# Arbitrary Lagrangian-Eulerian Discontinous Galerkin methods for nonlinear time-dependent first order partial differential equations

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# Abstract

The present thesis considers the development and analysis of arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) methods with time-dependent approximation spaces for conservation laws and the Hamilton-Jacobi equations.
Fundamentals about conservation laws, Hamilton-Jacobi equations and discontinuous Galerkin methods are presented. In particular, issues in the development of discontinuous Galerkin (DG) methods for the Hamilton-Jacobi equations are discussed.
The development of the ALE-DG methods based on the assumption that the distribution of the grid points is explicitly given for an upcoming time level. This assumption allows to construct a time-dependent local affine linear mapping to a reference cell and a time-dependent finite element test function space. In addition, a version of Reynolds' transport theorem can be proven.
For the fully-discrete ALE-DG method for nonlinear scalar conservation laws the geometric conservation law and a local maximum principle are proven. Furthermore, conditions for slope limiters are stated. These conditions ensure the total variation stability of the method. In addition, entropy stability is discussed. For the corresponding semi-discrete ALE-DG method, error estimates are proven. If a piecewise $\mathcal{P}^k$ polynomial approximation space is used on the reference cell, the sub-optimal $(k+\frac{1}{2})$ convergence for monotone fluxes and the optimal $(k+1)$ convergence for an upwind flux are proven in the $\mathrm{L}^2$-norm. The capability of the method is shown by numerical examples for nonlinear conservation laws.
Likewise, for the semi-discrete ALE-DG method for nonlinear Hamilton-Jacobi equations, error estimates are proven. In the one dimensional case the optimal $(k+1)$ convergence and in the two dimensional case the sub-optimal $(k+\frac{1}{2})$ convergence are proven in the $\mathrm{L}^2$-norm, if a piecewise $\mathcal{P}^k$ polynomial approximation space is used on the reference cell. For the fully-discrete method, the geometric conservation is proven and for the piecewise constant forward Euler step the convergence of the method to the unique physical relevant solution is discussed.

# Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Entwicklung und Analyse von arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) Methoden mit zeitabhängigen Testfunktionen Räumen für Erhaltungs- und Hamilton-Jacobi Gleichungen.
Grundlagen über Erhaltungsgleichungen, Hamilton-Jacobi Gleichungen und discontinuous Galerkin Methoden werden präsentiert. Insbesondere werden Probleme bei der Entwicklung von discontinuous Galerkin Methoden für die Hamilton-Jacobi Gleichungen untersucht.
Die Entwicklung der ALE-DG Methode basiert auf der Annahme, dass die Verteilung der Gitterpunkte zu einem kommenden Zeitpunkt explizit gegeben ist. Diese Annahme ermöglicht die Konstruktion einer zeitabhängigen lokal affin-linearen Abbildung auf eine Referenzzelle und eines zeitabhängigen Testfunktionen Raums. Zusätzlich kann eine Version des Reynolds'schen Transportsatzes gezeigt werden.
Für die vollständig diskretisierte ALE-DG Methode für nichtlineare Erhaltungsgleichungen werden der geometrischen Erhaltungssatz und ein lokales Maximumprinzip bewiesen. Des Weiteren werden Bedingungen für Limiter angegeben. Diese Bedingungen sichern die Stabilität der Methode im Sinne der totalen Variation. Zusätzlich wird die Entropie-Stabilität der Methode diskutiert. Für die zugehörige semi-diskretisierte ALE-DG Methode werden Fehlerabschätzungen gezeigt. Wenn auf der Referenzzelle ein Testfunktionen Raum, der stückweise Polynome vom Grad $k$ enthält verwendet wird, kann für einen monotonen Fluss die suboptimale Konvergenzordnung $(k+\frac{1}{2})$ und für einen upwind Fluss die optimale Konvergenzordnung $(k+1)$ in der $L^2$-Norm gezeigt werden. Die Leistungsfähigkeit der Methode wird anhand numerischer Beispiele für nichtlineare Erhaltungsgleichungen untersucht.
Ebenso werden für die semi-diskretisierte ALE-DG Methode für nichtlineare Hamilton-Jacobi Gleichungen Fehlerabschätzungen gezeigt. Wenn auf der Referenzzelle ein Testfunktionen Raum, der stückweise Polynome vom Grad $k$ enthält verwendet wird, kann im eindimensionalen Fall die optimale Konvergenzordnung $(k + 1)$ und im zweidimensionalen Fall die suboptimale Konvergenzordnung $(k + \frac{1}{2})$ in der $L^2$-Norm gezeigt werden. Für die vollständig diskretisierte ALE-DG Methode werden der geometrischen Erhaltungssatz bewiesen und für die stückweise konstante explizite Euler Diskretisierung wird die Konvergenz gegen die eindeutige physikalisch relevante Lösung diskutiert.

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# Contents

# Introduction

千里之行，始於足下。[2]

*(老子, 道德 )*

In the last decades the Runge-Kutta discontinuous Galerkin (RK-DG) finite element method, developed by Cockburn, Shu et. al. in a series of papers (cf. [10, 11, 12, 13, 14]), has been successfully applied to an amount of convection dominated problems like the shallow water equations for modeling waves in the atmosphere, rivers, lakes and oceans, the Euler equations of gas dynamics, the magnetohydrodynamics (MHD) equations, the Maxwell's equations in electromagnetism and many more. The RK-DG method can deal meshes with hanging notes (cf. Cockburn and Shu [16]), since the method uses piecewise polynomial test functions and thus there is no inter-element continuity required like in classical finite element methods. Therefore, the RK-DG methods are well suited for adaptive mesh refinement techniques.

For high order methods mesh refinement techniques are worthwhile, since an elaborate mesh refinement method can improve the solution behavior or the computational costs of a numerical scheme. For instance, it is well known that in the solution of a nonlinear conservation law, singularities like shock waves could appear (cf. Lax [53, Theorem 3.2, p. 12]). This kind of singularities are the cause of numerical artifacts like spurious oscillations in a high order method, since the variation of the solution changes immediately. Without taming these artifacts a numerical method will become unstable. In these situations, adaptive refinement methods could help to stabilize the numerical scheme by changing the size of the grid cells ($h$-refinement), reducing the polynomial order ($p$-refinement) or relocate a specific number of grid points close to the singularity ($r$-refinement). According to T. Tang [82], the $r$-refinement methods are also known as moving mesh methods, since in these methods the distribution of the grid points changes in time.

In practice, it is common to combine the $h$ and $p$-refinement methods to the $hp$-refinement methods. There are certain $hp$-refinement RK-DG methods for convection dominated problems in the literature. A small summary is given in Cockburn, Karniadakis and Shu's review article [9]. Furthermore, in the book [47] of Karniadakis and Sherwin theoretical as well as computational aspects of $hp$-refinement finite element methods are compiled. However, even if there are several interesting mathematical as well as computational aspects in $hp$-refinement RK-DG methods, these methods will omit in this thesis. This thesis is rather focused on a moving mesh method for scalar conservation laws and the Hamilton-Jacobi equations.

In general moving mesh methodologies are necessary to develop these methods. A common

[2] A journey of a thousand li starts with a single step. (Lǎozǐ, Tao Te Ching, Ch. 64, line 12, c. 6th-5th century BCE).

moving mesh methodology is the moving mesh partial differential equation (MMPDE) approach. This approach based on a posteriori error estimates or error indicators and has been developed and analyzed by Huang, Russell, T. Tang et. al. see e.g. [42], [43] and [82]. However, in this thesis, the development of a moving mesh methodology is skipped. Therefore, in order to describe the moving mesh method, the following universal assumption will be applied:

"*The distribution of the grid points is explicitly given for an upcoming time level.*"

This assumption allows to connect the cells of the partitions for the current and next time level by local affine linear mappings. These mappings yield time depending test functions for the DG discretization.

In the finite volume context a technique using local affine mappings was used by Fazio and LeVeque in [28]. In the method of Fazio and LeVeque, the distribution of the grid points is problem-oriented, for instance in a shock tube problem the distribution depends on the motion of a contact discontinuity. Later, Stockie, Mackenzie and Russell [78] combined the method of Fazio and LeVeque with a MMPDE approach to calculate the distribution of the grid points.

The moving mesh method, which is presented in this thesis, has an interesting feature. The grid is static, if the linear mappings are constant. In this case the motion of a fluid is described by the Eulerian description of motion. On the other hand, it is described by the Lagrangian description, if the linear mappings describe approximately the motion of the particles in a fluid. This observation justifies the classification in the class of arbitrary Lagrangian-Eulerian (ALE) methods. For a detailed description of ALE methods and comparisons with Lagrangian and Eulerian methods be referred to the article [25] by Donea et. al.

In the following, a few advantages of ALE methods will be briefly listed. Since in an Eulerian method the mesh is static, the method dissociates the mesh notes from the particles in a fluid. This results in a relative motion between the deforming fluid and the mesh, thereby convective effects could appear. In a Lagrangian method, convective effects cannot appear, since the mesh is moving with the fluid. Certainly, these methods are susceptible for geometric distortions in the computation mesh, since the deformation of the fluid will be transmitted to the mesh. These geometric distortions are often the source of instabilities in a numerical scheme. In general an ALE method is developed that the mesh is not exactly moving with the fluid like in a Lagrangian method, but it is more flexible to the deformation of the fluid than in an Eulerian method. Therefore, geometric distortions are avoided and convective effects could appear merely weak. Hence, ALE methods could preserve properties of the physical model, which got lost in an Eulerian method. For instance in [77] Springel developed an ALE finite volume method for the Euler equations, which maintains the Galilean-invariants of the physical system. Furthermore, these kind of methods are popular for the aeroelastic analysis of wings, since the Eulerian methods have difficulties following deforming material interfaces and mobile boundaries (cf. Donea et. al. [25]).

There are certain ALE-DG methods for equations with compressible viscous flows in the literature (cf. e.g. the publications of Lomtev et. al. [60], Persson et. al. [67] and Nguyen [63]). These papers are focused on the implementation and performance of the methods in aeroelastic applications. A more theoretical analysis of ALE methods has been done by Farhat et.

al. in the context of the geometric conservation law (GCL) (cf. [36, 27, 56]). The terminology geometric conservation law was introduced by Lombard and Thomas in [59]. The GCL governs the geometric parameters of a grid deformation method, such that the method provides for constant initial data the correct solution. This means, in the context of conservation laws, that constant states stay constant. The description of a GCL for a numerical scheme depends on the partial differential equation (PDE) as well as the numerical scheme itself. However, the GCL has a critical influence on the stability and accuracy of a numerical scheme. In [27] Grandmont, Guillard and Farhat proved for monotone ALE methods that the GCL is a necessary and sufficient condition to obtain a local maximum principle for the method. Furthermore, in [36] Guillard and Farhat proved for ALE-finite volume methods that the GCL is a necessary condition to ensure that the time discretization of the method is high order accurate. In addition, Lesoinne and Farhat showed in [56] that in particular for ALE methods in combination with a method of lines approach a geometric conservation law is not trivially satisfied. Therefore, in this thesis, it is one of the issues to develop ALE-DG methods, which are satisfying the GCL. It will be proven that the ALE-DG methods for scalar conservation laws and the Hamilton-Jacobi equations, which are presented in this thesis, satisfy the corresponding GCLs.

It has been mentioned that in the solution of conservation laws singularities could appear. These singularities are the source of numerical artifacts in the RK-DG method for static grids, which destabilize the scheme. A possible way to stabilize the RK-DG method are slope limiters. In [11, 13] Cockburn, Hou and Shu constructed slope limiters, such that the method stays high order accurate and the local mean values of the RK-DG solution become total variation stable. Furthermore, for some systems of conservation laws, like the Euler equations of gas dynamics, it is necessary that specific quantities stay positive. Therefore, the numerical method has to preserve bounds. In general it is not easy to prove that a high order method preserves bounds, even for non moving mesh methods. In [94, 95] X. Zhang and Shu developed a limiter for static grids, which ensures that the revised solution of a high order method preserves bounds. However, it is not clear, if these pleasant post processing techniques also work for an ALE-DG method. Hence, in this thesis it will be analyzed, how far the techniques of Cockburn, Hou and Shu [11, 13] as well as X. Zhang and Shu [94, 95] can be applied for the ALE-DG method, which is presented in this thesis. By following Cockburn and Shu's approach, conditions for slope limiters, which stabilize the ALE-DG method, will be stated. In numerical test examples the conditions will be validated. Moreover, for scalar conservation laws, it will be proven that X. Zhang and Shu's limiter provides a local maximum principle for the ALE-DG method.

A further point, which is important, for the capability of a numerical scheme for conservation laws is the entropy stability. This stability property ensures that the scheme converges to the physical relevant weak solution. It should be noted that even for the RK-DG method for static grids the issue of entropy stability is not completely solved. In [45] Jiang and Shu proved a cell entropy inequality with respect to the square entropy for the semi-discrete DG method and implicit RK-DG methods. Likewise, a cell entropy inequality for the backward Euler step of the ALE-DG method, which is presented in this thesis, will be proven. The proof for the ALE-DG method is slightly different from Jiang and Shu's proof, since the test functions as well as cells

are time-dependent in the ALE-DG method and a transport equation is necessary to evaluate the time derivatives of the volume integrals.
The cell entropy inequality of the ALE-DG method provides the $L^2$-stability of the method. This stability is for static grids the key to a priori error estimates for smooth solutions of conservation laws. These error estimates show the high order accuracy of the DG method for scalar conservation laws, since it is well known that up to a certain time level the solutions of nonlinear scalar conservation laws are smooth, if the initial data of the problem is a smooth function (cf. Kröner [49, Lemma 2.1.2, p. 17]). Certainly, at singularities the method degenerates to at most first order. In particular, Cockburn and Gremaud proved in [8] for the shock-capturing streamline diffusion DG finite element method for scalar conservation laws that the error behaves as $\mathcal{O}\left(h^{\frac{1}{4}}\right)$ for nonlinear problems, thereby is $h$ the maximal cell length and the piecewise polynomial test functions could be of arbitrary degree.
For DG methods for static grids, there are an amount of a priori error estimates for smooth solutions of conservation laws in the literature. In the following certain results will be listed: The first a priori error estimate for a DG method has been proven by LeSaint and Raviart [55]. Johnson and Pitkřanta proved in [46] that for linear conservation laws the discontinuous Galerkin a priori error behaves as $\mathcal{O}\left(h^{k+1}\right)$. Later, Peterson proved in [66] that the result of Johnson and Pitkäranta is the optimal a priori error for any DG method for conservation laws. For nonlinear scalar conservation laws and symmetrizable systems Q. Zhang and Shu proved a priori error estimates in [90], [91] and [92]. They proved for DG methods with a second and third order strong stability preserving Runge-Kutta (SSP-RK) time discretization method that the a priori error behaves as $\mathcal{O}\left(h^{k+\frac{1}{2}} + (\triangle t)^{\sigma}\right)$, $\sigma = 2, 3$, in the general case and $\mathcal{O}\left(h^{k+1} + (\triangle t)^{\sigma}\right)$, $\sigma = 2, 3$, by applying an upwind numerical flux.
Error estimates for ALE-DG methods are less common in the literatur. For the semi-discrete ALE-DG method, which is presented in this thesis, a priori error estimates for smooth solutions of scalar conservation laws will be proven. More precisely, the sub-optimal $(k+\frac{1}{2})$ convergence for the semi-discrete ALE-DG method with a monotone numerical flux and the optimal $(k+1)$ convergence for the method with an upwind numerical flux, if a piecewise $\mathcal{P}^k$ polynomial approximation space is used, will be proven.
Furthermore, in this thesis an ALE-DG method for the Hamilton-Jacobi equations will be presented. Hamilton-Jacobi equations have an important role in the optimal control theory, thereby these equations occur as Hamilton-Jacobi-Bellman equations. In addition, the time-independent Eikonal equations, which are applied in the theory of physical wave and ray optics, are related to Hamilton-Jacobi equations. Hamilton-Jacobi equations are closely related to conservation laws, since for smooth solutions these equations can be written as conservation laws. The solutions of the Hamilton-Jacobi equations are continuous functions and have in general discontinuous derivatives (cf. Crandall and Lions [17, Theorem V.I.2]. Hence, similar to conservation laws, a class of generalized solution is necessary to analyze the Hamilton-Jacobi equations. These solutions are the viscosity solutions, which were introduced by Crandall and Lions [17, Definition I.1]. They called the solutions viscosity solutions, since the vanishing

viscosity method was used in the existence proofs for equations of Hamilton-Jacobi type in several papers (cf. e.g. Fleming [30] and for the Eikonal equations Kružkov [51]).
Numerical schemes for solving the Hamilton-Jacobi equations are widespread in the literature. In the following, a few schemes are listed. In [18] Crandall and Lions developed a first order monotone finite difference method for solving the Hamilton-Jacobi equations. For this method, they proved an error estimate and the convergence to the unique viscosity solution. Later, for static grids, Jiang, Peng, Osher, Sethian and Shu [44, 65, 64] developed high order essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) schemes for solving the Hamilton-Jacobi equations. These methods are adaptations of ENO and WENO schemes for conservation laws. In [70] Sethian developed several moving mesh methods for solving the Hamilton-Jacobi equations considering propagating interfaces. These methods are combinations of ENO and WENO schemes with Eulerian and Lagrangian methods. A further moving mesh finite difference method has been developed by H.-Z. Tang, T. Tang and P. Zhang [81], thereby a MMPDE approach was used to describe the distribution of the mesh points. The Hamilton-Jacobi equations are pure nonlinear. Therefore the development of finite element methods for the Hamilton-Jacobi equations is not straightforward, since the integration by parts formula cannot be applied directly. Hence, finite element methods for the Hamilton-Jacobi equations are often adapted from finite element methods for conservation laws, since for smooth solutions the Hamilton-Jacobi equtions can be written as systems of conservation laws. One of these adaptations is Hu, Lepsky and Shu's method, which was developed in [40, 41, 54]. This method was extended by Mackenzie and Nicola in [61] to a moving mesh DG method for the Hamilton-Jacobi equtions. The distribution of the mesh points in this method is described by a MMPDE approach similar to H.-Z. Tang, T. Tang and P. Zhang's method. It should be noted that in general the systems of conservation laws, which arise from the Hamilton-Jacobi equations, are not strictly or strongly hyperbolic systems. Thus, a method for directly solving the Hamilton-Jacobi equations is desirable. In [2] Barth and Sethian consider the Hamilton-Jacobi equations with Hamiltonians, which are a homogeneous functions of degree $p > 0$. Then, by Euler's homogeneous function theorem the Hamilton-Jacobi equations can be written as quasilinear PDEs. For these type of Hamilton-Jacobi equations, Barth and Sethian developed a Petrov-Galerkin space-time method for triangular meshes. Furthermore, they combined this method with an $h$-refinement method. However, it is a little restrictive to require that the Hamiltonian need to be a homogeneous function. Nevertheless, for static grids, there are also DG methods for directly solving the Hamilton-Jacobi equations with less restrictive conditions for the Hamiltonians. Examples are the DG methods by Cheng and Shu [4] or the method by Yan and Osher [86]. In [84] Xiong, Shu and M. Zhang proved for the semi-discrete versions of these methods a priori error estimates for smooth solutions of the Hamilton-Jacobi equations. More precisely, for a piecewise $\mathcal{P}^k$ polynomial approximation space, they proved the optimal $(k+1)$ convergence for both methods in one dimension and for cartesian grids in two dimensions.
In the present thesis Yan and Osher's method [86] is of particular importance. In this method, the Hamilton-Jacobi equations are discretized such that locally in any cell a system of three

unknowns appears. One of the unknowns is the approximate DG solution for the Hamilton-Jacobi equations. The other two unknowns are quantities to evaluate the numerical flux. These unknowns are determined by solving upwind RK-DG schemes. A similar DG approach has been introduce by Cockburn and Shu in [15] for solving time-dependent convection diffusion systems. Cockburn and Shu called these extensions of the standard RK-DG method for conservation laws local discontinuous Galerkin (LDG) methods. Therefore, Yan and Osher called their method LDG method for directly solving the Hamilton-Jacobi equations. In addition, Yan and Osher proved that the piecewise constant forward Euler step of their method degenerates to Crandall and Lions [18] monotone finite difference method. This relationship suggests that the method provides the unique viscosity solution.
In this thesis, an ALE-LDG method for directly solving the Hamilton-Jacobi equations will be developed based on Yan and Osher's LDG method. Furthermore, it will be proven that the method satisfies the GCL and the piecewise constant forward Euler step of the method is a monotone finite difference method. In addition, for the semi-discrete ALE-LDG method a priori error estimates for smooth solutions of the Hamilton-Jacobi equations will be proven. More precisely, for a piecewise $\mathcal{P}^k$ polynomial approximation space, the optimal $(k+1)$ convergence for the one dimensional method and the supobtimal $(k+\frac{1}{2})$ for two dimensional method on simplicial meshes will be proven.

The organization of the thesis is as follows:
In the first chapter, the mathematical notion of some fundamental concepts of scalar conservation laws and the Hamilton-Jacobi equations, like the class of generalized solutions, are presented. Then, the strong stability preserving Runge-Kutta (SSP-RK) methods, the RK-DG method for one dimensional scalar conservation laws for static grids and DG methods for the Hamilton-Jacobi equations are discussed.
In the second chapter, the universal assumption for the distribution of the grid points is used to construct a tessellation of time-dependent simplex cells. A time-dependent test function space for the spatial DG discretization is constructed and several transport equations are proven for the test functions. Furthermore, it is justified that the classical interpolation, inverse and trace inequalities are compatible with the time-dependent simplex cells.
In the third chapter, the ALE-DG method for scalar conservation laws is developed. First of all, for the fully-discrete ALE-DG method the GCL and a local maximum principle are proven. Furthermore, conditions for the slope limiter are derived and the entropy stability is discussed. Then, a priori error estimates for the method with a monotone numerical flux and an upwind numerical flux are proven. Afterward, numerical results are presented to demonstrate the accuracy and capabilities of the method in one and two dimensions.
In the fourth chapter, the ALE-LDG method for the Hamilton-Jacobi equations is developed. First of all, for the fully-discrete ALE-LDG method the GCL is proven. Then, the piecewise constant forward Euler step of the method is analyzed. Afterward, a priori error estimates for the method in one and two dimensions are proven.
Finally, in the last chapter, some concluding remarks, ongoing projects and possible future

works are discussed.

# Notation

In this thesis, the common notation in the theory of conservation laws and associated numerical methods will be applied. In particular, for a real positive number $T$ we denote the interval $[0, T]$ time interval and a point $t \in [0, T]$ time point. The number $d \in \mathbb{N}$ will always represent the space dimension. We will merely consider partial differential equations (PDEs) posed over a simple domain $\Omega \subseteq \mathbb{R}^d$. More precisely, $\Omega$ is, throughout this thesis, a convex polyhedron and in the one dimensional case an open interval. The closure, interior, boundary and the convex hull of a set $M \subseteq \mathbb{R}^d$ are denoted by $\overline{M}$, int$M$, $\partial M := \overline{M} \setminus \text{int}M$ and conv$M$. A matrix will always be denoted by bold letters like $\mathbf{A} \in \mathbb{R}^{d \times d}$. In addition, for any subset $M \subseteq \Omega$ with a Lipschitz boundary, we denote the $\mathrm{L}^2(M)$-inner product for all $v, w \in \mathrm{L}^2(M)$ by $(v, w)_M := \int_M vw\, dx$. Likewise, for a subset $e \subseteq \partial M$ with nonzero $(d-1)$-Lebesgue measure $|e|_{d-1}$, we denote the $\mathrm{L}^2(e)$-inner product for all $v, w \in \mathrm{L}^2(e)$ by $\langle v, w \rangle_e := \int_e vw\, d\Gamma$. Furthermore, if there is no confusion, we apply the notation $u(t)$ instead of $u(x, t)$ for a function $u : \Omega \times [0, T] \to \mathbb{R}$.

### Constants

To avoid confusion with different constants, we denote, throughout the thesis, a positive constant, which is independent of the meshsize and the approximate solution of a PDE, by $C$. Note, that the constant may depend on the solution of a PDE and may have a different value in each occurrence.

# 1. Fundamentals

In this chapter, we present a brief introduction about the theory of scalar conservations laws and the Hamilton–Jacobi equations. Furthermore, the RK-DG method for one dimensional scalar conservation laws and two DG methods for the Hamilton–Jacobi equations will be discussed.

## 1.1. Time-dependent first order partial differential equations (PDEs)

In the upcoming section, we introduce some theoretical aspects of the PDEs, which are considered in the thesis.

### 1.1.1. Scalar conservation laws

One dimensional scalar conservation laws are time-dependent PDEs, which appear as the following initial value problem: For an open interval $\Omega \subseteq \mathbb{R}$, a suitable initial function $u_0 : \Omega \to \mathbb{R}$ and a continuous flux function $f : \mathbb{R} \to \mathbb{R}$, find a function $u : \Omega \times (0, T) \to \mathbb{R}$, $u = u(x, t)$, which satisfies

$$\partial_t u + \partial_x f(u) = 0, \quad \text{in } \Omega \times (0, T), \tag{1.1.1a}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.1.1b}$$

In general, the problem (1.1.1) appears with suitable boundary continuous, but for convenience we consider the problem with periodic boundary conditions in this thesis.
It is well known that even for smooth initial data the characteristic curves, associated with the equation (1.1.1a), could intersect at some time point (cf. Godlewski and Raviart [32, p. 25-26]. Hence, in general the solutions of these equations are not classical solutions. A generalized class of solutions is necessary to analyze these problems. For conservation laws the class of generalized solutions is given by the weak solutions (cf. e.g. Dafermos [19, Chapter IV, Section 4.3]).

**Definition 1.1.1** *Let $u_0 \in \mathrm{L}^\infty(\Omega)$. Then a function $u \in \mathrm{L}^\infty(\Omega \times (0, T))$, which satisfies*

$$\int_0^T \int_\Omega (u \partial_t \varphi + f(u) \partial_x \varphi) \, dx dt + \int_\Omega u_0(x) \varphi(x, 0) \, dx = 0$$

*for all* $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$, *is called a weak solution or a solution in the distributional sense of* (1.1.1).

However, for a finite time interval it is still possible to find a classical solution of the problem (1.1.1). Furthermore, in particular, the classical solution and the weak solution agree for this finite time interval. This result can be stated as the following proposition. A proof can be found in the book of Kröner [49, Lemma 2.1.2, p. 17].

**Proposition 1.1.1** *Let* $f \in \mathcal{C}^2(\mathbb{R})$ *and* $u_0 \in \mathcal{C}^1(\mathbb{R})$. *Furthermore, suppose that the function* $f'' : \mathbb{R} \to \mathbb{R}$ *as well as* $u_0 : \mathbb{R} \to \mathbb{R}$ *are bounded on* $\mathbb{R}$. *Then, there exists a* $T_0 > 0$, *such that the equation* (1.1.1) *has a classical solution* $u \in \mathcal{C}^1(\mathbb{R} \times [0, T_0))$.

The space $\mathrm{L}^\infty(\Omega \times (0,T))$ contains piecewise smooth functions. For these functions a weak solution can be characterized by the famous Rankine-Hugoniot condition.

**Proposition 1.1.2** *Let* $\Sigma : (0,T) \to \Omega \times (0,T)$, $t \mapsto (\sigma(t), t)$, *be a smooth curve, which separates the domain* $\Omega \times (0,T)$ *into two parts* $M_l$ *and* $M_r$. *Furthermore, let* $u \in \mathrm{L}^\infty(\Omega \times (0,T))$ *with* $u_l := u|_{M_l} \in \mathcal{C}^1\left(\overline{M_l}\right)$ *as well as* $u_l := u|_{M_l} \in \mathcal{C}^1\left(\overline{M_r}\right)$, *such that* $u_l$ *satisfies* (1.1.1a) *locally in* $M_l$ *and* $u_r$ *satisfies* (1.1.1a) *locally in* $M_r$ *in the classical sense. Then* $u$ *is a weak solution of* (1.1.1), *if and only if the Rankine-Hugoniot condition*

$$\Big(u_l(\sigma(t), t) - u_r(\sigma(t), t)\Big)\sigma'(t) = f(u_l(\sigma(t), t)) - f(u_r(\sigma(t), t))$$

*is satisfied.*

The following example provides that weak solutions are in general not unique.

**Example 1.1.1** *This example can also be found in the book of LeVeque [57, Chapter 3, Section 3.5]. We consider the Burgers' equation*

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0, \quad \textit{in } (-2,2) \times (0,T) \tag{1.1.2}$$

*with the initial data*

$$u_0(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{1.1.3}$$

*We would like to mention that the problem* (1.1.1) *with initial data of the type* (1.1.3) *is called Riemann problem. According to proposition 1.1.2, is*

$$u_1(x,t) = \begin{cases} 1, & x > \frac{t}{2}, \\ 0, & x \leq \frac{t}{2} \end{cases}$$

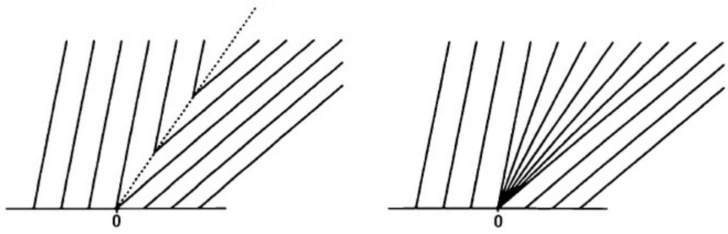

**Figure 1.1.1.** The characteristic curves for the solutions $u_1$ (left) and $u_2$ (right). (*Source: LeVeque [57, Figure 3.9]*)

*a weak solution of the Riemann problem above. Otherwise, the function*

$$u_2(x,t) = \begin{cases} 1, & x > t, \\ \frac{x}{t}, & 0 \leq x \leq t, \\ 0, & x < 0 \end{cases}$$

*is a weak solution of the problem above, too. The solution $u_1$ is a shock wave with the propagation speed $\sigma(t) = \frac{t}{2}$ and the solution $u_2$ is a rarefaction wave. It should be noted that for the solution $u_1$ the characteristic curves, associated with the equation* (1.1.2)*, go out of the shock (cf. figure 1.1.1). This behavior of the solution is physically not meaningful (cf. Godlewski and Raviart [32, Chapter 2, section 6]). Furthermore, the solution is not stable to perturbations (cf. LeVeque [57, Chapter 3, Section 3.5]).*

The example yields weak solutions are not unique and they can be physically meaningless. Therefore, an admissible condition is necessary to select the physical meaningful solution among the whole set of weak solutions. Admissible conditions for conservation laws are called entropy conditions and the physical relevant solution is the entropy solution of a conservation law. Common entropy conditions for scalar conservation laws, are the Lax, Oleĭnik or the Kružkov entropy condition. In the one dimensional case, these entropy conditions are equivalent, if the flux function in (1.1.1a) is convex (cf. Kröner [49, Theorem 2.1.22, p. 35]). In this thesis, the entropy solution of a scalar conservation law will be defined in the sense of Kružkov's entropy condition.

**Definition 1.1.2** *Let $u \in \mathrm{L}^\infty(\Omega \times (0,T))$ be a weak solution of* (1.1.1) *and $\eta_k(u) := |u-k|$ for all $k \in \mathbb{R}$. Then $u$ is called entropy solution, if it satisfies*

$$\int_0^T \int_\Omega \left(\eta_k(u)\,\partial_t\varphi + \eta_k'(u)\left[\left(f(u) - f(k)\right)\partial_x\varphi\right]\right)\,dxdt \geq 0 \tag{1.1.4}$$

*for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$ with $\varphi \geq 0$ and $k \in \mathbb{R}$. Note that the functions $\eta_k(\cdot)$, $k \in \mathbb{R}$ are known as Kružkov entropies.*

The existence and uniqueness of an unique entropy solution (1.1.4) for the problem (1.1.1) has been proven by Kružkov in [50, Theorem 2 and Theorem 5]. Kružkov's results can be summarized as the upcoming theorem.

**Theorem 1.1.2** *Let $u_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. Then there exists an unique entropy solution $u \in \mathrm{L}^\infty(\Omega \times (0,T))$ of* (1.1.1)*. Furthermore, the solution satisfies the estimates*

$$\|u(\cdot, t)\|_{\mathrm{L}^\infty(\Omega)} \leq \|u_0\|_{\mathrm{L}^\infty(\Omega)} \quad \textit{a.e. in } [0,T) \tag{1.1.5}$$

*and*

$$|u(\cdot, t)|_{\mathrm{BV}(\Omega)} \leq |u_0|_{\mathrm{BV}(\Omega)} \quad \textit{for all } t \in [0,T). \tag{1.1.6}$$

In [23] DiPerna has proven that the entropy condition (1.1.4) can be relaxed, if the flux function in (1.1.1) is convex. This result can be summarized as the proposition.

**Proposition 1.1.3** *Let $\eta, f, F \in \mathcal{C}(\mathbb{R})$, $\eta$, $f$ be convex functions and $F$ satisfies $F' = \eta' f'$. Then a weak solution $u \in \mathrm{L}^\infty(\Omega \times (0,T))$ of* (1.1.1) *is the unique entropy solution, if*

$$\int_0^T \int_\Omega \left(\eta(u)\, \partial_t \varphi + F(u)\, \partial_x \varphi\right) dxdt \geq 0 \tag{1.1.7}$$

*for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$ with $\varphi \geq 0$.*

The function $F$ in equation (1.1.7) is called entropy flux function and the tuple $(\eta, F)$ is called pair of entropy functions for (1.1.1).
We close this section with some comments about existence and uniqueness of solutions for systems of conservation laws. For scalar conservation laws theorem 1.1.2 ensures the existence and uniqueness of an entropy solution, for systems of conservation laws are the situation much more complicated. The existence and uniqueness theory for systems are still an open problem, although Glimm [31] proved the existence of a weak solution for one dimensional genuinely nonlinear systems and Liu [58] generalized Glimm's result to nonlinear strictly hyperbolic systems, which can be linearly degenerate. In particular, the results of Glimm and Liu ensure the existence of a weak solution for the one dimensional Euler equations. For two dimensional systems of conservation laws is the situation more worse. An introduction and a few results about systems of two dimensional conservation laws can be found in the book of Dafermos [19, Chapter XVII]. Nevertheless, we have to mention that recently C. de Lellis and L. Székelyhidi Jr. [21] as well as [22] proved that for the incompressible Euler equations with bounded compactly supported initial data admissibility criteria like global and local energy inequalities do not provide a unique weak solution. Therefore, the approved methods for scalar conservation laws to ensure the well-posedness are failing for systems of conservation laws in several space dimensions.

### 1.1.2. Hamilton-Jacobi equations

Hamilton-Jacobi equations appear in control theory as Hamilton-Jacobi-Bellman equations (cf. Evans [26, Chapter 10, section 3]). In this thesis, we will consider the following initial value problem, which based on a simple type of the Hamilton-Jacobi equations: For an open interval $\Omega \subseteq \mathbb{R}$, a suitable initial function $u_0 : \Omega \to \mathbb{R}$ and a continuous Hamiltonian $H : \mathbb{R} \to \mathbb{R}$, find a function $u : \Omega \times (0,T) \to \mathbb{R}$, $u = u(x,t)$, which satisfies

$$\partial_t u + H(\partial_x u) = 0, \quad \text{in } \Omega \times (0,T), \tag{1.1.8a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.1.8b}$$

Henceforward, we write $H = H(p)$ for the Hamiltonian in (1.1.8a), where $p$ is a variable for which we substitute the partial derivative $\partial_x u$ in $H$. The partial derivatives of the Hamiltonian will be denoted by $\partial_p H(p)$.
In addition, we also consider, the two dimensional counterpart of the problem (1.1.8). This is the following initial value problem: For a domain $\Omega \subseteq \mathbb{R}^2$, an initial function $u_0 : \Omega \to \mathbb{R}$ and a continuous flux function $H : \mathbb{R}^2 \to \mathbb{R}$ find a function $u : \Omega \times (0,T) \to \mathbb{R}$, which satisfies

$$\partial_t u + H(\nabla u) = 0, \quad \text{in } \Omega \times (0,T), \tag{1.1.9a}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y)^T \in \Omega. \tag{1.1.9b}$$

As in the one dimensional case, we substitute the partial derivatives $\partial_x u$ and $\partial_y u$ by $(p,q)^T \in \mathbb{R}^2$ and utilize the notation $H = H(p,q)$, $\partial_p H(p,q,x,y)$ as well as $\partial_q H(p,q,x,y)$ for the Hamiltonian and its partial derivatives. In general, the problems (1.1.8) and (1.1.9) appear with suitable boundary continuous, but for convenience we consider the problems with periodic boundary conditions in this thesis. Henceforth, all the definitions and theorems will be presented merely for the problem (1.1.9). It can be verified that these statements also hold for the problem (1.1.8).
Similar to scalar conservation laws, the initial value problems (1.1.8) and (1.1.9) do not have classical solutions. This can be noticed as follows:
We assume that $u$ is a classical solution of (1.1.8). Thus, $u \in \mathcal{C}^1(\Omega \times [0,T])$. We define $v := \partial_x u$. Then by Schwarz integrability condition and (1.1.8a) follows

$$\partial_t v + \partial_x H(v) = 0 \quad \text{in } \Omega \times (0,T). \tag{1.1.10}$$

Certainly, according to the discussion in the previous section, in general the solutions of the scalar conservation law (1.1.10) are not continuous. This result contradicts the assumption that $u$ is a classical solution of (1.1.8).
Likewise, if we assume that $u \in \mathcal{C}^1(\Omega \times [0,T])$ is a classical solution of (1.1.9) and define $v_1 := \partial_x u$ and $v_2 := \partial_y u$, it follows by Schwarz integrability condition and (1.1.9a)

$$\begin{aligned} \partial_t v_1 + \partial_x H(v_1, v_2) = 0, &\quad \text{in } \Omega \times (0,T), \\ \partial_t v_2 + \partial_y H(v_1, v_2) = 0, &\quad \text{in } \Omega \times (0,T). \end{aligned} \tag{1.1.11}$$

These equations are a system of conservation laws and therefore, in general, the functions $v_1$ and $v_2$ are not continuous. Hence, the function $u$ cannot be a classical solution of (1.1.9). Therefore, like for scalar conservation laws, a class of generalized solutions is necessary to analyze the initial value problems (1.1.8) and (1.1.9). In the literature, the right class of generalized solutions was discussed for a long time, since the equations (1.1.8a) and (1.1.9a) are not quasi-linear and thus it is not possible to define weak solutions in the sense of the definition (1.1.1). In the beginning, a generalized solution was defined as a $\mathrm{W}^{1,\infty}(\Omega \times (0,T))$ function, which satisfies (1.1.8) or (1.1.9) almost everywhere. Unfortunately, this definition is not practicable for an existence proof by the vanishing viscosity method. Moreover, the definition does not ensure uniqueness. This can be ascertained by the following example.

**Example 1.1.3** *This example was presented by Crandall and Lions in [17]. We consider the initial value problem*

$$\partial_t u + (\partial_x u)^2 = 0, \quad \text{in } (0,1) \times (0,T), \tag{1.1.12a}$$
$$u(x,0) = 0, \quad x \in (0,1). \tag{1.1.12b}$$

*Then is* $u_1(x,t) := 0$ *a solution of* (1.1.12)*, but on the other hand the function*

$$u_2(x,t) := \begin{cases} |x| - t, & 0 \le t \le |x|, \\ 0, & |x| < t \end{cases}$$

*is in* $\mathrm{W}^{1,\infty}(\Omega \times (0,T))$ *and satisfies* (1.1.12) *in the classical sense except on the lines*

$$\{(x,t) : \ t = \pm x \text{ or } x = 0\}.$$

Hence, a more appropriate definition for generalized solutions was introduced by Crandall and Lions in [17, Definition I.1]. They defined the class of generalized solution as follows.

**Definition 1.1.3** *A function* $u \in \mathcal{C}(\Omega \times (0,T))$ *is called viscosity solution of the initial value problem* (1.1.9)*, if* $u$ *satisfies* (1.1.9b) *and for all* $\varphi \in \mathcal{C}^\infty(\Omega \times (0,T))$ *hold:*

$$\begin{cases} \text{If } u - \varphi \text{ has a local maximum at a point } (x_0,y_0,t_0) \in \Omega \times (0,T), \text{ is} \\ \partial_t \varphi(x_0,y_0,t_0) + H(\partial_x \varphi(x_0,y_0,t_0)) \le 0 \end{cases} \tag{1.1.13}$$

*as well as*

$$\begin{cases} \text{if } u - \varphi \text{ has a local minimum at a point } (x_0,y_0,t_0) \in \Omega \times (0,T), \text{ is} \\ \partial_t \varphi(x_0,y_0,t_0) + H(\partial_x \varphi(x_0,y_0,t_0)) \ge 0. \end{cases} \tag{1.1.14}$$

Note that in the definition 1.1.3 a local maximum principle instead of the integration by parts formula has been used to shift the partial derivatives from $u$ to a test function. Therefore, this definition enables existence proofs for (1.1.8) and (1.1.9) by the vanishing viscosity method. However, a priori it is not clear that a viscosity solution is the physical relevant solution for the Hamilton-Jacobi equations. Crandall and Lions proved in [17, Proposition VI.1] that a viscosity solution is actually the physical relevant solution. This result can be summarized as follows.

**Proposition 1.1.4** *Let $u^\varepsilon$ be a solution of the viscose problem*

$$\partial_t u^\varepsilon + H^\varepsilon(\nabla u^\varepsilon) - \varepsilon \triangle u^\varepsilon = 0, \quad \text{in } \Omega \times (0,T) \tag{1.1.15}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \quad (x,y)^T \in \Omega \tag{1.1.16}$$

*with $\partial_t u^\varepsilon, \partial_x u^\varepsilon, \partial_y u^\varepsilon \in \mathcal{C}(\Omega \times (0,T))$. Suppose*

$$H^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} H \quad \text{in } \mathcal{C}(\Omega \times (0,T)), \qquad u_0^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u_0 \quad \text{in } \mathcal{C}\left(\overline{\Omega}\right)$$

*and there exists a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and a $u \in \mathcal{C}\left(\overline{\Omega \times (0,T)}\right)$ with*

$$u^{\varepsilon_n} \underset{\varepsilon_n \to 0}{\longrightarrow} u \quad \text{in } \mathcal{C}\left(\overline{\Omega \times (0,T)}\right).$$

*Then $u$ is a viscosity solution of the initial value problem* (1.1.9).

Furthermore, Crandall and Lions proved in [17, Theorem V.I.2] the existence and uniqueness of viscosity solutions for the problems (1.1.8) and (1.1.9). This result can be summarized as follows.

**Theorem 1.1.4** *Let $u_0 \in \mathcal{C}(\Omega)$ and $H \in \mathbb{R}^2(\Omega)$. Then there exists a unique function $u \in \mathcal{C}\left(\overline{\Omega \times (0,T)}\right)$ which satisfies* (1.1.13)*,* (1.1.14) *and*

$$\lim_{t\to 0} \|u(\cdot,t) - u_0(\cdot)\|_{\mathrm{L}^\infty(\Omega)} = 0.$$

*Moreover, it holds for all $x, y \in \Omega$ and $t \in [0,T]$*

$$|u(x,t) - u(y,t)| \leq \sup_{\xi\in\Omega} |u_0(\xi) - u_0(\xi + y - x)|.$$

Finally, we would like to mention that Hopf presented in [39] a closed formula for a solution of the problems (1.1.8) and (1.1.9). Hence, the question appears, if Hopf's solution is a viscosity solution. This question was responded by Barles in [1, Theorem I.2] with the following result.

**Proposition 1.1.5** *Let $H \in \mathcal{C}^2(\Omega)$ be a convex function and $u_0 \in \mathcal{C}(\Omega)$. Then the unique viscosity solution is given by Hopf's formula*

$$u(x,t) = \min_{y\in\Omega} \left\{ t \sup_{p\in\mathbb{R}^2} \left\{ p \cdot \left(\frac{x-y}{t}\right) - H(p) \right\} + u_0(y) \right\}, \quad (x,t) \in \overline{\Omega \times (0,T)}.$$

## 1.2. The Runge-Kutta discontinuous Galerkin (RK-DG) method

According to theorem 1.1.2 the problem (1.1.1) has an unique entropy solution, which satisfies the inequalities (1.1.5) and (1.1.6). Therefore, from a theoretical point of view it is desirable that a discretization of the problem (1.1.1) has comparable properties. In particular, we would like to have a numerical scheme for conservation laws with the properties:

i) It is in conservation form.

ii) It satisfies a discrete maximum principle.

iii) It is total variation stable.

iv) It is entropy stable.

v) It is high order accurate.

It should be noted that the conditions i) - iv) and compactness arguments provide for one dimensional first order finite difference schemes with Lipschitz continuous numerical flux functions the covergence to the unique entropy solution of the initial value problem (1.1.1) (cf. Kröner [49, Theorem 2.3.9, p. 60]). Furthermore, it should be noted that the meaning of the point v) could be confusing in the context of conservation laws, since the solution of these equations could be discontinuous. It is well known that at discontinuities like shock waves a numerical method is at most first order accurate. Therefore, the point v) has to be understood in the following sense: A numerical method for conservation laws is high order accurate, if it is a high resolution method in regions, where the solution of the conservation law is smooth.
If we understand the conditions ii), iii) and iv) in the sense of classical finite difference schemes, a scheme has to be total variation diminishing (TVD) and entropy stable in the sense of the Kružkov entropy functions. A class of finite difference schemes, which are in conservation form, TVD and entropy stable are the monotone schemes. Unfortunately, monotone schemes are merely first order methods (cf. Godlewski and Raviart [32, Theorem 3.1, p. 125]) and thus the condition v) is not satisfied. Furthermore, it is well known that TVD schemes are at most second order accurate at local extrema. This problematic is exemplified by an example in the book of Gottlieb, Ketcheson and Shu [35, Chapter 11, Section 11.2]. Hence, the concepts to analyze classical finite difference schemes are too restrictive to ensure that a scheme is high order accurate. An alternative to the TVD stability has been developed by Shu in [72]. This concept covers iii) and it is compatible with v), but in contrast to the TVD stability it does not cover ii). Therefore, in [94] X. Zhang and Shu introduced a high order accurate limiter, which provides a discrete maximum principle. In order to ensure the entropy stability, according to proposition 1.1.3 it is enough to prove the stability for one specific convex function instead of all Kružkov entropy functions.
A well known class of high order accurate finite difference schemes are the essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) schemes. Many of these schemes are constructed to satisfy the conditions i) - v). Among others Shu has summarized

the development, analysis and the computational behavior of ENO and WENO schemes in [75]. Besides many benefits, ENO and WENO schemes have the disadvantage that informations from several cells are necessary to construct a high degree polynomial as approximate solution. Furthermore, in general it is not easy to develop ENO and WENO schemes for unstructured grids. Thus, these schemes are not attractive for adaptive mesh refinement techniques. From this point of view, these schemes are unsuitable for parallel computing for instance on supercomputers with an amount of processors.
Towards a method, which is theoretical well suited in the sense of condition i) - iv) and more practical than the ENO and WENO schemes in the sense of changing mesh geometries, Cockburn and Shu developed and analyzed the Runge-Kutta discontinuous Galerkin (RK-DG) for the problem (1.1.1) in [10, 11]. Later, in [13] Cockburn, Hou and Shu extended the method to multidimensional scalar conservation laws and in [12, 14] Cockburn, Lin and Shu extended the method to hyperbolic systems of conservation laws.
The method based on a method of lines approach. For the spatial discretization a discontinuous Galerkin method with piecewise polynomial test functions is applied. Since this kind of test functions are not well defined in the cell interfaces, Cockburn and Shu applied monotone numerical flux functions to evaluate the surface integrals of the physical flux function on the cell interfaces. Therefore, the method is in conservation form and the piecewise polynomial version of the method degenerates to a monotone finite difference scheme. Furthermore, it should be noted that the numerical flux function depends merely on informations of adjacent cells and the choice of piecewise polynomial test functions makes the method attractive for changing mesh geometries. In particular, the method can deal meshes with hanging notes (cf. Cockburn and Shu [16]). Therefore, it is well-suited for adaptive mesh refinement techniques and parallel computing. For the time discretization the so called strong stability preserving Runge-Kutta (SSP-RK) methods are applied. These time integration methods have been developed by Shu and Osher in [73, 74] to carry the total variation stability of the forward Euler step to an explicit high oder RK method in the method of lines approach. In the section (1.2.1), the benefits of these methods will be presented.
Nevertheless, despite all its advantages, the RK-DG method has some drawbacks. In general Galerkin methods without any stabilization terms are not stable for convection dominated problems (cf. Kröner [49, Example 2.6.1, p. 117]). Therefore, Cockburn and Shu introduced a post process procedure by limiters to stabilize the method. In several space dimensions, the construction of limiters is nontrivial and is constrained to the mesh geometry (cf. Hou, Cockburn and Shu [13]). Therefore, the advantage of the method to handle changing mesh geometries is restricted by the post process procedure. Recently, Kuzmin [52] constructed limiters based on Taylor polynomials. These limiters are working for non simplicial or non cartesian meshes, too. However, Kuzmins limiting procedure is complex and thus numerical quite costly compared to the standard limiting procedure by Cockburn and Shu. As mentioned by Cockburn, Karniadakis and Shu in [9], it would be very useful to devise a RK-DG method that does not have a limiter and remains nonlinearly stable and high order accurate.
Another issue for RK-DG methods is the Courant-Friedrichs-Lewy (CFL) number. In the con-

text of linear scalar conservation laws, the CFL number for the RK-DG method with piecewise polynomials of degree $k$ should behave as $\frac{\triangle t}{\triangle_K} = \mathcal{O}\left((2k+1)^{-1}\right)$, for all cells $K$ with the diameter $\triangle_K$. For $k \in \{0,1\}$, this ratio has been proven by Cockburn and Shu in [10]. Furthermore, numerical experiments suggest that the number $(2k+1)^{-1}$ is less than 5% smaller than the numerically obtained estimates for the CFL number (cf. Cockburn and Shu [16, Table 2.2]). The ratio of the CFL condition has to be respected for nonlinear problems, too. It should be noted that the number could decrease even more when a post process procedure to stabilize the method is applied. Since the ratio of the CFL number is depending on the inverse value of the polynomial degree, the number has to be decreased, for high order RK-DG methods. A small CFL number aggravated the computation time costs of a numerical method. In order to relax the ratio of the CFL number, $p$-refinement techniques in combination with local time stepping techniques could be used. Local time stepping techniques for hyperbolic conservation laws have been developed e.g. by Flaherty et. al. in [29]. These techniques are necessary to prepare a numerical method for parallel computing.
However, issues concerning the CFL number are not a component of this thesis, instead we develop and analyze certain ALE-DG methods. For the ALE-DG methods, we want to prove that Cockburn, Shu and X. Zhang's post process procedures to stabilize the RK-DG method can be applied, too. These post process procedures were introduced and analyzed in [7, 11, 16, 94]. In this section, we present the main ingredients of Cockburn, Shu and X. Zhang's post process procedures based on the one dimensional weak RK-DG formulation for the problem (1.1.1). Furthermore, we discuss explicit SSP-RK time integration methods. Later, in chapter 3, we will analyze our one dimensional ALE-DG method with these tools.

### 1.2.1. Strong stability preserving Runge-Kutta (SSP-RK) methods

The right choice of the time integration method in a method of lines approach for conservation laws is an important ingredient to obtain a stable numerical method.The following numerical example by Gottlieb and Shu [33, Section 2] demonstrates this issues. In order to present the example, we partition the time interval $[0,T]$ equidistant. Henceforth, is the set $\{t_n\}_{n=0}^{L}$, $L \in \mathbb{N}$, given by

$$t_n := n\triangle t \quad \text{and} \quad \triangle t := \frac{T}{L} \tag{1.2.1}$$

an equidistant partition of the time interval $[0,T]$. In particular, we call $\triangle t$ the time step of the method.

**Example 1.2.1** *We consider the Burgers' equation* (1.1.2) *with the initial data*

$$u_0(x) = \begin{cases} -\frac{1}{2}, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

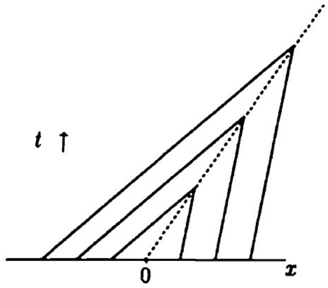


**Figure 1.2.1.** The characteristic curves for the solution $u$. (*Source: LeVeque [57, Figure 3.8]*)

*According to the previous sections*

$$u(x,t) = \begin{cases} -\frac{1}{2}, & x > \frac{t}{4}, \\ 1, & x \leq \frac{t}{4} \end{cases}$$

*is a weak solution of the Riemann problem. This solution is a shock wave with the propagation speed* $\sigma(t) = \frac{t}{4}$ *and it is the physically relevant solution of the Riemann problem, since the characteristic curves for the solution go into the shock (cf. figure 1.2.1).*
*Gottlieb and Shu applied a second order monotonic upstream centered scheme for conservation laws (MUSCL scheme) (cf. van Leer [83]) to solve the Riemann problem above. Details about the MUSCL scheme in this example, can be found in Gottlieb and Shu's article [33]. For the time integration, they compared the methods*

$$\begin{cases} u^{n,0} &= u^n, \\ u^{n,1} &= u^{n,0} + \triangle t\mathcal{L}\left(u^{n,0}\right), \\ u^{n+1} &= u^{n,2} = \frac{1}{2}u^{n,0} + \frac{1}{2}u^{n,1} + \frac{1}{2}\triangle t\mathcal{L}\left(u^{n,1}\right) \end{cases} \tag{1.2.2}$$

*and*

$$\begin{cases} u^{n,0} &= u^n, \\ u^{n,1} &= u^n - 20\triangle t\mathcal{L}\left(u^{n,0}\right), \\ u^{n+1} &= u^{n,2} = u^{n,0} + \frac{41}{40}\triangle t\mathcal{L}\left(u^{n,0}\right) - \frac{1}{40}\triangle t\mathcal{L}\left(u^{n,1}\right). \end{cases} \tag{1.2.3}$$

*It follows by Harten's Lemma (cf. Harten [37]) that the MUSCL scheme with the forward Euler step as time integration method is TVD stable. Therefore, according to proposition 1.2.1 the MUSCL scheme with the time integration method* (1.2.2) *is TVD stable. The MUSCL scheme with the time integration method* (1.2.3) *cannot be TVD stable, since the approximate solution starts to oscillate after 500 time steps (cf. figure 1.2.2). Furthermore, we can see in figure 1.2.2 that the solution of the MUSCL scheme with the time integration method* (1.2.2) *converges to the unique entropy solution* $u$ *and the MUSCL scheme with the time integration* (1.2.3) *method produced a wrong solution.*

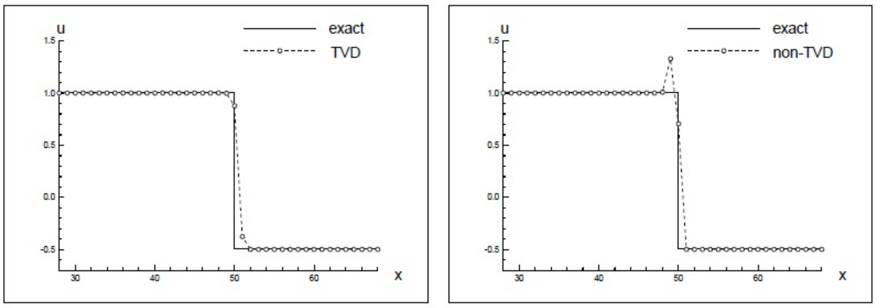


**Figure 1.2.2.** The solution of a second order TVD MUSCL spatial discretization after 500 time steps. Left: The SSP method (1.2.2). Right: The non-SSP method (1.2.3). (*Source: Gottlieb and Shu [33, Figure 1]*)

In the following, we present a certain class of Runge Kutta (RK) methods. This methods were first introduced by Shu and Osher in [73, 74] to ensure from the TVD stability of the forward Euler step the TVD stability of a high stage RK method. Hence, the methods were called TVD Runge-Kutta methods. Later, in [34] Gottlieb, Shu and Tadmor performed a systematic study of these methods and proved that the methods ensure also other stability properties than the TVD stability. Therefore, they changed the name of the TVD Runge-Kutta methods to Strong stability preserving Runge-Kutta (SSP-RK) methods.
We present the explicit SSP-RK time integration methods based on the ODE

$$u'(t) = \mathcal{L}\left(u\left(t\right), t\right), \quad t \in \left[0, T\right], \tag{1.2.4a}$$

$$u(0) = u^0, \tag{1.2.4b}$$

where $\mathcal{L} : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $u^0 \in \mathbb{R}^d$ are given. In the context of the method of lines approach, the function $\mathcal{L} = \mathcal{L}\left(u, t\right)$ is an approximation to the spatial differential operator in a PDE and $u^0$ is the initial data projected to a finite space. Shu and Osher introduced in [73, 74] the following way to describe a $s$-stage RK method for the ODE (1.2.4)

$$\begin{cases} u^{n,0} & = u^n; \\ \text{for } k & = 1, ..., s \text{ compute the intermediate functions :} \\ u^{n,k} & = \sum_{\ell=0}^{k-1} \alpha_{k\ell} w^{k,\ell}, \quad w^{k,\ell} = u^{n,\ell} + \frac{\beta_{k\ell}}{\alpha_{k\ell}} \triangle t \mathcal{L}\left(u^{n,\ell}, t_n + \delta_\ell \triangle t\right); \\ u^{n+1} & = u^{n,s}, \end{cases} \tag{1.2.5}$$

where for all $k = 1, ..., s$ and $\ell = 0, ..., k-1$

$$\alpha_{k\ell} \geq 0, \quad \text{if } \beta_{k\ell} \neq 0 \text{ then } \alpha_{k\ell} > 0, \quad \sum_{\ell=0}^{k-1} \alpha_{k\ell} = 1 \tag{1.2.6a}$$

| | $\alpha_{k\ell}$ | $\beta_{k\ell}$ | $\delta_\ell$ | $C_{\text{RK}}$ |
|---|---|---|---|---|
| Optimal second order method | $1$<br>$\frac{1}{2}, \frac{1}{2}$ | $1$<br>$0, \frac{1}{2}$ | $0$<br>$1$ | $1$ |
| Optimal third order method<br>(Shu's scheme) | $1$<br>$\frac{3}{4}, \frac{1}{4}$<br>$\frac{1}{3}, 0, \frac{2}{3}$ | $1$<br>$0, \frac{1}{4}$<br>$0, 0, \frac{2}{3}$ | $0$<br>$1$<br>$\frac{1}{2}$ | $1$ |

**Table 1.2.1.** The parameters of the optimal second and third order SSP-RK method.

and

$$\delta_0 = 0, \quad \delta_k = \sum_{\ell=0}^{k-1} \left(\alpha_{k\ell}\delta_\ell + \beta_{k\ell}\right), \quad \delta_s = 1. \tag{1.2.6b}$$

Note that by the conditions (1.2.6) the intermediate functions are convex combinations of first order forward Euler steps. This is the crucial feature of these classes of RK methods. Hence, if it is possible to adjust the time step adequate, a stability result for the forward Euler step provides a stability result for the high order RK methods. In fact, Gottlieb and Shu proved the following result in [33].

**Proposition 1.2.1** *Let $|\cdot|$ be a suitable seminorm on a function space. If for all $k = 1, ..., s$ as well as $\ell = 0, ..., k-1$ the conditions* (1.2.6) *are satisfied, $\beta_{k\ell} \geq 0$ and under the time step restriction $\triangle t \leq \triangle t_{FE}$ hold*

$$\left|u^{n,\ell} + \triangle t \mathcal{L}\left(u^{n,\ell}, t_n + \delta_\ell \triangle t\right)\right| \leq \left|u^{n,\ell}\right|. \tag{1.2.7}$$

*Then the solution obtained by the RK method* (1.2.5) *satisfies*

$$\left|u^{n+1}\right| \leq |u^n| \tag{1.2.8}$$

*under the time step restriction*

$$C_{RK} \triangle t \leq \triangle t_{FE}, \quad C_{RK} := \max\left\{\frac{\beta_{k\ell}}{\alpha_{k\ell}} : \ k = 1, ..., s, \ \ell = 0, ..., k-1\right\}. \tag{1.2.9}$$

It should be noted that the time step restriction (1.2.9) in proposition 1.2.1 is necessary to ensure the stability property (1.2.8) from the stability property (1.2.7) (cf. Gottlieb, Ketcheson and Shu [35, Theorem 3.3, p. 40]). Certainly, for different RK methods of the same order the coefficient $C_{\text{RK}}$ could be different. In order to minimize the cost of the time integration the coefficient $C_{\text{RK}}$ should be small. Hence, for a specific order the RK method in the form (1.2.5) with the smallest coefficient $C_{\text{RK}}$ is optimal. In this sense Gottlieb and Shu stated the optimal second and third order SSP-RK method in [33]. The parameters and the corresponding coefficients $C_{\text{RK}}$ of these methods are listed in the table (1.2.1).

Unfortunately, it is not straightforward to find explicit high order SSP-RK methods. In [33, Proposition 3.3] Gottlieb and Shu proved that a four stage, fourth order RK method in the form (1.2.5) with non-negative coefficients cannot exists. Admittedly, Spiteri and Ruuth constructed in [76] a fifth stage, fourth order SSP-RK method, but in [69] they proved that there is no fifth order SSP-RK method with non-negative coefficients. However, there are also investigations of implicit SSP-RK methods (cf. e.g. Gottlieb, Ketcheson and Shu [35]).
Finally, we would like to mention that a numerical method with a RK method of the form (1.2.5) as time integration method could be also stable, if the method is not stable for the forward Euler step. For instance Chavent and Cockburn proved in [3] by von Neumann analysis that for linear problems the forward Euler step of the RK-DG method is unconditionally $\mathrm{L}^2$-unstable. On the other hand, Q. Zhang and Shu proved in [90] that the RK-DG method with the optimal second order SSP-RK method is $\mathrm{L}^2$-stable under the time step restriction $\triangle t = \mathcal{O}\left(\triangle_K\right)$, where $\triangle_K$ is the diameter of a cell $K$, for piecewise polynomials of degree one and under the time step restriction $\triangle t = \mathcal{O}\left(\left(\triangle_K\right)^{\frac{4}{3}}\right)$ for piecewise polynomials of degree greater equal two.

### 1.2.2. The weak RK-DG formulation

In the following, we introduce the discontinuous Galerkin (DG) discretization of the problem (1.1.1), present the main ingredients for Cockburn, Shu and X. Zhang's post process procedures to stabilize the RK-DG method and discuss some theoretical results.
First of all we divide the domain $\Omega \subseteq \mathbb{R}$ by cells

$$K_j := \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad \triangle_j := x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} > 0, \quad j = 1, ..., N$$

such that $\overline{\Omega} = \bigcup_{j=1}^{N} \overline{K_j}$. Next, we define the following finite element test function space

$$\mathcal{V}_h := \left\{v \in \mathrm{L}^2\left(\Omega\right):\ v|_{K_j} \in \mathcal{P}^k\left(K_j\right)\ \forall j = 1, ..., N\right\}, \tag{1.2.10}$$

where $\mathcal{P}^k\left(K_j\right)$ denotes the space of polynomials in $K_j$ of degree at most $k$. Since the space $\mathcal{V}_h$ contains discontinuous test functions, we define for any $v \in \mathcal{V}_h$ and $j = 1, ..., N$ the quantities

$$v^+_{j-\frac{1}{2}} := \lim_{\varepsilon\to 0} v\left(x_{j-\frac{1}{2}} + \varepsilon\right), \quad v^-_{j-\frac{1}{2}} := \lim_{\varepsilon\to 0} v\left(x_{j-\frac{1}{2}} - \varepsilon\right), \quad [\![v]\!]_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}.$$

Furthermore, we approximate for all $t \in (0, T)$ the solution $u$ of the equation (1.1.1a) by the function $u_h\left(t\right) \in \mathcal{V}_h$ given by

$$u_h\left(t\right)|_{K_j} = \sum_{\ell=0}^{k} u^j_\ell\left(t\right)\phi^j_\ell, \tag{1.2.11}$$

where $\left\{\phi^j_0, ..., \phi^j_k\right\}$ is a basis of the space $\mathcal{P}^k\left(K_j\right)$. Note that for convenience the value $x \in K_j$ has been skipped in (1.2.11). The coefficients $u^j_0\left(t\right)$,...,$u^j_k\left(t\right)$ in (1.2.11) are the unknowns of the method. In order to determine these coefficients, we consider the problem: Find a function $u_h \in \mathcal{V}_h$, such that for all $v \in \mathcal{V}_h$ and $j = 1, ..., N$ the following equation is satisfied

$$(\partial_t u_h, v)_{K_j} - (f(u_h), v)_{K_j} + \widehat{f}(u_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \widehat{f}(u_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \tag{1.2.12}$$

where $\widehat{f}(u_h)_{j-\frac{1}{2}} := \widehat{f}\left(u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right)$. The function $\widehat{f}(\cdot,\cdot)$ is a numerical flux function, which is consistent with $f$, increasing in the first, decreasing in the second and Lipschitz continuous in both arguments. These kind of numerical flux functions are known as monotone fluxes. A collection of familiar numerical fluxes can be found in Cockburn's lecture notes [7, Section 3.4] or in the review article [16, Section 2.1] by Cockburn and Shu. The initial data to solve the ODE in (1.2.12) could be given by the $\mathrm{L}^2$-projection on the space $\mathcal{P}^k(K_j)$ of the initial function $u_0$ in problem (1.1.1). Then we obtain for all $v \in \mathcal{V}_h(t)$ the equation

$$(u_h(0), v)_{K_j} = (u_0, v)_{K_j}. \tag{1.2.13}$$

The ODE (1.2.12) is solved by a time integration method of the type (1.2.5) with the properties (1.2.6).
It should be noted that the RK-DG method has to be stabilized as we mentioned in the previous sections. Otherwise, the method is not stable and cannot satisfy the properties i) - v), which were introduced at the beginning of this section. According to proposition 1.2.1 it is enough to stabilize the method for the forward Euler step. In the following, we present the approach of Cockburn and Shu to stabilize the RK-DG method. This approach was published in [7, 11, 16]. Hence, first of all, we consider the forward Euler step of the weak formulation (1.2.12) with the test function $v = 1$ and obtain the scheme

$$\overline{u}^{n+1}_j = \overline{u}^n_j - \frac{\triangle t}{\triangle_j}\left(\widehat{f}(u_h)_{j+\frac{1}{2}} - \widehat{f}(u_h)_{j-\frac{1}{2}}\right), \tag{1.2.14}$$

where

$$\overline{u}^n_j := \frac{1}{\triangle_j}(u_h(t_n), v)_{K_j}$$

is the local mean value of the RK-DG solution $u_h$ at time level $t_n$ for the cell $K_j$. The local mean values provide the piecewise constant function $\overline{u}_h$ by $\overline{u}_h(x,t) = \overline{u}^n_j$, $x \in K_j$ and $t \in [t_n, t_{n+1})$. Note that the scheme (1.2.14) is of a similar structure as a classical three point finite difference scheme. A TVD stable finite difference scheme is not producing oscillatory solutions. Therefore, we are taming the RK-DG solution that the scheme (1.2.14) becomes TVD stable. It can be proven by Harten's Lemma (cf. Harten [37]) that the scheme (1.2.14) is TVD stable, if the solution $u_h$ satisfies for all $n = 1, ..., L$

$$u^-_{h,j+\frac{1}{2}}(t_n) = \overline{u}^n_j + m\left(u^-_{h,j+\frac{1}{2}}(t_n) - \overline{u}^n_j, \triangle_-\overline{u}^n_j, \triangle_+\overline{u}^n_j\right) \tag{1.2.15}$$

and

$$u_{h,j-\frac{1}{2}}^{+}(t_n) = \overline{u}_j^n - m\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{+}(t_n), \triangle_{-}\overline{u}_j^n, \triangle_{+}\overline{u}_j^n\right), \tag{1.2.16}$$

where $\triangle_{+}\overline{u}_j^n := \overline{u}_j^n - \overline{u}_{j-1}^n$, $\triangle_{+}\overline{u}_j^n := \overline{u}_{j+1}^n - \overline{u}_j^n$ and for all $(\alpha_1, ..., \alpha_s) \in \mathbb{R}^s$

$$m(\alpha_1, \cdots, \alpha_s) := \begin{cases} \sigma \min\limits_{1 \le \tau \le s} \alpha_\tau, & \text{if } \sigma = \operatorname{sgn}(\alpha_1) = \cdots = \operatorname{sign}(\alpha_s), \\ 0, & \text{else} \end{cases} \tag{1.2.17}$$

is the minmod function, which was introduced to construct the minmod limter for MUSCL schemes (cf. e.g. Sweby [79]). In general the solution of the RK-DG method does not satisfy the conditions (1.2.15) and (1.2.16). The solution has to be modified by a TVD slope limiter. A collection of TVD slope limiters are presented in Cockburn's lecture notes [7, p. 179-180]. If the modified RK-DG solution satisfies the conditions (1.2.15) and (1.2.16), the method is called TVD stable in the mean values (TVDM stable).

Unfortunately, the TVD stability is too restrictive to ensure that a method is high order accurate in regions, where the solution of the problem (1.1.1) is smooth. Hence, in order to maintain the high order accuracy at local extrema, a less restrictive TVB limiter has been introduced (c.f. Cockburn and Shu [11, 72]). A TVB limiter is taming the solution of the RK-DG method, such that the conditions (1.2.15) and (1.2.16) are satisfied for the modified minmod function instead of the minmod function (1.2.17). The modified minmod function is for all $(\alpha_1, ..., \alpha_s) \in \mathbb{R}^s$ given by

$$\overline{m}(\alpha_1, \cdots, \alpha_s) := \begin{cases} \alpha_1, & \text{if } |\alpha_1| \le \widetilde{M}(\triangle_j)^2, \\ m(\alpha_1, \cdots, \alpha_s), & \text{else.} \end{cases} \tag{1.2.18}$$

Note that in the modified minmod function the parameter $\widetilde{M}$ appears. This parameter is the Achilles' heel in the application of TVB limiter, since for systems of conservation laws it is not a priori clear how to chose this parameter. For scalar conservation laws is the situation simpler, since the solution of the problem (1.1.1) satisfies the maximum principle (1.1.5). Selection options for the parameter $\widetilde{M}$ have been discussed by Cockburn and Shu in [11, Lemma 2.2]. Furthermore, they proved that the TVB limiter provides the TVB stability of the scheme (1.2.14) (c.f. Cockburn and Shu [11, Lemma 2.3]). In accordance with the TVDM stability, the RK-DG method is called TVB stable in the mean values (TVBM stable), if the scheme (1.2.14) is TVB stable.

Admittedly, the TVBM stability is not enough to ensure that the method satisfies a maximum principle. In [94] X. Zhang and Shu suggested to modify the RK-DG solution for all $x \in K_j$ by

$$\widetilde{u}_h(x, t_n) = \theta\left(u_h(x, t_n) - \overline{u}_j^n\right) + \overline{u}_j^n, \quad \theta = \min\left\{\left|\frac{M - \overline{u}_j^n}{M_j - \overline{u}_j^n}\right|, \left|\frac{m - \overline{u}_j^n}{m_j - \overline{u}_j^n}\right|, 1\right\}, \tag{1.2.19}$$

where

$$m := \min_{x \in \Omega} u_0(x) \quad \text{and} \quad M := \max_{x \in \Omega} u_0(x) \tag{1.2.20}$$

as well as

$$m_j := \min_{x\in K_j} u_h(x) \quad \text{and} \quad M_j := \max_{x\in K_j} u_h(x). \tag{1.2.21}$$

Furthermore, X. Zhang and Shu [94, Lemma 2.4 and Theorem 2.5] proved that the limiter (1.2.19) is not affecting the accuracy of the method and the modified solution satisfies the following maximum principle

$$\max_{x\in\Omega} \widetilde{u}_h(x, t_{n+1}) \le \max_{x\in\Omega} \widetilde{u}_h(x, t_n) \quad \text{and} \quad \min_{x\in\Omega} \widetilde{u}_h(x, t_{n+1}) \ge \min_{x\in\Omega} \widetilde{u}_h(x, t_n).$$

It should be noted that X. Zhang and Shu's limiter could be also applied to systems of conservation laws. For example, the density and the pressure in the Euler equations need to be positive quantities. Several high order methods for the Euler equations do not ensure this restriction, but the approximate density and pressure revised by X. Zhang and Shu's limiter have this property, since this limiter preserves positive quantities (cf. X. Zhang and C.-W [95]).
Next, we discuss the entropy stability of the RK-DG method. On the one hand, Cockburn and Shu proved in [11, Lemma 2.8] that the RK-DG is entropy stable with respect to the Kružkov entropy condition (1.1.4), if the solution $u_h$ satisfies for all $n = 1, \ldots, L-1$

$$u^-_{h,j+\frac{1}{2}}(t_n) = \overline{u}^n_j + m\left(\widetilde{u}^n_j, \triangle_-\overline{u}^n_j, \triangle_+\overline{u}^n_j, \text{sign}\left(\widetilde{u}^n_j\right) C_{\text{E}} \left(\triangle_j\right)^{\beta}\right) \tag{1.2.22}$$

and

$$u^+_{h,j-\frac{1}{2}}(t_n) = \overline{u}^n_j - m\left(\widetilde{\widetilde{u}}^n_j, \triangle_-\overline{u}^n_j, \triangle_+\overline{u}^n_j, \text{sign}\left(\widetilde{\widetilde{u}}^n_j\right) C_{\text{E}} \left(\triangle_j\right)^{\beta}\right), \tag{1.2.23}$$

where $\widetilde{u}^n_j := u^-_{h,j+\frac{1}{2}}(t_n) - \overline{u}^n_j$, $\widetilde{\widetilde{u}}^n_j := \overline{u}^n_j - u^+_{h,j+\frac{1}{2}}(t_n)$ and $C_{\text{E}} > 0$ as well as $\beta > 0$ are fixed parameters. Of course, the RK-DG solution has to be modified by a limiter to satisfy (1.2.22) and (1.2.23). A collection of limiters, which enforce the entropy stability of the method, are presented in Cockburn's lecture notes [7, p. 179-180]. However, it should be noted that in practice the choice of the parameters $C_{\text{E}}$ and $\beta$ is not easy, since the parameters cannot be automatically adjusted. The parameters have to be tuned for each individual problem. Note that inappropriate choices of the parameters could smear out the profile of the solution or the limiter could fail to correct a weak entropy violating shock. On the other hand Jiang and Shu proved in [45] a cell entropy inequality for the semi-discrete DG method (1.2.12). They used the fact that monotone numerical flux functions satisfy the E-flux condition which is for all $a, b \in \mathbb{R}$ given by

$$\left(\widehat{f}(a, b) - f(\zeta)\right)(b - a) \le 0, \quad \text{for all } \zeta \in [a, b]. \tag{1.2.24}$$

Furthermore, they extended this result to certain time integration methods like the backward Euler and the Crank-Nicolson time integration method by applying the identity

$$(a - b)(\vartheta a + (1 - \vartheta) b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \left(\vartheta - \frac{1}{2}\right)(a - b)^2, \tag{1.2.25}$$

for all $a, b \in \mathbb{R}$ and $\vartheta \in [0, 1]$. Unfortunately it is not possible to extend this result to the forward Euler step, since in [3] Chavent and Cockburn proved that the forward Euler step of the RK-DG method is unconditionally $\mathrm{L}^2$-unstable, if for the spacial discretization non piecewise constant polynomials are used. Nevertheless, in [11, Section 4, Example 3] Cockburn and Shu showed by numerical experiments that the method for scalar conservation laws is entropy stable without any entropy correction by a limiter, which enforces the RK-DG solution to satisfy (1.2.22) and (1.2.23). Moreover, they showed that the method is also entropy stable for scalar conservation laws with a nonconvex flux like the Buckley-Leverett flux (cf. Cockburn and Shu [11, Section 4, Example 3]). Therefore, in the sense of entropy stability, there is a big gap between the theoretical results and the actual performance of the method in applications.
Finally, the theoretical aspects of the RK-DG method, which have been mentioned in this section, can be summarized in the following theorem. The theorem has been proven by Cockburn in [7].

**Theorem 1.2.2** *Suppose that the slope limiter in the RK-DK method ensures TVDM stability or that the limiter ensures TVBM stability and the limiter* (1.2.19) *is used. Then there is a supsequence* $(h_k)_{k\in\mathbb{N}}$ *generated by the method with*

$$\overline{u}_{h_k} \underset{h_k\to 0}{\longrightarrow} u \quad \textit{in}\ \mathrm{L}^\infty\left(0, T; \mathrm{L}^1(\Omega)\right)$$

*and* $u$ *is a weak solution of* (1.1.1). *Furthermore, if the slope limiter ensures entropy stability the whole supsequence* $(\overline{u}_h)_{h>0}$ *converges in* $\mathrm{L}^\infty(0, T; \mathrm{L}^1(\Omega))$ *to the unique entropy solution of* (1.1.1).

### 1.2.3. Discontinuous Galerkin methods for the Hamilton-Jacobi equations

Generalized solutions for the Hamilton-Jacobi equations are different from weak entropy solutions. Nevertheless, in [18] Crandall and Lions proved that monotone finite difference schemes for the problems (1.1.8) and (1.1.9) converge to the unique viscosity solution. In the previous sections, we have seen that the RK-DG method is in some sense a high order extension of a monotone finite difference scheme. Hence, it is meaningful to extend the RK-DG method to the Hamilton-Jacobi equations. Certainly, the development of RK-DG methods for the Hamilton-Jacobi equations is not straightforward, since these equations are not quasi-linear and thus we cannot apply the integration by parts formula. In the following, two RK-DG methods for the Hamilton-Jacobi equations will be listed.
**Hu and Shu's method:** We have seen that for sufficiently smooth functions the Hamilton-Jacobi equations can be written as the scalar conservation law (1.1.10) and the system (1.1.11). This relationship between conservation laws and Hamilton-Jacobi equations was utilized by Hu, Lepsky and Shu in a series of papers [40, 41, 54] to adapt the RK-DG method for conservation laws to the Hamilton-Jacobi equations.
In the one dimensional case, two test function spaces are used to describe the method. The first

one is the space (1.2.10) to describe the approximate solution $u_h$ by (1.2.11) the other space is given by

$$\mathcal{U}_h := \left\{ v \in \mathrm{L}^2(\Omega) : \ v|_{K_j} \in \mathcal{P}^{k-1}(K_j) \ \forall j = 1, ..., N \right\}. \tag{1.2.26}$$

These test function spaces are applied to the problem: Find a function $u_h \in \mathcal{V}_h$ such that for all $v \in \mathcal{U}_h$ and $j = 1, ..., N$ holds

$$(\partial_t \partial_x u_h, v)_{K_j} - (H(\partial_x u_h), v)_{K_j} + \widehat{H}(\partial_x u_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \widehat{H}(\partial_x u_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \tag{1.2.27}$$

thereby is

$$\widehat{H}(\partial_x u_h)_{j-\frac{1}{2}} := \widehat{H}\left( (\partial_x u_h)^-_{j-\frac{1}{2}}, (\partial_x u_h)^+_{j-\frac{1}{2}} \right)$$

a suitable numerical flux function. A common choice is a Lax Friedrichs type flux as suggested by Crandall and Lions in [18]. Other choices are the Godunov or Osher-Sethian flux (cf. Osher, Sethian and Shu [64] and [65]). The discrete weak formulation (1.2.27) determines $u_h$ up to a constant. To determine the missing constant, there are two ways:

a) It has to be required that for any cell $K_j$, $j = 1, ..., N$, the following equation is satisfied

$$(\partial_t u_h, 1)_{K_j} + (H(\partial_x u_h), 1)_{K_j} = 0. \tag{1.2.28}$$

b) The condition (1.2.28) is merely used in a few cells, for instance in $K_1$. Afterward, the missing constant for the cell $K_j$, $j = 1, ..., N$, could be determined by solving the equation

$$u_h(x_j, t) = u_h(x_1, t) + \int_{x_1}^{x_j} \partial_x u_h \, dx, \tag{1.2.29}$$

where $x_j \in K_j$ and $x_1 \in K_1$.

Both approaches are discussed and implemented by Hu and Shu in [40]. They mentioned that for smooth problems the approaches perform similar, but for problems with singularities the performance of the first approach is better, if the integral path in (1.2.29) is chosen adversely. Nevertheless, in the sense of computational coast is the second approach preferable to the first approach.
For the semi-discrete scheme (1.2.27), the cell entropy inequality, which was proven by Jiang and Shu in [45], provides the $\mathrm{L}^2$-stability of the sequence $(\partial_x u_h)_{h>0}$. Therefore, by the Rellich-Kondrachov theorem (cf. Ciarlet [5, Theorem 6.6-3, p. 333]) and Aubin-Lions-Simon lemma (cf. Simon [71, Section 8, Theorem 5]) exists a subsequence of $(u_h)_{h>0}$, which converges. It can be proven that the limes of the subsequence is a solution of the initial value problem (1.1.8), but in general it is not the unique viscosity solution.
In order to present the two dimensional extension of Hu and Shu's method, we assume that $\mathcal{T}_h$, $h > 0$, is a suitable family of tessellations of the domain $\Omega \subseteq \mathbb{R}^2$. Similar to the one dimensional case is the approximate solution $u_h$ a piecewise $\mathcal{P}^k$ polynomial. If the approximate solution has been determined at time level $t_n$, the approximate solution for the upcoming time level can be determined as follows:

- Apply the RK-DG method for hyperbolic systems of conservation laws (cf. Cockburn and Shu [14]) with a test function space of piecewise $\mathcal{P}^{k-1}$ polynomials for the system (1.1.11) to determine provisional functions $v_{1,h}$ and $v_{2,h}$ at time level $t_{n+1}$. These functions are approximations for $\partial_x u$ and $\partial_y u$ where $u$ denotes the unique viscosity solution of the initial value problem (1.1.9).
- For all cells $K \in \mathcal{T}_h$ determine the gradient $\nabla u_h$ at time level $t_{n+1}$ by

$$\|\nabla u_h - (v_{1,h}, v_{2,h})\|_{[\mathrm{L}^2(K)]^2} = \min_{\phi \in \mathcal{P}^k(K)} \|\nabla \phi - (v_{1,h}, v_{2,h})\|_{[\mathrm{L}^2(K)]^2}. \tag{1.2.30}$$

- Determine the missing constant by one of the following approaches:
  a) Require that for any cell $K \in \mathcal{T}_h$ holds

$$(\partial_t u_h, 1)_K + (H(\nabla u_h), 1)_K = 0. \tag{1.2.31}$$

  b) Apply the condition (1.2.31) merely in a few cells, for instance in the corner cell $K_{\text{corner}} \in \mathcal{T}_h$. Afterward, the missing constant for the cell $K \in \mathcal{T}_h$ could be determined by solving the path integral

$$u_h(x_K, t) = u_h(x_{K_{\text{corner}}}, t) + \int_\gamma (\partial_x u_h)\, dx + (\partial_y u_h)\, dy, \tag{1.2.32}$$

  where $\gamma$ is a path from the point $x_{K_{\text{corner}}} \in K_{\text{corner}}$ to the point $x_K \in K$.

Both approaches are discussed and implemented by Hu, Lepsky and Shu in [40] and [41]. The behavior of the approaches is similar to the one dimensional analogues. Furthermore, Lepsky, Hu and Shu analyzed in [54] the effect of the equation (1.2.30). They proved that the $\mathrm{L}^2$-norm of the quantity $\nabla u_h$ is not bigger than the $\mathrm{L}^2$-norm of $(v_{1,h}, v_{2,h})$ and that the mean values of $v_{1,h}$ and , $v_{2,h}$ are equal to the mean values of $\partial_x u_h$ and $\partial_y u_h$. Therefore, the equation (1.2.30) does not affect the stability properties of the RK-DG method.
Nevertheless, it should be noted that in general the system (1.1.11) is not strictly or strongly hyperbolic. Admittedly, the RK-DG method by Cockburn and Shu [14] was developed for strictly or strongly hyperbolic systems. Hence, a numerical method for directly solving the Hamilton-Jacobi equations without a consideration of the system (1.1.11) is preferable. However, the capability of Hu, Lepsky and Shu's method for one and two dimensions in the numerical experiments of Hu, Lepsky and Shu in [40, 41, 54] is magnificent.
**Yan and Osher's local discontinuous Galerkin (LDG) method:** This method was developed by Yan and Osher in [86] to solve directly the one and two dimensional Hamilton-Jacobi equations. In the following, for convenience, we present merely the one dimensional method. The method for the two dimensional Hamilton-Jacobi equations can be specified by similar arguments (cf. Yan and Osher [86]). Furthermore, we apply the same notation as in the section 1.2.2.

First of all, we approximate for all $t \in (0,T)$ the solution $u$ of the equation (1.1.8a) by a function $u_h(t) \in \mathcal{V}_h$ given by (1.2.11). Next, in order to determine the coefficients of the approximation, we consider the following problem: Find a function $u_h \in \mathcal{V}_h$, such that for all $v \in \mathcal{V}_h$ and $j = 1, ..., N$ the following equation is satisfied

$$(\partial_t u_h, v)_{K_j} - \left(\widehat{H}(p_1, p_2), v\right)_{K_j} = 0, \tag{1.2.33}$$

where the variables $p_1$ and $p_2$ are approximations for $\partial_x u_h$ and $\widehat{H}(p_1, p_2)$ is a local Lax-Friedrichs flux given by

$$\widehat{H}(p_1, p_2) = H\left(\frac{p_1 + p_2}{2}\right) - \frac{\lambda_j}{2}(p_2 - p_1) \tag{1.2.34}$$

with

$$\lambda_j = \max_{p \in D_j} |\partial_p H(p)|, \quad D_j := [\min(p_1, p_2), \max(p_1, p_2)]\,|_{K_j}. \tag{1.2.35}$$

In order to determine the variables $p_1$ and $p_2$ we consider the following problems: Find functions $p_1, p_2 \in \mathcal{V}_h$, such that for all $v \in \mathcal{V}_h$ and $j = 1, ..., N$ the following equations are satisfied

$$(p_1, v)_{K_j} + (u_h, \partial_x v)_{K_j} - u^-_{h,j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + u^-_{h,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{1.2.36}$$

and

$$(p_2, v)_{K_j} + (u_h, \partial_x v)_{K_j} - u^+_{h,j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + u^+_{h,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0. \tag{1.2.37}$$

The initial data to solve the ODE in (1.2.33) could be given by the $L^2$-projection of the function $u_0$ in problem (1.1.8). Then the ODE (1.2.33) is solved by a RK method of the type (1.2.5) with the properties (1.2.6). We would like to mention that the DG method (1.2.33), (1.2.36) and (1.2.37) has a similar structure as the local discontinuous Galerkin (LDG) method, which was developed by Cockburn and Shu in [15] to solve systems of time-dependent convection-diffusion equations. Hence, Yan and Osher called the method above LDG method for directly solving the Hamilton-Jacobi equations.

Next we consider the piecewise constant forward Euler step of Yan and Osher's LDG method. Let $u_h|_{K_j}(t_n) = \overline{u}^n_j$, $j = 1, ..., N$, be the piecewise constant solution of the method at time level $t = t_n$. Then the equations (1.2.36) and (1.2.37) degenerate to

$$\triangle_j p_1 - \overline{u}^n_j + \overline{u}^n_{j-1} = 0 \quad \text{and} \quad \triangle_j p_2 - \overline{u}^n_{j+1} + \overline{u}^n_j = 0. \tag{1.2.38}$$

Therefore, we obtain by (1.2.33), (1.2.34) and (1.2.38)

$$\overline{u}^{n+1}_j = \overline{u}^n_j - \triangle t H\left(\frac{\overline{u}^n_{j+1} - \overline{u}^n_{j-1}}{2\triangle_j}\right) + \triangle t \lambda_j \left(\frac{\overline{u}^n_{j+1} - 2\overline{u}^n_j + \overline{u}^n_{j-1}}{2\triangle_j}\right). \tag{1.2.39}$$

Next, we define for all $a, b, c \in \mathbb{R}$ the function

$$F\left(a,b,c\right) = \left(1 - \lambda_j \frac{\triangle t}{\triangle_j}\right) b - \triangle t H\left(\frac{c-a}{2\triangle_j}\right) + \triangle t \lambda_j \left(\frac{c+a}{2\triangle_j}\right).$$

Thus, the scheme (1.2.39) can be written as follows

$$\overline{u}_j^{n+1} = F\left(\overline{u}_{j-1}^n, \overline{u}_j^n, \overline{u}_{j+1}^n\right).$$

The function $F = F(a,b,c)$ is increasing in all arguments, if the following CFL condition is satisfied

$$\lambda_j \frac{\triangle t}{\triangle_j} \leq 1, \quad \text{for all } j = 1, ..., N.$$

Therefore, the scheme (1.2.39) is a monotone scheme. Crandall and Lions proved in [18] that monotone finite difference scheme for the problem (1.1.8) converge to the unique viscosity solution. Hence, the first order version of Yan and Osher's LDG method provides the right solution of the problem (1.1.8). Thus, Yan and Osher's LDG method differs from Cheng and Shu's DG method for directly solving the Hamilton-Jacobi equations, which was published in [4]. For the semi-discrete high order version of Yan and Osher's LDG method, Xiong, Shu and M. Zhang proved in [84] the optimal $(k+1)$ convergence in one dimension and for cartesian grids in two dimensions. In addition, in [86] Yan and Osher showed the capability of the high order method in numerical experiments.

Based on Yan and Osher's LDG method, we will present an ALE-LDG method for directly solving the Hamilton-Jacobi equations in chapter 4.

# 2. The discrete setting for the arbitrary Lagrangian-Eulerian (ALE) method

In this chapter, we present the main ingredients to describe our arbitrary Lagrangian-Eulerian (ALE) method. Later in the chapters 3 and 4 we will combine the tools of this chapter with discontinuous Galerkin methods to discretize the problems (1.1.1) and (1.1.8).
First of all, we construct time-dependent simplicial cells based on the assumption:

"*The distribution of the grid points is explicitly given for an upcoming time level.*"

Next, we define a time-dependent affine mapping to connect the time-dependent simplicial cells with a reference simplex. The mapping yields an explicit expression for the grid velocity field of the ALE method and is an important tool for the implementation of the method. Afterward, we define the test function space for the ALE-DG methods and prove transport equations. Finally, we discuss the one dimensional case, where the assumptions for the mesh are not as restrictive as in several space dimensions.
Furthermore, we present conditions, which ensure that for any time point $t$ the time-depending cells provide a regular family of simplicial meshes of the domain $\Omega$. In the following, we denote $\mathcal{T}_h$, $h > 0$, as a family of simplicial meshes of the domain $\Omega$, if

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$$

and for any $K \in \mathcal{T}_h$ there are $d + 1$ affinely independent points $v_0, ..., v_d \in \Omega$ such that $K := \text{int}\,(\text{conv}\,\{v_0, ..., v_d\})$. In the sense of Ciarlet [6] the family $\mathcal{T}_h$, $h > 0$, is called regular, if it has the properties:

($\mathcal{T}_h$1) Admissibility:
For any two elements $K_1, K_2 \in \mathcal{T}_h$ the set $e := \overline{K_1} \cap \overline{K_2}$ is either disjoint or $e \subseteq \partial K_1 \cap \partial K_2$. In the second case we call $e$ edge of $K_1$ and $K_2$.

($\mathcal{T}_h$2) Shape-regularity (cf. Ciarlet [6, p. 132, 140]):
For any cell $K \in \mathcal{T}_h$ are $\triangle_K$ the diameter of $K$, $h = h_{\mathcal{T}_h} := \max_{K \in \mathcal{T}_h} \triangle_K$ is the meshsize of the triangulation and $\rho_K$ is the radius of the largest ball contained in $K$. This quantities are connected as follows: There are constants $\sigma > 0$ and $\tau > 0$, independent of $h$, such that

$$\frac{\triangle_K}{\rho_K} \leq \sigma \quad \text{and} \quad \frac{h}{\triangle_K} \leq \tau.$$

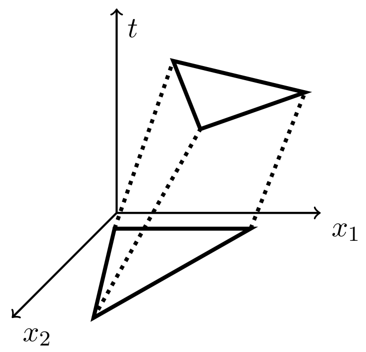


**Figure 2.1.1.** The vertices of a two dimensional simplex move to the vertices of a simplex at the next time level.

It is well known that for a regular static simplicial mesh the classical inverse, trace and interpolation inequalities (cf. e.g. Ciarlet [6]) can be applied. Therefore, if the time-dependent simplicial cells provide for any $t \in [0, T]$ a regular mesh, these inequalities could be also applied for the time-dependent cells.

## 2.1. The mesh for the ALE method

In order to present the ALE-DG methods, we have to describe the motion of the mesh. The PDEs, in this thesis, are considered merely with compactly supported initial data and periodic boundary conditions. Thus, we can assume that the boundary of the domain $\Omega$ is not moving and that the mesh topology is changing solely in the interior of the domain. Moreover, we assume that there are at time level $t_n$ and $t_{n+1}$ regular families of simplicial meshes $\mathcal{T}_{h_n}$ as well as $\mathcal{T}_{h_{n+1}}$, $h_n, h_{n+1} > 0$. Likewise, we assume that $\mathcal{T}_{h_n}$ and $\mathcal{T}_{h_{n+1}}$ have the same mesh topology. Furthermore, by these assumptions the vertices at time level $t_n$ and the vertices at time level $t_{n+1}$ can be connected by time-dependent straight lines (cf. figure 2.1.1). Physically, this setting can be interpreted as follows: The vertices at time level $t_n$ are moving to the vertices at time level $t_{n+1}$.

In the following, the motion of the grid will be described more precisely. Let $K^n \in \mathcal{T}_{h_n}$, $K^{n+1} \in \mathcal{T}_{h_{n+1}}$ be arbitrary fixed simplexes with vertices $v_0^n, ..., v_d^n \in \Omega$ and $v_0^{n+1}, ..., v_d^{n+1} \in \Omega$. For $\ell = n, n+1$ we define the matrix

$$\mathbf{A}_{K^\ell} := \left(v_1^\ell - v_0^\ell, ..., v_d^\ell - v_0^\ell\right) \tag{2.1.1}$$

and assume $\det(\mathbf{A}_{K^\ell}) > 0$. This means that the orientation of the corresponding simplex is positive. Therefore, we can define for all $j = 0, ..., d$ and $t \in [t_n, t_{n+1}]$ time-dependent straight lines

$$v_j(t) := v_j^n + \omega_{K^n,j}(t - t_n), \qquad \omega_{K^n,j} := \frac{1}{\triangle t}\left(v_j^{n+1} - v_j^n\right), \tag{2.1.2}$$

where the quantity $\omega_{K^n,\ell}$ describes the speed of motion. The straight lines (2.1.2) provide for any $t \in [t_n, t_{n+1}]$ time-dependent cells

$$K(t) := \text{int}\left(\text{conv}\{v_0(t), ..., v_d(t)\}\right). \tag{2.1.3}$$

For any cell (2.1.3), the diameter and radius of the largest ball, contained in $K(t)$, are denoted by $\triangle_{K(t)}$ as well as $\rho_{K(t)}$. Furthermore, for any $t \in [t_n, t_{n+1}]$, the family of all sets containing the cells (2.1.3) will be denoted by $\mathcal{T}_{h(t)}$, $h(t) > 0$. Additionally, we define the quantities

$$h(t) := h_{\mathcal{T}_{h(t)}} = \max_{K(t)\in\mathcal{T}_{h(t)}} \triangle_{K(t)}. \tag{2.1.4}$$

and

$$h := \max_{t\in[0,T]} h(t). \tag{2.1.5}$$

The quantity $h$ will be denoted as global length.
It seems to be meaningful that for any $t \in [t_n, t_{n+1}]$ the set $\mathcal{T}_{h(t)}$ is a regular simplicial mesh of the domain $\Omega$. Certainly, in general the cells (2.1.3) do not provide a regular simplicial mesh. In order to obtain a regular mesh, we need to introduce some assumptions. Therefore, we apply the upcoming notation: For any two elements $K_1^\ell, K_2^\ell \in \mathcal{T}_{h_\ell}$, $\ell = n, n+1$, we define $e^\ell_{K_1,K_2} := \overline{K_1^\ell} \cap \overline{K_2^\ell}$. If it is clear which cells are considered, we will just write $e^\ell$.
Next, we introduce some assumptions:

(A1) For any $t \in [t_n, t_{n+1}]$, there are constants $\gamma, \delta > 0$, independent of $h(t)$, such that

$$\forall K(t) \in \mathcal{T}_{h(t)}, \ \ \max\left(\triangle_{K^n}, \triangle_{K^{n+1}}\right) \leq \gamma \triangle_{K(t)} \ \text{ and } \ \max\left(\rho_{K^n}, \rho_{K^{n+1}}\right) \leq \delta \rho_{K(t)},$$

where $\triangle_{K^\ell}$, $\ell = n, n+1$, is the diameter of $K^\ell$ and $\rho_{K^\ell}$ is the radius of the largest ball contained in $K^\ell$.

(A2) If the cells $K_1^n$, $K_2^n$ are connected with the cells $K_1^{n+1}$, $K_2^{n+1}$ by (2.1.2), the sets $e^n$ and $e^{n+1}$ have the same cardinality and the set $e^n$ is an interface of the cells $K_1^n$, $K_2^n$ likewise the set $e^{n+1}$ is an interface of the cells $K_1^{n+1}$, $K_2^{n+1}$.

(A3) For any $t \in (t_n, t_{n+1})$ holds

$$\left\|\mathbf{A}_{K^n}^{-1}\mathbf{A}_{K^{n+1}}\right\|_{\mathcal{L}\left(\mathbb{R}^d,\mathbb{R}^d\right)} < \varepsilon(t) \quad \text{or} \quad \left\|\mathbf{A}_{K^{n+1}}^{-1}\mathbf{A}_{K^n}\right\|_{\mathcal{L}\left(\mathbb{R}^d,\mathbb{R}^d\right)} < (\varepsilon(t))^{-1}, \tag{2.1.6}$$

where $\varepsilon(t) := \frac{t_{n+1}-t}{t-t_n}$.

The assumptions above supply the following result.

**Lemma 2.1.1** *Suppose the assumptions* $(A1)$ *and* $(A2)$*. Then for any* $t \in [t_n, t_{n+1}]$ *is* $\mathcal{T}_{h(t)}$*,* $h(t) > 0$*, a regular family of simplicial meshes of the domain* $\Omega$*. Furthermore, if the assumption* $(A3)$ *is satisfied, is for any* $t \in [t_n, t_{n+1}]$ *the determinant of the matrix*

$$\boldsymbol{A}_{K(t)} := (v_1(t) - v_0(t), ..., v_d(t) - v_0(t)) \tag{2.1.7}$$

*a strictly positive real number.*

**Proof.** The definition of $\mathcal{T}_{h(t)}$ yield $\mathcal{T}_{h_n} = \mathcal{T}_{h(t_n)}$ and $\mathcal{T}_{h_{n+1}} = \mathcal{T}_{h(t_{n+1})}$. Accordingly, in the following is $t \in (t_n, t_{n+1})$ an arbitrary fixed point. By the assumption (A1) the largest ball, contained in a cell $K(t) \in \mathcal{T}_{h(t)}$, is not empty, since the tessellations $\mathcal{T}_{h_n}$ and $\mathcal{T}_{h_{n+1}}$ are shape-regular and the cells $K^n$ as well as $K^{n+1}$ are not empty. Therefore, the points $v_0(t), ..., v_d(t)$ are affinely independent. Hence, the cells (2.1.3) are simplexes. Thus, in particular, the straight lines (2.1.2) are not interacting. Furthermore, the cells $K(t) \in \mathcal{T}_{h(t)}$ cover exactly the domain $\Omega$, since $\Omega$ is a convex set.
The assumption (A2) supplies that there is no vertex of a cell in the interior of a neighboring cell. Therefore, the condition ($\mathcal{T}_h 1$) is satisfied. In order to prove the condition ($\mathcal{T}_h 2$), we denote the mesh regularity parameters of $\mathcal{T}_{h_n}$ as well as $\mathcal{T}_{h_{n+1}}$ by $\sigma_n, \tau_n$ and $\sigma_{n+1}, \tau_{n+1}$. Then we obtain by (2.1.2), assumption (A1) and the mesh regularity of $\mathcal{T}_{h_n}$ as well as $\mathcal{T}_{h_{n+1}}$

$$\begin{aligned}\triangle_{K(t)} &= \max_{0 \le i < j \le d} \left| \left(1 - \frac{t - t_n}{\triangle t}\right)\left(v_j^n - v_i^n\right) + \frac{t - t_n}{\triangle t}\left(v_j^{n+1} - v_i^{n+1}\right) \right| \\ &\le \left(1 - \frac{t - t_n}{\triangle t}\right) \triangle_{K^n} + \frac{t - t_n}{\triangle t} \triangle_{K^{n+1}} \\ &\le \sigma_n \rho_{K^n} + \sigma_{n+1} \rho_{K^{n+1}} \le 2\delta \max\{\sigma_n, \sigma_{n+1}\} \rho_{K(t)},\end{aligned} \tag{2.1.8}$$

where we used the fact that the diameter of a simplex equals the greatest Eucledian distance between two vertices of the simplex. In a similar way (2.1.2), assumption (A1) and the mesh regularity of $\mathcal{T}_{h_n}$ as well as $\mathcal{T}_{h_{n+1}}$ provide

$$\begin{aligned} h(t) &\le 2 \max\left(h_{\mathcal{T}_{h_n}}, h_{\mathcal{T}_{h_{n+1}}}\right) \le 2 \max(\tau_n, \tau_{n+1}) \max(\triangle_{K^n}, \triangle_{K^{n+1}}) \\ &\le 2\gamma \max\{\tau_n, \tau_{n+1}\} \triangle_{K(t)}. \end{aligned} \tag{2.1.9}$$

These two estimates yield the condition ($\mathcal{T}_h 2$).
By (2.1.2) follows for all $t \in (t_n, t_{n+1})$

$$\mathbf{A}_{K(t)} = \left(1 - \frac{t - t_n}{\triangle t}\right) \mathbf{A}_{K^n} + \frac{t - t_n}{\triangle t} \mathbf{A}_{K^{n+1}}.$$

Therefore, according to the assumption (A3) and lemma A.1.1 in the appendix is $\det\left(\mathbf{A}_{K(t)}\right) > 0$, for all $t \in (t_n, t_{n+1})$. □

**Remark 2.1.1** *In the proof of lemma 2.1.1, the shape-regularity of $\mathcal{T}_{h_n}$ and $\mathcal{T}_{h_{n+1}}$ yield that the vertices* (2.1.2) *are affinely independent for all $t \in (t_n, t_{n+1})$. We would like to mention that the affine independence can be also received by a condition on the angles of the cells $K^n \in \mathcal{T}_{h_n}$ as well as $K^{n+1} \in \mathcal{T}_{h_{n+1}}$. This is meaningful, since Zlámal's angle condition is equivalent to the shape-regularity condition $(T_h 2)$ (cf. Ciarlet [6, p. 461]).*

Next, we discuss the relationship between the quantities $\rho_{K(t)}$, $\triangle_{K(t)}$, $h(t)$ and the $d$-Lebesgue measure $|K(t)|_d$ of a cell $K(t) \in \mathcal{T}_{h(t)}$. The estimates (2.1.8) and (2.1.9) provide that the mesh regularity parameters for the family $\mathcal{T}_{h(t)}$, $h(t) > 0$, are given by

$$\varsigma := 2\delta \max\{\sigma_n, \sigma_{n+1}\} \quad \text{and} \quad \varrho := 2\gamma \max\{\tau_n, \tau_{n+1}\}.$$

Hence, we obtain for the $d$-Lebesgue measure of a cell $K(t) \in \mathcal{T}_{h(t)}$ the estimate

$$\frac{\alpha(d)}{\varrho\varsigma}(h(t))^d \leq \frac{\alpha(d)}{\varsigma}\triangle^d_{K(t)} \leq \alpha(d)\,\rho^d_{K(t)} \leq |K(t)|_d, \tag{2.1.10}$$

where $\alpha(d) := \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}$ is the volume of the $\mathbb{R}^d$ unit ball. Moreover, $\Gamma\left(\frac{d}{2}+1\right)$ denotes the gamma function evaluated in the point $\frac{d}{2}+1$. Likewise, we obtain by Hadamard's inequality

$$|K(t)|_d = \frac{1}{d!}\left|\det\left(\mathbf{A}_{K(t)}\right)\right| \leq \frac{1}{d!}\prod_{j=1}^{d}|v_j(t) - v_0(t)| \leq \frac{1}{d!}\triangle^d_{K(t)} \leq \frac{1}{d!}(h(t))^d, \tag{2.1.11}$$

where we used the fact that the diameter of a simplex equals the greatest Eucledian distance between two vertices of a simplex.
Henceforward, we assume that $\mathcal{T}_{h(t)}$, $h(t) > 0$, is a regular family of simplicial meshes. By this assumption an arbitrary cell $K(t) \in \mathcal{T}_{h(t)}$ can be connected with the simplex

$$\mathcal{K}_d := \left\{(\xi_1, ..., \xi_d) \in \mathbb{R}^d : \ \forall j \in \{1, ..., d\},\ \xi_j \geq 0 \text{ and } \sum_{j=1}^{d}\xi_j \leq 1\right\}, \tag{2.1.12}$$

because for any simplex exists an affine mapping to the simplex $\mathcal{K}_d$. Hence, we obtain for any cell $K(t) \in \mathcal{T}_{h(t)}$ a time-dependent affine mapping (cf. figure 2.1.2)

$$\chi_{K(t)} : \mathcal{K}_d \to \overline{K(t)}, \quad \xi \mapsto \chi_{K(t)}(\xi, t) := \mathbf{A}_{K(t)}\xi + v_0(t), \tag{2.1.13}$$

where the matrix $\mathbf{A}_{K(t)}$ is given by (2.1.7). Since $\mathbf{A}_{K(t)}$ is a regular matrix, the mapping is for any $t \in [t_n, t_{n+1}]$ bijective. The corresponding inverse is given by

$$\chi^{-1}_{K(t)} : \overline{K(t)} \to \mathcal{K}_d, \quad x \mapsto \chi^{-1}_{K(t)}(x, t) = \mathbf{A}^{-1}_{K(t)}(x - v_0(t)). \tag{2.1.14}$$

Moreover, in any time interval $(t_n, t_{n+1})$ are the straight lines (2.1.2) differentiable with respect to $t$. Thus, the mapping (2.1.13) is also differentiable with respect to $t$. Hence, the grid velocity in the point $x = \chi_{K(t)}(\xi, t)$ can be defined by

$$\omega_{K(t)}(x,t) := \partial_t \left(\chi_{K(t)}(\xi,t)\right). \tag{2.1.15}$$

Further, in the points $x = \chi_{K^n}(\xi, t_n)$ we define the grid velocity by

$$\omega_{K^n}(x,t_n) := \lim_{t \to t_n} \omega_{K(t)}(x,t). \tag{2.1.16}$$

Accordingly, in any arbitrary fixed space-time element $\overline{K(t)} \times [t_n, t_{n+1})$ we have a local description of the grid velocity field. Finally, it is obvious that our method is an ALE method. The grid is static, if the grid velocity is zero. In this case the motion of a fluid is described by the Eulerian representation of motion. On the other hand the motion of the particles in a fluid can be described approximately by the motion of the grid points, if the time step $\triangle t$ is chosen sufficiently small. In this case the motion is described by the Lagrangian representation. Next, we define the global grid velocity as a function $\omega : [0,T] \times \Omega \to \mathbb{R}^d$ with the properties

$$\omega(t)|_{K(t)} = \omega_{K(t)}(t), \ \forall t \in [0,T) \text{ and } K(t) \in \mathcal{T}_{h(t)}, \tag{2.1.17}$$

and

$$\omega(x,T) := \lim_{t \to T} \omega(x,t), \ \forall x \in \Omega. \tag{2.1.18}$$

If it is clear which cell $K(t) \in \mathcal{T}_{h(t)}$ is considered, we will omit the label $|_{K(t)}$ in (2.1.17). Note that the global grid velocity is piecewise constant in time and for all $t \in [0,T]$ is

$$\omega(t) \in \left\{ w \in \left[\mathrm{L}^2(\Omega)\right]^d : \ \forall K(t) \in \mathcal{T}_{h(t)}, \ w|_{K(t)} \in \mathrm{H}(\mathrm{div};\Omega) \right\}, \tag{2.1.19}$$

where the function space $\mathrm{H}(\mathrm{div};\Omega)$ is the set of all $\left[\mathrm{L}^2(\Omega)\right]^d$-vector fields with divergence in $\mathrm{L}^2(\Omega)$. This space is an important tool to analyze PDEs with diffusion. For a more detailed explanation of this function space and applications in the theory of incompressible Navier-Stokes equations, we refer to the books of Di Pietro and Ern [24, p. 16-18] as well as Temam [80, p. 4-15]. In the following, we assume:

(A4) There are constants $c_0, c_1 > 0$, independent of $h(t)$ for any $t \in [0,T]$, such that

$$\max_{(x,t) \in \Omega \times [0,T]} |\omega(x,t)| \le c_0 \quad \text{and} \quad \max_{(x,t) \in \Omega \times [0,T]} |\mathrm{div}(\omega(x,t))| \le c_1. \tag{2.1.20}$$

The differential operator in (2.1.20) has to be understood as a broken divergence operator in the sense of (2.1.19). Finally, we would like to mention that by Piola's identity (cf. Ciarlet [5, p. 461]) and (2.1.15) the divergence operator in (2.1.20) for any point $x = \chi_{K(t)}(\xi,t)$ can be written as follows

$$\mathrm{div}(\omega(x,t)) = \mathrm{div}_\xi \left(\mathbf{A}_{K(t)}^{-1} \left(\omega\left(\chi_{K(t)}(\xi,t), t\right)\right)\right), \tag{2.1.21}$$

where $\mathrm{div}_\xi(\cdot)$ is the divergence operator in the reference cell $\mathrm{int}(\mathcal{K}_d)$.

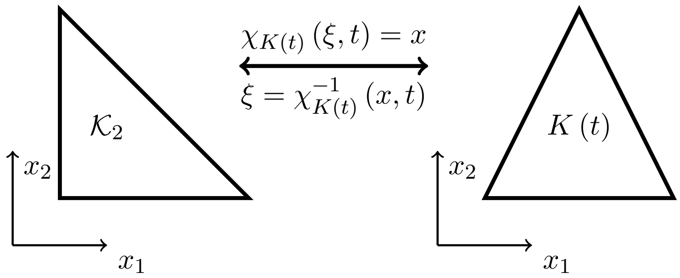


**Figure 2.1.2.** The simplex $\mathcal{K}_2$ mapped to a time-dependent simplex $K(t)$ by a one by one mapping $\chi(\xi,t)$ in two space dimensions.

## 2.2. The test function space for the ALE discontinuous Galerkin methods

In this section, we assume that the assumptions $(A1)$, $(A2)$ and $(A3)$ are satisfied. Thus, by lemma 2.1.1 is $\mathcal{T}_{h(t)}$, $h(t) > 0$ a regular family of simplicial meshes. Moreover, we apply the following notation for any function $v \in \mathrm{L}^2(\Omega)$ restricted to a cell $K(t) \in \mathcal{T}_{h(t)}$

$$v^*|_{K(t)}(\xi) := v\left(\chi_{K(t)}(\xi,t)\right), \quad \xi \in \mathcal{K}_d. \tag{2.2.1}$$

If it is clear which cell $K(t) \in \mathcal{T}_{h(t)}$ is considered, we will omit the label $|_{K(t)}$ in (2.2.1). By this notation, the following finite element test function space can be defined

$$\mathcal{V}_{h,d}(t) := \left\{v \in \mathrm{L}^2(\Omega) :\ v^*|_{K(t)} \in \mathcal{P}^k(\mathcal{K}_d)\ \forall K(t) \in \mathcal{T}_{h(t)}\right\}, \tag{2.2.2}$$

where $\mathcal{P}^k(\mathcal{K}_d)$ denotes the space of polynomials in $\mathcal{K}_d$ of degree at most $k$. We would like to mention that $\mathcal{V}_{h,d}(t)$ is a finite dimensional space, since $\mathcal{P}^k(\mathcal{K}_d)$ has dimension $\frac{(k+d)!}{k!d!}$ (cf. Di Pietro and Ern [24, p. 12]). Furthermore, it should be noted that the index $t$ in the definition (2.2.2) highlights that the test functions are time-dependent. The test function space (2.2.2) can be defined by a non polynomial space, too. A natural choice for other test functions are exponential monomials. This kind of test functions have been analyzed by Yuan and Shu in [89]. In general functions from the space $\mathcal{V}_{h,d}(t)$ are discontinuous along the interface of two adjacent cells. Certainly, it exists in any interface a two value trace for the test functions, since these functions are polynomials in the interior of the cells and the cells have Lipschitz boundaries (cf. Ciarlet [5, p. 334-335]). Therefore, for any function $v \in \mathcal{V}_{h,d}(t)$, two adjacent cells $K_1(t), K_2(t) \in \mathcal{T}_{h(t)}$ and a point $x \in e(t) := \overline{K_1(t)} \cap \overline{K_2(t)}$ the limits

$$v^{\mathrm{int}_{K_1(t)}}(x) := \lim_{\varepsilon\to 0} v\left(x - \varepsilon n_{K_1(t)}\right), \quad v^{\mathrm{ext}_{K_1(t)}}(x) := \lim_{\varepsilon\to 0} v\left(x - \varepsilon n_{K_2(t)}\right)$$

are well defined. The vector $n_{K_1(t)}$ is pointing from $K_1(t)$ to $K_2(t)$ and the vector $n_{K_2(t)}$ is pointing from $K_2(t)$ to $K_1(t)$. Furthermore, the vectors $n_{K_\ell(t)}$, $\ell = 1, 2$, are well defined for the point $x$. Accordingly, the average and jump of $v$ along the edge $e(t)$ are defined by

$$\{\!\{v\}\!\}_{e(t)} := \frac{1}{2}\left(v^{\mathrm{int}_{K_1(t)}} + v^{\mathrm{ext}_{K_1(t)}}\right), \quad [\![v]\!]_{e(t)} := v^{\mathrm{ext}_{K_1(t)}} n_{K_1(t)} + v^{\mathrm{int}_{K_1(t)}} n_{K_2(t)}. \tag{2.2.3}$$

Furthermore, the test functions satisfy the following transport equation.

**Lemma 2.2.1** *Let $u : \Omega \times [0,T] \to \mathbb{R}$ be a sufficiently smooth function in any cell $K(t) \in \mathcal{T}_{h(t)}$. Then for all $K(t) \in \mathcal{T}_{h(t)}$ and for all $v \in \mathcal{V}_{h,d}(t)$ hold the transport equation*

$$\frac{d}{dt}(u,v)_{K(t)} = (\partial_t u, v)_{K(t)} + (\operatorname{div}(\omega u), v)_{K(t)}. \tag{2.2.4}$$

**Proof.** Let $K(t) \in \mathcal{T}_{h(t)}$ be an arbitrary cell. Then the Jacobian matrix of the mapping (2.1.13) is the matrix $\mathbf{A}_{K(t)}$. Lemma 2.1.1 provides that the determinant of this matrix is non-negative, since we assume $(A3)$. Henceforth, the determinant will be denoted by $J_d(t) := \det\left(\mathbf{A}_{K(t)}\right)$. Next, since the function $u : \Omega \times [0,T] \to \mathbb{R}$ is sufficiently smooth in the cell $K(t)$, we obtain by Piola's identity (cf. Ciarlet [5, p. 461])

$$\partial_t u^* = \partial_t u + \omega \cdot \nabla u = \partial_t u + \mathbf{A}_{K(t)}^{-1}\omega \cdot \nabla_\xi u^*, \tag{2.2.5}$$

where $u^*$ is defined by (2.2.1) and $\cdot$ denotes the inner product of two $\mathbb{R}^d$ vectors. Moreover, $\nabla(\cdot)$ denotes the nabla operator in the cell $K_j(t)$ and $\nabla_\xi(\cdot)$ denotes the nabla operator in the reference cell $\operatorname{int}(\mathcal{K}_d)$. Next, by Piola's identity and Jacobi's formula (cf. Magnus and Neudecker [62, p. 200 et seq.]) follows

$$\frac{d}{dt}J_d(t) = \operatorname{div}_\xi\left(\mathbf{A}_{K(t)}^{-1}\omega\right)J_d(t). \tag{2.2.6}$$

Furthermore, for any test function $v \in \mathcal{V}_{h,d}(t)$ is $(u^*, \partial_t v^*)_{\mathcal{K}_d} = 0$, since the test functions are time-independent polynomials on the reference cell. Finally the equations (2.2.5) and (2.2.6) provide

$$\begin{aligned}(\partial_t u, v)_{K(t)} &= \frac{d}{dt}(u^*, v^* J_d(t))_{\mathcal{K}_d} - \left(\operatorname{div}_\xi\left(\left(\mathbf{A}_{K(t)}^{-1}\omega\right)u^*\right), v^* J_d(t)\right)_{\mathcal{K}_d} \\ &= \frac{d}{dt}(u,v)_{K(t)} - (\operatorname{div}(\omega u), v)_{K(t)}. \end{aligned} \tag{2.2.7}$$

□

Note that the transformation to the reference cell in (2.2.7) is not true for test functions, which are polynomials with time-dependent coefficients on the reference cell like the function (1.2.11). Thus, the transport equation (2.2.4) is not true for this class of functions. However, in any Galerkin method for PDEs with time evolution, the trial functions are linear combinations of the test functions and the coefficients of these linear combinations are time-dependent. Hence, for the evaluation of these functions the next auxiliary lemma is helpful.

**Lemma 2.2.2** *Let $\phi_1^*, ..., \phi_s^* \in \mathcal{P}^k(\mathcal{K}_d)$, $s = \frac{(k+d)!}{k!d!}$, be a basis of the space $\mathcal{P}^k(\mathcal{K}_d)$. Then any test function $v \in \mathcal{V}_{h,d}(t)$, restricted to a cell $K(t) \in \mathcal{T}_{h(t)}$, is contained in the linear hull of the functions*

$$\phi_\vartheta(x) := \phi_\vartheta^*\left(\boldsymbol{A}_{K(t)}^{-1}(x - v_0(t))\right), \quad x \in K(t) \quad \textit{and} \quad \vartheta = 1, ..., s. \tag{2.2.8}$$

*In particular, the functions* (2.2.8) *satisfy the equation*

$$\partial_t \phi_\vartheta = -\omega \cdot \nabla \phi_\vartheta. \tag{2.2.9}$$

*Furthermore, is $v(t) \in \mathcal{V}_{h,d}(t)$ for all $t \in [0,T]$ and there are functions $a_\vartheta \in \mathcal{C}^1([0,T])$, $\vartheta = 1, ..., s$, such that $v(t)$, restricted to a cell $K(t) \in \mathcal{T}_{h(t)}$, can be written as $v(t) := \sum_{\vartheta=1}^s a_\vartheta(t)\phi_\vartheta$. Then for any $t \in [0,T]$ the function $\partial_t v(t)$ is an element of the space $\mathcal{V}_{h,d}(t)$.*

**Proof.** The first part of the lemma follows by standard arguments from linear algebra, since for any $K(t) \in \mathcal{T}_{h(t)}$ the mapping (2.1.13) is an affine one by one mapping.
Next, we prove the equation (2.2.9). First of all, we obtain for any cell $K(t) \in \mathcal{T}_{h(t)}$ and any point $x = \chi_{K(t)}(\xi, t)$

$$\begin{aligned}
&\partial_t \left(\mathbf{A}_{K(t)}^{-1}(x - v_0(t))\right) \\
=& -\mathbf{A}_{K(t)}^{-1}\left(\left(\partial_t \mathbf{A}_{K(t)}\right)\mathbf{A}_{K(t)}^{-1}(x - v_0(t)) + \partial_t v_0(t)\right) \\
=& -\mathbf{A}_{K(t)}^{-1}(\omega(x,t)).
\end{aligned} \tag{2.2.10}$$

Hence, (2.2.10) and Piola's identity (cf. Ciarlet [5, p. 461]) supply for all $\vartheta = 1, ..., m$

$$\partial_t \phi_\vartheta = -\omega \cdot \left(\mathbf{A}_{K(t)}^{-1}\right)^T (\nabla_\xi \phi_\vartheta^*) = -\omega \cdot \nabla \phi_\vartheta. \tag{2.2.11}$$

Finally, we obtain by (2.2.11) for any arbitrary fixed $t \in [0,T]$

$$\partial_t v(t) = \sum_{\vartheta=1}^m (\partial_t a_\vartheta(t))\phi_\vartheta - \sum_{\vartheta=1}^m a_\vartheta(t)\,\omega\nabla\phi_\vartheta. \tag{2.2.12}$$

The first sum of the right hand side in equation (2.2.12) is obviously in $\mathcal{V}_{h,d}(t)$, since $a_\vartheta \in \mathcal{C}^1([0,T])$, $\vartheta = 1, ..., m$. The other sum is in $\mathcal{V}_{h,d}(t)$, since the grid velocity is for any cell $K(t) \in \mathcal{T}_{h(t)}$ a linear function with respect to $\xi$ and thus $\omega\nabla\phi_\vartheta \in \mathcal{V}_{h,d}(t)$, $\vartheta = 1, ..., m$. □

### 2.2.1. Inverse and trace inequalities

Lemma 2.1.1 provides that $\mathcal{T}_{h(t)}$, $h(t) > 0$, is a regular family of simplicial meshes, if the assumptions $(A1)$, $(A2)$ and $(A3)$ are satisfied. Moreover, for any cell $K(t) \in \mathcal{T}_{h(t)}$ there

exists by (2.1.13) an affine mapping to the simplex $\mathcal{K}_d$. Therefore, the classical inverse and trace inequalities (cf. e.g. Ciarlet [6, Theorem 3.2.6, p. 140-141] or Di Pietro and Ern [24, p. 27-30]) can be applied for the test function space (2.2.2).
In the following, is

$$\Gamma_{h(t)} := \bigcup_{K(t)\in\mathcal{T}_{h(t)}} \partial K\left(t\right)$$

and the corresponding broken $\mathrm{L}^2\left(\mathit{\Gamma}_{h(t)}\right)$-norm is defined by

$$\|v\|_{\mathit{\Gamma}_{h(t)}} := \left(\sum_{K(t)\in\mathcal{T}_{h(t)}} \sum_{e(t)\subseteq\partial K(t)} \|v\|^2_{\mathrm{L}^2(e(t))}\right)^{\frac{1}{2}}, \tag{2.2.13}$$

where $v \in \mathrm{L}^2\left(\partial K(t)\right)$ for all $K(t) \in \mathcal{T}_{h(t)}$ and the sets $e(t) \subseteq \partial K(t)$ are edges of $K(t)$. Likewise, we define the broken $\mathrm{H}^1\left(\mathcal{T}_{h(t)}\right)$-seminorm by

$$|v|_{\mathrm{H}^1\left(\mathcal{T}_{h(t)}\right)} := \left(\sum_{K(t)\in\mathcal{T}_{h(t)}} \|\nabla v\|^2_{[\mathrm{L}^2(K(t))]^d}\right)^{\frac{1}{2}}, \tag{2.2.14}$$

where $v \in \mathrm{H}^1\left(K(t)\right)$ for all $K(t) \in \mathcal{T}_{h(t)}$. Furthermore, we assume that there exists a constant $\kappa$, independent of $h$, such that for all $t \in [0, T]$

$$h \leq \kappa h\left(t\right), \tag{2.2.15}$$

where $h$ is given by (2.1.5). Then, the classical inverse and trace inequalities as well as (2.2.15) provide the upcoming inequalities.

**Lemma 2.2.3** *Suppose the assumption* (2.2.15) *is satisfied. Then for all* $v \in \mathcal{V}_{h,d}\left(t\right)$*, there exists a non-negative constant* $C$*, independent of* $v$ *and* $h$*, such that*

$$h\,|v|_{\mathrm{H}^1\left(\mathcal{T}_{h(t)}\right)} + h^{\frac{d}{2}}\,\|v\|_{\mathrm{L}^\infty(\Omega)} \leq C\,\|v\|_{\mathrm{L}^2(\Omega)}\,, \tag{2.2.16}$$

*where the quantity* $|\cdot|_{\mathrm{H}^1\left(\mathcal{T}_{h(t)}\right)}$ *is given by* (2.2.14).

Lemma 2.2.3 follows by a more general inverse inequality for regular grids (cf. Ciarlet [6, Theorem 3.2.6, p. 140-141]) combined with the regularity assumption (2.2.15). Therefore, we omit a proof.
Let $K\left(t\right) \in \mathcal{T}_{h(t)}$ be an arbitrary cell. Then for all $p \geq 1$ and a set $e\left(t\right) \in \partial K\left(t\right)$ with $(d-1)$-Lebesgue measure the function space $\mathrm{L}^p\left(e\left(t\right)\right)$ is continuously embedded in $\mathrm{W}^{1,p}\left(K\left(t\right)\right)$, since the cell $K(t)$ has a Lipschitz boundary (cf. Ciarlet [5, p. 334-335]). Therefore, for the mesh $\mathcal{T}_{h(t)}$ we may adopt the upcoming trace inequalities. Moreover, in combination with (2.2.15), we obtain the following estimates.

**Lemma 2.2.4** *Let $v \in \mathcal{V}_{h,d}(t)$ and $u \in \mathrm{H}^1(K(t))$, for all $K(t) \in \mathcal{T}_{h(t)}$. Furthermore, suppose the assumption* (2.2.15) *is satisfied. Then there are non-negative constants $C_{T0}$ and $C_{T1}$, independent of $u$, $v$ and $h$, such that*

$$h^{\frac{1}{2}} \|v\|_{\Gamma_{h(t)}} \leq C_{T0} \|v\|_{\mathrm{L}^2(\Omega)} \tag{2.2.17}$$

*and*

$$\|u\|^2_{\Gamma_{h(t)}} \leq C_{T1} d h^{-1} \|u\|^2_{\mathrm{L}^2(\Omega)} + 2C_{T0} \sum_{K(t)\in\mathcal{T}_{h(t)}} \Big( \|\nabla u\|_{[L^2(K(t))]^d} \|u\|_{\mathrm{L}^2(K(t))} \Big), \tag{2.2.18}$$

*where the quantity $\|\cdot\|_{\Gamma_{h(t)}}$ is given by* (2.2.13).

We omit a proof of lemma 2.2.4, since these inequalities are a consequence of well known results from approximation theory (cf. e.g. Di Pietro and Ern [24, p. 27-30]) in combination with (2.2.15).

### 2.2.2. The $\mathrm{L}^2$-projection and interpolation errors

For any cell $K(t) \in \mathcal{T}_{h(t)}$ the $\mathrm{L}^2$-projection $\mathcal{P}_h(u,t)$ of a function $u \in \mathrm{L}^2(\Omega)$ into the test function space $\mathcal{V}_{h,d}(t)$ is defined by

$$(\mathcal{P}_h(u,t), v)_{K(t)} = (u,v)_{K(t)}, \quad \forall v \in \mathcal{V}_{h,d}(t). \tag{2.2.19}$$

We would like to mention that the index $t$ in the definition of the $\mathrm{L}^2$-projection $\mathcal{P}_h(\cdot,t)$ highlights that the projection maps to a space with time-dependent functions. In addition, lemma 2.2.1, lemma 2.2.2 and the equation (2.2.19) yield for all $v \in \mathcal{V}_{h,d}(t)$

$$\Big(\partial_t (u - \mathcal{P}_h(u,t)), v\Big)_{K(t)} = -\Big(\mathrm{div}\Big(\omega (u - \mathcal{P}_h(u,t)) v\Big), 1\Big)_{K(t)}, \tag{2.2.20}$$

since according to lemma 2.2.1 is $(u - \mathcal{P}_h(u,t), \partial_t v)_{K(t)} = 0$, for all all test functions $v \in \mathcal{V}_{h,d}(t)$, which are polynomials with time-dependent coefficients on the reference cell.
In the section 2.2.1, we justified the utilization of the classical inverse and trace inequalities for the test function space (2.2.2). The same argumentation allows to apply the following classical interpolation error estimates for the $\mathrm{L}^2$-projection.

**Lemma 2.2.5** *Let $u \in \mathrm{H}^{k+1}(\Omega)$. Moreover, suppose the assumption* (2.2.15) *is satisfied. Then there are non-negative constants $C_{P0}$, $C_{P1}$, $C_{P2}$ and $C_{P3}$, such that*

$$\|u - \mathcal{P}_h(u,t)\|_{\Gamma_{h(t)}} \leq C_{P0} h^{k+\frac{1}{2}}, \tag{2.2.21}$$

$$\|u - \mathcal{P}_h(u,t)\|_{\mathrm{L}^2(\Omega)} \leq C_{P1} h^{k+1}, \tag{2.2.22}$$

$$\|u - \mathcal{P}_h(u,t)\|_{\mathrm{L}^\infty(\Omega)} \leq C_{P2} h^{k+1-\frac{d}{2}} \tag{2.2.23}$$

*and*

$$\sum_{K(t)\in\mathcal{T}_{h(t)}} \|\nabla (u - \mathcal{P}_h (u,t))\|_{[\mathrm{L}^2(K(t))]^d} \leq C_{P3} h^k, \tag{2.2.24}$$

*where* $h$ *is given by* (2.1.5) *and the quantity* $\|\cdot\|_{\Gamma_{h(t)}}$ *is given by* (2.2.13). *The constants are depending on* $u$*, but they are independent of* $h$.

**Proof.** The estimates (2.2.22) and (2.2.24) follow by a more general result (cf. Ciarlet [6, Theorem 3.1.6, p. 124-125]) in combination with (2.2.15), the projection theorem for inner product spaces (cf. Ciarlet [5, Theorem 4.3-1, p. 183-184]) and the inequality (2.2.16). Therefore, we omit a proof for (2.2.22) and (2.2.24). Likewise, the estimate (2.2.23) follows from Ciarlet's result in combination with the estimates (2.1.10), (2.1.11) for the $d$-Lebesgue measure of a cell $K(t) \in \mathcal{T}_{h(t)}$ as well as the regularity assumption (2.2.15). Furthermore, the estimates (2.2.18) and (2.2.22) provide (2.2.21). □

## 2.3. The one dimensional setup

In this section, we present concepts which were published by Klingenberg et al. in [48].
In the one dimensional case is the domain $\Omega \subseteq \mathbb{R}$ an open interval and a partition at time level $t_n$, $n = 0, ..., L$, can be characterized by a point set $\left\{x^n_{j-\frac{1}{2}}\right\}_{j=1}^{N}$. The points yield for any $n = 0, ..., L$ and $j = 1, ..., N$ cells

$$K^n_j := \left(x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}}\right) \quad \text{and} \quad \triangle^n_j := x^n_{j+\frac{1}{2}} - x^n_{j-\frac{1}{2}}. \tag{2.3.1}$$

In the following, we assume that for any $n = 0, ..., L$ the cells (2.3.1) provide a partition of the domain $\Omega$ with the properties:

(P1) The cells $K^n_j$, $j = 1, ..., N$, cover exactly the domain $\Omega$ such that $\overline{\Omega} = \bigcup_{j=1}^{N} \overline{K^\ell_j}$.

(P2) The meshsize of the partition is given by $h_n := \max_{1\leq j\leq N} \triangle^n_j$ and there exists a constant $\tau_n$, independent of $h_n$, such that for all $j = 1, ..., N$

$$0 < \triangle^n_j \leq h_n \leq \tau_n \triangle^n_j. \tag{2.3.2}$$

It should be noted that a partition of a one dimensional domain is shape-regular, if the inequality (2.3.2) is satisfied. In order to prove some properties of the ALE-DG method for scalar conservation laws in chapter 3, we introduce the following more restrictive regularity hypothesis:

(P2a) There exists a constant $\tau$ for all $L, N \in \mathbb{N}$ such that

$$0 < \triangle_j^n \leq h_n \leq \tau \triangle_j^n, \quad \text{for all } n = 0, ..., L \text{ and } j = 1, ..., N. \tag{2.3.3}$$

In particular, the constant $\tau$ is independent of $L$, $N$ and the meshsize $h_n$, $n = 0, ..., L$.

We would like to mention that a partition of the domain $\Omega$ with the property (P2a) satisfies the property (P2), too.

Henceforth, we assume that the point sets $\left\{x_{j-\frac{1}{2}}^n\right\}_{j=1}^N$ and $\left\{x_{j-\frac{1}{2}}^{n+1}\right\}_{j=1}^N$ provide partitions of $\Omega$ with the properties (P1) and (P2) at the time levels $t_n$ as well as $t_{n+1}$. Then, for any $j = 1, ..., N$, we can connect the points at time level $t_n$ and $t_{n+1}$ by the following time-dependent straight lines

$$x_{j-\frac{1}{2}}(t) := x_{j-\frac{1}{2}}^n + \omega_{j-\frac{1}{2}}^n (t - t_n), \quad \omega_{j-\frac{1}{2}}^n := \frac{x_{j-\frac{1}{2}}^{n+1} - x_{j-\frac{1}{2}}^n}{\triangle t}. \tag{2.3.4}$$

The straight lines (2.3.4) provide for any $t \in [t_n, t_{n+1}]$ and all $j = 1, ..., N$ time-dependent cells

$$K_j(t) := \left(x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)\right). \tag{2.3.5}$$

Note the equivalence to the definition of the cells (2.1.3). The length as well as maximal length of a cell (2.3.5) are denoted by

$$\triangle_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t), \quad \text{and} \quad h(t) := \max_{1 \leq j \leq N} \triangle_j(t).$$

The global length $h$ of the cells (2.3.5) is defined by (2.1.5). Next, we prove that the time-dependent cells provide a partition of the domain $\Omega$ with the properties (P1) and (P2).

**Lemma 2.3.1** *Suppose the point sets at time level $t_n$ as well as $t_{n+1}$ provide a partition of the domain $\Omega$ with the properties (P1) and (P2). Then the time-dependent cells* (2.3.5) *provide for any $t \in [t_n, t_{n+1}]$ a partition of the domain $\Omega$ with the properties (P1) and (P2).*

**Proof.** The time-dependent cells (2.3.5) satisfy $K_j^n = K_j(t_n)$ and $K_j^{n+1} = K_j(t_{n+1})$. Hence, in the following, it is enough to consider an arbitrary fixed point $t \in (t_n, t_{n+1})$. First of all, by the condition (P2) follows

$$\triangle_j(t) = \left(1 - \frac{t - t_n}{\triangle t}\right) \triangle_j^n + \left(\frac{t - t_n}{\triangle t}\right) \triangle_j^{n+1} > 0. \tag{2.3.6}$$

The inequality (2.3.6) ensures that the mesh topology does not change in time. Thus, for all $j = 2, ..., N$ is

$$\overline{K_{j-1}(t)} \cap \overline{K_j(t)} = x_{j-\frac{1}{2}}(t).$$

Furthermore, by the condition (P1) and convexity arguments follows $\overline{\Omega} = \bigcup_{j=1}^{N} \overline{K_j(t)}$. Next, the inequalities (2.3.2) and (2.3.6) supply the shape-regularity

$$h(t) \leq \left(1 - \frac{t - t_n}{\triangle t}\right) h_n + \left(\frac{t - t_n}{\triangle t}\right) h_{n+1} \leq \max(\tau_n, \tau_{n+1}) \triangle_j(t). \tag{2.3.7}$$

□

Since for any $t \in [t_n, t_{n+1}]$ and $j = 1, ..., N$ the conditions (P1) and (P2) provide the inequality (2.3.6), the cells $K_j(t)$ can be connected with a reference cell $[0, 1]$ by an affine one by one mapping

$$\chi_j : [0, 1] \to \overline{K_j(t)}, \quad \xi \mapsto \chi_j(\xi, t) := \triangle_j(t)\,\xi + x_{j-\frac{1}{2}}(t). \tag{2.3.8}$$

The corresponding inverse of (2.3.8) is the mapping

$$\chi_j^{-1} : \overline{K_j(t)} \to [0, 1], \quad x \mapsto \chi_j^{-1}(x, t) = \frac{x - x_{j-\frac{1}{2}}(t)}{\triangle_j(t)}. \tag{2.3.9}$$

Therefore, for any $t \in (t_n, t_{n+1})$ the local grid velocity in the point $x = \chi_j(\xi, t)$ is defined by

$$\omega_{K_j(t)}(x, t) = \partial_t \chi_j(\xi, t) \tag{2.3.10}$$

and in the points $x = \chi_j(\xi, t_n)$ by

$$\omega_{K_j^n}(x, t_n) := \lim_{t \to t_n} \omega_{K_j(t)}(x, t). \tag{2.3.11}$$

Note that by (2.3.4), (2.3.8), (2.3.9), (2.3.10) and (2.3.11) for all $t \in [t_n, t_{n+1})$ and $x \in K_j(t)$ the local grid velocity is given by

$$\omega(x, t) = \left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\left(\frac{x - x_{j-\frac{1}{2}}(t)}{\triangle_j(t)}\right) + \omega_{j-\frac{1}{2}}^n. \tag{2.3.12}$$

In addition, by (2.1.17) and (2.1.18) a global grid velocity function $\omega : \Omega \times [0, T] \to \mathbb{R}$ can be defined. Next, we define by the mapping (2.3.8) a finite dimensional test function space

$$\mathcal{V}_h(t) := \left\{ v \in \mathrm{L}^2(\Omega) \mid v^*|_{K_j(t)} \in \mathcal{P}^k([0, 1]),\ \forall j = 1, \cdots, N \right\}, \tag{2.3.13}$$

where $\mathcal{P}^k([0, 1])$ denotes the space of polynomials in $[0, 1]$ of degree at most $k$. Note that the function $v^*|_{K_j(t)}$ is defined by (2.2.1) and the value $t$ in the definition of (2.3.13) highlights that the test functions are time-dependent.
In the one dimensional case the intersection of two cells $\overline{K_{j-1}(t)}$ and $\overline{K_j(t)}$ contain merely the point $x_{j-\frac{1}{2}}(t)$. For these points we denote the left as well as right limit of a test function $v \in \mathcal{V}_h(t)$ by

$$v_{j-\frac{1}{2}}^{\pm} := \lim_{\varepsilon \to 0} v\left(x_{j-\frac{1}{2}}(t) \pm \varepsilon, t\right). \tag{2.3.14}$$

Therefore, the average and the jump of a test function are denoted by

$$\{\!\{v\}\!\}_{j-\frac{1}{2}} := \frac{1}{2}\left(v^+_{j-\frac{1}{2}} + v^-_{j-\frac{1}{2}}\right) \quad \text{and} \quad [\![v]\!]_{j-\frac{1}{2}} := v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}. \tag{2.3.15}$$

We would like to mention that the whole one dimensional setting presented in this section is just a special case of the more general multidimensional setting presented in the sections 2.1 and 2.2. In particular, the test function space $\mathcal{V}_h(t)$ corresponds to the space $\mathcal{V}_{h,1}(t)$ given by (2.2.2). Therefore, the transport equations presented in lemma 2.2.1 and lemma 2.2.2 hold for the test function space (2.3.13), too. Likewise, lemma 2.2.3 and lemma 2.2.4 hold for the test function space (2.3.13), where it should be noted that in one dimension surface integrals reduce to pointwise evaluations and thus the broken $\mathrm{L}^2(\varGamma_h)$-norm of a function $v \in \mathcal{V}_h(t)$ is defined by

$$\|v\|_{\varGamma_{h(t)}} := \left(\sum_{j=1}^{N} \left|v^+_{j-\frac{1}{2}}\right|^2 + \left|v^-_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}. \tag{2.3.16}$$

Moreover, for the cells (2.3.5) a one dimensional $\mathrm{L}^2$-projection can be defined similar to (2.2.19). Likewise, we consider Gauss-Radau projections beside the $\mathrm{L}^2$-projection. For $k \geq 1$, we define the Gauss-Radau projections $\mathcal{P}_h^{\pm}(u,t)$ of a function $u \in \mathrm{L}^2(\Omega)$ into $\mathcal{V}_h(t)$ for all $v \in \mathcal{V}_h(t)$ with $v^*|_{K_j(t)} \in \mathcal{P}^{k-1}([0,1])$ by

$$\left(\mathcal{P}_h^{\pm}(u,t), v\right)_{K_j(t)} = (u,v)_{K_j(t)} \tag{2.3.17}$$

and

$$\mathcal{P}_h^{+}(u,t)\left(x^+_{j-\frac{1}{2}}(t)\right) := u\left(x^+_{j-\frac{1}{2}}(t)\right), \quad \mathcal{P}_h^{-}(u,t)\left(x^-_{j+\frac{1}{2}}(t)\right) := u\left(x^-_{j+\frac{1}{2}}(t)\right), \tag{2.3.18}$$

where the function $v^*|_{K_j(t)}$ is defined by (2.2.1). Note that the index $t$ in the definition of the Gauss-Radau projections $\mathcal{P}_h^{\pm}(\cdot,t)$ highlights that the projections map to a time-dependent function space. For the one dimensional $\mathrm{L}^2$-projection and the Gauss-Radau projections hold the following interpolation error estimates.

**Lemma 2.3.2** *Let $u \in \mathrm{H}^{k+1}(\Omega)$ and $\mathcal{Q}_h(\cdot,t)$ be either the $\mathrm{L}^2$-projection or one of the Gauss-Radau projections. Furthermore, suppose the condition* (2.2.15). *Then there are non-negative constants $C_{R0}$, $C_{R1}$, $C_{R2}$ and $C_{R3}$, such that*

$$\|u - \mathcal{Q}_h(u,t)\|_{\varGamma_{h(t)}} \leq C_{R0} h^{k+\frac{1}{2}}, \tag{2.3.19}$$

$$\|u - \mathcal{Q}_h(u,t)\|_{\mathrm{L}^2(\Omega)} \leq C_{R1} h^{k+1}, \tag{2.3.20}$$

$$\|u - \mathcal{Q}_h(u,t)\|_{\mathrm{L}^\infty(\Omega)} \leq C_{R2} h^{k+\frac{1}{2}} \tag{2.3.21}$$

*and*

$$\sum_{j=1}^{N} \left\|\partial_x \left(u - \mathcal{Q}_h\left(u,t\right)\right)\right\|_{\mathrm{L}^2(K_j(t))} \leq C_{R3} h^k, \tag{2.3.22}$$

*where $h$ is given by* (2.1.5) *and the quantity $\|\cdot\|_{\Gamma_{h(t)}}$ is given by* (2.3.16)*. The constants are depending on $u$, but they are independent of $h$.*

**Proof.** For the one dimensional $\mathrm{L}^2$-projection, the inequalities in lemma 2.3.2 can be proven similar to the inequalities in lemma 2.2.5. For the Gauss-Radau projections, we utilize that the error $\mathcal{P}_h\left(u,t\right) - \mathcal{P}^{\pm}{}_h\left(u,t\right)$ behaves as $\mathcal{O}\left(h^{k+1}\right)$ in the $\mathrm{L}^2$-norm. A proof of this error estimate is given in the appendix of the thesis (cf. inequality (A.2.3)). Hence, the interpolation estimates for the Gauss-Radau projections in lemma 2.3.2 follow from the interpolation estimates for the $\mathrm{L}^2$-projection in lemma 2.2.5, the inverse inequality (2.2.16) and the trace inequality (2.2.17). □

For the Gauss-Radau projections, the equation (2.2.20) is not true. Nevertheless, Klingenberg et al. proved in [48] the following auxiliary lemma to evaluate the time derivative of these projections.

**Lemma 2.3.3** *Let $u : \Omega \times [0,T] \to \mathbb{R}$ be a sufficiently smooth function and $\mathcal{Q}_h\left(\cdot,t\right)$ be either $\mathcal{P}_h\left(\cdot,t\right)$ or $\mathcal{P}_h^{\pm}\left(\cdot,t\right)$. Then for any cell $K_j(t)$, $j = 1, ..., N$, given by* (2.3.5) *holds*

$$\begin{aligned} &\partial_t \mathcal{Q}_h(u\left(t\right),t) + \omega\left(t\right)\partial_x \mathcal{Q}_h(u\left(t\right),t) \\ =&\mathcal{Q}_h\left(\partial_t(u\left(t\right)),t\right) + \mathcal{Q}_h\left(\omega\left(t\right)\partial_x(u\left(t\right)),t\right). \end{aligned} \tag{2.3.23}$$

**Proof.** The proof based on Legendre polynomials to characterize the projections $\mathcal{P}_h\left(\cdot,t\right)$ as well as $\mathcal{P}_h^{\pm}\left(\cdot,t\right)$ in the cells (2.3.5). We denote the Legendre polynomials by $L_\vartheta$, $\vartheta = 0, \cdots, k$, where $\vartheta$ is the degree of the polynomials. They can be expressed by Rodrigues' formula (cf. Hochstrasser [38, p. 785]) and have the properties

$$\left(L_\vartheta, L_{\vartheta'}\right)_{[-1,1]} = \frac{2}{2\vartheta+1}\delta_{\vartheta\vartheta'}, \quad L_\vartheta(-1) = (-1)^\vartheta \quad \text{and} \quad L_\vartheta(1) = 1,$$

where $\delta_{\vartheta\vartheta'}$ denotes the Kronecker delta. Furthermore, the Legendre polynomials are an orthogonal basis of the space $\mathcal{P}^k\left([-1,1]\right)$. Therefore, the functions

$$\mathcal{L}_\vartheta\left(x,t\right) := L_\vartheta\left(\frac{2\left(x - x_{j-\frac{1}{2}}\left(t\right)\right)}{\triangle_j\left(t\right)} - 1\right), \quad \forall x \in K_j\left(t\right), \tag{2.3.24}$$

represent an orthogonal basis of the test function space $\mathcal{V}_h\left(t\right)$ in any cell $K_j\left(t\right)$, $j = 1, ..., N$. Thus, in $K_j\left(t\right)$, the $\mathrm{L}^2$-projection of a function $v \in \mathrm{L}^2\left(\Omega\right)$ can be written as

$$\mathcal{P}_h\left(v,t\right) = \sum_{\vartheta=0}^{k} c_{\vartheta,j}\left(v,t\right)\mathcal{L}_\vartheta\left(x,t\right), \quad c_{\vartheta,j}\left(v,t\right) = \left(\frac{2\vartheta+1}{\triangle_j\left(t\right)}\right)\left(v,\mathcal{L}_\vartheta\right)_{K_j(t)}.$$

Likewise, in $K_j(t)$ the Gauss-Radau projections of a function $v \in \mathrm{L}^2(\Omega)$ can be written as

$$\mathcal{P}_h^{\pm}(v,t) = \sum_{\vartheta=0}^{k} r_{\vartheta,j}^{\pm}(v,t)\,\mathcal{L}_{\vartheta}(x,t), \tag{2.3.25}$$

where the coefficients are given by

$$r_{\vartheta,j}^{\pm}(v,t) = \left(\frac{2\vartheta+1}{\triangle_j(t)}\right)(v,\mathcal{L}_{\vartheta})_{K_j(t)}, \quad \vartheta = 0,\cdots,k-1, \tag{2.3.26}$$

$$r_{k,j}^{+}(v,t) = (-1)^k\, v\left(x_{j-\frac{1}{2}}^{+}(t),t\right) - \sum_{\vartheta=0}^{k-1}(-1)^{k+\vartheta}\, r_{\vartheta,j}^{+}(v,t) \tag{2.3.27}$$

and

$$r_{k,j}^{-}(v,t) := v\left(x_{j+\frac{1}{2}}^{-}(t),t\right) - \sum_{\vartheta=0}^{k-1} r_{\vartheta,j}^{-}(v,t). \tag{2.3.28}$$

Let $u:\Omega\times[0,T]\to\mathbb{R}$ be a sufficiently smooth function in the space-time element $\overline{K_j(t)}\times(t_n,t_{n+1})$. Then we obtain by (2.3.4)

$$\frac{d}{dt}\left(u\left(x_{j\mp\frac{1}{2}}^{\pm}(t),t\right)\right) = \partial_t\left(u\left(x_{j\mp\frac{1}{2}}^{\pm}(t),t\right)\right) + \omega_{j\mp\frac{1}{2}}^{n}\partial_x\left(u\left(x_{j\mp\frac{1}{2}}^{\pm}(t),t\right)\right). \tag{2.3.29}$$

Next, by (2.3.4), (2.3.8) and (2.3.10) follows

$$\frac{\triangle_j^{n+1}-\triangle_j^{n}}{\triangle t} = \omega_{j+\frac{1}{2}}^{n} - \omega_{j-\frac{1}{2}}^{n} \quad \text{and} \quad \partial_x\omega(t) = \frac{\triangle_j'(t)}{\triangle_j(t)}, \tag{2.3.30}$$

where $\triangle_j^n$, $\triangle_j^{n+1}$ are given by (2.3.1) and $\triangle_j'(t)$ is the time derivative of the cell length $\triangle_j(t)$. Note that (2.3.4) provides

$$\triangle_j'(t) = \omega_{j+\frac{1}{2}}^{n} - \omega_{j-\frac{1}{2}}^{n}. \tag{2.3.31}$$

Thus, by (2.3.31) follows

$$\frac{d}{dt}\left(\frac{2\vartheta+1}{\triangle_j(t)}\right) = \frac{-(2\vartheta+1)\triangle_j'(t)}{(\triangle_j(t))^2} = -\left(\frac{2\vartheta+1}{\triangle_j(t)^2}\right)\left(\omega_{j+\frac{1}{2}}^{n} - \omega_{j-\frac{1}{2}}^{n}\right). \tag{2.3.32}$$

Therefore, we obtain by the transport equation (2.2.4), (2.3.30), (2.3.31) and (2.3.32)

$$\begin{aligned}\frac{d}{dt}\left(\frac{2\vartheta+1}{\triangle_j(t)}(u(t),\mathcal{L}_{\vartheta})_{K_j(t)}\right) &= \frac{2\vartheta+1}{\triangle_j(t)}\left(\partial_t(u(t)) + \partial_x(\omega(t)u(t)),\mathcal{L}_{\vartheta}\right)_{K_j(t)} \\ &\quad - \left(\frac{2\vartheta+1}{\triangle_j(t)}\right)\left(\frac{\omega_{j+\frac{1}{2}}^{n} - \omega_{j-\frac{1}{2}}^{n}}{\triangle_j(t)}\right)(u(t),\mathcal{L}_{\vartheta})_{K_j(t)} \\ &= \frac{2\vartheta+1}{\triangle_j(t)}\left(\partial_t(u(t)) + \omega(t)\,\partial_x(u(t)),\mathcal{L}_{\vartheta}\right)_{K_j(t)}.\end{aligned} \tag{2.3.33}$$

Hence, we obtain by (2.3.29) and (2.3.33) for the time derivatives of the coefficients of the projections $\mathcal{P}_h(u,t)$ and $\mathcal{P}_h^{\pm}(u,t)$

$$\partial_t \left(c_{\vartheta,j}\left(u\left(t\right),t\right)\right) = c_{\vartheta,j}\left(\partial_t\left(u\left(t\right)\right),t\right) + c_{\vartheta,j}\left(\omega\left(t\right)\partial_x\left(u\left(t\right)\right),t\right) \tag{2.3.34}$$

and

$$\partial_t \left(r^{\pm}_{\vartheta,j}\left(u\left(t\right),t\right)\right) = r^{\pm}_{\vartheta,j}\left(\partial_t\left(u\left(t\right)\right),t\right) + r^{\pm}_{\vartheta,j}\left(\omega\left(t\right)\partial_x\left(u\left(t\right)\right),t\right). \tag{2.3.35}$$

Let $\mathcal{Q}_h(u(t),t)$ be either $\mathcal{P}_h(u(t),t)$ or $\mathcal{P}_h^{\pm}(u(t),t)$. Then, in the cell $K_j(t)$, the projection $\mathcal{Q}_h(u(t),t)$ can be written as

$$\mathcal{Q}_h(u(t),t) = \sum_{\vartheta=0}^{k} q_{\vartheta,j}(u(t),t)\,\mathcal{L}_\vartheta(x,t),$$

where the coefficients $q_{\vartheta,j}(u(t),t)$ are $c_{\vartheta,j}(u(t),t)$ or $r^{\pm}_{\vartheta,j}(u(t),t)$. Therefore, we obtain by (2.3.34), (2.3.35) and (2.2.9)

$$\begin{aligned}\partial_t\left(\mathcal{Q}_h(u(t),t)\right) =& \sum_{\vartheta=0}^{k}\partial_t\left(q_\vartheta\left(u\left(t\right),t\right)\right)\mathcal{L}_\vartheta(x,t)\\ &-\sum_{\vartheta=0}^{k} q_\vartheta\left(u\left(t\right),t\right)\omega(x,t)\,\partial_x\left(\mathcal{L}_\vartheta(x,t)\right)\\ =&\mathcal{Q}_h\left(\partial_t\left(u\left(t\right)\right),t\right)+\mathcal{Q}_h\left(\omega\left(t\right)\partial_x\left(u\left(t\right)\right),t\right)\\ &-\omega\left(t\right)\partial_x\left(\mathcal{Q}_h\left(u\left(t\right),t\right)\right)\end{aligned}$$

□

# 3. An arbitrary Lagrangian-Eulerian discontinuous Galerkin (ALE-DG) method for scalar conservation laws

In this chapter, we introduce and analyze an ALE-DG method for solving the initial value problem (1.1.1). The content of this chapter based on concepts which were published by Klingenberg et al. in [48]. However, some of the results have been enhanced since then.

## 3.1. The ALE-DG method

In the following, a semi-discrete ALE-DG method will be presented, while we apply the time-dependent cells (2.3.5). Furthermore, the geometric conservation law (GCL) for the ALE-DG method will be discussed.

### 3.1.1. The one dimensional semi-discrete ALE-DG discretization

For the time interval $[t_n, t_{n+1}]$ and a cell $K_j(t)$, $j = 1, ..., N$, the description of the method ensued as follows. We approximate for all $t \in [t_n, t_{n+1})$ the solution $u$ of the problem (1.1.1) by the function $u_h \in \mathcal{V}_h(t)$ given by

$$u_h(x,t) = \sum_{\ell=0}^{k} u_\ell^j(t)\,\phi_\ell^j(x,t), \quad \text{for all } t \in [t_n, t_{n+1}) \text{ and } x \in K_j(t), \tag{3.1.1}$$

where $\left\{\phi_0^j(x,t), ..., \phi_k^j(x,t)\right\}$ is a basis of the space $\mathcal{V}_h(t)$ in the cell $K_j(t)$. For convenience we will skip the variables $t \in [t_n, t_{n+1})$ and $x \in K_j(t)$ to describe the function (3.1.1). The coefficients $u_0^j(t)$,...,$u_k^j(t)$ in (3.1.1) are the unknowns of the method. In order to determine these coefficients, we multiply the equation (1.1.1a) by a test function $v \in \mathcal{V}_h(t)$ and apply the transport equation (2.2.4) as well as the integration by parts formula. Hence, we obtain

$$\begin{aligned}\frac{d}{dt}(u_h, v)_{K_j(t)} = &(g(\omega, u_h), \partial_x v)_{K_j(t)} - g\left(\omega_{j+\frac{1}{2}}^n, u_h\left(x_{j+\frac{1}{2}}(t), t\right)\right) v_{j+\frac{1}{2}}^- \\ &+ g\left(\omega_{j-\frac{1}{2}}^n, u_h\left(x_{j-\frac{1}{2}}(t), t\right)\right) v_{j-\frac{1}{2}}^+,\end{aligned}$$

where the function $g(\omega, \cdot)$ is given by

$$g(\omega, u) := f(u) - \omega u. \tag{3.1.2}$$

In general the function $u_h$ is discontinuous in the cell interface points $x_{j-\frac{1}{2}}(t)$. Hence, we replace the flux $g(\omega^n_{j-\frac{1}{2}}, u_h(x_{j-\frac{1}{2}}(t), t))$ by a monotone numerical flux, which is a single valued function defined in the cell interface points and depends on $u^-_{h,j-\frac{1}{2}}$ in the second argument and $u^+_{h,j-\frac{1}{2}}$ in the third argument. Moreover, the numerical flux should have the properties:

($\widehat{g}1$) Consistence:
For any smooth function holds the identity $\widehat{g}(\omega, u, u) = g(\omega, u)$.

($\widehat{g}2$) Monotonicity:
The numerical flux function $\widehat{g}(\omega, \cdot, \cdot)$ is increasing in the second argument and decreasing in the third argument.

($\widehat{g}3$) Lipschitz continuity:
For all $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$, it holds the inequality

$$|\widehat{g}(\omega, a_1, b_1) - \widehat{g}(\omega, a_2, b_2)| \leq L^-_{\widehat{g}} |a_1 - a_2| + L^+_{\widehat{g}} |b_1 - b_2|, \tag{3.1.3}$$

where the Lipschitz constants $L^-_{\widehat{g}}$ and $L^+_{\widehat{g}}$ are independent of the global length $h$ given by (2.1.5).

The choice of a suitable numerical flux function completes the derivation of the semi-discrete ALE-DG method and the time-dependent coefficients of the function (3.1.1) are determined by the following problem: Seek a function $u_h \in \mathcal{V}_h(t)$, such that for all $v \in \mathcal{V}_h(t)$ and $j = 1, \cdots, N$ holds

$$\begin{aligned}\frac{d}{dt}(u_h, v)_{K_j(t)} = &(g(\omega, u_h), \partial_x v)_{K_j(t)} - \widehat{g}\left(\omega^n_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} \\ &+ \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}}.\end{aligned} \tag{3.1.4}$$

Furthermore, by the transport equation (2.2.4) the equation (3.1.4) can be rewritten. Hence, (3.1.4) is for all $v \in \mathcal{V}_h(t)$ and $j = 1, \cdots, N$ equivalent to

$$\begin{aligned}(\partial_t u_h, v)_{K_j(t)} = &(f(u_h), \partial_x v)_{K_j(t)} - (\partial_x(\omega u_h v), 1)_{K_j(t)} \\ &- \widehat{g}\left(\omega^n_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} + \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}}.\end{aligned} \tag{3.1.5}$$

Next, we present examples of numerical fluxes, which are satisfying the properties ($\widehat{g}1$), ($\widehat{g}2$) and ($\widehat{g}3$) in a point $x_{j-\frac{1}{2}}(t)$. In order to present these fluxes, we use the notation $u^-_h := u^-_{h,j-\frac{1}{2}}$

and $u_h^+ := u_{h,j-\frac{1}{2}}^+$.
*The upwind numerical flux*:

$$\widehat{g}^{\mathrm{U}}\left(\omega_{j-\frac{1}{2}}^n, u_h^-, u_h^+\right) := \begin{cases} g\left(\omega_{j-\frac{1}{2}}^n, u_h^-\right), & \text{if } \partial_u g\left(\omega_{j-\frac{1}{2}}^n, u\right) \geq 0,\ \forall u \in \left[u_h^-.u_h^+\right], \\ g\left(\omega_{j-\frac{1}{2}}^n, u_h^+\right), & \text{if } \partial_u g\left(\omega_{j-\frac{1}{2}}^n, u\right) < 0,\ \forall u \in \left[u_h^-.u_h^+\right]. \end{cases} \tag{3.1.6}$$

*The Godunov flux*:

$$\widehat{g}^{\mathrm{G}}\left(\omega_{j-\frac{1}{2}}^n, u_h^-, u_h^+\right) := \begin{cases} \min\left\{g\left(\omega_{j-\frac{1}{2}}^n, u\right) :\ u_h^- \leq u \leq u_h^+\right\}, & \text{if } u_h^- \leq u_h^+, \\ \max\left\{g\left(\omega_{j-\frac{1}{2}}^n, u\right) :\ u_h^+ \leq u \leq u_h^-\right\}, & \text{if } u_h^- > u_h^+. \end{cases} \tag{3.1.7}$$

*The Engquist-Osher flux*:

$$\begin{aligned} \widehat{g}^{\mathrm{EO}}\left(\omega_{j-\frac{1}{2}}^n, u_h^-, u_h^+\right) &:= \int_0^{u_h^+} \min\left\{\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u\right), 0\right\} du \\ &\quad + \int_0^{u_h^-} \max\left\{\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u\right), 0\right\} du + g\left(\omega_{j-\frac{1}{2}}^n, 0\right). \end{aligned} \tag{3.1.8}$$

*The Lax-Friedrichs flux*:

$$\widehat{g}^{\mathrm{LF}}\left(\omega_{j-\frac{1}{2}}^n, u_h^-, u_h^+\right) := \frac{1}{2}\left(g\left(\omega_{j-\frac{1}{2}}^n, u_h^-\right) + g\left(\omega_{j-\frac{1}{2}}^n, u_h^+\right)\right) - \frac{\lambda}{2}\left(u_h^+ - u_h^-\right), \tag{3.1.9}$$

where

$$\lambda := \max\left\{\left|\partial_u g\left(\omega\left(x,t\right), u\right)\right| :\ u \in [m, M],\ x \in \Omega\right\} \tag{3.1.10}$$

with $m$ and $M$ given by (1.2.20). Note that the grid velocity $\omega = \omega\left(x,t\right)$ in (3.1.10) has to be understood as the global grid velocity given by (2.1.17) and (2.1.18).

### 3.1.2. The geometric conservation law (GCL) for the ALE-DG method

If we consider the equation (1.1.1) with constant initial data $u_0 \in \mathbb{R}$, is $u = u_0$ the unique entropy solution of (1.1.1). Therefore, the GCL is satisfied for the ALE-DG method, if the method preserves constant states. Let $u_h = 1$ be the approximate solution of the method. Then $\partial_t u_h = 0$ and $\partial_x u_h = 0$. Thus, by (2.3.30) and (2.3.31) the equation (3.1.4) degenerates for all $v \in \mathcal{V}_h(t)$ and $j = 1, \ldots, N$ to

$$\frac{d}{dt}\left(1, v\right)_{K_j(t)} = \frac{\triangle_j'\left(t\right)}{\triangle_j\left(t\right)}\left(1, v\right)_{K_j(t)}. \tag{3.1.11}$$

This equation is the semi-discrete GCL condition for the ALE-DG method (3.1.4). Next, we transform the semi-discrete GCL condition (3.1.11) on the reference cell and obtain

$$\left(\frac{d}{dt}J\left(t\right)-\triangle_j'\left(t\right)\right)\left(1,v^*\right)_{(0,1)}=0, \tag{3.1.12}$$

where $J(t)=\Delta_j(t)$ is the Jacobian determinant of the mapping (2.3.8) and the functions $u_h^*$ as well as $v^*$ are given by (2.2.1). Note that the quantity $\Delta_j'(t)$ is time-independent by (2.3.31). Furthermore, $J(t)$ depends linear on $t$ and is independent of $x$. Thus, the ODE in (3.1.12) is satisfied for any first order time discretization method, e.g. the forward Euler step or a high order single step method. In particular, (3.1.12) is satisfied for any SSP-RK method, which has been presented in the section 1.2.1. Hence, we proved the following result.

**Proposition 3.1.1** *The fully discrete ALE-DG method* (3.1.4) *satisfies the discrete geometric conservation law for any first order time discretization method or high order single step method in which the stage order is equal or higher than first order.*

## 3.2. The non-linear stability of the fully-discrete method

It would be desirable that the fully-discrete ALE-DG method has the properties i) - v), which are listed in section 1.2. However, we cannot expect that the method (3.1.4) satisfies all these conditions without a stabilization term or a post process procedure. Even the RK-DG for static grids is not stable without a modification of the method. In this section, we adapted the post process procedures, which were briefly discussed in section 1.2.2, to the fully-discrete one dimensional ALE-DG method. First of all, we consider the forward Euler step of the ALE-DG method (3.1.4) and we state conditions for the TVDM and TVBM stability. These conditions are necessary to construct slope limiters for the fully-discrete ALE-DG method. Next, we discuss the stability of the ALE-DG method (3.1.4) discretized by an arbitrary high order SSP-RK time integration method and justify that X. Zhang and Shu's limiter (1.2.19) can be used for the ALE-DG method. This limiter provides a local maximum principle for the method. Finally, we consider the entropy stability of the fully-discrete ALE-DG method. The high order accuracy of the ALE-DG method for smooth solutions will be discussed merely for the one dimensional semi-discrete method (3.1.4) in the section 3.3.

### 3.2.1. The forward Euler step

We consider the forward Euler step of the ALE-DG method (3.1.4) with the Lax-Friedrichs flux given by (3.1.9) and (3.1.10). For any point $x^n_{j-\frac{1}{2}}$, $j=1,...,n$, the Lax-Friedrichs flux can be written as an increasing function

$$\widehat{g}_+\left(\omega^n_{j-\frac{1}{2}},u_h^-\right):=\frac{1}{2}\left(g\left(\omega^n_{j-\frac{1}{2}},u_h^-\right)+\lambda u_h^-\right) \tag{3.2.1}$$

and a decreasing function

$$\widehat{g}_-\left(\omega^n_{j-\frac{1}{2}}, u_h^+\right) := \frac{1}{2}\left(g\left(\omega^n_{j-\frac{1}{2}}, u_h^+\right) - \lambda u_h^+\right), \tag{3.2.2}$$

where $u_h^\pm := u^\pm_{h,j-\frac{1}{2}}$. Then, it follows

$$\widehat{g}\left(\omega^n_{j+\frac{1}{2}}, u_h^-, u_h^+\right) - \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u_h^-, u_h^+\right) = -\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)\frac{u_h^- + u_h^+}{2}. \tag{3.2.3}$$

Next, we define the local mean value of the ALE-DG solution $u_h$ at time level $t_n$ for the cell $K_j^n = K_j(t_n)$ by

$$\overline{u}_j^n := \frac{1}{\triangle_j^n}(u_h(t_n), 1)_{K_j^n}, \tag{3.2.4}$$

where $\triangle_j^n$ and $K_j^n$ are given by (2.3.1). Moreover, the forward as well as backward differential operators of the local mean value (3.2.4) are denoted by

$$\triangle_+\overline{u}_j^n := \overline{u}_{j+1}^n - \overline{u}_j^n \quad \text{and} \quad \triangle_-\overline{u}_j^n := \overline{u}_j^n - \overline{u}_{j-1}^n. \tag{3.2.5}$$

It follows by (2.3.30)

$$\triangle_j^{n+1}\overline{u}_j^{n+1} - \triangle_j^n\overline{u}_j^n = \triangle_j^{n+1}\left(\overline{u}_j^{n+1} - \overline{u}_j^n\right) + \triangle t\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)\overline{u}_j^n. \tag{3.2.6}$$

Therefore, by (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.6) the forward Euler step of the weak formulation (3.1.4) with a Lax-Friedrichs flux and the test function $v = 1$ can be written as follows

$$\begin{aligned}\overline{u}_j^{n+1} = \overline{u}_j^n &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j+\frac{1}{2}}\right) - \widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j-\frac{1}{2}}\right)\right)\\ &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, u^{n,-}_{h,j+\frac{1}{2}}\right) - \widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, u^{n,-}_{h,j-\frac{1}{2}}\right)\right)\\ &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)\left(\overline{u}_j^n - \frac{1}{2}\left(u^{n,-}_{h,j+\frac{1}{2}} + u^{n,+}_{h,j-\frac{1}{2}}\right)\right),\end{aligned} \tag{3.2.7}$$

where $u^{n,\pm}_{h,j-\frac{1}{2}} = \lim_{\varepsilon\to 0} u_h\left(x^n_{j-\frac{1}{2}} \pm \varepsilon, t_n\right)$. Note that the scheme (3.2.7) is of a similar structure as a classical three point finite difference scheme. It is well know that a TVD or TVB stable finite difference scheme is not producing spurious oscillations (cf. e.g. Harten [37]). Hence, we analyze the scheme (3.2.7) for TVD or TVB stability. Then, the forward Euler step of the ALE-DG method (3.1.4) becomes TVDM or TVBM stable. This means the ALE-DG solution $u_h^n = u_h(t_n)$ is stable in the sense of the seminorm

$$|u_h^n|_{\text{TVM}} := \sum_{j=1}^N \left|\triangle_+\overline{u}_j^n\right|. \tag{3.2.8}$$

In order to obtain the TVD or TVB stability of the scheme (3.2.7), we follow Cockburn's discussion in [7]. Therefore, we apply for all $v, w \in \mathbb{R}$ the notation

$$\eta(v,w) := \operatorname{sign}(v) - \operatorname{sign}(w). \tag{3.2.9}$$

First of all, we subtract the equation (3.2.7) for $j$ multiplied by $\operatorname{sign}\left(\triangle_+ \overline{u}_j^{n+1}\right)$ from the equations (3.2.7) for $j+1$ multiplied by $\operatorname{sign}\left(\triangle_+ \overline{u}_j^{n+1}\right)$. Then we sum the result from $j =$ to $N$ and obtain

$$\left|u_h^{n+1}\right|_{\mathrm{TVM}} - \left|u_h^n\right|_{\mathrm{TVM}} + \Theta + \Xi_\omega = 0. \tag{3.2.10}$$

The quantity $\Theta$ in (3.2.10) is given by

$$\begin{aligned}\Theta := &\sum_{j=1}^{N}\left(p\left(\overline{u}_{j+1}^n, u_{h,j+\frac{3}{2}}^{n,-}, u_{h,j+\frac{1}{2}}^{n,+}\right) - p\left(\overline{u}_j^n, u_{h,j+\frac{1}{2}}^{n,-}, u_{h,j-\frac{1}{2}}^{n,+}\right)\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,-}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,-}\right)\right)\eta\left(\triangle_-\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &-\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,+}\right)\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_-\overline{u}_j^{n+1}\right),\end{aligned} \tag{3.2.11}$$

where for all piecewise continuous functions $v, w \in \mathrm{L}^2(\Omega)$

$$p\left(v, w^-, w^+\right) := v - \frac{\triangle t}{\triangle_{j+1}^{n+1}}\widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, w^-\right) + \frac{\triangle t}{\triangle_j^{n+1}}\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, w^+\right).$$

The other quantity $\Xi_\omega$ in equation (3.2.10) results from the grid velocity. It is given by

$$\begin{aligned}\Xi_\omega := &\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_{j+1}^{n+1}}a_{j+1,\mp}\left(\omega_{j+\frac{3}{2}}^n - \omega_{j+\frac{1}{2}}^n\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}b_{j,\mp}\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\eta\left(\triangle_+\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right)\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right)c_{j,\pm}\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right)d_{j,\pm},\end{aligned} \tag{3.2.12}$$

where

$$\begin{aligned}a_{j,-} &:= -\left(\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}\right), \quad b_{j,-} := -\left(u_{h,j+\frac{1}{2}}^{n,-} - \overline{u}_j^n\right),\\ c_{j,+} &:= \eta\left(\triangle_+\overline{u}_j^n, \triangle_-\overline{u}_j^{n+1}\right), \quad d_{j,+} := \eta\left(\triangle_-\overline{u}_j^n, \triangle_+\overline{u}_j^{n+1}\right),\end{aligned}$$

if $\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right) \geq 0$ and

$$\begin{aligned} a_{j,+} &:= u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j, \quad b_{j,+} := \overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}, \\ c_{j,-} &:= -\eta\left(\triangle_-\overline{u}^n_j, \triangle_+\overline{u}^{n+1}_j\right), \quad d_{j,-} := -\eta\left(\triangle_+\overline{u}^n_j, \triangle_-\overline{u}^{n+1}_j\right), \end{aligned}$$

if $\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right) \leq 0$. Hence, the ALE-DG method is total variation diminishing stable in the mean values (TVDM stable), if we can ensure that $\Theta + \Xi_\omega \geq 0$. In fact the sum $\Theta + \Xi_\omega$ is positive, if the ALE-DG solution satisfies the following conditions:

$$\operatorname{sign}\left(\triangle_+\overline{u}^n_j\right) = \operatorname{sign}\left(r_{j,\mp} - s_{j,\mp}\right), \tag{3.2.13}$$

$$\operatorname{sign}\left(\triangle_-\overline{u}^n_j\right) = \operatorname{sign}\left(u^{n,-}_{h,j+\frac{1}{2}} - u^{n,-}_{h,j-\frac{1}{2}}\right), \tag{3.2.14}$$

$$\operatorname{sign}\left(\triangle_+\overline{u}^n_j\right) = \operatorname{sign}\left(u^{n,+}_{h,j+\frac{1}{2}} - u^{n,+}_{h,j-\frac{1}{2}}\right), \tag{3.2.15}$$

where

$$r_{j,\mp} := p\left(\overline{u}^n_{j+1}, u^{n,-}_{h,j+\frac{3}{2}}, u^{n,+}_{h,j+\frac{1}{2}}\right) + \frac{1}{2}\frac{\triangle t}{\triangle^{n+1}_{j+1}}\left(\omega^n_{j+\frac{3}{2}} - \omega^n_{j+\frac{1}{2}}\right) a_{j+1,\mp}$$

and

$$s_{j,\mp} := p\left(\overline{u}^n_j, u^{n,-}_{h,j+\frac{1}{2}}, u^{n,+}_{h,j-\frac{1}{2}}\right) - \frac{1}{2}\frac{\triangle t}{\triangle^{n+1}_j}\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right) b_{j,\mp}.$$

In addition,

$$\operatorname{sign}\left(\triangle_+\overline{u}^n_j\right) = \operatorname{sign}\left(u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j\right) \tag{3.2.16}$$

$$\operatorname{sign}\left(\triangle_-\overline{u}^n_j\right) = \operatorname{sign}\left(\overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}\right), \tag{3.2.17}$$

if $\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}} \geq 0$ and

$$\operatorname{sign}\left(\triangle_-\overline{u}^n_j\right) = \operatorname{sign}\left(u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j\right) \tag{3.2.18}$$

$$\operatorname{sign}\left(\triangle_+\overline{u}^n_j\right) = \operatorname{sign}\left(\overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}\right), \tag{3.2.19}$$

if $\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}} \leq 0$. In general the ALE-DG solution does not satisfy the conditions (3.2.13)-(3.2.19). The solution has to be revised by a limiter. In [11] Cockburn and Shu have developed TVD limiters for the RK-DG method for static grids. These limiters ensure that the revised RK-DG solution satisfies (1.2.15) as well as (1.2.16) and the method becomes TVDM stable. In the following, we prove that the forward Euler step of the ALE-DG method (3.1.4) is also stable, if the solution satisfies (1.2.15) and (1.2.16). This means that Cockburn and Shu's TVD limiters ensure the TVDM stability of the forward Euler step of the ALE-DG method.

**Lemma 3.2.1** *Let $u_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ and $u_h$ be the solution of the forward Euler step of the ALE-DG method* (3.1.4) *with a Lax-Friedrichs flux. The solution $u_h$ satisfies the conditions* (1.2.15) *and* (1.2.16)*. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$, the grid velocity satisfies the condition $(A4)$, $h_{n+1} \in (0,1)$ and it holds the CFL condition*

$$\frac{\triangle t}{h_{n+1}} \leq \frac{1}{\tau_{n+1}(c_1 + 4\lambda)}, \tag{3.2.20}$$

*where $h_{n+1}$ as well as $\tau_{n+1}$ are given by the condition $(P2)$, the parameter $\lambda$ is given by* (3.1.10) *and the constant $c_1$ is given by the condition* (2.1.20)*. Then, it holds for all $n = 0, ..., L$*

$$\left|u_h^{n+1}\right|_{TVM} \leq |u_0|_{\mathrm{BV}(\Omega)}.$$

**Proof.** First of all, we prove that $u_h$ satisfies the conditions (3.2.13)-(3.2.19). Then the sum $\Theta + \Xi_\omega$ is non-negative and thus the ALE-DG method becomes TVDM stable. Obviously, the conditions (3.2.16)-(3.2.19) are fulfilled, since we assume that the ALE-DG solution satisfies (1.2.15) and (1.2.16). Next, we notice that

$$u^{n,-}_{h,j+\frac{1}{2}} - u^{n,-}_{h,j-\frac{1}{2}} = u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j + \triangle_- \overline{u}^n_j - \left(u^{n,-}_{h,j-\frac{1}{2}} - \overline{u}^n_{j-1}\right) \tag{3.2.21}$$

and

$$u^{n,+}_{h,j+\frac{1}{2}} - u^{n,+}_{h,j-\frac{1}{2}} = -\left(\overline{u}^n_{j+1} - u^{n,+}_{h,j+\frac{1}{2}}\right) + \triangle_+ \overline{u}^n_j + \overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}. \tag{3.2.22}$$

By (1.2.15) is $u^{n,-}_{h,j+\frac{1}{2}} - u^{n,-}_{h,j-\frac{1}{2}} = 0$, if $\triangle_- \overline{u}^n_j = 0$. On the other hand follows by the definition of the minmod function and (1.2.15)

$$0 \leq \left(1 - \frac{u^{n,-}_{h,j-\frac{1}{2}} - \overline{u}^n_{j-1}}{\triangle_- \overline{u}^n_j} + \frac{u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j}{\triangle_- \overline{u}^n_j}\right) \leq 2, \tag{3.2.23}$$

if $\triangle_- \overline{u}^n_j \neq 0$. Hence, (3.2.14) is satisfied, since (3.2.21) and (3.2.23) provide

$$u^{n,-}_{h,j+\frac{1}{2}} - u^{n,-}_{h,j-\frac{1}{2}} = \left(1 - \frac{u^{n,-}_{h,j-\frac{1}{2}} - \overline{u}^n_{j-1}}{\triangle_- \overline{u}^n_j} + \frac{u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j}{\triangle_- \overline{u}^n_j}\right) \triangle_- \overline{u}^n_j. \tag{3.2.24}$$

Likewise, it follows $u^{n,+}_{h,j+\frac{1}{2}} - u^{n,+}_{h,j-\frac{1}{2}} = 0$, if $\triangle_+ \overline{u}^n_j = 0$. Otherwise, by the definition of the minmod function and (1.2.16) follows

$$0 \leq \left(1 - \frac{\overline{u}^n_{j+1} - u^{n,+}_{h,j+\frac{1}{2}}}{\triangle_+ \overline{u}^n_j} + \frac{\overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}}{\triangle_+ \overline{u}^n_j}\right) \leq 2, \tag{3.2.25}$$

if $\triangle_+\overline{u}_j^n \neq 0$. Next, by (3.2.22) and (3.2.25) follows

$$u_{h,j+\frac{1}{2}}^{n,+} - u_{h,j-\frac{1}{2}}^{n,+} = \left(1 + \frac{\overline{u}_j^n - u_{h,j-\frac{1}{2}}^{n,+}}{\triangle_+\overline{u}_j^n} - \frac{\overline{u}_{j+1}^n - u_{h,j+\frac{1}{2}}^{n,+}}{\triangle_+\overline{u}_j^n}\right)\triangle_+\overline{u}_j^n. \tag{3.2.26}$$

This equation verified the condition (3.2.15).
Thus, the condition (3.2.13) remains to prove. This proof is less straightforward. Note that $r_{j,\mp} - s_{j,\mp} = 0$, if $\triangle_+\overline{u}_j^n = 0$. Hence, we consider merely the case $\triangle_+\overline{u}_j^n \neq 0$. First of all, we define the quantity

$$\begin{aligned} q_{j,\mp} :=& \frac{\triangle t}{\triangle_{j+1}^{n+1}}\left(\widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{3}{2}}^{n,-}\right) - \widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,-}\right)\right) \\ &- \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,+}\right)\right) \\ &- \frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left(\omega_{j+\frac{3}{2}}^n - \omega_{j+\frac{1}{2}}^n\right)a_{j+1,\mp} \\ &- \frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)b_{j,\mp}. \end{aligned} \tag{3.2.27}$$

In the following, we prove that the absolute value of the quantity (3.2.27) can be estimated by $\left|\triangle_+\overline{u}_j^n\right|$. Afterward, we apply the following implication of the axioms for ordered fields:

$$\text{For all } a, b \in \mathbb{R} \text{ with } |a| \geq |b| \text{ follows } \operatorname{sign}(a) = \operatorname{sign}(a-b). \tag{3.2.28}$$

Since we assume that $u_h$ satisfies the equations (1.2.15) as well as (1.2.16) and $\triangle_-\overline{u}_{j+1}^n = \triangle_+\overline{u}_j^n$, it follows by the mean value theorem, the condition $(P2)$ and (3.2.23), (3.2.24), (3.2.25) as well as (3.2.26)

$$\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{3}{2}}^{n,-}\right) - \widehat{g}_+\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,-}\right)\right| \leq \frac{2\lambda\tau_{n+1}\triangle t}{h_{n+1}}\left|\triangle_+\overline{u}_j^n\right| \tag{3.2.29}$$

and

$$\frac{\triangle t}{\triangle_j^{n+1}}\left|\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,+}\right)\right| \leq \frac{2\lambda\tau_{n+1}\triangle t}{h_{n+1}}\left|\triangle_+\overline{u}_j^n\right|, \tag{3.2.30}$$

where $h_{n+1}$ as well as $\tau_{n+1}$ are given by $(P2)$. Furthermore, the definition of the minmod function and the equations (1.2.15) and (1.2.16) yield

$$|a_{j+1,\mp}| \leq \left|\triangle_+\overline{u}_j^n\right| \quad \text{and} \quad |b_{j,\mp}| \leq \left|\triangle_+\overline{u}_j^n\right|.$$

In addition, we obtain by (2.3.30), (2.3.31) and the condition $(A4)$ of the grid velocity

$$\left|\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right| \le \max_{x\in K_j^{n+1}} \left|\partial_x\left(\omega\left(x,t_{n+1}\right)\right)\right| \triangle_j^{n+1} \le c_1 h_{n+1} \le c_1, \tag{3.2.31}$$

since $h_{n+1} \in (0,1)$. Hence, by the condition $(P2)$ and the estimate (3.2.31) follows

$$\frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\left(\omega^n_{j+\frac{3}{2}} - \omega^n_{j+\frac{1}{2}}\right)a_{j+1,\mp}\right| \le \frac{c_1\tau_{n+1}\triangle t}{2h_{n+1}}\left|\triangle_+\overline{u}_j^n\right| \tag{3.2.32}$$

and

$$\frac{1}{2}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)b_{j,\mp}\right| \le \frac{c_1\tau_{n+1}\triangle t}{2h_{n+1}}\left|\triangle_+\overline{u}_j^n\right|, \tag{3.2.33}$$

since the grid velocity satisfies the condition $(A4)$. Thus, the CFL condition (3.2.20) and the estimates (3.2.29), (3.2.30), (3.2.32) as well as (3.2.33) provide

$$\begin{aligned}
\left|q_{j,\mp}\right| \le & \frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\widehat{g}_+\left(\omega^n_{j+\frac{1}{2}}, u^{n,-}_{h,j+\frac{3}{2}}\right) - \widehat{g}_+\left(\omega^n_{j+\frac{1}{2}}, u^{n,-}_{h,j+\frac{1}{2}}\right)\right| \\
& + \frac{\triangle t}{\triangle_{j}^{n+1}}\left|\widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j+\frac{1}{2}}\right) - \widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j-\frac{1}{2}}\right)\right| \\
& + \frac{1}{2}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\left(\omega^n_{j+\frac{3}{2}} - \omega^n_{j+\frac{1}{2}}\right)a_{j+1,\mp}\right| \\
& + \frac{1}{2}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\left(\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}\right)b_{j,\mp}\right| \\
& \le \tau_{n+1}\left(4\lambda + c_1\right)\frac{\triangle t}{h_{n+1}} \le \left|\triangle_+\overline{u}_j^n\right|.
\end{aligned} \tag{3.2.34}$$

Therefore, we obtain by (3.2.28)

$$\operatorname{sign}\left(\triangle_+\overline{u}_j^n\right) = \operatorname{sign}\left(\triangle_+\overline{u}_j^n - q_{j,\mp}\right) = \operatorname{sign}\left(r_{j,\mp} - s_{j,\mp}\right).$$

Hence, the sum $\Theta + \Xi_\omega$ is non-negative and we obtain by (3.2.10)

$$\left|u_h^{n+1}\right|_{\text{TVM}} \le \left|u_h^n\right|_{\text{TVM}}. \tag{3.2.35}$$

Thence, by applying successive the inequality (3.2.35) we conclude

$$\left|u_h^n\right|_{\text{TVM}} \le \left|u_h^0\right|_{\text{TVM}} \le \left|u_0\right|_{\text{BV}(\Omega)},$$

since the function $u_h^0$ is the $\mathrm{L}^2$-projection of the function $u_0 \in \mathrm{BV}(\Omega)$. □

Next, we prove that the forward Euler step of the ALE-DG method is TVBM stable, if there exists a constant $\widetilde{M}$, such that the solution $u_h$ satisfies the conditions

$$\left|u^{-}_{h,j+\frac{1}{2}}(t_n) - \overline{u}^n_j\right| \leq \widetilde{M}\left(\triangle^n_j\right)^2 \quad \text{and} \quad \left|\overline{u}^n_j - u^{+}_{h,j-\frac{1}{2}}(t_n)\right| \leq \widetilde{M}\left(\triangle^n_j\right)^2, \tag{3.2.36}$$

for all cells $K^n_j$, $j = 1, ..., N$, given by (2.3.1).

**Lemma 3.2.2** *Let* $u_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ *and* $u_h$ *be the solution of the forward Euler step of the ALE-DG method* (3.1.4) *with a Lax-Friedrichs flux. The solution* $u_h$ *satisfies the conditions* (3.2.36). *Furthermore, for any time level* $t = t_n$, $n = 0, ..., L$, *there exists a partition of the domain* $\Omega$ *with the properties* $(P1)$ *as well* $(P2a)$, *the grid velocity satisfies the condition* $(A4)$, $h_{n+1} \in (0, 1)$ *with* $h_n \leq h_{n+1}$ *and it holds the CFL condition*

$$\frac{\triangle t}{h_{n+1}} \leq \frac{1}{\tau(c_1 + 4\lambda)}, \tag{3.2.37}$$

*where* $h_n$ *as well as* $h_{n+1}$ *are given by* $(P2)$, $\tau$ *is given by* $(P2a)$, *the parameter* $\lambda$ *is given by* (3.1.10) *and the constant* $c_1$ *is given by the condition* (2.1.20). *Then, it holds for all* $n = 0, \cdots, L$

$$|u^n_h|_{TVM} \leq |u_0|_{\mathrm{BV}(\Omega)} + (4c_1 + 16\lambda)\widetilde{M}|\Omega|\tau T,$$

*where* $|\Omega|$ *denotes the Lebesgue measure of the set* $\Omega$, $T$ *denotes the final point of the time interval* $[0, T]$ *and the constant* $\widetilde{M}$ *is given by the condition* (3.2.36).

**Proof.** The function $\eta = \eta(a, b)$, $a, b \in \mathbb{R}$, given by (3.2.9) can be written as follows

$$\eta(a, b) = \begin{cases} 2, & \text{if } a > 0 \text{ and } b < 0, \\ -2, & \text{if } a < 0 \text{ and } b > 0, \\ 0, & \text{else.} \end{cases} \tag{3.2.38}$$

Next, we obtain by (3.2.21), (3.2.22) as well as (3.2.36) for all cells $K^n_j$

$$\begin{aligned} &\widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, u^{n,-}_{h,j+\frac{1}{2}}\right) - \widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, u^{n,-}_{h,j-\frac{1}{2}}\right) \\ \geq& -\lambda\left(\left|u^{n,-}_{h,j+\frac{1}{2}} - \overline{u}^n_j\right| + \left|\triangle_+\overline{u}^n_{j-1}\right| + \left|u^{n,-}_{h,j-\frac{1}{2}} - \overline{u}^n_{j-1}\right|\right) \\ \geq& -2\lambda\widetilde{M}\left(\triangle^n_j\right)^2 - \lambda\left|\triangle_+\overline{u}^n_{j-1}\right| \end{aligned} \tag{3.2.39}$$

and

$$\begin{aligned} &\widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j+\frac{1}{2}}\right) - \widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, u^{n,+}_{h,j-\frac{1}{2}}\right) \\ \geq& -\lambda\left(\left|\overline{u}^n_{j+1} - u^{n,+}_{h,j+\frac{1}{2}}\right| + \left|\triangle_+\overline{u}^n_j\right| + \left|\overline{u}^n_j - u^{n,+}_{h,j-\frac{1}{2}}\right|\right) \\ \geq& -2\lambda\widetilde{M}\left(\triangle^n_j\right)^2 - \lambda\left|\triangle_+\overline{u}^n_j\right|. \end{aligned} \tag{3.2.40}$$

The condition $(P2a)$ and the CFL condition (3.2.37) provide the inequality

$$\begin{aligned} &\sum_{j=1}^{N}\left|\triangle_{+}\overline{u}_{j}^{n}\right| - \lambda\sum_{j=1}^{N}\frac{\triangle t}{\triangle_{j+1}^{n+1}}\left|\triangle_{+}\overline{u}_{j}^{n}\right| \\ &-2\lambda\sum_{j=1}^{N}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\triangle_{+}\overline{u}_{j}^{n}\right| - \lambda\sum_{j=1}^{N}\frac{\triangle t}{\triangle_{j}^{n+1}}\left|\triangle_{+}\overline{u}_{j-1}^{n}\right| \\ &\geq\left(1-4\lambda\tau\frac{\triangle t}{h_{n+1}}\right)\sum_{j=1}^{N}\left|\triangle_{+}\overline{u}_{j}^{n}\right|\geq 0, \end{aligned} \tag{3.2.41}$$

since we consider the initial value problem (1.1.1) with periodic boundary conditions. Therefore, since we assume $h_n \leq h_{n+1}$, we obtain by the condition $(P2)$, (3.2.38), (3.2.39), (3.2.40) and (3.2.41) for the quantity $\Theta$ given by (3.2.11) the estimate

$$\begin{aligned} \Theta \geq& 2\sum_{j=1}^{N}\left|\triangle_{+}\overline{u}_{j}^{n}\right| - 8\lambda\tau\frac{\triangle t}{h_{n+1}}\sum_{j=1}^{N}\triangle_{+}\overline{u}_{j}^{n} \\ &-16\lambda\tau\frac{\triangle t}{h_{n+1}}\widetilde{M}\sum_{j=1}^{N}\left(\triangle_{j}^{n}\right)^{2} \\ \geq& -16\lambda\tau\triangle t\widetilde{M}\left|\Omega\right|\frac{h_n}{h_{n+1}} \\ \geq& -16\lambda\tau\widetilde{M}\left|\Omega\right|\triangle t. \end{aligned} \tag{3.2.42}$$

The conditions (3.2.36) supply $a_{j,\mp} \geq -\widetilde{M}\left(\triangle_{j}^{n}\right)^{2}$ and $b_{j,\mp} \geq -\widetilde{M}\left(\triangle_{j}^{n}\right)^{2}$, for all cells $K_j^n$. Hence, by the condition $(P2)$, the estimate (3.2.31), the conditions (3.2.36) and (3.2.38) follows for the quantity $\Xi_\omega$ given by (3.2.12)

$$\begin{aligned} \Xi_\omega \geq& -4c_1\tau\frac{\triangle t}{h_{n+1}}\widetilde{M}\sum_{j=1}^{N}\left(\triangle_{j}^{n}\right)^{2} \\ \geq& -4c_1\tau\triangle t\widetilde{M}\left|\Omega\right|\frac{h_n}{h_{n+1}} \\ \geq& -4c_1\tau\widetilde{M}\left|\Omega\right|\triangle t, \end{aligned} \tag{3.2.43}$$

since the grid velocity satisfies the condition $(A4)$, $h_{n+1} \in (0,1)$ and we assume $h_n \leq h_{n+1}$. Thus, we obtain by (3.2.10), (3.2.42) and (3.2.43)

$$\begin{aligned} 0 &= \left|u_h^{n+1}\right|_{\text{TVM}} - \left|u_h^{n}\right|_{\text{TVM}} + \Theta + \Xi_\omega \\ &\geq \left|u_h^{n+1}\right|_{\text{TVM}} - \left|u_h^{n}\right|_{\text{TVM}} - \left(16\lambda + 4c_1\right)\tau\widetilde{M}\left|\Omega\right|\triangle t. \end{aligned} \tag{3.2.44}$$

Hence, it follows for all $n = 0, ..., L$ by (1.2.1) and applying successively the inequality (3.2.44)

$$\begin{aligned}\left|u_h^{n+1}\right|_{\mathrm{TVM}} &\leq \left|u_h^0\right|_{\mathrm{TVM}} + L\left(16\lambda + 4c_1\right)\tau\widetilde{M}\left|\Omega\right|\triangle t\\ &\leq \left|u_0\right|_{\mathrm{BV}} + \left(16\lambda + 4c_1\right)\tau\widetilde{M}\left|\Omega\right|T,\end{aligned}$$

since $u_h^0$ is the $\mathrm{L}^2$-projection of the function $u_0 \in \mathrm{BV}\left(\Omega\right)$. □

In the section 1.2, we mentioned that TVD schemes are at most second order accurate at local extrema. In order to overcome this issue, Cockburn and Shu introduced in [11, Lemma 2.2] the modified minmod function (1.2.18) and proved that this function does not affect the accuracy of the RK-DG method. For the forward Euler step of the ALE-DG method (3.1.4) we define for all $(\alpha_1, ..., \alpha_s) \in \mathbb{R}^s$ the following modified minmod function

$$\widetilde{m}\left(\alpha_1, \cdots, \alpha_s\right) := \begin{cases}\alpha_1, & \text{if } \left|\alpha_1\right| \leq \widetilde{M}\left(\triangle_j^n\right)^2,\\ m\left(\alpha_1, \cdots, \alpha_s\right), & \text{else,}\end{cases} \tag{3.2.45}$$

where $\triangle_j^n$ is given by (2.3.1) and $\widetilde{M}$ is the same parameter as in the modified minmod function (1.2.18). Selection options for the parameter $\widetilde{M}$ have been discussed by Cockburn and Shu in [11, Lemma 2.2]. Note that by the same arguments as in Cockburn and Shu's proof [11, Lemma 2.2] it can be proven that the modified minmod function (3.2.45) does not affect the accuracy of the ALE-DG method.

Apparently, lemma 3.2.1 and lemma 3.2.2 ensure that the forward Euler step of the ALE-DG method (3.1.4) becomes TVBM stable, if the solution satisfies the conditions (1.2.15) and (1.2.16) for the modified minmod function (3.2.45). Therefore, the TVD slope limiters, which are presented in Cockburn's lecture notes [7, p. 179-180], could be used with the modified minmod function (3.2.45) instead of the minmod function (1.2.17) to ensure the TVBM stability without affecting the high order accuracy of the forward Euler step of the ALE-DG method (3.1.4). These inferences can be summarized as the following result.

**Proposition 3.2.1** *Let* $u_0 \in \mathrm{L}^\infty\left(\Omega\right) \cap \mathrm{BV}\left(\Omega\right)$ *and* $u_h$ *be the solution of the forward Euler step of the ALE-DG method* (3.1.4) *with a Lax-Friedrichs flux. The solution* $u_h$ *is revised by a TVB limiter, such that the conditions* (1.2.15) *and* (1.2.16) *are satisfied for the modified minmod function* (3.2.45) *instead of the classical minmod function* (1.2.17)*. Then under the same assumptions as in lemma 3.2.2 the ALE-DG method* (3.1.4) *is TVBM stable and for smooth solutions of the initial value problem* (1.1.1) *is* $u_h$ *a* $(k+1)$ *order accurate approximation, if the test function space* (2.3.13) *is defined by piecewise polynomials of degree* $k$.

### 3.2.2. Higher order time discretization

A method with the forward Euler step as time integration method is merely first order accurate in time. Hence, in order to obtain a high order accurate fully-discrete ALE-DG method, we apply the SSP-RK methods (1.2.5) presented in section 1.2.1. In this section, we discuss the stability of the ALE-DG method (3.1.4) discretized by an arbitrary SSP-RK method (1.2.5). Therefore, we define for all $k = 1, ..., s$, $\ell = 0, ..., k-1$ and $j = 1, ..., N$

$$t_{n,\ell} := t_n + \delta_\ell \triangle, \quad \triangle_j^{n,\ell} = \triangle_j \left(t_{n,\ell}\right) \tag{3.2.46}$$

and

$$u_{h,j-\frac{1}{2}}^{n,\ell,\pm} = \lim_{\varepsilon \to 0} u_h \left(x_{j-\frac{1}{2}} \left(t_{n,\ell}\right) \pm \varepsilon, t_{n,\ell}\right), \quad \overline{u}_j^{n,\ell} := \frac{1}{\triangle_j^{n,\ell}} \left(u_h(t_{n,\ell}), 1\right)_{K_j^{n,\ell}}, \tag{3.2.47}$$

where $K_j^{n,\ell} = K_j \left(t_{n,\ell}\right)$. Note that for all $k = 1, ..., s$, $\ell = 0, ..., k-1$ and $j = 1, ..., N$ follows

$$\triangle_j^{n,k} - \triangle_j^{n,\ell} = \left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right) \left(\delta_k - \delta_\ell\right) \triangle t. \tag{3.2.48}$$

Therefore, by (1.2.5), (3.2.1), (3.2.2), (3.2.3), (3.2.4), (3.2.47) and (3.2.48) the SSP-RK discretization of the weak ALE-DG formulation (3.1.4) with a Lax-Friedrichs flux and the test function $v = 1$ can be written as follows

$$\begin{cases} \overline{u}_j^{n,0} & = \overline{u}_j^n; \\ \text{for } k & = 1, ..., s \text{ compute the intermediate functions :} \\ \overline{u}_j^{n,k} & = \sum_{\ell=0}^{k-1} \alpha_{k\ell} w_j^{k,\ell}; \\ \overline{u}_j^{n+1} & = \overline{u}_j^{n,s}, \end{cases} \tag{3.2.49}$$

where for all $k = 1, ..., s$ and $\ell = 0, ..., k-1$ the coefficients satisfy the conditions (1.2.6) and

$$\begin{aligned} w_j^{k,\ell} = & \overline{u}_j^{n,\ell} - \frac{\beta_{k\ell}}{\alpha_{k\ell}} \frac{\triangle t}{\triangle_j^{n+1}} \left(\widehat{g}_- \left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,\ell,+}\right) - \widehat{g}_- \left(\omega_{j+\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,\ell,+}\right)\right) \\ & - \frac{\beta_{k\ell}}{\alpha_{k\ell}} \frac{\triangle t}{\triangle_j^{n+1}} \left(\widehat{g}_+ \left(\omega_{j-\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,\ell,-}\right) - \widehat{g}_+ \left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,\ell,-}\right)\right) \\ & + \frac{1}{2} \frac{\beta_{k\ell}}{\alpha_{k\ell}} \frac{\triangle t}{\triangle_j^{n+1}} \left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right) \left(u_{h,j+\frac{1}{2}}^{n,\ell,-} + u_{h,j-\frac{1}{2}}^{n,\ell,+}\right) \\ & - \left(\delta_k - \delta_\ell\right) \frac{\triangle t}{\triangle_j^{n+1}} \left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right) \overline{u}_j^{n,\ell}. \end{aligned} \tag{3.2.50}$$

It should be noted that by the conditions (1.2.6) the intermediate functions are convex combinations of schemes of the type (3.2.7), if the coefficients of the scheme (3.2.49) additionally have the properties

$$\beta_{k\ell} \geq 0 \quad \text{and} \quad \frac{\beta_{k\ell}}{\alpha_{k\ell}} = \delta_k - \delta_\ell, \tag{3.2.51}$$

for all $k = 1, ..., s$ and $\ell = 0, ..., k-1$ Thus, we obtain by proposition 1.2.1, lemma 3.2.1 and lemma 3.2.2 the following result.

**Theorem 3.2.1** *Let* $u_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ *and* $u_h$ *be the solution of the ALE-DG method* (3.1.4) *with a Lax-Friedrichs flux and discretized by a SSP-RK method* (1.2.5), *such that for all* $k = 1, ..., s$ *as well as* $\ell = 0, ..., k-1$ *the coefficients of the RK method satisfy the conditions* (1.2.6) *and* (3.2.51). *Then the fully-discrete ALE-DG method satisfies:*

*i) The method is TVDM stable, if the assumptions of lemma 3.2.1 are satisfied with the CFL condition*

$$C_{RK}\frac{\triangle t}{h_{n+1}} \leq \frac{1}{\tau_{n+1}\left(c_1 + 4\lambda\right)},$$

*where* $C_{RK}$ *is given by* (1.2.9).

*ii) The method is TVBM stable and for smooth solutions of the initial value problem* (1.1.1) *the ALE-DG solution* $u_h$ *is a* $(k+1)$ *order accurate approximation, if the assumptions of lemma 3.2.2 are satisfied with the CFL condition*

$$C_{RK}\frac{\triangle t}{h_{n+1}} \leq \frac{1}{\tau\left(c_1 + 4\lambda\right)}$$

*and the test function space* (2.3.13) *is defined by piecewise polynomials of degree* $k$.

We would like to mention that the condition (3.2.51) for the coefficients of the SSP-RK method is quite restrictive. As we can see in the table (1.2.1), the coefficients of Shu's third order scheme already contravene the conditions (3.2.51). However, this does not mean that a SSP-RK method (1.2.5), which contravenes (3.2.51), is TVDM or TVBM unstable. Moreover, the TVD or TVB limiter stabilize the RK-DG methods even for SSP-RK method which contravene the conditions (3.2.51). In fact, the numerical experiments for the Burgers' equation in example 3.4.1 and for the Euler equations in example 3.4.2 provide good results, if the ALE-DG method (3.1.4) is discretized by Shu's third order scheme and TVD or TVB limiter are used.

Next, we discuss a local maximum principle for the ALE-DG method. Therefore, we apply the $p$-point Gauss-Lobatto quadrature formula (cf. Davis and Polonsky [20, p. 888]) on the reference cell $[0,1]$ to rewrite the local mean values of the ALE-DG solution. Thus, we choose the parameter $p$ to be the smallest integer satisfying $p-3 \geq k$, when the test function space (2.3.13) is defined by piecewise polynomials of degree $k$. In the following, the points

$$-1 = \zeta_1 < \zeta_2 < \cdots < \zeta_p = 1$$

are the classical Gauss-Lobatto quadrature points for the interval $[-1,1]$ and the corresponding quadrature weights are $\sigma_\nu$, for all $\nu = 1, \cdots, p$. These quantities are listed by Davis and Polonsky in [20, p. 888]. It should be noted that $\sum_{\nu=1}^{p} \frac{\sigma_\nu}{2} = 1$. Next, we define for all $k = 1, ..., s$ and $\ell = 0, ..., k-1$

$$u_h^{n,\ell,1} := u_{h,j-\frac{1}{2}}^{n,\ell,+} = u_h^*|_{K_j^{n,\ell}}\left(\frac{\zeta_1 + 1}{2}\right), \quad u_h^{n,\ell,p} := u_{h,j+\frac{1}{2}}^{n,\ell,-} = u_h^*|_{K_j^{n,\ell}}\left(\frac{\zeta_p + 1}{2}\right) \tag{3.2.52}$$

and for all $\nu = 2, \cdots, p-1$

$$u_h^{n,\ell,\nu} := u_h^*|_{K_j^{n,\ell}} \left( \frac{\zeta_\nu + 1}{2} \right), \tag{3.2.53}$$

where $u_h^*|_{K_j^{n,\ell}}$ is given by (2.2.1). Thus, since the parameter $p$ satisfies $p - 3 \geq k$, the Gauss-Lobatto quadrature rule is exact for polynomials of degree $k$ and we obtain

$$\overline{u}_j^{n,\ell} = \int_0^1 u_h^*|_{K_j^{n,\ell}}(\xi)\, d\xi = \frac{1}{2} \int_{-1}^1 u_h^*|_{K_j^{n,\ell}} \left( \frac{\zeta + 1}{2} \right) d\zeta = \sum_{\nu=1}^{p} \frac{\sigma_\nu}{2} u_h^{n,\ell,\nu}. \tag{3.2.54}$$

These formulations enable to state the upcoming lemma.

**Lemma 3.2.3** *Let $u_h$ be the solution of the ALE-DG method* (3.1.4) *with a Lax-Friedrichs flux and discretized by a SSP-RK method* (1.2.5)*, such that for all $k = 1, ..., s$ as well as $\ell = 0, ..., k-1$ the coefficients of the RK method satisfy the conditions* (1.2.6) *and $\beta_{k\ell} \geq 0$. The test function space* (2.3.13) *is given by piecewise polynomials of degree $k$. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$ and the grid velocity satisfies the condition $(A4)$. In addition, the values $u_{h,j-\frac{1}{2}}^{n,\ell,-}, u_h^{n,\ell,1}, \cdots, u_h^{n,\ell,p}, u_{h,j+\frac{1}{2}}^{n,\ell,+}$ and $\overline{u}_j^{n,\ell}$ are in the interval $[m, M]$ for all $j = 1, \cdots, N$, $k = 1, ..., s$ as well as $\ell = 0, ..., k-1$. Moreover, $h_{n+1} \in (0, 1)$ and it holds the CFL condition*

$$\frac{\triangle t}{h_{n+1}} \leq \frac{\min\limits_{1 \leq \nu \leq p} \sigma_\nu}{\tau_{n+1} \left( c_1 \min\limits_{1 \leq \nu \leq p} \sigma_\nu + c_1 C_{RK} + 4\lambda C_{RK} \right)}, \tag{3.2.55}$$

*where $C_{RK}$ is given by* (1.2.9)*, $h_{n+1}$ as well as $\tau_{n+1}$ are given by the condition $(P2)$, the parameter $\lambda$ is given by* (3.1.10)*, the constant $c_1$ is given by the condition* (2.1.20) *and $\sigma_1, ..., \sigma_p$ are the Gauss-Lobatto quadrature weights. Then $\overline{u}_j^{n+1}$ is in the interval $[m, M]$, for all $j = 1, \cdots, N$.*

**Proof.** First of all, we define for all $k = 1, ..., s$ and $\ell = 0, ..., k-1$ the quantities

$$C_j^\ell := \begin{cases} -\left( \dfrac{\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n,\ell,+}\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u_h^{n,\ell,1}\right)}{u_{h,j+\frac{1}{2}}^{n,\ell,+} - u_h^{n,\ell,1}} \right), & \text{if } u_{h,j+\frac{1}{2}}^{n,\ell,+} \neq u_h^{n,\ell,1}, \\ 0, & \text{if } u_{h,j+\frac{1}{2}}^{n,\ell,+} = u_h^{n,\ell,1} \end{cases}$$

and

$$D_j^\ell := \begin{cases} \dfrac{\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, u_h^{n,\ell,p}\right) - \widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n,\ell,-}\right)}{u_h^{n,\ell,p} - u_{h,j-\frac{1}{2}}^{n,\ell,-}}, & \text{if } u_h^{n,\ell,p} \neq u_{h,j-\frac{1}{2}}^{n,\ell,-}, \\ 0, & \text{if } u_h^{n,\ell,p} = u_{h,j-\frac{1}{2}}^{n,\ell,-}. \end{cases}$$

Note that the mean value theorem provides

$$0 \leq C_j^\ell \leq \lambda \quad \text{and} \quad 0 \leq D_j^\ell \leq \lambda,$$

where the lower bound appears, since $\widehat{g}_-\left(\omega^n_{j+\frac{1}{2}},\cdot\right)$ is a decreasing and $\widehat{g}_+\left(\omega^n_{j-\frac{1}{2}},\cdot\right)$ is an increasing function. The Gauss-Lobatto quadrature weights are strictly positive. Therefore, we define for all $(a_0,...,a_{p+1}) \in \mathbb{R}^{p+2}$ the function

$$\begin{aligned}
&H_\ell\left(a_0,\cdots,a_{p+1}\right)\\
&:= \frac{\sigma_1}{2}\left(1-\frac{\triangle t}{\triangle_j^{n+1}}\left(\left(\omega^n_{j+\frac{1}{2}}-\omega^n_{j-\frac{1}{2}}\right)\left((\delta_k-\delta_\ell)-\frac{1}{\sigma_1}\frac{\beta_{k\ell}}{\alpha_{k\ell}}\right)+\frac{2}{\sigma_1}C_j^\ell\frac{\beta_{k\ell}}{\alpha_{k\ell}}\right)\right)a_1\\
&+\frac{\sigma_p}{2}\left(1-\frac{\triangle t}{\triangle_j^{n+1}}\left(\left(\omega^n_{j+\frac{1}{2}}-\omega^n_{j-\frac{1}{2}}\right)\left((\delta_k-\delta_\ell)-\frac{1}{\sigma_p}\frac{\beta_{k\ell}}{\alpha_{k\ell}}\right)+\frac{2}{\sigma_p}D_j^\ell\frac{\beta_{k\ell}}{\alpha_{k\ell}}\right)\right)a_p\\
&+\frac{\triangle t}{\triangle_j^{n+1}}\frac{\beta_{k\ell}}{\alpha_{k\ell}}\left(C_j^\ell a_{p+1}+D_j^\ell a_0\right)\\
&+\sum_{\nu=2}^{p-1}\frac{\sigma_\nu}{2}\left(1-\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega^n_{j+\frac{1}{2}}-\omega^n_{j-\frac{1}{2}}\right)(\delta_k-\delta_\ell)\right)a_\nu.
\end{aligned}$$

Hence, by (3.2.52), (3.2.53) and (3.2.54) the scheme given by (3.2.50) can be written as

$$w_j^{k,\ell} = H\left(u^{n,\ell,-}_{h,j-\frac{1}{2}},u_h^{n,\ell,1},\cdots,u_h^{n,\ell,p},u^{n,\ell,+}_{h,j+\frac{1}{2}}\right), \tag{3.2.56}$$

since the quadrature weights satisfy $\sum\limits_{\nu=1}^{p}\frac{\sigma_\nu}{2} = 1$. Likewise, since $|\delta_k-\delta_\ell| \leq 1$ and $h_{n+1} \in (0,1)$, it follows by (3.2.31) and the CFL number (3.2.55) for all $(a_0,...,a_{p+1}) \in \mathbb{R}^{p+2}$ and $\nu = 0,...,p+1$

$$\partial_{a_\nu}H_\ell\left(a_0,...,a_{p+1}\right) \geq 0. \tag{3.2.57}$$

Next, we define for all $k = 1,...,s$ and $a^\ell = \left(a_0^\ell,...,a_{p+1}^\ell\right) \in \mathbb{R}^{p+2}$, $\ell = 0,...,k-1$, the function

$$H_k\left(a^0,...,a^{k-1}\right) := \sum_{\ell=0}^{k-1}\alpha_{k\ell}H_\ell\left(a_0^\ell,...,a_{p+1}^\ell\right). \tag{3.2.58}$$

The conditions (1.2.6) for the coefficients of the SSP-RK method provide for all $k = 1,...,s$ and $\ell = 0,...,k-1$

$$\sum_{\ell=0}^{k-1}\alpha_{k\ell}\left((\delta_k-\delta_\ell)-\frac{\beta_{k\ell}}{\alpha_{k\ell}}\right) = \delta_k - \sum_{\ell=0}^{k-1}\left(\alpha_{k\ell}\delta_\ell+\beta_{k\ell}\right) = 0.$$

Hence, for all $a \in \mathbb{R}$ and $k = 1, ..., s$ is $H_k(a, ..., a) = a$. Furthermore, by (3.2.57) the functions $H_k$, $k = 1, ..., s$, are increasing in every argument, since the coefficients $\alpha_{k\ell}$ are non-negative. Moreover, by (3.2.58) the scheme (3.2.49) can be written as follows

$$\begin{cases} \overline{u}_j^{n,0} & = \overline{u}_j^n; \\ \text{for } k & = 1, ..., s \text{ compute the intermediate functions :} \\ \overline{u}_j^{n,k} & = H_k\left(u^0, ..., u^{k-1}\right); \\ \overline{u}_j^{n+1} & = \overline{u}_j^{n,s}, \end{cases}$$

where for all $\ell = 0, ..., k-1$ we denote

$$u^\ell := \left(u_{h,j-\frac{1}{2}}^{n,\ell,-}, u_h^{n,\ell,1}, ...u_h^{n,\ell,p}, u_{h,j+\frac{1}{2}}^{n,\ell,+}\right).$$

For all $k = 1, ..., s$ the schemes $\overline{u}_j^{n,k} = H_k\left(u^0, ..., u^{k-1}\right)$ are monotone schemes in conservation form and thus we obtain for $k = s$

$$m \leq H_s\left(m, ..., m\right) \leq \overline{u}_j^{n+1} \leq H_s\left(M, ..., M\right) = M.$$

This completes the proof. □

We can extract from lemma 3.2.3 that the ALE-DG method (3.1.4) discretized by a SSP-RK method (1.2.5) satisfies a local maximum principle, if the ALE-DG solution has certain properties. X. Zhang and Shu's limiter (1.2.19) ensures that the ALE-DG solution satisfies the conditions of lemma 3.2.3. Moreover, X. Zhang and Shu proved in [94, Lemma 2.4] that the limiter (1.2.19) does not affect the accuracy of a numerical method. Therefore, lemma 3.2.3 and X. Zhang and C.-W. Shu's limiter provide the following maximum principle for the ALE-DG method (3.1.4) discretized by a SSP-RK method (1.2.5).

**Theorem 3.2.2** *Suppose the same assumptions as in lemma 3.2.3. Then the solution of the ALE-DG method* (3.1.4) *discretized by a SSP-RK method* (1.2.5)*, revised by the limiter* (1.2.19)*, is for smooth solutions of the initial value problem* (1.1.1) *a* $(k+1)$ *order accurate approximation, if the test function space* (2.3.13) *is defined by piecewise polynomials of degree* $k$*. Furthermore, if* $\overline{u}_j^n$ *is in the interval* $[m, M]$*, for all* $j = 1, \cdots, N$*,* $\overline{u}_j^{n+1}$ *is also in the interval* $[m, M]$*, for all* $j = 1, \cdots, N$*.*

### 3.2.3. On cell entropy inequalities for the fully-discrete method

For finite difference schemes, cell entropy inequalities are commonly used tools to ensure the entropy stability of the scheme. A numerical method for the problem (1.1.1) is entropy stable, if it converges to the unique entropy solution. In this section, we discuss cell entropy inequalities for the fully-discrete ALE-DG method. Therefore, cell entropy inequalities for the time-dependent cells (2.3.5) need to be defined.

Let $\eta, F \in \mathcal{C}(\mathbb{R})$ be two functions such that the tuple $(\eta, F)$ is a pair of entropy functions for the initial value problem (1.1.1). Then an entropy inequality for the problem (1.1.1) is given by

$$\partial_t \eta(u) + \partial_x F(u) \leq 0, \text{ in } \Omega \times (0,T). \tag{3.2.59}$$

Note that the inequality (3.2.59) has to be understood in the sense of distributions. Since $(\eta, F)$ is a pair of entropy functions, it holds $F' = \eta' f'$. Hence, if we integrate the entropy inequality (3.2.59) over any cell $K_j(t)$, $j = 1, ..., N$, it follows by the transport equation (2.2.4)

$$\begin{aligned} 0 \geq & \frac{d}{dt}\left(\eta(u), 1\right)_{K_j(t)} + F\left(u^-_{j+\frac{1}{2}}\right) - \omega^n_{j+\frac{1}{2}} \eta\left(u^-_{j+\frac{1}{2}}\right) \\ & - F\left(u^+_{j-\frac{1}{2}}\right) + \omega^n_{j-\frac{1}{2}} \eta\left(u^+_{j-\frac{1}{2}}\right), \end{aligned} \tag{3.2.60}$$

where $u^{\pm}_{j-\frac{1}{2}} := \lim_{\varepsilon \to 0} u\left(x_{j-\frac{1}{2}}(t) \pm \varepsilon, t\right)$. Henceforth, we call the inequality (3.2.60) cell entropy inequality for the cells (2.3.5). Accordingly, in the sense of Godlewski and Raviart [32, Definition 4.1, p. 142], we call the ALE-DG method consistent with the entropy inequality (3.2.59), if it satisfies a discrete version of the cell entropy inequality (3.2.60).
First of all, we prove for the piecewise constant forward Euler step of the ALE-DG method with the Lax-Friedrichs flux given by (3.1.9) and (3.1.10) a discrete cell entropy inequality with respect to the Kružkov entropies and the corresponding entropy fluxes. These pairs of entropy functions are given by

$$\eta_k(u) := |u - k|, \quad F_k(u) = \int_k^u \eta_k'(v) f'(v)\, dv, \quad \text{for all } k \in \mathbb{R}. \tag{3.2.61}$$

The piecewise constant forward Euler step of the ALE-DG method with the Lax-Friedrichs flux (3.1.9) and (3.1.10) is given by

$$\begin{aligned} 0 = & \overline{u}^{n+1}_j - \overline{u}^n_j + \frac{\triangle t}{\triangle^{n+1}_j}\left(\widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, \overline{u}^n_{j+1}\right) - \widehat{g}_-\left(\omega^n_{j+\frac{1}{2}}, \overline{u}^n_j\right)\right) \\ & + \frac{\triangle t}{\triangle^{n+1}_j}\left(\widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, \overline{u}^n_j\right) - \widehat{g}_+\left(\omega^n_{j-\frac{1}{2}}, \overline{u}^n_{j-1}\right)\right), \end{aligned} \tag{3.2.62}$$

since the scheme (3.2.7) degenerates to (3.2.62), if the test function space (2.3.13) is given by piecewise constant polynomials. For the scheme (3.2.62), we have the following entropy inequality.

**Proposition 3.2.2** *For any time level $t = t_n$, $n = 0, ..., L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$. Consider the piecewise constant forward Euler step of the ALE-DG method with the Lax-Friedrichs flux given by* (3.1.9) *as well as* (3.1.10) *and suppose the CFL condition*

$$\frac{\triangle t}{h_{n+1}} \leq \frac{1}{2\tau_{n+1}\lambda}, \tag{3.2.63}$$

*where $h_{n+1}$ as well as $\tau_{n+1}$ are given by the condition $(P2)$ and the parameter $\lambda$ is given by* (3.1.10). *Then the solution of the scheme* (3.2.62) *satisfies for any cell $K_j^{n+1}$, $j = 1, ..., N$, the following cell entropy inequality*

$$\eta_k\left(\overline{u}_j^{n+1}\right) \leq \eta_k\left(\overline{u}_j^{n}\right) - \frac{\triangle t}{\triangle_j^{n+1}}\left(H_k\left(\omega, \overline{u}_j^n, \overline{u}_{j+1}^n\right) - H_k\left(\omega, \overline{u}_{j-1}^n, \overline{u}_j^n\right)\right), \tag{3.2.64}$$

*where $\eta_k(u)$, $k \in \mathbb{R}$ are the Kružkov entropies given by* (3.2.61) *and for all $a, b \in [m, M]$ is*

$$\begin{aligned} H_k\left(\omega, a, b\right) :=& \frac{1}{2}\int_k^a \eta_k'\left(v\right)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n, v\right) + \lambda\right) dv \\ &+ \frac{1}{2}\int_k^b \eta_k'\left(v\right)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv. \end{aligned} \tag{3.2.65}$$

**Proof.** The idea of the upcoming proof is to evaluate the scheme (3.2.62) multiplied by $\eta_k'\left(\overline{u}_j^{n+1}\right)$, $j = 1, ..., N$. Therefore, first of all, we derive some helpful identities.
The integration by parts formula provides

$$\begin{aligned} \left(\overline{u}_j^{n+1} - \overline{u}_j^n\right)\eta_k'\left(\overline{u}_j^{n+1}\right) =& \eta_k\left(\overline{u}_j^{n+1}\right) - \eta_k\left(\overline{u}_j^n\right) \\ &+ \int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n} \eta_k''\left(v\right)\left(\overline{u}_j^n - v\right) dv. \end{aligned} \tag{3.2.66}$$

Note that $\eta_k''(u)$ is a Dirac delta function. However, the product above is well defined, since the values $\overline{u}_j^n$ are constant. Next, we obtain by the definition of the function (3.2.1) as well as (3.2.2)

$$\partial_u \widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, u\right) = \frac{1}{2}\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u\right) + \lambda\right)$$

and

$$\partial_u \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, u\right) = \frac{1}{2}\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, u\right) - \lambda\right)$$

These identities as well as the integration by parts formula supply

$$\begin{aligned} &\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_{j+1}^n\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_j^n\right)\right)\eta_k'\left(\overline{u}_j^{n+1}\right) \\ =& -\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j+1}^n} \eta_k''\left(v\right)\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_{j+1}^n\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, v\right)\right) dv \\ &+ \frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j+1}^n} \eta_k'\left(v\right)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv \\ &+ \frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n} \eta_k''\left(v\right)\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \overline{u}_j^n\right) - \widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, v\right)\right) dv \\ &- \frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n} \eta_k'\left(v\right)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv \end{aligned} \tag{3.2.67}$$

as well as

$$\begin{aligned}
&\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_j^n\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)\right)\eta_k'\left(\overline{u}_j^{n+1}\right)\\
=&\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n}\eta_k''(v)\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,v\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_j^n\right)\right)dv\\
+&\frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n}\eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n,v\right)+\lambda\right)dv\\
+&\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j-1}^n}\eta_k''(v)\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,v\right)\right)dv\\
-&\frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j-1}^n}\eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n,v\right)+\lambda\right)dv.
\end{aligned}\tag{3.2.68}$$

It should be noted that the products above are well defined, since the functions (3.2.1) and (3.2.2) are continuous. For the next steps, we define for all $a,b\in[m,M]$ the quantities

$$\begin{aligned}
H_k(\omega,a,b):=&\frac{1}{2}\int_k^a\eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n,v\right)+\lambda\right)dv\\
&+\frac{1}{2}\int_k^b\eta_k'(v)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n,v\right)-\lambda\right)dv
\end{aligned}\tag{3.2.69}$$

and

$$\begin{aligned}
\Theta_j:=&\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n}\eta_k''(u)\left(p_j\left(\overline{u}_j^n\right)-p_j(v)\right)dv\\
&-\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j+1}^n}\eta_k''(v)\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,\overline{u}_{j+1}^n\right)-\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,v\right)\right)dv\\
&+\frac{\triangle t}{\triangle_j^{n+1}}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j-1}^n}\eta_k''(v)\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,v\right)\right)dv,
\end{aligned}\tag{3.2.70}$$

where

$$p_j(u):=u-\frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,u\right)-\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,u\right)\right).$$

The function (3.2.69) provides for all $k \in \mathbb{R}$ the identity

$$\begin{aligned}
&H_k\left(\omega, \overline{u}_j^n, \overline{u}_{j+1}^n\right) - H_k\left(\omega, \overline{u}_{j-1}^n, \overline{u}_j^n\right)\\
=&\frac{1}{2}\int_{\overline{u}_{j-1}^n}^{\overline{u}_{j+1}^n} \eta_k'(v)\, f'(v)\ dv - \frac{1}{2}\left(\omega_{j+\frac{1}{2}}^n + \lambda\right)\overline{u}_{j+1}^n\\
&+\frac{1}{2}\left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n + \lambda\right)\overline{u}_j^n + \frac{1}{2}\left(\omega_{j-\frac{1}{2}}^n - \lambda\right)\overline{u}_{j-1}^n\\
=&\frac{1}{2}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j+1}^n} \eta_k'(v)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv\\
&-\frac{1}{2}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n} \eta_k'(v)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv\\
&+\frac{1}{2}\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n} \eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n, v\right) + \lambda\right) dv\\
&-\frac{1}{2}\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j-1}^n} \eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n, v\right) + \lambda\right) dv. \qquad (3.2.71)
\end{aligned}$$

Next, we multiply the scheme (3.2.62) by $\eta_k'\left(\overline{u}_j^{n+1}\right)$, apply (3.2.66), (3.2.67), (3.2.67), (3.2.68), (3.2.70) as well as (3.2.71) and obtain

$$\eta_k\left(\overline{u}_j^{n+1}\right) = \eta_k\left(\overline{u}_j^n\right) - \frac{\triangle t}{\triangle_j^{n+1}}\left(H_k\left(\omega, \overline{u}_j^n, \overline{u}_{j+1}^n\right) - H_k\left(\omega, \overline{u}_{j-1}^n, \overline{u}_j^n\right)\right) - \Theta_j. \qquad (3.2.72)$$

For any $u \in [m, M]$, we obtain by (3.2.61)

$$\begin{aligned}
H_k(\omega, u, u) =&\frac{1}{2}\int_k^u \eta_k'(v)\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n, v\right) + \lambda\right) dv\\
&+\frac{1}{2}\int_k^u \eta_k'(v)\left(\partial_u g\left(\omega_{j+\frac{1}{2}}^n, v\right) - \lambda\right) dv\\
=&\int_k^u \eta_k'(v)\, f'(v)\ dv - \omega\left(\overline{x}_j^n, t_n\right)\int_k^u \eta_k'(v)\ dv\\
=&F_k(u) - \omega\left(\overline{x}_j^n, t_n\right)\eta_k(u), \qquad (3.2.73)
\end{aligned}$$

where

$$\omega\left(\overline{x}_j^n, t_n\right) = \frac{1}{2}\left(\omega_{j+\frac{1}{2}}^n + \omega_{j-\frac{1}{2}}^n\right), \quad \overline{x}_j^n = \frac{1}{2}\left(x_{j+\frac{1}{2}}^n + x_{j-\frac{1}{2}}^n\right).$$

Thus, the equation (3.2.72) provides an entropy inequality for the piecewise constant forward Euler step of the ALE-DG method (3.2.62), if the quantity (3.2.70) is non-negative. Since $\eta_k''(u)$ is a Dirac delta function, the function $\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n, \cdot\right)$ is decreasing and the function $\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n, \cdot\right)$

is increasing, it follows

$$
\begin{aligned}
&-\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j+1}^n} \eta_k''(v)\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,\overline{u}_{j+1}^n\right)-\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,v\right)\right)\,dv\\
&=\begin{cases}\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,k\right)-\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,\overline{u}_{j+1}^n\right)\right)\geq 0, & \text{if } \overline{u}_j^{n+1}\leq k\leq \overline{u}_{j+1}^n,\\
\left(\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,\overline{u}_{j+1}^n\right)-\widehat{g}_-\left(\omega_{j+\frac{1}{2}}^n,k\right)\right)\geq 0, & \text{if } \overline{u}_{j+1}^n\leq k\leq \overline{u}_j^{n+1},\\
0, & \text{else}\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\overline{u}_j^{n+1}}^{\overline{u}_{j-1}^n} \eta_k''(v)\left(\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,v\right)\right)\,dv\\
&=\begin{cases}\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,k\right)\geq 0, & \text{if } \overline{u}_j^{n+1}\leq k\leq \overline{u}_{j-1}^n,\\
\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,k\right)-\widehat{g}_+\left(\omega_{j-\frac{1}{2}}^n,\overline{u}_{j-1}^n\right)\geq 0, & \text{if } \overline{u}_{j-1}^n\leq k\leq \overline{u}_j^{n+1},\\
0, & \text{else,}\end{cases}
\end{aligned}
$$

Furthermore, the CFL condition (3.2.63) provides for all $u,v\in[m,M]$

$$
1-\frac{1}{2}\frac{\triangle t}{\triangle_j^{n+1}}\left(\partial_u g\left(\omega_{j-\frac{1}{2}}^n,u\right)-\partial_u g\left(\omega_{j+\frac{1}{2}}^n,v\right)+2\lambda\right)\geq 1-2\lambda\frac{\tau_{n+1}}{h_{n+1}}\geq 0,
$$

since the parameter $\lambda$ is given by (3.1.10). Hence, according to the mean value theorem follows

$$
\int_{\overline{u}_j^{n+1}}^{\overline{u}_j^n}\eta_k''(v)\left(p_j\left(\overline{u}_j^n\right)-p_j(v)\right)\,dv=\begin{cases}p_j\left(\overline{u}_j^n\right)-p_j(k)\geq 0, & \text{if } \overline{u}_j^{n+1}\leq k\leq \overline{u}_j^n,\\
p_j(k)-p_j\left(\overline{u}_j^n\right)\geq 0, & \text{if } \overline{u}_j^n\leq k\leq \overline{u}_j^{n+1},\\
0, & \text{else.}\end{cases}
$$

Thus, $\Theta_j\geq 0$ and this completes the proof. □

We would like to mention that for high order methods it is not easy to prove a cell entropy inequality with respect to the Kružkov entropy functions (3.2.61). In the sense of the RK-DG method, this issue was discussed by Cockburn and Shu in [11, Lemma 2.8 and Lemma 2.9].
Often it is common to prove for a high order method merely a cell entropy inequality with respect to the square entropy $\eta(u)=\frac{1}{2}u^2$. However, it should be noted that according to DiPerna's result (cf. proposition 1.1.3) a cell entropy inequality with respect to the square entropy can merely provide the convergence to the unique entropy solution of the problem (1.1.1), if the flux function in equation (1.1.1a) is convex.
We mentioned, in the section 1.2.2, that Jiang and Shu proved in [45] a cell entropy inequality with respect to the square entropy for certain RK-DG methods including the backward Euler

and the Crank-Nicolson time integration method. Since Jiang and Shu applied the equation (1.2.25) to prove the inequality, their result does not cover the forward Euler step of the RK-DG method. This is not surprising, since Chavent and Cockburn proved in [3] that the forward Euler step of the RK-DG method is unconditional $\mathrm{L}^2$-unstable. In the following, we will extend the result of Jiang and Shu to the backward Euler step of the ALE-DG method. Therefore, we apply the following equation: For all $a, b, c, d \in \mathbb{R}$ and $\vartheta \in [0,1]$ holds

$$\begin{aligned}(ac - bd)\left(\vartheta a + (1-\vartheta) b\right) =& \frac{1}{2}a^2 c - \frac{1}{2}b^2 d + \left(\vartheta - \frac{1}{2}\right)(a-b)^2 d \\ &+ \left(\left(\vartheta - \frac{1}{2}\right)a^2 + (1-\vartheta)ab\right)(c-d). \qquad (3.2.74)\end{aligned}$$

It should be noted that the equation (3.2.74) is more restrictive than (1.2.25), because (3.2.74) does not provide a cell entropy inequality for the Crank-Nicolson time integration method. Moreover, the backward Euler step is an implicit method. Hence, we can neglect the CFL condition (3.2.63). Otherwise for the implementation of an implicit time integration method an iterative solver is necessary. The distribution of the grid points is of particular importance for the efficiency, robustness and accuracy of an iterative solver (cf. e.g. Pollul and Reusken [68]). This restriction to the grid points can affect the flexibility of the ALE-DG method. However, from an academic point of view, the upcoming inequality is a much stronger result than the inequality presented in proposition 3.2.2, since the upcoming inequality holds for the ALE-DG method with an arbitrary monotone numerical flux and the test function space (2.3.13) can be given by piecewise polynomials with an arbitrary degree.

**Proposition 3.2.3** *Let $u_0 \in \mathrm{L}^2(\Omega)$ and $u_h$ be the solution of the backward Euler step of the ALE-DG method* (3.1.4)*. Furthermore, for any time level $t = t_n$, $n = 0, \dots, L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$. The initial data for the method is the function $\mathcal{P}_h(u_0, 0) = u_h^0$. Then $u_h$ satisfies for any cell $K_j^{n+1}$, $j = 1, \dots, N$, the following cell entropy inequality*

$$\begin{aligned}\left(\eta\left(u_h^{n+1}\right), 1\right)_{K_j^{n+1}} \leq& \left(\eta\left(u_h^n\right), 1\right)_{K_j^n} - \triangle t H\left(\omega, u_{h,j+\frac{1}{2}}^{n+1,-}, u_{h,j+\frac{1}{2}}^{n+1,+}\right) \\ &+ \triangle t H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right), \qquad (3.2.75)\end{aligned}$$

*where $\eta(u) := \frac{u^2}{2}$ is the square entropy and*

$$\begin{aligned}H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) :=& -\int^{u_{h,j-\frac{1}{2}}^{n+1,-}} f(u)\, du - \omega_{j-\frac{1}{2}}^n \eta\left(u_{h,j-\frac{1}{2}}^{n+1,-}\right) \\ &+ \widehat{g}\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) u_{h,j-\frac{1}{2}}^{n+1,-}.\end{aligned}$$

*In addition, it holds for all $n = 1, \dots, L$*

$$\left\| u_h^n \right\|_{\mathrm{L}^2(\Omega)} \leq \left\| u_0 \right\|_{\mathrm{L}^2(\Omega)}.$$

**Proof.** First of all we transform the backward Euler step of the ALE-DG method (3.1.4) to the reference cell $(0,1)$. Note that the functions of the space (2.3.13) are polynomials of degree $k \geq 1$ on the reference cell. Hence, we obtain for all $v^* \in \mathcal{P}^k$ and for all $j = 1, ..., n$

$$\begin{aligned}&\frac{1}{\triangle t}\left(\triangle_j^{n+1}u_h^{*,n+1} - \triangle_j^n u_h^{*,n}, v^*\right)_{(0,1)}\\&= \left(g\left(\omega, u_h^{*,n+1}\right), \partial_\xi v^*\right)_{(0,1)} - \widehat{g}\left(\omega_{j+\frac{1}{2}}^n, u_{h,j+\frac{1}{2}}^{n+1,-}, u_{h,j+\frac{1}{2}}^{n+1,+}\right) v_{j+\frac{1}{2}}^-\\&+\widehat{g}\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) v_{j-\frac{1}{2}}^+, \end{aligned} \tag{3.2.76}$$

where the functions $u_h^{*,n+1} = u_h^*(t_{n+1})$, $u_h^{*,n} = u_h^*(t_n)$ as well as $v^*$ are given by (2.2.1) and $\triangle_j^{n+1}$ as well as $\triangle_j^{n+1}$ are given by (2.3.1). Next, we consider the left-hand side of the equation (3.2.76) with the test $u_h^{*,n+1}$ and apply the equations (2.3.30), (2.3.31) and (3.2.74) with $\vartheta = 1$. Hence, we obtain

$$\begin{aligned}&\frac{1}{\triangle t}\left(\left(\triangle_j^{n+1}u_h^{*,n+1} - \triangle_j^n u_h^{*,n}, u_h^{*,n+1}\right)_{(0,1)}\right)\\&=\frac{1}{\triangle t}\left(\left(\eta\left(u_h^{*,n+1}\right), \triangle_j^{n+1}\right)_{(0,1)} - \left(\eta\left(u_h^{*,n}\right), \triangle_j^n\right)_{(0,1)}\right)\\&+\frac{1}{\triangle t}\left(\eta\left(u_h^{*,n+1} - u_h^{*,n}\right), \triangle_j^n\right)_{(0,1)} + \left((\partial_\xi\omega)\,\eta\left(u_h^{*,n+1}\right), 1\right)_{(0,1)}, \end{aligned} \tag{3.2.77}$$

since the square entropy is for all $u \in \mathbb{R}$ given by $\eta(u) = \frac{1}{2}u^2$. Likewise, it follows by (3.1.2)

$$\left(g\left(\omega, u_h^{*,n+1}\right), \partial_\xi u_h^{*,n+1}\right)_{(0,1)} = \int_{u_{h,j-\frac{1}{2}}^{n+1,-}}^{u_{h,j+\frac{1}{2}}^{n+1,+}} f(u)\, du - \frac{1}{2}\left(\omega, \partial_\xi\left(u_h^{*,n+1}\right)^2\right)_{(0,1)}. \tag{3.2.78}$$

Next, we define for all $u \in \left[u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j+\frac{1}{2}}^{n+1,+}\right]$

$$G(\omega, u) := \int^u f(v)\, dv - \omega\eta(u).$$

Furthermore, we define for all $j = 1, ..., N$

$$\begin{aligned}H\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) &= -\,G\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}\right)\\&+\widehat{g}\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) u_{h,j-\frac{1}{2}}^{n+1,-}\end{aligned} \tag{3.2.79}$$

and

$$\begin{aligned}\Theta_{j-\frac{1}{2}}^{n+1} :=&\, G\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,+}\right) - G\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,-}\right)\\&-\widehat{g}\left(\omega_{j-\frac{1}{2}}^n, u_{h,j-\frac{1}{2}}^{n+1,+}, u_{h,j-\frac{1}{2}}^{n+1,-}\right) [\![u_h^{n+1}]\!]_{j-\frac{1}{2}}.\end{aligned} \tag{3.2.80}$$

Next, we consider the equation (3.2.76) with the test $u_h^{*,n+1}$ and apply the equations (3.2.77), (3.2.78), (3.2.79) and (3.2.80). Thus, we obtain

$$\begin{aligned} 0 =& \frac{1}{\triangle t}\left(\left(\eta\left(u_h^{*,n+1}\right), \triangle_j^{n+1}\right)_{(0,1)} - \left(\eta\left(u_h^{*,n}\right), \triangle_j^{n}\right)_{(0,1)}\right) \\ &+ \frac{1}{\triangle t}\left(\eta\left(u_h^{*,n+1} - u_h^{*,n}\right), \triangle_j^{n}\right)_{(0,1)} + H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) \\ &- H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) + \Theta_{j-\frac{1}{2}}^{n+1}. \end{aligned} \tag{3.2.81}$$

For any $u \in \mathcal{C}\left([m, M] \times [0, T]\right)$ with $u_{j-\frac{1}{2}}^{n+1} := u\left(x_{j-\frac{1}{2}}^{n+1}, t_{n+1}\right)$, $j = 1, ..., N$, is

$$\widehat{g}\left(\omega_{j-\frac{1}{2}}^{n}, u_{j-\frac{1}{2}}^{n+1}, u_{j-\frac{1}{2}}^{n+1}\right) = f\left(u_{j-\frac{1}{2}}^{n+1}\right)\eta'\left(u_{j-\frac{1}{2}}^{n+1}\right) - 2\omega_{j-\frac{1}{2}}^{n}\eta\left(u_{j-\frac{1}{2}}^{n+1}\right).$$

Hence, the integration by parts formula provides

$$\begin{aligned} H\left(\omega_{j-\frac{1}{2}}^{n}, u_{j-\frac{1}{2}}^{n+1}, u_{j-\frac{1}{2}}^{n+1}\right) =& - G\left(\omega_{j-\frac{1}{2}}^{n}, u_{j-\frac{1}{2}}^{n+1}\right) + \widehat{g}\left(\omega_{j-\frac{1}{2}}^{n}, u_{j-\frac{1}{2}}^{n+1}, u_{j-\frac{1}{2}}^{n+1}\right) u_{j-\frac{1}{2}}^{n+1} \\ =& F\left(u_{j-\frac{1}{2}}^{n+1}\right) - \omega_{j-\frac{1}{2}}^{n}\eta\left(u_{j-\frac{1}{2}}^{n+1}\right), \end{aligned}$$

where $F(u) := \int^u \eta'(v) f'(v)\, dv$ is the entropy flux. Thus, the equation (3.2.81) provides an entropy inequality for the backward Euler step of the ALE-DG method, if the quantity (3.2.68) is non-negative.
The function $G(\omega, u)$ is differentiable in the second argument. Thus by the mean value theorem, there exists for all $j = 1, ..., N$ a $\vartheta \in [u_{h,j-\frac{1}{2}}^{-}, u_{h,j-\frac{1}{2}}^{+}]$, such that

$$G\left(\omega_{j-\frac{1}{2}}^{n}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) - G\left(\omega_{j-\frac{1}{2}}^{n}, u_{h,j-\frac{1}{2}}^{n+1,-}\right) = g\left(\omega_{j-\frac{1}{2}}^{n}, \vartheta\right) [\![u_h]\!]_{j-\frac{1}{2}},$$

since $\partial_u G(\omega, u) = f(u) - \omega u = g(\omega, u)$. Hence, by the properties ($\widehat{g}1$) and ($\widehat{g}2$) of the numerical flux $\widehat{g}(\omega, \cdot, \cdot)$ follows $\Theta_{j-\frac{1}{2}} \geq 0$, for all $j = 1, ..., N$. Furthermore, we obtain for all $\ell = n, n+1$ and $j = 1, ..., N$

$$\left(\eta\left(u_h^{*,\ell}\right), \triangle_j^{\ell}\right)_{(0,1)} = \left(\eta\left(u_h^{\ell}\right), 1\right)_{K_j^{\ell}},$$

where the cells $K_j^{\ell}$ and $\triangle_j^{\ell}$ are given by (2.3.1). Therefore, we obtain by (3.2.81)

$$\begin{aligned} \left(\eta\left(u_h^{n+1}\right), 1\right)_{K_j^{n+1}} \leq& \left(\eta\left(u_h^{n}\right), 1\right)_{K_j^{n}} - \triangle t H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right) \\ &+ \triangle t H\left(\omega, u_{h,j-\frac{1}{2}}^{n+1,-}, u_{h,j-\frac{1}{2}}^{n+1,+}\right). \end{aligned} \tag{3.2.82}$$

Since we consider the initial value problem (1.1.1) with periodic boundary conditions and the initial data for the backward Euler step of the ALE-DG method is given by $\mathcal{P}_h(u_0, 0) = u_h^0$, a summation from $j = 1$ to $N$ of the inequality (3.2.82) provides for all $n = 1, ..., L$

$$\|u_h^n\|_{L^2(\Omega)} \leq \left\|u_h^0\right\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.$$

□

Finally, we would like to mention that Klingenberg et al. proved in [48, Proposition 2.2] a cell entropy inequality for the semi-discrete ALE-DG method (3.1.4) by applying similar techniques as in the proof of proposition 3.2.3.

## 3.3. A priori error estimates for the one dimensional semi-discrete ALE-DG method

In this section, we present a priori error estimates for the one dimensional semi-discrete ALE-DG method with smooth solutions of the initial value problem (1.1.1). These error estimates indicate the high order accuracy of the semi-discrete ALE-DG method in regions where the solution of the initial value problem (1.1.1) is smooth. Note that we cannot expect the high order accuracy of a numerical method in regions, where the solution of the initial value problem (1.1.1) has singularities. This causality was proven e.g. by Cockburn and Gremaud in the context of streamline diffusion finite element methods for scalar conservation laws.
The a priori error estimates for the semi-discrete ALE-DG method will be estimated in the sense of the global length $h$ given by (2.1.5). Furthermore, we will utilize techniques, which were introduced by Q. Zhang and Shu in [90, 92]. However, since we apply time-dependent cells in the method, there are some differences in the proof. First of all, we cannot utilize the ALE-DG solution $u_h$ as a test function in the equation (3.1.4). Thus, we have to apply the equivalent equation (3.1.5) for the proof. In addition, we have to use the transport equation (2.2.4) to manage the differentiation of the time-dependent volume integrals. Finally, we compensate the nonlinear nature of the flux function in (1.1.1a) by a Taylor expansion like Q. Zhang and Shu. Therefore, we need the following a priori assumption for a smooth solution $u$ of the initial value problem (1.1.1) and the approximate solution $u_h$ given by the ALE-DG method

$$\max_{t\in[0,T]} \|u - u_h\|_{L^\infty(\Omega)} \leq C_{\mathcal{F}} h, \tag{3.3.1}$$

where the constant $C_{\mathcal{F}}$ is independent of $u_h$ and $h$. Note that for convenience we skipped the index $t$ in (3.3.1). Moreover, we need to ensure that the flux function $f(u)$ in (1.1.1a) and its derivatives are bounded. Since the solution of the initial value problem (1.1.1) satisfies the maximum principle (1.1.5), the flux function $f(u)$ in (1.1.1a) can be modified such that the modified flux and its derivatives are bounded in the interval $[m, M]$, where $m$ and $M$ are given by (1.2.20). The details of this modification can be found in Q. Zhang and Shu's article [90, Section 3.2]. Hence, the following constants are well defined

$$c_0^\star := \max_{(x,t)\in\Omega\times[0,T]} \left|f'\left(u\left(x,t\right)\right)\right|, \qquad c_1^\star := \max_{m\leq v\leq M} \left|f''\left(v\right)\right| \tag{3.3.2}$$

and

$$c_2^\star := \max_{(x,t)\in\Omega\times[0,T]} \left|\partial_x f'\left(u\left(x,t\right)\right)\right|, \tag{3.3.3}$$

if the function $u = u\left(x,t\right)$ in is sufficiently smooth. Furthermore, the numerical flux

$$\widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right), \quad \text{for all } j = 1, ..., N,$$

will be evaluated by a Taylor expansion. In order to evaluate the numerical flux function by a Taylor expansion we proceed as Q. Zhang and Shu [90] and apply a quantity to measure for all $j = 1, ..., N$ the difference between the functions

$$g\left(\omega^n_{j-\frac{1}{2}}, u\left(x_{j-\frac{1}{2}}\left(t\right), t\right)\right) \quad \text{as well as} \quad \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right).$$

This quantity was introduced by Harten in [37] and it is for any piecewise smooth function $v \in \mathrm{L}^2\left(\Omega\right)$ defined by

$$\widehat{a}\left(\widehat{g}; v\right) := \begin{cases} [\![v]\!]^{-1}\left(g\left(\omega, \{\!\{v\}\!\}\right) - \widehat{g}\left(\omega, v^-, v^+\right)\right), & \text{if} \quad [\![v]\!] \neq 0, \\ \left|\partial_u g\left(\omega, \{\!\{v\}\!\}\right)\right|, & \text{if} \quad [\![v]\!] = 0, \end{cases} \tag{3.3.4}$$

where $\partial_u g\left(\omega, \{\!\{v\}\!\}\right) = f'\left(u\right) - \omega$. In addition, for an arbitrary discontinuity of the piecewise smooth function $v$ we denote $v^-$ the left-sided and $v^+$ the right-sided limes. Thus, the average and jump operators are given by

$$\{\!\{v\}\!\} := \frac{1}{2}\left(v^+ + v^-\right) \quad \text{as well as} \quad [\![v]\!] := v^+ - v^-. \tag{3.3.5}$$

Next, we prove the following auxiliary lemma for the quantity (3.3.4).

**Lemma 3.3.1** *Suppose the numerical flux function $\widehat{g}$ has the properties ($\widehat{g}1$) - ($\widehat{g}3$). Then for any piecewise smooth function $v \in \mathrm{L}^2\left(\Omega\right)$ the quantity $\widehat{a}\left(\widehat{g}; v\right)$ given by* (3.3.4) *is non negative and bounded by the constant*

$$C_{\widehat{a}} := \frac{1}{2}\left(L^-_{\widehat{g}} + L^+_{\widehat{g}}\right), \tag{3.3.6}$$

*where $L^-_{\widehat{g}}$ and $L^+_{\widehat{g}}$ are defined in ($\widehat{g}3$). Moreover, it holds the inequality*

$$\frac{1}{2}\left|\partial_u g\left(\omega, \{\!\{v\}\!\}\right)\right| \leq \widehat{a}\left(\widehat{g}; v\right) + \frac{c_1^\star}{4}\left|[\![v]\!]\right|, \tag{3.3.7}$$

*where the $c_1^\star$ is given by* (3.3.2).

**Proof.** [3] Let $v \in \mathrm{L}^2(\Omega)$ be a piecewise smooth function. If $[\![v]\!] = 0$, the statement of lemma 3.3.1 is fulfilled. Hence, we assume that $[\![v]\!] \neq 0$.
The numerical flux is an E-flux, since it has the properties ($\widehat{g}1$) - ($\widehat{g}2$). Therefore, for the function $g(\omega, \cdot)$ given by (3.1.2) and the numerical flux function $\widehat{g}(\omega, \cdot, \cdot)$ holds the estimate (1.2.24). Thus, we obtain

$$\widehat{a}(\widehat{g}; v) = [\![v]\!]^{-1}\left(g(\omega, \{\!\{v\}\!\}) - \widehat{g}\left(\omega, v^-, v^+\right)\right) \geq 0. \tag{3.3.8}$$

Next, since

$$\{\!\{v\}\!\} - v^- = \frac{1}{2}[\![v]\!] \quad \text{and} \quad \{\!\{v\}\!\} - v^+ = -\frac{1}{2}[\![v]\!], \tag{3.3.9}$$

it follows by the property ($\widehat{g}3$) of the numerical flux

$$\left|g(\omega, \{\!\{v\}\!\}) - \widehat{g}\left(\omega, v^-, v^+\right)\right| \leq \frac{1}{2}\left(L^-_{\widehat{g}} + L^+_{\widehat{g}}\right)|[\![v]\!]|.$$

Hence, $\widehat{a}(\widehat{g}; v)$ is bounded by $\frac{1}{2}\left(L^-_{\widehat{g}} + L^+_{\widehat{g}}\right)$. It is $\partial^2_u g(\omega, \{\!\{v\}\!\}) = f''(\{\!\{v\}\!\})$. Therefore, a Taylor expansion on the function $g(\omega, v^-)$ up to second order and (3.3.9) yield

$$g(\omega, v^-) = g(\omega, \{\!\{v\}\!\}) - \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\})[\![v]\!] + \frac{1}{8}f''(\{\!\{v\}\!\})[\![v]\!]^2. \tag{3.3.10}$$

Likewise, a Taylor expansion on the function $g(\omega, v^+)$ up to second order and (3.3.9) supply

$$g\left(\omega, v^+\right) = g(\omega, \{\!\{v\}\!\}) + \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\})[\![v]\!] + \frac{1}{8}f''(\{\!\{v\}\!\})[\![v]\!]^2, \tag{3.3.11}$$

Thus, by the E-flux property of the numerical flux, (3.3.10) as well as (3.3.11) follows

$$\begin{aligned}0 \leq & \left(g(\omega, v^-) - \widehat{g}\left(\omega, v^-, v^+\right)\right)[\![v]\!]^{-1} \\ =&\widehat{a}(\widehat{g}; v) - \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\}) + \frac{1}{8}f''(\{\!\{v\}\!\})|[\![v]\!]| \\ \leq&\widehat{a}(\widehat{g}; v) - \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\})[\![v]\!] + \frac{c_1^\star}{8}|[\![v]\!]|\end{aligned} \tag{3.3.12}$$

and

$$\begin{aligned}0 \leq & \left(g\left(\omega, v^+\right) - \widehat{g}\left(\omega, v^-, v^+\right)\right)[\![v]\!]^{-1} \\ =&\widehat{a}(\widehat{g}; v) + \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\}) + \frac{1}{8}f''(\{\!\{v\}\!\})|[\![v]\!]| \\ \leq&\widehat{a}(\widehat{g}; v) + \frac{1}{2}\partial_u g(\omega, \{\!\{v\}\!\})[\![v]\!] + \frac{c_1^\star}{8}|[\![v]\!]|.\end{aligned} \tag{3.3.13}$$

Hence, we obtain by the estimates (3.3.12) and (3.3.13) the inequality (3.3.7). □

[3] The proof is guided by Q. Zhang and Shu's proof presented in [90, Lemma 3.1].

### 3.3.1. A suboptimal a priori error estimate

In this section we prove the following a priori error estimate for the semi-discrete ALE-DG method with an arbitrary numerical flux, which has the properties ($\widehat{g}1$) - ($\widehat{g}3$).

**Theorem 3.3.1** *Let $u \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{H}^{k+1}(\Omega)\right)$ be the exact solution of the initial value problem* (1.1.1) *and the flux function $f : \mathbb{R} \to \mathbb{R}$ be an element of the space $\mathcal{C}^2(\mathbb{R})$. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a regular partition of the domain $\Omega$, the conditions $(P1)$ as well $(P2)$ are satisfied and the grid velocity satisfies the condition $(A4)$. In addition, the condition* (2.2.15) *is satisfied for the global length $h$ given by* (2.1.5). *Let $u_h$ be the solution of the semi-discrete ALE-DG method* (3.1.4) *with the test function space* (2.3.13) *given by piecewise polynomials of degree $k \geq 2$. The initial data for the method is the function $\mathcal{P}_h(u_0, 0)$. Then there exists a constant $C$ independent of $u_h$ and $h$, such that*

$$\max_{t\in[0,T]} \|u - u_h\|_{\mathrm{L}^2(\Omega)} \leq C h^{k+\frac{1}{2}}.$$

Before we start with the actual proof of theorem 3.3.1, we make some preparations. Henceforth $t \in [0, T]$ is an arbitrary fixed point and for all $j = 1, ..., N$ and any piecewise smooth function $p \in \mathrm{L}^2(\Omega)$ we apply the notation

$$u_{j-\frac{1}{2}} := u\left(x_{j-\frac{1}{2}}(t), t\right) \quad \text{and} \quad \widehat{a}(\widehat{g}; p)_{j-\frac{1}{2}} := \widehat{a}\left(\widehat{g}; p^-_{j-\frac{1}{2}}, p^+_{j-\frac{1}{2}}\right), \tag{3.3.14}$$

where the function $\widehat{a}(\widehat{g}; p)$ is given by (3.3.4). In addition, we define the quantities

$$\psi_h := u - \mathcal{P}_h(u, t) \quad \text{and} \quad \varphi_h := u_h - \mathcal{P}_h(u, t). \tag{3.3.15}$$

Then, we obtain the error function

$$e_h := u - u_h = \psi_h - \varphi_h. \tag{3.3.16}$$

Note that the smooth solution $u$ of the initial value problem (1.1.1) and the approximate solution $u_h$ given by the ALE-DG method satisfy the equation (3.1.4) as well as the equivalent equation (3.1.5). Hence, the equation (3.1.5) provides for all test functions $v \in \mathcal{V}_h(t)$

$$\begin{aligned} 0 = &\left(\partial_t e_h, v\right)_{K_j(t)} + \left(\partial_x\left(\omega e_h v\right), 1\right)_{K_j(t)} - \left(f(u) - f(u_h), \partial_x v\right)_{K_j(t)} \\ &+ g\left(\omega^n_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} - g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}} \\ &- \widehat{g}\left(\omega^n_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} + \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}}. \end{aligned} \tag{3.3.17}$$

Next, by a Taylor expansion up to second order on the flux function $f(u_h)$ we obtain

$$f(u_h) = f(u) - f'(u)\, e_h + \frac{1}{2} f''(\Theta)\left(e_h\right)^2, \tag{3.3.18}$$

where $\Theta$ is a value between $u$ and $u_h$. For all cell interface points $x_{j-\frac{1}{2}}(t)$, $j = 1, ..., N$, is

$$[\![u_h]\!]_{j-\frac{1}{2}} = -[\![e_h]\!]_{j-\frac{1}{2}}, \tag{3.3.19}$$

since we assume that the solution $u$ of the initial value problem (1.1.1) is sufficiently smooth. Likewise, we obtain for all $j = 1, ..., N$ by a Taylor expansion up to second order

$$\begin{aligned} g\left(\omega^n_{j-\frac{1}{2}}, \{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right) =& g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) \\ & - \partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right) \\ & + \frac{1}{2}\partial_u^2 g\left(\omega^n_{j-\frac{1}{2}}, \Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2, \end{aligned} \tag{3.3.20}$$

where $\Theta_{j-\frac{1}{2}}$ is a value between $u_{j-\frac{1}{2}}$ and $\{\!\{u_h\}\!\}_{j-\frac{1}{2}}$. Note that

$$\partial_u^2 g\left(\omega^n_{j-\frac{1}{2}}, \Theta_{j-\frac{1}{2}}\right) = f''\left(\Theta_{j-\frac{1}{2}}\right), \quad \text{for all } j = 1, ..., N.$$

Therefore, the quantity (3.3.4), the equation (3.3.19) and (3.3.20) provide

$$\begin{aligned} & g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - \widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}, u^+_{h,j-\frac{1}{2}}\right) \\ =& g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - g\left(\omega^n_{j-\frac{1}{2}}, \{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right) \\ & + \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} [\![u_h]\!]_{j-\frac{1}{2}} \\ =& \partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)\{\!\{e_h\}\!\}_{j-\frac{1}{2}} - \frac{1}{2}f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2 \\ & - \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} [\![e_h]\!]_{j-\frac{1}{2}}. \end{aligned} \tag{3.3.21}$$

Next, we consider the equation (3.3.17) with the test function $\varphi_h$. A summation from $j = 1$ to $N$ and the equations (3.3.18) as well as (3.3.21) supply

$$\begin{aligned}
0 =& \sum_{j=1}^{N} (\partial_t e_h, \varphi_h)_{K_j(t)} + \sum_{j=1}^{N} (\partial_x (\omega e_h \varphi_h), 1)_{K_j(t)} \\
&- \sum_{j=1}^{N} (f'(u)\, e_h, \partial_x \varphi_h)_{K_j(t)} \\
&+ \frac{1}{2} \sum_{j=1}^{N} \left( f''(\Theta)\, (e_h)^2, \partial_x \varphi_h \right)_{K_j(t)} \\
&- \sum_{j=1}^{N} \partial_u g \left( \omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}} \right) \{\!\{e_h\}\!\}_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}} \\
&+ \frac{1}{2} \sum_{j=1}^{N} f'' \left( \Theta_{j-\frac{1}{2}} \right) \left( \{\!\{e_h\}\!\}_{j-\frac{1}{2}} \right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}} \\
&+ \sum_{j=1}^{N} \widehat{a}\, (\widehat{g}; u_h)_{j-\frac{1}{2}} [\![e_h]\!]_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}},
\end{aligned} \tag{3.3.22}$$

since we consider the initial value problem (1.1.1) with periodic boundary conditions. The transport equation (2.2.4) provides

$$\begin{aligned}
&- \sum_{j=1}^{N} (\partial_t \varphi_h, \varphi_h)_{K_j(t)} - \sum_{j=1}^{N} \left( \partial_x \left( \omega\, (\varphi_h)^2 \right), 1 \right)_{K_j(t)} \\
&= - \frac{1}{2} \frac{d}{dt} \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)} - \frac{1}{2} \sum_{j=1}^{N} \left( \partial_x \left( \omega\, (\varphi_h)^2 \right), 1 \right)_{K_j(t)}.
\end{aligned} \tag{3.3.23}$$

Likewise, by the properties (2.2.19) and (2.2.20) of the $\mathrm{L}^2$-projection and the transport equation (2.2.4) we obtain

$$\begin{aligned}
&\sum_{j=1}^{N} (\partial_t \psi_h, \varphi_h)_{K_j(t)} + \sum_{j=1}^{N} (\partial_x (\omega \psi_h \varphi_h), 1)_{K_j(t)} \\
&= \sum_{j=1}^{N} (\partial_t (\psi_h \varphi_h), 1)_{K_j(t)} + \sum_{j=1}^{N} (\partial_x (\omega \psi_h \varphi_h), 1)_{K_j(t)} \\
&= \sum_{j=1}^{N} \frac{d}{dt} (\psi_h, \varphi_h)_{K_j(t)} = 0.
\end{aligned} \tag{3.3.24}$$

Therefore, by (3.3.23) and (3.3.24) the equation (3.3.22) can be written as follows

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} = a_1\left(\psi_h,\varphi_h\right) + a_2\left(e_h,\varphi_h\right) + a_3\left(\omega,\psi_h,\varphi_h\right), \tag{3.3.25}$$

where

$$\begin{aligned} a_1\left(\psi_h,\varphi_h\right) &:= -\sum_{j=1}^{N}\left(f'\left(u\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)}, \\ a_2\left(e_h,\varphi_h\right) &:= \frac{1}{2}\sum_{j=1}^{N}\left(f''\left(\Theta\right)\left(e_h\right)^2,\partial_x\varphi_h\right)_{K_j(t)} \\ &+ \frac{1}{2}\sum_{j=1}^{N} f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} a_3\left(\omega,\psi_h,\varphi_h\right) &:= \frac{1}{2}\sum_{j=1}^{N}\left(f'\left(u\right),\partial_x\left(\varphi_h\right)^2\right)_{K_j(t)} \\ &- \frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right),1\right)_{K_j(t)} \\ &- \sum_{j=1}^{N}\partial_u g\left(\omega^n_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)\{\!\{e_h\}\!\}_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}} \\ &+ \sum_{j=1}^{N}\widehat{a}\left(\widehat{g};u_h\right)_{j-\frac{1}{2}}[\![e_h]\!]_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}. \end{aligned}$$

In the following, we present auxiliary lemmas to estimate the quantities $a_1\left(\psi_h,\varphi_h\right)$, $a_2\left(e_h,\varphi_h\right)$ and $a_3\left(\omega,\psi_h,\varphi_h\right)$. According to lemma 2.3.1 the time dependent cells (2.3.5) yield for any $t\in[0,T]$ a regular partition of the domain $\Omega$, since we suppose the conditions $(P1)$ and $(P2)$. Therefore, the interpolation and inverse inequalities presented in lemma 2.2.3, lemma 2.2.4 and lemma 2.2.5 can be applied.

**Lemma 3.3.2** *Suppose the same assumptions as in theorem 3.3.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$a_1\left(\psi_h,\varphi_h\right) \leq C\left(h^{2k+2} + \left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{3.3.26}$$

**Proof.** For any cell $K_j\left(t\right)$, $j=1,...,N$, we denote the mean value of the solution $u$ by

$$\overline{u}_j\left(t\right) := \frac{1}{\triangle_j\left(t\right)}\left(u,1\right)_{K_j(t)}. \tag{3.3.27}$$

Since the solution $u$ of the initial value problem (1.1.1) satisfies the inequality (1.1.5), it follows by the mean value theorem

$$\max_{x\in K_j(t)} \left|f'\left(u\left(x,t\right)\right)-f'\left(\overline{u}_j\left(t\right)\right)\right| \leq c_1^\star \triangle_j\left(t\right) \leq c_1^\star h, \tag{3.3.28}$$

where the constant $c_1^\star$ is given by (3.3.2). Hence, we obtain by the property (2.2.19) of the $\mathrm{L}^2$-projection, Young's inequality, the interpolation property (2.2.22), the inverse inequality (2.2.16) and the estimate (3.3.28)

$$\begin{aligned} a_1\left(\psi_h,\varphi_h\right) =& \sum_{j=1}^N \left(\left(f'\left(\overline{u}_j\left(t\right)\right)-f'\left(u\right)\right)\psi_h, \partial_x\varphi_h\right)_{K_j(t)} \\ \leq& \frac{c_1^\star}{2}\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)} + \frac{c_1^\star}{2}h^2\left\|\partial_x\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} \\ \leq& C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \end{aligned}$$

□

Next, we present an estimate for the quantity $a_2\left(e_h,\varphi_h\right)$.

**Lemma 3.3.3** *Suppose the same assumptions as in theorem 3.3.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$a_2\left(e_h,\varphi_h\right) \leq C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{3.3.29}$$

**Proof.** First of all Young's inequality, the a priori assumption (3.3.1), the interpolation property (2.2.22) and the inverse inequality (2.2.16) provide

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^N \left(f''\left(\Theta\right)\left(e_h\right)^2, \partial_x\varphi_h\right)_{K_j(t)} \\ \leq& \frac{c_1^\star C_{\mathcal{F}}}{2} h \sum_{j=1}^N \left(\left|e_h\right|, \left|\partial_x\varphi_h\right|\right)_{K_j(t)} \\ \leq& \frac{c_1^\star C_{\mathcal{F}}}{4}\left(\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)} + \left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right) + \frac{c_1^\star C_{\mathcal{F}}}{2}h^2\left\|\partial_x\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)} \\ \leq& C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \end{aligned} \tag{3.3.30}$$

Likewise, by Young's inequality, Cauchy-Schwarz's inequality, the a priori assumption (3.3.1), the interpolation property (2.2.21) and the trace inequality (2.2.17) follows

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N} f''\left(\Theta_{j-\frac{1}{2}}\right)\left(\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}} \\
\leq&\frac{c_1^\star C_{\mathcal{F}}}{2} h \sum_{j=1}^{N}\left|\{\!\{e_h\}\!\}_{j-\frac{1}{2}}\right|\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right| \\
\leq&\frac{1}{2}c_1^\star C_{\mathcal{F}} h\left(\left|\psi^+_{h,j-\frac{1}{2}}\right|^2+\left|\psi^-_{h,j-\frac{1}{2}}\right|^2\right) \\
+&\frac{3}{2}c_1^\star C_{\mathcal{F}} h\left(\left|\varphi^+_{h,j-\frac{1}{2}}\right|^2+\left|\varphi^-_{h,j-\frac{1}{2}}\right|^2\right) \\
\leq& C\left(h^{2k+2}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).
\end{aligned}
\tag{3.3.31}
$$

Thus by (3.3.30) and (3.3.31) we obtain

$$
a_2\left(e_h,\varphi_h\right)\leq C\left(h^{2k+2}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).
$$

□

The next lemma is the main ingredient to proof theorem 3.3.1.

**Lemma 3.3.4** *Suppose the same assumptions as in theorem 3.3.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$
a_3\left(\omega,\psi_h,\varphi_h\right)\leq C\left(h^{2k+1}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).
\tag{3.3.32}
$$

**Proof.** In this proof, we apply the following identity: For all piecewise smooth functions $v, w \in \mathrm{L}^2(\Omega)$ holds

$$
[\![vw]\!]=\{\!\{v\}\!\}[\![w]\!]+[\![v]\!]\{\!\{w\}\!\},
\tag{3.3.33}
$$

where the jump $[\![\cdot]\!]$ and average $\{\!\{\cdot\}\!\}$ operators are given by (3.3.5). Since we consider the initial value problem (1.1.1) with periodic boundary conditions, it follows by the integration by parts formula and the identity (3.3.33)

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N}\left(f'(u),\partial_x(\varphi_h)^2\right)_{K_j(t)}-\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega(\varphi_h)^2\right),1\right)_{K_j(t)} \\
\leq&-\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x f'(u),(\varphi_h)^2\right)_{K_j(t)}-\frac{1}{2}\sum_{j=1}^{N}\partial_u g\left(\omega^n_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)[\![|\varphi_h|^2]\!]_{j-\frac{1}{2}} \\
\leq&\frac{c_2^\star}{2}\|\varphi_h\|^2_{\mathrm{L}(\Omega)}-\sum_{j=1}^{N}\partial_u g\left(\omega^n_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)\{\!\{\varphi_h\}\!\}_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}},
\end{aligned}
\tag{3.3.34}
$$

where the constant $c_2^\star$ is given by (3.3.3) and $\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right) = f'\left(u_{j-\frac{1}{2}}\right) - \omega_{j-\frac{1}{2}}^n$. Furthermore, lemma 3.3.1 provides that the quantity $\widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}}$ is non-negative and thus Young's inequality provides

$$\begin{aligned}
&\sum_{j=1}^{N} \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} [\![e_h]\!]_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}} \\
=&\sum_{j=1}^{N} \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} \left([\![\psi_h]\!]_{j-\frac{1}{2}} [\![\varphi_h]\!]_{j-\frac{1}{2}} - \left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right) \\
\leq&\frac{1}{2} \sum_{j=1}^{N} \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} \left(\left|[\![\psi_h]\!]_{j-\frac{1}{2}}\right|^2 - \left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right).
\end{aligned} \tag{3.3.35}$$

Therefore, we obtain by a modified version of Young's inequality[4] and (3.3.34) as well as (3.3.35)

$$\begin{aligned}
a_3\left(\omega, \psi_h, \varphi_h\right) \leq &\sum_{j=1}^{N} \partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right) \left(\left|\{\!\{\psi_h\}\!\}_{j-\frac{1}{2}}\right|^2 + \frac{1}{4}\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right) \\
&+\frac{1}{2} \sum_{j=1}^{N} \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} \left(\left|[\![\psi_h]\!]_{j-\frac{1}{2}}\right|^2 - \left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right) + \frac{c_2^\star}{2} \left\|\varphi_h\right\|_{\mathrm{L}(\Omega)}^2.
\end{aligned} \tag{3.3.36}$$

Since the grid velocity satisfies the condition $(A4)$, it follows

$$\left|\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right)\right| \leq c_0^\star + \max_{x \in \Omega \times [0,T]} \left|\omega\left(x,t\right)\right| \leq c_0^\star + c_0,$$

where the constants $c_0$ and $c_0^\star$ are given by (2.1.20) and (3.3.2). Thus, we obtain by Young's inequality and the interpolation property (2.2.21)

$$\begin{aligned}
&\sum_{j=1}^{N} \left|\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right)\right| \left|\{\!\{\psi_h\}\!\}_{j-\frac{1}{2}}\right|^2 + \sum_{j=1}^{N} \frac{1}{2} \widehat{a}\left(\widehat{g}; u_h\right)_{j-\frac{1}{2}} \left|[\![\psi_h]\!]_{j-\frac{1}{2}}\right|^2 \\
\leq&\frac{1}{2}\left(c_0^\star + c_0 + 2C_{\widehat{a}}\right) \sum_{j=1}^{N} \left(\left|\psi_{h,j-\frac{1}{2}}^+\right|^2 + \left|\psi_{h,j-\frac{1}{2}}^-\right|^2\right) \leq C h^{2k+1},
\end{aligned} \tag{3.3.37}$$

where the constant $C_{\widehat{a}}$ is given by (3.3.6). Next, the mean value theorem and the a priori assumption (3.3.1) yield

$$\begin{aligned}
&\left|\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right) - \partial_u g\left(\omega_{j-\frac{1}{2}}^n, \{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right)\right| \\
=&\left|f'\left(u_{j-\frac{1}{2}}\right) - f'\left(\{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right)\right| \\
\leq&c_1^\star \left|u_{j-\frac{1}{2}} - \{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right| \leq c_1^\star C_{\mathcal{F}} h,
\end{aligned} \tag{3.3.38}$$

[4] For all $a, b \in [0, \infty)$ is $ab \leq a^2 + \frac{1}{4} b^2$.

where the constant $c_1^\star$ is given by (3.3.2). Hence, the reverse triangle inequality, the a priori assumption (3.3.1), the inequality (3.3.38), the inequality (3.3.7) for the quantity $\widehat{a}\left(\widehat{g};\cdot\right)$, the equation (3.3.19) and the trace inequality (2.2.17) provide

$$\begin{aligned}
&\sum_{j=1}^{N}\left(\frac{1}{4}\left|\partial_u g\left(\omega^n_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)\right|-\frac{1}{2}\widehat{a}\left(\widehat{g};u_h\right)_{j-\frac{1}{2}}\right)\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\\
\leq&\frac{1}{4}\sum_{j=1}^{N}\left|\partial_u g\left(\omega^n_{j-\frac{1}{2}},u_{j-\frac{1}{2}}\right)-\partial_u g\left(\omega^n_{j-\frac{1}{2}},\{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right)\right|\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\\
+&\frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{2}\left|\partial_u g\left(\omega^n_{j-\frac{1}{2}},\{\!\{u_h\}\!\}_{j-\frac{1}{2}}\right)\right|-\widehat{a}\left(\widehat{g};u_h\right)_{j-\frac{1}{2}}\right)\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\\
\leq&\frac{c_1^\star C_{\mathcal{F}}}{4}h\sum_{j=1}^{N}\left(\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right)+\frac{c_1^\star}{8}\sum_{j=1}^{N}\left|[\![e_h]\!]_{j-\frac{1}{2}}\right|\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\\
\leq&\frac{1}{2}c_1^\star C_{\mathcal{F}}h\sum_{j=1}^{N}\left(\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2\right)\leq C\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)},
\end{aligned}\tag{3.3.39}$$

where the constant $c_1^\star$ is given by (3.3.2). Finally we obtain by (3.3.36), (3.3.37) and (3.3.39)

$$a_3\left(\omega,\psi_h,\varphi_h\right)\leq C\left(h^{2k+1}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right).$$

□

Note that the inequality (3.3.37) is the reason why we can merely prove the suboptimal a priori error estimate for the semi-discrete ALE-DG method. In order to prove the optimal a priori error estimate, we need more assumptions. The optimal error estimate for semi-discrete ALE-DG method will be discussed in the section 3.3.2.

Finally, we come to the actual proof of theorem 3.3.1.

**Proof of theorem 3.3.1.** We apply the estimates (3.3.26), (3.3.29) and (3.3.32) to estimate the equation (3.3.25) and obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\leq C\left(h^{2k+1}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right).$$

Hence, Gronwall's inequality provides

$$\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\leq\exp\left(2CT\right)\left(h^{2k+1}+\left\|\varphi_h\left(0\right)\right\|^2_{\mathrm{L}^2(\Omega)}\right).\tag{3.3.40}$$

It is $\left\|\varphi_h\left(0\right)\right\|^2_{\mathrm{L}^2(\Omega)}=0$, since $u_h(0)=\mathcal{P}_h(u_0,0)$. Therefore, the interpolation inequality (2.2.22) and the estimate (3.3.40) supply

$$\left\|e_h\right\|_{\mathrm{L}^2(\Omega)}\leq\left\|\psi_h\right\|_{\mathrm{L}^2(\Omega)}+\left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)}\leq Ch^{2k+1}.$$

This completes the proof. □

### 3.3.2. An optimal error estimate for the semi-discrete ALE-DG method with an upwind numerical flux

In order to achieve the optimal a priori error estimate for the one dimensional semi-discrete ALE-DG method, we assume that for all cells $K_j(t)$, $j = 1, ..., N$, the function $g(\omega, \cdot)$ given by (3.1.2) satisfies

$$\partial_u g(\omega(x,t), u) \geq 0, \quad \text{for all } u \in [m, M], \ t \in [0, T] \text{ and } x \in K_j(t), \tag{3.3.41}$$

or

$$\partial_u g(\omega(x,t), u) < 0, \quad \text{for all } u \in [m, M], \ t \in [0, T] \text{ and } x \in K_j(t), \tag{3.3.42}$$

where $m$ and $M$ are given by (1.2.20). This assumption provides the upcoming a priori error estimate.

**Theorem 3.3.2** *Let $u \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{H}^{k+2}(\Omega)\right)$ be the exact solution of the initial value problem* (1.1.1) *and the flux function $f : \mathbb{R} \to \mathbb{R}$ be an element of the space $\mathcal{C}^2(\mathbb{R})$. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$ and the grid velocity satisfies the condition $(A4)$. In addition, the condition* (2.2.15) *is satisfied for the global length $h$ given by* (2.1.5)*. Let $u_h$ be the solution of the semi-discrete ALE-DG method* (3.1.4) *with the upwind numerical flux* (3.1.6) *and the test function space* (2.3.13) *given by piecewise polynomials of degree $k \geq 1$. Moreover, for all cells $K_j(t)$, $j = 1, ..., N$, the function $g(\omega, \cdot)$ given by* (3.1.2) *satisfies* (3.3.41) *or* (3.3.42) *and the initial data for the method is the function $\mathcal{P}^-{}_h(u_0, 0)$, if the function $g(\omega, \cdot)$ satisfies* (3.3.41) *and $\mathcal{P}^+{}_h(u_0, 0)$, if function $g(\omega, \cdot)$ satisfies* (3.3.42)*. Then there exists a constant $C$ independent of $u_h$ and $h$, such that*

$$\max_{t \in [0,T]} \|u - u_h\|_{\mathrm{L}^2(\Omega)} \leq C h^{k+1}.$$

In the following, we assume that the function $g(\omega, \cdot)$ given by (3.1.2) satisfies (3.3.41) for all cells $K_j(t)$, $j = 1, ..., N$. The case in which the function $g(\omega, \cdot)$ satisfies (3.3.42) for all cells $K_j(t)$, can be analyzed similar. Since the function $g(\omega, \cdot)$ satisfies (3.3.41), the upwind numerical flux function (3.1.6) degenerates to

$$\widehat{g}\left(\omega^n_{j-\frac{1}{2}}, u_h^-, u_h^+\right) = g\left(\omega^n_{j-\frac{1}{2}}, u_h^-\right). \tag{3.3.43}$$

Hence, we apply the Gauss-Radau projection $\mathcal{P}_h^-(u, t)$ of the exact solution.Henceforth, $t \in [0, T]$ is an arbitrary fixed point and for all $j = 1, ..., N$ we apply the notation (3.3.14) and

$$\psi_h := u - \mathcal{P}_h^-(u, t) \quad \text{and} \quad \varphi_h := u_h - \mathcal{P}_h^-(u, t). \tag{3.3.44}$$

In addition, we denote the error function $e_h$ by (3.3.16). Then, the equation (3.1.5) yields for all test functions $v \in \mathcal{V}_h(t)$ the error equation

$$\begin{aligned}0 =& \left(\partial_t e_h, v_h\right)_{K_j(t)} + \left(\partial_x\left(\omega e_h v_h\right), 1\right)_{K_j(t)} - \left(f(u) - f(u_h), \partial_x v_h\right)_{K_j(t)}\\&+g\left(\omega^n_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} - g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}}\\&-g\left(\omega^n_{j+\frac{1}{2}}, u^-_{h,j+\frac{1}{2}}\right) v^-_{j+\frac{1}{2}} + g\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}\right) v^+_{j-\frac{1}{2}},\end{aligned} \tag{3.3.45}$$

since the smooth solution $u$ of the initial value problem (1.1.1) and the approximate solution $u_h$ given by the ALE-DG method satisfy the equation (3.1.4) as well as the equivalent equation (3.1.5). Next, a Taylor expansion on the function $g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)$ up to second order supplies

$$\begin{aligned}&g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) - g\left(\omega^n_{j-\frac{1}{2}}, u^-_{h,j-\frac{1}{2}}\right)\\=&\partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) e^-_{h,j-\frac{1}{2}} - \frac{1}{2} f''\left(\Theta^-_{j-\frac{1}{2}}\right)\left(e^n_{h,j-\frac{1}{2}}\right)^2,\end{aligned} \tag{3.3.46}$$

where $\Theta^-_{j-\frac{1}{2}}(t)$ is a value between $u_{j-\frac{1}{2}}$ and $u^-_{h,j-\frac{1}{2}}(t)$. Next, we consider the equation (3.3.45) with the test function $\varphi_h$. A summation from $j = 1$ to $N$ and the equations (3.3.18) as well (3.3.46) provide

$$\begin{aligned}0 =& \sum_{j=1}^{N}\left(\partial_t e_h, v_h\right)_{K_j(t)} + \sum_{j=1}^{N}\left(\partial_x\left(\omega e_h v_h\right), 1\right)_{K_j(t)}\\&-\sum_{j=1}^{N}\left(f'(u)\, e_h, \partial_x\varphi_h\right)_{K_j(t)}\\&+\frac{1}{2}\sum_{j=1}^{N}\left(f''(\Theta)\left(e_h\right)^2, \partial_x\varphi_h\right)_{K_j(t)}\\&+\frac{1}{2}\sum_{j=1}^{N}\left(f''\left(\Theta^-_{j-\frac{1}{2}}\right)\left(e^-_{h,j-\frac{1}{2}}\right)^2, [\![\varphi_h]\!]_{j-\frac{1}{2}}\right)\\&-\sum_{j=1}^{N}\left(\partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right) e^-_{h,j-\frac{1}{2}}, [\![\varphi_h]\!]_{j-\frac{1}{2}}\right),\end{aligned} \tag{3.3.47}$$

since we consider the initial value problem (1.1.1) with periodic boundary conditions. The property (2.3.18) of the Gauss-Radau projection $\mathcal{P}^-_h(u, t)$ provides

$$\sum_{j=1}^{N}\left(\partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)\psi^-_{h,j-\frac{1}{2}}, [\![\varphi_h]\!]_{j-\frac{1}{2}}\right) = 0. \tag{3.3.48}$$

Therefore, by (3.3.23) and (3.3.48) the equation (3.3.47) can be written as

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)}^2 = b_1\left(e_h, \varphi_h\right) + b_2\left(\omega, \varphi_h, \varphi_h\right) + b_3\left(\omega, \psi_h, \varphi_h\right), \tag{3.3.49}$$

where

$$\begin{aligned} b_1\left(e_h, \varphi_h\right) =& \frac{1}{2}\sum_{j=1}^{N}\left(f''\left(\Theta\right)\left(e_h\right)^2, \partial_x\varphi_h\right)_{K_j(t)} \\ &+ \frac{1}{2}\sum_{j=1}^{N} f''\left(\Theta_{j-\frac{1}{2}}^-\right)\left(e_{h,j-\frac{1}{2}}^-\right)^2 [\![\varphi_h]\!]_{j-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} b_2\left(\omega, \varphi_h, \varphi_h\right) =& \frac{1}{2}\sum_{j=1}^{N}\left(f'\left(u\right), \partial_x\left(\varphi_h\right)^2\right)_{K_j(t)} - \frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(\omega\left(\varphi_h\right)^2\right), 1\right)_{K_j(t)} \\ &+ \sum_{j=1}^{N}\partial_u g\left(\omega_{j-\frac{1}{2}}^n, u_{j-\frac{1}{2}}\right)\varphi_{h,j-\frac{1}{2}}^-[\![\varphi_h]\!]_{j-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} b_3\left(\omega, \psi_h, \varphi_h\right) =& \sum_{j=1}^{N}\left(\partial_t\psi_h, \varphi_h\right)_{K_j(t)} + \sum_{j=1}^{N}\left(\partial_x\left(\omega\psi_h\varphi_h\right), 1\right)_{K_j(t)} \\ &- \sum_{j=1}^{N}\left(f'\left(u\right)\psi_h, \partial_x\varphi_h\right)_{K_j(t)}. \end{aligned}$$

In the following, we present estimates for $b_1\left(e_h, \varphi_h\right)$, $b_2\left(\omega, \varphi_h, \varphi_h\right)$ and $b_3\left(\omega, \psi_h, \varphi_h\right)$. Since there exists for any time level $t = t_n$, $n = 0, ..., L$, a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$, the cells (2.3.5) have the properties $(P1)$ and $(P2)$ (cf. lemma 2.3.1), too. Therefore, we have for any $t \in [0, T]$ a regular partition of the domain $\Omega$. Hence, we can apply the interpolation and inverse inequalities presented in lemma 2.2.3, lemma 2.2.4 and lemma 2.3.2.

The quantity $b_1\left(e_h, \varphi_h\right)$ is similar to the quantity $a_2\left(e_h, \varphi_h\right)$ in the section 3.3.1. Thus, by the same techniques as in the proof of lemma 3.3.3 the following estimate can be proven

$$b_1\left(e_h, \varphi_h\right) \leq C\left(h^{2k+2} + \left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)}^2\right). \tag{3.3.50}$$

For any piecewise smooth function $v \in \mathrm{L}^2\left(\Omega\right)$ we obtain by (3.3.9)

$$\left(\{\!\{v\}\!\} - v^-\right)[\![v]\!] = \frac{1}{2}[\![v]\!]^2. \tag{3.3.51}$$

Hence, since the function $g(\omega, \cdot)$ given by (3.1.2) satisfies (3.3.41), the integration by parts formula and the identities (3.3.33) as well as (3.3.51) provide the following estimate for the quantity $b_2(\omega, \varphi_h, \varphi_h)$

$$\begin{aligned} b_2(\omega, \varphi_h, \varphi_h) =& -\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(f'(u)\right), (\varphi_h)^2\right) \\ &-\frac{1}{2}\sum_{j=\frac{1}{2}}^{N}\left(f\left(u_{j-\frac{1}{2}}\right)+\omega^n_{j-\frac{1}{2}}\right)[\![|\varphi_h|^2]\!]_{j-\frac{1}{2}} \\ &+\sum_{j=1}^{N}\partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)\varphi^-_{h,j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}} \\ =& -\sum_{j=1}^{N}\partial_u g\left(\omega^n_{j-\frac{1}{2}}, u_{j-\frac{1}{2}}\right)\left|[\![\varphi_h]\!]_{j-\frac{1}{2}}\right|^2 \\ &-\frac{1}{2}\sum_{j=1}^{N}\left(\partial_x\left(f'(u)\right), (\varphi_h)^2\right)_{K_j(t)} \le \frac{c_2^\star}{2}\|\varphi_h\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.3.52}$$

where the constant $c_2^\star$ is given by (3.3.3).
Next, we present an auxiliary lemma to estimate the quantity $b_3(\omega, \psi_h, \varphi_h)$.

**Lemma 3.3.5** *Suppose the same assumptions as in theorem 3.3.2. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$b_3(\omega, \psi_h, \varphi_h) \le C\left(h^{2k+2} + \|\varphi_h\|^2_{L^2(\Omega)}\right). \tag{3.3.53}$$

**Proof.** The equation (2.3.23) in lemma 2.3.3 provides

$$\begin{aligned} b_3(\omega, \psi_h, \varphi_h) =& \sum_{j=1}^{N}\left(\partial_t u - \mathcal{P}_h^-\left(\partial_t u, t\right), \varphi_h\right)_{K_j(t)} \\ &+\sum_{j=1}^{N}\left(\omega\partial_x u - \mathcal{P}_h^-\left(\omega\partial_x u, t\right), \varphi_h\right)_{K_j(t)} \\ &-\sum_{j=1}^{N}\left(\left(\omega\partial_x\psi_h, \varphi_h\right)_{K_j(t)} + \left(\partial_x\left(\omega\psi_h\varphi_h\right), 1\right)_{K_j(t)}\right) \\ &-\sum_{j=1}^{N}\left(f'(u)\,\psi_h, \partial_x\varphi_h\right)_{K_j(t)}. \end{aligned} \tag{3.3.54}$$

In the following, we will estimate the right-hand side of the equation (3.3.54).
First of all, it follows by Young's inequality and the interpolation property (2.2.22)

$$\begin{aligned}
&\sum_{j=1}^{N}\left(\partial_t u-\mathcal{P}_h^-\left(\partial_t u,t\right),\varphi_h\right)_{K_j(t)}\\
&+\sum_{j=1}^{N}\left(\omega\partial_x u-\mathcal{P}_h^-\left(\omega\partial_x u,t\right),\varphi_h\right)_{K_j(t)}\\
\le&\frac{1}{2}\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\\
&+\frac{1}{2}\left\|\omega\partial_x u-\mathcal{P}_h^-\left(\omega\partial_x u,t\right)\right\|^2_{\mathrm{L}^2(\Omega)}\\
\le&Ch^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)},
\end{aligned}\tag{3.3.55}$$

since $u\in \mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^{k+2}\left(\Omega\right)\right)$ and the grid velocity satisfies the condition $(A4)$. Next, the integration by parts formula provides

$$\begin{aligned}
&-\sum_{j=1}^{N}\left(\left(\omega\partial_x\psi_h,\varphi_h\right)_{K_j(t)}+\left(\partial_x\left(\omega\psi_h\varphi_h\right),1\right)_{K_j(t)}\right)\\
&-\sum_{j=1}^{N}\left(f'\left(u\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)}\\
=&\sum_{j=1}^{N}\left(\left(\partial_x\omega\right)\psi_h,\varphi_h\right)_{K_j(t)}-\sum_{j=1}^{N}\left(\partial_u g\left(\omega,u\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)},
\end{aligned}\tag{3.3.56}$$

since the function $g\left(\omega,\cdot\right)$ is given by (3.1.2). Furthermore, it follows by the inequality (3.3.28)

$$\begin{aligned}
&\max_{x\in K_j(t)}\left|\partial_u g\left(\omega,u\left(x,t\right)\right)-\partial_u g\left(\omega,\overline{u}_j\left(t\right)\right)\right|\\
\le&\max_{x\in K_j(t)}\left|f'\left(u\left(x,t\right)\right)-f'\left(\overline{u}_j\left(t\right)\right)\right|\le c_1^\star h,
\end{aligned}\tag{3.3.57}$$

where the constant $c_1^\star$ is given by (3.3.2) and $\overline{u}_j\left(t\right)$ is given by (3.3.27). Hence, we obtain by the property (2.3.17) of the Gauss-Radau projection, Young's inequality, the condition $(A4)$ for the grid velocity, the interpolation property (2.2.22), the inverse inequality (2.2.16) and the inequality (3.3.57)

$$\begin{aligned}
&\sum_{j=1}^{N}\left(\left(\left(\partial_x\omega\right)\psi_h,\varphi_h\right)_{K_j(t)}-\left(\partial_u g\left(\omega,u\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)}\right)\\
\leq&\sum_{j=1}^{N}\left(\partial_u\left(g\left(\omega,\overline{u}_j\left(t\right)\right)-g\left(\omega,u\right)\right)\psi_h,\partial_x\varphi_h\right)_{K_j(t)}\\
&+\frac{c_1}{2}\left(\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right)\\
\leq&\frac{c_1+c_1^\star}{2}\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)}+\frac{c_1}{2}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}+\frac{c_1^\star h^2}{2}\left\|\partial_x\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\\
\leq&C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right), \qquad (3.3.58)
\end{aligned}$$

where the constant $c_1$ is given by the condition (2.1.20). Finally, (3.3.54), (3.3.55), (3.3.56) and (3.3.58) provide

$$b_3\left(\omega,\psi_h,\varphi_h\right)\leq C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \qquad (3.3.59)$$

□

Now we come to the proof of theorem 3.3.2.
**Proof of theorem 3.3.2.** The equation (3.3.49) and the estimates (3.3.50), (3.3.52) and (3.3.53) provide

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\leq C\left(h^{2k+2}+\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\right). \qquad (3.3.60)$$

Next, we proceed similar to the proof of theorem 3.3.1. We evaluate the estimate (3.3.60) by Gronwall's inequality and obtain

$$\left\|\varphi_h\right\|^2_{\mathrm{L}^2(\Omega)}\leq\exp\left(2CT\right)h^{2k+2}. \qquad (3.3.61)$$

This estimate and the interpolation property (2.3.20) provide the desired error estimate. □

### 3.3.3. Justification of the a priori assumption

In this section, we justify the a priori assumption (3.3.1). We will proceed similar to Xu and Shu in [85]. However, we have to mention that the proof in this thesis is slightly different from Xu and Shu's proof, since we stated the a priori assumption (3.3.1) in the $\mathrm{L}^\infty$-norm instead of the $\mathrm{L}^2$-norm.
First of all, by the interpolation inequality (2.2.23), the inverse inequality (2.2.16) and (3.3.40) or (3.3.61), follows for all $t\in[0,T]$

$$\begin{aligned}
\left\|e_h\left(t\right)\right\|_{\mathrm{L}^\infty(\Omega)}&\leq\left\|\psi_h\left(t\right)\right\|_{\mathrm{L}^\infty(\Omega)}+\left\|\varphi_h\left(t\right)\right\|_{\mathrm{L}^\infty(\Omega)}\\
&\leq C\left(h^{k+\frac{1}{2}}+h^{-\frac{1}{2}}\left\|\varphi_h\left(t\right)\right\|_{\mathrm{L}^2(\Omega)}\right)\leq Ch^{k+\sigma}, \qquad (3.3.62)
\end{aligned}$$

where $\sigma = \frac{1}{2}$, if the assumptions of theorem 3.3.2 are satisfied, otherwise $\sigma = 0$. Next, we define

$$t^* := \sup \left\{ t \in [0, \infty) : \|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq C_{\mathcal{F}} h \right\}.$$

The a priori assumption (3.3.1) is verified, if $t^* = \infty$. If $t^* < \infty$, it follows by continuity

$$\|u(t^*) - u_h(t^*)\|_{L^\infty(\Omega)} = C_{\mathcal{F}} h. \tag{3.3.63}$$

The quantity $h \in (0, 1)$ can be chosen such that

$$Ch^{k+\sigma} \leq \frac{1}{2} C_{\mathcal{F}} h. \tag{3.3.64}$$

Therefore, (3.3.62), (3.3.63) and (3.3.64) yield the contradiction

$$\|u(t^*) - u_h(t^*)\|_{L^\infty(\Omega)} \leq Ch^{k+\sigma} \leq \frac{1}{2} C_{\mathcal{F}} h < \|u(t^*) - u_h(t^*)\|_{L^\infty(\Omega)},$$

if $t^* < T$. Hence, the a priori assumption (3.3.1) is true for the interval $[0, T]$.
The estimate (3.3.64) shows the reason why theorem 3.3.1 is not true, if the test function space (2.3.13) is given by piecewise polynomials of degree smaller than two.

## 3.4. Numerical experiments

In this section, we show the performance of the ALE-DG method. For the time discretization of the ALE-DG method we apply Shu's third order SSP-RK method. The coefficients of this method are listed in the table 1.2.1. We present numerical experiments for the one dimensional ALE-DG method (3.1.4) and verify numerically the theoretical results of the previous sections. Furthermore, we present the capability of the ALE-DG method for simplicial meshes in two dimensions by solving the two dimensional Euler equations with different initial data. The experiments for the one dimensional ALE-DG method were published by Klingenberg et al. in [48]. The two dimensional experiments have been done by Yinhua Xia.[5]

### 3.4.1. One dimensional Experiments

We consider two examples and compare the solution $u_h^S$ of the standard DG method for static grids, developed by Cockburn, Lin and Shu in [10, 11, 12], with the solution $u_h^M$ of the ALE-DG method. Moreover, For both methods we adopt a Lax-Friedrichs numerical flux. The standard DG method is applied on a static uniform grid. We have not developed a moving mesh methodology for the ALE-DG method. Thus, we apply for the ALE-DG method the following two different grid point distributions:

$$x_{j+\frac{1}{2}}(t) = x_{j+\frac{1}{2}}(0) + 0.4 \sin(t)(x_{j+\frac{1}{2}}(0) - 1)x_{j+\frac{1}{2}}(0) \tag{3.4.1}$$

[5] The author sincerely thanks Yinhua Xia that he provided his numerical experiments for this thesis.

and

$$x_{j+\frac{1}{2}}(t) = x_{j+\frac{1}{2}}(0) + 0.4\sin(t)H(x_{j+\frac{1}{2}}(0) - 0.5)(x_{j+\frac{1}{2}}(0) - 1))x_{j+\frac{1}{2}}(0), \tag{3.4.2}$$

where for all $x \in \mathbb{R}$

$$H(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0 \end{cases}$$

is the Heaviside function. In both scenarios the evolution of the grid points starts from a static uniform grid. It should be noted that the grid velocity, which corresponds to the grid points (3.4.2), does not satisfy the condition $(A4)$. However, we will see that even for the grid points (3.4.2) the method provides the optimal rate of convergence in smooth regions. This attested the capability of the ALE-DG method.

**Example 3.4.1 (Burgers' equation)** *We solve the following initial value problem in the domain $\Omega = (0, 1)$ with periodic boundary conditions*

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0, \quad u(x, 0) = \frac{1}{4} + \frac{1}{2}\sin(\pi(2x - 1)). \tag{3.4.3}$$

*The unique entropy solution of the problem* (3.4.3) *is smooth at time level $t = 0.1$ and has a well developed shock at time level $t = 0.4$. In order to demonstrate the behavior of the spatial error, we choose the time step small enough and apply a TVB limiter with the parameter $\widetilde{M} = 20$. Moreover, for both methods we apply piecewise $\mathcal{P}^2$ and $\mathcal{P}^3$ polynomials.*

*The table 3.4.1 shows the convergence history of the standard DG method as well as the ALE-DG method at time level $t = 0.1$ and the table 3.4.2 shows the convergence history when the shock is developed. In both tables, the convergence is presented with respect to the cell number $N$. For both methods, numerically the optimal rate of convergence can be reached. Furthermore, the TVB limiter is not affecting the high order accuracy in smooth regions. Moreover, in the figure 3.4.1 the exact solution of the equation* (3.4.3)*, the solution of the DG method for static grids and the ALE-DG solutions are plotted at time level $t = 0.4$. The DG and ALE-DG solution in 3.4.1 are calculated for $N = 80$ cells and piecewise polynomials of degree $k = 4$. The figure shows that for both methods the shock is captured and no spurious oscillations appear.*

*The table 3.4.3 shows the convergence history of the DG method for static grids and the ALE-DG method with $N = 40$ cells at time level $t = 0.1$ and $t = 0.4$. The convergence history is presented with respect to the degree $k$ of the piecewise polynomial test functions. We can see that the ALE-DG method maintains the spectral convergence property of the DG method for static grids. This indicates that the ALE-DG method is compatible with $p$-refinement.*

*In all these examples, the distribution of the grid points for the ALE-DG method was given by* (3.4.1)*. Thus, we present in table 3.4.4 the accuracy of the ALE-DG solutions $u_h^M$ for the grid point*

| | | $u-u_h^S$ | | $u-u_h^S$ | | $u-u_h^M$ | | $u-u_h^M$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $L^\infty$ norm | order | $L^2$ norm | order | $L^\infty$ norm | order | $L^2$ norm | order |
| $\mathcal{P}^2$ | 10 | 4.34E-03 | – | 9.10E-04 | – | 4.74E-03 | – | 9.87E-04 | – |
| | 20 | 7.53E-04 | 2.53 | 1.25E-04 | 2.86 | 8.10E-04 | 2.55 | 1.28E-04 | 2.95 |
| | 40 | 1.14E-04 | 2.72 | 1.70E-05 | 2.88 | 1.25E-04 | 2.70 | 1.72E-05 | 2.90 |
| | 80 | 1.60E-05 | 2.83 | 2.28E-06 | 2.90 | 1.76E-05 | 2.83 | 2.32E-06 | 2.89 |
| | 160 | 2.13E-06 | 2.91 | 3.00E-07 | 2.93 | 2.36E-06 | 2.90 | 3.08E-08 | 2.91 |
| $\mathcal{P}^3$ | 10 | 5.55E-04 | – | 7.46E-05 | – | 5.10E-04 | – | 7.47E-05 | – |
| | 20 | 4.16E-05 | 3.74 | 5.21E-06 | 3.84 | 3.58E-05 | 3.83 | 5.09E-06 | 3.88 |
| | 40 | 3.12E-06 | 3.74 | 3.66E-07 | 3.83 | 2.71E-06 | 3.72 | 3.51E-07 | 3.86 |
| | 80 | 2.11E-07 | 3.89 | 2.49E-08 | 3.88 | 1.83E-07 | 3.89 | 2.43E-08 | 3.85 |
| | 160 | 1.37E-08 | 3.94 | 1.66E-09 | 3.91 | 1.19E-08 | 3.94 | 1.64E-09 | 3.89 |

**Table 3.4.1.** The spatial errors of the DG solutions $u_h^S$ and the ALE-DG solutions $u_h^M$ at time level $t = 0.1$ for the Burgers' equation (3.4.3). The grid point distribution is given by (3.4.1).

| | | $u-u_h^S$ | | $u-u_h^S$ | | $u-u_h^M$ | | $u-u_h^M$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $L^\infty$ error | order | $L^2$ error | order | $L^\infty$ error | order | $L^2$ error | order |
| $\mathcal{P}^2$ | 10 | 5.81E-03 | – | 1.11E-03 | – | 1.72E-02 | – | 2.45E-03 | – |
| | 20 | 1.75E-04 | 5.05 | 3.81E-05 | 4.86 | 8.61E-04 | 4.32 | 8.47E-05 | 4.85 |
| | 40 | 2.41E-05 | 2.86 | 4.05E-06 | 3.23 | 3.26E-05 | 4.72 | 4.12E-06 | 4.36 |
| | 80 | 3.33E-06 | 2.86 | 4.59E-07 | 3.14 | 4.58E-06 | 2.83 | 4.90E-07 | 3.07 |
| | 160 | 4.37E-07 | 2.93 | 5.41E-08 | 3.08 | 6.09E-07 | 2.91 | 5.57E-08 | 3.14 |
| $\mathcal{P}^3$ | 10 | 2.26E-03 | – | 4.27E-04 | – | 5.39E-03 | – | 8.85E-04 | – |
| | 20 | 9.99E-06 | 7.82 | 1.51E-06 | 8.14 | 1.83E-04 | 4.88 | 1.81E-05 | 5.61 |
| | 40 | 8.40E-07 | 3.57 | 8.83E-08 | 4.10 | 1.25E-06 | 7.19 | 9.66E-08 | 7.55 |
| | 80 | 6.19E-08 | 3.76 | 5.37E-09 | 4.04 | 9.52E-08 | 3.71 | 5.95E-09 | 4.02 |
| | 160 | 4.17E-09 | 3.89 | 3.29E-10 | 4.03 | 6.49E-09 | 3.87 | 3.63E-10 | 4.03 |

**Table 3.4.2.** The spatial errors of the DG solutions $u_h^S$ and the ALE-DG solutions $u_h^M$ in smooth regions $\Omega = \{x : |x - shock| \geq 0.1\}$ at time level $t = 0.4$ for the Burgers' equation (3.4.3). The grid point distribution is given by (3.4.1).

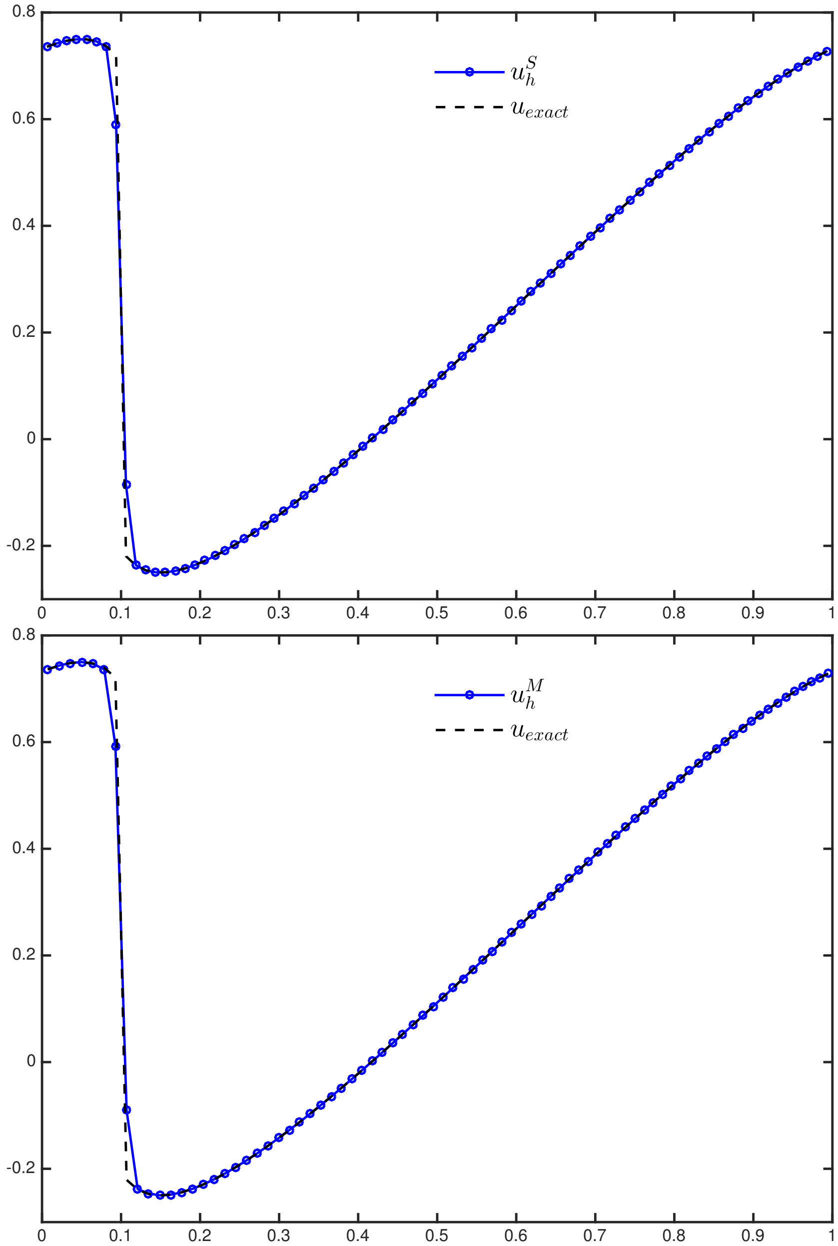


**Figure 3.4.1.** Comparison of the exact solution (dashed line) of the Burgers' equation (3.4.3), the DG solution $u_h^S$ (top) and the ALE-DG solution $u_h^M$ (bottom) with $N = 80$, $k = 4$, at time level $t = 0.4$. The grid point distribution is given by (3.4.1).

| | $t = 0.1$ | | $t = 0.4$ | |
|---|---|---|---|---|
| $k$ | $u - u_h^S$ | $u - u_h^M$ | $u - u_h^S$ | $u - u_h^M$ |
| 1 | 1.89E-03 | 1.77E-03 | 7.21E-04 | 7.25E-04 |
| 2 | 1.14E-04 | 1.25E-04 | 2.41E-05 | 3.25E-05 |
| 3 | 3.12E-06 | 2.71E-06 | 8.40E-07 | 1.25E-06 |
| 4 | 1.21E-07 | 1.44E-07 | 3.33E-08 | 5.47E-08 |
| 5 | 4.25E-09 | 3.40E-09 | 1.30E-09 | 2.40E-09 |
| 6 | 1.53E-10 | 1.97E-10 | 5.36E-11 | 1.09E-10 |
| 7 | 3.98E-12 | 3.55E-12 | 2.42E-12 | 5.07E-12 |
| 8 | 1.38E-13 | 1.52E-13 | 1.45E-13 | 3.50E-13 |
| 9 | 7.88E-15 | 4.14E-14 | 1.61E-14 | 7.70E-14 |

**Table 3.4.3.** The spatial $\mathrm{L}^\infty$-errors of the DG solutions $u_h^S$ and the ALE-DG solutions $u_h^M$ at time level $t = 0.1$ and $t = 0.4$ in a smooth region for the Burgers' equation (3.4.3) with $N = 40$. The grid point distribution is given by (3.4.1).

| | | $t = 0.1$ | | $t = 0.1$ | | $t = 0.4$ | | $t = 0.4$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $\mathrm{L}^\infty$ error | order | $\mathrm{L}^2$ error | order | $\mathrm{L}^\infty$ error | order | $\mathrm{L}^2$ error | order |
| | 10 | 4.46E-03 | – | 9.28E-04 | – | 5.76E-03 | – | 1.10E-03 | – |
| $\mathcal{P}^2$ | 20 | 7.61E-04 | 2.55 | 1.25E-04 | 2.89 | 1.76E-04 | 5.03 | 3.24E-05 | 5.09 |
| | 40 | 1.14E-04 | 2.74 | 1.67E-05 | 2.90 | 2.41E-05 | 2.87 | 3.51E-06 | 3.21 |
| | 80 | 1.60E-05 | 2.83 | 2.26E-06 | 2.89 | 3.33E-06 | 2.86 | 4.00E-07 | 3.13 |
| | 160 | 2.43E-06 | 2.72 | 3.05E-07 | 2.89 | 4.37E-07 | 2.93 | 5.05E-08 | 2.99 |
| $\mathcal{P}^3$ | 10 | 5.10E-04 | – | 7.08E-05 | – | 2.02E-03 | – | 3.40E-04 | – |
| | 20 | 3.58E-05 | 3.83 | 4.82E-06 | 3.88 | 9.78E-06 | 7.69 | 1.19E-06 | 8.16 |
| | 40 | 2.71E-06 | 3.72 | 3.40E-07 | 3.83 | 8.15E-07 | 3.58 | 7.30E-08 | 4.03 |
| | 80 | 1.83E-07 | 3.89 | 2.31E-08 | 3.88 | 6.19E-08 | 3.72 | 4.42E-09 | 4.05 |
| | 160 | 1.18E-08 | 3.95 | 1.55E-09 | 3.90 | 4.17E-09 | 3.89 | 2.70E-10 | 4.03 |

**Table 3.4.4.** The spatial errors of the ALE-DG solutions $u_h^M$ in smooth regions at time level $t = 0.1$ and $t = 0.4$ for the Burgers' equation (3.4.3). The grid point distribution is given by (3.4.1).

*distribution* (3.4.2). *The table shows that in this case the method also maintains numerically the high order accuracy in smooth regions.*

**Example 3.4.2 (Euler's equations in one dimension)** *In this example, the grid points for the ALE-DG method are given by* (3.4.1).
*For the one dimensional Euler equations, the unknowns are the mass density $\rho$, the velocity $v$, the energy density per unit mass $e$ and the pressure $p$. For these primitive variables the one dimensional Euler equations for a polytropic ideal gas are given by*

$$\partial_t \mathbf{w} + \mathbf{B}(\mathbf{w})\partial_x \mathbf{w} = 0 \tag{3.4.4}$$

*with*

$$\mathbf{w} := \begin{pmatrix} \rho \\ v \\ p \end{pmatrix}, \quad \mathbf{B}(\mathbf{w}) := \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \rho c^2 & v \end{pmatrix}, \quad c := \sqrt{\frac{\gamma p}{\rho}}$$

*and*

$$p = (\gamma - 1)\left(\rho E - \frac{\rho v^2}{2}\right), \quad \gamma > 1. \tag{3.4.5}$$

*The matrix $\mathbf{B}(\mathbf{w})$ has the eigenvalues $v$ and $v \pm c$. Hence, the system* (3.4.4) *is strictly hyperbolic. Furthermore, the system is symmetrizable (cf. Godlewski and Raviart [32, Chapter 1, Theorem 3.2 and Example 3.3]). The Euler equations* (3.4.4) *can be also written as the following system of conservation laws*

$$\begin{aligned} &\partial_t \mathbf{u} + \partial_x\Big(\mathbf{f}(\mathbf{u})\Big) = 0, \\ &\boldsymbol{u} = (\rho, m, E)^T, \quad \mathbf{f}(\mathbf{u}) = v\mathbf{u} + (0, p, pv)^T, \quad m := \rho v \end{aligned}$$

*with the equation* (3.4.5) *to determine the pressure. In the following, we solve the Euler equations in the domain $\Omega = (0,1)$ with periodic boundary conditions for two different initial functions. For the computation, we consider the equation* (3.4.5) *with $\gamma = 1.4$.*

*First of all, we solve the equations for a smooth initial function (plain wave) given by*

$$\begin{pmatrix} \rho(x,0) \\ v(x,0) \\ p(x,0) \end{pmatrix} = \begin{pmatrix} 1 + 0.5\sin(2\pi x) \\ 1 \\ 1 \end{pmatrix}.$$

*This initial value problem has the exact solution*

$$\begin{pmatrix} \rho(x,t) \\ v(x,t) \\ p(x,t) \end{pmatrix} = \begin{pmatrix} 1 + 0.5\sin(2\pi(x - t)) \\ 1 \\ 1 \end{pmatrix}.$$

| | | $\rho-\rho_h^S$ | | $\rho-\rho_h^S$ | | $\rho-\rho_h^M$ | | $\rho-\rho_h^M$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $L^\infty$ norm | order | $L^2$ norm | order | $L^\infty$ norm | order | $L^2$ norm | order |
| $\mathcal{P}^2$ | 10 | 2.63E-03 | – | 9.95E-04 | – | 5.14E-03 | – | 1.48E-03 | – |
| | 20 | 3.87E-04 | 2.77 | 1.42E-04 | 2.78 | 7.88E-04 | 2.70 | 2.20E-04 | 2.75 |
| | 40 | 5.10E-05 | 2.92 | 1.87E-05 | 2.93 | 1.06E-04 | 2.89 | 2.94E-05 | 2.91 |
| | 80 | 6.46E-06 | 2.98 | 2.38E-06 | 2.98 | 1.36E-05 | 2.96 | 3.75E-06 | 2.97 |
| | 160 | 8.08E-07 | 3.00 | 2.98E-07 | 2.99 | 1.71E-06 | 2.99 | 4.71E-07 | 2.99 |
| $\mathcal{P}^3$ | 10 | 7.23E-05 | – | 1.92E-05 | – | 1.91E-04 | – | 3.60E-05 | – |
| | 20 | 4.40E-06 | 4.04 | 1.07E-06 | 4.16 | 1.27E-05 | 3.90 | 1.97E-06 | 4.19 |
| | 40 | 2.74E-07 | 4.01 | 6.65E-08 | 4.01 | 8.07E-07 | 3.98 | 1.15E-07 | 4.10 |
| | 80 | 1.71E-08 | 4.00 | 4.14E-09 | 4.00 | 5.10E-08 | 3.98 | 6.99E-09 | 4.04 |
| | 160 | 1.07E-09 | 4.00 | 2.59E-10 | 4.00 | 3.20E-09 | 3.99 | 4.30E-10 | 4.02 |

**Table 3.4.5.** The spatial errors at time level $t = 1.2$ of the density $\rho_h^S$ given by the DG method and the errors at time level $t = 1.2$ for the density $\rho_h^M$ given by the ALE-DG method. The distribution of the grid points for the ALE-DG method is given by (3.4.1).

*The table 3.4.5 shows the convergence history at time $t = 1.2$ of the density given by the DG method for static grids and by the ALE-DG method with piecewise $\mathcal{P}^2$ and $\mathcal{P}^3$ polynomials. We can see that numerically the optimal rate of convergence can be reached.*

*Next, we solve the Euler equations for a discontinuous initial function given by*

$$\begin{pmatrix} \rho_L \\ v_L \\ p_L \end{pmatrix} = \begin{pmatrix} 1 \\ 0.75 \\ 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \rho_R \\ v_R \\ p_R \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0 \\ 1 \end{pmatrix}.$$

*This Riemann problem is a modified Sod shock tube problem. The exact solution of this problem is discontinuous. Hence, we apply the DG method for static grids and the ALE-DG method with a TVB limiter with $\widetilde{M} = 20$. The exact solution, the solution of the DG method and the solution of the ALE-DG method for $N = 200$ cells and piecewise polynomials of degree $k = 4$ are plotted at time level $t = 0.2$ in the figures 3.4.2, 3.4.3 and 3.4.4.*

The numerical experiments in the examples 3.4.1 and 3.4.2 show that the ALE-DG method maintains the properties of the DG method for static grids, like uniformly high order accuracy and shock capturing. Furthermore, the table 3.4.6 shows numerically that the ALE-DG method satisfies the geometric conservation law.

### 3.4.2. Two dimensional Experiments

In this section, we show the capability of the ALE-DG method for the two dimensional Euler equations. For these equations the unknowns are the mass density $\rho$, the velocity $(v, w)^T$, the

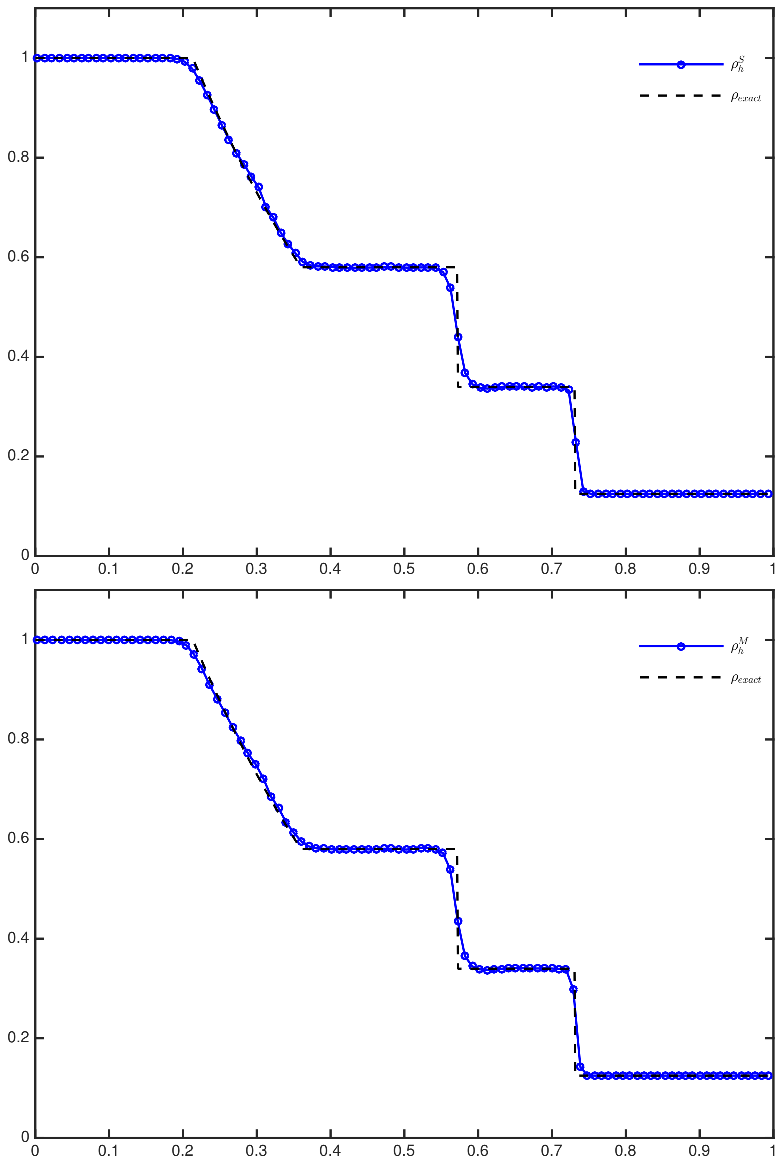


**Figure 3.4.2.** Comparison of the exact density (dashed line) for the modified Sod shock tube problem with the density $\rho_h^S$ (top) given by the DG method for static grids and the density $\rho_h^M$ (bottom) given by the ALE-DG method at time level $t = 0.2$. The DG and the ALE-DG method are applied with $N = 200$ cells, piecewise polynomials of degree $k = 4$ and a TVB limiter with $\widetilde{M} = 20$.

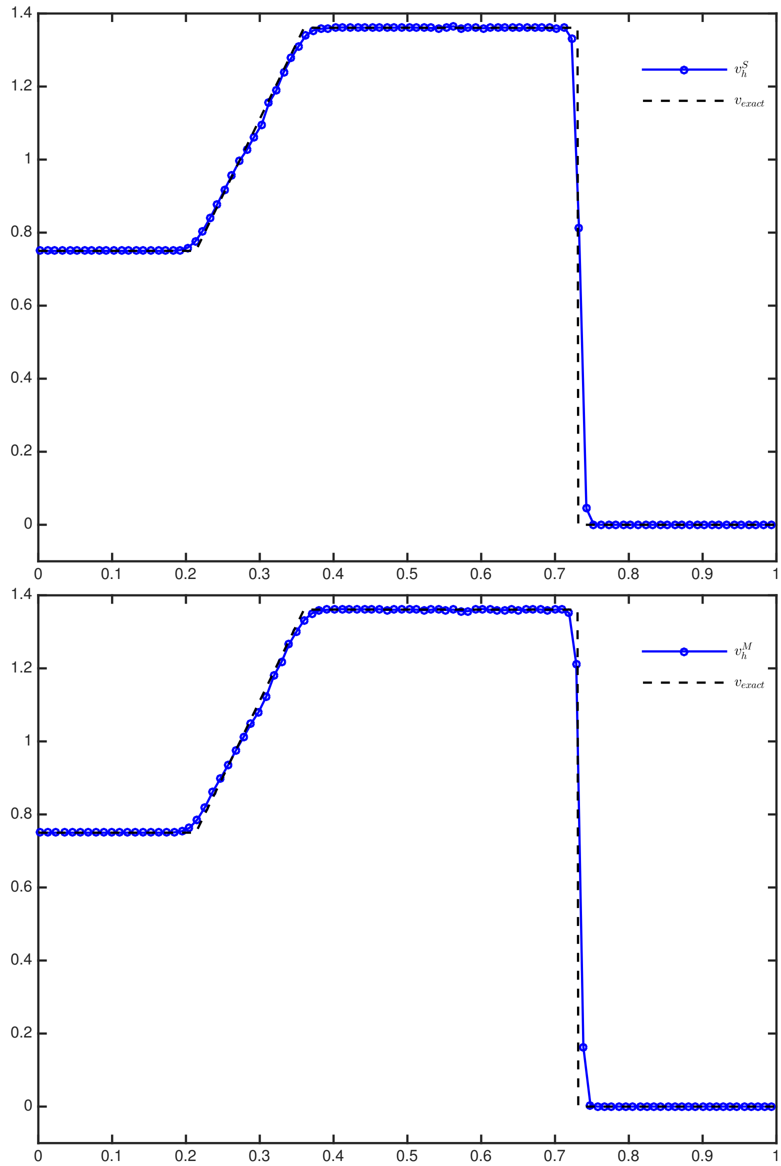

**Figure 3.4.3.** Comparison of the exact velocity (dashed line) for the modified Sod shock tube problem with the velocity $v_h^S$ (top) given by the DG method for static grids and the velocity $v_h^M$ (bottom) given by the ALE-DG method at time level $t = 0.2$. The DG and the ALE-DG method are applied with $N = 200$ cells, piecewise polynomials of degree $k = 4$ and a TVB limiter with $\widetilde{M} = 20$.

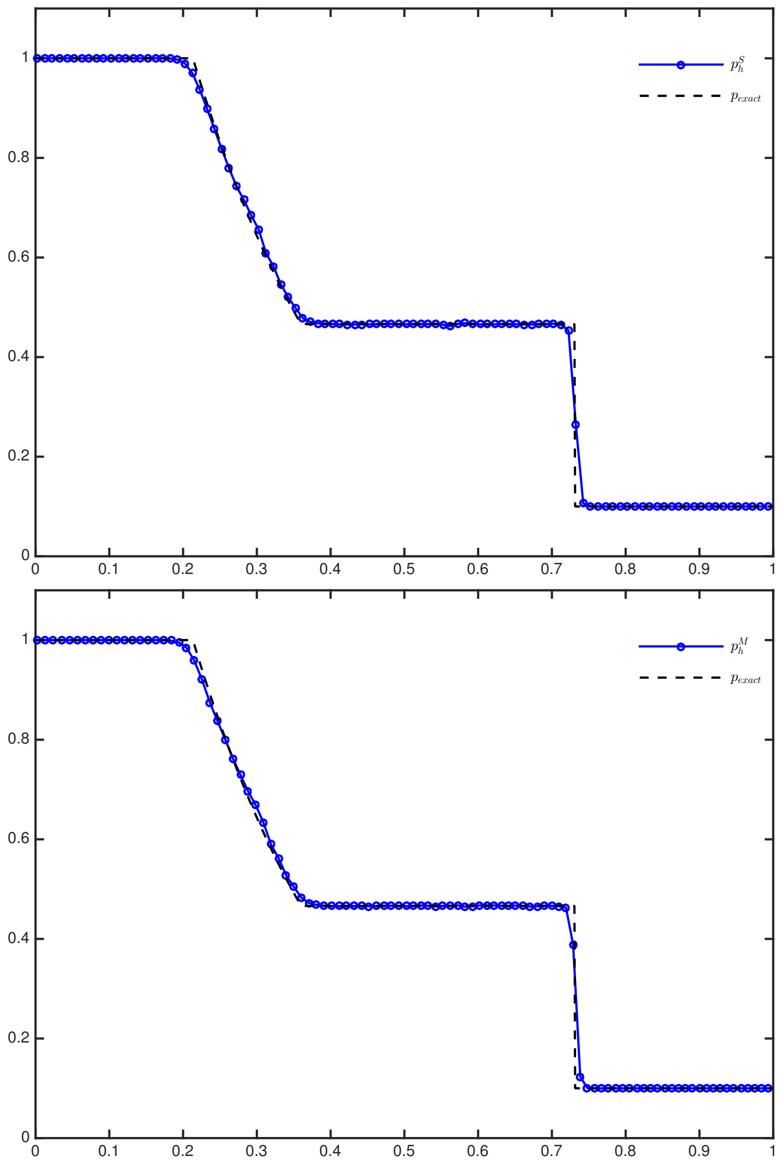


**Figure 3.4.4.** Comparison of the exact pressure (dashed line) for the modified Sod shock tube problem with the pressure $p_h^S$ (top) given by the DG method for static grids and the pressure $p_h^M$ (bottom) given by the ALE-DG method at time level $t = 0.2$. The DG and the ALE-DG method are applied with $N = 200$ cells, piecewise polynomials of degree $k = 4$ and a TVB limiter with $\widetilde{M} = 20$.

| $N$ | $\mathcal{P}^2$ | $\mathcal{P}^3$ | $\mathcal{P}^2$ | $\mathcal{P}^3$ |
|---|---|---|---|---|
| 10 | 4.44E-15 | 9.77E-15 | 4.44E-15 | 5.77E-15 |
| 20 | 9.99E-15 | 1.24E-14 | 5.77E-15 | 9.66E-15 |
| 40 | 1.24E-14 | 1.89E-14 | 9.55E-15 | 1.78E-14 |
| 80 | 2.22E-14 | 2.51E-14 | 1.77E-14 | 2.45E-14 |
| 160 | 2.80E-14 | 3.62E-14 | 3.24E-14 | 3.30E-14 |

**Table 3.4.6.** $\mathrm{L}^{\infty}$-errors at time level $t = 1.2$ for Burgers' equation (3.4.3) with the constant solution $u = 1$ and Euler's equations with the constant solution $(\rho, v, p) = (1, 1, 1)$. The distribution of the grid points for the ALE-DG method is given by (3.4.1).

energy density per unit mass $e$ and the pressure $p$. The two dimensional Euler equations for the primitive variables $\mathbf{w} = (\rho, v, w, p)^T$ are given by

$$\partial_t \mathbf{w} + \mathbf{B}_1(\mathbf{w})\partial_x \mathbf{w} + \mathbf{B}_2(\mathbf{w})\partial_y \mathbf{w} = 0 \tag{3.4.6}$$

with

$$\mathbf{B}_1(\mathbf{w}) := \begin{pmatrix} v & \rho & 0 & 0 \\ 0 & v & 0 & \frac{1}{\rho} \\ 0 & 0 & v & 0 \\ 0 & \rho c^2 & 0 & v \end{pmatrix}, \quad \mathbf{B}_1(\mathbf{w}) := \begin{pmatrix} w & 0 & \rho & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & \frac{1}{\rho} \\ 0 & 0 & \rho c^2 & w \end{pmatrix}, \quad c := \sqrt{\frac{\gamma p}{\rho}}$$

and

$$p = (\gamma - 1)\left(\rho E - \frac{\rho}{2}\left(v^2 + w^2\right)\right), \quad \gamma > 1. \tag{3.4.7}$$

The matrix $\mathbf{B}_1(\mathbf{w})$ has the eigenvalues $v$, $v$ as well as $v \pm c$ and the matrix $\mathbf{B}_2(\mathbf{w})$ has the eigenvalues $w$, $w$ as well as $w \pm c$. Hence, the system (3.4.6) is not a strictly hyperbolic, but it can be proven that the system is symmetrizable (cf. Godlewski and Raviart [32, Chapter 1, Theorem 3.2 and Example 3.3]). Moreover, the equations (3.4.6) can be written as the following system of conservation laws

$$\partial_t \mathbf{u} + \partial_x\Big(\mathbf{f}(\mathbf{u})\Big) + \partial_y\Big(\mathbf{g}(\mathbf{u})\Big) = 0 \tag{3.4.8}$$

with

$$\mathbf{u} := \begin{pmatrix} \rho \\ \rho v \\ \rho w \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) := \begin{pmatrix} \rho v \\ \rho v^2 + p \\ \rho v w \\ v\,(E + p) \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) := \begin{pmatrix} \rho w \\ \rho v w \\ \rho w^2 + p \\ w\,(E + p) \end{pmatrix},$$

and the equation (3.4.7) to determine the pressure.
We solve the two dimensional Euler equations for two different initial value functions and

compare the solution $\mathbf{u}_h^S$ of the standard DG method for static grids developed by Cockburn, Shu et. al. in [13, 14] with the solution $\mathbf{u}_h^M$ of the ALE-DG method. For both methods we adopt a Lax-Friedrichs numerical flux. The standard DG method is applied on a uniform simplicial mesh with the cell size $\triangle x = \triangle y = h$ given by

$$h := \begin{cases} h_0/2^m, & \text{if } m \geq 1, \\ \frac{1}{2} & \text{if } m = 0. \end{cases} \tag{3.4.9}$$

Since we have not developed a moving mesh methodology for the ALE-DG method, we apply the following grid point distribution:

$$x_{j+\frac{1}{2}}(t) = x_{j+\frac{1}{2}}(0) + 0.4\sin(t)(x_{j+\frac{1}{2}}(0) - 1)x_{j+\frac{1}{2}}(0) \tag{3.4.10a}$$

and

$$y_{j+\frac{1}{2}}(t) = y_{j+\frac{1}{2}}(0) + 0.4\sin(t)(y_{j+\frac{1}{2}}(0) - 1)y_{j+\frac{1}{2}}(0). \tag{3.4.10b}$$

The evolution of the grid points will start from a static uniform simplicial mesh with the cell size $h$ given by (3.4.9). The shape of the simplicial mesh for the ALE-DG method is shown in figure 3.4.5 for the start point and the final time point. The setup of the test examples for the two dimensional Euler equations can be found in the articles [88] and [93].

**Example 3.4.3 (Two dimensional plain wave)** *In this example, we consider the two dimensional Euler equations in the domain* $\Omega = (-1,1) \times (-1,1)$ *with periodic boundary conditions and the smooth initial function*

$$\begin{pmatrix} \rho\,(x,y,0) \\ v\,(x,y,0) \\ w\,(x,y,0) \\ p\,(x,y,0) \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\sin\left(\pi\left(x+y\right)\right) \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

*Furthermore, we apply* $\gamma = 1.4$ *in the equation* (3.4.7) *for the pressure. The exact solution of this initial value problem is the function*

$$\begin{pmatrix} \rho\,(x,y,t) \\ v\,(x,y,t) \\ w\,(x,y,t) \\ p\,(x,y,t) \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\sin\left(\pi\left(x+y-2t\right)\right) \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

*The table 3.4.7 shows the convergence history of the approximate density function given by the standard DG and the ALE-DG method at time level* $t = 1$ *in the* $\mathrm{L}^\infty$*-norm. For both methods, numerically the optimal rate of convergence can be reached.*

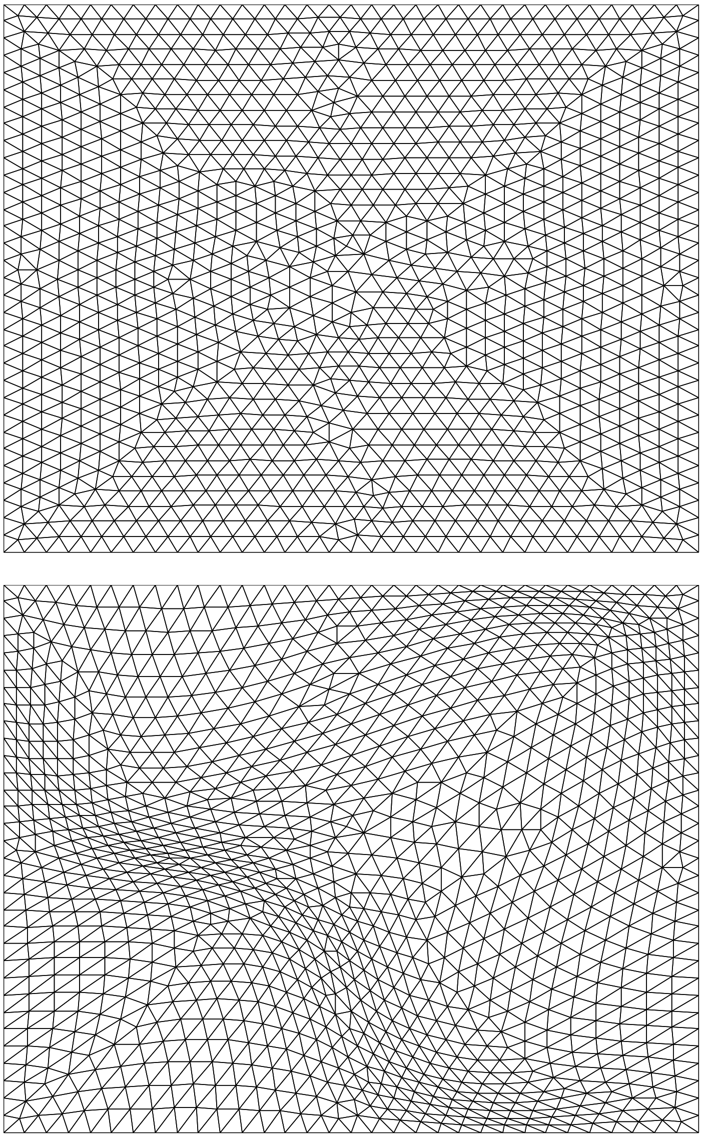

**Figure 3.4.5.** The simplicial mesh for the two dimensional ALE-DG method at time $t = 0$ (top) and the final time level (bottom).

| | $m$ | $\rho - \rho_h^M$ L$^\infty$-error | order | $\rho - \rho_h^S$ L$^\infty$-error | order | | $\rho - \rho_h^M$ L$^\infty$-error | order | $\rho - \rho_h^S$ L$^\infty$-error | order |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{P}^1$ | 1 | 2.94E-01 | – | 2.72E-01 | – | $\mathcal{P}^3$ | 3.39E-02 | – | 2.23E-02 | – |
| | 2 | 1.18E-01 | 1.32 | 6.14E-02 | 2.15 | | 6.65E-03 | 2.35 | 2.02E-03 | 3.46 |
| | 3 | 2.93E-02 | 2.01 | 1.49E-02 | 2.04 | | 4.66E-04 | 3.83 | 1.33E-04 | 3.92 |
| | 4 | 8.47E-03 | 1.79 | 4.55E-03 | 1.71 | | 2.60E-05 | 4.16 | 8.58E-06 | 3.95 |
| $\mathcal{P}^2$ | 1 | 1.31E-01 | – | 7.72E-02 | – | $\mathcal{P}^4$ | 5.56E-03 | – | 2.44E-03 | – |
| | 2 | 3.62E-02 | 1.86 | 1.93E-02 | 2.00 | | 6.12E-04 | 3.18 | 1.52E-04 | 4.00 |
| | 3 | 5.52E-03 | 2.71 | 3.25E-03 | 2.57 | | 2.85E-05 | 4.42 | 6.63E-06 | 4.52 |
| | 4 | 1.01E-03 | 2.31 | 4.33E-04 | 2.91 | | 9.18E-07 | 4.96 | 2.63E-07 | 4.67 |

**Table 3.4.7.** The spatial L$^\infty$-errors at time level $t = 1$ for the density $\rho_h^S$ given by the DG method and the errors at time level $t = 1$ for the density $\rho_h^M$ given by the ALE-DG method. The methods are used with the cell length (3.4.9).

| | $m$ | $\rho - \rho_h^M$ L$^\infty$-error | order | $\rho - \rho_h^S$ L$^\infty$-error | order | | $\rho - \rho_h^M$ L$^\infty$-error | order | $\rho - \rho_h^S$ L$^\infty$-error | order |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{P}^1$ | 1 | 1.15E-02 | – | 8.33E-03 | – | $\mathcal{P}^3$ | 3.21E-03 | – | 1.90E-03 | – |
| | 2 | 6.92E-03 | 0.73 | 2.90E-03 | 1.52 | | 1.18E-03 | 1.44 | 4.21E-04 | 2.17 |
| | 3 | 1.17E-03 | 2.56 | 1.32E-03 | 1.14 | | 1.12E-04 | 3.40 | 5.53E-05 | 2.93 |
| | 4 | 3.70E-04 | 1.66 | 2.62E-04 | 2.33 | | 2.59E-06 | 5.43 | 4.70E-06 | 3.56 |
| $\mathcal{P}^2$ | 1 | 5.48E-03 | – | 3.67E-03 | – | $\mathcal{P}^4$ | 1.37E-03 | – | 8.86E-04 | – |
| | 2 | 4.26E-03 | 0.36 | 8.84E-04 | 2.05 | | 2.32E-04 | 2.56 | 1.04E-04 | 3.09 |
| | 3 | 5.36E-04 | 2.99 | 2.63E-04 | 1.75 | | 8.59E-06 | 4.76 | 6.20E-06 | 4.07 |
| | 4 | 1.86E-04 | 1.53 | 4.40E-05 | 2.58 | | 4.45E-07 | 4.27 | 2.95E-07 | 4.39 |

**Table 3.4.8.** The spatial L$^\infty$-errors for the vortex problem at time level $t = 1$ for the density $\rho_h^S$ given by the DG method and the errors at time level $t = 1$ for the density $\rho_h^M$ given by the ALE-DG method. The methods are used with the cell length (3.4.9).

**Example 3.4.4 (Isentropic vortex evolution)** *We consider the two dimensional Euler equations in the domain* $\Omega = (0, 10) \times (0, 10)$ *with periodic boundary conditions. In addition, we apply* $\gamma = 1.4$ *in the equation* (3.4.7) *for the pressure.*
*The vortex evolution problem can be described as follows: The mean flow is* $(\rho, v, w, p)^T = (1, 1, 1, 1)^T$*. We add to the mean flow an isentropic vortex with no perturbation in the entropy* $S = \frac{p}{\rho^\gamma}$*. Then the perturbation values for the velocity* $(v, w)^T$*, the temperature* $T = \frac{p}{\rho}$ *and the entropy* $S = \frac{p}{\rho^\gamma}$ *are given by*

$$(\delta u, \delta v)^T = \frac{\varepsilon}{2\pi} \exp\left(\frac{1}{2}\left(1 - r^2\right)\right)(-\overline{y}, \overline{x})^T \tag{3.4.11}$$

*and*

$$\delta T = -\frac{(\gamma - 1)\,\varepsilon^2}{8\delta\pi^2} \exp\left(1 - r^2\right), \quad \delta S = 0, \tag{3.4.12}$$

*where* $\varepsilon = 5$ *is the vortex strength,* $(\overline{x}, \overline{y})^T = (x - 5, y - 5)^T$ *and* $r^2 = \overline{x}^2 + \overline{y}^2$*. Next, we solve the two dimensional Euler equations with the perturbed mean flow as initial function. This initial value problem has an exact smooth solution. The solution corresponds to a pure advection of the vortex at the mean flow (cf. Yee, Sandham and Djomehri [88, Section 3.1]). In the table* (3.4.8)*, the* $\mathrm{L}^\infty$*-errors at time level* $t = 1$ *for the Euler vortex problem are listed. We can recognize that the errors for the ALE-DG solution behave similar to the errors for the standard DG solution. Furthermore, both methods provide high order accurate approximations for the exact solution of the vortex problem.*

The numerical experiments in this section show that in two dimensions the ALE-DG method also maintains the uniformly high order accuracy of the DG method for static grids. In a future work, the capability of the method to handle shocks needs to be analyzed.

# 4. An arbitrary Lagrangian-Eulerian local discontinuous Galerkin (ALE-LDG) method for directly solving the Hamilton-Jacobi equations

In this chapter, we introduce and analyze an ALE-DG method for the initial value problems (1.1.8) and (1.1.9). The ALE method will degenerate on static grids to Yan and Osher's LDG method, which has been introduced in the section (1.2.3). Therefore, we call our method ALE-LDG method. Furthermore we prove a priori error estimates for the ALE-DG method. The optimal rate of converge for the one dimensional semi-discrete ALE-LDG method and the sub-optimal rate of converge for the two dimensional semi-discrete ALE-LDG method for regular simplicial meshes will be proven. For the proofs of the error estimates, we apply techniques, which have been introduce by Xiong, Shu and M. Zhang in [84] to proof a priori error estimates for the semi-discrete version of Yan and Osher's method.

## 4.1. The ALE-LDG for the Hamilton–Jacobi equations

In this section, the one dimensional as well as two dimensional ALE-LDG method are discussed. Therefore, we apply the time-dependent cells (2.3.5) for the one dimensional method and the cells (2.1.3) for the two dimensional analogue. Moreover, the geometric conservation law for the method will be discussed. Finally, we pay extra attention to the piecewise constant forward Euler step of the ALE-LDG method and show that the method is consistent to Crandall and Lions finite difference scheme, which was presented in [18].

### 4.1.1. The one dimensional semi-discrete ALE-LDG discretization

For the time interval $[t_n, t_{n+1}]$ and a cell $K_j(t)$, $j = 1, \ldots, N$, the description of the method ensues as follows. We approximate for all $t \in [t_n, t_{n+1})$ the solution $u$ of the equation (1.1.8) by the function $u_h \in \mathcal{V}_h(t)$, which is given by (3.1.1). The coefficients $u_0^j(t)$,...,$u_k^j(t)$ in (3.1.1) are the unknowns of the method. In order to determine these coefficients, we multiply the equation (1.1.8a) by a test function $v \in \mathcal{V}_h(t)$ and apply the transport equation (2.2.4). In general the function $u_h$ is discontinuous along the interface of two adjacent cells and thus $\partial_x u_h$ is merely defined in the interior of the cells. Therefore, we replace the quantity $G(\omega, \partial_x u_h)$ by a numerical flux $\widehat{G}\left(\omega,p_1,p_2\right)$, where

$$G\left(\omega,p\right)=H\left(p\right)-\omega p. \tag{4.1.1}$$

It should be noted that the function $G\left(\omega,p\right)$ depends on $t\in[t_n,t_{n+1})$ and $x\in K_j(t)$, but for convenience we skipped these variables in the definition. However, in some situations, we will highlight that $G\left(\omega,p\right)$ depends on a special point $x\in K_j(t)$ by the notation $G\left(\omega,p\right)|_x$. Finally, the semi-discrete method can be written as follows: Find a function $u_h\in\mathcal{V}_h(t)$, such that for all $v\in\mathcal{V}_h(t)$ and $j=1,...,N$ holds

$$\frac{d}{dt}\left(u_h,v\right)_{K_j(t)}-\left(\left(\partial_x\omega\right)u_h,v\right)_{K_j(t)}+\left(\widehat{G}\left(\omega,p_1,p_2\right),v\right)_{K_j(t)}=0. \tag{4.1.2}$$

The transport equation (2.2.4) provides that the equation (4.1.2) is for all $v\in\mathcal{V}_h(t)$ and all $j=1,...,N$ equivalent to the equation

$$\left(\partial_t u_h,v\right)_{K_j(t)}+\left(\omega\left(\partial_x u_h\right),v\right)_{K_j(t)}+\left(\widehat{G}\left(\omega,p_1,p_2\right),v\right)_{K_j(t)}=0. \tag{4.1.3}$$

Furthermore, the numerical flux is given by the following Lax Friedrichs flux

$$\widehat{G}\left(\omega,p_1,p_2\right):=G\left(\omega,\frac{p_1+p_2}{2}\right)-\frac{\lambda_j}{2}\left(p_2-p_1\right), \tag{4.1.4}$$

where

$$\lambda_j:=\max\left\{\left|\partial_p G\left(\omega,p\right)|_x\right|:\ p\in D_j\text{ and }x\in K_j\left(t\right)\right\} \tag{4.1.5}$$

with $D_j:=\left[\min\left(p_1,p_2\right),\max\left(p_1,p_2\right)\right]|_{K_j(t)}$. We would like to mention that the parameter $\lambda_j$ is constant in any cell $K_j\left(t\right)$. This property of the numerical flux will be essential in the proof of the a priori error estimates for the method. However, it should be noted that the dissipation of the Lax Friedrichs flux (4.1.4) is larger than for the local Lax Friedrichs flux, which Yan and Osher suggested for their LDG scheme in [86].
The variables $p_1$ and $p_2$ in (4.1.4) are used to approximate $\partial_x u_h$. We obtain these variables by solving the following two upwind schemes: Seek functions $p_1,p_2\in\mathcal{V}_h(t)$, such that for all $v_1,v_2\in\mathcal{V}_h(t)$ and all $j=1,...,N$ holds

$$\left(p_1,v_1\right)_{K_j(t)}+\left(u_h,\partial_x v_1\right)_{K_j(t)}-\left(u_h\right)^-_{j+\frac{1}{2}}v^-_{1,j+\frac{1}{2}}+\left(u_h\right)^-_{j-\frac{1}{2}}v^+_{1,j-\frac{1}{2}}=0 \tag{4.1.6}$$

and

$$\left(p_2,v_2\right)_{K_j(t)}+\left(u_h,\partial_x v_2\right)_{K_j(t)}-\left(u_h\right)^+_{j+\frac{1}{2}}v^-_{2,j+\frac{1}{2}}+\left(u_h\right)^+_{j-\frac{1}{2}}v^+_{2,j-\frac{1}{2}}=0. \tag{4.1.7}$$

### 4.1.2. The two dimensional semi-discrete ALE-LDG discretization for simplicial meshes

For the time interval $[t_n, t_{n+1}]$ and a cell $K(t) \in \mathcal{T}_{h(t)}$, the description of the two dimensional semi-discrete ALE-DG method ensues similar to the description of the one dimensional method, since the transport equation (2.2.4) holds for regular simplicial meshes. However, the two dimensional ALE-LDG method is a system of five equations, since a two dimensional Hamiltonian depends on more variables than its one dimensional analogue. Therefore, in two dimensions, we obtain the following method: Seek a function $u_h \in \mathcal{V}_{h,2}(t)$, such that for all $v \in \mathcal{V}_{h,2}(t)$ and all cells $K(t) \in \mathcal{T}_{h(t)}$ holds

$$\frac{d}{dt}(u_h, v)_{K(t)} - ((\text{div}\omega)\, u_h, v)_{K(t)} + \left(\widehat{G}(\omega, p_1, p_2, q_1, q_2), v\right)_{K(t)} = 0, \tag{4.1.8}$$

where the function $\widehat{G}(\omega, p_1, p_2, q_1, q_2)$ is a two dimensional Lax Friedrichs flux. For any cell $K(t) \in \mathcal{T}_{h(t)}$ and all $(x, y) \in K(t)$ this numerical flux is given by

$$\begin{aligned}&\widehat{G}(\omega, p_1, p_2, q_1, q_2)\\ :=&G\left(\omega, \frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) - \frac{\lambda_{K(t)}}{2}(p_2 - p_1) - \frac{\mu_{K(t)}}{2}(q_2 - q_1),\end{aligned} \tag{4.1.9}$$

where

$$G(\omega, p, q) := H(p, q) - \omega(x, y, t) \cdot (p, q)^T, \tag{4.1.10}$$

$$\lambda_{K(t)} := \max\left\{\left|\partial_p G(\omega, p, q)\,|_{(x,y)}\right| : \ p \in D_{K(t)},\ q \in E_{K(t)} \text{ and } (x, y) \in K(t)\right\} \tag{4.1.11}$$

and

$$\mu_{K(t)} := \max\left\{\left|\partial_q G(\omega, p, q)\,|_{(x,y)}\right| : \ p \in D_{K(t)},\ q \in E_{K(t)} \text{ and } (x, y) \in K(t)\right\} \tag{4.1.12}$$

with

$$D_{K(t)} := [\min(p_1, p_2), \max(p_1, p_2)]|_{K(t)}, \quad E_{K(t)} := [\min(q_1, q_2), \max(q_1, q_2)]|_{K(t)}.$$

Note that the function $G(\omega, p, q)$ depends on $(x, y)$, but for convenience we skipped this variable in the definition (4.1.10). Nevertheless, in some situations, we will highlight the dependence on a special point $(x, y) \in K(t)$ by the notation $G(\omega, p, q)|_{(x,y)}$ like in the definition of (4.1.11) and (4.1.12). Moreover, we would like to mention that the parameters $\lambda_{K(t)}$ and $\mu_{K(t)}$ are constant in any cell $K(t) \in \mathcal{T}_{h(t)}$.

The variables $p_1$ and $p_2$ in (4.1.9) are used to approximate $\partial_x u_h$. We obtain these variables by solving the following two upwind schemes:

Seek functions $p_1, p_2 \in \mathcal{V}_{h,2}(t)$, such that for all $v_1, v_2 \in \mathcal{V}_{h,2}(t)$ and all cells $K(t) \in \mathcal{T}_{h(t)}$ holds

$$(p_1, v_1)_{K(t)} + (u_h, \partial_x v_1)_{K(t)} - \left\langle u_h^-, v_1^{\text{int}_{K(t)}} n_{K(t),x}\right\rangle_{\partial K(t)} = 0, \tag{4.1.13}$$

$$(p_2, v_2)_{K(t)} + (u_h, \partial_x v_2)_{K(t)} - \left\langle u_h^+, v_2^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)} = 0, \tag{4.1.14}$$

where $n_{K(t),x}$ denotes the $x$-component of the outward unit normal for the cell $K(t)$ along the cell boundary $\partial K(t)$ and

$$u_h^+ := \begin{cases} u_h^{\mathrm{ext}_{K(t)}}, & \text{if } n_{K(t),x} > 0, \\ u_h^{\mathrm{int}_{K(t)}}, & \text{else,} \end{cases} \quad \text{as well as} \quad u_h^- := \begin{cases} u_h^{\mathrm{int}_{K(t)}}, & \text{if } n_{K(t),x} > 0, \\ u_h^{\mathrm{ext}_{K(t)}}, & \text{else.} \end{cases}$$

Likewise, the variables $q_1$ and $q_2$ are used to approximate $\partial_y u_h$. These variables are given by the following two upwind schemes: Seek functions $q_1, q_2 \in \mathcal{V}_{h,2}(t)$, such that for all $v_1, v_2 \in \mathcal{V}_{h,2}(t)$ and all cells $K(t) \in \mathcal{T}_{h(t)}$ holds

$$(q_1, v_1)_{K(t)} + (u_h, \partial_y v_1)_{K(t)} - \left\langle u_h^-, v_1^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} = 0, \tag{4.1.15}$$

$$(q_2, v_2)_{K(t)} + (u_h, \partial_y v_2)_{K(t)} - \left\langle u_h^+, v_2^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} = 0, \tag{4.1.16}$$

where $n_{K(t),y}$ denotes the $y$-component of the outward unit normal for the cell $K(t)$ along the cell boundary $\partial K(t)$ and

$$u_h^+ := \begin{cases} u_h^{\mathrm{ext}_{K(t)}}, & \text{if } n_{K(t),y} > 0, \\ u_h^{\mathrm{int}_{K(t)}}, & \text{else,} \end{cases} \quad \text{as well as} \quad u_h^- := \begin{cases} u_h^{\mathrm{int}_{K(t)}}, & \text{if } n_{K(t),y} > 0, \\ u_h^{\mathrm{ext}_{K(t)}}, & \text{else.} \end{cases}$$

Finally, we would like to mention that by the transport equation (2.2.4) for all $v \in \mathcal{V}_{h,2}(t)$ and all cells $K(t) \in \mathcal{T}_{h(t)}$ the equation (4.1.8) is equivalent to the following equation

$$(\partial_t u_h, v)_{K(t)} + (\omega \cdot \nabla u_h, v)_{K(t)} + \left(\widehat{G}(\omega, p_1, p_2, q_1, q_2), v\right)_{K(t)} = 0. \tag{4.1.17}$$

### 4.1.3. The GCL for the ALE-LDG method

In this section, we discuss the GCL for the ALE-LDG method. This will be done merely for the one dimensional ALE-LDG method, but the same arguments hold for the two dimensional method, too. However, the two dimensional calculation leads to a different result. We will mention the reasons for this during the one dimensional calculation.

According to theorem 1.1.4 $u = u_0 - H(0)\, t$ is the unique viscous solution of the problem (1.1.8), if the initial data $u_0$ is a constant function. Hence, the ALE-LDG method satisfies the GCL, if for constant initial data $u_0$ the approximate solution $u_h$ is given by $u_h = u_0 - H(0)\, t$. In the following, we assume that the approximate solution of the method is $u_h = u_0 - H(0)\, t$, where $u_0 \in \mathbb{R}$. Then $\partial_t u_h = -H(0)$ and $\partial_x u_h = 0$. Therefore, by (4.1.6) and (4.1.7) follows $(p_1, v)_{K_j(t)} = 0$ as well as $(p_2, v)_{K_j(t)} = 0$ for all $v \in \mathcal{V}_h(t)$ and $j = 1, ..., N$. Hence, we obtain by a density argument $p_1 = 0$ and $p_2 = 0$. Thus, (4.1.2) and (2.3.30) provide for all $v \in \mathcal{V}_h(t)$ and $j = 1, ..., N$

$$\frac{d}{dt}\left(u_h, v\right)_{K_j(t)} - \left(\frac{\triangle_j'(t)}{\triangle_j(t)} u_h, v\right)_{K_j(t)} + \left(H(0), v\right)_{K_j(t)} = 0. \tag{4.1.18}$$

This equation is the semi-discrete GCL condition for the ALE-LDG method (4.1.2). Next, we obtain

$$\frac{d}{dt}\left(u_h, v\right)_{K_j(t)} = -\left(H(0), v^* J(t)\right)_{(0,1)} + \left(\frac{d}{dt} J(t)\right)\left(u_h^*, v^*\right)_{(0,1)},$$

where $J(t) = \Delta_j(t)$ is the Jacobian determinant of the mapping (2.3.8) and the functions $u_h^*$ as well as $v^*$ are given by (2.2.1). Therefore, the semi-discrete GCL condition (4.1.18) projected to the reference cell provides the following ODE

$$\left(\frac{d}{dt} J(t)\right)\left(u_h^*, v^*\right)_{(0,1)} = \triangle_j'(t)\left(u_h^*, v^*\right)_{(0,1)}. \tag{4.1.19}$$

It is $u_h^* = u_h$, since $u_h$ does not depend on $x$. Thus, in the one dimensional case, the same arguments as in section (3.1.2) yield that the discrete geometric conservation law is satisfied for any first order time discretization method, e.g. the forward Euler step or a high order single step method. In the two dimensional case the right hand side in equation (4.1.19) depends linear on $t$. Hence, in the two dimensional case, the discrete geometric conservation law is satisfied for any second order time discretization method. Therefore, we proved the following result.

**Proposition 4.1.1** *The fully discrete ALE-LDG method* (4.1.2) *satisfies the discrete geometric conservation law for any first order time discretization method or high order single step method in which the stage order is equal or higher than first order. Furthermore, the fully discrete ALE-LDG method* (4.1.8) *satisfies the discrete geometric conservation law for any second order time discretization method.*

### 4.1.4. The ALE-LDG piecewise constant forward Euler step

Proposition 4.1.1 provides that the forward Euler step is compatible with the semi-discrete ALE-LDG methods (4.1.2) and (4.1.8). In the following, the one dimensional piecewise constant forward Euler step of the ALE-LDG method will be considered. We would like to mention that the two dimensional method can be analyzed by similar arguments.
Let $\overline{u}_j^n$ be a piecewise constant approximation for the solution $u$ of (1.1.8) at time level $t_n$ in the cell $K_j^n$, $j = 1, ..., N$. Then for any $j = 1, ..., N$ the equations (4.1.6) and (4.1.7) provide

$$p_1^n = \frac{\overline{u}_j^n - \overline{u}_{j-1}^n}{\triangle_j^n} \quad \text{and} \quad p_2^n = \frac{\overline{u}_{j+1}^n - \overline{u}_j^n}{\triangle_j^n}. \tag{4.1.20}$$

The identities (4.1.20) provide

$$p_1^n + p_2^n = \frac{1}{2\triangle_j^n}\left(\overline{u}_{j+1}^n - \overline{u}_{j-1}^n\right), \quad p_2^n - p_1^n = \frac{1}{\triangle_j^n}\left(\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n\right). \tag{4.1.21}$$

Furthermore, by (2.3.30), (2.3.31) and (3.2.6) follows

$$\begin{aligned}
&\frac{1}{\triangle_j^{n+1}}\left(\triangle_j^{n+1}\overline{u}_j^{n+1} - \triangle_j^n\overline{u}_j^n\right) - \frac{\triangle t}{\triangle_j^{n+1}}\left(\left(\partial_x\omega\right),1\right)_{K_j^n}\overline{u}_j^n \\
=&\overline{u}_j^{n+1} - \overline{u}_j^n + \frac{\triangle t}{\triangle_j^{n+1}}\left(\frac{\triangle_j^{n+1} - \triangle_j^n}{\triangle t} - \left(\omega_{j+\frac{1}{2}}^n - \omega_{j-\frac{1}{2}}^n\right)\right)\overline{u}_j^n \\
=&\overline{u}_j^{n+1} - \overline{u}_j^n.
\end{aligned} \tag{4.1.22}$$

The second order accurate midpoint quadrature method provides

$$(\omega,1)_{K_j^n} = \omega\left(\overline{x}_j^n, t_n\right)\triangle_j^n, \quad \overline{x}_j^n := \frac{1}{2}\left(x_{j+\frac{1}{2}}^n + x_{j-\frac{1}{2}}^n\right), \tag{4.1.23}$$

since the grid velocity is given by (2.3.12) in the space-time element $\overline{K(t)} \times [t_n, t_{n+1})$. Therefore, by (4.1.21), (4.1.22) and (4.1.23) the forward Euler step of the ALE-LDG method for piecewise constant approximations can be written as

$$\begin{aligned}
\overline{u}_j^{n+1} =&\overline{u}_j^n - \frac{\triangle t}{\triangle_j^{n+1}}\left(\widehat{G}\left(\omega^n, p_1^n, p_2^n\right), 1\right)_{K^n} \\
=&\left(1 - \lambda_j^n\frac{\triangle t}{\triangle_j^{n+1}}\right)\overline{u}_j^n - \triangle_j^n\frac{\triangle t}{\triangle_j^{n+1}}H\left(\frac{\overline{u}_{j+1}^n - \overline{u}_{j-1}^n}{2\triangle_j^n}\right) \\
&+\frac{1}{2}\triangle_j^n\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega\left(\overline{x}_j^n, t_n\right)\left(\frac{\overline{u}_{j+1}^n - \overline{u}_{j-1}^n}{\triangle_j^n}\right) + \lambda_j^n\left(\frac{\overline{u}_{j+1}^n + \overline{u}_{j-1}^n}{\triangle_j^n}\right)\right),
\end{aligned} \tag{4.1.24}$$

where

$$\lambda_j^n := \max\left\{\left|\partial_p G\left(\omega\left(x, t_n\right), p\right)\right| : \ p \in D_j^n \text{ and } x \in K_j^n\right\}$$

with $D_j^n := \left[\min\left(p_1^n, p_2^n\right), \max\left(p_1^n, p_2^n\right)\right]|_{K_j^n}$. Next, we define for all $a, b, c \in \mathbb{R}$ the function

$$\begin{aligned}
F\left(a,b,c\right) :=& \left(1 - \lambda_j^n\frac{\triangle t}{\triangle_j^{n+1}}\right)b - \triangle_j^n\frac{\triangle t}{\triangle_j^{n+1}}H\left(\frac{c-a}{2\triangle_j^n}\right) \\
&+\frac{1}{2}\triangle_j^n\frac{\triangle t}{\triangle_j^{n+1}}\left(\omega\left(\overline{x}_j^n, t_n\right)\left(\frac{c-a}{\triangle_j^n}\right) + \lambda_j^n\left(\frac{a+c}{\triangle_j^n}\right)\right).
\end{aligned}$$

The function $F = F(a,b,c)$ is increasing in all arguments, if the following CFL condition is satisfied

$$\frac{\triangle t}{h^{n+1}} \leq \frac{1}{\tau_{n+1}\lambda_j^n}, \tag{4.1.25}$$

where $h_{n+1}$ as well as $\tau_{n+1}$ are given by the condition $(P2)$. Hence, the scheme (4.1.24) is a monotone scheme, if the CFL condition (4.1.25) is satisfied. Moreover, in general the mesh parameters are chosen such that there exists a constant $C > 0$, independent of $\triangle t$, $\triangle_j^n$ and $\triangle_j^{n+1}$, with $\triangle_j^n \leq C\triangle_j^{n+1}$. Therefore, the first order version of the ALE-LDG method will converge to the viscous solution, since in [18] Crandall and Lions have proven that monotone schemes for Hamilton-Jacobi equations converge to the unique viscous solution.

## 4.2. A priori error estimates

In this section, we present a priori error estimates for the one and two dimensional semi-discrete ALE-LDG method with smooth solutions of the initial value problems (1.1.8) and (1.1.9). The a priori error estimates for the semi-discrete ALE-LDG method will be estimated in the sense of the global length $h$ given by (2.1.5). Furthermore, we will apply techniques, which were introduced by Xiong, Shu and M. Zhang in [84]. However, since we apply time-dependent cells in the method, there are some differences in the proof. First of all, we cannot utilize the ALE-LDG solution $u_h$ as a test function in the equations (4.1.2) and (4.1.8). Thus, we have to apply the equivalent equations (4.1.3) and (4.1.17) for the proofs. Furthermore, we have to use the transport equation (2.2.4) to manage the differentiation of the time-dependent volume integrals. Finally, we compensate the nonlinear nature of the Hamiltonian in (1.1.8a) and (1.1.9a) by a Taylor expansion like Xiong et al. Therefore, we need the following a priori assumptions for a smooth solution $u$ of the initial value problem (1.1.8) or (1.1.9) and the approximate solution $u_h$ given by the ALE-LDG method

$$\max_{t\in[0,T]} \left( \|\partial_x u - p_1\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_x u - p_2\|_{\mathrm{L}^\infty(\Omega)} \right) \leq C_{\mathcal{H}_1} h, \tag{4.2.1}$$

and

$$\max_{t\in[0,T]} \left( \|\partial_y u - q_1\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_y u - q_2\|_{\mathrm{L}^\infty(\Omega)} \right) \leq C_{\mathcal{H}_2} h, \tag{4.2.2}$$

where the constants $C_{\mathcal{H}_1}$ and $C_{\mathcal{H}_2}$ are independent of $u_h$ and $h$. Note that for convenience we skipped the index $t$ in (4.2.1) and (4.2.2). This will be also done in all the upcoming estimates, except in the justification of (4.2.1) and (4.2.2) at the end of this section. Furthermore, we would like to mention that the a priori assumptions (4.2.1) and (4.2.2) are slightly different from the a priori assumption in [84]. However, the a priori assumptions above supply

$$\max_{t\in[0,T]} \left( \left\| \partial_x u - \frac{p_1 + p_2}{2} \right\|_{\mathrm{L}^\infty(\Omega)} + \left\| \partial_y u - \frac{q_1 + q_2}{2} \right\|_{\mathrm{L}^\infty(\Omega)} \right) \leq Ch. \tag{4.2.3}$$

This is the a priori assumption, which was used by Xiong et al. in [84] to prove a priori error estimates for Yan and Osher's LDG scheme on a static cartesian grid. In order to evaluate the Hamiltonian in (1.1.8a) and (1.1.9a) by a Taylor expansion, we assume that $H \in \mathcal{C}^2(\mathbb{R})$ in the one dimensional case and $H \in \mathcal{C}^2(\mathbb{R}^2)$ in the two dimensional case. Furthermore, we assume that the Hamiltonian and its derivatives are bounded.

### 4.2.1. An optimal error estimate for the one dimensional method

In order to reach the optimal a priori error estimate for the ALE-LDG method, we introduce a further projection. For any cell $K_j(t) \in \mathcal{T}_{h(t)}$, $j = 1, ..., N$, we define the $\mathcal{Q}$-projection $\mathcal{Q}_h(u, t)$ of a function $u \in \mathrm{L}^2(\Omega)$ into the test function space $\mathcal{V}_h(t)$ by

$$\mathcal{Q}_h(u,t) := \begin{cases} \mathcal{P}_h(u,t), & \text{if } \partial_p G(\omega, \partial_x u) \text{ changes the sign in } K_j(t), \\ \mathcal{P}_h^-(u,t), & \text{if } \partial_p G(\omega, \partial_x u) > 0 \text{ in } K_j(t), \\ \mathcal{P}_h^+(u,t), & \text{if } \partial_p G(\omega, \partial_x u) < 0 \text{ in } K_j(t), \end{cases} \tag{4.2.4}$$

where $\mathcal{P}_h(\cdot, t)$ is the $\mathrm{L}^2$-projection (2.2.19) and $\mathcal{P}_h^{\pm}(\cdot, t)$ are the Gauss-Radau projections (2.3.17) as well as (2.3.18). In addition, the function $G(\omega, \cdot)$ is given by (4.1.1). The $\mathcal{Q}$-projection provides the following a priori error estimate.

**Theorem 4.2.1** *Let $u \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{H}^{k+2}(\Omega)\right)$ be the exact solution of the initial value problem* (1.1.8). *Suppose the Hamiltonian $H : \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathcal{C}^2(\mathbb{R})$, is bounded and has bounded derivatives. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$ and the grid velocity satisfies the condition $(A4)$. In addition, the condition* (2.2.15) *is satisfied for the global length $h$ given by* (2.1.5). *Let $u_h$ be the solution of the semi-discrete ALE-LDG method* (4.1.2), (4.1.6) *and* (4.1.7) *with the test function space* (2.3.13) *given by piecewise polynomials of degree $k \geq 2$. The initial data for the method is the function $\mathcal{Q}_h(u_0, 0)$. Then there exists a constant $C$ independent of $u_h$ and $h$, such that*

$$\max_{t \in [0,T]} \|u - u_h\|_{\mathrm{L}^2(\Omega)} \leq C h^{k+1}.$$

Before we start with the actual proof of theorem 4.2.1, we make some preparations. First of all, we define the quantities

$$\psi_h := u - \mathcal{Q}_h(u,t), \quad \varphi_h := u_h - \mathcal{Q}_h(u,t), \quad \eta_h := \partial_x u_h - \frac{p_1 + p_2}{2}. \tag{4.2.5}$$

Then the error function $e_h := u - u_h$ and the quantity $\partial_x u - \frac{p_1+p_2}{2}$ can be written as

$$e_h = \psi_h - \varphi_h \quad \text{and} \quad \partial_x u - \frac{p_1 + p_2}{2} = \partial_x(\psi_h - \varphi_h) + \eta_h. \tag{4.2.6}$$

Since we assume that the exact solution $u$ of the problem (1.1.8) is sufficiently smooth, we obtain for any cell interface point $x_{j-\frac{1}{2}}(t)$ the equation

$$[\![u_h]\!]_{j-\frac{1}{2}} = -[\![e_h]\!]_{j-\frac{1}{2}}. \tag{4.2.7}$$

Next, by a Taylor expansion on the Hamiltonian $H\left(\frac{p_1+p_2}{2}\right)$ up to second order, the Lax Friedrichs flux (4.1.4) can be written as

$$\begin{aligned}\widehat{G}\left(\omega, p_1, p_2\right) = &G\left(\omega, \partial_x u\right) - \partial_p G\left(\omega, \partial_x u\right)\left(\partial_x u - \frac{p_1+p_2}{2}\right)\\ &+ \frac{1}{2}\partial_p^2 H\left(\Theta\right)\left(\partial_x u - \frac{p_1+p_2}{2}\right)^2 - \frac{\lambda_j}{2}\left(p_2 - p_1\right),\end{aligned} \tag{4.2.8}$$

where $\Theta$ is a value between $\partial_x u$ and $\frac{p_1+p_2}{2}$. In addition, the transport equation (2.2.4) provides

$$\begin{aligned}&\left(\partial_t \varphi_h, \varphi_h\right)_{K_j(t)} - \left(\omega \partial_x \varphi_h, \varphi_h\right)_{K_j(t)}\\ =&\frac{1}{2}\frac{d}{dt}\left(\varphi_h, \varphi_h\right)_{K_j(t)} + \frac{1}{2}\left(\partial_x \omega, \left(\varphi_h\right)^2\right)_{K_j(t)}.\end{aligned} \tag{4.2.9}$$

The Lax Friedrichs flux (4.1.4) is consistent. Thus, the exact solution $u$ and the approximation solution $u_h$ satisfy the equation (4.1.2) and the equivalent equation (4.1.3). Therefore, we obtain by (4.1.3), (4.2.5), (4.2.6), (4.2.8) and (4.2.9) the following error equation

$$\begin{aligned}\frac{d}{dt}\left(\varphi_h, \varphi_h\right)_{K_j(t)} =&2a_{1,j}\left(\psi_h, \varphi_h\right) + 2a_{2,j}\left(\psi_h, \varphi_h, \eta_h\right)\\ &+2a_{3,j}\left(\psi_h, \varphi_{h)}\right),\end{aligned} \tag{4.2.10}$$

where

$$\begin{aligned}a_{1,j}\left(\psi_h, \varphi_h\right) :=& \left(\partial_t \psi_h, \varphi_h\right)_{K_j(t)} + \left(\omega \partial_x \psi_h, \varphi_h\right)_{K_j(t)}\\ &+\frac{1}{2}\left(\partial_x \omega, \left(\varphi_h\right)^2\right)_{K_j(t)},\end{aligned} \tag{4.2.11}$$

$$\begin{aligned}a_{2,j}\left(\psi_h, \varphi_h, \eta_h\right) :=& \left(\partial_p G\left(\omega, \partial_x u\right)\left(\partial_x\left(\psi_h - \varphi_h\right) + \eta_h\right), \varphi_h\right)_{K_j(t)}\\ &+\frac{1}{2}\left(\lambda_j\left(p_2 - p_1\right), \varphi_h\right)_{K_j(t)}\end{aligned} \tag{4.2.12}$$

and

$$a_{3,j}\left(\psi_h, \varphi_h\right) := -\frac{1}{2}\left(\partial_p^2 H\left(\Theta\right)\left(\partial_x u - \frac{p_1+p_2}{2}\right)^2, \varphi_h\right)_{K_j(t)}. \tag{4.2.13}$$

Moreover, since $\left(\psi_h, \partial_x \varphi_h\right)_{K_j(t)} = 0$, the equations (4.1.6) and (4.1.7) yield the error equations

$$\begin{aligned}0 =& \left(\partial_x\left(\psi_h - \varphi_h\right), \varphi_h\right)_{K_j(t)} + \left(\partial_x u_h - p_1, \varphi_h\right)_{K_j(t)}\\ &- \left(\psi^-_{h,j+\frac{1}{2}}\varphi^-_{h,j+\frac{1}{2}} - \psi^-_{h,j-\frac{1}{2}}\varphi^-_{h,j-\frac{1}{2}}\right) + \psi^-_{h,j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}\\ &+\frac{1}{2}\left(\left(\varphi^-_{h,j+\frac{1}{2}}\right)^2 - \left(\varphi^-_{h,j-\frac{1}{2}}\right)^2\right) + \frac{1}{2}\left([\![\varphi_h]\!]_{j-\frac{1}{2}}\right)^2\end{aligned} \tag{4.2.14}$$

and

$$\begin{aligned}0 =& \left(\partial_x\left(\psi_h-\varphi_h\right),\varphi_h\right)_{K_j(t)}+\left(\partial_x u_h-p_2,\varphi_h\right)_{K_j(t)}\\&-\left(\psi^+_{h,j+\frac{1}{2}}\varphi^-_{h,j+\frac{1}{2}}-\psi^+_{h,j-\frac{1}{2}}\varphi^-_{h,j-\frac{1}{2}}\right)+\psi^+_{h,j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}\\&+\frac{1}{2}\left(\left(\varphi^+_{h,j+\frac{1}{2}}\right)^2-\left(\varphi^+_{h,j-\frac{1}{2}}\right)^2\right)-\frac{1}{2}\left([\![\varphi_h]\!]_{j+\frac{1}{2}}\right)^2.\end{aligned} \tag{4.2.15}$$

In the following, we present estimates for the quantities (4.2.11), (4.2.12) and (4.2.13). During the proofs of the estimates, we will apply the inverse, trace and interpolation estimates presented in lemma 2.2.3, lemma 2.2.4 and lemma 2.3.2. This can be done, since the conditions $(P1)$ and $(P2)$ ensure that we have for any $t\in[0,T]$ a regular partition of the domain $\Omega$ (cf. lemma 2.3.1). Moreover, the a priori assumptions and the property (A4) of the grid velocity provide the following auxiliary result.

**Lemma 4.2.1** *Suppose $u\in \mathrm{W}^{1,\infty}\left(0,T;\mathrm{H}^2\left(\Omega\right)\right)$ and the Hamiltonian $H:\mathbb{R}\to\mathbb{R}$ belongs to the space $\mathcal{C}^2\left(\mathbb{R}\right)$, is bounded and has bounded derivatives. Moreover, the grid velocity satisfies the condition $(A4)$. Then there are constants $C_1^\star$, $C_2^\star$, $C_3^\star$ and $C_4^\star$, independent of $h$ and $u_h$, such that for any $j=1,\ldots,N$*

$$\left\|\partial_p G\left(\omega,\partial_x u\right)-\lambda_j\right\|_{\mathrm{L}^\infty(K_j(t))}\le C_1^\star h, \tag{4.2.16}$$

*if $\partial_p G\left(\omega,\partial_x u\right)>0$ in the cell $K_j\left(t\right)$,*

$$\left\|\partial_p G\left(\omega,\partial_x u\right)+\lambda_j\right\|_{\mathrm{L}^\infty(K_j(t))}\le C_2^\star h, \tag{4.2.17}$$

*if $\partial_p G\left(\omega,\partial_x u\right)<0$ in the cell $K_j\left(t\right)$,*

$$\left\|\partial_p G\left(\omega,\partial_x u\right)\right\|_{\mathrm{L}^\infty(K_j(t))}+\lambda_j\le C_3^\star h, \tag{4.2.18}$$

*if $\partial_p G\left(\omega,\partial_x u\right)$ changes the sign in the cell $K_j\left(t\right)$, where the function $G(\omega,\cdot)$ is given by* (4.1.1). *Moreover for all $j=2,\ldots,N$ holds*

$$\left|\lambda_j-\lambda_{j-1}\right|\le C_4^\star h. \tag{4.2.19}$$

**Proof.** Let $K_j(t)$ and $K_{j-1}(t)$, $j=2,\ldots,N$, be arbitrary cells. The function $G\left(\omega\left(t\right),\cdot\right)$ belongs to the space $\mathcal{C}^2\left(\mathbb{R}\right)$ for any $t\in[0,T]$, is bounded and has bounded derivatives, since we suppose that the Hamiltonian $H:\mathbb{R}\to\mathbb{R}$ has these properties and the grid velocity satisfies the condition $(A4)$. For convenience, in the following, we will skip the quantity $t\in[0,T]$ and write just $G\left(\omega,\cdot\right)$ for the function above.
Note that there are points $(\hat{p},\hat{x})\in D_j\times K_j\left(t\right)$ and $(\check{p},\check{x})\in D_{j-1}\times K_{j-1}\left(t\right)$, such that

$$\lambda_j=\left|\left.\partial_p G\left(\omega,\hat{p}\right)\right|_{\hat{x}}\right| \quad\text{and}\quad \lambda_{j-1}=\left|\left.\partial_p G\left(\omega,\check{p}\right)\right|_{\check{x}}\right|. \tag{4.2.20}$$

If $\partial_p G(\omega, \partial_x u) > 0$ in the cell $K_j(t)$, it follows for all $x \in K_j(t)$ by the reverse triangle inequality and (4.2.20)

$$\begin{aligned}
\left|\, \partial_p G(\omega, \partial_x u)|_x - \lambda_j \right| &= \left|\left|\, \partial_p G(\omega, \partial_x u)|_x\right| - \left|\, \partial_p G(\omega, \hat{p})|_{\hat{x}}\right|\right| \\
&\leq \left| \partial_p G(\omega, \partial_x u)|_x - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x \right| \\
&\quad + \left| \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} \right| \\
&\quad + \left| \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} - \partial_p G(\omega, \hat{p})|_{\hat{x}} \right|. 
\end{aligned} \tag{4.2.21}$$

Next, we obtain by the mean value theorem and the inequality (4.2.3)

$$\begin{aligned}
&\left| \partial_p G(\omega, \partial_x u)|_x - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x \right| \\
\leq &\max_{p \in \mathbb{R}} \left| \partial_p^2 G(\omega, p) \right| \left\| \partial_x u - \frac{p_1 + p_2}{2} \right\|_{\mathrm{L}^\infty(\Omega)} \leq Ch.
\end{aligned} \tag{4.2.22}$$

Likewise, it follows by the mean value theorem and (2.1.20)

$$\begin{aligned}
&\left| \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} \right| \\
= &\left| \omega(\hat{x}, t) - \omega(x, t) \right| \leq c_1 h.
\end{aligned} \tag{4.2.23}$$

In addition, we obtain by the mean value theorem and the a priori assumptions (4.2.1) as well as (4.2.2)

$$\begin{aligned}
&\left| \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} - \partial_p G(\omega, \hat{p})|_{\hat{x}} \right| \\
\leq &\max_{p \in \mathbb{R}} \left| \partial_p^2 G(\omega, p) \right| \left( \left\| \frac{p_1 + p_2}{2} - \hat{p} \right\|_{\mathrm{L}^\infty(\Omega)} \right) \\
\leq &\max_{p \in \mathbb{R}} \left| \partial_p^2 G(\omega, p) \right| \left( \| p_2 - \partial_x u \|_{\mathrm{L}^\infty(\Omega)} + \| \partial_x u - p_1 \|_{\mathrm{L}^\infty(\Omega)} \right) \leq Ch,
\end{aligned} \tag{4.2.24}$$

since $\hat{p} \in D_j := [\min(p_1, p_2), \max(p_1, p_2)]$. Therefore, by (4.2.21), (4.2.22), (4.2.23) and (4.2.24) follows

$$\| \partial_p G(\omega, \partial_x u) - \lambda_j \|_{\mathrm{L}^\infty(K_j(t))} \leq Ch.$$

If $\partial_p G(\omega, \partial_x u) < 0$ in the cell $K_j(t)$, it follows by the reverse triangle inequality and (4.2.20)

$$\begin{aligned}
\left|\partial_p G(\omega, \partial_x u)|_x + \lambda_j\right| &= \left|\left|\partial_p G(\omega, \hat{p})|_{\hat{x}}\right| - \left|\partial_p G(\omega, \partial_x u)|_x\right|\right| \\
&\leq \left|\partial_p G(\omega, \hat{p})|_{\hat{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}}\right| \\
&+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x\right| \\
&+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_x - \partial_p G(\omega, \partial_x u)|_x\right|. \qquad (4.2.25)
\end{aligned}$$

Thus, by (4.2.25), (4.2.22), (4.2.23) and (4.2.24) follows

$$\|\partial_p G(\omega, \partial_x u) + \lambda_j\|_{\mathrm{L}^\infty(K_j(t))} \leq Ch.$$

If $\partial_p G(\omega, \partial_x u)$ change the sign in the cell $K_j(t)$, exists at least one point $\tilde{x} \in K_j(t)$, such that $\partial_p G(\omega, \partial_x u)|_{\tilde{x}} = 0$. Hence, by a Taylor expansion in the point $\tilde{x}$ follows for all $x \in K_j(t)$

$$\partial_p G(\omega, \partial_x u)|_x = \frac{d}{dx}\left(\partial_p G(\omega, \partial_x u)|_\theta\right)(x - \tilde{x}),$$

where $\theta$ is a value between $x$ and $\tilde{x}$. Therefore, we obtain

$$\|\partial_p G(\omega, \partial_x u)\|_{\mathrm{L}^\infty(K_j(t))} \leq Ch.$$

Moreover, we obtain by the reverse triangle inequality and (4.2.20)

$$\begin{aligned}
\lambda_j &= \left|\left|\partial_p G(\omega, \hat{p})|_{\hat{x}}\right| - \left|\partial_p G(\omega, \partial_x u)|_{\tilde{x}}\right|\right| \\
&\leq \left|\partial_p G(\omega, \hat{p})|_{\hat{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}}\right| \\
&+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\tilde{x}}\right| \\
&+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\tilde{x}} - \partial_p G(\omega, \partial_x u)|_{\tilde{x}}\right|. \qquad (4.2.26)
\end{aligned}$$

The terms on the right hand side of the estimate (4.2.26) can be estimated similar to (4.2.22), (4.2.23) and (4.2.24). Therefore, we obtain $\lambda_j \leq Ch$.
The function $|\partial_p G(\omega, \partial_x u)|$ is for all $x \in \Omega$ continuous, since $u \in \mathrm{W}^{1,\infty}(0, T; \mathrm{H}^2(\Omega))$ and

$H \in \mathcal{C}^2(\mathbb{R})$. Thus, we obtain by the reverse triangle inequality

$$\begin{aligned}|\lambda_j - \lambda_{j-1}| &= \left|\left|\partial_p G(\omega, \hat{p})\right|_{\hat{x}}\right| - \left|\partial_p G(\omega, \check{p})\right|_{\check{x}}\right|\right| \\ &\leq \left|\partial_p G(\omega, \hat{p})|_{\hat{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}}\right| \\ &+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\hat{x}} - \partial_p G(\omega, \partial_x u)|_{\hat{x}}\right| \\ &+ \left|\partial_p G(\omega, \partial_x u)|_{\hat{x}} - \partial_p G(\omega, \partial_x u)|_{x_{j-\frac{1}{2}}(t)}\right| \\ &+ \left|\partial_p G(\omega, \partial_x u)|_{x_{j-\frac{1}{2}}(t)} - \partial_p G(\omega, \partial_x u)|_{\check{x}}\right| \\ &+ \left|\partial_p G(\omega, \partial_x u)|_{\check{x}} - \partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\check{x}}\right| \\ &+ \left|\partial_p G\left(\omega, \frac{p_1 + p_2}{2}\right)\Big|_{\check{x}} - \partial_p G(\omega, \check{p})|_{\check{x}}\right|. \end{aligned} \tag{4.2.27}$$

The terms on the right hand side of the estimate (4.2.27) can be estimated similar to (4.2.22), (4.2.23) and (4.2.24). Hence, it follows $|\lambda_j - \lambda_{j-1}| \leq Ch$. □

Next, we present an estimate for the quantity (4.2.11).

**Lemma 4.2.2** *Suppose the same assumptions as in theorem 4.2.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\sum_{j=1}^{N} a_{1,j}(\psi_h, \varphi_h) \leq C\left(h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.28}$$

**Proof.** The equation (2.3.23) provides

$$\begin{aligned} a_{1,j}(\psi_h, \varphi_h) &= (\partial_t u - \mathcal{Q}_h(\partial_t u, t), \varphi_h)_{K_j(t)} \\ &+ (\omega \partial_x u - \mathcal{Q}_h(\omega \partial_x u, t), \varphi_h)_{K_j(t)} \\ &+ \frac{1}{2}\left(\partial_x \omega, (\varphi_h)^2\right)_{K_j(t)}. \end{aligned} \tag{4.2.29}$$

Next, we sum the equation (4.2.29) from $j = 1$ to $N$ and utilize Young's inequality as well as the interpolation estimate (2.3.20). This provides the inequality (4.2.28), since the condition (A4) ensures that $\omega$ and $\partial_x \omega$ are bounded by constants, which are independent of $h$, and the initial value problem (1.1.8) is considered with periodic boundary conditions. □

The upcoming lemma is motivated by a result of Xiong, Shu and M. Zhang [84, Lemma 3.2].

**Lemma 4.2.3** *Suppose the same assumptions as in theorem 4.2.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\|\partial_x u_h - p_1\|_{\mathrm{L}^2(\Omega)} + \|\partial_x u_h - p_2\|_{\mathrm{L}^2(\Omega)} \leq C\left(h^k + h^{-1}\|\varphi_h\|_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.30}$$

**Proof.** We consider for any cell $K_j(t)$, $j = 1, ..., N$, and for all $v \in \mathcal{V}_h(t)$ the equation

$$(\partial_x u_h, v)_{K_j(t)} + (u_h, \partial_x v)_{K_j(t)} - u^-_{h,j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - u^+_{h,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0. \tag{4.2.31}$$

Next, we subtract the equation (4.1.6) from (4.2.31) and sum the result from $j = 1$ to $N$. Hence, we obtain by (4.2.7) for all $v \in \mathcal{V}_h(t)$

$$\sum_{j=1}^{N} (\partial_x u_h - p_1, v)_{K_j(t)} = -\sum_{j=1}^{N} [\![u_h]\!]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = \sum_{j=1}^{N} [\![e_h]\!]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}, \tag{4.2.32}$$

since we consider the initial value problem (1.1.8) with periodic boundary conditions. Next, we consider the equation (4.2.32) with the test function $\partial_x u_h - p_1$ and apply the equations (4.2.6), Cauchy-Schwarz's inequality, Young's inequality, (2.3.19) and (2.2.17). This provides

$$\begin{aligned} \|\partial_x u_h - p_1\|^2_{\mathrm{L}^2(\Omega)} &\leq \left( \|\psi_h\|_{\Gamma_{h(t)}} + \|\varphi_h\|_{\Gamma_{h(t)}} \right) \|\partial_x u_h - p_1\|_{\Gamma_{h(t)}} \\ &\leq C h^{-1} \left( h^{k+1} + \|\varphi_h\|_{\mathrm{L}^2(\Omega)} \right) \|\partial_x u_h - p_1\|_{\mathrm{L}^2(\Omega)} . \end{aligned} \tag{4.2.33}$$

Next, we subtract the equation (4.1.7) from (4.2.31) and sum the result from $j = 1$ to $N$. This yields

$$\|\partial_x u_h - p_2\|^2_{\mathrm{L}^2(\Omega)} \leq C h^{-1} \left( h^{k+1} + \|\varphi_h\|_{\mathrm{L}^2(\Omega)} \right) \|\partial_x u_h - p_2\|_{\mathrm{L}^2(\Omega)} . \tag{4.2.34}$$

Therefore, the estimates (4.2.33) and (4.2.34) supply (4.2.30). □

It should be noted that the inequality (4.2.30) provides

$$\|\eta_h\|_{\mathrm{L}^2(\Omega)} \leq C \left( h^k + h^{-1} \|\varphi_h\|_{\mathrm{L}^2(\Omega)} \right). \tag{4.2.35}$$

This inequality yields the essential auxiliary lemma to prove theorem 4.2.1.

**Lemma 4.2.4** *Suppose the same assumptions as in theorem 4.2.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\sum_{j=1}^{N} a_{2,j} \left( \psi_h, \varphi_h, \eta_h \right) \leq C \left( h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)} \right). \tag{4.2.36}$$

**Proof.** This proof is guided by a proof of Xiong, Shu and M. Zhang, which was published in [84, Theorem 3.1].
We will prove the inequality (4.2.36) in two steps. In the first step, we consider different families of consecutive cells $\{K_j(t)\}_{j=j_1}^{j_2}$ with $1 \leq j_1 \leq j_2 \leq N$ and estimate the quantity (4.2.12) summed over the consecutive cells. In the course of the proof, we multiply the equations (4.2.14) and (4.2.15) with the parameter $\lambda_j$ given by (4.1.5) several times. This is possible, since

the parameter $\lambda_j$ is constant in any cell $K_j(t)$, $j = 1, ..., N$. In the final step, we apply the estimates, which have been proven in the first step, to estimate the quantity (4.2.12) summed over all cells.

***Case 1)*** We consider an arbitrary family of consecutive cells $\{K_j(t)\}_{j=j_1}^{j_2}$ with $1 \leq j_1 \leq j_2 \leq N$ and $\partial_p G(\omega, \partial_x u)$ changes the sign in any cell $K_j(t)$, $j = j_1, ..., j_2$. First of all, for any cell $K_j(t)$, $j = j_1, ..., j_2$, we add the equations (4.2.14) multiplied by $-\frac{\lambda_j}{2}$ and (4.2.15) multiplied by $\frac{\lambda_j}{2}$ to the quantity $a_{2,j}(\psi_h, \varphi_h, \eta_h)$. Then it follows

$$\begin{aligned}
&a_{2,j}(\psi_h, \varphi_h, \eta_h) \\
=&\left(\partial_p G(\omega, \partial_x u)\left(\partial_x(\psi_h - \varphi_h) + \eta_h\right), \varphi_h\right)_{K_j(t)} \\
&-\frac{\lambda_j}{2}[\![\psi_h]\!]_{j+\frac{1}{2}}\varphi^-_{h,j+\frac{1}{2}} + \frac{\lambda_j}{2}[\![\psi_h]\!]_{j-\frac{1}{2}}\varphi^-_{h,j-\frac{1}{2}} \\
&+\frac{\lambda_j}{4}\left[\!\!\left[(\varphi_h)^2\right]\!\!\right]_{j+\frac{1}{2}} - \frac{\lambda_j}{4}\left[\!\!\left[(\varphi_h)^2\right]\!\!\right]_{j-\frac{1}{2}} \\
&-\frac{\lambda_j}{4}\left([\![\varphi_h]\!]_{j+\frac{1}{2}}\right)^2 + \frac{\lambda_j}{4}\left([\![\varphi_h]\!]_{j-\frac{1}{2}}\right)^2 \\
&+\frac{\lambda_j}{2}[\![\psi_h]\!]_{j-\frac{1}{2}}[\![\varphi_h]\!]_{j-\frac{1}{2}}.
\end{aligned} \tag{4.2.37}$$

Next, we sum the equation (4.2.37) from $j = j_1$ to $j_2$, utilize Young's inequality as well as (4.2.18) and obtain

$$\begin{aligned}
&\sum_{j=j_1}^{j_2} a_{2,j}(\psi_h, \varphi_h, \eta_h) \\
\leq& Ch^2 \sum_{j=j_1}^{j_2}\left(\|\partial_x\psi_h\|^2_{L^2(K_j(t))} + \|\partial_x\varphi_h\|^2_{L^2(K_j(t))}\right) \\
&+C\sum_{j=j_1}^{j_2}\left(\|\varphi_h\|^2_{L^2(K_j(t))} + h^2\|\eta_h\|^2_{L^2(K_j(t))}\right) \\
&+Ch\sum_{j=j_1}^{j_2+1}\left(\left(\psi^+_{h,j-\frac{1}{2}}\right)^2 + \left(\psi^-_{h,j-\frac{1}{2}}\right)^2\right) \\
&+Ch\sum_{j=j_1}^{j_2+1}\left(\left(\varphi^+_{h,j-\frac{1}{2}}\right)^2 + \left(\varphi^-_{h,j-\frac{1}{2}}\right)^2\right).
\end{aligned} \tag{4.2.38}$$

***Case 2)*** We consider an arbitrary family of consecutive cells $\{K_j(t)\}_{j=j_1}^{j_2}$ with $1 \leq j_1 \leq j_2 \leq N$ and $\partial_p G(\omega, \partial_x u) > 0$ for any cell $K_j(t)$, $j = j_1, ..., j_2$. Then is $\mathcal{Q}_{h(t)} = \mathcal{P}^-_h$ and thus $\psi^-_{h,j-\frac{1}{2}} = 0$, $j = j_1, ..., j_2$. We add the equation (4.2.14) multiplied by $-\lambda_j$ to the quantity

$a_{2,j}(\psi_h, \varphi_h, \eta_h)$ and obtain

$$\begin{aligned} & a_{2,j}(\psi_h, \varphi_h, \eta_h) \\ =& \left( \left( \partial_p G(\omega, \partial_x u) - \lambda_j \right) \left( \partial_x (\psi_h - \varphi_h) + \eta_h \right), \varphi_h \right)_{K_j(t)} \\ & - \frac{\lambda_j}{2} \left( \left( \varphi^-_{h,j+\frac{1}{2}} \right)^2 - \left( \varphi^-_{h,j-\frac{1}{2}} \right)^2 \right) - \frac{\lambda_j}{2} \left( [\![\varphi_h]\!]_{j-\frac{1}{2}} \right)^2. \end{aligned} \tag{4.2.39}$$

Furthermore according to (4.2.18), there are constants $C_1$ and $C_2$, independent of $h$, such that $\lambda_{j_2+1} \leq C_1 h$ and $\lambda_{j_1-1} \leq C_2 h$, since $\partial_p G(\omega, \partial_x u)$ changes the sign in the cells $K_{j_1-1}(t)$ as well as $K_{j_2+1}(t)$. Therefore, a summation by parts and (4.2.19) provide

$$\begin{aligned} & - \sum_{j=j_1}^{j_2} \left( \frac{\lambda_j}{2} \left( \left( \varphi^-_{h,j+\frac{1}{2}} \right)^2 - \left( \varphi^-_{h,j-\frac{1}{2}} \right)^2 \right) \right) \\ =& - \frac{1}{2} \left( \lambda_{j_2+1} \left( \varphi^-_{h,j_2+\frac{1}{2}} \right)^2 - \lambda_{j_1-1} \left( \varphi^-_{h,j_1-\frac{1}{2}} \right)^2 \right) \\ & + \frac{1}{2} \left( \lambda_{j_1} - \lambda_{j_1-1} \right) \left( \varphi^-_{h,j_1-\frac{1}{2}} \right)^2 \\ & + \frac{1}{2} \sum_{j=j_1}^{j_2} \left( \lambda_j - \lambda_{j-1} \right) \left( \varphi^-_{h,j-\frac{1}{2}} \right)^2 \leq Ch \sum_{j=j_1}^{j_2+1} \left( \varphi^-_{h,j-\frac{1}{2}} \right)^2. \end{aligned} \tag{4.2.40}$$

Next, we sum the equation (4.2.39) from $j = j_1$ to $j_2$ and apply Young's inequality, (4.2.16) as well as (4.2.40) and obtain the inequality (4.2.38), since $-\sum_{j=j_1}^{j_2} \frac{\lambda_j}{2} \left( [\![\varphi_h]\!]_{j-\frac{1}{2}} \right)^2 \leq 0$.

***Case 3)*** We consider an arbitrary family of consecutive cells $\{K_j(t)\}_{j=j_1}^{j_2}$ with $1 \leq j_1 \leq j_2 \leq N$ and $\partial_p G(\omega, \partial_x u) < 0$ for any cell $K_j(t)$, $j = j_1, ..., j_2$. In this case $\mathcal{Q}_h = \mathcal{P}_h^+$ and thus $\psi^+_{h,j-\frac{1}{2}} = 0$, $j = j_1, ..., j_2$. Therefore we add the equation (4.2.15) multiplied by $\lambda_j$ to the quantity $a_{2,j}(\psi_h, \varphi_h, \eta_h)$. This yields a similar equation as (4.2.39). Hence, by (4.2.17) and the same arguments as in case 2) we obtain the inequality (4.2.38).

Finally, we sum the equation (4.2.12) from $j = 1$ to $N$ and apply the inverse inequality (2.2.16), the trace inequality (2.2.17), the interpolation estimates (2.3.19), (2.3.20) as well as (2.3.22) and the inequality (4.2.38), which has been proven in the cases 1), 2) and 3). This provides

$$\begin{aligned} & \sum_{j=1}^{N} a_{2,j}(\psi_h, \varphi_h, \eta_h) \\ \leq & Ch^2 \left( \|\partial_x \psi_h\|^2_{L^2(\Omega)} + \|\partial_x \varphi_h\|^2_{L^2(\Omega)} \right) \\ & + C \|\varphi_h\|^2_{L^2(\Omega)} + Ch^2 \|\eta_h\|^2_{L^2(\Omega)} \\ & + Ch \left( \|\psi_h\|^2_{\Gamma_{h(t)}} + \|\varphi_h\|^2_{\Gamma_{h(t)}} \right) \\ \leq & C \left( h^{2k+2} + \|\varphi_h\|^2_{L^2(\Omega)} \right), \end{aligned} \tag{4.2.41}$$

since the initial value problem (1.1.8) is considered with periodic boundary conditions. Note, that in the equation (4.2.41) each boundary term has been counted at most twice. This does not affect the error estimate. □

Finally, we present an auxiliary lemma to estimate the remainder of the Taylor expansion on the Hamiltonian $H\left(\frac{p_1+p_2}{2}\right)$.

**Lemma 4.2.5** *Suppose the same assumptions as in theorem 4.2.1. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\sum_{j=1}^{N} a_{3,j}\left(\psi_h, \varphi_h\right) \le C\left(h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.42}$$

**Proof.** The inequality (4.2.3), (4.2.6), Cauchy-Schwarz inequality, Young's inequality, the interpolation inequality (2.3.22), (4.2.35) and the inverse inequality (2.2.16) supply

$$\begin{aligned}\sum_{j=1}^{N} a_{3,j}\left(\psi_h, \varphi_h\right) \le & C\left\|\partial_p^2 H\right\|_{\mathrm{L}^\infty(\mathbb{R})} h^2\left(\|\partial_x\psi_h\|^2_{\mathrm{L}^2(\Omega)} + \|\partial_x\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right)\\ &+C\left\|\partial_p^2 H\right\|_{\mathrm{L}^\infty(\mathbb{R})}\left(h^2\|\eta_h\|^2_{\mathrm{L}^2(\Omega)} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right)\\ \le & C\left(h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right),\end{aligned}$$

since the problem (1.1.8) is considered with periodic boundary conditions. □

Now we come to the actual proof of theorem 4.2.1.

**Proof of theorem 4.2.1.** We sum the equation (4.2.10) from $j = 1$ to $N$ and apply (4.2.28), (4.2.36), (4.2.42) and Gronwall's inequality. This yields

$$\|\varphi_h\|_{\mathrm{L}^2(\Omega)} \le Ch^{k+1}. \tag{4.2.43}$$

Hence, we obtain by (4.2.43) and the interpolation inequality (2.3.20) the desired error estimate. □

### 4.2.2. A suboptimal error estimate for the two dimensional method

For the two dimensional ALE-LDG method we have the following error estimate.

**Theorem 4.2.2** *Let $u \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{H}^{k+1}(\Omega)\right)$ be the exact solution of the initial value problem* (1.1.9). *Suppose the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ belongs to the space $\mathcal{C}^2(\mathbb{R}^2)$, is bounded and has bounded derivatives. Furthermore, for any time level $t = t_n$, $n = 0, ..., L$, there exists a simplicial mesh $\mathcal{T}_{h_n}$ of the domain $\Omega$. The simplicial meshes $\mathcal{T}_{h_n}$ have the same mesh topology* *and the properties* $(A1)$, $(A2)$ *as well as* $(A3)$. *In addition, the grid velocity satisfies the condition* $(A4)$. *Moreover, the condition* (2.2.15) *is satisfied for the global length* $h$ *given by* (2.1.5). *Let* $u_h$ *be the solution of the semi-discrete ALE-LDG method* (4.1.8), (4.1.13), (4.1.14), (4.1.15) *and* (4.1.16) *with the test function space* (2.2.2) *given by piecewise polynomials of degree* $k \geq 3$. *The initial data for the method is the function* $\mathcal{P}_h(u_0, 0)$. *Then there exists a constant* $C$ *independent of* $u_h$ *and* $h$, *such that*

$$\max_{t\in[0,T]} \|u - u_h\|_{\mathrm{L}^2(\Omega)} \leq Ch^{k+\frac{1}{2}}.$$

We introduce some notation, before we start with the actual proof of theorem 4.2.2. First of all, we define the quantities

$$\psi_h := u - \mathcal{P}_h(u,t), \quad \varphi_h := u_h - \mathcal{P}_h(u,t) \tag{4.2.44}$$

and

$$\eta_h := \partial_x u_h - \frac{p_1+p_2}{2}, \quad \zeta_h := \partial_y u_h - \frac{q_1+q_2}{2}, \tag{4.2.45}$$

where $\mathcal{P}_h(u,t)$ is the $\mathrm{L}^2$-projection of the solution of the problem (1.1.9). Note that the $\mathrm{L}^2$-projection maps to the time-dependent test function space $\mathcal{V}_{h,2}(t)$ given by (2.2.2). Then the error function $e_h := u - u_h$ can be written as

$$e_h = \psi_h - \varphi_h, \tag{4.2.46}$$

and the quantities $\partial_x u - \frac{p_1+p_2}{2}$ as well as $\partial_y u - \frac{q_1+q_2}{2}$ can be written as

$$\partial_x u - \frac{p_1+p_2}{2} = \partial_x(\psi_h - \varphi_h) + \eta_h, \ \ \partial_y u - \frac{q_1+q_2}{2} = \partial_y(\psi_h - \varphi_h) + \zeta_h. \tag{4.2.47}$$

Since we suppose that the exact solution $u$ of equation (1.1.8) is sufficiently smooth, we obtain for any edge $e(t) \subseteq \partial K(t)$

$$[\![u_h]\!]_{e(t)} = -[\![e_h]\!]_{e(t)}. \tag{4.2.48}$$

Next, by a Taylor expansion on the Hamiltonian $H\left(\frac{p_1+p_2}{2}, \frac{q_1+q_2}{2}\right)$ up to second order, the Lax Friedrichs flux (4.1.9) can be written as

$$\begin{aligned}
\widehat{G}(\omega, p_1, p_2, q_1, q_2) =& G(\omega, \nabla u) - \partial_p G(\omega, \nabla u)\left(\partial_x u - \frac{p_1+p_2}{2}\right) \\
&-\partial_q G(\omega, \nabla u)\left(\partial_y u - \frac{q_1+q_2}{2}\right) \\
&+\frac{1}{2}\partial_p^2 H(\Theta_1, \Theta_2)\left(\partial_x u - \frac{p_1+p_2}{2}\right)^2 \\
&+\frac{1}{2}\partial_q^2 H(\Theta_1, \Theta_2)\left(\partial_y u - \frac{p_1+p_2}{2}\right)^2 \\
&+\partial_p\partial_q H(\Theta_1, \Theta_2)\left(\partial_x u - \frac{p_1+p_2}{2}\right)\left(\partial_y u - \frac{q_1+q_2}{2}\right) \\
&-\frac{\lambda_{K(t)}}{2}(p_2 - p_1) - \frac{\mu_{K(t)}}{2}(q_2 - q_1),
\end{aligned} \tag{4.2.49}$$

where the function $G(\omega,\cdot,\cdot)$ is given by (4.1.10), $\Theta_1$ is a value between $\partial_x u$ as well as $\frac{p_1+p_2}{2}$ and $\Theta_2$ is a value between $\partial_y u$ as well as $\frac{q_1+q_2}{2}$. In addition, by (2.2.20) follows

$$\left(\partial_t\psi_h,\varphi_h\right)_{K(t)}+\left(\omega\cdot\nabla\psi_h,\varphi_h\right)_{K(t)}=-\left(\mathrm{div}\omega,\psi_h\varphi_h\right)_{K(t)}-\left(\psi_h,\omega\cdot\nabla\varphi_h\right)_{K(t)} \tag{4.2.50}$$

and the transport equation (2.2.4) provides

$$\left(\partial_t\varphi_h,\varphi_h\right)_{K(t)}+\left(\omega\cdot\nabla\varphi_h,\varphi_h\right)_{K(t)}=\frac{1}{2}\frac{d}{dt}\left(\varphi_h,\varphi_h\right)_{K(t)}-\left(\mathrm{div}\omega,\left(\varphi_h\right)^2\right)_{K(t)}. \tag{4.2.51}$$

The Lax Friedrichs flux (4.1.9) is consistent. Thus, the exact solution $u$ and the approximate solution $u_h$ satisfy the equation (4.1.8) and the equivalent equation (4.1.17). Therefore, we obtain by (4.1.17), (4.2.44), (4.2.45), (4.2.46), (4.2.47), (4.2.49), (4.2.50) and (4.2.51) the error equation

$$\begin{aligned}\frac{d}{dt}\left(\varphi_h,\varphi_h\right)_{K(t)}=&2a_{1,K(t)}\left(\psi_h,\varphi_h\right)+2a_{2,K(t)}\left(\psi_h,\varphi_h,\eta_h\right)\\&+2a_{3,K(t)}\left(\psi_h,\varphi_h,\zeta_h\right),\end{aligned} \tag{4.2.52}$$

where

$$\begin{aligned}&a_{1,K(t)}\left(\psi_h,\varphi_h\right)\\:=&-\left(\mathrm{div}\omega,\psi_h\varphi_h\right)_{K(t)}-\left(\psi_h,\omega\cdot\nabla\varphi_h\right)_{K(t)}+\frac{1}{2}\left(\mathrm{div}\omega,\left(\varphi_h\right)^2\right)_{K(t)}\\&-\frac{1}{2}\left(\partial_p^2H\left(\Theta_1,\Theta_2\right)\left(\partial_x u-\frac{p_1+p_2}{2}\right)^2,\varphi_h\right)_{K(t)}\\&-\frac{1}{2}\left(\partial_q^2H\left(\Theta_1,\Theta_2\right)\left(\partial_y u-\frac{q_1+q_2}{2}\right)^2,\varphi_h\right)_{K(t)}\\&-\left(\partial_p\partial_qH\left(\Theta_1,\Theta_2\right)\left(\partial_x u-\frac{p_1+p_2}{2}\right)\left(\partial_y u-\frac{q_1+q_2}{2}\right),\varphi_h\right)_{K(t)},\end{aligned} \tag{4.2.53}$$

$$\begin{aligned}&a_{2,K(t)}\left(\psi_h,\varphi_h,\eta_h\right)\\:=&\left(\partial_pG\left(\omega,\nabla u\right)\left(\partial_x\left(\psi_h-\varphi_h\right)+\eta_h\right),\varphi_h\right)_{K(t)}+\left(\frac{\lambda_{K(t)}}{2}\left(p_2-p_1\right),\varphi_h\right)_{K(t)}\end{aligned} \tag{4.2.54}$$

and

$$\begin{aligned}&a_{3,K(t)}\left(\psi_h,\varphi_h,\zeta_h\right)\\:=&\left(\partial_qG\left(\omega,\nabla u\right)\left(\partial_y\left(\psi_h-\varphi_h\right)+\zeta_h\right),\varphi_h\right)_{K(t)}+\left(\frac{\mu_{K(t)}}{2}\left(q_2-q_1\right),\varphi_h\right)_{K(t)}.\end{aligned} \tag{4.2.55}$$

Moreover, since $(\psi_h, \partial_x \varphi_h)_{K(t)} = 0$ and $(\psi_h, \partial_y \varphi_h)_{K(t)} = 0$, the equations (4.1.13), (4.1.14), (4.1.15) and (4.1.16) yield the error equations

$$\begin{aligned} 0 = &\left(\partial_x \left(\psi_h - \varphi_h\right), \varphi_h\right)_{K(t)} + \left(\partial_x u_h - p_1, \varphi_h\right)_{K(t)} \\ &- \left\langle \psi_h^-, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)} + \left\langle \varphi_h^-, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)} \\ &- \frac{1}{2} \left\langle \varphi_h^{\mathrm{int}_{K(t)}}, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)}, \end{aligned} \tag{4.2.56}$$

$$\begin{aligned} 0 = &\left(\partial_x \left(\psi_h - \varphi_h\right), \varphi_h\right)_{K(t)} + \left(\partial_x u_h - p_2, \varphi_h\right)_{K(t)} \\ &- \left\langle \psi_h^+, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)} + \left\langle \varphi_h^+, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)} \\ &- \frac{1}{2} \left\langle \varphi_h^{\mathrm{int}_{K(t)}}, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),x} \right\rangle_{\partial K(t)}, \end{aligned} \tag{4.2.57}$$

$$\begin{aligned} 0 = &\left(\partial_y \left(\psi_h - \varphi_h\right), \varphi_h\right)_{K(t)} + \left(\partial_y u_h - q_1, \varphi_h\right)_{K(t)} \\ &- \left\langle \psi_h^-, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} + \left\langle \varphi_h^-, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} \\ &- \frac{1}{2} \left\langle \varphi_h^{\mathrm{int}_{K(t)}}, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} \end{aligned} \tag{4.2.58}$$

and

$$\begin{aligned} 0 = &\left(\partial_y \left(\psi_h - \varphi_h\right), \varphi_h\right)_{K(t)} + \left(\partial_y u_h - q_2, \varphi_h\right)_{K(t)} \\ &- \left\langle \psi_h^+, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} + \left\langle \varphi_h^+, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)} \\ &- \frac{1}{2} \left\langle \varphi_h^{\mathrm{int}_{K(t)}}, \varphi_h^{\mathrm{int}_{K(t)}} n_{K(t),y} \right\rangle_{\partial K(t)}. \end{aligned} \tag{4.2.59}$$

In the following, we present estimates for the quantities (4.2.53), (4.2.54) and (4.2.55). According to lemma 2.1.1, $\mathcal{T}_{h(t)}$ is for any $t \in [0, T]$ a regular tesselation of the domain $\Omega$, since we suppose the conditions $(A1)$-$(A3)$. Thus, the interpolation and inverse inequalities presented in lemma 2.2.3, lemma 2.2.4 and lemma 2.2.5 can be applied. Furthermore, the same arguments as in the proof of lemma 4.2.1 lead to the following two dimensional extension of lemma 4.2.1. Since the lemma can be proven similar to its one dimensional analogue, we skip the two dimensional proof.

**Lemma 4.2.6** *Suppose $K(t) \in \mathcal{T}_{h(t)}$ is an arbitrary cell, $u \in \mathrm{W}^{1,\infty}\left(0, T; \mathrm{H}^2(\Omega)\right)$, the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ belongs to the space $\mathcal{C}^2\left(\mathbb{R}^2\right)$, is bounded and has bounded derivatives.*

*Furthermore, the grid velocity satisfies the condition* $(A4)$*. Then there are constants* $C_1^\star$, $C_2^\star$, $C_3^\star$ *and* $C_4^\star$*, independent of* $h$ *and* $u_h$*, such that*

$$\left\|\partial_\vartheta G\left(\omega, \nabla u\right) - \alpha_{K(t)}\right\|_{\mathrm{L}^\infty(K(t))} \leq C_1^\star h, \tag{4.2.60}$$

*if* $\partial_\vartheta G\left(\omega, \nabla u\right) > 0$*, for* $\vartheta = p$ *and* $\alpha_{K(t)} = \lambda_{K(t)}$ *or* $\vartheta = q$ *and* $\alpha_{K(t)} = \mu_{K(t)}$*, in the cell* $K\left(t\right)$*,*

$$\left\|\partial_\vartheta G\left(\omega, \nabla u\right) + \alpha_{K(t)}\right\|_{\mathrm{L}^\infty(K(t))} \leq C_2^\star h, \tag{4.2.61}$$

*if* $\partial_\vartheta G\left(\omega, \nabla u\right) < 0$*, for* $\vartheta = p$ *and* $\alpha_{K(t)} = \lambda_{K(t)}$ *or* $\vartheta = q$ *and* $\alpha_{K(t)} = \mu_{K(t)}$*, in the cell* $K\left(t\right)$*,*

$$\left\|\partial_\vartheta G\left(\omega, \nabla u\right)\right\|_{\mathrm{L}^\infty(K(t))} + \alpha_{K(t)} \leq C_3^\star h, \tag{4.2.62}$$

*if* $\partial_\vartheta G\left(\omega, \nabla u\right)$*, for* $\vartheta = p$ *and* $\alpha_{K(t)} = \lambda_{K(t)}$ *or* $\vartheta = q$ *and* $\alpha_{K(t)} = \mu_{K(t)}$*, changes the sign in the cell* $K\left(t\right)$*, where the function* $G(\omega,\cdot,\cdot)$ *is given by* (4.1.10)*. Moreover for two arbitrary adjacent cells* $K_1\left(t\right), K_2\left(t\right) \in \mathcal{T}_{h(t)}$ *holds*

$$\left|\lambda_{K_1(t)} - \lambda_{K_2(t)}\right| + \left|\mu_{K_1(t)} - \mu_{K_2(t)}\right| \leq C_4^\star h. \tag{4.2.63}$$

Next, we prove the upcoming two dimensional extension of lemma 4.2.3.

**Lemma 4.2.7** *Suppose the same assumptions as in theorem 4.2.2. Then there exists a constant* $C$*, independent of* $u_h$ *and* $h$*, such that*

$$\left\|\partial_x u_h - p_1\right\|_{\mathrm{L}^2(\Omega)} + \left\|\partial_x u_h - p_2\right\|_{\mathrm{L}^2(\Omega)} \leq C\left(h^k + h^{-1}\left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)}\right) \tag{4.2.64}$$

*and*

$$\left\|\partial_y u_h - q_1\right\|_{\mathrm{L}^2(\Omega)} + \left\|\partial_y u_h - q_2\right\|_{\mathrm{L}^2(\Omega)} \leq C\left(h^k + h^{-1}\left\|\varphi_h\right\|_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.65}$$

**Proof.** We consider for any cell $K(t) \in \mathcal{T}_{h(t)}$ and for all $v \in \mathcal{V}_{h,2}(t)$ the equations

$$\left(\partial_x u_h, v\right)_{K(t)} + \left(u_h, \partial_x v\right)_{K(t)} - \left\langle u_h^{\mathrm{int}_{K(t)}}, v^{\mathrm{int}_{K(t)}} n_{K(t),x}\right\rangle_{\partial K(t)} = 0 \tag{4.2.66}$$

as well as

$$\left(\partial_y u_h, v\right)_{K(t)} + \left(u_h, \partial_y v\right)_{K(t)} - \left\langle u_h^{\mathrm{int}_{K(t)}}, v^{\mathrm{int}_{K(t)}} n_{K(t),y}\right\rangle_{\partial K(t)} = 0. \tag{4.2.67}$$

First of all, we subtract the equation (4.1.13) from (4.2.66) and obtain for any cell $K(t) \in \mathcal{T}_{h(t)}$ and all $v \in \mathcal{V}_{h,2}(t)$

$$\left(\partial_x u_h - p_1, v\right)_{K(t)} - \left\langle u_h^{\mathrm{int}_{K(t)}} - u_h^-, v^{\mathrm{int}_{K(t)}} n_{K(t),x}\right\rangle_{\partial K(t)} = 0. \tag{4.2.68}$$

Next, we consider two arbitrary adjacent cells $K_1(t), K_2(t) \in \mathcal{T}_{h(t)}$ with the edge $e(t) := \overline{K_1(t)} \cap \overline{K_2(t)}$. The outward normals of $K_1(t)$ as well as $K_2(t)$ are denoted by $n_{K_1(t)}$ and $n_{K_2(t)}$. It should be noted that for any function $w \in \mathrm{L}^2(\Omega)$ with $w|_{K(t)} \in \mathrm{H}^1(K(t))$, for all $K(t) \in \mathcal{T}_{h(t)}$, it follows

$$w^{\mathrm{ext}_{K_2(t)}} = w^{\mathrm{int}_{K_1(t)}} \quad \text{and} \quad w^{\mathrm{ext}_{K_1(t)}} = w^{\mathrm{int}_{K_2(t)}}, \tag{4.2.69}$$

since $n_{K_1(t)} = -n_{K_2(t)}$. Therefore, if $n_{K_1(t),x} > 0$, are $w^- = w^{\mathrm{int}_{K_1(t)}}$ as well as $w^+ = w^{\mathrm{int}_{K_2(t)}}$ and if $n_{K_1(t),x} < 0$, are $w^- = w^{\mathrm{int}_{K_2(t)}}$ as well as $w^+ = w^{\mathrm{int}_{K_1(t)}}$. Hence, for all $v \in \mathcal{V}_{h,2}(t)$ follows

$$\begin{aligned}
&- \left\langle u_h^{\mathrm{int}_{K_1(t)}} - u_h^-, v^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x} \right\rangle_{e(t)} \\
&- \left\langle u_h^{\mathrm{int}_{K_2(t)}} - u_h^-, v^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x} \right\rangle_{e(t)} \\
&= \begin{cases} \left\langle [\![u_h]\!]_{e(t),x}, v^{\mathrm{int}_{K_1(t)}} \right\rangle_{e(t)}, & \text{if } n_{K_1(t),x} > 0, \\ \left\langle [\![u_h]\!]_{e(t),x}, v^{\mathrm{int}_{K_2(t)}} \right\rangle_{e(t)}, & \text{if } n_{K_1(t),x} < 0, \end{cases}
\end{aligned} \tag{4.2.70}$$

where $[\![u_h]\!]_{e(t),x}$ is the $x$ component of the jump vector given by (2.2.3). Thus, if we sum the equation (4.2.68) over all cells $K(t) \in \mathcal{T}_{h(t)}$ and apply (4.2.46), (4.2.48) as well as (4.2.70), it follows for all $v \in \mathcal{V}_{h,2}(t)$

$$\begin{aligned}
&\sum_{K(t) \in \mathcal{T}_{h(t)}} (\partial_x u_h - p_1, v)_{K(t)} \\
&\leq \sum_{K(t) \in \mathcal{T}_{h(t)}} \sum_{e(t) \in \partial K(t)} \left\langle \left| [\![u_h]\!]_{e(t),x} \right|, \left| v^{\mathrm{int}_{K(t)}} \right| + \left| v^{\mathrm{ext}_{K(t)}} \right| \right\rangle_{e(t)} \\
&\leq \sum_{K(t) \in \mathcal{T}_{h(t)}} \sum_{e(t) \in \partial K(t)} \left\langle \left| [\![\psi_h]\!]_{e(t),x} \right| + \left| [\![\varphi_h]\!]_{e(t),x} \right|, \left| v^{\mathrm{int}_{K(t)}} \right| + \left| v^{\mathrm{ext}_{K(t)}} \right| \right\rangle_{e(t)},
\end{aligned} \tag{4.2.71}$$

since we consider the initial value problem (1.1.9) with periodic boundary conditions. Next, we apply $\partial u_h - p_1$ as test function in (4.2.71). Then Cauchy-Schwarz's inequality, Young's inequality, the trace inequality (2.2.17) and the interpolation inequality (2.2.21) provide

$$\begin{aligned}
\|\partial_x u_h - p_1\|_{\mathrm{L}^2(\Omega)}^2 &\leq \left( \|\psi_h\|_{\Gamma_{h(t)}} + \|\varphi_h\|_{\Gamma_{h(t)}} \right) \|\partial_x u - p_1\|_{\Gamma_{h(t)}} \\
&\leq C h^{-1} \left( h^{k+1} + \|\varphi_h\|_{\mathrm{L}^2(\Omega)} \right) \|\partial_x u_h - p_1\|_{\mathrm{L}^2(\Omega)}.
\end{aligned} \tag{4.2.72}$$

Likewise, by subtracting the equation (4.1.14) from (4.2.66) and summing the result over all cells $K(t) \in \mathcal{T}_{h(t)}$, it follows

$$\|\partial_x u_h - p_2\|_{\mathrm{L}^2(\Omega)} \leq C h^{-1} \left( h^{k+1} + \|\varphi_h\|_{\mathrm{L}^2(\Omega)} \right). \tag{4.2.73}$$

Similarly, the inequality (4.2.65) can be proven by applying (4.2.67), (4.2.58), (4.2.59) and the arguments above. □

It should be noted that the inequalities (4.2.64) and (4.2.65) yield

$$\|\eta_h\|_{\mathrm{L}^2(\Omega)} + \|\zeta_h\|_{\mathrm{L}^2(\Omega)} \leq C\left(h^k + h^{-1}\|\varphi_h\|_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.74}$$

This inequality provides the following estimate for the quantity (4.2.53) summed over all cells $K(t) \in \mathcal{T}_{h(t)}$.

**Lemma 4.2.8** *Suppose the same assumptions as in theorem 4.2.2. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\sum_{K(t)\in\mathcal{T}_{h(t)}} a_{1,K(t)}(\psi_h, \varphi_h) \leq C\left(h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.75}$$

**Proof.** First of all we introduce for all $j = 1, ..., N$ the following notation

$$\overline{\omega}(t) := \omega(\overline{x}_j(t), t), \quad \overline{x}_j(t) := \frac{1}{2}\left(x_{j+\frac{1}{2}}(t) + x_{j-\frac{1}{2}}(t)\right).$$

It should be noted that for all $x \in K(t)$ by the condition (A4) follows

$$|\omega(x,t) - \overline{\omega}(t)| \leq c_1 \triangle_{K(t)} \leq c_1 h, \tag{4.2.76}$$

where we used the fact that the diameter of a simplex equals the greatest Eucledian distance between two vertices of a simplex. Therefore, we obtain by the Cauchy-Schwarz inequality, Young's inequality, the property (2.2.19) of the L2-projection, the inverse inequality (2.2.3), the interpolation inequality (2.2.22) and (4.2.76)

$$\begin{aligned}\sum_{K(t)\in\mathcal{T}_{h(t)}} (\psi_h, \omega\cdot\nabla\varphi_h)_{K(t)} =& \sum_{K(t)\in\mathcal{T}_{h(t)}} (\psi_h, (\omega-\overline{\omega})\cdot\nabla\varphi_h)_{K(t)} \\ \leq& \frac{c_1}{2}\left(\|\psi_h\|^2_{\mathrm{L}^2(\Omega)} + h^2\|\nabla\varphi_h\|^2_{[\mathrm{L}^2(\Omega)]^2}\right) \\ \leq& C\left(h^{2k+2} + \|\varphi_h\|^2_{[\mathrm{L}^2(\Omega)]}\right). \end{aligned} \tag{4.2.77}$$

Next, the Cauchy-Schwarz inequality, Young's inequality, the interpolation inequality (2.2.22), (4.2.77) and the condition (A4) supply

$$\begin{aligned}&-\sum_{K(t)\in\mathcal{T}_{h(t)}} \left((\mathrm{div}\omega, \psi_h\varphi_h)_{K(t)} + (\psi_h, \omega\cdot\nabla\varphi_h)_{K(t)} - \frac{1}{2}\left(\mathrm{div}\omega, (\varphi_h)^2\right)_{K(t)}\right) \\ \leq& C\left(h^{2k+2} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right), \end{aligned} \tag{4.2.78}$$

since we consider the initial value problem (1.1.9) with periodic boundary conditions. Likewise, we obtain by (4.2.3), (4.2.46), (4.2.47), Cauchy-Schwarz's inequality, Young's inequality, (4.2.74), the inverse inequality (2.2.16) and the interpolation inequality (2.2.24)

$$\begin{aligned}
&-\frac{1}{2}\sum_{K(t)\in\mathcal{T}_{h(t)}}\left(\partial_p^2 H\left(\Theta_1,\Theta_2,x,y\right)\left(\partial_x u-\frac{p_1+p_2}{2}\right)^2,\varphi_h\right)_{K(t)}\\
\leq & C\left\|\partial_p^2 H\right\|_{\mathrm{L}^\infty(\mathbb{R}^2)} h^2\left(\|\partial_x\psi_h\|^2_{\mathrm{L}^2(\Omega)}+\|\partial_x\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right)\\
+&C\left\|\partial_p^2 H\right\|_{\mathrm{L}^\infty(\mathbb{R}^2)}\left(h^2\|\eta_h\|^2_{\mathrm{L}^2(\Omega)}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right)\leq C\left(h^{2k+2}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).
\end{aligned}\tag{4.2.79}$$

Likewise, we obtain

$$\begin{aligned}
&-\sum_{K(t)\in\mathcal{T}_{h(t)}}\frac{1}{2}\left(\partial_q^2 H\left(\Theta_1,\Theta_2,x,y\right)\left(\partial_y u-\frac{q_1+q_2}{2}\right)^2,\varphi_h\right)_{K(t)}\\
\leq & C\left(h^{2k+2}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right)
\end{aligned}\tag{4.2.80}$$

and

$$\begin{aligned}
&-\sum_{K(t)\in\mathcal{T}_{h(t)}}\left(\partial_p\partial_q H\left(\Theta_1,\Theta_2\right)\left(\partial_x u-\frac{p_1+p_2}{2}\right)\left(\partial_y u-\frac{q_1+q_2}{2}\right),\varphi_h\right)_{K(t)}\\
\leq & C\left(h^{2k+2}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).
\end{aligned}\tag{4.2.81}$$

Finally, the estimates (4.2.78), (4.2.79), (4.2.80) and (4.2.81) provide (4.2.75). ☐

Furthermore, the inequality (4.2.74) yields the following estimate for the quantities (4.2.54) and (4.2.55) summed over all cells $K(t)\in\mathcal{T}_{h(t)}$.

**Lemma 4.2.9** *Suppose the same assumptions as in theorem 4.2.2. Then there exists a constant $C$, independent of $u_h$ and $h$, such that*

$$\sum_{K(t)\in\mathcal{T}_{h(t)}}\left(a_{2,K(t)}\left(\psi_h,\varphi_h,\eta_h\right)+a_{3,K(t)}\left(\psi_h,\varphi_h,\zeta_h\right)\right)\leq C\left(h^{2k+1}+\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right).\tag{4.2.82}$$

**Proof.** The proof ensues in two steps. First of all we estimate the quantities (4.2.54) and (4.2.55) summed over two arbitrary adjacent cells. This requires a case analysis. Afterward we apply the results of the case analysis to estimate the quantities (4.2.54) and (4.2.55) summed over all cells $K(t)\in\mathcal{T}_{h(t)}$.

Henceforth, $K_1(t), K_2(t)\in\mathcal{T}_{h(t)}$ are two arbitrary adjacent cells and the set $e(t):=\overline{K_1(t)}\cap\overline{K_2(t)}$ is an edge of $K_1(t)$ as well as $K_2(t)$. Moreover, the outward normals of $K_1(t)$ and $K_2(t)$ are denoted by $n_{K_1(t)}$ as well as $n_{K_2(t)}$.

Before we start with the case analysis, we would like to mention that the parameters $\lambda_{K_\ell(t)}$

and $\mu_{K_\ell(t)}$ given by (4.1.11) and (4.1.12) are constant in the cells $K_\ell(t)$, $\ell = 1, 2$. Thus, there is no harm to multiply the equations (4.2.56), (4.2.57), (4.2.58) and (4.2.59) by $\lambda_{K_\ell(t)}$ and $\mu_{K_\ell(t)}$.
***Case 1)*** We assume that the function $\partial_p G(\omega, \nabla u)$ changes the sign in both cells $K_1(t)$ and $K_2(t)$.
First of all, for $\ell = 1, 2$, we add the equation (4.2.56) multiplied by $-\frac{\lambda_{K_\ell(t)}}{2}$ and the equation (4.2.57) multiplied by $\frac{\lambda_{K_\ell(t)}}{2}$ to $a_{2,K_\ell(t)}(\psi_h, \varphi_h, \eta_h)$. This provides the equation

$$\begin{aligned} & a_{2,K_\ell(t)}(\psi_h, \varphi_h, \eta_h) \\ =& \left(\partial_p G(\omega, \nabla u)\left(\partial_x(\psi_h - \varphi_h) + \eta_h\right), \varphi_h\right)_{K_\ell(t)} \\ -& \frac{\lambda_{K_\ell(t)}}{2}\left\langle \left(\psi_h^+ - \psi_h^-\right) - \left(\varphi_h^+ - \varphi_h^-\right), \varphi_h^{\mathrm{int}_{K_\ell(t)}} n_{K_\ell(t),x}\right\rangle_{\partial K_\ell(t)}. \end{aligned} \tag{4.2.83}$$

The Cauchy-Schwarz inequality, Young's inequality and (4.2.62) supply

$$\begin{aligned} & \left(\partial_p G(\omega, \nabla u)\left(\partial_x(\psi_h - \varphi_h) + \eta_h\right), \varphi_h\right)_{K_\ell(t)} \\ \leq & Ch^2\left(\|\partial_x \psi_h\|^2_{\mathrm{L}^2(K_\ell(t))} + \|\partial_x \varphi_h\|^2_{\mathrm{L}^2(K_\ell(t))}\right) \\ +& C\left(h^2\|\eta_h\|^2_{\mathrm{L}^2(K_\ell(t))} + \|\varphi_h\|^2_{\mathrm{L}^2(K_\ell(t))}\right). \end{aligned} \tag{4.2.84}$$

Next, for the edge $e(t) := \overline{K_1(t)} \cap \overline{K_2(t)}$, we estimate the sum

$$\begin{aligned} & \frac{\lambda_{K_1(t)}}{2}\left\langle \left(\psi_h^+ - \psi_h^-\right) - \left(\varphi_h^+ - \varphi_h^-\right), \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{e(t)} \\ +& \frac{\lambda_{K_2(t)}}{2}\left\langle \left(\psi_h^+ - \psi_h^-\right) - \left(\varphi_h^+ - \varphi_h^-\right), \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)}. \end{aligned}$$

Hence, we obtain by (4.2.69), Young's inequality and (4.2.62)

$$\begin{aligned} & -\frac{\lambda_{K_1(t)}}{2}\left\langle [\![\psi_h]\!]_{e(t),x} - [\![\varphi_h]\!]_{e(t),x}, \varphi_h^{\mathrm{int}_{K_1(t)}}\right\rangle_{e(t)} \\ +& \frac{\lambda_{K_2(t)}}{2}\left\langle [\![\psi_h]\!]_{e(t),x} - [\![\varphi_h]\!]_{e(t),x}, \varphi_h^{\mathrm{int}_{K_2(t)}}\right\rangle_{e(t)} \\ \leq & Ch\left\langle \left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\psi_h^{\mathrm{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_2(t)}}\right)^2, 1\right\rangle_{e(t)}, \end{aligned} \tag{4.2.85}$$

if $n_{K_1(t),x} > 0$. Likewise, it follows by (4.2.69), Young's inequality and (4.2.62)

$$\begin{aligned} & \frac{\lambda_{K_1(t)}}{2}\left\langle [\![\psi_h]\!]_{e(t),x} - [\![\varphi_h]\!]_{e(t),x}, \varphi_h^{\mathrm{int}_{K_1(t)}}\right\rangle_{e(t)} \\ -& \frac{\lambda_{K_2(t)}}{2}\left\langle [\![\psi_h]\!]_{e(t),x} - [\![\varphi_h]\!]_{e(t),x}, \varphi_h^{\mathrm{int}_{K_2(t)}}\right\rangle_{e(t)} \\ \leq & Ch\left\langle \left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\psi_h^{\mathrm{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_2(t)}}\right)^2, 1\right\rangle_{e(t)}, \end{aligned} \tag{4.2.86}$$

if $n_{K_1(t),x} < 0$.
***Case 2)*** We assume that the function $\partial_p G(\omega, \nabla u)$ changes the sign in $K_1(t)$ and is positive in $K_2(t)$. Before we start to analyze case 2), we would like to mention that the cases in which $\partial_p G(\omega, \nabla u)$ changes the sign in one cell and is positive or negative in the other cell can be analyzed by similar arguments. Hence, we skip all the other cases of this type and consider only the case 2) as a model case.
For the cell $K_1(t)$ we can proceed similar to case 1). This provides the estimates (4.2.83) and (4.2.84). For the cell $K_2(t)$ we multiply the equation (4.2.56) by $-\lambda_{K_2(t)}$ and add the result to $a_{2,K_2(t)}(\psi_h, \varphi_h, \eta_h)$. This provides

$$\begin{aligned}
&a_{2,K_2(t)}(\psi_h, \varphi_h, \eta_h) \\
=&\left(\left(\partial_p G(\omega, \nabla u) - \lambda_{K_2(t)}\right)\left(\partial_x(\psi_h - \varphi_h) + \eta_h\right), \varphi_h\right)_{K_2(t)} \\
&+\lambda_{K_2(t)}\left\langle \psi_h^- - \varphi_h^- + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{\partial K_2(t)}.
\end{aligned} \tag{4.2.87}$$

The Cauchy-Schwarz inequality, Young's inequality and (4.2.63) supply

$$\begin{aligned}
&\left(\partial_p G(\omega, \nabla u)\left(\partial_x(\psi_h - \varphi_h) + \eta_h\right), \varphi_h\right)_{K_2(t)} \\
\le& Ch^2\left(\|\partial_x \psi_h\|^2_{\mathrm{L}^2(K_2(t))} + \|\partial_x \varphi_h\|^2_{\mathrm{L}^2(K_2(t))}\right) \\
&+C\left(h^2\|\eta_h\|^2_{\mathrm{L}^2(K_2(t))} + \|\varphi_h\|^2_{\mathrm{L}^2(K_2(t))}\right).
\end{aligned} \tag{4.2.88}$$

Next, for the edge $e(t) := \overline{K_1(t)} \cap \overline{K_2(t)}$, we estimate the sum

$$\begin{aligned}
&\frac{\lambda_{K_1(t)}}{2}\left\langle\left(\psi_h^+ - \psi_h^-\right) - \left(\varphi_h^+ - \varphi_h^-\right), \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{e(t)} \\
&+\lambda_{K_2(t)}\left\langle \psi_h^- - \varphi_h^- + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)}.
\end{aligned}$$

Hence, we obtain by (4.2.69), Young's inequality, (4.2.62) and (4.2.63)

$$\begin{aligned}
&-\frac{\lambda_{K_1(t)}}{2}\left\langle \psi_h^{\mathrm{int}_{K_2(t)}} - \psi_h^{\mathrm{int}_{K_1(t)}} - \varphi_h^{\mathrm{int}_{K_2(t)}} + \varphi_h^{\mathrm{int}_{K_1(t)}}, \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{e(t)} \\
&+\lambda_{K_1(t)}\left\langle \psi_h^{\mathrm{int}_{K_1(t)}} - \varphi_h^{\mathrm{int}_{K_1(t)}} + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)} \\
&+\left(\lambda_{K_2(t)} - \lambda_{K_1(t)}\right)\left\langle \psi_h^{\mathrm{int}_{K_1(t)}} - \varphi_h^{\mathrm{int}_{K_1(t)}} + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)} \\
\le& Ch\left\langle\left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\psi_h^{\mathrm{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_2(t)}}\right)^2, 1\right\rangle_{e(t)},
\end{aligned} \tag{4.2.89}$$

if $n_{K_1(t),x} > 0$. Likewise, it follows by (4.2.69), Young's inequality, (4.2.62) and (4.2.63)

$$\begin{aligned}
&\frac{\lambda_{K_1(t)}}{2}\left\langle \psi_h^{\mathrm{int}_{K_2(t)}} - \psi_h^{\mathrm{int}_{K_1(t)}} - \varphi_h^{\mathrm{int}_{K_2(t)}} + \varphi_h^{\mathrm{int}_{K_1(t)}}, \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{e(t)} \\
&+\lambda_{K_1(t)}\left\langle \psi_h^{\mathrm{int}_{K_2(t)}} - \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)} \\
&+\left(\lambda_{K_2(t)} - \lambda_{K_1(t)}\right)\left\langle \psi_h^{\mathrm{int}_{K_2(t)}} - \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{e(t)} \\
&\leq Ch\left\langle \left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\psi_h^{\mathrm{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_2(t)}}\right)^2, 1\right\rangle_{e(t)}, \qquad (4.2.90)
\end{aligned}$$

if $n_{K_1(t),x} < 0$.
***Case 3)*** We assume that the function $\partial_p G(\omega, \nabla u)$ is positive in both cells $K_1(t)$ and $K_2(t)$. We would like to mention that the case in which $\partial_p G(\omega, \nabla u)$ is negative in both cells can be analyzed by similar arguments. Hence, we skip this case.
First of all, similar to case 2), we obtain for both cells the estimates (4.2.87) and (4.2.88). Moreover, there exists a constant $C$ independent of $h$, such that $\left|\lambda_{K_\ell(t)}\right| \leq C$, $\ell = 1, 2$. Next, for the edge $e(t) := \overline{K_1(t)} \cap \overline{K_2(t)}$, we estimate the sum

$$\begin{aligned}
&\lambda_{K_1(t)}\left\langle \psi_h^- - \varphi_h^- + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_1(t)}}, \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{\partial K_1(t)} \\
&+\lambda_{K_2(t)}\left\langle \psi_h^- - \varphi_h^- + \frac{1}{2}\varphi_h^{\mathrm{int}_{K_2(t)}}, \varphi_h^{\mathrm{int}_{K_2(t)}} n_{K_2(t),x}\right\rangle_{\partial K_2(t)}.
\end{aligned}$$

Hence, we obtain by (4.2.69), Young's inequality and (4.2.63)

$$\begin{aligned}
&-\left(\lambda_{K_2(t)} - \lambda_{K_1(t)}\right)\left\langle \psi_h^{\mathrm{int}_{K_1(t)}} - \frac{1}{2}\varphi_h^{\mathrm{int}_{K_1(t)}}, \varphi_h^{\mathrm{int}_{K_1(t)}} n_{K_1(t),x}\right\rangle_{e(t)} \\
&-\lambda_{K_2(t)}\left\langle \psi_h^{\mathrm{int}_{K_1(t)}}, \left(\varphi_h^{\mathrm{int}_{K_2(t)}} - \varphi_h^{\mathrm{int}_{K_1(t)}}\right) n_{K_1(t),x}\right\rangle_{e(t)} \\
&-\frac{\lambda_{K_2(t)}}{2}\left\langle \left(\varphi_h^{\mathrm{int}_{K_2(t)}} - \varphi_h^{\mathrm{int}_{K_1(t)}}\right)^2, n_{K_1(t),x}\right\rangle_{e(t)} \\
&\leq C\left(h\left\langle \left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2 + \left(\varphi_h^{\mathrm{int}_{K_1(t)}}\right)^2, 1\right\rangle_{e(t)} + \left\langle \left(\psi_h^{\mathrm{int}_{K_1(t)}}\right)^2, 1\right\rangle_{e(t)}\right), \qquad (4.2.91)
\end{aligned}$$

if $n_{K_1(t),x} > 0$. Likewise, it follows by (4.2.69), Young's inequality and (4.2.63)

$$\begin{aligned}
&\left(\lambda_{K_2(t)} - \lambda_{K_1(t)}\right) \left\langle \psi_h^{\text{int}_{K_2(t)}} - \frac{1}{2}\varphi_h^{\text{int}_{K_2(t)}}, \varphi_h^{\text{int}_{K_2(t)}} n_{K_2(t),x} \right\rangle_{e(t)} \\
&+\lambda_{K_1(t)} \left\langle \psi_h^{\text{int}_{K_2(t)}}, \left(\varphi_h^{\text{int}_{K_2(t)}} - \varphi_h^{\text{int}_{K_1(t)}}\right) n_{K_2(t),x} \right\rangle_{e(t)} \\
&-\frac{\lambda_{K_1(t)}}{2} \left\langle \left(\varphi_h^{\text{int}_{K_2(t)}} - \varphi_h^{\text{int}_{K_1(t)}}\right)^2, n_{K_2(t),x} \right\rangle_{e(t)} \\
&\leq C \left( h \left\langle \left(\psi_h^{\text{int}_{K_2(t)}}\right)^2 + \left(\varphi_h^{\text{int}_{K_2(t)}}\right)^2, 1 \right\rangle_{e(t)} + \left\langle \left(\psi_h^{\text{int}_{K_2(t)}}\right)^2, 1 \right\rangle_{e(t)} \right),
\end{aligned} \tag{4.2.92}$$

if $n_{K_1(t),x} < 0$. Furthermore, since the Hamiltonian belongs to the space $\mathcal{C}^2(\Omega)$, the case, where $\partial_p G(\omega, \nabla u)$ is positive in a cell and negative in an adjacent cell, can not appear.
It should be noted that for the edges of $K_1(t)$ and $K_2(t)$, which are interacting with other neighboring cells, the estimates (4.2.85), (4.2.86) (4.2.89) (4.2.90) (4.2.91) (4.2.92) follow by the same analysis as in the cases 1, 2 and 3. Therefore, a summation of the quantity (4.2.54) over all cells $K(t) \in \mathcal{T}_{h(t)}$ and the inverse inequality (2.2.16), the trace inequality (2.2.17) and the interpolation inequalities (2.2.21), (2.2.22) as well as (2.2.24) provide

$$\begin{aligned}
&\sum_{K(t)\in\mathcal{T}_{h(t)}} a_{2,K(t)}\left(\psi_h, \varphi_h, \eta_h\right) \\
&\leq C \sum_{K(t)\in\mathcal{T}_{h(t)}} \sum_{e(t)\in\partial K(t)} \left\langle \left(\psi_h^{\text{ext}_{K(t)}}\right)^2 + \left(\psi_h^{\text{int}_{K(t)}}\right)^2, 1 \right\rangle_{e(t)} \\
&+Ch \sum_{K(t)\in\mathcal{T}_{h(t)}} \sum_{e(t)\in\partial K(t)} \left\langle \left(\varphi_h^{\text{ext}_{K(t)}}\right)^2 + \left(\varphi_h^{\text{int}_{K(t)}}\right)^2, 1 \right\rangle_{e(t)} \\
&+Ch^2 \left( \|\partial_x \psi_h\|^2_{\mathrm{L}^2(\Omega)} + \|\partial_x \varphi_h\|^2_{\mathrm{L}^2(\Omega)} + \|\eta_h\|^2_{\mathrm{L}^2(\Omega)} \right) + C\|\varphi_h\|^2_{\mathrm{L}^2(\Omega)} \\
&\leq C\left(h^{2k+1} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right),
\end{aligned} \tag{4.2.93}$$

since we consider the initial value problem (1.1.9) with periodic boundary conditions. Note that in the inequalities (4.2.91) and (4.2.92) an extra $h$ to control the loss of accuracy by the trace inequality is missing. Thus, the interpolation estimate (2.2.21) provides merely the sub-optimal order $h^{2k+1}$ instead of the optimal order $h^{2k+2}$. Furthermore, in equation (4.2.93) each boundary term has been counted at most twice. This does not affect the error estimate.
Similarly, by applying the error equations (4.2.58) and (4.2.59) follows

$$\sum_{K(t)\in\mathcal{T}_{h(t)}} a_{3,K(t)}\left(\psi_h, \varphi_h, \zeta_h\right) \leq C\left(h^{2k+1} + \|\varphi_h\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{4.2.94}$$

Therefore, we obtain the inequality (4.2.82) by (4.2.93) and (4.2.94). □

All these auxiliary lemmas ensure the proof of theorem 4.2.2.
**Proof of theorem 4.2.2.** We sum the equation (4.2.10) over all cells $K(t) \in \mathcal{T}_{h(t)}$ and apply (4.2.75), (4.2.82) and Gronwall's inequality. This yields

$$\|\varphi_h\|_{\mathrm{L}^2(\Omega)} \leq Ch^{k+\frac{1}{2}}. \tag{4.2.95}$$

Hence, we obtain by (4.2.95) and (2.2.22) the desired error estimate. □

### 4.2.3. Justification of the a priori assumptions

It remains to verify the a priori assumptions (4.2.1) and (4.2.2). This will be done merely for the one dimensional version of the a priori assumption. The justification of the two dimensional analogues ensues by similar arguments. Nevertheless, there are a few differences in the verification of the two dimensional a priori assumptions. These differences appear, since there are $d$-dependent bounds in the inverse inequality (2.2.16) and the interpolation inequality (2.2.23). At the end of this section this issue and its effect will be discussed.
In the following, we proceed similar to Xiong et al. [84] or Xu and Shu [85]. First of all, by the interpolation inequality (2.3.21), the inverse inequality (2.2.16), (4.2.30) and (4.2.43) follows for all $t \in [0, T]$

$$\begin{aligned}
&\|\partial_x u(t) - p_1(t)\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_x u(t) - p_2(t)\|_{\mathrm{L}^\infty(\Omega)} \\
\leq& 2\|\partial_x \psi_h(t)\|_{\mathrm{L}^\infty(\Omega)} + 2\|\partial_x \varphi_h(t)\|_{\mathrm{L}^\infty(\Omega)} \\
&+ \|\partial_x u_h(t) - p_1(t)\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_x u_h(t) - p_2(t)\|_{\mathrm{L}^\infty(\Omega)} \\
\leq& C\left(h^{k-\frac{1}{2}} + h^{-\frac{1}{2}}\|\partial_x \varphi_h(t)\|_{\mathrm{L}^2(\Omega)}\right) \\
&+ Ch^{-\frac{1}{2}}\left(\|\partial_x u_h(t) - p_1(t)\|_{\mathrm{L}^2(\Omega)} + \|\partial_x u_h(t) - p_2(t)\|_{\mathrm{L}^2(\Omega)}\right) \\
\leq& C\left(h^{k-\frac{1}{2}} + h^{-\frac{3}{2}}\|\varphi_h(t)\|_{\mathrm{L}^2(\Omega)}\right) \leq Ch^{k-\frac{1}{2}}.
\end{aligned} \tag{4.2.96}$$

Next, we define

$$t^* := \sup\left\{t \in [0, \infty) : \|\partial_x u(t) - p_1(t)\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_x u(t) - p_2(t)\|_{\mathrm{L}^\infty(\Omega)} \leq C_{\mathcal{H}_1} h\right\}.$$

The a priori assumption (4.2.1) is verified, if $t^* = \infty$. If $t^* < \infty$, it follows by continuity

$$\|\partial_x u(t^*) - p_1(t^*)\|_{\mathrm{L}^\infty(\Omega)} + \|\partial_x u(t^*) - p_2(t^*)\|_{\mathrm{L}^\infty(\Omega)} = C_{\mathcal{H}_1} h. \tag{4.2.97}$$

Since $k > 2$, the quantity $h$ can be chosen that

$$h^{k-\frac{3}{2}} \leq \frac{C_{\mathcal{H}_1}}{2C}, \tag{4.2.98}$$

where the constant $C$ is given by the inequality (4.2.96). Hence, (4.2.96), (4.2.97) and (4.2.98) provide the contradiction

$$\begin{aligned}&\left\|\partial_x u\left(t^*\right)-p_1\left(t^*\right)\right\|_{\mathrm{L}^\infty(\Omega)}+\left\|\partial_x u\left(t^*\right)-p_2\left(t^*\right)\right\|_{\mathrm{L}^\infty(\Omega)}\\ \leq&\frac{1}{2}C_{\mathcal{H}_1}h<C_{\mathcal{H}_1}h=\left\|\partial_x u\left(t^*\right)-p_1\left(t^*\right)\right\|_{\mathrm{L}^\infty(\Omega)}+\left\|\partial_x u\left(t^*\right)-p_2\left(t^*\right)\right\|_{\mathrm{L}^\infty(\Omega)},\end{aligned}$$

if $t^* < T$. Therefore, the a priori assumption is true for the interval $[0, T]$.
In a similar way, the two dimensional a priori assumptions can be ensured by the interpolation inequalities (2.2.23), (2.2.16), (4.2.64), (4.2.65) and (4.2.95). However, it should be noted that the restriction $k \geq 3$ is necessary to verify the a priori assumptions in two dimensions, since the bounds in the inequalities (2.2.23) and (2.2.16) are $d$-dependent.

# 5. Conclusions

I look back, so much time
What once was, now it's gone
Now it's fine, now it's fine

*(mind.in.a.box[6])*

In this thesis, we developed and analyzed an ALE-DG method for scalar conservation and an ALE-LDG method for directly solving the Hamilton-Jacobi equations. We presented these methods without a moving mesh methodology. Therefore, in order to discribe the ALE kinematics of our methods, we needed to apply the assumption:

"*The distribution of the grid points is explicitly given for an upcoming time level.*"

Based on this assumption we constructed time-dependent local affine linear mappings. This mappings supplied time-depend cells, local grid velocities and a time-dependent test function space. It should be noted that the computational effort to obtain these quantities is small. Furthermore, merely local data has to be applied to calculate these quantities, which makes our ALE-DG methods attractive for parallel computing. In future works, this aspect of the ALE-DG methods need to be investigated. In this sense, it is necessary to develop a moving mesh methodology, which ensures the above assumption with a minor calculation effort.

We proved, in the one and multidimensional case that under certain assumptions the time-dependent cells yield a regular tesselation of the spatial domain. Therefore, the classical interpolation, inverse and trace inequalities for static grids (cf. Ciarlet [6] and Di Pietro and Ern [24]) can be applied. Moreover, we proved transport equations for the time-dependent test function space and for the $L^2$-projection as well as the Gauss-Radau projections. These results are the ingredients to prove a suboptimal and optimal a priori error estimate for the one dimensional semi-discrete ALE-DG method for scalar conservation laws. Likewise, these results provided the optimal a-priori error estimate for the one dimensional semi-discrete ALE-LDG method for the Hamilton-Jacobi equations and the suboptimal a priori error estimate for the two dimensional equivalent.

In addition, we proved that the fully discrete ALE-DG and ALE-LDG method satisfy the geometric conservation law. For the fully discrete ALE-DG method, a local maximum principle was proven and conditions for TVD/TVB limiter were stated. Certainly, it should be noted that we did not prove the compatibility of the conditions for TVD/TVB limiter with the ALE-DG method discretized by Shu's third order SSP-RK method (cf. table 1.2.1). However, in our numerical experiments the ALE-DG method was discretized by Shu's third order SSP-RK method

[6] mind.in.a.box is an Austrian music band. The quote is from the song *Redefined* (Metropolis Records, 2007).

compatible with the conditions for TVD/TVB limiter.
Finally, we showed the capability of the ALE-DG for scalar conservation laws by some numerical experiments. For the one dimensional ALE-DG method, by solving the Burgers' equation and the Euler equations, it was shown that the method is uniformly high order accurate and shock capturing. For the two dimensional ALE-DG method, by solving the Euler equations, it was shown that the method is uniformly high order accurate. In a future work, the capability of the two dimensional ALE-DG method to handle strong singularities, like in the Mach-three wind tunnel test problem (cf. Kröner [49, Example 1.0.4, p. 6-7]), need to be considered. Therefore, limiter to stabilize the two dimensional ALE-DG method need to be developed.

## Ongoing and future works

Two projects for future works were already mentioned. First of all, it is necessary to develop a moving mesh methodology for the ALE-DG and ALE-LDG method. Furthermore, we mentioned that for the two dimensional ALE-DG method for conservation laws limiter to stabilize the method need to be developed. This is a challenge, since Cockburn, Hou and Shu already noticed in [13] the complexity to construct limiter for the two dimensional RK-DG method, which are flexible to the mesh topology. Indeed, Kuzmin developed in [52] a slope limiter based on Taylor polynomials, which has a better flexibility than the classical limiter of Cockburn, Hou and Shu, but it is not clear, if this limiter can handle a moving mesh topology. Therefore, it would be worthwhile to analyze the two dimensional ALE-DG method with suitable limiter.
Next, it should be noted that even in one dimension further numerical experiments are desirable. In [87] Yang and Shu analyzed the RK-DG method for hyperbolic equations involving $\delta$-singularities. For these kind of problems, it would be interesting to test the capability of the ALE-DG method in numerical experiments, since the ALE-DG method gives the freedom to relocate a specific number of grid points close to the singularity.
Furthermore, it is necessary to do numerical experiments with the ALE-LDG method for the Hamilton-Jacobi equations and compare the capability of the method with the capability of Cheng and Shu's method [4] as well as Yan and Osher's method [86].
In this thesis, we considered the initial value problems (1.1.1) and (1.1.8) with periodic boundary conditions. Thus, it was possible to assume that the boundary of the domain does not move with the grid. For problems with boundary conditions, it is common to apply isoparametric families of finite elements (cf. Ciarlet [6, p. 224]). This kind of elements were used e.g. by Nguyen in [63]. However, it is not clear, if the ALE-DG and ALE-LDG method satisfy the geometric conservation law for these kind of elements. Therefore, it would be interesting to consider the ALE-DG methods with isoparametric families of finite elements.
There are still certain theoretical problems for the ALE-DG and ALE-LDG method. The a priori error estimates were proven merely for the semi-discrete methods. From the perspective of the analysis, it would be worthwhile to prove a priori error estimates for the ALE-DG and ALE-LDG method discretized by Shu's third order SSP-RK method. The proofs for a priori

error estimates for the fully discrete RK-DG method are highly nontrivial and very technical, since the analysis for finite difference methods has to be combined with the analysis for finite element methods (cf. Q. Zhang and Shu [90, 92]). For the analysis of the ALE-DG and ALE-LDG method, we need additionally a discrete version of the transport equation (2.2.1), which is compatible with Shu's third order SSP-RK method and does not hurt the geometric conservation law. Another theoretical problem for the future work could be a convergence proof for the ALE-LDG method. The convergence to the unique viscous solution is not proven for many high order methods for the Hamilton-Jacobi equations. Hence, even for Yan and Osher's method this could be an attractive problem.

# A. Appendix

## A.1. A determinant inequality

**Lemma A.1.1** *Let* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}$ *with* $\det \boldsymbol{A} > 0$, $\det \boldsymbol{B} > 0$ *and* $\left\|\boldsymbol{A}^{-1}\boldsymbol{B}\right\|_{\mathcal{L}\left(\mathbb{R}^d,\mathbb{R}^d\right)} < 1$. *Then is* $\det\left(\boldsymbol{A}+\boldsymbol{B}\right)$ *strictly positive.*

**Proof.** First of all, we obtain

$$\mathbf{A}+\mathbf{B} = \mathbf{A}\left(\mathbf{id}_{\mathbb{R}^d} - \left(-\mathbf{A}^{-1}\mathbf{B}\right)\right), \tag{A.1.1}$$

where $\mathbf{id}_{\mathbb{R}^d}$ is the identity matrix in $\mathbb{R}^{d\times d}$.
Henceforth, the cardinality of a set is denoted by the quantity $|\cdot|_c$. Let $I, J \subseteq \mathbb{N}$ be index sets, such that $|I|_c$ is the number of pairwise different real eigenvalues and $|J|_c$ is the number of pairwise different complex eigenvalues of the matrix $\left(-\mathbf{A}^{-1}\mathbf{B}\right)$. The real and the complex eigenvalues of the matrix $\left(-\mathbf{A}^{-1}\mathbf{B}\right)$ are denoted by $\mu_j$, $j \in 1, ..., |I|_c$, and $\lambda_\ell$, $\ell \in 1, ..., |J|_c$. It should be noted that the complex conjugate $\overline{\lambda}$ of an eigenvalue $\lambda$ is also an eigenvalue of the matrix $\left(-\mathbf{A}^{-1}\mathbf{B}\right)$, since $\left(-\mathbf{A}^{-1}\mathbf{B}\right) \in \mathbb{R}^{d\times d}$. Hence, the complex eigenvalues are given by $\lambda_\ell, \overline{\lambda_\ell}$, for all $\ell = 1, ..., \frac{1}{2}|J|_c$. Note that

$$0 < |\mu_j| < 1 \quad \text{and} \quad 0 < |\lambda_\ell| < 1, \tag{A.1.2}$$

for all $j \in 1, ..., |I|_c$, and $\ell \in 1, ..., |J|_c$, since $\left\|\mathbf{A}^{-1}\mathbf{B}\right\|_{\mathcal{L}\left(\mathbb{R}^d,\mathbb{R}^d\right)} < 1$. In particular, the Neumann series $\sum\limits_{k=0}^{\infty}\left(-\mathbf{A}^{-1}\mathbf{B}\right)^k$ converges and thus we obtain

$$\sum_{k=0}^{\infty}\left(-\mathbf{A}^{-1}\mathbf{B}\right)^k = \left(\mathbf{id}_{\mathbb{R}^d} - \left(-\mathbf{A}^{-1}\mathbf{B}\right)\right)^{-1}.$$

This matrix has the eigenvalues $\frac{1}{1-\mu_j}$, $\frac{1}{1-\lambda_\ell}$ and $\frac{1}{1-\overline{\lambda_\ell}}$, $j = 1, ..., |I|_c$ and $\ell = 1, ..., \frac{1}{2}|J|_c$. In the following, the indices $r_j$, $j = 1, ..., |I|_c$, denote the algebraic multiplicity of the real eigenvalues $\frac{1}{1-\mu_j}$. Likewise, the indices $s_\ell$, $\ell = 1, ..., \frac{1}{2}|J|_c$, denote the algebraic multiplicity of the complex eigenvalues $\frac{1}{1-\lambda_\ell}$ and $\frac{1}{1-\overline{\lambda_\ell}}$. Hence, by (A.1.2) follows

$$\det\left(\left(\mathbf{id}_{\mathbb{R}^d} - \left(-\mathbf{A}^{-1}\mathbf{B}\right)\right)^{-1}\right) = \left(\prod_{j=1}^{|I|_c}\left(\frac{1}{1-\mu_j}\right)^{r_j}\right)\left(\prod_{\ell=1}^{\frac{1}{2}|J|_c}\left(\frac{1}{|1-\lambda_\ell|^2}\right)^{s_\ell}\right) > 0, \tag{A.1.3}$$

where we used the fact that the Jordan normal form of a matrix has the same determinant as the corresponding Matrix. Thus, by (A.1.1) and (A.1.3) follows

$$\det\left(\mathbf{A}+\mathbf{B}\right)=\det\left(\mathbf{A}\right)\det\left(\mathbf{id}_{\mathbb{R}^d}-\left(-\mathbf{A}^{-1}\mathbf{B}\right)\right)>0.$$

□

## A.2. Properties of the Gauss-Radau projections

In the proof of lemma 2.3.2 we used that for all $u \in \mathrm{H}^{k+1}\left(\Omega\right)$ holds

$$\left\|\mathcal{P}_h\left(u,t\right)-\mathcal{P}_h^{\pm}\left(u,t\right)\right\|_{\mathrm{L}^2(\Omega)}=\mathcal{O}\left(h^{k+1}\right), \tag{A.2.1}$$

where $h$ is given by (2.1.5). This relationship between the $\mathrm{L}^2$-projection and the Gauss-Radau projection follows from the upcoming auxiliary result.

**Lemma A.2.1** *Suppose there exists for any time level $t = t_n$, $n = 0, ..., L$, a partition of the domain $\Omega$ with the properties $(P1)$ as well $(P2)$ and the regularity condition* (2.2.15) *is satisfied for the global length $h$ given by* (2.1.5)*. Then holds*

$$\mathcal{P}_h^{\pm}\left(v,t\right)=v \quad \textit{for all } v\in\mathcal{V}_h\left(t\right). \tag{A.2.2}$$

*Moreover, there exists a constant $C$, such that for all $u \in \mathrm{H}^{k+1}\left(\Omega\right)$ holds*

$$\left\|\mathcal{P}_h^{\pm}\left(u,t\right)-\mathcal{P}_h\left(u,t\right)\right\|_{\mathrm{L}^2(\Omega)}\leq Ch^{k+1}. \tag{A.2.3}$$

*The constant $C$ dependents on $u$, but is independent of $h$.*

**Proof.** The identity (A.2.2) follows by a simple calculation. Hence, we present merely the proof of the inequality (A.2.3).
Henceforth, is $\psi_h := u - \mathcal{P}_h\left(u,t\right)$ and we apply the representation (2.3.25) of the Gauss-Radau projection for the cell $K_j(t)$. For the functions (2.3.24), it follows by the properties of the Legendre polynomials

$$\left(\mathcal{L}_{\vartheta},\mathcal{L}_{\vartheta'}\right)_{K_j(t)}=\frac{\triangle_j\left(t\right)}{2\vartheta+1}\delta_{\vartheta\vartheta'} \quad \text{for all } \vartheta,\vartheta'=0,...,k, \tag{A.2.4}$$

where $\delta_{\vartheta\vartheta'}$ denotes the Kronecker delta. Hence, the Cauchy-Schwarz inequality provides for all $j = 1, ..., N$ and $\vartheta = 0, ..., k$

$$\left(\psi_h,\mathcal{L}_{\vartheta}\right)_{K_j(t)}\leq\sqrt{\frac{h}{2\vartheta+1}}\left\|\psi_h\right\|_{\mathrm{L}^2(K_j(t))}. \tag{A.2.5}$$

Furthermore, we obtain by (2.3.25) and (A.2.4)

$$\left\|\mathcal{P}_h^{\pm}\left(\psi_h,t\right)\right\|^2_{\mathrm{L}^2(\Omega)}=\sum_{j=1}^{N}\sum_{\vartheta=0}^{k}\frac{\triangle_j\left(t\right)}{2\vartheta+1}\left|r^{\pm}_{\vartheta,j}\left(\psi_h,t\right)\right|^2. \tag{A.2.6}$$

Thus, by (2.3.25) and (A.2.6) follows

$$\begin{aligned}\left(\psi_h,\mathcal{P}_h^{\pm}\left(\psi_h,t\right)\right)_{\Omega}=&\left\|\mathcal{P}_h^{\pm}\left(\psi_h,t\right)\right\|^2_{\mathrm{L}^2(\Omega)}\\&-\sum_{j=1}^{N}\frac{\triangle_j\left(t\right)}{2\vartheta+1}\left|r^{\pm}_{k,j}\left(\psi_h,t\right)\right|^2\\&+\sum_{j=1}^{N}r^{\pm}_{k,j}\left(\psi_h,t\right)\left(\psi_h,\mathcal{L}_k\right)_{K_j(t)}.\end{aligned} \tag{A.2.7}$$

Next, the equations (A.2.6) and (A.2.7) provide

$$\begin{aligned}0\le&\left\|\psi_h-\mathcal{P}_h^{\pm}\left(\psi_h,t\right)\right\|^2_{\mathrm{L}^2(\Omega)}\\=&\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)}-\left\|\mathcal{P}_h^{\pm}\left(\psi_h\right)\right\|^2_{\mathrm{L}^2(\Omega)}\\&+\sum_{j=1}^{N}2r^{\pm}_{k,j}\left(\psi_h,t\right)\left(\frac{\triangle_j\left(t\right)}{2k+1}r^{\pm}_{k,j}\left(\psi_h,t\right)-\left(\psi_h,\mathcal{L}_k\right)_{K_j(t)}\right).\end{aligned} \tag{A.2.8}$$

In one dimension, surface integrals reduce to pointwise evaluations. Therefore, we obtain by Cauchy-Schwarz's inequality, (A.2.5), (2.2.21) and (2.2.22)

$$\begin{aligned}&\sum_{j=1}^{N}\frac{\triangle_j\left(t\right)}{2k+1}\left|r^{\pm}_{k,j}\left(\psi_h,t\right)\right|^2\\\le&\frac{2h}{2k+1}\sum_{j=1}^{N}\left|\psi^{\pm}_{h,j\mp\frac{1}{2}}\right|^2+\frac{2kh}{2k+1}\sum_{\vartheta=0}^{k-1}\sum_{j=1}^{N}\left|r^{\pm}_{\vartheta,j}\left(\psi_h,t\right)\right|^2\\\le&\frac{2h}{2k+1}\left\|\psi_h\right\|^2_{\Gamma_{h(t)}}+2k\sum_{\vartheta=0}^{k-1}\frac{2\vartheta+1}{2k+1}\left\|\psi_h\right\|^2_{\mathrm{L}^2(\Omega)}\le Ch^{2k+2}.\end{aligned} \tag{A.2.9}$$

Likewise, by Cauchy-Schwarz's inequality, (A.2.5), (2.2.21) and (2.2.22) follows

$$\begin{aligned}
&\sum_{j=1}^{N} \left| r_{k,j}^{\pm} \left( \psi_h, t \right) \left( \psi_h, \mathcal{L}_k \right)_{K_j(t)} \right| \\
\leq &\sqrt{\frac{h}{2k+1}} \left\| \psi_h \right\|_{\Gamma_{h(t)}} \left\| \psi_h \right\|_{\mathrm{L}^2(\Omega)} \\
+ &\sum_{\vartheta=0}^{k-1} \frac{2\vartheta+1}{2k+1} \left\| \psi_h \right\|_{\mathrm{L}^2(\Omega)}^2 \leq C h^{2k+2}.
\end{aligned} \tag{A.2.10}$$

Next, we obtain by the identity (A.2.2) in lemma A.2.1 the identity

$$\left\| \mathcal{P}_h^{\pm} \left( u, t \right) - \mathcal{P}_h \left( u, t \right) \right\|_{\mathrm{L}^2(\Omega)} = \left\| \mathcal{P}_h^{\pm} \left( \psi_h, t \right) \right\|_{\mathrm{L}^2(\Omega)}. \tag{A.2.11}$$

Thus, (A.2.8), (A.2.9) and (A.2.10) and (A.2.11) yield

$$\begin{aligned}
&\left\| \mathcal{P}_h^{\pm} \left( u, t \right) - \mathcal{P}_h \left( u, t \right) \right\|_{\mathrm{L}^2(\Omega)} \\
\leq &\left\| \psi_h \right\|_{\mathrm{L}^2(\Omega)}^2 + \frac{2h}{2k+1} \sum_{j=1}^{N} \left| r_{k,j}^{\pm} \left( \psi_{h(t)}, t \right) \right|^2 \\
+2 &\sum_{j=1}^{N} \left| r_{k,j}^{\pm} \left( \psi_h, t \right) \right| \left| \left( \psi_h, \mathcal{L}_k \right)_{K_j(t)} \right| \leq C h^{2k+2}.
\end{aligned}$$

This estimate completes the proof. □

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