Loewner equations in multiply connected domains

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Für meine Eltern und für Laura

Contents

Nomenclature

$\mathbb{A}_r$	$:= \mathbb{A}_{r,1} = \{z \in \mathbb{C} \mid r < \lvert z\rvert < 1\}$, annulus with radius $r \in (0,1)$
$\mathbb{A}_{r,R}$	$:= \{z \in \mathbb{C} \mid r < \lvert z\rvert < R\}$, annulus with radii $0 < r < R < \infty$
$\mathbb{C}$	complex plane
$\mathbb{C}_\infty$	$:= \mathbb{C} \cup \{\infty\}$, the Riemann sphere
$\mathfrak{c}(g)$	appropriate capacity of g, i.e. $\mathfrak{c}(g) = \mathrm{lmr}(g)$ (radial case), $\mathfrak{c}(g) = \mathrm{lcm}(g)$ (bilateral case), $\mathfrak{c}(g) = \mathrm{hcap}(g)$ (chordal case), page 40
$\mathfrak{c}_\Omega(\mathfrak{H})$	$:= \mathfrak{c}(g)$, where g denotes the normalised appropriate mapping function on $\Omega \setminus \mathfrak{H}$, page 40
χ	chordal metric
$\mathrm{cl}(A)$	closure of a set $A \subseteq \mathbb{C}$ with respect to the standard topology in $\mathbb{C}$
$\mathrm{cl}_\infty(A)$	closure of a set $A \subseteq \mathbb{C}_\infty$ with respect to the standard topology in $\mathbb{C}_\infty$
$\mathrm{con}(\Omega)$	connectivity of a domain Ω
$\mathbb{D}$	$:= \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$, the unit disk
$\mathbb{D}_r$	$:= \{z \in \mathbb{C} \mid \lvert z\rvert < r\}$, $r \in (0,\infty)$
$\mathrm{diam}(A)$	$:= \sup\{\lvert a-b\rvert \mid a,b \in A\}$, diameter of a set $A \subseteq \mathbb{C}$
$\mathrm{dist}(A,B)$	$:= \inf\{\lvert a-b\rvert \mid a \in A, b \in B\}$, distance of two sets $A, B \subseteq \mathbb{C}$
$\mathrm{ext}(\Gamma)$	exterior of the Jordan curve Γ
$\mathrm{hcap}(g)$	half-plane capacity of g, page 35
$\mathrm{hcap}_\Omega(\mathfrak{H})$	$:= \mathrm{hcap}(g)$, where g is the normalised chordal mapping function on $\Omega \setminus \mathfrak{H}$, page 35
$\mathrm{int}(\Gamma)$	interior of the Jordan curve Γ
$\mathrm{lcm}(g)$	logarithmic conformal modulus, page 32

$\mathrm{lcm}_\Omega(\mathfrak{H})$	$:=\mathrm{lcm}(g)$, where g denotes the bilateral mapping function on $\Omega \setminus \mathfrak{H}$, page 32
$\mathrm{lmr}(g)$	$:= \ln g'(0)$, logarithmic mapping radius of g, page 25
$\mathrm{lmr}_\Omega(\mathfrak{H})$	$:= \mathrm{lmr}(g)$, where g is the normalised radial mapping function on $\Omega \setminus \mathfrak{H}$, page 25
$\mathcal{C}(I)$	set of continuous functions $f : I \to \mathbb{C}$, $I \subseteq \mathbb{R}$
$\mathcal{C}^k(I)$	set of k times continuously differentiable functions $f : I \to \mathbb{C}$, $I \subseteq \mathbb{R}$, $k \in \mathbb{N}$
$\Omega_n \xrightarrow{\mathrm{k}} \Omega$	Ω is the kernel of each subsequence of $(\Omega_n)_{n\in\mathbb{N}}$, page 15
$\partial\Omega$	$:= \mathrm{cl}(\Omega) \setminus \Omega$, the boundary of a domain Ω
$\partial_\infty\Omega$	$:= \mathrm{cl}_\infty(\Omega) \setminus \Omega$, the boundary of a domain Ω on the Riemann sphere
$\mathbb{R}$	real numbers
$\mathbb{T}$	$:= \{z \in \mathbb{C} \mid \lvert z\rvert = 1\} = \partial\mathbb{D}$, the unit circle
$\mathbb{T}_r$	$:= \{z \in \mathbb{C} \mid \lvert z\rvert = r\} = \partial\mathbb{D}$, $r \in (0,\infty)$
$\mathbb{H}$	$:= \{z \in \mathbb{C} \mid \Im(z) > 0\}$, the upper half-plane
$A \subseteq B$	A is a subset of B
$A \subsetneq B$	$:\Leftrightarrow A \subseteq B \wedge A \neq B$, A is a strict subset of B
$B_r(\infty)$	$:= \{z \in \mathbb{C}_\infty \mid \chi(z,\infty) < r\}$, ball around ∞ with radius $r > 0$ with respect to the chordal metric
$B_r(w)$	$:= \{z \in \mathbb{C}_\infty \mid \lvert z - w\rvert < r\}$, ball around $w \in \mathbb{C}$ with radius $r > 0$
$f_n \xrightarrow{\text{l.u.}} f$	$(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f, page 16

Chapter 1

Introduction to Loewner theory

In 1923, Charles Loewner (born as Karel Löwner) laid the foundation of a theory, nowadays known as *Loewner theory*, see [Löw23]. In this context, Loewner considered conformal mappings from the unit disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ onto $\mathbb{D}$ minus a single slit, also known as single slit mappings. For any domain Ω, we will call a function $g : \Omega \to D$ *conformal*[1] (or *conformal mapping from* Ω *onto* D) if g is analytic and one-to-one and $g(\Omega) = D$. Moreover, Loewner proved that the class of single slit mappings is dense in the class of all conformal maps on $\mathbb{D}$. Loewner found out how to parametrise the slit in order to describe the conformal single slit mappings by a differential equation known as the *Loewner differential equation*. The long-term goal of his approach was to use the differential equation in order to attack Bieberbach's conjecture.

1.1 Radial Loewner equation and Bieberbach's conjecture

First of all, let $\gamma : [0, T] \to \mathrm{cl}(\mathbb{D})$ be a simple and continuous curve having $\gamma(0) \in \mathbb{T} := \partial\mathbb{D}$ and $\gamma(0, T] \subseteq \mathbb{D} \setminus \{0\}$. Using Riemann's well-known mapping theorem, we find for each $t \in [0, T]$, a unique conformal mapping g_t from $\mathbb{D} \setminus \gamma(0, t]$ onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. Then it is an easy consequence of Schwarz lemma to see that $t \mapsto g_t'(0)$ is strictly increasing and $g_0'(0) = 1$. Moreover, $t \mapsto g_t'(0)$ is continuous on $[0, T]$. Note that this follows immediately from the kernel theorem due to Carathéodory (see Section 2.1). Summarising, it is not a great constraint to assume that γ is parametrised in such a way that $g_t'(0) = e^t$ for all $t \in [0, T]$. Obviously, for all $t \in [0, T]$, $h_t := g_{T-t}^{-1}$ satisfies $h_t'(0) = e^{t-T}$ and h_t is the unique conformal mapping from $\mathbb{D}$ onto $\Omega_{T-t} = \mathbb{D} \setminus \gamma(0, T - t]$ having the same normalisation as g_t. In 1923, Loewner proved the following theorem.

Theorem A. *Let $\gamma : [0, T] \to \mathbb{D} \cup \mathbb{T}$ be a simple and continuous curve satisfying $\gamma(0) \in \mathbb{T}$ and $\gamma(0, T] \subseteq \mathbb{D} \setminus \{0\}$. For each $t \in [0, T]$, h_t denotes the unique conformal mapping that maps $\mathbb{D}$ onto $\Omega_{T-t} := \mathbb{D} \setminus \gamma(0, T - t]$ with the normalisation $h_t(0) > 0$ and $h_t(0) = 0$. Assume $h_t'(0) = e^{t-T}$ for all $t \in [0, T]$. Then, for each $z \in \mathbb{D}$, $t \mapsto h_t(z)$ is differentiable*[2]

[1] If $f : \Omega \to D$ is analytic and one-to-one but not necessarily onto, we call f *univalent*. Thus an univalent function $f : \Omega \to D$ is conformal if and only if $f(\Omega) = D$.

[2] We will always indicate the derivative w.r.t. t by $\dot{h}_t(z)$, while we write $h_t'(z)$ to denote the derivative w.r.t. z.

on $[0, T]$ *and satisfies*

$$\dot{h}_t(z) = h_t'(z) \cdot z \cdot \frac{\kappa_t + z}{\kappa_t - z} \qquad \text{for all } t \in [0, T] \text{ and all } z \in \mathbb{D},$$

with a continuous function $t \mapsto \kappa_t \in \mathbb{T}$ *on* $[0, T]$.

The differential equation in Theorem A is called *(single-slit) radial Loewner partial differential equation*. It is a straightforward calculation to find the following corollary.

Corollary B. *Let* $\gamma : [0, T] \to \mathbb{D} \cup \mathbb{T}$ *be a simple and continuous curve satisfying* $\gamma(0) \in \mathbb{T}$ *and* $\gamma(0, T] \subseteq \mathbb{D}$. *For each* $t \in [0, T]$, g_t *denotes the unique conformal mapping that maps* $\Omega_t := \mathbb{D} \setminus \gamma(0, t]$ *onto* $\mathbb{D}$ *with the normalisation* $g_t(0) = 0$ *and* $g_t'(0) > 0$. *Assume* $g_t'(0) = e^t$ *for all* $t \in [0, T]$. *Then* $t \mapsto g_t(z)$ *is differentiable on* $[0, T]$ *for each* $z \in \mathbb{D}$ *and satisfies*

$$\dot{g}_t(z) = g_t(z) \cdot \frac{U_t + g_t(z)}{U_t - g_t(z)} \qquad \text{for all } t \in [0, T] \text{ and all } z \in \Omega_T, \tag{1.1}$$

with a continuous function $t \mapsto U_t \in \mathbb{T}$ *on* $[0, T]$.

The previous differential equation is called *(single-slit) radial Loewner ordinary differential equation*. $\kappa : [0, T] \to \mathbb{T}$ from Theorem A and $U : [0, T] \to \mathbb{T}$ from Corollary B are called *driving functions* or *driving terms*. It is not hard to show that $\kappa_t = U_{T-t}$ for all $t \in [0, T]$ in the previous context.

As mentioned before, Loewner's work was heavily motivated by the so called Bieberbach conjecture. Therefore, let us consider the following famous class:

$$\mathcal{S} := \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ univalent}, f'(0) = 1\}.$$

If $f \in \mathcal{S}$, we have $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ around 0. In 1916, see [Bie16], L. Bieberbach conjectured $|a_k| \leq k$ for all $k \in \mathbb{N}$, while equality holds for some $k \geq 2$ if and only if f is a rotation of the Koebe function. Using the previous differential equations, Loewner was able to prove $|a_3| \leq 3$, see [Löw23]. Although this was a great breakthrough, Loewner was a little bit disappointed that he was not able to solve Bieberbach's conjecture completely. Maybe this is the reason that the paper [Löw23] was Loewner's first and only paper concerning this field[3]. Finally, almost seven decades later, the problem was solved by L. de Branges, see [DB85]. While de Branges did not use Loewner's differential equation directly, C. FitzGerald and C. Pommerenke found an easier proof based on Loewner's results, see [FP85].

After Loewner published his paper in 1923, his ideas were developed and generalised a lot. Herein, important contributions are due to C. Pommerenke, see [Pom65], and P. Kufarev, see [Kuf43], leading to 'general Loewner equations'. The difference between Loewner original equation, see Equation (1.1), and more general Loewner type equations is the kernel on the right-hand side. Nevertheless, most recent applications are based on Loewner's original equation, so we continue with that.

[3]This was mentioned by P. Duren during some lectures at Würzburg university in May 2014, see [Dur14].

One might ask the natural question if there are other families of domains $(\Omega_t)_{t\in[0,T]}$ having $\Omega_t \subsetneq \Omega_s$, whenever $t < s$, with corresponding functions $g_t : \Omega_t \to \mathbb{D}$ normalised by $g_t(0) = 0$ and $g_t'(0) = e^t$ such that $t \mapsto g_t$ fulfils a single-slit Loewner differential equation with a continuous function $t \mapsto U_t$. By 'other families of domains' we mean domains such that Ω_T is not a slit domain. This question goes back to Loewner (see [Löw23], page 117):

'Es sei hier noch bemerkt, daß dieser Satz nicht umgekehrt werden kann, d.h. es gibt stetige Funktionen $t \mapsto U_t$, wo die Lösung von (1.1) keine Schlitzabbildungen liefert. Es ist mir jedoch nicht bekannt, welche Bereiche außer den Schlitzbereichen auf diese Art noch entstehen können.'

The first example of a non-slit mapping is due to P.P. Kufarev, see [Kuf47].

In 1966, Pommerenke gave an answer to Loewner's previous question, see [Pom66], where he proved the following theorem (even for unbounded Ω_t).

Theorem C (Theorem 1 in [Pom66]). *Let $(\Omega_t)_{t\in[0,T]}$ be a family of simply connected domains such that $0 \in \Omega_T$, $\Omega_0 = \mathbb{D}$ and $\Omega_s \subsetneq \Omega_t$ whenever $0 \le t < s \le T$. Assume, for all $t \in [0,T]$, $g_t : \Omega_t \to \mathbb{D}$ is the unique conformal mapping with the normalisation $g_t'(0) = e^t$ and $g_t(0) = 0$. Then the following two statements are equivalent:*

(a) For each $z \in \Omega_t$, $t \mapsto g_t(z)$ is differentiable on $[0,T]$ and fulfils differential equation (1.1) with a continuous function $t \mapsto U_t \in \mathbb{T}$.

(b) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $s, t \in [0,T]$ and $0 < s-t < \delta$, some cross-cut E of Ω_t with $\operatorname{diam}(E) < \varepsilon$ separates 0 from $\Omega_t \setminus \Omega_s$.

See Section 2.4 in [Pom92] or Chapter 5 for the definition of a cross-cut. Theorem C is helpful to understand how domains coming from a Loewner equation have to look like. Moreover, it leads easily to non-slit mappings satisfying a Loewner equation.

In particular, it is possible to find families of domains $(\Omega_t)_{t\in[0,T]}$ such that the corresponding family $(g_t)_{t\in[0,T]}$ satisfies Equation (1.1), while $\partial\Omega_t$ is not even locally connected for some $t \in [0,T]$. This case was studied intensively by J. Lind, D. Marshall and S. Rohde, see [LMR10] where several examples are given.

1.2 Chordal Loewner equation and SLE

Recently, Loewner's equation was used with great success in probability theory, as we will see later. In this context, instead of Loewner's original setting a different setting, introduced originally by Kufarev considering slits growing in the upper half-plane, is used mostly.

Therefore, let $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ denote the upper half-plane and denote by $\gamma : [0,T] \to \mathbb{H} \cup \mathbb{R}$ a simple and continuous curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(0,T] \subseteq \mathbb{H}$. Analogously to Section 1.1, Riemann's mapping theorem gives us a conformal mapping g_t from $\Omega_t := \mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$ for each $t \in [0,T]$. This mapping is unique if we consider the normalisation

$$g_t(z) = z + \frac{a_t}{z} + \mathcal{O}(|z|^{-2}) \qquad \text{around } \infty, \tag{1.2}$$

known as the *hydrodynamic normalisation*. It is easy to see that $a_t \geq 0$, while a_t is called the *half-plane capacity*. Unsurprisingly, the half-plane capacity a_t plays a similar role as $g_t'(0)$ in the radial case. Indeed, $t \mapsto a_t$ is strictly increasing and continuous, so we may assume that γ is parametrised in such a way that $a_t = 2t$ for all $t \in [0,T]$[4]. Then P. Kufarev, V. Sobolev and L. Sporyševa proved the following result, see [KSS68].

Theorem D. *Let $\gamma : [0,T] \to \mathbb{H} \cup \mathbb{R}$ denote a simple and continuous curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(0,T] \in \mathbb{H}$. For each $t \in [0,T]$, g_t denotes the unique conformal mapping from $\Omega_t := \mathbb{H} \setminus \gamma(0,T]$ onto $\mathbb{H}$ having the hydrodynamic normalisation, see Equation* (1.2)*. Moreover, for each $t \in [0,T]$, assume $a_t = 2t$ where a_t denotes the half-plane capacity. Then $t \mapsto g_t$ is differentiable on $[0,T]$ and fulfils*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t} \qquad \text{for all } t \in [0,T] \text{ and all } z \in \Omega_T,$$

with a continuous function $t \mapsto U_t \in \mathbb{R}$ on $[0,T]$.

Note this theorem was published in Russian and it got studied a lot from the Soviet school. As a result of the cold war, the radial case on the one hand and the chordal case on the other case were often developed further independently from each other.

As indicated already before, Loewner's differential equation, in particular Theorem D, became of big interest once again in 2000. Before going into detail, it is important to notice that for each fixed $z \in \mathbb{H}$, the initial value problem

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t} \qquad g_0(z) = z,$$

with a given continuous function $U_t : \mathbb{R} \to \mathbb{R}$, does has a unique solution $g_t(z)$ up to a time T_z. Then g_t is the unique conformal mapping from $\Omega_t := \{z \in \mathbb{H} \mid T_z > t\}$ onto $\mathbb{H}$ satisfying the hydrodynamic normalisation, see Theorem 4.6 in [Law05] for a detailed proof. Notice that same inverse result holds in the radial case as well, see Theorem 4.14 in [Law05].

In 2000, O. Schramm had the fruitful idea to replace the driving term U_t by a Brownian motion $\sqrt{\kappa}B_t$, i.e. B_t is a standard Brownian motion and $\kappa \geq 0$. This leads to the definition of chordal Schramm–Loewner evolution (SLE) with parameter κ, see [Sch00] for more details. Schramm realised that the (random) domains Ω_t differ heavily from the choice of κ. If $\kappa \in [0,4]$, with probability 1 Ω_t is given by $\mathbb{H}$ minus a simple curve. In the case $\kappa \in (4,8)$, w.p.1 Ω_t comes from $\mathbb{H}$ minus a random curve hitting itself and the real axis infinitely often. Finally, if $\kappa \geq 8$, w.p.1 Ω_t is generated by a space-filling curve. In probability theory, Schramm–Loewner evolution was used with great success, e.g. to prove the Mandelbrot conjecture, see [LSW01], or to find scaling limits of discrete random processes.

1.3 Multiple slit Loewner equations

As we have seen in the previous sections, single slit mappings play a huge role in complex analysis. Nevertheless, there are many models involving several of slits. Therefore, let

[4]For historical reasons, γ is most often parametrised in such a way that $a_t = 2t$. Moreover the value 2 plays a (hidden) role in the radial case as well, see for example Proposition 2.5 and Remark 2.2.

$\gamma_1, \ldots, \gamma_m : [0,T] \to \mathbb{D} \cup \mathbb{T}$, $m \in \mathbb{N}$, denote disjoint simple and continuous curves such that $\gamma_k(0,T] \subseteq \mathbb{D} \setminus \{0\}$ and $\gamma_k(0) \in \mathbb{T}$ for each $k \in \{1, \ldots, m\}$. Analogously to Section 1.1, we define for each $t \in [0,T]$, g_t as the unique conformal mapping from $\Omega_t := \mathbb{D} \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ onto $\mathbb{D}$ having the normalisation $g_t(0) = 0$ and $g_t'(0) > 0$. Moreover, we set $\Gamma_k := \gamma_k[0,T]$, $k \in \{1, \ldots, m\}$ and we call $(\Gamma_1, \ldots, \Gamma_m)$ *tuple of disjoint radial unparametrised slit in* $\mathbb{D}$. Then one might ask the question if there are parametrisations $\gamma_k : [0,T] \to \Gamma_k$ with $k \in \{1, \ldots, m\}$ such that the corresponding $t \mapsto g_t$ are differentiable on $[0,T]$. Moreover, it would be nice to find a characterisation of the parametrisations leading to differentiable $t \mapsto g_t$.

The first person who studied this case was E. Peschl, see [Pes36]. He proposed the following theorem (see Theorem 12 in [Pes36]).

Theorem E. *Let* $(\Gamma_1, \ldots, \Gamma_m)$, $m \in \mathbb{N}$, *denote a tuple of disjoint radial unparametrised slits in* $\mathbb{D}$.

Then there are parametrisations $\gamma_k : [0,T] \to \Gamma_k$ *with* $k \in \{1, \ldots, m\}$ *and* $T > 0$ *such that for each* $z \in \mathbb{D} \setminus \bigcup_{k=1}^m \Gamma_k$, *the corresponding* $g_t(z)$ *is differentiable w.r.t* t *on* $[0,T]$ *and fulfils*

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^m \lambda_k(t) \frac{U_k(t) + g_t(z)}{U_k(t) - g_t(z)} \qquad \text{for all } z \in \Omega \setminus \bigcup_{k=1}^m \Gamma_k \text{ and all } t \in [0,T],$$

where for each $k \in \{1, \ldots, m\}$, $t \mapsto U_k(t) \in \mathbb{T}$ *and* $t \mapsto \lambda_k(t) \geq 0$ *are continuous on* $[0,T]$.

Here, for all $t \in [0,T]$, g_t *denotes the unique conformal mapping from* $\Omega_t := \mathbb{D} \setminus \bigcup_{k=1}^m \gamma(0,t]$ *onto* $\mathbb{D}$ *having the normalisation* $g_t(0) = 0$ *and* $g_t'(0) > 0$.

The previous differential equation is called *multiple slit radial Loewner ODE*. Analogously to Section 1.1, it is easy to see that the inverse function $h_t := g_t^{-1}$ satisfies a partial different equation, called *multiple slit radial Loewner PDE*. Obviously, it is possible to consider multiple slits in the chordal setting as well. Moreover, it is important to note that Peschl considered already slits having branch points, for example two slits have a branch point on $\mathbb{T}$ if $\gamma_1(0) = \gamma_2(0)$ but $\gamma_1(0,T]$ and $\gamma_2(0,T]$ are still disjoint.

Nowadays multiple slit Loewner equations have several applications. First let us have a rough look at Mathematical Physics. Herein, multiple slit Loewner equations are used to describe Laplacian growth models. For example, see [Sel99] or [CM02] where it is assumed that $\gamma_k(t)$ expands in terms relative to the Laplacian field. Moreover, the considered slits can have branch points as well. One can see that the functions $\lambda_k(t)$, $k \in \{1, \ldots, m\}$, represent growth factors indicating 'how fast a slit grows'. Models where the growth of multiple slits is restricted to a channel were studied in [GS08].

Another application of multiple slit Loewner equations is due to D. Prokhorov. Herein, Prokhorov used multiple slit Loewner equations from an control-theoretical point of view to study coefficient extremal problems for univalent functions, see [Pro93]. An important theorem for his approach is the following.

Theorem F. *Let* (Γ_1, Γ_2) *denote a tuple of disjoint radial unparametrised slits in* $\mathbb{D}$. *Moreover, assume* Γ_1, Γ_2 *are piecewise analytic.*

Then there is a unique $T > 0$, unique parametrisations $\gamma_1, \gamma_2 : [0,T] \to \Gamma_k$ and unique constants $\lambda_1, \lambda_2 \in (0,1)$ with $\lambda_1 + \lambda_2 = 1$ such that for each $z \in \mathbb{D} \setminus (\Gamma_1 \cup \Gamma_2)$, $t \mapsto g_t(z)$ is differentiable on $[0,T]$ and fulfils the following differential equation

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^{2} \lambda_k \frac{U_k(t) + g_t(z)}{U_k(t) - g_t(z)} \qquad \text{for all } z \in \mathbb{D} \setminus (\Gamma_1 \cup \Gamma_2) \text{ and all } t \in [0,T],$$

where, for each $k \in \{1,2\}$, $t \mapsto U_k(t) \in \mathbb{T}$ is continuous on $[0,T]$. Herein, for each $t \in [0,T]$, g_t denotes the unique conformal mapping from $\Omega_t := \mathbb{D} \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$ onto $\mathbb{D}$ having the normalisation $g_t(0) = 0$ and $g_t'(0) > 0$.

Under the conditions of Theorem F, the differential equation gives us easily $g_t'(0) = e^t$ for all $t \in [0,T]$.

Note that the unique parametrisations from the previous theorem can be seen as a canonical parametrisation of the two unparametrised slits Γ_1 and Γ_2. In the single slit case, there is a unique parametrisation satisfying $g_t'(0) = e^t$ for all $t \in [0,T]$ as well. Thus Theorem F represents the natural generalization of Loewner's original theorem to multiple slits, see also the introduction of Chapter 3. Recently, O. Roth and S. Schleißinger found a proof of Theorem F in the chordal case. Herein, they were able to drop the assumption of Γ_1 and Γ_2 to be piecewise analytic, see [RS14].

Finally, the multiple slit equation was used to define Schramm–Loewner evolution for multiple slits, see [KL07] for more details. There are a lot of papers concerning SLE for multiple slits, see [dMS15] for a recent reference.

1.4 Loewner equations in multiply connected domains

Nowadays, there are several generalizations of Loewner's differential equation to multiply connected domains. The first person who considered multiply connected domains was Yûsaku Komatu (2 January 1914 – 30 July 2004), see his doctoral thesis [Kom43], which was supervised by M. Tsuji. Detailed informations concerning Komatu's life and mathematical work can be found in [Tak05]. In [Kom43], Komatu established a Loewner equation in a doubly connected annulus and he used this result to study distortion properties (see §4, §5 and §6 in [Kom43]). Later Komatu considered a general $\mathfrak{n}$-connected slit annulus case, $\mathfrak{n} \in \mathbb{N}$, as well, see [Kom50].

In this context, let Ω denote a circular slit annulus, i.e. an annulus $\mathbb{A}_Q := \{z \in \mathbb{C} \mid Q < |z| < 1\}$ minus $\mathfrak{n} - 2$ disjoint proper concentric circular arcs (centred at 0). Moreover, let $\gamma : [0,T] \to \Omega \cup \mathbb{T}$ be a simple continuous curve such that $\gamma(0) \in \mathbb{T}$ and $\gamma(0,T] \subseteq \Omega$. Note that there are mapping theorems for multiply connected domains analogously to Riemann's mapping theorem for simply connected domains, see Section 2.1 for more details. Using a suitable normalisation (see the next theorem), there is a unique conformal mapping from $\Omega_t := \Omega \backslash \gamma(0,t]$ onto a circular slit annulus D_t with inner radius $q_t > 0$ for each $t \in [0,T]$. Komatu found out that it is important to parametrise $\Gamma := \gamma[0,T]$ in such a way that $t \mapsto q_t$ is differentiable. In particular, it is possible to find a unique parametrisation such that $q_t = Qe^t$. Then Komatu proposed the following theorem, see page 30 in [Kom50].

Theorem G. *Let Ω denote a circular slit annulus with inner radius Q and $\gamma : [0,T] \to \Omega \cup \mathbb{T}$ is a continuous and simply curve such that $\gamma(0) \in \mathbb{T}$ and $\gamma(0,T] \subseteq \Omega$. Moreover, for each $t \in [0,T]$, we denote by g_t the unique conformal mapping from $\Omega_t := \Omega \setminus \gamma(0,t]$ onto the circular slit annulus D_t such that g_t associates $\mathbb{T}_Q$ with $\mathbb{T}_{q_t}$, g_t associates the outer boundary of Ω_t with $\mathbb{T}$, and $g_t(Q) = q_t$. Assume γ is parametrised in such a way that $q_t = Q \cdot e^t$ for all $t \in [0,T]$.*

Then, for each $t \in [0,T)$ and each $z \in \Omega_t$, $t \mapsto g_t(z)$ is differentiable from the left and satisfies the following differential equation

$$\partial_t^- g_t(z) = -g_t(z)\left(\frac{\partial F_{D_t}(U_t, g_t(z))}{\partial n_1} - \frac{\partial F_{D_t}(U_t, q_t)}{\partial n_1} + \sum_{k=1}^{\mathfrak{n}-1} R_{D_t;k}(g_t(z))\frac{\partial_t^- m_k(t)}{m_k(t)}\right) \tag{1.3}$$

for all $z \in \Omega_T$ and $t \in [0,T]$. Herein, for each $t \in [0,T]$, $F_{D_t}(\cdot, w)$ is a multivalent function such that the real part denotes Green's function of D_t with pole at w. For each $k \in \{1,\ldots,\mathfrak{n}\}$ and $t \in [0,T]$, $R_{D_t;k}$ is a multivalent function such that the real part is the harmonic measure of D_t where $\Re R_{D_t;k} \equiv 1$ on $C_k(t)$ and 0 otherwise. $C_k(t)$, $k \in \{1,\ldots,\mathfrak{n}\}$, describes the boundary components of D_t where $C_1(t) = \mathbb{T}_{q_t}$ and $C_\mathfrak{n}(t) = \mathbb{T}$. $m_k(t)$ denotes the radius of $C_k(t)$, i.e. $m_k(t) = \operatorname{dist}(0, C_k(t))$ and $t \mapsto m_k(t)$ is differentiable from the left. Finally, $\frac{\partial}{\partial n_1}$ denotes the derivative along the unit inner normal of the first variable.

The previous differential equation is called *bilateral (single-slit) Komatu-Loewner ODE*[5].

In 2005, R. Bauer and R. Friedrich found similar results in the radial and chordal case, see [BF06] and [BF08]. In the radial case the canonical class is the unit disk minus disjoint proper concentric circular arcs, while in the chordal case one takes the upper half-plane minus disjoint proper closed line segments slits parallel to the real axis, see also Figure 2.1 in Section 2.1. The annulus case (see Theorem G), is also called bilateral case. Moreover, in [BF08], Bauer and Friedrich gave the first rigorous proof of Theorem G. Following Komatu's ideas they only proved differentiability in the left sense.

On top of this Bauer and Friedrich used their results to define candidates for SLE in multiply connected domains. In this context, they considered all the three different cases (radial, bilateral and chordal). Recently, there are several papers concerning SLE in multiply connected domains, see for example [Law11] and [Dre11].

Simultaneously to the authors research, Z. Chen, M. Fukushima and S. Rhode found a way to establish (left and right) differentiability of g_t in the chordal setting, see [CFR13]. Note that their proof is based on probabilistic arguments. A new proof of the doubly connected bilateral case, i.e. the annulus without interior slits, using methods related to Komatu's original ideas was given recently by M. Fukushima and H. Kaneko, see [FK14]. In the general $\mathfrak{n}$-connected bilateral case they proved differentiability analogously to the approach of Bauer and Friedrich, only in the left sense. This problem led them to the following question, see Section 6 in [FK14]:

[5]Throughout in this thesis, Komatu-Loewner equations represent the multiply connected case, while Loewner equations represent the simply connected case.

'In the case where $\mathfrak{n} > 2$ so that the degree of the multiplicity of the circular slit annulus Ω is equal or greater than 3, the problem of proving the equation (1.3) to be a genuine ODE remains open, although Komatu [Kom50] tried to do so by an induction in $\mathfrak{n} \geq 2$ not quite successfully.'

Beside the previously mentioned works, there were many other contributions to Loewner's differential equation in multiply connected domains. For example, in 1951, G. Goluzin found a way how to prove Komatu's results in the doubly connected case without using the theory of elliptic functions, see [Gol51]. A completely different setting was considered by Kufarev in terms of covering maps of the unit disk, see [KK55].

In the annulus case, M. Contreras, S. Díaz-Madrigal and P. Gumenyuk established recently a general Loewner theory, see [CDMG13] and [CDMG11].

Finally, let us mention a survey paper about the evolution of Loewner theory, see [ABCDM10].

1.5 Outline of the thesis

The main object of this thesis is to generalise all previously mentioned theorems to multiply connected domains and multiple slits. Concerning this matter, our approach is purely function-theoretic. As far as possible, we will prove the theorems in all three cases (radial, bilateral and chordal) simultaneously. Moreover, we are going to separate between disjoint and branched slits and surprisingly, we will see that some statements are not valid in the branch point case any more.

In **Chapter 2** we are going to generalise Theorem A, G and D to multiply connected domains and multiple slits. First of all, we summarise some important tools and notations, see Section 2.1. Herein, we discuss the concept of kernel convergence for multiply connected domains. This method will be key for our approach. In Section 2.2 we will study the kernel function $\Phi_{a,\zeta,\Omega}$ in all three cases, which appeared in the bilateral case already on the right-hand side of the differential equation in Theorem G. Beside an analytic representation in terms of relatives to Green's function we will mainly use a geometric characterisation of $\Phi_{a,\zeta,\Omega}$, see Proposition 2.17. Together with the extended kernel theorem, this representation will give us more flexibility, in particular for proving the right differentiability.

Next, we are going to prove Theorem G, see Section 2.6. In this context, we will prove left and right differentiability, so this will solve the previous question by Fukushima and Kaneko. Note that we will give a universal proof, i.e. we prove the radial, bilateral and chordal case simultaneously. On top of this, in the context of this proof we will consider already multiple slits. Finally, we obtain Theorem 2.30, 2.31 and 2.36 generalizing Theorem A, G and D to multiply connected domains and multiple slits, see also Remark 2.6. Preliminary to Section 2.6, we prepare all the important facts regarding the radial case in Section 2.3, the bilateral case in Section 2.4 and the chordal case in Section 2.5. In order to prove Theorem 2.30, 2.31 and 2.36 simultaneously, we summarise these facts in the beginning of Section 2.6, see Subsection 2.6.1.

Subsequently, we are going to consider arbitrary parametrisation of multiple slit. In this regard, we will show that differentiability still holds almost everywhere, see Theorem

2.52, 2.53 and 2.54 in Section 2.7. Finally, we will discuss an important subadditivity property known to be true in the simply connected cases, see Section 2.8. This property will be an important tool for our further approach. Unfortunately, we do not know if this is also true in case of multiply connected domains, see Question 1.

The goal of **Chapter 3** is to generalise Theorem F to the radial, bilateral and chordal case for multiply connected domains. We will be able to drop the assumption of piecewise analytic slits in here as well. In Section 3.1 we are going to discuss the disjoint case, see Theorem 3.2, 3.3 and 3.4. Note that the given proof will be universal, i.e. we will prove the radial, bilateral and chordal case simultaneously.

Subsequently, see Section 3.2, we consider the branch point case. Herein, we study all three cases simultaneously as well and we are going to prove the existence of constant coefficients in case of multiply connected domains. Unfortunately, we were not able to prove the uniqueness of these constant coefficients and their corresponding parametrisations in the multiply connected setting. However, we will give a uniqueness proof in the simply connected case. The reason for this is that we have the subadditivity property of the appropriate capacity available for simply connected domains only, see Section 2.8.

The major goal of **Chapter 4** is to generalise Theorem E to the radial, bilateral and chordal case in multiply connected domains. In Section 4.1 we will consider the disjoint case where we will describe all parametrisations in the multiply connected multiple slit case leading to (continuously) differentiable mapping functions. In this context, we will obtain a characterisation of the differentiability set in the multiply connected multiple slit case by differentiability sets in simplified single slit cases, see Theorem 4.1 and Corollary 4.2. In the following, we will use this Theorem to construct parametrisations leading to continuously differentiable mapping functions, see Proposition 4.4 and Theorem 4.6.

Next, we are going to discuss the branch point case, see Section 4.2. We will see that the previous characterisation is not true in general, see Theorem 4.8. The given counterexamples are based on self-similarity.

Note that the previous chapters considered slit mappings only, so in the final **Chapter 5** we will study the growth of general hulls in multiply connected domains. As a reason of technical difficulties, we will restrict this to the radial case. Nevertheless, it is possible to establish analogous results in the bilateral and chordal case, in a similar way, as well. In Section 5.2 we are going to generalise Theorem C to multiply connected domains, see Theorem 5.1. Unfortunately, we need to restrict this theorem to hulls that do not swallow interior boundary components, see Example 5.1 pointing out a reason why this is necessary. Finally, it is possible to sharpen one direction of Theorem 5.1 to general hulls, allowing them to swallow interior boundary components, see Theorem 5.2.

Chapter 2

Komatu–Loewner equations for canonical domains

First of all, let us define the following classes of domains:

(a) A *circular slit disk* is the unit disk $\mathbb{D}$ minus a finite number of disjoint proper concentric circular arcs centred at 0 with radii in $(r, 1)$.

(b) A *circular slit annulus* is an annulus $\mathbb{A}_r := \{z \in \mathbb{C} \mid r < |z| < 1\}$, with $r \in (0, 1)$, minus a finite number of disjoint proper concentric circular arcs centred at 0 with radii in $(r, 1)$.

(c) An *upper (or right) parallel slit half-plane* is the upper (right) half-plane minus a finite number of disjoint proper closed line segments parallel to the real (imaginary) axis.

A domain Ω is called *canonical* if it is a circular slit disk, a circular slit annulus or an upper parallel slit half-plane, see Figure 2.1.

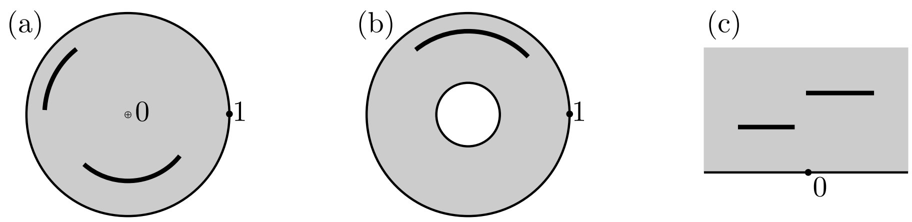

Figure 2.1: *Triply connected canonical domains: circular slit disk, circular slit annulus and upper parallel slit half-plane*

2.1 Some important tools and notations

We denote by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ the Riemann sphere. Let $\Omega \subseteq \mathbb{C}_\infty$ be a finitely connected domain. Then Ω is called *nondegenerate* if each boundary component of Ω with respect to $\mathbb{C}_\infty$ consists of more than a single point[6]. Obviously, each canonical domain is nondegenerate. In the following we denote by $\mathrm{con}(\Omega)$ the connectivity of the domain Ω. Moreover, we use the abbreviations $\mathrm{cl}(\Omega)$ and $\mathrm{cl}_\infty(\Omega)$ to indicate the closure of the domain Ω with respect to the standard topology on $\mathbb{C}$ and $\mathbb{C}_\infty$, respectively. Note that $\mathrm{cl}(\Omega) = \mathrm{cl}_\infty(\Omega)$ if $\Omega \subseteq \mathbb{C}$ is bounded and $\mathrm{cl}(\Omega) \cup \{\infty\} = \mathrm{cl}_\infty(\Omega)$ if $\Omega \subseteq \mathbb{C}$ is unbounded. A *(parametrised) Jordan curve (in $\Omega \subseteq \mathbb{C}$)* is a continuous $\gamma : [a, b] \to \Omega$, $a < b$, such that $\gamma(t) = \gamma(s)$ if and only if $s = t$ or $|s - t| = b - a$. Moreover, $\gamma : [a, b] \to \mathbb{C}_\infty$ is a *(parametrised) Jordan curve (in $\mathbb{C}_\infty$)* if there is a $c \in \mathbb{C}$ such that $t \mapsto 1/(\gamma(t) - c)$ is a parametrised Jordan curve in $\mathbb{C}$. Sometimes we call the trace $\gamma[a, b]$ a Jordan curve as well. The Jordan curve theorem shows that every Jordan curve Γ in $\mathbb{C}$ divides the plane $\mathbb{C}$ into two domains: the interior $\mathrm{int}(\Gamma)$ and the exterior $\mathrm{ext}(\Gamma)$. Finally, a bounded domain $\Omega \subseteq \mathbb{C}$ is called *analytic Jordan domain* if each boundary component is an analytic Jordan curve in $\mathbb{C}$.

Riemann mapping theorems and extremal properties

The well-known Riemann mapping theorem shows that each simply connected domain $\Omega \neq \mathbb{C}$ can be mapped by a conformal mapping g onto the unit disk. Moreover, this mapping is unique if we claim $a \mapsto g(a) = 0$ and $g'(a) > 0$ with some arbitrary $a \in \Omega$. An iteratively application of this theorem gives us the following lemma.

Lemma 2.1 ([Con95], Theorem 15.2.1)**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$. Then there is a conformal mapping $g : \Omega \to D$ such that D is an analytic Jordan domain where $\mathbb{T}$ is the outer boundary component of D.*

Obviously, the mapping from Lemma 2.1 mapping is not unique. For instance, we find for every boundary component A of Ω a conformal mapping g such that g associates A with $\mathbb{T}$. Nevertheless, in case of multiply connected domains, there are analogous theorems to Riemann's mapping theorem for simply connected domains.

Proposition 2.2 ([Con95], Theorem 15.6.2)**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$, $a \in \Omega$ and E is a connected component of $\partial_\infty\Omega$. Then there is a unique circular slit disk D and a unique conformal mapping $g : \Omega \to D$ such that g associates E with $\mathbb{T}$, $g(a) = 0$ and $g'(a) > 0$.*

Proposition 2.3 ([Con95], Theorem 15.5.1)**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \geq 2$, $a \in \Omega$ and E and F are two different connected components of $\partial_\infty\Omega$. Then there is a unique $r > 0$, a unique circular slit annulus D with inner radius r and a unique conformal mapping $g : \Omega \to D$ such that g associates E with $\mathbb{T}$, g associates F with $\mathbb{T}_r$ and $g'(a) > 0$.*

Proposition 2.4. *Let $\Omega \subseteq \mathbb{H}$ be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$ and assume $\mathbb{H} \setminus \Omega$ is bounded. Then there is a unique upper parallel slit half-plane D, a unique $a \in \mathbb{C}$ and a unique conformal mapping $g : \Omega \to D$ with*

$$g(z) = z + \frac{a}{z} + \mathcal{O}(|z|^{-2}) \qquad \textit{around } \infty.$$

[6] For example $\mathbb{C}$ is not nondegenerate.

Proof. This theorem follows easily from Theorem 3.5.2 in [Gru78]. First let $\Omega^* \subseteq \mathbb{C}_\infty$ be the domain that arise from Ω by a reflection along the real axis. Consequently, $\infty \in \Omega^*$ and $\text{con}(\Omega^*) = 2\mathfrak{n} - 1$ where $\mathfrak{n} := \text{con}(\Omega)$. Using Theorem 3.5.2 in [Gru78], we find a unique conformal mapping $g^* : \Omega^* \to D^*$ where D^* is the Riemann sphere minus $2\mathfrak{n} - 1$ bounded disjoint proper closed line segments parallel to real axis and $g^*(z) - z \to 0$ as $z \to \infty$. Note that $g^*(z) = \overline{g^*(\bar{z})}$ for all $z \in \Omega^*$. Otherwise, $h(z) := \overline{g^*(\bar{z})}$ would contradict the uniqueness of g^*. Finally, $g := g^*|_\Omega$ is the unique mapping function we were looking for. □

Remark 2.1. Note that the previous proof showed that the function g in Proposition 2.4 is well defined in a neighbourhood around ∞ by the Schwarz reflection principle. Moreover, $a \geq 0$ as real values around ∞ are mapped by $g(z) = z + \frac{a}{z} + \mathcal{O}(|z|^{-2})$ to real values around ∞ and the orientation is preserved. The previous normalisation is called *hydrodynamic normalisation*.

These three mapping theorems will build our foundation for studying expanding families of multiply connected domains:

(a) Proposition 2.2 will be used in Section 2.3 in order to establish a radial Komatu–Loewner equation.

(b) Proposition 2.3 will be used in Section 2.4 in order to establish a bilateral Komatu–Loewner equation.

(c) Proposition 2.4 will be used in Section 2.5 in order to establish a chordal Komatu–Loewner equation.

Beside these theorems we need one further canonical mapping that will represent the kernel in Komatu–Loewner equations.

Proposition 2.5 (Theorem 2.3 in [Cou77])**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$ such that $\partial\Omega$ is locally connected, and the outer or unbounded boundary component $C_\mathfrak{n}$ is an analytic Jordan curve in $\mathbb{C}_\infty$. Assume $\zeta \in C_\mathfrak{n} \setminus \{\infty\}$ and $a \in \text{cl}_\infty(\Omega) \setminus \{\zeta\}$.*

Then there is a unique conformal mapping $w \mapsto \Phi_{a,\zeta,\Omega}(w)$ that maps Ω onto a right parallel slit half-plane with the normalisation $\Phi_{a,\zeta,\Omega}(a) \geq 0$ and $|\Phi_{a,\zeta,\Omega}(w)(w - \zeta)| \to 2$ as $w \to \zeta$.

It is easy to see that $\Phi_{a,\zeta,\Omega} - \Phi_{b,\zeta,\Omega} \equiv ic$ with $c \in \mathbb{R}$ whenever $a, b \in \text{cl}_\infty(\Omega)$ and $\zeta \in C_\mathfrak{n} \setminus \{a, b, \infty\}$. Note that $\Phi_{a,\zeta,\Omega}(a)$ is well defined since $\partial_\infty\Omega$ is locally connected, see Theorem 2.1 in [Pom92]. Moreover, the limit $\lim_{w\to\zeta} |\Phi_{a,\zeta,\Omega}(w)(w - \zeta)|$ is well-defined as well. To see this let us have a look at the function $h(w) := 1/\Phi_{a,\zeta,\Omega}(w)$, $w \in \Omega$. Since $C_\mathfrak{n}$ is an analytic Jordan curve, we are able to reflect h along $C_\mathfrak{n}$, so h has an analytic extension to an open neighbourhood of ζ. Finally, an easy calculation shows $\lim_{w\to\zeta} \Phi_{a,\zeta,\Omega}(w)(w - \zeta) = 1/h'(\zeta)$ where $h'(\zeta) \neq 0$ as a consequence of the univalence.

From now on and for the rest of this thesis, we will use the notation $\Phi_{a,\zeta,\Omega}(w)$ in order to represent the conformal map from the previous proposition.

Remark 2.2. In case of $\Omega = \mathbb{D}$ and $\zeta \in \mathbb{T}$ we get

$$\Phi_{0,\zeta,\mathbb{D}}(w) = \frac{\zeta + w}{\zeta - w} \quad \text{for all } w \in \mathbb{D}.$$

Analogously, with $\Omega = \mathbb{H}$ and $\zeta \in \mathbb{R}$, we find

$$\Phi_{\infty,\zeta,\mathbb{H}}(w) = \frac{2\mathrm{i}}{w - \zeta} \quad \text{for all } w \in \mathbb{H}.$$

Finally, we are going to discuss some extremal properties related to the conformal mappings from Proposition 2.2, 2.3 and 2.4.

Lemma 2.6 (Theorem IX.26 in [Tsu75])**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected bounded domain, $\mathfrak{n} \in \mathbb{N}$ and $a \in \Omega$. Assume $E \subseteq \partial\Omega$ denotes the outer boundary component of Ω and*

$$\mathcal{F} := \{f : \Omega \to \mathbb{D} \,|\, f \text{ univalent, } f(a) = 0,\ f'(a) > 0,\ f \text{ associates } E \text{ with } \mathbb{T}\}.$$

Then the unique mapping $g \in \mathcal{F}$ from Proposition 2.2 fulfils the extremal property $g'(a) = \max_{f\in\mathcal{F}} f'(a)$.

Alternatively see Chapter VII.2 in [Neh52].

Remark 2.3. In the previous definition of $\mathcal{F}$ we require f to be univalent. What if we drop the univalence? Let us consider the following class

$$\mathcal{F} := \{f : \Omega \to \mathbb{D} \,|\, f \text{ analytic, } f(a) = 0,\ f'(a) > 0\}$$

and consider the extremal problem $\sup_{f\in\mathcal{F}} f'(a)$. Using Montel's theorem, it is easy to see that there is an analytic extremal function f^*. If $\Omega \neq \mathbb{C}$ is simply connected, f^* coincides with g, but if $\mathfrak{n} = \mathrm{con}(\Omega) > 1$ this is not the case. In particular, f^* is the so called *Ahlfors function* that maps Ω onto the $\mathfrak{n}$-sheeted unit disk, see Theorem XI.3.1 in [Gol69]. Consequently, f^* is not univalent if $\mathfrak{n} > 1$.

Lemma 2.7 (Theorem IX.29 in [Tsu75])**.** *Let Ω be a nondegenerate $\mathfrak{n}$-connected bounded domain with $\mathfrak{n} \geq 2$ and E and F are two different boundary components of Ω. Moreover, we set*

$$\mathcal{F} := \bigcup_{r\in(0,1)} \mathcal{F}_r, \quad \mathcal{F}_r := \{f : \Omega \to \mathbb{D} \mid f \text{ univalent,} \\ f \text{ associates } E \text{ with } \mathbb{T} \text{ and } F \text{ with } \mathbb{T}_r\}.$$

Then the unique mapping $g \in \mathcal{F}$ from Proposition 2.3 fulfils the extremal property $g \in \mathcal{F}_{r_0}$ with $\mathcal{F} = \bigcup_{r\in(0,r_0]} \mathcal{F}_r$ and $r_0 \in (0,1)$.

Lemma 2.8. *Let $\Omega \subseteq \mathbb{H}$ be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$ such that $\mathbb{H} \setminus \Omega$ is bounded. Moreover, we set*

$$\mathcal{F} := \{f : \Omega \to \mathbb{H} \mid f \text{ univalent, } \mathbb{R} \text{ is the unbounded connected component of } \partial f(\Omega), \\ f(z) = z + \tfrac{a_f}{z} + \mathcal{O}(|z|^{-2}) \text{ around } \infty\}.$$

Then the unique mapping $g \in \mathcal{F}$ from Proposition 2.4 fulfils the extremal property $a_g = \max_{f\in\mathcal{F}} a_f$.

Proof. Obviously, each $f \in \mathcal{F}$ can be extended to a function $f : \Omega^* \to \mathbb{C}_\infty$ such that Ω^* arise from the reflection of Ω along the real axis. Thus ∞ is an inner point of Ω^*. Note that the extended mapping g from Proposition 2.4 maps Ω^* onto $\mathbb{C}_\infty$ minus disjoint proper closed line segments parallel to the real axis. Finally, we get the asserted extremal property by applying Theorem 3.5.2 in [Gru78] to the class $\mathcal{F}^* := \{f : \Omega^* \to \mathbb{C} \mid f|_\Omega \in \mathcal{F}\}$ and the extended function $g : \Omega^* \to \mathbb{C}_\infty$. □

Kernel convergence

Kernel convergence due to Carathèodory is a very powerful and important tool in geometric function theory, see Section 1.4 in [Pom75] or Section 3.1 in [Dur83] where the concept is explained in case of simply connected domains.

In case of multiply connected domains we refer to Section 15.4. in [Con95]. Let $(\Omega_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ be a sequence of domains such that $0 \in \Omega_n$ for almost all $n \in \mathbb{N}$. The *kernel (with respect to* 0*) of the sequence* $(\Omega_n)_{n\in\mathbb{N}}$ is the connected component of the set

$$K := \{z \in \mathbb{C} \mid \exists_{r>0}\, \exists_{N\in\mathbb{N}}\, \forall_{n\geq N} :\ B_r(z) \subseteq \Omega_n\} \tag{2.1}$$

that contains 0 if there is a connected component that contains 0. Otherwise the sequence does not have a kernel. We say that the sequence $(\Omega_n)_{n\in\mathbb{N}}$ converges to Ω in terms of $\Omega_n \xrightarrow{\text{k}} \Omega$ if Ω is the kernel of each subsequence of $(\Omega_n)_{n\in\mathbb{N}}$. Moreover, let $(\Omega_t)_{t\in[0,T]}$ be a family of domains. Then $t \mapsto \Omega_t$ is *continuous at* t_0 *(with respect to kernel convergence)* if $\Omega_{t_n} \xrightarrow{\text{k}} \Omega_{t_0}$ for each sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ with $t_n \to t_0$. In this case we write $\Omega_t \xrightarrow{\text{k}} \Omega_{t_0}$ as well. On top of this we call the family $(\Omega_t)_{t\in[0,T]}$ *continuous (with respect to kernel convergence)* if $t \mapsto \Omega_t$ is continuous at each $t \in [0,T]$.

According to this definition it is not surprising that monotone sequences converge to their kernels.

Lemma 2.9. *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *satisfy* $0 \in \Omega_n \subseteq \Omega_{n+1}$ *for all* $n \in \mathbb{N}$ *or* $\mathbb{D}_\varepsilon \subseteq \Omega_{n+1} \subseteq \Omega_n$ *for all* $n \in \mathbb{N}$ *with an arbitrary* $\varepsilon > 0$*. Then the sequence* $(\Omega_n)_{n\in\mathbb{N}}$ *does have a kernel* Ω *and* $\Omega_n \xrightarrow{k} \Omega$*.*

Proof. First of all, it is clear that $K = \bigcup_{n\in\mathbb{N}} \Omega_n$ if $\Omega_n \subseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$ and $K = \bigcap_{n\in\mathbb{N}} \Omega_n$ if $\Omega_{n+1} \subseteq \Omega_n$ where K is defined as in Equation (2.1). Moreover, we obtain the same set K if we consider subsequences of $(\Omega_n)_{n\in\mathbb{N}}$.

Consequently, there is an $\varepsilon > 0$ such that $\mathbb{D}_\varepsilon \subseteq \Omega_n$ for all $n \in \mathbb{N}$ in either case. Thus $\mathbb{D}_\varepsilon \subseteq K$ as well. Summarising, $(\Omega_n)_{n\in\mathbb{N}}$ converges to the connected component of K that contains 0, which we denote by Ω. □

A very important property of kernel convergence is the following.

Lemma 2.10. *Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be a sequence of domains such that* $\Omega_n \xrightarrow{k} \Omega$*. Assume* $a \in \partial\Omega$ *is fixed. Then we find a sequence* $(a_n)_{n\in\mathbb{N}}$ *with* $a_n \in \partial\Omega_n$ *such that* $a_n \to a$ *as* $n \to \infty$*.*

Proof. This follows immediately from Exercise 15.4.5 from [Con95]. □

Next, we are going to combine the concept of kernel convergence with sequences of analytic functions. Therefore, let us assume that $f_n : \Omega_n \to D_n$ is a conformal mapping for each $n \in \mathbb{N}$ and $\Omega_n \xrightarrow{\text{k}} \Omega$. Then we say that $(f_n)_{n\in\mathbb{N}}$ *converges locally uniformly on* Ω *to* $f : \Omega \to \mathbb{C}$ or *uniformly on compact sets of* Ω *to* $f : \Omega \to \mathbb{C}$ if for every compact subset $K \subseteq \Omega$ and for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in K$ and all $n \geq N$. When this happens $f : \Omega \to D$ is either conformal or constant and we write $f_n \xrightarrow{\text{l.u.}} f$ on Ω.

On top of this, let $(\Omega_t)_{t\in[0,T]}$ be a continuous family with respect to kernel convergence and $(f_t)_{t\in[0,T]}$ with $f_t : \Omega_t \to \mathbb{C}$ analytic. Then $t \mapsto f_t$ is called *continuous at* t_0 *(with respect to compact convergence)* if $f_t \xrightarrow{\text{l.u.}} f_{t_0}$ on Ω_{t_0}, i.e. if we have $f_{t_n} \xrightarrow{\text{l.u.}} f_{t_0}$ on Ω_{t_0} for each sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0, T]$ with $t_n \to t_0$. Furthermore, $t \mapsto f_t$ with $t \in [0, T]$ is called *continuous (with respect to compact convergence)* if $t \mapsto f_t$ is continuous at each $t \in [0, T]$.

An interesting question is if there is any connection between the convergence of $(f_n)_{n\in\mathbb{N}}$ and the convergence of the image domains $(D_n)_{n\in\mathbb{N}} = (f_n(\Omega_n))_{n\in\mathbb{N}}$. The following proposition gives an answer.

Proposition 2.11 (Theorem 15.4.10 in [Con95])**.** *Assume* $f_n : \Omega_n \to D_n$ *is conformal for each* $n \in \mathbb{N}$, $\Omega_n \xrightarrow{k} \Omega \neq \mathbb{C}$, *and* $f_n(0) = 0$ *and* $f_n'(0) > 0$ *for almost all* $n \in \mathbb{N}$. *Then there is a conformal mapping* $f : \Omega \to D$ *such that* $f_n \xrightarrow{l.u.} f$ *on* Ω *if and only if* $D_n \xrightarrow{k} D$.

What if the sequence Ω_n does not have a kernel or does not satisfy $0 \in \Omega_n$? Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of domains (not necessarily having $0 \in \Omega_n$ for all or at least almost all $n \in \mathbb{N}$). Then we can not define the kernel of $(\Omega_n)_{n\in\mathbb{N}}$ as we did previously. Nevertheless, we can still define the set K from Equation (2.1) what we call the *weak kernel of* $(\Omega_n)_{n\in\mathbb{N}}$ if K is non-empty.

Let Ω be the weak kernel of $(\Omega_n)_{n\in\mathbb{N}}$ and denote by A an arbitrary connected component of Ω. Then we find easily, $\Omega_n - a \xrightarrow{\text{k}} A - a$ for each $a \in A$. Herein, $A - a := \{z \in \mathbb{C} \mid z + a \in A\}$. Moreover, let $f_n : \Omega_n \to D_n$ be a conformal mapping for each $n \in \mathbb{N}$. Then we say $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly on A to $f : A \to \mathbb{C}$ if there is an $a \in A$ such that $h_n \xrightarrow{\text{l.u.}} h$ on $A - a$ with $h_n : \Omega_n - a \to D_n$, $h_n(z) := f_n(z + a)$ and $h : \Omega - a \to \mathbb{C}$, $h(z) := f(z + a)$. If this happens, we write $f_n \xrightarrow{\text{l.u.}} f$ on A as well.

Consequently, Proposition 2.11 gives us the following corollary.

Corollary 2.12. *Let* $A \neq \mathbb{C}$ *and* Ω_n, $n \in \mathbb{N}$, *be domains such that* $\Omega_n - a \xrightarrow{k} A - a$ *for some* $a \in A$. *For each* $n \in \mathbb{N}$, *we denote by* $f_n : \Omega_n \to D_n$ *a conformal mapping. Moreover, assume* $f_n \xrightarrow{l.u.} f$ *on* A *with a conformal mapping* $f : A \to D$. *Then* $D_n - b \xrightarrow{k} D - b$ *for all* $b \in D$.

Proof. Let $a \in A$ and $\Omega_n' := \Omega_n - a$. Then $\Omega_n' \xrightarrow{\text{k}} A' := A - a$. Note that $f_n(a) \to f(a)$ and $f_n'(a) \to f'(a) \neq 0$. This gives us $h_n \xrightarrow{\text{l.u.}} h$ on A' where $h_n(z) := (f_n(z + a) - f_n(a))/f_n'(a)$ for all $z \in \Omega_n'$ and $h(z) := (f(z + a) - f(a))/f'(a)$ for all $z \in A'$. It is necessary to choose $n \in \mathbb{N}$ large enough in order to guarantee $a \in \Omega_n$. Then $h_n(0) = 0$ and $h_n'(0) = 1 > 0$, so we find together with Proposition 2.11 $(D_n - f_n(a))/f_n'(a) \xrightarrow{\text{k}} (D - f(a))/f'(a)$. This shows $D_n - b \xrightarrow{\text{k}} D - b$ for all $b \in \mathbb{D}$ as well. □

Finally, a convergent sequence of canonical domains converges to a canonical domain:

Lemma 2.13. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of circular slit disks with $D_n \xrightarrow{k} D$ and $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$. Assume D is nondegenerate. Then D is a circular slit disk.*

Proof. Note that D is bounded by 1 as all D_n are bounded by 1. We denote by $E_1, \dots E_{\mathfrak{n}}$ the connected components of $\mathbb{C} \setminus D$ where $E_{\mathfrak{n}}$ is the unbounded connected component. For each E_k, $k \in \{1, \dots, \mathfrak{n}\}$, we find a Jordan curve $\Delta_k \subseteq D$ such that Δ_k separates E_k from E_j with $j \in \{1, \dots, \mathfrak{n}\} \setminus \{k\}$. Moreover, we can choose Δ_k in such a way that $\operatorname{dist}(\Delta_k, \Delta_j) > \delta$ whenever $j \neq k$. We set $E_k^{\Delta} := \Delta_k \cup \operatorname{int}(\Delta_k)$, $k \in \{1, \dots, \mathfrak{n}-1\}$ and $E_{\mathfrak{n}}^{\Delta} := \Delta_{\mathfrak{n}} \cup \operatorname{ext}(\Delta_{\mathfrak{n}})$. Consequently, $\operatorname{dist}(E_k^{\Delta}, E_j^{\Delta}) > \delta$ for all $k \neq j$ as well. Note that $D^{\Delta} := D \setminus \bigcup_{k=1}^{\mathfrak{n}} E_k^{\Delta}$ is an $\mathfrak{n}$-connected domain. D is the kernel of the sequence $(D_n)_{n\in\mathbb{N}}$ and $\operatorname{cl}(D^{\Delta})$ is a compact set in D, so we find $\operatorname{cl}(D^{\Delta}) \subseteq D_n$ for all $n \geq N$ with $N \in \mathbb{N}$ large.

Next, let be $a \in \partial E_k$ with $k \in \{1, \dots, \mathfrak{n}\}$. Using Lemma 2.10, we find a sequence $a_n \in \partial D_n$ such that $a_n \to a$. Since $D^{\Delta} \subseteq D_n$ and $\operatorname{con}(D_n) = \operatorname{con}(D)$, it is necessary that for all large $n \geq N$, the $\mathfrak{n}-1$ bounded connected components of ∂D_n are distributed one-to-one to the bounded connected components of $\mathbb{C} \setminus D$, i.e. if $F_1, \dots, F_{\mathfrak{n}-1}$ denote the bounded connected components of $\mathbb{C} \setminus D_n$, then $F_k \subseteq E_{I(k)}^{\Delta}$ for all $k \in \{1, \dots, \mathfrak{n}-1\}$ where $I : \{1, \dots, \mathfrak{n}-1\} \to \{1, \dots, \mathfrak{n}-1\}$ is one-to-one. This gives us

$$\mathbb{D} \setminus \left(\bigcup_{k=1}^{\mathfrak{n}-1} E_k^{\Delta} \right) \subseteq D_n \quad \text{for all large} n \geq N.$$

Consequently, we get $E_{\mathfrak{n}} = \mathbb{C} \setminus \mathbb{D}$, i.e. $\mathbb{T}$ is the outer boundary of D.

On top of this, let E_k, $k \in \{1, \dots, \mathfrak{n}-1\}$ be an arbitrary bounded connected component of $\mathbb{C} \setminus D$. Thus we proved already that for each large $n \in \mathbb{N}$ exactly one (bounded) connected component of $\mathbb{C} \setminus D_n$ is a subset of E_k^{Δ}. Note that all the bounded connected components of $\mathbb{C} \setminus D_n$ are disjoint proper concentric circular arcs. As mentioned before, for each $a \in \partial E_k$, we find a sequence $a_n \in \partial D_n$ such that $a_n \to a$. Finally, all sequences $|a_n|$ are independent of a, so $|a|$ is constant for each $a \in E_k$, i.e. E_k is a concentric circular arc. □

Lemma 2.14. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of circular slit annuli, D is a nondegenerate domain with $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$, and $D_n - a \xrightarrow{k} D - a$ for some $a \in D$. Then D is a circular slit annulus.*

Lemma 2.15. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of upper (or right) parallel slit half-planes, D is a nondegenerate domain with $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$, and $D_n - a \xrightarrow{k} D - a$ for some $a \in D$. Then D is an upper (or right) parallel slit half-plane.*

Proof of Lemma 2.14 and 2.15. This works in the same way as the proof of Lemma 2.13 □

Normal families

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of conformal maps $f_n : \Omega_n \to D_n$ and $\Omega_n \xrightarrow{\text{k}} \Omega$. In order to find locally uniformly convergent sequences or at least subsequences, a great tool is the concept of normal families, see [Sch93] as a useful reference.

By $\mathcal{F} \subseteq \{f : \Omega \to \mathbb{C} \mid f \text{ analytic}\}$ we denote a locally bounded family, i.e. for each compact set $K \subseteq \Omega$ we find an $M > 0$ such that $\|f\|_K \leq M$ for all $f \in \mathcal{F}$. Herein, $\|f\|_K := \max_{z\in K} |f(z)|$. Then Montel's famous theorem states that $\mathcal{F}$ is a normal family, i.e. we find for each sequence $(f_n)_{n\in\mathbb{N}} \subseteq F$ a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{l.u.}} f$ on Ω where $f : \Omega \to \mathbb{C}$ is an analytic function or f_{n_k} tends uniformly on compacts of Ω to infinity. If $(f_n)_{n\in\mathbb{N}}$ is locally bounded, the second case can not occur, so $f_{n_k} \xrightarrow{\text{l.u.}} f$ on Ω where $f : \Omega \to \mathbb{C}$ is analytic.

In our case we have to deal very often with functions $f_n : \Omega_n \to D_n$, so we can not apply Montel's theorem directly. Nevertheless, it is not hard to adapt the fundamental concept to our case. Therefore, let $f_n : \Omega_n \to \mathbb{C}$ be a sequence of analytic functions with $\Omega_n \xrightarrow{\text{k}} \Omega$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is called *locally bounded* if for every compact set $K \subseteq \Omega$, we find an $N \in \mathbb{N}$ and $M > 0$ such that $\|f_n\|_K \leq M$ for all $n \geq N$.

When this happens it is not hard to see that there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{l.u.}} f$ on Ω with an analytic function $f : \Omega \to \mathbb{C}$. To prove this, it is enough to study an increasing sequence of compact sets $(K_l)_{l\in\mathbb{N}}$ such that $K_l \subseteq K_{l+1}$ for all $l \in \mathbb{N}$ and $\Omega = \bigcup_{l=1}^{\infty} K_l$. Finally, we obtain the stated subsequence by using a diagonal argument combined with Montel's theorem applied to each K_l.

Some useful harmonic functions

Let $\Omega \subseteq \mathbb{C}$ be a domain and $z_0 \in \Omega$. Then $G : \Omega \times \Omega \to \text{cl}_\infty(\mathbb{R})$ is called *Green's function of* Ω if the following three conditions are satisfied.

(i) For each $z \in \Omega$, $\zeta \mapsto G(\zeta, z)$ is harmonic on $\Omega \setminus \{z\}$.

(ii) For each $z \in \Omega$, $\zeta \mapsto G(\zeta, z) + \ln|\zeta - z|$ is harmonic on Ω.

(iii) For each $z \in \Omega$, $\lim_{\zeta\to\partial_\infty\Omega} G(\zeta, z) = 0$.

Note that there is at least one Green function corresponding to a domain $\Omega \subseteq \mathbb{C}$. Moreover, it is not hard to show that each nondegenerate finitely connected domain does have a Green function. This is mainly based on the fact that $(\zeta, z) \mapsto G(f(\zeta), f(z))$ represents Green's function of the domain Ω' where $f : \Omega' \to \Omega$ is a conformal mapping and G is Green's function of Ω. An important property of Green's function is it's symmetry property, i.e. $G(\zeta, z) = G(z, \zeta)$ for all $(\zeta, z) \in \Omega \times \Omega$, see [GM05], Chapter II.2 for more details. On top of this, Green's function can be used to generalise Poisson's formula for $\mathbb{D}$ as follows.

Proposition 2.16 (Generalised Poisson formula, see Theorem II.2.5 in [GM05])**.** *Let Ω be an $\mathfrak{n}$-connected analytic Jordan domain with $\mathfrak{n} \in \mathbb{N}$. Moreover, the function u : $\text{cl}(\Omega) \to \mathbb{R}$ is continuous on $\text{cl}(\Omega)$ and harmonic on Ω. Then*

$$u(z) = -\frac{1}{2\pi} \int\limits_{\partial\Omega} \frac{\partial G(\zeta, z)}{\partial n_\zeta} u(\zeta) |\mathrm{d}\zeta| \qquad \textit{for all } z \in \Omega$$

where n_ζ denotes the unit outer normal at $\zeta \in \partial\Omega$.

Note that $\partial G(\zeta, z)/\partial n_\zeta$ is well-defined for each $\zeta \in \partial\Omega$, as $\zeta \mapsto G(\zeta, z)$ has a harmonic extension to $\mathrm{cl}(\Omega)$.

Let Ω be an $\mathfrak{n}$-connected analytic Jordan domain with the boundary components $C_1, \ldots, C_\mathfrak{n}$, $\mathfrak{n} \in \mathbb{N}$. Here, $C_\mathfrak{n}$ denotes the outer boundary component. In general, Green's function of a finitely connected domain Ω does not have a (single-valued) conjugate harmonic function. A reason for this is a nonvanishing period of $G(\cdot, z)$ around z. To be more precise, the period is 2π, i.e.

$$\int_{\partial B_\varepsilon(z)} \frac{\partial G(\zeta, z)}{\partial n_\zeta} |\mathrm{d}\zeta| = 2\pi, \tag{2.2}$$

with a small $\varepsilon > 0$ and n_ζ pointing towards z. On top of this, there are additional nonvanishing periods if $\mathfrak{n} \geq 2$:

$$-2\pi\omega_k(z) := \int_{\gamma_k} \frac{\partial G(\zeta, z)}{\partial n_\zeta} |\mathrm{d}\zeta|, \qquad k \in \{1, \ldots, \mathfrak{n} - 1\}.$$

Here, γ_k is a circuit around C_k and n_ζ denotes the unit outer normal, i.e. n_ζ points towards C_k. Next, we are going to remove these additional periods. Note that this is only necessary in the case $\mathfrak{n} \geq 2$.

For each $k \in \{1, \ldots, \mathfrak{n} - 1\}$, the function $z \mapsto \omega_k(z)$ is harmonic on Ω and is called *harmonic measure of Ω with respect to C_k*. Using Proposition 2.16, $\omega_k(z)$ tends to 1 if z approaches C_k and $\omega_k(z)$ tends to 0 if z approaches $\partial\Omega \setminus C_k$. The vector $\vec{\omega}(z) := (\omega_1(z), \ldots, \omega_{\mathfrak{n}-1}(z))^T$ is called *harmonic measure vector of* Ω.

Next, we denote the periods of ω_k around C_j by $2\pi P_{j,k}$, i.e.

$$2\pi P_{j,k} = \int_{\gamma_j} \frac{\partial \omega_k(z)}{\partial n} |\mathrm{d}z|.$$

Using the symmetry property of Green's function, it is a straightforward calculation to see that the matrix $P := (p_{j,k})_{j,k=1,\ldots,\mathfrak{n}-1}$ is symmetric and positive definite. This matrix is called *period matrix of* Ω.

Finally, assume $\zeta \in \Omega$ and let us have a deeper look at the function

$$z \mapsto -G(\zeta, z) - \vec{\omega}(z)^T P^{-1} \vec{\omega}(\zeta), \quad z \in \Omega.$$

It is not hard to show that this function is harmonic and has vanishing periods around each C_j with $j \in \{1, \ldots, \mathfrak{n}-1\}$. The only remaining nonvanishing period (like in the case $\mathfrak{n} = 1$) appears on circuits around ζ. Using the symmetry property of Green's function and Equation (2.2), we can see that this period is precisely 2π. In the case $\mathfrak{n} = 1$ it is not necessary to add additional functions to Green's function, so we may define $\vec{\omega} \equiv 0$ and $P \equiv 1$ in this case.

Nevertheless, the harmonic conjugate of $z \mapsto -G(\zeta, z) - \vec{\omega}(z)^T P^{-1} \vec{\omega}(\zeta)$ is multiple-valued in either case. As we have seen before, the conjugate function changes by 2π if z

describes a small circle around ζ. Thus by applying the exponential function we get a (single-valued) analytic function $g : \Omega \to \mathbb{C}$ with

$$\big|g(z)\big| = \exp\big(-G(\zeta, z) - \vec{\omega}(z)^T P^{-1}\vec{\omega}(\zeta)\big) \quad \text{for all } z \in \Omega. \tag{2.3}$$

It is not hard to prove that the function $g : \Omega \to \mathbb{C}$ is univalent, so $g : \Omega \to g(\Omega)$ is conformal. Moreover, $|g(z)|$ is constant with values $c_j \leq 1$ if z approaches an arbitrary point on C_j with $j \in \{1, \ldots, \mathfrak{n}\}$. In particular $|g(z)| = 1$ if z approaches $C_\mathfrak{n}$, so $\mathbb{T}$ is the outer boundary component of D. On top of this, $g(\zeta) = 0$ and the conjugate function can be chosen in such a way that $g'(\zeta) > 0$ holds. Summarising, g coincides with the unique Riemann mapping function form Proposition 2.2. This construction goes back to M. Schiffer, see Chapter 1 of the Appendix of [Cou77] for more details.

Obviously, the representation $|g(z)| = \exp\big(-G(\zeta, z) - \vec{\omega}(z)^T P^{-1}\vec{\omega}(\zeta)\big)$ holds for arbitrary nondegenerate $\mathfrak{n}$-connected domains Ω' as well. For this, Lemma 2.1 shows that there is a conformal mapping $f : \Omega' \to \Omega$ such that Ω is an analytic Jordan domain. Then it is easy to see that $G(f(\zeta), f(z))$ represents Green's function of Ω', $\omega_k(f(z))$ is the harmonic measure of Ω' with respect to the k-th boundary component of Ω' and the period matrix P is invariant under conformal mappings.

2.2 The kernel function $\Phi_{a,\zeta,\Omega}$

The goal of this section is to describe $\Phi_{a,\zeta,\Omega}$ in terms of relatives to Green's function.

Proposition 2.17. *Let Ω be a nondegenerate $\mathfrak{n}$-connected domain with $\mathfrak{n} \in \mathbb{N}$ such that $\partial\Omega$ is locally connected and the outer or unbounded boundary component $C_\mathfrak{n}$ is an analytic Jordan curve in $\mathbb{C}_\infty$. Assume $\zeta \in C_\mathfrak{n} \setminus \{\infty\}$ and $a \in \mathrm{cl}_\infty(\Omega) \setminus \zeta$. Then we have*

$$\Re\big(\Phi_{a,\zeta,\Omega}(w)\big) = -\frac{\partial G(\zeta, w)}{\partial n_\zeta} - \vec{\omega}(w)^T P^{-1} \frac{\partial \vec{\omega}(\zeta)}{\partial n_\zeta} \qquad \textit{for all } w \in \Omega.$$

Note that the right-hand side does not depend on a, as $\Phi_{a,\zeta,\Omega}(a) \geq 0$ determines the additive imaginary constant in a unique way.

Proof. 1) First of all, we assume that Ω is an analytic Jordan domain having $\mathbb{T}$ as it outer boundary component.

Let us denote by $G(\zeta, z)$ Green's function, $\vec{\omega}(z)$ is the harmonic measure vector and P stands for the period matrix of Ω. Moreover, we set

$$H(\zeta, z) := -\ln\left|\frac{\zeta - z}{1 - \zeta\bar{z}}\right| \qquad \text{with } \zeta, z \in \mathbb{D}.$$

Thus the function $\zeta \mapsto F(\zeta, z) := H(\zeta, z) - G(\zeta, z)$ is harmonic and positive on D for each $z \in D$, as $F(\zeta, z) = 0$ for all $\zeta \in \mathbb{T}$ and $F(\zeta, z) > 0$ if $\zeta \in \partial D \setminus \mathbb{T}$. Moreover, $F(\zeta, z) = F(z, \zeta)$ for all $\zeta, z \in D$, as G and H are Green functions, so $z \mapsto F(\zeta, z)$ is harmonic and positive on D for each $\zeta \in D$ as well.

Let be $\zeta_0 \in \mathbb{T}$. We extend $z \mapsto F(\zeta, z)$ to a harmonic function on $B_\varepsilon(\zeta_0)$ for all $\zeta \in D$ by using the Schwarz reflection principle if $\varepsilon > 0$ is small enough. To be more precise,

$$z \mapsto F(\zeta, z) := \begin{cases} -F(\zeta, 1/\bar{z}) & \text{for all } z \in B_\varepsilon(\zeta_0) \cap \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}, \\ 0 & \text{for all } z \in B_\varepsilon(\zeta_0) \cap \mathbb{T}. \end{cases}$$

Analogously, we reflect the function $\zeta \mapsto F(\zeta, z)$ to $B_\varepsilon(\zeta_0) \cap \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}$ by $\zeta \mapsto F(\zeta, z) := -F(1/\bar{\zeta}, z)$ for all $z \in B_\varepsilon(\zeta_0)$. Consequently, the function $z \mapsto F(\zeta, z)$ is harmonic on $B_\varepsilon(\zeta_0)$ for all $\zeta \in B_\varepsilon(\zeta_0) \setminus \mathbb{T}$. Moreover, $z \mapsto F(\zeta_0, z) \equiv 0$ if $\zeta_0 \in B_\varepsilon(\zeta_0) \cap \mathbb{T}$, so $z \mapsto F(\zeta, z)$ is harmonic on $B_\varepsilon(\zeta_0)$ for all $\zeta \in B_\varepsilon(\zeta_0)$. Conversely, $\zeta \mapsto F(\zeta, z)$ is harmonic on $B_\varepsilon(\zeta_0)$ for all $z \in B_\varepsilon(\zeta_0)$ as well.

Let be $\zeta_0 \in \mathbb{T}$ and denote by h_n a positive sequence converging to 0. Then

$$z \mapsto -\frac{F(\zeta_0 + h_n\zeta_0, z) - F(\zeta_0, z)}{h_n} = -\frac{F\big(\zeta_0(1 + h_n), z\big)}{h_n}$$

is a sequence of positive harmonic functions, which is normal by Montel's theorem. See [Sch93], Theorem 5.4.3 for further details. Thus we find a locally uniformly convergent subsequence converging to the function $z \mapsto -\partial/\partial n_{\zeta_0} F(\zeta_0, z)$, which needs to be harmonic in $B_\varepsilon(\zeta_0)$ as well. Herein, $\partial/\partial n_{\zeta_0}$ stands for the outward pointing derivative with respect to the unit circle. Moreover, $-\partial/\partial n_{\zeta_0} F(\zeta_0, z) = 0$ if $z \in \mathbb{T}$. Note that an easy calculation yields $\partial/\partial n_{\zeta_0} H(\zeta_0, z) = -\Re(\frac{\zeta_0+z}{\zeta_0-z})$ for all $z \in \mathbb{D}$. Consequently, we find

$$\begin{aligned} z \mapsto V(\zeta_0, z) &:= -\frac{\partial G(\zeta_0, z)}{\partial n_{\zeta_0}} - \vec{\omega}(z)^T P^{-1} \frac{\partial \vec{\omega}(\zeta_0)}{\partial n_{\zeta_0}} \\ &= \Re\left(\frac{\zeta_0 + z}{\zeta_0 - z}\right) + \frac{\partial F(\zeta_0, z)}{\partial n_{\zeta_0}} - \vec{\omega}(z)^T P^{-1} \frac{\partial \vec{\omega}(\zeta_0)}{\partial n_{\zeta_0}} \end{aligned}$$

with $z \in \Omega$. It is important to mention that $\vec{\omega}$ can be continued along $\mathbb{T}$, so the derivative of the harmonic measure vector is well-defined.

A straightforward calculation shows that $z \mapsto V(\zeta_0, z)$ has vanishing periods, so there exists a harmonic conjugate. Summarising, we have an analytic function $\Psi : \Omega \to \mathbb{C}$ with $\Re(\Psi(z)) = V(\zeta_0, z)$ for all $z \in \Omega$.

On top of this $z \mapsto V(\zeta_0, z)$ is constant on each boundary component of Ω. This can be seen by using the definition of V in case of the inner boundary components of Ω and the alternative representation of V (involving F) in case of the outer boundary component $\mathbb{T}$. Herein, $z \mapsto V(\zeta_0, z)$ has the constant value 0 on $C_\mathfrak{n}$. Finally, by using the argument principle together with the previous results it is not hard to see that $z \mapsto \Psi(z)$ maps Ω conformal onto a right parallel slit half-plane with $|\Psi(z)(z - \zeta_0)| \to 2$ if z tends to ζ_0.

2) Next, let us assume that Ω is an $\mathfrak{n}$-connected domain such that the outer boundary component $C_\mathfrak{n}$ is an analytic Jordan curve in $\mathbb{C}_\infty$ and $\xi \in C_\mathfrak{n} \setminus \{\infty\}$. By Lemma 2.1, there is a conformal map $T : \Omega \to \Omega'$ such that Ω' is an $\mathfrak{n}$-connected analytic Jordan domain with $\mathbb{T}$ as the outer boundary component. Without loss of generality we may assume $T(\xi) = 1 \in \mathbb{T}$. In particular T associates $C_\mathfrak{n}$ with $\mathbb{T}$. Note that T can be extended to an analytic function on $\Omega \cup C_\mathfrak{n}$ by the Schwarz reflection principle.

By $G_{\Omega'}(\zeta, z)$ we denote Green's function of Ω' and $G_\Omega(\xi, w)$ is the Green function of Ω. Obviously, we have $G_\Omega(\xi, w) = G_{\Omega'}(T(\xi), T(w)) = G_{\Omega'}(\zeta, z)$. We have similar relations for the harmonic measure and the period matrix, i.e. $\omega_\Omega(w) = \omega_{\Omega'}(T(w)) = \omega_{\Omega'}(z)$ and $P_\Omega = P_{\Omega'}$. Consequently, we get

$$\begin{aligned} V_\Omega(\xi, w) &:= -\frac{\partial G_\Omega(\xi, w)}{\partial n_\xi} - \vec{\omega}_\Omega(w)^T P_\Omega^{-1} \frac{\partial \vec{\omega}_\Omega(\xi)}{\partial n_\xi} \\ &= -\frac{\partial G_{\Omega'}(T(\xi), T(w))}{\partial n_\xi} - \vec{\omega}_{\Omega'}(T(w))^T P_{\Omega'}^{-1} \frac{\partial \vec{\omega}_{\Omega'}(T(\xi))}{\partial n_\xi} \\ &= \left(-\frac{\partial G_{\Omega'}(1, T(w))}{\partial n_\zeta} - \vec{\omega}_{\Omega'}(T(w))^T P_{\Omega'}^{-1} \frac{\partial \vec{\omega}_{\Omega'}(1)}{\partial n_\zeta} \right) |T'(\xi)| =: |T'(\xi)| \cdot V_{\Omega'}(1, T(w)). \end{aligned}$$

Using the first part, $w \mapsto V_\Omega(\xi, w)$ is the real part of a conformal mapping Ψ from Ω onto a right parallel slit half-plane. Moreover, we have

$$\begin{aligned} V_{\Omega'}(1, T(w)) &= \Re \left(\frac{1 + T(w)}{1 - T(w)} + \sum_{k=0}^{\infty} a_k (w - \xi)^k \right) \\ &= \Re \left(\frac{2}{T'(\xi)} \frac{1}{w - \xi} + \sum_{k=0}^{\infty} b_k (w - \xi)^k \right), \end{aligned}$$

for all $w \in B_\varepsilon(\xi)$ with a small $\varepsilon > 0$ and $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{C}$. Combining this with the previous equation we get

$$V_\Omega(\xi, w) = \Re \left(\frac{2|T'(\xi)|}{T'(\xi)} \frac{1}{w - \xi} + \sum_{k=0}^{\infty} b_k (w - \xi)^k \right).$$

Consequently, we get $\lim_{w\to\xi} \Psi(w)(w - \xi) = 2\frac{|T'(\xi)|}{T'(\xi)} = 2e^{\mathrm{i}\phi}$ with $\xi = re^{\mathrm{i}\phi}$. Summarising, $\Re\Psi \equiv \Re\Phi_{a,\xi,\Omega}$.

□

Lemma 2.18. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of circular slit disks with $D_n \xrightarrow{k} D$ and $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$. Moreover, assume D is nondegenerate and $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathbb{T}$ with $\zeta_n \to \zeta_0$. Then $\Phi_{0,\zeta_n,D_n} \xrightarrow{l.u.} \Phi_{0,\zeta_0,D}$ on the circular slit disk D as $n \to \infty$.*

Lemma 2.19. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of circular slit annuli, for each $n \in \mathbb{N}$, q_n is the inner radius of D_n, and D is a nondegenerate domain having $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$. Moreover, $D_n - a \xrightarrow{k} D - a$ for some $a \in D$. Assume $\zeta_n \to \zeta_0$ with $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathbb{T}$. Then $q_n \to q \in (0,1)$ and $\Phi_{q_n,\zeta_n,D_n} \xrightarrow{l.u.} \Phi_{q,\zeta_0,D}$ on the circular slit annulus D as $n \to \infty$.*

Lemma 2.20. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of upper parallel slit half-planes and D is a nondegenerate domain having $\operatorname{con}(D_n) = \operatorname{con}(D)$ for all $n \in \mathbb{N}$. Moreover, $D_n - a \xrightarrow{k} D - a$ for some $a \in D$ and assume $\zeta_n \to \zeta_0 \in \mathbb{R}$ with $(\zeta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Then $\Phi_{\infty,\zeta_n,D_n} \xrightarrow{l.u.} \Phi_{\infty,\zeta_0,D}$ on the upper parallel slit half-plane D as $n \to \infty$.*

Proof of Lemma 2.18, 2.19 and 2.20. We denote by Φ_n the corresponding mapping function, i.e. $\Phi_n = \Phi_{0,\zeta_n,D_n}$, $\Phi_n = \Phi_{q_n,\zeta_n,D_n}$ or $\Phi_n = \Phi_{\infty,\zeta_n,D_n}$ for all $n \in \mathbb{N}$. Using Lemma 2.13, 2.14 or 2.15, D is a circular slit disk, a circular slit annulus or an upper parallel slit half-plane. $h_n := 1/(\Phi_n + 1)$ is a bounded sequence, so we find with Montel's theorem a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ converging locally uniformly to the analytic function $h : D \to \mathbb{C}$. Note that h is either univalent or constant.

Using the proof of Lemma 2.13, each $z \mapsto h_{n_k}(z)$ can be extended by the Schwarz reflection principle to a univalent function on $B_\varepsilon(\zeta_0)$ with a small $\varepsilon > 0$. We calculate $|h'_{n_k}(\zeta_{n_k})| = \frac{1}{2}$ for all $k \in \mathbb{N}$. Note that h_{n_k} converges locally uniformly on the reflection as well. This is based on the fact, that h_{n_k} is bounded on $B_\varepsilon(\zeta_0)$ by Koebe's distortion theorem for univalent functions. Consequently, h fulfils $|h'(\zeta_0)| = \frac{1}{2}$, so h can not be constant. We have $\Phi_{n_k} \xrightarrow{\text{l.u.}} \Phi$ on D as well where $\Phi := 1/h - 1$, so $\Phi : D \to R$ is conformal as well. Consequently, we find

$$\Phi(w) = \frac{2e^{i\phi}}{w - \zeta_0} + \mathcal{O}(1) \qquad \text{around } \zeta_0,$$

so $|\Phi(w)(w - \zeta_0)| \to 2$ as $w \to \zeta$. Since $\Phi : D \to R$ is a conformal mapping, R is necessarily a nondegenerate domain having $\operatorname{con}(R) = \operatorname{con}(D) = \operatorname{con}(D_n) = \operatorname{con}(R_n)$ with $R_n := \Phi_n(D_n)$ for all $n \in \mathbb{N}$.

Using Corollary 2.12, we find $R_{n_k} - a \xrightarrow{\text{k}} R - a$ for all $a \in R$ as $k \to \infty$. On top of this Lemma 2.15 yields that R is a right parallel slit half-plane. It is easy to see that $\Phi(a) \geq 0$ if $a = 0$, $a = q$ or $a = \infty$. Summarising, $\Phi \equiv \Phi_{a,\zeta_0,D}$.

As all convergent subsequences $(\Phi_{n_k})_{k\in\mathbb{N}}$ converge to the same function $\Phi_{a,\zeta_0,D}$ also the whole sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges locally uniformly to $\Phi_{a,\zeta_0,D}$ on D. □

Finally, we are going to prove an extension of Schwarz integral formula to multiply connected domains.

Proposition 2.21. *Let Ω be a nondegenerate $\mathfrak{n}$-connected bounded domain with $\mathfrak{n} \in \mathbb{N}$ and $C_1, \ldots, C_{\mathfrak{n}}$ representing the boundary components of Ω. Assume $\partial\Omega$ is locally connected and the outer boundary component $C_{\mathfrak{n}}$ is an analytic Jordan curve. Moreover, $F : \Omega \to \mathbb{C}$ is analytic, $\Re(F)$ is continuous on $\operatorname{cl}(\Omega)$ and $\Re(F)$ is constant on each C_k with $k \in \{1, \ldots, \mathfrak{n} - 1\}$.*

Then the following representation holds for each $a \in \operatorname{cl}(\Omega) \setminus C_{\mathfrak{n}}$:

$$F(z) = \frac{1}{2\pi} \int\limits_{C_{\mathfrak{n}}} \Re\big(F(\zeta)\big) \cdot \Phi_{a,\zeta,\Omega}(z) |\mathrm{d}\zeta| + \mathrm{i}c \qquad \textit{for all } z \in \Omega.$$

In this context, the constant $c \in \mathbb{R}$ depends only on the choice of a.

Note that for each $z \in \Omega$, $\zeta \mapsto \Phi_{a,\zeta,\Omega}(z)$ is continuous on $C_{\mathfrak{n}}$. To see this let $a \in \operatorname{cl}(\Omega) \setminus C_{\mathfrak{n}}$ and $T : \Omega \to \Omega'$ be a conformal mapping such that Ω' is a circular slit disk. We find T in such a way that T associates $C_{\mathfrak{n}}$ with $\mathbb{T}$. Moreover we set $R := T^{-1} : \Omega' \to \Omega$. An easy calculation gives us $\Phi_{a,\zeta,\Omega}(z) = T'(\zeta)\Phi_{T(a),T(\zeta),\Omega'}(z)$. By a reflection, $\zeta \mapsto T(\zeta)$ is analytic on $C_{\mathfrak{n}}$, so $\zeta \mapsto T'(\zeta)$ as well as $\zeta \mapsto T(\zeta)$ are continuous on $C_{\mathfrak{n}}$. Thus $\zeta \mapsto \Phi_{a,\zeta,\Omega}(z)$ is continuous on $C_{\mathfrak{n}}$ by Lemma 2.18.

The proof of Proposition 2.21 follows the ideas of [Kom50] and the proof of Theorem 5.1 in [BF06]

Proof. 1) First of all, we going to assume that Ω is an analytic Jordan domain where the outer boundary component of Ω is $C_{\mathfrak{n}} = \mathbb{T}$. The other boundary components are denoted by $C_1, \dots, C_{\mathfrak{n}-1}$.

Then $z \mapsto \Re F(z)$ is harmonic on Ω and continuous on $\mathrm{cl}(\Omega)$, so Poisson's formula (see Proposition 2.16) gives us

$$\Re\big(F(z)\big) = -\frac{1}{2\pi} \int\limits_{\partial\Omega} \Re\big(F(\zeta)\big) \frac{\partial G(\zeta, z)}{\partial n_\zeta} |\mathrm{d}\zeta| \qquad \text{for all } z \in \Omega,$$

where $G(\zeta, z)$ denotes the Green function of Ω with pole at z. Since F is an analytic function, the periods with respect to C_k, $k \in \{1, \dots, \mathfrak{n}-1\}$, vanish. Thus, for all $k \in \{1, \dots, \mathfrak{n}-1\}$, we get

$$0 = \int\limits_{C_k} \frac{\partial \Re F}{\partial n_\zeta}(\zeta) |\mathrm{d}\zeta| = \int\limits_{\partial\Omega} \omega_k(\zeta) \frac{\partial \Re F}{\partial n_\zeta}(\zeta) |\mathrm{d}\zeta| = \int\limits_{\partial\Omega} \Re F(\zeta) \frac{\partial \omega_k(\zeta)}{\partial n_\zeta} |\mathrm{d}\zeta|,$$

where $\omega_k(\zeta)$ denotes the harmonic measure of Ω w.r.t. C_k. Note that the last equation is an application of Green's theorem. By combining these two equations we find

$$\Re\big(F(z)\big) = -\frac{1}{2\pi} \int\limits_{\partial\Omega} \Re\big(F(\zeta)\big) \left(\frac{\partial G(\zeta, z)}{\partial n_\zeta} + \vec{\omega}(z)^T P^{-1} \frac{\partial \vec{\omega}(\zeta)}{\partial n_\zeta} \right) |\mathrm{d}\zeta| \quad \text{for all } z \in \Omega,$$

where $\vec{\omega}$ denotes the harmonic measure vector. The matrix P is the period matrix. Using Proposition 2.17, we find

$$-\frac{\partial G(\zeta, z)}{\partial n_\zeta} - \vec{\omega}(z)^T P^{-1} \frac{\partial \vec{\omega}(\zeta)}{\partial n_\zeta} = \Re\big(\Phi_{a,\zeta,\Omega}(z)\big) \qquad \text{for each } z \in \Omega.$$

Herein, $\Phi_{a,\zeta,\Omega}(z)$ denotes the unique mapping from Proposition 2.5 with some $a \in \mathrm{cl}(\Omega) \setminus C_{\mathfrak{n}}$. Obviously, $\zeta \mapsto \Re(F(\zeta))$ is constant on each C_k and $\zeta \mapsto G(\zeta, z) + \vec{\omega}(z) P^{-1} \omega(\zeta)$ has vanishing periods on circuits around each C_k with $k \in \{1, \dots, \mathfrak{n}-1\}$. Consequently, we get

$$\Re\big(F(z)\big) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \Re\big(F(\zeta)\big) \Re\big(\Phi_{a,\zeta,\Omega}(z)\big) |\mathrm{d}\zeta| \qquad \text{for all } z \in \Omega.$$

Using the open mapping theorem, we find

$$F(z) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \Re\big(F(\zeta)\big) \Phi_{a,\zeta,\Omega}(z) |\mathrm{d}\zeta| + \mathrm{i}c \qquad \text{for all } z \in \Omega,$$

where $c \in \mathbb{R}$ (depends on the choice of a).

2) Next let Ω be an $\mathfrak{n}$-connected bounded domain such that the outer boundary component $C_{\mathfrak{n}}$ is an analytic Jordan curve and $\partial\Omega$ is locally connected. Using Lemma 2.1, we find a conformal mapping $T : \Omega \to \Omega'$ where Ω' is an analytic Jordan domain having $\mathbb{T}$ as the outer boundary component. Moreover, we can find T in such a way that T associates $C_{\mathfrak{n}}$ with $\mathbb{T}$. We denote the inverse function by R, i.e. $R := T^{-1} : \Omega' \to \Omega$.

Using the first part, we find with some $a \in \mathrm{cl}(\Omega) \setminus C_\mathfrak{n}$:

$$(F \circ R)(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \Re\big((F \circ R)(\zeta)\big) \Phi_{T(a),\zeta,\Omega'}(z) |\mathrm{d}\zeta| + \mathrm{i}c \qquad \text{for all } z \in \Omega'.$$

Note that T can be extended to an analytic function on $C_\mathfrak{n}$. Proposition 2.17 yields $\Phi_{T(a),\zeta,\Omega'}(z) = |R'(\zeta)| \Phi_{a,R(\zeta),\Omega}(R(z))$ for each $\zeta \in \mathbb{T}$. Consequently, we get with $z = T(w)$ and a simple substitution

$$F(w) = \frac{1}{2\pi} \int_{C_\mathfrak{n}} \Re\big(F(\zeta)\big) \Phi_{a,\zeta,\Omega}(w) |\mathrm{d}\zeta| + \mathrm{i}c \qquad \text{for all } z \in \Omega,$$

so the proof is complete.

□

2.3 Radial case

In order to study radial Komatu–Loewner equations we take an arbitrary circular slit disk Ω as our initial domain. A subset $\mathfrak{H} \subseteq \Omega \setminus \{0\}$ is called *(compact) radial hull in* Ω or *(compact) radial* Ω*-hull* if $\Omega \cap \mathrm{cl}(\mathfrak{H}) = \mathfrak{H}$, $\Omega \setminus \mathfrak{H}$ is a domain and $\mathbb{T} \cup \mathrm{cl}(\mathfrak{H})$ is connected[7]. By $g_\mathfrak{H}$ we denote the unique conformal mapping that maps $\Omega \setminus \mathfrak{H}$ onto a circular slit disk $D_\mathfrak{H}$ normalised in such a way that $g_\mathfrak{H}(0) = 0$, $g'_\mathfrak{H}(0) > 0$ and $g_\mathfrak{H}$ associates the outer boundary component of $\Omega \setminus \mathfrak{H}$ with $\mathbb{T}$, see Proposition 2.2. We will call this function *normalised radial mapping function on* $\Omega \setminus \mathfrak{H}$. On top of this we define the so called *logarithmic mapping radius* by $\mathrm{lmr}(g) := \ln g'(0)$ for each function g that is analytic at 0 with $g'(0) > 0$. Sometimes we also write $\mathrm{lmr}_\Omega(\mathfrak{H}) := \mathrm{lmr}(g_\mathfrak{H})$ where $g_\mathfrak{H}$ is the normalised radial mapping function on $\Omega \setminus \mathfrak{H}$.

Next, let $(\mathfrak{H}_t)_{t\in[0,T]} \subseteq \Omega$ be a family of radial Ω-hulls, i.e. $\mathfrak{H}_t$ is a radial Ω-hull for each $t \in [0,T]$. Then we say $(\mathfrak{H}_t)_{t\in[0,T]}$ is an *increasing family of radial* Ω*-hulls* if $\mathfrak{H}_t \subsetneq \mathfrak{H}_s$ whenever $0 \le t < s \le T$ and $\mathfrak{H}_0 = \emptyset$. Moreover, $(\mathfrak{H}_t)_{t\in[0,T]}$ is called *continuous family of radial* Ω*-hulls* if $(\Omega_t)_{t\in[0,T]}$, with $\Omega_t := \Omega \setminus \mathfrak{H}_t$, is continuous with respect to kernel convergence on $[0,T]$.

2.3.1 Single slit Komatu–Loewner equation

For now let us restrict ourself to slits, i.e. we do not treat general hulls (which we will study in Chapter 5). Let $\gamma : [0,T] \to \mathrm{cl}(\Omega) \setminus \{0\}$ be simple and continuous with $\gamma(0,T] \subseteq \Omega$ and $\gamma(0) \in \mathbb{T}$. Obviously, $(\gamma(0,t])_{t\in[0,T]}$ is an increasing continuous family of radial Ω-hulls. For each $t \in [0,T]$, we set $\Omega_t := \Omega \setminus \gamma(0,t]$ and denote by g_t the normalised radial mapping function from Ω_t onto the circular slit disk D_t.

Later (see Lemma 2.24 and 2.25) we will see that the function $t \mapsto \mathrm{lmr}(g_t)$ is strictly increasing and continuous on [0,T]. Since $g_0 \equiv \mathrm{id}$, i.e $\mathrm{lmr}(g_0) = 0$, it is not a great restriction to assume $t \mapsto \mathrm{lmr}(g_t) = t$ for all $t \in [0,T]$. Otherwise we can reparametrise

[7] In the simply connected case this definition is equivalent to the usual definition of a radial $\mathbb{D}$-hull, see [Law05], Section 3.5.

γ. Moreover, we are going to show later that $t \mapsto U_t := g_t(\gamma(t))$ is continuous on $[0,T]$, see Lemma 2.29. Note that $g_t(\gamma(t))$ is well-defined. Finally, by $w \mapsto \Phi_{0,U_t,D_t}(w)$ we denote the mapping function from Proposition 2.5 that maps the circular slit disk D_t onto a right parallel slit half-plane with $a = 0$, see Figure 2.2.

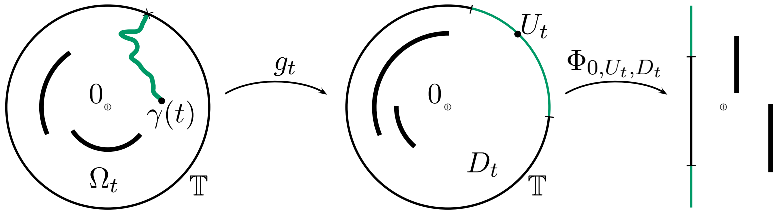

FIGURE 2.2: *Mapping behaviour of $z \mapsto g_t(z)$ and $w \mapsto \Phi_{0,U_t,D_t}(w)$ in the radial single slit case*

Then we have the following theorem.

Theorem 2.22. *Let Ω be a circular slit disk and $\gamma : [0,T] \to \mathrm{cl}(\Omega) \setminus \{0\}$ be simple and continuous with $\gamma(0,T] \subseteq \Omega$ and $\gamma(0) \in \mathbb{T}$. Moreover, we set $\Omega_t := \Omega \setminus \gamma(0,t]$. Assume $g_t : \Omega_t \to D_t$ is the normalised radial mapping function from Ω_t onto D_t with $\mathrm{lmr}(g_t) = t$ for each $t \in [0,T]$.*

Then $t \mapsto g_t(z)$ is continuously differentiable on $[0,T]$ for each $z \in \Omega_T$, and we get

$$\dot{g}_t(z) = g_t(z) \cdot \Phi_{0,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } t \in [0,T] \text{ and all } z \in \Omega_T, \tag{2.4}$$

where $t \mapsto U_t := g_t(\gamma(t)) \in \mathbb{T}$ is continuous on $[0,T]$.

Equation (2.4) is called *radial (single slit) Komatu–Loewner ordinary differential equation*, whereas $t \mapsto U_t$ is called *driving term.*

Remark 2.4. As mentioned in the introduction, see Section 1.4, Bauer and Friedrich proved the differential equation (2.4) in the left sense, see Theorem 5.1 in [BF06]. Moreover, they used the different representation of the kernel, in terms of relatives to Green's function, on the right-hand side of Equation (2.4) to work with, see Proposition 2.17.

If we do not assume $\mathrm{lmr}(g_t) = t$ for all $t \in [0,T]$, we get the following theorem, which can be seen as a pointwise version of the previous theorem.

Theorem 2.23. *Let Ω be a circular slit disk and $\gamma : [0,T] \to \mathrm{cl}(\Omega) \setminus \{0\}$ be simple and continuous with $\gamma(0,T] \subseteq \Omega$ and $\gamma(0) \in \mathbb{T}$. Moreover, for each $t \in [0,T]$, we set $\Omega_t := \Omega \setminus \gamma(0,t]$. $g_t : \Omega_t \to D_t$ is the normalised radial mapping function from Ω_t onto D_t for all $t \in [0,T]$ and assume $t \mapsto c(t) := \mathrm{lmr}(g_t)$ is differentiable at t_0.*

Then the function $t \mapsto g_t(z)$ is differentiable at t_0 for each $z \in \Omega_{t_0}$ and satisfies

$$\dot{g}_{t_0}(z) = \dot{c}(t_0) \cdot g_{t_0}(z) \cdot \Phi_{0,U_{t_0},D_{t_0}}\big(g_{t_0}(w)\big) \qquad \text{for all } z \in \Omega_{t_0},$$

with a continuous function $t \mapsto U_t =: g_t(\gamma(t)) \in \mathbb{T}$ for all $t \in [0,T]$.

Obviously, Theorem 2.22 follows immediately from Theorem 2.23. Here, the continuity of $t \mapsto \dot{g}_t(z)$ comes from Lemma 2.18. Before we are able to prove Theorem 2.23 we need some preliminary lemmas.

Lemma 2.24. *Let Ω be a circular slit disk and $\mathfrak{A}, \mathfrak{B} \subseteq \Omega \setminus \{0\}$ be radial Ω-hulls with $\mathfrak{A} \subsetneq \mathfrak{B}$. Then $\mathrm{lmr}(g_{\mathfrak{A}}) < \mathrm{lmr}(g_{\mathfrak{B}})$ where $g_{\mathfrak{A}}$ and $g_{\mathfrak{A}}$ denote the normalised radial mapping functions on $\Omega \setminus \mathfrak{A}$ and $\Omega \setminus \mathfrak{B}$, respectively.*

Proof. First of all, we denote the unbounded connected component of $\mathbb{C} \setminus g_{\mathfrak{A}}(\Omega \setminus \mathfrak{B})$ by F. Note that $\mathbb{C} \setminus F \subsetneq \mathbb{D}$ is a simply connected domain, so there is a unique conformal mapping $h : \mathbb{C} \setminus F \to \mathbb{D}$ with $h(0) = 0$ and $h'(0) > 0$. Since h^{-1} fulfils the condition of Schwarz lemma, we necessarily get $h'(0) > 1$. Thus we have $h \circ g_{\mathfrak{A}} \in \mathcal{F}$ where

$$\mathcal{F} := \{f : \Omega \setminus \mathfrak{B} \to \mathbb{D} \,|\, f \text{ univalent, } f(0) = 0,\ f'(0) > 0,\ f \text{ associates } \partial F \text{ with } \mathbb{T}\}.$$

Using the extremal property corresponding to $\mathcal{F}$, see Lemma 2.6, we find

$$h'(0) \cdot g'_{\mathfrak{A}}(0) = (h \circ g_{\mathfrak{A}})'(0) \leq g'_{\mathfrak{B}}(0),$$

i.e. $\mathrm{lmr}(h) + \mathrm{lmr}(g_{\mathfrak{A}}) \leq \mathrm{lmr}(g_{\mathfrak{B}})$ with $\mathrm{lmr}(h) > 0$. □

Lemma 2.25. *Let Ω be a circular slit disk, $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of radial Ω-hulls and g_t denotes the normalised radial mapping function from $\Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t for each $t \in [0, T]$. Let $(t_n)_{n\in\mathbb{N}} \subseteq [0, T]$ with $t_n \to t_0 \in [0, T]$ and let $\Omega_{t_n} \xrightarrow{k} \Omega_{t_0}$. Moreover, assume $\mathrm{con}(\Omega_{t_n}) = \mathrm{con}(\Omega_{t_0})$ for all $n \in \mathbb{N}$. Then $g_{t_n} \xrightarrow{l.u.} g_{t_0}$ on Ω_{t_0} as $n \to \infty$. Moreover, $\mathrm{lmr}(g_{t_n}) \to \mathrm{lmr}(g_{t_0})$ as $n \to \infty$.*

Corollary 2.26. *Let Ω be a circular slit disk, $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing and continuous family of radial Ω-hulls and g_t denotes the normalised radial mapping function on $\Omega \setminus \mathfrak{H}_t$ for each $t \in [0, T]$. Furthermore, we assume $\mathrm{con}(\Omega_t) = \mathrm{con}(\Omega)$ for all $t \in [0, T]$. Then $t \mapsto g_t$ is continuous on $[0, T]$. Moreover, $t \mapsto \mathrm{lmr}(g_t)$ is continuous on $[0, T]$ as well.*

Remark 2.5. Later we will see that the assumption

$$\mathrm{con}(\Omega_{t_n}) = \mathrm{con}(\Omega_{t_0}) \text{ for all } n \in \mathbb{N}$$

in Lemma 2.25 can be dropped without substitution, see Proposition 5.6. In order to do so, we will need a stronger version of Lemma 2.13 as well, see Lemma 5.5.

Proof of Lemma 2.25. By Montel's theorem $h_n := g_{t_n}$ is normal in Ω_{t_0}, so we find a locally uniformly convergent subsequence $(h_{n_k})_{k\in\mathbb{N}}$ on Ω_{t_0}. The limit function $h : \Omega_{t_0} \to \mathbb{C}$ is either univalent or constant. Using Lemma 2.24, $h'_n(0) \geq 1$ for all $n \in \mathbb{N}$, so h can not be constant, i.e. $h : \Omega_{t_0} \to D =: h(\Omega_{t_0})$ is conformal. This shows that D is nondegenerate. Next, Proposition 2.11 yields $D_{t_{n_k}} \xrightarrow{k} D$ where $D_{t_{n_k}} := h_{n_k}(\Omega_{t_{n_k}})$. Using Lemma 2.13, D needs to be a circular slit disk, as $\mathrm{con}(D_{t_{n_k}}) = \mathrm{con}(D)$ for all $k \in \mathbb{N}$.

Summarising, h is a conformal mapping from Ω_{t_0} onto the circular slit disk D with $h(0) = 0$ and $h'(0) > 0$. Moreover, h associates the outer boundary component of Ω_{t_0} with $\mathbb{T}$. To see this let us consider a circuit K around an inner boundary component of Ω_{t_0}, say $K = \{z \in \mathbb{D} \mid \mathrm{dist}(C, z) = \delta\}$ where $\delta > 0$ is small. Herein, we choose δ small enough such that K is a Jordan curve in Ω_{t_0} and the winding number of K around 0 is 0. Then the compact set K is mapped by h to $h(K)$, which surrounds an inner boundary component of D, as the winding number of $h(K)$ is 0 as well. Using the pigeonhole principle, h associates the inner boundary component of Ω_{t_0} with the inner boundary components of D.

Finally, $h \equiv g_{t_0}$. As all convergent subsequences $(h_{n_k})_{k\in\mathbb{N}}$ converge to the same function g_{t_0}, also the whole sequence $(g_{t_n})_{n\in\mathbb{N}}$ converges locally uniformly on Ω_{t_0} to g_{t_0}. □

Lemma 2.27. *Let Ω be a circular slit disk and $\mathfrak{H}$ is a radial Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then*

$$\operatorname{lmr}(g_{\mathfrak{H}}) = -\frac{1}{2\pi}\int\limits_{\mathbb{T}} \ln|g_{\mathfrak{H}}^{-1}(\zeta)||\mathrm{d}\zeta|,$$

where $g_{\mathfrak{H}}$ is the normalised radial mapping function from $\Omega_{\mathfrak{H}}$ onto the circular slit disk $D_{\mathfrak{H}}$.

Note that the integral is well defined as $g_{\mathfrak{H}}$ has a continuous extension to the boundary. This is a consequence of the local connectedness of $\partial(\Omega \setminus \mathfrak{H})$, see Theorem 2.1 in [Pom92].

Proof. Let $\mathfrak{n} := \operatorname{con}(\Omega_{\mathfrak{H}})$. Note that there is an analytic branch of the logarithm such that $z \mapsto \log(g_{\mathfrak{H}}^{-1}(z)/z)$ is an analytic function. This follows immediately from the mapping behaviour of $g_{\mathfrak{H}}$ together with simple calculations of winding numbers. By Cauchy's integral formula, we find

$$\begin{aligned}
-\operatorname{lmr}(g_{\mathfrak{H}}) &= \log\left(\frac{\mathrm{d}}{\mathrm{d}z}\, g_{\mathfrak{H}}^{-1}(z)\Big|_{z=0}\right) = \log\left(\frac{g_{\mathfrak{H}}^{-1}(z)}{z}\right)\Big|_{z=0} \\
&= \frac{1}{2\pi\mathrm{i}}\int\limits_{\partial D_{\mathfrak{H}}} \log\left(\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right)\frac{\mathrm{d}\zeta}{\zeta} = \frac{1}{2\pi}\int\limits_{\partial D_{\mathfrak{H}}} \log\left(\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right)\mathrm{d}\arg\zeta \\
&= \frac{1}{2\pi}\int\limits_{\partial D_{\mathfrak{H}}} \ln\left|\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta.
\end{aligned}$$

Note that the last equality is a consequence of $\operatorname{lmr}(g_{\mathfrak{H}}) \geq 0$. The boundary $\partial D_{\mathfrak{H}}$ consists of $\mathbb{T} = C_{\mathfrak{n}}$ and disjoint proper concentric circular arcs $C_1, \dots C_{\mathfrak{n}-1}$. Herein, the function $\zeta \mapsto \ln|g_{\mathfrak{H}}^{-1}(\zeta)/\zeta|$ is constant on each C_k with $k \in \{1, \dots, \mathfrak{n}-1\}$. Thus we find

$$\frac{1}{2\pi}\int\limits_{C_k} \ln\left|\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta = \frac{1}{2\pi}\ln\left|\frac{g_{\mathfrak{H}}^{-1}(\zeta_0)}{\zeta_0}\right|\int\limits_{C_k}\mathrm{d}\arg\zeta = 0$$

for each $k \in \{1, \dots, \mathfrak{n}-1\}$, as we integrate on both sides of the arc C_k. Here, ζ_0 is arbitrarily chosen from C_k. Summarising we find

$$-\operatorname{lmr}(g_{\mathfrak{H}}) = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \ln\left|\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \ln\left|g_{\mathfrak{H}}^{-1}(\zeta)\right||\mathrm{d}\zeta|.$$

□

Lemma 2.28. *Let Ω be a circular slit disk and $\mathfrak{H}$ is a radial Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then we have*

$$\log \frac{g_{\mathfrak{H}}^{-1}(z)}{z} = \frac{1}{2\pi} \int_{\mathbb{T}} \ln |g_{\mathfrak{H}}^{-1}(\zeta)| \, \Phi_{0,\zeta,D_{\mathfrak{H}}}(z) \, |\mathrm{d}\zeta| \qquad \textit{for each } z \in D_{\mathfrak{H}},$$

where $g_{\mathfrak{H}}$ is the normalised radial mapping function from $\Omega_{\mathfrak{H}}$ onto $D_{\mathfrak{H}}$.

Like in the proof of previous lemma there is an analytic branch of the logarithm on the left-hand side.

Proof. $D_{\mathfrak{H}}$ is a circular slit disk with boundary components $C_1, \ldots, C_{\mathfrak{n}} = \mathbb{T}$ and we consider the function

$$z \mapsto F(z) := \log \frac{g_{\mathfrak{H}}^{-1}(z)}{z}, \quad z \in D_{\mathfrak{H}},$$

which is analytic in $D_{\mathfrak{H}}$. Moreover, there is a continuous extension of F to $\partial\Omega$, as $\partial\Omega$ is locally connected. Then $\Re(F)$ is constant on C_k for each $k \in \{1, \ldots, \mathfrak{n}-1\}$, so we can apply Proposition 2.21 with $a = 0$ to get

$$\log \frac{g_{\mathfrak{H}}^{-1}(z)}{z} = \frac{1}{2\pi} \int_{\mathbb{T}} \ln \left| \frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta} \right| \Phi_{0,\zeta,D_{\mathfrak{H}}}(z) \, |\mathrm{d}\zeta| + \mathrm{i}c \qquad \text{for each } z \in D_{\mathfrak{H}}$$

with $c \in \mathbb{R}$. Finally, we set $z = 0$ to get $c = 0$, as $\Phi_{0,\zeta,D_{\mathfrak{H}}}(0) > 0$ for all $\zeta \in \mathbb{T}$ and $\log(g_{\mathfrak{H}}^{-1}(z)/z)|_{z=0} = -\operatorname{lmr}(g_{\mathfrak{H}}) < 0$. □

Lemma 2.29. *Let $\gamma : [0,T] \to \mathrm{cl}(\Omega)\setminus\{0\}$ be simple and continuous with $\gamma(0,T] \subseteq \Omega$ and $\gamma(0) \in \mathbb{T}$, and for each $t \in [0,T]$, $g_t : \Omega_t \to D_t$ denotes the normalised radial mapping function on $\Omega_t := \Omega \setminus \gamma(0,t]$. Moreover, we set*

$$U_t := g_t(\gamma(t)), \quad s_{\underline{t},\bar{t}} := g_{\bar{t}}(\gamma[\underline{t},\bar{t}]), \qquad 0 \le \underline{t} < \bar{t} \le T.$$

Then $s_{\underline{t},\bar{t}} \to U_{t_0}$ as $\bar{t} \to t_0 \leftarrow \underline{t}$. On top of this $t \mapsto U_t$ is continuous on $[0,T]$.

The image $g_{\bar{t}}(\gamma[\underline{t},\bar{t}])$ represents the image of both sides of the slit, i.e. $s_{\underline{t},\bar{t}} = \{a \in \mathbb{T} \mid g_{\bar{t}}^{-1}(a) \in \gamma[\underline{t},\bar{t}]\}$, see also Remark 2.9.

Proof of Lemma 2.29. This is a special case of Lemma 2.43, which we are going to prove later. □

Proof of Theorem 2.23. Using Proposition 2.11 and Corollary 2.26, $(D_t)_{t\in[0,T]}$ is a continuous family, since $(\Omega_t)_{t\in[0,T]}$ is a continuous family with $\mathrm{con}(\Omega_t) = \mathrm{con}(\Omega)$ for all $t \in [0,T]$. Let us define $g_{\underline{t},\bar{t}} := g_{\bar{t}} \circ g_{\underline{t}}^{-1}$ with $0 \le \underline{t} < \bar{t} \le T$. Thus $g_{\underline{t},\bar{t}}$ maps $D_{\underline{t}} \setminus S_{\underline{t},\bar{t}}$ onto the circular slit disk $D_{\bar{t}}$ where $S_{\underline{t},\bar{t}} := g_{\underline{t}}(\gamma(\underline{t},\bar{t}])$ is a slit starting in $U_{\underline{t}}$. Obviously, $S_{\underline{t},\bar{t}}$ is a locally connected radial $D_{\underline{t}}$-hull, so we are able to apply Lemma 2.28 to get

$$\log \frac{g_{\underline{t},\bar{t}}^{-1}(z)}{z} = \frac{1}{2\pi} \int_{\mathbb{T}} \ln |g_{\underline{t},\bar{t}}^{-1}(\zeta)| \, \Phi_{0,\zeta,D_{\bar{t}}}(z) \, |\mathrm{d}\zeta| \qquad \text{for all } z \in D_{\bar{t}}.$$

Next, we set $s_{\underline{t},\bar{t}} := g_{\bar{t}}(\gamma[\underline{t},\bar{t}]) = \{a \in \mathbb{T} \mid g_{\bar{t}}^{-1} \in \gamma_k[\underline{t},\bar{t}]\}$, so $s_{\underline{t},\bar{t}}$ is a compact connected subset of $\mathbb{T}$. Applying $z = g_{\bar{t}}(w)$ gives us

$$\log \frac{g_{\underline{t}}(w)}{g_{\bar{t}}(w)} = \frac{1}{2\pi} \int_{s_{\underline{t},\bar{t}}} \ln |g_{\underline{t},\bar{t}}^{-1}(\zeta)|\, \Phi_{0,\zeta,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big)\, |\mathrm{d}\zeta| \qquad \text{for each } w \in \Omega_{\bar{t}}.$$

Note that $\zeta \mapsto \Phi_{0,\zeta,D_{\bar{t}}}\big(g_{\underline{t}}(w)\big)$ is continuous by Lemma 2.18 and $\zeta \mapsto \ln|g_{\underline{t},\bar{t}}^{-1}(\zeta)| \leq 0$ for all $\zeta \in s_{\underline{t},\bar{t}}$, so we find with the mean value theorem

$$\log \frac{g_{\underline{t}}(w)}{g_{\bar{t}}(w)} = \Big(\Re\Phi_{0,\zeta_1,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big) + \mathrm{i}\Im\Phi_{0,\zeta_2,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big)\Big)\frac{1}{2\pi}\int_{s_{\underline{t},\bar{t}}} \ln|g_{\underline{t},\bar{t}}^{-1}(\zeta)|\,|\mathrm{d}\zeta|$$

for all $w \in \Omega_t$ where $\zeta_1, \zeta_2 \in s_{\underline{t},\bar{t}}$. Using Lemma 2.27, we see that the remaining integral on the right-hand side coincides with $-\operatorname{lmr}(g_{\underline{t},\bar{t}}) = -\ln g'_{\underline{t},\bar{t}}(0) = \operatorname{lmr}(g_{\underline{t}}) - \operatorname{lmr}(g_{\bar{t}})$. Let $w \in \Omega_{t_0}$ be fix. If we choose $\underline{t}$ and $\bar{t}$ close to t_0, we get $w \in \Omega_{\bar{t}}$, and using Lemma 2.25 we find a branch of the logarithm in order to get

$$\frac{\log g_{\bar{t}}(w) - \log g_{\underline{t}}(w)}{\bar{t} - \underline{t}} = \Big(\Re\Phi_{0,\zeta_1,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big) + \mathrm{i}\Im\Phi_{0,\zeta_2,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big)\Big)\frac{\operatorname{lmr}(g_{\bar{t}}) - \operatorname{lmr}(g_{\underline{t}})}{\bar{t} - \underline{t}}.$$

Using Lemma 2.29, we see $s_{\underline{t},\bar{t}} \to U_{t_0} =: g_{t_0}(\gamma(t_0))$ as $\underline{t} \nearrow t_0 = \bar{t}$ or $\bar{t} \searrow t_0 = \underline{t}$. Consequently, $\zeta_j \to U_{t_0}$ $(j \in \{1,2\})$. Finally, we find with Lemma 2.25 and 2.18 and $D_{\bar{t}} \xrightarrow{\mathrm{k}} D_{t_0}$ as $\bar{t} \searrow t_0$:

$$\dot{g}_{t_0}(w) = g_{t_0}(w) \cdot \Phi_{0,U_{t_0},D_{t_0}}\big(g_{t_0}(w)\big) \cdot \dot{c}(t_0) \qquad \text{for all } w \in \Omega_{t_0}$$

and $c(t) := \operatorname{lmr}(g_t)$. Note that the continuity of $t \mapsto U_t$ follows immediately from Lemma 2.29. □

2.3.2 Multiple slit Komatu–Loewner equation

Next, we are going to extend the previous theorems to multiple slits. Let Ω be an arbitrary circular slit disk and $T > 0$. For each $k \in \{1, \ldots, m\}$ with $m \in \mathbb{N}$, let $\gamma_k : [0,T] \to \operatorname{cl}(\Omega)\setminus\{0\}$ be simple and continuous with $\gamma_k(0,T] \subseteq \Omega$ and $\gamma_k(0) \in \mathbb{T}$. Moreover, assume $\gamma_j[0,T] \cap \gamma_k[0,T] = \emptyset$ whenever $k \neq j$. Then we call $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ a *tuple of disjoint radial (parametrised) slits in* Ω. Obviously, $(\mathfrak{H}_t)_{t\in[0,T]}$, with $\mathfrak{H}_t := \bigcup_{k=1}^m \gamma_k(0,t]$, is an increasing and continuous family of radial Ω-hulls.

We denote by $g_t : \Omega_t := \Omega \setminus \mathfrak{H}_t \to D_t$ the normalised radial mapping function on Ω_t for all $t \in [0,T]$. Using Corollary 2.26 and Lemma 2.24, the function $t \mapsto \operatorname{lmr}(g_t)$ is continuous and strictly increasing. Later we will see that in this case $t \mapsto g_t$ is not necessarily differentiable at a point t_0 if $t \mapsto \operatorname{lmr}(g_t)$ is differentiable at t_0, see Example 4.1.

In order to give a necessary condition we need some further abbreviation. Therefore, for each $t, \tau \in [0,T]$ and $k \in \{1, \ldots, m\}$, we set $\mathfrak{H}_k(t,\tau) := \bigcup_{j=1, j\neq k}^m \gamma_j(0,\tau] \cup \gamma_k(0,t]$. Since $\mathfrak{H}_k(t,\tau)$ is a radial Ω-hull, we may define

$$f_{k;t,\tau} : \Omega_k(t,\tau) := \Omega \setminus \mathfrak{H}_k(t,\tau) \to D_k(t,\tau),$$

as the normalised radial mapping function from $\Omega_k(t,\tau) := \Omega \setminus \mathfrak{H}_k(t,\tau)$ onto the circular slit disk $D_k(t,\tau)$, see Figure 2.3. Note that in this case $g_t \equiv f_{k;t,t}$, $\Omega_t = \Omega_k(t,t)$ and $D_t = D_k(t,t)$ (independent of k).

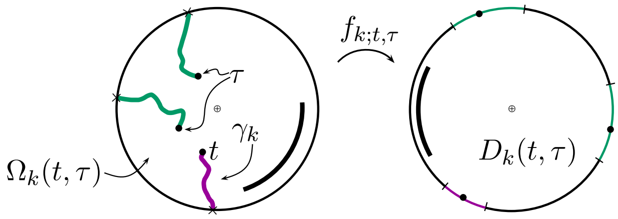

FIGURE 2.3: *Normalised radial mapping function* $f_{k;t,\tau} : \Omega_k(t,\tau) \to D_k(t,\tau)$

Theorem 2.30. *Let Ω be a circular slit disk, $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint radial slits in Ω with $m \in \mathbb{N}$, and let $t_0 \in [0,T]$. For each $t, \tau \in [0,T]$ and $k \in \{1,\dots,m\}$, $f_{k;t,\tau} : \Omega_k(t,\tau) \to D_k(t,\tau)$ and $g_t : \Omega_t \to D_t$ denote the normalised radial mapping functions on $\Omega_k(t,\tau) := \Omega \setminus (\gamma_k(0,t] \cup \bigcup_{j\neq k} \gamma_j(0,\tau])$ and $\Omega_t := \Omega_k(t,t)$, respectively. Then the following three statements are equivalent.*

(i) The limit $\lambda_k(t_0) := \lim_{t\to t_0} \frac{\mathrm{lmr}(f_{k;t,t_0}) - \mathrm{lmr}(f_{k;t_0,t_0})}{t-t_0}$ exists for each $k \in \{1,\dots,m\}$.

(ii) The function $t \mapsto g_t(z)$ is differentiable at t_0 for every $z \in \Omega_{t_0}$.

(iii) The function $t \mapsto g_t(z)$ is differentiable at t_0 for each $z \in \Omega_{t_0}$ and fulfils

$$\dot{g}_{t_0}(z) = g_{t_0}(z) \sum_{k=1}^{m} \lambda_k(t_0) \Phi_{0,U_k(t_0),D_{t_0}}\big(g_{t_0}(z)\big) \qquad \text{for all } z \in \Omega_{t_0},$$

where for all $k \in \{1,\dots,m\}$, $\lambda_k(t_0) \geq 0$ and the driving term $U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0,T]$.

When this happens, $t \mapsto \mathrm{lmr}(g_t)$ is differentiable at t_0 with derivative $\sum_{k=1}^m \lambda_k(t_0)$.

Remark 2.6. In case of one slit, i.e. $m = 1$, this theorem is more or less equivalent to Theorem 2.23. To be more precise, Theorem 2.30 generalises Theorem 2.23 in the case $m = 1$ slightly, as it shows that $t \mapsto g_t$ is differentiable at t_0 if and only if $t \mapsto \mathrm{lmr}(g_t)$ is differentiable at t_0. Obviously, Theorem 2.30 contains Theorem 2.22 as well. Consequently, we will discuss only the multiple slit version in the upcoming bilateral and chordal case.

The proof of Theorem 2.30 can be found in Section 2.6.

2.4 Bilateral case

Next, let us switch to the bilateral case where we take a circular slit annulus Ω as our initial domain. Let $Q \in (0,1)$ denote the inner radius of Ω. A subset $\mathfrak{H} \subseteq \Omega$ is called *(compact) bilateral hull in* Ω or *(compact) bilateral Ω-hull* if $\Omega \cap \mathrm{cl}(\mathfrak{H}) = \mathfrak{H}$, $\Omega \setminus \mathfrak{H}$ is a domain, $\mathbb{T} \cup \mathrm{cl}(\mathfrak{H})$ is connected and $\mathrm{dist}(\mathfrak{H}, \mathbb{T}_Q) > 0$. By $g_{\mathfrak{H}}$ we denote the unique conformal mapping that maps $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$ onto a circular slit annulus $D_{\mathfrak{H}}$ with inner radius $q_{\mathfrak{H}} \in (0,1)$ such that $g_{\mathfrak{H}}$ associates the outer boundary of $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$ with $\mathbb{T}$ and $g_{\mathfrak{H}}(Q) = q_{\mathfrak{H}}$, see Proposition 2.3. Then we call $g_{\mathfrak{H}}$ the *normalised bilateral mapping function on* $\Omega \setminus \mathfrak{H}$.

Next, let $(\mathfrak{H}_t)_{t\in[0,T]} \subseteq \Omega$ be a family of bilateral Ω-hulls, i.e. $\mathfrak{H}_t$ is a bilateral Ω-hull for each $t \in [0,T]$. Then we say $(\mathfrak{H}_t)_{t\in[0,T]}$ is an *increasing family of bilateral* Ω*-hulls* if $\mathfrak{H}_t \subsetneq \mathfrak{H}_s$ whenever $0 \le t < s \le T$ and $\mathfrak{H}_0 = \emptyset$. Moreover, $(\mathfrak{H}_t)_{t\in[0,T]}$ is called *continuous family of bilateral* Ω*-hulls* if $(\Omega_t - a)_{t\in[0,T]}$, with $\Omega_t := \Omega \setminus \mathfrak{H}_t$ and some $a \in \Omega_T$, is continuous on $[0,T]$ with respect to kernel convergence. This definition ensures that we consider the connected component of the weak kernel that has $\mathbb{T}_Q$ as a boundary component.

Let $A, B \subseteq \mathbb{D}$ be domains where the inner boundary component is a circle (with radii q_A and q_B) and f is a conformal mapping from A onto B that associates $\mathbb{T}_{q_A}$ with $\mathbb{T}_{q_B}$. Then we set $\mathrm{lcm}(f) := \ln q_B - \ln q_A$, what we call the *logarithmic conformal modulus*. Let Ω be a circular slit annulus and $\mathfrak{H}$ be a bilateral Ω-hull. Then we use the abbreviation $\mathrm{lcm}_\Omega(\mathfrak{H}) := \mathrm{lcm}(g_{\mathfrak{H}})$ as well where $g_{\mathfrak{H}}$ denotes the normalised bilateral mapping function on $\Omega \setminus \mathfrak{H}$.

As mentioned in Subsection 2.3.2, see Remark 2.6, Theorem 2.23 follows from Theorem 2.30, so we will skip the single slit case in the bilateral setting. Consequently, will go directly to the multiple slit case. Let Ω be an arbitrary circular slit annulus and for each $k \in \{1, \ldots, m\}$ with $m \in \mathbb{N}$, $\gamma_k : [0,T] \to \mathbb{C}$ is continuous and simple with $\gamma_k(0,T] \subseteq \Omega$ and $\gamma_k(0) \in \mathbb{T}$. Moreover, assume $\gamma_j[0,T] \cap \gamma_k[0,T] = \emptyset$ whenever $k \neq j$. Then we call $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ a *tuple of disjoint bilateral (parametrised) slits in* Ω. Obviously, $(\mathfrak{H}_t)_{t\in[0,T]}$, with $\mathfrak{H}_t := \bigcup_{k=1}^m \gamma_k(0,t]$, is a family of increasing and continuous bilateral hulls in Ω.

Next, let $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint bilateral slits in a circular slit annulus Ω. Then we set $\mathfrak{H}_k(t,\tau) := \bigcup_{j=1, j\neq k}^m \gamma_j(0,\tau] \cup \gamma_k(0,t]$ with $t, \tau \in [0,T]$ and $k \in \{1, \ldots, m\}$. Since $\mathfrak{H}_k(t,\tau)$ is a bilateral Ω-hull as well, we may define

$$f_{k;t,\tau} : \Omega_k(t,\tau) := \Omega \setminus \mathfrak{H}_k(t,\tau) \to D_k(t,\tau)$$

as the normalised bilateral mapping function from $\Omega_k(t,\tau) := \Omega \setminus \mathfrak{H}_k(t,\tau)$ onto the circular slit annulus $D_k(t,\tau)$ with $t, \tau \in [0,T]$ and $k \in \{1, \ldots, m\}$. Herein, the inner radius of $D_k(t,\tau)$ is denoted by $q_k(t,\tau)$. Moreover, we set $g_t := f_{k;t,t}$, $\Omega_t := \Omega_k(t,t)$, $D_t := D_k(t,t)$ and $q_t := q_k(t,t)$ (independent of k) for each $t \in [0,T]$.

Then we have the following theorem

Theorem 2.31. *Let Ω be a circular slit annulus, $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint bilateral slits in Ω with $m \in \mathbb{N}$, and let $t_0 \in [0,T]$. For each $t, \tau \in [0,T]$ and $k \in \{1, \ldots, m\}$, $f_{k;t,\tau} : \Omega_k(t,\tau) \to D_k(t,\tau)$ and $g_t : \Omega_t \to D_t$ denote the normalised bilateral mapping functions on $\Omega_k(t,\tau) := \Omega \setminus (\gamma_k(0,t] \cup \bigcup_{j\neq k} \gamma_j(0,\tau])$ and $\Omega_t := \Omega_k(t,t)$, respectively. Then the following three statements are equivalent.*

(i) The limit $\lambda_k(t_0) := \lim_{t\to t_0} \frac{\mathrm{lcm}(f_{k;t,t_0}) - \mathrm{lcm}(f_{k;t_0,t_0})}{t - t_0}$ exists for each $k \in \{1, \ldots, m\}$.

(ii) The function $t \mapsto g_t(z)$ is differentiable at t_0 for each $z \in \Omega_{t_0}$.

(iii) The function $t \mapsto g_t(z)$ is differentiable at t_0 for each $z \in \Omega_{t_0}$ and fulfils

$$\dot{g}_{t_0}(z) = g_{t_0}(z) \sum_{k=1}^{m} \lambda_k(t_0) \Phi_{q_{t_0}, U_k(t_0), D_{t_0}}\big(g_{t_0}(z)\big) \quad \text{for all } z \in \Omega_{t_0}, \tag{2.5}$$

where for all $k \in \{1, \dots, m\}$, $\lambda_k(t_0) \geq 0$ and the driving term $U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0, T]$. Here q_t denotes the inner radius of D_t, $t \in [0, T]$.

When this happens, $t \mapsto \operatorname{lcm}(g_t)$ is differentiable at t_0 with derivative $\sum_{k=1}^{m} \lambda_k(t_0)$.

As in the previous section, $w \mapsto \Phi_{q_{t_0}, U_k(t_0), D_{t_0}}(w)$ denotes the unique mapping from Proposition 2.5, see also Figure 2.4.

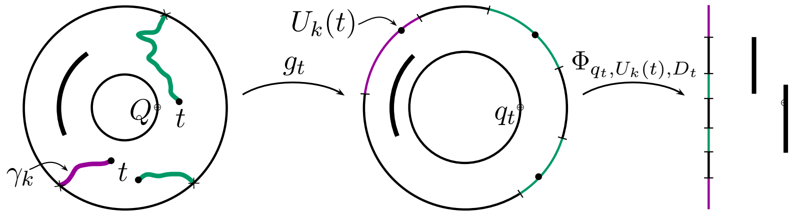

FIGURE 2.4: *Mapping behaviour of $z \mapsto g_t(z)$ and $w \mapsto \Phi_{q_t, U_k(t), D_t}(w)$ in the bilateral multiple slit case*

The proof of Theorem 2.31 can be found in Section 2.6. In order to do so, we need some preliminary lemmas.

Lemma 2.32. *Let Ω be a circular slit annulus and $\mathfrak{A}, \mathfrak{B} \subseteq \Omega$ be bilateral Ω-hulls with $\mathfrak{A} \subsetneq \mathfrak{B}$. Then $\operatorname{lcm}(g_{\mathfrak{A}}) < \operatorname{lcm}(g_{\mathfrak{B}})$ where $g_{\mathfrak{A}}$ and $g_{\mathfrak{B}}$ denote the normalised bilateral mapping function on $\Omega \setminus \mathfrak{A}$ and $\Omega \setminus \mathfrak{B}$, respectively.*

Proof. First of all, we note that the functions $g_{\mathfrak{A}}$ and $g_{\mathfrak{B}}$ are related to an extremal property, see Lemma 2.7. Moreover, we denote by Q the inner radius of Ω

Note that $\mathfrak{C} := g_{\mathfrak{A}}(\mathfrak{B} \setminus \mathfrak{A})$ is a compact bilateral hull in $D_{\mathfrak{A}} = g_{\mathfrak{A}}(\Omega \setminus \mathfrak{A})$. Herein, $q_{\mathfrak{A}}$ is the inner radius of the circular slit annulus $D_{\mathfrak{A}}$. Next, we find a unique conformal mapping $h : \mathbb{A}_{q_{\mathfrak{A}}} \setminus \mathfrak{C} \to \mathbb{A}_{q^*}$ having $h(q_{\mathfrak{A}}) = q^* > 0$, see also Proposition 2.3. Then $q_{\mathfrak{A}} < q^*$ by Theorem 3, Chapter V.1. of [Gol69].

Using the notation from Lemma 2.7 with $\Omega \setminus \mathfrak{B}$ as the initial domain, E as the outer boundary component of $\Omega \setminus \mathfrak{B}$ and $F := \mathbb{T}_Q$ we find $h \circ g_{\mathfrak{A}}, g_{\mathfrak{B}} \in \mathcal{F}$. Consequently, we find $q^* \leq q_{\mathfrak{B}}$ where $q_{\mathfrak{B}}$ is the inner radius of $D_{\mathfrak{B}} = g_{\mathfrak{B}}(\Omega \setminus \mathfrak{B})$. Summarising, we get $q_{\mathfrak{A}} < q_{\mathfrak{B}}$. □

Lemma 2.33. *Let Ω be a circular slit annulus, $(\mathfrak{H}_t)_{t \in [0,T]}$ be an increasing family of bilateral Ω-hulls and g_t denotes the normalised bilateral mapping function on $\Omega \setminus \mathfrak{H}_t$ for each $t \in [0, T]$. Let $(t_n)_{n \in \mathbb{N}} \subseteq [0, T]$ with $t_n \to t_0$, assume $\operatorname{con}(\Omega_{t_n}) = \operatorname{con}(\Omega_{t_0})$ for all $n \in \mathbb{N}$ and $\Omega_{t_n} - a \xrightarrow{k} \Omega_{t_0} - a$ for some $a \in \Omega_T$.*

Then $g_{t_n} \xrightarrow{l.u.} g_{t_0}$ on Ω_{t_0} as $n \to \infty$. Moreover, $\operatorname{lcm}(g_{t_n}) \to \operatorname{lcm}(g_{t_0})$ as well.

Proof. Let Q denote the inner radius of Ω. By definition $\Omega_{t_n}-a \xrightarrow{\text{k}} \Omega_{t_0}-a$ if $t_n \to t_0$ for some $a \in \Omega_T$.

By Montel's theorem $h_n := g_{t_n}$ is normal, so we find a locally uniform convergent subsequence $(h_{n_k})_{k\in\mathbb{N}}$ on Ω_{t_0}. The limit function $h : \Omega_{t_0} \to \mathbb{C}$ is either univalent or constant.

Note that $\operatorname{dist}(\mathfrak{H}_t, \mathbb{T}_Q) \geq \operatorname{dist}(\mathfrak{H}_T, \mathbb{T}_Q) > 0$ for all $t \in [0,T]$. Using the Schwarz reflection principle, we can extend each h_n analytically to $\mathbb{A}_{Q-\delta,Q}$ with some $\delta > 0$ small. Moreover, $g_{t_n}(\mathbb{T}_Q) = \mathbb{T}_{q_{t_n}}$ where $q_{t_n} \in [Q,1)$ by Lemma 2.32. Herein, q_{t_n} denotes the inner radius of $D_{t_n} = g_{t_n}(\Omega \setminus \mathfrak{H}_{t_n})$. Consequently, h_{n_k} converges uniformly on $\mathbb{T}_Q$ to $\mathbb{T}_{q^*}$ with $q^* \geq Q$, so the limit function h can not be constant. Since $h : \Omega_{t_0} \to D$ is a conformal mapping, Corollary 2.12 yields $D_{t_{n_k}}-a \xrightarrow{\text{k}} D-a$ for some $a \in D$. Note that $\operatorname{con}(D) = \operatorname{con}(\Omega_{t_0}) = \operatorname{con}\Omega_{t_n} = \operatorname{con}(D_{t_n})$ for all $n \in \mathbb{N}$, so D is a circular slit annulus by Lemma 2.14. Since h_{n_k} converges uniformly on $\mathbb{T}_Q$ to $\mathbb{T}_{q^*}$, h associates $\mathbb{T}_Q$ with the inner boundary component $\mathbb{T}_{q^*}$ of D and $h(Q) = q^* > 0$.

On top of this each (interior) proper concentric circular arc of Ω_{t_0} is mapped by h to an (interior) proper concentric circular arc of D. This can be seen by using the argument principle, see for instance the proof of Lemma 2.25. Hence, h associates the outer boundary of Ω_{t_0} with $\mathbb{T}$. Summarising, $h \equiv g_{t_0}$. As all convergent subsequences $(h_{n_k})_{k\in\mathbb{N}}$ converge to the same function g_{t_0}, also the whole sequence $(g_{t_n})_{n\in\mathbb{N}}$ converges locally uniformly to g_{t_0} on Ω_{t_0}. Obviously, $q_n \to q^* = \exp(\operatorname{lcm}(g_{t_0}))$ as well. □

Lemma 2.34. *Let Ω be a circular slit annulus and $\mathfrak{H}$ be a bilateral Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then*

$$\operatorname{lcm}(g_{\mathfrak{H}}) = -\frac{1}{2\pi}\int\limits_{\mathbb{T}} \ln\left|g_{\mathfrak{H}}^{-1}(\zeta)\right||\mathrm{d}\zeta|,$$

where $g_{\mathfrak{H}}$ is the normalised bilateral mapping function from $\Omega_{\mathfrak{H}}$ onto the circular slit annulus $D_{\mathfrak{H}}$.

Proof. Herein, we denote by $q \in (0,1)$ the inner radius of $D_{\mathfrak{H}}$. In particular we have $q \in (Q,1)$ by Lemma 2.32. Cauchy's theorem yields

$$0 = \frac{1}{2\pi\mathrm{i}}\int\limits_{\partial D_{\mathfrak{H}}} \log\left(\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right)\frac{\mathrm{d}\zeta}{\zeta} = \frac{1}{2\pi}\int\limits_{\partial D_{\mathfrak{H}}} \ln\left|\frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta.$$

Like in the radial case, the logarithm is well-defined. The last equality is an immediate consequence of the fact that each connected component of $\partial D_{\mathfrak{H}}$ is a concentric circular arc centred at 0. Note that $\log|g_{\mathfrak{H}}^{-1}(\zeta)/\zeta|$ is constant on each connected component of $\partial D_{\mathfrak{H}}$, so we find

$$0 = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \ln\left|\frac{h_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta - \frac{1}{2\pi}\int\limits_{\mathbb{T}_q} \ln\left|\frac{h_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta.$$

Finally, we get $-\frac{1}{2\pi}\int_{\mathbb{T}_q} \ln\left|\frac{h_{\mathfrak{H}}^{-1}(\zeta)}{\zeta}\right|\mathrm{d}\arg\zeta = -\ln\frac{Q}{q} = \operatorname{lcm}(g_{\mathfrak{H}})$. □

Lemma 2.35. *Let Ω be a circular slit annulus and $\mathfrak{H}$ be a bilateral Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then we have*

$$\log \frac{g_{\mathfrak{H}}^{-1}(z)}{z} = \frac{1}{2\pi} \int_{\mathbb{T}} \ln |g_{\mathfrak{H}}^{-1}(\zeta)|\, \Phi_{q_{\mathfrak{H}},\zeta,D_{\mathfrak{H}}}(z)\, |\mathrm{d}\zeta| \qquad \textit{for all } z \in D_{\mathfrak{H}},$$

where $g_{\mathfrak{H}}$ denotes the normalised bilateral mapping function from $\Omega_{\mathfrak{H}}$ onto $D_{\mathfrak{H}}$. Herein, $q_{\mathfrak{H}}$ denotes the inner radius of $D_{\mathfrak{H}}$.

Proof. Let us consider the function

$$F(z) := \log \frac{g_{\mathfrak{H}}^{-1}(z)}{z}, \quad z \in D_{\mathfrak{H}},$$

which is analytic on $D_{\mathfrak{H}}$. We denote by $C_1, \ldots, C_{\mathfrak{n}} = \mathbb{T}$ the boundary components of $D_{\mathfrak{H}}$. Note that F can be extended continuously to $\partial D_{\mathfrak{H}}$ and $\Re(F)$ is constant on each C_k, $k \in \{1, \ldots, \mathfrak{n}-1\}$. Hence we find with Proposition 2.21 and $a = q_{\mathfrak{H}}$

$$\log \frac{g_{\mathfrak{H}}^{-1}(z)}{z} = \frac{1}{2\pi} \int_{\mathbb{T}} \ln \left| \frac{g_{\mathfrak{H}}^{-1}(\zeta)}{\zeta} \right| \Phi_{q_{\mathfrak{H}},\zeta,D_{\mathfrak{H}}}(z)\, |\mathrm{d}\zeta| + \mathrm{i}c$$

where $c \in \mathbb{R}$. Finally, let us apply $z = q_{\mathfrak{H}}$ to get $c = 0$, as $\Phi_{q_{\mathfrak{H}},\zeta,D_{\mathfrak{H}}}(q_{\mathfrak{H}}) \geq 0$ and $\log(g_{\mathfrak{H}}^{-1}(z)/z)|_{z=q_{\mathfrak{H}}} = -\operatorname{lcm}(g_{\mathfrak{H}}) < 0$. □

2.5 Chordal case

Finally, we are going to discuss the chordal case. In this context we take an upper parallel slit half-plane Ω as our initial domain. A bounded subset $\mathfrak{H} \subseteq \Omega$ is called *(compact) chordal hull in* Ω or *(compact) chordal Ω-hull* if $\Omega \cap \operatorname{cl}(\mathfrak{H}) = \mathfrak{H}$, $\Omega \setminus \mathfrak{H}$ is a domain and $\mathbb{R} \cup \operatorname{cl}(\mathfrak{H})$ is connected. By $g_{\mathfrak{H}}$ we denote the unique conformal mapping that maps $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$ onto an upper parallel slit half-plane $D_{\mathfrak{H}}$ such that

$$g_{\mathfrak{H}}(z) = z + \frac{a_{g_{\mathfrak{H}}}}{z} + \mathcal{O}(|z|^{-2}), \qquad \text{around } \infty.$$

We call this function *normalised chordal mapping function on* $\Omega \setminus \mathfrak{H}$. Herein, the value $\operatorname{hcap}(g_{\mathfrak{H}}) := a_{\mathfrak{H}} := a_{g_{\mathfrak{H}}}$ is called *half-plane capacity of* $g_{\mathfrak{H}}$. Sometimes we write $\operatorname{hcap}_{\Omega}(\mathfrak{H}) := \operatorname{hcap}(g_{\mathfrak{H}})$ as well if $g_{\mathfrak{H}}$ is the normalised chordal mapping function on $\Omega \setminus \mathfrak{H}$. Moreover, let g be a function that is analytic on $B_{\varepsilon}(\infty)$, with some $\varepsilon > 0$, having an expansion $g(z) = z + \frac{a_g}{z} + \mathcal{O}(|z|^{-2})$ around ∞. Then we call $\operatorname{hcap}(g) := |a_g|$ the half-plane capacity of g as well. On top of this, $a_g \geq 0$ if there are constants $\delta_1, \delta_2 > 0$ such that $B_{\delta_1}(\infty) \cap \mathbb{H} \subseteq g(B_{\varepsilon}(\infty) \cap \mathbb{H}) \subseteq B_{\delta_2}(\infty) \cap \mathbb{H}$.

Next, let $(\mathfrak{H}_t)_{t\in[0,T]} \subseteq \Omega$ be a family of chordal Ω-hulls, i.e. $\mathfrak{H}_t$ is a chordal Ω-hull for each $t \in [0,T]$. Then $(\mathfrak{H}_t)_{t\in[0,T]}$ is called *increasing family of chordal Ω-hulls* if $\mathfrak{H}_t \subsetneq \mathfrak{H}_s$ whenever $0 \leq t < s \leq T$ and $\mathfrak{H}_0 = \emptyset$. Moreover, $(\mathfrak{H}_t)_{t\in[0,T]} \subseteq \Omega$ is called *continuous family of chordal Ω-hulls* if $(\Omega_t - a)_{t\in[0,T]}$, with $\Omega_t := \Omega \setminus \mathfrak{H}_t$ and some $a \in \Omega_T$, is continuous with respect to kernel convergence.

As in the bilateral setting, we will go directly to the multiple slit case. Let Ω be an upper parallel slit half-plane and for each $k \in \{1,\dots,m\}$ with $m \in \mathbb{N}$, $\gamma_k : [0,T] \to \mathbb{C}$ is continuous and simple with $\gamma_k(0,T] \subseteq \Omega$ and $\gamma_k(0) \in \mathbb{R}$. Moreover, $\gamma_j[0,T] \cap \gamma_k[0,T] = \emptyset$ whenever $k \neq j$. Then we call $(\gamma_1,\dots,\gamma_m)_{t\in[0,T]}$ a *tuple of disjoint chordal (parametrised) slits in* Ω. Obviously, $(\mathfrak{H}_t)_{t\in[0,T]}$, with $\mathfrak{H}_t := \bigcup_{k=1}^m \gamma_k(0,t]$ is a family of increasing continuous chordal Ω-hulls.

Let $(\gamma_1,\dots,\gamma_m)_{t\in[0,T]}$ be a tuple of disjoint chordal slits in an upper parallel slit half-plane Ω. For each $t,\tau \in [0,T]$ and $k \in \{1,\dots,m\}$, we set $\mathfrak{H}_k(t,\tau) := \bigcup_{j=1, j\neq k}^m \gamma_j(0,\tau] \cup \gamma_k(0,t]$. Since $\mathfrak{H}_k(t,\tau)$ is a chordal Ω-hull as well, we may define

$$f_{k;t,\tau} : \Omega_k(t,\tau) := \Omega \setminus \mathfrak{H}_k(t,\tau) \to D_k(t,\tau),$$

as the normalised chordal mapping function from $\Omega \setminus \mathfrak{H}_k(t,\tau)$ onto the upper parallel slit half-plane $D_k(t,\tau)$ with $t,\tau \in [0,T]$ and $k \in \{1,\dots,m\}$. Moreover, for each $t \in [0,T]$, we set independently of $k \in \{1,\dots,m\}$, $g_t := f_{k;t,t}$, $\Omega_t := \Omega_k(t,t)$ and $D_t := D_k(t,t)$.

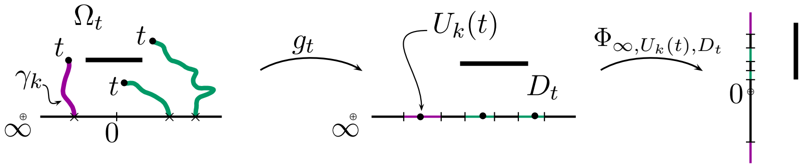

FIGURE 2.5: *Mapping behaviour of* $z \mapsto g_t(z)$ *and* $w \mapsto \Phi_{\infty,U_k(t),D_t}(w)$ *in the chordal multiple slit case*

Then we have the following theorem

Theorem 2.36. *Let* Ω *be an upper parallel slit half-plane,* $(\gamma_1,\dots,\gamma_m)_{t\in[0,T]}$ *be a tuple of disjoint chordal slits in* Ω *with* $m \in \mathbb{N}$*, and let* $t_0 \in [0,T]$*. For each* $t,\tau \in [0,T]$ *and* $k \in \{1,\dots,m\}$*,* $f_{k;t,\tau} : \Omega_k(t,\tau) \to D_k(t,\tau)$ *and* $g_t : \Omega_t \to D_t$ *denote the normalised chordal mapping functions on* $\Omega_k(t,\tau) := \Omega \setminus (\gamma_k(0,t] \cup \bigcup_{j\neq k} \gamma_j(0,\tau])$ *and* $\Omega_t := \Omega_k(t,t)$*, respectively. Then the following three statements are equivalent.*

(i) The limit $\lambda_k(t_0) := \lim_{t\to t_0} \frac{\operatorname{hcap}(f_{k;t,t_0}) - \operatorname{hcap}(f_{k;t_0,t_0})}{t-t_0}$ *exists for each* $k \in \{1,\dots,m\}$*.*

(ii) The function $t \mapsto g_t(z)$ *is differentiable at* t_0 *for each* $z \in \Omega_{t_0}$*.*

(iii) The function $t \mapsto g_t(z)$ *is differentiable at* t_0 *for each* $z \in \Omega_{t_0}$ *and fulfils*

$$\dot{g}_{t_0}(z) = -\frac{\mathrm{i}}{2} \sum_{k=1}^m \lambda_k(t_0) \Phi_{\infty,U_k(t_0),D_{t_0}}\big(g_{t_0}(z)\big) \qquad \text{for all } z \in \Omega_{t_0},$$

where for each $k \in \{1,\dots,m\}$*,* $\lambda_k(t_0) \geq 0$ *and the driving term* $U_k(t) := g_t(\gamma_k(t))$ *is continuous on* $[0,T]$*.*

When this happens, $t \mapsto \operatorname{hcap}(g_t)$ *is differentiable at* t_0 *with derivative* $\sum_{k=1}^m \lambda_k(t_0)$*.*

As before, $w \mapsto \Phi_{\infty,U_k(t_0),D_{t_0}}(w)$ denotes the unique mapping function from Proposition 2.5, see also Figure 2.5.

Remark 2.7. Let $\Omega = \mathbb{H}$ and $(\gamma)_{t\in[0,T]}$ be a chordal slit in $\mathbb{H}$ parametrised in such a way that $\operatorname{hcap}(g_t) = 2t$ for all $t \in [0,T]$ where, for each $t \in [0,T]$, g_t denotes the normalised chordal mapping function on $\Omega \setminus \gamma(0,t]$. Then $\lambda(t_0) := \lim_{t\to t_0} \frac{\operatorname{hcap}(g_t)-\operatorname{hcap}(g_{t_0})}{t-t_0} = 2$ for all $t_0 \in [0,T]$, so Theorem 2.36 and Remark 2.2 give us Theorem D.

The proof of Theorem 2.36 can be found in Section 2.6. In order to do so, we need some preliminary lemmas.

Lemma 2.37. *Let Ω be an upper parallel slit half-plane and $\mathfrak{A}, \mathfrak{B} \subseteq \Omega$ be chordal Ω-hulls with $\mathfrak{A} \subsetneq \mathfrak{B}$. Then $\operatorname{hcap}(g_{\mathfrak{A}}) < \operatorname{hcap}(g_{\mathfrak{B}})$ where $g_{\mathfrak{A}}$ and $g_{\mathfrak{B}}$ denote the normalised chordal mapping function on $\Omega \setminus \mathfrak{A}$ and $\Omega \setminus \mathfrak{B}$, respectively.*

Proof. First of all, we note that the functions $g_{\mathfrak{A}}$ and $g_{\mathfrak{B}}$ are related to an extremal property, see Lemma 2.8.

$\mathfrak{C} := g_{\mathfrak{A}}(\mathfrak{B} \setminus \mathfrak{A})$ is a compact chordal hull in $D_{\mathfrak{A}} = g_{\mathfrak{A}}(\Omega \setminus \mathfrak{A})$. Using Riemann's mapping theorem (for simply connected domains), we find a unique conformal mapping $h : \mathbb{H} \setminus \mathfrak{C} \to \mathbb{H}$ having $h(z) - z \to 0$ as $z \to \infty$. Hence $h(z) = z + \frac{a_h}{z} + \mathcal{O}(|z|^2)$ around ∞ with $a_h > 0$, see [Law05], Section 3.4. Consequently, $(h \circ g_{\mathfrak{A}})(z) = z + \frac{a_h + a_{\mathfrak{A}}}{z} + \mathcal{O}(|z|^2)$ around ∞ where $a_{\mathfrak{A}} = \operatorname{hcap}(g_{\mathfrak{A}})$.

Next, let us use the notation from Lemma 2.8 with $\Omega \setminus \mathfrak{B}$ as the initial domain. Then $g_{\mathfrak{B}}, h \circ g_{\mathfrak{A}} \in \mathcal{F}$ and we find $\operatorname{hcap}(g_{\mathfrak{A}}) + a_h \leq \operatorname{hcap}(g_{\mathfrak{B}})$, so $\operatorname{hcap}(g_{\mathfrak{A}}) < \operatorname{hcap}(g_{\mathfrak{B}})$. □

Lemma 2.38. *Let Ω be an upper parallel slit half-plane, $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of chordal Ω-hulls and for each $t \in [0,T]$, g_t denotes the normalised chordal mapping function on $\Omega \setminus \mathfrak{H}_t$. Let $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ with $t_n \to t_0$, assume $\operatorname{con}(\Omega_{t_n}) = \operatorname{con}(\Omega_{t_0})$ for all $n \in \mathbb{N}$ and $\Omega_{t_n} - a \xrightarrow{k} \Omega_{t_0} - a$ for some $a \in \Omega_T$.*

Then $g_{t_n} \xrightarrow{l.u.} g_{t_0}$ on Ω_{t_0} as $n \to \infty$. Moreover, $\operatorname{hcap}(g_{t_n}) \to \operatorname{hcap}(g_{t_0})$ as well.

Proof. First of all, by definition $\Omega_{t_n} - a \xrightarrow{\text{k}} \Omega_{t_0} - a$ if $t_n \to t_0$ for some $a \in \Omega_T$. Next, we set $g_n := g_{t_n}$ and note that each g_n has an analytic continuation to $B_\varepsilon(\infty)$ with $\varepsilon > 0$ small. Thus we find with $h_n(z) := 1/(g_n(1/z) + \mathrm{i})$:

$$h_n(z) = z - iz^2 - (\operatorname{hcap}(g_n) + 1)z^3 + \mathcal{O}(|z|^4) \text{ around } z = 0.$$

Using Koebe's distortion theorem, $(h_n)_{n\in\mathbb{N}}$ is a bounded sequence on $1/\Omega_{t_0} \cup U$ where $1/\Omega_{t_0} := \{z \in \mathbb{C} \mid 1/z \in \Omega_{t_0}\}$ and U is a small neighbourhood of 0. Consequently, we find a locally uniform convergent subsequence $(h_{n_k})_{k\in\mathbb{N}}$ where $h_{n_k} \xrightarrow{\text{l.u.}} h$ on $1/\Omega_{t_0} \cup U$. h can not be constant as we have $h(z) = z - \mathrm{i}z^2 + \mathcal{O}(|z^3|)$ around 0. Thus $g_{n_k}(z) = 1/h_{n_k}(1/z) - \mathrm{i}$ converges locally uniformly on Ω_{t_0} to the univalent function $g(z) = 1/h(1/z) - \mathrm{i}$. An easy calculation yields $g(z) - z \to 0$ as $z \to \infty$.

Next, we set $D_n := g_n(\Omega \setminus \mathfrak{H}_{t_n})$, i.e. D_n is an upper parallel slit half-plane. Since $g : \Omega_{t_0} \to D := g(\Omega_{t_0})$ is conformal we find, by Corollary 2.12, $D_{n_k} - a \xrightarrow{\text{k}} D - a$ for some $a \in D$. Note that $\operatorname{con}(D) = \operatorname{con}(\Omega) = \operatorname{con}(\Omega_{t_n}) = \operatorname{con}(D_{t_n})$, so D is an upper parallel slit half-plane, by Lemma 2.15. Together with the previous calculation, we find $g \equiv g_{t_0}$.

As all convergent subsequences $(g_{n_k})_{k\in\mathbb{N}}$ converge to the same function g_{t_0}, also the whole sequence $(g_n)_{n\in\mathbb{N}}$ converges locally uniformly to g_{t_0} on Ω_{t_0}. In this case $\operatorname{hcap}(g_n) \to \operatorname{hcap}(g_{t_0})$ as well, as $g_{t_0}(z) = 1/h(1/z) - \mathrm{i}$. □

Lemma 2.39. *Let Ω be an upper parallel slit half-plane and $\mathfrak{H}$ be a chordal Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then we have*

$$\operatorname{hcap}(g_{\mathfrak{H}}) = \frac{1}{\pi} \int\limits_{\partial\mathbb{H}} \Im\big(g_{\mathfrak{H}}^{-1}(\zeta)\big)|\mathrm{d}\zeta|,$$

where $g_{\mathfrak{H}}$ denotes the normalised chordal mapping function on $\Omega_{\mathfrak{H}}$.

Note that there is an $R_0 > 0$ such that $\Im(g_{\mathfrak{H}}^{-1}(\zeta)) = 0$ for all $\zeta \in \partial\mathbb{H}$ with $|\zeta| > R_0$, so the previous integral is well-defined.

Proof. First of all, we note that $g_{\mathfrak{H}} : \Omega \setminus \mathfrak{H} \to D_{\mathfrak{H}}$ can be reflected along the real line to a function $g_* : \Omega^* \setminus \mathfrak{H}^* \to D^*$. Herein, Ω^*, D^* and $\mathfrak{H}^*$ come out of reflecting Ω, $D_{\mathfrak{H}}$ and $\mathfrak{H}$ on the real line, respectively . Thus $\operatorname{con}(\Omega^*) = 2\mathfrak{n} - 1$ where $\mathfrak{n} = \operatorname{con}(\Omega)$. Consequently, ∞ is an inner point of $\Omega^* \setminus \mathfrak{H}^*$ and D^*.

Together with Cauchy's formula we find

$$z \cdot \big(g_*^{-1}(z) - z\big) = z\frac{1}{2\pi\mathrm{i}} \int\limits_{\partial D^*} \frac{g_*^{-1}(\zeta) - \zeta}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \int\limits_{\partial D^*} \frac{g_*^{-1}(\zeta) - \zeta}{\frac{\zeta}{z} - 1}\mathrm{d}\zeta$$

for each $z \in D^*$. Next, we apply $z = \infty$. Alternatively we could substitute $z = \frac{1}{w}$ and apply $w = 0$. In either way we find

$$\operatorname{hcap}(g_*^{-1}) = -\frac{1}{2\pi\mathrm{i}} \int\limits_{\partial D^*} g_*^{-1}(\zeta) - \zeta \,\mathrm{d}\zeta = \frac{1}{2\pi} \int\limits_{\partial D^*} \mathrm{i}(g_*^{-1}(\zeta) - \zeta)\,\mathrm{d}\zeta.$$

Moreover, we denote the connected components of ∂D^* by $C_0, \dots, C_{2\mathfrak{n}-1}$ where $\Im(C_0) = 0$. Since $\operatorname{hcap}(g_*^{-1}) > 0$ we find

$$\operatorname{hcap}(g_*^{-1}) = -\frac{1}{2\pi} \int\limits_{\partial D^*} \Im(g_*^{-1}(\zeta) - \zeta)\mathrm{d}\zeta = -\frac{1}{2\pi} \sum_{k=0}^{2\mathfrak{n}-1} \int\limits_{C_k} \Im(g_*^{-1}(\zeta) - \zeta)\mathrm{d}\zeta.$$

Moreover, for each $k \neq 0$, $\zeta \mapsto \Im(g_*^{-1}(\zeta) - \zeta)$ is constant on C_k, so the integrals over C_k vanish, as we have to consider both sides of the line segments C_k. Consequently, we find by symmetry

$$\operatorname{hcap}(g_{\mathfrak{H}}) = -\operatorname{hcap}(g_*^{-1}) = \frac{1}{2\pi} \int\limits_{C_0} \Im(g_{\mathfrak{H}}^{-1}(\zeta) - \zeta)|\mathrm{d}\zeta| = \frac{1}{\pi} \int\limits_{\partial\mathbb{H}} \Im(g_{\mathfrak{H}}^{-1}(\zeta))|\mathrm{d}\zeta|.$$

□

Lemma 2.40. *Let Ω be an upper parallel slit half-plane and $\mathfrak{H}$ be a chordal Ω-hull such that $\partial\Omega_{\mathfrak{H}}$ is locally connected with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$. Then we have*

$$g_{\mathfrak{H}}^{-1}(z) - z = \frac{\mathrm{i}}{2\pi} \int\limits_{\partial\mathbb{H}} \Im\big(g_{\mathfrak{H}}^{-1}(\zeta)\big)\, \Phi_{\infty,\zeta,D_{\mathfrak{H}}}(z)\, |\mathrm{d}\zeta| \qquad \text{for all } z \in D_{\mathfrak{H}},$$

where $g_{\mathfrak{H}}$ denotes the normalised chordal mapping function from $\Omega_{\mathfrak{H}}$ onto $D_{\mathfrak{H}}$.

Like in the previous lemma, there is an $R_0 > 0$ such that $\Im(g_{\mathfrak{H}}^{-1}(\zeta)) = 0$ for all $\zeta \in \partial\mathbb{H}$ with $|\zeta| > R_0$, so the previous integral is well-defined.

Proof. Let $T(z) := \frac{1}{z} - \mathrm{i}$, $F(z) := -\mathrm{i}\big(g_{\mathfrak{H}}^{-1}(T(z)) - T(z)\big)$ and $R(w) := T^{-1}(w) = \frac{1}{w+\mathrm{i}}$ mapping $D_{\mathfrak{H}}$ onto $D' \subseteq B_{1/2}(-\mathrm{i}/2)$. Then Proposition 2.21 gives us with some $a \in D'$:

$$F(z) = \frac{1}{2\pi} \int\limits_{\partial B_{1/2}(-\mathrm{i}/2)} \Re\big(F(\zeta)\big) \cdot \Phi_{a,\zeta,D'}(z)|\mathrm{d}\zeta| + \mathrm{i}c \qquad \text{for all } z \in D'.$$

Note that we find a connected subset $s \subseteq \partial B_{1/2}(-\mathrm{i}/2)$ such that

$$\Re\big(F(\zeta)\big) = \Im\big(g_{\mathfrak{H}}^{-1}(T(\zeta)) - T(\zeta)\big) = \Im\big(g_{\mathfrak{H}}^{-1}(T(\zeta))\big) = 0 \quad \text{for all } \zeta \in B_{1/2}(-\mathrm{i}/2) \setminus s.$$

As T maps $\partial B_{1/2}(-\mathrm{i}/2) \setminus \{0\}$ onto $\mathbb{R}$ and 0 to ∞, we choose s in such a way that $\mathrm{dist}(s, 0) > 0$. Consequently, we find

$$F(z) = \frac{1}{2\pi} \int\limits_{s} \Im\big(g_{\mathfrak{H}}^{-1}(T(\zeta))\big) \cdot \Phi_{0,\zeta,D'}(z)|\mathrm{d}\zeta| + \mathrm{i}d \qquad \text{for all } z \in D'$$

where $d \in \mathbb{R}$. Note that $\Phi_{0,\zeta,D'}$ and $\Phi_{a,\zeta,D'}$ differ only in an imaginary constant and $0 \notin s$. It is easy to see that $\Phi_{0,\zeta,D'}(z) = |T'(\zeta)| \cdot \Phi_{\infty,T(\zeta),D_{\mathfrak{H}}}(T(z))$. Hence an easy substitution yields for all $z \in D'$

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int\limits_{s} \Im\big(g_{\mathfrak{H}}^{-1}(T(\zeta))\big) \cdot \Phi_{\infty,T(\zeta),D_{\mathfrak{H}}}(T(z))|T'(\zeta)||\mathrm{d}\zeta| + \mathrm{i}d \\ &= \frac{1}{2\pi} \int\limits_{T(s)} \Im\big(g_{\mathfrak{H}}^{-1}(\zeta)\big) \cdot \Phi_{\infty,\zeta,D_{\mathfrak{H}}}(T(z))|\mathrm{d}\zeta| + \mathrm{i}d. \end{aligned}$$

We apply $z = 0$ to get $d = 0$. Finally, a substitution $w = T(z)$ completes the proof. □

2.6 A universal proof for multiple slit Komatu–Loewner equations

As mentioned previously we are going to prove Theorem 2.30, 2.31 and 2.36 simultaneously. Herein, let Ω be a canonical domain, i.e. Ω is a circular slit disk, a circular slit annulus or an upper parallel slit half-plane. We say $\mathfrak{H}$ is an *appropriate hull in* Ω if $\mathfrak{H}$ is a radial Ω-hull when Ω is a circular slit disk, $\mathfrak{H}$ is a bilateral Ω-hull if Ω is a circular slit annulus and $\mathfrak{H}$ is a chordal Ω-hull if Ω is an upper parallel slit half-plane. In particular, $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is called *tuple of disjoint appropriate slits in* Ω if $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint radial slits in a circular slit disk Ω, $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint bilateral slits whenever Ω is a circular slit annulus, or $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint chordal slits if Ω is an upper parallel slit half-plane. Obviously, $\mathfrak{H} := \bigcup_{k=1}^m \gamma_k(0, t_k]$, with $t_k \in [0, T]$, is an appropriate Ω-hull if $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint appropriate slits in Ω.

A family $(\mathfrak{H}_t)_{t\in[0,T]}$ of appropriate Ω-hulls is called increasing if $\mathfrak{H}_t \subsetneq \mathfrak{H}_s$ whenever $t < s$ and $\mathfrak{H}_0 = \emptyset$. $(\mathfrak{H}_t)_{t\in[0,T]}$ is called continuous if $(\Omega_t - a)_{t\in[0,T]}$, with $\Omega_t := \Omega \setminus \mathfrak{H}_t$ and some $a \in \Omega_T$, is continuous. If $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint appropriate slit in Ω, then $(\bigcup_{k=1}^m \gamma_k(0,t])_{t\in[0,T]}$ is clearly a family of increasing continuous Ω-hulls.

Let $\mathfrak{H}$ be an appropriate hull in Ω. We call $g_{\mathfrak{H}} : \Omega\setminus\mathfrak{H} \to D_{\mathfrak{H}}$ the *normalised appropriate mapping function on* $\Omega\setminus\mathfrak{H}$ if $g_{\mathfrak{H}}$ is the normalised radial mapping function on $\Omega\setminus\mathfrak{H}$ when Ω is a circular slit disk, $g_{\mathfrak{H}}$ is the normalised bilateral mapping function on $\Omega\setminus\mathfrak{H}$ if Ω is a circular slit annulus, and $g_{\mathfrak{H}}$ is the normalised chordal mapping function on $\Omega\setminus\mathfrak{H}$ if Ω is an upper parallel slit half-plane. Consequently, $D_{\mathfrak{H}} := g_{\mathfrak{H}}(\Omega\setminus\mathfrak{H})$ and Ω do always have the same canonical type. Next, let $g_{\mathfrak{H}}$ be the normalised appropriate mapping function on $\Omega\setminus\mathfrak{H}$. Analogously, we denote by $\mathfrak{c}(g_{\mathfrak{H}})$ the logarithmic mapping radius of $g_{\mathfrak{H}}$ if underlying we have the radial case, the logarithmic conformal modulus of $g_{\mathfrak{H}}$ in the bilateral case, and the half-plane capacity of $g_{\mathfrak{H}}$ if underlying we have the chordal case, respectively. In this context, we call $\mathfrak{c}(g_{\mathfrak{H}})$ *appropriate capacity of* $g_{\mathfrak{H}}$. Moreover, we use the abbreviation $\mathfrak{c}_\Omega(\mathfrak{H}) := \mathfrak{c}(g_{\mathfrak{H}})$ as well where $g_{\mathfrak{H}}$ denotes the normalised appropriate mapping function on $\Omega\setminus\mathfrak{H}$.

Note that the implication (iii)⇒(ii) of Theorem 2.30, 2.31 and 2.36 is trivial. We are going to prove the implication (i)⇒(iii) in Subsection 2.6.2 and the implication (ii)⇒(i) in Subsection 2.6.3.

Before we can do so we need some preliminary lemmas.

2.6.1 Some preliminary lemmas

Summarising Lemma 2.24, 2.32 and 2.37 we find the following lemma.

Lemma 2.41. *Let Ω be a canonical domain, $\mathfrak{A}, \mathfrak{B}$ be appropriate hulls in Ω satisfying $\mathfrak{A} \subsetneq \mathfrak{B}$, and $g_{\mathfrak{A}}$ and $g_{\mathfrak{B}}$ denote the normalised appropriate mapping function on $\Omega\setminus\mathfrak{A}$ and $\Omega\setminus\mathfrak{B}$, respectively. Then $\mathfrak{c}(g_{\mathfrak{A}}) < \mathfrak{c}(g_{\mathfrak{B}})$.*

In the same way we find with Lemma 2.25, 2.33 and 2.38 the following.

Lemma 2.42. *Let Ω be a canonical domain, $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of appropriate Ω-hulls, and for each $t \in [0,T]$, g_t denotes the normalised appropriate mapping function from $\Omega_t := \Omega\setminus\mathfrak{H}_t$ onto the canonical domain D_t. Let $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ with $t_n \to t_0$, assume $\operatorname{con}(\Omega_{t_n}) = \operatorname{con}(\Omega_{t_0})$ for all $n \in \mathbb{N}$, and $\Omega_{t_n} - a \xrightarrow{k} \Omega_{t_0} - a$ for some $a \in \Omega_T$.*

Then $g_{t_n} \xrightarrow{l.u.} g_{t_0}$ on Ω_{t_0} and $D_{t_n} - a \xrightarrow{k} D_{t_0} - a$ for all $a \in D_{t_0}$ as $n \to \infty$. Moreover, $\mathfrak{c}(g_{t_n}) \to \mathfrak{c}(g_{t_0})$ as well. Additionally, assume $t \mapsto \mathfrak{H}_t$ is continuous on $[0,T]$ and $\operatorname{con}(\Omega_t) = \operatorname{con}(\Omega)$ for all $t \in [0,T]$. Then $t \mapsto g_t$ is continuous on $[0,T]$ and there is a $\delta > 0$ such that for each $t \in [0,T]$, $\operatorname{dist}(C_j(t), C_k(t)) > \delta$ whenever $j \neq k$. Here, $C_1(t), \ldots C_{\mathfrak{n}}(t)$ denote the boundary components of D_t.

Proof. Note that $D_{t_n} - a \xrightarrow{k} D_{t_0} - a$ for all $a \in D_{t_0}$ follows immediately from Corollary 2.12, so it only remains to prove the second part. This can be done by using the same idea as in the proof of Lemma 2.25 where we proved that the inner boundary components are mapped to the inner boundary components.

In order to do so let $C_1, \ldots, C_{\mathfrak{n}-1}$ denote the inner boundary components of Ω. For each small $\rho > 0$, we set $C_k^\rho := \{z \in \mathbb{C} \mid \text{dist}(z, C_k) = \rho\}$. Since $\text{dist}(\mathfrak{H}_T, C_k) > 0$ for each $k \in \{1, \ldots, \mathfrak{n}-1\}$, we find a small $\rho > 0$ such that $C_j^\rho \cap C_k^\rho = \emptyset$ and each C_k^ρ is a Jordan curve in Ω separating C_k from C_j, $j \neq k$.

Suppose there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\min_{j\neq k} \text{dist}(C_j(t_n), C_k(t_n)) \to 0$. Without loss of generality we can assume that $t_n \to t_0 \in [0, T]$. Since C_k^ρ is a compact set, we get $g_{t_n}(C_k^\rho) \to g_{t_0}(C_k^\rho)$, so there is an $N \in \mathbb{N}$ such that for each $k \in \{1, \ldots \mathfrak{n}-1\}$ and all $n \geq N$, $C_k(t_n)$ is surrounded by $g_{t_0}(C_{I(k)}^\rho)$. Herein, $I : \{1, \ldots, \mathfrak{n}-1\} \to \{1, \ldots, \mathfrak{n}-1\}$ is one-to-one. Consequently, $\min_{j\neq k} \text{dist}(C_j(t_n), C_k(t_n)) > \delta =: \min_{j\neq k} \text{dist}(g_{t_0}(C_j^\rho), g_{t_0}(C_k^\rho)) > 0$, so this yields a contradiction. □

Lemma 2.43. *Let Ω be a canonical domain and $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint appropriate slits in Ω. Assume $f_{k;t,\tau}$, with $k \in \{1, \ldots, m\}$ and $t, \tau \in [0, T]$, is the normalised appropriate mapping function from $\Omega_k(t, \tau) := \Omega \setminus (\bigcup_{j=1, j\neq k}^m \gamma_j(0, \tau] \cup \gamma_k(0, t])$ onto the canonical domain $D_k(t, \tau)$. Next, we set $U_k(t, \tau) := f_{k;t,\tau}(\gamma_k(t))$ and*

$$S_{k;\underline{t},\bar{t},\tau} := f_{k;\underline{t},\tau}\big(\gamma_k(\underline{t}, \bar{t}]\big), \quad s_{k;\underline{t},\bar{t},\tau} := f_{k;\bar{t},\tau}\big(\gamma_k[\underline{t}, \bar{t}]\big)$$

for all $k \in \{1, \ldots, m\}$, $0 \leq \underline{t} < \bar{t} \leq T$ and $\tau \in [0, T]$. Then the function $(t, \tau) \mapsto U_k(t, \tau)$ is continuous on $[0, T]^2$ and

$$S_{k;t,t_0,\tau} \to U_k(t_0, \tau_0) \text{ as } (t, \tau) \to (t_0, \tau_0) \quad (\text{where } t \nearrow t_0),$$
$$s_{k;t_0,t,\tau} \to U_k(t_0, \tau_0) \text{ as } (t, \tau) \to (t_0, \tau_0) \quad (\text{where } t \searrow t_0).$$

Remark 2.8. Obviously, the same is true if we consider the image of γ_j under $f_{k;t,\tau}$ with $j \neq k$, i.e. $f_{k;t,\tau}(\gamma_j(\tau, \tau_0]) \to f_{k;t_0,\tau_0}(\gamma_j(\tau_0))$ if $(t, \tau) \to (t_0, \tau_0)$ with $\tau \nearrow \tau_0$, and $f_{k;t,\tau}(\gamma_j[\tau_0, \tau]) \to f_{k;t_0,\tau_0}(\gamma_j(\tau_0))$ if $(t, \tau) \to (t_0, \tau_0)$ with $\tau \searrow \tau_0$. Analogously, we receive the continuity of $(t, \tau) \mapsto f_{k;t,\tau}(\gamma_j(\tau))$, with $j \neq k$, as well.

Proof. Since there is no risk of confusion, we omit the index k. We will only show $S_{t,t_0,\tau} \to U(t_0, \tau_0)$ as $(t, \tau) \to (t_0, \tau_0)$ where $t \nearrow t_0$. The other case $s_{t_0,t,\tau} \to U(t_0, \tau_0)$ as $(t, \tau) \to (t_0, \tau_0)$ where $t \searrow t_0$ follows in the same way. Since $U(t, \tau) \in S_{t,t_0,\tau}$ and $U(t, \tau) \in s_{t_0,t,\tau}$, the continuity of U follows immediately.

Let $t_0 \in (0, T]$. As mentioned before, we will show that for every $\varepsilon > 0$, there is a $\delta > 0$ with $S_{t,t_0,\tau} \subseteq B_\varepsilon(U(t_0, \tau_0))$ for all $t \in [t_0 - \delta, t_0]$ and $\tau \in [\tau_0 - \delta, \tau_0 + \delta] \cap [0, T]$. Note that $z \mapsto f_{t_0,\tau_0}(z)$ has a continuous extension to the boundary with respect to the two sides of the slit, see also Remark 2.9. Thus for each small $\varepsilon > 0$, we find a $\delta_1 > 0$ such that $s_{t,t_0,\tau_0} \subseteq B_\varepsilon(U(t_0, \tau_0))$ for all $t \in [t_0 - \delta_1, t_0]$. Moreover, the function $f_{t,\tau} \circ f_{t_0,\tau_0}^{-1}$ converges by Lemma 2.42 locally uniformly to the identity if (t, τ) tends to (t_0, τ_0). Using the Schwarz reflection principle and Lemma 2.42, we see that these functions can be extended analytically to $B_{2\varepsilon}(U(t_0, \tau_0)) \setminus s_{t,t_0,\tau_0}$ if ε and $|t_0 - t|$ are small enough, see also Figure 2.6. Considering the uniform convergence on $\partial B_\varepsilon(U(t_0, \tau_0))$, we find a $\delta \in (0, \delta_1)$ such that $S_{t,t_0,\tau} \subseteq B_\varepsilon(U(t_0, \tau_0))$ for all $\tau \in [\tau_0 - \delta, \tau_0 + \delta] \cup [0, T]$ and all $t \in [t_0 - \delta, t_0]$.

The proof of the remark works in the same way. □

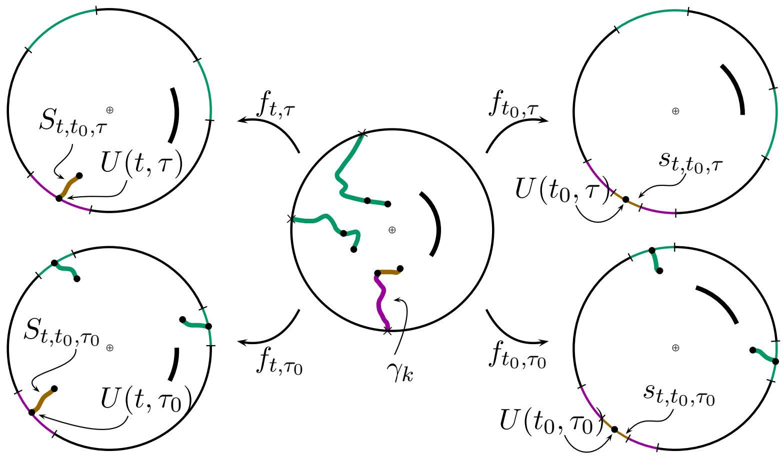

FIGURE 2.6: *Mapping behaviour of $f_{t,\tau}$, $f_{t_0,\tau}$ f_{t_0,τ_0} and f_{t,τ_0} in the proof of Lemma 2.43 in the radial case.*

The previous lemma shows that the image of each tip $f_{k;t,\tau}(\gamma_k(t))$ is continuous w.r.t t. This is true also for every other boundary point a on the outer or unbounded boundary component of Ω_t. Consequently, we have the following lemma

Lemma 2.44. *Let Ω be a canonical domain and $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint appropriate slits in Ω. For each $k \in \{1, \ldots, m\}$ and $t, \tau \in [0, T]$, $f_{k;t,\tau}$ is the normalised appropriate mapping function from $\Omega_k(t, \tau) := \Omega \setminus (\bigcup_{j=1,j\neq k}^{m} \gamma_j(0, \tau] \cup \gamma_k(0, t])$ onto the canonical domain $D_k(t, \tau)$. Moreover, $(t_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ are convergent sequences with limits t_0 and τ_0, respectively. Assume $a \in C$ (with respect to prime ends) where C denotes the outer or unbounded boundary component of $\Omega_k(t_0, \tau_0)$.*

Then $f_{k;t_n,\tau_n}(a) \to f_{k;t_0,\tau_0}(a)$ when $n \to \infty$.

Remark 2.9. If $a \in \gamma_k[0, t_0)$ or $a \in \gamma_j[0, \tau_0)$ with $j \neq k$, then a is either on the one or on the other side of the slit, so $f_{k;t_n,\tau_n}(a)$ and $f_{k;t_0,\tau_0}(a)$ are well-defined. An extensive discussion of the boundary behaviour of slit mappings can be found in Section 2.3 in [dMG13].

Proof. Note that the case $a = \gamma_k(t_0)$ or $a = \gamma_j(t_0)$ with $j \neq k$ follows immediately from Lemma 2.43 and Remark 2.8.

For the rest let us consider the function $h_n := f_{k;t_n,\tau_n} \circ f^{-1}_{k;t_0,\tau_0}$, which tends locally uniformly on $D_k(t_0, \tau_0)$ to the identity, see Lemma 2.42. Moreover Lemma 2.43 gives us

$$f_{k;t_0,\tau_0}(\gamma_k[\min(t_n, t_0), \max(t_n, t_0)]) \to f_{k;t_0,\tau_0}(\gamma_k(t_0)) = U_k(t_0, \tau_0)$$
$$f_{k;t_0,\tau_0}(\gamma_j[\min(\tau_n, \tau_0), \max(\tau_n, \tau_0)]) \to f_{k;t_0,\tau_0}(\gamma_j(\tau_0)), \quad j \neq k.$$

Note that we find an $N \in \mathbb{N}$ and an $\varepsilon > 0$ such that there is an analytic continuation of h_n to $B_\varepsilon(f_{k;t_0,\tau_0}(a))$. Herein, h_n converges locally uniformly on $B_\varepsilon(a)$ to the identity as well. Consequently, $f_{k;t_n,\tau_n}(a) = h_n(f_{k;t_0,\tau_0}(a)) \to f_{k;t_0,\tau_0}(a)$. □

Summarising Lemma 2.27, 2.34 and 2.39 we get the following result.

Lemma 2.45. *Let Ω be a canonical domain and $\mathfrak{H}$ is an appropriate hull in Ω such that $\partial\Omega_{\mathfrak{H}}$, with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$, is locally connected. By $g_{\mathfrak{H}}$ we denote the normalised appropriate mapping function from $\Omega_{\mathfrak{H}}$ onto the canonical domain $D_{\mathfrak{H}}$. Then we have*

$$\mathfrak{c}(g_{\mathfrak{H}}) = -\frac{1}{2\pi} \int\limits_{C} \Re F\big(g_{\mathfrak{H}}^{-1}(\zeta)\big) |\mathrm{d}\zeta|.$$

Herein, C denotes the outer or unbounded boundary component of $D_{\mathfrak{H}}$, and $F(w) := 2\mathrm{i}w$ with $w \in \mathbb{C}$ in the chordal case and $F(w) := \log(w)$ with $w \in \mathbb{C} \setminus \{0\}$ in the radial or bilateral case.

Note that, $w \mapsto \Re \log(w) = \log|w|$ does not depend of the branch of the logarithm. Finally, we get with Lemma 2.28, 2.35 and 2.40

Lemma 2.46. *Let Ω be a canonical domain and $\mathfrak{H}$ is an appropriate hull in Ω such that $\partial\Omega_{\mathfrak{H}}$, with $\Omega_{\mathfrak{H}} := \Omega \setminus \mathfrak{H}$, is locally connected. By $g_{\mathfrak{H}} : \Omega_{\mathfrak{H}} \to D_{\mathfrak{H}}$ we denote the normalised appropriate mapping function from $\Omega_{\mathfrak{H}}$ onto the canonical domain $D_{\mathfrak{H}}$. Then we have*

$$F(g_{\mathfrak{H}}^{-1}(z)) - F(z) = \frac{1}{2\pi} \int\limits_{C} \Re F\big(g_{\mathfrak{H}}^{-1}(\zeta)\big) \cdot \Phi_{a,\zeta,D_{\mathfrak{H}}}(z) |\mathrm{d}\zeta| \quad \text{for all } z \in D_{\mathfrak{H}}.$$

Herein, C denotes the outer or unbounded boundary component of $D_{\mathfrak{H}}$, and $F(w) := 2\mathrm{i}w$ with $w \in \mathbb{C}$ in the chordal case and $F(w) := \log(w)$ with $w \in \mathbb{C} \setminus \{0\}$ in the radial or bilateral case. Moreover, $a := 0$ in the radial case, $a := q$ in the bilateral case where q is the inner radius of the circular slit annulus $D_{\mathfrak{H}}$ and $a := \infty$ in the chordal case.

Note that there is always a branch of the logarithm in order to get, independently of the branch of the logarithm, an analytic function on the left side.

Lemma 2.47. *Let $A, B \subseteq \mathbb{D}$ be bounded domains. Assume there exists an $R > 0$ such that $A \cap B_R(1) = \mathbb{D} \cap B_R(1)$ and $B \cap B_R(1) = \mathbb{D} \cap B_R(1)$. Moreover, let $T : A \to B$ be a conformal mapping from A onto B satisfying $T(1) = 1$ and*

$$\left| \frac{\mathrm{d}}{\mathrm{d}z} \Big(\log\big(T(z)\big) - c \log z \Big) \right| < \delta \qquad \text{for all } z \in B_\varepsilon(1) \cap A,$$

with some small $\varepsilon > 0$, $\delta > 0$ and $c := T'(1)$. Then $c = T'(1) > 0$ and the inequality

$$|z|^{c+\delta} \leq |T(z)| \leq |z|^{c-\delta}$$

holds for all $z \in A \cap B_\varepsilon(1)$.

If $\varepsilon > 0$ is small enough we do always find a branch of the logarithm in order to get an analytic function $z \mapsto \log(T(z)) - c \log z$. Moreover, the derivative does not depend on a particular branch, so we can see $z \mapsto \log(T(z)) - c \log(z)$ as a multiple-valued function as well.

Proof. First of all, we extend the function T to an analytic map on $B_\varepsilon(1)$ for a small $\varepsilon > 0$, by using the Schwarz reflection principle. The small arc $\mathbb{T} \cap B_\varepsilon(1)$ is mapped into $\mathbb{T}$ with $T(1) = 1$, so the property $c := T'(1) > 0$ is obviously true.

Next, we set $\gamma_\theta(r) := r \cdot e^{\mathrm{i}\theta}$ for all $r \in [r_0, 1]$ and all $|\theta| < \phi$. In this context we can choose $r_0 < 1$ close enough to 1 and $\phi > 0$ small enough to get $\gamma_\theta(r) \in B_\varepsilon(1)$ for all $r \in [r_0, 1]$ and all $\theta \in (-\phi, \phi)$. Moreover, for each $\theta \in (-\phi, \phi)$, we define

$$h_\theta(r) := \Re\left(\log \frac{T(\gamma_\theta(r))}{(\gamma_\theta(r))^c}\right) = \ln\left|\frac{T(\gamma_\theta(r))}{(\gamma_\theta(r))^c}\right|, \quad r \in [r_0, 1].$$

Some simple calculations give us for all $\theta \in (-\phi, \phi)$ and all $r \in [r_0, 1]$:

$$\begin{aligned} \left|\frac{\partial}{\partial r} h_\theta(r)\right| &= \left|\Re\left(\frac{\mathrm{d}}{\mathrm{d}z} \log\left(\frac{T(z)}{z^c}\right)\Big|_{z=\gamma_\theta(r)} \cdot \dot{\gamma}_\theta(r)\right)\right| \\ &= \left|\Re\left(\left(\frac{T'(z)}{T(z)} - \frac{c}{z}\right)\Big|_{z=\gamma_\theta(r)} \cdot e^{\mathrm{i}\theta}\right)\right| \\ &\le \left|\frac{T'(z)}{T(z)} - \frac{c}{z}\Big|_{z=\gamma_\theta(r)}\right| \le \delta. \end{aligned}$$

We have $h_\theta(1) = 0$, so we find

$$\ln(r^\delta) = \delta \ln(r) \le h_\theta(r) \le -\delta \ln(r) = \ln(r^{-\delta}) \quad \text{for all } \theta \in (-\phi, \phi),\ r \in [r_0, 1].$$

Finally, we get $\ln(|z|^\delta) \le \left|\frac{T(z)}{z^c}\right| \le \ln(|z|^{-\delta})$ for all $z \in \{r \cdot e^{\mathrm{i}\theta} \mid r \in [r_0, 1],\ \theta \in (-\phi, \phi)\}$, so the proof is complete. □

Lemma 2.48. *Let $A, B \subseteq \mathbb{H}$ be domains and assume there exists an $R > 0$ such that $A \cap \mathbb{D}_R = \mathbb{H} \cap \mathbb{D}_R$, $B \cap \mathbb{D}_R = \mathbb{H} \cap \mathbb{D}_R$. Moreover, let $T : A \to B$ be a conformal mapping from A onto B with $T(0) = 0$ and*

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\big(T(z) - cz\big)\right| < \delta, \qquad \textit{for all } z \in \mathbb{D}_\varepsilon \cap A,$$

where $\varepsilon > 0$ is small, $\delta > 0$ and $c := T'(1)$. Then $c = T'(0) > 0$ and the inequality

$$(c - \delta)\Im(z) < \Im T(z) < (c + \delta)\Im(z)$$

holds for all $z \in A \cap B_\varepsilon(0)$.

Proof. First of all, we can extend T along $\mathbb{D}_R$ to an analytic function. Herein, it is easy to see that $c = T'(0) > 0$ holds. Let $\gamma_a(t) := a + \mathrm{i}t$ and $h_a(t) := \Im\big(T(\gamma_a(t)) - c\gamma_a(t)\big)$ for all $a \in [-a_0, a_0]$ and $t \in [0, t_0]$ with $a_0, t_0 > 0$. We choose a_0 and t_0 in such a way that $\gamma_a(t) \in \mathbb{D}_\varepsilon \cap A$ for all $t \in [0, t_0]$ and all $a \in [-a_0, a_0]$. Consequently, we find

$$\left|\frac{\partial}{\partial t} h_a(t)\right| = \left|\Im\Big((T'(\gamma_a(t)) - c) \cdot \mathrm{i}\Big)\right| \le |T'(\gamma_a(t)) - c| < \delta.$$

Hence, $-\delta t \le h_a(t) \le \delta t$. Finally, the proof is complete by substituting $z = \gamma_a(t)$. □

Summarising Lemma 2.47 and 2.48 we find the following lemma.

Lemma 2.49. *Let $G := \mathbb{D}$ or $G := \mathbb{H}$ and $\zeta_1, \zeta_2 \in \partial G$. $A, B \subseteq G$ are domains and assume there is an $R > 0$ such that $A \cap B_R(\zeta_1) = G \cap B_R(\zeta_1)$ and $B \cap B_R(\zeta_2) = G \cap B_R(\zeta_2)$. Moreover, let $T : A \to B$ be a conformal mapping where $T(\zeta_1) = \zeta_2$ and*

$$\left| \frac{\mathrm{d}}{\mathrm{d}z}\Big(F\big(T(z)\big) - |c|F(z)\Big)\right| < \delta \qquad \text{for all } z \in B_\varepsilon(\zeta_1) \cap G, \tag{2.6}$$

where $\varepsilon > 0$ is small, $\delta > 0$ and $c := T'(\zeta_1)$. Herein, $F(w) := 2\mathrm{i}w$ with $w \in \mathbb{C}$ if $G = \mathbb{H}$ and $F(w) := \log(w)$ with $w \in \mathbb{C} \setminus \{0\}$ if $G = \mathbb{D}$

Then the inequality

$$(|c| + \delta)\Re F(z) \le \Re F\big(T(z)\big) \le (|c| - \delta)\Re F(z)$$

holds for all $z \in B_\varepsilon(\zeta_1) \cap G$.

Remark 2.10. In the chordal case Equation (2.6) is equivalent to

$$2\big|T'(z) - |c|\big| = 2|T'(z) - c| = 2|T'(z) - T'(\zeta_1)| < \delta \quad \text{ for all } z \in B_\varepsilon(\zeta_1) \cap \mathbb{H},$$

as $c := T'(\zeta_1) > 0$.

In the radial case Equation (2.6) is equivalent to

$$\left|\frac{T'(\zeta_1 z)}{T(\zeta_1 z)} - \frac{|c|}{\zeta_1 z}\right| = \left|\frac{\hat{T}'(z)}{\hat{T}(z)} - \frac{|c|}{z}\right| = \left|\frac{\hat{T}'(z)}{\hat{T}(z)} - \frac{\hat{T}'(1)}{z}\right| < \delta \quad \text{ for all } z \in B_\varepsilon(1) \cap \mathbb{D},$$

with $\hat{T}(z) := \frac{T(\zeta_1 z)}{\zeta_2}$, i.e. $\hat{T}(1) = 1$.

2.6.2 Proof of Theorem 2.30, 2.31 and 2.36: (i)⇒(iii)

Proof for $t \searrow t_0$. Let be $t_0 < t$ and for each $t, \tau \in [0, T]$ and $k \in \{1, \ldots, m\}$, $f_{k;t,\tau}$ denotes the normalised appropriate mapping function from $\Omega_k(t,\tau) := \Omega \setminus (\bigcup_{j=1, j\neq k}^m \gamma_j(0,\tau] \cup \gamma_k(0,t])$ onto the canonical domain $D_k(t,\tau)$. Moreover, we write $g_t := f_{k;t,t}$, $\Omega_t := \Omega_k(t,t)$, $D_t := D_k(t,t)$ and

$$S_{k;t_0,t,\tau} := f_{k;t_0,\tau}\big(\gamma_k(t_0,t]\big), \quad s_{k;t_0,t,\tau} := f_{k;t,\tau}\big(\gamma_k[t_0,t]\big).$$

On top of this we define $s_k(t_0,t) := s_{k;t_0,t,t}$ and $S_k(t_0,t) := S_{k;t_0,t,t_0}$. Finally, we set $g_{t_0,t} := g_t \circ g_{t_0}^{-1}$, so this is the normalised appropriate mapping function from $D_{t_0} \setminus \bigcup_{k=1}^m S_k(t_0,t)$ onto D_t. Using Lemma 2.46, we find

$$F(g_{t_0,t}^{-1}(z)) - F(z) = \frac{1}{2\pi}\sum_{k=1}^m \int_{s_k(t,t_0)} \Re F\big(g_{t_0,t}^{-1}(\zeta)\big) \cdot \Phi_{a_t,\zeta,D_t}(z)|\mathrm{d}\zeta| \quad \text{for all } z \in D_t,$$

with $F(w) := \log(w)$, $w \in \mathbb{C} \setminus \{0\}$, in the radial and bilateral case and $F(w) := 2\mathrm{i}w$, $w \in \mathbb{C}$, in the chordal case. $a_t := 0$ in the radial case, $a_t := q_t$ in the bilateral case where q_t is the inner radius of the circular slit annulus D_t, and $a_t := \infty$ in the chordal case. Note that $\zeta \mapsto \Phi_{a_t,\zeta,D_t}(z)$ is continuous on $s_k(t_0,t)$ by Lemma 2.18, 2.19 or 2.20. $\zeta \mapsto \Re F(g_{t_0,t}^{-1}(\zeta))$ is continuous on $s_k(t_0,t)$ as well and $\Re F(g_{t_0,t}^{-1}(\zeta)) \leq 0$, so the mean value theorem gives us

$$F(g_{t_0,t}^{-1}(z)) - F(z) = \sum_{k=1}^{m} \Big(\Re\big(\Phi_{a_t,\zeta_k^1,D_t}(z)\big) + \mathrm{i}\,\Im\big(\Phi_{a_t,\zeta_k^2,D_t}(z)\big)\Big) \frac{1}{2\pi} \int\limits_{s_k(t_0,t)} \Re F(g_{t_0,t}^{-1}(\zeta))|\mathrm{d}\zeta| \quad (2.7)$$

for all $z \in D_t$ and some $\zeta_k^j \in s_k(t_0,t)$, $j \in \{1,2\}$. For each $k \in \{1,\dots,m\}$, we denote the remaining integral by $2\pi c_k(t_0,t)$.

For now let us fix $k \in \{1,\dots,m\}$ and $t > t_0$, and consider the function $z \mapsto h_t^{-1} := f_{k;t,t_0} \circ g_t^{-1}$, which can be extended analytically to $B_\varepsilon(U_k(t_0))$, with $\varepsilon > 0$ small, by using the Schwarz reflection principle and Lemma 2.42 and 2.43. In this context, Lemma 2.42 ensures that the interior boundary components of D_t come not to close to $\mathbb{T}$. Moreover,

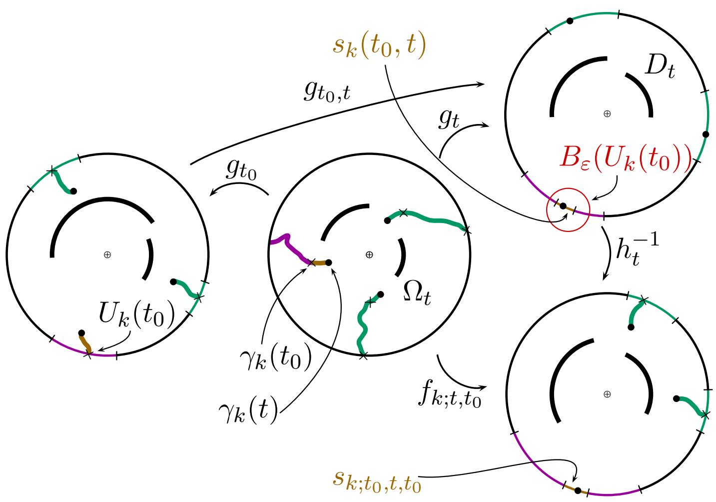

FIGURE 2.7: *Radial mappings g_{t_0}, g_t, h_t and $f_{k;t,t_0}$ in the proof of Theorem 2.30, 2.31, 2.36 (i)⇒(iii) in the case $t > t_0$*

Lemma 2.42 shows that h_t^{-1} (as well as h_t) tends to the identity locally uniformly on D_{t_0} as $t \searrow t_0$. On top of this the local uniform convergence holds on $B_\varepsilon(U_k(t_0))$ as well. If t is close to t_0, we get by substitution

$$\begin{aligned} c_k(t_0,t) &= \frac{1}{2\pi} \int\limits_{s_k(t_0,t)} \Re F\big(g_{t_0,t}^{-1}(\zeta)\big)\,|\mathrm{d}\zeta| = \frac{1}{2\pi} \int\limits_{s_{k;t_0,t,t_0}} \Re F\big(g_{t_0,t}^{-1}(h_t(\zeta))\big) \cdot |h_t'(\zeta)|\,|\mathrm{d}\zeta| \\ &= \frac{1}{2\pi} \int\limits_{s_{k;t_0,t,t_0}} \Re F\big(g_{t_0} \circ f_{k;t,t_0}^{-1}(\zeta)\big) \cdot |h_t'(\zeta)|\,|\mathrm{d}\zeta|. \end{aligned}$$

Moreover, $\zeta \mapsto \Re F(g_{t_0} \circ f_{k;t,t_0}^{-1}(\zeta))$ and $\zeta \mapsto |h_t'(\zeta)|$ are continuous on $s_{k;t_0,t,t_0}$ and $\Re F(g_{t_0} \circ$

$f_{k;t,t_0}^{-1}(\zeta)) \le 0$ so the mean value theorem yields

$$c_k(t_0,t) = |h_t'(\zeta^*)| \frac{1}{2\pi} \int\limits_{s_{k;t_0,t,t_0}} \Re F\big(g_{t_0} \circ f_{k;t,t_0}^{-1}(\zeta)\big)\, |\mathrm{d}\zeta|$$

where $\zeta^* \in s_{k;t_0,t,t_0}$. Note that $f_{k;t,t_0} \circ g_{t_0}^{-1}$ is the normalised appropriate mapping function from $D_{t_0} \setminus S_k(t_0,t)$ onto $D_k(t,t_0)$, so we find with Lemma 2.45

$$c_k(t_0,t) = |h_t'(\zeta^*)| \big(-\mathfrak{c}(f_{k;t,t_0} \circ g_{t_0}^{-1})\big) = -|h_t'(\zeta^*)| \big(\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})\big). \tag{2.8}$$

We have $\zeta^* \in s_{k;t_0,t,t_0} \subseteq B_\varepsilon(\zeta_k(t_0))$ if t is close to t_0, so we find $h_t'(\zeta^*) \to 1$ as $t \searrow t_0$. Summarising, we get

$$\lim_{t \searrow t_0} \frac{c_k(t_0,t)}{t-t_0} = \lim_{t \searrow t_0} -\frac{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}{t-t_0} = -\lambda_k(t_0). \tag{2.9}$$

Obviously, we can do this for each $k \in \{1, \dots, m\}$.

Next, Equation (2.7) with $z := g_t(w)$ and $w \in \Omega_t$ yields

$$\frac{F(g_{t_0}(w)) - F(g_t(w))}{t-t_0} = \sum_{k=1}^m \Big(\Re\big(\Phi_{a_t,\zeta_k^1,D_t}(g_t(w))\big) + \mathrm{i}\, \Im\big(\Phi_{a_t,\zeta_k^2,D_t}(g_t(w))\big) \Big) \frac{c_k(t_0,t)}{t-t_0}.$$

for all $t > t_0$. As mentioned before, for each $j \in \{1,2\}$, $\zeta_k^j \in s_k(t_0,t)$ and $s_k(t_0,t) \to U_k(t_0)$, see Lemma 2.43. Consequently, $\zeta_k^j \to U_k(t_0)$ as $t \searrow t_0$. Using Lemma 2.42, we get $D_t - b \xrightarrow{\text{k}} D_{t_0} - b$ for each $b \in D_{t_0}$. Thus we find with Lemma 2.18, 2.19 or 2.20 in either case

$$\Phi_{a_t,\zeta_k^j,D_t} \xrightarrow{\text{l.u.}} \Phi_{a_{t_0},U_k(t_0),D_{t_0}} \quad \text{on } D_{t_0}.$$

As mentioned already, Lemma 2.42 gives us $g_t \xrightarrow{\text{l.u.}} g_{t_0}$ on Ω_{t_0} as $t \searrow t_0$, so we find

$$\lim_{t \searrow t_0} \frac{F(g_t(w)) - F(g_{t_0}(w))}{t-t_0} = \sum_{k=1}^m \lambda_k(t_0) \cdot \Phi_{a_{t_0},U_{t_0},D_{t_0}}\big(g_{t_0}(w)\big) \qquad \text{for all } w \in \Omega_{t_0}.$$

Finally, note that $g_{t_0,t} = g_t \circ g_{t_0}^{-1}$ is the normalised appropriate mapping function from $D_{t_0} \setminus \bigcup_{k=1}^m S_k(t_0,t)$ onto D_t, so we can apply Lemma 2.45 to get

$$\mathfrak{c}(g_t) - \mathfrak{c}(g_{t_0}) = \mathfrak{c}(g_{t_0,t}) = -\frac{1}{2\pi} \int\limits_{\mathbb{T}} \Re F\big(g_{t_0,t}^{-1}(\zeta)\big) |\mathrm{d}\zeta| = \\ -\frac{1}{2\pi} \sum_{k=1}^m \int\limits_{s_k(t_0,t)} \Re F\big(g_{t_0,t}^{-1}(\zeta)\big) |\mathrm{d}\zeta| = \sum_{k=1}^m -c_k(t_0,t)$$

for all $t > t_0$. Using Equation (2.9), we find

$$\lim_{t \searrow t_0} \frac{\mathfrak{c}(g_t) - \mathfrak{c}(g_{t_0})}{t-t_0} = \sum_{k=1}^m \lambda_k(t_0),$$

so the proof is complete. □

As a nice side effect, Equation (2.8) immediately yields the following lemma.

Lemma 2.50. *Let Ω be a canonical domain and denote by $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ a tuple of disjoint appropriate slits in Ω. For each $t, \tau \in [0,T]$ and $k \in \{1,\dots,m\}$, denote by $f_{k;t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0,t] \cup \bigcup_{j\neq k} \gamma_j(0,\tau])$. Moreover, we set $g_t := f_{k;t,t}$ for all $t \in [0,T]$ and $s_k(t_0,t) := g_t(\gamma_k[t_0,t])$ for all $t > t_0$ and $k \in \{1,\dots,m\}$. Then for all $k \in \{1,\dots,m\}$, we find*

$$\frac{-c_k(t_0,t)}{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})} \xrightarrow{t\searrow t_0} 1 \qquad \text{with } c_k(t_0,t) := \frac{1}{2\pi} \int\limits_{s_k(t_0,t)} \Re F\big((g_{t_0} \circ g_t^{-1})(\zeta)\big)|\mathrm{d}\zeta|,$$

where $F(w) := 2\mathrm{i}w$ for all $w \in \mathbb{C}$ in the chordal case and $F(w) := \log(w)$ for all $w \in \mathbb{C} \setminus \{0\}$ in the radial and bilateral case.

Proof for $t \nearrow t_0$. Assume $t < t_0$ and $k \in \{1,\dots,m\}$. We use the same abbreviation as in the previous case $t \searrow t_0$, so for each $t, \tau \in [0,T]$ and $k \in \{1,\dots,m\}$, $f_{k;t,\tau}$ is the normalised appropriate mapping function from $\Omega_k(t,\tau) := \Omega \setminus (\bigcup_{j=1, j\neq k}^m \gamma_j(0,\tau] \cup \gamma_k(0,t])$ onto the canonical domain $D_k(t,\tau)$ and $g_t := f_{k;t,t}$, $\Omega_t := \Omega_k(t,t)$, $D_t := D_k(t,t)$. Moreover, we write

$$S_{k;t,t_0,\tau} := f_{k;t,\tau}\big(\gamma_k(t,t_0]\big), \quad s_{k;t,t_0,\tau} := f_{k;t_0,\tau}\big(\gamma_k[t,t_0]\big), \quad \tau \in [0,T].$$

On top of this we set $s_k(t,t_0) := s_{k;t,t_0,t_0}$ and $S_k(t,t_0) := S_{k;t,t_0,t}$. Finally, we set $g_{t,t_0} := g_{t_0} \circ g_t^{-1}$, so this is the normalised appropriate mapping function from $D_t \setminus \bigcup_{k=1}^m S_k(t,t_0)$ onto D_{t_0}. Like the previous case we find by using Lemma 2.46 and the mean value theorem

$$\begin{aligned} &F(g_{t,t_0}^{-1}(z)) - F(z) = \\ &\qquad \sum_{k=1}^m \Big(\Re\big(\Phi_{a_{t_0},\zeta_k^1,D_{t_0}}(z)\big) + \mathrm{i}\,\Im\big(\Phi_{a_{t_0},\zeta_k^2,D_{t_0}}(z)\big)\Big)\frac{1}{2\pi} \int\limits_{s_k(t,t_0)} \Re F(g_{t,t_0}^{-1}(\zeta))|\mathrm{d}\zeta| \end{aligned} \tag{2.10}$$

with $\zeta_k^j \in s_k(t,t_0)$, $j \in \{1,2\}$. Herein, $a_{t_0} := 0$ in the radial case, $a_{t_0} := q$ in the bilateral case where q is the inner radius of the circular slit annulus D_{t_0} and $a_{t_0} := \infty$ in the chordal case. We denote the remaining integral on the right-hand side in Equation (2.10) by $2\pi c_k(t,t_0)$.

Next, let us consider the function $h_t^{-1} := f_{k;t,t_0} \circ g_t^{-1}$, which is the normalised appropriate mapping function from $D_t \setminus \bigcup_{j\neq k} S_j(t,t_0)$ onto $D_k(t,t_0)$. Using Lemma 2.43 and 2.42, we find a small $\varepsilon > 0$ such that there is an analytic continuation of h_t to $B_\varepsilon(U_k(t_0))$ for all $t < t_0$ close enough to t_0. Moreover, h_t tends locally uniformly on D_{t_0} (as well as on the extension $B_\varepsilon(U_k(t_0))$) to the identity if $t \nearrow t_0$. Obviously, we have

$$c_k(t,t_0) = \frac{1}{2\pi} \int\limits_{s_k(t,t_0)} \Re F\big(g_{t,t_0}^{-1}(\zeta)\big)|\mathrm{d}\zeta| = \frac{1}{2\pi} \int\limits_{s_k(t,t_0)} \Re F(h_t \circ f_{k;t,t_0} \circ g_{t_0}^{-1})(\zeta)(\mathrm{d}\zeta).$$

Note that $(f_{k;t,t_0} \circ g_{t_0}^{-1})(\zeta) \in \mathrm{cl}(S_{k;t,t_0,t})$ if $\zeta \in s_k(t,t_0)$. On top of this $S_{k;t,t_0,t} \to U_k(t_0) =: \zeta_0$ if $t \nearrow t_0$ by Lemma 2.43. Hence we find a compact set $K \subseteq B_\varepsilon(\zeta_0)$ such

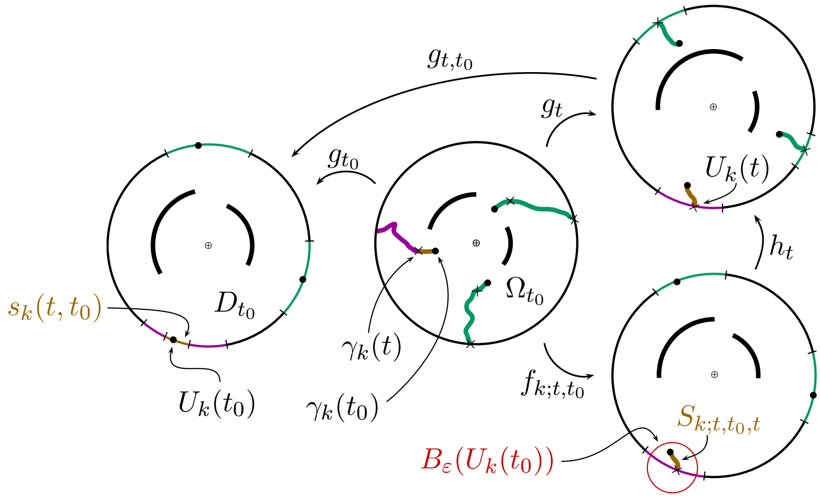

FIGURE 2.8: *Radial mappings g_{t_0}, g_t, h_t and $f_{k;t,t_0}$ in the proof of Theorem 2.30, 2.31, 2.36 (i)⇒(iii) in the case $t < t_0$*

that $S_{k;t,t_0,t} \subseteq K$ for all t close enough to t_0. h_t converges uniformly on K to the identity as $t \nearrow t_0$, so using Remark 2.10, we find for each $\delta > 0$ a $t^* < t_0$ such that $\left|\frac{\mathrm{d}}{\mathrm{d}z}\big(F\big(h_t(z)\big) - |h_t'(\zeta_0)|F(z)\big)\right| < \delta$ for all $t \in [t^*, t_0]$ and all $z \in K$. Using Lemma 2.49, we find

$$(|h_t'(\zeta_0)| + \delta)\Re F(z) \leq \Re F(h_t(z)) \leq (|h_t'(\zeta_0)| - \delta)\Re F(z)$$

for all $z \in K$ and all $t \in [t^*, t_0]$. Note that $|h_t'(\zeta_0)| \to 1$ as $t \nearrow t_0$. Summarising, we get

$$\big(|h_t'(\zeta_0)| + \delta\big)\frac{1}{2\pi}\int\limits_{s_k(t,t_0)} \Re F(f_{k;t,t_0} \circ g_{t_0}^{-1})(\zeta)(\mathrm{d}\zeta) \leq c_k(t,t_0) \leq \big(|h_t'(\zeta_0)| - \delta\big)\frac{1}{2\pi}\int\limits_{s_k(t,t_0)} \Re F(f_{k;t,t_0} \circ g_{t_0}^{-1})(\zeta)(\mathrm{d}\zeta) \tag{2.11}$$

for all $t \in [t^*, t_0]$. Like in the previous case, $g_{t_0} \circ f_{k;t,t_0}^{-1}$ is the normalised appropriate mapping function from $D_k(t,t_0) \setminus S_{k;t,t_0,t}$ onto D_{t_0}, so we get with Lemma 2.45

$$\frac{1}{2\pi}\int\limits_{s_k(t,t_0)} \Re F(f_{k;t,t_0} \circ g_{t_0}^{-1})(\zeta)(\mathrm{d}\zeta) = -\mathfrak{c}(g_{t_0} \circ f_{k;t,t_0}^{-1}) = -\big(\mathfrak{c}(f_{k;t_0,t_0}) - \mathfrak{c}(f_{k;t,t_0})\big).$$

Hence, $\lim_{t\nearrow t_0} \frac{c_k(t,t_0)}{t_0 - t} = -\lambda_k(t_0)$. Obviously, we can do this for each $k \in \{1, \dots, m\}$.

Next, Equation (2.10) with $z := g_{t_0}(w)$ and $w \in \Omega_{t_0}$ gives us

$$\frac{F(g_t(w)) - F(g_{t_0}(w))}{t - t_0} = \sum_{k=1}^{m}\Big(\Re\big(\Phi_{a_{t_0},\zeta_k^1,D_{t_0}}(g_{t_0}(w))\big) + \mathrm{i}\,\Im\big(\Phi_{a_{t_0},\zeta_k^2,D_{t_0}}(g_{t_0}(w))\big)\Big)\frac{c_k(t_0,t)}{t - t_0}.$$

As mentioned before $\zeta_k^j \in s_k(t,t_0)$, $j \in \{1,2\}$ and $s_k(t,t_0) \to U_k(t_0)$, see Lemma 2.43. Consequently, $\zeta_k^j \to U_k(t_0)$ as $t \searrow t_0$. Using Lemma 2.18, 2.19 or 2.20, we find in either case $\Phi_{a_{t_0},\zeta_k^j,D_{t_0}} \xrightarrow{\text{l.u.}} \Phi_{a_{t_0},U_k(t_0),D_{t_0}}$ on D_{t_0} as $t \nearrow t_0$. Finally, we find

$$\lim_{t\nearrow t_0} \frac{F(g_t(w)) - F(g_{t_0}(w))}{t - t_0} = \sum_{k=1}^{m} \lambda_k(t_0) \cdot \Phi_{a_{t_0},U_k(t_0),D_{t_0}}(g_{t_0}(w))$$

for all $w \in \Omega_{t_0}$. □

As a nice side effect, Equation (2.11) immediately yields the following lemma.

Lemma 2.51. *Let Ω be a canonical domain and denote by $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ a tuple of disjoints appropriate slits in Ω. For each $t, \tau \in [0, T]$ and $k \in \{1, \dots, m\}$, $f_{k;t,\tau}$ denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0, t] \cup \bigcup_{j\neq k} \gamma_j(0, \tau])$. $g_t := f_{k;t,t}$ for all $t \in [0, T]$ and $s_k(t_0, t) := g_{t_0}(\gamma_k[t, t_0])$ for all $t < t_0$ and $k \in \{1, \dots, m\}$. Then we find*

$$\frac{-c_k(t, t_0)}{\mathfrak{c}(f_{k;t_0,t_0}) - \mathfrak{c}(f_{k;t,t_0})} \xrightarrow{t\nearrow t_0} 1 \qquad \text{with } c_k(t, t_0) := \frac{1}{2\pi} \int_{s_k(t,t_0)} \Re F\big((g_t \circ g_{t_0}^{-1})(\zeta)\big)|\mathrm{d}\zeta|,$$

where $F(w) := 2\mathrm{i}w$ for all $w \in \mathbb{C}$ in the chordal case and $F(w) := \log(w)$ for all $w \in \mathbb{C} \setminus \{0\}$ in the radial and bilateral case.

2.6.3 Proof of Theorem 2.30, 2.31 and 2.36: (ii)⇒(i)

Proof for $t \searrow t_0$. We use the same notations as in the proof of (i)⇒(iii), i.e. for each $t, \tau \in [0, T]$ and $k \in \{1, \dots, m\}$, $f_{k;t,\tau}$ denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0, t] \cup \bigcup_{j\neq k} \gamma_j(0, \tau])$. Moreover, $g_t := f_{k;t,t}$, $s_k(t_0, t) := g_t(\gamma_k[t_0, t])$ and $U_k(t) := g_t(\gamma_k(t))$ for all $t \in [0, T]$ and all $k \in \{1, \dots, m\}$.

Let $t_0 < t$. Applying the real part on Equation (2.7) with $z = g_t(w)$, for each $k \in \{1, \dots, m\}$ and all $w \in \Omega_t$, we get

$$\begin{aligned}\Re F(g_t(w)) - \Re F(g_{t_0}(w)) &= -\frac{1}{2\pi} \sum_{k=1}^{m} \Re\Phi_{a_t,\zeta_k,D_t}(g_t(w)) \int_{s_k(t_0,t)} \Re F(g_{t_0} \circ g_t^{-1})(\zeta)\,|\mathrm{d}\zeta| \\ &\geq -\frac{1}{2\pi} \Re\Phi_{a_t,\zeta_k,D_t}(g_t(w)) \int_{s_k(t_0,t)} \Re F(g_{t_0} \circ g_t^{-1})(\zeta)\,|\mathrm{d}\zeta| \geq 0.\end{aligned}$$

Here $\zeta_k \in s_k(t_0, t)$, and $F(w) := 2\mathrm{i}w$ for all $w \in \mathbb{C}$ in the chordal case and $F(w) := \log(w)$ for all $w \in \mathbb{C} \setminus \{0\}$ in the radial and bilateral case. $a_t := 0$ in the radial case, $a_t := q_t$ in the bilateral case where q_t is the inner radius of the circular slit annulus D_t and $a_t := \infty$ in the chordal case. Next, let us denote the remaining integral on the right-hand side by $2\pi c_k(t_0, t)$, so we find with $t > t_0$

$$\frac{\Re F(g_t(z)) - \Re F(g_{t_0}(z))}{t - t_0} \geq -\Re\Phi_{a_t,\zeta_k,D_t}(g_t(z)) \frac{c_k(t_0, t)}{t - t_0} \geq 0.$$

Analogously to the proof of (i)⇒(iii), for all $k \in \{1, \dots, m\}$, we have

$$\Phi_{a_t,\zeta_k,D_t} \circ g_t \xrightarrow{\text{l.u.}} \Phi_{a_{t_0},U_k(t_0),D_{t_0}} \circ g_{t_0} \quad \text{on } \Omega_{t_0}$$

when $t \searrow t_0$, as $\zeta_k \in s_k(t_0,t) \to U_k(t_0)$. Since $t \mapsto \Re F(g_t(z))$ is differentiable at t_0, each $t \mapsto \frac{c_k(t_0,t)}{t-t_0}$, $k \in \{1,\dots,m\}$, is bounded on $(t_0,T]$. Summarising, for each $w \in \Omega_t$, we find

$$\Re F(g_t(w)) - \Re F(g_{t_0}(w)) = -\sum_{k=1}^{m} \Re\Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(w)\big)\, c_k(t_0,t) + o(|t-t_0|).$$

Using Lemma 2.50, we see that

$$\lim_{t\searrow t_0} \frac{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}{t-t_0} \quad \text{exists if and only if} \quad \lim_{t\searrow t_0} \frac{c_k(t_0,t)}{t-t_0} \text{ exists.}$$

Consequently, we are going to prove the existence of the limits $\lim_{t\searrow t_0} \frac{c_k(t,t_0)}{t-t_0}$, $k \in \{1,\dots,m\}$.

For this purpose, we show that we can find $w_1,\dots,w_m \in \Omega_{t_0}$ such that each $c_k(t,t_0)$ can be represented as a linear combination of the functions

$$\Re F(g_t(w_1)) - \Re F(g_{t_0}(w_1)), \quad \dots, \quad \Re F(g_t(w_m)) - \Re F(g_{t_0}(w_m)).$$

This is equivalent to the question whether it is possible to find $w_1,\dots,w_m \in \Omega_{t_0}$ such that the vectors $v_1,\dots,v_m \in \mathbb{R}^m$ are linear independently where

$$v_k := \Big(\Re\Phi_{a_{t_0},U_1(t_0),D_{t_0}}\big(g_{t_0}(w_k)\big),\dots,\Re\Phi_{a_{t_0},U_m(t_0),D_{t_0}}\big(g_{t_0}(w_k)\big)\Big).$$

Since $\Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(\gamma_k(t_0))\big) = \infty$ and $\Re\Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(\gamma_j(t_0))\big) = 0$ if $j \neq k$, we find $w_k \in \Omega(t_0)$ close enough to $\gamma_k(t_0)$ in order to get

$$\Re\Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(w_k)\big) = 1, \qquad \Re\Phi_{a_{t_0},U_j(t_0),D_{t_0}}\big(g_{t_0}(w_k)\big) < \tfrac{1}{m} \text{ for all } j \neq k.$$

This is based on the fact that we may consider the preimage of the curve $\delta(x) := 1 + \mathrm{i}x$, $x > 0$ under the mapping $z \mapsto \Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(z)\big)$ and choose x large enough in order to find a suitable $w_k \in \Omega_{t_0}$, see Figure 2.9. Consequently, the matrix $(v_1^T,\dots,v_m^T)$ is a

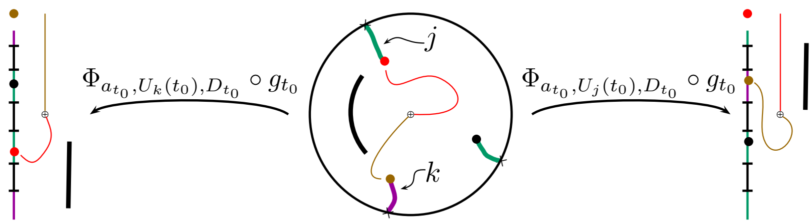

FIGURE 2.9: *The preimages of* $\delta(x) := 1 + x$ *under the mapping* $z \mapsto \Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(z)\big)$ *in the radial case*

diagonally dominant matrix, so it is invertible as well. □

Proof for $t \nearrow t_0$. This works in the same way as in the case $t \searrow t_0$ with Lemma 2.51 instead of Lemma 2.50. □

2.7 Almost everywhere differentiability

In this section we are going to show that a family $g_t : \Omega_t \to D_t$, with $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ and disjoint appropriate slits $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ in Ω, is least for almost every $t \in [0,T]$ differentiable.

Theorem 2.52. *Let Ω be a circular slit disk and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint radial slits in Ω with $m \in \mathbb{N}$. For each $t \in [0,T]$, $g_t : \Omega_t \to D_t$ denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ onto the circular slit disk D_t.*

Then there is a null set $\mathcal{N}$ of $[0,T]$ such that $t \mapsto g_t(z)$ is differentiable on $[0,T] \setminus \mathcal{N}$ for each $z \in \Omega_T$ and satisfies

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^m \lambda_k(t) \Phi_{0,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } t \in [0,T] \setminus \mathcal{N} \text{ and all } z \in \Omega_T,$$

where, for each $k \in \{1, \dots, m\}$, the driving term $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0,T]$ and $\lambda_k(t) \geq 0$ for each $t \in [0,T] \setminus \mathcal{N}$. Moreover, $t \mapsto \operatorname{lmr}(g_t)$ is differentiable on $[0,T] \setminus \mathcal{N}$ with derivative $\sum_{k=1}^m \lambda_k(t)$.

We have the same in the bilateral case.

Theorem 2.53. *Let Ω be a circular slit annulus and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint bilateral slits in Ω with $m \in \mathbb{N}$. For each $t \in [0,T]$, $g_t : \Omega_t \to D_t$ denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ onto the circular slit annulus D_t with inner radius q_t.*

Then there is a null set $\mathcal{N}$ of $[0,T]$ such that $t \mapsto g_t(z)$ is differentiable on $[0,T] \setminus \mathcal{N}$ for each $z \in \Omega_T$ and satisfies

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^m \lambda_k(t) \Phi_{q_t,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } t \in [0,T] \setminus \mathcal{N} \text{ and all } z \in \Omega_T,$$

where, for each $k \in \{1, \dots, m\}$, the driving term $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0,T]$ and $\lambda_k(t) \geq 0$ for each $t \in [0,T] \setminus \mathcal{N}$. Moreover, $t \mapsto \operatorname{lcm}(g_t)$ is differentiable on $[0,T] \setminus \mathcal{N}$ with derivative $\sum_{k=1}^m \lambda_k(t)$.

Finally, we have the following theorem in the chordal case.

Theorem 2.54. *Let Ω be an upper parallel slit half-plane and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint chordal slits in Ω with $m \in \mathbb{N}$. For each $t \in [0,T]$, $g_t : \Omega_t \to D_t$ denotes the normalised chordal mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ onto the upper parallel slit half-plane D_t.*

Then there is a null set $\mathcal{N}$ of $[0,T]$ such that $t \mapsto g_t(z)$ is differentiable on $[0,T] \setminus \mathcal{N}$ for each $z \in \Omega_T$ and satisfies

$$\dot{g}_t(z) = -\frac{\mathrm{i}}{2} \sum_{k=1}^m \lambda_k(t) \Phi_{\infty,U_k(t),D_t}\big(g_t(z)\big), \quad \text{for all } t \in [0,T] \setminus \mathcal{N} \text{ and all } z \in \Omega_T,$$

where, for each $k \in \{1, \dots, m\}$, the driving term $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0,T]$ and $\lambda_k(t) \geq 0$ for each $t \in [0,T] \setminus \mathcal{N}$. Moreover, $t \mapsto \operatorname{hcap}(g_t)$ is differentiable on $[0,T] \setminus \mathcal{N}$ with derivative $\sum_{k=1}^m \lambda_k(t)$.

Before we are able to prove this theorems, we need a preliminary proposition. Therefore, we introduce some notation as follows.

Let Ω be a canonical domain. Then we set $\Omega^{\mathcal{S}} := \mathbb{D}$ if Ω is a circular slit disk, $\Omega^{\mathcal{S}} := \mathbb{A}_Q$ if Ω is a circular slit annulus with inner radius $Q \in (0,1)$ and $\Omega^{\mathcal{S}} := \mathbb{H}$ if Ω is an upper parallel slit half-plane. We call $\Omega^{\mathcal{S}}$ the *simplification of* Ω. Note that $\Omega^{\mathcal{S}}$ comes out of Ω by erasing all concentric circular arcs of Ω in the radial and bilateral case or by erasing all bounded parallel arcs of Ω in the chordal case. Consequently, $\Omega^{\mathcal{S}}$ is a canonical domain as well whereas $\operatorname{con}(\Omega^{\mathcal{S}}) \leq \operatorname{con}(\Omega)$.

Let $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint appropriate slits in Ω. $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ is a tuple of disjoint appropriate slits in $\Omega^{\mathcal{S}}$ as well. For each $t \in [0,T]$ and $k \in \{1, \ldots, m\}$, we denote by $h_{k;t}$ the normalised appropriate mapping function on $\Omega^{\mathcal{S}} \setminus \gamma_k(0,t]$. Hence, $h_{k;t} : \mathbb{D} \setminus \gamma_k(0,t] \to \mathbb{D}$ in the radial case, $h_{k;t} : \mathbb{A}_Q \setminus \gamma_k(0,t] \to \mathbb{A}_{r_k}$ in the bilateral case and $h_{k;t} : \mathbb{H} \setminus \gamma_k(0,t] \to \mathbb{H}$ in the chordal case.

Proposition 2.55. *Let Ω be canonical domain and $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ be a tuple of disjoint appropriate slits in Ω with $m \in \mathbb{N}$. For each $t, \tau \in [0,T]$ and $k \in \{1, \ldots, m\}$, $f_{k;t,\tau}$ is the normalised appropriate mapping function from $\Omega_k(t,\tau) := \Omega \setminus \big(\bigcup_{j=1,j\neq k}^{m} \gamma_j(0,\tau] \cup \gamma_k(0,t]\big)$ onto $D_k(t,\tau)$. Moreover, for each $t \in [0,T]$ and $k \in \{1, \ldots, m\}$, $h_{k;t}$ denotes the normalised appropriate mapping function on $\Omega^{\mathcal{S}} \setminus \gamma_k(0,t]$. Assume $(\bar{t}_n)_{n\in\mathbb{N}}$, $(\underline{t}_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ are convergent sequences in $[0,T]$ with $\tau_n \to \tau_0$, $\underline{t}_n \to t_0 \leftarrow \bar{t}_n$ and $\underline{t}_n < \bar{t}_n$ for all $n \in \mathbb{N}$. Then*

$$\frac{\mathfrak{c}(f_{k;\bar{t}_n,\tau_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\tau_n})}{\mathfrak{c}(h_{k;\bar{t}_n}) - \mathfrak{c}(h_{k;\underline{t}_n})} \xrightarrow{n\to\infty} |\alpha_k^2(t_0,\tau_0)| \quad \text{for all } k \in \{1, \ldots, m\}.$$

For each $k \in \{1, \ldots, m\}$, $(t,\tau) \mapsto |\alpha_k(t,\tau)| := |(f_{k;t,\tau} \circ h_{k;t}^{-1})'(\Upsilon_k(t))|$, with $\Upsilon_k(t) := h_{k;t}(\gamma_k(t))$, is continuous and positive on $[0,T]^2$.

Note that $\alpha_k(t,\tau) := (f_{k;t,\tau} \circ h_{k;t}^{-1})'(\Upsilon_k(t))$ is well-defined, as $f_{k;t,\tau} \circ h_{k;t}^{-1}$ can be extended analytically to $B_\varepsilon(\Upsilon_k(t))$ with $\varepsilon > 0$ small, see Lemma 2.42. Moreover, see Figure 4.2 illustrating $\alpha_k(t,t)$.

Remark 2.11. In the chordal single slit case a similar result was established by S. Drenning, see Proposition 6.25 in [Dre11], where the proof is based on probabilistic arguments.

Proof of Proposition 2.55. Let $k \in \{1, \ldots, m\}$ be fix. First of all, $f_{k;\bar{t}_n,\tau_n} \circ f_{k;\underline{t}_n,\tau_n}^{-1}$ is the normalised appropriate mapping function from $D_k(\underline{t}_n,\tau_n) \setminus S_{k;\underline{t}_n,\bar{t}_n,\tau_n}$ onto $D_k(\bar{t}_n,\tau_n)$, so we find by Lemma 2.45

$$\mathfrak{c}(f_{k;\bar{t}_n,\tau_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\tau_n}) = \mathfrak{c}(f_{k;\bar{t}_n,\tau_n} \circ f_{k;\underline{t}_n,\tau_n}^{-1}) = -\frac{1}{2\pi} \int\limits_{s_{k;\underline{t}_n,\bar{t}_n,\tau_n}} \Re F(f_{k;\underline{t}_n,\tau_n} \circ f_{k;\bar{t}_n,\tau_n}^{-1})(\zeta)|\mathrm{d}\zeta|.$$

Herein, for each $\underline{t}, \bar{t}, \tau \in [0,T]$ with $\underline{t} < \bar{t}$, $s_{k;\underline{t},\bar{t},\tau} := f_{k;\bar{t},\tau}(\gamma_k[\underline{t},\bar{t}])$ and $S_{k;\underline{t},\bar{t},\tau} := f_{k;\underline{t},\tau}(\gamma_k(\underline{t},\bar{t}])$ are defined like in Lemma 2.43.

Next, we consider the function $R_n := f_{k;\bar{t}_n,\tau_n} \circ h_{k;\bar{t}_n}^{-1}$ and for each $\underline{t}, \bar{t} \in [0,T]$ with $\underline{t} < \bar{t}$, we set $\sigma_{k;\underline{t},\bar{t}} := h_{k;\bar{t}}(\gamma_k[\underline{t},\bar{t}])$ and $\Sigma_{k;\underline{t},\bar{t}} := h_{k;\underline{t}}(\gamma_k(\underline{t},\bar{t}])$. Note that Lemma 2.43 is

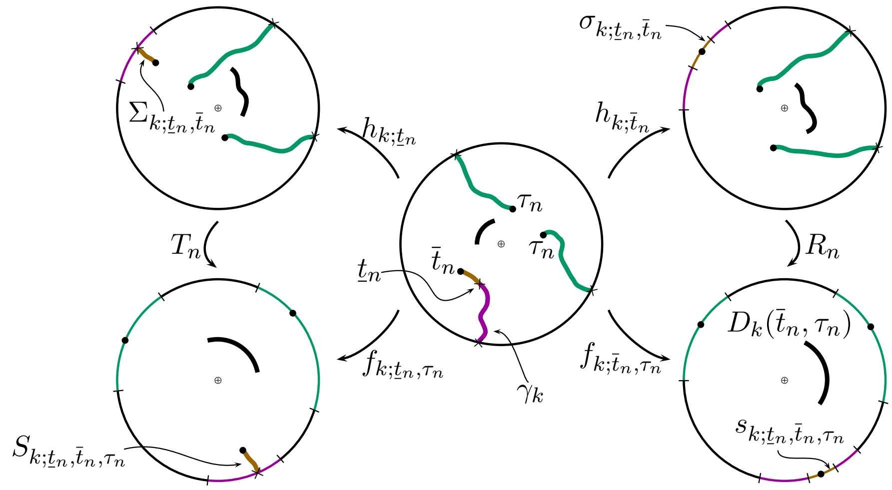

FIGURE 2.10: *Radial mappings $f_{k;\underline{t}_n,\tau_n}$, $h_{k;\underline{t}_n}$, $h_{k;\bar{t}_n}$ and $f_{k;\bar{t}_n,\tau_n}$ in the proof of Proposition 2.55*

applicable to $\Upsilon_k(t)$, $\sigma_{k;\underline{t},\bar{t}}$ and $\Sigma_{k;\underline{t},\bar{t}}$ as well. Lemma 2.42 and 2.44 show that each R_n can be extended analytically to $B_\varepsilon(\Upsilon_k(t_0))$ if $\varepsilon > 0$ is small and n is large enough. $\sigma_{k;\underline{t}_n,\bar{t}_n} \to \Upsilon_k(t_0)$, so we can assume $\sigma_{k;\underline{t}_n,\bar{t}_n} \subseteq B_\varepsilon(\Upsilon_k(t_0))$ for all large n as well. Consequently, we find with an easily substitution and the mean value theorem

$$\begin{aligned}\mathfrak{c}(f_{k;\bar{t}_n,\tau_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\tau_n}) &= -\frac{1}{2\pi}\int\limits_{\sigma_{k;\underline{t}_n,\bar{t}_n}} \Re F(f_{k;\underline{t}_n,\tau_n}\circ f^{-1}_{k;\bar{t}_n,\tau_n}\circ R_n)(\zeta)|R_n'(\zeta)||\mathrm{d}\zeta| \\ &= -\frac{1}{2\pi}|R_n'(\zeta_n)|\int\limits_{\sigma_{k;\underline{t}_n,\bar{t}_n}} \Re F(f_{k;\underline{t}_n,\tau_n}\circ h^{-1}_{k;\bar{t}_n})(\zeta)|\mathrm{d}\zeta|\end{aligned}$$

with $\zeta_n \in \sigma_{k;\underline{t}_n,\bar{t}_n}$. Thus $\zeta_n \to \Upsilon_k(t_0)$, i.e. and $|R_n'(\zeta_n)| \to |\alpha_k(t_0,\tau_0)|$ as R_n tends to $f_{k;t_0,\tau_0}\circ h^{-1}_{k;t_0}$ locally uniformly on $B_\varepsilon(\Upsilon_k(t_0))$, see Lemma 2.42.

Next, let us define $T_n := f_{k;\underline{t}_n,\tau_n}\circ h^{-1}_{k;\underline{t}_n}$. Analogously, we are able to extend T_n analytically to $B_\varepsilon(\Upsilon_{t_0})$ with a small $\varepsilon > 0$ for all large $n \in \mathbb{N}$. Using Lemma 2.42, T_n converges locally uniformly on $B_\varepsilon(\Upsilon_k(t_0))$ to $f_{k;t_0,\tau_0}\circ h^{-1}_{k;t_0}$ as well. On top of this, we find $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that $\Sigma_{k;\underline{t}_n,\bar{t}_n} \subseteq B_{\varepsilon_n}(\Upsilon_k(t_0))$ for all large $n \in \mathbb{N}$, as Lemma 2.43 yields $\Sigma_{k;\underline{t}_n,\bar{t}_n} \to \Upsilon_k(t_0)$. Using Remark 2.10, we find for each $\delta > 0$ an $N \in \mathbb{N}$ such that for all $z \in B_{\varepsilon_n}(\Upsilon_k(t_0))$ and all $n \geq N$, $\left|\frac{\mathrm{d}}{\mathrm{d}z}\big(F(T_n(z)) - |c|F(z)\big)\right| < \delta$ with $c := T_n'(\Upsilon_k(t_0))$. This is based on the fact that $z \mapsto (f_{k;t_0,\tau_0}\circ h^{-1}_{k;t_0})(z)$ as well as $z \mapsto (f_{k;t_0,\tau_0}\circ h^{-1}_{k;t_0})'(z)$ are continuous on $B_\varepsilon(\Upsilon_k(t_0))$ and $T_n \xrightarrow{\text{l.u.}} f_{k;t_0,\tau_0}\circ h^{-1}_{k;t_0}$ on

$B_\varepsilon(\Upsilon_k(t_0))$. Hence, Lemma 2.49 yields

$$-\frac{1}{2\pi}|R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|-\delta\big)\int\limits_{\sigma_{k;\underline{t}_n,\bar{t}_n}}\Re F(h_{k;\underline{t}_n}\circ h_{k;\bar{t}_n}^{-1})(\zeta)|\mathrm{d}\zeta|$$
$$\leq \mathfrak{c}(f_{k;\bar{t}_n,\tau_n})-\mathfrak{c}(f_{k;\underline{t}_n,\tau_n})=-\frac{1}{2\pi}|R_n'(\zeta_n)|\int\limits_{\sigma_{k;\underline{t}_n,\bar{t}_n}}\Re F(T_n\circ h_{k;\underline{t}_n}\circ h_{k;\bar{t}_n}^{-1})(\zeta)|\mathrm{d}\zeta|$$
$$\leq -\frac{1}{2\pi}|R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|+\delta\big)\int\limits_{\sigma_{k;\underline{t}_n,\bar{t}_n}}\Re F(h_{k;\underline{t}_n}\circ h_{k;\bar{t}_n}^{-1})(\zeta)|\mathrm{d}\zeta|.$$

Note that $h_{k;\bar{t}_n}\circ h_{k;\underline{t}_n}^{-1}$ is the normalised appropriate mapping function on $\Omega^{\mathcal{S}}\setminus\Sigma_{k;\underline{t}_n,\bar{t}_n}$, so Lemma 2.45 gives us

$$\begin{aligned}|R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|-\delta\big)\big(\mathfrak{c}(h_{k;\bar{t}_n}\circ h_{k;\underline{t}_n}^{-1})\big)&\leq \mathfrak{c}(f_{k;\bar{t}_n,\tau_n})-\mathfrak{c}(f_{k;\underline{t}_n,\tau_n})\\&\leq R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|+\delta\big)\big(\mathfrak{c}(h_{k;\bar{t}_n}\circ h_{k;\underline{t}_n}^{-1})\big).\end{aligned}$$

Summarising, $\mathfrak{c}(h_{k;\bar{t}_n}\circ h_{k;\underline{t}_n}^{-1})=\mathfrak{c}(h_{k;\bar{t}_n})-\mathfrak{c}(h_{k;\underline{t}_n})>0$ yields

$$|R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|-\delta\big)\leq\frac{\mathfrak{c}(f_{k;\bar{t}_n,\tau_n})-\mathfrak{c}(f_{k;\underline{t}_n,\tau_n})}{\mathfrak{c}(h_{k;\bar{t}_n})-\mathfrak{c}(h_{k;\underline{t}_n})}\leq|R_n'(\zeta_n)|\,\big(\big|T_n'(\Upsilon_k(t_0))\big|+\delta\big).$$

Note that $R_n'(\zeta_n)\to\alpha_k(t_0,\tau_0)$ as well as $T_n'(\Upsilon_k(t_0))\to\alpha_k(t_0,\tau_0)$.

Finally, as a consequence of the univalence on the continuation, $\alpha_k(t,\tau)\neq 0$ for all $t,\tau\in[0,T]$. On top of this, $(t,\tau)\mapsto\alpha_k(t,\tau)$ is continuous on $[0,T]^2$. This follows immediately from Lemma 2.42 and 2.43. □

Now it is very easy to prove the three theorems.

Proof of Theorem 2.52, 2.53 and 2.54. First of all, note that the continuity of $t\mapsto U_k(t)$ follows immediately from Lemma 2.43.

Let Ω be a canonical domain and $(\gamma_1,\dots,\gamma_m)_{t\in[0,T]}$ be a tuple of disjoint appropriate slits in Ω. Obviously, $(\gamma_1,\dots,\gamma_m)$ is a tuple of disjoint appropriate slits in $\Omega^{\mathcal{S}}$ as well where $\Omega^{\mathcal{S}}$ is the simplification of Ω.

As before, for each $k\in\{1,\dots,m\}$ and $t\in[0,T]$, we denote by $h_{k;t}$ the normalised appropriate mapping function on $\Omega^{\mathcal{S}}\setminus\gamma_k(0,t]$. Moreover, for each $k\in\{1,\dots,m\}$ and $t,\tau\in[0,T]$, $f_{k;t,\tau}$ is the normalised appropriate mapping function on $\Omega\setminus(\bigcup_{j\neq k}\gamma_j(0,\tau]\cup\gamma_k(0,t])$.

For now let us fix $k\in\{1,\dots,m\}$. Using Lemma 2.41, the function $t\mapsto\mathfrak{c}(h_{k;t})$ is strictly increasing. Thus we find a null set $\mathcal{N}_k$ such that $t\mapsto\mathfrak{c}(h_{k;t})$ is differentiable on $[0,T]\setminus\mathcal{N}_k$. Let be $t_0\in[0,T]\setminus\mathcal{N}_k$ and denote by $\mu_k(t_0)$ the derivative of $t\mapsto\mathfrak{c}(h_{k;t})$ at t_0. Note that $\mu_k(t_0)\geq 0$. Assume $(t_n)_{n\in\mathbb{N}}\subseteq[0,T]$ is a sequence with $t_n\to t_0$. Using Proposition 2.55 with $\tau_n:=t_0$, $\underline{t}_n:=t_0$ and $\bar{t}_n:=t_n$, we find

$$\begin{aligned}\frac{\mathfrak{c}(f_{k;t_n,t_0})-\mathfrak{c}(f_{k;t_0,t_0})}{t_n-t_0}&=\frac{\mathfrak{c}(f_{k;t_n,t_0})-\mathfrak{c}(f_{k;t_0,t_0})}{\mathfrak{c}(h_{k;t_n})-\mathfrak{c}(h_{k;t_0})}\cdot\frac{\mathfrak{c}(h_{k;t_n})-\mathfrak{c}(h_{k;t_0})}{t_n-t_0}\\&\xrightarrow{n\to\infty}|\alpha_k^2(t_0,t_0)|\cdot\mu_k(t_0)\geq 0.\end{aligned}$$

Consequently, the limit $\lambda_k(t_0) := \lim_{t\to t_0} \frac{\mathfrak{c}(f_{k;t_n,t_0})-\mathfrak{c}(f_{k;t_0,t_0})}{t_n-t_0}$ exists for all $t_0 \in [0,T]\setminus\mathcal{N}_k$.

We can do this for each $k \in \{1,\dots,m\}$, so the limit $\lambda_k(t_0)$ exists for each $k \in \{1,\dots,m\}$ and all $t_0 \in [0,T]\setminus\mathcal{N}$ where $\mathcal{N} := \bigcup_{k=1}^m \mathcal{N}_k$. Finally, we apply Theorem 2.30, 2.31 or 2.36 what completes the proof. □

2.8 A subadditivity property in simply connected domains

The following lemma is well-known, see [Law05], Proposition 3.42.

Lemma 2.56 (Proposition 3.42 in [Law05])**.** *Let $\Omega := \mathbb{H}$ and denote by $\mathfrak{A}$ and $\mathfrak{B}$ two chordal hulls in $\mathbb{H}$ such that $\mathfrak{A}\cup\mathfrak{B}$ is a chordal $\mathbb{H}$-hull as well. Assume $g_{\mathfrak{A}}$, $g_{\mathfrak{B}}$ and $g_{\mathfrak{A}\cup\mathfrak{B}}$ denote the normalised chordal mapping functions on $\mathbb{H}\setminus\mathfrak{A}$, $\mathbb{H}\setminus\mathfrak{B}$ and $\mathbb{H}\setminus(\mathfrak{A}\cup\mathfrak{B})$, respectively. Then*

$$\operatorname{hcap}(g_{\mathfrak{A}\cup\mathfrak{B}}) \leq \operatorname{hcap}(g_{\mathfrak{A}}) + \operatorname{hcap}(g_{\mathfrak{B}}).$$

We have an analogous subadditivity property in the radial case as well.

Lemma 2.57. *Let $\Omega := \mathbb{D}$ and denote by $\mathfrak{A}$ and $\mathfrak{B}$ two radial hulls in $\mathbb{D}$ such that $\mathfrak{A}\cup\mathfrak{B}$ is a radial $\mathbb{D}$-hull as well. Assume $g_{\mathfrak{A}}$, $g_{\mathfrak{B}}$ and $g_{\mathfrak{A}\cup\mathfrak{B}}$ denote the normalised radial mapping functions on $\mathbb{D}\setminus\mathfrak{A}$, $\mathbb{D}\setminus\mathfrak{B}$ and $\mathbb{D}\setminus(\mathfrak{A}\cup\mathfrak{B})$, respectively. Then*

$$\operatorname{lmr}(g_{\mathfrak{A}\cup\mathfrak{B}}) \leq \operatorname{lmr}(g_{\mathfrak{A}}) + \operatorname{lmr}(g_{\mathfrak{B}}).$$

Proof. Let $A := \mathbb{D}\cup\mathbb{T}\cup\{z\in\mathbb{C} \mid 1/z \in \mathfrak{A}\}$ and $B := \mathbb{D}\cup\mathbb{T}\cup\{z\in\mathbb{C} \mid 1/z\in\mathfrak{B}\}$, so A, B are bounded connected compact sets. Using Renggli's theorem in [Ren61], we find

$$\operatorname{cap}(A\cap B)\cdot\operatorname{cap}(A\cup B) \leq \operatorname{cap}(A)\cdot\operatorname{cap}(B), \tag{2.12}$$

whereas cap denotes the logarithmic capacity, see Chapter 9.3 in [Pom92] for a definition. Note that

$$z \mapsto \frac{1}{g_{\mathfrak{A}}^{-1}\left(\frac{1}{z}\right)} = g'_{\mathfrak{A}}(0)z + \mathcal{O}(1) \qquad \text{around } \infty$$

maps $\{|z|>1\}\cup\{\infty\}$ conformally onto $\mathbb{C}_\infty\setminus A$. Using Corollary 9.9. from [Pom92], we find $\operatorname{cap}(A) = g'_{\mathfrak{A}}(0)$. Analogously, we find $\operatorname{cap}(B) = g'_{\mathfrak{B}}(0)$ and $\operatorname{cap}(A\cup B) = g'_{\mathfrak{A}\cup\mathfrak{B}}(0)$. Moreover, using the monotonicity of the logarithmic capacity, see Equation (9) from Chapter 9.3 in [Pom92], $\operatorname{cap}(A\cap B) \geq \operatorname{cap}(\mathbb{T}\cup\mathbb{D}) = 1$, so by applying the logarithm the proof is complete. □

Remark 2.12. As we have seen in the previous proof, Lemma 2.57 is an easy consequence of the strong submultiplicativity of the logarithmic capacity, see Equation (2.12). To the best of our knowledge, the first proof of this property goes back to Renggli, see [Ren61].

Unfortunately, we have the connection (Corollary 9.9, from [Pom92]) between the logarithmic mapping radius lmr and the logarithmic capacity cap only in the case of simply connected domains.

Quite recently O. Roth and D. Kraus found a new proof of the strong submultiplicativity of the Poincaré metric, see [KR14]. This leads to a definition of the so-called *Poincaré capacity* pcap that coincides with the logarithmic capacity cap in the case of simply connected domains and has a strong submultiplicativity property as well, see Remark 1.4 and Corollary 1.2 in [KR14]. Moreover, the Poincaré capacity has a connection with conformal maps (even in multiply connected domains). Unfortunately (for our purpose), an equivalent way to define pcap is related to the universal covering map, so we can not use this result to find a formulation of Lemma 2.57 for multiply connected domains.

Right now we do not know if there is a generalization of Lemma 2.56 and 2.57 to multiply connected domains as well:

Question 1. Do we have a subadditivity property in multiply connected domains as well? This leads to the following three cases:

(i) Let Ω be a circular slit disk, $\mathfrak{A}$ and $\mathfrak{B}$ be radial Ω-hulls such that $\mathfrak{A} \cup \mathfrak{B}$ is a radial Ω-hull as well, and denote by $g_{\mathfrak{A}}$, $g_{\mathfrak{B}}$ and $g_{\mathfrak{A}\cup\mathfrak{B}}$ the normalised radial mapping functions on $\Omega \setminus \mathfrak{A}$, $\Omega \setminus \mathfrak{B}$ and $\Omega \setminus (\mathfrak{A} \cup \mathfrak{B})$, respectively. Do we have

$$\operatorname{lmr}(g_{\mathfrak{A}\cup\mathfrak{B}}) \leq \operatorname{lmr}(g_{\mathfrak{A}}) + \operatorname{lmr}(g_{\mathfrak{B}})?$$

(ii) Let Ω be a circular slit annulus, $\mathfrak{A}$ and $\mathfrak{B}$ be bilateral Ω-hulls such that $\mathfrak{A} \cup \mathfrak{B}$ is a bilateral Ω-hull as well, and denote by $g_{\mathfrak{A}}$, $g_{\mathfrak{B}}$ and $g_{\mathfrak{A}\cup\mathfrak{B}}$ the normalised bilateral mapping functions on $\Omega \setminus \mathfrak{A}$, $\Omega \setminus \mathfrak{B}$ and $\Omega \setminus (\mathfrak{A} \cup \mathfrak{B})$, respectively. Do we have

$$\operatorname{lcm}(g_{\mathfrak{A}\cup\mathfrak{B}}) \leq \operatorname{lcm}(g_{\mathfrak{A}}) + \operatorname{lcm}(g_{\mathfrak{B}})?$$

(iii) Let Ω be an upper parallel slit half-plane, $\mathfrak{A}$ and $\mathfrak{B}$ be chordal Ω-hulls such that $\mathfrak{A} \cup \mathfrak{B}$ is a chordal Ω-hull as well, and denote by $g_{\mathfrak{A}}$, $g_{\mathfrak{B}}$ and $g_{\mathfrak{A}\cup\mathfrak{B}}$ the normalised chordal mapping functions on $\Omega \setminus \mathfrak{A}$, $\Omega \setminus \mathfrak{B}$ and $\Omega \setminus (\mathfrak{A} \cup \mathfrak{B})$, respectively. Do we have

$$\operatorname{hcap}(g_{\mathfrak{A}\cup\mathfrak{B}}) \leq \operatorname{hcap}(g_{\mathfrak{A}}) + \operatorname{hcap}(g_{\mathfrak{B}})?$$

Chapter 3

Constant Coefficients

Let Ω be a canonical domain. A tuple $(\Gamma_1, \ldots, \Gamma_m)$, with $m \in \mathbb{N}$, is called *tuple of disjoint appropriate unparametrised slits in* Ω if there is a $T > 0$ and a tuple $(\gamma_1, \ldots, \gamma_m)_{[0,T]}$ of disjoint appropriate slits in Ω such that $\gamma_k[0,T] = \Gamma_k$ for each $k \in \{1, \ldots, m\}$. In this case, each $(\gamma_1, \ldots, \gamma_m)_{[0,T]}$ is called *admissible parametrisation of* $(\Gamma_1, \ldots, \Gamma_m)$.

Let $m = 1$, i.e. Γ is an appropriate unparametrised slit in the canonical domain Ω. First of all, let $\gamma : [0,T] \to \Gamma$ denote an arbitrary admissible parametrisation of Γ. Moreover, for each $t \in [0,T]$, we denote by $g_t : \Omega \setminus \gamma(0,t] \to D_t$ the normalised appropriate mapping function from $\Omega_t := \Omega \setminus \gamma(0,t]$ onto the canonical domain D_t.

Using Theorem 2.52, 2.53 or 2.54, we find a null set $\mathcal{N}$ such that $t \mapsto g_t(z)$ is differentiable on $[0,T] \setminus \mathcal{N}$ for each $z \in \Omega_T$ and satisfies

$$\dot{g}_t(z) = E\big(g_t(z)\big) \cdot \lambda(t) \cdot \Phi_{a_t,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } t \in [0,T] \setminus \mathcal{N} \text{ and all } z \in \Omega_T,$$

with values $\lambda(t) \geq 0$ for each $t \in [0,T] \setminus \mathcal{N}$ and $U_t := g_t(\gamma(t))$. Herein, $a_t := 0$ in the radial case, a_t is the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case. Moreover, for all $w \in \mathbb{C}$, we set $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2i}$ in the chordal case.

One might ask the natural question if there is a reparametrisation $v(s) : [0,L] \to [0,T]$ with $L > 0$ such that $s \mapsto g_{v(s)}$ is (everywhere) differentiable on $[0,L]$ with '*nice*' values $\lambda(t)$. In the single slit case we may argue as follows: Using Lemma 2.42 and 2.41, we see that $t \mapsto \mathfrak{c}(g_t)$ is strictly increasing and continuous with $\mathfrak{c}(g_0) = 0$ and $\mathfrak{c}(g_T) =: L$. Let $v^{-1}(t) := \mathfrak{c}(g_t)$ for all $t \in [0,T]$, so $\mathfrak{c}(g_{v(s)}) = s$ for all $s \in [0,L]$. Then Theorem 2.22 yields that $s \mapsto g_{v(s)}(z)$ is differentiable on $[0,L]$ for each $z \in \Omega \setminus \Gamma$ with

$$\frac{\mathrm{d}}{\mathrm{d}s} g_{v(s)}(z) = E\big(g_{v(s)}(z)\big) \cdot \Phi_{0,U_{v(s)},D_{v(s)}}\big(g_{v(s)}(z)\big) \qquad \text{for all } s \in [0,L] \text{ and all } z \in \Omega \setminus \Gamma,$$

where $s \mapsto U_{v(s)} = g_{v(s)}(\gamma(v(s)))$ is continuous on $[0,L]$. Note that the reparametrisation $v(s)$ is unique with respect of getting $\lambda \equiv 1$.

Summarising, we have the following corollary.

Corollary 3.1. *Let* Ω *be a canonical domain and denote by* Γ *an appropriate unparametrised slit in* Ω*. Then there is a unique* $L > 0$ *and a unique admissible parametrisation*

$\gamma : [0, L] \to \Gamma$ of Γ such that for each $z \in \Omega \setminus \Gamma$, $t \mapsto g_t(z)$ is continuously differentiable on $[0, L]$ and satisfies

$$\dot{g}_t(z) = E\big(g_t(z)\big) \cdot \Phi_{a_t,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } t \in [0, L] \text{ and all } z \in \Omega \setminus \Gamma$$

where, for each $t \in [0, L]$, g_t is the normalised appropriate mapping function from $\Omega \setminus \gamma(0, t]$ onto D_t and $U_t := g_t(\gamma(t))$ is the continuous driving term. Herein, $a_t := 0$ in the radial case, a_t is the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case. Moreover, for all $w \in \mathbb{C}$, we set $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case.

Sometimes this parametrisation is called *Loewner parametrisation of* Γ.

The follow-up question is: what if we do have more than one slit, i.e. $m > 1$? Do we have parametrisations like the Loewner parametrisation in the single slit case? We will give an answer to this question in the following section.

3.1 Disjoint slits

Theorem 3.2. *Let Ω be a circular slit disk and (Γ_1, Γ_2) be a tuple of disjoint radial unparametrised slits in Ω. Then there is a unique $L > 0$, unique (constants) $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, and a unique admissible parametrisation $(\gamma_1, \gamma_2)_{t\in[0,L]}$ of (Γ_1, Γ_2) such that for each $z \in \Omega_L$, $t \mapsto g_t(z)$ is continuously differentiable on $[0, L]$ and satisfies*

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^{2} \lambda_k \Phi_{0,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } t \in [0, L] \text{ and all } z \in \Omega_L.$$

Herein, for each $t \in [0, L]$, g_t is the normalised radial mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^{2} \gamma_k(0, t]$ onto the circular slit disk D_t. Moreover, for each $k \in \{1, 2\}$, the driving function $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0, L]$.

Remark 3.1. As mentioned already in the introduction, this theorem goes back to Prokhorov, see Theorem F. He considered the simply connected case, i.e. $\Omega = \mathbb{D}$ and piecewise analytic slits Γ_1, Γ_2, see Theorem 1 and 2 in [Pro93].

Theorem 3.3. *Let Ω be a circular slit annulus with inner radius $Q \in (0, 1)$ and (Γ_1, Γ_2) be a tuple of disjoint bilateral unparametrised slits in Ω. Then there is a unique $L > 0$, unique (constants) $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, and a unique admissible parametrisation $(\gamma_1, \gamma_2)_{t\in[0,L]}$ of (Γ_1, Γ_2) such that, for each $z \in \Omega_L$, $t \mapsto g_t(z)$ is continuously differentiable on $[0, L]$ and satisfies*

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^{2} \lambda_k \Phi_{q_t,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } t \in [0, L] \text{ and all } z \in \Omega_L.$$

Herein, for each $t \in [0, L]$, g_t is the normalised bilateral mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^{2} \gamma_k(0, t]$ onto the circular slit annulus D_t. q_t is the inner radius of D_t, $t \in [0, L]$, and for each $k \in \{1, 2\}$, the driving function $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0, L]$.

Theorem 3.4. *Let Ω be an upper parallel slit half-plane and (Γ_1, Γ_2) be a tuple of disjoint chordal unparametrised slits in Ω. Then there is a unique $L > 0$, unique (constants) $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, and a unique admissible parametrisation $(\gamma_1, \gamma_2)_{t\in[0,L]}$ of (Γ_1, Γ_2) such that for each $z \in \Omega_L$, $t \mapsto g_t(z)$ is continuously differentiable on $[0, L]$ and satisfies*

$$\dot{g}_t(z) = -\frac{\mathrm{i}}{2}\sum_{k=1}^{2} \lambda_k \Phi_{\infty,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } t \in [0, L] \text{ and all } z \in \Omega_L.$$

Herein, for each $t \in [0, L]$, g_t is the normalised chordal mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^{2} \gamma_k(0, t]$ onto the upper parallel slit half-plane D_t. Moreover, for each $k \in \{1, 2\}$, the driving function $t \mapsto U_k(t) := g_t(\gamma_k(t))$ is continuous on $[0, L]$.

Remark 3.2. O. Roth and S. Schleißinger found the first proof of Theorem 3.4 in case of simply connected domains without assuming piecewise analytic slits, see [RS14]. During a summer school in Sevilla ('Complex Analysis and Related Areas', February 2013) Sebastian Schleißinger presented their proof. This was the beginning of a collaboration of S. Schleißinger and the author of this thesis, see [BS15a]. In this context, ideas from [RS14] were combined with methods from [BL14] resulting in a proof of Theorem 3.2. The advantage of this approach is that the proof is universal, in the sense that the proof in the radial, bilateral and chordal case differs not really. See Subsection 3.1.2 where we prove Theorem 3.2, 3.3 and 3.4 simultaneously.

Remark 3.3. Note that Theorem 3.2, 3.3 and 3.4 prove the existence and uniqueness of constant coefficients for two disjoint unparametrised slits (Γ_1, Γ_2). One might ask the question if the same is true for more than two disjoint unparametrised slits $(\Gamma_1, \dots, \Gamma_m)$ with $m > 2$. Following the steps of the existence proof (see Subsection 3.1.2) we can see that the existence of constant coefficients can be received in the same way as in the two slit case. Unfortunately, the uniqueness of constant coefficients in the case $m > 2$ can not be reasoned in the same way as in two slit case. Moreover, we do not know how to prove the uniqueness otherwise, so there is still the following open problem.

Question 2. Is there a similar result of Theorem 3.2, 3.3 and 3.4 in the case of more than two slits?

Finally, let us mention [Sch15], Subsection 3.6.5 with a lot of useful remarks about constant coefficients in the simply connected chordal case. Most of these remarks hold in the multiply connected cases as well.

3.1.1 Some preliminary lemmas

Lemma 3.5. *Let Ω be a canonical domain and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ denotes a tuple of disjoint appropriate slits in Ω. For each $k \in \{1, \dots, m\}$ and $t, \tau \in [0, T]$, $f_{k;t,\tau}$ denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0, t] \bigcup_{j\neq k} \gamma_j(0, \tau])$.*

Then for each $\varepsilon > 0$, we find a $\delta > 0$ such that

$$1 - \varepsilon \leq \frac{\mathfrak{c}(f_{k;\bar{t},\bar{\tau}}) - \mathfrak{c}(f_{k;\underline{t},\bar{\tau}})}{\mathfrak{c}(f_{k;\bar{t},\underline{\tau}}) - \mathfrak{c}(f_{k;\underline{t},\underline{\tau}})} \leq 1 + \varepsilon$$

for all $\underline{t}, \bar{t}, \underline{\tau}, \bar{\tau} \in [0, T]$ with $0 < \bar{t} - \underline{t} < \delta$ and $0 \leq \bar{\tau} - \underline{\tau} < \delta$ and all $k \in \{1, \dots, m\}$.

Proof. Suppose the opposite is true, so there is a $k \in \{1, \ldots, m\}$ and sequences $(\underline{t}_n)_{n\in\mathbb{N}}$, $(\bar{t}_n)_{n\in\mathbb{N}}$, $(\underline{\tau}_n)_{n\in\mathbb{N}}$ and $(\overline{\tau}_n)_{n\in\mathbb{N}}$ with $\underline{t}_n < \bar{t}_n$ and $\underline{\tau}_n \le \overline{\tau}_n$ such that

$$\left| \frac{\mathfrak{c}(f_{k;\bar{t}_n,\overline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\overline{\tau}_n})}{\mathfrak{c}(f_{k;\bar{t}_n,\underline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\underline{\tau}_n})} - 1 \right| > \varepsilon$$

for all $n \in \mathbb{N}$. Obviously, we can assume without loss of generality that each sequence is convergent, i.e. $\underline{t}_n \to t_0 \leftarrow \bar{t}_n$ and $\underline{\tau}_n \to \tau_0 \leftarrow \overline{\tau}_n$. For all $k \in \{1, \ldots, m\}$ and all $t \in [0,T]$, let us denote by $h_{k;t}$ the normalised appropriate mapping functions on $\Omega \setminus \gamma_k(0,t]$. Using Proposition 2.55, we find

$$\frac{\mathfrak{c}(f_{k;\bar{t}_n,\overline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\overline{\tau}_n})}{\mathfrak{c}(h_{k;\bar{t}_n}) - \mathfrak{c}(h_{k;\underline{t}_n})} \xrightarrow{n\to\infty} |\alpha_k^2(t_0,\tau_0)| \xleftarrow{n\to\infty} \frac{\mathfrak{c}(f_{k;\bar{t}_n,\underline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\underline{\tau}_n})}{\mathfrak{c}(h_{k;\bar{t}_n}) - \mathfrak{c}(h_{k;\underline{t}_n})}.$$

Consequently, we get

$$\frac{\mathfrak{c}(f_{k;\bar{t}_n,\overline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\overline{\tau}_n})}{\mathfrak{c}(f_{k;\bar{t}_n,\underline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\underline{\tau}_n})} = \frac{\mathfrak{c}(f_{k;\bar{t}_n,\overline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\overline{\tau}_n})}{\mathfrak{c}(h_{k;\bar{t}_n}) - \mathfrak{c}(h_{k;\underline{t}_n})} \cdot \frac{\mathfrak{c}(h_{k;\bar{t}_n}) - \mathfrak{c}(h_{k;\underline{t}_n})}{\mathfrak{c}(f_{k;\bar{t}_n,\underline{\tau}_n}) - \mathfrak{c}(f_{k;\underline{t}_n,\underline{\tau}_n})} \xrightarrow{n\to\infty} 1.$$

This is a contradiction, so the proof is complete. □

Lemma 3.6. *Let Ω be a canonical domain and $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ denotes a tuple of disjoint appropriate slits in Ω. For each $k \in \{1, \ldots, m\}$ and $t, \tau \in [0,T]$, $f_{k;t,\tau}$ is the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0,t] \bigcup_{j\neq k} \gamma_j(0,\tau])$. Moreover, we set $g_t := f_{k;t,t}$ for each $t \in [0,T]$ independently of $k \in \{1, \ldots, m\}$. Assume $Z = \{t_0, \ldots, t_n\}$, with $t_0 = 0$ and $t_n = t$, is a partition of the interval $[0,t]$, i.e. $t_0 < t_1 < \ldots < t_n$ and*

$$\mathcal{S}(f_k, t, Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l}).$$

Then for each $t \in [0,T]$ and $k \in \{1, \ldots, m\}$, $\mathcal{S}(f_k, t, Z) \to c_k(t) \ge 0$ as $|Z| \to 0$, whereas $|Z|$ denotes the norm of the partition Z, i.e. $|Z| := \max_{l=0,\ldots,n-1} t_{l+1} - t_l$. Moreover, each $t \mapsto c_k(t)$, $k \in \{1, \ldots, m\}$, is continuous and strictly increasing on $[0,T]$, and for each $t_0 \in [0,T]$,

$$\frac{c_k(t) - c_k(t_0)}{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})} \to 1 \qquad \text{as } t \to t_0.$$

Finally, assume $\mathfrak{c}(g_t) = t$ for all $t \in [0,T]$. Then each $t \mapsto c_k(t)$, $k \in \{1, \ldots, m\}$, is Lipschitz continuous on $[0,T]$ and $\sum_{k=1}^m c_k(t) = t$ for all $t \in [0,T]$.

Proof. 1) First of all, we are going to show $\mathcal{S}(f_k, t, Z) \to c_k(t)$ as $|Z| \to 0$. Therefore, let $k \in \{1, \ldots, m\}$ and $t \in [0,T]$ be fix. Let us consider two partitions $Z_1 = \{t_0^*, \ldots, t_{n_1}^*\}$ and Z_2 of the interval $[0,t]$ with $|Z_1|, |Z_2| < \delta$ where $\delta > 0$. Denote by $Z = \{t_0, \ldots, t_n\}$ the union of Z_1 and Z_2. By adding zeros we achieve

$$|\mathcal{S}(f_k,t,Z) - \mathcal{S}(f_k,t,Z_1)| \le \sum_{l=0}^{n-1} \left| [\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})] - [\mathfrak{c}(f_{k;t_{l+1},\phi(t_l)}) - \mathfrak{c}(f_{k;t_l,\phi(t_l)})] \right|,$$

where $\phi(t_l) := t_p^*$ if $t_l \in [t_p^*, t_{p+1}^*)$ with $l \in \{0, \dots, n-1\}$ and $p \in \{0, \dots, n_1 - 1\}$. Consequently, $|\phi(t_l) - t_l| \le |Z_1| \le \delta$. Thus we get

$$|\mathcal{S}(f_k, t, Z) - \mathcal{S}(f_k, t, Z_1)| \le \sum_{l=0}^{n-1} |\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})| \cdot \left| 1 - \frac{\mathfrak{c}(f_{k;t_{l+1},\phi(t_l)}) - \mathfrak{c}(f_{k;t_l,\phi(t_l)})}{\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})} \right|.$$

For any given $\varepsilon > 0$, we can choose $\delta > 0$ (by using Lemma 3.5) in such a way that

$$1 - \varepsilon < \frac{\mathfrak{c}(f_{k;t_{l+1},\phi(t_l)}) - \mathfrak{c}(f_{k;t_l,\phi(t_l)})}{\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})} < 1 + \varepsilon$$

holds for all $l \in \{0, \dots, n-1\}$. Lemma 2.41 gives us $\mathfrak{c}(f_{k;t_{l+1},t_{l+1}}) > \mathfrak{c}(f_{k;t_{l+1},t_l}) > \mathfrak{c}(f_{k;t_l,t_l})$ for all $l \in \{1, \dots, n-1\}$. Thus we have

$$\begin{aligned}
|\mathcal{S}(f_k, t, Z) - \mathcal{S}(f_k, t, Z_1)| &\le \varepsilon \sum_{l=0}^{n-1} \big(\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l}) \big) \\
&< \varepsilon \sum_{l=0}^{n-1} \big(\mathfrak{c}(f_{k;t_{l+1},t_{l+1}}) - \mathfrak{c}(f_{k;t_l,t_l}) \big) \\
&= \varepsilon \cdot \big(\mathfrak{c}(f_{k;t_n,t_n}) - \mathfrak{c}(f_{k;t_0,t_0}) \big) = \varepsilon \cdot \big(\mathfrak{c}(g_t) - \mathfrak{c}(g_0) \big).
\end{aligned}$$

Replacing Z_1 with Z_2 we get $|\mathcal{S}(f_k, t, Z) - \mathcal{S}(f_k, t, Z_2)| \le \varepsilon\big(\mathfrak{c}(g_t) - \mathfrak{c}(g_0)\big)$ as well. Consequently, we have $|\mathcal{S}(f_k, t, Z_1) - \mathcal{S}(f_k, t, Z_2)| \le 2\varepsilon\big(\mathfrak{c}(g_t) - \mathfrak{c}(g_0)\big)$, so $\mathcal{S}(f_k, t, Z)$ converges to a value $c_k(t) \in [0, \infty)$ if $|Z| \to 0$.

2) Next, we are going to prove that $t \mapsto c_k(t)$ is strictly increasing. Therefore, we fix $k \in \{1, \dots, m\}$. Let $\varepsilon > 0$, $t_0 \in [0, T]$ and choose $t \in [0, T]$ in such a way that $0 < |t - t_0| < \delta$ where $\delta > 0$ is chosen according to Lemma 3.5 with respect to ε. Assume $Z = \{t_k := t_0 + \frac{k}{n}(t - t_0) \mid k \in \{0, \dots n\}\}$. Thus we get

$$\begin{aligned}
&\left| \mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0}) - \sum_{l=0}^{n-1} \mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l}) \right| \\
&= \left| \sum_{l=0}^{n-1} \big([\mathfrak{c}(f_{k;t_{l+1},t_0}) - \mathfrak{c}(f_{k;t_l,t_0})] - [\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})] \big) \right| = *
\end{aligned}$$

where the first equality follows by adding zeros. Using Lemma 3.5, we find

$$\begin{aligned}
* &\le \sum_{l=0}^{n-1} |\mathfrak{c}(f_{k;t_{l+1},t_0}) - \mathfrak{c}(f_{k;t_l,t_0})| \cdot \left| 1 - \frac{\mathfrak{c}(f_{k;t_{l+1},t_l}) - \mathfrak{c}(f_{k;t_l,t_l})}{\mathfrak{c}(f_{k;t_{l+1},t_0}) - \mathfrak{c}(f_{k;t_l,t_0})} \right| \\
&< \varepsilon \sum_{l=0}^{n-1} \big| \mathfrak{c}(f_{k;t_{l+1},t_0}) - \mathfrak{c}(f_{k;t_l,t_0}) \big|.
\end{aligned}$$

Letting $n \to \infty$ gives us

$$\big| c_k(t) - c_k(t_0) - \big(\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0}) \big) \big| < \varepsilon |\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})|.$$

Consequently, we find

$$\left|1 - \frac{c_k(t) - c_k(t_0)}{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}\right| \to 0 \tag{3.1}$$

as $t \to t_0$ and $\varepsilon \to 0$ simultaneously such that $|t - t_0| < \delta(\varepsilon)$ with $\delta(\varepsilon) > 0$ depending on ε, see Lemma 3.5. Moreover, this shows that $t \mapsto c_k(t)$ can not be constant on a small interval $[t, t_0]$ or $[t_0, t]$, as $t \mapsto \mathfrak{c}(f_{k;t,t_0})$ is strictly increasing on $[0, T]$. Otherwise the previous limit would not be zero.

3) Now we will show that the function $t \mapsto c_k(t)$ is continuous. Let $k \in \{1, \dots, m\}$, $0 < t_1 < t_2 \le T$, $Z_1(n) := \{0, \frac{t_1}{n}, \frac{2t_1}{n}, \dots, t_1\}$, $Z_2(n) := \{t_1, t_1 + \frac{t_2-t_1}{n}, t_1 + 2\frac{t_2-t_1}{n}, \dots, t_2\} =: \{t_0^*, \dots, t_n^*\}$ with $t_1 = t_0^* < t_1^* < \dots < t_n^* = t_2$, and $Z(n) := Z_1(n) \cup Z_2(n)$. Thus we have

$$\begin{aligned} c_k(t_2) - c_k(t_1) &= \lim_{n\to\infty} \mathcal{S}\big(f_k, t_1, Z(n)\big) - \mathcal{S}\big(f_k, t_2, Z_1(n)\big) \\ &= \lim_{n\to\infty} \sum_{l=0}^{n-1} \mathfrak{c}(f_{k;t_{l+1}^*,t_l^*}) - \mathfrak{c}(f_{k;t_l^*,t_l^*}) < \sum_{l=0}^{n-1} \mathfrak{c}(f_{k;t_{l+1}^*,t_{l+1}^*}) - \mathfrak{c}(f_{k;t_l^*,t_l^*}) = \mathfrak{c}(g_{t_2}) - \mathfrak{c}(g_{t_1}). \end{aligned}$$

Note that $t \mapsto \mathfrak{c}(g_t)$ is continuous on $[0, T]$, see Lemma 2.42. Consequently, $t \mapsto c_k(t)$ is a continuous real-valued function.

Next, let us assume $\mathfrak{c}(g_t) = t$ for all $t \in [0, T]$. Using the previous estimation, $\mathfrak{c}(g_{t_2}) - \mathfrak{c}(g_{t_1}) = t_2 - t_1$, so $t \mapsto c_k(t)$ is Lipschitz continuous on $[0, T]$. Consequently, $t \mapsto c_k(t)$ is almost everywhere differentiable, i.e. there is a null set $\mathcal{N}_k$ such that $t \mapsto c_k(t)$ is differentiable on $[0, T] \setminus \mathcal{N}_k$. Moreover, Equation (3.1) gives us

$$\lambda_k(t_0) := \lim_{t\to t_0} \frac{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}{t - t_0} = \dot{c}_k(t_0) \qquad \text{for all } t_0 \in [0, T] \setminus \mathcal{N}_k.$$

Obviously, we get this for each $k \in \{1, \dots, m\}$, so each limit $\lambda_k(t_0)$, $k \in \{1, \dots, m\}$, exits for all $t_0 \in [0, T] \setminus \mathcal{N}$ with $\mathcal{N} := \bigcup_{k=1}^m \mathcal{N}_k$. Using Theorem 2.30, 2.31 or 2.36, we find $\sum_{k=1}^m \lambda_k(t_0) = 1$ for all $t \in [0, T] \setminus \mathcal{N}$. Summarising, $\sum_{k=1}^m \dot{c}_k(t_0) \equiv 1$ for all $t_0 \in [0, T] \setminus \mathcal{N}$, so $\sum_{k=1}^m c_k(t) = t$ for all $t \in [0, T]$.

□

Remark 3.4. Note that Lemma 3.6 leads to an alternative proof of Theorem 2.52, 2.53 and 2.54:

Let Ω be a canonical domain and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ denotes a tuple of disjoint appropriate slits in Ω. Using the same notation as in Lemma 3.6, for each $k \in \{1, \dots, m\}$, we get an increasing functions $t \mapsto c_k(t)$ on $[0, T]$. Consequently, each $t \mapsto c_k(t)$, $k \in \{1, \dots, m\}$, is differentiable on $[0, T] \setminus \mathcal{N}_k$ where $\mathcal{N}_k$ is a null set of $[0, T]$. Summarising, $t \mapsto c_k(t)$ is differentiable on $[0, T] \setminus \mathcal{N}$ for each $k \in \{1, \dots, m\}$ where $\mathcal{N} := \bigcup_{k=1}^m \mathcal{N}_k$. Using Lemma 3.6, each limit

$$\lim_{t\to t_0} \frac{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}{t - t_0}, \qquad k \in \{1, \dots, m\},\ t_0 \in [0, T] \setminus \mathcal{N}$$

exists. Finally, Theorem 2.30, 2.31 or 2.36 completes the proof.

3.1.2 Proof of Theorem 3.2, 3.3 and 3.4

Let Ω be a canonical domain and $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be an admissible parametrisation of (Γ_1, Γ_2). Moreover, for each $t, \tau \in [0, T]$, we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$ and we define $g_t := f_{t,t}$ for all $t \in [0, T]$. On top of this we set $\mathfrak{c}(t, \tau) := \mathfrak{c}(f_{t,\tau})$, so $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$ is strictly increasing in each variable and continuous on $[0, T]^2$, see Lemma 2.41 and 2.42.

Note that all functions $t \mapsto g_t$ that satisfy a multiple slit Komatu–Loewner equation with normalised weights, i.e. for each $t \in [0, T]$, $\lambda_1(t) + \lambda_2(t) \equiv 1$, fulfil $\mathfrak{c}(g_t) = t$ for all $t \in [0, T]$, see Theorem 2.30, 2.31 or 2.36.

With the notation from Theorem 3.2, 3.3 and 3.4, we find $L = \mathfrak{c}(T, T)$ independently of T.

Let $u, v : [0, L] \to [0, T]$ be increasing homeomorphisms, $\underline{t}, \bar{t} \in [0, L]$, $\underline{t} < \bar{t}$, and Z denotes an arbitrary partition of the interval $[\underline{t}, \bar{t}]$. During this subsection we will use the following abbreviations

$$\begin{aligned} \mathcal{S}_1(u, v, [\underline{t}, \bar{t}], Z) &:= \sum_{l=0}^{n-1} \mathfrak{c}\big(u(t_{l+1}), v(t_l)\big) - \mathfrak{c}\big(u(t_l), v(t_l)\big), \\ \mathcal{S}_2(u, v, [\underline{t}, \bar{t}], Z) &:= \sum_{l=0}^{n-1} \mathfrak{c}\big(u(t_l), v(t_{l+1})\big) - \mathfrak{c}\big(u(t_l), v(t_l)\big). \end{aligned} \tag{3.2}$$

Moreover, we set $\mathcal{S}_k(u, v, t, Z) := \mathcal{S}_k(u, v, [0, t], Z)$ with $k \in \{1, 2\}$ and a partition Z of the interval $[0, t]$. Note that for each $k \in \{1, 2\}$, $t \mapsto \mathcal{S}_k(u, v, t, Z)$ tends pointwise to an increasing and continuous function on $[0, L]$ if $|Z| \to 0$, see Lemma 3.6.

In order to proof Theorem 3.2, 3.3 and 3.4, we split the proof into the existence part and the uniqueness part. We will discuss both parts separately. First, we will prove the existence part. In this context, we are going to show that we find increasing homeomorphisms $u, v : [0, L] \to [0, T]$ such that the admissible parametrisation $(\gamma_1 \circ u, \gamma_2 \circ v)_{t\in[0,L]}$ satisfies a Komatu–Loewner equation with $\lambda_1(t) = \lambda_0$ and $\lambda_2(t) = 1 - \lambda_0$ for all $t \in [0, L]$. The proceeding of this proof is as follows.

1) First of all, we will use a Bang-Bang method introduced in [RS14] to construct two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of increasing homeomorphisms of $u_n, v_n : [0, L] \to [0, T]$.

2) By using a diagonal argument on u_n and v_n, we will find two subsequences $(u_n^*)_{n\in\mathbb{N}}$ and $(v_n^*)_{n\in\mathbb{N}}$ which converge pointwise on a dense set $S \subseteq [0, L]$ to increasing functions u and v respectively. The functions u and v can be extended to continuous functions defined on $[0, L]$ with $u(L) = T = v(L)$. Furthermore, we will get $\lambda_0 \in [0, 1]$ by the construction of u and v.

3) On top of this we show $\lambda_0 \in (0, 1)$ and the strict monotonicity of $t \mapsto u(t)$ and $t \mapsto v(t)$ on $[0, L]$.

4) Next, we will derive a connection between the sum $\mathcal{S}_1(u_n^*, v_n^*, t, Z)$ and the sum $\mathcal{S}_1(u, v, t, Z)$ for a given partition Z of the interval $[0, t]$.

5) Moreover, we will find a connection between $\mathcal{S}_1(u_n^*, v_n^*, t, Z)$ and λ_0.

6) By combining these results, we will find $\mathcal{S}_1(u, v, t, Z) \to \lambda_0 t$ if $|Z| \to 0$.

7) Finally, we will obtain a Komatu-Loewner equation with constant coefficients λ_0 and $1 - \lambda_0$ for the admissible parametrisations u and v.

Proof (Existence). 1) Let $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be an arbitrary admissible parametrisation of (Γ_1, Γ_2). Moreover, for each $t, \tau \in [0, T]$, we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$. On top of this we write $\mathfrak{c}(t, \tau) := \mathfrak{c}(f_{t,\tau})$. Assume $u, v : [0, L] \to [0, T]$ are increasing homeomorphisms and Z denotes an arbitrary partition of the interval $[\underline{t}, \bar{t}] \subseteq [0, T]$. Then for each $k \in \{1, 2\}$, we define $\mathcal{S}_k(u, v, [\underline{t}, \bar{t}], Z)$ in the same way as in Equation (3.2) and we set $\mathcal{S}_k(u, v, t, Z) := \mathcal{S}_k(u, v, [0, t], Z)$ for any partition Z of the interval $[0, t] \subseteq [0, T]$. To construct u_n and v_n, we first extend γ_1 and γ_2 to an interval $[0, T^*]$, $T^* > T$, such that $\gamma_1[0, T^*]$ and $\gamma_2[0, T^*]$ are still disjoint slits and $\mathfrak{c}(T^*, 0) \geq L$, $\mathfrak{c}(0, T^*) \geq L$. Let $n \in \mathbb{N}$ and $\lambda \in [0, 1]$. We let $t_{0,n} = \tau_{0,n} = 0$, and for $k \in \{1, \dots, n\}$, we define $t_{k,n} > 0$ and $\tau_{k,n} > 0$ recursively as the unique values with

$$\mathfrak{c}(t_{k,n}, \tau_{k-1,n}) - \mathfrak{c}(t_{k-1,n}, \tau_{k-1,n}) = L\frac{\lambda}{n}, \quad \mathfrak{c}(t_{k,n}, \tau_{k,n}) - \mathfrak{c}(t_{k,n}, \tau_{k-1,n}) = L\frac{1-\lambda}{n}.$$

Since $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$ is strictly increasing in both variables, see Lemma 2.41, we get

$$\begin{aligned} \mathfrak{c}(t_{n,n}, \tau_{n,n}) &= L \leq \mathfrak{c}(T^*, 0) < \mathfrak{c}(T^*, \tau_{n,n}), \\ \mathfrak{c}(t_{n,n}, \tau_{n,n}) &= L \leq \mathfrak{c}(0, T^*) < \mathfrak{c}(t_{n,n}, T^*). \end{aligned}$$

Consequently, $t_{n,n}, \tau_{n,n} \leq T^*$.
Furthermore, note that the values $t_{k,n} = t_{k,n}(\lambda)$ and $\tau_{k,n} = \tau_{k,n}(\lambda)$ depend continuously on λ: This follows easily by induction and the continuity and strict monotonicity of the function $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$, see Lemma 2.42 and 2.41. Consequently, for every $n \in \mathbb{N}$, we find a value $\lambda_n \in (0, 1)$ with $t_{n,n}(\lambda_n) = T$. Now we define functions $u_n : [0, L] \to [0, t_{n,n}]$ and $v_n : [0, L] \to [0, \tau_{n,n}]$. For each $n \in \mathbb{N}$, we set

$$u_n\Big(L\frac{k}{2^n}\Big) := t_{k,2^n}(\lambda_{2^n}), \quad v_n\Big(L\frac{k}{2^n}\Big) := \tau_{k,2^n}(\lambda_{2^n})$$

for all $k \in \{0, \dots, 2^n\}$. The values of u_n and v_n between the supporting points are defined by linear interpolation. An immediate consequence of this construction is

$$\mathfrak{c}\bigg(u_n\Big(L\frac{k}{2^n}\Big), v_n\Big(L\frac{k}{2^n}\Big)\bigg) = \mathfrak{c}\big(t_{k,2^n}(\lambda_{2^n}), \tau_{k,2^n}(\lambda_{2^n})\big) = L\frac{k}{2^n}. \tag{3.3}$$

for all $k \in \{0, \dots, 2^n\}$.

2) λ_{2^n} is bounded, so we find a subsequence $(m_{k,0})_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that $(\lambda_{2^{m_{k,0}}})_{k\in\mathbb{N}}$ is convergent with the limit $\lambda_0 \in [0, 1]$. Next, we set

$$S := \bigcup_{n=1}^{\infty} S_n, \quad S_n := \Big\{L\frac{k}{2^n} \,\big|\, k \in \{0, \dots, 2^n\}\Big\}.$$

S is a dense and countable subset of $[0, L]$. Denote by $a : \mathbb{N} \to S$ a bijective mapping. Since the sequences $(u_{m_{k,0}}(a_1))_{k\in\mathbb{N}}$ and $(v_{m_{k,0}}(a_1))_{k\in\mathbb{N}}$ are bounded (by T^*), we find a subsequence $(m_{k,1})_{k\in\mathbb{N}}$ of $(m_{k,0})_{k\in\mathbb{N}}$ such that $(u_{m_{k,1}}(a_1))_{k\in\mathbb{N}}$ and $(v_{m_{k,1}}(a_1))_{k\in\mathbb{N}}$ are convergent.

Inductively, we define $(m_{k,l})_{k\in\mathbb{N}}$, $l \in \mathbb{N}$, to be a subsequence of $(m_{k,l-1})_{k\in\mathbb{N}}$ such that $(u_{m_{k,l}}(a_l))_{k\in\mathbb{N}}$ and $(v_{m_{k,l}}(a_l))_{k\in\mathbb{N}}$ are convergent.

Consequently, we define sequences $u_n^* := u_{m_{n,n}}$ and $v_n^* := v_{m_{n,n}}$, which are (pointwise) convergent on S. We denote by u and v the limit functions, i.e.

$$u(t) := \lim_{n\to\infty} u_n^*(t), \quad v(t) := \lim_{n\to\infty} v_n^*(t) \qquad \text{for all } t \in S.$$

Moreover, we set $\lambda_n^* := \lambda_{2^{m_{n,n}}}$ and $S_n^* := S_{m_{n,n}}$. By using Equation (3.3), we get $\mathfrak{c}\big(u_n^*(t), v_n^*(t)\big) = t$ for $t \in S$ if n is big enough. Consequently, Lemma 2.42 yields

$$\mathfrak{c}\big(u(t), v(t)\big) = \lim_{n\to\infty} \mathfrak{c}\big(u_n^*(t), v_n^*(t)\big) = t \qquad \text{for all } t \in S. \tag{3.4}$$

Furthermore, since $t \mapsto u_n^*(t)$ and $t \mapsto v_n^*(t)$ are strictly increasing, the functions $t \mapsto u(t)$ and $t \mapsto v(t)$ are increasing too. Moreover, u and v can be extended in a continuous and unique way to $[0, L]$. To see this, let $t_0 \in (0, L)$ and define

$$t_1 := \lim_{\substack{t\nearrow t_0\\ t\in S}} u(t), \quad t_2 := \lim_{\substack{t\searrow t_0\\ t\in S}} u(t), \quad \tau_1 := \lim_{\substack{t\nearrow t_0\\ t\in S}} v(t), \quad \tau_2 := \lim_{\substack{t\searrow t_0\\ t\in S}} v(t).$$

Thus we find by Lemma 2.42 and Equation (3.4)

$$\mathfrak{c}(t_1, \tau_1) = \lim_{\substack{t\nearrow t_0\\ t\in S}} \mathfrak{c}(u(t), v(t)) = t_0 = \lim_{\substack{t\searrow t_0\\ t\in S}} \mathfrak{c}(u(t), v(t)) = \mathfrak{c}(t_2, \tau_2).$$

Since $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$ is strictly increasing in both variables and $t_1 \le t_2$ and $\tau_1 \le \tau_2$, we find $t_1 = t_2$ and $\tau_1 = \tau_2$. If $t_0 \in \{0, L\}$ we can argue in the same way, so $t \mapsto u(t)$ and $t \mapsto v(t)$ are continuous on $[0, L]$. Summarising, u and v are continuous and increasing on $[0, L]$ with $u(L) = T$ and $v(L) = T$. For later use, we define $h_{t,\tau}^{[n]} := f_{u_n^*(t),v_n^*(\tau)}$ and $h_{t,\tau} := f_{u(t),v(\tau)}$ for all $t, \tau \in [0, L]$.

3) Next, we are going to show that $t \mapsto u(t)$ and $t \mapsto v(t)$ are strictly increasing on $[0, L]$ and $\lambda_0 \in (0, 1)$.

Using Lemma 3.5, we find a $\delta > 0$ corresponding to $\varepsilon = \frac{1}{2}$. The functions $t \mapsto u(t)$ and $t \mapsto v(t)$ are (uniformly) continuous on $[0, L]$, so we have:

$$\exists \mu > 0 : |\bar{t} - \underline{t}| < \mu \Rightarrow |u(\bar{t}) - u(\underline{t})|,\ |v(\bar{t}) - v(\underline{t})| < \tfrac{\delta}{2}.$$

Assume $\underline{t}, \bar{t} \in S$ with $0 < \bar{t} - \underline{t} < \mu$. Consequently, $|u(\bar{t}) - u(\underline{t})| < \frac{\delta}{2}$ and $|v(\bar{t}) - v(\underline{t})| < \frac{\delta}{2}$, so we get:

$$\exists n_0 \in \mathbb{N}\, \forall n \ge n_0 : |u_n^*(\bar{t}) - u_n^*(\underline{t})|,\ |v_n^*(\bar{t}) - v_n^*(\underline{t})| < \delta,$$

as $u_n^*(t)$ and $v_n^*(t)$ are pointwise convergent on S. Moreover, we choose n_0 large enough to satisfy $\underline{t}, \bar{t} \in S_{n_0}^*$. Then for any $n \ge n_0$ we get

$$\frac{1}{2} = 1 - \varepsilon \le \frac{\mathfrak{c}(u_n^*(t_{l+1}), v_n^*(\underline{t})) - \mathfrak{c}(u_n^*(t_l), v_n^*(\underline{t}))}{\mathfrak{c}(u_n^*(t_{l+1}), v_n^*(t_l)) - \mathfrak{c}(u_n^*(t_l), v_n^*(t_l))} \le 1 + \varepsilon = \frac{3}{2}$$

for all $l \in \{0, \dots, s-1\}$ where $S_n^*([\underline{t}, \bar{t}]) := \{t_1, \dots, t_s\} := [\underline{t}, \bar{t}] \cap S_n^*$. Consequently, we get by summing over all $l \in \{0, \dots, s-1\}$

$$\tfrac{1}{2}\lambda_n^*(\bar{t} - \underline{t}) = \tfrac{1}{2}\mathcal{S}_1(u_n^*, v_n^*, [\underline{t}, \bar{t}], S_n^*([\underline{t}, \bar{t}])) \leq \mathfrak{c}(u_n^*(\bar{t}), v_n^*(\underline{t})) - \mathfrak{c}(u_n^*(\underline{t}), v_n^*(\underline{t})) \leq \tfrac{3}{2}\mathcal{S}_1(u_n^*, v_n^*, [\underline{t}, \bar{t}], S_n^*([\underline{t}, \bar{t}])) = \tfrac{3}{2}\lambda_n^*(\bar{t} - \underline{t}).$$

Letting $n \to \infty$ we find with Lemma 2.42

$$\tfrac{1}{2}\lambda_0(\bar{t} - \underline{t}) \leq \mathfrak{c}(u(\bar{t}), v(\underline{t})) - \mathfrak{c}(u(\underline{t}), v(\underline{t})) \leq \tfrac{3}{2}\lambda_0(\bar{t} - \underline{t}). \tag{3.5}$$

Note that $t \mapsto u(t)$ is continuous and increasing with $u(0) = 0$ and $u(L) = T$ so we find $\underline{t}, \bar{t} \in S$ with $0 < \bar{t} - \underline{t} < \mu$ and $u(\underline{t}) < u(\bar{t})$. Using Lemma 2.41, $\mathfrak{c}(u(\bar{t}), v(\underline{t})) - \mathfrak{c}(u(\underline{t}), v(\underline{t})) > 0$, so (3.5) yields $\lambda_0 \neq 0$. On top of this Equation (3.5) gives us $\mathfrak{c}(u(\bar{t}), v(\underline{t})) - \mathfrak{c}(u(\underline{t}), v(\underline{t})) > 0$ whenever $0 < \bar{t} - \underline{t} < \mu$, i.e. $t \mapsto u(t)$ is strictly increasing on $[0, L]$. Analogously, we find

$$\tfrac{1}{2}(1 - \lambda_0)(\bar{t} - \underline{t}) \leq \mathfrak{c}(u(\underline{t}), v(\bar{t})) - \mathfrak{c}(u(\underline{t}), v(\underline{t})) \leq \tfrac{3}{2}(1 - \lambda_0)(\bar{t} - \underline{t}),$$

for all $\underline{t}, \bar{t} \in S$ with $0 \leq \bar{t} - \underline{t} \leq \mu$ so $t \mapsto v(t)$ is strictly increasing on $[0, L]$ and $1 - \lambda_0 \neq 0$ as well. Summarising, $t \mapsto u(t)$, $t \mapsto v(t)$ are strictly increasing and $\lambda_0 \in (0, 1)$.

4) Next, we show that for every fixed $\varepsilon > 0$, fixed $t \in S$ and fixed partition $Z \subseteq S$ of the interval $[0, t]$, there exists an $n_0 \in \mathbb{N}$ such that

$$|\mathcal{S}_1(u_n^*, v_n^*, t, Z) - \mathcal{S}_1(u, v, t, Z)| < \varepsilon$$

holds for all $n \geq n_0$.

Fix $\varepsilon > 0$ and $Z = \{t_0, t_1, \dots, t_s\}$. As the function $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$ is (uniformly) continuous on $[0, T^*]^2$ by Lemma 2.42, there exists $\delta > 0$ such that

$$|\mathfrak{c}(\underline{t}, \underline{\tau}) - \mathfrak{c}(\bar{t}, \bar{\tau})| < \frac{\varepsilon}{2s} \quad \text{whenever} \quad |\underline{t} - \bar{t}|,\ |\underline{\tau} - \bar{\tau}| < \delta.$$

Since $Z \subseteq S$ is finite, we find an $n_0 \in \mathbb{N}$ such that $|u_n^*(t_l) - u(t_l)|$, $|v_n^*(t_l) - v(t_l)| < \delta$ holds for all $l \in \{0, \dots, s\}$ and all $n \geq n_0$. Consequently, for all $n \geq n_0$, we find

$$\begin{aligned}
&|\mathcal{S}_1(u_n^*, v_n^*, t, Z) - \mathcal{S}_1(u, v, t, Z)| \\
&\quad = \Big| \sum_{l=0}^{s-1} \mathfrak{c}(h_{t_{l+1}, t_l}^{[n]}) - \mathfrak{c}(h_{t_l, t_l}^{[n]}) - \sum_{l=0}^{s-1} \mathfrak{c}(h_{t_{l+1}, t_l}) - \mathfrak{c}(h_{t_l, t_l}) \Big| \\
&\quad \leq \sum_{l=0}^{s-1} \big| \mathfrak{c}(h_{t_{l+1}, t_l}^{[n]}) - \mathfrak{c}(h_{t_{l+1}, t_l}) \big| + \sum_{l=0}^{s-1} \big| \mathfrak{c}(h_{t_l, t_l}^{[n]}) - \mathfrak{c}(h_{t_l, t_l}) \big| \leq 2s \frac{\varepsilon}{2s} = \varepsilon.
\end{aligned}$$

5) For now we fix $t \in S$, $t > 0$. We show that for all $\varepsilon > 0$, we find a $\mu > 0$ such that for all partitions $Z \subseteq S$ of $[0, t]$ with $|Z| < \mu$, there exists an $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$, we have

$$|\mathcal{S}_1(u_n^*, v_n^*, t, Z) - \lambda_0 t| < \varepsilon.$$

Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that the inequality from Lemma 3.5 holds. Since the functions $t \mapsto u(t)$ and $t \mapsto v(t)$ are (uniformly) continuous on $[0, t]$, we get:

$$\exists \mu > 0 : \ |\underline{t} - \bar{t}| < \mu \Rightarrow |u(\underline{t}) - u(\bar{t})|, |v(\underline{t}) - v(\bar{t})| < \tfrac{\delta}{2}.$$

Denote by $Z = \{t_0, \ldots, t_s\}$ a partition of $[0, t]$ with $|Z| < \mu$ and $Z \subseteq S$. Then we find an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we get $Z \subseteq S_n^*$ and

$$|u_n^*(t_l) - u(t_l)|,\ |v_n^*(t_l) - v(t_l)| < \tfrac{\delta}{4} \quad \text{for all } l \in \{0, \ldots, s\} \text{ and all } n \geq n_0.$$

As a consequence we get

$$\begin{aligned} &|u_n^*(t_{l+1}) - u_n^*(t_l)| \\ &\qquad \leq |u_n^*(t_{l+1}) - u(t_{l+1})| + |u(t_{l+1}) - u(t_l)| + |u(t_l) - u_n^*(t_l)| < \tfrac{\delta}{4} + \tfrac{\delta}{2} + \tfrac{\delta}{4} = \delta \end{aligned}$$

for all $n \geq n_0$ and all $l \in \{0, \ldots, s\}$. In the same way we find $|v_n^*(t_{l+1}) - v_n^*(t_l)| < \delta$ for all $n \geq n_0$ and all $l \in \{0, \ldots, s\}$. Next, for each $n \in \mathbb{N}$, we set $S_n^*(t) := S_n^* \cap [0, t]$. $S_n^*(t)$ is a partition of the interval $[0, t]$ and we write $S_n^*(t) = \{t_0^*, \ldots, t_{s^*}^*\}$. For each $n \geq n_0$, we find

$$\begin{aligned} |\lambda_n^* t - &\mathcal{S}_1(u_n^*, v_n^*, t, Z)| = |\mathcal{S}_1(u_n^*, v_n^*, t, S_n^*(t)) - \mathcal{S}_1(u_n^*, v_n^*, t, Z)| \\ &= \sum_{p=0}^{s^*-1} \big|[\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_p}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})] - [\mathfrak{c}(h^{[n]}_{t^*_{p+1}, \phi(t^*_p)}) - \mathfrak{c}(h^{[n]}_{t^*_p, \phi(t^*_p)})]\big|, \end{aligned}$$

where $\phi(t_p^*) := t_l$ if $t_p^* \in [t_l, t_{l+1})$ with $p \in \{0, \ldots, s^* - 1\}$ and $l \in \{0, \ldots, s - 1\}$. Thus we get

$$\begin{aligned} &|\lambda_n^* t - \mathcal{S}_1(u_n^*, v_n^*, t, Z)| \\ &\quad = \sum_{p=0}^{s^*-1} \big|\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_p}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})\big| \cdot \left| 1 - \frac{\mathfrak{c}(h^{[n]}_{t^*_{p+1}, \phi(t^*_p)}) - \mathfrak{c}(h^{[n]}_{t^*_p, \phi(t^*_p)})}{\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_p}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})} \right|. \end{aligned}$$

Since $|u_n^*(t_{p+1}^*) - u_n^*(t_p^*)| < \delta$ and $|v_n^*(\phi(t_p^*)) - v_n^*(t_p^*)| < \delta$ for all $n \geq n_0$ and all $p \in \{0, \ldots, s^*\}$, Lemma 3.5 gives us

$$|\lambda_n^* t - \mathcal{S}_1(u_n^*, v_n^*, t, Z)| \leq \varepsilon \sum_{p=0}^{s^*-1} \big(\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_p}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})\big) \leq L\varepsilon$$

for all $n \geq n_0$. The last inequality is a consequence of the monotonicity of $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$:

$$\sum_{p=0}^{s^*-1} \big(\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_p}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})\big) \leq \sum_{p=0}^{s^*-1} \big(\mathfrak{c}(h^{[n]}_{t^*_{p+1}, t^*_{p+1}}) - \mathfrak{c}(h^{[n]}_{t^*_p, t^*_p})\big) = t \leq L.$$

The assertion follows now, since λ_n^* converges to λ_0.

6) If we put 4) and 5) together, we find for every $t \in [0,T]$ and $\varepsilon > 0$, a $\mu > 0$ such that for all partitions $Z \subseteq S$ of the interval $[0,t]$ with $|Z| < \mu$, the inequality

$$|\mathcal{S}_1(u,v,t,Z) - \lambda_0 t| < \varepsilon \tag{3.6}$$

holds.

7) Since u and v are strictly increasing homeomorphisms of $[0,L]$ to $[0,T]$, we can apply Lemma 3.6 to the admissible parametrisation $(\gamma_1 \circ u, \gamma_2 \circ v)_{t\in[0,L]}$ to get Lipschitz continuous and increasing functions c_1 and c_2 with $c_1 + c_2 \equiv \mathrm{id}$, as $\mathfrak{c}(u(t), v(t)) = t$ for all $t \in [0,L]$. Equation (3.6) gives us

$$c_1(t) = \lambda_0 t, \quad c_2(t) = t - c_1(t) = (1-\lambda_0)t$$

for all $t \in S$. Since S is dense on $[0,L]$, this relation holds for all $t \in [0,L]$. On top of this we find with Lemma 3.6

$$\lim_{t\to t_0} \frac{\mathfrak{c}(h_{t,t_0}) - \mathfrak{c}(h_{t_0,t_0})}{t-t_0} = \lambda_0, \qquad \lim_{t\to t_0} \frac{\mathfrak{c}(h_{t_0,t}) - \mathfrak{c}(h_{t_0,t_0})}{t-t_0} = 1-\lambda_0$$

for all $t_0 \in [0,L]$. For each $t \in [0,L]$, we set $g_t := h_{t,t}$, $D_t := g_t\big(\Omega \setminus (\gamma_1(0,u(t)] \cup \gamma_2(0,v(t)])\big)$, and $U_1(t) = g_t(\gamma_1(u(t)))$ and $U_2(t) := g_t(\gamma_2(v(t)))$. Finally, we can apply Theorem 2.30, 2.31 or 2.36 to find

$$\dot{g}_t(z) = E\big(g_t(z)\big)\Big(\lambda_0 \Phi_{a_t,U_1(t),D_t}\big(g_t(z)\big) + (1-\lambda_0)\Phi_{a_t,U_2(t),D_t}\big(g_t(z)\big)\Big), \quad t \in [0,L].$$

with continuous driving terms $t \mapsto U_k(t)$, $k \in \{1,2\}$, see Lemma 2.43. Herein, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. Moreover, for all $t \in [0,T]$, $a_t := 0$ in the radial case, $a_t := q_t$ in the bilateral case where q_t is the inner radius of the circular slit annulus D_t and $a_t := \infty$ in the chordal case.

□

Proof (Uniqueness). Let $(\gamma_1, \gamma_2)_{[0,T]}$ be a tuple of disjoint appropriate slits in Ω. For each $t, \tau \in [0,T]$, $f_{t,\tau} : \Omega(t,\tau) \to D(t,\tau)$ denotes the normalised appropriate mapping function from $\Omega(t,\tau) := \Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,\tau])$ onto the canonical domain $D(t,\tau)$ and $\mathfrak{c}(t,\tau) := \mathfrak{c}(f_{t,\tau})$. Moreover, we set $U_1(t,\tau) := f_{t,\tau}(\gamma_1(t))$ and $U_2(t,\tau) := f_{t,\tau}(\gamma_2(\tau))$ for all $t,\tau \in [0,T]$. Let $u, v : [0,L] \to [0,T]$ be increasing homeomorphisms and Z denotes an arbitrary partition of the interval $[\underline{t}, \bar{t}] \subseteq [0,T]$. Then for each $k \in \{1,2\}$, we define $\mathcal{S}_k(u,v,[\underline{t},\bar{t}],Z)$ in the same way as in Equation (3.2) and we write $\mathcal{S}_k(u,v,t,Z) := \mathcal{S}_k(u,v,[0,t],Z)$ for any partition Z of the interval $[0,t] \subseteq [0,T]$. Moreover, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case.

Assume u_1, v_1 and u_2, v_2 are increasing homeomorphisms from $[0,L]$ onto $[0,T]$ such that the functions $g_t := f_{u_1(t),v_1(t)}$ and $h_t := f_{u_2(t),v_2(t)}$ satisfy the differential equations

$$\dot{g}_t(z) = E\big(g_t(z)\big)\Big(\lambda_1 \Phi_{a_t,\xi_1(t),G_t}\big(g_t(z)\big) + (1-\lambda_1)\Phi_{a_t,\xi_2(t),G_t}\big(g_t(z)\big)\Big),$$
$$\dot{h}_t(z) = E\big(h_t(z)\big)\Big(\lambda_2 \Phi_{b_t,\zeta_1(t),H_t}\big(h_t(z)\big) + (1-\lambda_2)\Phi_{b_t,\zeta_2(t),H_t}\big(h_t(z)\big)\Big)$$

for all $t \in [0, L]$ with coefficients $\lambda_1, \lambda_2 \in (0,1)$, continuous driving functions $\xi_k(t) := U_k(u_1(t), v_1(t))$, $\zeta_k(t) := U_k(u_2(t), v_2(t))$, $k \in \{1,2\}$, and $G_t := g_t(\Omega(u_1(t), v_1(t))) = D(u_1(t), v_1(t))$, $H_t := g_t(\Omega(u_2(t), v_2(t))) = D(u_2(t), v_2(t))$ for all $t \in [0,T]$. Moreover, for each $t \in [0,T]$, $a_t := 0$ in the radial case, a_t is the inner radius of G_t in the bilateral case and $a_t =: \infty$ in the chordal case. Analogously, $b_t := 0$ in the radial case, b_t is the inner radius of H_t in the bilateral case and $b_t := \infty$ in the chordal case for all $t \in [0, L]$.

Using Theorem 2.30, 2.31 or 2.36, we find $\mathfrak{c}(g_t) = \mathfrak{c}(h_t) = t$ for all $t \in [0, L]$ and

$$\begin{aligned} \lambda_1 &= \lim_{t \to t_0} \frac{\mathfrak{c}(u_1(t), v_1(t_0)) - \mathfrak{c}(u_1(t_0), v_1(t_0))}{t - t_0}, \\ \lambda_2 &= \lim_{t \to t_0} \frac{\mathfrak{c}(u_2(t), v_2(t_0)) - \mathfrak{c}(u_2(t_0), v_2(t_0))}{t - t_0} \end{aligned} \tag{3.7}$$

for all $t_0 \in [0, L]$.

1) First of all we will show $\lambda_1 = \lambda_2$, so suppose $\lambda_1 > \lambda_2$. Therefore, we set

$$x_k(t) := \mathfrak{c}\big(u_k(t), v_k(0)\big) = \mathfrak{c}\big(u_k(t), 0\big), \qquad k \in \{1,2\}.$$

Denote by $0 < t_0 \leq L$ the first positive time when $u_1(t_0) = u_2(t_0)$. Consequently, $v_1(t_0) = v_2(t_0)$ and $x_1(t_0) = x_2(t_0)$ by normalisation and the monotonicity of $(t, \tau) \mapsto \mathfrak{c}(t, \tau)$ in each variable. Equation (3.7) gives us

$$\dot{x}_1(0) = \lambda_1 > \lambda_2 = \dot{x}_2(0),$$

so we have $x_1(t) > x_2(t)$ and $u_1(t) > u_2(t)$ for all $t \in (0, t_0)$. Consequently, we have also $\mathfrak{c}\big(u_1(t), v_1(t_0)\big) > \mathfrak{c}\big(u_2(t), v_2(t_0)\big)$ for all $t \in (0, t_0)$. Thus we get

$$\mathfrak{c}\big(u_2(t), v_2(t_0)\big) < \mathfrak{c}\big(u_1(t), v_1(t_0)\big) < \mathfrak{c}\big(u_1(t_0), v_1(t_0)\big) = t_0 = \mathfrak{c}\big(u_2(t_0), v_2(t_0)\big)$$

if $t < t_0$. This implies

$$\frac{\mathfrak{c}\big(u_1(t_0), v_1(t_0)\big) - \mathfrak{c}\big(u_1(t), v_1(t_0)\big)}{t_0 - t} < \frac{\mathfrak{c}\big(u_2(t_0), v_2(t_0)\big) - \mathfrak{c}\big(u_2(t), v_2(t_0)\big)}{t_0 - t}$$

for all $t < t_0$. If t tends to t_0, we get $\lambda_1 \leq \lambda_2$ by Equation (3.7). This is a contradiction, so $\lambda_1 = \lambda_2 =: \lambda$.

2) Next, we are going to prove the uniqueness of the parametrisation. The following idea goes back to a work of O. Roth and S. Schleißinger, see [RS14]

Let $(u, v)(t) := (u_1, v_1)(t)$ for all $t \in [0, L]$, or $(u, v)(t) := (u_2, v_2)(t)$ for all $t \in [0, L]$. We are going to derive a differential equation for $x(t) := \mathfrak{c}(u(t), 0)$ on $[0, L]$. Note that $v(t)$ is uniquely determined by $x(t)$ and t, as $\mathfrak{c}(u(t), v(t)) = t$ for all $t \in [0, L]$. Consequently, we write $\tilde{u}(x)$ and $\tilde{v}(x, t)$ such that $\mathfrak{c}(\tilde{u}(x), \tilde{v}(x, t)) = t$ and $\mathfrak{c}(\tilde{u}(x), 0) = x$ for all $x, t \in [0, L]$ with $x \leq t$.[8] Obviously, $\tilde{u}$ and $\tilde{v}$ are continuous.

[8]In order to get $\tilde{u}$ and $\tilde{v}$ well-defined for each $x, t \in [0, L]$ with $x \leq t$, we extend γ_1 and γ_2 to an interval $[0, L^*]$ such that $\mathfrak{c}(L^*, 0) > L$ and $\mathfrak{c}(0, L^*) > L$.

Using Proposition 2.55, we find immediately

$$\frac{\dot{x}(t)}{\lambda} = \lim_{\tau\to t} \frac{\mathfrak{c}(f_{u(\tau),0}) - \mathfrak{c}(f_{u(t),0})}{\mathfrak{c}(f_{u(\tau),v(t)}) - \mathfrak{c}(f_{u(t),v(t)})} = \frac{|\alpha_1(u(t),0)|^2}{|\alpha_1(u(t),v(t))|^2} = \frac{1}{\left|(f_{u(t),v(t)} \circ f_{u(t),0}^{-1})'(z)\right|^2}$$

with $z = U_1(u(t), 0)$. Note that we can interpret the right-hand side as a function of (x, t), so we write

$$C(x,t) := \left|(f_{u(t),v(t)} \circ f_{u(t),0}^{-1})'(z)\right| = \left|(f_{\tilde{u}(x),\tilde{v}(x,t)} \circ f_{\tilde{u}(x),0}^{-1})'(z_x)\right|$$

with $x \le t$.

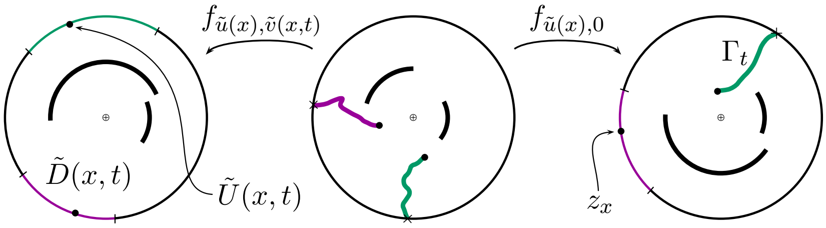

FIGURE 3.1: *Radial mapping functions $f_{\tilde{u}(x),\tilde{v}(x,t)}$ and $f_{\tilde{u}(x),0}$ in the uniqueness proof of Theorem 3.2, 3.3 and 3.4*

It is easy to see that $(x,t) \mapsto C(x,t)$ is continuous and positive on $\{(x,t) \in [0,L]^2 \mid x \le t\}$.

For now let us fix $x \in [0, L)$ and we set $h_t := f_{\tilde{u}(x),\tilde{v}(x,t)} \circ f_{\tilde{u}(x),0}^{-1}$. Consequently, for each $t \in [x, L]$, h_t is the normalised appropriate mapping function on $D(\tilde{u}(x), 0) \setminus \Gamma_t$ with $\Gamma_t := f_{\tilde{u}(x),0}(\gamma_2[0, \tilde{v}(x,t)])$ for all $t \ge x$. Moreover, $\mathfrak{c}(h_t) = \mathfrak{c}(f_{\tilde{u}(x),\tilde{v}(x,t)}) - \mathfrak{c}(f_{\tilde{u}(x),0}) = t - x$. Using Theorem 2.30, 2.31 or 2.36, $t \mapsto h_t(z)$ is differentiable for all $t \in [x, L]$ and all $z \in D(\tilde{u}(x), 0) \setminus \Gamma_L$ and satisfies

$$\dot{h}_t(z) = E\big(h_t(z)\big)\Phi_{c_t,\tilde{U}(x,t),\tilde{D}(x,t)}\big(h_t(z)\big) \quad \text{for all } t \in [x, L] \text{ and all } z \in D(\tilde{u}(x), 0) \setminus \Gamma_L, \tag{3.8}$$

where $\tilde{U}(x,t) := U_2(\tilde{u}(x), \tilde{v}(x,t))$ and $\tilde{D}(x,t) := D(\tilde{u}(x), \tilde{v}(x,t))$. Herein, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. Moreover, $c_t := 0$ in the radial case, c_t is the inner radius of $\tilde{D}(x,t)$ in the bilateral case and $c_t := \infty$ in the chordal case. Using Schwarz reflection principle, Equation (3.8) holds for all $z \in B_\varepsilon(z_x)$ as well with a small $\varepsilon > 0$. This gives us

$$\dot{h}_t(z) = E\big(h_t(z)\big)\Phi_{c_t,\tilde{U}(x,t),\tilde{D}(x,t)}\big(h_t(z)\big) \quad \text{for all } z \in B_\varepsilon(z_x).$$

Note that the right-hand side is continuous: assume $x_n \to x_0$ and $t_n \to t_0$, so we get

$$\Phi_{c_{t_n},\tilde{U}(x_n,t_n),\tilde{D}(x_n,t_n)} \circ h_{t_n} \xrightarrow{\text{l.u.}} \Phi_{c_{t_0},\tilde{U}(x_0,t_0),\tilde{D}(x_0,t_0)} \circ h_{t_0} \quad \text{on } B_\varepsilon(z_{x_0}),$$

see Lemma 2.18, 2.19 or 2.20. Obviously, the same is true for the derivative w.r.t. z, and by Lemma 2.43, $x \mapsto z_x$ is continuous as well. Summarising, $(x,t) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}h_t'(z_x)$ is continuous on $\Delta := \{(x,t) \in [0,L]^2 \mid x \leq t\}$. Consequently, $(x,t) \mapsto C(x,t) = |h_t'(z_x)|$ is continuously differentiable on Δ w.r.t. t (uniformly in x), as $(x,t) \mapsto h_t'(z_x)$ is continuous and positive on Δ.

Finally, x_1 and x_2 satisfy the differential equation $\dot{x}(t) = \lambda / C^2(x,t)$. Using Theorem 2.7 from [CP03], the solution needs to be unique, i.e. $x_1 \equiv x_2$. Obviously, $u_1 \equiv u_2$ and $v_1 \equiv v_2$.

□

Remark 3.5. In the simply connected case it is possible to give an '*easier*' proof of the uniqueness of constant coefficients. The reason for this is an additional tool (see Lemma 2.57 and 2.56) available in simply connected domains only. We refer to the proof of Theorem 3.8 where we will prove the uniqueness in the simply connected case for slits having branch points. Note that this proof holds in the disjoint case word by word as well.

3.2 Slits having branch points

Next, let us consider slits that may have a branch point. In particular, we are going to study the case where two slits start at a common point. Let Ω be a canonical domain and denote by C the outer or unbounded boundary component of Ω. Moreover, each $\gamma_k : [0,T] \to \mathrm{cl}(\Omega)$, $k \in \{1,2\}$, is continuous and simple, $\gamma_1(0,T] \cup \gamma_2(0,T]$ is an appropriate Ω-hull, $\gamma_1(0) = \gamma_2(0) = U_0 \in C$ and $\gamma_1(0,T] \cap \gamma_2(0,T] = \emptyset$. Then we call $(\gamma_1, \gamma_2)_{t\in[0,T]}$ a *tuple of branched appropriate slits in* Ω. Obviously, $(\gamma_1(0,t] \cup \gamma_2(0,t])_{t\in[0,T]}$ is an increasing and continuous family of appropriate Ω-hulls. Moreover, (Γ_1, Γ_2) (with $\Gamma_1, \Gamma_2 \subseteq \mathrm{cl}(\Omega)$) is called *tuple of branched appropriate unparametrised slits in* Ω if there is a $T > 0$ and a tuple $(\gamma_1, \gamma_2)_{t\in[0,T]}$ of branched appropriate slits in Ω such that $\Gamma_k = \gamma_k[0,T]$ with $k \in \{1,2\}$. In this case $(\gamma_1, \gamma_2)_{t\in[0,T]}$ is called *admissible parametrisation of* (Γ_1, Γ_2).

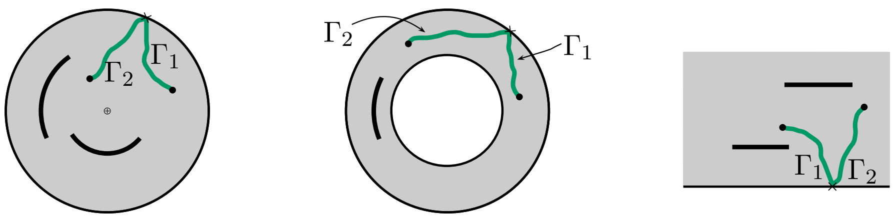

FIGURE 3.2: *Tuple of branched unparametrised slits in canonical domains*

Theorem 3.7. *Let Ω be a canonical domain and (Γ_1, Γ_2) be a tuple of branched appropriate unparametrised slits in Ω. Then there is a unique $L > 0$, constants $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, and an admissible parametrisation $(\gamma_1, \gamma_2)_{t\in[0,L]}$ of (Γ_1, Γ_2) such that for each $z \in \Omega_L$, $t \mapsto g_t(z)$ is continuously differentiable on $[0,L]$ and satisfies*

$$\dot{g}_t(z) = E\big(g_t(z)\big) \sum_{k=1}^{2} \lambda_k \Phi_{a_t, U_k(t), D_t}\big(g_t(z)\big) \quad \textit{for all } t \in [0,L] \textit{ and all } z \in \Omega_L.$$

Herein, for each $t \in [0, L]$, $g_t : \Omega_t \to D_t$ is the normalised radial mapping function on $\Omega_t := \Omega \setminus \bigcup_{k=1}^{2} \gamma_k(0, t]$ and for each $k \in \{1, 2\}$, $U_k(t) := g_t(\gamma_k(t))$ denotes a continuous driving function on $[0, L]$. Moreover, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. For each $t \in [0, T]$, $a_t := 0$ in the radial case, $a_t := q_t$ where q_t denotes the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case.

Note that Theorem 3.7 gives an existence statement only. Unfortunately, we are able to prove uniqueness only for simply connected domains.

Theorem 3.8. *Let Ω be a simply connected canonical domain and (Γ_1, Γ_2) be a tuple of branched appropriate unparametrised slits in Ω. Assume $(\gamma_1, \gamma_2)_{t\in[0,L]}$ is an admissible parametrisation of (Γ_1, Γ_2) from Theorem 3.7 with constant coefficients λ and $1 - \lambda$, $\lambda \in (0, 1)$. Then the weight λ and the admissible parametrisation $(\gamma_1, \gamma_2)_{t\in[0,L]}$ is unique.*

In order to prove these theorems we need some preliminary lemmas.

3.2.1 Some preliminary lemmas

Lemma 3.9. *Let Ω be a canonical domain and denote by $(\gamma_1, \gamma_2)_{t\in[0,T]}$ a tuple of branched appropriate slits in Ω. For each $t, \tau \in [0, T]$, $f_{t,\tau}$ denotes the normalised appropriate mapping function from $\Omega(t, \tau) := \Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$ onto the canonical domain $D(t, \tau)$. Moreover, for each $t \in [0, T]$, we set $g_t := f_{t,t}$, $\Omega_t := \Omega(t, t)$ and $D_t := D(t, t)$. Assume $t_0 \in (0, T]$.*

Then the following two statements are equivalent.

(i) For each $z \in \Omega_{t_0}$, $t \mapsto g_t(z)$ is differentiable at t_0 and fulfils

$$\dot{g}_{t_0}(z) = E\big(g_{t_0}(z)\big) \sum_{k=1}^{2} \lambda_k(t_0) \Phi_{a_{t_0}, U_k(t_0), D_{t_0}}\big(g_{t_0}(z)\big) \quad \text{for all } z \in \Omega_{t_0},$$

where each $U_k(t) := g_t(\gamma_k(t))$, $k \in \{1, 2\}$, is continuous on $[0, T]$ and $\lambda_k(t_0) \geq 0$, $k \in \{1, 2\}$.

(ii) $\lambda_1(t_0) := \lim_{t\to t_0} \frac{\mathfrak{c}(f_{t,t_0}) - \mathfrak{c}(f_{t_0,t_0})}{t - t_0}$ and $\lambda_2(t_0) := \lim_{t\to t_0} \frac{\mathfrak{c}(f_{t_0,t}) - \mathfrak{c}(f_{t_0,t_0})}{t - t_0}$ exist.

When this happens, $t \mapsto \mathfrak{c}(g_t)$ is differentiable at t_0 with derivative $\lambda_1(t_0) + \lambda_2(t_0)$.

Herein, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. Moreover, for each $t \in [0, T]$, $a_t := 0$ in the radial case, a_t is the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case.

Proof. This proof is quite easy. First of all, we choose $\varepsilon > 0$ in such a way that $\varepsilon < t_0$. Next, we set $h := g_\varepsilon$, $G := h(\Omega_\varepsilon)$ and $\Delta_k := h(\gamma_k[\varepsilon, T])$ with $k \in \{1, 2\}$. Consequently, (Δ_1, Δ_2) is a tuple of unparametrised disjoint slits in the canonical domain G and $(\delta_1, \delta_2)_{s\in[0,T-\varepsilon]}$ is an admissible parametrisation of (Δ_1, Δ_2) where $\delta_k(s) := h(\gamma_k(s + \varepsilon))$ for all $s \in [0, T - \varepsilon]$ and $k \in \{1, 2\}$. For each $s, \sigma \in [0, T - \varepsilon]$, we denote by $h_{s,\sigma}$ the normalised appropriate mapping function on $G_{s,\sigma} := G \setminus (\delta_1(0, s] \cup \delta_2(0, \sigma])$, and for each

$s \in [0, T-\varepsilon]$, h_s is the normalised appropriate mapping function from $G_s := G_{s,s}$ onto the canonical domain H_s. Obviously, we have

$$\mathfrak{c}(f_{t_0+s-s_0,t_0}) - \mathfrak{c}(f_{t_0,t_0}) = \mathfrak{c}(h_{s,s_0}) - \mathfrak{c}(h_{s_0,s_0}) \qquad \text{for all } s \in [0, T-\varepsilon],$$

with $s_0 := t_0 - \varepsilon$, as $f_{s+\varepsilon,\sigma+\varepsilon} = h_{s,\sigma} \circ h$ for all $s, \sigma \in [0, T-\varepsilon]$. Note that $U_k(t_0) = h_{s_0}(\delta_k(s_0))$, $D_{t_0} = H_{s_0}$. Using Theorem 2.30, 2.31 or 2.36, the function $s \mapsto h_s(w)$ is differentiable at $s_0 = t_0 - \varepsilon$ for all $w \in G_{s_0}$ with

$$\dot{h}_{s_0}(w) = E\big(h_{s_0}(w)\big) \sum_{k=1}^{2} \mu_k(s_0) \Phi_{a_{t_0}, U_k(t_0), D_{t_0}}\big(h_{s_0}(w)\big) \quad \text{for all } w \in G_{s_0} \tag{3.9}$$

and $\mu_1(s_0), \mu_2(s_0) \geq 0$ if and only if the following two limits exist:

$$\mu_1(s_0) := \lim_{s \to s_0} \frac{\mathfrak{c}(h_{s,s_0}) - \mathfrak{c}(h_{s_0,s_0})}{s - s_0}, \qquad \mu_2(s_0) := \lim_{s \to s_0} \frac{\mathfrak{c}(h_{s_0,s}) - \mathfrak{c}(h_{s_0,s_0})}{s - s_0}.$$

Finally, by substituting $w = h(z)$ in Equation (3.9) we get the stated equivalence. □

At $t_0 = 0$ we have the following lemma.

Lemma 3.10. *Let Ω be a canonical domain, $(\gamma_1, \gamma_2)_{t \in [0,T]}$ be a tuple of branched appropriate slits in Ω and denote by g_t, $t \in [0, T]$, the normalised appropriate mapping function from $\Omega_t := \Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$ onto the canonical domain D_t.*

Then the following two statements are equivalent.

(i) For each $z \in \Omega$, $t \mapsto g_t(z)$ is differentiable at 0 and fulfils

$$\dot{g}_0(z) = \lambda\, E(z)\, \Phi_{a,\gamma_1(0),\Omega}\big(z\big) \quad \text{for all } z \in \Omega$$

with some $\lambda \geq 0$

(ii) $t \mapsto \mathfrak{c}(g_t)$ is differentiable at 0 with derivative λ.

Herein, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. Moreover, $a := 0$ in the radial case, $a := Q$ where Q is the inner radius of Ω in the bilateral case and $a := \infty$ in the chordal case.

Proof. First of all, using Lemma 2.46 we find

$$F(g_t^{-1}(w)) - F(w) = \frac{1}{2\pi} \int_C \Re F\big(g_t^{-1}(\zeta)\big) \cdot \Phi_{a_t,\zeta,D_t}(w) |\mathrm{d}\zeta| \quad t > 0,\ w \in D_t.$$

Here, C denotes the outer or unbounded boundary component of D_t. $F(w) := \log(w)$, $w \in \mathbb{C} \setminus \{0\}$, in the radial and bilateral case, and $F(w) := 2\mathrm{i}w$, $w \in \mathbb{C}$, in the chordal case. Moreover, $a_t := 0$ in the radial case, a_t denotes the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case.

Let $\varepsilon > 0$ be small. We can choose $t_0 > 0$ small enough in order to have $S_t := \gamma_1(0,t] \cap \gamma_2(0,t] \subseteq B_\varepsilon(U_0)$ for all $t \in [0, t_0]$ with $U_0 := \gamma_1(0) = \gamma_2(0)$. Using the

Schwarz reflection principle, for each $t \le t_0$, we extend g_t analytically to a neighbourhood $\partial B_\varepsilon(U_0)$. Using Lemma 2.42, we find $g_t \xrightarrow{\text{l.u.}} \text{id}$ on Ω. We have uniform convergence on $\partial B_\varepsilon(U_0)$ as well, so we find $s_t := g_t(S_t) \subseteq B_\varepsilon(U_0)$ for all t small enough. This gives us $s_t \to U_0$ if $t \searrow 0$.

Let $t > 0$. Together with the mean value theorem and Lemma 2.45, we get

$$\begin{aligned} F(g_t^{-1}(w)) - F(w) &= \Big(\Re\Phi_{a_t,\zeta_1,D_t}(w) + \Im\Phi_{a_t,\zeta_2,D_t}(w)\Big)\frac{1}{2\pi}\int_{s_t} \Re F\big(g_t^{-1}(\zeta)\big)|\mathrm{d}\zeta| \\ &= -\Big(\Re\Phi_{a_t,\zeta_1,D_t}(w) + \Im\Phi_{a_t,\zeta_2,D_t}(w)\Big)\mathfrak{c}(g_t) \end{aligned}$$

for all $w \in D_t$ and some $\zeta_1, \zeta_2 \in s_t$. Next, let us substitute $w = g_t(z)$ so we get for each $z \in \Omega_t$

$$F(g_t(z)) - F(g_0(z)) = \Big(\Re\Phi_{a_t,\zeta_1,D_t}(g_t(z)) + \Im\Phi_{a_t,\zeta_2,D_t}(g_t(z))\Big) \cdot (\mathfrak{c}(g_t) - \mathfrak{c}(g_0)). \quad (3.10)$$

Using Lemma 2.18, 2.19 or 2.20, for each $k \in \{1,2\}$, we find $\Phi_{a_t,\zeta_k,D_t} \xrightarrow{\text{l.u.}} \Phi_{a,U_0,\Omega}$ on Ω. Summarising, as $\zeta_1, \zeta_2 \in s_t \to U_0$, the proof is complete. □

Remark 3.6. Using Equation (3.10), we easily see that the following statement is equivalent to (i) and (ii) of Lemma 3.10.

(iii) There is a $z_0 \in \Omega \setminus \{0\}$ such that $t \mapsto g_t(z_0)$ is differentiable at $t = 0$.

Note that in the chordal and bilateral case we can write $z_0 \in \Omega$ instead of $z_0 \in \Omega \setminus \{0\}$.

Moreover, the proof of Lemma 3.10 shows that $s_t := g_t(\gamma_1[0,t] \cup \gamma_2[0,t]) \to U_0 := \gamma_1(0) = \gamma_2(0)$ as $t \searrow 0$.

Lemma 3.11. *Let Ω be a canonical domain and $(\gamma_1, \gamma_2)_{t\in[0,T]}$ denote a tuple of branched appropriate slits in Ω. For each $t, \tau \in [0,T]$, $f_{t,\tau}$ is the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,\tau])$. Moreover, we set $g_t := f_{t,t}$ for all $t \in [0,T]$. Assume $Z = \{t_0, \dots, t_n\}$ with $t_0 = 0$ and $t_n = t$ is a partition of the interval $[0,t] \subseteq [0,T]$, i.e. $t_0 < t_1 < \dots < t_n$, and*

$$\mathcal{S}_1(f,t,Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{t_{l+1},t_l}) - \mathfrak{c}(f_{t_l,t_l}), \quad \mathcal{S}_2(f,t,Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{t_l,t_{l+1}}) - \mathfrak{c}(f_{t_l,t_l}).$$

Then for each $t \in [0,T]$ and $k \in \{1,2\}$, $\mathcal{S}_k(f,t,Z) \to c_k(t) \ge 0$ as $|Z| \to 0$ whereas $|Z|$ denotes the norm of the partition Z, i.e. $|Z| := \max_{l=0,\dots,n-1} t_{l+1} - t_l$. Moreover, each $t \mapsto c_k(t)$ is continuous and strictly increasing on $[0,T]$, $c_k(0) = 0$, and for each $t_0 \in (0,T]$ and $k \in \{1,2\}$,

$$\frac{c_1(t) - c_1(t_0)}{\mathfrak{c}(f_{t,t_0}) - \mathfrak{c}(f_{t_0,t_0})} \to 1 \text{ as } t \to t_0 \quad \text{and} \quad \frac{c_2(t) - c_2(t_0)}{\mathfrak{c}(f_{t_0,t}) - \mathfrak{c}(f_{t_0,t_0})} \to 1 \text{ as } t \to t_0.$$

Finally, assume $\mathfrak{c}(g_t) = t$ for all $t \in [0,T]$. Then each $t \mapsto c_k(t)$, $k \in \{1,2\}$, is Lipschitz continuous on $[0,T]$ and $\sum_{k=1}^{2} c_k(t) = t$ for all $t \in [0,T]$.

Proof. 1) First of all, we will prove the existence of the functions c_k, $k \in \{1,2\}$. In order to do so, we set

$$\mathcal{S}_1(f,[\underline{t},\bar{t}],Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{t_{l+1},t_l}) - \mathfrak{c}(f_{t_l,t_l}), \quad \mathcal{S}_2(f,[\underline{t},\bar{t}],Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{t_l,t_{l+1}}) - \mathfrak{c}(f_{t_l,t_l}),$$

with $0 \le \underline{t} < \bar{t} \le T$ and a partition $Z := \{t_0,\dots,t_n\}$ of the interval $[\underline{t},\bar{t}]$. Let $\rho > 0$. By Lemma 2.42, the function $t \mapsto \mathfrak{c}(g_t)$ is continuous on $[0,T]$, so we find an $\varepsilon > 0$ such that $\mathfrak{c}(g_t) < \frac{\rho}{4}$ holds for all $t \in [0,\varepsilon]$. Next, we set $h := g_\varepsilon$. Consequently, Δ_1, Δ_2 with $\Delta_k := h(\gamma_k[\varepsilon,T])$ are disjoint appropriate unparametrised slits in the canonical domain $G := h(\Omega \setminus (\gamma_1(\varepsilon,T] \cup \gamma_2(\varepsilon,T]))$. On top of this $(\delta_1,\delta_2)_{s\in[0,T-\varepsilon]}$, with $\delta_k(s) := h(\gamma_k(s+\varepsilon))$ for all $s \in [0,T-\varepsilon]$, is an admissible parametrisation of (Δ_1,Δ_2). Moreover, for each $s,\sigma \in [0,T-\varepsilon]$, we denote by $h_{s,\sigma}$ the normalised appropriate mapping function on $G \setminus (\delta_1(0,s] \cup \delta_2(0,\sigma])$.

Let $t \in (\varepsilon,T]$ be fix. Obviously, this allows us to apply Lemma 3.6, so we find a $\mu > 0$ such that

$$|\mathcal{S}_1(f,[\varepsilon,t],Z_1) - \mathcal{S}_1(f,[\varepsilon,t],Z_2)| = |\mathcal{S}_1(h,t-\varepsilon,Z_1^\varepsilon) - \mathcal{S}_1(h,t-\varepsilon,Z_2^\varepsilon)| < \tfrac{\rho}{2}$$

for all partitions Z_1, Z_2 of the interval $[\varepsilon,t]$ with $|Z_1|,|Z_2| < \mu$ and $Z_k^\varepsilon := Z_k - \varepsilon$. Finally, let Z_1, Z_2 be partitions of the interval $[0,t] \subseteq [0,T]$ with $|Z_1|,|Z_2| < \mu$, so we get

$$\begin{aligned} |\mathcal{S}_1(f,[0,t],Z_1) - \mathcal{S}_1(f,[0,t],Z_2)| \le{}& \\ &|\mathcal{S}_1(f,[0,\varepsilon],Z_1 \cap [0,\varepsilon])| + |\mathcal{S}_1(f,[0,\varepsilon],Z_2 \cap [0,\varepsilon])| \\ &+ |\mathcal{S}_1(f,[\varepsilon,t],Z_1 \cap [\varepsilon,T]) - \mathcal{S}_1(f,[\varepsilon,t],Z_2 \cap [\varepsilon,T])| < \tfrac{\rho}{4} + \tfrac{\rho}{4} + \tfrac{\rho}{2} = \rho, \end{aligned}$$

as we can assume without loss of generality $\varepsilon \in Z_1 \cap Z_2$. We can do the same for $\mathcal{S}_2$ instead of $\mathcal{S}_1$ to get the existence of c_2.

2) Next, let us fix $t_0 \in (0,T]$ and $0 < \varepsilon < t_0$. We use the same notations as in the first part, i.e. $h := g_\varepsilon$, and for all $s,\sigma \in [0,T-\varepsilon]$, $h_{s,\sigma}$ is the normalised appropriate mapping function on $G \setminus (\delta_1[0,s] \cup \delta_2[0,\sigma])$ with $G := h(\Omega \setminus (\gamma_1(0,\varepsilon] \cup \gamma_2(0,\varepsilon]))$ and $\delta_k(s) := h(\gamma_k(\varepsilon+s))$. On top of this we set $c_k^\varepsilon(s) := \lim_{|Z|\to 0} \mathcal{S}_k(h,s,Z)$ for all $s \in [0,T-\varepsilon]$ and $c_k(t) = \lim_{|Z|\to 0} \mathcal{S}_k(f,t,Z)$ for all $t \in [0,T]$ with $k \in \{1,2\}$. Obviously, $c_k(s+\varepsilon) = c_k^\varepsilon(s) + c_k(\varepsilon)$ for all $s \in [0,T-\varepsilon]$. Thus we find with Lemma 3.6

$$\frac{c_1(t) - c_1(t_0)}{\mathfrak{c}(f_{t,t_0}) - \mathfrak{c}(f_{t_0,t_0})} = \frac{c_1^\varepsilon(t-\varepsilon) - c_1^\varepsilon(t_0-\varepsilon)}{\mathfrak{c}(h_{t-\varepsilon,t_0-\varepsilon}) - \mathfrak{c}(f_{t_0-\varepsilon,t_0-\varepsilon})} = \frac{c_1^\varepsilon(s) - c_1^\varepsilon(s_0)}{\mathfrak{c}(h_{s,s_0}) - \mathfrak{c}(f_{s_0,s_0})} \xrightarrow{s\to s_0} 1$$

with $s_0 := t_0 - \varepsilon$ and $s := t - \varepsilon$. Moreover Lemma 3.6 shows that for each fixed $\varepsilon > 0$ and $k \in \{1,2\}$, $s \mapsto c_k^\varepsilon(s)$ is continuous on $[0,T-\varepsilon]$. Together with $c_k(s+\varepsilon) = c_k^\varepsilon(s) + c_k(\varepsilon)$ for all $s \in [0,T-\varepsilon]$ and $k \in \{1,2\}$, we easily see that each $t \mapsto c_k(t)$ is continuous and strictly increasing on $[0,T]$. Finally, assume $\mathfrak{c}(g_t) = t$ for all $t \in [0,T]$. Then $\mathfrak{c}(h_s) = s$ for all $s \in [0,T-\varepsilon]$, so Lemma 3.6 gives us $\sum_{k=1}^2 c_k(s+\varepsilon) = \sum_{k=1}^2 (c_k^\varepsilon(s) + c_k(\varepsilon)) = s + \sum_{k=1}^2 c_k(\varepsilon)$ for all $s \in [0,T-\varepsilon]$. Letting $\varepsilon \to 0$ and by using the continuity, we find $\sum_{k=1}^2 c_k(t) = t$ for all $t \in [0,T]$. Obviously, $t \mapsto c_k(t)$, $k \in \{1,2\}$, is Lipschitz continuous on $[0,T]$ in this case as well, as $c_k(t) \ge 0$ for all $t \in [0,T]$.

□

3.2.2 Proof of Theorem 3.7 and 3.8

In order to prove Theorem 3.7, we are going to use the disjoint case (and the corresponding proof) in a major way.

Note that we can apply step 1) and 2) of the existence proof of Theorem 3.2, 3.3 and 3.4 on branched slits as well. We call $(u_n^*, v_n^*, u, v)_{t\in[0,L]}$ *bang–bang functions corresponding to* $(\gamma_1, \gamma_2)_{t\in[0,T]}$. Using the notation of Theorem 3.7, step 1) and 2) give us $L = \mathfrak{c}_\Omega(\Gamma_1 \cup \Gamma_2)$ like in the disjoint case.

Unfortunately, we can not apply step 3), 4) and 5) (directly) in the branch point case. The reason for this is that we proved Proposition 2.55 in the disjoint case only. Consequently, one might ask the question if Proposition 2.55 is true in the branch point case as well. In general, this is not the case, so there are counterexamples, see Section 4.2. Nevertheless, we can use step 3), 4) and 5) in order to find the following lemma.

Lemma 3.12. *Let* Ω *be a canonical domain,* $(\gamma_1, \gamma_2)_{t\in[0,T]}$ *be a tuple of branched appropriate slits in* Ω *and for each* $t, \tau \in [0, T]$*,* $f_{t,\tau}$ *denotes the normalised appropriate mapping function on* $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$*. Assume* $(u_n^*, v_n^*, u, v)_{t\in[0,L]}$ *are bang–bang functions corresponding to* $(\gamma_1, \gamma_2)_{t\in[0,T]}$*.*

Then $u, v : [0, L] \to [0, T]$ *are continuous and strictly increasing,* $\mathfrak{c}(f_{u(t),v(t)}) = t$ *for all* $t \in [0, L]$ *and* $\mathcal{S}_1(u, v, [\underline{t}, \bar{t}], Z) \to (\bar{t} - \underline{t})\lambda$ *and* $\mathcal{S}_2(u, v, [\underline{t}, \bar{t}], Z) \to (\bar{t} - \underline{t})(1 - \lambda)$ *with* $\underline{t}, \bar{t} \in S := \bigcup_{n\in\mathbb{N}}\{\frac{k}{2^n}L \mid k \in \{0, \dots, 2^n\}\}$*,* $0 < \underline{t} < \bar{t}$*,* $\lambda \in (0, 1)$ *and* $L = \mathfrak{c}(f_{T,T})$*.*

Herein, for the definition of $\mathcal{S}_k(u, v, [\underline{t}, \bar{t}], Z)$ *see Equation* (3.2)*.*

Proof. Summarising, step 1) and 2) yield that $t \mapsto u(t)$ and $t \mapsto v(t)$ are continuous and increasing on $[0, L]$ and $\mathfrak{c}(f_{u(t),v(t)}) = t$ for all $t \in [0, L]$. As mentioned before, this gives us $L = \mathfrak{c}(f_{T,T})$.

Let $\underline{t}, \bar{t} \in S$ with $0 < \underline{t} < \bar{t}$. Then either $u(\underline{t}) \neq 0$ or $v(\underline{t}) \neq 0$. Thus we find an $\varepsilon > 0$ such that $\varepsilon < u(\underline{t})$ or $\varepsilon < v(\underline{t})$. Without loss of generality we assume $\varepsilon < u(\underline{t})$. Then we set $h := f_{\varepsilon,0}$, $G := h(\Omega \setminus \gamma_1[0, \varepsilon])$, $\Delta_1 := h(\gamma_1[\varepsilon, T])$ and $\Delta_2 := h(\gamma_2[0, T])$. Obviously, Δ_1, Δ_2 are disjoint appropriate unparametrised slits in the canonical domain G. Moreover, we find an $n_0 \in \mathbb{N}$ in order to get $u_n^*(\underline{t}) > \varepsilon$ for all $n \geq n_0$. Since Δ_1 and Δ_2 are disjoint we can apply step 3), 4) and 5), so $t \mapsto u(t)$ and $t \mapsto v(t)$ are strictly increasing on $[\underline{t}, L]$, $\mathcal{S}_1(u, v, [\underline{t}, \bar{t}], Z) \to (\bar{t} - \underline{t})\lambda$ as $|Z| \to 0$ with $\lambda \in (0, 1)$ and $\mathcal{S}_2(u, v, [\underline{t}, \bar{t}], Z) \to (\bar{t} - \underline{t})(1 - \lambda)$ as $|Z| \to 0$. Note that $\underline{t} > 0$ is arbitrary, so $t \mapsto u(t)$ and $t \mapsto v(t)$ are strictly increasing on $[0, L]$. □

Now we are able to prove Theorem 3.7.

Proof of Theorem 3.7. Let Ω be a canonical domain and $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be appropriate branched slits in Ω. For each $t, \tau \in [0, T]$ we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega(t, \tau) := \Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$. Assume $(u_n^*, v_n^*, u, v)_{t\in[0,L]}$ are bang–bang functions corresponding to $(\gamma_1, \gamma_2)_{t\in[0,T]}$ with $L = \mathfrak{c}(f_{T,T})$. We set $h_{t,\tau} := f_{u(t),v(\tau)}$ for all $t, \tau \in [0, L]$. Using Lemma 3.11, we find strictly increasing and continuous functions

$$c_k(t) := \lim_{|Z|\to 0} \mathcal{S}_k(h, t, Z) := \lim_{|Z|\to 0} \mathcal{S}_k(u, v, t, Z) \qquad \text{for all } t \in [0, L] \text{ and } k \in \{1, 2\}.$$

Next, Lemma 3.12 gives us $c_1(t) - c_1(\varepsilon) = (t-\varepsilon)\lambda$ and $c_2(t) - c_2(\varepsilon) = (t-\varepsilon)(1-\lambda)$ for all $\varepsilon, t \in S$ with $0 < \varepsilon < t$, and some $\lambda \in (0,1)$. Letting $\varepsilon \searrow 0$ we get $c_1(t) = \lambda t$ and $c_2(t) = (1-\lambda)t$ for all $t \in S$. $t \mapsto c_k(t)$ is continuous on $[0, L]$, so we find $c_1(t) = \lambda t$ and $c_2(t) = (1-\lambda)t$ for all $t \in [0, L]$. Using Lemma 3.11, for each $t_0 \in (0, L]$, we find

$$\lim_{t\to t_0} \frac{\mathfrak{c}(h_{t,t_0}) - \mathfrak{c}(h_{t_0,t_0})}{t - t_0} = \lambda, \quad \lim_{t\to t_0} \frac{\mathfrak{c}(h_{t_0,t}) - \mathfrak{c}(h_{t_0,t_0})}{t - t_0} = 1 - \lambda.$$

For all $t \in [0, L]$, we set $g_t := h_{t,t}$ and $D_t := g_t(\Omega(t, t))$. Consequently, Lemma 3.9 yields

$$\dot{g}_t(z) = E\big(h_t(z)\big) \sum_{k=1}^{2} \lambda_k \Phi_{a_t,U_k(t),D_t}\big(g_t(z)\big), \quad \text{for all } t \in (0, L] \text{ and all } z \in \Omega_L,$$

with $\lambda_1 := \lambda$ and $\lambda_2 = 1 - \lambda$. Herein, for all $t \in [0, L]$, $a_t := 0$ in the radial case, a_t denotes the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case. Herein, for all $w \in \mathbb{C}$, $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2\mathrm{i}}$ in the chordal case. Moreover, $U_1(t) := g_t(\gamma_1(u(t)))$ and $U_2(t) := g_t(\gamma_2(v(t)))$ for all $t \in [0, L]$. Note that $\mathfrak{c}(h_t) = t$, so Lemma 3.10 gives us

$$\dot{g}_0(z) = E(z)\Phi_{a_0,\gamma_1(0),\Omega}\big(z\big) \quad \text{for all } z \in \Omega.$$

$\gamma_1(0) = \gamma_2(0) = U_1(0) = U_2(0)$, $D_0 = \Omega$, so we find

$$\dot{g}_0(z) = E\big(g_0(z)\big) \sum_{k=1}^{2} \lambda_k \Phi_{a_0,U_k(0),D_0}\big(g_0(z)\big) \quad \text{for all } z \in \Omega.$$

Summarising, $g_t(z)$ satisfies a Komatu–Loewner equation with constant coefficients. □

Lemma 3.13. *Let Ω be a canonical simply connected domain and denote by $(\gamma_1, \gamma_2)_{t\in[0,T]}$ a tuple of disjoint or branched slits in Ω. For each $t, \tau \in [0, T]$, we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$ and we set $\mathfrak{c}(t, \tau) := \mathfrak{c}(f_{t,\tau})$. Assume $0 \le \underline{t} \le \bar{t} \le T$ and $0 \le \underline{\tau} \le \bar{\tau} \le T$. Then*

$$\mathfrak{c}(\bar{t}, \bar{\tau}) - \mathfrak{c}(\underline{t}, \bar{\tau}) \le \mathfrak{c}(\bar{t}, \underline{\tau}) - \mathfrak{c}(\underline{t}, \underline{\tau}).$$

Proof. Assume $0 \le \underline{t} < \bar{t} \le T$ and $0 \le \underline{\tau} < \bar{\tau} \le T$. Let $h := f_{\underline{t},\underline{\tau}}$, $G := h(\Omega \setminus (\gamma_1(0, \underline{t}] \cup \gamma_2(0, \underline{\tau}]))$ and $\Delta_1 := h(\gamma_1(\underline{t}, \bar{t}])$ and $\Delta_2 := h(\gamma_2(\underline{\tau}, \bar{\tau}])$. Note that Δ_1, Δ_2 and $\Delta_1 \cup \Delta_2$ are appropriate hulls in the canonical domain G and we denote by h_{Δ_1}, h_{Δ_2} and $h_{\Delta_1\cup\Delta_2}$ the normalised appropriate mapping functions on $G \setminus \Delta_1$, $G \setminus \Delta_2$ and $G \setminus (\Delta_1 \cup \Delta_2)$, respectively.

Consequently, we can apply Lemma 2.56 and 2.57 to get

$$\mathfrak{c}(h_{\Delta_1\cup\Delta_2}) \le \mathfrak{c}(h_{\Delta_1}) + \mathfrak{c}(h_{\Delta_2}).$$

Note that $\mathfrak{c}(h_{\Delta_1\cup\Delta_2}) = \mathfrak{c}(\bar{t}, \bar{\tau}) - \mathfrak{c}(\underline{t}, \underline{\tau})$, $\mathfrak{c}(h_{\Delta_1}) = \mathfrak{c}(\bar{t}, \underline{\tau}) - \mathfrak{c}(\underline{t}, \underline{\tau})$ and $\mathfrak{c}(h_{\Delta_2}) = \mathfrak{c}(\underline{t}, \bar{\tau}) - \mathfrak{c}(\underline{t}, \underline{\tau})$, so the proof is complete. □

Proof of Theorem 3.8. 1) First of all, let Ω be a simply connected canonical domain, $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be appropriate branched slits in Ω and let $L := \mathfrak{c}_\Omega(\Gamma_1 \cup \Gamma_2)$. For each $t, \tau \in [0, T]$, we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0,t]\cup\gamma_2(0,t])$. Assume $u_1, v_1 : [0, L] \to [0, T]$ and $u_2, v_2 : [0, L] \to [0, T]$ are increasing homeomorphisms having $\mathfrak{c}(u_1(t), v_1(t)) = t = \mathfrak{c}(u_2(t), v_2(t))$ for all $t \in [0, L]$, and for each $t_0 \in [0, L]$ and $k \in \{1, 2\}$, $t \mapsto \mathfrak{c}(u_k(t), v_k(t_0))$ and $t \mapsto \mathfrak{c}(u_k(t_0), v_k(t))$ are differentiable at t_0 with constant derivatives λ_k and $1 - \lambda_k$, respectively. Using Lemma 3.9 and 3.10, this is equivalent to claim that each $t \mapsto f_{u_k(t),v_k(t)}$ fulfils a multiple slit Loewner equation with constant coefficients λ_k and $1 - \lambda_k$, $k \in \{1, 2\}$.

2) Next, suppose $\lambda_1 > \lambda_2$. Note that $t \mapsto \mathfrak{c}(u_k(t), T)$ is differentiable at $t = L$ with derivative λ_k, $k \in \{1, 2\}$. Moreover, $\mathfrak{c}(u_1(L), T) = L = \mathfrak{c}(u_2(L), T)$, so we find $\mathfrak{c}(u_1(t), T) < \mathfrak{c}(u_2(t), T)$ for all $t \in (L - \varepsilon, L)$ with a small $\varepsilon > 0$, as $\lambda_1 > \lambda_2$. Using Lemma 2.41, we find $u_1(t) < u_2(t)$ for all $t \in (L - \varepsilon, L)$ as well. Let us denote by $t_0 \in [0, L)$ the unique time such that $t_0 := \sup\{t \in [0, L) \mid u_1(t) = u_2(t)\}$. Using $u_1(t) < u_2(t)$ for all $t \in (L - \varepsilon, L)$, we find $t_0 < L$. Consequently, $u_1(t) < u_2(t)$ for all $t \in (t_0, L)$.

Next, let $Z_2 := \{t_0, \dots, t_n\}$ be a partition of the interval $[t_0, L]$, say $t_l = t_0 + \frac{l}{n}(L - t_0)$ for all $l \in \{0, \dots, n\}$. Moreover, we find unique values $\tau_0, \dots, \tau_n \in [t_0, L]$ such that $u_1(\tau_l) = u_2(t_l)$ for all $l \in \{1, \dots, n\}$. Thus $Z_1 := \{\tau_0, \dots \tau_n\}$ is a partition of the interval $[t_0, L]$. Using $u_1(t) < u_2(t)$ for all $t \in (t_0, L)$, we find $\tau_l \geq t_l$ for all $l \in \{0, \dots, n\}$. Since $\mathfrak{c}(u_2(t_l), v_2(t_l)) = t_l \leq \tau_l = \mathfrak{c}(u_1(\tau_l), v_1(\tau_l))$, Lemma 2.41 gives us $v_1(\tau_l) \geq v_2(t_l)$ for all $l \in \{0, \dots, n\}$. Using Lemma 3.13, we find

$$\begin{aligned}\mathfrak{c}\big(u_2(t_{l+1}), v_2(t_l)\big) - \mathfrak{c}\big(u_2(t_l), v_2(t_l)\big) &\geq \mathfrak{c}\big(u_2(t_{l+1}), v_1(\tau_l)\big) - \mathfrak{c}\big(u_2(t_l), v_1(\tau_l)\big)\\ &= \mathfrak{c}\big(u_1(\tau_{l+1}), v_1(\tau_l)\big) - \mathfrak{c}\big(u_1(\tau_l), v_1(\tau_l)\big)\end{aligned}$$

for all $l \in \{0, \dots, n - 1\}$. Consequently, we get

$$\sum_{l=0}^{n-1} \mathfrak{c}\big(u_2(t_{l+1}), v_2(t_l)\big) - \mathfrak{c}\big(u_2(t_l), v_2(t_l)\big) \geq \sum_{l=0}^{n-1} \mathfrak{c}\big(u_1(\tau_{l+1}), v_1(\tau_l)\big) - \mathfrak{c}\big(u_1(\tau_l), v_1(\tau_l)\big).$$

Using Lemma 3.11, we see that the term on the left-hand side tends to $\lambda_2(L - t_0)$, while the right-hand side tends to $\lambda_1(L - t_0)$, so we find $\lambda_2 \geq \lambda_1$. This is a contradiction, as $\lambda_2 < \lambda_1$, so $\lambda_1 = \lambda_2 =: \lambda$.

3) Finally, we are going to show $u_1(t) = u_2(t)$ for all $t \in [0, L]$. Let $t \in [0, L]$ be fix and suppose $u_1(t) < u_2(t)$. As before we find a unique $t_0 := \sup\{\tau \in [0, t) \mid u_1(\tau) = u_2(\tau)\}$. Using the continuity of u_1 and u_2, we find $t_0 < t$. Consequently, $u_1(\tau) < u_2(\tau)$ for all $\tau \in (t_0, t]$.

Next, let $\{t_0, \dots, t_n\}$ be a partition of the interval $[t_0, t]$, say $t_l = t_0 + \frac{l}{n}(t - t_0)$ with $l \in \{0, \dots, n\}$ and some $n \in \mathbb{N}$. Moreover, we find unique values $\tau_0, \dots, \tau_n \in [t_0, L]$ such that $u_2(t_l) = u_1(\tau_l)$ for all $l \in \{0, \dots, n\}$. Note that $Z_1 := \{\tau_0, \dots, \tau_n\}$ is a partition of the interval $[t_0, \tau]$ where $\tau \in (t_0, L]$ satisfies $u_1(\tau) = u_2(t)$. Consequently, $\tau > t$ as well as $\tau_l \geq t_l$ for all $l \in \{0, \dots, n\}$. Like in the previous part, we get $v_2(t_l) \leq v_1(\tau_l)$ for all

$l = \{0, \dots, n\}$, so, together with Lemma 3.13, we find

$$\sum_{l=0}^{n-1} \mathfrak{c}(u_2(t_{l+1}), v_2(t_l)) - \mathfrak{c}(u_2(t_l), v_2(t_l)) \geq \sum_{l=0}^{n-1} \mathfrak{c}(u_1(\tau_{l+1}), v_1(\tau_l)) - \mathfrak{c}(u_1(\tau_l), v_1(\tau_l)).$$

Using Lemma 3.11, the left-hand side tends to $\lambda(t - t_0)$, while the right-hand side tends to $\lambda(\tau - t_0)$, so we get $t \geq \tau$. This is a contradiction as $t < \tau$, so the proof is complete. $\square$

Chapter 4

Komatu–Loewner equations vs. Loewner equations

Let Ω be a canonical domain and denote by $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ a tuple of disjoint appropriate slits in Ω. For each $t \in [0, T]$, we denote by g_t the normalised appropriate mapping function on $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0, t]$. Moreover, for each $k \in \{1, \ldots, m\}$ and $t \in [0, T]$, we set $h_{k;t}$ as the normalised appropriate mapping function on $\Omega^S \setminus \gamma_k(0, t]$. In this context Ω^S is the simplification of Ω, i.e. $\Omega^S := \mathbb{D}$ if Ω is a circular slit disk, $\Omega^S := \mathbb{H}$ if Ω is an upper parallel slit half-plane, and $\Omega^S := \mathbb{A}_Q = \{z \in \mathbb{C} \mid Q < |z| < 1\}$ if Ω is a circular slit annulus with inner radius $Q \in (0, 1)$, see also Section 2.7.

Then one might ask whether there is a connection between differentiability of $t \mapsto g_t$ and differentiability $t \mapsto h_{k;t}$ with $k \in \{1, \ldots, m\}$? See also Figure 4.1 where we put g_t side by side to $h_{k;t}$.

We will discuss this question in the disjoint case and in the branch point case separately.

4.1 Disjoint slits

Theorem 4.1. *Let Ω be a canonical domain and denote by $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ a tuple of disjoint appropriate slits in Ω. For each $t \in [0, T]$, we denote by g_t the normalised appropriate mapping function on $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0, t]$. Moreover, for each $k \in \{1, \ldots, m\}$ and $t \in [0, T]$, we set $h_{k;t}$ as the normalised appropriate mapping function on $\Omega^S \setminus \gamma_k(0, t]$. Let $t_0 \in [0, T]$. Then the following two statements are equivalent.*

(i) $t \mapsto g_t(z)$ is differentiable at $t = t_0$ for each $z \in \Omega_{t_0}$.

(ii) Each $t \mapsto h_{k;t}(z)$, $k \in \{1, \ldots, m\}$, is differentiable at $t = t_0$ for each $z \in \Omega^S \setminus \gamma_k(0, t_0]$.

Proof. Basically, this follows immediately from Proposition 2.55 and Theorem 2.30, 2.31 and 2.36. Therefore, for each $t, \tau \in [0, T]$, $f_{k;t,\tau}$ denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_k(0, t] \cup \bigcup_{j\neq k} \gamma_j(0, \tau])$. Let $t_0 \in [0, T]$. Using Proposition

2.55, we get for each $k \in \{1, \ldots, m\}$:

$$\lambda_k(t_0) := \lim_{t \to t_0} \frac{\mathfrak{c}(f_{k;t,t_0}) - \mathfrak{c}(f_{k;t_0,t_0})}{t - t_0} \text{ exists } \Leftrightarrow \ \mu_k(t_0) := \lim_{t \to t_0} \frac{\mathfrak{c}(h_{k;t}) - \mathfrak{c}(h_{k;t_0})}{t - t_0} \text{ exists.}$$

Using (i)⇔(ii) from Theorem 2.30, 2.31 or 2.36, we find: $t \mapsto g_t(z)$ is differentiable at t_0 for each $z \in \Omega_{t_0}$ if and only if each limit $\lambda_k(t_0)$, $k \in \{1, \ldots m\}$, exists. In the same way, for each $k \in \{1, \ldots, m\}$, $t \mapsto h_{k;t}(z)$ is differentiable at t_0 for all $z \in \Omega^S \setminus \gamma_k(0, t_0]$ if and only if the limit $\mu_k(t_0)$ exists. □

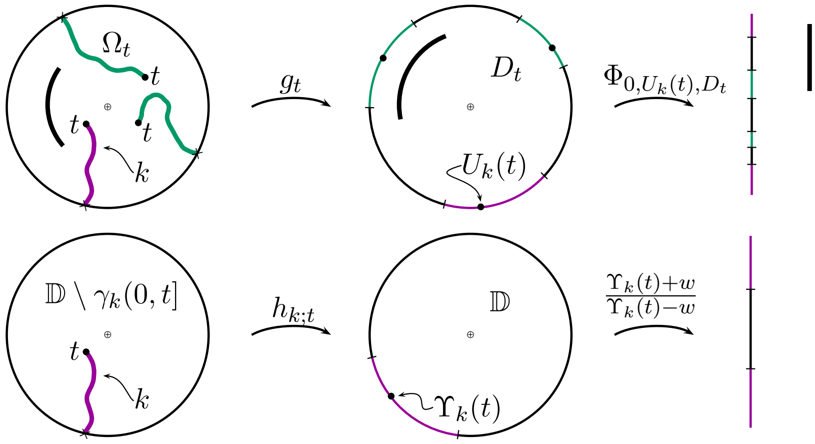

FIGURE 4.1: *The mapping functions g_t and $h_{k;t}$ in the radial case; note that $\Omega^S = \mathbb{D}$ and $\Phi_{b_t,\Upsilon_k(t),\Delta_k(t)}(w) = (\Upsilon_k(t) + w)/(\Upsilon_k(t) - w)$*

We find easily with (ii)⇔(iii) from Theorem 2.30, 2.31 or 2.36:

Corollary 4.2. *Let Ω be a canonical domain and denote by $(\gamma_1, \ldots, \gamma_m)_{t\in[0,T]}$ a tuple of disjoint appropriate slits in Ω. For each $t, \tau \in [0, T]$ and $k \in \{1, \ldots, m\}$, we denote by $f_{k;t,\tau}$ the normalised appropriate mapping function from $\Omega_k(t, \tau) := \Omega \setminus (\gamma_k(0, t] \cup \bigcup_{j\neq k} \gamma_j(0, \tau])$ onto the canonical domain $D_k(t, \tau)$. Independently of $k \in \{1, \ldots, m\}$, we set $g_t := f_{k;t,t}$, $\Omega_t := \Omega_k(t, t)$ and $D_t := D_k(t, t)$ for all $t \in [0, T]$. Moreover, for each $k \in \{1, \ldots, m\}$, we set $h_{k;t}$ as the normalised appropriate mapping function from $\Omega^S \setminus \gamma_k(0, t]$ onto $\Delta_k(t)$ with $t \in [0, T]$. Finally, let $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2i}$ in the chordal case. Let $t_0 \in [0, T]$. Then the following two statements are equivalent.*

(i) $t \mapsto g_t(z)$ is differentiable at $t = t_0$ for each $z \in \Omega_{t_0}$ and satisfies

$$\dot{g}_{t_0}(z) = E\big(g_{t_0}(z)\big) \sum_{k=1}^{m} \lambda_k(t_0) \Phi_{a_{t_0},U_k(t_0),D_{t_0}}\big(g_{t_0}(z)\big) \qquad \text{for all } z \in \Omega_{t_0},$$

with $\lambda_k(t_0) \geq 0$ and $U_k(t) := g_t(\gamma_k(t))$ continuous on $[0, T]$. Herein, $a_t := 0$ in the radial case, a_t is the inner radius of D_t in the bilateral case and $a_t := \infty$ in the chordal case.

(ii) *Each $t \mapsto h_{k;t}(z)$, $k \in \{1, \dots, m\}$, is differentiable at $t = t_0$ for all $z \in \Omega^S \setminus \gamma_k(0, t_0]$, and satisfies*

$$\dot{h}_{k;t_0}(z) = E\big(h_{k;t_0}(z)\big)\mu_k(t_0)\Phi_{b_{t_0},\Upsilon_k(t_0),\Delta_k(t_0)}\big(h_{k;t_0}(z)\big) \quad \text{for all } z \in \Omega^S \setminus \gamma_k(0, t_0],$$

with $\mu_k(t_0) \geq 0$ and $\Upsilon_k(t) := h_{k;t}(\gamma_k(t))$ continuous on $[0, T]$. Herein, $b_t := 0$ in the radial case, b_t is the inner radius of $\Delta_k(t)$ in the bilateral case and $b_t := \infty$ in the chordal case.

When this happens, $\lambda_k(t_0) = |\alpha_k^2(t_0)|\,\mu_k(t_0)$ for all $k \in \{1, \dots, m\}$ where each $t \mapsto |\alpha_k(t)| := |(g_t \circ h_{k;t}^{-1})'(\Upsilon_k(t))|$ is a positive continuous function on $[0, T]$.

In this context, each $t \mapsto |\alpha_k^2(t)|$ represents a distortion factor. Note that positivity and continuity of $t \mapsto |\alpha_k(t)|$ is an immediate consequence of Proposition 2.55.

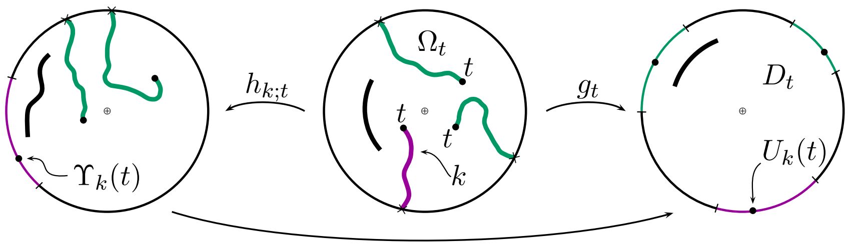

FIGURE 4.2: *The mapping $g_t \circ h_{k;t}^{-1}$ involved in the distortion factor $t \mapsto |\alpha_k(t)|$*

Next, we will use the previous results to give an idea how to find admissible parametrisations of unparametrised slits, in order to get Komatu–Loewner equations with differentiability everywhere.

Corollary 4.3. *Let Ω be a canonical domain and denote by $(\Gamma_1, \dots, \Gamma_m)$ a tuple of disjoint unparametrised appropriate slits in Ω. Then we find an admissible parametrisation $(\gamma_1, \dots, \gamma_m)_{t\in[0,L]}$ such that for each $z \in \Omega_t$, $t \mapsto g_t(z)$ is (continuously) differentiable on $[0, L]$ and satisfies*

$$\dot{g}_t(z) = E\big(g_t(z)\big) \cdot \sum_{k=1}^{m} \lambda_k(t)\Phi_{a_t,U_k(t),D_t}\big(g_t(z)\big) \qquad \text{for all } z \in \Omega_T \text{ and all } t \in [0, L],$$

where $t \mapsto U_k(t) := g_t(\gamma_k(t))$ and $t \mapsto \lambda_k(t) > 0$ are continuous on $[0, L]$ for each $k \in \{1, \dots, m\}$. Moreover, $\lambda_1, \dots, \lambda_m$ are normalised in the following sense: $\sum_{k=1}^m \lambda_k(t) = 1$ for all $t \in [0, L]$.

For each $t \in [0, L]$, g_t denotes the normalised appropriate mapping function from $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0, t]$ onto the canonical domain D_t. Moreover, for all $w \in \mathbb{C}$, we set $E(w) := w$ in the radial and bilateral case and $E(w) := \frac{1}{2i}$ in the chordal case. $a_t := 0$ in the radial case, a_t is the inner radius of D_t in the bilateral case, and $a_t := \infty$ in the chordal case for all $t \in [0, L]$.

Proof. First of all, let us fix $k \in \{1, \dots, m\}$ and $T > 0$. Note that we find an admissible parametrisation $\delta_k : [0, T] \to \Gamma_k$ such that $t \mapsto \mathfrak{c}(h_{k;t}) = L_k \frac{t}{T}$ with $L_k = \mathfrak{c}_\Omega(\Gamma_k) > 0$

for all $t \in [0,T]$. For each $t \in [0,T]$, $h_{k;t}$ denotes the normalised appropriate mapping function on $\Omega^S \setminus \delta_k(0,t]$. To see this, let $\tilde{\delta}_k$ be an arbitrary parametrisation of Γ_k, so $\tilde{\delta}_k : [0,T_k] \to \Gamma_k$ with $T_k > 0$. Let $\tilde{h}_{k;t}$ be the normalised appropriate mapping function on $\Omega^S \setminus \tilde{\delta}_k(0,t]$. Then $t \mapsto c_k(t) := \mathfrak{c}(\tilde{h}_{k;t})$ is an increasing homeomorphism from $[0,T_k]$ onto $[0,L_k]$. Next, let $\delta_k(t) := (\tilde{\delta}_k \circ c_k^{-1})(L_k \frac{t}{T})$, so $\mathfrak{c}(h_{k;t}) = L_k \frac{t}{T}$ for all $t \in [0,T]$.

Note that we can do this for each $k \in \{1,\dots,m\}$, so $(\delta_1,\dots,\delta_m)_{[0,T]}$ is an admissible parametrisation of the tuple $(\Gamma_1,\dots,\Gamma_m)$. Using Theorem 2.30, 2.31 or 2.36 applied to the single slit case, $h_{1;t},\dots,h_{m;t}$ satisfy condition (ii) of Corollary 4.2 for each $t_0 \in [0,T]$. For each $t \in [0,T]$, $\tilde{g}_t$ denotes the normalised appropriate mapping function on $\Omega \setminus \bigcup_{k=1}^m \delta_k(0,t]$ and we set $c_t := \mathfrak{c}(\tilde{g}_t)$. Using Corollary 4.2, $t \mapsto \tilde{g}_t(z)$ is differentiable on $[0,T]$ for all $z \in \Omega \setminus \bigcup_{k=1}^m \Gamma_k$. Moreover, $t \mapsto c_t$ is an increasing homeomorphism of $[0,T]$ onto $[0,L]$ with $L > 0$ and $t \mapsto c_t$ is continuously differentiable with positive derivative on $[0,T]$. Note that the positivity and continuity is a consequence of the positivity and continuity of the distortion factor together with Theorem 2.30, 2.31 or 2.36. We set $d_t := c_t^{-1}$ for all $t \in [0,L]$.

Finally, let us define $\gamma_k(t) := \delta_k(d_t)$ for all $t \in [0,L]$ and $k \in \{1,\dots,m\}$. Note that, for each $k \in \{1,\dots,m\}$, $t \mapsto \mathfrak{c}(h_{k;d_t}) = \frac{L_k}{T} d_t$ is continuously differentiable on $[0,L]$. Again $h_{1;d_t},\dots,h_{m;d_t}$ satisfy condition (ii) of Corollary 4.2, so using Corollary 4.2, $g_t := \tilde{g}_{d_t}$ satisfies condition (i). Using the same notations as in Corollary 4.2, each $t \mapsto \mu_k(t)$ is continuous on $[0,T]$, as $\mu_k(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{c}(h_{k;d_t}) = \frac{L_k}{T}\dot{d}_t > 0$. Hence, $t \mapsto \lambda_k(t)$ is continuous and positive on $[0,L]$ as well. Moreover, $\mathfrak{c}(g_t) = \mathfrak{c}(\tilde{g}_{d_t}) = (c \circ d)(t) = t$ for all $t \in [0,T]$, so $\sum_{k=1}^m \lambda_k \equiv 1$. □

The previous corollary gives a construction how to find tuples of multiple slits that lead to continuous normalised Komatu–Loewner equations, i.e. the mapping function g_t fulfils a differential equation everywhere (and not only almost everywhere) with normalised λ_k. The idea was to start with $m \in \mathbb{N}$ single slit Loewner equations that are everywhere differentiable. Using Corollary 4.2, we get differentiability in the multiple slit setting as well. Finally, a normalisation afterwards gives us normalised weights λ_k.

Next, we are going to construct tuples of multiple slits leading to continuous Komatu–Loewner equations that are already normalised. Again, this is based on single slit Loewner equations. Unfortunately, we can do this in simply connected domains only. The reason for this is that we need the subadditivity of $\mathfrak{c}$, see Lemma 2.57 and 2.56.

Proposition 4.4. *Let Ω be a simply connected canonical domain and denote by (Γ_1,Γ_2) a tuple of branched or disjoint unparametrised appropriate slits in Ω. Moreover, $L := \mathfrak{c}_\Omega(\Gamma_1 \cup \Gamma_2)$. Assume $(\gamma_1)_{t\in[0,L]}$ is an admissible parametrisation of Γ_1 such that $t \mapsto \mathfrak{c}(h_{1;t})$ is Lipschitz continuous on $[0,L]$ with a Lipschitz constant $K < 1$. Herein, for each $t \in [0,L]$, $h_{1;t}$ denotes the normalised appropriate mapping function on $\Omega \setminus \gamma_1(0,t]$.*

Then we find a unique admissible parametrisation $(\gamma_2)_{t\in[0,L]}$ of Γ_2 such that $\mathfrak{c}(g_t) = t$ for all $t \in [0,L]$ where g_t denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$ for all $t \in [0,L]$.

Moreover, assume $\Gamma_1 \cap \Gamma_2 = \emptyset$, i.e. (Γ_1,Γ_2) is a tuple of disjoint unparametrised appropriate slits in Ω. For each $t \in [0,L]$, $h_{2;t}$ denotes the normalised appropriate mapping function on $\Omega \setminus \gamma_2(0,t]$. Then $t \mapsto h_{1;t}$ is differentiable at t_0 if and only if

$t \mapsto h_{2;t}$ or $t \mapsto g_t$ is differentiable[9] *at t_0.*

Proof. 1) First of all, let $(\delta_2)_{t\in[0,L]}$ be an arbitrary admissible parametrisation of Γ_2. Moreover, we denote by $\tilde{f}_{t,\tau}$ the normalised mapping function on $\Omega \setminus (\gamma_1(0,t] \cup \delta_2(0,\tau])$ with $t, \tau \in [0, L]$. Note that $\mathfrak{c}(h_{1;t}) < t$ for all $t \in [0, L]$. Consequently, for each $t \in [0, L]$, we find a unique $\tau_t \in [0, L]$ such that $\mathfrak{c}(\tilde{f}_{t,\tau_t}) = t$. Hence, we get a unique continuous function $\tau : [0, L] \to [0, L]$ such that $\mathfrak{c}(\tilde{f}_{t,\tau_t}) = t$ for all $t \in [0, L]$, and $\tau_0 = 0$ and $\tau_L = L$. Note that the continuity is an immediate consequence Lemma 2.42.

Next, we set $\gamma_2(t) := \delta_2(\tau_t)$ for all $t \in [0, L]$. Consequently, it remains to prove that $\gamma_2 : [0, L] \to \Gamma_2$ is bijective. In order to prove the bijective correspondence let $0 \le t_1 < t_2 \le L$ and assume $\gamma_2(t_1) = \gamma_2(t_2)$. For each $t, \tau \in [0, T]$, we denote by $f_{t,\tau}$ the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, \tau])$. Lemma 3.13 gives us

$$\begin{aligned} t_2 - t_1 &= \mathfrak{c}(f_{t_2,t_2}) - \mathfrak{c}(f_{t_1,t_1}) = \mathfrak{c}(f_{t_2,t_1}) - \mathfrak{c}(f_{t_1,t_1}) \\ &\le \mathfrak{c}(f_{t_2,0}) - \mathfrak{c}(f_{t_1,0}) = \mathfrak{c}(h_{1;t_2}) - \mathfrak{c}(h_{1;t_1}) < t_2 - t_1. \end{aligned}$$

This is a contradiction, so γ_2 needs to be bijective.

2) Additionally, assume (Γ_1, Γ_2) is a tuple of disjoint unparametrised appropriate slits in Ω.

Let be $Z = \{s_0, \dots, s_n\}$ be a partition of the interval $[0, t] \subseteq [0, L]$ and we set:

$$\mathcal{S}_1(f, t, Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{s_{l+1},s_l}) - \mathfrak{c}(f_{s_l,s_l}), \quad \mathcal{S}_2(f, t, Z) := \sum_{l=0}^{n-1} \mathfrak{c}(f_{s_l,s_{l+1}}) - \mathfrak{c}(f_{s_l,s_l}).$$

By Lemma 3.6, each limit $c_k(t) := \lim_{|Z|\to 0} \mathcal{S}_k(f, t, Z)$, $k \in \{1, 2\}$, exists and forms an increasing and Lipschitz continuous function $t \mapsto c_k(t)$. Moreover, Lemma 3.6 gives us $c_1(t) + c_2(t) = t$ for all $t \in [0, L]$ as $\mathfrak{c}(g_t) = t$ for all $t \in [0, L]$. Using Proposition 2.55 and Lemma 3.6, for each $k \in \{1, 2\}$, $t \mapsto c_k(t)$ is differentiable at t_0 if and only if $t \mapsto h_{k;t}$ is differentiable at t_0. For each $t \in [0, T]$, we have $c_2(t) = t - c_1(t)$, so $t \mapsto c_2(t)$ is differentiable at t_0 if and only if $t \mapsto c_1(t)$ is differentiable at t_0. Summarising, $t \mapsto h_{2;t}$ is differentiable at t_0 if and only if $t \mapsto h_{1;t}$ is differentiable at t_0. Using Theorem 4.1, $t \mapsto g_t$ is differentiable if and only if $t \mapsto h_{1;t}$ and $t \mapsto h_{2;t}$ are differentiable at t_0

□

Example 4.1. Let Ω be a simply connected canonical domain and denote by (Γ_1, Γ_2) a tuple of disjoint unparametrised slits in Ω with $\mathfrak{c}_\Omega(\Gamma_1 \cup \Gamma_2) = 1$. Using Lemma 2.41, $L_1 := \mathfrak{c}_\Omega(\Gamma_1) < 1$ as well as $L_2 := \mathfrak{c}_\Omega(\Gamma_2) < 1$. Consequently, we find an $\varepsilon > 0$ such that $L_1 + \varepsilon < 1$ as well. Then we define

$$u_1 : [0, 1] \to [0, L_1], \quad t \mapsto u_1(t) := \begin{cases} (L_1 + \varepsilon)t & \text{if } t \in [0, \frac{1}{2}], \\ (L_1 - \varepsilon)t + \varepsilon & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

[9]For each $t \in [0, T]$, let f_t be analytic on Ω_t, and assume $(\Omega_t)_{t\in[0,T]}$ is continuous. We say $t \mapsto f_t$ is differential at t_0 if for every $z \in \Omega_{t_0}$, $t \mapsto f_t(z)$ is differentiable at t_0.

Obviously, we find a unique admissible parametrisation $(\gamma_1)_{t\in[0,1]}$ of Γ_1 such that $\mathfrak{c}(h_{1;t}) = u_1(t)$ for all $t \in [0,1]$. Again, for each $t \in [0,1]$, $h_{1;t}$ denotes the normalised appropriate mapping function on $\mathbb{D} \setminus \gamma_1(0,t]$. Obviously, $\mathfrak{c}(h_{1;t}) = u_1(t)$ is Lipschitz continuous with Lipschitz constant $L_1 + \varepsilon < 1$. Using Proposition 4.4, we find a unique admissible parametrisation $(\gamma_2)_{t\in[0,1]}$ of Γ_2 such that $\mathfrak{c}(g_t) = t$ for all $t \in [0,1]$. Herein, g_t is the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$, $t \in [0,1]$.

Then the function $t \mapsto h_{1;t}$ is not differentiable at $t = \frac{1}{2}$, see Theorem 2.30, 2.31 and 2.36. Using Proposition 4.4, $t \mapsto h_{2;t}$ and $t \mapsto g_t$ are not differentiable at t_0 as well. Nevertheless, $\mathfrak{c}(g_t) = t$ is differentiable at t_0, so this example shows that differentiability of $t \mapsto \mathfrak{c}(g_t)$ at a point t_0 is not sufficient to get differentiability of $t \mapsto g_t$ at t_0.

Next, we will have a deeper look at '*nice*' slits, i.e. slits that are two times continuously differentiable and regular. In this context, regular means that the first derivative does not vanish. In the radial simply connected single slit case the following result, due to C. Earle and A. Epstein, see [EE01], is already known.

Lemma 4.5 (Theorem 3, see [EE01]). *Let $(\gamma)_{t\in[0,T]}$ be a radial slit in $\mathbb{D}$ with $\gamma \in \mathcal{C}^2([0,T])$ and γ regular, i.e. $t \mapsto \gamma(t)$ is two times continuously differentiable on $[0,T]$ with $\dot\gamma(t) \neq 0$ for all $t \in [0,T]$. For each $t \in [0,T]$, we denote by h_t the normalised radial mapping function on $\mathbb{D} \setminus \gamma(0,t]$.*

Then $t \mapsto h_t$ is (continuously) differentiable on $[0,T]$ and satisfies

$$\dot h_t(z) = h_t(z)\mu(t)\Phi_{0,\Upsilon_t,\mathbb{D}}\big(h_t(z)\big) = h_t(z)\mu(t)\frac{\Upsilon_t + h_t(z)}{\Upsilon_t - h_t(z)}, \quad z \in \mathbb{D} \setminus \gamma(0,T],\ t \in [0,T]$$

where, for all $t \in [0,T]$, $\Upsilon_t := h_t(\gamma(t)) \in \mathbb{T}$ and $\mu(t) > 0$. On top of this $\Upsilon \in \mathcal{C}^1([0,T])$ and $\mu \in \mathcal{C}([0,T])$.

Next, we are going to generalise this result to multiply connected domains and several slits.

Theorem 4.6. *Let Ω be a circular slit disk and $(\gamma_1, \dots, \gamma_m)_{t\in[0,T]}$ be radial slits in Ω. For each $k \in \{1, \dots, m\}$, assume $\gamma_k \in \mathcal{C}^2([0,T])$ and γ_k is regular. Moreover, we denote by g_t the normalised radial mapping function on $\Omega_t := \Omega \setminus \bigcup_{k=1}^m \gamma_k(0,t]$ for all $t \in [0,T]$.*

Then for each $z \in \Omega_T$, $t \mapsto g_t(z)$ is continuously differentiable on $[0,T]$ and satisfies

$$\dot g_t(z) = g_t(z) \sum_{k=1}^m \lambda_k(t)\Phi_{0,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } z \in \Omega_T \text{ and all } t \in [0,T],$$

where, for each $k \in \{1, \dots, m\}$ and $t \in [0,T]$, $U_k(t) := g_t(\gamma_k(t))$ and $\lambda_k(t) > 0$. On top of this, for each $k \in \{1, \dots, m\}$, $U_k \in \mathcal{C}^1([0,T])$ and $\lambda_k \in \mathcal{C}([0,T])$.

Proof. For each $t \in [0,T]$, we denote by $h_{k;t}$ the normalised radial mapping function on $\mathbb{D} \setminus \gamma_k(0,t]$ onto $\mathbb{D}$. Using Lemma 4.5, for each $k \in \{1, \dots, m\}$ and $z \in \Omega_T$, $t \mapsto h_{k;t}(z)$ is continuous differentiable on $[0,T]$ and satisfies

$$\dot h_{k;t}(z) = h_{k;t}(z)\mu_k(t)\Phi_{0,\Upsilon_k(t),\mathbb{D}}\big(h_{k;t}(z)\big) \quad \text{for all } z \in \Omega_T \text{ and } t \in [0,T], \tag{4.1}$$

with $\Upsilon_k(t) := h_{k;t}(\gamma_k(t))$ and $\mu_k(t) > 0$. Moreover, $\Upsilon_k \in \mathcal{C}^1([0,T])$ and $\mu_k \in \mathcal{C}([0,T])$ for each $k \in \{1, \dots, m\}$. Then Corollary 4.2 shows that for each $z \in \Omega_T$, $t \mapsto g_t(z)$ is differentiable on $[0,T]$ as well and satisfies

$$\dot{g}_t(z) = g_t(z) \sum_{k=1}^{m} \lambda_k(t) \Phi_{0,U_k(t),D_t}\big(g_t(z)\big) \quad \text{for all } z \in \Omega_T \text{ and } t \in [0,T], \tag{4.2}$$

where $\lambda_k(t) = |\alpha_k(t)|^2 \mu_k(t)$ and $|\alpha_k|$ is positive and continuous on $[0,T]$. Consequently, each $t \mapsto \lambda_k(t)$, $k \in \{1, \dots, m\}$, is continuous and positive on $[0,T]$.

Finally, for each $k \in \{1, \dots, m\}$, we are going to prove $U_k \in \mathcal{C}^1([0,T])$. Therefore, we fix $k \in \{1, \dots, m\}$. Note that $U_k(t) = g_t(h_{k;t}^{-1}(\Upsilon_k(t)))$ holds for all $t \in [0,L]$, and $\Upsilon_k \in \mathcal{C}^1([0,T])$. Let $t_0 \in [0,T]$. Using Lemma 2.42 and 2.44, there is an $\varepsilon > 0$ and a $\delta > 0$ such that $z \mapsto (g_t \circ h_{k;t}^{-1})(z)$ has an analytic continuation to $B_\varepsilon(\Upsilon_k(t_0))$ for all $t \in (t_0 - \delta, t_0 + \delta) \cap [0,T]$. For each $z \in B_\varepsilon(\Upsilon_k(t_0)) \cap \mathbb{D}$, $t \mapsto (g_t \circ h_{k;t}^{-1})(z)$ is continuously differentiable on $(t_0 - \delta, t_0 + \delta) \cap [0,T]$. An easy calculation together with Equation (4.1) and (4.2) shows that $t \mapsto (g_t \circ h_{k;t}^{-1})(z)$ is continuous differentiable on $B_\varepsilon(\Upsilon_k(t_0))$ as well. Thus $t \mapsto U_k(t)$ needs to be continuously differentiable on $(t_0 - \delta, t_0 + \delta) \cap [0,T]$ as well. Summarising, $U_k \in \mathcal{C}^1([0,T])$ □

4.2 Slits having branch points

Next, let us consider the branch point case. Is there a theorem like Theorem 4.1 as well? In order to simplify the notations we take into consideration two branched slits only.

Let (Γ_1, Γ_2) be a tuple of branched unparametrised appropriate slits in Ω. Here, Ω is a canonical domain and Ω^S denotes the simplification of Ω. Assume $(\gamma_1, \gamma_2)_{t \in [0,T]}$ is an admissible parametrisation, and for each $t \in [0,T]$, denote by $h_{1;t}$, $h_{2;t}$ and g_t the normalised appropriate mapping functions on $\Omega^S \setminus \gamma_1(0,t]$, $\Omega^S \setminus \gamma_2(0,t]$ and $\Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$, respectively. Let $t_0 > 0$. Then it is easy to see that $t \mapsto g_t$ is differentiable at t_0 if and only if $t \mapsto h_{1;t}$ and $t \mapsto h_{2;t}$ are differentiable at t_0. Note that we can trace this problem back to the disjoint case. In particular, we apply g_ε with $\varepsilon < t_0$ to get two disjoint slits $\delta_k(t) := g_\varepsilon(\gamma_k(t + \varepsilon))$, $k \in \{1,2\}$ and $t \in [0, T - \varepsilon]$. Then we use Theorem 4.1 to get the desired statement. We used this method already in Section 3.2. Hence, we have the following corollary.

Corollary 4.7. *Let Ω be a canonical domain and denote by $(\gamma_1, \gamma_2)_{t \in [0,T]}$ a tuple of branched appropriate slits in Ω. For each $t \in [0,T]$, we denote by g_t the normalised appropriate mapping function on $\Omega_t := \Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$. Moreover, for each $k \in \{1,2\}$ and $t \in [0,T]$, we set $h_{k;t}$ as the normalised appropriate mapping function on $\Omega^S \setminus \gamma_k(0,t]$. Let $t_0 \in (0,T]$. Then the following two statements are equivalent.*

(i) $t \mapsto g_t(z)$ is differentiable at $t = t_0$ for each $z \in \Omega_{t_0}$.

(ii) For each $k \in \{1,2\}$, $t \mapsto h_{k;t}(z)$ is differentiable at $t = t_0$ for all $z \in \Omega^S \setminus \gamma_k(0,t_0]$.

It remains to have a look at the case $t_0 = 0$. In this case Theorem 4.1 is not true in general, as we have the following result.

Theorem 4.8. *Let $\Omega = \mathbb{H}$. Then we find a tuple $(\gamma_1, \gamma_2)_{t\in[0,T]}$ of branched chordal slits in $\mathbb{H}$ such that for each $z \in \mathbb{H} \setminus \gamma_k(0,T]$ and $k \in \{1,2\}$, $t \mapsto h_{k;t}(z)$ is continuously differentiable on $[0,T]$, while, for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is not differentiable at $t = 0$. Herein, for each $t \in [0,T]$, $h_{1;t}$, $h_{2;t}$ and g_t denote the normalised chordal mapping functions on $\mathbb{H} \setminus \gamma_1(0,t]$, $\mathbb{H} \setminus \gamma_2(0,t]$ and $\mathbb{H} \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$, respectively.*

Proof. Let $\Omega = \mathbb{H}$, $0 \le \varepsilon < \frac{1}{2}$ and let A be the closed set that connects the points

$$\frac{\varepsilon}{2} + \frac{1}{2}\mathrm{i}, \qquad \frac{\varepsilon}{2} + \frac{3}{4}\mathrm{i}, \qquad \frac{1}{4} + \frac{3}{4}\mathrm{i}, \qquad \frac{1}{4} + \mathrm{i}, \qquad \varepsilon + \mathrm{i}$$

by straight line segments, so A is the union of four closed straight line segments. Then we set $\Gamma_1 := \{0\} \cup \bigcup_{n=0}^{\infty} \frac{1}{2^n} A$. Note that $\frac{1}{2}A \cap A = \{\frac{i}{2} + \frac{\varepsilon}{2}\}$, so Γ_1 is a chordal unparametrised slit in Ω, see Figure 4.3. Then we find an admissible parametrisation of $\gamma_1 : [0,T] \to \Gamma_1$ such that $\operatorname{hcap}(h_{1;t}) = t$ for all $t \in [0,T]$. In this context, for each $t \in [0,T]$, $h_{1;t}$ denotes the normalised chordal mapping function on $\mathbb{H} \setminus \gamma_1(0,t]$. Obviously, $T = \operatorname{hcap}_{\mathbb{H}}(\Gamma_1)$ in this case.

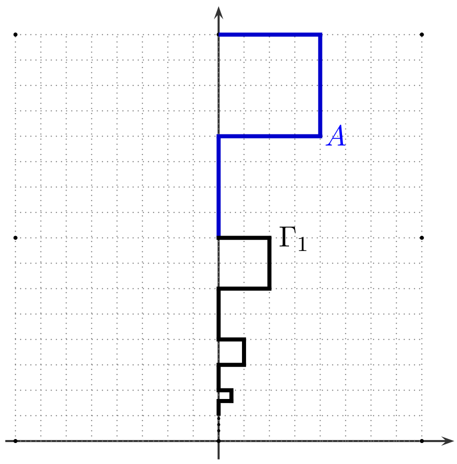

FIGURE 4.3: *A and Γ_1 for $\varepsilon = 0$*

Next, we reflect Γ_1 along the imaginary axis, so $\Gamma_2 := \{z \in \mathbb{C} \mid -\bar{z} \in \Gamma_1\}$. We parametrise $\gamma_2 : [0,T] \to \Gamma_2$ in the same way as Γ_1, i.e. $\operatorname{hcap}(h_{2;t}) = t$ for all $t \in [0,T]$. Analogously, $h_{2;t}$ denotes the normalised chordal mapping function on $\mathbb{H} \setminus \gamma_2(0,t]$. For reasons of symmetry, $\gamma_2(t)$, with $t \in [0,T]$, is the reflection of $\gamma_1(t)$ along the imaginary axis. Thus, for each $k \in \{1,2\}$ and $z \in \Omega \setminus \Gamma_k$, $t \mapsto h_{k;t}(z)$ is continuously differentiable on $[0,T]$, see Theorem 2.36 applied to the single slit case.

On top of this, Γ_1 and Γ_2 are self-similar, i.e. $\frac{1}{2}\Gamma_k \subseteq \Gamma_k$ with $k \in \{1,2\}$. Let $k \in \{1,2\}$. For each $t \in [0,T]$, there is a $t^* \in [0,T]$ such that $\gamma_k(0,t^*] = \frac{1}{2}\gamma_k(0,t]$. Note that

$$\operatorname{hcap}_{\mathbb{H}}(d\mathfrak{H}) = d^2 \operatorname{hcap}_{\mathbb{H}}(\mathfrak{H}) \quad \text{for all } d > 0 \text{ and all chordal hulls } \mathfrak{H} \text{ in } \mathbb{H}. \tag{4.3}$$

Consequently, $t^* = \operatorname{hcap}(h_{k;t^*}) = \frac{1}{4}\operatorname{hcap}(h_{k;t}) = \frac{1}{4}t$. Thus we have $\gamma_k(\frac{t}{4}) = \frac{1}{2}\gamma_k(t)$ for all $t \in [0,T]$. Inductively, we get $\gamma_k(\frac{t}{4^n}) = \frac{1}{2^n}\gamma_k(t)$ for all $t \in [0,T]$ and all $n \in \mathbb{N}$.

Next, for each $t \in [0,T]$, let us denote by g_t the normalised chordal mapping function on $\Omega \setminus \mathfrak{H}(t)$ where $\mathfrak{H}(t)$ is the smallest chordal $\mathbb{H}$-hull containing $\gamma_1(0,t] \cup \gamma_2(0,t]$. Note that $\mathfrak{H}(t) = \gamma_1(0,t] \cup \gamma_2(0,t]$ whenever $\varepsilon > 0$. On the other hand, the complement of the union has bounded connected components if $\varepsilon = 0$. In either case, for each $t \in [0,T]$, $\mathfrak{H}(t)$ is self-similar in the sense that $\frac{1}{2}\mathfrak{H}(t) \subseteq \mathfrak{H}(t)$. To be more precise, $\mathfrak{H}(\frac{t}{4^n}) = \frac{1}{2^n}\mathfrak{H}(t)$ for all $t \in [0,T]$ and all $n \in \mathbb{N}$, as $\gamma_k(\frac{t}{4^n}) = \frac{1}{2^n}\gamma_k(t)$, $k \in \{1,2\}$. Next, let us define $c(t) := \mathrm{hcap}(g_t)$ for all $t \in [0,T]$. Again Equation (4.3) gives us $c(\frac{t}{4^n}) = \frac{1}{4^n}c(t)$ for all $t \in [0,T]$ and all $n \in \mathbb{N}$. Thus we may write

$$\frac{c(\frac{t}{4^n})}{\frac{t}{4^n}} = \frac{c(t)}{t} \qquad \text{for all } n \in \mathbb{N} \text{ and } t \in (0,T]. \tag{4.4}$$

Suppose $t \mapsto c(t)$ is differentiable at $t = 0$. Then Equation (4.4) gives us $c(t) = \dot{c}(0) \cdot t$ for all $t \in [0,T]$, i.e. c is linear. As $T = \mathrm{hcap}(h_{1;T}) < \mathrm{hcap}(g_T) = c(T) = \dot{c}(0) \cdot T$ we have $\dot{c}(0) > 1$.

Let t_2 and t_1 be defined by $\gamma_1(t_1) = \frac{1}{2}i + \frac{\varepsilon}{2}$ and $\gamma_1(t_2) = \frac{3}{4}i + \frac{\varepsilon}{2}$. From [LMR10], Lemma 4.10, it follows that $t_2, t_1, c(t_2), c(t_1)$ depend continuously on ε. For $\varepsilon = 0$ we have $\mathfrak{H}_{t_2} \setminus \mathfrak{H}_{t_1} = \gamma_1(t_1,t_2]$ and we set $\mathfrak{A} := h_{1;t_1}(\mathfrak{H}_{t_1} \setminus \gamma_1(0,t_1])$ and $\mathfrak{B} := h_{1;t_1}(\gamma_1(0,t_2] \setminus \gamma_1(0,t_1]) = h_{1;t_1}(\gamma_1(t_1,t_2])$. Note that $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A} \cup \mathfrak{B}$ are chordal hulls in $\mathbb{H}$. Using Lemma 2.56, we get $\mathrm{hcap}_{\mathbb{H}}(\mathfrak{A} \cup \mathfrak{B}) \leq \mathrm{hcap}_{\mathbb{H}}(\mathfrak{A}) + \mathrm{hcap}_{\mathbb{H}}(\mathfrak{B})$. Moreover, $\mathrm{hcap}_{\mathbb{H}}(\mathfrak{A}) = \mathrm{hcap}(g_{t_1}) - \mathrm{hcap}(h_{1;t_1})$, $\mathrm{hcap}_{\mathbb{H}}(\mathfrak{B}) = \mathrm{hcap}(h_{1;t_2}) - \mathrm{hcap}(h_{1;t_1})$ and $\mathrm{hcap}_{\mathbb{H}}(\mathfrak{A} \cup \mathfrak{B}) = \mathrm{hcap}(g_{t_2}) - \mathrm{hcap}(h_{1;t_1})$. Summarising, we find

$$c(t_2) - c(t_1) = \mathrm{hcap}(g_{t_2}) - \mathrm{hcap}(g_{t_1}) \leq \mathrm{hcap}(h_{1;t_2}) - \mathrm{hcap}(h_{1;t_1}) = t_2 - t_1.$$

Now choose $\varepsilon > 0$ small enough sucht that we still have

$$\frac{c(t_2) - c(t_1)}{t_2 - t_1} < \dot{c}(0) \in (1, \infty).$$

This is a contradiction as $c(t) = \dot{c}(0)t$ for all $t \in [0,T]$. Thus $t \mapsto c(t) := \mathfrak{c}(g_t)$ can not be differentiable at t_0. Finally, Lemma 3.10 and Remark 3.6 show that for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is not differentiable at $t = 0$. $\square$

Note that Theorem 4.8 is restricted to the chordal case. One reason for this is that Equation (4.3), which is known as *scaling property of* hcap, is available only in the chordal case. Another reason is Lemma 2.56 that is only available in the simply connected case. Summarising, the previous counterexample is restricted to $\mathbb{H}$. Nevertheless, we will use this counterexample to find counterexamples in all other (even multiply connected) cases as well. In order to to so let us have a look at the next lemma.

Lemma 4.9. *Let $(\delta_1, \delta_2)_{t \in [0,T]}$ be a tuple of branched chordal slit in $\mathbb{H}$ with $\delta_1(0) = 0 = \delta_2(0)$. Assume Ω is a canonical domain. For each $k \in \{1,2\}$, we set $\gamma_k(t) := \exp\left(\sqrt{2}\mathrm{i}\delta_k(t)\right)$ in the radial and bilateral case and $\gamma_k(t) := \delta_k(t)$ in the chordal case.*

Then we find a $t_0 \in (0,T]$ such that $(\gamma_1, \gamma_2)_{t \in [0,t_0]}$ is a tuple of branched appropriate slits in Ω. Moreover, let us consider one of the following two cases.

(i) $\mathfrak{H}_t := \gamma_k(0,t]$ and $\tilde{\mathfrak{H}}_t := \delta_k(0,t]$ for all $t \leq t_0$ and some $k \in \{1,2\}$.

(ii) $\mathfrak{H}_t := \gamma_1(0,t] \cup \gamma_2(0,t]$ *and* $\tilde{\mathfrak{H}}_t := \delta_1(0,t] \cup \delta_2(0,t]$ *for all* $t \leq t_0$.

In either case, $t \mapsto c(t) := \mathfrak{c}_\Omega(\mathfrak{H}_t)$ *is differentiable at* $t = 0$ *if and only if* $t \mapsto d(t) := \mathrm{hcap}_{\mathbb{H}}(\tilde{\mathfrak{H}}_t)$ *is differentiable at* $t = 0$*. When this happens* $\dot{c}(0) = \dot{d}(0)$*.*

Proof. Obviously, we find a $t_0 \in (0,T]$ such that $\gamma_k(0,t_0] \cap \partial\Omega = \emptyset$ for all $k \in \{1,2\}$. For each $t \in [0,t_0]$, we denote by g_t the normalised appropriate mapping function on $\Omega \setminus \mathfrak{H}_t$. Moreover, $\tilde{g}_t$ is the normalised chordal mapping function on $\mathbb{H} \setminus \tilde{\mathfrak{H}}_t$ with $t \in [0,t_0]$. On top of this we set $s_t := g_t(\mathfrak{H}_t)$ and $\tilde{s}_t := \tilde{g}_t(\tilde{\mathfrak{H}}_t)$ with $t \in [0,t_0]$. Using Remark 3.6, $s_t \to \gamma_1(0) = \gamma_2(0)$ and $\tilde{s}_t \to \delta_1(0) = \delta_2(0) = 0$ as $t \searrow 0$.

Let $\varepsilon > 0$ be small and let us consider the function

$$T_t(\zeta) := \begin{cases} g_t\Big(\exp\big(\sqrt{2}\mathrm{i} \cdot \tilde{g}_t^{-1}(\zeta)\big)\Big) & \text{in the radial or bilateral case,} \\ g_t\big(\tilde{g}_t^{-1}(\zeta)\big) & \text{in the chordal case,} \end{cases}$$

which is, by reflection and Lemma 2.42, univalent on $\mathbb{D}_\varepsilon$ for all $t \in [0,t^*]$ with a small $t^* < t_0$ and small $\varepsilon > 0$. In the radial and bilateral case we are able to write $\tilde{g}_t^{-1}(\zeta) = -\mathrm{i}\frac{1}{\sqrt{2}}\log\big(g_t^{-1}(T_t(\zeta))\big)$ with a suitable branch of the logarithm and small t. Using Lemma 2.39, we find with a substitution and the mean value theorem

$$\begin{aligned} d(t) &= \frac{1}{\pi}\int_{\tilde{s}_t} \Im\big(\tilde{g}_t^{-1}(\zeta)\big)|\mathrm{d}\zeta| = \frac{1}{\pi}\int_{\tilde{s}_t} \Im\Big(-\tfrac{\mathrm{i}}{\sqrt{2}}\log\big(g_t^{-1}(T_t(\zeta))\big)\Big)|\mathrm{d}\zeta| \\ &= -\frac{1}{\sqrt{2}\pi}\int_{\tilde{s}_t} \ln\big|g_t^{-1}(T_t(\zeta))\big||\mathrm{d}\zeta| = -\frac{1}{\sqrt{2}\pi}\int_{s_t} \ln\big|g_t^{-1}(\xi)\big|\frac{1}{|T_t'(T_t^{-1}(\xi))|}|\mathrm{d}\xi| \\ &= -\frac{\sqrt{2}}{|T_t'(\zeta_t)|}\frac{1}{2\pi}\int_{s_t} \ln\big|g_t^{-1}(\xi)\big||\mathrm{d}\xi| = \frac{\sqrt{2}}{|T_t'(\zeta_t)|}c(t) \end{aligned}$$

for all small $t < t^*$ where $\zeta_t \in \tilde{s}_t$. Note that the last equality follow immediately from Lemma 2.27 and 2.34. Analogously, we have in the chordal case together with Lemma 2.39

$$d(t) = \frac{1}{\pi}\int_{\tilde{s}_t} \Im\big(\tilde{g}_t^{-1}(\zeta)\big)|\mathrm{d}\zeta| = \frac{1}{\pi}\int_{\tilde{s}_t} \Im\Big(g_t^{-1}\big(T_t(\zeta)\big)\Big)|\mathrm{d}\zeta| = \frac{1}{|T_t'(\zeta_t)|}c(t)$$

and $\zeta_t \in \tilde{s}_t$. Note that $g_t \xrightarrow{\text{l.u.}} \mathrm{id}$ on Ω and $\tilde{g}_t \xrightarrow{\text{l.u.}} \mathrm{id}$ on $\mathbb{H}$ as $t \searrow 0$, so it is easy to see that $|T_t'(\zeta_t)| \to \sqrt{2}$ as $t \searrow 0$ in the radial and bilateral case, and $|T_t'(\zeta_t)| \to 1$ as $t \searrow 0$ in the chordal case. □

Combining Lemma 4.9, Theorem 4.8 and Lemma 3.10, we find the following corollary.

Corollary 4.10. *Let* Ω *be a canonical domain and denote by* Ω^S *the simplification of* Ω*. Then we find a tuple* $(\gamma_1,\gamma_2)_{t\in[0,T]}$ *of branched appropriate slits in* Ω *such that each* $t \mapsto h_{k;t}(z)$, $k \in \{1,2\}$, *is differentiable at* 0 *for all* $z \in \Omega^S$*, while, for each* $z \in \Omega \setminus \{0\}$, $t \mapsto g_t$ *is not differentiable at* $t = 0$*. Herein, for each* $t \in [0,T]$, $h_{1;t}$, $h_{2;t}$ *and* g_t *denote the normalised appropriate mapping functions on* $\Omega^S \setminus \gamma_1(0,t]$, $\Omega^S \setminus \gamma_2(0,t]$ *and* $\Omega \setminus (\gamma_1(0,t] \cup \gamma_2(0,t])$*, respectively.*

On the other hand we also give a condition that ensures differentiability of $t \mapsto g_t$ at $t = 0$ whenever $t \mapsto h_{1;t}$ and $t \mapsto h_{2;t}$ are differentiable at $t = 0$. In order to do so, we need the following definition. Therefore, let $\phi \in (0, \pi)$. Assume $(\gamma)_{t\in[0,T]}$ is a chordal slit in the upper parallel slit half-plane Ω. Then we say γ *approaches* $\mathbb{R}$ *at* $x \in \mathbb{R}$ *in* ϕ*-direction* if for every $\varepsilon > 0$, there is a $t_0 > 0$ such that

$$\gamma(0, t_0] \subseteq \{z \in \mathbb{H} \mid \phi - \varepsilon < \arg(z - x) < \phi + \varepsilon\}.$$

Analogously, let Ω be a circular slit disk or circular slit annulus and let $(\gamma)_{t\in[0,T]}$ be an appropriate silt in Ω. Then we say γ *approaches* $\mathbb{T}$ *at* $\xi \in \mathbb{T}$ *in* ϕ*-direction* if for every $\varepsilon > 0$, there is a $t_0 > 0$ such that

$$\gamma(0, t_0] \subseteq \{z \in \mathbb{D} \mid \phi - \varepsilon < \arg(\gamma(0) - z) + \arg(\gamma(0)) - \tfrac{\pi}{2} < \phi + \varepsilon\}.$$

Theorem 4.11. *Let $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be branched chordal slits in $\mathbb{H}$. Assume γ_1 and γ_2 approach $\mathbb{R}$ at $\gamma_1(0) = \gamma_2(0)$ in α_k-direction with $\alpha_k \in (0, \pi)$, $k \in \{1, 2\}$. For each $k \in \{1, 2\}$, we denote by $h_{k;t}$ the normalised chordal mapping function on $\mathbb{H} \setminus \gamma_k(0, t]$ with $t \in [0, T]$, and assume that each $t \mapsto h_{k;t}(z)$, $k \in \{1, 2\}$, is differentiable at $t = 0$ for all $z \in \mathbb{H}$. Then for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is differentiable at $t = 0$ where g_t denotes the normalised chordal mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, t])$ with $t \in [0, T]$.*

Proof. See Theorem 3 in [BS15b] □

Obviously, using Lemma 4.9 and 3.10 we find the following corollary.

Corollary 4.12. *Let Ω be a canonical domain and let $(\gamma_1, \gamma_2)_{t\in[0,T]}$ be branched appropriate slits in Ω. Assume γ_1 and γ_2 approach the outer or unbounded boundary of Ω at $\gamma_1(0) = \gamma_2(0)$ in α_k-direction with $\alpha_k \in (0, \pi)$, $k \in \{1, 2\}$. For each $k \in \{1, 2\}$, we denote by $h_{k;t}$ the normalised appropriate mapping function on $\Omega \setminus \gamma_k(0, t]$ with $t \in [0, T]$, and assume that each $t \mapsto h_{k;t}(z)$, $k \in \{1, 2\}$, is differentiable at $t = 0$ for all $z \in \Omega$. Then for each $z \in \Omega$, $t \mapsto g_t(z)$ is differentiable at $t = 0$ where g_t denotes the normalised appropriate mapping function on $\Omega \setminus (\gamma_1(0, t] \cup \gamma_2(0, t])$ with $t \in [0, T]$.*

Finally, it is worth to mention that the inverse of Theorem 4.11 or Corollary 4.12 is not true.

Example 4.2. Let (Γ_1, Γ_2) be a tuple of branched chordal unparametrised slits in $\mathbb{H}$, and assume there is an admissible parametrisation $(\delta_1, \delta_2)_{t\in[0,T]}$ such that for each $k \in \{1, 2\}$, δ_k approaches $\mathbb{R}$ at $\delta_1(0) = \delta_2(0)$ in α_k-direction with $\alpha_k \in (0, \pi)$. By definition γ_k approaches $\mathbb{R}$ at $\gamma_1(0) = \gamma_2(0)$ in α_k-direction as well if $(\gamma_1, \gamma_2)_{t\in[0,L]}$ is another admissible parametrisation of (Γ_1, Γ_2).

Without loss of generality we may assume $L := \operatorname{hcap}_{\mathbb{H}}(\Gamma_1 \cup \Gamma_2) = 1$. Moreover, let $L_k := \operatorname{hcap}_{\mathbb{H}}(\Gamma_k)$ with $k \in \{1, 2\}$. Then $L_k < 1$, so we find an $\varepsilon > 0$ such that $L_1 + \varepsilon < 1$. Next, we define:

$$\tilde{u} : [0, 1] \to [0, L_1], \quad t \mapsto \tilde{u}(t) := \begin{cases} (L_1 + \varepsilon)t & \text{if } t \in [0, \frac{1}{2}], \\ (L_1 - \varepsilon)t + \varepsilon & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

We will use $\tilde{u}$ to construct another increasing homeomorphism $u : [0,1] \to [0, L_1]$:

$$u(t) := \begin{cases} \frac{1}{2^n}\tilde{u}(2^n t - 1) + \frac{L_1}{2^n} & \text{if } t \in (\frac{1}{2^n}, \frac{2}{2^n}] \text{ with } n \in \mathbb{N}, \\ 0 & \text{if } t = 0, \end{cases}$$

see Figure 4.4. We have $|u(t_2) - u(t_1)| \leq (L_1 + \varepsilon)(t_2 - t_1)$ for all $0 \leq t_1 \leq t_2 \leq 1$, so u is strictly increasing and Lipschitz continuous on $[0,1]$ with Lipschitz constant $L_1 + \varepsilon < 1$. Then we find a unique admissible parametrisation $(\gamma_1)_{t\in[0,1]}$ of Γ_1 such that $\mathrm{hcap}(h_{1;t}) = u(t)$ for all $t \in [0,1]$. In this context, for each $t \in [0,T]$, $h_{1;t}$ denotes the normalised chordal mapping function on $\Omega \setminus \gamma_1(0,t]$. Using Proposition 4.4, we find a unique admissible parametrisation $(\gamma_2)_{t\in[0,1]}$ of Γ_2 such that $\mathrm{hcap}(g_t) = t$ for all $t \in [0,1]$. Analogously, g_t denotes the normalised chordal mapping function on $\Omega\setminus(\gamma_1(0,t]\cup\gamma_2(0,t])$ with $t \in [0,1]$. Note that $\mathfrak{c}(g_t) = t$ is differentiable at $t = 0$, so using Lemma 3.10, for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is differentiable at $t = 0$. However, using Remark 3.6, $t \mapsto h_{1;t}(z)$ is not differentiable at t_0 for any $z \in \mathbb{H}$, as $t \mapsto \mathrm{hcap}(h_{1;t}) = u(t)$ is not differentiable at $t = 0$.

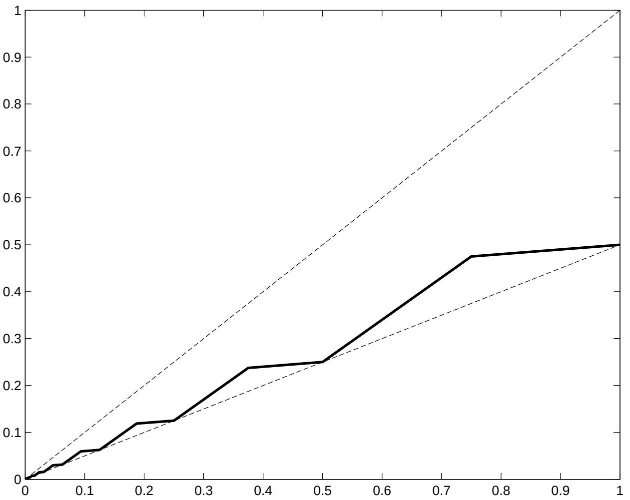

FIGURE 4.4: *The function u from Example 4.2*

Chapter 5

Generalization to hulls with local growth

Theorem 5.1. *Let Ω be a circular slit disk and denote by $(\mathfrak{H}_t)_{t\in[0,T]}$ a family of increasing radial Ω-hulls such that $\operatorname{con}(\Omega \setminus \mathfrak{H}_t) = \operatorname{con}(\Omega)$ for all $t \in [0,T]$. For each $t \in [0,T]$, g_t denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t. Moreover, assume $\operatorname{lmr}(g_t) = t$ for all $t \in [0,T]$. Then the following two statements are equivalent:*

(i) For each $t \in [0,T]$, $t \mapsto g_t$ is (continuously) differentiable and fulfils the differential equation

$$\dot{g}_t(z) = g_t(z) \cdot \Phi_{0,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } t \in [0,T] \text{ and all } z \in \Omega_T,$$

with a continuous function $t \mapsto U_t \in \mathbb{T}$.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $t \in [0, T-\delta]$, some cross-cut E of Ω_t with $\operatorname{diam}(E) < \varepsilon$ separates 0 from $\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t$.

In this context, a *cross-cut* E of the domain Ω is an open Jordan arc in Ω[10] such that $\operatorname{cl}(E) = E \cup \{a, b\}$ with $a, b \in \partial\Omega$. Let Ω be a circular slit disk and let $(\mathfrak{H}_t)_{t\in[0,T]}$ be a family of increasing radial Ω-hulls. Then we say $(\mathfrak{H}_t)_{t\in[0,T]}$ satisfies the *local growth property* if condition (ii) from Theorem 5.1 is fulfilled. If $\varepsilon > 0$ in condition (ii) is sufficiently small[11], the two endpoints a, b of the cross-cut E need to be part of the outer boundary component of Ω_t, see Theorem V.11.7 and Exercise V.11.4 in [New52].

Unfortunately, Theorem 5.1, in particular the direction (i)⇒(ii), does only hold for hulls satisfying $\operatorname{con}(\Omega\setminus\mathfrak{H}_t) = \operatorname{con}(\Omega)$ for all $t \in [0,T]$. We will give an example of a family $(\mathfrak{H}_t)_{t\in[0,T]}$ of increasing hulls such that $t \mapsto g_t$ is differentiable, while $(\mathfrak{H}_t)_{t\in[0,T]}$ does not satisfy the local growth property. See Example 5.1 for more details. Nevertheless, the direction (ii)⇒(i) is true in general, see the next theorem.

[10] An open Jordan arc in Ω is the trace of a simple and continuous $\gamma : (a,b) \to \Omega$, $a < b$.

[11] Let $C_1, \ldots, C_{\mathfrak{n}}$ denote the connected components of Ω where $C_{\mathfrak{n}} = \mathbb{T}$. For each $j \in \{1, \ldots, \mathfrak{n} - 1\}$, we denote by z_j, w_j the two tips of C_j. Moreover, we set $m_1 := \min_{j=1}^{\mathfrak{n}-1} |z_j - w_j|$ and $m_2 := \min_{j=1}^{\mathfrak{n}-1} \operatorname{dist}(C_j, 0)$. Then $\varepsilon > 0$ is *sufficiently small* if $\varepsilon < \min(m_1, m_2)$.

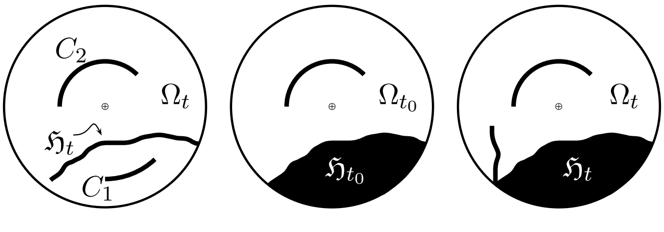

FIGURE 5.1: *The concentric circular arc C_1 gets swallowed by the hull $\mathfrak{H}_{t_0}$*

Theorem 5.2. *Let Ω be a circular slit disk and denote by $(\mathfrak{H}_t)_{t\in[0,T]}$ a family of increasing radial Ω-hulls satisfying the local growth property. For each $t \in [0,T]$, g_t denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t, and assume $\operatorname{lmr}(g_t) = t$.*

Then, for each $z \in \Omega_T$, $t \mapsto g_t(z)$ is (continuously) differentiable on $[0,T]$ and satisfies the differential equation

$$\dot{g}_t(z) = g_t(z) \cdot \Phi_{0,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } t \in [0,T] \text{ and all } z \in \Omega_T,$$

with a continuous driving function $t \mapsto U_t \in \mathbb{T}$, $t \in [0,T]$.

Example 5.1. Let $\Omega := \mathbb{D} \setminus C$ with $C := \{(1-r)e^{i\phi} \mid \phi \in [-\alpha,\alpha]\}$, and $\alpha \in (0,\pi)$ and $r \in (0,1)$. Thus Ω is a (doubly connected) circular slit disk. Moreover, we define an increasing family of radial $\mathbb{D}$-hulls $(\mathfrak{H}_t)_{t\in[0,r]}$ as follows. We set $\mathfrak{H}_t := \{1-\tau \mid \tau \in (0,t]\}$ if $t \in [0,r)$ and $\mathfrak{H}_r := (r,1)$. Note that $\Omega_t := \Omega \setminus \mathfrak{H}_t$ is doubly connected whenever $t \in [0,r)$, while $\Omega_r := \Omega \setminus (r,1)$ is simply connected. As usual, for each $t \in [0,r]$, we denote by g_t the normalised radial mapping function from Ω_t onto the circular slit disk D_t. Obviously, $(\mathfrak{H}_t)_{t\in[0,r]}$ is continuous, so using Proposition 5.6, $t \mapsto g_t$ and $t \mapsto \operatorname{lmr}(g_t)$ are continuous on $[0,r]$ as well. On top of this $t \mapsto \operatorname{lmr}(g_t)$ is strictly increasing on $[0,r]$, see Lemma 2.24. Without loss of generality, we may assume $\operatorname{lmr}(g_t) = ct$ for all $t \in [0,r]$ with $c := \mathfrak{c}_\Omega(\mathfrak{H}_r)/r > 0$. Otherwise we reparametrise $\mathfrak{H}_t$. Using Theorem 2.30 (applied to the single slit case) or Theorem 2.23, we find

$$\dot{g}_t(z) = c g_t(z)\Phi_{0,U_t,D_t}\big(g_t(z)\big) \qquad \text{for all } z \in \Omega_r \text{ and all } t \in [0,r). \tag{5.1}$$

For symmetry reasons, we get $U_t = g_t(\mathfrak{H}_t) = 1$ for each $t \in [0,r)$. Moreover, using Proposition 5.6 together with Proposition 2.11, we find $D_t \xrightarrow{\text{k}} \mathbb{D}$ as $t \nearrow r$.

Let $(t_n)_{n\in\mathbb{N}} \subseteq [0,r)$ be a sequence with $t_n \to r$ and we set $h_n := \Phi_{0,1,D_{t_n}}$ for all $n \in \mathbb{N}$. Montel's theorem gives us a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ such that $h_{n_k} \xrightarrow{\text{l.u.}} h$ on $\mathbb{D}$. The limit function h is either univalent or constant. Suppose h is constant. For each $w \in \mathbb{C} \setminus \{-1\}$, we write $T(w) := \frac{w-1}{w+1}$, so each $T \circ h_{n_k}$ maps $D_{t_{n_k}}$ univalent into $\mathbb{D}$ where $\mathbb{T}$ is associated with $\mathbb{T}$. Using Equation (5.1), we find $\Phi_{0,1,D_t}(0) = 1$ for all $t \in [0,r)$. This gives us $h \equiv 1$, and $T \circ h \equiv 0$ as well. This is a contradiction to Wolff's lemma[12]. To see this let $\zeta_0 \in \mathbb{T}$ be fix and define $E_k(\varepsilon) := h_{n_k}(\partial B_\varepsilon(\zeta_0) \cap D_{t_{n_k}})$ for all $k \in \mathbb{N}$. Then Wolff's lemma gives us $\inf_{r\in(\varepsilon,\sqrt{\varepsilon})} \operatorname{diam}\big(E_k(r)\big) < 2\pi/\sqrt{\log 1/\varepsilon}$ for each $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$. We choose $\varepsilon \in (0,1)$ small enough in order to get $2\pi\sqrt{\log 1/\varepsilon} < \frac{1}{2}$. On the other

[12] See [Pom92], Proposition 2.2.

hand $h_{n_k}(K)$ tends uniformly to 0 for any compact set in $K \subseteq \mathbb{D}$. In particular there is a $k \in \mathbb{N}$ such that dist $\big(h_{n_k}(\mathbb{T}_{1-\varepsilon}), 0\big) < 1/2$, contradicting $\inf_{r\in(\varepsilon,\sqrt{\varepsilon})} \operatorname{diam}(E_k(r)) < \frac{1}{2}$. Summarising, h can not be constant, so h is univalent.

Next, we will show that $h(\mathbb{D}) = \{z \in \mathbb{C} \mid \Re(z) > 0\}$. Therefore, let $R_n := h_n(D_{t_n})$ for all $n \in \mathbb{N}$, so R_n is doubly connected. In particular it is easy to see that $R_n = \{z \in \mathbb{C} \mid \Re(z) > 0\} \setminus E_n$ where $E_n = \{x_n + \mathrm{i}y \mid |y| \le y_n\}$ and $x_n, y_n > 0$, i.e. E_n is a proper closed line segments parallel to the imaginary axis. Thus it is enough to prove $x_n \to \infty$. $D_{t_n} = \mathbb{D} \setminus C_n$ is a circular slit disk where $r_n := \operatorname{dist}(C_n, 0) \to 1$ if $n \to \infty$. Note that this follows immediately from $D_t \xrightarrow{\text{k}} \mathbb{D}$ if $t \nearrow r$. Thus $T(x_n) = (T \circ h_n)(r_n) \to 1$ by Wolff's lemma used in the same way as before.

Summarising, h maps $\mathbb{D}$ conformal onto $\mathbb{H}$ with $h(0) = 1$, so $h(w) = (1 + w)/(1 - w)$ for all $w \in \mathbb{D}$. Note that we can do this for each locally uniformly convergent subsequence of $(h_n)_{n\in\mathbb{N}}$, so the whole sequence h_n tends to h, i.e. $h_n \xrightarrow{\text{l.u.}} h$ on $\mathbb{D}$. Thus

$$cg_t(z)\Phi_{0,U_t,D_t}\big(g_t(z)\big) \xrightarrow{\text{l.u.}} cg_r(z)\frac{1 + g_r(z)}{1 - g_r(z)}, \qquad \text{on } \mathbb{D} \text{ as } t \nearrow r.$$

Finally, we find together with the mean value theorem

$$\dot{g}_r(z) = cg_r(z)\frac{1 + g_r(z)}{1 - g_r(z)} \qquad \text{for all } z \in \Omega \setminus \mathfrak{H}_r.$$

Consequently, $t \mapsto g_t$ is continuously differentiable on $[0, r]$, while the corresponding family of radial $\mathbb{D}$-hulls $(\mathfrak{H}_t)_{t\in[0,r]}$ does not satisfy the local growth property.

5.1 Some preliminary lemmas

Lemma 5.3. *Let Ω be a circular slit disk and let $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of radial Ω-hulls satisfying the local growth property. Then the family $(\mathfrak{H}_t)_{t\in[0,T]}$ is continuous.*

Proof. First of all, let us define $\Omega_t := \Omega \setminus \mathfrak{H}_t$ with $t \in [0, T]$. Using the monotonicity, see Lemma 2.24, we need to study only the following two cases: $t_n \nearrow t_0$ and $t_n \searrow t_0$.

1) $t_n \searrow t_0$: Using Lemma 2.9, the increasing sequence $(\Omega_{t_n})_{n\in\mathbb{N}}$ has a kernel K. Obviously, $\Omega_{t_n} \subseteq K \subseteq \Omega_{t_0}$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Then the local growth property gives us an $N \in \mathbb{N}$ such that whenever $n \ge N$, some cross-cut E of Ω_{t_0} with $\operatorname{diam}(E) < \varepsilon$ separates $\Omega_{t_0} \setminus \Omega_{t_n} = \mathfrak{H}_{t_n} \setminus \mathfrak{H}_{t_0}$ from 0. Note that $\Omega_{t_0} \setminus K \subseteq \Omega_{t_0} \setminus \Omega_{t_n}$ for all $n \in \mathbb{N}$, so E separates $\Omega_{t_0} \setminus K$ from 0 as well. Letting $\varepsilon \to 0$ we find $K = \Omega_{t_0}$.

2) $t_n \nearrow t_0$: Again using Lemma 2.9, the decreasing sequence $(\Omega_{t_n})_{n\in\mathbb{N}}$ has a kernel K. Obviously, $\Omega_{t_0} \subseteq K \subseteq \Omega_{t_n}$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Using the local growth property, we find an $N \in \mathbb{N}$ such that whenever $n \ge N$, some cross-cut E of Ω_{t_n} with $\operatorname{diam}(E) < \varepsilon$ separates $\Omega_{t_n} \setminus \Omega_{t_0}$ from 0. Note that $K \setminus \Omega_{t_0} \subseteq \Omega_{t_n} \setminus \Omega_{t_0}$ for all $n \in \mathbb{N}$, so E separates $K \setminus \Omega_{t_0}$ from 0 as well. Letting $\varepsilon \to 0$ we find $K = \Omega_{t_0}$.

Summarising, Ω_{t_0} is the kernel of $(\Omega_{t_n})_{n\in\mathbb{N}}$. □

Let Ω be a circular slit disk and let $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of radial hulls in Ω. Moreover, we denote by $C_1, \ldots, C_{\mathfrak{n}-1}$, with $\mathfrak{n} = \text{con}(\Omega) \in \mathbb{N}$, the interior boundary components of Ω. Let $C \in \{C_1, \ldots, C_{\mathfrak{n}-1}\}$ and let $t_0 \in (0, T]$. Then we say C is *swallowed by* $\mathfrak{H}_{t_0}$ if $\text{dist}(\mathfrak{H}_{t_0}, C) = 0$, see also Figure 5.1

Lemma 5.4. *Let Ω be a circular slit disk, $(\mathfrak{H}_t)_{t\in[0,T]}$ be a family of continuous and increasing radial Ω-hulls, and we set $\Omega_t := \Omega \setminus \mathfrak{H}_t$ for all $t \in [0, T]$. Then the step function $t \mapsto \text{con}(\Omega_t)$, $t \in [0, T]$, is decreasing, continuous from the right and of finite range.*

Proof. First of all, the monotonicity is an immediate consequence of the property $\mathfrak{H}_t \subseteq \mathfrak{H}_s$ for all $0 \le t \le s \le T$. The fact that $t \mapsto \text{con}(\Omega_t)$ is a step function of finite range is trivial.

Next, let be $t_0 \in [0, T)$ and denote by $C \in \{C_1, \ldots, C_{\mathfrak{n}}\}$ an arbitrary boundary component satisfying $\text{dist}(\mathfrak{H}_{t_0}, C) > 0$, i.e. C is not swallowed by the hull $\mathfrak{H}_{t_0}$. Consequently, we find a small $\delta > 0$ such that $C^\delta := \{z \in \mathbb{D} \mid \text{dist}(z, C) \le \delta\}$ is not swallowed by $\mathfrak{H}_{t_0}$ as well, i.e. $\text{dist}(\mathfrak{H}_{t_0}, C^\delta) > 0$. Consequently, $\partial C^\delta \subseteq \Omega_{t_0}$ if δ is small enough. Assume $(t_n)_{n\in\mathbb{N}} \subseteq [0, T]$ with $t_n \searrow t_0$. Since Ω_{t_0} is the kernel of the sequence Ω_{t_n}, we find $\partial C^\delta \subseteq \Omega_{t_n}$ for all $n \ge N \in \mathbb{N}$. Thus we have $0 < \text{dist}(\mathfrak{H}_{t_n}, C^\delta) < \text{dist}(\mathfrak{H}_{t_n}, C)$ for all $n \ge N$. Using the monotonicity of the family $(\mathfrak{H}_t)_{t\in[0,T]}$, we get $\text{dist}(\mathfrak{H}_t, C) > 0$ for all $t \in [t_0, t_N]$. Finally, since we are able to do this for each C that is not swallowed by $\mathfrak{H}_{t_0}$, we find $\text{con}(\Omega_t) = \text{con}(\Omega_{t_0})$ for all $t \in [t_0, t^*]$ with $t^* > t_0$, so $t \mapsto \text{con}(\Omega_t)$ is continuous from the right. □

Lemma 5.5. *Let Ω be a circular slit disk and $(\mathfrak{H}_t)_{t\in[0,T]}$ is an increasing family of radial Ω-hulls. For each $t \in [0, T]$, g_t denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t. Let $(t_n)_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence converging to $t_0 \in [0, T]$ and assume $\Omega_{t_n} \xrightarrow{k} \Omega_{t_0}$ and $D_{t_n} \xrightarrow{k} D$. Then D is a circular slit disk.*

Proof. First of all, we set $m := \text{con}(\Omega_{t_0})$ and $s := \lim_{n\to\infty} \text{con}(\Omega_{t_n})$. Using Lemma 5.4, we get $s \ge m$. We will separate the following two cases:

1) $s = m$: In this case $\text{con}(\Omega_{t_n}) = \text{con}(\Omega_{t_0})$ for all n large enough. By assumption $\Omega_{t_n} \xrightarrow{\text{k}} \Omega_{t_0}$, so $g_{t_n} \xrightarrow{\text{l.u.}} g_{t_0}$ on Ω_{t_0}, see Lemma 2.25. Using Proposition 2.11, we find $g_{t_n}(\Omega_{t_n}) = D_{t_n} \xrightarrow{\text{k}} D_{t_0} = g_{t_0}(\Omega_{t_0})$, so $D_{t_0} = D$ is a circular slit disk.

2) $s > m$: In this context we use the same abbreviation as in the proof of Lemma 2.13. Since $t \mapsto \text{con}(\Omega_t)$ is a step function, we are able to find an $N \in \mathbb{N}$ such that $\text{con}(\Omega_{t_n}) = s$ for all $n \ge N$. Again, Lemma 5.4 gives us $t_n < t_0$ for all $n \ge N$.

First of all, we are going to show that there is an $r \in (0, 1]$ such that $\frac{1}{r}D$ is a circular slit disk. We denote by $E_1, \ldots E_m$ the connected components of $\mathbb{C} \setminus D$ where E_m is the unbounded connected component. Analogously to the proof of Lemma 2.13, we find for each E_k, $k \in \{1, \ldots, m\}$, a Jordan curve $\Delta_k \subseteq D$ such that Δ_k separates E_k from E_j with $j \in \{1, \ldots, m\} \setminus \{k\}$. Moreover, we can choose Δ_k in such a way that $\text{dist}(\Delta_k, \Delta_j) > \delta$ whenever $j \neq k$. We set $E_k^\Delta := \Delta_k \cup \text{int}(\Delta_k)$, $k \in \{1, \ldots, m-1\}$ and

$E_m^\Delta := \Delta_m \cup \mathrm{ext}(\Delta_m)$. Then $D^\Delta := D \setminus \bigcup_{k=1}^m E_k^\Delta$ is an m-connected domain. Note that D is the kernel of the sequence $(D_{t_n})_{n\in\mathbb{N}}$ and $\mathrm{cl}(D^\Delta)$ is a compact set in D, so we find $\mathrm{cl}(D^\Delta) \subseteq D_{t_n}$ for all $n \geq N$ with some $N \in \mathbb{N}$.

Next, we denote by $F_1, \ldots, F_s$ the connected components of $\mathbb{C} \setminus D_{t_n}$ where $F_s = \{z \in \mathbb{C} \mid |z| \geq 1\}$ is the unbounded connected component. Consequently, $F_1, \ldots, F_{s-1}$ are concentric circular arcs. Obviously, we find $F_k \subseteq E_{I(k)}^\Delta$ for all $k \in \{1, \ldots, s\}$ where $I : \{1, \ldots, s\} \to \{1, \ldots, m\}$ is onto. Let E be an arbitrary connected component of $\mathbb{C} \setminus D$. Then for each $a \in \partial E$ we find a sequence $a_n \in \partial D_{t_n}$ with $a_n \to a$, see Lemma 2.10. Suppose $a, b \in \partial E$ with $|a| \neq |b|$. Lemma 2.10 gives us sequences $(a_n), (b_n) \subseteq \partial D_{t_n}$ such that $a_n \to a$ and $b_n \to b$. Consequently, $|a_n| \neq |b_n|$ for all $n \geq M$ with $M \in \mathbb{N}$. Since $F_1, \ldots, F_s$ are circular arcs, there are at most s different sequences $(|a_n|)_{n\geq M}$, so the set $\{|a| \mid a \in \partial E\}$ is finite. This proves that $|a|$ is constant for each $a \in \partial E$. Since $D \subseteq \mathbb{D}$, $E_1 \ldots E_{m-1}$ are circular arcs, while $\partial E_m = \mathbb{T}_r$ with $r \in (0, 1]$.

Finally, we are going to show $r = 1$, so D is a circular slit disk. Suppose $r < 1$. We set $h_n := g_{t_n}^{-1}$ for all $n \in \mathbb{N}$. Using Proposition 2.11, we find $h_n \xrightarrow{\text{l.u.}} h$ on D. Moreover, $h : D \to \Omega_{t_0}$ is conformal, as $g_t'(0) \in [1, g_T'(0)]$ for all $t \in [0, T]$, see Lemma 2.24. Then we are able to find a subsequence $(D_{t_{n_k}})_{k\in\mathbb{N}}$ of $(D_{t_n})_{n\in\mathbb{N}}$ such that for some $r_1, r_2 \in (r, 1)$ with $r_1 < r_2$, $\mathbb{A}_{r_1,r_2} := \{z \in \mathbb{D} \mid r_1 < |z| < r_2\} \subseteq D_{t_{n_k}}$ for all $k \in \mathbb{N}$. Using Montel's theorem, we find a subsequence $(h_{m_k})_{k\in\mathbb{N}}$ of $(h_{n_k})_{k\in\mathbb{N}}$ such that $(h_{m_k})_{k\in\mathbb{N}}$ converges locally uniformly on $\mathbb{A}_{r_1,r_2}$ to the function $h^* : \mathbb{A}_{r_1,r_2} \to \mathbb{C}$, which is either univalent or constant. In order to show that h^* can not be constant, we set $\gamma(\tau) := r_0 e^{\mathrm{i}\tau}$ for all $\tau \in [0, 2\pi]$ and some $r_0 \in (r_1, r_2)$, so $\Gamma := \gamma[0, 2\pi]$ is a compact set in $\mathbb{A}_{r_1,r_2}$. Moreover, γ has winding number 1 around 0. Suppose h^* is constant. Then $(h_{m_k})_{k\in\mathbb{N}}$ converges uniformly on Γ to 0. This is a contradiction to the fact that h_n is conformal. On the other hand, suppose h^* is univalent. Then Γ is mapped to the Jordan curve $h^*(\Gamma)$. Note that $h_{m_k}(\Gamma)$ separates $\mathbb{T} \cup \mathfrak{H}_{m_k}$ from 0 and $\mathfrak{H}_{m_k}(\Gamma)$ converges uniformly to $h^*(\Gamma)$, so $h^*(\Gamma) \subseteq \Omega_{t_0}$. On the other hand $g_{t_{m_k}}(h^*(\Gamma))$ converges uniformly to a compact set $K \subseteq D$. This is a contradiction, since $\Gamma \subseteq \mathbb{A}_{r_1,r_2}$ and $g_{t_{m_k}}$ is univalent.

□

Proposition 5.6. *Let Ω be a circular slit disk, $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing family of radial Ω-hulls and let $t_0 \in [0, T]$. For each $t \in [0, T]$, g_t denotes the normalised radial mapping function from $\Omega_t := \Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t. Then the following three statements are equivalent.*

(i) $t \mapsto \Omega_t$ *is continuous at* t_0.

(ii) The real-valued function $t \mapsto \mathrm{lmr}(g_t)$ *is continuous at* t_0.

(iii) $t \mapsto g_t$ *is continuous at* t_0.

Proof. First of all, note that (iii)⇒(ii) is trivial.

In order to prove (i)⇒(iii) let us assume $\Omega_{t_n} \xrightarrow{\text{k}} \Omega_{t_0}$ for some sequence $t_n \to t_0$. By Montel's theorem we find a subsequence $(\Omega_{t_{n_k}})_{k\in\mathbb{N}}$ of $(\Omega_{t_n})_{n\in\mathbb{N}}$ such that $g_{t_{n_k}} \xrightarrow{\text{l.u.}} h$ on Ω_{t_0}. Herein, $h : \Omega_{t_0} \to D$ is a conformal map, as $g_{t_n}'(0) \geq 1$ for all $n \in \mathbb{N}$. Using Proposition 2.11, $D_{t_{n_k}} \xrightarrow{\text{k}} D$ as $k \to \infty$. Thus Lemma 5.5 shows that D is a circular slit disk. Analogously to the proof of Lemma 2.25 we easily see $h \equiv g_{t_0}$. Thus $g_{t_n} \xrightarrow{\text{l.u.}} g_{t_0}$ on Ω_{t_0} as well.

Finally, let us have a look at (ii)⇒(i). Let us assume $t \mapsto \operatorname{lmr}(g_t)$ is continuous at t_0. Denote by $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ a sequence converging to $t_0 \in [0,T]$. Without loss of generality we may assume $t_n \nearrow t_0$ or $t_n \searrow t_0$. In either case, $(\Omega_{t_n})_{n\in\mathbb{N}}$ has a kernel K, see Lemma 2.9. As a consequence of Montel's theorem we find a subsequence $(\Omega_{t_{n_k}})_{k\in\mathbb{N}}$ of $(\Omega_{t_n})_{n\in\mathbb{N}}$ such that $g_{t_{n_k}}$ convergences locally uniformly to $g : K \to D$. Since $\operatorname{lmr}(g_{t_m}) \to \operatorname{lmr}(g_{t_0})$, we find $\operatorname{lmr}(g_{t_0}) = \operatorname{lmr}(g)$. Moreover, $K \subseteq \Omega_{t_0}$ or $\Omega_{t_0} \subseteq K$, so we find $K = \Omega_{t_0}$ together with Lemma 2.24. □

Lemma 5.7. *Let Ω be a circular slit disk and denote by $C_1, \dots, C_{\mathfrak{n}-1}$, $\mathfrak{n} \in \mathbb{N}$, the interior boundary components of Ω. Moreover, let $(\mathfrak{H}_t)_{t\in[0,T]}$ be an increasing and continuous family of radial Ω-hulls, and for each $t \in [0,T]$, we denote by g_t the normalised radial mapping function on $\Omega_t := \Omega \setminus \mathfrak{H}_t$. Assume $C \in \{C_1, \dots, C_{\mathfrak{n}-1}\}$ with $\operatorname{cl}(\mathfrak{H}_t) \cap C = \emptyset$ for all $t < t_0$. Then $t \mapsto \operatorname{dist}(0, g_t(C))$ is continuous on $[0, t_0)$.*

Proof. Let $t^* < t_0$. Thus we find a Jordan curve $\Gamma \subseteq \Omega_{t^*}$ around C with Γ close enough to C such that $\operatorname{dist}(g_{t^*}(z), g_{t^*}(C)) < \varepsilon/2$ for all $z \in \Gamma$ with some small $\varepsilon > 0$. Using Proposition 5.6, we get $g_t \xrightarrow{\text{l.u.}} g_{t^*}$ on Ω_{t^*} as $t \to t^*$. In particular, we find $g_t \to g_{t^*}$ uniformly on Γ. So there is a $\delta > 0$ such that $|g_t(z) - g_{t^*}(z)| < \varepsilon/2$ for all $t \in (t^*-\delta, t^*+\delta)$ and all $z \in \Gamma$. Consequently, $\operatorname{dist}(g_t(z), g_{t^*}(C)) < \varepsilon$ for all $t \in (t^*-\delta, t^*+\delta)$ and all $z \in \Gamma$. Since $g_t(C)$ is part of the interior of $g_t(\Gamma)$, we find $\operatorname{dist}(g_t(C), g_{t^*}(C)) < \varepsilon$ for all $t \in (t^*-\delta, t^*+\delta)$ as well. Both sets $g_t(C)$ and $g_{t^*}(C)$ are circular arcs, so the proof is complete. □

Lemma 5.8. *Let Ω be a circular slit disk. Assume $(\mathfrak{H}_t)_{t\in[0,T]}$ is an increasing family of radial Ω-hulls. Moreover, for each $t \in [0,T]$, g_t is the normalised radial mapping function from $\Omega_t := \Omega \setminus \mathfrak{H}_t$ onto the circular slit disk D_t.*

Then the following two statements are equivalent.

(i) $(\mathfrak{H}_t)_{t\in[0,T]}$ satisfies the local growth property.

(ii) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $t \in [0, T-\delta]$, some cross-cut F of D_t with $\operatorname{diam}(F) < \varepsilon$ separates 0 from $g_t(\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t)$.

Proof. 1) (i)⇒(ii): Let $(\mathfrak{H}_t)_{t\in[0,T]}$ be a family of increasing Ω-hulls satisfying the local growth property. Let $\varepsilon > 0$ be small. We find a $\delta > 0$ such that whenever $t \in [0, T-\delta]$, some cross-cut E of Ω_t with $\operatorname{diam}(E) < \varepsilon$ separates 0 from $\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t$. Assume $a \in E$. For each $r \in [\varepsilon, \sqrt{\varepsilon}]$, $C_r := \partial B_r(a) \cap \Omega_t$ separates E in Ω_t from 0. Obviously, C_r separates $K_{t+\delta} \setminus K_t$ in Ω_t from 0 as well. Using Wolff's lemma, we find $\inf_{r\in(\varepsilon,\sqrt{\varepsilon})} \operatorname{diam}(g_t(C_r)) < 4\pi/\sqrt{\log 1/\varepsilon}$. Let $F := h_t(E)$, so we get $\operatorname{diam}(F) < 4\pi/\sqrt{\log 1/\varepsilon}$ as well.

2) (ii)⇒(i): This works in the same way by Wolff's lemma.

□

Remark 5.1. In the simply connected case, (i) and (ii) from Lemma 5.8 are equivalent to the statement *(iii) For each* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that whenever* $t \in [0, T-\delta]$*,* $\operatorname{diam}(g_t(\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t)) < \varepsilon$*.*

Unfortunately, this is not the case if we consider multiply connected domains, see Example 5.1 where condition (iii) is satisfied, while (i) or (ii) are not.

Obviously, the implication (ii)⇒(iii) is true. When this happens, we are able to define $U_t := \bigcap_{\delta>0} \operatorname{cl}\left(g_t(\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t)\right)$ for all $t \in [0, T)$. Analogously to the previous chapters $t \mapsto U_t$ is called *driving term* or *driving function*.

Lemma 5.9. *Let* Ω *be a circular slit disk and let* $(\mathfrak{H}_t)_{t\in[0,T]}$ *be an increasing family of radial* Ω*-hulls satisfying the local growth property. Then for each* $t \in [0, T]$*,* $\operatorname{cl}(\mathfrak{H}_t)$ *is connected.*

Proof. First of all keep in mind that $\mathfrak{H}_0 = \emptyset$ as $(\mathfrak{H}_t)_{t\in[0,T]}$ is an increasing family of radial Ω-hulls. Suppose there is a $t_0 \in (0, T]$ such that $\operatorname{cl}(\mathfrak{H}_{t_0})$ is not connected. Then there are proper compact sets A and B such that $A \cap B = \emptyset$ and $A \cup B = \operatorname{cl}(\mathfrak{H}_{t_0})$. We set $t_A := \inf\{t \in [0, t_0] \mid A \cap \operatorname{cl}(\mathfrak{H}_t) \neq \emptyset\}$ and $t_B := \inf\{t \in [0, t_0] \mid B \cap \operatorname{cl}(\mathfrak{H}_t) = \emptyset\}$. Without restricting generality we may assume $t_A \geq t_B$. Note that $t_A < t_0$. Otherwise $t \mapsto \Omega_t$, with $\Omega_t := \Omega \setminus \mathfrak{H}_t$, is not continuous at $t = t_0$ contradicting Lemma 5.3. Using the same argument, $A \cap \mathfrak{H}_{t_A} = \emptyset$.

If $t_A = 0$, we immediately find $g_0(\mathfrak{H}_\varepsilon) \cap A \neq \emptyset$ and $g_0(\mathfrak{H}_\varepsilon) \cap B \neq \emptyset$. Consequently, $\operatorname{diam}(\mathfrak{H}_\varepsilon) > \operatorname{dist}(A, B)$ for all $\varepsilon > 0$. This yields a contradiction to the local growth property, see also Remark 5.1.

Next, assume $t_A > 0$. Obviously, we find an open set $E \subseteq \Omega_{t_A}$ such that $\operatorname{cl}(A \cap \mathfrak{H}_{t_A+\varepsilon'}) \subseteq E \cup \mathbb{T}$ and $\operatorname{dist}(\partial E \setminus \mathbb{T}, A \cap \mathfrak{H}_{t_A+\varepsilon'}) > 0$ with some small $\varepsilon' > 0$. For each $t \in [0, t_A]$, we reflect g_t on $T_E := \mathbb{T} \cap \operatorname{cl}(E)$, so g_t is analytic on $\Omega_{t_A} \cup \bar{E}$ with $\bar{E} := \{w \in \mathbb{C} \mid 1/\bar{w} \in E\}$. Then Proposition 5.6 gives us $g_{t_A-\varepsilon} \xrightarrow{\text{l.u.}} g_{t_A}$ on $\Omega_{t_A} \cup \bar{E} \cup T_E$ as $\varepsilon \searrow 0$. Let $E' := g_{t_A}(E)$. This shows $\operatorname{diam}(g_{t_A-\varepsilon}(\mathfrak{H}_{t_A+\varepsilon} \setminus \mathfrak{H}_{t_A-\varepsilon})) \geq \operatorname{dist}(g_{t_A}(\mathfrak{H}_{t_A+\varepsilon'} \cap A), \partial E')$ for all small $\varepsilon < \varepsilon'$ yielding a contradiction to the local growth property, see Remark 5.1. □

Lemma 5.10. *Let* Ω *be a circular slit disk and* $(\mathfrak{H}_t)_{t\in[0,T]}$ *be an increasing family of radial* Ω*-hulls satisfying the local growth property. For each* $t \in [0, T]$*,* g_t *denotes the normalised radial mapping function on* $\Omega_t := \Omega \setminus \mathfrak{H}_t$*. Assume* $U_t := \bigcap_{\delta>0} \operatorname{cl}\left(g_t(\mathfrak{H}_{t+\delta} \setminus \mathfrak{H}_t)\right)$ *with* $t \in [0, T)$*.*

Then $t \mapsto U_t$ *is uniformly continuous on* $[0, T)$*. Furthermore, the limit* $\lim_{t\nearrow T} U_t =: U_T$ *exists.*

Proof. We are going to prove the following statement:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall 0 \leq t < s < T \text{ with } s - t < \delta : \ |U_t - U_s| < \varepsilon.$$

For each $0 \leq t < s < T$, we set $S_{t,s} := g_t(\mathfrak{H}_s \setminus \mathfrak{H}_t)$.

Let us assume the opposite, so there are sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ with $t_n < s_n$ and $|t_n - s_n| \to 0$ such that $|U_{s_n} - U_{t_n}| > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Without loss of generality we assume that the sequences $(t_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$ and $(U_{t_n})_{n\in\mathbb{N}}$ are convergent with limits $t_0 = s_0$ and U^*, respectively.

Next, we denote by $C_1, \dots, C_{\mathfrak{n}-1}$ the interior concentric circular arcs of $\partial\Omega$. Each C_k that gets not swallowed by $\mathfrak{H}_{t_0}$ fulfils $\operatorname{dist}(g_{t_n}(C_k), 0) > \rho$ for all large $n \in \mathbb{N}$ and some $\rho > 0$ with $\rho < \varepsilon$. If C_k gets swallowed by $\mathfrak{H}_{t_0}$ such that $\operatorname{dist}(C_k, \mathfrak{H}_{t_n}) > 0$ for at least infinite $n \in \mathbb{N}$, we get $\operatorname{dist}(S_{t_n,t_0}, g_{t_n}(C)) = 0$ and $t_n < t_0$ for almost all $n \in \mathbb{N}$. Thus any small cross-cut F that separates S_{t_n,t_0} from 0 separates $g_{t_n}(C)$ from 0 as well.

Using Lemma 5.8, we choose $N \in \mathbb{N}$ large enough such that whenever $n \geq N$, some cross-cut F_n of D_{t_n} separates $S_{t_n,\max(s_n,t_0)}$ from 0 with $\operatorname{diam}(F_n) < \frac{\rho}{4}$. Moreover, we enlarge N in such a way to get $|U_{t_n} - U^*| < \frac{\rho}{4}$ for all $n \geq N$.

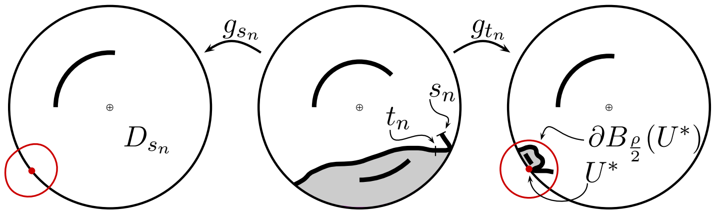

FIGURE 5.2: *Mapping behaviour of g_{s_n} and g_{t_n}*

As mentioned before, for any $k \in \{1, \dots, \mathfrak{n}-1\}$ and large $n \geq N$, $\operatorname{dist}(g_{t_n}(C_k), \mathbb{T}) = 0$ (i.e. C_k was already swallowed by the hull), $\operatorname{dist}(g_{t_n}(C_k), \mathbb{T}) > \rho$, or F_n separates $g_{t_n}(C_k)$ from 0

In either case, $h_n := g_{s_n} \circ g_{t_n}^{-1}$ can be continued in an analytic way to a neighbourhood V of $\partial B_{\rho/2}(U^*)$ for all $n \geq N$, as $|U^* - U_{t_n}| < \frac{\rho}{4}$, $U_{t_n} \in \operatorname{cl}(S_{t_n,s_n})$ and $\operatorname{diam}(S_{t_n,s_n}) < \operatorname{diam}(F_n) < \frac{\rho}{4}$ for all large $n \geq N$. Using Proposition 5.6, h_n convergences uniformly on $\partial B_{\rho/2}(U^*)$ to the identity. Thus we find $h_n(\partial B_{\rho/2}(U^*)) \subseteq B_{3\rho/4}(U^*)$ for all large n. Moreover, we set

$$s_{t_n,s_n} := \operatorname{cl}\left\{z \in \mathbb{T} \,\middle|\, \exists r > 0 \; \exists (z_k)_{k\in\mathbb{N}} \subseteq D_{s_n} \colon \; z_k \to z \text{ and } |h_n^{-1}(z_k)| < 1 - r\right\},$$

so we have $s_{t_n,s_n} \subseteq B_{3\rho/4}(U^*)$ and $U_{s_n} \in s_{t_n,s_n}$ for all large n. Consequently, we find $|U_{s_n} - U_{t_n}| \leq |U_{s_n} - U^*| + |U^* - U_{t_n}| < \frac{3\rho}{4} + \frac{\rho}{4} = \rho < \varepsilon$. This is a contradiction, so the proof is complete. □

Remark 5.2. As another consequence, we have seen in the previous proof that $s_{t_n,s_n} \to U_{t_0}$ whenever $t_n \to t_0 \leftarrow s_n$. Herein, s_{t_n,s_n} is defined in the same way as before.

Let Ω be a circular slit disk and let $\mathfrak{H}$ be a radial Ω-hull. Note that we can not apply Lemma 2.27 or 2.28, as $\Omega \setminus \mathfrak{H}$ is not necessarily locally connected. In the following we will deduce a way to circumvent this problem. Therefore, we denote by g the normalised radial mapping function from $\Omega \setminus \mathfrak{H}$ onto the circular slit disk D. Moreover, let us assume $\operatorname{cl}(\mathfrak{H})$ is connected. Using Lemma 5.9, this is always the case if the hull $\mathfrak{H}$ comes from a family that satisfies the local growth property. Next we set

$$s_{\mathfrak{H}} := \operatorname{cl}\left\{z \in \mathbb{T} \,\middle|\, \exists r > 0 \; \exists (z_k)_{k\in\mathbb{N}} \subseteq D \colon \; z_k \to z \text{ and } |g^{-1}(z_k)| < 1 - r\right\}. \tag{5.2}$$

$s_{\mathfrak{H}}$ is a connected and compact subset of $\mathbb{T}$. On top of this we define

$$\mathfrak{H}^{\varepsilon} := \mathfrak{H} \cup \{g^{-1}(z) \mid z \in D, \; \operatorname{dist}(z, s_{\mathfrak{H}}) \leq \varepsilon\},$$

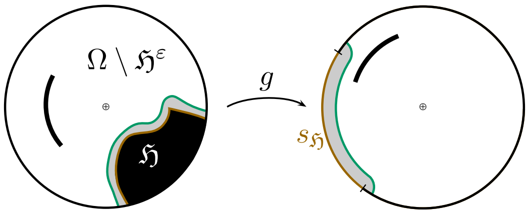

FIGURE 5.3: *The ε-extension of a hull $\mathfrak{H}$*

what we call the *ε-extension of $\mathfrak{H}$ in Ω*, see Figure 5.3. Note that $\mathfrak{H}^\varepsilon$ is a radial Ω-hull as well if $\varepsilon > 0$ is small enough.

In contrast to $\mathfrak{H}$, $\mathfrak{H}^\varepsilon$ is locally connected. This allows us to apply Lemma 2.27 and 2.28 followed by the limit process $\varepsilon \to 0$. See the following three lemmas for more details

Lemma 5.11. *Let Ω be a circular slit disk, $\mathfrak{H}$ be a radial Ω-hull such that $\mathrm{cl}(\mathfrak{H})$ is connected, and for each small $\varepsilon > 0$, $\mathfrak{H}^\varepsilon$ denotes the ε-extension of $\mathfrak{H}$. Moreover, we denote by g and g^ε the normalised radial mapping function on $\Omega \setminus \mathfrak{H}$ and $\Omega \setminus \mathfrak{H}^\varepsilon$, respectively.*

Then $g^\varepsilon \xrightarrow{\text{l.u.}} g$ on Ω as $\varepsilon \to 0$. Moreover, $s_{\mathfrak{H}^\varepsilon} \to s_\mathfrak{H}$ as $\varepsilon \to 0$ where $s_\mathfrak{H}$ and $s_{\mathfrak{H}^\varepsilon}$ are defined by Equation (5.2)*.*

Proof. Obviously, $\Omega \setminus \mathfrak{H}^\varepsilon \xrightarrow{\text{k}} \Omega \setminus \mathfrak{H}$ as $\varepsilon \to 0$, so Proposition 5.6 gives us $g^\varepsilon \xrightarrow{\text{l.u.}} g$ on $\Omega \setminus \mathfrak{H}$ as $\varepsilon \to 0$. Consequently, $h_\varepsilon := g \circ (g^\varepsilon)^{-1}$ tends to the identity, so $s_{\mathfrak{H}^\varepsilon} \to s_\mathfrak{H}$ follows immediately with an reflection of h_ε on $\mathbb{T}$. □

Lemma 5.12. *Let Ω be a circular slit disk and let $\mathfrak{H}$ be a radial Ω-hull such that $\mathrm{cl}(\mathfrak{H})$ is connected. Moreover, g denotes the normalised radial mapping function from $\Omega \setminus \mathfrak{H}$ onto the circular slit disk D. Then*

$$\log \frac{g^{-1}(z)}{z} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int\limits_{s_{\mathfrak{H}^\varepsilon}} \ln \left|(g^\varepsilon)^{-1}(\zeta)\right| \cdot \Phi_{0,\zeta,D^\varepsilon}(z)\, |\mathrm{d}\zeta| \quad \text{for all } z \in D,$$

where $\mathfrak{H}^\varepsilon$ denotes the ε-extension of $\mathfrak{H}$, g^ε is the normalised radial mapping function from $\Omega \setminus \mathfrak{H}^\varepsilon$ onto the circular slit disk D^ε, and $s_{\mathfrak{H}^\varepsilon}$ is defined by Equation (5.2)*.*

Proof. This follows immediately from Lemma 2.28 and 5.11. □

Lemma 5.13. *Let Ω be a circular slit disk and let $\mathfrak{H}$ be a radial hull in Ω such that $\mathrm{cl}(\mathfrak{H})$ is connected. Moreover, g denotes the normalised radial mapping function from $\Omega \setminus \mathfrak{H}$ onto the circular slit disk D. Then*

$$\mathrm{lmr}(g) = -\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int\limits_{s_{\mathfrak{H}^\varepsilon}} \ln \left|(g^\varepsilon)^{-1}(\zeta)\right| |\mathrm{d}\zeta|,$$

where $\mathfrak{H}^\varepsilon$ denotes the ε-extension of $\mathfrak{H}$, g^ε is the normalised radial mapping function on $\Omega \setminus \mathfrak{H}^\varepsilon$, and $s_{\mathfrak{H}^\varepsilon}$ is defined by Equation (5.2)*.*

Proof. This follows immediately from Lemma 2.27 and Lemma 5.11. □

5.2 Proof of Theorem 5.1 and 5.2

Proof of Theorem 5.2. First of all, for each $t \in [0, T]$, $\mathrm{cl}(\mathfrak{H}_t)$ is connected, as $(\mathfrak{H}_t)_{t\in[0,T]}$ satisfies the local growth property, see Lemma 5.9. Moreover, U_t is defined as in Lemma 5.10 for each $t \in [0, T]$. Let $0 \leq \underline{t} < \bar{t} \leq T$ be fixed, so $g_{\underline{t},\bar{t}} := g_{\mathfrak{A}} := g_{\bar{t}} \circ g_{\underline{t}}^{-1}$ is the normalised radial mapping function on $D_{\underline{t}} \setminus \mathfrak{A}$, with $\mathfrak{A} := g_{\underline{t}}(\mathfrak{H}_{\bar{t}} \setminus \mathfrak{H}_{\underline{t}})$, onto the circular slit disk $D_{\bar{t}}$. Obviously, $\mathfrak{A}$ is a radial $D_{\underline{t}}$-hull, so the mapping function is well-defined. Using Lemma 5.12, we find

$$\log \frac{g_{\underline{t},\bar{t}}^{-1}(z)}{z} = \log \frac{g_{\mathfrak{A}}^{-1}(z)}{z} = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int\limits_{s_{\mathfrak{A}^\varepsilon}} \ln \left| g_{\mathfrak{A}^\varepsilon}^{-1}(\zeta) \right| \cdot \Phi_{0,\zeta,D^\varepsilon}(z) \, |\mathrm{d}\zeta| \quad \text{for all } z \in D_{\bar{t}},$$

where $\mathfrak{A}^\varepsilon$ denotes the ε-extension extension of $\mathfrak{A}$, $g_{\mathfrak{A}^\varepsilon}$ denotes the normalised radial mapping function from $D_{\underline{t}} \setminus \mathfrak{A}^\varepsilon$ onto the circular slit disk D^ε and $s_{\mathfrak{H}^\varepsilon}$ is defined by Equation (5.2). As a consequence of Lemma 2.18, $\zeta \mapsto \Phi_{0,\zeta,D^\varepsilon}(z)$ is continuous on $s_{\mathfrak{A}^\varepsilon}$, so the mean value theorem yields

$$\log \frac{g_{\underline{t},\bar{t}}^{-1}(z)}{z} = \lim_{\varepsilon\to 0} \Big(\Re\big(\Phi_{0,\zeta_1^\varepsilon,D^\varepsilon}(z)\big) + \mathrm{i}\Im\big(\Phi_{0,\zeta_2^\varepsilon,D^\varepsilon}(z)\big) \Big) \frac{1}{2\pi} \int\limits_{s_{\mathfrak{A}^\varepsilon}} \ln \left| g_{\mathfrak{A}^\varepsilon}^{-1}(\zeta) \right| |\mathrm{d}\zeta|, \quad z \in D_{\bar{t}}$$

where $\zeta_1^\varepsilon, \zeta_2^\varepsilon \in s_{\mathfrak{A}^\varepsilon}$. Note that $\zeta_1^\varepsilon, \zeta_2^\varepsilon$ are bounded, so we find a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $\zeta_j^{\varepsilon_n} \to \zeta_j$, $j \in \{1, 2\}$, as $n \to \infty$. Moreover, Lemma 5.11 yields $\zeta_j \in s_{\mathfrak{A}}$, $j \in \{1, 2\}$. Using Lemma 5.11, we find $D^{\varepsilon_n} \xrightarrow{\text{k}} D_{\bar{t}}$ with $\mathrm{con}(D^{\varepsilon_n}) = \mathrm{con}(D_{\bar{t}})$ if n is large enough. Letting $n \to \infty$, Lemma 2.18 and 5.13 give us

$$\log \frac{g_{\underline{t},\bar{t}}^{-1}(z)}{z} = -\Big(\Re\big(\Phi_{0,\zeta_1,D_{\bar{t}}}(z)\big) + \mathrm{i}\Im\big(\Phi_{0,\zeta_2,D_{\bar{t}}}(z)\big) \Big) \operatorname{lmr}(g_{\mathfrak{A}}) \quad \text{for all } z \in D_{\bar{t}}.$$

Using $\operatorname{lmr}(g_{\mathfrak{A}}) = \operatorname{lmr}(g_{\underline{t},\bar{t}}) = \operatorname{lmr}(g_{\bar{t}}) - \operatorname{lmr}(g_{\underline{t}}) = \bar{t} - \underline{t}$ and by applying $z = g_{\bar{t}}(w)$, we find

$$\frac{-\log \frac{g_{\underline{t}}(w)}{g_{\bar{t}}(w)}}{\bar{t} - \underline{t}} = \Re\big(\Phi_{0,\zeta_1,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big)\big) + \mathrm{i}\Im\big(\Phi_{0,\zeta_2,D_{\bar{t}}}\big(g_{\bar{t}}(w)\big)\big) \quad \text{for all } w \in \Omega_{\bar{t}}.$$

For each $j \in \{1, 2\}$, $\zeta_j \in s_{\mathfrak{A}}$, so $\zeta_j \to U_{\underline{t}}$ if $\bar{t} \searrow \underline{t}$ and $\zeta_j \to U_{\bar{t}}$ if $\underline{t} \nearrow \bar{t}$, see Remark 5.2. Letting $\underline{t} \nearrow \bar{t}$, Lemma 2.18 shows

$$\Phi_{0,\zeta_j,D_{\bar{t}}} \circ g_{\bar{t}} \xrightarrow{\text{l.u.}} \Phi_{0,U_{\bar{t}},D_{\bar{t}}} \circ g_{\bar{t}} \text{ on } \Omega_{\bar{t}} \quad \text{as } \underline{t} \nearrow \bar{t}.$$

On the other hand let $\bar{t} \searrow \underline{t}$. Then Lemma 5.4 yields $\mathrm{con}(D_{\underline{t}}) = \mathrm{con}(D_{\bar{t}})$ if $\bar{t}$ is close enough to $\underline{t}$. Consequently, we can use Lemma 2.18 once again together with Proposition 5.6 to obtain

$$\Phi_{0,\zeta_j,D_{\bar{t}}} \circ g_{\bar{t}} \xrightarrow{\text{l.u.}} \Phi_{0,U_{\underline{t}},D_{\underline{t}}} \circ g_{\underline{t}} \text{ on } \Omega_{\underline{t}} \quad \text{as } \bar{t} \searrow \underline{t}.$$

Note that the continuity of $t \mapsto \Phi_{0,U_t,D_t}$ follows analogously to the proof of Lemma 2.18 combined with Wolff's lemma (applied in the same way as in Example 5.1). Summarising, the proof is complete as $t \mapsto U_t$ is continuous by Lemma 5.10. □

Proof of Theorem 5.1. Note that the previous proof showed already (ii)$\Rightarrow$ (i), so we need to prove (i)$\Rightarrow$(ii) only.

1) First of all, $t \mapsto \Omega_t$ and $t \mapsto g_t$ are continuous on $[0,T]$ by (ii)$\Rightarrow$(i),(iii) from Proposition 5.6. Let us denote by $C_1(t), \ldots, C_{\mathfrak{n}}(t)$ the boundary components of D_t where $C_{\mathfrak{n}}(t) = \mathbb{T}$. As $\mathrm{con}(\Omega_t) = \mathrm{con}(\Omega)$ for all $t \in [0,T]$, Lemma 2.42 gives us a $\rho > 0$ such that $\mathrm{dist}(C_k(t), \mathbb{T}) > \rho$ for all $t \in [0,T]$ and all $k \in \{1, \ldots, \mathfrak{n}-1\}$.

2) Next, we are going to prove $|\Phi_{0,U_t,D_t}(w)| \leq \frac{K}{|U_t - w|}$ for all $w \in D_t$ and all $t \in [0,T]$ with some $K > 0$. Using the definition of Φ_{0,U_t,D_t}, we find $R_t(w) := \Phi_{0,U_t,D_t}(w) \cdot (w - U_t)$ is bounded on D_t. Suppose there is no $K > 0$ fulfilling the previous condition. Thus there is a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ and a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D_{t_n}$ such that $R_{t_n}(w_n) \to \infty$. By boundedness, we may assume $t_n \to t_0 \in [0,T]$. Using Proposition 2.11, $D_{t_n} \xrightarrow{\mathrm{k}} D_{t_0}$, so together with $\mathrm{con}(D_{t_n}) = \mathrm{con}(D_{t_0})$ and Lemma 2.18 we find $\Phi_{0,U_t,D_t} \xrightarrow{\mathrm{l.u.}} \Phi_{0,U_{t_0},D_{t_0}}$ on D_{t_0}. Thus $R_t \xrightarrow{\mathrm{l.u.}} R_{t_0}$ on D_{t_0} as well. Since $\mathrm{dist}(C_k(t), \mathbb{T}) > \rho$ for all $t \in [0,T]$, we are able to reflect each Φ_{0,U_t,D_t}, $t \in [0,T]$, along $\mathbb{T}$, so we are able to continue each R_t analytically to $\mathbb{A}_{1,1+\rho}$ with $\rho > 0$ defined in Part 1.

Let $r \in (1, 1+\rho)$, so R_{t_n} converges uniformly on $\mathbb{T}_r$ to $R_{t_0}(\mathbb{T}_r)$. Obviously, $R_{t_0}(\mathbb{T}_r)$ is bounded and we have $|R_{t_n}(w_n)| \leq \max_{z\in\mathrm{cl}(\mathbb{D}_r)} |R_{t_n}(z)| = \max_{\zeta\in\mathbb{T}_r} |R_{t_n}(\zeta)|$. This is a contradiction.

3) In order to continue the proof, we follow the first part of Pommerenke's proof, see proof of Theorem 1 in [Pom66]. Therefore, we set $S_{t,s} := g_t(\mathfrak{H}_s \setminus \mathfrak{H}_t)$, whenever $0 \leq t < s \leq T$, and let $\varepsilon > 0$. $t \mapsto U_t$ is uniformly continuous on $[0,T]$, so we find a $\delta < \frac{\varepsilon^2}{8K}$ such that $|U_t - U_s| < \frac{\varepsilon}{4}$ for all $0 \leq t \leq s \leq T$ with $s - t \leq \delta$. Here $K > 0$ is defined like in Part 2

We are going to show $\nu(s) := |U_t - (g_s \circ g_t^{-1})(w)| > \frac{\varepsilon}{2}$ whenever $w \in D_t \setminus S_{t,s}$, $|w - U_t| > \varepsilon$ and $0 \leq s - t \leq \delta$. Suppose this is false, i.e. we find $t, s \in [0,T]$ with $0 < s - t < \delta$ and $w \in D_t \setminus S_{t,s}$, $|w - U_t| > \varepsilon$ such that $\nu(s) \leq \frac{\varepsilon}{2}$. Note that $\nu(t) = |U_t - w| > \varepsilon$, so we find a first time $t_1 \in (t, s]$ such that $\nu(t_1) = \frac{\varepsilon}{2}$. This follows immediately from the fact that $\tau \mapsto \nu(\tau)$ is continuous on $[t, s]$. Consequently, we get

$$\begin{aligned} \left|U_\tau - (g_\tau \circ g_t^{-1})(w)\right| &\geq \left|U_t - (g_\tau \circ g_t^{-1})(w)\right| - \left|U_\tau - U_t\right| \\ &= \nu(\tau) - |U_\tau - U_t| \geq \tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{4} = \tfrac{\varepsilon}{4} \end{aligned}$$

for all $\tau \in [t, t_1]$. Using the differential equation, we find together with the previous part

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}\tau}\nu(\tau)\right| \leq \left|(g_\tau \circ g_t^{-1})(w)\right| \cdot \left|\Phi_{0,U_\tau,D_\tau}\big(g_\tau \circ g_t^{-1}(w)\big)\right| \leq 1 \cdot \tfrac{4K}{\varepsilon}$$

for all $\tau \in [t, t_1]$. Summarising, we find the following contradiction

$$\tfrac{\varepsilon}{2} = \varepsilon - \tfrac{\varepsilon}{2} \leq \nu(t) - \nu(t_1) = \left|\int_{t_1}^{t} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\nu(\tau)\mathrm{d}\tau\right| \leq (t_1 - t)\tfrac{4K}{\varepsilon} \leq \delta\tfrac{4K}{\varepsilon} < \tfrac{\varepsilon^2}{8K} \cdot \tfrac{4K}{\varepsilon} = \tfrac{\varepsilon}{2}.$$

Unfortunately, we can not apply further parts of Pommerenke's proof, so we need to argue in an another way.

4) Next, we are going to show that for each $\varepsilon > 0$, there is a $\mu > 0$ such that $\mathrm{diam}(S_{t,s}) < 3\varepsilon$, whenever $t, s \in [0,T]$ with $0 < s - t < \mu$. In order to prove this, suppose there are sequences $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subseteq [0,T]$ and an $\varepsilon > 0$ such that $t_n - s_n \to 0$ and $\mathrm{diam}(S_{t_n,s_n}) \geq 3\varepsilon$ for all $n \in \mathbb{N}$. By boundedness, we may assume that (t_n) and (s_n) are convergent with limit $t_0 \in [0,T]$. Thus we find $w_n \in D_{t_n} \setminus S_{t_n,s_n}$ (close enough to S_{t_n,s_n}) such that $|w_n - U_{t_n}| > \varepsilon$ and $|(g_{s_n} \circ g_{t_n}^{-1})(w_n)| \geq \sqrt{|w_n|}$ for all $n \in \mathbb{N}$. Moreover, we write $g_{t_n}(z_n) = w_n$, so we have $|g_{s_n}(z_n)| \geq \sqrt{|g_{t_n}(z_n)|}$. Using Part 3, we are able to choose n large enough in order to get $0 < s_n - t_n < \delta$ where $\delta < \frac{\varepsilon^2}{8K}$ is defined as in Part 3. Thus we find

$$\nu(s) = |U_{t_n} - (g_s \circ g_{t_n})^{-1}(w_n)| = |U_{t_n} - g_s(z_n)| > \tfrac{\varepsilon}{2} \quad \text{for all } s \in [t_n, s_n].$$

Moreover, using $|U_t - U_s| < \frac{\varepsilon}{4}$ whenever $|t - s| < \delta$, we find $|U_s - g_s(z_n)| \geq \frac{\varepsilon}{4}$ for all $s \in [t_n, s_n]$. For each $n \in \mathbb{N}$, we get

$$\tfrac{1}{4}\left(\tfrac{1}{|g_{t_n}(z_n)|} - 1\right) \leq \tfrac{1}{2}\tfrac{1}{2}\ln\left|\tfrac{1}{g_{t_n}(z_n)}\right| \leq \ln\left|\tfrac{g_{s_n}(z_n)}{g_{t_n}(z_n)}\right| = \\ \ln|g_{s_n}(z_n)| - \ln|g_{t_n}(z_n)| = (s_n - t_n)\Re\Phi_{0,U_{\xi_n},D_{\xi_n}}\big(g_{\xi_n}(z_n)\big) \quad (5.3)$$

with $\xi_n \in [t_n, s_n]$. Here, the last equality is a consequence of the differential equation. In particular we used the mean value theorem applied to the real part of the logarithmic derivative. Notice, Equation (5.3) together with Part 2 show that $|g_{t_n}(z_n)| \to 1$ if $n \to \infty$, as $U_{\xi_n} - g_{\xi_n}(z_n) \geq \frac{\varepsilon}{4}$. Moreover, $|g_{t_n}(z_n)| \leq |g_{\xi_n}(z_n)|$ for all $n \in \mathbb{N}$. This follows from the fact that $t \mapsto \ln|g_t(z)|$ is increasing, as g_t satisfies the given differential equation while $\Re\Phi_{0,U_t,D_t}(z) \geq 0$. Consequently $|g_{\xi_n}(z_n)| \to 1$ if $n \to \infty$.

Obviously, we have $\xi_n \to t_0$, and we may assume without loss of generality $g_{\xi_n}(z_n) \to \zeta_0 \in \mathbb{T}$. Consequently, $|\zeta_0 - U_{t_0}| > \frac{\varepsilon}{4}$. Once again the mean value theorem yields

$$\left|\frac{\Re\Phi_{0,U_{\xi_n},D_{\xi_n}}\big(g_{\xi_n}(z_n)\big) - \Re\Phi_{0,U_{\xi_n},D_{\xi_n}}\left(\frac{g_{\xi_n}(z_n)}{|g_{\xi_n}(z_n)|}\right)}{g_{\xi_n}(z_n) - \frac{g_{\xi_n}(z_n)}{|g_{\xi_n}(z_n)|}}\right| \leq \left|\Phi'_{0,U_{\xi_n},D_{\xi_n}}(\zeta_n)\right| \quad (5.4)$$

with $\zeta_n \in \left\{\big(g_{\xi_n}(z_n)/|g_{\xi_n}(z_n)| - g_{\xi_n}(z_n)\big)t + g_{\xi_n}(z_n) \,\middle|\, t \in [0,1]\right\}$ and $n \in \mathbb{N}$ large. Herein, $\zeta_n \to \zeta_0$ as well. Using Lemma 2.18, we find $\Phi_{0,U_{\xi_n},D_{\xi_n}} \xrightarrow{\text{l.u.}} \Phi_{0,U_{t_0},D_{t_0}}$ on D_{t_0}. Using the fact $|U_{t_0} - \zeta_0| > 0$, each $w \mapsto \Phi_{0,U_{\xi_n},D_{\xi_n}}(w)$ can be extended analytically to a small neighbourhood around ζ_0 if n is large enough. Thus $\Phi'_{0,U_{\xi_n},D_{\xi_n}}(\zeta_n) \to \Phi'_{0,U_{t_0},D_{t_0}}(\zeta_0)$, so we get $\left|\Phi'_{0,U_{\xi_n},D_{\xi_n}}(\zeta_n)\right| \leq L$ for all $n \in \mathbb{N}$ large enough with $L > 0$.

Combined with Equation (5.3) and (5.4), we find

$$\tfrac{1}{4}\left(\tfrac{1}{|g_{t_n}(z_n)|} - 1\right) \leq (s_n - t_n)L|g_{\xi_n}(z_n)|\left(\tfrac{1}{|g_{\xi_n}(z_n)|} - 1\right) \leq (s_n - t_n)L\left(\tfrac{1}{|g_{t_n}(z_n)|} - 1\right). \quad (5.5)$$

Finally, Equation (5.5) yields a contradiction, as $s_n - t_n \to 0$ when n tends to infinity.

5) Using the previous part, we find a $\mu > 0$ such that $\mathrm{diam}(S_{t,s}) < \varepsilon$, whenever $0 \leq t \leq s \leq T$ with $0 \leq s - t \leq \mu$. Here, we can choose $\varepsilon < \rho$ where ρ is defined according

to Part 1. Let $a \in \mathbb{T} \cup \mathrm{cl}(S_{t,s})$. Consequently, $S_{t,s} \subseteq B_\varepsilon(a)$ and $B_\varepsilon(a) \cap D_t = B_\varepsilon(a) \cap \mathbb{D}$. Thus $\partial B_\varepsilon(a) \cap \mathbb{D}$ is a cross-cut in D_t separating $S_{t,s}$ from 0. Using Lemma 5.8, the proof is complete.

□

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