tfrac{1}{M_{1,2}^{\mathrm{e}}}d_1\right). \tag{5.51}$$

Proof. For the proof of (5.50), let $q_1 < \min\{d_1,\, M_{1,2}^{\mathrm{d}}d_2\}$ and $q_2 < \min\{d_2,\, M_{2,1}^{\mathrm{d}}d_1\}$.

Then Lemma 3.7, Remark 3.6 and Lemma A.1 yield

$$P(V_1^{\mathrm{d}} \leq q_1) = P(A_1 + M_{1,2}^{\mathrm{d}} A_2 \leq (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}) q_1), \tag{5.52}$$

$$P(V_2^{\mathrm{d}} \leq q_2) = P(A_2 + M_{2,1}^{\mathrm{d}} A_1 \leq (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}) q_2), \tag{5.53}$$

$$P(V_1^{\mathrm{d}} \leq q_1, V_2^{\mathrm{d}} \leq q_2) = P(A_1 + M_{1,2}^{\mathrm{d}} A_2 \leq (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}) q_1, \\ A_2 + M_{2,1}^{\mathrm{d}} A_1 \leq (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}) q_2). \tag{5.54}$$

Let $u < \min\{F_{V_1^{\mathrm{d}}}(d_1), F_{V_1^{\mathrm{d}}}(M_{1,2}^{\mathrm{d}} d_2)\}$ and $v < \min\{F_{V_2^{\mathrm{d}}}(d_2), F_{V_2^{\mathrm{d}}}(M_{2,1}^{\mathrm{d}} d_1)\}$. Setting $A_1' := (A_1 + M_{1,2}^{\mathrm{d}} A_2)/(1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})$ and $A_2' := (A_2 + M_{2,1}^{\mathrm{d}} A_1)/(1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})$, Corollary 5.4 and (5.52)–(5.54) imply

$$C_{V_1^{\mathrm{d}}, V_2^{\mathrm{d}}}(u,v) = F_{V_1^{\mathrm{d}}, V_2^{\mathrm{d}}}\left(F_{V_1^{\mathrm{d}}}^{-1}(u), F_{V_2^{\mathrm{d}}}^{-1}(v)\right) = F_{A_1', A_2'}\left(F_{A_1'}^{-1}(u), F_{A_2'}^{-1}(v)\right) = C_{A_1', A_2'}(u,v), \tag{5.55}$$

and (5.50) follows from Proposition 5.5.

For the proof of (5.51), let $q_1 > d_1 + d_2/M_{2,1}^{\mathrm{e}}$ and $q_2 > d_2 + d_1/M_{1,2}^{\mathrm{e}}$. Lemma 3.7 implies

$$P(V_1^{\mathrm{e}} \leq q_1) = P(A_1 + M_{1,2}^{\mathrm{e}} A_2 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) q_1 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_1 + M_{1,2}^{\mathrm{e}} d_2), \tag{5.56}$$

since $q_1 \geq q_1 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}(d_1 - q_1) + M_{1,2}^{\mathrm{e}}(d_2 - A_2) \Leftrightarrow M_{2,1}^{\mathrm{e}} q_1 \geq M_{2,1}^{\mathrm{e}} d_1 + d_2 - A_2$, which holds because of $q_1 > d_1 + d_2/M_{2,1}^{\mathrm{e}}$ and $A_2 \geq 0$. Similarly, Remark 3.6 and Lemma A.2 yield

$$P(V_2^{\mathrm{e}} \leq q_2) = P(A_2 + M_{2,1}^{\mathrm{e}} A_1 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) q_2 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_2 + M_{2,1}^{\mathrm{e}} d_1), \tag{5.57}$$

$$P(V_1^{\mathrm{e}} \leq q_1, V_2^{\mathrm{e}} \leq q_2) = P(A_1 + M_{1,2}^{\mathrm{e}} A_2 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) q_1 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_1 + M_{1,2}^{\mathrm{e}} d_2, \\ A_2 + M_{2,1}^{\mathrm{e}} A_1 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) q_2 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_2 + M_{2,1}^{\mathrm{e}} d_1). \tag{5.58}$$

Hence, by Corollary 5.4,

$$C_{V_1^{\mathrm{e}}, V_2^{\mathrm{e}}}(u,v) = C_{A_1'', A_2''}(u,v) \text{ if } u > F_{V_1^{\mathrm{e}}}(d_1 + d_2/M_{2,1}^{\mathrm{e}}) \text{ and } v > F_{V_2^{\mathrm{e}}}(d_2 + d_1/M_{1,2}^{\mathrm{e}}), \tag{5.59}$$

with $A_1'' := (A_1 + M_{1,2}^{\mathrm{e}} A_2 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_1 - M_{1,2}^{\mathrm{e}} d_2)/(1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}})$ and $A_2'' := (A_2 + M_{2,1}^{\mathrm{e}} A_1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_2 - M_{2,1}^{\mathrm{e}} d_1)/(1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}})$, and (5.51) follows from Theorem 5.5. □

Proposition 5.7 and Proposition 5.8 will be crucial for the derivation of the tail dependence coefficient under cross-ownership in Section 5.4 and Section 5.5.

5.2 Definition of Tail Dependence

Following Frahm et al. [2005], we define the lower tail dependence coefficient (lower TDC) of two random variables X and Y as

$$\lambda_{\mathrm{L}} = \lim_{u \to 0} P(F_X(X) \leq u \,|\, F_Y(Y) \leq u), \tag{5.60}$$

provided the limit exists. Analogously, the upper tail dependence coefficient (upper TDC) is given by

$$\lambda_{\mathrm{U}} = \lim_{u \to 1} P(F_X(X) > u \,|\, F_Y(Y) > u), \tag{5.61}$$

provided the limit exists. Note that the lower and upper TDC is symmetric with respect to X and Y (cf. (5.62) and (5.63)). As Frahm et al. [2005] note, "the TDC roughly corresponds to the probability that one margin exceeds a high/low threshold under the condition that the other margin exceeds a high/low threshold"[Frahm et al., 2005, p. 3]. If $\lambda_{\mathrm{L}} = 0$, X and Y are called lower tail independent, if $\lambda_{\mathrm{L}} = 1$, they are called perfectly lower tail dependent. Similarly, for $\lambda_{\mathrm{U}} = 0$ and $\lambda_{\mathrm{U}} = 1$ they are called upper tail independent and perfectly upper tail dependent, respectively.

If X and Y have continuous distributions, Theorem 5.3, Theorem 5.5, (5.9) and (5.10) imply that λ_{L} and λ_{U} can also be expressed in terms of the copula $C_{X,Y}$:

$$\lambda_{\mathrm{L}} = \lim_{u \to 0} \frac{P(F_X(X) \leq u, F_Y(Y) \leq u)}{P(F_Y(Y) \leq u)} = \lim_{u \to 0} \frac{C_{X,Y}(u,u)}{u}, \tag{5.62}$$

$$\lambda_{\mathrm{U}} = \lim_{u \to 1} \frac{P(F_X(X) > u, F_Y(Y) > u)}{P(F_Y(Y) > u)} = \lim_{u \to 1} \frac{\bar{C}_{X,Y}(1-u,1-u)}{1-u} \tag{5.63}$$

$$= \lim_{u \to 1} \frac{1 - 2u + C_{X,Y}(u,u)}{1-u} \tag{5.64}$$

$$= \lim_{u \to 0} \frac{-1 + 2u + C_{X,Y}(1-u,1-u)}{u}. \tag{5.65}$$

5.3 Tail Dependence without Cross-Ownership

In order to have a comparison, we first consider the lower and upper tail dependence coefficient for firms not linked by cross-ownership. By Definition 3.1 we then have $V_i = A_i$, i.e. the copula of firm values V_1 and V_2 of plain firms equals the copula of the exogenous asset values A_1 and A_2. Since we assume (A_1, A_2) to follow a bivariate lognormal distribution (cf. Assumption 5.6), we have $A_i = \exp(Z_i)$ with $Z_i \sim \mathcal{N}(\mu_i, \sigma_i)$, $i = 1, 2$. Then Theorem 5.5 yields

$$C_{V_1,V_2} = C_{A_1,A_2} = C_{\exp(Z_1),\exp(Z_2)} = C_{Z_1,Z_2} = C_{\frac{Z_1-\mu_1}{\sigma_1},\frac{Z_2-\mu_2}{\sigma_2}} = C_{\mathrm{G}}, \tag{5.66}$$

where C_{G} stands for the Gaussian copula, which is defined as follows. Let (X, Y) be a bivariate standard normally distributed random vector with correlation coefficient ρ

($|\rho| < 1$). By Theorem 5.3 there exists a unique copula C_{G} such that

$$\Phi_\rho(x, y) = C_{\mathrm{G}}(\Phi(x), \Phi(y)), \quad x, y \in \mathbb{R}, \tag{5.67}$$

and Definition 5.1 and Corollary 5.4 yield

$$C_{\mathrm{G}}(u, v) = \begin{cases} \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), & u, v \in (0, 1), \\ 0, & u = 0 \text{ or } v = 0, \\ u, & v = 1, \\ v, & u = 1. \end{cases} \tag{5.68}$$

Hence, under the assumption of bivariate lognormally distributed exogenous asset values, the copula of firm values of plain firms equals the Gaussian copula.
If $\lambda_{\mathrm{L,G}}$ and $\lambda_{\mathrm{U,G}}$ stand for the lower and upper TDC of C_{G}, respectively, straightforward calculations show (cf. Example 3.4 and Section 5.2 in Embrechts et al. [2003]) that the Gaussian copula with $|\rho| < 1$ is lower and upper tail independent, i.e.

$$\lambda_{\mathrm{L,G}} = \lambda_{\mathrm{U,G}} = 0. \tag{5.69}$$

Hence, under Assumption 5.6, firm values of firms not linked by cross-ownership are lower and upper tail independent. In the following, we will derive the lower and upper TDC under cross-ownership of debt only and under cross-ownership of equity only, which we will denote by $\lambda_{\mathrm{L}}^{\mathrm{d}}$, $\lambda_{\mathrm{U}}^{\mathrm{d}}$, $\lambda_{\mathrm{L}}^{\mathrm{e}}$ and $\lambda_{\mathrm{U}}^{\mathrm{e}}$. Recall that we assume $M_{1,2}^{\mathrm{d}}, M_{2,1}^{\mathrm{d}} > 0$ under cross-ownership of debt only and $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} > 0$ under cross-ownership of equity only, and that exogenous asset values are distributed as in Assumption 5.6.

5.4 Tail Dependence under XOS of Debt only

5.4.1 $\lambda_{\mathrm{L}}^{\mathrm{d}}$

By Corollary 5.4, Proposition 5.8 and (5.62),

$$\lambda_{\mathrm{L}}^{\mathrm{d}} = \lim_{u \to 0} \frac{C_{A_1 + M_{1,2}^{\mathrm{d}} A_2, M_{2,1}^{\mathrm{d}} A_1 + A_2}(u, u)}{u} \tag{5.70}$$

$$= \lim_{u \to 0} \frac{F_{A_1 + M_{1,2}^{\mathrm{d}} A_2, M_{2,1}^{\mathrm{d}} A_1 + A_2}\left(F^{-1}_{A_1 + M_{1,2}^{\mathrm{d}} A_2}(u), F^{-1}_{M_{2,1}^{\mathrm{d}} A_1 + A_2}(u)\right)}{u}, \tag{5.71}$$

provided the limit exists. Note that due to Assumption 5.6, $F_{A_1 + M_{1,2}^{\mathrm{d}} A_2}$ and $F_{M_{2,1}^{\mathrm{d}} A_1 + A_2}$ are strictly increasing on $\mathbb{R}_0^+$, i.e. $F^{-1}_{A_1 + M_{1,2}^{\mathrm{d}} A_2}$ and $F^{-1}_{M_{2,1}^{\mathrm{d}} A_1 + A_2}$ are ordinary inverses. For the evaluation of (5.71) we will employ the insights of Section A.4.1 by identifying m_1 and m_2 of Section A.4.1 with $M_{2,1}^{\mathrm{d}}$ and $M_{1,2}^{\mathrm{d}}$, respectively. We proceed with the notation in terms of m_1 and m_2 for better readability and since the following considerations hold beyond the context of cross-ownership (cf. Section 5.5.3). Hence, with $m_1 = M_{2,1}^{\mathrm{d}}$ and

$m_2 = M^{\mathrm{d}}_{1,2}$, $y := F^{-1}_{m_1 A_1 + A_2}(u)$ and

$$\vartheta(y) := \frac{F^{-1}_{A_1+m_2A_2}(F_{m_1A_1+A_2}(y))}{y}, \tag{5.72}$$

we have

$$\lambda^{\mathrm{d}}_{\mathrm{L}} = \lim_{y\to 0} \frac{P\left(A_1 + m_2A_2 \le \vartheta(y)\, y,\ m_1A_1 + A_2 \le y\right)}{P(m_1A_1 + A_2 \le y)}, \tag{5.73}$$

provided the limit exists. Since the asymptotic distributions of $A_1+m_2A_2$ and $m_1A_1+A_2$ and the limit of $\vartheta(y)$ for $y \to 0$ strongly depend on ρ, we need to distinguish between the following three cases. In doing so, we always assume $m_1, m_2 \in (0,1)$.

5.4.1.1 Determination of $\lambda^{\mathrm{d}}_{\mathrm{L}}$ if $-1 < \rho < \sigma_1/\sigma_2$

In this section we always suppose $\rho < \sigma_1/\sigma_2$ in Assumption 5.6, i.e. by Lemma A.11,

$$\lim_{y\to 0} \vartheta(y) = m_2^{\frac{s_1}{s_1+s_2}} \left(\frac{1}{m_1}\right)^{\frac{s_2}{s_1+s_2}} \in \left(m_2, \frac{1}{m_1}\right), \tag{5.74}$$

with $s_1, s_2 > 0$ defined as in Lemma A.10. Let $\gamma \in (m_2, 1/m_1)$ be such that

$$m_2^{\frac{s_1}{s_1+s_2}} \left(\frac{1}{m_1}\right)^{\frac{s_2}{s_1+s_2}} < \gamma < \frac{s_1}{s_1+s_2} m_2 + \frac{s_2}{s_1+s_2}\frac{1}{m_1}. \tag{5.75}$$

Note that such a γ exists since the inequality of geometric and arithmetic means is strict because of $s_1, s_2 > 0$. Then

$$\frac{P\left(A_1 + m_2A_2 \le \vartheta(y)\, y,\ m_1A_1 + A_2 \le y\right)}{P(m_1A_1 + A_2 \le y)} \le \frac{P\left(A_1 + m_2A_2 \le \gamma y,\ m_1A_1 + A_2 \le y\right)}{P(m_1A_1 + A_2 \le y)} \tag{5.76}$$

for y sufficiently small, and

$$\begin{aligned}
&\frac{P\left(A_1 + m_2A_2 \le \gamma y,\ m_1A_1 + A_2 \le y\right)}{P(m_1A_1 + A_2 \le y)} \\
&= \frac{1}{F_{m_1A_1+A_2}(y)} \int_0^y P(A_1 \le \gamma y - m_2 x,\ A_1 \le (y-x)/m_1 \,|\, A_2 = x)\, f_{A_2}(x)\,\mathrm{d}x && (5.77)\\
&= \frac{1}{F_{m_1A_1+A_2}(y)} \left(\int_0^{\delta y} P(A_1 \le \gamma y - m_2 x \,|\, A_2 = x)\, f_{A_2}(x)\,\mathrm{d}x \right. \\
&\qquad \left. + \int_{\delta y}^{y} P(A_1 \le (y-x)/m_1 \,|\, A_2 = x)\, f_{A_2}(x)\,\mathrm{d}x \right) && (5.78)
\end{aligned}$$

with

$$\delta := \frac{\frac{1}{m_1} - \gamma}{\frac{1}{m_1} - m_2} \in (0,1). \tag{5.79}$$

In order to be able to apply L'Hôpital's rule to (5.78) for $y \to 0$, we need to see that both integrals are differentiable with respect to y. For that we will employ the Leibniz rule for the differentiation of parameter integrals (cf. Theorem 3 of Swartz [1994], for example).

For $x \in (0, a_1)$ for some $a_1 > 0$, and for $y > 0$, let

$$g_1(x,y) := P\left(A_1 \le \gamma y - m_2 x \,|\, A_2 = x\right) \tag{5.80}$$

$$= 1_{\{y > m_2 x/\gamma\}} \times \Phi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho\frac{\ln(x) - \mu_2}{\sigma_2}\right)\right). \tag{5.81}$$

Obviously, for an arbitrary $y > 0$, $g_1(\cdot, y) \times f_{A_2}(\cdot)$ is Lebesgue-integrable on $(0, a_1)$. Furthermore, for a given $x \in (0, a_1)$,

$$\frac{\partial}{\partial y}\Phi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho\frac{\ln(x) - \mu_2}{\sigma_2}\right)\right) \tag{5.82}$$

$$= \frac{1}{\sqrt{1-\rho^2}\sigma_1}\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho\frac{\ln(x) - \mu_2}{\sigma_2}\right)\right)\frac{\gamma}{\gamma y - m_2 x}, \tag{5.83}$$

which implies $\lim_{y \to m_2 x/\gamma} D_2 g_1(x,y) = 0$, i.e. $g_1(x, \cdot)$ is continuously differentiable with respect to $y > 0$. Furthermore, $|D_2 g_1(x,y)| \le \text{const.} \times x^{-\rho\sigma_1/\sigma_2}$, as (5.83) implies

$$\left|\frac{\partial}{\partial y}\Phi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho\frac{\ln(x) - \mu_2}{\sigma_2}\right)\right)\right| \tag{5.84}$$

$$= \text{const.} \times \exp\left(-\frac{(\sigma_2(\ln(\gamma y - m_2 x) - \mu_1) - \rho\sigma_1(\ln(x) - \mu_2))^2}{2(1-\rho^2)\sigma_1^2\sigma_2^2} - \ln(\gamma y - m_2 x)\right) \tag{5.85}$$

and straightforward calculations show that the argument of the exp-term as a quadratic function in $\ln(\gamma y - m_2 x)$ is bounded from above by some constant times $x^{-\rho\sigma_1/\sigma_2}$. Furthermore, $\lim_{x \to 0} x^{-\rho\sigma_1/\sigma_2} f_{A_2}(x) = 0$, i.e. $x^{-\rho\sigma_1/\sigma_2} f_{A_2}(x)$ is integrable on $(0, a_1)$. Hence, the integral $\int_0^{a_1} P\left(A_1 \le \gamma y - m_2 x \,|\, A_2 = x\right) f_{A_2}(x)\,\mathrm{d}x$ is differentiable with respect to $y > 0$ by the Leibniz rule for the differentiation of parameter integrals. Similarly, let

$$g_2(x,y) := P\left(A_1 \le (y - x)/m_1 \,|\, A_2 = x\right) \tag{5.86}$$

$$= 1_{\{y > x\}} \times \Phi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln((y - x)/m_1) - \mu_1}{\sigma_1} - \rho\frac{\ln(x) - \mu_2}{\sigma_2}\right)\right), \tag{5.87}$$

where $x \in (0, a_2)$ for some $a_2 > 0$ and with $y > 0$. Then we have $\lim_{y \to x} D_2 g_2(x,y) = 0$, i.e. $g_2(x, \cdot)$ is continuously differentiable with respect to $y > 0$ and again, its derivative is bounded by some constant times $x^{-\rho\sigma_1/\sigma_2}$, with $x^{-\rho\sigma_1/\sigma_2} f_{A_2}(x)$ being integrable on $(0, a_2)$. Thus, the integral $\int_0^{a_2} P\left(A_1 \le (y - x)/m_1 \,|\, A_2 = x\right) f_{A_2}(x)\,\mathrm{d}x$ is differentiable with respect to $y > 0$ by the Leibniz rule. Hence, by L'Hôpital's rule, the Leibniz rule and the chain rule,

$$\lim_{y\to 0} \frac{1}{F_{m_1A_1+A_2}(y)} \left(\int_0^{\delta y} P(A_1 \le \gamma y - m_2 x \,|\, A_2 = x)\, f_{A_2}(x)\,\mathrm{d}x + \int_{\delta y}^{y} P(A_1 \le (y-x)/m_1 \,|\, A_2 = x)\, f_{A_2}(x)\,\mathrm{d}x \right) \tag{5.88}$$

$$= \lim_{y\to 0} \frac{1}{f_{m_1A_1+A_2}(y)} \times \Bigg(\int_0^{\delta y} \frac{1}{\sqrt{1-\rho^2}\sigma_1} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho \frac{\ln(x) - \mu_2}{\sigma_2} \right) \right) \frac{\gamma f_{A_2}(x)}{\gamma y - m_2 x}\,\mathrm{d}x$$
$$+ \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln(\gamma y - m_2 \delta y) - \mu_1}{\sigma_1} - \rho \frac{\ln(\delta y) - \mu_2}{\sigma_2} \right) \right) f_{A_2}(\delta y)\,\delta$$
$$+ \int_{\delta y}^{y} \frac{1}{\sqrt{1-\rho^2}\sigma_1} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln((y-x)/m_1) - \mu_1}{\sigma_1} - \rho \frac{\ln(x) - \mu_2}{\sigma_2} \right) \right) \frac{f_{A_2}(x)}{y-x}\,\mathrm{d}x$$
$$- \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln((y-\delta y)/m_1) - \mu_1}{\sigma_1} - \rho \frac{\ln(\delta y) - \mu_2}{\sigma_2} \right) \right) f_{A_2}(\delta y)\,\delta \Bigg) \tag{5.89}$$

$$= \lim_{y\to 0} \frac{1}{f_{m_1A_1+A_2}(y)} \times \frac{1}{\sqrt{1-\rho^2}\sigma_1} \times \Bigg(\int_0^{\delta y} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln(\gamma y - m_2 x) - \mu_1}{\sigma_1} - \rho \frac{\ln(x) - \mu_2}{\sigma_2} \right) \right) \frac{\gamma f_{A_2}(x)}{\gamma y - m_2 x}\,\mathrm{d}x$$
$$+ \int_{\delta y}^{y} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln((y-x)/m_1) - \mu_1}{\sigma_1} - \rho \frac{\ln(x) - \mu_2}{\sigma_2} \right) \right) \frac{f_{A_2}(x)}{y-x}\,\mathrm{d}x \Bigg) \tag{5.90}$$

$$= \lim_{y\to 0} \frac{1}{f_{m_1A_1+A_2}(y)} \times \frac{1}{\sqrt{1-\rho^2}\sigma_1} \times \Bigg(\int_0^{\delta} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln(\gamma y - m_2 x y) - \mu_1}{\sigma_1} - \rho \frac{\ln(xy) - \mu_2}{\sigma_2} \right) \right) \frac{\gamma f_{A_2}(xy)}{\gamma - m_2 x}\,\mathrm{d}x$$
$$+ \int_{\delta}^{1} \varphi\left(\frac{1}{\sqrt{1-\rho^2}} \left(\frac{\ln((y-xy)/m_1) - \mu_1}{\sigma_1} - \rho \frac{\ln(xy) - \mu_2}{\sigma_2} \right) \right) \frac{f_{A_2}(xy)}{1-x}\,\mathrm{d}x \Bigg), \tag{5.91}$$

where (5.90) follows from the fact that $\gamma y - m_2\delta y = (y - \delta y)/m_1$ by the definition of δ, i.e. the Φ-terms in (5.89) cancel. Furthermore, (5.91) follows from the substitution $x \mapsto xy$. Next, we want to pull the limit in (5.91) into the integrals. In order to be able to apply the dominated convergence theorem, we need to see that both integrands divided by $f_{m_1A_1+A_2}(y)$ have an integrable majorant. Note that the term $(\gamma - m_2 x)^{-1}$ can be neglected as it is bounded by $0 < \gamma - m_2\delta < \gamma - m_2 x < \gamma$. By Lemma A.10, for

$x > 0$ and $y \to 0$,

$$\frac{f_{A_2}(xy)}{f_{m_1A_1+A_2}(y)}\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2xy)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right) \tag{5.92}$$

$$\sim \text{const.}\times\frac{\sqrt{-\ln(y)}}{x}\times\exp\Bigg(S\ln(y) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\Big[-(s_1+s_2)\ln(y)^2 + 2\ln(y)\Big(s_1\mu_2 + s_2(\mu_1+\ln(m_1))\Big) + (1-\rho^2)\sigma_1^2\Big(\ln(x)+\ln(y)-\mu_2\Big)^2 + \Big((\sigma_2-\rho\sigma_1)\ln(y) + \sigma_2(\ln(\gamma - m_2x)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\Big)^2\Big]\Bigg) \tag{5.93}$$

$$= \text{const.}\times\frac{\sqrt{-\ln(y)}}{x}\times\exp\Bigg(S\ln(y) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\times \Big[\ln(y)^2\underbrace{\Big(-(s_1+s_2)+(1-\rho^2)\sigma_1^2+(\sigma_2-\rho\sigma_1)^2\Big)}_{=0} + 2\ln(y)\Big\{s_1\mu_2 + s_2(\mu_1+\ln(m_1)) + (1-\rho^2)\sigma_1^2(\ln(x)-\mu_2) + (\sigma_2-\rho\sigma_1)\big(\sigma_2(\ln(\gamma-m_2x)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\big)\Big\} + (1-\rho^2)\sigma_1^2(\ln(x)-\mu_2)^2 + \Big(\sigma_2(\ln(\gamma-m_2x)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\Big)^2\Big]\Bigg) \tag{5.94}$$

$$= \text{const.}\times\frac{\sqrt{-\ln(y)}}{x}\times\exp\Bigg(S\ln(y) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\times \Big[2\ln(y)\Big\{s_1\ln(x) + s_2\ln(\gamma-m_2x) + s_2\ln(m_1)\Big\} + (1-\rho^2)\sigma_1^2(\ln(x)-\mu_2)^2 + \Big(\sigma_2(\ln(\gamma-m_2x)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\Big)^2\Big]\Bigg) \tag{5.95}$$

$$\leq \text{const.}\times\frac{\sqrt{-\ln(y)}}{x}\times\exp\Bigg(\frac{-1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\times \Big[2\ln(y)\left\{-s_1\ln\left(\frac{s_1}{s_1+s_2}\right) - s_2\ln\left(\frac{s_2}{s_1+s_2}\right) + s_1\ln(\delta) + s_2\ln(1-\delta)\right\} + (1-\rho^2)\sigma_1^2(\ln(x)-\mu_2)^2 + \Big(\sigma_2(\ln(\gamma-m_2x)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\Big)^2\Big]\Bigg) \tag{5.96}$$

$$\to 0, \quad y \to 0, \tag{5.97}$$

where (5.96) and (5.97) can be seen as follows. Straightforward calculations show that the term in curly brackets in (5.95) is strictly increasing in $x \in (0, \frac{s_1 \times \gamma/m_2}{s_1+s_2})$ and in particular for $x \in (0, \delta)$ due to (5.75) and (5.79). Hence, the term in curly brackets becomes maximal for x equal to the upper bound of integration δ, with corresponding maximum value $s_1 \ln(\delta) + s_2 \ln\left(\frac{\gamma - m_2}{1/m_1 - m_2}\right) = s_1 \ln(\delta) + s_2 \ln(1-\delta)$. This and (A91) yield (5.96). Since the function $s_1 \ln(z) + s_2 \ln(1-z)$, $z \in (0,1)$, takes its maximum value for $z = s_1/(s_1+s_2)$, but $\delta \neq s_1/(s_1+s_2)$ by (5.75) and (5.79), the term in curly brackets in (5.96) is strictly negative, and (5.97) follows for any $x \in (0, \delta]$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\delta] \times (0,1]$ with limit $(0, y^*)$, $y^* \in [0,1]$. Then, if we denote the RHS of (5.96) by $g_3(x,y)$, we have $\lim_{n\to\infty} g_3(x_n, y_n) = 0$, since the expression $\sqrt{-\ln(y)}/x$ is dominated by the exp-term. Altogether, it follows that $\tilde{g}_3$ defined on $[0,\delta] \times [0,1]$ with $\tilde{g}_3(x,y) = g_3(x,y)$ if $x, y \neq 0$ and with $\tilde{g}_3(x,y) = 0$ otherwise is continuous on $[0,\delta] \times [0,1]$. Hence, $\tilde{g}_3$ has an absolute maximum on $[0,\delta] \times [0,1]$, i.e. the integrand of the first integral of (5.91) divided by $f_{m_1A_1+A_2}(y)$ is bounded by this maximum value and therefore has an integrable majorant, if $y \leq 1$. Thus, by the dominated convergence theorem,

$$\lim_{y\to 0} \int_0^\delta \frac{\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 xy)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right)\frac{\gamma f_{A_2}(xy)}{\gamma - m_2 x}}{f_{m_1A_1+A_2}(y)}\,\mathrm{d}x$$

$$= \int_0^\delta \lim_{y\to 0} \frac{\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln(\gamma y - m_2 xy)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right)\frac{\gamma f_{A_2}(xy)}{\gamma - m_2 x}}{f_{m_1A_1+A_2}(y)}\,\mathrm{d}x \tag{5.98}$$

$$= \int_0^\delta 0\,\mathrm{d}x = 0 \tag{5.99}$$

since the limit of (5.92) is 0 for $y \to 0$ by (5.93)–(5.97).

For the second integral of (5.91) we proceed analogously, and we obtain for $x \in [\delta, 1)$

$$\frac{f_{A_2}(xy)}{f_{m_1A_1+A_2}(y)}\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln((y-xy)/m_1)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right)\frac{1}{1-x}$$

$$\sim \text{const.} \times \frac{\sqrt{-\ln(y)}}{x(1-x)} \times \exp\left(\frac{-1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\times \right.$$

$$\left[2\ln(y)\left\{-s_1\ln\left(\frac{s_1}{s_1+s_2}\right) - s_2\ln\left(\frac{s_2}{s_1+s_2}\right) + s_1\ln(x) + s_2\ln(1-x)\right\}\right.$$

$$\left.\left. + (1-\rho^2)\sigma_1^2(\ln(x)-\mu_2)^2 + \Big(\sigma_2(\ln((1-x)/m_1)-\mu_1) - \rho\sigma_1(\ln(x)-\mu_2)\Big)^2\right]\right) \tag{5.100}$$

$$\to 0, \quad y \to 0, \tag{5.101}$$

since the term in curly brackets is negative for all $x \in [\delta, 1)$, because $s_1/(s_1+s_2)$ is smaller than the lower bound of integration δ by (5.75) and (5.79). Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $[\delta, 1) \times (0, 1]$ with limit $(1, y^*)$, $y^* \in [0, 1]$. Then, if we denote the RHS of (5.100) by $g_4(x, y)$, we have $\lim_{n\to\infty} g_4(x_n, y_n) = 0$, i.e. together with (5.101) it follows that $\tilde{g}_4$ defined on $[\delta, 1] \times [0, 1]$ with $\tilde{g}_4(x, y) = g_4(x, y)$ if $x \neq 1$ and $y \neq 0$ and with $\tilde{g}_4(x, y) = 0$ otherwise is continuous on $[\delta, 1]\times[0, 1]$. Hence, $\tilde{g}_4$ has an absolute maximum on $[\delta, 1]\times[0, 1]$, i.e. the integrand of the second integral of (5.91) divided by $f_{m_1A_1+A_2}(y)$ has an integrable majorant and thus, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{y\to 0}\int_\delta^1 \frac{\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln((y-xy)/m_1)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right)\frac{f_{A_2}(xy)}{1-x}}{f_{m_1A_1+A_2}(y)}\,\mathrm{d}x \\
&= \int_\delta^1 \lim_{y\to 0}\frac{\varphi\left(\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\ln((y-xy)/m_1)-\mu_1}{\sigma_1} - \rho\frac{\ln(xy)-\mu_2}{\sigma_2}\right)\right)\frac{f_{A_2}(xy)}{1-x}}{f_{m_1A_1+A_2}(y)}\,\mathrm{d}x \qquad (5.102)\\
&= \int_\delta^1 0\,\mathrm{d}x = 0 \qquad (5.103)
\end{aligned}$$

by (5.100)–(5.101). Therefore, together with (5.98)–(5.99), we have (5.91)=0, i.e. the limit of (5.78) is 0 for $y \to 0$. Hence, by (5.73) and (5.76),

$$\lambda_{\mathrm{L}}^{\mathrm{d}} = 0 \quad \text{for } -1 < \rho < \sigma_1/\sigma_2. \qquad (5.104)$$

5.4.1.2 Determination of $\lambda_{\mathrm{L}}^{\mathrm{d}}$ if $\sigma_1/\sigma_2 < \rho < 1$

For $\sigma_1/\sigma_2 < \rho < 1$, $\lambda_{\mathrm{L}}^{\mathrm{d}}$ can be determined as follows. If we read the inequalities in the numerator of (5.73) as equalities, straightforward calculations show that these two linear functions in A_1 intersect for A_1 equal to

$$\frac{\vartheta(y) - m_2}{1 - m_1m_2} \times y =: \kappa(y) \times y, \qquad (5.105)$$

with $\kappa(y) \in (0, 1/m_1)$ by Lemma A.9. Then, as becomes clear in Figure 5.1,

$$\begin{aligned}
&\frac{P\left(A_1 + m_2A_2 \leq \vartheta(y)\,y,\, m_1A_1 + A_2 \leq y\right)}{P(m_1A_1 + A_2 \leq y)} \qquad (5.106)\\
&\geq 1 - \frac{P\left(A_1 > \kappa(y)\,y,\, m_1A_1 + A_2 \leq y\right)}{P(m_1A_1 + A_2 \leq y)} \qquad (5.107)\\
&\geq 1 - \frac{P\left(\kappa(y)\,y < A_1 \leq y/m_1\right)}{P(m_1A_1 + A_2 \leq y)} \qquad (5.108)\\
&= 1 - \frac{P\left(A_1 \leq y/m_1\right)}{P(m_1A_1 + A_2 \leq y)}\,\frac{P\left(\kappa(y)\,y < A_1 \leq y/m_1\right)}{P\left(A_1 \leq y/m_1\right)} \qquad (5.109)\\
&= 1 - \frac{P\left(A_1 \leq y/m_1\right)}{P(m_1A_1 + A_2 \leq y)}\left(1 - \frac{F_{A_1}(\kappa(y)\,y)}{F_{A_1}\left(y/m_1\right)}\right), \qquad (5.110)
\end{aligned}$$

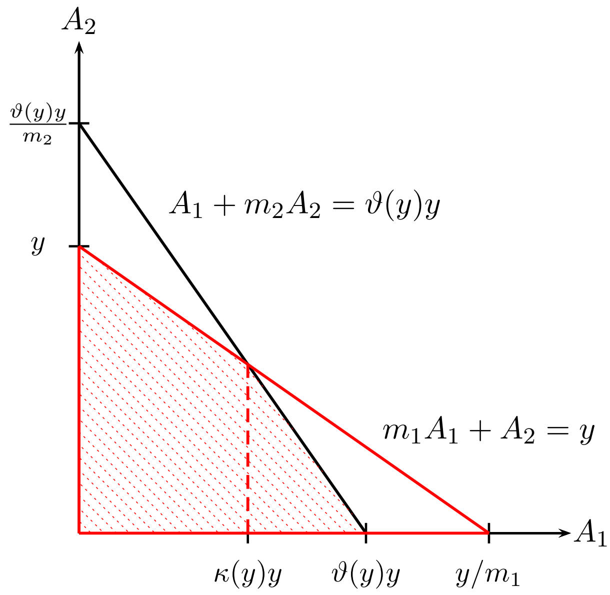


Figure 5.1: Sketch of the probabilities in (5.106). Dotted area: numerator of (5.106), red triangle: denominator of (5.106).

where the limit of the first ratio of (5.110) is 1 for $y \to 0$ by L'Hôpital's rule and Lemma A.10. Furthermore, by L'Hôpital's rule,

$$\lim_{y\to 0} \frac{F_{A_1}(\kappa(y)\,y)}{F_{A_1}(y/m_1)} \tag{5.111}$$

$$= \lim_{y\to 0} \frac{f_{A_1}(\kappa(y)\,y) \times \frac{\partial}{\partial y}(\kappa(y)\,y)}{f_{A_1}(y/m_1)/m_1} \tag{5.112}$$

$$= \lim_{y\to 0} \frac{\frac{\partial}{\partial y}(\kappa(y)\,y)}{\kappa(y)} \exp\left(-\frac{1}{2\sigma_1^2}\Big((\ln(\kappa(y)) + \ln(y) - \mu_1)^2 - (\ln(y) - \ln(m_1) - \mu_1)^2\Big)\right) \tag{5.113}$$

$$= \lim_{y\to 0} \frac{\frac{\partial}{\partial y}(\kappa(y)\,y)}{\kappa(y)} \exp\left(-\frac{1}{2\sigma_1^2}\Big(2(\ln(y) - \mu_1)\ln(m_1\kappa(y)) + \ln(\kappa(y))^2 - \ln(m_1)^2\Big)\right) \tag{5.114}$$

$$= 1, \tag{5.115}$$

since $\lim_{y\to 0} \kappa(y) = \lim_{y\to 0} \frac{\vartheta(y)-m_2}{1-m_1 m_2} = \frac{1}{m_1}$ by Lemma A.11, and due to Lemma 5.9. Thus, the limits of (5.110) and (5.106) are 1 for $y \to 0$, which means that

$$\lambda_L^d = 1 \quad \text{for } \sigma_1/\sigma_2 < \rho < 1. \tag{5.116}$$

Lemma 5.9. *Let* (A_1, A_2) *follow a bivariate lognormal distribution as in Assumption 5.6 with* $\sigma_1/\sigma_2 < \rho < 1$*. Then for* $\kappa(y)$ *as defined in* (5.105),

$$\lim_{y\to 0} \ln(y) \times \ln(m_1\kappa(y)) = 0. \tag{5.117}$$

Furthermore, $\lim_{y\to 0} \frac{\partial}{\partial y}\left(\kappa(y)\, y\right) = 1/m_1$.

Proof. We have

$$\lim_{y\to 0} \ln(y) \times \ln(m_1\kappa(y)) = \lim_{y\to 0} \ln(y) \times \ln(m_1\vartheta(y)) \times \frac{\ln(m_1\kappa(y))}{\ln(m_1\vartheta(y))}. \tag{5.118}$$

Since $\lim_{y\to 0}\kappa(y) = \lim_{y\to 0}\vartheta(y) = 1/m_1$ by Lemma A.11, L'Hôpital's rule yields

$$\lim_{y\to 0} \frac{\ln(m_1\kappa(y))}{\ln(m_1\vartheta(y))} = \lim_{y\to 0} \frac{\vartheta(y)}{\kappa(y)} \times \frac{\kappa'(y)}{\vartheta'(y)} = \lim_{y\to 0} \frac{\vartheta(y)}{\vartheta(y) - m_2} = \frac{1}{1 - m_1 m_2}. \tag{5.119}$$

In order to determine the limit of $\ln(y) \times \ln(m_1\vartheta(y))$, first note that by (5.72),

$$\frac{F_{A_1+m_2A_2}(\vartheta(y)\, y)}{F_{A_1+m_2A_2}(y/m_1)} = \frac{F_{m_1A_1+A_2}(y)}{F_{A_1+m_2A_2}(y/m_1)} \to 1, \quad y \to 0, \tag{5.120}$$

since by L'Hôpital's rule and Lemma A.10,

$$\lim_{y\to 0} \frac{F_{m_1A_1+A_2}(y)}{F_{A_1+m_2A_2}(y/m_1)} = \lim_{y\to 0} \frac{f_{m_1A_1+A_2}(y)}{f_{A_1+m_2A_2}(y/m_1)/m_1} = \lim_{y\to 0} \frac{f_{A_1}(y/m_1)}{f_{A_1}(y/m_1)} = 1. \tag{5.121}$$

On the other hand, the limit in (5.120) can also be determined by using Theorem 1 of Gulisashvili and Tankov [forthcoming], which yields[4]

$$\lim_{y\to 0} \frac{F_{A_1+m_2A_2}(\vartheta(y)\, y)}{F_{A_1+m_2A_2}(y/m_1)} \tag{5.122}$$

$$= \lim_{y\to 0} \frac{\ln(y/m_1)}{\ln(\vartheta(y)\, y)} \, \frac{(\vartheta(y)\, y)^{\mu_1/\sigma_1^2}}{(y/m_1)^{\mu_1/\sigma_1^2}} \, \frac{\exp\left(\frac{-1}{2\sigma_1^2}\ln(\vartheta(y)\, y)^2\right)}{\exp\left(\frac{-1}{2\sigma_1^2}\ln(y/m_1)^2\right)} \, \frac{1 + O(-\ln(\vartheta(y)\, y)^{-1})}{1 + O(-\ln(y/m_1)^{-1})} \tag{5.123}$$

$$= \lim_{y\to 0} \frac{\ln(y/m_1)}{\ln(\vartheta(y)\, y)} (m_1\vartheta(y))^{\mu_1/\sigma_1^2} \exp\left(\frac{-1}{2\sigma_1^2}\left(\left(\ln(y) + \ln(\vartheta(y))\right)^2 - \left(\ln(y) - \ln(m_1)\right)^2\right)\right) \tag{5.124}$$

$$= \lim_{y\to 0} (m_1\vartheta(y))^{\mu_1/\sigma_1^2} \exp\left(\frac{-1}{2\sigma_1^2}\left(2\ln(y)\ln(m_1\vartheta(y)) + \ln(\vartheta(y))^2 - \ln(m_1)^2\right)\right). \tag{5.125}$$

[4] With $\mathfrak{B} = (\mathfrak{B}_{ij})_{1\le i,j\le n}$ of Gulisashvili and Tankov [forthcoming] identical to our $\mathbf{\Sigma}$ of Assumption 5.6, straightforward calculations yield $\bar{w} = (1, 0)$ for $\rho > \sigma_1/\sigma_2$, and therefore $\bar{n} = 1$ and $\bar{I} = \{1\} =: \{\bar{k}(1)\}$ (cf. equations (6) and (8) of Gulisashvili and Tankov [forthcoming]). Then $\bar{\mathfrak{B}} = \mathfrak{B}_{11} = \sigma_1^2$ and $\bar{A}_1 = \bar{\mathfrak{B}}^{-1} = 1/\sigma_1^2$. Furthermore, their Assumption $(\mathcal{A})$ is met for $\rho > \sigma_1/\sigma_2$, since we only need to consider $i = 2$, and $e^2 = (0, 1)^T$, implying $(e^i - \bar{w})^T \mathfrak{B}\bar{w} = \rho\sigma_1\sigma_2 - \sigma_1^2 \neq 0$. As we consider the distribution function of $A_1 + m_2A_2$, their $\bar{\mu}_1$ equals our μ_1.

The last ratio in (5.123) converges to 1 for $y \to 0$ because

$$\limsup_{y\to 0} O(-\ln(\vartheta(y)\,y)^{-1}) \times (-\ln(\vartheta(y)\,y)) < \infty \tag{5.126}$$

by (2.1), i.e. $\lim_{y\to 0} O(-\ln(\vartheta(y)\,y)^{-1}) = 0$. Similarly, $\lim_{y\to 0} O(-\ln(y/m_1)^{-1}) = 0$. Due to $\lim_{y\to 0}\vartheta(y) = 1/m_1$, (5.120) implies that the limit of the exp-term in (5.125) is 1, and therefore $\lim_{y\to 0}\ln(y) \times \ln(m_1\vartheta(y)) = 0$. Then (5.117) follows from (5.118) and (5.119). Moreover, by Lemma A.10,

$$\frac{\partial}{\partial y}(\vartheta(y)\,y) = \frac{\partial}{\partial y}F^{-1}_{A_1+m_2A_2}\left(F_{m_1A_1+A_2}(y)\right) \tag{5.127}$$

$$= \frac{f_{m_1A_1+A_2}(y)}{f_{A_1+m_2A_2}\left(\vartheta(y)\,y\right)} \tag{5.128}$$

$$\sim \vartheta(y)\,\frac{\varphi\left(\frac{1}{\sigma_1}\left(\ln(y) - \mu_1 - \ln(m_1)\right)\right)}{\varphi\left(\frac{1}{\sigma_1}\left(\ln(\vartheta(y)\,y) - \mu_1\right)\right)} \tag{5.129}$$

$$= \vartheta(y)\,\exp\left(-\frac{1}{2\sigma_1^2}\left(\left(\ln(y) - \mu_1 - \ln(m_1)\right)^2 - \left(\ln(y) + \ln(\vartheta(y)) - \mu_1\right)^2\right)\right) \tag{5.130}$$

$$= \vartheta(y)\,\exp\left(-\frac{1}{2\sigma_1^2}\Big(-2\left(\ln(y) - \mu_1\right)\ln(m_1\vartheta(y)) + \ln(m_1)^2 - \ln(\vartheta(y))^2\Big)\right), \tag{5.131}$$

i.e. $\lim_{y\to 0}\frac{\partial}{\partial y}(\vartheta(y)\,y) = 1/m_1$ by (5.117) and due to $\lim_{y\to 0}\vartheta(y) = 1/m_1$, and therefore

$$\lim_{y\to 0}\frac{\partial}{\partial y}(\kappa(y)\,y) = \frac{1}{1-m_1m_2}\lim_{y\to 0}\left(\frac{\partial}{\partial y}(\vartheta(y)\,y) - m_2\right) = \frac{1}{m_1}. \tag{5.132}$$

□

5.4.1.3 The Case $\rho = \sigma_1/\sigma_2$

For the, as they call it, exceptional case $\rho = \sigma_1/\sigma_2$, Gulisashvili and Tankov [forthcoming] remark that the left tail behaviour of the density of the sum of lognormals derived by Gao et al. [2009] is qualitatively different from the cases where $\rho \neq \sigma_1/\sigma_2$. This might explain the fact that to our knowledge, the asymptotic density provided by Gao et al. [2009] (cf. Lemma A.10) is the only result on the left tail behaviour of the sum of lognormals for $\rho = \sigma_1/\sigma_2$ in the literature (the asymptotic distribution function of the sum of lognormals provided by Gulisashvili and Tankov [forthcoming] holds for $\rho \neq \sigma_1/\sigma_2$ only). Unfortunately, we could not prove the existence of the limit $\lim_{y\to 0}\vartheta(y)$ in this situation, but if it exists, it can be shown that it is $1/m_1$. However, even if the existence of this limit was known, the methods employed for $\rho < \sigma_1/\sigma_2$ and $\rho > \sigma_1/\sigma_2$ cannot be used to determine the limit of $\frac{P(A_1+m_2A_2\le\vartheta(y)\,y,\,m_1A_1+A_2\le y)}{P(m_1A_1+A_2\le y)}$ for $y \to 0$ for $\rho = \sigma_1/\sigma_2$, since there would neither be a $\gamma \in \mathbb{R}$ such that $\lim_{y\to 0}\vartheta(y) < \gamma < 1/m_1$ (cf. (5.75)), nor could the limit of $\ln(y) \times \ln(m_1\kappa(y))$ (cf. Lemma 5.9) be determined. Hence, we need to skip the case $\rho = \sigma_1/\sigma_2$ in this work, but fortunately, this case is negligible from a practical point of view.

5.4.2 λ_U^d

Under cross-ownership of debt only with bivariate lognormally distributed exogenous asset values as in Assumption 5.6, the coefficient of upper tail dependence λ_U^d between firm values remains unchanged compared to the scenario of plain firms, i.e. $\lambda_U^d = 0$, independently of the value of ρ. This can be seen as follows.

$$\lambda_U^d = \lim_{u\to 1} \frac{1-2u+C_{A_1,A_2}(u,u)}{1-u} + \lim_{u\to 1} \frac{C_{V_1^d,V_2^d}(u,u) - C_{A_1,A_2}(u,u)}{1-u} \tag{5.133}$$

$$= \lambda_{U,G} + \lim_{u\to 1} \frac{C_{V_1^d,V_2^d}(u,u) - C_{A_1,A_2}(u,u)}{1-u} \tag{5.134}$$

$$= \lim_{u\to 1} \frac{C_{V_1^d,V_2^d}(u,u) - C_{A_1,A_2}(u,u)}{1-u} \tag{5.135}$$

by (5.64) and (5.69), and Proposition 5.7 yields for $u \to 1$

$$\frac{|C_{V_1^d,V_2^d}(u,u) - C_{A_1,A_2}(u,u)|}{1-u} = \frac{O(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(u)))}{1-u} \tag{5.136}$$

$$= \underbrace{\frac{O(f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(u)))}{f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(u))}}_{\leq \text{ const. for } u\to 1} \frac{f_{A_1}(F_{A_1}^{-1}(u)) + f_{A_2}(F_{A_2}^{-1}(u))}{1-u} \to 0, \tag{5.137}$$

since for the lognormal distribution by L'Hôpital's rule,

$$\lim_{u\to 1} \frac{f_{A_i}(F_{A_i}^{-1}(u))}{1-u} = \lim_{x\to\infty} \frac{f_{A_i}(x)}{1-F_{A_i}(x)} = \lim_{x\to\infty} \frac{f_{A_i}(x)\left(\frac{\ln(x)-\mu_i}{\sigma_i^2}+1\right)}{x \times f_{A_i}(x)} = 0, \quad i = 1,2. \tag{5.138}$$

5.4.3 Summary of Results under XOS of Debt only

Our results on tail dependence of firm values under cross-ownership of debt only are summarized in the following proposition.

Proposition 5.10. *Under cross-ownership of debt only with bivariate lognormally distributed exogenous asset values as in Assumption 5.6, the coefficient of lower tail dependence λ_L^d is given by*

$$\lambda_L^d = \begin{cases} 0, & -1 < \rho < \sigma_1/\sigma_2, \\ 1, & \sigma_1/\sigma_2 < \rho < 1, \end{cases} \tag{5.139}$$

and the coefficient of upper tail dependence λ_U^d is 0.

Proof. This follows from (5.104), (5.116) and Section 5.4.2. □

Hence, if the correlation between the logarithmized exogenous asset values is smaller than σ_1/σ_2, firm values remain lower tail independent also in the presence of cross-ownership. If the correlation lies above this bound, however, the lower tail dependence coefficient of firm values takes its maximum possible value of 1, i.e. there is perfect lower tail dependence between the firm values under cross-ownership with bivariate log-normally distributed exogenous asset values. Hence, for a portfolio made up of two indices representing these two firms' values, if the holder of this portfolio ignores the presence of financial interconnections, he might not be aware of the fact that given a relatively big loss in one index, the other index is likely to decline strongly in value as well. However, he only needs to be conscious about the mere presence of cross-holdings, but not the realized level of cross-ownership, as the lower tail dependence coefficient does not depend on the exact values of the cross-ownership fractions. Even if they are very close to 1, firm values remain tail lower independent if $\rho < \sigma_1/\sigma_2$. This condition can be interpreted as follows. Due to (cf. Nalbach-Leniewska [1979], for example)

$$\mathrm{Corr}(A_1, A_2) = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{(\exp(\sigma_1^2) - 1)(\exp(\sigma_2^2) - 1)}}, \tag{5.140}$$

the condition $\rho < \sigma_1/\sigma_2$ is equivalent to

$$\mathrm{Corr}(A_1, A_2) < \sqrt{\frac{\exp(\sigma_1^2) - 1}{\exp(\sigma_2^2) - 1}} = \frac{\mathrm{CV}(A_1)}{\mathrm{CV}(A_2)}, \tag{5.141}$$

with $\mathrm{CV}(X)$ denoting the coefficient of variation of a random variable X. Hence, the lower tail dependence coefficient of firm values under cross-ownership of debt only (which coincides with the lower tail dependence coefficient of $A_1 + M^{\mathrm{d}}_{1,2}A_2$ and $M^{\mathrm{d}}_{2,1}A_1 + A_2$ by Proposition 5.8 and (5.62)) is 0 resp. 1 if the correlation between A_1 and A_2 is smaller resp. bigger than the ratio of the coefficients of variation of A_1 and A_2. In particular, the lower tail dependence coefficient does not depend on the correlation between $A_1+M^{\mathrm{d}}_{1,2}A_2$ and $M^{\mathrm{d}}_{2,1}A_1 + A_2$.
As for the case of two firms not linked by cross-ownership, firm values are upper tail independent under cross-ownership of debt only, i.e. relatively big firm values occur independently of each other.

5.5 Tail Dependence under XOS of Equity only

5.5.1 $\lambda^{\mathrm{e}}_{\mathrm{L}}$

Under cross-ownership of equity only with bivariate lognormally distributed exogenous asset values as in Assumption 5.6, the coefficient of lower tail dependence $\lambda^{\mathrm{e}}_{\mathrm{L}}$ between firm values remains unchanged compared to the scenario of plain firms, i.e. $\lambda^{\mathrm{e}}_{\mathrm{L}} = 0$. This can be seen as follows. If $\rho > 0$, we obtain from (5.62), (5.69) and Proposition 5.7 that

$$\lambda_{\mathrm{L}}^{\mathrm{e}} = \lim_{u\to 0} \frac{C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{u} \tag{5.142}$$

$$= \lim_{u\to 0} \frac{C_{A_1,A_2}(u,u)}{u} + \lim_{u\to 0} \frac{C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u) - C_{A_1,A_2}(u,u)}{u} \tag{5.143}$$

$$= \lambda_{\mathrm{L,G}} + 0 = 0. \tag{5.144}$$

If $\rho \le 0$, let $q_1, q_2 \in \mathbb{R}$ be such that $u = F_{V_1^{\mathrm{e}}}(q_1) = F_{V_2^{\mathrm{e}}}(q_2)$. Then by Theorem 5.3 and (5.44) for u sufficiently small,

$$C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u) = F_{A_1,A_2}(q_1,q_2) = \int_{-\infty}^{\frac{\ln(q_1)-\mu_1}{\sigma_1}} \int_{-\infty}^{\frac{\ln(q_2)-\mu_2}{\sigma_2}} \varphi_\rho(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \tag{5.145}$$

with

$$\varphi_\rho(x_1,x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\Big(x_1^2 - 2\rho x_1 x_2 + x_2^2\Big)\right). \tag{5.146}$$

Let now $\rho < 0$. Since we can assume $x_1, x_2 < 0$ as we are interested in the limit $u \to 0$, it is straightforward to see that then $\varphi_\rho(x_1,x_2) \le \varphi_{-\rho}(x_1,x_2)$. Hence, if we label every copula with the underlying correlation ρ, we have

$$C^{\rho}_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u) \le C^{-\rho}_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u) \quad \text{for } \rho < 0, \tag{5.147}$$

and thus

$$0 \le \lim_{u\to 0} \frac{C^{\rho}_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{u} \le \lim_{u\to 0} \frac{C^{-\rho}_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{u} = 0 \tag{5.148}$$

by (5.144). For $\rho = 0$ and u sufficiently small we have $C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u) = F_{A_1,A_2}(q_1,q_2) = F_{A_1}(q_1)\,F_{A_2}(q_2)$ by (5.44) and Theorem 5.3. Furthermore, (5.35) yields

$$\frac{F_{A_1}(q_1)}{F_{V_1^{\mathrm{e}}}(q_1)} = \frac{1}{1 - \frac{p_1(q_1)}{F_{A_1}(q_1)}} \le \frac{1}{F_{A_2}(d_2)} \tag{5.149}$$

since $p_1(q_1) \le F_{A_1}(q_1) \times (1 - F_{A_2}(d_2))$ by (5.38) for $\rho = 0$. Hence,

$$\frac{C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{u} = \frac{F_{A_1}(q_1)\,F_{A_2}(q_2)}{F_{V_1^{\mathrm{e}}}(q_1)} \le \frac{1}{F_{A_2}(d_2)} F_{A_2}\left(F^{-1}_{V_2^{\mathrm{e}}}(u)\right), \tag{5.150}$$

which implies $\lim_{u\to 0} \frac{C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{u} = 0$ for $\rho = 0$. Altogether, $\lambda_{\mathrm{L}}^{\mathrm{e}} = 0$ for all $-1 < \rho < 1$.

Case	σ_1, σ_2	$\mu_{A_1+m_2A_2}$	$m_{A_1+m_2A_2}$	$\mu_{m_1A_1+A_2}$	$m_{m_1A_1+A_2}$
1	$\sigma_1 < \sigma_2$	$\mu_2 + \ln(m_2)$	1	μ_2	1
2	$\sigma_1 = \sigma_2$	$\mu_1 = \mu_2 + \ln(m_2)$	2	μ_2	1
3	$\sigma_1 = \sigma_2$	μ_1	1	$\mu_1 + \ln(m_1) = \mu_2$	2
4a	$\sigma_1 = \sigma_2$	$\mu_2 + \ln(m_2)$	1	μ_2	1
4b	$\sigma_1 = \sigma_2$	μ_1	1	$\mu_1 + \ln(m_1)$	1
4c	$\sigma_1 = \sigma_2$	μ_1	1	μ_2	1

Table 5.1: Parameter values of the case differentiation in the proof of Lemma A.13 with $\mu_{A_1+m_2A_2}$, $m_{A_1+m_2A_2}$, $\mu_{m_1A_1+A_2}$ and $m_{m_1A_1+A_2}$ as defined in (A123).

5.5.2 $\lambda_{\mathrm{U}}^{\mathrm{e}}$

By (5.63), (5.64), Proposition 5.8 and (5.11),

$$\lambda_{\mathrm{U}}^{\mathrm{e}} = \lim_{u\to 1} \frac{1-2u+C_{V_1^{\mathrm{e}},V_2^{\mathrm{e}}}(u,u)}{1-u} = \lim_{u\to 1} \frac{1-2u+C_{A_1+M_{1,2}^{\mathrm{e}}A_2, M_{2,1}^{\mathrm{e}}A_1+A_2}(u,u)}{1-u} \tag{5.151}$$

$$= \lim_{u\to 1} \frac{\bar{C}_{A_1+M_{1,2}^{\mathrm{e}}A_2, M_{2,1}^{\mathrm{e}}A_1+A_2}(1-u,1-u)}{1-u} \tag{5.152}$$

$$= \lim_{u\to 1} \frac{P\left(A_1+M_{1,2}^{\mathrm{e}}A_2 > \bar{F}^{-1}_{A_1+M_{1,2}^{\mathrm{e}}A_2}(1-u),\ M_{2,1}^{\mathrm{e}}A_1+A_2 > \bar{F}^{-1}_{M_{2,1}^{\mathrm{e}}A_1+A_2}(1-u)\right)}{1-u} \tag{5.153}$$

$$= \lim_{u\to 1} \frac{P\left(A_1+M_{1,2}^{\mathrm{e}}A_2 > F^{-1}_{A_1+M_{1,2}^{\mathrm{e}}A_2}(u),\ M_{2,1}^{\mathrm{e}}A_1+A_2 > F^{-1}_{M_{2,1}^{\mathrm{e}}A_1+A_2}(u)\right)}{1-u}, \tag{5.154}$$

provided the limit exists. For the evaluation of (5.154) we will use the insights of Section A.4.2 by identifying $M_{1,2}^{\mathrm{e}}$ with m_2 and $M_{2,1}^{\mathrm{e}}$ with m_1, and as in Section 5.4.1 we proceed with the notation in terms of m_1 and m_2 for better readability and since the following considerations hold beyond the context of cross-ownership. Hence, with $m_1 = M_{2,1}^{\mathrm{e}}$ and $m_2 = M_{1,2}^{\mathrm{e}}$, $y = F^{-1}_{m_1A_1+A_2}(u)$ and $\vartheta(y)$ defined as in (5.72) we obtain

$$\lambda_{\mathrm{U}}^{\mathrm{e}} = \lim_{y\to\infty} \frac{P\left(A_1+m_2A_2 > \vartheta(y)\,y,\ m_1A_1+A_2 > y\right)}{P(m_1A_1+A_2 > y)}, \tag{5.155}$$

provided the limit exists. For the evaluation of (5.155) the case differentiation of the proof of Lemma A.13 will be helpful. For better readability, Table 5.1 contains the corresponding parameter values.

First, note that by Theorem A.12 and Table 5.1,

$$P(m_1A_1+A_2 > y) \sim m_{m_1A_1+A_2} P(A_2 > y), \quad y\to\infty, \tag{5.156}$$

if we are not in Case 4b.

If we are in Case 1 or Case 4a, $\lim_{y\to\infty}\vartheta(y) = m_2$ by Lemma A.13. Then, since

$\vartheta(y) > m_2$ for all $y > 0$ by Lemma A.9, and because of (5.156) and Theorem A.12,

$$1 \geq \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \tag{5.157}$$

$$\geq \frac{P\left(m_1 A_1 + A_2 > \frac{\vartheta(y)\, y}{m_2}\right)}{P(m_1 A_1 + A_2 > y)} \sim \frac{P\left(A_2 > \frac{\vartheta(y)\, y}{m_2}\right)}{P(A_2 > y)} \tag{5.158}$$

$$\sim \frac{\ln(y) - \mu_2}{\ln(y) + \ln\left(\frac{\vartheta(y)}{m_2}\right) - \mu_2} \exp\left(\frac{-1}{2\sigma_2^2}\left(\left(\ln(y) + \ln\left(\frac{\vartheta(y)}{m_2}\right) - \mu_2\right)^2 - (\ln(y) - \mu_2)^2\right)\right) \tag{5.159}$$

$$\sim \exp\left(\frac{-1}{2\sigma_2^2}\left(2(\ln(y) - \mu_2)\ln\left(\frac{\vartheta(y)}{m_2}\right) + \ln\left(\frac{\vartheta(y)}{m_2}\right)^2\right)\right), \quad y \to \infty, \tag{5.160}$$

$$\to 1, \quad y \to \infty, \tag{5.161}$$

where (5.161) can be seen as follows. For $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$, let $\bar{F}_{\mu_X, \sigma_X^2}$ denote the survival function of X. In Case 1 and Case 4a Theorem A.12 then yields

$$\frac{\bar{F}_{A_1+m_2A_2}(\vartheta(y)\, y)}{\bar{F}_{A_1+m_2A_2}(m_2 y)} \sim \frac{\bar{F}_{\mu_{A_1+m_2A_2}, \sigma_2^2}(\vartheta(y)\, y)}{\bar{F}_{\mu_{A_1+m_2A_2}, \sigma_2^2}(m_2 y)} = \frac{\bar{F}_{\mu_2+\ln(m_2), \sigma_2^2}(\vartheta(y)\, y)}{\bar{F}_{\mu_2+\ln(m_2), \sigma_2^2}(m_2 y)} \tag{5.162}$$

$$\sim \frac{\ln(y) - \mu_2}{\ln(y) + \ln(\vartheta(y)) - \mu_2 - \ln(m_2)} \times \exp\left(\frac{-1}{2\sigma_2^2}\left(\left(\ln(y) + \ln(\vartheta(y)) - \mu_2 - \ln(m_2)\right)^2 - (\ln(y) - \mu_2)^2\right)\right) \tag{5.163}$$

$$\sim \exp\left(\frac{-1}{2\sigma_2^2}\left(2(\ln(y) - \mu_2)\ln\left(\frac{\vartheta(y)}{m_2}\right) + \ln\left(\frac{\vartheta(y)}{m_2}\right)^2\right)\right), \quad y \to \infty. \tag{5.164}$$

On the other hand, (5.72) and Theorem A.12 yield

$$\frac{\bar{F}_{A_1+m_2A_2}(\vartheta(y)\, y)}{\bar{F}_{A_1+m_2A_2}(m_2 y)} = \frac{\bar{F}_{m_1A_1+A_2}(y)}{\bar{F}_{A_1+m_2A_2}(m_2 y)} \tag{5.165}$$

$$\sim \frac{m_{m_1A_1+A_2}\bar{F}_{\mu_2, \sigma_2^2}(y)}{m_{A_1+m_2A_2}\bar{F}_{\mu_2+\ln(m_2), \sigma_2^2}(m_2 y)} = \frac{m_{m_1A_1+A_2}}{m_{A_1+m_2A_2}} = 1, \quad y \to 0, \tag{5.166}$$

in Case 1 and Case 4a, and (5.161) follows. Hence, in Case 1 and Case 4a,

$$\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} = 1. \tag{5.167}$$

In Case 3 and Case 4b Theorem A.12 yields

$$P(m_1 A_1 + A_2 > z) \sim m_{m_1A_1+A_2}\bar{F}_{\mu_1+\ln(m_1), \sigma_1^2}(z) = m_{m_1A_1+A_2} P(A_1 > z/m_1), \quad z \to \infty. \tag{5.168}$$

Hence, because of $\vartheta(y) < 1/m_1$ for all $y > 0$ by Lemma A.9,

$$1 \geq \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \geq \frac{P(A_1 > y/m_1)}{P(m_1 A_1 + A_2 > y)} \tag{5.169}$$

$$\sim \frac{P(A_1 > y/m_1)}{m_{m_1 A_1 + A_2} P(A_1 > y/m_1)} = \frac{1}{m_{m_1 A_1 + A_2}}, \quad y \to \infty, \tag{5.170}$$

and therefore

$$\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} = 1, \qquad \text{Case 4b}, \tag{5.171}$$

$$\liminf_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \geq 0.5, \quad \text{Case 3}. \tag{5.172}$$

Furthermore,

$$\begin{aligned} &\frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \\ &= \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y,\, A_2 \leq y\right)}{P(m_1 A_1 + A_2 > y)} \\ &\quad + \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y,\, A_2 > y\right)}{P(m_1 A_1 + A_2 > y)}, \end{aligned} \tag{5.173}$$

and in the following, we will evaluate the limit of (5.173) for $y \to \infty$ for Case 3 and Case 4c. For the first ratio we have

$$\frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y, A_2 \leq y\right)}{P(m_1 A_1 + A_2 > y)}$$

$$\leq \frac{P\left(m_1 A_1 + A_2 > y,\, A_2 \leq y\right)}{P(m_1 A_1 + A_2 > y)} = \frac{P\left(m_1 A_1 + A_2 > y\right) - P\left(A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \tag{5.174}$$

$$= 1 - \frac{P(A_2 > y)}{P(m_1 A_1 + A_2 > y)} \to 1 - \frac{1}{m_{m_1 A_1 + A_2}}, \quad y \to \infty, \tag{5.175}$$

by (5.156). The evaluation of the limit of the second ratio in (5.173) is more elaborate.

$$\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, m_1 A_1 + A_2 > y, A_2 > y\right)}{P(m_1 A_1 + A_2 > y)}$$

$$= \lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\, A_2 > y\right)}{m_{m_1 A_1 + A_2} P(A_2 > y)} \tag{5.176}$$

$$= \lim_{y\to\infty} \frac{1}{m_{m_1 A_1 + A_2} \bar{F}_{A_2}(y)} \int_y^{\infty} P\left(A_1 > \vartheta(y)\, y - m_2 x \,|\, A_2 = x\right)\, f_{A_2}(x)\, \mathrm{d}x \tag{5.177}$$

$$= \lim_{y\to\infty} \frac{1}{m_{m_1 A_1 + A_2} \bar{F}_{A_2}(y)} \int_y^{\frac{\vartheta(y)y}{m_2}} P\left(A_1 > \vartheta(y)\, y - m_2 x \,|\, A_2 = x\right)\, f_{A_2}(x)\, \mathrm{d}x, \tag{5.178}$$

provided the limits exist, since as in (5.158)–(5.160),

$$\begin{aligned}
&\lim_{y\to\infty} \frac{1}{\bar{F}_{A_2}(y)} \int_{\frac{\vartheta(y)y}{m_2}}^{\infty} P\left(A_1 > \vartheta(y)\, y - m_2 x \,|\, A_2 = x\right)\, f_{A_2}(x)\,\mathrm{d}x \\
&= \lim_{y\to\infty} \frac{1}{\bar{F}_{A_2}(y)} \int_{\frac{\vartheta(y)y}{m_2}}^{\infty} f_{A_2}(x)\,\mathrm{d}x = \lim_{y\to\infty} \frac{\bar{F}_{A_2}\left(\frac{\vartheta(y)\, y}{m_2}\right)}{\bar{F}_{A_2}(y)} \qquad (5.179)\\
&= \lim_{y\to\infty} \exp\left(\frac{-1}{2\sigma_2^2}\left(2(\ln(y)-\mu_2)\ln\left(\frac{\vartheta(y)}{m_2}\right) + \ln\left(\frac{\vartheta(y)}{m_2}\right)^2\right)\right) = 0, \qquad (5.180)
\end{aligned}$$

where the last equality follows from the fact that $\lim_{y\to\infty}\ln(\vartheta(y)/m_2) > 0$ in Case 3 and Case 4c by Lemma A.13. Then by Theorem A.12 and with the substitution $x \mapsto xy$,

$$\begin{aligned}
&\lim_{y\to\infty} \frac{1}{m_{m_1A_1+A_2}\bar{F}_{A_2}(y)} \int_{y}^{\frac{\vartheta(y)y}{m_2}} P\left(A_1 > \vartheta(y)\, y - m_2 x \,|\, A_2 = x\right)\, f_{A_2}(x)\,\mathrm{d}x \\
&= \text{const.} \times \lim_{y\to\infty} \frac{\ln(y)-\mu_2}{y \times f_{A_2}(y)} \int_{y}^{\frac{\vartheta(y)y}{m_2}} P\left(A_1 > \vartheta(y)\, y - m_2 x \,|\, A_2 = x\right)\, f_{A_2}(x)\,\mathrm{d}x \qquad (5.181)\\
&= \text{const.} \times \lim_{y\to\infty} \frac{\ln(y)-\mu_2}{f_{A_2}(y)} \int_{1}^{\frac{\vartheta(y)}{m_2}} P\left(A_1 > \vartheta(y)\, y - m_2 xy \,|\, A_2 = xy\right)\, f_{A_2}(xy)\,\mathrm{d}x \qquad (5.182)\\
&= \text{const.} \times \lim_{y\to\infty} \int_{1}^{2c} 1_{\left\{x\le\frac{\vartheta(y)}{m_2}\right\}} P\left(A_1 > \vartheta(y)\, y - m_2 xy \,|\, A_2 = xy\right) f_{A_2}(xy)\, \frac{\ln(y)-\mu_2}{f_{A_2}(y)}\mathrm{d}x \qquad (5.183)
\end{aligned}$$

with some $c \in (\max\{1,\, 0.5\bar{\vartheta}/m_2)\}, \bar{\vartheta}/m_2)$ and $\bar{\vartheta} := \lim_{y\to\infty}\vartheta(y)$. Note that such a c exists due to $\bar{\vartheta}/m_2 > 1$ in Case 3 and Case 4c. In particular, we have $c > 1$. In order to be able to swap the limit and the integral in (5.183), we show that the integrand is bounded by some constant. If $1 < x \le c$, there is an $\epsilon_1 > 0$ such that $\vartheta(y) - m_2 x \ge \vartheta(y) - m_2 c > \epsilon_1$ for y sufficiently big. Furthermore, $f_{A_2}(xy)/f_{A_2}(y) < 1$ for y sufficiently big, i.e. for y sufficiently big (recall that $\sigma_1 = \sigma_2$ in Case 3 and Case 4c),

$$\begin{aligned}
&P\left(A_1 > \vartheta(y)\, y - m_2 xy \,|\, A_2 = xy\right) f_{A_2}(xy) \frac{\ln(y)-\mu_2}{f_{A_2}(y)} \qquad (5.184)\\
&\le \Phi\left(\frac{-1}{\sqrt{1-\rho^2}\sigma_2}\Big(\ln(y) + \ln(\vartheta(y) - m_2 x) - \mu_1 - \rho\big(\ln(x) + \ln(y) - \mu_2\big)\Big)\right)(\ln(y)-\mu_2) \qquad (5.185)\\
&\le \Phi\left(\frac{-1}{\sqrt{1-\rho^2}\sigma_2}\Big((1-\rho)\ln(y) + \ln(\epsilon_1) - \mu_1 - \epsilon_2\Big)\right)(\ln(y)-\mu_2) \qquad (5.186)
\end{aligned}$$

with $\epsilon_2 = \sup_{1<x\le c}|\rho(\ln(x) - \mu_2)|$. Furthermore, L'Hôpital's rule yields for $a > 0$ and

$b \in \mathbb{R}$

$$\lim_{z\to\infty} \Phi\left(-az+b\right) \times z = \lim_{z\to\infty} \frac{\Phi\left(-az+b\right)}{1/z} = \lim_{z\to\infty} \frac{-a\,\varphi\left(-az+b\right)}{-1/z^2} \tag{5.187}$$

$$= \lim_{z\to\infty} a/\sqrt{2\pi} \times \exp(-(az+b)^2/2 + 2\ln(z)) = 0. \tag{5.188}$$

Hence,

$$\lim_{y\to\infty} \Phi\left(\frac{-1}{\sqrt{1-\rho^2}\sigma_2}\Big((1-\rho)\ln(y) + \ln(\epsilon_1) - \mu_1 - \epsilon_2\Big)\right)(\ln(y) - \mu_2) = 0, \tag{5.189}$$

i.e. in Case 3 and Case 4c for y sufficiently big, (5.184) and therefore the integrand of (5.183) is on $(1, c]$ smaller than some constant which only depends on c. If $x > c$, the integrand of (5.183) is bounded from above as follows.

$$P\left(A_1 > \vartheta(y)\,y - m_2xy \,|\, A_2 = xy\right) f_{A_2}(xy)\,\frac{\ln(y)-\mu_2}{f_{A_2}(y)}$$

$$\leq f_{A_2}(xy)\,\frac{\ln(y)-\mu_2}{f_{A_2}(y)} \tag{5.190}$$

$$= \frac{1}{x}\exp\left(\frac{-1}{2\sigma_2^2}\Big((\ln(x)+\ln(y)-\mu_2)^2 - (\ln(y)-\mu_2)^2\Big) + \ln(\ln(y)-\mu_2)\right) \tag{5.191}$$

$$= \frac{1}{x}\exp\left(\frac{-1}{2\sigma_2^2}\Big(2\ln(x)(\ln(y)-\mu_2) + \ln(x)^2\Big) + \ln(\ln(y)-\mu_2)\right), \tag{5.192}$$

and straightforward calculations show that the last expression as a function in $\ln(y)-\mu_2$ becomes maximal for $\ln(y)-\mu_2 = \sigma_2^2/\ln(x)$, i.e. for $x > c > 1$,

$$P\left(A_1 > \vartheta(y)\,y - m_2xy \,|\, A_2 = xy\right) f_{A_2}(xy)\frac{\ln(y)-\mu_2}{f_{A_2}(y)}$$

$$\leq \frac{1}{x}\exp\left(\frac{-1}{2\sigma_2^2}\Big(2\sigma_2^2 + \ln(x)^2\Big) + \ln\left(\sigma_2^2/\ln(x)\right)\right) \tag{5.193}$$

$$\leq \sigma_2^2/\ln(x) \leq \sigma_2^2/\ln(c). \tag{5.194}$$

Altogether, we have shown that the integrand of (5.183) is smaller than some constant for all $x \in (1, 2c)$, and the dominated convergence theorem yields

$$\lim_{y\to\infty}\int_1^{2c} 1_{\left\{x\leq\frac{\vartheta(y)}{m_2}\right\}} P\left(A_1 > \vartheta(y)\,y - m_2xy \,|\, A_2 = xy\right)\, f_{A_2}(xy)\frac{\ln(y)-\mu_2}{f_{A_2}(y)}\,\mathrm{d}x$$

$$= \int_1^{2c} \lim_{y\to\infty} 1_{\left\{x\leq\frac{\vartheta(y)}{m_2}\right\}} P\left(A_1 > \vartheta(y)\,y - m_2xy \,|\, A_2 = xy\right)\, f_{A_2}(xy)\frac{\ln(y)-\mu_2}{f_{A_2}(y)}\,\mathrm{d}x \tag{5.195}$$

$$= \int_1^{2c} 0\,\mathrm{d}x = 0, \tag{5.196}$$

where (5.196) follows from (5.185)–(5.186), (5.189)–(5.192) and since $\lim_{y\to\infty}$ (5.192) $= 0$ due to $x > 1$. Hence, the limit of the second ratio of (5.173) is 0 for $y \to \infty$ in Case 3 and Case 4c. Together with (5.175) this implies (cf. Table 5.1)

$$\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\; m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} = 0, \qquad \text{Case 4c,} \tag{5.197}$$

$$\limsup_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\; m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \le 0.5, \quad \text{Case 3.} \tag{5.198}$$

Hence, by (5.172),

$$\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\; m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} = 0.5 \tag{5.199}$$

in Case 3. Due to

$$\begin{aligned} &\lim_{y\to\infty} \frac{P\left(A_1 + m_2 A_2 > \vartheta(y)\, y,\; m_1 A_1 + A_2 > y\right)}{P(m_1 A_1 + A_2 > y)} \\ &= \lim_{u\to 1} \frac{P(A_1 + m_2 A_2 > F^{-1}_{A_1 + m_2 A_2}(u),\; m_1 A_1 + A_2 > F^{-1}_{m_1 A_1 + A_2}(u))}{1-u}, \end{aligned} \tag{5.200}$$

which is perfectly symmetric between $A_1 + m_2 A_2$ and $m_1 A_1 + A_2$ (recall that $\sigma_1 = \sigma_2$ in Case 2 and Case 3), the final result for Case 3 also holds for Case 2. Altogether, by (5.155), (5.167), (5.171), (5.197), (5.199) and (5.200),

$$\lambda_{\mathrm{U}}^{\mathrm{e}} = \begin{cases} 1, & \text{Case 1, Case 4a and Case 4b,} \\ 0.5, & \text{Case 2 and Case 3,} \\ 0, & \text{Case 4c.} \end{cases} \tag{5.201}$$

5.5.3 Summary of Results under XOS of Equity only

Our results on tail dependence of firm values under cross-ownership of equity only are summarized in the following proposition. For the sake of clarity, we put brackets in the corresponding case differentiation.

Proposition 5.11. *Under cross-ownership of equity only with bivariate lognormally distributed exogenous asset values as in Assumption 5.6, the coefficient of lower tail dependence* $\lambda_{\mathrm{L}}^{\mathrm{e}}$ *is 0 and the coefficient of upper tail dependence* $\lambda_{\mathrm{U}}^{\mathrm{e}}$ *is given by*

$$\lambda_{\mathrm{U}}^{\mathrm{e}} = \begin{cases} 1, & \sigma_1 < \sigma_2 \text{ or } (\sigma_1 = \sigma_2 \text{ and } (\mu_1 < \mu_2 + \ln(M_{1,2}^{\mathrm{e}}) \text{ or } \mu_2 < \mu_1 + \ln(M_{2,1}^{\mathrm{e}}))), \\ 0.5, & \sigma_1 = \sigma_2 \text{ and } (\mu_1 = \mu_2 + \ln(M_{1,2}^{\mathrm{e}}) \text{ or } \mu_2 = \mu_1 + \ln(M_{2,1}^{\mathrm{e}})), \\ 0, & \sigma_1 = \sigma_2 \text{ and } \mu_1 > \mu_2 + \ln(M_{1,2}^{\mathrm{e}}) \text{ and } \mu_2 > \mu_1 + \ln(M_{2,1}^{\mathrm{e}}). \end{cases} \tag{5.202}$$

Proof. This follows from Section 5.5.1, (5.201) and Table 5.1. □

By Proposition 5.11 the presence of cross-ownership of equity only can lead to both, upper tail independence and perfect upper tail dependence, depending on the exact levels of cross-ownership. If both cross-ownership fractions lie below certain bounds, upper tail independence is preserved, but if one of them exceeds the corresponding threshold, firm values are perfectly upper tail dependent under cross-ownership of equity only. This is in stark contrast to Proposition 5.10 for cross-ownership of debt only, where not the cross-ownership fractions, but the correlation between the logarithmized exogenous asset values was crucial. The content of Proposition 5.11 can be explained as follows.
Due to

$$P(V_1^{\mathrm{e}} \leq v_1) = P(A_1 + M_{1,2}^{\mathrm{e}} A_2 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) v_1 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_1 + M_{1,2}^{\mathrm{e}} d_2), \quad (5.203)$$

$$P(V_2^{\mathrm{e}} \leq v_2) = P(M_{2,1}^{\mathrm{e}} A_1 + A_2 \leq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) v_2 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_2 + M_{2,1}^{\mathrm{e}} d_1) \quad (5.204)$$

for v_i sufficiently big by Lemma 3.7 and Remark 3.6, the right tail of the distribution of a firm's value is determined by the right tail of the distribution of the sum of the firm's own exogenous asset value and the value of the fraction it holds of the other firm's exogenous asset. If for example $\sigma_1 = \sigma_2$, $\mu_1 > \mu_2 + \ln(M_{1,2}^{\mathrm{e}})$ and $\mu_2 > \mu_1 + \ln(M_{2,1}^{\mathrm{e}})$, Theorem A.12 yields

$$P(A_1 + M_{1,2}^{\mathrm{e}} A_2 > x) \sim P(A_1 > x), \quad (5.205)$$

$$P(M_{2,1}^{\mathrm{e}} A_1 + A_2 > x) \sim P(A_2 > x), \quad x \to \infty, \quad (5.206)$$

and therefore, by (5.203)–(5.204), $P(V_i^{\mathrm{e}} \geq v_i) \sim P(A_i \geq (1 - M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}) v_i + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}} d_i + M_{ij}^{\mathrm{e}} d_j)$ for $v_i \to \infty$ ($i, j \in \{1, 2\}$, $i \neq j$). This means that for each firm, its firm value is almost completely determined by its own exogenous asset value, if the firm value is sufficiently big. Since exogenous asset values are upper tail independent by (5.69) under Assumption 5.6, it is plausible that firm values are upper tail independent as well, if $\sigma_1 = \sigma_2$, $\mu_1 > \mu_2 + \ln(M_{1,2}^{\mathrm{e}})$ and $\mu_2 > \mu_1 + \ln(M_{2,1}^{\mathrm{e}})$. If $\sigma_1 < \sigma_2$, or if $\sigma_1 = \sigma_2$ and $\mu_1 < \mu_2 + \ln(M_{1,2}^{\mathrm{e}})$, or if $\sigma_1 = \sigma_2$ and $\mu_2 < \mu_1 + \ln(M_{2,1}^{\mathrm{e}})$, Theorem A.12 and (5.203)–(5.204) yield that the right tail of the distribution of, say, firm i's value is determined by the right tail of the distribution of its own exogenous asset value, whereas the right tail of the distribution of firm j's value ($i \neq j$) is determined by the value of the fraction it holds of firm i's exogenous asset value. Hence, for both firms, the right tail behaviour of their firm values is determined by the right tail behaviour of A_i, which illustrates the fact that firm values are perfectly upper tail dependent in these cases by Proposition 5.11.
In contrast to that, firm values are always lower tail independent under cross-ownership of equity only, irrespectively of the underlying parameter values of the lognormal distribution of exogenous asset values and the realized levels of cross-ownership.

The considerations of Section 5.4.1 and Section 5.5.2 not only yield the lower tail dependence coefficient under cross-ownership of debt only and the upper tail dependence coefficient under cross-ownership of equity only (provided that exogenous asset values follow a bivariate lognormal distribution with $\rho \neq \sigma_1/\sigma_2$), they also allow a more general interpretation in terms of two portfolios made up of the same lognormal assets.

Imagine two portfolios built from securities A_1 and A_2 with bivariate lognormal distribution as in Assumption 5.6 in that one portfolio contains A_1 and A_2 at the ratio of $1 : m_2$ and the other portfolio at the ratio $m_1 : 1$ $(m_1, m_2 \in (0, 1))$. Then the results of Section 5.4.1 and Section 5.5.2 show that these portfolios might be lower and/or upper tail independent, despite the fact that they are made up of the same securities. Depending on the underlying parameter constellation, various scenarios may occur: if for example $\sigma_1 = \sigma_2$, we have $\rho < \sigma_1/\sigma_2$, i.e. the two portfolios are lower tail independent, but the upper tail dependence coefficient can be 0, 0.5 or 1. In the more realistic case that $\sigma_1 < \sigma_2$, the portfolios are perfectly upper tail dependent, whereas the lower tails can be both, independent and perfectly dependent. If each of the two portfolios is diversified in the sense that A_1 and A_2 are negatively correlated (which is equivalent to $\rho < 0$ by (5.140)), this ensures lower tail independence of the two portfolios. Lower tail dependence is also preserved if the coefficients of variations of the two securities roughly coincide (cf. (5.141)), as this implies $\sigma_1/\sigma_2 \approx 1$ and therefore $\rho < \sigma_1/\sigma_2$ for most values of ρ. However, if σ_1 and σ_2 differ strongly, the portfolios may be lower tail dependent even for weakly positively correlated A_1 and A_2. These considerations show that in a sound management of the overall risk arising from holding both portfolios, a thorough estimation of the underlying parameters is necessary.

6 Sensitivities of Firm Values

As we have seen in Lemma 3.4, the solution $(\mathbf{r}, \mathbf{s})$ of the system (3.13)–(3.14) for the no-arbitrage prices of debt and equity under cross-ownership is continuous in the model parameters $\mathbf{a}$, $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$. In this section we consider derivatives and the monotonicity behaviour of $\mathbf{r}$ and $\mathbf{s}$ with respect to these input parameters. From Proposition 2 of Gouriéroux et al. [2012] we know that r_i and s_i are non-decreasing functions of the exogenous asset value a_j $(i, j \in \{1, \dots n\})$ for any given level of nominal liabilities $\mathbf{d}$ and any cross-ownership structure. Furthermore, Liu and Staum [2010] provide exact formulae for the related derivatives under cross-ownership of debt only. In the remainder of this section we extend the existing literature by analyzing sensitivities of $\mathbf{r}$ and $\mathbf{s}$ with respect to the model parameters $\mathbf{a}$, $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ for an arbitrary scenario of cross-ownership. We start with the cross-ownership fractions.

6.1 Influence of the Cross-Ownership Fractions

6.1.1 Monotonicity of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{v}$ in the Cross-Ownership Fractions

Monotonicity of $\mathbf{r}$ and $\mathbf{s}$ as solutions of (3.13)–(3.14) in the cross-ownership fractions is rather straightforward to see from (3.17). This yields the following proposition.

Proposition 6.1. *For n firms linked by cross-ownership of possibly both, debt and equity, let*

$$\mathbf{r} = \min\{\mathbf{d}, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\}, \tag{6.1}$$

$$\mathbf{s} = (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^{+}. \tag{6.2}$$

Then r_i and s_i are non-decreasing in M^{d}_{kj} and M^{e}_{kj} for all $i, j, k \in \{1, \dots, n\}$ $(k \neq j)$.

Proof. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be defined as $\mathbf{\Phi}$ in (3.17) with $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ exhibiting identical values of $\mathbf{a}$ and $\mathbf{d}$, but with cross-ownership matrices $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ such that exactly one of their $2n(n-1)$ entries (recall that $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ have zeros on the diagonals) differs between $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$. We assume that the matrix with the smaller entry belongs to $\mathbf{\Phi}_1$. Then

$$\mathbf{0} \leq \mathbf{\Phi}_1 \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \leq \mathbf{\Phi}_2 \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \qquad \text{for all } \mathbf{r}, \mathbf{s} \geq \mathbf{0}. \tag{6.3}$$

Furthermore, for $i = 1, 2$,

$$\boldsymbol{\Phi}_i \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{s}_1 \end{pmatrix} \leq \boldsymbol{\Phi}_i \begin{pmatrix} \mathbf{r}_2 \\ \mathbf{s}_2 \end{pmatrix} \qquad \text{for all } \mathbf{r}_1 \leq \mathbf{r}_2 \text{ and } \mathbf{s}_1 \leq \mathbf{s}_2. \tag{6.4}$$

Let $((\mathbf{r}_1^*)^T, (\mathbf{s}_1^*)^T)^T$ and $((\mathbf{r}_2^*)^T, (\mathbf{s}_2^*)^T)^T$ denote the fixed points of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$. The Picard Iteration for the fixed points of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ (cf. Suzuki [2002] and Fischer [2014]), (6.3) and (6.4) then yield for arbitrary $\mathbf{r}, \mathbf{s} \geq \mathbf{0}$

$$\begin{pmatrix} \mathbf{r}_1^* \\ \mathbf{s}_1^* \end{pmatrix} = \lim_{m\to\infty} \underbrace{\boldsymbol{\Phi}_1 \circ \ldots \circ \boldsymbol{\Phi}_1}_{m} \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} \leq \lim_{m\to\infty} \underbrace{\boldsymbol{\Phi}_2 \circ \ldots \circ \boldsymbol{\Phi}_2}_{m} \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_2^* \\ \mathbf{s}_2^* \end{pmatrix}, \tag{6.5}$$

and the assertion follows. □

Corollary 6.2. *In the situation of Proposition 6.1, v_i is non-decreasing in M_{kj}^{d} and M_{kj}^{e} for all $i, j, k \in \{1, \ldots, n\}$ $(k \neq j)$, i.e. the value of any firm in the system will not decrease if any cross-ownership fraction in the system increases.*

Proof. This directly follows from $v_i = r_i + s_i$ (cf. (3.16)) and Proposition 6.1. □

In the following section we will examine in more detail under what circumstances and to what extent the entries of $\mathbf{r}$ and $\mathbf{s}$ react to chances in the cross-ownership matrices. For that, we will consider derivatives within a Suzuki area.

6.1.2 Derivatives with Respect to the Cross-Ownership Fractions

6.1.2.1 The Implicit Function Theorem

Let $\mathbf{F} : \mathbb{R}^{2n^2} \to \mathbb{R}^{2n}$ be defined as

$$\mathbf{F}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) := \begin{pmatrix} \min\{\mathbf{d},\, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\} - \mathbf{r} \\ (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^+ - \mathbf{s} \end{pmatrix}. \tag{6.6}$$

Of course, $\mathbf{F}$ also depends on $\mathbf{a}$ and $\mathbf{d}$, but as we will assume them to be constant in this section, we suppress them in the notation. In the following we will refer to $\mathbf{r}$ and $\mathbf{s}$ as the variables of $\mathbf{F}$, and to the entries of the cross-ownership matrices $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ as the parameters of $\mathbf{F}$. Thus, there are $2n$ variables and $2n(n-1)$ parameters. Since the ith and $(n+i)$th component of $\mathbf{F}$ refer to r_i and s_i, respectively, and thus to the same firm, we will sometimes denote the components of $\mathbf{F}$ as $(F_1, \ldots, F_i, \ldots, F_n, F_{n+1}, \ldots, F_{n+i}, \ldots, F_{2n})$.

Clearly, (3.13)–(3.14) is equivalent to $\mathbf{F}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \mathbf{0}$, and since the solution $(\mathbf{r}^*, \mathbf{s}^*)$ of $\mathbf{F}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \mathbf{0}$ depends on $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, one is tempted to write $(\mathbf{r}^*, \mathbf{s}^*) = (\mathbf{r}^*(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}), \mathbf{s}^*(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}))$. For a strictly mathematical derivation of the existence of such a functional relationship and its properties, a version of the Implicit Function Theorem will prove useful. In contrast to the "classical" version of the Implicit Function Theorem (see for example Theorem 9.28 of Rudin [1974]), the version of Halkin [1974] does not require that $\mathbf{F}$ is globally continuously differentiable. We adopt its notation and prerequisites to our set-up.

Theorem 6.3. *Let* $\mathbf{X} \subset \mathbb{R}^k$, $\mathbf{P} \subset \mathbb{R}^m$, *and let* $\mathbf{G} : \mathbf{X} \times \mathbf{P} \to \mathbb{R}^k$, $(\mathbf{x}, \mathbf{p}) \mapsto \mathbf{G}(\mathbf{x}, \mathbf{p})$, *be continuous. For* $(\mathbf{x}^*, \mathbf{p}^*) \in \mathbf{X} \times \mathbf{P}$, *suppose that the derivative* $D\mathbf{G} \in \mathbb{R}^{k\times(k+m)}$ *of* $\mathbf{G}$ *with respect to* $(\mathbf{x}, \mathbf{p})$ *exists at* $(\mathbf{x}^*, \mathbf{p}^*)$. *For* $D\mathbf{G}|_{(\mathbf{x},\mathbf{p})=(\mathbf{x}^*,\mathbf{p}^*)} = (\mathbf{A}, \mathbf{B})$ *with* $\mathbf{A} \in \mathbb{R}^{k\times k}$ *and* $\mathbf{B} \in \mathbb{R}^{k\times m}$, *let* $\mathbf{A}$ *be invertible. Then there exists a neighbourhood* $\mathbf{U} \subset \mathbf{P}$ *of* $\mathbf{p}^*$ *and a function* $\boldsymbol{\psi} : \mathbf{U} \to \mathbf{X}$ *such that*

1. $\boldsymbol{\psi}(\mathbf{p}^*) = \mathbf{x}^*$,
2. $\mathbf{G}(\boldsymbol{\psi}(\mathbf{p}), \mathbf{p}) = \mathbf{G}(\mathbf{x}^*, \mathbf{p}^*)$ *for all* $\mathbf{p} \in \mathbf{U}$, *and*
3. $\boldsymbol{\psi}$ *is continuously differentiable at* $\mathbf{p}^*$ *with*

$$D\boldsymbol{\psi}(\mathbf{p}^*) = -\mathbf{A}^{-1}\mathbf{B}. \tag{6.7}$$

The notation of Theorem 6.3 can be transferred to our set-up as follows. First, we can identify k with the number of variables $2n$, m with the number of parameters $2n(n-1)$, and $\mathbf{G}$ with $\mathbf{F}$ as defined in (6.6). Furthermore, we can choose $\mathbf{X} = \mathbb{R}^{2n}$ and $\mathbf{P}$ as the set of all possible pairs of cross-ownership matrices $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$, which can be interpreted as a subset of $\mathbb{R}^{2n(n-1)}$, and we can identify $\mathbf{x}$ with $(\mathbf{r}, \mathbf{s})$ and $\mathbf{p}$ with a vector containing the off-diagonal elements of $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$.
Of course, $\mathbf{F}$ is continuous in all its variables and parameters. In order to be able to use Theorem 6.3, we need to see under what conditions $\mathbf{F}$ (i.e. each of the $2n$ components of $\mathbf{F}$) is continuously partially differentiable with respect to the components of $\mathbf{r}$ and $\mathbf{s}$ and the entries of $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$. As becomes immediately clear from (6.6), $\mathbf{F}$ is continuously partially differentiable with respect to each variable and parameter, if and only if

$$d_i \neq a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j \quad \text{for all } i \in \{1, \ldots, n\}. \tag{6.8}$$

For a given $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$, let $(\mathbf{r}^*, \mathbf{s}^*)$ be such that $\mathbf{F}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \mathbf{0}$, i.e. $(\mathbf{r}^*, \mathbf{s}^*)$ is a solution to (3.13)–(3.14). In the following, we always assume $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$ to be such that $\mathbf{a}$ lies in the inner of some Suzuki area $A_{\mathbf{z}}$ for some $\mathbf{z} \in \{\mathrm{s}, \mathrm{d}\}^n$. Then $r_i^* + s_i^* = v_i^* \neq d_i$ for all $i \in \{1, \ldots, n\}$ (cf. (3.18)), and thus by (3.16),

$$d_i \neq a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j^* + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j^* \quad \text{for all } i \in \{1, \ldots, n\}, \tag{6.9}$$

which means that $\mathbf{F}$ is continuously partially differentiable in all variables and parameters at $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$. The Jacobian matrix of $\mathbf{F}$ with respect to $\mathbf{r}$ and $\mathbf{s}$ can be derived from the following lemma.

Lemma 6.4. *For* $\mathbf{F}$ *as defined in* (6.6)*, let* $d_i \neq a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j$ *for all* $i \in \{1, \ldots, n\}$*. Then, for* $i, j \in \{1, \ldots, n\}$,

$$\frac{\partial F_i}{\partial r_j}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} -1, & i = j, \\ 0, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ M_{ij}^{\mathrm{d}}, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.10}$$

$$\frac{\partial F_i}{\partial s_j}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} 0, & i = j, \\ 0, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ M_{ij}^{\mathrm{e}}, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.11}$$

$$\frac{\partial F_{n+i}}{\partial r_j}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} 0, & i = j, \\ M_{ij}^{\mathrm{d}}, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ 0, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.12}$$

$$\frac{\partial F_{n+i}}{\partial s_j}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} -1, & i = j, \\ M_{ij}^{\mathrm{e}}, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ 0, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j. \end{cases} \tag{6.13}$$

Proof. This follows from straightforward calculations and the assumption that $M_{ii}^{\mathrm{d}} = M_{ii}^{\mathrm{e}} = 0$ for all $i = 1, \ldots, n$. □

Let us now assume that for a cross-ownership scenario given by $\mathbf{a}$, $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, q of the n firms are solvent ($q \in \{0, 1, \ldots, n\}$), w.l.o.g. we assume these firms to be the firms $1, \ldots, q$. Under the assumption that (6.9) holds, we have

$$d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j^* + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j^* \quad \Leftrightarrow \quad i \leq q. \tag{6.14}$$

By Lemma 6.4, the Jacobian matrix of $\mathbf{F}$ with respect to $(\mathbf{r}, \mathbf{s})$ evaluated at $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$ is then given as

$$\mathbf{J} = \left(\begin{array}{ccc|ccc} \frac{\partial F_1}{\partial r_1} & \cdots & \frac{\partial F_1}{\partial r_n} & \frac{\partial F_1}{\partial s_1} & \cdots & \frac{\partial F_1}{\partial s_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial r_1} & \cdots & \frac{\partial F_n}{\partial r_n} & \frac{\partial F_n}{\partial s_1} & \cdots & \frac{\partial F_n}{\partial s_n} \\ \hline \frac{\partial F_{n+1}}{\partial r_1} & \cdots & \frac{\partial F_{n+1}}{\partial r_n} & \frac{\partial F_{n+1}}{\partial s_1} & \cdots & \frac{\partial F_{n+1}}{\partial s_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_{2n}}{\partial r_1} & \cdots & \frac{\partial F_{2n}}{\partial r_n} & \frac{\partial F_{2n}}{\partial s_1} & \cdots & \frac{\partial F_{2n}}{\partial s_n} \end{array} \right) (\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) \tag{6.15}$$

$$
= \left(\begin{array}{c:c|c:c}
\begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 & 0 & 0 \\ \hdashline
M^{\mathrm{d}} & \begin{matrix} -1 & & M^{\mathrm{d}} \\ & \ddots & \\ M^{\mathrm{d}} & & -1 \end{matrix} & M^{\mathrm{e}} & \begin{matrix} 0 & & M^{\mathrm{e}} \\ & \ddots & \\ M^{\mathrm{e}} & & 0 \end{matrix} \\ \hline
\begin{matrix} 0 & & M^{\mathrm{d}} \\ & \ddots & \\ M^{\mathrm{d}} & & 0 \end{matrix} & M^{\mathrm{d}} & \begin{matrix} -1 & & M^{\mathrm{e}} \\ & \ddots & \\ M^{\mathrm{e}} & & -1 \end{matrix} & M^{\mathrm{e}} \\ \hdashline
0 & 0 & 0 & \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix}
\end{array}\right)
\begin{matrix} \}\, q \\ \}\, n-q \\ \}\, q \\ \}\, n-q \end{matrix}
$$

$$
\qquad\; \underbrace{\qquad}_{q} \quad \underbrace{\qquad}_{n-q} \quad \underbrace{\qquad}_{q} \quad \underbrace{\qquad}_{n-q} \tag{6.16}
$$

$$
=: \left(\begin{array}{c:c|c:c}
\mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} \\ \hdashline
\mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} \\ \hline
\mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} \\ \hdashline
\mathbf{M}_{41} & \mathbf{M}_{42} & \mathbf{M}_{43} & \mathbf{M}_{44}
\end{array}\right) - \mathbf{I}_{2n} =: \mathbf{M} - \mathbf{I}_{2n}, \tag{6.17}
$$

where in both, rows and columns of (6.16) and (6.17), the dashed lines separate the q solvent firms from the $n-q$ firms in default, and with

$$
\begin{pmatrix} \mathbf{M}_{31} & \mathbf{M}_{32} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} = \mathbf{M}^{\mathrm{d}}, \quad \begin{pmatrix} \mathbf{M}_{33} & \mathbf{M}_{34} \\ \mathbf{M}_{23} & \mathbf{M}_{24} \end{pmatrix} = \mathbf{M}^{\mathrm{e}}. \tag{6.18}
$$

This also clarifies how the entries M^{d} and M^{e} of (6.16) should be understood.
The matrices $\mathbf{M}_{kl}$ $(1 \leq k, l \leq 4)$ defined in (6.17) have non-negative entries, and from (6.16) and (6.18) it becomes clear that the column sums of $\mathbf{M}$ defined in (6.17) are strictly smaller than 1, since $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ are strictly left sub-stochastic. Then Lemma A.1 of Gouriéroux et al. [2012] implies that

$$
\det(\mathbf{J}) = \det(\mathbf{M} - \mathbf{I}_{2n}) = (-1)^{2n} \det(\mathbf{I}_{2n} - \mathbf{M}) > 0. \tag{6.19}
$$

In particular, $\mathbf{J}$ is invertible (see also Elsinger [2009] and Fischer [2014]). This is necessary for the applicability of Theorem 6.3 as $\mathbf{J}$ corresponds to $\mathbf{A}$ of Theorem 6.3. Laplace expansion (cf. equation (0.3.1.1) of Horn and Johnson [2013], for example) of $\mathbf{J}$ along the first q rows and the last $n-q$ rows yields

$$
0 < \det(\mathbf{J}) = (-1)^n \det\left(\begin{pmatrix} \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{32} & \mathbf{M}_{33} \end{pmatrix} - \mathbf{I}_n\right) = \det\left(\mathbf{I}_n - \begin{pmatrix} \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{32} & \mathbf{M}_{33} \end{pmatrix}\right). \tag{6.20}
$$

Hence, the determinant of $\mathbf{J}$ depends on half of all cross-ownership fractions only.

We now consider the partial derivatives of $\mathbf{F}$ with respect to the entries of the cross-ownership matrices $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$.

Lemma 6.5. *For* $\mathbf{F}$ *as defined in* (6.6)*, let* $d_i \neq a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j$ *for all* $i \in \{1, \ldots, n\}$*. Then, for* $i \neq j$ $(i, j \in \{1, \ldots, n\})$*,*

$$\frac{\partial F_i}{\partial M_{ij}^{\mathrm{d}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} 0, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ r_j, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.21}$$

$$\frac{\partial F_i}{\partial M_{ij}^{\mathrm{e}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} 0, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ s_j, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.22}$$

$$\frac{\partial F_{n+i}}{\partial M_{ij}^{\mathrm{d}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} r_j, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ 0, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \end{cases} \tag{6.23}$$

$$\frac{\partial F_{n+i}}{\partial M_{ij}^{\mathrm{e}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} s_j, & i \neq j \text{ and } d_i < a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j, \\ 0, & i \neq j \text{ and } d_i > a_i + \sum_{j=1}^n M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^n M_{ij}^{\mathrm{e}} s_j. \end{cases} \tag{6.24}$$

For $i \neq k$ *and* $k \neq j$ $(i, j, k \in \{1, \ldots, n\})$*,*

$$\frac{\partial F_i}{\partial M_{kj}^{\mathrm{d}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \frac{\partial F_i}{\partial M_{kj}^{\mathrm{e}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = 0, \tag{6.25}$$

$$\frac{\partial F_{n+i}}{\partial M_{kj}^{\mathrm{d}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \frac{\partial F_{n+i}}{\partial M_{kj}^{\mathrm{e}}}(\mathbf{r}, \mathbf{s}, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = 0. \tag{6.26}$$

Proof. This follows from straightforward calculations. □

We do not consider derivatives of $\mathbf{F}$ with respect to M_{kk}^{d} and M_{kk}^{e}, $k = 1, \ldots, n$, since these cross-ownership fractions are constant 0 in our model.

Corollary 6.6. *Under the assumption that* (6.9) *holds, for* $k \neq j$ $(j, k \in \{1, \ldots, n\})$*,*

$$\left(\frac{\partial F_l}{\partial M_{kj}^{\mathrm{d}}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) \right)_{1 \leq l \leq 2n} = \begin{cases} (\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, 0}_{k-1}, r_j^*, \underbrace{0, \ldots, 0}_{n-k}), & \text{firm } k \text{ solvent}, \\ (\underbrace{0, \ldots, 0}_{k-1}, r_j^*, \underbrace{0, \ldots, 0}_{n-k}, \underbrace{0, \ldots, 0}_{n}), & \text{firm } k \text{ in def.}, \end{cases} \tag{6.27}$$

$$\left(\frac{\partial F_l}{\partial M_{kj}^{\mathrm{e}}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) \right)_{1 \leq l \leq 2n} = \begin{cases} (\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, 0}_{k-1}, s_j^*, \underbrace{0, \ldots, 0}_{n-k}), & \text{firm } k \text{ solvent}, \\ (\underbrace{0, \ldots, 0}_{k-1}, s_j^*, \underbrace{0, \ldots, 0}_{n-k}, \underbrace{0, \ldots, 0}_{n}), & \text{firm } k \text{ in def.} \end{cases} \tag{6.28}$$

In particular, for $l \in \{1, \ldots, q\} \cup \{n+q+1, \ldots, 2n\}$,

$$\frac{\partial F_l}{\partial M^{\mathrm{d}}_{kj}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \frac{\partial F_l}{\partial M^{\mathrm{e}}_{kj}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = 0. \tag{6.29}$$

Proof. This is an immediate consequence of Lemma 6.5 and the fact that if (6.9) holds, firm k is in default if and only if $k > q$. □

Hence, under the assumption that (6.9) holds, the Jacobian matrix of $\mathbf{F}$ with respect to all variables and parameters exists. As we can identify our (invertible) $\mathbf{J}$ with $\mathbf{A}$ of Theorem 6.3, we can apply Theorem 6.3 to our problem and we obtain a neighbourhood $\mathbf{U} \subseteq \mathbf{P}$ of $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$ and a function $\boldsymbol{\psi} : \mathbf{U} \to \mathbb{R}^{2n}$ such that $\mathbf{F}(\boldsymbol{\psi}(\tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}}), \tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}}) = \mathbf{0}$ for all $(\tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}}) \in \mathbf{U}$, i.e. for a given pair $(\tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}})$ of cross-ownership fractions in the neighbourhood of $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$, $\boldsymbol{\psi}$ yields the corresponding solution $(\tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ to the problem $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}}) = \mathbf{0}$, which exists and is unique by Theorem 3.8 of Fischer [2014]. Hence,

$$\boldsymbol{\psi}(\tilde{\mathbf{M}}^{\mathrm{d}}, \tilde{\mathbf{M}}^{\mathrm{e}}) = (\tilde{\mathbf{r}}^T, \tilde{\mathbf{s}}^T)^T, \tag{6.30}$$

which is why we will identify $\psi_1, \ldots, \psi_n$ with $r_1, \ldots, r_n$ and $\psi_{n+1}, \ldots, \psi_{2n}$ with $s_1, \ldots, s_n$. Furthermore, $\boldsymbol{\psi}$ is differentiable at $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$, i.e. we can examine how slight changes in the entries of $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$ affect $\mathbf{r}^*$ and $\mathbf{s}^*$. This will be done in the following section. Note that since we assume $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ to be such that $\mathbf{a}$ lies in the inner of some Suzuki area and since $\mathbf{v}$ is continuous in $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ by Lemma 3.4, slight changes in $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ do not alter the financial status of any firm in the system.

6.1.2.2 The Jacobian Matrix of ψ

By Theorem 6.3, the Jacobian matrix of $\boldsymbol{\psi}$ in $(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$ with respect to the entries of the cross-ownership matrices, in symbols $\mathbf{J}_{\boldsymbol{\psi}}$, is given as

$$\mathbf{J}_{\boldsymbol{\psi}} = -\mathbf{J}^{-1}\mathbf{K} \in \mathbb{R}^{2n \times 2n(n-1)}, \tag{6.31}$$

where $\mathbf{K} \in \mathbb{R}^{2n \times 2n(n-1)}$ stands for the Jacobian matrix of $\mathbf{F}$ with respect to the $2n(n-1)$ entries of the cross-ownership matrices evaluated at $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$. $\mathbf{K}$ exists by Corollary 6.6. However, (6.31) can only be applied if the inverse of $\mathbf{J}$ is known. Instead of calculating $\mathbf{J}^{-1}$ explicitly, we determine the entries of $\mathbf{J}_{\boldsymbol{\psi}}$ by use of Cramer's rule (see Theorem 4.7D of Thrall and Tornheim [1957], for example), stating that the solution of a linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^m$ can be calculated as

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}, \quad 1 \le i \le m, \tag{6.32}$$

where the matrix $\mathbf{A}_i$ is obtained from $\mathbf{A}$ by replacing the ith column of $\mathbf{A}$ with $\mathbf{b}$.
For some $k, j \in \{1, \ldots, n\}$, $k \neq j$, let us consider the column of $\mathbf{J}_{\boldsymbol{\psi}}$ containing the derivatives of $\boldsymbol{\psi}$ with respect to M_{kj}, where M_{kj} stands for M^{d}_{kj} or M^{e}_{kj}. Then (6.31)

yields

$$\mathbf{J} \times \left(\frac{\partial \psi_1}{\partial M_{kj}}, \ldots, \frac{\partial \psi_{2n}}{\partial M_{kj}} \right)^T = - \left(\frac{\partial F_1}{\partial M_{kj}}, \ldots, \frac{\partial F_{2n}}{\partial M_{kj}} \right)^T, \tag{6.33}$$

where all derivatives are evaluated at $(\mathbf{r}^*, \mathbf{s}^*)$ resp. $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$. By Cramer's rule, the entries of $\big(\frac{\partial \psi_1}{\partial M_{kj}}, \ldots, \frac{\partial \psi_{2n}}{\partial M_{kj}}\big)(\mathbf{r}^*, \mathbf{s}^*)$ can be determined as

$$0 \le \frac{\partial \psi_l}{\partial M_{kj}}(\mathbf{r}^*, \mathbf{s}^*) = -\frac{\det(\mathbf{J}_l)}{\det(\mathbf{J})}, \quad 1 \le l \le 2n, \tag{6.34}$$

where $\mathbf{J}_l$ is obtained from $\mathbf{J}$ by replacing the lth column of $\mathbf{J}$ with $\big(\frac{\partial F_1}{\partial M_{kj}}, \ldots, \frac{\partial F_{2n}}{\partial M_{kj}}\big)^T$ evaluated at $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})$. Note that the inequality in (6.34) follows from Proposition 6.1.

In the following, we will further analyze $\det(\mathbf{J}_l)$. By Corollary 6.6, the inserted column $\big(\frac{\partial F_l}{\partial M_{kj}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})\big)_{1 \le l \le 2n}$ of $\mathbf{J}_l$ has at most one non-zero element, denoted by

$$m_j := \begin{cases} r_j^*, & M_{kj} = M_{kj}^{\mathrm{d}}, \\ s_j^*, & M_{kj} = M_{kj}^{\mathrm{e}}. \end{cases} \tag{6.35}$$

Of course, m_j might be 0, but as it is the only element of $\big(\frac{\partial F_l}{\partial M_{kj}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})\big)_{1 \le l \le 2n}$ that can be different from 0, we will refer to m_j as the non-zero element of $\big(\frac{\partial F_l}{\partial M_{kj}}(\mathbf{r}^*, \mathbf{s}^*, \mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}})\big)_{1 \le l \le 2n}$. Furthermore, let

$$\mathbf{J}_l =: \left(\begin{array}{c|c|c} \multicolumn{3}{c}{\mathbf{J}_{1,l}} \\ \hline \mathbf{J}_{21,l} & \mathbf{J}_{22,l} & \mathbf{J}_{23,l} \\ \hline \multicolumn{3}{c}{\mathbf{J}_{3,l}} \end{array} \right) \in \mathbb{R}^{2n \times 2n}, \tag{6.36}$$

where the dashed lines exactly correspond to the dashed lines in (6.16), i.e. $\mathbf{J}_{1,l} \in \mathbb{R}^{q \times 2n}$, $\mathbf{J}_{21,l} \in \mathbb{R}^{n \times q}$, $\mathbf{J}_{22,l} \in \mathbb{R}^{n \times n}$, $\mathbf{J}_{23,l} \in \mathbb{R}^{n \times (n-q)}$ and $\mathbf{J}_{3,l} \in \mathbb{R}^{(n-q) \times 2n}$.

By Corollary 6.6 the non-zero element m_j of the lth column of $\mathbf{J}_l$ cannot stand in the first q rows or the last $n - q$ rows of $\mathbf{J}_l$. Let us now assume m_j stands in $\mathbf{J}_{21,l}$ or $\mathbf{J}_{23,l}$, i.e. $l \in \{1, \ldots, q\} \cup \{n + q + 1, \ldots, 2n\}$. Apart from m_j, the lth column of $\mathbf{J}_l$ consists of zeros only. Then it can be seen from (6.16) that the lth row of $\mathbf{J}_l$ is a zero row. Hence,

$$\frac{\partial \psi_l}{\partial M_{kj}}(\mathbf{r}^*, \mathbf{s}^*) = -\frac{\det(\mathbf{J}_l)}{\det(\mathbf{J})} = 0, \quad l \in \{1, \ldots, q\} \cup \{n + q + 1, \ldots, 2n\}. \tag{6.37}$$

If we assume that m_j stands in $\mathbf{J}_{22,l}$, i.e. $l \in \{q + 1, \ldots, n + q\}$, successive Laplace expansion along the first q rows and last $n - q$ rows of $\mathbf{J}_l$ yields

$$\det(\mathbf{J}_l) = (-1)^n \det(\mathbf{J}_{22,l}). \tag{6.38}$$

Hence, we have the following proposition.

Proposition 6.7. *For n firms linked by cross-ownership of possibly both, debt and equity, let* $\mathbf{d}$*,* $\mathbf{M}^{\mathrm{d}}$ *and* $\mathbf{M}^{\mathrm{e}}$ *be such that* $\mathbf{a}$ *lies in the inner of some Suzuki area* $A_{\mathbf{z}}$*, and let* $(\mathbf{r}, \mathbf{s})$ *be the such that*

$$\mathbf{r} = \min\{\mathbf{d},\, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\}, \tag{6.39}$$

$$\mathbf{s} = (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^{+}. \tag{6.40}$$

Then, for $i, j, k \in \{1, \ldots, n\}$ $(k \neq j)$*,*

$$\frac{\partial r_i}{\partial M_{kj}}(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} 0, & \textit{firm } i \textit{ solvent}, \\ \frac{(-1)^{n+1}\det(\mathbf{J}_{22,i})}{\det(\mathbf{J})} \geq 0, & \textit{firm } i \textit{ in default}, \end{cases} \tag{6.41}$$

$$\frac{\partial s_i}{\partial M_{kj}}(\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}) = \begin{cases} \frac{(-1)^{n+1}\det(\mathbf{J}_{22,n+i})}{\det(\mathbf{J})} \geq 0, & \textit{firm } i \textit{ solvent}, \\ 0, & \textit{firm } i \textit{ in default}, \end{cases} \tag{6.42}$$

with $\mathbf{J}$*,* $\mathbf{J}_{22,i}$ *and* $\mathbf{J}_{22,n+i}$ *defined in* (6.16) *and* (6.36)*.*

Proof. The derivatives follow from (6.34), (6.37), (6.38) and the definition of $\boldsymbol{\psi}$. Proposition 6.1 yields that the derivatives are non-negative. □

If firm i is solvent, we have $r_i = d_i$, and it is clear that within a Suzuki area (where the financial status of any firm in the system is unchanged, and in particular, firm i stays solvent), this recovery value of debt is invariant under changes in any cross-ownership fraction. Within a Suzuki area, the firm's equity value s_i is non-decreasing in any cross-ownership fraction. From (6.20), the definition of $\mathbf{J}_{22,n+i}$ and the following Remark 6.8 it becomes clear that the corresponding derivative only depends on m_j (cf. (6.35)), the fractions of debt of the bankrupt firms held within the system and the fractions of the equity of the solvent firms held within the system.

Similarly, if firm i is in default, we have $s_i = 0$, i.e. the value of firm i's equity remains 0 also if any cross-ownership fraction in the model changes, as long as such changes do not alter the financial status of any firm in the system (in particular firm i remains in default) and as above, the derivative of firm i's recovery value of debt with respect to any cross-ownership fraction only depends on m_j, the fractions of debt of the bankrupt firms held within the system and the fractions of the equity of the solvent firms held within the system.

Remark 6.8. The fact that the corresponding derivatives of Proposition 6.7 are non-negative, can also be shown by considering the sign of $\det(\mathbf{J}_{22,l})$ for $l \in \{q+1, \ldots, n+q\}$. Recall that $\det(\mathbf{J}) > 0$ by (6.19). By (6.16) and (6.36), apart from the inserted column with one non-zero entry m_j only, the diagonal elements of $\mathbf{J}_{22,l} \in \mathbb{R}^{n\times n}$ equal -1 and the off-diagonal elements are entries of $\mathbf{M}^{\mathrm{d}}$ and/or $\mathbf{M}^{\mathrm{e}}$ (that depends on q, the number of solvent firms).

If m_j stands on the diagonal of $\mathbf{J}_{22,l}$, we can expand $\mathbf{J}_{22,l}$ along the column containing

m_j to obtain

$$\det(\mathbf{J}_{22,l}) = m_j \det(\mathbf{J}'_{22,l}), \tag{6.43}$$

where $\mathbf{J}'_{22,l}$ is the $(n-1)\times(n-1)-$submatrix of $\mathbf{J}_{22,l}$ where the row and column containing m_j have been deleted. Of course, the diagonal entries of $\mathbf{J}'_{22,l}$ equal -1, and by (6.18) and (6.36), the sum of the remaining entries is strictly smaller than 1 in each column. Hence, setting $\mathbf{J}''_{22,l} := \mathbf{J}'_{22,l} + \mathbf{I}_{n-1}$ we can write

$$(-1)^{n+1}\det(\mathbf{J}_{22,l}) = (-1)^{n+1} m_j \det(\mathbf{J}''_{22,l} - \mathbf{I}_{n-1}) = (-1)^{2n} m_j \det(\mathbf{I}_{n-1} - \mathbf{J}''_{22,l}) \geq 0, \tag{6.44}$$

since the last determinant is positive by Lemma A.1 of Gouriéroux et al. [2012].
If m_j stands in the kth row and pth column of $\mathbf{J}_{22,l}$ $(k \neq p)$, it is straightforward to see that $\mathbf{J}_{22,l}$ meets the assumptions of Lemma A.15, and Laplace expansion of $\mathbf{J}_{22,l}$ along the pth column yields (recall that $m_j \geq 0$ is the only non-zero element in this column)

$$(-1)^{n+1}\det(\mathbf{J}_{22,l}) = (-1)^{n+1}(-1)^{k+p} m_j \det(\mathbf{J}'''_{22,l}) \geq 0, \tag{6.45}$$

with $\mathbf{J}'''_{22,l}$ denoting the $(n-1)\times(n-1)-$submatrix of $\mathbf{J}_{22,l}$, where the kth row and pth column have been deleted.

6.2 Influence of the Nominal Level of Liabilities d

6.2.1 Derivatives with Respect to d

6.2.1.1 Arbitrary Cross-Ownership Structure

Having analyzed the influence of the cross-ownership fractions on $(\mathbf{r}, \mathbf{s})$ as solutions of (3.13)–(3.14) in the previous sections, we now examine how r_i and s_i change with d_j, $1 \leq i, j \leq n$, for a given value of $\mathbf{a}$ and fixed cross-ownership fractions $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$. In contrast to the analysis with respect to the cross-ownership fractions (cf. Proposition 6.1), sensitivities of $\mathbf{r}$ and $\mathbf{s}$ with respect to $\mathbf{d}$ cannot be directly obtained from the definition of $\mathbf{\Phi}$ in (3.17). Instead, we will first calculate derivatives within the Suzuki areas and infer global monotonicity properties from them. These derivatives will not be deduced by the Implicit Function Theorem as in Section 6.1.2, because in analogy to (6.34) this would yield a matrix $\bar{\mathbf{J}}_l$ containing derivatives of $\mathbf{F}$ with respect to some d_j, among others, but we would not be able to derive the sign of $\det(\bar{\mathbf{J}}_l)$ and thus not of $\frac{\partial F_l}{\partial d_j}$ for a certain scenario, whereas this sign can be determined with the approach of this section.
In the following, $\mathbf{r}$ and $\mathbf{s}$ are always solutions of (3.13)–(3.14), and we will always assume that slight changes in an entry of $\mathbf{d}$ do not alter the financial status (solvency or default) of any firm in the system, i.e. we stay within a certain Suzuki area. However, this implies that within a certain Suzuki area there might be values of d_j for which only one-sided derivatives of r_i and s_i with respect to d_j exist. Henceforth, if the two-sided derivative does not exist for a particular value of d_j, we mean the value of the one-sided

derivative. One-sided derivatives always exist because within a certain Suzuki area, the system (3.13)–(3.14) is linear.

Again, we will assume w.l.o.g. that the firms $1, \ldots, q$ are solvent and that the firms $q+1, \ldots, n$ are in default ($q \in \{0, 1, \ldots, n\}$). Then

$$r_i = d_i \text{ for } 1 \leq i \leq q, \quad s_i = 0 \text{ for } q < i \leq n, \tag{6.46}$$

and obviously, for $j \in \{1, \ldots, n\}$,

$$\frac{\partial r_i}{\partial d_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{for } 1 \leq i \leq q, \tag{6.47}$$

$$\frac{\partial s_i}{\partial d_j} = 0 \qquad \text{for } q < i \leq n. \tag{6.48}$$

The values of s_i $(1 \leq i \leq q)$ and r_i $(q < i \leq n)$ can be calculated by setting the $(q+1)$th to $(n+q)$th component of $\mathbf{F}$ to $\mathbf{0}$ (cf. (6.6)) and making use of (6.46). Rearranging the order of the equations in $\mathbf{F} = \mathbf{0}$ we obtain the following system of equations (see also Hain and Fischer [2015]).

$$\begin{aligned} \left(a_i + \sum_{j=1}^{q} M_{ij}^{\mathrm{d}} d_j + \sum_{j=q+1}^{n} M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^{q} M_{ij}^{\mathrm{e}} s_j - d_i - s_i \right)_{1 \leq i \leq q} &= \mathbf{0}_q \\ \left(a_i + \sum_{j=1}^{q} M_{ij}^{\mathrm{d}} d_j + \sum_{j=q+1}^{n} M_{ij}^{\mathrm{d}} r_j + \sum_{j=1}^{q} M_{ij}^{\mathrm{e}} s_j - r_i \right)_{q < i \leq n} &= \mathbf{0}_{n-q}. \end{aligned} \tag{6.49}$$

Equivalently,

$$\begin{aligned} \left(\sum_{j=1}^{q} M_{ij}^{\mathrm{e}} s_j + \sum_{j=q+1}^{n} M_{ij}^{\mathrm{d}} r_j - s_i \right)_{1 \leq i \leq q} &= - \left(a_i + \sum_{j=1}^{q} M_{ij}^{\mathrm{d}} d_j - d_i \right)_{1 \leq i \leq q} \\ \left(\sum_{j=1}^{q} M_{ij}^{\mathrm{e}} s_j + \sum_{j=q+1}^{n} M_{ij}^{\mathrm{d}} r_j - r_i \right)_{q < i \leq n} &= - \left(a_i + \sum_{j=1}^{q} M_{ij}^{\mathrm{d}} d_j \right)_{q < i \leq n}, \end{aligned} \tag{6.50}$$

which can be written as

$$\mathbf{A}\mathbf{x} = -\mathbf{b} \tag{6.51}$$

with

$$\mathbf{A} = \left(\left(M_{ij}^{\mathrm{e}} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq q}}, \left(M_{ij}^{\mathrm{d}} \right)_{\substack{1 \leq i \leq n \\ q < j \leq n}} \right) - \mathbf{I}_n \in \mathbb{R}^{n \times n}, \tag{6.52}$$

$$\mathbf{x} = (x_i)_{1 \leq i \leq n} = (s_1, \ldots, s_q, r_{q+1}, \ldots, r_n)^T \in \mathbb{R}^n, \tag{6.53}$$

$$\mathbf{b} = (b_i)_{1\le i\le n} = \left(a_i + \sum_{j=1}^{q} M^{\mathrm{d}}_{ij} d_j\right)_{1\le i\le n} - (d_1, \ldots, d_q, \underbrace{0, \ldots, 0}_{n-q})^T \in \mathbb{R}^n. \tag{6.54}$$

Of course, $\mathbf{x}$ depends on $\mathbf{a}$, $\mathbf{d}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$, but we will suppress them in our notation for better readability.

As $d_{q+1}, \ldots, d_n$ do not occur in (6.51), we have for a firm j in default,

$$\frac{\partial x_i}{\partial d_j} = 0 \quad \text{for all } i \in \{1, \ldots, n\}, \tag{6.55}$$

which means that the equity value of a solvent firm and the recovery value of debt of a firm in default are independent of the nominal level of liabilities of any firm of the system being in default, provided that slight changes in these nominal level of liabilities do not alter the financial status of any firm in the system.

Let us now consider $\frac{\partial x_i}{\partial d_j}$ ($i \in \{1, \ldots, n\}$) for a solvent firm j (i.e. $j \le q$), which is the derivative whose sign we could not have determined by the Implicit Function Theorem. Since x_i is linear in d_j, this derivative exists. By Cramer's rule, $x_i = -\frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$, where $\mathbf{A}_i$ is obtained from $\mathbf{A}$ by replacing the ith column of $\mathbf{A}$ with $\mathbf{b}$. Note that $\det(\mathbf{A}) \neq 0$ by Lemma A.1 of Fischer [2014]. Laplace expansion of $\mathbf{A}_i$ along the ith column yields

$$\det(\mathbf{A}_i) = \sum_{k=1}^{n} (-1)^{i+k}\, b_k \det(\mathbf{A}_{ki}), \tag{6.56}$$

where $\mathbf{A}_{ki} \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the submatrix of $\mathbf{A}$ where the kth row and ith column have been deleted. Since $\mathbf{d}$ does not occur in $\mathbf{A}$ and $\mathbf{A}_{ki}$,

$$\frac{\partial x_i}{\partial d_j} = \frac{-1}{\det(\mathbf{A})} \frac{\partial \det(\mathbf{A}_i)}{\partial d_j} = \frac{-1}{\det(\mathbf{A})} \left(\sum_{k=1}^{n} (-1)^{i+k} \frac{\partial b_k}{\partial d_j} \det(\mathbf{A}_{ki}) \right), \tag{6.57}$$

with, for $j \le q$ and due to $M_{kk} = 0$,

$$\frac{\partial b_k}{\partial d_j} = \begin{cases} -1, & k = j, \\ M^{\mathrm{d}}_{kj}, & k \neq j. \end{cases} \tag{6.58}$$

Let now $i = j$ (i.e. $i \le q$), meaning we consider $\frac{\partial s_i}{\partial d_i}$ for a solvent firm i. Then (6.57) and (6.58) yield

$$\frac{\partial s_i}{\partial d_i} = \frac{-1}{\det(\mathbf{A})} \left(\sum_{\substack{k=1\\k\neq i}}^{n} (-1)^{i+k} M^{\mathrm{d}}_{ki} \det(\mathbf{A}_{ki}) + (-1)^{2i}(-1)\det(\mathbf{A}_{ii}) \right) \tag{6.59}$$

$$= \frac{-1}{\det(\mathbf{A})} \det(\tilde{\mathbf{A}}_i), \tag{6.60}$$

where $\tilde{\mathbf{A}}_i$ equals $\mathbf{A}$, except that the off-diagonal elements M^{e}_{ki} $(1 \leq k \leq n,\ i \neq k)$ of the ith column of $\mathbf{A}$ have been replaced with M^{d}_{ki} $(1 \leq k \leq n, i \neq k)$. Then it follows from Lemma A.1 of Gouriéroux et al. [2012] that

$$\frac{\partial s_i}{\partial d_i} = -\frac{(-1)^n(>0)}{(-1)^n(>0)} < 0, \quad 1 \leq i \leq q, \tag{6.61}$$

where (>0) stands for a strictly positive factor, which may be different in the numerator and denominator of (6.61). Hence, as it was to be expected, ceteris paribus the equity value of a solvent firm increases if the nominal value of liabilities of this firm decreases.

The value of $\frac{\partial s_i}{\partial d_i}$ is in general not -1, which might have been expected from an intuitive point of view. Since Laplace expansion of $\tilde{\mathbf{A}}_i$ and $\mathbf{A}$ along the ith column yields (cf. (6.59))

$$\det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) = \sum_{\substack{k=1\\k\neq i}}^{n} (-1)^{i+k} \left(M^{\mathrm{d}}_{ki} - M^{\mathrm{e}}_{ki}\right) \det(\mathbf{A}_{ki}), \tag{6.62}$$

a sufficient condition for $\frac{\partial s_i}{\partial d_i} = -1$ is $M^{\mathrm{d}}_{ki} = M^{\mathrm{e}}_{ki}$ for all $1 \leq k \leq n$. Let us now examine under what circumstances firm i's equity value is prone to stronger or weaker changes in the firm's nominal level of liabilities. By Lemma A.15, for $k \neq i$,

$$(-1)^{i+k} \det(\mathbf{A}_{ki}) \begin{cases} \geq 0, & n \text{ odd},\\ \leq 0, & n \text{ even}. \end{cases} \tag{6.63}$$

Let n be even. Then both, $\det(\mathbf{A})$ and $\det(\tilde{\mathbf{A}}_i)$ are strictly positive by Lemma A.1 of Gouriéroux et al. [2012], and hence by (6.60),

$$\frac{\partial s_i}{\partial d_i} \leq -1 \quad \Leftrightarrow \quad \det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \geq 0. \tag{6.64}$$

Because of $(-1)^{i+k} \det(\mathbf{A}_{ki}) \leq 0$ for $k \neq i$ and n even by (6.63), a sufficient condition for $\det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \geq 0$ is $M^{\mathrm{d}}_{ki} \leq M^{\mathrm{e}}_{ki}$ for all $k \neq i$. Similarly, $\det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \leq 0$, i.e. $\frac{\partial s_i}{\partial d_i} \geq -1$, if $M^{\mathrm{d}}_{ki} \geq M^{\mathrm{e}}_{ki}$ for all $k \neq i$. For n odd, both $\det(\mathbf{A})$ and $\det(\tilde{\mathbf{A}}_i)$ are strictly negative and hence by (6.60), $\frac{\partial s_i}{\partial d_i} \leq -1 \Leftrightarrow \det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \leq 0$. Since $(-1)^{i+k} \det(\mathbf{A}_{ki}) \geq 0$ for $k \neq i$ and n odd by (6.63), a sufficient condition for $\det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \leq 0$ is $M^{\mathrm{d}}_{ki} \leq M^{\mathrm{e}}_{ki}$ for all $k \neq i$. Similarly, $\det(\tilde{\mathbf{A}}_i) - \det(\mathbf{A}) \geq 0$, i.e. $\frac{\partial s_i}{\partial d_i} \geq -1$, if $M^{\mathrm{d}}_{ki} \geq M^{\mathrm{e}}_{ki}$ for all $k \neq i$. Altogether, by (6.61),

$$\frac{\partial s_i}{\partial d_i} \in \begin{cases} (-\infty, -1], & M^{\mathrm{d}}_{ki} \leq M^{\mathrm{e}}_{ki} \text{ for all } k \neq i,\\ [-1, 0), & M^{\mathrm{d}}_{ki} \geq M^{\mathrm{e}}_{ki} \text{ for all } k \neq i. \end{cases} \tag{6.65}$$

Hence, if a solvent firm i is cross-held primarily via equity (i.e. if each of the other firms in the system holds at least as much of firm i's equity as of its debt, expressed in terms of the corresponding cross-ownership fractions), an increase in firm i's nominal level of liabilities affects firm i's equity value at least as strong (and probably stronger) as without cross-ownership, since in this case, we have ${}_{\mathrm{M}}s_i = a_i - d_i$, i.e. $\frac{\partial_{\mathrm{M}} s_i}{\partial d_i} = -1$, where the index M stands for Merton (cf. Section 3.1.1). On the other hand, if firm i is mainly cross-held via debt, cross-ownership is likely to mitigate the dependency of firm i's equity value on the firm's nominal level of liabilities.

In particular, for firm i solvent, (6.65) implies

$$\frac{\partial s_i}{\partial d_i} \in \begin{cases} (-\infty, -1], & \text{XOS of equity only,} \\ [-1, 0), & \text{XOS of debt only.} \end{cases} \tag{6.66}$$

In general, whether $\frac{\partial s_i}{\partial d_i}$ is bigger or smaller than -1, depends on the entries of $\mathbf{A}$ and $\mathbf{A}_i$ and thus in general on the financial statuses of the other firms in the system, which is illustrated in the following example.

Example 6.9. For three firms linked by cross-ownership, let

$$\mathbf{M}^{\mathrm{d}} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{\mathrm{e}} := \begin{pmatrix} 0 & 0 & 0.3 \\ 0.4 & 0 & 0.1 \\ 0 & 0.3 & 0 \end{pmatrix} \tag{6.67}$$

and $\mathbf{a} := (1, 3, 11)^T$, $\mathbf{d} := (4, 1, d_3)^T$ with $d_3 \geq 0$. We will show that for firm 2 and firm 3 solvent, $\frac{\partial s_3}{\partial d_3} < -1$ if firm 1 is solvent and $\frac{\partial s_3}{\partial d_3} > -1$ if firm 1 is in default. First,

$$r_1 = \min\{4,\, 1 + 0.3 s_3\}, \qquad s_1 = (-3 + 0.3 s_3)^+, \tag{6.68}$$

$$r_2 = \min\{1,\, 3 + 0.2 r_3 + 0.4 s_1 + 0.1 s_3\} = 1, \quad s_2 = 2 + 0.2 r_3 + 0.4 s_1 + 0.1 s_3, \tag{6.69}$$

$$r_3 = \min\{d_3,\, 11 + 0.3 s_2\}, \qquad s_3 = (11 + 0.3 s_2 - d_3)^+, \tag{6.70}$$

and it is straightforward to see that all the three firms are solvent with $s_i > 0$ $(i = 1, 2, 3)$ for $d_3 = 0$. For d_3 slightly bigger than 0, all firms remain solvent for reasons of continuity and (3.14) yields

$$\mathbf{s} = (\mathbf{I}_3 - \mathbf{M}^{\mathrm{e}})^{-1}(\mathbf{a} + (\mathbf{M}^{\mathrm{d}} - \mathbf{I}_3)\mathbf{d}) \tag{6.71}$$

$$= \tfrac{1}{0.934}(0.57 - 0.282 d_3, 3.22 - 0.02 d_3, 11.24 - 0.94 d_3)^T. \tag{6.72}$$

Hence, if all the three firms are solvent, $\frac{\partial s_3}{\partial d_3} = -0.94/0.934 < -1$. With d_3 increasing, (6.72) shows that firm 1 is the first firm to be in default, this is the case for $d_3 = 2\frac{1}{47}$. For d_3 slightly bigger than $2\frac{1}{47}$, firm 1 is in default and firm 2 and firm 3 are solvent with

$$r_2 = 1, \quad s_2 = 2 + 0.2 d_3 + 0.1 s_3, \tag{6.73}$$

$$r_3 = d_3, \quad s_3 = 11 + 0.3 s_2 - d_3, \tag{6.74}$$

which implies

$$s_3 = 11 + 0.3(2 + 0.2d_3 + 0.1s_3) - d_3 = \tfrac{1}{0.97}(11.6 - 0.94d_3), \tag{6.75}$$
$$s_2 = 2 + 0.2d_3 + 0.1\left(\tfrac{1}{0.97}(11.6 - 0.94d_3)\right) = \tfrac{1}{0.97}(3.1 + 0.1d_3), \tag{6.76}$$

i.e. $\frac{\partial s_3}{\partial d_3} = -0.94/0.97 > -1$ for $d_3 \in (2\frac{1}{47}, 12\frac{16}{47})$ (for $d_3 = 12\frac{16}{47}$, s_3 would become 0, i.e. we would leave the Suzuki area $\mathrm{A}_{\mathrm{dss}}$). Hence, for firm 2 and firm 3 solvent, the size of $\frac{\partial s_3}{\partial d_3}$ compared to -1 depends on the financial status of firm 1.

Having analyzed $\frac{\partial x_i}{\partial d_j}$ for $i = j \leq q$ (recall that the case $j > q$ is trivial by (6.55)), we now consider $\frac{\partial x_i}{\partial d_j}$ for $i \neq j$ $(1 \leq i \leq n,\ j \leq q)$. By (6.57) and (6.58), these derivatives are given as

$$\frac{\partial x_i}{\partial d_j} = \frac{-1}{\det(\mathbf{A})}\left(\sum_{\substack{k=1\\k\neq j}}^{n}(-1)^{i+k}M^{\mathrm{d}}_{kj}\det(\mathbf{A}_{ki}) - (-1)^{i+j}\det(\mathbf{A}_{ji})\right), \tag{6.77}$$

which can be both, positive and negative, depending on the entries of the cross-ownership matrices and the financial status of other firms in the system. This becomes clear from Example 6.9, where

$$\frac{\partial s_2}{\partial d_3} = \begin{cases} \frac{-0.02}{0.934}, & d_3 < 2\frac{1}{47}, \\ \frac{0.1}{0.97}, & d_3 \in (2\frac{1}{47}, 12\frac{16}{47}), \end{cases} \tag{6.78}$$

i.e. the sign of $\frac{\partial s_2}{\partial d_3}$ depends on whether firm 1 is solvent, or not. Analogously, there are examples where the sign of $\frac{\partial r_i}{\partial d_j}$ for firm i in default and firm j solvent depends on the financial status of a third firm in the system.

Our findings are summarized in the following proposition.

Proposition 6.10. *For n firms linked by cross-ownership of possibly both, debt and equity, let $\mathbf{a}$, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ be such that the financial status (solvency or default) of firm i remains unchanged for slight changes of d_i and d_j, and let $(\mathbf{r}, \mathbf{s})$ be the such that*

$$\mathbf{r} = \min\{\mathbf{d},\ \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s}\}, \tag{6.79}$$
$$\mathbf{s} = (\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{r} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d})^{+}. \tag{6.80}$$

Then, for firm j in default,

$$\frac{\partial r_i}{\partial d_j} = \frac{\partial s_i}{\partial d_j} = 0, \quad i \in \{1, \ldots, n\}. \tag{6.81}$$

If firm j is solvent,

$$\frac{\partial r_i}{\partial d_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \text{ and firm } i \text{ solvent}, \\ \text{depends}, & i \neq j \text{ and firm } i \text{ in default}, \end{cases} \tag{6.82}$$

$$\frac{\partial s_i}{\partial d_j} = \begin{cases} \text{const.} < 0, & i = j, \\ \text{depends}, & i \neq j \text{ and firm } i \text{ solvent}, \\ 0, & i \neq j \text{ and firm } i \text{ in default}, \end{cases} \tag{6.83}$$

where "depends" means that the derivative can be 0, or a positive or negative constant, contingent upon the realized cross-ownership structure and the financial status of the other firms in the system. Also the exact value of $\frac{\partial s_i}{\partial d_i}$ depends on the financial status of the other firms in the system.

Proof. This follows from (6.47), (6.48), (6.55), (6.61) and (6.78). □

For the special cases of cross-ownership of debt only and cross-ownership of equity only, the indeterminate signs in Proposition 6.10 can be further specified. This will be done in the next section.

Note that the approach used in this section, namely determining some of the r_i and s_i via (6.46) and by solving the reduced system of equations (6.50) with the help of Cramer's rule and then calculating the derivative of the solution with respect to $\mathbf{d}$, would not be expedient in order to determine the derivatives of r_i and s_i with respect to the cross-ownership fractions, since in this case, also the determinant of $\mathbf{A}$ would depend on the half of all cross-ownership fractions, which would make the calculation of $\frac{\partial x_i}{\partial M_{kj}}$ as in (6.57) difficult. So we prefer for that our approach using the Implicit Function Theorem (cf. Section 6.1.2).

Remark 6.11. The analysis of this section also yields derivatives of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{v}$ with respect to a_i within a certain Suzuki area for any constellation of cross-ownership. This complements the results of Liu and Staum [2010] valid under cross-ownership of debt only. By (6.46),

$$\frac{\partial r_i}{\partial a_j} = 0 \quad \text{for firm } i \text{ solvent}, \tag{6.84}$$

$$\frac{\partial s_i}{\partial a_j} = 0 \quad \text{for firm } i \text{ in default}. \tag{6.85}$$

Furthermore, due to

$$\frac{\partial b_k}{\partial a_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \tag{6.86}$$

by (6.54), we have in analogy to (6.57)

$$\frac{\partial x_i}{\partial a_j} = \frac{-1}{\det(\mathbf{A})}\frac{\partial \det(\mathbf{A}_i)}{\partial a_j} = \frac{-1}{\det(\mathbf{A})}\left(\sum_{k=1}^{n}(-1)^{i+k}\frac{\partial b_k}{\partial a_j}\det(\mathbf{A}_{ki})\right) \tag{6.87}$$

$$= \frac{-(-1)^{i+j}\det(\mathbf{A}_{ji})}{\det(\mathbf{A})} = \frac{-(-1)^{i+j}\det(\mathbf{A}_{ji})}{(-1)^n(>0)} \geq 0, \tag{6.88}$$

where the inequality follows from Proposition 2 of Gouriéroux et al. [2012]. For $i = j$, the inequality is strict by Lemma A.1 of Gouriéroux et al. [2012]. Hence, the recovery value of debt of a firm in default and the equity value of a solvent firm are strictly increasing in the firm's own exogenous asset value.

6.2.1.2 XOS of Debt only and XOS of Equity only

In this section we consider the sign of $\frac{\partial r_i}{\partial d_j}$ for firm i in default and firm j solvent, and $\frac{\partial s_i}{\partial d_j}$ for firm i and firm j solvent ($i \neq j$), which are not uniquely determined under a general scenario of cross-ownership (cf. Proposition 6.10), for the special cases of cross-ownership of debt only and cross-ownership of equity only. In particular, we always have $j \leq q$. Furthermore, we again assume that slight changes in d_j do not alter the financial status of any firm in the system.

Under cross-ownership of debt only the system (6.51) is given as

$$\left(\begin{array}{ccc|ccc} -1 & & 0 & & & \\ & \ddots & & & M^{\mathrm{d}} & \\ 0 & & -1 & & & \\ \hline & & & -1 & & M^{\mathrm{d}} \\ & 0 & & & \ddots & \\ & & & M^{\mathrm{d}} & & -1 \end{array}\right)\begin{pmatrix} s_1 \\ \vdots \\ s_q \\ r_{q+1} \\ \vdots \\ r_n \end{pmatrix} = -\mathbf{b}, \tag{6.89}$$

with M^{d} standing for entries of $\mathbf{M}^{\mathrm{d}}$ (cf. (6.52)), which is equivalent to solving

$$\underbrace{\left(\left(M^{\mathrm{d}}_{kl}\right)_{\substack{q<k\leq n\\ q<l\leq n}} - \mathbf{I}_{n-q}\right)}_{=:\mathbf{A}^{n-q}}\begin{pmatrix} r_{q+1} \\ \vdots \\ r_n \end{pmatrix} = -\left(a_k + \sum_{l=1}^{q} M^{\mathrm{d}}_{kl} d_l\right)_{q<k\leq n} \tag{6.90}$$

and calculating $(s_1, \ldots, s_q)$ as

$$\begin{pmatrix} s_1 \\ \vdots \\ s_q \end{pmatrix} = \left(a_k + \sum_{l=1}^{q} M^{\mathrm{d}}_{kl} d_l - d_k\right)_{1\leq k\leq q} + \left(M^{\mathrm{d}}_{kl}\right)_{\substack{1\leq k\leq q\\ q<l\leq n}}\begin{pmatrix} r_{q+1} \\ \vdots \\ r_n \end{pmatrix}. \tag{6.91}$$

By Cramer's rule, $r_i = -\frac{\det(\mathbf{A}^{n-q}_{i-q})}{\det(\mathbf{A}^{n-q})}$ $(i > q)$, where $\mathbf{A}^{n-q}_{i-q}$ is obtained from $\mathbf{A}^{n-q}$ by replacing the $(i-q)$th column of $\mathbf{A}^{n-q}$ with $\left(a_k + \sum_{l=1}^{q} M^{\mathrm{d}}_{kl} d_l\right)_{q<k\leq n}$. Expanding $\mathbf{A}^{n-q}_{i-q}$ along the $(i-q)$th column and deriving r_i with respect to d_j yields

$$\frac{\partial r_i}{\partial d_j} = \frac{-1}{\det(\mathbf{A}^{n-q})} \frac{\partial}{\partial d_j} \left(\sum_{k=q+1}^{n} (-1)^{(i-q)+(k-q)} \left(a_k + \sum_{l=1}^{q} M^{\mathrm{d}}_{kl} d_l \right) \det\left(\mathbf{A}^{n-q}_{k-q,i-q}\right) \right) \tag{6.92}$$

$$= \frac{-1}{\det(\mathbf{A}^{n-q})} \sum_{k=q+1}^{n} (-1)^{i+k} M^{\mathrm{d}}_{kj} \det\left(\mathbf{A}^{n-q}_{k-q,i-q}\right) \tag{6.93}$$

$$= \sum_{k=q+1}^{n} M^{\mathrm{d}}_{kj} \frac{(-1)^{i+k} \det\left(\mathbf{A}^{n-q}_{k-q,i-q}\right)}{(-1)^{n-q+1}(>0)} \geq 0, \tag{6.94}$$

where $\mathbf{A}^{n-q}_{k-q,i-q}$ is obtained from $\mathbf{A}^{n-q}$ by deleting the $(k-q)$th row and $(i-q)$th column. The equality in (6.94) follows from Lemma A.1 of Gouriéroux et al. [2012], and the inequality in (6.94) from Lemma A.15 applied to $\mathbf{A}^{n-q}$ for $k \neq i$ and Lemma A.1 of Gouriéroux et al. [2012] for $k = i$. Thus, we obtain from (6.91) for a solvent firm i,

$$\frac{\partial s_i}{\partial d_j} = M^{\mathrm{d}}_{ij} + \sum_{l=q+1}^{n} M^{\mathrm{d}}_{il} \underbrace{\frac{\partial r_l}{\partial d_j}}_{\geq 0 \text{ by (6.94)}} \geq 0. \tag{6.95}$$

These results can be explained as follows. If the nominal level of liabilities of a solvent firm j increases, this firm's recovery value of debt strictly increases by Proposition 6.10, since $r_j = d_j$ in this case. Hence, the equity value of solvent firms and the recovery value of debt of firms in default might increase by direct holding of r_j or indirectly by holding a part of another defaulted firm's recovery value of debt possibly affected by the change in d_j.

Under cross-ownership of equity only, (6.77) yields for $i \neq j$ $(j \leq q)$

$$\frac{\partial x_i}{\partial d_j} = \frac{(-1)^{i+j} \det(\mathbf{A}_{ji})}{\det(\mathbf{A})} = \frac{(-1)^{i+j} \det(\mathbf{A}_{ji})}{(-1)^n(>0)}, \tag{6.96}$$

where the denominator of (6.96) follows from Lemma A.1 of Gouriéroux et al. [2012]. Hence, by Lemma A.15,

$$\frac{\partial x_i}{\partial d_j} \leq 0 \quad \text{for } i \neq j \text{ and firm } j \text{ solvent.} \tag{6.97}$$

This can be interpreted as follows. If the nominal level of liabilities of a solvent firm j increases, its equity value s_j strictly decreases by Proposition 6.10. For a solvent firm i

(i.e. $x_i = s_i$), firm i's equity value could decrease as well, either via direct cross-holding of s_j, or via cross-holding of other firms' equity affected by a decrease in s_j. Similarly, for a firm i in default (i.e. $x_i = r_i$), firm i's recovery value of debt could decrease, since firm i's endogenous asset value might have declined directly or indirectly by the drop in s_j.

Altogether, we have the following proposition clarifying the derivatives declared as "depends" in Proposition 6.10.

Proposition 6.12. *In the situation of Proposition 6.10, let firm j be solvent. If the n firms are linked by cross-ownership of debt only,*

$$\frac{\partial r_i}{\partial d_j} \geq 0 \quad \textit{for firm } i \textit{ in default,} \tag{6.98}$$

$$\frac{\partial s_i}{\partial d_j} \geq 0 \quad \textit{for firm } i \textit{ solvent } (i \neq j). \tag{6.99}$$

Under cross-ownership of equity only,

$$\frac{\partial r_i}{\partial d_j} \leq 0 \quad \textit{for firm } i \textit{ in default,} \tag{6.100}$$

$$\frac{\partial s_i}{\partial d_j} \leq 0 \quad \textit{for firm } i \textit{ solvent } (i \neq j). \tag{6.101}$$

Since under cross-ownership of debt only and cross-ownership of equity only, a change in d_j has opposite effects on the considered r_i and s_i $(i \neq j)$, it plausible that these derivatives can be positive and negative under cross-ownership of both, debt and equity, and hence they are indeterminate in Proposition 6.10.

6.2.2 Monotonicity of $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{v}$ in $\mathbf{d}$

So far, our analysis was confined to derivatives within a certain Suzuki area. In this section we put these results together to obtain global monotonicity properties of r_i, s_i and v_i with respect to d_j $(i, j \in \{1, \dots, n\})$, where possible. For that, we need to distinguish between $i = j$ and $i \neq j$.

6.2.2.1 Case $i = j$

Let us first examine how the financial status of firm i evolves with d_i increasing. For that, we need the following definition, which is partly taken from Liu and Staum [2010].

Definition 6.13. *For a certain setting of cross-ownership, let firm i $(i \in \{1, \dots, n\})$ be solvent, i.e. $r_i = d_i$. Firm i is strictly solvent if $s_i > 0$, and a borderline firm if $s_i = 0$, i.e. firm i is a borderline firm if and only if $v_i = d_i$.*
For a single varying model parameter, a non-degenerate interval of values of this parameter is called a borderline interval of firm i, if firm i is a borderline firm for all parameter values in this interval.

As the following lemma shows, firm i has no borderline interval with respect to d_i.

Lemma 6.14. *For a system of* n *firms linked by cross-ownership and all other parameters held fixed, there is exactly one* $d_i' > 0$ *such that firm* i *is a borderline firm for* $d_i = d_i'$. *Firm* i *is strictly solvent for all* $d_i < d_i'$ *and in default for all* $d_i > d_i'$.

Proof. By Lemma 3.4 and Proposition 6.10 and with all other parameters held fixed, s_i as a part of the solution of (3.13)–(3.14) is continuous and piecewise linear in d_i. With d_i increasing, $\mathbf{a}$ might lie in several Suzuki areas (recall that the Suzuki areas depend on d_i), and the "pieces" of s_i as a function of d_i exactly correspond to these Suzuki areas, since within a Suzuki area, the derivative of s_i with respect to d_i is constant. In all Suzuki areas, we have $\frac{\partial s_i}{\partial d_i} \leq 0$ by Proposition 6.10. Hence, s_i is non-increasing in d_i. Let $s_i(d_i)$ denote the equity value of firm i in dependence of d_i. Since $s_i(0) > 0$ (otherwise, all entries of firm i's balance sheet would be zero), firm i is strictly solvent for $d_i = 0$, i.e. for $d_i = 0$ we are in one of the 2^{n-1} Suzuki areas where firm i is solvent. Let m denote the maximum of the corresponding 2^{n-1} values of $\frac{\partial s_i}{\partial d_i}$. Of course, $m < 0$ by (6.61), i.e. m is the least negative of these derivatives. Since s_i is continuous and piecewise linear in d_i, it follows that, with d_i increasing and as long as firm i is solvent,

$$0 \leq s_i(d_i) \leq s_i(0) + m \times d_i, \tag{6.102}$$

and therefore, firm i becomes a borderline firm for some $d_i' > 0$. Because of $\frac{\partial s_i}{\partial d_i} \leq 0$ for all $d_i \geq 0$ and due to $s_i \geq 0$ for all $d_i \geq 0$, we have $s_i = 0$ for all $d_i \geq d_i'$, i.e. $\frac{\partial s_i}{\partial d_i} = 0$ for all $d_i > d_i'$. By (6.61), this implies that firm i is in default for all $d_i > d_i'$. □

Corollary 6.15. *With* d_i' *as defined in Lemma 6.14,*

$$\frac{\partial r_i}{\partial d_i} = \begin{cases} 1, & d_i < d_i', \\ 0, & d_i > d_i', \end{cases} \tag{6.103}$$

$$\frac{\partial s_i}{\partial d_i} = \begin{cases} < 0, & d_i < d_i', \\ 0, & d_i > d_i', \end{cases} \tag{6.104}$$

where, for $d_i < d_i'$, $\frac{\partial s_i}{\partial d_i}$ *is piecewise constant, depending on the financial status (solvency and default) of the other firms in the system. In particular,* r_i *is non-decreasing and* s_i *is non-increasing in* d_i.

Proof. This immediately follows from Proposition 6.10 and Lemma 6.14. □

Let d_i' be defined as in Lemma 6.14. As we have seen in Example 6.9, the exact value of $\frac{\partial s_i}{\partial d_i}$ for $d_i < d_i'$ can be bigger or smaller than -1, i.e. it is impossible to make a general statement with respect to the monotonicity of $v_i = r_i + s_i$ with respect to d_i under a cross-ownership of both, debt and equity. For the special cases cross-ownership of debt only and cross-ownership of equity only, (6.66) yields the following corollary to Corollary 6.15.

Corollary 6.16. *For a system of n firms, v_i is non-decreasing in d_i under cross-ownership of debt only and non-increasing in d_i under cross-ownership of equity only. Under cross-ownership of both, debt and equity, v_i is in general not monotone in d_i.*

In the one firm Merton model, the firm value is invariant with respect to the debt level, and therefore also with respect to (partial) debt cancellation. Corollary 6.16 shows that under cross-ownership of debt only, a firm's value might increase (and will never decrease) with its nominal debt level. Hence, if for example all creditors of a firm (inside and outside the system of n firms) simultaneously cancel a part of the debt of this firm (leaving the financial status of all firms unchanged), this cancellation of debt might lower the firm value since the increase in the equity can be less than the loss in the debt in such a case (compare this with the comment after (6.65)). In contrast to that, such a debt cancellation under cross-ownership of equity only might increase the firm value, and will never decrease it.

6.2.2.2 Case $i \neq j$

In contrast to the case $i = j$, there might be a borderline interval of firm i with respect to d_j ($i \neq j$), since we cannot exclude the possibility that $\frac{\partial s_i}{\partial d_j} = 0$ for values of d_j where firm i is solvent. This becomes clear from (6.57), where M^{d}_{kj} and/or $\det(\mathbf{A}_{ki})$ might be 0, or the following simple example. For two firms linked by cross-ownership of debt only, let firm 1 be a borderline firm and firm 2 in default, i.e. $r_1 = d_1$, $r_2 = a_2 + M^{\mathrm{d}}_{21} d_1 < d_2$ and $s_1 = s_2 = 0$. Hence, with d_2 increasing, firm 2 will remain in default, i.e. the value of endogenous assets of firm 1 remains unchanged as well, and firm 1 is still a borderline firm.
Furthermore, in contrast to the case $i = j$, where we have seen in Lemma 6.14 that with d_i increasing, firm i cannot leave the status of default and become solvent again, there is no predefined "direction" for the development of firm i with d_j increasing ($i \neq j$). This can be seen from the following rather simple example.

Example 6.17. For three firms linked by cross-ownership, let

$$\mathbf{M}^{\mathrm{d}} := \begin{pmatrix} 0 & 0 & 0 \\ 0.6 & 0 & 0.1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{\mathrm{e}} := \begin{pmatrix} 0 & 0 & 0.4 \\ 0.2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.105}$$

and $\mathbf{a} := (3, 7.2, 19)^T$, $\mathbf{d} := (6, 11.8, d_3)^T$ with $d_3 \geq 0$. We will show that with d_3 increasing, firm 2 is first in default, then solvent and finally in default again. First,

$$r_1 = \min\{6,\ 3 + 0.4 s_3\}, \qquad s_1 = (-3 + 0.4 s_3)^+, \tag{6.106}$$

$$r_2 = \min\{11.8,\ 7.2 + 0.6 r_1 + 0.1 r_3 + 0.2 s_1\}, \quad s_2 = (-4.6 + 0.6 r_1 + 0.1 r_3 + 0.2 s_1)^+, \tag{6.107}$$

$$r_3 = \min\{d_3,\ 19\}, \qquad s_3 = (19 - d_3)^+. \tag{6.108}$$

It is straightforward to see that firm 3 is solvent if and only if $d_3 \leq 19$ and that firm 1

		j solvent	j in default
$\frac{\partial r_i}{\partial d_j}$	i in default	depends*/≥ 0**/≤ 0***	0
	i solvent	0	0
$\frac{\partial s_i}{\partial d_j}$	i in default	0	0
	i solvent	depends*/≥ 0**/≤ 0***	0

Table 6.1: Derivatives of r_i and s_i with respect to d_j ($i \neq j$); *general case, **XOS of debt only, ***XOS of equity only.

is solvent if and only if $d_3 \leq 11.5$. For $d_3 \leq 11.5$,

$$r_2 = \min\{11.8,\, 7.2 + 0.6 \times 6 + 0.1 d_3 + 0.2(-3 + 0.4(19 - d_3))\} \tag{6.109}$$

$$= \min\{11.8,\, 11.72 + 0.02 d_3\}, \tag{6.110}$$

i.e. for $d_3 < 4$, firm 2 is in default, and for $d_3 \in [4, 11.5]$, firm 2 is solvent. Furthermore, for $d_3 \in (11.5, 19]$, $r_2 = \min\{11.8,\, 7.2+0.6(3+0.4(19-d_3))+0.1d_3\} = \min\{11.8,\, 13.56-0.14d_3\}$, i.e. for $d_3 \in (11.5, 12\frac{4}{7}]$, firm 2 is solvent, and for $d_3 \in (12\frac{4}{7}, 19]$, firm 2 is in default. Finally, for $d_3 > 19$, $r_2 = \min\{11.8,\, 7.2 + 0.6 \times 3 + 0.1 \times 19\} = 10.9$, i.e. all the three firms are in default. Altogether,

$$\mathbf{a} \in \begin{cases} A_{\text{sds}}, & d_3 < 4, \\ A_{\text{sss}}, & 4 \leq d_3 \leq 11.5, \\ A_{\text{dss}}, & 11.5 < d_3 \leq 12\frac{4}{7}, \\ A_{\text{dds}}, & 12\frac{4}{7} < d_3 \leq 19, \\ A_{\text{ddd}}, & d_3 > 19, \end{cases} \tag{6.111}$$

with the Suzuki areas $A_{...}$ as defined in (3.18). In particular, with d_3 increasing, firm 2 is first in default, then solvent and finally in default again, whereas the financial status of firm 1 declines from solvency to default. Hence, there is no "monotonicity" of the financial status of firm i under changes in d_j ($i \neq j$), i.e. firm i may both, profit and suffer from changes in the nominal level of liabilities of another firm j.

As became clear in (6.78) and Example 6.17, s_i and r_i are in general not monotone in in another firm's nominal level of liabilities, and neither is v_i. However, such a monotonicity holds for the special cases of one type of cross-ownership only.

Proposition 6.18. *For a system of n firms and $i \neq j$, r_i, s_i and v_i are non-decreasing in d_j under cross-ownership of debt only and non-increasing in d_j under cross-ownership of equity only. Under cross-ownership of both, debt and equity, r_i, s_i and v_i are in general not monotone in d_j.*

Proof. This immediately follows from Table 6.1 and the fact that $v_i = r_i + s_i$. Table 6.1 is a direct consequence of Proposition 6.10 and Proposition 6.12. □

Hence, if the nominal level of debt of a firm decreases, the remaining firms in the system

cannot profit from this reduction under cross-ownership of debt only, and they cannot suffer from this reduction under cross-ownership of equity only.
Compared to a scenario without cross-ownership, where $v_i = a_i$ by Definition 3.1 and consequently $\frac{\partial v_i}{\partial d_j} = 0$ for all $1 \leq i, j \leq n$, changes in other firms' nominal level of liabilities may affect the considered firm's value both, positively and negatively, depending on the realized level of cross-ownership and the financial status of the other firms in the system.

6.3 Risk-Neutral Firm Values before Maturity

In the previous sections we analyzed how the recovery values of debt $\mathbf{r}$ and the equity values $\mathbf{s}$ react on changes in cross-ownership fractions and the face values of liabilities in a deterministic set-up (i.e. for a given value of exogenous assets at maturity T). In contrast to that, we now consider exogenous asset prices that follow a stochastic process. Then we can determine the firms' prices in $t < T$ by employing the risk-neutral valuation formula (cf. Theorem 6.2.3 of Bingham and Kiesel [2004]). As we will see, the results of Section 6.1 and Section 6.2 on the monotonicity of $\mathbf{v}$ (evaluated at maturity) transfer to risk-neutral firm prices before maturity.

We assume random exogenous asset values to follow a multivariate geometric Brownian motion $\mathbf{A} = (A_i)_{1 \leq i \leq n}$ (cf. equation (3.23) of Benth [2004]), i.e.

$$dA_i(t) = A_i(t) \left(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right), \quad 0 \leq t \leq T,\ 1 \leq i \leq n, \tag{6.112}$$

with a standard Brownian motion $\mathbf{W} = (W_j)_{1 \leq j \leq n}$ (cf. Bingham and Kiesel [2004], p. 160) on some probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Note that we set the dimension of the Brownian motion $\mathbf{W}$ to n as well. Furthermore, $\boldsymbol{\mu} = (\mu_i)_{1 \leq i \leq n} \in \mathbb{R}^n$, and we assume $\boldsymbol{\sigma} = (\sigma_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ to be invertible. Then Theorem 6.2.2 of Bingham and Kiesel [2004] implies the existence of a unique equivalent martingale measure $\mathbb{Q}$ (cf. Definition 6.1.4 of Bingham and Kiesel [2004]) for the discounted asset value process $\tilde{\mathbf{A}}(t) := \exp(-rt)\mathbf{A}(t)$, with r denoting the constant force of interest. Then by the product rule,

$$d\tilde{A}_i(t) = \exp(-rt) A_i(t) \left(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right) - r A_i(t) \exp(-rt) dt \tag{6.113}$$

$$= \tilde{A}_i(t) \left((\mu_i - r) dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right). \tag{6.114}$$

By Girsanov's Theorem (cf. Theorem 5.7.1 of Bingham and Kiesel [2004]), $dW_j(t) = d\tilde{W}_j(t) - \gamma_j(t) dt$ for all $1 \leq j \leq n$, where $\tilde{\mathbf{W}}$ is an n-dimensional Brownian motion under

$\mathbb{Q}$, and with $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n$. $\boldsymbol{\gamma}$ will be explicitly determined later. Hence,

$$d\tilde{A}_i(t) = \tilde{A}_i(t) \left((\mu_i - r)dt + \sum_{j=1}^{n} \sigma_{ij} \left(d\tilde{W}_j(t) - \gamma_j(t)dt \right) \right) \tag{6.115}$$

$$= \tilde{A}_i(t) \left(\left(\mu_i - r - \sum_{j=1}^{n} \sigma_{ij}\gamma_j(t) \right) dt + \sum_{j=1}^{n} \sigma_{ij} d\tilde{W}_j(t) \right). \tag{6.116}$$

Since $\mathbb{Q}$ is an equivalent martingale measure for $\tilde{\mathbf{A}}$, $\tilde{\mathbf{A}}$ is a $\mathbb{Q}-$local martingale by Definition 6.1.4 of Bingham and Kiesel [2004], it follows that

$$\mu_i - r - \sum_{j=1}^{n} \sigma_{ij}\gamma_j(t) = 0, \quad 0 \le t \le T \; a.s., \; 1 \le i \le n, \tag{6.117}$$

The equation in (6.117) can be written as $\boldsymbol{\sigma\gamma}(t) = \boldsymbol{\mu} - r\mathbf{1}_n$, $0 \le t \le T \; a.s.$, where $\mathbf{1}_n$ denotes an n-dimensional vector with all entries equal to 1. Since we assume $\boldsymbol{\sigma}$ to be invertible, $\boldsymbol{\gamma}(t)$ is uniquely defined and the function $\boldsymbol{\gamma}$ is constant for almost all $t \in [0, T]$, so we write $\boldsymbol{\gamma}$ instead of $\boldsymbol{\gamma}(t)$. By Girsanov's Theorem, $\mathbb{Q}$ can be obtained from $\mathbb{P}$ via $\frac{\partial \mathbb{Q}}{\partial \mathbb{P}} = \exp(-\boldsymbol{\gamma} W(T) - \frac{1}{2}\boldsymbol{\gamma\gamma}^T T)$ (with $\boldsymbol{\gamma}^T$ denoting $\boldsymbol{\gamma}$ transposed).

Since the firm value is a derivative of exogenous asset values, we can calculate the price of a firm i in $t < T$, in symbols $v_{i,t}$, as the discounted expected value of its firm value V_i at maturity T, where the expectation is taken with respect to the risk-neutral pricing measure $\mathbb{Q}$ derived above, i.e.

$$v_{i,t} := \exp(-r(T-t)) E_{\mathbb{Q}}(V_i \,|\, \mathcal{F}_t). \tag{6.118}$$

Let us now consider two identical systems of n firms each, where identical shall mean that both, the parameters of the firms themselves and their cross-ownership structure are identical between the two networks, with the only exception that in system 2, a parameter M_{kj} exceeds the corresponding parameter of system 1 by an amount of $\epsilon > 0$, provided that the corresponding cross-ownership matrix is still strictly left substochastic. For a parameter p, let ${}_l p$ stand for p obtained from system l, $l = 1, 2$.
Because of ${}_2V_i \ge {}_1V_i$ by Proposition 6.1, we also have $E_{\mathbb{Q}}({}_2V_i \,|\, \mathcal{F}_t) \ge E_{\mathbb{Q}}({}_1V_i \,|\, \mathcal{F}_t)$ and thus ${}_2v_{i,t} \ge {}_1v_{i,t}$ for all $1 \le i \le n$ and all $0 \le t \le T$, i.e. the monotonicity of firm values in the cross-ownership fractions transfers to risk-neutral prices of these firms for any time $t \in [0, T]$.
We have a similar result for all $v_{j,t}$, $1 \le j \le n$, $0 \le t \le T$, if the two systems differ by the value of a d_i only ($i \in \{1, \ldots, n\}$). If ${}_1d_i < {}_2d_i$, Corollary 6.16 and Proposition 6.18 imply ${}_1v_{j,t} \le {}_2v_{j,t}$ under cross-ownership of debt only and ${}_1v_{j,t} \ge {}_2v_{j,t}$ under cross-ownership of equity only. In particular, the "direction" of the effect is the same for all firms $1, \ldots, n$. However, for cross-ownership of possibly both, debt and equity, such a monotonicity of firm values is not given.

7 Contagion

By use of Remark 6.11 it can be evaluated how a firm's value reacts to marginal changes in an arbitrary firm's exogenous asset value immediately before maturity, but it does not become apparent whether the firm is driven into default, or not. In this section we extend this analysis in that we consider how firm values and probabilities of default react to multiple and possibly correlated changes in exogenous asset values and how the realized cross-ownership structure propagates or mitigates such changes throughout the system of firms.

In the wide-spread literature on financial networks, for example the interbank market, where banks cross-hold their liabilities, correlated asset values and cross-holdings are considered to be the two main sources of systemic risk (cf. Furfine [2003], Boss et al. [2006], Elsinger et al. [2006a], Degryse and Nguyen [2007], Frisell et al. [2007], Martínez-Jaramillo et al. [2010] and Eboli [2013], among others), where "[e]xplicitly or implicitly, systemic risk is usually understood as the failure or risk of failure of a significant part of the financial system" [Cifuentes, 2004, p. 367], or as Kaufman [2000] writes, "[s]ystemic risk refers to the risk or probability of breakdowns (losses) in an entire system as opposed to breakdowns in individual parts or components" [Kaufman, 2000, p. 92]. In addition to correlated asset values and bilateral exposures, Nier et al. [2007] list "feedback effects from endogenous fire-sale[s] of assets by distressed institutions" and "informational contagion" [Nier et al., 2007, p. 2034] as further sources of systemic risk. However, they do not incorporate these additional issues in their study, and also the exposure of banks to common risk factors is considered at the margin only. Of course, these considerations analogously hold for networks of more general financial firms, rather than just banks, with cross-ownership of equity as a possible additional channel of shock transmission. According to Staum [2013], these two main sources of systemic risk (correlated asset values and contagion due to cross-holdings) could be easily separated within structural models, to which Suzuki's model belongs, but in a realistic set-up, "it is necessary to study correlated shocks to appreciate the true impact of contagion ... and to understand the true effect of network structure on systemic risk" [Staum, 2013, p. 525], i.e. both aspects should be considered simultaneously. Nevertheless, a great part of the theoretical and simulation literature deals with idiosyncratic shocks hitting a single firm only.

In the remainder of this section, we will first propose a measure for contagion resulting from simultaneous shocks on a system of n firms linked by cross-ownership of possibly both, debt and equity, and study its properties for the two firms case. Although they employ a different operationalization of the effect of contagion, the work of Gouriéroux et al. [2012] is closest to our approach. Extending the results of Glasserman and Young [2015] we will see how firms can protect themselves against spillover effects leading to a default. Complementing the broad spectrum of theoretical considerations (see Eboli

[2013] and Acemoglu et al. [2015], among others) and simulation studies for both, existing banking systems and hypothetical networks (for an overview, see Upper [2011], Chinazzi and Fagiolo [2013], Cont et al. [2013] and Staum [2013]), Section 7.2 contains the results of our simulation study on how negative shocks are propagated through a system of n firms linked by cross-ownership of possibly both, debt and equity, leading to defaults and losses in firm values.
In the remainder of this section, variables equipped with a tilde stand for values *after* the shock.

7.1 Theoretical Considerations on Contagion

7.1.1 Contagion and the Change in Endogenous Asset Values

In this section we first consider a general network of n firms linked by cross-ownership of debt and/or equity, and then confine ourselves to the case of two firms only in order to be able to employ Suzuki's formulae provided in Lemma 3.5.
Following Gouriéroux et al. [2012] we assume that very close to maturity, the exogenous asset values $\mathbf{a}$ are subject to a shock such that the new values of the exogenous assets are given by

$$\tilde{\mathbf{a}} := \mathbf{a} + \boldsymbol{\delta}, \quad \boldsymbol{\delta} \geq -\mathbf{a}, \tag{7.1}$$

i.e. we assume that the shock $\boldsymbol{\delta} = (\delta_i)_{1 \leq i \leq n}$ is constrained in the sense that exogenous asset values after the shock, $\tilde{\mathbf{a}}$, are still non-negative, which means that the shock can wipe out all the exogenous assets of a firm, but it cannot transform assets into liabilities. Note that in our context, the word "shock" does not necessarily imply a decline of asset values, it just stands for an immediate change in any direction. Furthermore, we assume fixed cross-ownership matrices, and as Gouriéroux et al. [2012] remark, "[f]rom an economic point of view, this might be interpreted as the effect of an immediate not anticipated shock" [Gouriéroux et al., 2012, p. 1287], since the established cross-ownership structure cannot be changed instantaneously. The same assumption is made by Gai and Kapadia [2010] and May and Arinaminpathy [2010], among others.

By Definition 3.3, firm i's value ($i \in \{1, \ldots, n\}$) before the shock equals

$$v_i = a_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}}\, r_j + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}\, s_j, \tag{7.2}$$

with $\mathbf{r}$ and $\mathbf{s}$ being solutions of (3.13)–(3.14). In the remainder, we will sometimes write $\mathbf{r}(\mathbf{a})$, $\mathbf{s}(\mathbf{a})$ and $\mathbf{v}(\mathbf{a})$, where necessary. Hence, it is clear that the effect of the shock on the firm value v_i consists of two parts: first, a direct effect on v_i, caused by a change in value of the firm's exogenous asset, and an indirect effect stemming from a change in endogenous asset values. As we will see in the following, this indirect effect on firm values exists under Suzuki's model, but not under Merton's model.
Recall that under Merton's model (cf. Section 3.1.1) all assets of a firm are considered to

be exogenous, i.e. instead of distinguishing between exogenous and endogenous assets as in (7.2), we could imagine the total assets of firm i to consist of two classes of exogenous assets with total value $a_i + \bar{a}_i = {}_{\mathrm{M}}v_i$, where the index M stands for Merton and where $\bar{a}_i$ equals the value of endogenous assets before the shock, i.e.

$$\bar{a}_i = \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}). \tag{7.3}$$

After the shock $\boldsymbol{\delta}$ affecting $\mathbf{a}$ only, the firm value of firm i equals ${}_{\mathrm{M}}\tilde{v}_i = a_i + \delta_i + \bar{a}_i$. Hence, if we apply Merton's approach, as it was to be expected, the effect of the shock, measured as the difference of firm i's value before and after the shock, amounts to

$${}_{\mathrm{M}}\tilde{v}_i - {}_{\mathrm{M}}v_i = \delta_i. \tag{7.4}$$

Note that this approach is equivalent to the approach of Gouriéroux et al. [2012] and of Glasserman and Young [2015], who assume that immediately before the shock the two firms cash their stocks and bonds that they are holding of each other. Then the sum of asset values, i.e. the exogenous asset value and cash from unwinding cross-holdings, equals ${}_{\mathrm{M}}v_i$ and we can identify $\bar{a}_i$ with the value of the cash held. After that, the two firms are not linked by cross-ownership any more, and Merton's model can be applied. Hence, δ_i can be seen as the direct effect of the shock on firm i, where contagion effects due to cross-ownership have been eliminated.

Under Suzuki's model the value of firm i after the shock is given by

$$\tilde{v}_i = a_i + \delta_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}+\boldsymbol{\delta}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}+\boldsymbol{\delta}). \tag{7.5}$$

The difference of firm i's value before and after the shock thus equals

$$\tilde{v}_i - v_i = \delta_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}+\boldsymbol{\delta}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}+\boldsymbol{\delta}) - \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}) - \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}). \tag{7.6}$$

In the following we will mainly be concerned with the effect of the shock on the firms' endogenous asset values, since (7.4) and (7.6) show that the existence of this impact under Suzuki's model constitutes the main difference to Merton's approach. Hence, let c_i denote the change in firm i's endogenous asset value due to the shock $\boldsymbol{\delta}$, i.e.

$$\begin{aligned} c_i &:= \sum_{j=1}^{n} \Big(M_{ij}^{\mathrm{d}} r_j(\mathbf{a}+\boldsymbol{\delta}) + M_{ij}^{\mathrm{e}} s_j(\mathbf{a}+\boldsymbol{\delta}) \Big) - \sum_{j=1}^{n} \Big(M_{ij}^{\mathrm{d}} r_j(\mathbf{a}) + M_{ij}^{\mathrm{e}} s_j(\mathbf{a}) \Big) && (7.7) \\ &= \tilde{v}_i - (a_i + \delta_i) - (v_i - a_i) && (7.8) \\ &= \underbrace{\tilde{v}_i - v_i}_{\substack{\text{total effect of the shock} \\ \text{=effect of the shock under Suzuki's model}}} - \underbrace{\delta_i,}_{\substack{\text{direct effect of the shock} \\ \text{=effect of the shock under Merton's approach}}} && (7.9) \end{aligned}$$

where (7.8) follows from (7.2) and (7.5). In the following, the effect of contagion on firm i will be identified with c_i.

In the remainder of this section we will analyze for the case of two firms how the effect of contagion depends on the parameters $\mathbf{a}$, $\mathbf{d}$, $M^{\mathrm{d}}_{1,2}$, $M^{\mathrm{d}}_{2,1}$, $M^{\mathrm{e}}_{1,2}$, $M^{\mathrm{e}}_{2,1}$ and $\boldsymbol{\delta}$. However, we will only consider c_1 for reasons of symmetry. We assume that both firms are solvent before the shock, i.e. $\mathbf{a} \in A_{\mathrm{ss}}$. By Lemma A.16,

$$c_1 = \begin{cases} \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}, & \tilde{\mathbf{a}} \in A_{\mathrm{ss}}, \\ \frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}} + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})\frac{M^{\mathrm{e}}_{2,1}a_1+a_2+(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1-(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{sd}}, \\ \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1}} + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})\frac{a_1+M^{\mathrm{e}}_{1,2}a_2-(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})d_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2}{(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{ds}}, \\ \frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}} + \frac{(M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})a_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2}+M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1}))a_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})} & \\ \qquad - \frac{M^{\mathrm{e}}_{1,2}(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1+(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{dd}}. \end{cases} \tag{7.10}$$

Note that $\boldsymbol{\delta} = \mathbf{0}$ implies $\tilde{\mathbf{a}} \in A_{\mathrm{ss}}$ and $c_1 = 0$. Except for $\tilde{\mathbf{a}} \in A_{\mathrm{ss}}$, c_1 in general consists of two components. The first summand is directly influenced by the shock $\boldsymbol{\delta}$, with the impact of $\boldsymbol{\delta}$ depending on the realized cross-ownership fractions, which might be zero. The second component is not affected by the shock itself, but it depends on the realized cross-ownership situation before the shock, namely the value of exogenous assets before the shock, the level of liabilities and the cross-ownership structure. This second term can be both, positive and negative in all the three relevant areas. Similarly, the former asset values $\mathbf{a}$ can both, increase and decrease the change in the endogenous asset value of a firm due to contagion, depending on the relative sizes of $M^{\mathrm{d}}_{1,2}$ and $M^{\mathrm{e}}_{1,2}$ ($\tilde{\mathbf{a}} \in A_{\mathrm{sd}}$), $M^{\mathrm{d}}_{2,1}$ and $M^{\mathrm{e}}_{2,1}$ ($\tilde{\mathbf{a}} \in A_{\mathrm{ds}}$), and $M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}$ and $M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}$ ($\tilde{\mathbf{a}} \in A_{\mathrm{dd}}$). The nominal level of debt $\mathbf{d}$ can as well lead to both, higher or lower values of c_1. Hence, depending on the realized cross-ownership structure, a high value of the own exogenous asset and a low level of liabilities, indicating financial health before the shock, can amplify both, positive and negative effects of contagion. Maybe somewhat unexpected, we cannot generally say that higher levels of cross-ownership result in larger absolute values of c_1. Only if the shock is small, in the sense that $\tilde{\mathbf{a}} \in A_{\mathrm{ss}}$, and of the same direction for both firms, we can conclude that the higher the realized level of cross-ownership of equity, the larger the absolute change in endogenous asset values resulting from this shock. As we will see later, more detailed results can be derived for cross-ownership of debt only and cross-ownership of equity only.

A special situation occurs for a symmetrical cross-ownership structure, where symmetrical means symmetry with respect to cross-holdings of debt and equity for each firm, i.e. a firm holds the same fraction of the other firm's debt and equity such that

$$M_{1,2} := M^{\mathrm{d}}_{1,2} = M^{\mathrm{e}}_{1,2}, \quad M_{2,1} := M^{\mathrm{d}}_{2,1} = M^{\mathrm{e}}_{2,1}. \tag{7.11}$$

Then (7.10) reduces to

$$c_1 = \frac{M_{1,2}(M_{2,1}\delta_1 + \delta_2)}{1 - M_{1,2}M_{2,1}} \quad \text{for all } \tilde{\mathbf{a}} \geq \mathbf{0}, \tag{7.12}$$

i.e. under a symmetrical cross-ownership structure as described above and a given shock $\boldsymbol{\delta}$, the change in the endogenous asset value of a firm is independent of the former asset value $\mathbf{a}$, the level of nominal liabilities $\mathbf{d}$ and the financial status (solvency or default) of either firm after the shock, it only depends on the realized cross-ownership structure. A symmetrical cross-ownership structure enhances both, entirely positive and entirely negative shocks (i.e. shocks with $\delta_1, \delta_2 > 0$ or $\delta_1, \delta_2 < 0$), in the sense that the indirect effect goes into the same direction as the direct effect of $\boldsymbol{\delta}$, i.e. a symmetrical cross-ownership structure amplifies shocks of the described type. The tighter the cross-ownership structure, the stronger this effect.

Under cross-ownership of debt only firm 1 does not suffer or profit from contagion at all, if firm 2 is still solvent after the shock, i.e.

$$c_1 = 0, \quad \tilde{\mathbf{a}} \in A_{\mathrm{ss}} \cup A_{\mathrm{ds}}, \tag{7.13}$$

independently of the exact cross-ownership structure. In this case, the value of the debt of firm 2 remains d_2 also after the shock, so the value of endogenous assets of firm 1 is left unchanged by the shock. If firm 2 is in default after the shock, we have

$$0 > c_1 = \begin{cases} M^{\mathrm{d}}_{1,2}(a_2 + \delta_2 + M^{\mathrm{d}}_{2,1}d_1 - d_2), & \tilde{\mathbf{a}} \in A_{\mathrm{sd}}, \\ \frac{M^{\mathrm{d}}_{1,2}}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}}(M^{\mathrm{d}}_{2,1}(a_1+\delta_1) + a_2 + \delta_2 - (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2), & \tilde{\mathbf{a}} \in A_{\mathrm{dd}}, \end{cases} \tag{7.14}$$

where the inequality follows from (3.26) and (3.28). Altogether, $c_1 \leq 0$, i.e. positive shocks on exogenous assets do not increase the value of endogenous assets, but negative shocks can reduce both, exogenous and endogenous asset values. For $\tilde{\mathbf{a}} \in A_{\mathrm{sd}} \cup A_{\mathrm{dd}}$, c_1 is strictly increasing in $M^{\mathrm{d}}_{2,1}$ since straightforward calculations yield $\frac{\partial c_1}{\partial M^{\mathrm{d}}_{2,1}} > 0$ in the inner of both Suzuki areas and since c_1 is continuous in $M^{\mathrm{d}}_{2,1}$ by Lemma 3.4 and (7.9). Hence, ceteris paribus, the more firm 2 possesses of firm 1's debt, the less negative is the effect of contagion, i.e. for firm 1, the risk of losing a great part of its endogenous assets due to the default of firm 2 is less severe. In contrast to that, increasing $M^{\mathrm{d}}_{1,2}$ (with all other parameters fixed) can have diverse effects on firm 1's loss in endogenous assets, depending on whether firm 1 is itself in default after the shock and the exact parameter constellation.

If the two firms have established cross-ownership of equity only, it is straightforward to see from (7.10) that

$$c_1 = -M^{\mathrm{e}}_{1,2}\frac{M^{\mathrm{e}}_{2,1}a_1 + a_2 - M^{\mathrm{e}}_{2,1}d_1 - d_2}{1 - M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}} \leq 0, \quad \tilde{\mathbf{a}} \in A_{\mathrm{sd}} \cup A_{\mathrm{dd}}, \tag{7.15}$$

where the inequality follows from the fact that $\mathbf{a} \in A_{\text{ss}}$ (cf. (3.25)). Hence, the change in firm 1's endogenous asset value caused by contagion is independent of the actual size and direction of the shock $\boldsymbol{\delta}$, if this shock forces firm 2 to declare default. This can be explained as follows. In the described scenario, the equity value of firm 2 becomes 0, and hence the endogenous asset value of firm 1 declines to 0, and clearly, the size of this decline, which exactly equals $-c_1$ with c_1 given in (7.15), is independent of the size of the shock $\boldsymbol{\delta}$, provided that the shock ruins firm 2. Furthermore,

$$c_1 = M^{\text{e}}_{1,2}\delta_2 - M^{\text{e}}_{1,2}M^{\text{e}}_{2,1}\frac{a_1 + M^{\text{e}}_{1,2}a_2 - d_1 - M^{\text{e}}_{1,2}d_2}{1 - M^{\text{e}}_{1,2}M^{\text{e}}_{2,1}}, \quad \tilde{\mathbf{a}} \in A_{\text{ds}}, \tag{7.16}$$

where the numerator is non-negative because of the assumption $\mathbf{a} \in A_{\text{ss}}$. If we now assume $\delta_2 > 0$, it follows that $\delta_1 < 0$ because otherwise, $\tilde{\mathbf{a}} \notin A_{\text{ds}}$ (cf. Figure 7.1(d)). It is straightforward to see that for suitably chosen parameters, we can obtain $c_1 < 0$, i.e. under cross-ownership of equity only, the negative shock on firm 1's exogenous asset can affect the firm twofold, namely by the direct and indirect effect, even though firm 2, whose equity is cross-held by firm 1, experiences a positive shock. On the other hand, there are parameter constellations such that $\tilde{\mathbf{a}} \in A_{\text{ds}}$, $\delta_1 < 0$ and $c_1 > 0$, i.e. in this case, the effect of contagion cushions the direct effect of the shock on firm 1. For $\tilde{\mathbf{a}} \in A_{\text{ss}}$, c_1 is given as in (7.12) with $M_{1,2} = M^{\text{e}}_{1,2}$ and $M_{2,1} = M^{\text{e}}_{2,1}$, and we can argue similarly that cross-ownership of equity then amplifies both, entirely positive and entirely negative shocks, i.e. the firms profit resp. suffer twofold from such shocks. Altogether, for a firm linked to another firm by cross-ownership of equity only, the direction and the size of the effect of contagion c_1 strongly depend on the underlying parameters, and the indirect effect can both, reduce and amplify the total effect of the shock. Furthermore, an analysis of the corresponding derivatives shows that c_1 is non-decreasing in the realized cross-ownership fractions, i.e. in a closely linked system of two firms holding each other's equity, the change in the endogenous asset value of a firm caused by a given shock $\boldsymbol{\delta}$ is at least as big as in a less closely linked system.

Summary

In general, and in particular under cross-ownership of equity only, the change in firm i's ($i = 1, 2$) endogenous asset value due to a shock on the exogenous assets of a system of firms, can be positive and negative, i.e. it can both, mitigate and exacerbate the total effect of the shock. It is even possible that financial soundness (i.e. $a_i \gg d_i$) increases negative effects of contagion. Furthermore, we cannot generally say that a tighter cross-ownership structure leads to bigger absolute changes in endogenous asset values. If the network structure is symmetrical, i.e. if each of the two firms holds an identical fraction of the other firm's debt and equity, this presence of cross-ownership serves as a multiplicator in the sense that entirely positive or entirely negative shocks are amplified, i.e. there are gains or losses in both, exogenous and endogenous asset values. For two firms linked by cross-ownership of debt only the effect of contagion is always non-positive, i.e. such firms cannot profit from positive shocks beyond the direct effect, but they can suffer from negative shocks further amplified by cross-ownership.

7.1.2 Transfer of Negative Shocks

Cross-ownership of any type and intensity can lead to scenarios where an initially solvent firm goes bankrupt just because another firm in the system was subject to a negative shock, although the firm itself experienced a non-negative shock. Under which circumstances such a default might occur will be analyzed in the following. We start with the case of two firms.

As we are interested in situations where the effect of the negative shock hitting firm 2 on firm 1 is strong enough to drive firm 1 into default, although firm 1 experienced a non-negative shock itself, we assume $\delta_1 \geq 0$, $\delta_2 < 0$ and $\tilde{\mathbf{a}} \in A_{\text{ds}} \cup A_{\text{dd}}$. In order to see which parameter constellations allow such a serious transmission of the shock and how firm 2 bears the shock, we consider a series of figures, which are based on Figure 3.1, modified for the different types of cross-ownership and levels of liabilities. In all the four subfigures of Figure 7.1, the four (or three) Suzuki areas are defined by the black lines and the bold red lines, with the bold red lines separating the areas where firm 1 is in default, or not. On the bold red line itself, firm 1 is not in default. Since we assume both firms to be solvent before the shock, we have $\mathbf{a} \in A_{\text{ss}}$. Let us now consider the situation of firm 1 after a shock of the described type. In Figure 7.1(a), (b) and (d), the subarea of A_{ss} marked with the red solid lines is exactly the part of A_{ss} where a shift downwards or in down-right direction (indicated by the arrows) can lead to exogenous asset values left of the bold red line, i.e. exogenous asset values after the shock, where firm 1 is in default. Such possible exogenous asset values after the shock are indicated by the subarea with the red dashed lines. Note that the dotted lines at the boundaries of the dashed areas do not belong to the dashed areas.

We first consider the special case of cross-ownership of debt only, and we assume that $d_1 > M^{\text{d}}_{1,2} d_2$ and $d_2 > M^{\text{d}}_{2,1} d_1$. This leads to Suzuki areas as in Figure 7.1(a), and it becomes clear that the subarea with possible exogenous asset values after a shock of the described type completely lies in A_{dd}, i.e. if this shock is strong enough to ruin firm 1, firm 2 is ruined as well. This means that under cross-ownership of debt only, if firm 1 has experienced a positive shock, firm 2 cannot pass on a negative shock to firm 1 and remain solvent itself. Furthermore, Figure 7.1(a) shows that if firm 1's initial exogenous asset value is at least $d_1(1 - M^{\text{d}}_{1,2} M^{\text{d}}_{2,1})$, it cannot be ruined just because of a negative shock on firm 2's asset, while its own asset might have gained in value by the shock. Somewhat surprisingly, this threshold decreases with the cross-ownership structure getting tighter, i.e. more the two firms hold of each other's debt, the smaller own exogenous asset values are sufficient to be protected against a default merely caused by a negative shock on the other firm's asset. If $d_1 \leq M^{\text{d}}_{1,2} d_2$, the situation is qualitatively the same (cf. Figure 7.1(b)). If $d_2 \leq M^{\text{d}}_{2,1} d_1$, firm 1 cannot go bankrupt through a shock of the considered type at all (cf. Figure 7.1(c)).

The fact that under cross-ownership of debt only, a firm cannot pass its negative shock to another firm (having experienced a non-negative shock) and remain solvent itself also holds for systems of $n \geq 2$ firms. Let firm i be the only firm in the system subject to a negative shock. If firm i remained solvent, its recovery value of debt r_i would still be

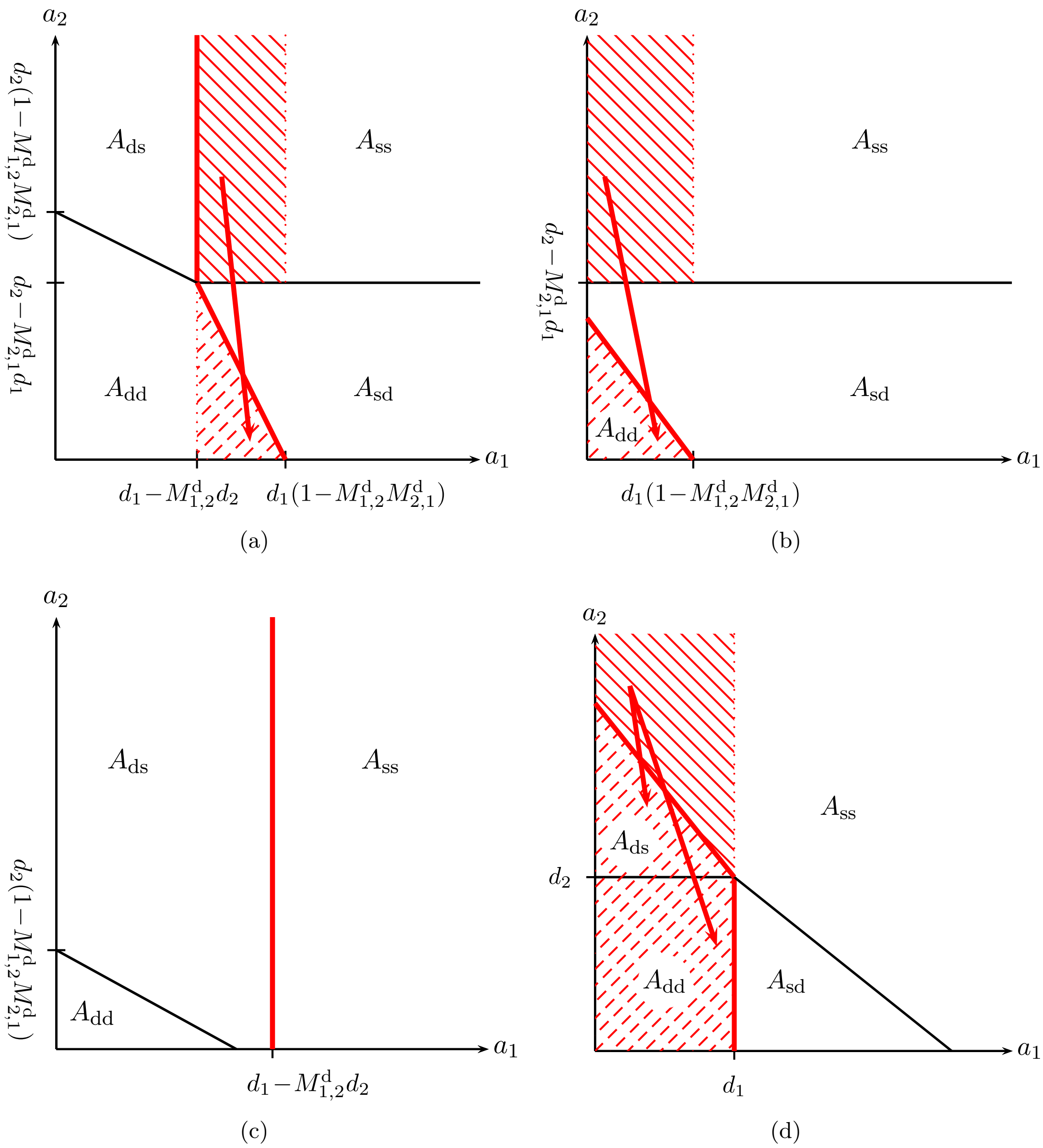


Figure 7.1: Effects of a shock with $\delta_1 \geq 0$ and $\delta_2 < 0$ on exogenous asset values of two initially healthy firms; areas with red solid lines: exogenous asset values before the shock for which firm 1 might be in default after such a shock; areas with red dashed lines: possible exogenous asset values after such a shock with firm 1 in default; (a) XOS of debt only with $d_1 > M^{\mathrm{d}}_{1,2}d_2$ and $d_2 > M^{\mathrm{d}}_{2,1}d_1$; (b) XOS of debt only with $d_1 < M^{\mathrm{d}}_{1,2}d_2$; (c) XOS of debt only with $d_2 < M^{\mathrm{d}}_{2,1}d_1$; (d) XOS of equity only.

equal to d_i, i.e. the endogenous asset values of the other firms in the systems would be unaffected by the shock, and they would remain solvent as well. Expressed differently, if other firms in the system go bankrupt due to the shock on firm i, firm i must be in default as well.
For two firms linked by cross-ownership of equity only it is indeed possible that the negative shock on firm 2's exogenous asset does not ruin firm 2, but firm 1 instead, although the shock affected firm 1's exogenous asset in a non-negative way. Figure 7.1(d) shows that among all shocks with $\delta_1 \geq 0$, $\delta_2 < 0$ and $\tilde{\mathbf{a}} \in A_{\text{ds}} \cup A_{\text{dd}}$, firm 2 is still solvent after the shock, if the negative shock on its exogenous asset has been mild in the sense that $\tilde{a}_2 \geq d_2$, whereas firm 1 is in default. Under cross-ownership of equity only, an exogenous asset value of at least d_1 is sufficient for firm 1 to be protected against default caused by the spillover of negative shocks, which is more than under cross-ownership of debt only and which cannot be modified by changing the level of equity cross-held.
Of course, defaults caused by such spillover effects can also happen in the presence of further firms in the system. The same holds for cross-ownership of both, debt and equity.

Having seen that a negative shock on a firm can lead to the default of other firms in the system, especially if we do not require that the firm directly hit by the shock remains solvent, we will now consider in a system of $n \geq 2$ firms linked by cross-ownership of debt and/or equity, for which shocks such a transfer can actually occur. Again, we assume that all firms are solvent before the shock and that a subset $\emptyset \neq S \subsetneq \{1, \ldots, n\}$ of firms is subject to a non-positive shock, whereas firms in a subset $\emptyset \neq D \subsetneq \{1, \ldots, n\}$ ($D \cap S = \emptyset$) receive a non-negative shock. The exogenous assets of the remaining firms (if any) do not experience any shock. In the following, we will analyze under what conditions on $\boldsymbol{\delta}$ the firms in D will be simultaneously in default after the shock. In doing so, we will not pose any restrictions on the financial status of the firms in S after the shock. Our approach can be seen as a generalization of Proposition 1 of Glasserman and Young [2015], who consider in a model identical to our model for cross-ownership of debt only a system of n firms, where exactly one firm gets a negative shock ruining all firms in a given set D, which do not experience shocks themselves. Then Glasserman and Young [2015] derive a lower bound for the size of a shock triggering such a chain reaction. As already mentioned, under cross-ownership of debt only, the shock on a single firm i can affect other firms in the system only if firm i is in default after the shock itself, because otherwise the recovery value of debt of firm i, which is cross-held by other firms, would remain unchanged equal to d_i. However, if the firms are linked by cross-ownership of equity as well, it is not necessary to assume that firm i or the firms in S are in default after the shock.
Consistent with the above considerations and in contrast to the definition of $\boldsymbol{\delta}$ in (7.1), let the shock $\boldsymbol{\delta}' = (\delta_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ be given as

$$\delta_i' \begin{cases} \in [-a_i, 0], & i \in S, \\ \geq 0, & i \in D, \\ = 0, & i \notin S \cup D. \end{cases} \tag{7.17}$$

Again we assume that negative shocks cannot turn exogenous assets into liabilities. For a given shock $\boldsymbol{\delta}'$ as in (7.17), let all the firms in D be in default after the shock. In the following, we will derive a condition on $\boldsymbol{\delta}'$ necessary for such a scenario. For this, the following lemma, valid for any shock $\boldsymbol{\delta}$ as in (7.1), will be useful.

Lemma 7.1. *After a shock $\boldsymbol{\delta} \geq -\mathbf{a}$, where all firms are assumed to be solvent before the shock, the difference of the recovery values of debt before and after the shock and the difference of the equity values before and after the shock are given as*

$$\mathbf{d} - \tilde{\mathbf{r}} = \Big(-\boldsymbol{\delta} - \mathbf{s} + \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) + \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big)^{+}, \tag{7.18}$$

$$\mathbf{s} - \tilde{\mathbf{s}} = \min\Big\{\mathbf{s},\, -\boldsymbol{\delta} + \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) + \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big\}. \tag{7.19}$$

Proof. By (3.13),

$$\begin{aligned}
\mathbf{d} - \tilde{\mathbf{r}} &= \mathbf{d} - \min\Big\{\mathbf{d},\, \mathbf{a} + \boldsymbol{\delta} + \mathbf{M}^{\mathrm{d}}\tilde{\mathbf{r}} + \mathbf{M}^{\mathrm{e}}\tilde{\mathbf{s}}\Big\} &(7.20)\\
&= \mathbf{d} - \min\Big\{\mathbf{d},\, \mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{d} + \mathbf{M}^{\mathrm{e}}\mathbf{s} + \boldsymbol{\delta} - \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) - \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big\} &(7.21)\\
&= \mathbf{d} - \min\Big\{\mathbf{d},\, \mathbf{d} + \mathbf{s} + \boldsymbol{\delta} - \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) - \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big\} &(7.22)\\
&= \Big(-\boldsymbol{\delta} - \mathbf{s} + \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) + \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big)^{+}, &(7.23)
\end{aligned}$$

where (7.22) follows from the fact that $\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{d} + \mathbf{M}^{\mathrm{e}}\mathbf{s} = \mathbf{v} = \mathbf{d} + \mathbf{s}$, which holds since we assume all firms to be solvent before the shock. Similarly, by (3.14),

$$\begin{aligned}
\mathbf{s} - \tilde{\mathbf{s}} &= \mathbf{s} - \Big(\mathbf{a} + \boldsymbol{\delta} + \mathbf{M}^{\mathrm{d}}\tilde{\mathbf{r}} + \mathbf{M}^{\mathrm{e}}\tilde{\mathbf{s}} - \mathbf{d}\Big)^{+} &(7.24)\\
&= \mathbf{s} - \Big(\mathbf{a} + \mathbf{M}^{\mathrm{d}}\mathbf{d} + \mathbf{M}^{\mathrm{e}}\mathbf{s} + \boldsymbol{\delta} - \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) - \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}}) - \mathbf{d}\Big)^{+} &(7.25)\\
&= \mathbf{s} - \Big(\mathbf{d} + \mathbf{s} + \boldsymbol{\delta} - \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) - \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}}) - \mathbf{d}\Big)^{+} &(7.26)\\
&= \min\Big\{\mathbf{s},\, -\boldsymbol{\delta} + \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) + \mathbf{M}^{\mathrm{e}}(\mathbf{s} - \tilde{\mathbf{s}})\Big\}. &(7.27)
\end{aligned}$$

□

For a vector $\mathbf{x} \in \mathbb{R}^n$ and a set $\emptyset \neq M \subseteq \{1, \ldots, n\}$, let $\mathbf{x}_M$ stand for the subvector of $\mathbf{x}$ obtained by removing all entries of $\mathbf{x}$ whose index is not in M, i.e. $\mathbf{x}_M \in \mathbb{R}^{|M|}$. For an $n \times n$-matrix $\mathbf{A}$, the $|M_1| \times |M_2|$-submatrix $\mathbf{A}_{M_1,M_2}$ is obtained by deleting the rows and columns with the corresponding index not in $\emptyset \neq M_1 \subseteq \{1, \ldots, n\}$ and $\emptyset \neq M_2 \subseteq \{1, \ldots, n\}$, respectively. If $|M_1| = n$ or $|M_2| = n$, we write $\mathbf{A}_{\cdot,M_2}$ and $\mathbf{A}_{M_1,\cdot}$, respectively.

The following proposition clarifies which shocks might lead to the default of all firms in D. More specifically, a necessary condition for the shock to cause such an event is given.

Proposition 7.2. *Let a system of n firms linked by cross-ownership be subject to a shock $\boldsymbol{\delta}'$ as defined in (7.17). All firms are assumed to be solvent before the shock. If all the firms in $D \neq \emptyset$ with $D \cap S = \emptyset$ are in default after the shock, then $\boldsymbol{\delta}'$ is such that*

$$-\sum_{i=1}^{n} \delta'_i \geq \sum_{j \in D} (1 - \beta^{\mathrm{e}}_j) s_j, \tag{7.28}$$

with $\beta^{\mathrm{e}}_k := \sum_{i=1}^{n} M^{\mathrm{e}}_{ik}$, $k \in \{1, \ldots, n\}$, i.e. the negative sum of all shock components exceeds or equals the sum of the equity values of the firms in D that are held outside the system of the n firms.

Proof. Let $\bar{D}$ denote the subset of firms that are in default after the shock $\boldsymbol{\delta}'$, i.e. $\bar{D} \supseteq D \neq \emptyset$ with $\tilde{\mathbf{s}}_{\bar{D}} = \mathbf{0}$ and $\tilde{\mathbf{r}}_{\bar{D}} \ll \mathbf{d}_{\bar{D}}$. Let $\bar{S} := \{1, \ldots, n\} \backslash \bar{D}$ denote the set of firms that remain solvent.
Let us first assume $k := |\bar{S}| > 0$. Since the firms in $\bar{S}$ are solvent after the shock, we have $\tilde{\mathbf{r}}_{\bar{S}} = \mathbf{d}_{\bar{S}}$, and Lemma 7.1 implies

$$\mathbf{s}_{\bar{S}} - \tilde{\mathbf{s}}_{\bar{S}} = -\boldsymbol{\delta}'_{\bar{S}} + \mathbf{M}^{\mathrm{d}}_{\bar{S},\bar{D}}(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) + \mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{S}}(\mathbf{s}_{\bar{S}} - \tilde{\mathbf{s}}_{\bar{S}}) + \mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{D}}\mathbf{s}_{\bar{D}} \tag{7.29}$$

$$= \underbrace{\left(\mathbf{I}_k - \mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{S}}\right)^{-1}}_{=:\tilde{\mathbf{M}}} \left(-\boldsymbol{\delta}'_{\bar{S}} + \mathbf{M}^{\mathrm{d}}_{\bar{S},\bar{D}}(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) + \mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{D}}\mathbf{s}_{\bar{D}}\right), \tag{7.30}$$

$$\mathbf{0} \ll \mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}} = -\boldsymbol{\delta}'_{\bar{D}} - \mathbf{s}_{\bar{D}} + \mathbf{M}^{\mathrm{d}}_{\bar{D},\bar{D}}(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) + \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{D}}\mathbf{s}_{\bar{D}} + \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{S}}(\mathbf{s}_{\bar{S}} - \tilde{\mathbf{s}}_{\bar{S}}). \tag{7.31}$$

Plugging (7.30) into (7.31) yields, with $l := |\bar{D}| > 0$,

$$\begin{aligned}\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}} = &-\boldsymbol{\delta}'_{\bar{D}} - \mathbf{s}_{\bar{D}} + \mathbf{M}^{\mathrm{d}}_{\bar{D},\bar{D}}(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) + \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{D}}\mathbf{s}_{\bar{D}} \\ &+ \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{S}}\,\tilde{\mathbf{M}}\left(-\boldsymbol{\delta}'_{\bar{S}} + \mathbf{M}^{\mathrm{d}}_{\bar{S},\bar{D}}(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) + \mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{D}}\mathbf{s}_{\bar{D}}\right)\end{aligned} \tag{7.32}$$

and therefore

$$\begin{aligned}-\boldsymbol{\delta}'_{\bar{D}} - \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{S}}\,\tilde{\mathbf{M}}\,\boldsymbol{\delta}'_{\bar{S}} = &\left(\mathbf{I}_l - \mathbf{M}^{\mathrm{d}}_{\bar{D},\bar{D}} - \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{S}}\,\tilde{\mathbf{M}}\,\mathbf{M}^{\mathrm{d}}_{\bar{S},\bar{D}}\right)(\mathbf{d}_{\bar{D}} - \tilde{\mathbf{r}}_{\bar{D}}) \\ &+ \left(\mathbf{I}_l - \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{D}} - \mathbf{M}^{\mathrm{e}}_{\bar{D},\bar{S}}\,\tilde{\mathbf{M}}\,\mathbf{M}^{\mathrm{e}}_{\bar{S},\bar{D}}\right)\mathbf{s}_{\bar{D}}.\end{aligned} \tag{7.33}$$

Setting $\bar{S} := \{i_1, i_2, \ldots, i_k\}$ and $\bar{D} := \{j_1, j_2, \ldots, j_l\}$, the sum of the l lines of the LHS of (7.33) equals

$$-\sum_{j \in \bar{D}} \delta'_j - \sum_{s=1}^{l}\sum_{t=1}^{k}\sum_{v=1}^{k} M^{\mathrm{e}}_{j_s i_t}\tilde{M}_{tv}\delta'_{i_v} = -\sum_{j \in \bar{D}} \delta'_j - \sum_{v=1}^{k} \underbrace{\left(\sum_{s=1}^{l}\sum_{t=1}^{k} M^{\mathrm{e}}_{j_s i_t}\tilde{M}_{tv}\right)}_{\leq 1 \text{ by Corollary A.18}} \underbrace{\delta'_{i_v}}_{\leq 0} \tag{7.34}$$

$$\leq -\sum_{j=1}^{n} \delta'_j, \tag{7.35}$$

where $\delta'_{i_v} \leq 0$ because of $i_v \in \bar{S}$ and $\bar{S} \cup D = \emptyset$. Twofold application of Lemma A.17 shows that the sum over the l lines of the RHS of (7.33) is greater than or equal to

$$\sum_{j \in \bar{D}} (1 - \beta^{\mathrm{d}}_j) \underbrace{(d_j - \tilde{r}_j)}_{>0} + \sum_{j \in \bar{D}} (1 - \beta^{\mathrm{e}}_j) s_j \geq \sum_{j \in D} (1 - \beta^{\mathrm{e}}_j) s_j, \tag{7.36}$$

with

$$\beta^{\mathrm{d}}_k := \sum_{i=1}^{n} M^{\mathrm{d}}_{ik} < 1, \quad \beta^{\mathrm{e}}_k := \sum_{i=1}^{n} M^{\mathrm{e}}_{ik} < 1, \quad k \in \{1, \dots, n\}. \tag{7.37}$$

Combining (7.35) and (7.36) according to (7.33), we obtain

$$-\sum_{j=1}^{n} \delta'_j \geq \sum_{j \in D} (1 - \beta^{\mathrm{e}}_j) s_j, \tag{7.38}$$

i.e. the assertion is shown for $\bar{S} \neq \emptyset$. If $\bar{S} = \emptyset$, we have $\bar{D} = \{1, \dots, n\}$ and $\tilde{\mathbf{s}} = \mathbf{0}$. Then Lemma 7.1 implies $\mathbf{0} \ll \mathbf{d} - \tilde{\mathbf{r}} = -\boldsymbol{\delta}' - \mathbf{s} + \mathbf{M}^{\mathrm{d}}(\mathbf{d} - \tilde{\mathbf{r}}) + \mathbf{M}^{\mathrm{e}}\mathbf{s}$, i.e.

$$-\boldsymbol{\delta}' = (\mathbf{I}_n - \mathbf{M}^{\mathrm{d}})(\mathbf{d} - \tilde{\mathbf{r}}) + (\mathbf{I}_n - \mathbf{M}^{\mathrm{e}})\, \mathbf{s}. \tag{7.39}$$

Summing up the n lines of (7.39) we obtain

$$-\sum_{j=1}^{n} \delta'_j = \sum_{j=1}^{n} (1 - \beta^{\mathrm{d}}_j) \underbrace{(d_j - \tilde{r}_j)}_{>0} + \sum_{j=1}^{n} (1 - \beta^{\mathrm{e}}_j) s_j \geq \sum_{j \in D} (1 - \beta^{\mathrm{e}}_j) s_j, \tag{7.40}$$

and the assertion follows. □

Note that the bound provided in Proposition 7.2 holds independently of the realized cross-ownership structure of debt, the exact holdings of a firm's equity by other firms, the nominal levels of liabilities $\mathbf{d}$ and the exogenous asset values $\mathbf{a}$.

Let us now consider a random shock $\boldsymbol{\Delta}' : (\Omega, \mathcal{A}, P) \to (\mathbb{R}^n, \mathbb{B}^n, \lambda_n)$, with $\mathbb{B}^n$ and λ_n denoting the $n-$dimensional Borel-σ-algebra and Lebesgue-measure, respectively. The following corollary to Proposition 7.2 does not depend on the exact multivariate distribution of $\boldsymbol{\Delta}'$, and thus it does not depend on the correlations between its components, either.

Corollary 7.3. *Let a system of n firms linked by cross-ownership be subject to a random shock $\boldsymbol{\Delta}' := (\Delta'_i)_{1 \leq i \leq n} : (\Omega, \mathcal{A}, P) \to (\mathbb{R}^n, \mathbb{B}^n, \lambda_n)$ defined in analogy to $\boldsymbol{\delta}'$ in (7.17). All firms are assumed to be solvent before the shock. Then the probability that all the firms in D with $D \cap S = \emptyset$ are in default after the shock is at most*

$$P\left(-\sum_{i=1}^{n} \Delta'_i \geq \sum_{j \in D} (1 - \beta^{\mathrm{e}}_j) s_j \right). \tag{7.41}$$

Simultaneous default of all firms in D caused by $\mathbf{\Delta}'$ is impossible if

$$\sum_{i\in S} a_i < \sum_{j\in D}(1-\beta^{\mathrm{e}}_j)s_j. \tag{7.42}$$

Proof. For all the firms in D to be in default after the shock $\mathbf{\Delta}'$, Lemma 7.2 shows that it is necessary that $-\sum_{i=1}^n \Delta'_i \geq \sum_{j\in D}(1-\beta^{\mathrm{e}}_j)s_j$, i.e. the set $\{\omega \in \Omega : -\sum_{i=1}^n \Delta'_i(\omega) \geq \sum_{j\in D}(1-\beta^{\mathrm{e}}_j)s_j\}$ is a superset of the set $\{\omega \in \Omega :$ all the firms in D are in default after the shock $\mathbf{\Delta}'(\omega)\}$. This proves (7.41). Since we assume $-\Delta'_i \leq a_i$ for all $i \in S$ and $\Delta_i \geq 0$ for all $i \in D$, we have $-\sum_{i=1}^n \Delta'_i \leq \sum_{j\in S} a_j$ and (7.42) follows. □

As it was to be expected, positive shocks on the firms in D mitigate the impact of negative shocks on firms in S in that they decrease the shock sum and thus the related upper boundary of the probability of joint default of the firms in D. By (7.41), this upper boundary of the probability of a spillover of negative shocks on firms in S ruining all firms in D can be influenced by the share of equity of the firms in D held within the system of the n firms. From a practical point of view, however, it is questionable to what extent firm j can control β^{e}_j. All other parameters of the system held fixed, we know from Proposition 6.1 that s_j is non-decreasing in M^{d}_{kl} and M^{e}_{kl} for all $k, l \in \{1, \dots, n\}$ $(k \neq l)$. This means that low values of M^{e}_{kj}, leading to a low value of β^{e}_j, not necessarily decrease the probability in (7.41), as they tend to be associated with a low value of s_j. However, it is unclear which effect dominates. If we assume fixed share prices $\mathbf{s}$, the upper boundary of the probability of a spillover is the smaller the more of the equity of the firms in D is held outside the system. Moreover, high values of M^{d}_{jk}, M^{e}_{jk} and M^{d}_{kj} may in general reduce this maximum probability as well. As we are considering an upper bound, though, this does not imply that the actual probability of such a serious spillover decreases as well. Nevertheless, by (7.42), a suitably chosen cross-ownership structure might protect against a simultaneous default of all firms in D caused by shocks of the type $\mathbf{\Delta}'$. Of course, since we assume the firms in D to receive non-negative shocks, a firm i in the set D cannot default due to the negative shocks on the firms in S if $d_i \leq a_i$, because its exogenous asset will be sufficient to repay all of its liabilities after the shock as well.

Corollary 7.3 extends Proposition 1 of Glasserman and Young [2015] in three ways, namely we consider cross-ownership of both, debt and equity instead of cross-ownership of debt only, we allow that more than one firm receives a negative shock, and we allow the firms in D to receive a positive shock. In the following, we will compare our Corollary 7.3, imposed with the restrictions $S = \{i\}$ and $\Delta'_j = 0$ for all $j \in D$ for better comparability, to Proposition 1 of Glasserman and Young [2015] in order to see how the additional possibility of equity cross-holdings changes the bounds on $\boldsymbol{\delta}'$ and $\mathbf{\Delta}'$. For that, we consider a shock $\boldsymbol{\delta}'' \geq -\mathbf{a}$ such that $\delta''_i < 0$ and $\delta''_j = 0$ for all $j \neq i$. Under these assumptions, (7.28) reduces to

$$-\delta''_i \geq \sum_{j\in D}(1-\beta^{\mathrm{e}}_j)s_j, \tag{7.43}$$

which is strictly smaller than the bound provided by Glasserman and Young [2015], given as

$$-\delta_i'' \geq s_i + \frac{1}{\beta_i^{\mathrm{d}}} \sum_{j \in D} s_j. \tag{7.44}$$

As already mentioned, Glasserman and Young [2015] have to assume that the single triggering firm i is in default after the shock itself, since otherwise, there would be no implications for the remaining firms. If we adopt this for better comparability, yielding $S = \{i\} \subset \bar{D}$, (7.33), (7.35) and (7.36) imply

$$-\delta_i'' \geq \sum_{j \in \bar{D}} (1 - \beta_j^{\mathrm{d}})(d_j - \tilde{r}_j) + \sum_{j \in \bar{D}} (1 - \beta_j^{\mathrm{e}}) s_j \tag{7.45}$$

$$\geq (1 - \beta_i^{\mathrm{d}})(d_i - \tilde{r}_i) + (1 - \beta_i^{\mathrm{e}}) s_i + \sum_{j \in D} (1 - \beta_j^{\mathrm{e}}) s_j. \tag{7.46}$$

By Lemma 7.1, $d_i - \tilde{r}_i \geq -\delta_i'' - s_i$, because $\mathbf{s} - \tilde{\mathbf{s}} \geq \mathbf{0}$ by Proposition 2 of Gouriéroux et al. [2012], since we consider non-positive shocks. Hence,

$$-\delta_i'' \geq \frac{\beta_i^{\mathrm{d}} - \beta_i^{\mathrm{e}}}{\beta_i^{\mathrm{d}}} s_i + \frac{1}{\beta_i^{\mathrm{d}}} \sum_{j \in D} (1 - \beta_j^{\mathrm{e}}) s_j. \tag{7.47}$$

Comparing (7.44) and (7.47), we see that the presence of cross-ownership of equity decreases the lower bound on $-\delta_i''$, i.e. there are potentially more values of $\boldsymbol{\delta}''$ leading to a spillover of the negative shock on firm i to the firms in D. However, this does not necessarily mean that the additional presence of cross-holdings of equity increases the exact probability of such spillover effects. Without cross-ownership of equity, (7.44) and (7.47) coincide.

7.1.3 Fundamental and Contagious Defaults

In the previous section we analyzed situations where firms default just because spillover effects originating from negative shocks on other firms reduce the value of these firms' endogenous assets, whereas these firms would have stayed solvent if they had experienced the non-negative shock on their exogenous asset only. Hence, analogously to the decomposition of the impact of the shock into a direct and indirect effect (cf. (7.9)), we can identify two types of defaults, driven by changes of the exogenous and endogenous asset value of a firm, respectively. The probability of default of a firm can be decomposed in two parts as well. Our approach for firms linked by cross-ownership of possibly both, debt and equity, can be seen as an extension of a similar distinction for banks on the interbank market with cross-ownership of debt only. For such systems of banks, Elsinger et al. [2006a] define fundamental defaults and contagious defaults, where the former occur under the assumption that all other banks in the system are still healthy and with the latter resulting from spillover effects only. A similar distinction is made by Cont et al. [2013]. Angelini et al. [1996], who consider the Italian interbank clearing system, mention defaults triggered by external events and defaults caused by the default of another bank in the system, which corresponds to the contagious default of Cont et al. [2013]. Under cross-ownership of debt only, the value of a firm's endogenous assets is affected by the shock if and only if one or more of the cross-held firms is bankrupt after the shock. Hence, fundamental defaults as described by Elsinger et al. [2006a] for example are based on the idea that a firm defaults due to the loss in its exogenous asset value, while the value of its endogenous assets remains unchanged. In contrast to that, contagious defaults occur under the assumption that a firm could have borne the shock on its own exogenous asset, i.e. it is not in a fundamental default, but default occurs since other firms in the system cannot meet their liabilities, which means that the considered firm's endogenous asset value has decreased. With cross-ownership of equity being present, however, it is clear that endogenous asset values will in general react to any shocks on the system, no matter whether the firms directly hit by the shock default, or not. This leads us to the following definition of fundamental and contagious defaults under cross-ownership of possibly both, debt and equity, valid for any shock $\boldsymbol{\delta}$ as in (7.1).

Definition 7.4. *For a system of n firms linked by cross-ownership of possibly both, debt and equity, firm i ($1 \leq i \leq n$) is in a fundamental default, if*

$$\tilde{a}_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}) < d_i \text{ and } \tilde{a}_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\tilde{\mathbf{a}}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\tilde{\mathbf{a}}) < d_i, \quad (7.48)$$

where the second condition indicates that firm i is indeed in default after the shock. Firm i is in a contagious default, if

$$\tilde{a}_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\mathbf{a}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\mathbf{a}) \geq d_i \text{ and } \tilde{a}_i + \sum_{j=1}^{n} M_{ij}^{\mathrm{d}} r_j(\tilde{\mathbf{a}}) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}} s_j(\tilde{\mathbf{a}}) < d_i, \quad (7.49)$$

where again the second condition indicates that firm i is indeed in default after the shock.

Fundamental defaults are driven by what we called the direct effect in Section 7.1.1, since the firm would be in default also under the assumption of a fixed endogenous asset value, whereas contagious defaults are caused by changes of the endogenous asset values due to the shock, i.e. for a frozen endogenous asset value, the firm could have borne the shock on its exogenous asset. In particular, all firms in a fundamental default are directly hit by the shock. In the wide-spread literature on networks with cross-holdings of debt only under an idiosyncratic shock, the term contagious default mostly stands for the default of firms not directly hit by the shock[5]. For the described set-up, this definition of a contagious default coincides with our Definition 7.4, but in general, contagious defaults as in Definition 7.4 can also occur for firms directly affected by the shock.
Note that in the situation of Proposition 7.2, the firms in D are in a contagious default,

[5] Eboli [2013] calls such defaults secondary defaults, whereas a firm is in a primary default if and only if it has been directly hit by the shock.

since the first inequality of (7.49) is met because all firms are assumed to be solvent before the shock and $\tilde{a}_i \geq a_i$ $(i \in D)$, and the second inequality in (7.49) is met because the firms in D are supposed to be in default after the shock.

We now consider random shocks $\mathbf{\Delta} = (\Delta_i)_{1\leq i\leq n} : (\Omega, \mathcal{A}, P) \to (\mathbb{R}^n, \mathbb{B}^n, \lambda_n)$ and the probability that a firm is in default after the shock. Similarly to (7.1) we assume $\mathbf{\Delta} \geq -\mathbf{a}$ $P-$a.s.. Let $\tilde{\mathbf{A}} = (\tilde{A}_i)_{1\leq i\leq n} := \mathbf{a} + \mathbf{\Delta} \geq \mathbf{0}$ $P-$a.s. denote the random value of the exogenous assets immediately after the shock.
By (7.48) and (7.49), any default of a firm in a system of n firms linked by cross-ownership due to a shock is either a fundamental or a contagious default. Hence, the probability that firm i is in default after the shock is the sum of the probability that firm i is in a fundamental default after the shock and the probability that firm i is in a contagious default after the shock. This means that we can quantify to what extent the probability of default of a firm in a system of n firms originates from the effect of contagion. An obvious measure is

$$k_i := \frac{\text{probability of a contagious default of firm } i}{\text{probability of default of firm } i}, \quad i = 1, \ldots, n. \tag{7.50}$$

With the random firm value of firm i after the shock given as

$$\tilde{V}_i = \tilde{A}_i + \sum_{j=1}^{n} M^{\mathrm{d}}_{ij} r_j(\tilde{\mathbf{A}}) + \sum_{j=1}^{n} M^{\mathrm{e}}_{ij} s_j(\tilde{\mathbf{A}}), \tag{7.51}$$

(7.3) and (7.49) yield

$$k_i = \frac{P(\tilde{A}_i + \bar{a}_i \geq d_i, \tilde{V}_i < d_i)}{P(\tilde{V}_i < d_i)} = P(\tilde{A}_i + \bar{a}_i \geq d_i \,|\, \tilde{V}_i < d_i). \tag{7.52}$$

By Definition 7.4, firms in a contagious default could have borne a possible shock on their exogenous assets, provided that the value of their endogenous assets had left unchanged. Hence, without due consideration of the realized network structure and treating endogenous asset values as fixed instead, such firms might have thought to be rather save from default due to shocks impending with a given probability and size on the system, as they could have withstood the direct effect on their exogenous asset values. This makes clear that contagious defaults could be interpreted as defaults not anticipated, and in this sense, k_i stands for the proportion of defaults not foreseeable if the network structure is not taken into account appropriately.
A resembling approach to catch the effects of shock transmission on the probability of default is proposed by Gouriéroux et al. [2012]. They compare the probability of default of a firm, i.e. the probability of default under Suzuki's model, with the probability of default of this firm without contagion, i.e. after having cashed possible cross-holdings, so that endogenous assets are converted into exogenous assets. This exactly corresponds to the above "mistake" of treating endogenous asset values as fixed instead of considering their dependence on the shock. In our notation, the latter probability can be written

as $P(\tilde{A}_i + \bar{a}_i < d_i)$ (cf. (7.3)), and as a measure of the impact of contagion on firm i, Gouriéroux et al. [2012] then define

$$K_i := \frac{\text{probability of default of firm } i}{\text{probability of default of firm } i \text{ without contagion}} \tag{7.53}$$

$$= \frac{P(\tilde{V}_i < d_i)}{P(\tilde{A}_i + \bar{a}_i < d_i)}. \tag{7.54}$$

Comparing k_i and K_i, we see that k_i is more appropriate, if we want to further analyze the components constituting the probability of default under Suzuki's model, whereas K_i relates the probability of default under Suzuki's model to the probability of default without contagion obtained from treating endogenous asset values as fixed. Hence, K_i can be interpreted as a correction factor that this "erroneous" probability of default needs to be multiplied with to obtain the actual probability of default.

If we consider non-positive shocks only, k_i and K_i are related as follows. Under non-positive shocks, a firm i with $\tilde{a}_i + \bar{a}_i = \tilde{a}_i + v_i - a_i < d_i$ is automatically in default, i.e. $\tilde{v}_i < d_i$. If we had $\tilde{v}_i \geq d_i$, it would follow that $\tilde{v}_i - \tilde{a}_i - (v_i - a_i) > 0$. However, by (7.2) and (7.51), $\tilde{v}_i - \tilde{a}_i - (v_i - a_i) = \sum_{j=1}^{n} M_{ij}^{\mathrm{d}}(\tilde{r}_j - d_j) + \sum_{j=1}^{n} M_{ij}^{\mathrm{e}}(\tilde{s}_j - s_j) \leq 0$, where the inequality follows from Proposition 2 of Gouriéroux et al. [2012], if we restrict ourselves to non-positive shocks. Hence, if $\tilde{a}_i + \bar{a}_i < d_i$, firm i is automatically in a fundamental default, and therefore $P(\tilde{A}_i + \bar{a}_i < d_i) \leq P(\tilde{A}_i + \bar{a}_i < d_i, \tilde{V}_i < d_i)$. Of course, we also have $P(\tilde{A}_i + \bar{a}_i < d_i) \geq P(\tilde{A}_i + \bar{a}_i < d_i, \tilde{V}_i < d_i)$. This yields

$$K_i = \frac{P(\tilde{V}_i < d_i)}{P(\tilde{A}_i + \bar{a}_i < d_i)} = \frac{P(\tilde{V}_i < d_i)}{P(\tilde{A}_i + \bar{a}_i < d_i, \tilde{V}_i < d_i)} \tag{7.55}$$

$$= \frac{1}{P(\tilde{A}_i + \bar{a}_i < d_i \,|\, \tilde{V}_i < d_i)} = \frac{1}{1 - k_i}. \tag{7.56}$$

Thus, under non-positive shocks, it is sufficient to consider either k_i or K_i. Furthermore, $K_i \geq 1$ under such shocks by (7.56), i.e. the above-mentioned "erroneous" probability of default underestimates the actual probability of default.

Simulated values of K_i, $i = 1, \ldots, 5$, for a system of five French banks, which are at least in part connected by cross-holdings of both, debt and equity, can be derived from Table 4 of Gouriéroux et al. [2012], where it is assumed that $\ln(\tilde{\mathbf{A}}) \sim \ln(\mathbf{a}) + \mathbf{u}$ with $\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_5)$ for some $\sigma^2 > 0$, i.e. the random exogenous asset values after the shock follow a lognormal distribution. Hence, positive shocks are possible as well, leading to simulated values of K_i between 0 and 1.04.

Recall that in Section 3.2.2.1 we considered a ratio of probabilities roughly resembling the reciprocal of (7.54), namely the ratio of the probability of default of (w.l.o.g) firm 1 under Suzuki's model ("p_{S}") and the lognormal model ("p_{L}"), which we called the relative risk RR, i.e. for firm 1,

$$\mathrm{RR} = \frac{p_{\mathrm{L}}}{p_{\mathrm{S}}} = \frac{P(W_1 < d_1)}{P(V_1 < d_1)}, \tag{7.57}$$

provided that $p_S > 0$, with W_1 following a lognormal distribution with parameters determined such that the first two moments of the firm values V_1 and W_1 obtained from Suzuki's model and the lognormal model coincided. Furthermore, we assumed exogenous asset values to follow a multivariate geometric Brownian motion, resulting in lognormally distributed exogenous asset values $\mathbf{A}$ at maturity.
In contrast to that, we now start from given exogenous asset values $\mathbf{a}$ at or infinitesimally close to maturity and impose an immediate shock on them, leading to random exogenous asset values $\tilde{\mathbf{A}}$ at maturity. All probabilities considered in this section relate to the size of the shock, whereas all probabilities in Section 3.2.2.1 refer to the value of exogenous assets at maturity evolving from the underlying geometric Brownian motion. Hence, neither these probabilities nor any ratios related to them cannot be directly compared.

In the next section we will see by means of our simulation study how the occurrence of fundamental and contagious defaults and the values of k_i and K_i depend on the realized type and intensity of cross-ownership and the frequency and size of shocks impending on the system.

7.2 Simulation Study on Contagion

7.2.1 Existing Literature

In the last 15 years, many simulation studies on the resilience of real-world systems of banks linked by cross-holdings of debt have been conducted. Comprehensive overviews are provided by Upper [2011] and Cont et al. [2013], for example. As Upper [2011] reports, most of the papers he considered use the sequential default algorithm developed by Furfine [2003], which is described by Upper [2011] as follows. First, some bank i fails by assumption. Any bank j fails if the nominal value of debt of bank i owed to bank j, multiplied by an exogenously given loss rate θ, exceeds the nominal equity value of bank j. If there is a bank k ($k \notin \{i, j\}$) such that the loss, given as θ times the nominal sizes of bank k's loans granted to bank i and bank j, exceeds bank k's equity, bank k is in default as well. Further rounds of defaults are possible. The algorithm stops if no further defaults occur. As Upper [2011] remarks, "this algorithm does not solve the simultaneity problem since it does not recognise that higher order defaults increase losses at the banks that have failed previously, which in turn raises the θ_i's in their liabilities" [Upper, 2011, p. 115], i.e. there are no feedback loops of losses. In contrast to sequential default algorithms used in cascade models (cf. Section 1 and Staum [2013]), in the fictitious default algorithm of Eisenberg and Noe [2001] and its successors employed in clearing models, "a market clearing equilibrium is defined through a clearing payment vector with proportional sharing of losses among counterparties in case of default", which "leads to an endogenous recovery rate which corresponds to a hypothetical situation where all bank portfolios are simultaneously liquidated" [Cont et al., 2013, p. 331]. However, also Glasserman and Young [2015] note that the application of the Eisenberg-Noe model does not necessarily mean that actual payments are made at the end of the period, so they prefer to call it a valuation model rather than a clearing model. In contrast to Upper

[2011], Cont et al. [2013] arrive at the conclusion that in most studies, the model of Eisenberg and Noe [2001] is used as a theoretical background[6].
A common finding of a great part of such studies can be subsumed as "[c]ontagion is a low probability high impact event" [Elsinger et al., 2006a, p. 1302], see also the results of Degryse and Nguyen [2007], Gai and Kapadia [2010], Cont et al. [2013], the survey paper of Upper [2011] and references in there. However, Upper [2011] and Cont et al. [2013] also bring up shortcoming of such studies, for example that they often consider a shock on a single firm only, and that complete data on the cross-ownership structure needed for simulation studies for existing banking systems are often unavailable, but are estimated by the method of entropy maximization (cf. Blien and Graef [1998], for example), which is based on the idea that banks or firms seek to spread their engagements as wide as possible. However, Upper [2011] warns that entropy maximization "will not be able to reproduce a number of stylized facts on interbank markets", for example the circumstance that most banks are linked to only a few other banks in the system and the presence of tiering, "where lower tier banks do not lend to each other but transact only with top tier banks, which tend to be tightly linked" [Upper, 2011, p. 117]. For empirical evidence see the references provided in Nier et al. [2007] and Upper [2011]. In our simulation study we will take these findings into account by analyzing an incomplete network where not all possible links are present, as it is also done by Nier et al. [2007], Gai and Kapadia [2010] and Elliott et al. [2014]. Furthermore, we consider a so-called core-periphery network as proposed by Nier et al. [2007] and Elliott et al. [2014], for example. According to Gouriéroux et al. [2012], missing data on cross-holdings might be available soon, due to new regulations for the financial sector ('Basel III'), which would make the use of estimation methods such as entropy maximization obsolete, at least in the context of banks.
Furthermore, although Nier et al. [2007] acknowledge that "[t]his line of research [i.e. simulation studies on existing banking systems] is valuable in providing insights on the empirical importance of interbank contagion for real world networks", they critically remark that "the results are invariably driven by the particulars of the banking system under study and cannot therefore provide easily generalizable insights into the drivers of systemic risk" [Nier et al., 2007, p. 2037]. In their own simulation study they examine the influence of several parameters related to the size and structure of a random network of banks on the number of defaults under cross-ownership of debt only. The underlying model consists of a random graph with nodes and directed and weighted edges representing firms and financial linkages (measured as the nominal capital outstanding), respectively. From that, balance sheets of all firms and the cross-ownership matrix $\mathbf{M}^{\mathrm{d}}$ can be derived, and it can be shown that by construction of these balance sheets, all firms are solvent in the framework of Fischer [2014] as well. As such, the model is structural, but after a shock on the exogenous asset of a single firm, defaults are determined using a sequential default algorithm instead of the model of Eisenberg and Noe [2001], i.e. it

[6]This discrepancy might be explained by the fact that the last revision of Upper's paper took place in 2009, and afterwards, as he remarks, more than 15 simulation studies on contagion in the interbank market have been published.

belongs to the class of cascade models. If a bank's nominal equity value is not sufficient to absorb the shock on its exogenous asset, the bank is in default and the residual loss (calculated as shock size minus equity value) is transferred to the creditors of this bank. This may trigger further rounds of defaults. In extreme cases where the residual loss exceeds a bank's nominal interbank liabilities, the remaining amount is transferred to external depositors, i.e. in contrast to the model of Eisenberg and Noe [2001] and its successors, external debt is senior to interbank liabilities.
In a similar framework as Nier et al. [2007], but with a slightly different rationale of constructing balance sheets and with zero recovery values of debt, Gai and Kapadia [2010] examine how the probability of occurrence and the size of a cascade, measured as the percentage of firms in default after the shock, depends on the chosen network parameters. A detailed comparison of the models and a theoretical confirmation of the results of Nier et al. [2007] and Gai and Kapadia [2010] can be found in May and Arinaminpathy [2010].
Closer to the set-up of Eisenberg and Noe [2001], Elliott et al. [2014] consider a network of firms cross-holding their firm values. In contrast to our model, Elliott et al. [2014] analyze market values of firms, which are calculated as the proportion of the firm value held outside the system of n firms. However, as they remark, this does not qualitatively alter the results. A firm i defaults as soon as its market value falls below a certain threshold $\underline{v}_i$, calculated as a fixed proportion of its initial market value. If default occurs, this firm experiences an additional loss in its market value due to failure costs, i.e. firm values and thus also market values are discontinuous in exogenous asset values. This discontinuity may lead to multiple equilibria, and Elliott et al. [2014] always consider the solution with the minimum number of defaults, as these defaults are unavoidable. Defaults are determined by an algorithm similar to the fictitious default algorithm of Eisenberg and Noe [2001]. Then Elliott et al. [2014] analyze in a simulation study with random networks how the number of defaults caused by an idiosyncratic shock depends on the realized network structure.
The studies of Nier et al. [2007], Gai and Kapadia [2010] and Elliott et al. [2014] will be the main references to our study.

To our knowledge, a comprehensive simulation study based on Eisenberg and Noe [2001] or, as an extension with cross-holdings of equity also, based on Suzuki [2002], Elsinger [2009] and Fischer [2014], that examines the influence of the network size and structure, determined by cross-holdings of possibly both, debt and equity, the frequency and magnitude of multiple shocks and the ratio of nominal liabilities to exogenous asset values on the effect of contagion and the resulting number of defaults, has not been published yet. Hence, our simulation study designed to analyze these effects will offer further insights complementing the existing literature. Its detailed set-up und the obtained results are presented in the following.

7.2.2 Set-Up and Parameter Values

7.2.2.1 Input Parameters and Algorithm

The main input parameters concern the structure of the system and the nature of the shocks. As in the previous sections and in line with the literature, we will only consider systems with all firms being solvent before the shock.

Remark 7.5. Under cross-ownership of equity only, if all firms are solvent before the shock, this implies $\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} d_i$. This can be seen as follows. First, if all firms are solvent before the shock, $\mathbf{s} = \mathbf{a} + \mathbf{M}^{\mathrm{e}}\mathbf{s} - \mathbf{d}$, i.e. $(\mathbf{I}_n - \mathbf{M}^{\mathrm{e}})\mathbf{s} = \mathbf{a} - \mathbf{d}$. Adding up the n rows shows that then

$$\underbrace{\sum_{i=1}^{n}(1-\beta_i^{\mathrm{e}})s_i}_{\geq 0} = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} d_i, \tag{7.58}$$

with β_i^{e} as defined in (7.37).

Network Structure

In our simulations we analyze three types of networks, namely the incomplete network, the core-periphery network, and the ring network. As in Nier et al. [2007], Gai and Kapadia [2010] and Elliott et al. [2014], we interpret the system of n firms as a random graph with the firms as nodes and their cross-holdings representing the directed edges, which means that mutual liabilities are not netted. However, as we allow cross-ownership of both, debt and equity, we actually need to consider two networks with identical nodes, but different path structures, one for each type of cross-ownership. In both networks, each directed link from firm i to firm j stands for a holding of firm i in firm j's debt or equity. In the incomplete network and the core-periphery network, all these links are present or absent independently of each other. Furthermore, in the incomplete network, each link has an identical probability of occurrence, which is often called the Erdös-Rényi probability, as Erdös and Rényi [1959] were among the first to study the properties of such random graphs [Nier et al., 2007]. Hence, incomplete networks exhibit symmetry between the n firms with respect to the expected numbers of ingoing and outgoing links. Following Elliott et al. [2014], a cross-ownership matrix $\mathbf{M} = (M_{ij})_{1\leq i,j\leq n}$ (standing for $\mathbf{M}^{\mathrm{d}}$ or $\mathbf{M}^{\mathrm{e}}$) of the incomplete network was created as follows. First, we generated a random graph with Erdös-Rényi probability p_θ, denoted in terms of a so-called adjacency matrix $\mathbf{G} \in \{0,1\}^{n\times n}$ [Elliott et al., 2014] with i.i.d. entries $G_{ij} \sim \mathrm{Ber}(p_\theta)$ for $i \neq j$ and $G_{ii} = 0$, $1 \leq i,j \leq n$. Then we set

$$M_{ij} := \frac{\beta \times G_{ij}}{\sum_{i=1}^{n} G_{ij}}, \quad 1 \leq i,j \leq n, \tag{7.59}$$

i.e. the strictly positive entries within a column of $\mathbf{M}$ have identical values and their respective sum always equals β. Thus, β stands for the fraction of any firm's debt or equity held within the system of the n firms, see also (7.37). As in Elliott et al. [2014], we will refer to β as the integration of the system with cross-ownership matrix

M. Furthermore,

$$E\left(\sum_{i=1}^{n} G_{ij}\right) = (n-1)\, p_\theta =: \theta \tag{7.60}$$

denotes the expected number of strictly positive entries within each column of **M**, i.e. θ equals the expected number of debtholders or shareholders of a firm. Based on Elliott et al. [2014] we will refer to θ as the diversification of the system, whereas in Nier et al. [2007] and Gai and Kapadia [2010], p_θ resp. θ are called the connectivity of the system. In our simulations, we considered every combination of

$$\beta \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95\}, \tag{7.61}$$

$$p_\theta \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}, \tag{7.62}$$

where, within a given scenario under cross-ownership of both, debt and equity, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ exhibited identical values of β and p_θ for the sake of simplicity, but their entries do not coincide in general. In the following, we will refer to this set-up as the incomplete network. Note that for $p_\theta = 1$, we obtain a complete network with all possible links being present. Such networks will be theoretically analyzed in Section 7.2.5.3 and Section A.8, see also Eboli [2013] and Acemoglu et al. [2015].
As already mentioned, real-world interbank networks with cross-holdings of debt exhibit asymmetry in the sense that so-called core firms have a higher probability of being connected to other firms in the system than peripheral firms (cf. Nier et al. [2007] and references therein), i.e. core firms tend to have more outgoing links than peripheral firms, which means that core firms tend to hold a part of more firms than peripheral firms. In our second network structure, called core-periphery network as in Elliott et al. [2014][7], we take this finding into account by dividing the set of n firms into two disjoint and non-empty subsets $C = \{1, \ldots, n_C\}$ and $P = \{n_C + 1, \ldots, n\}$ of core firms and peripheral firms, respectively. Note that under cross-ownership of both, debt and equity, a firm belongs to the same subset for both types of cross-holdings. In our simulations, we had 10% core firms and 90% peripheral firms, i.e. $n_C/n = 0.1$. This is in line with Elliott et al. [2014] and moreover a compromise between the empirical findings of Craig and von Peter [2014] and Fricke and Lux [2012], who estimated this fraction with about 2.5% for the German banking system and with about 23%-28% for the Italian overnight interbank market e-MID, respectively. Then core firms have a higher probability $p_{\theta,C}$ of holding a part of some other firm in the system than peripheral firms with corresponding probability $p_{\theta,P}$. Following Nier et al. [2007], for a better comparability, $p_{\theta,C}$ and $p_{\theta,P}$ were chosen such that the expected number of directed links is the same as for the incomplete network with Erdös-Rényi probability p_θ. In the core-periphery network, the expected number of links amounts to $p_{\theta,C}\, n_C\, (n-1) + p_{\theta,P}\, (n - n_C)\, (n-1) = (n-1)\, (p_{\theta,C}\, n_C + p_{\theta,P}\, (n - n_C))$, for the incomplete network the expected number of

[7] Nevertheless, our core-periphery network is more oriented towards the tiered network of Nier et al. [2007] than the core-periphery network of Elliott et al. [2014], which consists of 10 completely connected core firms and 90 peripheral firms linked to exactly one (core) firm such that the core firm holds a part of the peripheral firm.

links equals $p_\theta\, n\,(n-1)$. Equating these two expressions yields[8]

$$p_{\theta,P} = \frac{p_\theta\, n - p_{\theta,C}\, n_C}{n - n_C}. \tag{7.63}$$

Following Nier et al. [2007], the calculation of $p_{\theta,P}$ was based on[9] $p_\theta = 0.2$, and we let

$$p_{\theta,C} \in \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}, \tag{7.64}$$

implying

$$p_{\theta,P} \in \left\{\tfrac{17}{90}, \tfrac{16}{90}, \tfrac{15}{90}, \tfrac{14}{90}, \tfrac{13}{90}, \tfrac{12}{90}, \tfrac{11}{90}, \tfrac{10}{90}\right\}. \tag{7.65}$$

This implies that diversification in the core-periphery network is constant $0.2 \times (n-1)$ for all considered scenarios, even if $p_{\theta,C}$ is varied. For given values of $p_{\theta,C}$ and $p_{\theta,P}$ we then generated the adjacency matrix $\mathbf{G}$ such that independently $G_{ij} \sim \mathrm{Ber}(p_{\theta,C})$ for $1 \le i \le n_C$, $i \neq j$, and $G_{ij} \sim \mathrm{Ber}(p_{\theta,P})$ for $n_C + 1 \le i \le n$, $i \neq j$, and $G_{jj} = 0$ $(1 \le j \le n)$. Then $\mathbf{M}$ was created from $\mathbf{G}$ as for the incomplete network, with integration β taking the same values as in the incomplete network. Again, within a scenario under cross-ownership of both, debt and equity, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ were built from the same values of $p_{\theta,C}$ and β, but they were generally not identical. In contrast to Elliott et al. [2014], who equip core firms and peripheral firms with initial exogenous asset values of 8 and 1, respectively, we refrained from mixing up the two criteria "coreness" and initial size. Instead, in all our simulations on the three network structures, exogenous assets values did not differ between the n firms before the shock.

Apart from random graphs with independently created links, we also considered ring networks as described by Eboli [2013] and Acemoglu et al. [2015] for example, where for both, debt and equity, each firm partially owns exactly one firm and is partially owned by exactly one other firm, so that a chain of cross-holdings comprising all the n firms emerges. Hence, the resulting matrices $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ have exactly one positive entry in each row and column, but $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ do not need to coincide. In a ring network, we always have $\theta = 1$ and thus a fixed level of diversification. Integration β was varied in analogy to the incomplete network, where again, within a scenario, $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ had the same level of integration.

Remark 7.6. Besides p_θ and instead of integration β, Nier et al. [2007] use bank capitalization[10] $\gamma := \frac{\sum_{i=1}^n s_i}{\sum_{i=1}^n v_i}$ and the size of interbank exposures $\zeta := \frac{\sum_{i=1}^n (v_i - a_i)}{\sum_{i=1}^n v_i}$ as input parameters for the construction of the network, and they analyze how these parameters influence the number of defaults caused by a shock. However, in our set-up (i.e. our

[8] Although the underlying rationale is the same, this formula differs from the formula used by Nier et al. [2007], which might not be correct as it does not yield the correct result if $p_\theta = p_{\theta,C} = p_{\theta,P}$.

[9] As we will see on page 132, this corresponds to the benchmark value of p_θ in the incomplete network.

[10] In the model of Nier et al. [2007], the n banks exhibit the same level of capitalization (expressed as the ratio of equity value to total bank value), but as this cannot be guaranteed in our model, we identify bank capitalization of Nier et al. [2007] with $\frac{\sum_{i=1}^n s_i}{\sum_{i=1}^n v_i}$, since in their model, $\frac{s_j}{v_j} = \frac{\sum_{i=1}^n s_i}{\sum_{i=1}^n v_i}$ for all $j \in \{1, \ldots, n\}$.

method of network construction), γ and ζ are both endogenously determined and monotone transformations of β: since all firms are assumed to be solvent before the shock and therefore $v_i = d_i + s_i$, it follows that

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \beta_i^{\mathrm{d}} d_i + \sum_{i=1}^{n} \beta_i^{\mathrm{e}} s_i \tag{7.66}$$

$$= \begin{cases} \sum_{i=1}^n a_i + \beta \sum_{i=1}^n d_i, & \text{XOS of debt only,} \\ \sum_{i=1}^n a_i + \beta \sum_{i=1}^n (v_i - d_i) = \frac{\sum_{i=1}^n a_i - \beta \sum_{i=1}^n d_i}{1-\beta}, & \text{XOS of equity only,} \\ \sum_{i=1}^n a_i + \beta \sum_{i=1}^n (d_i + s_i) = \frac{\sum_{i=1}^n a_i}{1-\beta}, & \text{XOS of both, debt and equity,} \end{cases} \tag{7.67}$$

with β_i^{d} and β_i^{e} as defined in (7.37). Since all the three expressions $\big($and thus also $\gamma = \frac{\sum_{i=1}^n (v_i - d_i)}{\sum_{i=1}^n v_i} = 1 - \frac{\sum_{i=1}^n d_i}{\sum_{i=1}^n v_i}$ and $\zeta = 1 - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n v_i}\big)$ are monotone in β for given values of $\sum_{i=1}^n a_i$ and $\sum_{i=1}^n d_i$, we will display all results in terms of β only. Note that γ and ζ are non-decreasing in β. For cross-ownership of equity only this follows from Remark 7.5. Furthermore, (7.67) reveals that in our set-up, for any type of cross-ownership and any realization of any network type, $\sum_{i=1}^n v_i$ only depends on the integration β, $\sum_{i=1}^n a_i$ and $\sum_{i=1}^n d_i$, i.e. even for random networks, $\sum_{i=1}^n v_i$ and thus also $\sum_{i=1}^n s_i$, γ and ζ are deterministic. This will prove useful for some theoretical considerations on our output parameters in Section 7.2.3. However, individual firm values v_i do depend on the realized network structure.

Shocks

The literature on contagious effects of shocks on a system of n firms or banks linked by cross-ownership mainly employs one of the two following types of events or shocks. The majority of them (among many others, Angelini et al. [1996], Furfine [2003], Degryse and Nguyen [2007], Nier et al. [2007], and Gai and Kapadia [2010]) considers scenarios where the default of a single firm, mostly caused by a shock on the exogenous asset of this firms (often reducing its value to 0) might trigger a chain reaction leading to the default of other firms in the system. In the opinion of Upper [2011], this is a clear shortcoming limiting the usefulness of such models. Instead, he recommends to allow simultaneous shocks on some or all the n firms, as it is done for example by Boss et al. [2006], Elsinger et al. [2006a], Elsinger et al. [2006b], Frisell et al. [2007], Martínez-Jaramillo et al. [2010], and Gouriéroux et al. [2012]. In simulation studies on existing banking systems, such shocks follow either a theoretical multivariate distribution, sometimes with parameters calibrated to real data, or a multivariate empirical distribution completely estimated from existing data. This seems plausible, as it is the aim of such studies to gain a realistic impression of the impending danger of contagious defaults. In contrast to that, our study targets at identifying potential influence factors on the occurrence of defaults in a systematic way. Hence, based on a rationale provided in the online appendix[11] of

[11] cf. `https://www.aeaweb.org/aer/app/10410/20130115_app.pdf`.

Elliott et al. [2014], in our simulations shocks are such that each firm receives a negative shock with probability p_π, wiping out $\pi 100\%$ of its exogenous asset. Thus, the random exogenous value after the shock equals

$$\tilde{\mathbf{A}} = (\mathbf{I}_n - \pi \operatorname{diag}(B_1, \ldots, B_n))\mathbf{a}, \tag{7.68}$$

with i.i.d. Bernoulli distributed random variables B_i with parameter p_π. Within a scenario, p_π and π were identical for all the n firms, and we considered every combination of

$$p_\pi \in \{0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}, \tag{7.69}$$

$$\pi \in \{0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}, \tag{7.70}$$

where $\pi = 1$ corresponds to a total loss in a firm's exogenous asset value.

Further Input Parameters

All the n firms were endowed with exogenous assets of value $a_i = a = 1$ before the shock. The nominal level of liabilities d, likewise identical for all the n firms, was set to 0.95 or 1.1. For $d > a$, it is impossible to generate scenarios with all firms being solvent before the shock under cross-ownership of equity only by Remark 7.5. Hence, we skipped scenarios with $d = 1.1$ under cross-ownership of equity only. For $d < a$, only firms that actually receive a strictly negative shock can be in default after the shock. This is another reason why we consider multiple shocks instead of idiosyncratic shocks, since otherwise, at most one firm would be in default after the shock, if $d < a$. Under multiple shocks, defaults can be both, fundamental and contagious, even if $d < a$. Note that Nier et al. [2007] consider a higher level of total nominal debt to total exogenous asset values in their benchmark scenario, which can be calculated as $\frac{d}{a} = \frac{\sum_{i=1}^n d_i}{\sum_{i=1}^n a_i} = \frac{1-\gamma}{1-\zeta} = \frac{0.95}{0.8} = 1.1875$ by Remark 7.6. We chose smaller values of d/a in order to facilitate the generation of systems with all firms being solvent before the shock. Furthermore, as in Gai et al. [2011], our system consisted of $n = 250$ firms.

Generation of Scenarios, Benchmark Values and Clearing Algorithm

To each of the three types of cross-ownership (cf. Definition 3.2), we independently applied the following procedure for the generation of scenarios.

In a first attempt, we considered every combination of the above input parameters. However, this led to a huge number of scenarios, with results difficult to display clearly. Hence, we decided to follow the approach of Nier et al. [2007], i.e. we defined a benchmark scenario with fixed parameter values and based on that, we simultaneously varied two parameters out of three or four (shock probability p_π, shock size π, integration β and, for the incomplete network, Erdös-Rényi probability p_θ, and, for the core-periphery network, $p_{\theta,C}$). This was done for all possible pairs of these parameters. Hence, all results obtained for this set-up would need to be confirmed by simulations with additional combinations of parameter values. The benchmark values of the respective fixed parameters, indicated with superscript b, were chosen as follows. By Remark 7.6 for cross-ownership of debt only, $\zeta = 1 - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i + \beta \sum_{i=1}^n d_i}$, which implies $\beta \approx 0.21$ if we plug in the above-mentioned benchmark values of Nier et al. [2007], given as $\zeta = 0.2$ and $\frac{\sum_{i=1}^n d_i}{\sum_{i=1}^n a_i} = 1.1875$. Hence, we set $\beta^{\mathrm{b}} = 0.2$. Furthermore, we adopted $p_\theta^{\mathrm{b}} = 0.2$, yielding $\theta^{\mathrm{b}} = 49.8$. In the core-periphery network we set $p_{\theta,C}^{\mathrm{b}} = 0.5$, implying $p_{\theta,P}^{\mathrm{b}} = 1/6$. Since Nier et al. [2007] consider idiosyncratic shocks completely wiping out a firm's exogenous asset, benchmark values for p_π and π could not be derived from their work, and Elliott et al. [2014] do not work with benchmark values. Due to the lack of further sources in the literature and since our main goal is not to estimate real-world probabilities of default, but to gain an impression of the influence of our input parameters on this probability, we just set $p_\pi^{\mathrm{b}} = 0.2$ and $\pi^{\mathrm{b}} = 0.4$.

For a fixed network type (incomplete, core-periphery, ring), a given type of cross-ownership and a given pair of parameters to be varied, we first simulated $m = 1{,}000$ realizations of $\tilde{\mathbf{A}}$ for all combinations of the corresponding parameter values of p_π and π according to (7.68), and, depending on the considered type of cross-ownership, $m_1 = 2{,}000$ realizations of $\mathbf{M}^{\mathrm{d}}$ and/or $\mathbf{M}^{\mathrm{e}}$ for all combinations of the corresponding parameter values of β and p_θ, or β and $p_{\theta,C}$, if applicable.

Next, we fixed the value of d. Then, together with $a = 1$, we calculated values of $\mathbf{r}$ and $\mathbf{s}$ before the shock for the simulated cross-ownership matrix/matrices. For $d = 0.95$, all the 250 firms were automatically solvent before the shock due to $d < a$. If, for $d = 1.1$, it turned out that for some pair of matrices not all of the n firms were solvent before the shock, this realization of the cross-ownership matrix/matrices was skipped. This is why we simulated 2,000 realizations of $\mathbf{M}^{\mathrm{d}}$ and/or $\mathbf{M}^{\mathrm{e}}$ instead of 1,000 only. The procedure ended as soon as we had 1,000 solvent networks or all the 2,000 realizations had been tested. Hence, it was possible that for some scenarios, fewer than or even none of the intended 1,000 repetitions were at hand. For details, see Section 7.2.3.1.

In order to get the values $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{s}}$ after the shock for the obtained m_2 ($m_2 \leq m$) valid networks (in the sense that all the firms are solvent before the shock for the given value of d), we combined the kth valid realization of the cross-ownership matrix/matrices with the kth realization of $\tilde{\mathbf{A}}$ ($k = 1, \dots, m_2$) and from that, we determined $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{s}}$.

The values $\mathbf{r}$ and $\mathbf{s}$, as well as $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{s}}$, were calculated as fixed points of $\mathbf{\Phi}_{\mathbf{a},\mathbf{d},\mathbf{M}^{\mathrm{d}},\mathbf{M}^{\mathrm{e}}}$ of Fischer [2014] (cf. (3.17)) with tolerance $\epsilon = 1 \times 10^{-6}$ and starting vector $(\mathbf{r}_0^T, \mathbf{s}_0^T)^T := \mathbf{r}_{\mathrm{small}} := (\min\{\mathbf{d}, \mathbf{a}\}^T, ((\mathbf{a} - \mathbf{d})^+)^T)^T$ taken from Hain and Fischer [2015]. Recall that in contrast to our approach based on simultaneous clearing, Nier et al. [2007] and Gai and Kapadia [2010] employ sequential default procedures (cf. Section 7.2.1).

7.2.2.2 Output Parameters

Based on $\mathbf{r}$, $\mathbf{s}$, $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{s}}$, the following output parameters were calculated for each of the m_2 repetitions of a given combination of parameter values, quantifying the impact of the shock in several ways.

The most natural measure is the total number of defaults, which is also employed by Nier et al. [2007], Martínez-Jaramillo et al. [2010], Gai and Kapadia [2010], Eboli [2013], Elliott et al. [2014] and Acemoglu et al. [2015], among others. Additionally and similarly

to Elsinger et al. [2006a] and Elsinger et al. [2006b], we distinguished between fundamental defaults and contagious defaults as in Definition 7.4. If at least one firm was in default, we calculated the proportion of contagious defaults to total defaults among the 250 firms, in symbols $\hat{k}$, as an estimate of k_i defined in (7.52)[12]. As we consider non-positive shocks only, we did not estimate K_i due to (7.55)–(7.56).
In order to get an impression of the losses caused by the shock, we calculated the relative change in the sum of firm values

$$\eta := \frac{\sum_{i=1}^{n}(\tilde{v}_i - v_i)}{\sum_{i=1}^{n} v_i} \tag{7.72}$$

and the relative change in the value of endogenous assets

$$\xi := \frac{\sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} v_i} \tag{7.73}$$

with c_i as defined in (7.7).
Next, we summarized the total number of defaults, $\hat{k}$, η and ξ over the m_2 repetitions of a certain scenario by calculating their respective mean and standard deviation. Following the recommendation of Martínez-Jaramillo et al. [2010], we also estimated the $(1-\alpha)100\%$-VaR and the expected $(1-\alpha)100\%$ shortfall of the relative change in the sum of firm values. For the former, we calculated the negative empirical α-quantile ($\alpha \in (0,1)$) of $\frac{\sum_{i=1}^{n}(\tilde{v}_{ik}-v_{ik})}{\sum_{i=1}^{n} v_{ik}}$, $k = 1,\ldots,m_2$, where v_{ik} and $\tilde{v}_{ik}$ denote the firm values of firm i before and after the shock, respectively, obtained from the kth repetition of a certain combination of parameter values. As in Section 3.2.3, empirical quantiles were calculated with the default method of the function 'quantile' implemented in R.
For an integrable random variable X with CDF F_X, the expected $(1-\alpha)100\%$ shortfall $\mathrm{ES}_{1-\alpha}(X)$ is defined as (cf. Acerbi and Tasche [2002a])

$$\mathrm{ES}_{1-\alpha}(X) := -\frac{1}{\alpha}\Big(E\left(X 1_{\{X \le -\mathrm{VaR}_{1-\alpha}(X)\}}\right) + \mathrm{VaR}_{1-\alpha}(X)\left[P\left(X \le -\mathrm{VaR}_{1-\alpha}(X)\right) - \alpha\right]\Big), \tag{7.74}$$

which can be written as $\mathrm{ES}_{1-\alpha}(X) = -\frac{1}{\alpha}\int_0^\alpha F_X^{-1}(p)\mathrm{d}p$ [Acerbi and Tasche, 2002a]. As they remark, this last expression illustrates that the expected $(1-\alpha)100\%$ shortfall

[12] Since except for the core-periphery network, our scenarios exhibit symmetry between the n firms with respect to expected values of all network and shock parameters, we initially tried to estimate the probabilities in the numerator and denominator of k_i by calculating the proportion of the m_2 repetitions where firm 1 met the related criterion (i.e. $i = 1$). Hence, if the denominator was strictly positive,

$$\hat{k}_1 := \frac{\#\{j \in \{1,\ldots,m_2\} : \tilde{a}_{1j} + \bar{a}_{1j} \ge d, \tilde{v}_{1j} < d\}}{\#\{j \in \{1,\ldots,m_2\} : \tilde{v}_{1j} < d\}} \tag{7.71}$$

with $\tilde{a}_{1j}$, $\bar{a}_{1j}$ and $\tilde{v}_{1j}$ denoting the realized exogenous asset value after the shock, the realized endogenous asset value before the shock and the realized firm value after the shock, respectively, for firm 1 in the jth repetition. Otherwise, $\hat{k}_1$ was set to NA. However, with this approach, $\hat{k}_1$ would not have been available for about 40% of all existing combinations of parameter values, since firm 1 was solvent for all the m_2 repetitions. Hence, we used $\hat{k}$ instead, which reduced the fraction of combinations of parameter values with missing estimate of k_i to about 24%.

can be interpreted as the negative expected value (one could say, the expected loss) of the $\alpha 100\%$ worst case scenarios of X (i.e. scenarios with small values of X), whereas the $(1-\alpha)100\%$-VaR refers to the *minimum* loss associated with the $\alpha 100\%$ worst case scenarios of X. For a random sample $X_1, X_2, \ldots, X_m$ of i.i.d. copies of X, let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(m)}$ denote the corresponding order statistics. Then, according to Acerbi and Tasche [2002a], a natural estimator of $\mathrm{ES}_{1-\alpha}(X)$ is given by

$$\mathrm{ES}_{1-\alpha,m}(X) = -\frac{\sum_{k=1}^{w} X_{(k)}}{w} \tag{7.75}$$

with $w := \lfloor m\alpha \rfloor$. As Acerbi and Tasche [2002b] show, $\lim_{m\to\infty} \mathrm{ES}_{1-\alpha,m}(X) = \mathrm{ES}_{1-\alpha}(X)$. In our set-up, we had $X = \frac{\sum_{i=1}^{n}(\tilde{V}_i - v_i)}{\sum_{i=1}^{n} v_i}$ (recall that by Remark 7.6, the sum of firm values before the shock is deterministic by construction, even if we consider random networks). As in Section 3.2.3 we always set $\alpha = 0.01$. Because of $m_2 \leq m = 1{,}000$, we had $w \leq 10$ for all scenarios. Thus, in order to get stable estimates of $\mathrm{VaR}_{1-\alpha}$ and $\mathrm{ES}_{1-\alpha}$, these estimates were calculated for scenarios with $m_2 = m = 1{,}000$ only.

All output parameters were analyzed univariately or bivariately where necessary (i.e. with all parameters held fixed except of one or two, respectively). For the univariate analysis in terms of a certain parameter, we merged all data obtained from the bivariate simulations where this parameter had been varied and with the remaining parameters equal to their benchmark values.

7.2.3 Results

7.2.3.1 Missing Scenarios and Focus of Subsequent Analysis

As already mentioned, for $d = 0.95$, all firms were always solvent before the shock, and thus always $m_2 = 1000$. For $d = 1.1$, there were no valid scenarios under cross-ownership of equity only (cf. Remark 7.5). For the other two types of cross-ownership, for 10.3% of all combinations of parameter values no valid scenarios could be generated (i.e. $m_2 = 0$), 86.3% of all combinations of parameter values had $m_2 = 1{,}000$ and for the remaining 3.4% combinations of parameter values, we had $m_2 = 285.8$ on average (SD=295.0).

For both, cross-ownership of debt only and cross-ownership of both, debt and equity, a graphical comparison showed that the relationship between the input parameters and the output parameters was qualitatively the same for $d = 0.95$ and $d = 1.1$, with only few exceptions that will be mentioned explicitly. Hence, unless stated otherwise, we will consider $d = 0.95$ in the remainder, since in this case, we also have results for cross-ownership of equity only. However, the absolute number of defaults and the absolute value of the proportion of contagious defaults were not necessarily comparable between $d = 0.95$ and $d = 1.1$, since for $d = 0.95$, only firms that actually experience a negative shock can be in default after the shock (this default can be fundamental or contagious), whereas for $d = 1.1$, also firms without a shock can be in default after the shock. Hence,

the related results primarily refer to the shape of the curves, but not to the obtained absolute values of the output parameters.
For η and the estimated values of $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.99}$, a univariate analysis of the influence of the four resp. three input parameters is sufficient, since for all network types and all types of cross-ownership, the shape of the curve with respect a certain parameter did not depend on the other parameter varied. Furthermore, we saw that the results for the estimated values of $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.99}$ were qualitatively highly similar to the results for η (recall that these estimates are calculated from η) within each network type and for each type of cross-ownership, so we will only consider η in the following. In contrast to η, the total number of defaults, the proportion of contagious defaults $\hat{k}$, and ξ will be analyzed in due consideration of both parameters varied.

7.2.3.2 Analysis of Defaults

For $\pi = 0.05$, there are no defaults for $d = 0.95$, because in this case, $\tilde{a}_i = 0.95a = 0.95 = d_i$ for all firms i hit by the shock, i.e. all firms' exogenous assets are sufficient to repay their corresponding debt. Hence $\pi = 0.05$ is omitted in all subfigures of Figure 7.2–Figure 7.6 related to π.

Mean Number of Defaults
A graphical analysis of the mean numbers of defaults for $d = 0.95$ revealed that there are no substantial differences between the three network types, neither with respect to the mean total number of defaults (for a more detailed analysis, see Section 7.2.4) nor with respect to the influence of the input parameter on this mean total number of defaults. Hence, the corresponding results will be described for the incomplete network only.
If the 250 firms are linked by cross-ownership of debt only, Figure 7.2(e) shows that, for the shock probability p_π fixed to its benchmark value 0.2 and a given value of the shock size π, a suitably high level of integration β can totally avert defaults caused by negative shocks, as long as π is not too large. For $\pi = 0.4$ (the benchmark value of π), $\beta \geq 0.5$ is sufficient, for example, see also Figure 7.2(c) and (f) where $\pi = 0.4$. Similarly, in Figure 7.2(a) with $\beta = 0.2$, defaults only occur for $\pi \geq 0.15$. Hence, for a given level of integration, only shock sizes above a certain threshold seem to lead to defaults in the system. The mean number of defaults then very much depends on the value of p_π (cf. Figure 7.2(a)–(c)). As it was to be expected, a higher probability of being hit by a negative shock raises the mean total number of defaults in the system. As becomes apparent in Figure 7.2(b), (d) and (f), diversification θ is irrelevant to the mean number of defaults. Altogether, our simulations suggest that firms linked by cross-ownership of debt only are – ceteris paribus – better protected against defaults under a high level of integration. It does not matter whether this is achieved by many small links or few links with a high weight.
As becomes clear in Figure 7.3, the mean number of defaults under cross-ownership of equity only is virtually entirely determined by p_π, whereas π, p_θ and β do not seem to have much influence, with the only exception that high levels of integration β can mitigate the effects of small shocks π, at least for $p_\pi = 0.2$. For $d < a$, the number

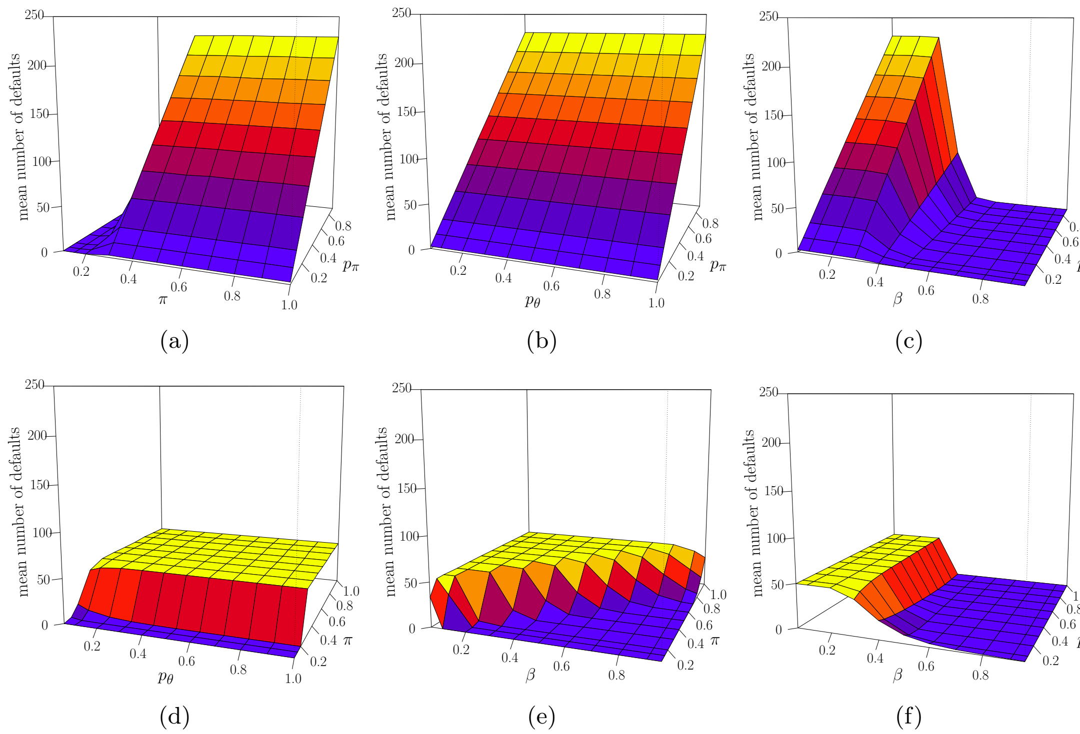


Figure 7.2: Mean number of defaults (taken over the $m_2 = 1{,}000$ repetitions) in dependence of input parameters under cross-ownership debt only (without $\pi = 0.05$); incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values.

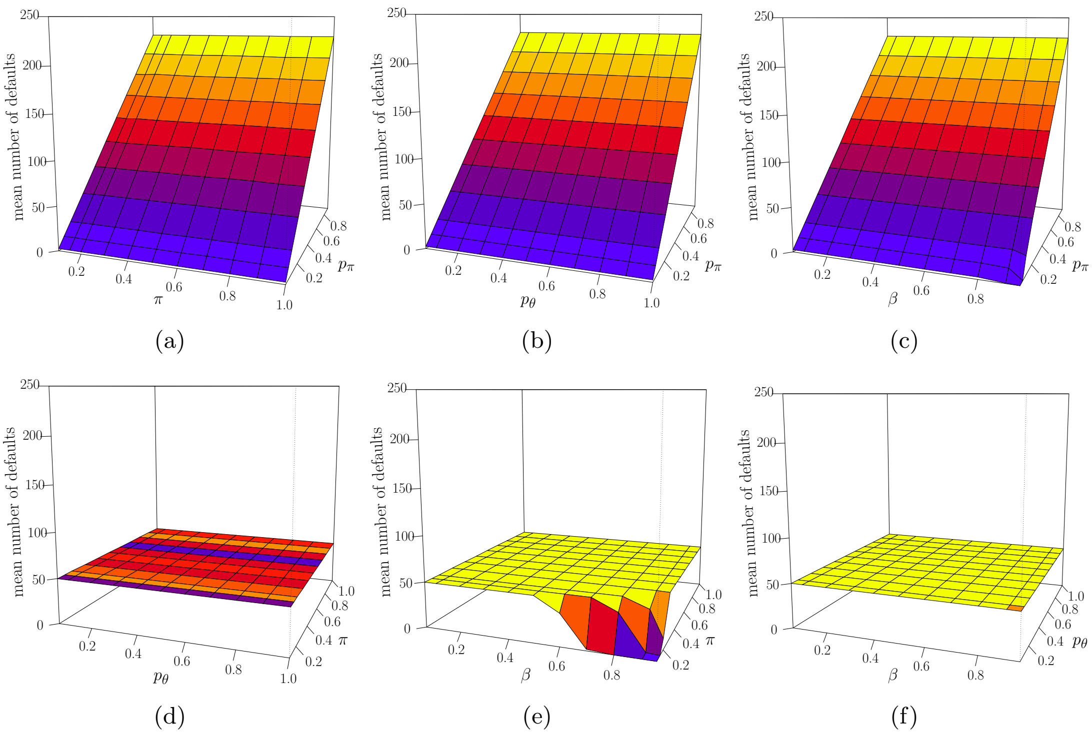


Figure 7.3: Mean number of defaults (taken over the $m_2 = 1{,}000$ repetitions) in dependence of input parameters under cross-ownership equity only (without $\pi = 0.05$); incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values.

of defaults is bounded from above by the number of firms hit by a shock, which has an expected value of $p_\pi \times 250$ in our set-up. In Figure 7.3, the actual mean numbers of defaults are quite in line with this expectation (except of Figure 7.3(e); recall that $p_\pi = 0.2$ in Figure 7.3(d)–(f)), i.e. nearly all the firms hit by a shock are in default after the shock. This means that firms linked by cross-ownership of equity only seem to have barely any option to reduce the mean number of default by a deliberately chosen cross-ownership structure that might absorb negative shocks (under the restriction that the same fraction of each firm's equity is held within the system). Only if the shocks are relatively small, a high level of integration seems to reduce the number of defaults in the system. However, further simulations are needed to examine this relationship for other values of p_π. Again, diversification θ does not seem to influence the occurrence of defaults (cf. Figure 7.3(b), (d) and (f)).
Figure 7.2 and Figure 7.3 give rise to the assumption that defaults are easier to prevent under cross-ownership of debt only than under cross-ownership of equity only. A possible explanation will be provided in Section 7.2.4.
For a system of firms linked by cross-ownership of both, debt and equity, the figures were qualitatively the same as for cross-ownership of debt only, hence the above insights under cross-ownership of debt only can be directly transferred to the case of cross-ownership of both, debt and equity. As to the relationship between the model parameters and the number of defaults, this might serve as a justification to ignore the additional presence of cross-ownership of equity (as it is done in most studies on banking systems), if for example reliable data on equity cross-holdings are not available. However, as we will see in Section 7.2.4, the mean numbers of default tend to be smaller under cross-ownership of both, debt and equity. Furthermore, recall that in our model, the matrices $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ are simulated with identical underlying parameters, i.e. other results might occur if this restriction is skipped.

As already mentioned, for $d = 0.95$, all conclusions derived for the incomplete network analogously hold for the core-periphery network and the ring network. Under cross-ownership of equity only they also hold if we analyze core firms and peripheral firms separately. However, we observe the following differences between these two subsystems for cross-ownership of debt only and cross-ownership of both, debt and equity. Generally, core firms are more resilient in the sense that ceteris paribus a lower level of integration is needed to avoid any default and higher shocks can be absorbed. The shape of the corresponding graphs is qualitatively the same as in Figure 7.2(a), (c) and (e). Furthermore, core firms can neutralize a high shock probability by holding a part of a sufficiently high number of other firms in the system (i.e. high level of $p_{\theta,C}$), whereas the mean number of defaults among peripheral firms does not vary with the number of links established by themselves (cf. Figure 7.4(a) and (d)), which admittedly run through a rather limited range (expected values of 27.67–47.03) in our simulations (recall that in our model, the higher the probability of a core firm having a directed link to another firm, the lower the corresponding probability of a peripheral firm, cf. (7.64) and (7.65)). The higher the possible shock itself, the more connections to (i.e. investments in) other firms are required for core firms to avoid default (cf. Figure 7.4(b)). Thus, for core firms, many

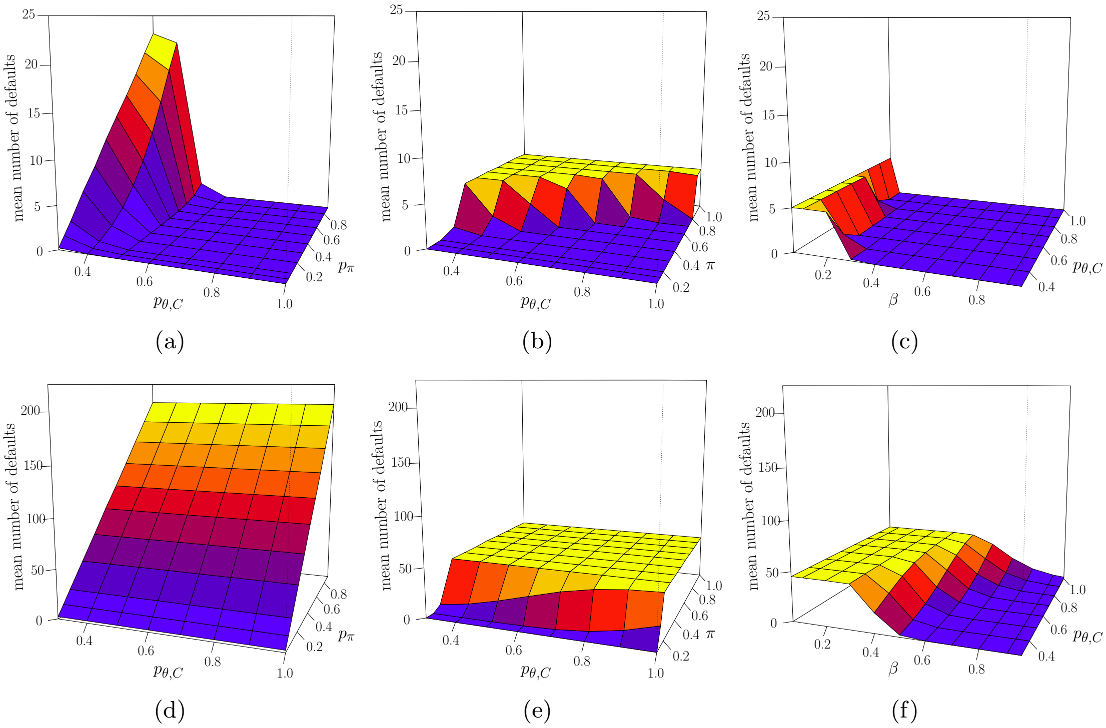


Figure 7.4: Mean number of defaults (taken over the $m_2 = 1{,}000$ repetitions) in dependence of $p_{\theta,C}$ under cross-ownership debt only (without $\pi = 0.05$); core-periphery network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values; (a)–(c) core firms; (d)–(f) peripheral firms.

links to other firms are generally advisable, whereas, depending on the shock parameters π and p_π, peripheral firms tend to suffer from high values of $p_{\theta,C}$ (cf. Figure 7.4(e) and (f)). Altogether, as to the mean number of defaults in relation to the size of the corresponding subgroup of firms, core firms are better off than peripheral firms. For both subsets of firms, a high level of integration is desirable under any type of cross-ownership.

For $d = 1.1$, the shapes of all figures were roughly comparable, but the mean numbers of defaults tended to be higher than $p_\pi \times 250$ (which means that probably also firms without a shock were in default after the shock) for large shocks of high probability in the incomplete and the core-periphery network. In the incomplete network, this was also the case for very low levels of diversification. In the ring network mean numbers of default bigger than $p_\pi \times 250$ occurred for very low levels of integration and/or big shocks. We will gain further insights into the occurrence of contagious defaults in dependence of π and β in Section 7.2.5.3.

Mean Proportion of Contagious Defaults

The mean proportion of contagious defaults, estimated by the mean of $\hat{k}$, strongly depends on the underlying network type and the type of cross-ownership, hence we will consider each of them separately. However, we skip the core-periphery network, since the results aggregated over all firms in the system often reflected its dichotomous structure, which makes a general description difficult. On the other hand, we refrained from conducting separate analyses of core firms and peripheral firms, since for core firms with their few defaults (if p_π is set to its benchmark value 0.2, together with the fact that we had 10% core firms, in such scenarios at most $0.2 \times 0.1 \times 250 = 5$ firms are in default), it was questionable whether we would obtain reliable results with respect to the occurrence of contagious defaults.

We begin with the incomplete network and $d = 0.95$. Under cross-ownership of debt only, contagious defaults occur with a maximal mean proportion of 7.0% of all defaults. Comparing Figure 7.5(a)–(c) and Figure 7.6(a)–(c) to Figure 7.2(c), (e) and (f), it becomes clear that contagious defaults especially occur in the transition area between the maximum possible number of defaults given as $p_\pi \times 250$ and no defaults. The same holds for cross-ownership of both, debt and equity. In the presence of cross-holdings of equity, our simulations suggest that contagious defaults are far more common with a maximal mean proportion of 100% of all defaults. As Figure 7.5(d)–(i) reveals, defaults under a certain combination of parameter values are mostly either solely fundamental defaults or solely contagious defaults. For $\beta = 0.2$ the mean proportions of contagious defaults were close to 0 under cross-ownership of either debt or equity with $d/a = 0.95$ (cf. Figure 7.6(a)–(f)) for any combination of parameters varied. Comparing Figure 7.7(a) and (b) to Figure 7.6(a) and (g) it becomes clear that for $d = 1.1$, contagious defaults additionally occur in scenarios with large shocks of high probability. Recall that for these parameter constellations the number of defaults was higher than the expected number of defaults of $p_\pi \times 250$. Altogether, we could conclude for the incomplete network that the occurrence of contagious defaults is not restricted to a certain level of integration and that under cross-ownership of debt only, contagious defaults are relatively rare. Under the ring network for $d = 0.95$, there were no contagious defaults under cross-ownership of debt only. For the other two types of cross-ownership, for any combination of parameter values we either had $\hat{k} = 0$ or $\hat{k} = 1$. However, it is not possible to derive a general rule as to whether contagious defaults occur or not. We can only state that in our set-up, contagious defaults only occurred for $\beta \geq 0.6$. In particular, there were no contagious defaults if β was set to its benchmark value 0.2. For p_π or π set to its benchmark value and the corresponding remaining parameters varied, the shape of the graphs were comparable to those of the incomplete network (cf. Figure 7.5(d), (e), (g) and (h)), except that only the values 0 and 1 were taken. For $d = 1.1$, $\hat{k}$ also took values between 0 and 1, and as it was to be expected, contagious defaults especially occurred for such parameter constellations where more than the expected number of firms were in default. As already mentioned, this was the case for very low levels of integration and/or big shocks.

7.2.3.3 Analysis of η

Since we consider non-positive shocks, we have $\tilde{v}_i \leq v_i$ for all $i \in \{1, \ldots, n\}$ by Proposition 2 of Gouriéroux et al. [2012] for any repetition within any network realization. Hence, within a repetition, $\eta = \frac{\sum_{i=1}^n \tilde{v}_i - \sum_{i=1}^n v_i}{\sum_{i=1}^n v_i} \leq 0$, with $\eta = 0$ indicating no losses in firm values. We start with some theoretical considerations on the random variable $H := \frac{\sum_{i=1}^n \tilde{V}_i - \sum_{i=1}^n v_i}{\sum_{i=1}^n v_i}$ (recall that in our set-up, $\sum_{i=1}^n v_i$ is deterministic by Remark 7.6) generating the values η. Note that randomness in $\tilde{V}_i$ is induced by both, the random network as well as the random shock. Similarly to (7.67) we obtain

$$\sum_{i=1}^n \tilde{V}_i = \sum_{i=1}^n \tilde{A}_i + \sum_{i=1}^n \beta_i^{\mathrm{d}} \tilde{R}_i + \sum_{i=1}^n \beta_i^{\mathrm{e}} \tilde{S}_i \tag{7.76}$$

$$= \begin{cases} \sum_{i=1}^n \tilde{A}_i + \beta \sum_{i=1}^n \tilde{R}_i, & \text{XOS of debt only,} \\ \sum_{i=1}^n \tilde{A}_i + \beta \sum_{i=1}^n (\tilde{V}_i - \tilde{R}_i) = \frac{\sum_{i=1}^n \tilde{A}_i - \beta \sum_{i=1}^n \tilde{R}_i}{1-\beta}, & \text{XOS of equity only,} \\ \sum_{i=1}^n \tilde{A}_i + \beta \sum_{i=1}^n (\tilde{R}_i + \tilde{S}_i) = \frac{\sum_{i=1}^n \tilde{A}_i}{1-\beta}, & \text{XOS of both, debt and equity,} \end{cases} \tag{7.77}$$

with $\tilde{R}_i$ and $\tilde{S}_i$ denoting the random recovery value of debt and the random equity value of firm i after the shock, where again randomness is induced by both, the random network, as well as the random shock. Combining (7.67) and (7.77) it follows that

$$H = \begin{cases} \frac{\sum_{i=1}^n (\tilde{A}_i - a_i) - \beta \sum_{i=1}^n (d_i - \tilde{R}_i)}{\sum_{i=1}^n a_i + \beta \sum_{i=1}^n d_i}, & \text{XOS of debt only,} \\ \frac{\sum_{i=1}^n (\tilde{A}_i - a_i) + \beta \sum_{i=1}^n (d_i - \tilde{R}_i)}{\sum_{i=1}^n a_i - \beta \sum_{i=1}^n d_i}, & \text{XOS of equity only,} \\ \frac{\sum_{i=1}^n (\tilde{A}_i - a_i)}{\sum_{i=1}^n a_i}, & \text{XOS of both, debt and equity,} \end{cases} \tag{7.78}$$

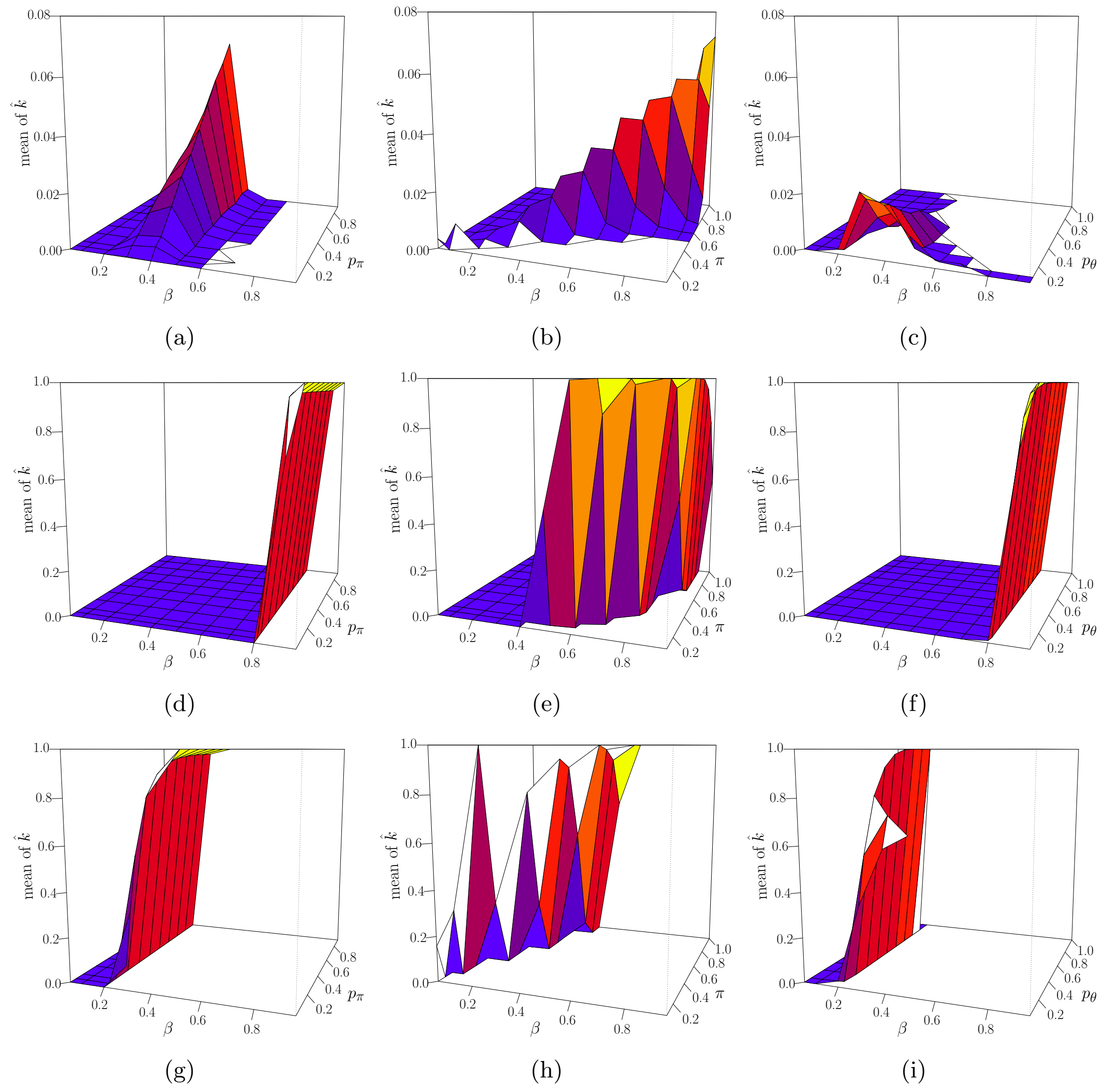


Figure 7.5: Mean of $\hat{k}$ (taken over the repetitions with at least one firm in default) in dependence of β (without $\pi = 0.05$); incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values; (a)–(c) XOS of debt only; (d)–(f) XOS of equity only; (g)–(i) XOS of both, debt and equity.

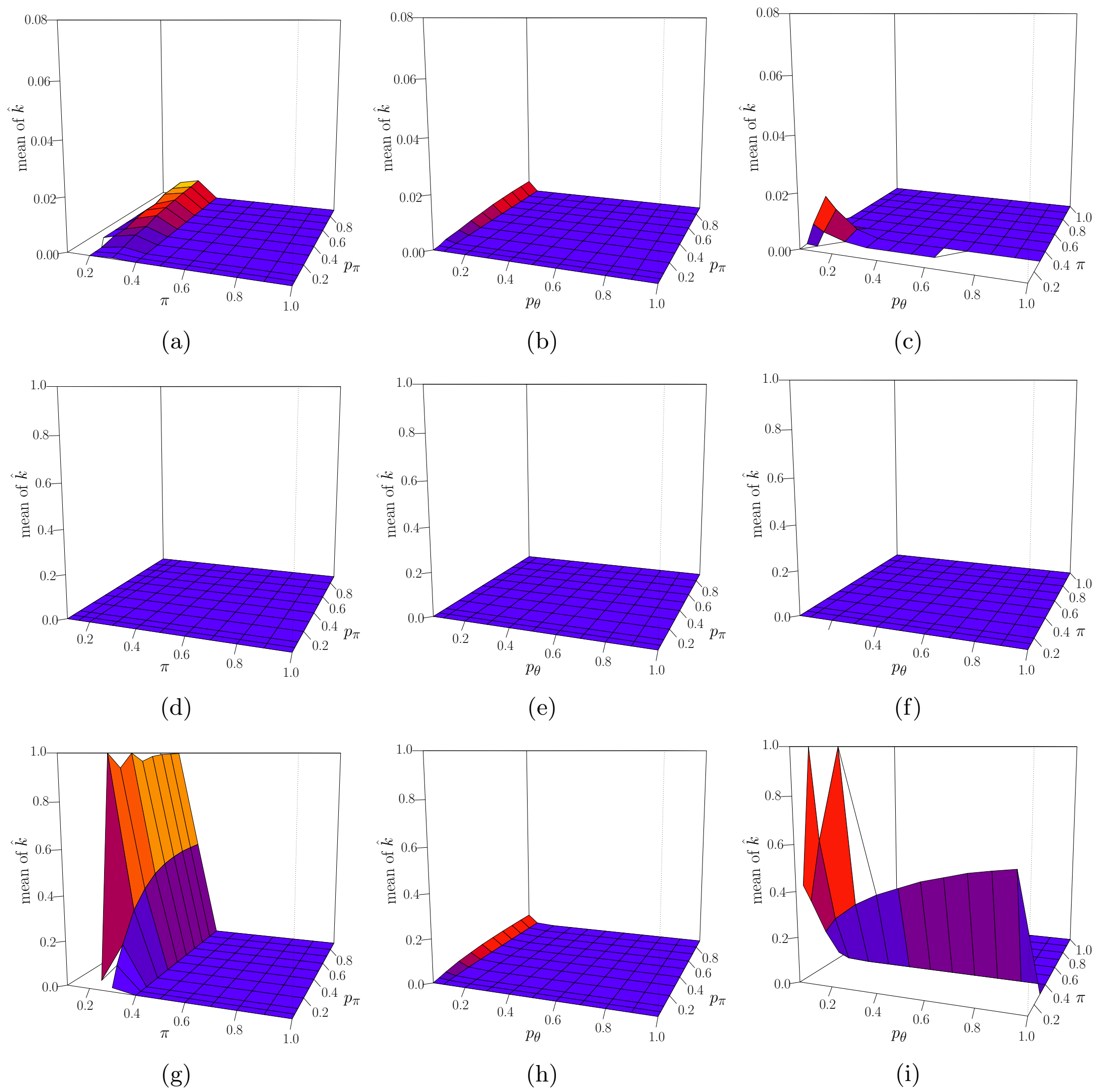
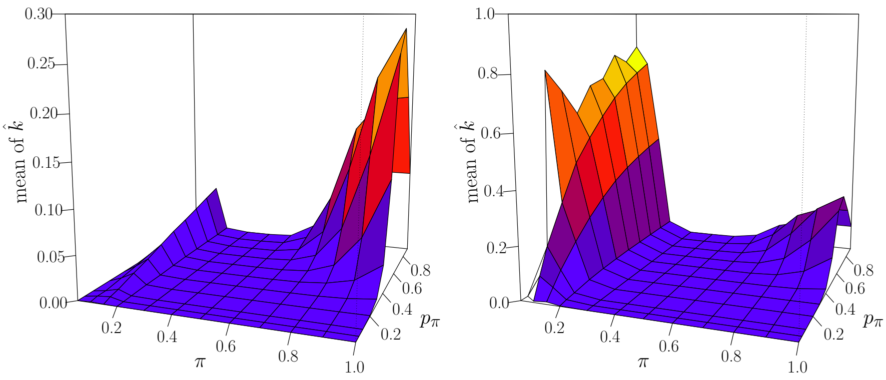


Figure 7.6: Mean of $\hat{k}$ (taken over the repetitions with at least one firm in default) in dependence of p_π, π (without $\pi = 0.05$) and p_θ; incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values; (a)–(c) XOS of debt only; (d)–(f) XOS of equity only; (g)–(i) XOS of both, debt and equity.



(a) (b)

Figure 7.7: Mean of $\hat{k}$ (taken over the repetitions with at least one firm in default) in dependence of p_π and π; incomplete network, $d/a = 1.1$, respective non-varied parameters equal to their benchmark values; (a) XOS of debt only; (b) XOS of both, debt and equity.

i.e. under cross-ownership of both, debt and equity, the relative total loss in firm values equals the relative total loss in exogenous asset values, provided that all column sums of $\mathbf{M}^{\mathrm{d}}$ and $\mathbf{M}^{\mathrm{e}}$ coincide. Due to

$$E(\tilde{A}_i) = (1 - \pi\, E(B_i))a_i = (1 - \pi\, p_\pi)a_i \tag{7.79}$$

by (7.68), we obtain

$$E(H) = \begin{cases} \frac{-\pi\, p_\pi \sum_{i=1}^n a_i - \beta \sum_{i=1}^n (d_i - E(\tilde{R}_i))}{\sum_{i=1}^n a_i + \beta \sum_{i=1}^n d_i}, & \text{XOS of debt only,} \\ \frac{-\pi\, p_\pi \sum_{i=1}^n a_i + \beta \sum_{i=1}^n (d_i - E(\tilde{R}_i))}{\sum_{i=1}^n a_i - \beta \sum_{i=1}^n d_i}, & \text{XOS of equity only,} \\ -\pi\, p_\pi, & \text{XOS of both, debt and equity.} \end{cases} \tag{7.80}$$

Hence, under cross-ownership of both, debt and equity, for any network type and any level of liabilities, the expected relative total loss in firm values equals the expected percentage loss of an individual firm's exogenous asset value, and it is strictly increasing in the shock size π and the shock probability p_π. In particular, the expected relative total loss in firm values does not depend on the levels of integration and diversification. By Proposition 2 of Gouriéroux et al. [2012] $E(\tilde{R}_i)$ is non-increasing in π and p_π, which implies that the expected relative total loss in firm values is non-increasing in π and p_π under cross-ownership of debt only as well. However, all these considerations strongly rely on the fact that all column sums of the corresponding cross-ownership matrix/matrices coincide. Further relationships become apparent in our simulations.

For both, cross-ownership of debt only and cross-ownership of equity only, the results

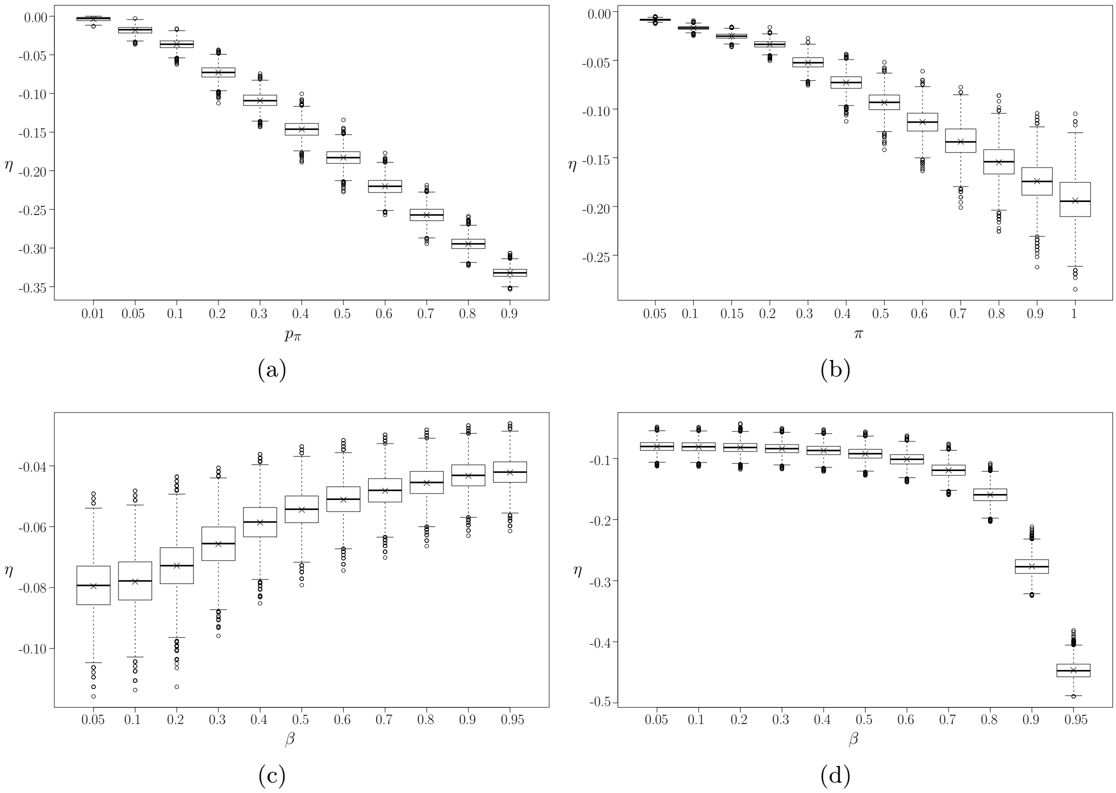


Figure 7.8: η in dependence of p_π, π and β; incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values; $\times$ indicate mean values; (a)–(c) XOS of debt only; (d) XOS of equity only.

on η did not differ much between $d/a = 0.95$ and $d/a = 1.1$, if applicable. As it was to be expected, η was strictly decreasing in p_π and π for all the three network types under cross-ownership of debt only (cf. Figure 7.8(a) and (b) for the incomplete network). We observed the same relationship under cross-ownership of equity only for any network type. Furthermore, neither p_θ in the incomplete network nor $p_{\theta,C}$ in the core-periphery network seem to affect η. For any network type, the influence of integration β depends on the considered type of cross-ownership as follows. If the n firms are linked by cross-ownership of debt only (cf. Figure 7.8(c) for the incomplete network), η was the bigger (i.e. the smaller the relative total loss) the higher the level of integration. Hence, firms can protect themselves against high relative losses due to negative shocks on the system by establishing a system where the majority of each firm's debt is held within the system. Under cross-ownership of equity only, we had the opposite effect, cf. Figure 7.8(d) for the incomplete network, i.e. high levels of integration foster high relative losses. However, we could not find a theoretical or an intuitive explanation for this difference between the two types of cross-ownership. In the core-periphery network the above insights hold likewise if we analyze core firms and peripheral firms separately.

As already mentioned, the results for η transfer to the estimated values of $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.99}$. Since both, $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.99}$ refer to losses in firm values, whereas η refers to gains in firm values, all relationships between the input parameters and η must be reversed to obtain the influence on the estimates of $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.99}$, but they lead to the same conclusions.

7.2.3.4 Analysis of ξ

Let $\Xi := \frac{\sum_{i=1}^n C_i}{\sum_{i=1}^n v_i}$ with $C_i := \tilde{V}_i - V_i - (\tilde{A}_i - a_i)$ (cf. (7.7)–(7.9)) denote the random variable generating the values ξ defined in (7.73). Then $\Xi = H - \frac{\sum_{i=1}^n (\tilde{A}_i - a_i)}{\sum_{i=1}^n v_i}$ and thus, by (7.67), (7.79) and (7.80),

$$E(\Xi) = \begin{cases} E(H) + \pi\, p_\pi \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i + \beta \sum_{i=1}^n d_i}, & \text{XOS of debt only,} \\ E(H) + (1-\beta)\, \pi\, p_\pi \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i - \beta \sum_{i=1}^n d_i}, & \text{XOS of equity only,} \\ -\beta\, \pi\, p_\pi, & \text{XOS of both, debt and equity,} \end{cases} \tag{7.81}$$

under the assumption that all n or $2n$ columns of the corresponding cross-ownership matrices exhibit the same column sum. Thus, under cross-ownership of both, debt and equity and for any network type and any level of liabilities, the expected relative total change in endogenous asset values becomes smaller (i.e. more negative), if the level of integration and/or the size of the shocks and/or the probability of a firm being hit by the shock increases, but it is invariant under changes in p_θ or $p_{\theta,C}$.

For cross-ownership of debt only and cross-ownership of equity only under any network type, our simulations revealed that again p_θ and $p_{\theta,C}$ do not affect the mean value

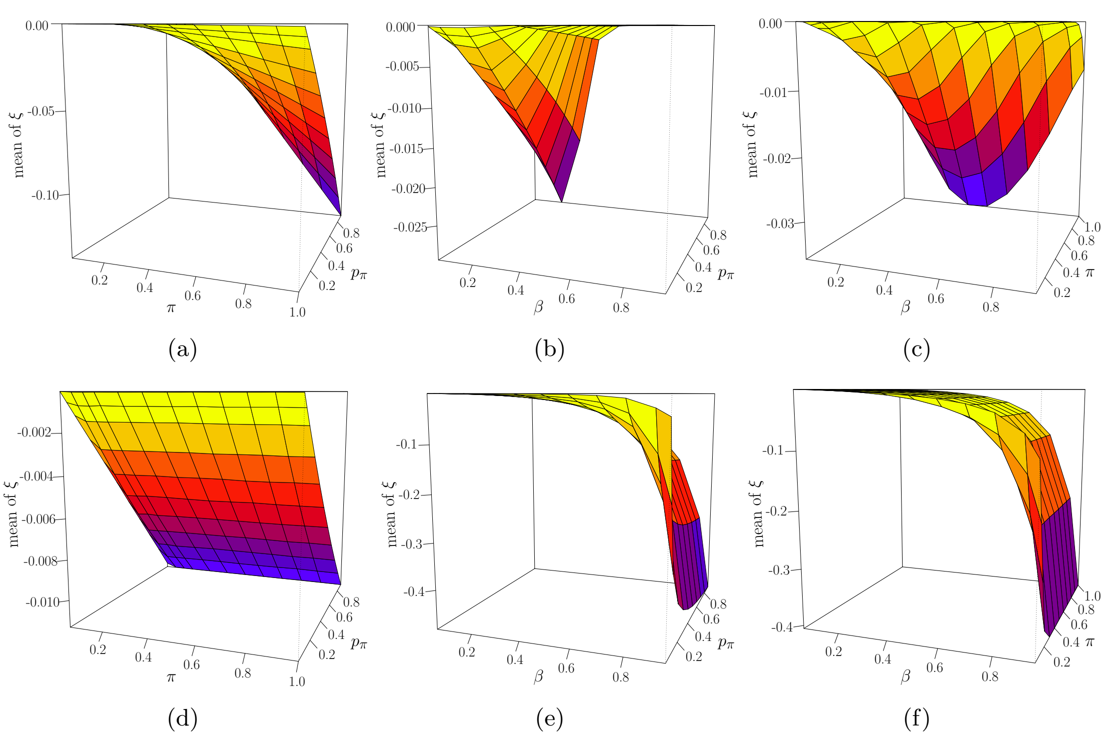


Figure 7.9: Mean of ξ (taken over the $m_2 = 1{,}000$ repetitions) in dependence of p_π, π and β; incomplete network, $d/a = 0.95$, respective non-varied parameters equal to their benchmark values; (a)–(c) XOS of debt only; (d)–(f) XOS of equity only.

of ξ. This was to be expected from (7.81) and the simulation results on η. Under cross-ownership of debt only, the second summand of (7.81) is strictly decreasing in β, whereas $E(H)$ seems to be increasing in β (cf. Figure 7.8(c)). This might explain the u-shaped course with respect to β in Figure 7.9(b) and (c). Hence, for any level of p_π and π, relatively low and relatively high levels of integration β could mitigate the negative effects of p_π and π, whereas the biggest mean relative loss in endogenous asset values occurs for medium levels of integration. If $d < a$, straightforward calculations show that under cross-ownership of equity only the derivative of the second summand of (7.81) with respect to integration β is non-positive. Hence, together with the fact that η seems to decrease in β, we have explained the shape of Figure 7.9(e) and (f) with respect to β. Similarly to η, a higher level of integration results in larger mean relative losses in endogenous asset values. Somewhat surprisingly, compared to β, p_π and π are of very limited influence only (cf. Figure 7.9(e) and (f)). For $d/a = 1.1$, the graphs were comparable to those for $d/a = 0.95$ given in Figure 7.9.
These insights gained for the incomplete network likewise hold for the ring network and the core-periphery network. In the latter, if we additionally distinguish between core and peripheral firms, the results were qualitatively the same, although the theoretical considerations on $E(H)$ and $\sum_{i=1}^n V_i$ (cf. Remark 7.6) do not hold for subsystems of firms. Discrepancies to the above insights only occurred with respect to the influence of $p_{\theta,C}$, the probability of a core firm being linked to some other firm in the system. If $p_{\theta,C}$ was varied, for most parameter combinations the mean relative total loss in endogenous asset values was non-increasing in $p_{\theta,C}$ for core firms and non-decreasing in $p_{\theta,C}$ (i.e. non-increasing in $p_{\theta,P}$) for peripheral firms. Hence, with respect to ξ, roughly speaking, both, core firms and peripheral firms should seek not to interlink with too many other firms in the system, provided that the total level of diversification and all other parameters remain fixed. Furthermore, with p_π or π increasing, core firms face a higher mean relative total loss in endogenous asset values than peripheral firms. Of course, core firms tend to have higher endogenous assets values than peripheral firms, but on the other hand, they also have a higher firm value, so the result is not trivial.

7.2.4 Summary and Conclusions

Based on the results of our simulation study described in the previous section, we will summarize our findings with respect to the different output parameters and derive recommendations as to the network architecture that should be aspired, comprising the network type, the levels of integration and diversification, and as to the most favourable type of cross-ownership. Recall that all our results rely on the assumption that the column sums of the corresponding cross-ownership matrix/matrices are identical.

Within a certain type of cross-ownership, there are nearly no differences with respect to the mean values of the considered output parameters between the three network types (cf. Table 7.1) for $d = 0.95$, especially if we take the relatively large standard deviations into account. Note that in Table 7.1, we excluded all scenarios where $p_\theta \neq 0.2$ in the incomplete network, as this parameter is constant in the ring network and core-periphery

parameter	network type	XOS of debt only	XOS of equity only	XOS of both, debt and equity
defaults	incomplete	44.9 ± 66.5	78.7 ± 66.5	40.5 ± 63.7
	core-periphery	48.5 ± 69.6	73.9 ± 73.2	43.3 ± 70.2
	ring	43.1 ± 64.0	79.6 ± 67.8	39.6 ± 63.6
$\hat{k}$	incomplete	0.005 ± 0.019	0.096 ± 0.288	0.123 ± 0.292
	core-periphery	0.008 ± 0.021	0.074 ± 0.193	0.091 ± 0.230
	ring	0 ± 0	0.109 ± 0.311	0.133 ± 0.339
η	incomplete	-0.125 ± 0.141	-0.187 ± 0.169	-0.149 ± 0.148
	core-periphery	-0.117 ± 0.149	-0.168 ± 0.172	-0.136 ± 0.157
	ring	-0.127 ± 0.145	-0.191 ± 0.173	-0.152 ± 0.152
ξ	incomplete	-0.010 ± 0.021	-0.048 ± 0.100	-0.054 ± 0.062
	core-periphery	-0.010 ± 0.021	-0.040 ± 0.080	-0.045 ± 0.060
	ring	-0.011 ± 0.021	-0.050 ± 0.101	-0.056 ± 0.064

Table 7.1: Mean values and standard deviations of the number of defaults, $\hat{k}$, η and ξ for all network types and all types of cross-ownership; $d/a = 0.95$, incomplete network with scenarios with $p_\theta = 0.2$ only.

network (1/249 resp. 0.2). Altogether, we cannot conclude for our parameter values that some network type is preferable with respect to either the number and type of defaults or the impact of the shock on mean relative losses in firm values and endogenous assets values[13]. Furthermore, ceteris paribus also the influence of the input parameters on the output parameters was roughly the same between the three network types, except of the proportion of contagious defaults.

Comparing the types of cross-ownership, it becomes clear from Table 7.1, Figure 7.2 and Figure 7.3 that under cross-ownership of equity only, firms are more prone to defaults (i.e. shocks are less well absorbed) than under the other two types of cross-ownership. In order to derive a possible explanation, we assume that $\mathbf{M}^{\mathrm{d}} = \mathbf{M}^{\mathrm{e}} =: \mathbf{M}$ and that all firms are solvent before the shock, i.e. by Remark 7.5, we assume $\sum_{i=1}^n a_i \geq \sum_{i=1}^n d_i$, which reduces to $a \geq d$ in our set-up. Straightforward calculations based on (7.67) show that before the shock, the sum of firm values under cross-ownership of equity only is smaller than the sum of firm values under cross-ownership of debt only, if and only if

$$\beta < 2 - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n d_i}. \tag{7.82}$$

Hence, in our set-up with $d = 0.95a$, the latter inequality holds for all $\beta < 2 - 1/0.95 \approx 0.947$, i.e. for all considered values of β except of $\beta = 0.95$. For $\beta \leq 0.9$, cross-ownership of debt only yields a higher sum of firm values before the shock. Since the construction of the incomplete network (however, not the related realizations) exhibits symmetry

[13] However, such a comparison has to be interpreted with caution, as only systems that can be transformed into each other in a self-financing way can actually be directly compared.

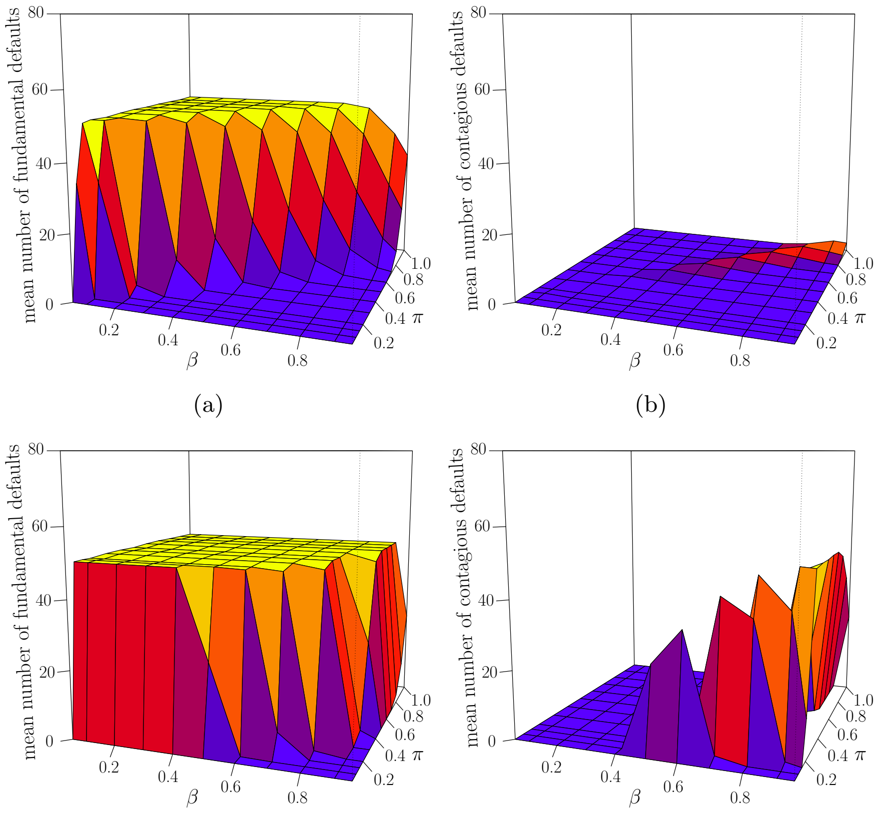


Figure 7.10: Mean numbers of fundamental and contagious defaults (taken over the $m_2 =$ 1,000 repetitions) in dependence of π and β; incomplete network, $d/a =$ 0.95, respective non-varied parameters equal to their benchmark values; (a)–(b) XOS of debt only; (c)–(d) XOS of equity only.

between the n firms, this gives rise to the supposition that also individual firm values v_i tend to be higher under cross-ownership of debt only than under cross-ownership of equity only, if integration β meets (7.82). In this case, as fundamental defaults occur if and only if $v_i - a_i + \tilde{A}_i < d_i$ (this holds because we consider non-positive shocks, cf. the derivation of (7.55)–(7.56)), ceteris paribus fundamental defaults are more likely under cross-ownership of equity only than under cross-ownership of debt only. This is confirmed by Figure 7.10(a) and (c), showing fundamental defaults in dependence of π and β under both types of cross-ownership. This gap is further increased by the occurrence of contagious defaults as follows. Under cross-ownership of equity only, since a decline of the exogenous asset values of initially healthy firms directly reduces the equity values of these firms (no matter whether they default, or not), other firms holding a part of these equity values experience a drop in their total asset values and thus in their equity values. Depending on the entries of $\mathbf{M}^{\mathrm{e}}$, this in turn might reduce the

equity values of the triggering firms even more, i.e. under cross-ownership of equity only, this indirect effect can finally drive the affected firms into default, even if they could have borne the initial shock. This is exactly what we defined as a contagious default in Definition 7.4. In contrast to that, under cross-ownership of debt only, firms that remain solvent despite a shock on their exogenous assets do not cause a loss in other firms' endogenous asset values, which suggests that contagious defaults are rarer under cross-ownership of debt only (cf. Figure 7.10(b) and (d)). Altogether, this might explain the fact that we observed fewer defaults under cross-ownership of debt only. In addition, under cross-ownership of debt only also the mean proportion of contagious defaults to total defaults was much smaller (max. 7.0%) than under cross-ownership of equity only (up to 100%).
Since the presence of cross-ownership of both, debt and equity, leads to higher firm values than cross-ownership of one type only, it is clear that under the former, we observe fewer defaults than under the latter. This is also confirmed by the mean number of defaults in Table 7.1. However, cross-ownership of both, debt and equity, provides two possible channels of contagion, leading to the biggest mean relative loss in endogenous asset values and the biggest proportion of contagious defaults (cf. Table 7.1). In contrast to that, the biggest relative total losses in firm values occurred for cross-ownership of equity only, i.e. compared to the two other types of cross-ownership, firm values under cross-ownership of equity only tend to be smallest also after the shock, which corresponds to the fact that we observed the highest mean number of defaults under this type of cross-ownership.

Under cross-ownership of debt only, as we have seen in Figure 7.2(c), (e) and (f), Figure 7.4(c) and (f), Figure 7.8(c) and Figure 7.9(b) and (c), a high level of integration β is generally advisable, with the minimum suitable value of β depending on the shock size π (cf. Figure 7.2(e)), and in the core-periphery network, on $p_{\theta,C}$ (cf. Figure 7.4(f)). As we have seen above, it does not matter which network type is realized, and diversification is irrelevant in the incomplete network (recall that diversification is constant in the other network types). In contrast to that, a low level of integration is recommended under cross-ownership of equity only in order to mitigate the financial losses in firm values and endogenous asset values due to the shock (cf. Figure 7.8(d) and Figure 7.9(e) and (f)). Hence, as to a suitable level integration, cross-ownership of debt only and cross-ownership of equity only once more have opposite effects. However, as it was the case under cross-ownership of debt only, defaults under cross-ownership of equity only are most likely avoided for high levels of integration (cf. Figure 7.3(e)), if at all, although high levels of integration lead to the biggest relative losses in firm values. This can be explained by Remark 7.6, where we saw that under cross-ownership of equity only, firm values before the shock are strictly increasing in β, i.e. for high levels of β, firm values before the shock are relatively big and thus the firms might withstand a relatively large percentual loss in firm values due to the shock.

Let us now imagine an investor having to decide in which of several systems of firms to invest a certain amount of money in form of a combined index of debt and equity of the firms within a system (we identify the value of this index with $\sum_{i=1}^{n} v_i$). We assume that we are very close to maturity and that some immediate negative shock on the exogenous assets might hit each system of firms. Then our simulations suggest that a system of firms linked by cross-ownership of debt only is preferable in that negative shocks lead to a moderate number of defaults and in particular to very few (unanticipated) contagious defaults (cf. Section 7.1.3) and the smallest mean relative financial losses. Furthermore, the system should exhibit a high level of integration.

7.2.5 Comparison to the Literature

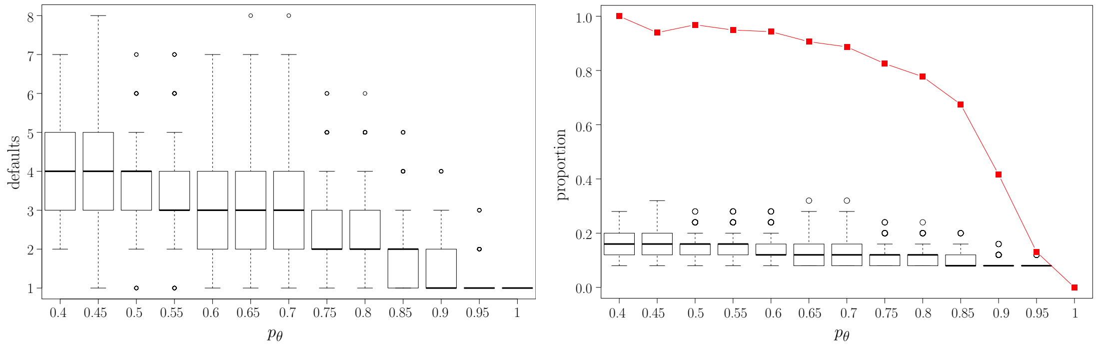
7.2.5.1 Influence of the Network Parameters

Since Nier et al. [2007] and Gai and Kapadia [2010] consider banks linked by cross-ownership of debt only, we will only refer to this type of cross-ownership in the following comparison.

By Remark 7.6, in our model the parameters bank capitalization γ and interbank exposures ζ are strictly monotone transformations of β, i.e. in our model, γ and ζ cannot be varied one at a time, as it is done by Nier et al. [2007]. There (cf. Figure 1 and Figure 2 therein), γ and ζ have a different impact on the number of defaults in that high values of γ lead to fewer defaults than low values, whereas high values of ζ increase the number of defaults. Our results on the influence of β under cross-ownership of debt only resemble their findings on γ (cf. Figure 7.2(c), (e) and (f)) in that high levels of integration and thus high ratios of equity values to firm values before the shock can prevent any default in the system. In contrast to Nier et al. [2007], we can hardly detect any relationship between diversification θ, or equivalently p_θ, and the number of defaults in the incomplete network. According to Figure 3 of Nier et al. [2007], in their model the strength of this relationship depends on the realized level of γ, and as our levels of γ (0.09–0.5) were considerably higher than theirs (0–0.1), we conducted a short additional simulation under cross-ownership of debt only with the following parameter values. With γ of Nier et al. [2007] varying between 0.01 and 0.07, as it is done in their Figure 3, our β and d/a change only slightly (cf. Section 7.2.2.1), so we set $\beta = 0.2$ and $d/a = 1.2$ and varied p_θ between 0.05 and 1. In order to get valid scenarios for small values of p_θ, we set $m = 1{,}000$ and $m_1 = 5,000$ (cf. Section 7.2.2.1). Nevertheless, we obtained no valid scenarios for $p_\theta \leq 0.3$ and only two valid scenarios for $p_\theta = 0.35$, which is why we skipped $p_\theta = 0.35$ in Figure 7.11(a)[14]. For better comparability we applied an idiosyncratic shock randomly hitting one of $n = 25$ firms with shock size $\pi = 1$. As becomes clear in Figure 7.11(a), the obtained numbers of default depend on diversification θ, but since we could not generate valid scenarios for smaller levels of diversification, we can only partly confirm the roughly inverted u-shaped relationship found by Nier et al. [2007]. Recall that in their method of network construction (cf. Section 7.2.1), all firms are automatically solvent before the shock.

Also Gai and Kapadia [2010] report an inverted u-shaped relationship between p_θ and the probability of a cascade of defaults comprising more than 5% of all banks after

[14] For p_θ equal to 0.4, 0.45 and 0.5, the obtained number of valid scenarios were 16, 99 and 413, respectively. For $p_\theta \geq 0.55$, 1,000 valid scenarios were available.



(a) (b)

Figure 7.11: Occurrence of defaults in dependence of p_θ under cross-ownership debt only with idiosyncratic shocks; incomplete network, $\pi = 1$, $\beta = 0.2$, $d/a = 1.2$, $n = 25$; (a) number of defaults; (b) red: proportion of scenarios with contagious defaults, boxplots: proportion of firms in default in scenarios with contagious defaults.

an idiosyncratic shock on a single firm. As already mentioned the model of Gai and Kapadia [2010] is rather similar to the model of Nier et al. [2007], so we compare the results of the above additional simulations to the results of Gai and Kapadia [2010] as well. As becomes clear from the red line in Figure 7.11(b), the probability of occurrence of at least one contagious default decreases with p_θ increasing. Note that due to $n = 25$ in the simulations underlying Figure 7.11, more than 5% defaults is equivalent to more than one default, which in turn is equivalent to the existence of at least one contagious default as we consider an idiosyncratic shock and cross-ownership of debt only. Since we did not obtain valid scenarios for smaller values of p_θ, we can only partly confirm the results of Gai and Kapadia [2010] with respect to the occurrence of a cascade of defaults. Furthermore, in our simulations at most 8 of the 25 firms in the system were in default in case a cascade occurred, whereas Gai and Kapadia [2010] report that with p_θ increasing, up to 100% of all firms are in default. This might be explained by the fact that in their model (and in contrast to ours and to the model of Nier et al. [2007]), the recovery value of debt of any defaulting bank is set to zero, which makes a far-spreading cascade of defaults more likely. Altogether, the results of our additional simulations on the incomplete network under idiosyncratic shocks do not support the "robust-yet-fragile tendency" of financial systems found by Gai and Kapadia [2010], as the occurrence of contagious defaults does not necessarily imply that all firms are simultaneously in default. However, this changes if we (theoretically) analyze complete instead of incomplete networks, provided that integration β is small enough (cf. Section 7.2.5.3).

In the simulations of Elliott et al. [2014] for an incomplete network exposed to an idiosyncratic shock, a cascade of defaults most likely occurs for medium levels of integration in conjunction with medium levels of diversification. Again, we tried to come closer to their set-up by implementing a further short simulation for the incomplete network under cross-ownership of both, debt and equity with identical cross-ownership matrices of debt and equity, since in the work of Elliott et al. [2014], banks are connected by holding a part of the other banks' values (recall that $v = r + s$). Furthermore, we set $n = 100$ and $d = 1.2$, and considered an idiosyncratic shock completely wiping out the exogenous asset of a firm. Nevertheless, this did not alter our findings that under cross-ownership of both, debt and equity, the mean numbers of default seem to decrease with integration β increasing and that diversification θ has minor influence only. However, it should be noted that it was impossible to generate solvent networks for low levels of integration in conjunction with low levels of diversification, whereas this can be easily achieved in the model of Elliott et al. [2014] by choosing the failure threshold on firm values sufficiently small. Hence, as in the comparison with the results of Nier et al. [2007], we have possibly seen the falling part of the inverted u-shaped relationship between integration and the numbers of default claimed by Elliott et al. [2014]. Furthermore, as Elliott et al. [2014] incorporate failure costs that reduce a firm's value to 0 in case of default, in their model a cascade of defaults is generally more likely to occur than in ours.

7.2.5.2 Fundamental and Contagious Defaults

Although the set-up of Elsinger et al. [2006a] differs from our approach as they consider a given network of cross-holdings of debt (determined by the structure of the Austrian interbank market) and a different mechanism generating exogenous shocks[15], our results under cross-ownership of debt only coincide with their long run results (zero bankruptcy costs) in that in scenarios with a higher number of fundamental defaults, contagious defaults occur with a higher probability than in scenarios with a low number of fundamental defaults, cf. Figure 7.12(a) with the sizes of the categories in Table 4 of Elsinger et al. [2006a] adapted to our network size. Under non-positive shocks, firm i is in a fundamental default if and only if $\tilde{A}_i + \bar{a}_i < d_i$ (cf. the derivation of (7.55)–(7.56)), i.e. if a firm has discerned a high danger of a fundamental default by considering endogenous asset values as fixed, it should be aware that especially for higher ratios of debt to exogenous asset values (empty squares in Figure 7.12(a)), the probability of at least one contagious default in the system might not be negligible under cross-ownership of debt only, which implies that the actual probability of default of a firm might be substantially higher than expected. However, this relationship is reverse under cross-ownership of equity only (cf. Figure 7.12(b)). As the probability of at least one contagious default is highest for scenarios with few fundamental defaults, it is especially important to be aware of the possibility of contagious defaults in such scenarios, since otherwise, firms might arrive at the conclusion not to be threatened by the impact of negative shocks at all. Altogether, we can "confirm the intuition that contagion is relatively more likely in scenarios where many banks face fundamental default simultaneously" [Elsinger et al.,

[15] Elsinger et al. [2006a] simulate shocks following a multivariate distribution derived from historical data on both, market risk and credit risk. In particular, the underlying parameters remain fixed for the whole simulation study. Although it is not explicitly mentioned, their Figure 4 gives rise to the assumption that shocks can be positive and negative.

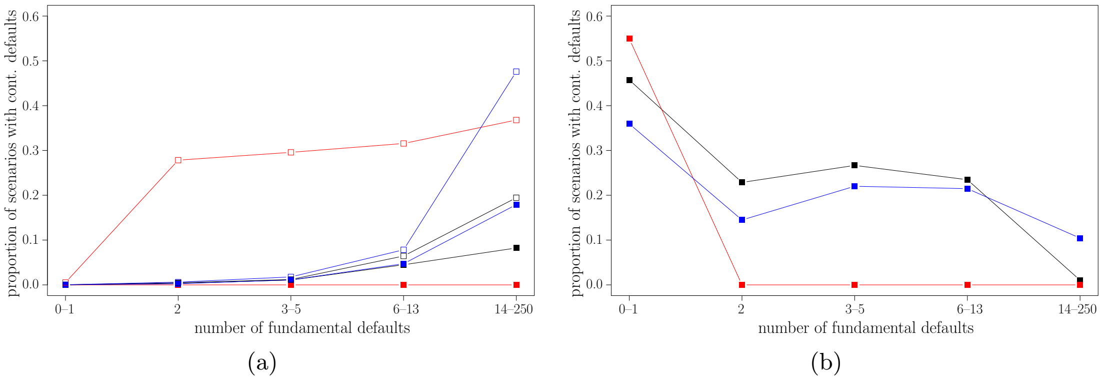


Figure 7.12: Proportion of scenarios with contagious defaults grouped by fundamental defaults; black: incomplete network, blue: core-periphery network, red: ring network; solid squares: $d/a = 0.95$, empty squares: $d/a = 1.1$; (a) XOS of debt only; (b) XOS of equity only.

2006a, p. 1310] only for cross-ownership of debt only.

7.2.5.3 Comparison of the Complete and the Ring Network

In their seminal paper, Allen and Gale [2000] examine a network consisting of four identical banks where each bank makes deposits in $t = 0$ in other banks to protect itself against liquidity shocks in $t = 1$. Then there are parameter constellations where all firms are simultaneously bankrupt (i.e. even the premature liquidation of long-term assets does not provide enough liquidity in the system) in the ring network, whereas the same parameter constellation in the complete network renders the firms not directly hit by the liquidity shocks insolvent (here, in departure from footnote 1, the term insolvency means that a partial premature liquidation of the long-term asset yields enough liquidity), but not bankrupt. In this sense, the complete network of Allen and Gale [2000] is more robust with respect to liquidity shocks on a single bank than the ring network. In the following we will compare the resilience of the two network types to an idiosyncratic shock in our model. For this, we examine the occurrence of fundamental defaults and contagious defaults in dependence of the shock size π and the level of integration β (as in our simulations, we assume that all column sums of $\mathbf{M}^{\mathrm{d}}$ are equal to β) under cross-ownership of debt only and under the assumption that the network exhibits symmetry between the n firms in terms of a and d, i.e. $a_i = a$ and $d_i = d$ for all $i \in \{1, \dots, n\}$ as our simulations. After that, we will see from our simulations whether the results might be transferred to multiple shocks.
In our theoretical analysis we suppose w.l.o.g. that firm 1 is hit by the shock. Furthermore, we assume $n > 2$ since for $n = 2$, the ring network and the complete network coincide. In the complete network, $M^{\mathrm{d}}_{ij} = \frac{\beta}{n-1}$ for all $i \neq j \in \{1, \dots, n\}$, and in the ring network, the non-zero elements of $\mathbf{M}^{\mathrm{d}}$ equal β. Let all the firms be strictly solvent

before the shock. Then $d = r_i = \min\{d_i,\, a_i + \sum_{j=1}^n M^{\mathrm{d}}_{ij} d_j\} = \min\{d,\, a + (n-1)\frac{\beta}{n-1} d\} = \min\{d,\, a + \beta d\}$ in the complete network, and $d = r_i = \min\{d,\, a + \beta d\}$ also in the ring network, i.e. this assumption is equivalent to $a > (1-\beta)d$[16].
Let $\tilde{a}_1$ denote the value of firm 1's exogenous asset after the shock, i.e. $\tilde{a}_1 = (1-\pi)a$. If $\tilde{a}_1 = (1-\beta)d$, the other firms are still strictly solvent, and for reasons of continuity, also for $\tilde{a}_1$ slightly smaller than this value, i.e. firm 1 is the first firm to default in both network types. As long as firm 1 is the only firm in default, its recovery value of debt equals $\tilde{r}_1 = \tilde{a}_1 + \beta d$ in both network types.
In the complete network the recovery values of debt of the remaining firms are identical for reasons of symmetry, and as long as firm 1 is the only firm in default they equal

$$d = \tilde{r}_j = \min\left\{d,\, a + (n-2)\frac{\beta}{n-1}d + \frac{\beta}{n-1}\tilde{r}_1\right\} \tag{7.83}$$

$$= \min\left\{d,\, a + (n-2)\frac{\beta}{n-1}d + \frac{\beta}{n-1}(\tilde{a}_1 + \beta d)\right\} \tag{7.84}$$

$$= \min\left\{d,\, a + \frac{\beta}{n-1}\tilde{a}_1 + \beta\frac{n-2+\beta}{n-1}d\right\}, \quad j \in \{2, \ldots, n\}. \tag{7.85}$$

Hence, in the complete network, firm j ($j \in \{2, \ldots, n\}$) stays solvent as long as

$$\tilde{a}_1 \geq \frac{(1-\beta)(n-1+\beta)}{\beta}d - \frac{n-1}{\beta}a. \tag{7.86}$$

Otherwise, all the n firms are simultaneously in default. Note that the RHS of (7.86) is strictly smaller than $(1-\beta)d$, since we assume $a > (1-\beta)d$. Thus, in the complete network with a given shock size π,

$$\begin{cases} \text{all firms are solvent,} & \text{if } \tilde{a}_1 \geq (1-\beta)d, \\ \text{only firm 1 is in default,} & \text{if } \tilde{a}_1 \in \left[\frac{(1-\beta)(n-1+\beta)}{\beta}d - \frac{n-1}{\beta}a, (1-\beta)d\right), \\ \text{all firms are in default,} & \text{if } \tilde{a}_1 < \frac{(1-\beta)(n-1+\beta)}{\beta}d - \frac{n-1}{\beta}a. \end{cases} \tag{7.87}$$

Since in our set-up the default of firm 1 is always fundamental, this means that if contagious defaults occur, they pertain to the whole system. Note that the bound on $\tilde{a}_1$ given in (7.86) might be negative (a sufficient condition is $d < a$, since in this case, firms not hit by the shock will always stay solvent), i.e. in this case, it is impossible that a second firm and thus the complete system goes bankrupt due to a shock on a single firm.
In the ring network we assume w.l.o.g. that firm $k+1$ holds a part of firm k's debt ($k \in \{1, \ldots, n-1\}$) and that firm 1 hold a part of firm n's debt. Since the recovery value of debt of firm 1 equals $\tilde{r}_1 = \tilde{a}_1 + \beta d$ as long as firm 1 is the only firm in default, the recovery value of debt of firm 2 then equals $d = \tilde{r}_2 = \min\{d,\, a + \beta(\tilde{a}_1 + \beta d)\}$ since

[16] If all firms are borderline firms before the shock, i.e. $a = (1-\beta)d$, any value of $\tilde{a}_1 < (1-\beta)d$ will drive all firms in the system into default, since the value of endogenous assets of any firm will decrease, no matter whether we consider the complete or the ring network.

due to $n \geq 3$, firm 1 does not hold firm 2's debt. Hence, in the ring network, firm 2 remains solvent as long as

$$\tilde{a}_1 \geq \frac{(1-\beta^2)d - a}{\beta}. \tag{7.88}$$

The RHS of (7.88) is strictly smaller than $(1-\beta)d$ since we assume $a > (1-\beta)d$, and it can be both, positive and negative. If $\tilde{a}_1$ falls below this bound, we have at least one contagious default. Again, $d < a$ is sufficient to prevent any contagious default. If we assume $d > a$, straightforward calculations show that the RHS of (7.86) and of (7.88) are strictly decreasing in β, i.e. the higher the level of integration, the smaller the bounds on $\tilde{a}_1$ and thus the fewer contagious defaults occur in both networks. For β sufficiently big, any contagious defaults can be prevented. A high level of β also exacerbates the occurrence of a default of the firm hit by the shock.

As we have seen, the bound on $\tilde{a}_1$ for the first default (which is fundamental) is identical between the two network types and equals $(1-\beta)d$. In contrast to that, comparing the bounds on $\tilde{a}_1$ for the first contagious default given in (7.86) and (7.88) leads to

$$\frac{(1-\beta)(n-1+\beta)}{\beta}d - \frac{n-1}{\beta}a < \frac{(1-\beta^2)d - a}{\beta} \tag{7.89}$$

$$\Leftrightarrow \qquad (1-\beta)d < a, \tag{7.90}$$

where the last inequality holds because we assume all firms to be strictly solvent before the shock, i.e. in the ring network, the first contagious default already occurs for bigger values of $\tilde{a}_1$ than in the complete network (if at all), if we assume symmetry between the n firms with respect to a and d. Hence, in line with Allen and Gale [2000], the ring network is more susceptible to contagious defaults than the complete network in the sense that smaller shocks are sufficient to cause contagious defaults. A similar result is provided in Corollary 2 of Eboli [2013], whose analysis is based on flow networks instead of a clearing algorithm.

In the complete network, each of the remaining firms in the system carries a part (a fraction of size $\beta/(n-1)$) of the decline in the recovery value of debt of the firm hit by the shock, and if the shock is small, the firms not hit by the shock can withstand this relatively small loss in their endogenous asset value and remain solvent. In contrast to that, in the ring network the loss in firm 1's recovery value of debt is completely transferred to a single other firm, which of course bears a higher risk that this receiving firm will also default, than in the complete network. If this firm defaults as well, the corresponding loss is again transferred to a single firm, which might also default. Hence, similarly to the results of Eboli [2013], we have a sequence of defaults in the ring network, i.e. with $\tilde{a}_1$ decreasing, the firms default one by one (if at all), but it is also possible that this chain ends before all firms are in default, whereas in the complete network, all the remaining firms default simultaneously as soon as $\tilde{a}_1$ falls below the bound given in (7.86). In the literature, this property that in complete networks shocks either lead to a default of the shocked firm(s) only or to a default of all firms in the system is often described as "robust yet fragile" (cf. Gai and Kapadia [2010], Eboli [2013], Acemoglu et al. [2015], and Staum [2013] and references therein, for example). Or, to put it in the words of Haldane [2009], picked up by Eboli [2013], interconnected networks "exhibit a knife-edge or tipping point property", in the sense that "within a certain range, connections serve as shock-absorbers [and] connectivity engenders robustness" [Haldane, 2009, p. 5]. However, as Acemoglu et al. [2015] remark, "beyond that range, interconnections start to serve as a mechanism for the propagation of shocks" [Acemoglu et al., 2015, p. 566]. Which network type leads to more expected defaults in total depends on the distribution of $\tilde{A}_1$. For multiple shocks with $\tilde{\mathbf{A}}$ as in (7.68), our simulation results will provide further insight.

Remark 7.7. The complete and the ring network are special cases of what we will call a balanced incomplete network, i.e. an incomplete network where every firm has exactly θ cross-holders and holds a part of θ other firms itself, $\theta \in \{1, \dots, n-1\}$[17]. Hence, every non-zero entry of $\mathbf{M}^{\mathrm{d}}$ equals β/θ. Then, for firm 1 hit by a small shock leaving the remaining firms solvent, $\tilde{r}_1 = \min\{d,\, \tilde{a}_1 + \theta\,\beta/\theta\, d\} = \min\{d,\, \tilde{a}_1 + \beta d\}$ as for the complete and for the ring network, i.e. again, firm 1 is in default as soon as $\tilde{a}_1 < (1-\beta)d$. W.l.o.g. let firm $2, 3, \dots, \theta+1$ hold a part of firm 1's debt. For $\tilde{a}_1$ slightly smaller than $(1-\beta)d$, all firms except of firm 1 are still solvent. In particular, we have for firm $2, 3, \dots, \theta+1$,

$$d = \tilde{r}_j = \min\left\{d,\, a + \frac{\beta}{\theta}\tilde{r}_1 + (\theta - 1)\frac{\beta}{\theta}d\right\} \tag{7.91}$$

$$= \min\left\{d,\, a + \frac{\beta}{\theta}(\tilde{a}_1 + \beta d) + (\theta - 1)\frac{\beta}{\theta}d\right\}, \quad j \in \{2, \dots, \theta+1\}. \tag{7.92}$$

Of course, firms holding a part of firm 1 will default earlier (i.e. already for smaller shocks) than firms not holding a part of firm 1, since under cross-ownership of debt only, a decline in a firm's endogenous asset value influences the financial health of other firms only if this firm is actually in default. Hence, firms $2, 3, \dots, \theta - 1$ are simultaneously in default as soon as

$$\tilde{a}_1 < \frac{(1-\beta)(\theta+\beta)}{\beta}d - \frac{\theta}{\beta}a, \tag{7.93}$$

and it becomes clear that (7.86) and (7.88) are special cases of (7.93) with $\theta = n-1$ and $\theta = 1$, respectively. Again, contagious defaults are impossible if $d < a$. The derivative of the RHS of (7.93) with respect to θ is given as $((1-\beta)d - a)/\beta < 0$, i.e. the bound on $\tilde{a}_1$ is strictly decreasing in θ. Hence, the more diversification, the bigger idiosyncratic shocks can be withstood in the sense that no contagious defaults occur, provided that the incomplete network is balanced in the above sense. Furthermore, the derivative of the RHS of (7.93) with respect to β equals $\theta(a-d)/\beta^2 - d$, which is negative if $d > a$, i.e. in this case, a higher level of integration exacerbates the occurrence of contagious defaults. Furthermore, for β sufficiently big, any contagious default can be prevented.

[17] A similar approach is employed by May and Arinaminpathy [2010] to evaluate the simulation results of Nier et al. [2007] and Gai and Kapadia [2010].

For the complete network under cross-ownership of debt only with symmetry between the n firms in terms of a and d, the above results remain valid if we consider multiple identical shocks instead of idiosyncratic shocks (except that (7.86) is replaced by a different formula), i.e. if the shock size is small, no firm is in default, for medium shocks only the shocked firms are in default and if the shock size exceeds a certain threshold, all firms are simultaneously in default. A similar result is derived in Theorem 5 of Eboli [2013]. In contrast to that, in the ring network and in a balanced incomplete network the occurrence of defaults under multiple shocks depends on the exact "chain" or structure of cross-holdings, so we do not derive a general formula. Instead, recall that for $p_\theta = 1$ our incomplete network is actually complete, so we use our simulation results on the complete network and the ring network to compare these network types with respect to the mean number of defaults and the occurrence of fundamental defaults and contagious defaults under multiple shocks in dependence of π and β[18]. Since for $d < a$ only the shocked firms might default, we analyze the results for $d = 1.1$. As becomes clear in Table 7.2, the ring network is slightly more susceptible to defaults than the complete network. This coincides with the theoretical results of Acemoglu et al. [2015], who show for a system of banks linked by cross-ownership of debt only in an economy lasting for three dates that the complete network is more stable with respect to the mean number of defaults than the ring network, provided that the shock is small. This holds for both, idiosyncratic and multiple shocks.
In line with our theoretical results on π and β for idiosyncratic shocks, the thresholds as to the occurrence of fundamental defaults under multiple shocks coincide between the two network types for both, π and β (cf. Table 7.2). Furthermore, as for idiosyncratic shocks, with π increasing, the ring network is more prone to contagious defaults than the complete network, in the sense that in the former, contagious defaults already occur for smaller values of π. However, in line with our theoretical results, if contagious defaults occurred in the complete network in our simulations, all firms were simultaneously in default. Already a relatively small level of integration prevented any contagious default in either network type. Hence, our simulations suggest the assumption that our theoretical results on idiosyncratic shocks can be directly transferred to multiple shocks.
Altogether, in accordance to the literature, our theoretical analysis on idiosyncratic shocks and our simulations on multiple shocks under cross-ownership of debt only show that, depending on the shock size, a different network structure is preferable with respect to the occurrence of defaults.

For a similar analysis under cross-ownership of equity only we need to assume $d \leq a$ in order to have a solvent system before the shock (cf. Remark 7.5). However, in this case only the firm hit by the shock might default, i.e. a cascade through the whole system is impossible under an idiosyncratic shock, i.e. the only interesting question is for which network type this default occurs earlier (i.e. for smaller shocks). For the

[18] For that, since in our set-up the restriction $p_\theta = 1$ in the complete network and variation of π or β means that p_π always equals its benchmark value 0.2 (cf. Section 7.2.2.1), we confine ourselves to scenarios of the ring network with $p_\pi = 0.2$ for better comparability.

parameter varied	network type	number of defaults	fundamental defaults	contagious defaults
π	complete	41.72 ± 19.54	iff $\pi \geq 0.15$	–
	ring	52.34 ± 29.47	iff $\pi \geq 0.15$	iff $\pi \geq 0.7$
β	complete	20.24 ± 26.41	iff $\beta \leq 0.4$	if $\beta \leq 0.1$
	ring	23.95 ± 31.67	iff $\beta \leq 0.4$	iff $\beta \leq 0.1$

Table 7.2: Mean values and standard deviations of the number of defaults and values of π and β leading to fundamental and contagious defaults under multiple shocks; $p_\pi = 0.2$, the respective non-varied parameter β or π equals its benchmark value; $d/a = 1.1$; "iff" stands for "if and only if".

sake of completeness the corresponding analysis can be found in Section A.8. There, we see that in contrast to systems of firms linked by cross-ownership of debt only, under cross-ownership of equity only the two networks types differ in that a default of the shocked firm already occurs for smaller shocks in the ring network than in the complete network. Hence, under cross-ownership of equity only and a symmetrical firm structure, the complete network is more resilient to idiosyncratic shocks than the ring network, independently of the realized level of integration β. Under multiple shocks however, our simulations reveal that the two network types do not differ in the mean numbers of default (complete: 45.93 ± 15.11, ring: 45.77 ± 15.06), and for both network types, defaults occurred if and only if $\pi \geq 0.1$. These numbers were calculated in complete analogy to Table 7.2, except that $d = 0.95$.

8 Final Remarks

For systems of $n \geq 2$ firms linked by mutual cross-holdings of debt and/or equity, our simulations as well as our theoretical analysis show that it is crucial to take the cross-ownership structure between firms correctly into account when it comes to firm valuation, instead of applying Merton's model to each firm separately. Otherwise, as we have seen in Section 3.2.2 and Section 4.1 for the two firms case, the probability of default of a firm might be grossly misestimated, with the direction of the effect depending on the realized type of cross-ownership. Roughly speaking, the lognormal model tends to overestimate resp. underestimate the actual probabilities of default under cross-ownership of debt only resp. cross-ownership of equity only, provided that the cross-ownership structure is sufficiently tight. This holds for both, univariate and bivariate probabilities of default. Under cross-ownership of both, debt and equity, either effect may be dominant, depending on the detailed structure of cross-holdings.

As becomes clear in Proposition 5.10 and Proposition 5.11, financial interconnectedness can alter tail dependence of firm values from tail independence to perfect tail dependence, at least under the assumption of bivariate lognormally distributed exogenous assets values. Again, cross-ownership of debt only and cross-ownership of equity only have opposed effects in a certain sense. Under cross-ownership of debt only, firm values remain upper tail independent, whereas they become perfectly lower tail dependent if the correlation between exogenous asset values exceeds a certain positive threshold, which does not depend on the exact level of cross-ownership. Under cross-ownership of equity only, the situation is reverse in that firm values always remain lower tail independent, but upper tail independence is preserved if and only if the right tail behaviour of both firms' values is determined by the right tail behaviour of the firms' own exogenous asset value instead of the respective other firm's exogenous asset value. Hence, it is crucial that the cross-ownership structure and the parameter values of the distribution of exogenous asset values are carefully analyzed, because otherwise the presence of perfect tail dependence of firm values might be overlooked, which might have serious consequences in terms of risk management. Consider for example a portfolio consisting of two indices representing the values of two firms linked by cross-ownership of debt only, where extreme losses in one firm's value might go hand in hand with extreme losses of the other firm's value. As a by-product valid beyond the context of cross-ownership, our analysis yields the lower and upper tail dependence coefficient of two portfolios built from the same two lognormally distributed securities (cf. Section 5.5.3).

A firm linked to other firms by cross-ownership is not only threatened by sudden changes in its own exogenous asset values, but may suffer from losses in its endogenous asset value caused by shocks on the other firms' exogenous assets. As our analysis shows, this effect of contagion can be positive as well as negative, i.e. it can both, mitigate and exacerbate

the loss in the firm's value. Both, the recovery value of debt and the equity value of any firm in the system are non-decreasing in any cross-ownership fraction in the model by Proposition 6.1, but we cannot generally say that a tighter cross-ownership structure leads to bigger absolute contagion effects.
In our simulation study on contagion, simultaneous shocks on the system caused the smallest number of defaults under cross-ownership of debt only in conjunction with a high level of integration, whereas the level of diversification and the realized network type were irrelevant to the number of defaults. However, a system of firms where each firm holds a fraction of every other firm's debt is "robust yet fragile" [Gai and Kapadia, 2010, p. 2403], meaning that as soon as the size of an idiosyncratic shock exceeds a certain threshold, the whole system tips from being completely solvent to being completely in default. Cross-ownership of debt only also leads to the lowest proportion of contagious defaults among all defaults. Contagious defaults can be interpreted as defaults that are not foreseeable if the cross-ownership structure is not properly taken into account in the valuation process (cf. Section 7.1.3), i.e. in this respect one could arrive at the conclusion that the neglect of financial interconnectedness in the valuation procedure is less severe under cross-ownership of debt only than under other types of cross-ownership.

As cross-ownership is actually present in the world's financial markets, our findings suggest that this phenomenon should not be ignored by both, academics and practitioners, even though our considerations are based on a relatively simple model of financial interconnectedness, imposed with the assumption of lognormally distributed exogenous asset values in a great part of the analysis. Sometimes (cf. Section 3.2, Section 4.1 and Section 5) we had to limit ourselves to the two firms case in order to be able to derive theoretical results. This raises the question if similar results hold for the n firms case and/or under weaker distributional assumptions. Apart from that, future research could aim at extending the analysis to systems of firms with multiple levels of seniorities, as they are considered by Elsinger [2009] and Fischer [2014]. Furthermore, one could allow debt of differing seniority. To this end, the algorithm provided by Pokutta et al. [2011] could serve as a starting point. However, both approaches strongly increase the number of parameters related to the network of cross-holdings, which impede the derivation of both, theoretical and simulation results.

Appendix

A.1 Bivariate Distribution of Firm Values under Cross-Ownership

The following two lemmas deal with the bivariate distribution of firm values under cross-ownership of debt only and cross-ownership of equity only. We do not assume that (A_1, A_2) follows a particular distribution.

Lemma A.1. *Let* V_1^{d} *and* V_2^{d} *be defined as in* (3.33) *and Remark 3.6. If we assume* $(A_1, A_2) \gg \mathbf{0}$ $P-$*a.s. with continuous bivariate distribution, then*

$$P(V_1^{\mathrm{d}} \le v_1, V_2^{\mathrm{d}} \le v_2)$$
$$= \begin{cases} P(A_1 \le (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})v_1 - M_{1,2}^{\mathrm{d}} A_2, \\ \quad A_2 \le (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})v_2 - M_{2,1}^{\mathrm{d}} A_1), & v_1 \le d_1, v_2 \le d_2, \\ P(A_1 \le \max\{v_1 - M_{1,2}^{\mathrm{d}} d_2,\, (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})v_1 - M_{1,2}^{\mathrm{d}} A_2\}, \\ \quad A_2 \le v_2 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_2 - M_{2,1}^{\mathrm{d}} A_1), & v_1 \le d_1, v_2 > d_2, \\ P(A_1 \le v_1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_1 - M_{1,2}^{\mathrm{d}} A_2, \\ \quad A_2 \le \max\{v_2 - M_{2,1}^{\mathrm{d}} d_1,\, (1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}})v_2 - M_{2,1}^{\mathrm{d}} A_1\}), & v_1 > d_1, v_2 \le d_2, \\ P(A_1 \le \max\{v_1 - M_{1,2}^{\mathrm{d}} d_2,\, v_1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_1 - M_{1,2}^{\mathrm{d}} A_2\}, \\ \quad A_2 \le \max\{v_2 - M_{2,1}^{\mathrm{d}} d_1,\, v_2 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_2 - M_{2,1}^{\mathrm{d}} A_1\}), & v_1 > d_1, v_2 > d_2. \end{cases} \tag{A1}$$

Furthermore, if $v_1 > d_1$ *and* $v_2 > d_2$*,*

$$\begin{aligned} P(V_1^{\mathrm{d}} \le v_1, V_2^{\mathrm{d}} \le v_2) &= P(A_1 \le v_1 - M_{1,2}^{\mathrm{d}} d_2,\, A_2 \le v_2 - M_{2,1}^{\mathrm{d}} d_1) \\ &\quad + P(v_1 - M_{1,2}^{\mathrm{d}} d_2 < A_1 \le v_1 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_1 - M_{1,2}^{\mathrm{d}} A_2) \\ &\quad + P(v_2 - M_{2,1}^{\mathrm{d}} d_1 < A_2 \le v_2 - M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}} d_2 - M_{2,1}^{\mathrm{d}} A_1). \end{aligned} \tag{A2}$$

Proof. By (3.33) and Remark 3.6,

$$\begin{aligned} &P(V_1^{\mathrm{d}} \le v_1, V_2^{\mathrm{d}} \le v_2) \\ &= P(V_1^{\mathrm{d}} \le v_1,\, V_2^{\mathrm{d}} \le v_2,\, A_{\mathrm{ss}}) + P(V_1^{\mathrm{d}} \le v_1,\, V_2^{\mathrm{d}} \le v_2,\, A_{\mathrm{sd}}) \\ &\quad + P(V_1^{\mathrm{d}} \le v_1,\, V_2^{\mathrm{d}} \le v_2,\, A_{\mathrm{ds}}) + P(V_1^{\mathrm{d}} \le v_1,\, V_2^{\mathrm{d}} \le v_2,\, A_{\mathrm{dd}}) \end{aligned} \tag{A3}$$

$$
\begin{aligned}
&= P(A_1 \le v_1 - M^{\mathrm{d}}_{1,2}d_2,\, A_1 \ge d_1 - M^{\mathrm{d}}_{1,2}d_2, \\
&\qquad A_2 \le v_2 - M^{\mathrm{d}}_{2,1}d_1,\, A_2 \ge d_2 - M^{\mathrm{d}}_{2,1}d_1) \qquad \text{(A4)}\\
&\quad + P(A_1 \le v_1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_1 - M^{\mathrm{d}}_{1,2}A_2,\, A_1 \ge (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_1 - M^{\mathrm{d}}_{1,2}A_2, \\
&\qquad A_2 \le v_2 - M^{\mathrm{d}}_{2,1}d_1,\, A_2 < d_2 - M^{\mathrm{d}}_{2,1}d_1) \qquad \text{(A5)}\\
&\quad + P(A_1 \le v_1 - M^{\mathrm{d}}_{1,2}d_2,\, A_1 < d_1 - M^{\mathrm{d}}_{1,2}d_2, \\
&\qquad A_2 \le v_2 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_2 - M^{\mathrm{d}}_{2,1}A_1,\, A_2 \ge (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2 - M^{\mathrm{d}}_{2,1}A_1) \qquad \text{(A6)}\\
&\quad + P(A_1 \le (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_1 - M^{\mathrm{d}}_{1,2}A_2,\, A_1 < (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_1 - M^{\mathrm{d}}_{1,2}A_2, \\
&\qquad A_2 \le (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_2 - M^{\mathrm{d}}_{2,1}A_1,\, A_2 < (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2 - M^{\mathrm{d}}_{2,1}A_1). \qquad \text{(A7)}
\end{aligned}
$$

Let $v_1 \le d_1$ and $v_2 \le d_2$. Then the probabilities in (A4)–(A6) vanish since we assume the distribution of (A_1, A_2) to be continuous, and the probability in (A7) reduces to $P(A_1 \le (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_1 - M^{\mathrm{d}}_{1,2}A_2,\, A_2 \le (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_2 - M^{\mathrm{d}}_{2,1}A_1)$, which shows the assertion for $v_1 \le d_1$ and $v_2 \le d_2$. If $v_1 \le d_1$ and $v_2 > d_2$, the probabilities in (A4) and (A5) vanish and we obtain (cf. Figure A.1(b))

$$
\begin{aligned}
&P(V^{\mathrm{d}}_1 \le v_1, V^{\mathrm{d}}_2 \le v_2) \\
&= P(A_1 \le v_1 - M^{\mathrm{d}}_{1,2}d_2, \\
&\qquad A_2 \le v_2 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_2 - M^{\mathrm{d}}_{2,1}A_1,\, A_2 \ge (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2 - M^{\mathrm{d}}_{2,1}A_1) \qquad \text{(A8)}\\
&\quad + P(A_1 \le (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_1 - M^{\mathrm{d}}_{1,2}A_2,\, A_2 < (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})d_2 - M^{\mathrm{d}}_{2,1}A_1) \\
&= P(A_1 \le \max\{v_1 - M^{\mathrm{d}}_{1,2}d_2,\, (1 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})v_1 - M^{\mathrm{d}}_{1,2}A_2\}, \\
&\qquad A_2 \le v_2 - M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}d_2 - M^{\mathrm{d}}_{2,1}A_1). \qquad \text{(A9)}
\end{aligned}
$$

Similarly, one can show the assertions for $v_1 > d_1$ and $v_2 \le d_2$ (cf. Figure A.1(c)), and for $v_1 > d_1$ and $v_2 > d_2$ (cf. Figure A.1(d)). Also (A2) follows from Figure A.1(d).
Note that if $M^{\mathrm{d}}_{2,1}d_1 \ge d_2$ or $M^{\mathrm{d}}_{1,2}d_2 \ge d_1$, the probabilities in (A5) resp. (A6) are always 0 (cf. Figure 7.1(b) and (c) for the related Suzuki areas). Hence, also the second resp. third probability in (A2) is zero. □

Lemma A.2. *Let V^{e}_1 and V^{e}_2 be defined as in* (3.34) *and Remark 3.6. If we assume $(A_1, A_2) \gg \mathbf{0}$ $P-$a.s. with continuous bivariate distribution, then*

$$
\begin{aligned}
&P(V^{\mathrm{e}}_1 \le v_1,\, V^{\mathrm{e}}_2 \le v_2) \\
&= \begin{cases}
P(A_1 \le v_1,\, A_2 \le v_2), & v_1 \le d_1, v_2 \le d_2, \\
P(A_1 \le \min\{v_1, v_1 + M^{\mathrm{e}}_{1,2}(d_2 - A_2)\},\, A_2 \le v_2), & v_1 \le d_1, v_2 > d_2, \\
P(A_1 \le v_1,\, A_2 \le \min\{v_2,\, v_2 + M^{\mathrm{e}}_{2,1}(d_1 - A_1)\}), & v_1 > d_1, v_2 \le d_2, \\
P(A_1 \le \min\{v_1,\, v_1 + M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}(d_1 - v_1) + M^{\mathrm{e}}_{1,2}(d_2 - A_2)\}, & \\
\quad A_2 \le \min\{v_2,\, v_2 + M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}(d_2 - v_2) + M^{\mathrm{e}}_{2,1}(d_1 - A_1)\}), & v_1 > d_1, v_2 > d_2.
\end{cases}
\end{aligned} \qquad \text{(A10)}
$$

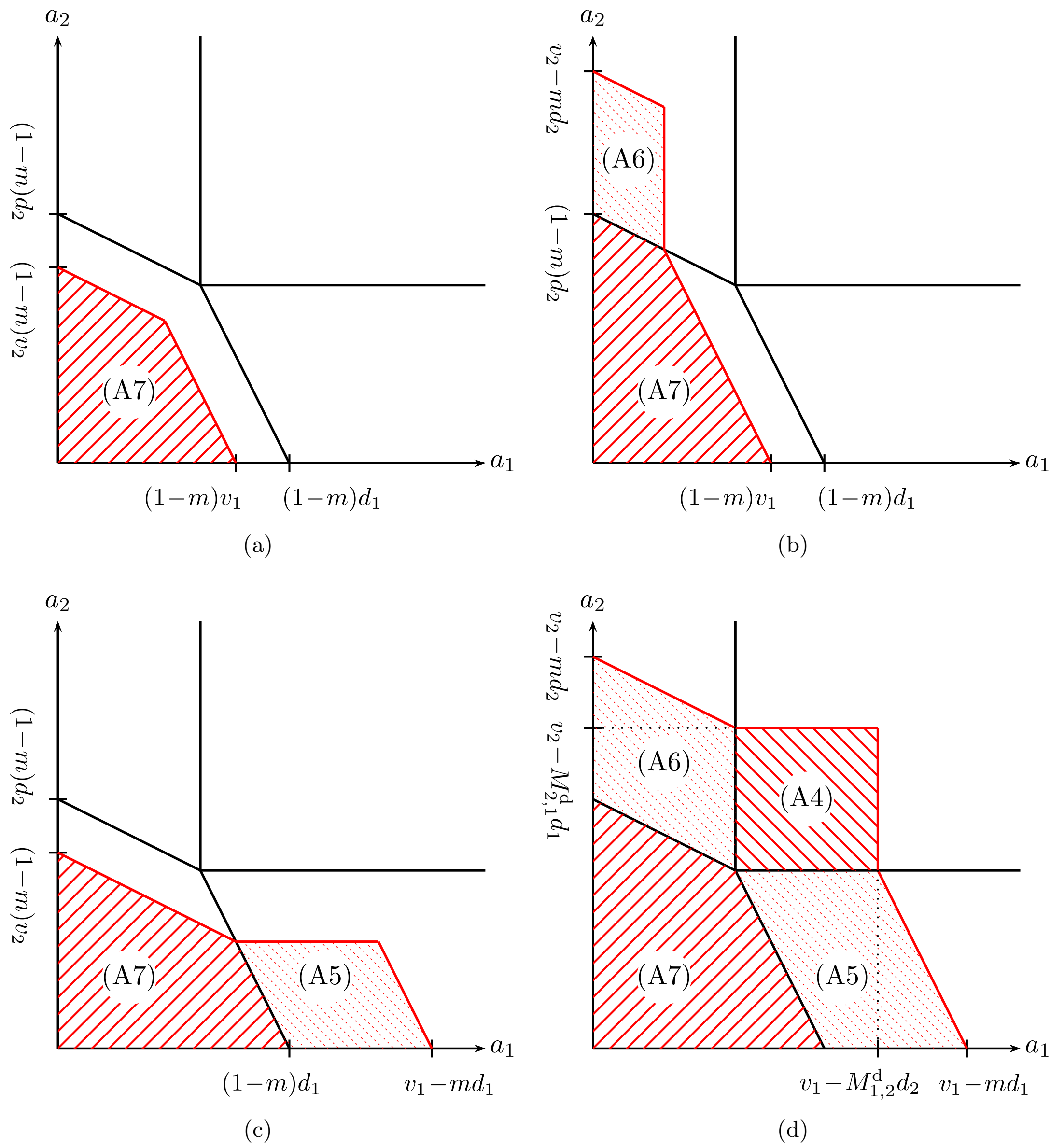


Figure A.1: Probability $P(V_1^{\mathrm{d}} \leq v_1, V_2^{\mathrm{d}} \leq v_2)$ decomposed according to (A4)–(A7) for $M_{2,1}^{\mathrm{d}} d_1 < d_2$ and $M_{1,2}^{\mathrm{d}} d_2 < d_1$, $m := M_{1,2}^{\mathrm{d}} M_{2,1}^{\mathrm{d}}$, the black solid lines separate the four Suzuki areas (cf. Figure 7.1(a)) and they cross in $(d_1 - M_{1,2}^{\mathrm{d}} d_2, d_2 - M_{2,1}^{\mathrm{d}} d_1)$; (a) $v_1 \leq d_1$, $v_2 \leq d_2$; (b) $v_1 \leq d_1$, $v_2 > d_2$; (c) $v_1 > d_1$, $v_2 \leq d_2$; (d) $v_1 > d_1$, $v_2 > d_2$.

Proof. By (3.34) and Remark 3.6,

$$
\begin{aligned}
&P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2) \\
&= P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2,\, A_{\mathrm{ss}}) + P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2,\, A_{\mathrm{sd}}) \\
&\quad + P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2,\, A_{\mathrm{ds}}) + P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2,\, A_{\mathrm{dd}}) \qquad \text{(A11)} \\
&= P(A_1 \le v_1 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}(d_1 - v_1) + M_{1,2}^{\mathrm{e}}(d_2 - A_2),\, A_1 \ge d_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2), \\
&\qquad A_2 \le v_2 + M_{1,2}^{\mathrm{e}} M_{2,1}^{\mathrm{e}}(d_2 - v_2) + M_{2,1}^{\mathrm{e}}(d_1 - A_1),\, A_2 \ge d_2 + M_{2,1}^{\mathrm{e}}(d_1 - A_1)) \qquad \text{(A12)} \\
&\quad + P(A_1 \le v_1,\, A_1 \ge d_1,\, A_2 \le v_2 + M_{2,1}^{\mathrm{e}}(d_1 - A_1),\, A_2 < d_2 + M_{2,1}^{\mathrm{e}}(d_1 - A_1)) \qquad \text{(A13)} \\
&\quad + P(A_1 \le v_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2),\, A_1 < d_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2),\, A_2 \le v_2,\, A_2 \ge d_2) \qquad \text{(A14)} \\
&\quad + P(A_1 \le v_1,\, A_1 < d_1,\, A_2 \le v_2,\, A_2 < d_2). \qquad \text{(A15)}
\end{aligned}
$$

Let $v_1 \le d_1$ and $v_2 \le d_2$. Then the probabilities in (A12)–(A14) vanish since we assume the bivariate distribution of (A_1, A_2) to be continuous, and the assertion follows. For $v_1 \le d_1$ and $v_2 > d_2$, the probabilities in (A12) and (A13) vanish, and we obtain (cf. Figure A.2(b))

$$
\begin{aligned}
&P(V_1^{\mathrm{e}} \le v_1,\, V_2^{\mathrm{e}} \le v_2) \\
&= P(A_1 \le v_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2),\, d_2 \le A_2 \le v_2) + P(A_1 \le v_1,\, A_2 < d_2) \qquad \text{(A16)} \\
&= P(A_1 \le \min\{v_1,\, v_1 + M_{1,2}^{\mathrm{e}}(d_2 - A_2)\},\, A_2 \le v_2). \qquad \text{(A17)}
\end{aligned}
$$

Similarly, one can show the assertion for $v_1 > d_1$ and $v_2 \le d_2$ (cf. Figure A.2(c)), and $v_1 > d_1$ and $v_2 > d_2$ (cf. Figure A.2(d)). □

A.2 The Fenton–Wilkinson Method

A.2.1 Original Rationale and Adaption to our Set-Up

According to Marlow [1967], the general idea was first proposed, but not published, by Wilkinson in 1934. In 1960, Fenton took up Wilkinson's idea [Dufresne, 2008] of fitting a lognormal distribution to a sum of lognormals by matching the corresponding first and second central moments.

Let Y_i $(i = 1, \ldots, k)$ be independent and (not necessarily identically) lognormally distributed random variables. Then the sum $\sum_{i=1}^k Y_i$ is approximated with a lognormally distributed random variable Y such that

$$
E(Y) = E\left(\sum_{i=1}^k Y_i\right), \quad \mathrm{Var}(Y) = \mathrm{Var}\left(\sum_{i=1}^k Y_i\right). \qquad \text{(A18)}
$$

In our set-up of Section 3.2.2, firm values V_1 (cf. (3.23)) are section-wise defined on the four Suzuki areas, and on each area, V_1 is a sum of lognormally distributed random variables plus some constant. Hence, we are not exactly in the framework considered by

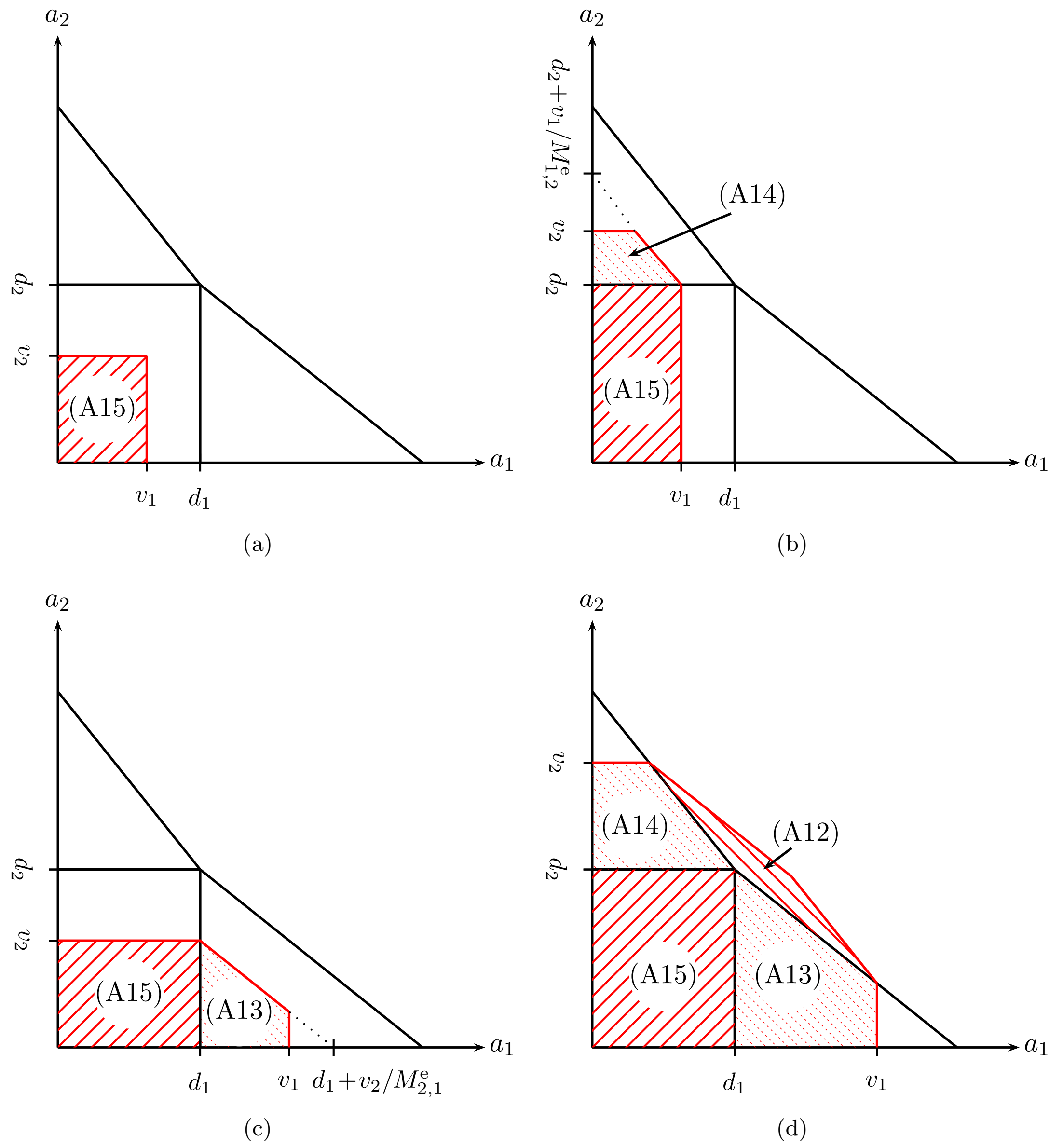


Figure A.2: Probability $P(V_1^{\mathrm{e}} \le v_1, V_2^{\mathrm{e}} \le v_2)$ decomposed according to (A12)–(A15), the black solid lines separate the four Suzuki areas (cf. Figure 7.1(d)) and they cross in $(d_1; d_2)$; (a) $v_1 \le d_1$, $v_2 \le d_2$; (b) $v_1 \le d_1$, $v_2 > d_2$; (c) $v_1 > d_1$, $v_2 \le d_2$; (d) $v_1 > d_1$, $v_2 > d_2$.

Fenton [1960], but the rationale can be applied analogously. Note that by applying the Fenton–Wilkinson method to the empirical distribution function $\hat{F}_{V_1}$, we use empirical moments of V_1 instead of theoretical moments, as the latter cannot be calculated exactly in general, if exogenous asset values follow a lognormal distribution.

A.2.2 The Fenton–Wilkinson Method applied to $(A_1 + m_2A_2)/(1 - m_1m_2)$

Under certain special cases (cf. Section 3.2.2.2 and Section 3.2.2.3) the firm value V_1 equals or can be approximated with $A := (A_1 + m_2A_2)/(1 - m_1m_2)$ for some $m_1, m_2 \in [0, 1)$. In the following, we will analyze in a short simulation study how well the Fenton-Wilkinson method works for different values of m_1, m_2 and σ^2 when applied to A. For that, let A_1, A_2 be independent random variables with

$$A_i \sim \mathcal{LN}(-0.5\sigma^2, \sigma^2), \tag{A19}$$

i.e. $E(A_i) = 1$ and $\mathrm{Var}(A_i) = \exp(\sigma^2) - 1$, $i = 1, 2$, as in Section 3.2.2 and Section 3.2.3. Hence, the logarithms of the two summands of A are normally distributed with different means, identical variances and correlation 0. We consider $\sigma^2 \in$ {0.00995, 0.22314, 0.44629, 0.69315, 1, 1.17865, 1.60944, 1.98100, 2.30259, 3.25810, 4.04743, 4.61512, 10, 20, 30, 40} and $(m_1, m_2) \in \{0.1, 0.2, \ldots, 0.9\}^2$. Within each scenario, we simulated 100,000 values of (A_1, A_2) and from that, we calculated the resulting empirical mean and empirical variance of A. These moments were taken as the moments of a fitted lognormal distribution (cf. Section A.2.1). Finally, we calculated the Kolmogorov–Smirnov statistic (cf. (3.46)) between the empirical distribution function of A and the approximated lognormal distribution function. This yielded the following results.

As becomes clear in Figure A.3, the bigger σ^2, the higher the level of the Kolmogorov–Smirnov values, i.e. the bigger the discrepancy between the distribution of $(A_1 + m_2A_2)/(1 - m_1m_2)$ and the lognormal distribution. For very small values of σ^2, we observed nearly no difference between the two distributions. For $m_1 = m_2 = 0$ one would expect the Kolmogorov–Smirnov values to be close to 0 for any value of σ^2 as $A = A_1$ in this case. As some additional simulations revealed, the estimation of the parameters of the lognormal distribution underlying $A = A_1$ deteriorates with σ^2 increasing and thus the fit between the actual lognormal CDF of A_1 and the fitted lognormal CDF gets worse. This makes the influence of σ^2 in Figure A.3 plausible. For all considered values of σ^2, the parameters m_1 and m_2 do not seem to influence to what extent the distribution of $(A_1 + m_2A_2)/(1 - m_1m_2)$ differs from the lognormal distribution.

A.3 Auxiliary Results for the Limiting Probabilities of Default

Lemma A.3. *Let $d_1, d_2 > 0$ and let A_{ss}, A_{sd}, A_{ds} and A_{dd} be given by (3.25)–(3.28). Under cross-ownership of debt only, the pointwise limits of their indicator functions $1_{A_{\mathrm{ss}}}$, $1_{A_{\mathrm{sd}}}$, $1_{A_{\mathrm{ds}}}$ and $1_{A_{\mathrm{dd}}}$ exist for $M^{\mathrm{d}}_{1,2}, M^{\mathrm{d}}_{2,1} \to 1$.*

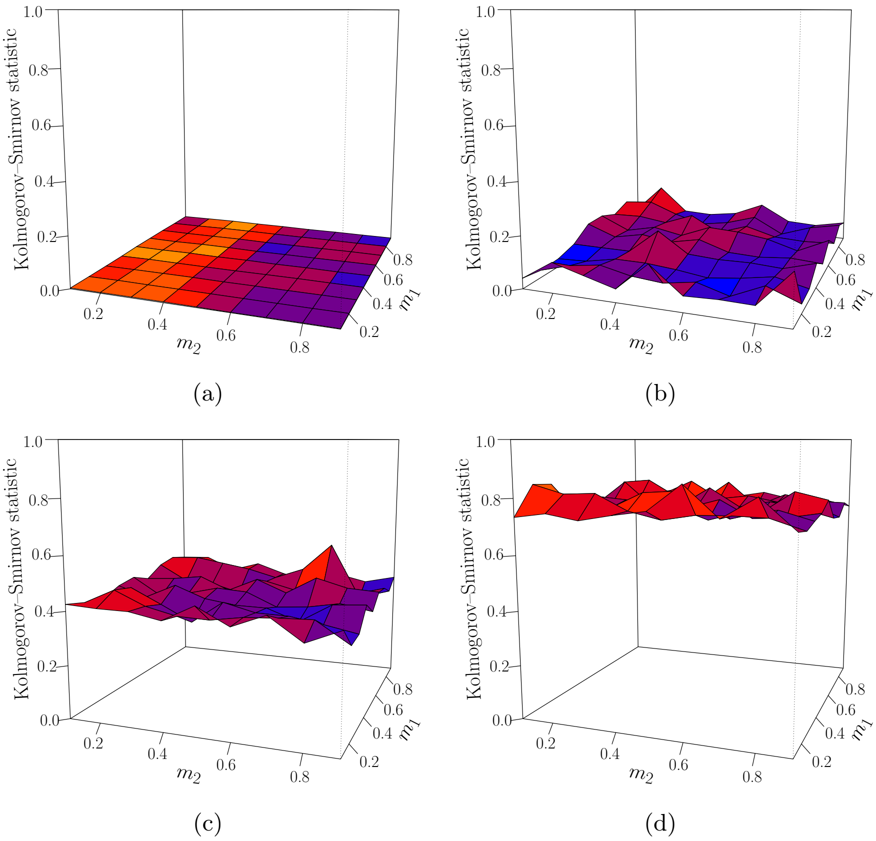


Figure A.3: Kolmogorov–Smirnov statistics in dependence of m_1, m_2 and σ^2; (a) $\sigma^2 = 0.00995$; (b) $\sigma^2 = 3.2581$; (c) $\sigma^2 = 20$; (d) $\sigma^2 = 40$.

Proof. Under cross-ownership of debt only we have

$$A_{\text{ss}} = \{(a_1, a_2) \geq \mathbf{0} : a_1 \geq d_1 - M^{\text{d}}_{1,2} d_2,\ a_2 \geq d_2 - M^{\text{d}}_{2,1} d_1\}, \tag{A20}$$

$$A_{\text{sd}} = \{(a_1, a_2) \geq \mathbf{0} : a_1 + M^{\text{d}}_{1,2} a_2 \geq (1 - M^{\text{d}}_{1,2} M^{\text{d}}_{2,1}) d_1,\ a_2 < d_2 - M^{\text{d}}_{2,1} d_1\}, \tag{A21}$$

$$A_{\text{ds}} = \{(a_1, a_2) \geq \mathbf{0} : a_1 < d_1 - M^{\text{d}}_{1,2} d_2,\ M^{\text{d}}_{2,1} a_1 + a_2 \geq (1 - M^{\text{d}}_{1,2} M^{\text{d}}_{2,1}) d_2\}, \tag{A22}$$

$$\begin{aligned} A_{\text{dd}} = \{(a_1, a_2) \geq \mathbf{0} : a_1 + M^{\text{d}}_{1,2} a_2 &< (1 - M^{\text{d}}_{1,2} M^{\text{d}}_{2,1}) d_1, \\ M^{\text{d}}_{2,1} a_1 + a_2 &< (1 - M^{\text{d}}_{1,2} M^{\text{d}}_{2,1}) d_2\}. \end{aligned} \tag{A23}$$

Let $(M^{\text{d}}_{1,2,n_1})_{n_1 \in \mathbb{N}}$ and $(M^{\text{d}}_{2,1,n_2})_{n_2 \in \mathbb{N}}$ be arbitrary, but strictly increasing sequences in $(0, 1)$ with limit 1, and let A_{ss,n_1,n_2}, A_{sd,n_1,n_2}, A_{ds,n_1,n_2} and A_{dd,n_1,n_2} stand for the Suzuki areas associated with the n_1th and n_2th element the above sequences.
First, it is easy to see from (A20) that A_{ss,n_1,n_2} is strictly increasing in both, n_1 and n_2. Hence, also the sequence of indicator functions $1_{A_{\text{ss},n_1,n_2}}$ is pointwise strictly increasing in n_1 and n_2, i.e. $\lim_{n_1,n_2\to\infty} 1_{A_{\text{ss},n_1,n_2}}$ exists. Next,

$$A_{\text{sd},n_1,n_2} = \underbrace{\{a_1 + M^{\text{d}}_{1,2,n_1} a_2 \geq (1 - M^{\text{d}}_{1,2,n_1} M^{\text{d}}_{2,1,n_2}) d_1\}}_{:={}_1A_{\text{sd},n_1,n_2}} \cap \underbrace{\{a_2 < d_2 - M^{\text{d}}_{2,1,n_2} d_1\}}_{:={}_2A_{\text{sd},n_2}}, \tag{A24}$$

where ${}_1A_{\text{sd},n_1,n_2}$ increases in both, n_1 and n_2 and ${}_2A_{\text{sd},n_2}$ decreases in n_2. Hence, the limits of the associated (separate) indicator functions exist, and because of $1_{A_{\text{sd},n_1,n_2}} = 1_{{}_1A_{\text{sd},n_1,n_2}} \times 1_{{}_2A_{\text{sd},n_2}}$ for all $n_1, n_2 \in \mathbb{N}$ by (A24), the limit of $1_{A_{\text{sd},n_1,n_2}}$ exists as well. Analogously we can write

$$A_{\text{ds},n_1,n_2} = \underbrace{\{a_1 < d_1 - M^{\text{d}}_{1,2,n_1} d_2\}}_{:={}_1A_{\text{ds},n_1}} \cap \underbrace{\{M^{\text{d}}_{2,1,n_2} a_1 + a_2 \geq (1 - M^{\text{d}}_{1,2,n_1} M^{\text{d}}_{2,1,n_2}) d_2\}}_{:={}_2A_{\text{ds},n_1,n_2}}, \tag{A25}$$

with ${}_1A_{\text{ds},n_1}$ decreasing in n_1 and ${}_2A_{\text{ds},n_1,n_2}$ increasing in both, n_1 and n_2. Hence, the limits of the related indicator functions exist, and thus also the limit of $1_{A_{\text{ds},n_1,n_2}}$, if n_1, n_2 converge to infinity. Furthermore, A_{dd,n_1,n_2} is strictly decreasing in both, n_1 and n_2, i.e. the limit of the associated indicator function for $n_1, n_2 \to \infty$ exists. ☐

Lemma A.4. *Let $d_1, d_2 > 0$. With A^*_{ss}, A^*_{sd}, A^*_{ds} and A^*_{dd} as defined in* (4.4), *we have*

$$A^*_{\text{dd}} = \{(0, 0)\}, \tag{A26}$$

$$A^*_{\text{sd}} = \emptyset \quad \Leftrightarrow \quad d_2 < d_1, \tag{A27}$$

$$A^*_{\text{ds}} = \emptyset \quad \Leftrightarrow \quad d_1 < d_2. \tag{A28}$$

If $d_1 = d_2$, A^*_{sd} and A^*_{ds} equal the strictly positive a_1-axis and a_2-axis, respectively.

Proof. Let $(M^{\text{d}}_{1,2,n_1})_{n_1 \in \mathbb{N}}$, $(M^{\text{d}}_{2,1,n_2})_{n_2 \in \mathbb{N}}$, A_{ss,n_1,n_2}, A_{sd,n_1,n_2}, A_{ds,n_1,n_2} and A_{dd,n_1,n_2} be defined as in the proof of Lemma A.3.
Since $(0, 0) \in A_{\text{dd},n_1,n_2}$ for all $n_1, n_2 \in \mathbb{N}$, i.e. $1_{A_{\text{dd},n_1,n_2}}(0, 0) = 1$ for all $n_1, n_2 \in \mathbb{N}$, we have $\lim_{n_1,n_2\to\infty} 1_{A_{\text{dd},n_1,n_2}}(0, 0) = 1$, i.e. $(0, 0) \in A^*_{\text{dd}}$. Let us now assume $(a^*_1, a^*_2) \in A^*_{\text{dd}}$

with $a_1^*, a_2^* \geq 0$ and $a_1^* + a_2^* > 0$, w.l.o.g. let $a_1^* > 0$. Since A_{dd,n_1,n_2} is strictly decreasing in n_1 and n_2 (cf. (A23)), we have for all $n_1, n_2 \in \mathbb{N}$,

$$0 < a_1^* + M_{1,2,n_1}^{\mathrm{d}} a_2^* < (1 - M_{1,2,n_1}^{\mathrm{d}} M_{2,1,n_2}^{\mathrm{d}}) d_1, \tag{A29}$$

and as the RHS of (A29) converges to 0 if n_1 and n_2 go to infinity, (A26) follows. Let us now assume $d_2 < d_1$ and $(a_1^*, a_2^*) \in A_{\mathrm{sd}}^*$. Then, by (A24),

$$a_2^* < d_2 - M_{2,1,n_2}^{\mathrm{d}} d_1 \quad \text{for all } n_2 \in \mathbb{N}. \tag{A30}$$

Since the limit of the RHS of (A30) for $n_2 \to \infty$ is negative, such an $(a_1^*, a_2^*) \geq \mathbf{0}$ does not exist. If $d_2 \geq d_1$, it is straightforward to see that $A_{\mathrm{sd}}^* = \{(a_1, a_2) \geq \mathbf{0} : a_1 + a_2 > 0,\ a_2 \leq d_2 - d_1\}$, and (A27) follows. In particular, we have for $d_1 = d_2$ that $A_{\mathrm{sd}}^* = \{(a_1, a_2) \geq \mathbf{0} : a_1 > 0,\ a_2 = 0\}$. Analogously, one can show (A28) with the help of (A25), and we obtain for $d_1 \geq d_2$ that $A_{\mathrm{ds}}^* = \{(a_1, a_2) \geq \mathbf{0} : a_1 + a_2 > 0,\ a_1 \leq d_1 - d_2\}$, and $A_{\mathrm{ds}}^* = \{(a_1, a_2) \geq \mathbf{0} : a_1 = 0,\ a_2 > 0\}$ if $d_1 = d_2$. □

Lemma A.5. *Let $\mu, \tilde{\mu} \in \mathbb{R}$, $\sigma, \tilde{\sigma}, d_2 \in \mathbb{R}^+$, $\sigma > \tilde{\sigma}$, be such that*

$$\exp(\tilde{\mu} + 0.5\tilde{\sigma}^2) = \exp(\mu + 0.5\sigma^2) + d_2, \tag{A31}$$

$$(\exp(\tilde{\sigma}^2) - 1)\exp(2\tilde{\mu} + \tilde{\sigma}^2) = (\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2), \tag{A32}$$

which exactly corresponds to the definition of $\tilde{\mu}$ and $\tilde{\sigma}$ in (4.7). *Then*

$$\exp(\tilde{\sigma}\mu - \sigma\tilde{\mu}) < \frac{\left(\frac{\tilde{\sigma}}{\sigma - \tilde{\sigma}} d_2\right)^{\tilde{\sigma}}}{\left(\frac{\sigma}{\sigma - \tilde{\sigma}} d_2\right)^{\sigma}}. \tag{A33}$$

Proof. First, (A32) implies $\exp(\tilde{\mu} + 0.5\tilde{\sigma}^2) = \sqrt{\frac{\exp(\sigma^2)-1}{\exp(\tilde{\sigma}^2)-1}} \exp(\mu + 0.5\sigma^2)$, i.e.

$$\tilde{\mu} = \mu + 0.5\sigma^2 - 0.5\tilde{\sigma}^2 + \ln\left(\sqrt{\frac{\exp(\sigma^2) - 1}{\exp(\tilde{\sigma}^2) - 1}}\right), \tag{A34}$$

$$\ln(d_2) = \mu + 0.5\sigma^2 + \ln\left(\sqrt{\frac{\exp(\sigma^2) - 1}{\exp(\tilde{\sigma}^2) - 1}} - 1\right). \tag{A35}$$

Hence,

$$\text{(A33)} \Leftrightarrow\ 0 > \tilde{\sigma}\mu - \tilde{\mu}\sigma + (\sigma - \tilde{\sigma})\ln(d_2) - (\sigma - \tilde{\sigma})\ln(\sigma - \tilde{\sigma}) - \tilde{\sigma}\ln(\tilde{\sigma}) + \sigma\ln(\sigma) \tag{A36}$$

$$\begin{aligned}\Leftrightarrow\ 0 > &- \sigma\ln\left(\sqrt{\frac{\exp(\sigma^2) - 1}{\exp(\tilde{\sigma}^2) - 1}}\right) + 0.5\tilde{\sigma}^2\sigma - 0.5\tilde{\sigma}\sigma^2 - \tilde{\sigma}\ln(\tilde{\sigma}) + \sigma\ln(\sigma) \\ &+ (\sigma - \tilde{\sigma})\ln\left(\sqrt{\frac{\exp(\sigma^2) - 1}{\exp(\tilde{\sigma}^2) - 1}} - 1\right) - (\sigma - \tilde{\sigma})\ln(\sigma - \tilde{\sigma})\end{aligned} \tag{A37}$$

$$\begin{aligned}\Leftrightarrow\ 0 > (\sigma-\tilde{\sigma})&\left(\ln\left(\sqrt{\frac{\exp(\sigma^2)-1}{\exp(\tilde{\sigma}^2)-1}}-1\right)-\ln(\sigma-\tilde{\sigma})-0.5\sigma\tilde{\sigma}\right)\\ &-\tilde{\sigma}\ln(\tilde{\sigma})+\sigma\ln\left(\sigma\sqrt{\frac{\exp(\tilde{\sigma}^2)-1}{\exp(\sigma^2)-1}}\right).\end{aligned}\tag{A38}$$

Due to $\sigma > \tilde{\sigma}$, it is sufficient for (A33) to show that

$$(\sigma-\tilde{\sigma})\left(\ln\left(\sqrt{\frac{\exp(\sigma^2)-1}{\exp(\tilde{\sigma}^2)-1}}-1\right)-\ln(\sigma-\tilde{\sigma})\right)-\tilde{\sigma}\ln(\tilde{\sigma})+\sigma\ln\left(\sigma\sqrt{\frac{\exp(\tilde{\sigma}^2)-1}{\exp(\sigma^2)-1}}\right)\tag{A39}$$

is smaller than 0, or equivalently

$$\begin{aligned}&(\sigma-\tilde{\sigma})\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right)-\sigma\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}}{\sigma}\right)\\ &<-\tilde{\sigma}\ln\left(\frac{\sqrt{\exp(\tilde{\sigma}^2)-1}}{\tilde{\sigma}}\right).\end{aligned}\tag{A40}$$

For that, we consider the derivative of the LHS of (A40) with respect to σ:

$$\begin{aligned}&\frac{\partial}{\partial\sigma}\left[(\sigma-\tilde{\sigma})\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right)-\sigma\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}}{\sigma}\right)\right]\\ &=\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right)\\ &\quad+\frac{(\sigma-\tilde{\sigma})^2\left(\frac{\exp(\sigma^2)\sigma}{\sqrt{\exp(\sigma^2)-1}}(\sigma-\tilde{\sigma})-\left(\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}\right)\right)}{\left(\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}\right)(\sigma-\tilde{\sigma})^2}\\ &\quad-\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}}{\sigma}\right)-\frac{\sigma^2\left(\frac{\exp(\sigma^2)\sigma}{\sqrt{\exp(\sigma^2)-1}}-\sqrt{\exp(\sigma^2)-1}\right)}{\sqrt{\exp(\sigma^2)-1}\,\sigma^2}\end{aligned}\tag{A41}$$

$$\begin{aligned}&=\ln\left(\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}\right)-\ln\left(\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}\right)\\ &\quad+\frac{\exp(\sigma^2)\sigma}{\sqrt{\exp(\sigma^2)-1}}\left(\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}-\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}\right).\end{aligned}\tag{A42}$$

Because of $\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}-\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}<0$, this derivative is negative if and only if

$$\frac{\ln\left(\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}\right)-\ln\left(\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}\right)}{\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}-\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}} < \frac{\exp(\sigma^2)\sigma}{\sqrt{\exp(\sigma^2)-1}}. \tag{A43}$$

Since the LHS of (A43) can be interpreted as the difference quotient of the concave logarithmic function in $x_0 = \frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}$ and $x = \frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}$, the LHS of (A43) is strictly decreasing in x and thus strictly increasing in $\tilde{\sigma}$. From

$$\lim_{\tilde{\sigma}\nearrow\sigma}\frac{\sigma-\tilde{\sigma}}{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}} = \left(\lim_{\tilde{\sigma}\nearrow\sigma}\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right)^{-1} \tag{A44}$$

$$= \left(\frac{\partial}{\partial\sigma}\sqrt{\exp(\sigma^2)-1}\right)^{-1} = \frac{\sqrt{\exp(\sigma^2)-1}}{\exp(\sigma^2)\sigma} \tag{A45}$$

it follows that the LHS of (A43) is smaller than

$$\frac{\ln\left(\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}\right)-\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}}{\exp(\sigma^2)\sigma}\right)}{\frac{\sigma}{\sqrt{\exp(\sigma^2)-1}}-\frac{\sqrt{\exp(\sigma^2)-1}}{\exp(\sigma^2)\sigma}} < \frac{\exp(\sigma^2)\sigma}{\sqrt{\exp(\sigma^2)-1}}, \tag{A46}$$

where the last inequality follows from straightforward calculations and the fact that $\ln(x) < x-1$ for all $x > 0$. Thus, (A43) is met for all $\tilde{\sigma} < \sigma$, i.e. the LHS of (A40) is strictly decreasing in σ for all $\tilde{\sigma} > 0$. Hence, for (A40) it only remains to show that (A40) holds in the limit of $\sigma \searrow \tilde{\sigma}$. Because of

$$\left|\lim_{\sigma\searrow\tilde{\sigma}}\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right| = \left|\frac{\partial}{\partial\tilde{\sigma}}\sqrt{\exp(\tilde{\sigma}^2)-1}\right| \tag{A47}$$

$$= \left|\frac{\exp(\tilde{\sigma}^2)\tilde{\sigma}}{\sqrt{\exp(\tilde{\sigma}^2)-1}}\right| < \infty, \tag{A48}$$

we have

$$\lim_{\sigma\searrow\tilde{\sigma}}\left[(\sigma-\tilde{\sigma})\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}-\sqrt{\exp(\tilde{\sigma}^2)-1}}{\sigma-\tilde{\sigma}}\right)-\sigma\ln\left(\frac{\sqrt{\exp(\sigma^2)-1}}{\sigma}\right)\right]$$

$$= -\tilde{\sigma}\ln\left(\frac{\sqrt{\exp(\tilde{\sigma}^2)-1}}{\tilde{\sigma}}\right), \tag{A49}$$

i.e. (A40) and (A33) follow. □

Lemma A.6. *Let* (A_1, A_2) *follow a bivariate lognormal distribution as in* (4.1) *and let* $\tilde{\mu}$ *and* $\tilde{\sigma}$ *be defined as in* (4.36) *and* (4.37)*, respectively. Then*

$$\lim_{M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1} \tilde{\sigma} < \infty \quad \textit{and} \quad \tilde{\mu} \to \infty \;\; \textit{for } M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1. \tag{A50}$$

Proof. From $V^{\mathrm{e}}_1\big|_{A_{\mathrm{ss}}} = (A_1 + M^{\mathrm{e}}_{1,2} A_2 - M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1} d_1 + d_2))/(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})$ by (3.34) we obtain

$$E(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}}) = \tfrac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}} E([A_1 + M^{\mathrm{e}}_{1,2} A_2 - M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1} d_1 + d_2)] \times 1_{A_{\mathrm{ss}}}), \tag{A51}$$

$$\begin{aligned} &\mathrm{Var}(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}}) \\ &= \left(\tfrac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}}\right)^2 \mathrm{Var}([A_1 + M^{\mathrm{e}}_{1,2} A_2] \times 1_{A_{\mathrm{ss}}}) &\text{(A52)}\\ &= \left(\tfrac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}}\right)^2 \left(E([A_1 + M^{\mathrm{e}}_{1,2} A_2]^2 \times 1_{A_{\mathrm{ss}}}) - E([A_1 + M^{\mathrm{e}}_{1,2} A_2] \times 1_{A_{\mathrm{ss}}})^2\right). &\text{(A53)} \end{aligned}$$

Let $1_{A^*_{\mathrm{ss}}}$ denote the limit of $1_{A_{\mathrm{ss}}}$ if $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1$. For its existence, see Lemma A.7. In particular, $1_{A^*_{\mathrm{ss}}} \geq 1_{A_{\mathrm{ss}}}$ for all $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \in (0,1)$. Because of

$$[A_1 + M^{\mathrm{e}}_{1,2} A_2 - M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1} d_1 + d_2)] \times 1_{A_{\mathrm{ss}}} \leq [A_1 + A_2] \times 1_{A_{\mathrm{ss}}} \leq [A_1 + A_2] \times 1_{A^*_{\mathrm{ss}}}, \tag{A54}$$

$$[A_1 + M^{\mathrm{e}}_{1,2} A_2]^2 \times 1_{A_{\mathrm{ss}}} \leq (A_1 + A_2)^2 \times 1_{A^*_{\mathrm{ss}}} \tag{A55}$$

for all $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \in (0,1)$, where the RHS of both, (A54) and (A55) are integrable, the dominated convergence theorem yields for $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1$

$$E([A_1 + M^{\mathrm{e}}_{1,2} A_2 - M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1} d_1 + d_2)] \times 1_{A_{\mathrm{ss}}}) \to E([A_1 + A_2 - d_1 - d_2] \times 1_{A^*_{\mathrm{ss}}}) < \infty, \tag{A56}$$

$$E([A_1 + M^{\mathrm{e}}_{1,2} A_2] \times 1_{A_{\mathrm{ss}}}) \to E([A_1 + A_2] \times 1_{A^*_{\mathrm{ss}}}) < \infty, \tag{A57}$$

$$E([A_1 + M^{\mathrm{e}}_{1,2} A_2]^2 \times 1_{A_{\mathrm{ss}}}) \to E([A_1 + A_2]^2 \times 1_{A^*_{\mathrm{ss}}}) < \infty, \tag{A58}$$

i.e. $\mathrm{Var}([A_1 + M^{\mathrm{e}}_{1,2} A_2] \times 1_{A_{\mathrm{ss}}}) \to \mathrm{Var}([A_1 + A_2] \times 1_{A^*_{\mathrm{ss}}})$ for $M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1$. Note that both, $E([A_1 + A_2 - d_1 - d_2] \times 1_{A^*_{\mathrm{ss}}})$ and $\mathrm{Var}([A_1 + A_2] \times 1_{A^*_{\mathrm{ss}}})$, are strictly positive due to the lognormal distribution of (A_1, A_2) which yields $P(A^*_{\mathrm{ss}}) > 0$ (cf. Lemma A.7). Hence, (A51)–(A58) imply

$$E(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}}) \to \infty, \tag{A59}$$

$$\mathrm{Var}(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}}) \to \infty, \tag{A60}$$

$$\frac{\mathrm{Var}(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}})}{E(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}})} = \frac{\mathrm{Var}([A_1 + M^{\mathrm{e}}_{1,2} A_2] \times 1_{A_{\mathrm{ss}}})}{(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})\, E([A_1 + M^{\mathrm{e}}_{1,2} A_2 - M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1} d_1 + d_2)] \times 1_{A_{\mathrm{ss}}})} \to \infty, \tag{A61}$$

$$\frac{\mathrm{Var}(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}})}{E(V^{\mathrm{e}}_1 \times 1_{A_{\mathrm{ss}}})^2} \to \frac{\mathrm{Var}([A_1 + A_2] \times 1_{A^*_{\mathrm{ss}}})}{E([A_1 + A_2 - d_1 - d_2] \times 1_{A^*_{\mathrm{ss}}})^2} < \infty, \quad M^{\mathrm{e}}_{1,2}, M^{\mathrm{e}}_{2,1} \to 1. \tag{A62}$$

Then

$$E(V_1^{\mathrm{e}}) = E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}}) + E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}}) \to \infty \quad \text{for } M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1, \tag{A63}$$

$$\mathrm{Var}(V_1^{\mathrm{e}}) = \mathrm{Var}(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}}) + \mathrm{Var}(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}}) - 2E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}}) \times E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}}), \tag{A64}$$

where

$$\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}}) < \infty, \qquad \lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} \mathrm{Var}(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}}) < \infty, \tag{A65}$$

since straightforward calculations show that $V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}^{\mathrm{c}}} < d_1 + d_2/M_{2,1}^{\mathrm{e}}$. By (A61), $\mathrm{Var}(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}})$ diverges faster than $E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}})$, and therefore $\mathrm{Var}(V_1^{\mathrm{e}}) \to \infty$ for $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1$. Furthermore,

$$\frac{\mathrm{Var}(V_1^{\mathrm{e}})}{E(V_1^{\mathrm{e}})^2} \sim \frac{\mathrm{Var}(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}})}{E(V_1^{\mathrm{e}} \times 1_{A_{\mathrm{ss}}})^2}, \quad M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1, \tag{A66}$$

because all the other terms in (A63) and (A64) are dominated by the expressions on the RHS of (A66), which go to infinity. Hence, by (A62),

$$\frac{\mathrm{Var}(V_1^{\mathrm{e}})}{E(V_1^{\mathrm{e}})^2} \to \frac{\mathrm{Var}([A_1 + A_2] \times 1_{A_{\mathrm{ss}}^*})}{E([A_1 + A_2 - d_1 - d_2] \times 1_{A_{\mathrm{ss}}^*})^2} < \infty, \qquad M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1. \tag{A67}$$

Altogether, by (A63) and (A67),

$$\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} \tilde{\sigma} = \ln \left(\lim_{M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1} \frac{\mathrm{Var}(V_1^{\mathrm{e}})}{E(V_1^{\mathrm{e}})^2} + 1 \right)^{0.5} < \infty \tag{A68}$$

and

$$\tilde{\mu} = -\frac{1}{2} \ln \left(\underbrace{\frac{\mathrm{Var}(V_1^{\mathrm{e}})}{E(V_1^{\mathrm{e}})^4} + \frac{1}{E(V_1^{\mathrm{e}})^2}}_{\to 0} \right) \to \infty \ \text{ for } M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1. \tag{A69}$$

□

Lemma A.7. *Let* A_{ss} *be defined as in* (3.25). *Under cross-ownership of equity only, the pointwise limit of* $1_{A_{\mathrm{ss}}}$ *for* $M_{1,2}^{\mathrm{e}}, M_{2,1}^{\mathrm{e}} \to 1$ *exists and is given by* $1_{A_{\mathrm{ss}}^*}$ *with*

$$A_{\mathrm{ss}}^* := \{(a_1, a_2) \geq \mathbf{0} : \ a_1 + a_2 \geq d_1 + d_2\}. \tag{A70}$$

Proof. Under cross-ownership of equity only, the formula of A_{ss} reduces to

$$\{(a_1, a_2) \geq \mathbf{0} : \ a_1 + M_{1,2}^{\mathrm{e}} a_2 \geq d_1 + M_{1,2}^{\mathrm{e}} d_2, \ M_{2,1}^{\mathrm{e}} a_1 + a_2 \geq M_{2,1}^{\mathrm{e}} d_1 + d_2\}. \tag{A71}$$

Let $(M^{\mathrm{e}}_{1,2,n_1})_{n_1\in\mathbb{N}}$ and $(M^{\mathrm{e}}_{2,1,n_2})_{n_2\in\mathbb{N}}$ be arbitrary, but strictly increasing sequences in $(0,1)$ with limit 1, and let A_{ss,n_1,n_2} stand for A_{ss} associated with the n_1th and n_2th element the above sequences. Then it is easy to see from (A71) that A_{ss,n_1,n_2} is strictly increasing in both, n_1 and n_2. Hence, the indicator function of A_{ss,n_1,n_2} is pointwise strictly increasing in n_1 and n_2, and its pointwise limit exists and is a function with values in $\{0,1\}$ only. As such, this limit is of the form 1_A for some set $A\subseteq\mathbb{R}^+_0\times\mathbb{R}^+_0$. In order to show $A=A^*_{\mathrm{ss}}$, we first assume $1_A(a^*_1,a^*_2)=1$, i.e. there is an $N_1\in\mathbb{N}$ such that $(a^*_1,a^*_2)\in A_{\mathrm{ss},n_1,n_2}$ for all $n_1,n_2\geq N_1$, i.e.

$$a^*_1+M^{\mathrm{e}}_{1,2,n_1}a^*_2\geq d_1+M^{\mathrm{e}}_{1,2,n_1}d_2\quad\text{for all } n_1\geq N_1, \tag{A72}$$

$$M^{\mathrm{e}}_{2,1,n_2}a^*_1+a^*_2\geq M^{\mathrm{e}}_{2,1,n_2}d_1+d_2\quad\text{for all } n_2\geq N_1. \tag{A73}$$

In the limit of $n_1,n_2\to\infty$, this means that $a^*_1+a^*_2\geq d_1+d_2$, i.e. $A\subseteq A^*_{\mathrm{ss}}$. Let now $1_A(a^*_1,a^*_2)=0$, i.e. there is an $N_2\in\mathbb{N}$ such that

$$a^*_1+M^{\mathrm{e}}_{1,2,n_1}a^*_2< d_1+M^{\mathrm{e}}_{1,2,n_1}d_2\quad\text{for all } n_1,n_2\geq N_2, \tag{A74}$$

$$M^{\mathrm{e}}_{2,1,n_2}a^*_1+a^*_2< M^{\mathrm{e}}_{2,1,n_2}d_1+d_2\quad\text{for all } n_1,n_2\geq N_2. \tag{A75}$$

In the limit of $n_1,n_2\to\infty$, we obtain from (A74) and (A75) that $a^*_1+a^*_2\leq d_1+d_2$. If we had $a^*_1+a^*_2=d_1+d_2$, (A74) and (A75) would imply $a^*_2>d_2$ and $a^*_1>d_1$, in contradiction to $a^*_1+a^*_2=d_1+d_2$. Hence, $a^*_1+a^*_2<d_1+d_2$, i.e. $(a^*_1,a^*_2)\notin A^*_{\mathrm{ss}}$, and the assertion follows. □

Lemma A.8. *Let $x,y\geq 0$. Then*

$$\exp(\sqrt{xy})-1\leq\sqrt{(\exp(x)-1)(\exp(y)-1)}. \tag{A76}$$

Proof. W.l.o.g. let $x\leq y$. The assertion is clear for $x=y$. For $x<y$, we show

$$\frac{\partial}{\partial y}\left(\sqrt{(\exp(x)-1)(\exp(y)-1)}-\exp(\sqrt{xy})+1\right) \tag{A77}$$

$$=\underbrace{\sqrt{\frac{\exp(x)-1}{\exp(y)-1}}\exp(y)-\sqrt{\frac{x}{y}}\exp(\sqrt{xy})}_{=:f(x,y)}\geq 0 \tag{A78}$$

for all $x<y$, which proves the assertion. The absolute minimum of $f(\cdot,y)$ $(y>0)$ is taken either for $x\in(0,y)$ or for $x=0$ or $x=y$, and in the latter two cases, it is straightforward to see that (A78) holds. If the minimum is taken in $(0,y)$, say in $x=\tilde{x}$, the derivative of $f(\cdot,y)$ with respect to x must be 0 in $x=\tilde{x}$, which is equivalent to

$$\frac{\exp(\tilde{x})\exp(y)}{\sqrt{(\exp(\tilde{x})-1)(\exp(y)-1)}}=\exp(\sqrt{\tilde{x}y})\left(1+\frac{1}{\sqrt{\tilde{x}y}}\right). \tag{A79}$$

Replacing $\exp(\sqrt{xy})$ in $f(x,y)$ by use of (A79), we obtain

$$f(x,y) \geq 0 \quad \Leftrightarrow \quad \sqrt{\frac{\exp(\tilde{x})-1}{\exp(y)-1}}\exp(y) - \sqrt{\frac{\tilde{x}}{y}}\frac{\exp(\tilde{x})\exp(y)}{\sqrt{(\exp(\tilde{x})-1)(\exp(y)-1)}}\frac{1}{\left(1+\frac{1}{\sqrt{\tilde{x}y}}\right)} \geq 0 \tag{A80}$$

$$\Leftrightarrow \quad \exp(\tilde{x}) - 1 \geq \exp(\tilde{x})\frac{\tilde{x}}{1+\sqrt{\tilde{x}y}}, \tag{A81}$$

which is true for all values of $y > \tilde{x}$, since the RHS of (A81) is strictly decreasing in y and for $y = \tilde{x}$, (A81) is equivalent to $\exp(\tilde{x}) \geq 1 + \tilde{x}$, which holds because of the power series representation of $\exp(\tilde{x})$. Hence, we have shown that (A78) even holds if $f(\cdot, y)$ takes its minimum value with respect to x, and the assertion follows. □

A.4 Limiting Behaviour of $\vartheta(y)$

Let $\vartheta(y)$ be defined as in (5.72), i.e.

$$\vartheta(y) = \frac{F^{-1}_{A_1+m_2A_2}(F_{m_1A_1+A_2}(y))}{y}. \tag{A82}$$

Lemma A.9. *Let A_1, A_2 be positive random variables with strictly increasing continuous distribution functions, and let $m_1, m_2 \in (0,1)$. Then*

$$\vartheta(y) \in \left(m_2, \frac{1}{m_1}\right) \quad \textit{for all } y > 0. \tag{A83}$$

Proof. Obviously, since we assume the distribution functions of A_1 and A_2 to be strictly increasing, the distribution functions and inverse distribution functions of $A_1 + m_2A_2$ and $m_1A_1 + A_2$ are strictly increasing. Then

$$m_2 < \frac{F^{-1}_{A_1+m_2A_2}(F_{m_1A_1+A_2}(y))}{y}$$

$$\Leftrightarrow \quad F_{A_1+m_2A_2}(m_2y) < F_{m_1A_1+A_2}(y) \tag{A84}$$

$$\Leftrightarrow \quad P\left(A_1/m_2 + A_2 \leq y\right) < P\left(m_1A_1 + A_2 \leq y\right), \tag{A85}$$

where the last inequality holds because of $m_1, m_2 \in (0,1)$ and due to $A_1 > 0$. Similarly,

$$\frac{F^{-1}_{A_1+m_2A_2}(F_{m_1A_1+A_2}(y))}{y} < \frac{1}{m_1}$$

$$\Leftrightarrow \quad F_{m_1A_1+A_2}(y) < F_{A_1+m_2A_2}\left(y/m_1\right) \tag{A86}$$

$$\Leftrightarrow \quad P\left(m_1A_1 + A_2 \leq y\right) < P\left(m_1A_1 + m_1m_2A_2 \leq y\right), \tag{A87}$$

where the last inequality holds because of $m_1, m_2 \in (0,1)$ and due to $A_2 > 0$. □

A.4.1 Limit of $\vartheta(y)$ for $y \to 0$

Under Assumption 5.6 we do not have closed formulae for the densities of $A_1 + m_2 A_2$ and $m_1 A_1 + A_2$, but the work of Gao et al. [2009] provides expressions asymptotically equivalent to $f_{\ln(A_1+m_2A_2)}$ and $f_{\ln(m_1A_1+A_2)}$, leading to the following lemma.

Lemma A.10. *Let (A_1, A_2) be distributed as in Assumption 5.6 and let $m_1, m_2 \in (0,1)$. Then*

$$f_{A_1+m_2A_2}(x) \sim \frac{f_{\text{Gao}}(\ln(x), 1, m_2)}{x}, \quad x \to 0^+, \tag{A88}$$

$$f_{m_1A_1+A_2}(x) \sim \frac{f_{\text{Gao}}(\ln(x), m_1, 1)}{x}, \quad x \to 0^+, \tag{A89}$$

with

$$f_{\text{Gao}}(z, q_1, q_2) = \begin{cases} c(q_1, q_2) \times \frac{1}{\sqrt{-z}} \exp(-S \times z - G_1(z, q_1, q_2)), & \rho < \frac{\sigma_1}{\sigma_2}, \\ \frac{1}{\sqrt{\ln(-z)}} \varphi_{\mu_1+\ln(q_1),\sigma_1^2}(z)\, G_2(z, q_1, q_2), & \rho = \frac{\sigma_1}{\sigma_2}, \\ \varphi_{\mu_1+\ln(q_1),\sigma_1^2}(z), & \rho > \frac{\sigma_1}{\sigma_2}, \end{cases} \tag{A90}$$

and

$$S := \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2}\left(s_1 \ln\left(\frac{s_1}{s_1+s_2}\right) + s_2 \ln\left(\frac{s_2}{s_1+s_2}\right)\right), \tag{A91}$$

$$s_1 := \sigma_1^2 - \rho\sigma_1\sigma_2, \tag{A92}$$

$$s_2 := \sigma_2^2 - \rho\sigma_1\sigma_2, \tag{A93}$$

$$G_1(z, q_1, q_2) := \frac{z^2(s_1+s_2) - 2z\big(s_1(\mu_2 + \ln(q_2)) + s_2(\mu_1 + \ln(q_1))\big)}{2(1-\rho^2)\sigma_1^2\sigma_2^2}, \tag{A94}$$

$$\begin{aligned} G_2(z, q_1, q_2) := \exp\Bigg(\frac{1}{2(\sigma_2^2 - \sigma_1^2)}\Big(&- \big(\ln((1-\rho^{-2})z) - \ln(\ln((1-\rho^{-2})z)) \\ &+ \mu_2 + \ln(q_2) - \mu_1 - \ln(q_1)\big)^2 - 2\ln((1-\rho^{-2})z)\Big)\Bigg), \end{aligned} \tag{A95}$$

and some positive constant $c(q_1, q_2)$ depending on q_1 and q_2.

Proof. By Theorem 2.3 of Gao et al. [2009], the density of $\ln(A_1 + m_2A_2)$ as a function in z is asymptotically equivalent to $f_{\text{Gao}}(z, 1, m_2)$ for $z \to -\infty$, and the density of $\ln(m_1A_1 + A_2)$ is asymptotically equivalent to $f_{\text{Gao}}(z, m_1, 1)$ for $z \to -\infty$. Furthermore,

$$f_{\ln(A_1+m_2A_2)}(z) = \frac{\partial}{\partial z} P(\ln(A_1 + m_2A_2) \le z) = \frac{\partial}{\partial z} P(A_1 + m_2A_2 \le \exp(z)) \tag{A96}$$

$$= f_{A_1+m_2A_2}(\exp(z)) \times \exp(z), \tag{A97}$$

i.e. $f_{A_1+m_2A_2}(z) = f_{\ln(A_1+m_2A_2)}(\ln(z))/z \sim f_{\text{Gao}}(\ln(z), 1, m_2)/z$, $z \to 0^+$. (A89) can be shown similarly. ☐

Lemma A.11. *Let (A_1, A_2) be distributed as in Assumption 5.6 and let $m_1, m_2 \in (0,1)$. Then*

$$\lim_{y\to 0} \vartheta(y) = \begin{cases} m_2^{s_1/(s_1+s_2)} \left(\frac{1}{m_1}\right)^{s_2/(s_1+s_2)}, & \rho < \frac{\sigma_1}{\sigma_2}, \\ \frac{1}{m_1}, & \rho > \frac{\sigma_1}{\sigma_2}, \end{cases} \tag{A98}$$

with s_1, s_2 defined as in Lemma A.10.

Proof. Let $\rho \neq \sigma_1/\sigma_2$. Setting $u = F_{m_1 A_1 + A_2}(y)$ we have

$$\lim_{y\to 0} \vartheta(y) = \lim_{y\to 0} \frac{F^{-1}_{A_1+m_2A_2}(F_{m_1A_1+A_2}(y))}{y} = \lim_{u\to 0} \frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{m_1A_1+A_2}(u)}. \tag{A99}$$

For $x < 1$ let $F_{\text{Gao}}(x, q_1, q_2) := \int_{-\infty}^{\ln(x)} f_{\text{Gao}}(z, q_1, q_2)\,\mathrm{d}z$ (cf. (A90)), i.e.

$$\frac{\partial}{\partial x} F_{\text{Gao}}(x, q_1, q_2) = \frac{f_{\text{Gao}}(\ln(x), q_1, q_2)}{x}, \tag{A100}$$

and for $x > 1$, let

$$h_1(x) := \frac{1}{F_{\text{Gao}}\left(\frac{1}{x}, m_1, 1\right)}, \quad h_2(x) := \frac{1}{P\left(m_1A_1 + A_2 \le \frac{1}{x}\right)}. \tag{A101}$$

Then L'Hôpital's rule, (A100) and Lemma A.10 yield for $\lambda \in (0,1)$ and $\rho > \sigma_1/\sigma_2$

$$\lim_{x\to\infty} \frac{h_1(\lambda x)}{h_1(x)} = \lim_{x\to\infty} \frac{f_{\text{Gao}}\left(\ln\left(\frac{1}{x}\right), m_1, 1\right)}{f_{\text{Gao}}\left(\ln\left(\frac{1}{\lambda x}\right), m_1, 1\right)} \tag{A102}$$

$$= \lim_{x\to\infty} \exp\left(\frac{-1}{2\sigma_1^2}\Big(\big(-\ln(x) - \mu_1 - \ln(m_1)\big)^2 - \big(-\ln(x) - \ln(\lambda) - \mu_1 - \ln(m_1)\big)^2\Big)\right) \tag{A103}$$

$$= \lim_{x\to\infty} \exp\left(\frac{1}{2\sigma_1^2}\Big(2\ln(\lambda)(\ln(x) + \mu_1 + \ln(m_1)) + \ln(\lambda)^2\Big)\right) = 0. \tag{A104}$$

Similarly, for $\rho < \sigma_1/\sigma_2$,

$$\lim_{x\to\infty} \frac{h_1(\lambda x)}{h_1(x)} = \lim_{x\to\infty} \frac{f_{\text{Gao}}\left(\ln\left(\frac{1}{x}\right), m_1, 1\right)}{f_{\text{Gao}}\left(\ln\left(\frac{1}{\lambda x}\right), m_1, 1\right)} \tag{A105}$$

$$\begin{aligned} = \lim_{x\to\infty} \sqrt{\frac{\ln(\lambda x)}{\ln(x)}} \exp\Bigg(& -S\left(-\ln(x) + \ln(\lambda x)\right) \\ & - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\Big[(s_1+s_2)\left(\ln(x)^2 - \ln(\lambda x)^2\right) \\ & - 2\big(-\ln(x) + \ln(\lambda x)\big)\big(s_1\mu_2 + s_2(\mu_1 + \ln(m_1))\big)\Big]\Bigg) \end{aligned} \tag{A106}$$

$$= \lim_{x\to\infty} \exp\Bigg(- S\ln(\lambda) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\Big[(s_1+s_2)\left(-2\ln(\lambda)\ln(x) - \ln(\lambda)^2\right) \\ - 2\ln(\lambda)\big(s_1\mu_2 + s_2(\mu_1 + \ln(m_1))\big)\Big]\Bigg) \tag{A107}$$

$$= 0, \tag{A108}$$

since $s_1 + s_2 > 0$. Hence, h_1 is weakly pseudo-monotone of positive variation (WPMPV) (cf. Definition 2.2 of Buldygin et al. [2005a]). Due to $\lim_{x\to\infty} h_1(x)/h_2(x) = 1$ by L'Hôpital's rule and Lemma A.10, and since h_1 and h_2 are strictly increasing in x, Theorem 9.1 of Buldygin et al. [2005b] can be applied, yielding that the inverses of h_1 and h_2 are asymptotically equivalent, i.e.

$$1 = \lim_{x\to\infty} \frac{h_1^{-1}(x)}{h_2^{-1}(x)} = \lim_{x\to\infty} \frac{F^{-1}_{m_1A_1+A_2}(1/x)}{F^{-1}_{\text{Gao}}(1/x, m_1, 1)} = \lim_{u\to 0} \frac{F^{-1}_{m_1A_1+A_2}(u)}{F^{-1}_{\text{Gao}}(u, m_1, 1)}. \tag{A109}$$

Analogously, one can show that the inverses of $F_{\text{Gao}}(\cdot, 1, m_2)$ and $F_{A_1+m_2A_2}$ are asymptotically equivalent, i.e.

$$\lim_{u\to 0} \frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{\text{Gao}}(u, 1, m_2)} = 1. \tag{A110}$$

If $\rho > \sigma_1/\sigma_2$, we have $F_{\text{Gao}}(x, m_1, 1) = F_{A_1}(x/m_1)$ and $F_{\text{Gao}}(x, 1, m_2) = F_{A_1}(x)$, and therefore $F^{-1}_{\text{Gao}}(u, m_1, 1) = m_1 F^{-1}_{A_1}(u)$ and $F^{-1}_{\text{Gao}}(u, 1, m_2) = F^{-1}_{A_1}(u)$. Hence, by (A109) and (A110),

$$\lim_{u\to 0} \frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{m_1A_1+A_2}(u)} = \frac{1}{m_1} \tag{A111}$$

if $\rho > \sigma_1/\sigma_2$. If $\rho < \sigma_1/\sigma_2$, we need some further reasoning. As we can write

$$\frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{m_1A_1+A_2}(u)} = \frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{\text{Gao}}(u, 1, m_2)} \times \frac{F^{-1}_{\text{Gao}}(u, m_1, 1)}{F^{-1}_{m_1A_1+A_2}(u)} \times \frac{F^{-1}_{\text{Gao}}(u, 1, m_2)}{F^{-1}_{\text{Gao}}(u, m_1, 1)}, \tag{A112}$$

it remains to calculate the limit of $\frac{F^{-1}_{\text{Gao}}(u,1,m_2)}{F^{-1}_{\text{Gao}}(u,m_1,1)}$, $u \to 0$. For that, we set

$$\tau := (m_2)^{s_1/(s_1+s_2)} \left(\frac{1}{m_1}\right)^{s_2/(s_1+s_2)}, \tag{A113}$$

and by L'Hôpital's rule and Lemma A.10,

$$\lim_{y\to 0} \frac{F_{\text{Gao}}(\tau y, 1, m_2)}{F_{\text{Gao}}(y, m_1, 1)} = \lim_{y\to 0} \frac{f_{\text{Gao}}(\ln(\tau y), 1, m_2)}{f_{\text{Gao}}(\ln(y), m_1, 1)} \tag{A114}$$

$$= \frac{c(1,m_2)}{c(m_1,1)} \lim_{y\to 0} \sqrt{\frac{\ln(y)}{\ln(\tau y)}} \exp\Big(- S\big(\ln(\tau y) - \ln(y)\big) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\Big[(s_1+s_2)\left(\ln(\tau y)^2 - \ln(y)^2\right) - 2\ln(\tau y)\big(s_1(\mu_2+\ln(m_2)) + s_2\mu_1\big) + 2\ln(y)\big(s_1\mu_2 + s_2(\mu_1+\ln(m_1))\big)\Big]\Big) \tag{A115}$$

$$= \frac{c(1,m_2)}{c(m_1,1)} \lim_{y\to 0} \exp\Big(- S\ln(\tau) - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\Big[(s_1+s_2)\big(2\ln(\tau)\ln(y) + \ln(\tau)^2\big) - 2\ln(y)\big(s_1\ln(m_2) - s_2\ln(m_1)\big) - 2\ln(\tau)\big(s_1(\mu_2+\ln(m_2)) + s_2\mu_1\big)\Big]\Big) \tag{A116}$$

$$= \frac{c(1,m_2)}{c(m_1,1)} \lim_{y\to 0} \exp\Big(\text{const.} - \frac{1}{2(1-\rho^2)\sigma_1^2\sigma_2^2} \times \Big[2\ln(y)\underbrace{\left(\ln(\tau)(s_1+s_2) - s_1\ln(m_2) + s_2\ln(m_1)\right)}_{=0 \text{ by (A113)}}\Big]\Big) \tag{A117}$$

$$=: \iota > 0, \tag{A118}$$

i.e. with $h_3(x) := 1/F_{\text{Gao}}(\tau/x, 1, m_2)$ and $h_4(x) := h_1(x)/\iota$ it follows that

$$1 = \lim_{y\to 0} \frac{F_{\text{Gao}}(\tau y, 1, m_2)}{\iota \times F_{\text{Gao}}(y, m_1, 1)} = \lim_{x\to\infty} \frac{h_4(x)}{h_3(x)}. \tag{A119}$$

By (A105)–(A108) h_4 is WPMPV and by Theorem 9.1 of Buldygin et al. [2005b],

$$1 = \lim_{x\to\infty} \frac{h_4^{-1}(x)}{h_3^{-1}(x)} = \lim_{x\to\infty} \frac{F_{\text{Gao}}^{-1}(1/x, 1, m_2)}{\tau \times F_{\text{Gao}}^{-1}(1/(\iota x), m_1, 1)} = \lim_{u\to 0} \frac{F_{\text{Gao}}^{-1}(u, 1, m_2)}{\tau \times F_{\text{Gao}}^{-1}(u/\iota, m_1, 1)}, \tag{A120}$$

i.e.

$$\lim_{u\to 0} \frac{F_{\text{Gao}}^{-1}(u, 1, m_2)}{F_{\text{Gao}}^{-1}(u/\iota, m_1, 1)} = \tau. \tag{A121}$$

Furthermore, (A105)–(A108) show that $\lim_{x\to\infty} h_1(\lambda x)/h_1(x) = 0$ for all $\lambda \in (0,1)$ and $\lim_{x\to\infty} h_1(\lambda x)/h_1(x) = \infty$ for all $\lambda > 1$, i.e. h_1 is rapidly varying. Due to $\lim_{x\to\infty} h_1(x) = \infty$ and since h_1 is strictly increasing in x for x sufficiently big, we can apply part b) of the remark on p. 1400 of Djurčić and Torgašev [2007], implying that h_1^{-1} is slowly varying, i.e. $\lim_{x\to\infty} h_1^{-1}(\lambda x)/h_1^{-1}(x) = 1$ for all $\lambda > 0$. Hence, for $\lambda = \iota$,

$$1 = \lim_{x\to\infty} \frac{F_{\text{Gao}}^{-1}(1/x, m_1, 1)}{F_{\text{Gao}}^{-1}(1/(\iota x), m_1, 1)} = \lim_{u\to 0} \frac{F_{\text{Gao}}^{-1}(u, m_1, 1)}{F_{\text{Gao}}^{-1}(u/\iota, m_1, 1)}. \tag{A122}$$

Then the assertion for $\rho < \sigma_1/\sigma_2$ follows from (A109), (A110), (A112), (A121) and (A122). $\square$

A.4.2 Limit of $\vartheta(y)$ for $y \to \infty$

Theorem A.12. *Let (X_1, X_2) be distributed as in Assumption 5.6, and let*

$$\mu_{X_1+X_2} := \max_{k:\sigma_k=\sigma_2} \mu_k, \quad m_{X_1+X_2} := \#\{k : \sigma_k = \sigma_2, \, \mu_k = \mu_{X_1+X_2}\} \in \{1, 2\}. \tag{A123}$$

Then

$$P(X_1 + X_2 > x) \sim m_{X_1+X_2} \times P(Z > x), \quad x \to \infty, \tag{A124}$$

where $Z \sim \mathcal{LN}(\mu_{X_1+X_2}, \sigma_2^2)$ and

$$\bar{F}_{\mu_{X_1+X_2},\sigma_2^2}(z) := P(Z > x) \sim \frac{\sigma_2}{\sqrt{2\pi}(\ln(x) - \mu_{X_1+X_2})} \exp\left(\frac{-1}{2\sigma_2^2}\Big(\ln(x) - \mu_{X_1+X_2}\Big)^2\right) \tag{A125}$$

for $x \to \infty$. Furthermore,

$$f_{X_1+X_2}(x) \sim m_{X_1+X_2} \times f_Z(x), \quad x \to \infty. \tag{A126}$$

Proof. Cf. Theorem 1 and Remark 1 of Asmussen and Rojas-Nandayapa [2008] and Theorem 2.4 of Gao et al. [2009]. $\square$

Using Theorem A.12, we can show the following lemma.

Lemma A.13. *Let (A_1, A_2) be distributed as in Assumption 5.6 and let $m_1, m_2 \in (0, 1)$. Then*

$$\lim_{y\to\infty} \vartheta(y) = \begin{cases} m_2, & \sigma_1 < \sigma_2, \text{ or } \sigma_1 = \sigma_2 \text{ and } \mu_1 \le \mu_2 + \ln(m_2), \\ \frac{1}{m_1}, & \sigma_1 = \sigma_2 \text{ and } \mu_2 \le \mu_1 + \ln(m_1), \\ \exp(\mu_1 - \mu_2) \in \left(m_2, \frac{1}{m_1}\right), & \sigma_1 = \sigma_2 \text{ and } \mu_1 > \mu_2 + \ln(m_2) \\ & \text{and } \mu_2 > \mu_1 + \ln(m_1). \end{cases} \tag{A127}$$

Proof. First, with $u = F_{m_1A_1+A_2}(y)$,

$$\lim_{y\to\infty} \vartheta(y) = \lim_{u\to 1} \frac{F^{-1}_{A_1+m_2A_2}(u)}{F^{-1}_{m_1A_1+A_2}(u)} = \lim_{u\to 1} \frac{\bar{F}^{-1}_{A_1+m_2A_2}(1-u)}{\bar{F}^{-1}_{m_1A_1+A_2}(1-u)} = \lim_{u\to 0} \frac{\bar{F}^{-1}_{A_1+m_2A_2}(u)}{\bar{F}^{-1}_{m_1A_1+A_2}(u)}. \tag{A128}$$

By Theorem A.12,

$$\lim_{x\to\infty} \frac{\bar{F}_{m_1A_1+A_2}(x)}{m_{m_1A_1+A_2}\bar{F}_{\mu_{m_1A_1+A_2},\sigma_2^2}(x)} = 1. \tag{A129}$$

In order to see whether the corresponding inverses are asymptotically equivalent as well, we will again make use of Theorem 9.1 of Buldygin et al. [2005b]. For that, let

$$h_1(x) := \frac{1}{m_{m_1A_1+A_2}\bar{F}_{\mu_{m_1A_1+A_2},\sigma_2^2}(x)}, \quad h_2(x) := \frac{1}{\bar{F}_{m_1A_1+A_2}(x)}, \tag{A130}$$

i.e. $\lim_{x\to\infty}\frac{h_1(x)}{h_2(x)} = 1$. Furthermore, for $\lambda \in (0,1)$, (A125) implies

$$\lim_{x\to\infty}\frac{h_1(\lambda x)}{h_1(x)} = \lim_{x\to\infty}\frac{\bar{F}_{\mu_{m_1A_1+A_2},\sigma_2^2}(x)}{\bar{F}_{\mu_{m_1A_1+A_2},\sigma_2^2}(\lambda x)} \tag{A131}$$

$$= \lim_{x\to\infty}\frac{\ln(x)+\ln(\lambda)-\mu_{m_1A_1+A_2}}{\ln(x)-\mu_{m_1A_1+A_2}}\times \exp\left(\frac{-1}{2\sigma_2^2}\Big(\big(\ln(x)-\mu_{m_1A_1+A_2}\big)^2-\big(\ln(x)+\ln(\lambda)-\mu_{m_1A_1+A_2}\big)^2\Big)\right) \tag{A132}$$

$$= \lim_{x\to\infty}\exp\left(\frac{-1}{2\sigma_2^2}\Big(-2\ln(\lambda)(\ln(x)-\mu_{m_1A_1+A_2})-\ln(\lambda)^2\Big)\right) = 0, \tag{A133}$$

i.e. h_1 is WPMPV. Since h_1 and h_2 are strictly increasing in x, Theorem 9.1 of Buldygin et al. [2005b] can be applied, yielding

$$1 = \lim_{x\to\infty}\frac{h_1^{-1}(x)}{h_2^{-1}(x)} = \lim_{x\to\infty}\frac{\bar{F}^{-1}_{\mu_{m_1A_1+A_2},\sigma_2^2}\left(\frac{1}{m_{m_1A_1+A_2}\,x}\right)}{\bar{F}^{-1}_{m_1A_1+A_2}\left(\frac{1}{x}\right)} = \lim_{u\to 0}\frac{\bar{F}^{-1}_{\mu_{m_1A_1+A_2},\sigma_2^2}\left(u/m_{m_1A_1+A_2}\right)}{\bar{F}^{-1}_{m_1A_1+A_2}(u)}. \tag{A134}$$

Similarly, $\lim_{u\to 0}\frac{\bar{F}^{-1}_{A_1+m_2A_2}(u)}{\bar{F}^{-1}_{\mu_{A_1+m_2A_2},\sigma_2^2}(u/m_{A_1+m_2A_2})} = 1$, i.e.

$$\lim_{u\to 0}\frac{\bar{F}^{-1}_{A_1+m_2A_2}(u)}{\bar{F}^{-1}_{m_1A_1+A_2}(u)} = \lim_{u\to 0}\frac{\bar{F}^{-1}_{\mu_{A_1+m_2A_2},\sigma_2^2}\left(u/m_{A_1+m_2A_2}\right)}{\bar{F}^{-1}_{\mu_{m_1A_1+A_2},\sigma_2^2}\left(u/m_{m_1A_1+A_2}\right)} \tag{A135}$$

$$= \lim_{u\to 0}\exp\Big(\sigma_2\big(\Phi^{-1}\left(u/m_{m_1A_1+A_2}\right)-\Phi^{-1}\left(u/m_{A_1+m_2A_2}\right)\big)+\mu_{A_1+m_2A_2}-\mu_{m_1A_1+A_2}\Big), \tag{A136}$$

provided the limit exists. In order to determine this limit, we need to distinguish between several cases.

Case 1: $\sigma_1 < \sigma_2$. Then (cf. (A123)) $\mu_{A_1+m_2A_2} = \mu_2 + \ln(m_2)$, $m_{A_1+m_2A_2} = 1$, $\mu_{m_1A_1+A_2} = \mu_2$ and $m_{m_1A_1+A_2} = 1$, and the assertion follows.
Case 2: $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2 + \ln(m_2)$. Then $\mu_1 + \ln(m_1) < \mu_2$, i.e. $\mu_{A_1+m_2A_2} = \mu_2 + \ln(m_2)$, $m_{A_1+m_2A_2} = 2$, $\mu_{m_1A_1+A_2} = \mu_2$ and $m_{m_1A_1+A_2} = 1$. Thus, the exp-term in (A136) equals $\exp\big(\sigma_2\big(\Phi^{-1}(u)-\Phi^{-1}(u/2)\big)+\ln(m_2)\big)$, which has a limes superior of at

most $1/m_1$ by Lemma A.9, (A128) and (A135)–(A136), and therefore

$$1 \leq \liminf_{u\to 0} \exp\Big(\sigma_2\big(\Phi^{-1}(u) - \Phi^{-1}(u/2)\big)\Big) \tag{A137}$$

$$\leq \limsup_{u\to 0} \exp\Big(\sigma_2\big(\Phi^{-1}(u) - \Phi^{-1}(u/2)\big)\Big) \leq \frac{1}{m_1 m_2} \quad \text{for all } m_1, m_2 \in (0,1). \tag{A138}$$

Hence, since the term within the limes superior does not depend on m_1 or m_2, we have

$$\lim_{u\to 0} \exp\Big(\sigma_2\big(\Phi^{-1}(u) - \Phi^{-1}(u/2)\big)\Big) = 1, \tag{A139}$$

implying that the limit in (A136) exists and equals m_2.
Case 3: $\sigma_1 = \sigma_2$ and $\mu_2 = \mu_1 + \ln(m_1)$. Then $\mu_2 + \ln(m_2) < \mu_1$, i.e. $\mu_{A_1+m_2A_2} = \mu_1$, $m_{A_1+m_2A_2} = 1$, $\mu_{m_1A_1+A_2} = \mu_1 + \ln(m_1)$ and $m_{m_1A_1+A_2} = 2$. Thus, the exp-term in (A136) equals $\exp\big(\sigma_2\big(\Phi^{-1}(u/2) - \Phi^{-1}(u)\big) - \ln(m_1)\big)$, i.e. by (A139) the limit in (A136) exists and equals $1/m_1$.
Case 4: $\sigma_1 = \sigma_2$ and $\mu_1 \neq \mu_2 + \ln(m_2)$ and $\mu_2 \neq \mu_1 + \ln(m_1)$. Then $m_{A_1+m_2A_2} = m_{m_1A_1+A_2} = 1$ and the exp-term in (A136) reduces to $\exp(\mu_{A_1+m_2A_2} - \mu_{m_1A_1+A_2})$. We distinguish between the following three subcases:
Case 4a: $\mu_1 < \mu_2 + \ln(m_2)$. Then $\mu_{A_1+m_2A_2} = \mu_2 + \ln(m_2)$ and $\mu_{m_1A_1+A_2} = \mu_2$, i.e. the limit in (A136) is m_2.
Case 4b: $\mu_2 < \mu_1 + \ln(m_1)$. Then $\mu_{A_1+m_2A_2} = \mu_1$ and $\mu_{m_1A_1+A_2} = \mu_1 + \ln(m_1)$, i.e. the limit in (A136) is $1/m_1$.
Case 4c: $\mu_1 > \mu_2 + \ln(m_2)$ and $\mu_2 > \mu_1 + \ln(m_1)$. Then $\mu_{A_1+m_2A_2} = \mu_1$ and $\mu_{m_1A_1+A_2} = \mu_2$, i.e. the limit in (A136) is $\exp(\mu_1 - \mu_2) \in (m_2, 1/m_1)$. □

A.5 Some Useful Determinants

Lemma A.14. *Let* $\mathbf{A} = (a_{ij})_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m}$, $m \geq 2$, *be such that* $a_{ii} = -1$ *for all* $i \geq 2$. *Furthermore, let the remaining entries of* $\mathbf{A}$ *be non-negative and let*

$$\sum_{i=2, i\neq j}^{m} a_{ij} < 1 \quad \textit{for all } j \geq 2. \tag{A140}$$

Then

$$\det(\mathbf{A}) \begin{cases} \geq 0, & m \textit{ odd}, \\ \leq 0, & m \textit{ even}. \end{cases} \tag{A141}$$

Proof. By induction. For $m = 2$,

$$\det(\mathbf{A}) = -a_{11} - a_{12}a_{21} \leq 0. \tag{A142}$$

Let now (A141) hold for some $m > 2$. We show that (A141) then also holds for an $(m+1) \times (m+1)-$matrix $\mathbf{B}$ of the described type. For that, we expand $\mathbf{B}$ along the first row, yielding $\det(\mathbf{B}) = \sum_{j=1}^{m+1}(-1)^{1+j} b_{1j} \det(\mathbf{B}_{1j})$, where $\mathbf{B}_{1j} \in \mathbb{R}^{m\times m}$ denotes the reduced matrix corresponding to the expansion by b_{1j}.

For $j = 1$, the diagonal elements of $\mathbf{B}_{11}$ are all equal to -1, and the sum of the remaining elements is strictly smaller than 1 in each column by (A140). Writing $\det(\mathbf{B}_{11}) = \det(\mathbf{B}'_{11} - \mathbf{I}_m) = (-1)^m \det(\mathbf{I}_m - \mathbf{B}'_{11})$, the last determinant is strictly positive by Lemma A.1 of Gouriéroux et al. [2012], i.e. the sign of $\det(\mathbf{B}_{11})$ depends on m only.

Let now $j > 1$. Deleting the first row of $\mathbf{B}$ shifts all entries one row upwards, in addition the elements right of column j are also shifted one column to the left. Hence, the entries equal to -1 to the lower right of b_{1j} remain on the diagonal, whereas the entries equal to -1 to the upper left of b_{1j} are shifted to the superdiagonal. Since there are $j-2$ entries equal to -1 to the upper left of b_{1j}, $j-2$ row swaps are needed to shift them back to the diagonal. The resulting $m \times m-$matrix $\mathbf{B}'_{1j}$ then has $m-1$ entries equal to -1, which can all be found on the diagonal. Furthermore, by one row swap and one column swap, an entry equal to -1 can be moved within the diagonal, so these operations do not alter the determinant of $\mathbf{B}'_{1j}$, and we can assume w.l.o.g. that the first diagonal element is not -1. Furthermore, this entry is non-negative, i.e. we are exactly in the situation of the induction assumption, and it follows that $\det(\mathbf{B}'_{1j})$ is non-negative if m is odd and non-positive if m is even. Altogether,

$$\det(\mathbf{B}) = \sum_{j=1}^{m+1} (-1)^{1+j}\, b_{1j} \det(\mathbf{B}_{1j}) \tag{A143}$$

$$= (-1)^{m+2}\, b_{11} \det(\mathbf{I}_m - \mathbf{B}'_{11}) + \sum_{j=2}^{m+1} (-1)^{1+j}(-1)^{j-2}\, b_{1j} \det(\mathbf{B}'_{1j}) \tag{A144}$$

$$= (-1)^m \underbrace{b_{11}}_{\geq 0} \underbrace{\det(\mathbf{I}_m - \mathbf{B}'_{11})}_{>0} - \sum_{j=2}^{m+1} \underbrace{b_{1j}}_{\geq 0} \underbrace{\det(\mathbf{B}'_{1j})}_{\substack{\geq 0 \text{ if } m \text{ odd,} \\ \leq 0 \text{ if } m \text{ even}}} \tag{A145}$$

$$\begin{cases} \leq 0, & m \text{ odd, i.e. } m+1 \text{ even,} \\ \geq 0, & m \text{ even, i.e. } m+1 \text{ odd.} \end{cases} \tag{A146}$$

□

Lemma A.15. *Let $k, p \in \{1, \ldots, n\}$ ($n \geq 2$, $k \neq p$) and let $\mathbf{A} = (a_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ be such that $a_{ii} = -1$ for all $i \notin \{k, p\}$. Furthermore, let the non-diagonal entries of $\mathbf{A}$ be non-negative with*

$$\sum_{i=1, i\neq j}^{n} a_{ij} < 1 \quad \text{for all } j \in \{1, \ldots, n\}. \tag{A147}$$

If $\mathbf{A}_{kp}$ denotes the submatrix of $\mathbf{A}$ where the kth row and pth column of $\mathbf{A}$ have been deleted,

$$(-1)^{k+p} \det(\mathbf{A}_{kp}) \begin{cases} \geq 0, & n \text{ odd,} \\ \leq 0, & n \text{ even.} \end{cases} \tag{A148}$$

Proof. For the purpose of illustration, let us first consider the following segment of $\mathbf{A}$ before and after deleting the kth row and pth column:

$$\begin{pmatrix} -1 & & & & & & & \\ & -1 & & & & & & \\ & & & & & & & \\ & & & -1 & & & & \\ & & & \leftarrow & -1 & & & \\ & & a_{kp} & & & & & \\ & & & & & & -1 & \\ & & & & & & \nwarrow & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & & & & & & \\ & -1 & & & & & \\ & & \cdot & & & & \\ & & -1 & \cdot & & & \\ & & & -1 & \cdot & & \\ & & & & & -1 & \\ & & & & & & -1 \end{pmatrix}$$

Hence, elements equal to -1 to the upper left of a_{kp} or lower right of a_{kp} remain on the diagonal. If a_{kp} stands below the diagonal as in our example, elements to the upper right of a_{kp} are shifted to the left by one column, i.e. the elements equal to -1 are shifted from the diagonal to the subdiagonal. This is also indicated by the arrows in the non-reduced matrix. If a_{kp} stands above the diagonal, elements equal to -1 to the lower left of a_{kp} are shifted from the diagonal to the superdiagonal. The reduced matrix $\mathbf{A}_{kp} \in \mathbb{R}^{(n-1)\times(n-1)}$ contains $n-2$ entries equal to -1, and the number of entries equal to -1 not on the diagonal depends on the "distance" of a_{kp} to the diagonal of A. It is straightforward to see that $|k-p|-1$ entries equal to -1 are shifted to the subdiagonal or superdiagonal. Hence, $|k-p|-1$ row swaps or column swaps are necessary to move them to the diagonal of the reduced matrix. We denote the matrix resulting from these swaps with $\mathbf{A}'_{kp}$, i.e. all $n-2$ entries of $\mathbf{A}'_{kp} \in \mathbb{R}^{(n-1)\times(n-1)}$ equal to -1 stand on the diagonal, and we can assume w.l.o.g. that the first diagonal entry is not -1, since one row swap and one column swap (i.e. an even number of swaps) is necessary to move an entry by one place within the diagonal. Hence, $\mathbf{A}'_{kp}$ meets the assumptions of Lemma A.14 with $m = n-1$, and we obtain

$$(-1)^{k+p}\det(\mathbf{A}_{kp}) = \underbrace{(-1)^{k+p}(-1)^{|k-p|-1}}_{=-1}\det(\mathbf{A}'_{kp}) \begin{cases} \geq 0, & n \text{ odd}, \\ \leq 0, & n \text{ even}. \end{cases} \tag{A149}$$

□

A.6 Derivation of the Effect of Contagion c_1

The following lemma contains the effect of contagion under Suzuki's model for the two firms case.

Lemma A.16. *In the set-up of Section 7.1.1, c_1 defined in (7.7) with $\mathbf{a} \in A_{\mathrm{ss}}$ equals*

$$
c_1 = \begin{cases}
\frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}, & \tilde{\mathbf{a}} \in A_{\mathrm{ss}},\\
\frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}} + (M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})\frac{M^{\mathrm{e}}_{2,1}a_1+a_2+(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1-(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{sd}},\\
\frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1}} + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})\frac{a_1+M^{\mathrm{e}}_{1,2}a_2-(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})d_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2}{(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{ds}},\\
\frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}} + \frac{(M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})a_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2}+M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1}))a_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})} & \\
\qquad - \frac{M^{\mathrm{e}}_{1,2}(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1+(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{d}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}, & \tilde{\mathbf{a}} \in A_{\mathrm{dd}}.
\end{cases}
\tag{A150}
$$

Proof. For $\tilde{\mathbf{a}} \in A_{\mathrm{ss}}$, (3.23) and (7.9) imply

$$
\begin{aligned}
c_1 &= \frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}\Big(a_1+\delta_1+M^{\mathrm{e}}_{1,2}(a_2+\delta_2)+M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2\\
&\qquad -\Big(a_1+M^{\mathrm{e}}_{1,2}a_2+M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2\Big)\Big)-\delta_1 &\text{(A151)}\\
&= \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}. &\text{(A152)}
\end{aligned}
$$

If $\tilde{\mathbf{a}} \in A_{\mathrm{sd}}$,

$$
\begin{aligned}
c_1 &= \frac{1}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}\Big(a_1+\delta_1+M^{\mathrm{d}}_{1,2}(a_2+\delta_2)+M^{\mathrm{d}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1\Big)\\
&\quad -\frac{1}{1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1}}\Big(a_1+M^{\mathrm{e}}_{1,2}a_2+M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1+(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2\Big)-\delta_1 &\text{(A153)}\\
&= \frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}+\frac{M^{\mathrm{e}}_{2,1}(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})a_1}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}\\
&\quad +\frac{(M^{\mathrm{d}}_{1,2}(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})-M^{\mathrm{e}}_{1,2}(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}))a_2-(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}\\
&\quad +\frac{(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})(M^{\mathrm{d}}_{1,2}(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})-M^{\mathrm{e}}_{1,2}(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}))d_1}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})} &\text{(A154)}\\
&= \frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{e}}_{2,1}\delta_1+\delta_2)}{1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1}}\\
&\quad +(M^{\mathrm{d}}_{1,2}-M^{\mathrm{e}}_{1,2})\frac{M^{\mathrm{e}}_{2,1}a_1+a_2+(M^{\mathrm{d}}_{2,1}-M^{\mathrm{e}}_{2,1})d_1-(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})d_2}{(1-M^{\mathrm{d}}_{1,2}M^{\mathrm{e}}_{2,1})(1-M^{\mathrm{e}}_{1,2}M^{\mathrm{e}}_{2,1})}. &\text{(A155)}
\end{aligned}
$$

If $\tilde{\mathbf{a}} \in A_{\mathrm{ds}}$,

$$\begin{aligned} c_1 &= \frac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1}} \Big(a_1 + \delta_1 + M^{\mathrm{e}}_{1,2}(a_2 + \delta_2) + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2}) d_2 \Big) \\ &\quad - \frac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}} \Big(a_1 + M^{\mathrm{e}}_{1,2} a_2 + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) d_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2}) d_2 \Big) - \delta_1 \end{aligned} \tag{A156}$$

$$\begin{aligned} &= \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1 + \delta_2)}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1}} + \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) a_1 + (M^{\mathrm{e}}_{1,2})^2 (M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) a_2}{(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1})(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})} \\ &\quad + \frac{-M^{\mathrm{e}}_{1,2}(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) d_1 + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2})(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) d_2}{(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1})(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})} \end{aligned} \tag{A157}$$

$$\begin{aligned} &= \frac{M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1 + \delta_2)}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1}} \\ &\quad + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) \frac{a_1 + M^{\mathrm{e}}_{1,2} a_2 - (1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1}) d_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2}) d_2}{(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{d}}_{2,1})(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})}. \end{aligned} \tag{A158}$$

If $\tilde{\mathbf{a}} \in A_{\mathrm{dd}}$,

$$\begin{aligned} c_1 &= \frac{1}{1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1}} \Big(a_1 + \delta_1 + M^{\mathrm{d}}_{1,2}(a_2 + \delta_2) \Big) \\ &\quad - \frac{1}{1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}} \Big(a_1 + M^{\mathrm{e}}_{1,2} a_2 + M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) d_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2}) d_2 \Big) - \delta_1 \end{aligned} \tag{A159}$$

$$\begin{aligned} &= \frac{M^{\mathrm{d}}_{1,2}(M^{\mathrm{d}}_{2,1}\delta_1 + \delta_2)}{1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1}} \\ &\quad + \frac{(M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1}) a_1 + (M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2} + M^{\mathrm{d}}_{1,2} M^{\mathrm{e}}_{1,2}(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1})) a_2}{(1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1})(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})} \\ &\quad - \frac{M^{\mathrm{e}}_{1,2}(1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{2,1} - M^{\mathrm{e}}_{2,1}) d_1 + (1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1})(M^{\mathrm{d}}_{1,2} - M^{\mathrm{e}}_{1,2}) d_2}{(1 - M^{\mathrm{d}}_{1,2} M^{\mathrm{d}}_{2,1})(1 - M^{\mathrm{e}}_{1,2} M^{\mathrm{e}}_{2,1})}. \end{aligned} \tag{A160}$$

□

A.7 Column Sums of the Schur Complement of Strictly Column Diagonally Dominant Matrices

For a matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ and non-empty sets $M_1, M_2 \subseteq N := \{1, \ldots, n\}$, $n \in \mathbb{N}$, let $\mathbf{A}_{M_1,M_2}$ denote the $|M_1| \times |M_2|$-submatrix of $\mathbf{A}$ where only the rows with index in M_1 and the columns with index in M_2 have been retained. If $M_1 = N$ or $M_2 = N$, we write

$\mathbf{A}_{\cdot,M_2}$ and $\mathbf{A}_{M_1,\cdot}$, respectively.
Let N be divided into two disjoint and non-empty subsets N_1 and N_2 and let $\mathbf{A}_{N_1,N_1}$ be invertible. Then the Schur complement (cf. Puntanen and Styan [2005], for example) of $\mathbf{A}$ with respect to $\mathbf{A}_{N_1,N_1}$ is defined as

$$\mathbf{A}' := \mathbf{A}/\mathbf{A}_{N_1,N_1} := \mathbf{A}_{N_2,N_2} - \mathbf{A}_{N_2,N_1}\mathbf{A}_{N_1,N_1}^{-1}\mathbf{A}_{N_1,N_2} \in \mathbb{C}^{|N_2|\times|N_2|}. \tag{A161}$$

The Schur complement is for example useful for the calculation of the determinant of the block matrix $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{N_1,N_1} & \mathbf{A}_{N_1,N_2} \\ \mathbf{A}_{N_2,N_1} & \mathbf{A}_{N_2,N_2} \end{pmatrix}$, given as $\det(\mathbf{A}) = \det(\mathbf{A}_{N_1,N_1})\det(\mathbf{A}/\mathbf{A}_{N_1,N_1})$.
Furthermore, an arbitrary complex matrix $\mathbf{B} = (b_{ij})_{1\leq i,j\leq n} \in \mathbb{C}^{n\times n}$ is called strictly row diagonally dominant (cf. Liu and Zhang [2005], for example), if

$$|b_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{n} |b_{ij}| > 0 \quad \text{for all } i \in \{1,\ldots,n\}. \tag{A162}$$

Similarly, $\mathbf{B}$ is called strictly column diagonally dominant if

$$|b_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{n} |b_{ji}| > 0 \quad \text{for all } i \in \{1,\ldots,n\}. \tag{A163}$$

The comparison matrix $\tilde{\mathbf{C}} = (\tilde{c}_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ of a complex matrix $\mathbf{C} = (c_{ij})_{1\leq i,j\leq n} \in \mathbb{C}^{n\times n}$, as defined by for example Varga [2000], is given as

$$\tilde{c}_{ii} := |c_{ii}|, \quad \tilde{c}_{ij} := -|c_{ij}| \text{ for } i \neq j, \quad 1 \leq i,j \leq n. \tag{A164}$$

The following considerations are mainly based on the work of Liu and Zhang [2005], but in contrast to them, we will denote all results for strictly column diagonally dominant matrices instead of rows. Moreover, we will employ the notation of Lemma 7.2, since we will apply the results of the present section only in the proof of Lemma 7.2. Hence, $\bar{D}$ and $\bar{S}$ stand for the subsets of firms that are in default and solvent after the shock $\boldsymbol{\delta}'$ (cf. (7.17)), respectively. Note that due to $D \subseteq \bar{D}$, $\bar{D}$ is non-empty. Furthermore, we assume $\bar{S} \neq \emptyset$ since Lemma A.17 is needed for this case only.

Lemma A.17. *Let $k := |\bar{S}| > 0$ and $l := |\bar{D}| > 0$. Then the jth column sum of $\mathbf{I}_l - \mathbf{M}^{\mathrm{y}}_{\bar{D},\bar{D}} - \mathbf{M}^{\mathrm{x}}_{\bar{D},\bar{S}}\big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big)^{-1}\mathbf{M}^{\mathrm{y}}_{\bar{S},\bar{D}}$, $j \in 1,\ldots,l$, is greater than or equal to 1 minus the jth column sum of $\mathbf{M}^{\mathrm{y}}_{\cdot,\bar{D}}$, with $\mathbf{M}^{\mathrm{x}}, \mathbf{M}^{\mathrm{y}} \in \{\mathbf{M}^{\mathrm{d}}, \mathbf{M}^{\mathrm{e}}\}$.*
Furthermore, the diagonal elements of $\mathbf{I}_l - \mathbf{M}^{\mathrm{y}}_{\bar{D},\bar{D}} - \mathbf{M}^{\mathrm{x}}_{\bar{D},\bar{S}}\big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big)^{-1}\mathbf{M}^{\mathrm{y}}_{\bar{S},\bar{D}}$ are strictly positive.

Proof. With $n = k + l$, let

$$\mathbf{A} := (a_{ij})_{1\leq i,j\leq n} := \begin{pmatrix} \mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}} & -\mathbf{M}^{\mathrm{y}}_{\bar{S},\bar{D}} \\ -\mathbf{M}^{\mathrm{x}}_{\bar{D},\bar{S}} & \mathbf{I}_l - \mathbf{M}^{\mathrm{y}}_{\bar{D},\bar{D}} \end{pmatrix}. \tag{A165}$$

Obviously, $\mathbf{A}' = (a'_{ij})_{1\le i,j\le l} := \mathbf{I}_l - \mathbf{M}^{\mathrm{y}}_{\bar{D},\bar{D}} - \mathbf{M}^{\mathrm{x}}_{\bar{D},\bar{S}}\big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big)^{-1}\mathbf{M}^{\mathrm{y}}_{\bar{S},\bar{D}}$ is the Schur complement of $\mathbf{A}$ with respect to $\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}$. By Lemma A.1 of Fischer [2014], $\big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big)^{-1} \ge 0$, and therefore $a'_{ij} \le 0$ for $1 \le i,j \le l$, $i \ne j$. It is straightforward to see that $\mathbf{A}$ is strictly column diagonally dominant and that the comparison matrix $\tilde{\mathbf{A}}$ of $\mathbf{A}$ coincides with $\mathbf{A}$. Following the proof of Theorem 1 of Liu and Zhang [2005], we will now show that the jth column sum of $\mathbf{A}'$ ($j \in \{1,\ldots,l\}$) is greater than or equal to the $(k+j)$th column sum of $\mathbf{A}$, i.e. $\sum_{s=1}^{l} a'_{sj} \ge \sum_{s=1}^{n} a_{s,k+j}$. For that, we set $\tilde{\mathbf{M}} := \big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big)^{-1}$ and

$$w_t := \min_{1\le v\le k} \frac{\sum_{u=1}^{n} a_{uv}}{a_{vv}} \sum_{u=1}^{k} |a_{ut}| = \min_{1\le v\le k} \sum_{u=1}^{n} a_{uv} \times \sum_{u=1}^{k} |a_{ut}|, \quad t \in \{k+1,\ldots,n\}. \tag{A166}$$

Note that this definition of w_t exactly corresponds to (3.1) of Liu and Zhang [2005], except that we take column sums instead of row sums. Since $\sum_{u=1}^{n} a_{uv} > 0$ for all v because $\mathbf{A}$ is strictly column diagonally dominant, we have $w_t \ge 0$. Then

$$\sum_{s=1}^{l} a'_{sj} = \sum_{s=1}^{l} \Big(a_{k+s,k+j} - (a_{k+s,1}, a_{k+s,2}\ldots, a_{k+s,k})\,\tilde{\mathbf{M}}\,(a_{1,k+j}, a_{2,k+j},\ldots,a_{k,k+j})^T\Big) \tag{A167}$$

$$= \sum_{s=1}^{n} a_{s,k+j} + w_{k+j} - \sum_{s=1}^{k} a_{s,k+j} - w_{k+j} - \left(\sum_{s=1}^{l} a_{k+s,1}, \sum_{s=1}^{l} a_{k+s,2}\ldots, \sum_{s=1}^{l} a_{k+s,k}\right)\tilde{\mathbf{M}}\,(a_{1,k+j}, a_{2,k+j},\ldots,a_{k,k+j})^T \tag{A168}$$

$$= \sum_{s=1}^{n} a_{s,k+j} + w_{k+j} + \frac{1}{\det(\tilde{\mathbf{M}}^{-1})}\det(\mathbf{B}) \tag{A169}$$

with

$$\mathbf{B} := \begin{pmatrix} -\sum_{s=1}^{k} a_{s,k+j} - w_{k+j} & \sum_{s=1}^{l} a_{k+s,1} \;\; \sum_{s=1}^{l} a_{k+s,2} \;\cdots\; \sum_{s=1}^{l} a_{k+s,k} \\ a_{1,k+j} & \\ a_{2,k+j} & \tilde{\mathbf{M}}^{-1} \\ \vdots & \\ a_{k,k+j} & \end{pmatrix}, \tag{A170}$$

since the determinant of the quadratic block matrix $\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \tilde{\mathbf{M}}^{-1}\end{pmatrix}$ can be calculated as $\det(\mathbf{B}) = \det(\tilde{\mathbf{M}}^{-1})\det(\mathbf{B}_1 - \mathbf{B}_2\tilde{\mathbf{M}}\mathbf{B}_3)$. Because of $w_{k+j} \ge 0$ and $\det(\tilde{\mathbf{M}}^{-1}) = \det\big(\mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}\big) > 0$ by Lemma A.1 of Gouriéroux et al. [2012], it remains to show that $\det(\mathbf{B}) \ge 0$. Since the matrix $\tilde{\mathbf{M}}^{-1} = \mathbf{I}_k - \mathbf{M}^{\mathrm{x}}_{\bar{S},\bar{S}}$ coincides with its comparison matrix, we can apply Lemma 4 of Liu and Zhang [2005] (adopted for column sums instead of row sums, i.e. the matrices of Liu and Zhang [2005] must be transposed) to

$$\mathbf{B} = \begin{pmatrix} -\sum_{s=1}^{k} a_{s,k+j} - w_{k+j} & -\sum_{s=1}^{l} |a_{k+s,1}| \;\; -\sum_{s=1}^{l} |a_{k+s,2}| \;\; \ldots \;\; -\sum_{s=1}^{l} |a_{k+s,k}| \\ -|a_{1,k+j}| & \\ -|a_{2,k+j}| & \tilde{\mathbf{M}}^{-1} \\ \vdots & \\ -|a_{k,k+j}| & \end{pmatrix}. \tag{A171}$$

It follows that $\det(\mathbf{B}) \geq 0$ since

$$-\sum_{s=1}^{k} a_{s,k+j} - w_{k+j} = \sum_{s=1}^{k} |a_{s,k+j}| \left(1 - \min_{1\leq v\leq k} \sum_{u=1}^{n} a_{uv}\right) = \sum_{s=1}^{k} |a_{s,k+j}| \max_{1\leq v\leq k} \sum_{\substack{u=1\\u\neq v}}^{n} (-a_{uv}) \tag{A172}$$

$$= \sum_{s=1}^{k} |a_{s,k+j}| \max_{1\leq v\leq k} \sum_{\substack{u=1\\u\neq v}}^{n} |a_{uv}|, \tag{A173}$$

where the last line exactly corresponds to the RHS of the inequality (2.3) of Liu and Zhang [2005]. Hence,

$$\sum_{s=1}^{l} a'_{sj} \geq \sum_{s=1}^{n} a_{s,k+j} > 0, \tag{A174}$$

since $\mathbf{A}$ is strictly column diagonally dominant. Due to $a'_{sj} \leq 0$ for $s \neq j$, this implies $a'_{jj} > 0$ for all $j \in \{1, \ldots, l\}$. □

Corollary A.18. *In the situation of Lemma A.17, let* $\bar{S} = \{i_1, i_2, \ldots, i_k\}$, $\bar{D} = \{j_1, j_2, \ldots, j_l\}$ *and* $\tilde{\mathbf{M}} = (\tilde{M}_{tv})_{1\leq t,v\leq k}$. *Then*

$$\sum_{s=1}^{l} \sum_{t=1}^{k} M^{\mathrm{x}}_{j_s i_t} \tilde{M}_{tv} \leq 1 \quad \textit{for all } v \in \{1, \ldots, k\}. \tag{A175}$$

Proof. Let $\mathbf{A}$ and $\mathbf{A}'$ be defined as in the proof of Lemma A.17. Straightforward calculations and Lemma A.17 with x=y yield for $w \in \{1, \ldots, l\}$

$$1 - \sum_{s=1}^{l} M^{\mathrm{x}}_{j_s j_w} - \sum_{u=1}^{k} M^{\mathrm{x}}_{i_u j_w} = \sum_{s=1}^{n} a_{s,k+w} \leq \sum_{s=1}^{l} a'_{sw} \tag{A176}$$

$$= 1 - \sum_{s=1}^{l} M^{\mathrm{x}}_{j_s j_w} - \sum_{s=1}^{l} \sum_{t=1}^{k} \sum_{u=1}^{k} M^{\mathrm{x}}_{j_s i_t} \tilde{M}_{tu} M^{\mathrm{x}}_{i_u j_w} \tag{A177}$$

$$= 1 - \sum_{s=1}^{l} M^{\mathrm{x}}_{j_s j_w} - \sum_{u=1}^{k} \left(\sum_{s=1}^{l} \sum_{t=1}^{k} M^{\mathrm{x}}_{j_s i_t} \tilde{M}_{tu} \right) M^{\mathrm{x}}_{i_u j_w} \tag{A178}$$

and thus

$$\sum_{u=1}^{k}\left(1-\sum_{s=1}^{l}\sum_{t=1}^{k}M^{\mathrm{x}}_{j_s i_t}\tilde{M}_{tu}\right)M^{\mathrm{x}}_{i_u j_w}\geq 0. \tag{A179}$$

Since (A179) holds for any cross-ownership matrix $\mathbf{M}^{\mathrm{x}}$ and since the entries $M^{\mathrm{x}}_{i_u j_w}$, $u\in\{1,\ldots,k\}$, do not occur in the term in brackets, we obtain

$$\left(1-\sum_{s=1}^{l}\sum_{t=1}^{k}M^{\mathrm{x}}_{j_s i_t}\tilde{M}_{tv}\right)M^{\mathrm{x}}_{i_v j_w}\geq 0 \quad \text{for } v\in\{1,\ldots,k\}. \tag{A180}$$

This proves the assertion. □

A.8 Network Comparison under Cross-Ownership of Equity only

This section extends the analysis of Section 7.2.5.3 to cross-ownership of equity only. In detail, we will compare the occurrence of defaults in the complete network and the ring network for a system consisting of $n\geq 3$ initially identical firms with $d<a$ (cf. Remark 7.5; for $d=a$ all firms would be borderline firms and firm 1 would default for any negative shock on its exogenous asset) under an idiosyncratic shock on firm 1 leading to an exogenous asset value of firm 1 after the shock of size $\tilde{a}_1$. Since for $d<a$ only the shocked firm might default, we will only consider for which network type the first default occurs earlier (i.e. for bigger values of $\tilde{a}_1$). As in Section 7.2.5.3 we assume that the column sums of all matrices are equal to β.

We start with the complete network. Let us first assume that the shock is small so that all firms remain solvent after the shock. Such a shock exists for reasons of continuity and since all firms are strictly solvent before the shock. Clearly, firm 2, firm 3,..., and firm n exhibit identical equity values $\tilde{s}$ after the shock, which implies

$$\tilde{s}_1=\tilde{a}_1+\sum_{j=2}^{n}\frac{\beta}{n-1}\tilde{s}_j-d=\tilde{a}_1+(n-1)\frac{\beta}{n-1}\tilde{s}-d=\tilde{a}_1+\beta\tilde{s}-d. \tag{A181}$$

$$\tilde{s}=a+(n-2)\frac{\beta}{n-1}\tilde{s}+\frac{\beta}{n-1}\tilde{s}_1-d. \tag{A182}$$

Straightforward calculations yield $\tilde{s}_1=\frac{(n-1)(1-\beta)+\beta}{(n-1+\beta)(1-\beta)}\tilde{a}_1+\frac{\beta(n-1)}{(n-1+\beta)(1-\beta)}a-\frac{1}{1-\beta}d$, i.e. in the complete network, default of firm 1 occurs as soon as

$$\tilde{a}_1<\frac{(n-1+\beta)d-\beta(n-1)a}{(n-1)(1-\beta)+\beta}. \tag{A183}$$

In the ring network, we again assume that firm $k+1$ holds a part of firm k's equity, $k\in\{1,\ldots,n-1\}$, and that firm 1 hold a part of firm n's equity, i.e. after a small shock leaving all firms solvent,

$$\begin{aligned}
\tilde{s}_1 &= \tilde{a}_1 + \frac{\beta}{n-1}\tilde{s}_n - d \\
\tilde{s}_2 &= a + \frac{\beta}{n-1}\tilde{s}_1 - d \\
&\vdots \\
\tilde{s}_n &= a + \frac{\beta}{n-1}\tilde{s}_{n-1} - d,
\end{aligned} \tag{A184}$$

or equivalently,

$$\tilde{\mathbf{s}} = \underbrace{\begin{pmatrix} 1 & & & & & -\frac{\beta}{n-1} \\ -\frac{\beta}{n-1} & 1 & & & & \\ & -\frac{\beta}{n-1} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\frac{\beta}{n-1} & 1 \end{pmatrix}^{-1}}_{=:\left(\mathbf{I}_n - \frac{\beta}{n-1}\mathbf{C}_n\right)^{-1}} \left(\begin{pmatrix} \tilde{a}_1 \\ a \\ \vdots \\ a \end{pmatrix} - \begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix} \right) \tag{A185}$$

with n-dimensional cyclic permutation matrix $\mathbf{C}_n$ (cf. equation (0.9.6.2) of Horn and Johnson [2013], for example). For $b := \frac{\beta}{n-1}$, let

$$\mathbf{B} = \frac{1}{1-b^n} \begin{pmatrix} 1 & b^{n-1} & b^{n-2} & \dots & b^2 & b \\ b & 1 & b^{n-1} & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & b^{n-1} \\ b^{n-1} & \dots & \dots & \dots & b & 1 \end{pmatrix}, \tag{A186}$$

and by equation (0.9.6.3) of Horn and Johnson [2013], $\mathbf{B} = \frac{1}{1-b^n}\sum_{k=0}^{n-1} b^k \mathbf{C}_n^k$. Then $\mathbf{B} \times (\mathbf{I}_n - b\mathbf{C}_n) = \mathbf{B} - \frac{1}{1-b^n}\sum_{k=1}^{n} b^k \mathbf{C}_n^k = \mathbf{B} - (\mathbf{B} - \frac{1}{1-b^n}(\mathbf{I}_n - b^n\mathbf{I}_n)) = \mathbf{I}_n$ due to $\mathbf{C}_n^n = \mathbf{I}_n$, as the n-fold application of a cyclic permutation of length n is the identity permutation. Hence, $(\mathbf{I}_n - \frac{\beta}{n-1}\mathbf{C}_n)^{-1} = \mathbf{B}$ and thus

$$\tilde{s}_1 = \frac{1}{1-b^n}\left(\tilde{a}_1 - d + \sum_{j=1}^{n-1} b^j (a-d)\right) \tag{A187}$$

$$= \frac{1}{1-b^n}\left(\tilde{a}_1 - d + \left(\frac{1-b^n}{1-b} - 1\right)(a-d)\right) \tag{A188}$$

$$= \frac{\tilde{a}_1 - a}{1-b^n} + \frac{a-d}{1-b}, \tag{A189}$$

i.e. in the ring network, default of firm 1 occurs as soon as

$$\tilde{a}_1 < a - \frac{1-b^n}{1-b}(a-d) = \frac{-\beta\left(1-\left(\frac{\beta}{n-1}\right)^{n-1}\right)}{n-1-\beta}a + \frac{(n-1)\left(1-\left(\frac{\beta}{n-1}\right)^{n}\right)}{n-1-\beta}d. \tag{A190}$$

We will now show that the RHS of (A183) is smaller than the RHS of (A190) for all possible values of $d < a$, β and n. First, note that for $d = a$, the RHS of (A183) and (A190) coincide. The assertion follows if we can prove that the derivative of the RHS of (A183) with respect to a, given as $\frac{-\beta(n-1)}{(n-1)(1-\beta)+\beta}$, is smaller than the derivative of the RHS of (A190) with respect to a, given as $\frac{-\beta}{n-1-\beta}\left(1-\left(\frac{\beta}{n-1}\right)^{n-1}\right)$, for all $a > d$. Due to

$$\frac{-\beta(n-1)}{(n-1)(1-\beta)+\beta} < \frac{-\beta}{n-1-\beta}\left(1-\left(\frac{\beta}{n-1}\right)^{n-1}\right) \tag{A191}$$

$$\Leftrightarrow \quad \frac{n-1}{(n-1)(1-\beta)+\beta} > \frac{1-\left(\frac{\beta}{n-1}\right)^{n-1}}{n-1-\beta} \tag{A192}$$

$$\Leftrightarrow \quad (n-1)(n-1-\beta) > \underbrace{\left(1-\left(\frac{\beta}{n-1}\right)^{n-1}\right)}_{\in(0;1)}((n-1)(1-\beta)+\beta), \tag{A193}$$

it is sufficient to show that $(n-1)(n-1-\beta) > (n-1)(1-\beta)+\beta$, which is equivalent to $(n-1)(n-2) > \beta$. Since we assume $n \geq 3$ and due to $\beta < 1$, the assertion follows. Thus, the RHS of (A190) is greater than the RHS of (A183) for $d < a$, which means that in the ring network, firm 1 is in default already for bigger values of $\tilde{a}_1$ and thus smaller shocks than in the complete network. Hence, under cross-ownership of equity only, a complete network of n identical firms with $d < a$ is more resilient to idiosyncratic shocks than the ring network in the sense that the default of firm 1 occurs for smaller values of $\tilde{a}_1$. Straightforward calculations show that for both network types, the critical bound on $\tilde{a}_1$ is strictly decreasing in β, i.e. a high level of integration can delay the default of the firm hit by the shock, but in contrast to systems linked by cross-ownership of debt only, there are parameter constellations where it cannot be completely prevented under cross-ownership of equity only, since even for $\beta = 1$, the RHS of (A183) and (A190) can be positive.

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