Value Ranges for Schlicht Functions

Dissertationsschrift zur Erlangung des naturwissenschaftlichen Doktorgrades der Julius-Maximilians-Universität Würzburg

vorgelegt von

Julia Diana Koch

aus

Berlin, Deutschland

Würzburg 2016

Eingereicht am 18.11.2016

Erster Gutachter: Prof. Dr. Oliver Roth

Zweiter Gutachter: Prof. Dr. Dmitri Valentinovich Prokhorov

Tag der mündlichen Prüfung: 3.2.2017

Acknowledgements

They say it takes a village to raise a child, and it certainly took advice and encouragement from at least a small village for me to write this thesis. It is my pleasure to thank all those who supported me during the research for and writing of this thesis.

My deepest gratitude goes to Prof. Oliver Roth for his continuous support, advice, patience, sense of humour, and kindness. His knowledge and guidance helped me at all times, and I am especially grateful for the trust he put in me during my years as a PhD student.
I would like to thank Prof. Stephan Ruscheweyh for giving me a position in his group, for many interesting discussions, and for making possible my stay in Chile, and Prof. Luis Salinas for his hospitality and support during this stay.

Furthermore, I thank my favourite co-author Dr. Sebastian Schleißinger for fruitful discussions, for very pleasant collaboration, and also for him being a good friend through all of these years.
I will always think back fondly to my time at our group and department, a big thank you goes out to those who made it such a friendly and inspiring environment, especially Christoph, Daniel, Daniela, Jörn, Nicola, Richard, Thomas, and Uta.
And, last but not least, I would like to take this opportunity to thank those who helped me keep my sanity, in particular my parents and Christian.

Contents

Nomenclature

$\mathbb{H}$ $:= \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, the upper half-plane

$\mathbb{C}$ the complex plane

$\mathbb{D}$ $:= \{z \in \mathbb{C} \mid |z| < 1\}$, the unit disk

$\mathbb{D}_r$ $:= \{z \in \mathbb{C} \mid |z| < r\}$, $r \in (0, \infty)$

$\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{R}$ the set of real numbers

$\mathcal{A}(M)$ the set of functions $f : M \to \mathbb{C}$ analytic on a domain $M \subseteq \mathbb{C}$

$\mathcal{H}$ $:= \{f : \mathbb{H} \to \mathbb{H} : f$ schlicht, $f(z) = z - \frac{c}{z} + \gamma(z)$, where $\angle \lim_{z\to\infty} z \cdot \gamma(z) = 0\}$, the class of schlicht self-mappings of $\mathbb{H}$ with hydrodynamic normalisation

$\mathcal{H}(T)$ $:= \{f : \mathbb{H} \to \mathbb{H} : f$ schlicht, $f(z) = z - \frac{2T}{z} + \gamma(z)$, where $\angle \lim_{z\to\infty} z \cdot \gamma(z) = 0\}$, the class of schlicht self-mappings of $\mathbb{H}$ with hydrodynamic normalisation and fixed half-plane capacity

$\mathcal{S}$ $:= \{f : \mathbb{D} \to \mathbb{C} : f$ schlicht, $f(0) = 0$, $f'(0) = 1\}$, the class of schlicht normalised functions

$\mathcal{S}^{>}$ $:= \{f : \mathbb{D} \to \mathbb{D} : f$ schlicht, $f(0) = 0$, $f'(0) > 0\}$, the class of schlicht, bounded, and normalised functions with positive derivative at the origin

$\mathcal{S}^{\geq}$ $:= \mathcal{S}^{>} \cup \{0\}$, the compactification of $S^{>}$

$\mathcal{S}_T$ $:= \{f : \mathbb{D} \to \mathbb{D} : f$ schlicht, $f(0) = 0$, $f'(0) = e^{-T}\}$, $T > 0$, the class of schlicht, bounded, and normalised functions with fixed derivative at the origin

$\mathcal{T}$ $:= \{f : \mathbb{D} \to \mathbb{D} : f \in \mathcal{A}(\mathbb{D})$, $f(0) = 0$, $f'(0) > 0$, $\mathrm{Im}(f(z))\,\mathrm{Im}(z) \geq 0$ for all $z \in \mathbb{D}\}$, the class of analytic, bounded, and normalised typically real functions

$\mathcal{U}$ $:= \{f : \mathbb{D} \to \mathbb{D} \,:\, f$ schlicht, $f(0) = 0,\ f'(0) > 0,\ f$ has only real coefficients in its Taylor expansion around $0\}$, the class of schlicht, bounded, and normalised functions with real coefficients

$\mathcal{R}$ $:= \{f : \mathbb{D} \to \mathbb{D} \,:\, f \in \mathcal{A}(\mathbb{D}),\ f(0) = 0, f$ has only real coefficients in its Taylor expansion around $0\}$, the class of analytic normalised functions with real coefficients

$\ker A$ $:= \{x \in \mathbb{Z}^n \,:\, Ax = 0\}$, the kernel of a linear mapping $A : \mathbb{Z}^n \to \mathbb{Z}^m$

M^o the interior of a set M

S_1 $:= \{f : \mathbb{D} \to \mathbb{D} \,:\, f$ schlicht, $f(0) = 0\}$, the class of schlicht, bounded, and normalised functions

$S_1(\zeta, \omega)$ $:= \{f \in S_1 \,:\, f(\zeta) = \omega\}$, the class of schlicht, bounded, and normalised functions with fixed value at $z = \zeta$, $\zeta \in \mathbb{D}$

Chapter 1

Introduction and outline of this thesis

Around the turn of the last century, the question of where an analytic function defined on the unit disc $\mathbb{D}$[1] can map points $z_0 \in \mathbb{D}$ was considered for different classes of such functions[2].

A first answer was given in the 19th century, by Hermann Armandus Schwarz, in what is today known as the Schwarz lemma (see [Boa10] for the interesting history of this lemma and its name):

Theorem 1.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic, with $f(0) = 0$. Then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$, and if equality holds in one $z \in \mathbb{D}$, f is a rotation of the Koebe function, i.e. $f(z) = \frac{z}{(1-\eta z)^2}$ with $\eta \in \partial\mathbb{D}$.*

Of course, the question of determining the value range of a class of functions $f \in \mathcal{A}(\mathbb{D})$ is intrinsically related to the question of the coefficient region of this class, since $f(z) = \sum_{k=0}^{\infty} a_k z^k$ if and only if $f^{(k)}(0) = a_k k!$.

Among the most famous questions of this kind is the so-called Bieberbach Conjecture (see [Koe94] for an overview of the history of this conjecture):

Conjecture 1.2 ([Bie16])**.** *Let*

$$f \in \mathcal{S} := \{f : \mathbb{D} \to \mathbb{C} \,:\, f \text{ schlicht},\ f(0) = 0,\ f'(0) = 1\}$$

[1] Since by the Riemann mapping theorem, any simply connected domain Ω which is a proper subset of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$, we can always limit ourselves to this case.

[2] For a detailed survey of the history of value ranges see [SS50, Ch. I] or, especially for the case of bounded functions, [Pro02].

have Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *at the origin. Then*

$$|a_k| \leq k$$

where strict inequality holds for every n *unless* f *is a rotation of the Koebe function.*

The case of $k = 2$ was proven by Bieberbach himself by applying the area principle [Gro14]. He did not know about Grönwall's result and proved it independently in his paper. He further defined

$$k_n := \max\{|a_n| \,:\, f(z) = z + \sum_{j=2}^{\infty} a_j z^j,\ f \in \mathcal{S}\}$$

(note that this maximum exists since the class $\mathcal{S}$ is compact) and tentatively remarked: „Vielleicht ist überhaupt $k_n = n$“[3].

In the following years, partial results for subclasses of $\mathcal{S}$ were proven: Loewner [Löw17] showed that $|a_k| \leq 1$ for convex functions f, Nevanlinna [Nev20] proved the Bieberbach conjecture for starlike functions, Dieudonné [Die31] and Rogosinski [Rog32] for functions with real coefficients, and Reade [Rea55] for close-to-convex functions.
The first notable progress in the general question goes back to a revolutionary idea by Charles Loewner [Löw23], namely, to express schlicht functions as solutions to a certain differential equation, see chapter 2. Using his method, he was able to prove that $|a_3| \leq 3$.

Through Loewner's equation, it is possible to interpret an optimisation problem for classes of schlicht functions as the problem of finding a control that steers the trajectory of a dynamic system to the boundary of its reachable set. Thus, powerful tools from the theory of optimal control can be applied to tackle the question of the value range of functions which can be expressed via the Loewner equation.
In recent years, it became a topic of much interest again after Schramm [Sch00] introduced the notion of SLE (stochastic or Schramm-Loewner evolution), where a Brownian motion takes the place of the driving term in Loewner's equation, and which has a range of application in statistical physics. Two of the Fields medal recipients of the last years (Werner in 2006 and Smirnov in 2010) have been awarded for their work in this field.
See also [ABC10] and [BC14] for a survey of the history and ongoing evolution of

[3] "Perhaps $k_n = n$ holds in general"

Loewner theory.
Large parts of this thesis rely on this technique.

When decades later finally a complete proof of Bieberbach's conjecture was given by DeBranges [DeB85] and shortly afterwards a simpler version by FitzGerald and Pommerenke [FP85], they both relied on Loewner's results. Both actually proved the Milin conjecture, which implies the Bieberbach conjecture:

Conjecture 1.3 ([Mil71], proof in [DeB85], [FP85]). *Let $f \in \mathcal{S}$ and define its* logarithmic coefficients c_k *by*

$$\log \frac{f(z)}{z} =: \sum_{k=1}^{\infty} c_k z^k.$$

Then, for $n \geq 1$,

$$\sum_{k=1}^{n} (n-k+1) \left(k|c_k|^2 - \frac{4}{k} \right) \leq 0.$$

Another field in the theory of univalent functions was also boosted by Bieberbach's conjecture: Garabedian and Schiffer [GS55] proved that the conjecture holds for the fourth coefficient, i.e. $|a_4| \leq 4$ by making use of a variation technique[4]. The basic idea is very simple: if a function is locally extremal for a real-valued functional, then slight perturbations (or *variations*) of this function will yield a smaller value of that functional.
This general idea was already used by Grötzsch [Grö30] and Marty [Mar34], but the technique could not unfold its full potential until Schiffer [Sch38] developed a method, his boundary variation, which gives a more systematic approach to constructing powerful variation families. He proved that a function that is extremal in the class $\mathcal{S}$ for a real-valued functional satisfies a certain differential equation, the so-called *Schiffer equation*.
Goluzin [Gol46] gave a different approach to deriving the same results as Schiffer, and Schiffer [Sch39], [Sch43] went on to develop his interior variation, which is based on potential theory.
In general, variational methods yield necessary conditions for functions to be extremal for a problem. Teichmüller [Tei38] developed a method to prove that these necessary conditions are sufficient, as well. Since then, Schiffer's differential equation has been applied to a wide range of problems for classes of schlicht functions,

[4]Roth [Rot98, Ch. II.2.2] showed that, in fact, for functions in $\mathcal{S}$, applying the Pontryagin Maximum Principle to the corresponding Loewner equation is more or less equivalent to using Schiffer's equation.

see [Sch58] for a survey.
The result in Chapter 5 will be mainly based on a variational approach.

In the following, Chapter 2 will give a short overview over one of the most important techniques used, the Pontryagin Maximum Principle in combination with Loewner theory, and then introduce complex versions of Pontryagin's theorem.
Chapter 3 will deal with variants of Rogosinski's lemma [Rog34] and describe value sets for bounded functions with interior normalization, i.e. functions $\mathbb{D} \to \mathbb{D}$ with certain prescribed Taylor coefficient(s) at $z = 0$. We will determine the sets

$$\{f(z_0) \,:\, f \in \mathcal{S}_T\},$$

where

$$\mathcal{S}_T := \{f : \mathbb{D} \to \mathbb{D} \text{ schlicht, } f(0) = 0,\ f'(0) = e^{-T}\} \text{ for } T > 0;$$

the analogue for the inverse functions, and that for typically real (though not necessarily schlicht) functions, and finally that for functions with real coefficients.
In Chapter 4, we concern ourselves with functions with boundary normalization; for convenience, we switch from $\mathbb{D}$ to the upper half-plane $\mathbb{H}$. We discuss known results and describe the value sets for symmetric functions.
Chapter 5 discusses value ranges for the derivative of functions from certain classes. We give a necessary condition for a schlicht function $\mathbb{D} \to \mathbb{D}$ to have extremal derivative in a point ζ where $f(\zeta)$ is fixed.

Chapter 2

The Pontryagin Maximum Principle

2.1 The PMP and the radial Loewner equation

The Pontryagin maximum principle (or shortly PMP) was developed in the 1950s by Lev Semyonovich Pontryagin and his group at the Steklov Institute ([Pon78], cited in [PP09]), although research in the direction was also undertaken in the United States around the same time (see [PP09] for a historical survey).

We consider a differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1.1}$$

and ask for the maximum a functional J can attain over the set of all trajectories of (2.1.1), i.e. we want to determine

$$\max_x J(x), \text{ where } x \text{ is a solution to (2.1.1).}$$

The Pontryagin maximum principle then states that for an extremal solution $\tilde{x}$, i.e.

$$\frac{d}{dt}\tilde{x}(t) = f(\tilde{x}(t), \tilde{u}(t)) \text{ and } \max_x J(x) = J(\tilde{x}),$$

there has to be a so-called adjoint response $\tilde{\eta}$, i.e. a solution to the equation

$$\dot{\eta}(t) = -\eta D_x f(\tilde{x}(t), \tilde{u}(t)),$$

such that the function

$$H(\tilde{\eta}(t), \tilde{x}(t), u(t)) := \tilde{\eta}(t) \cdot f(\tilde{x}(t), u(t))$$

takes its maximum with respect to u at $\tilde{u}(t)$ for each $t \geq 0$.
Thus, the PMP shifts the problem of finding an extremal trajectory x for J to

that of finding a driving term u extremal for H, which is often much easier.

Note that the principle only yields a necessary condition for an extremal x; it is therefore often important to know in the first place that such an optimum exists. In our setting, we will always deal with continuous functionals and compact sets so that existence of an extremum is obvious.

Classically, the principle was formulated for problems in $\mathbb{R}^n$, though it has of course been used in complex settings for decades, especially in connection with the so-called *Loewner differential equation.* In his seminal 1923 paper [Löw23], Charles Loewner (born as Karel or Karl Löwner) introduced his approach to solving the Bieberbach conjecture and showed that there is a connection between schlicht functions and control theory. Namely, schlicht functions on $\mathbb{D}$ can be approximated by so-called one-slit functions, that is, functions which map $\mathbb{D}$ onto $\mathbb{D}$ minus a simple continuous curve, and slit functions f which fulfil $f(0) = 0$ and $f'(0) > 0$ can be described as solutions to the so-called Loewner differential equation

$$\dot{f}(z,t) = -f(z,t)\frac{\kappa(t) - f(z,t)}{\kappa(t) + f(z,t)}, \ t \geq 0, \qquad f(z,0) = z \tag{2.1.2}$$

where $\kappa : [0, \infty) \to \partial\mathbb{D}$ is a measurable function.
Note that the functions $w \mapsto \frac{\kappa+w}{\kappa-w}$ are the extreme points of the set of *Herglotz functions*

$$\mathcal{P} := \{p \in \mathcal{A}(\mathbb{D}) \, : \, \operatorname{Re} p(z) > 0 \text{ for } z \in \mathbb{D}, \, p(0) = 1\},$$

and in fact, general normalised schlicht functions (which do not necessarily map $\mathbb{D}$ onto slit regions) can be expressed as solutions to

$$\dot{f}(z,t) = -f(z,t)p(f(z,t),t), \quad f(z,0) = z, \tag{2.1.3}$$

where $z \mapsto p(z,t) \in \mathcal{P}$ for almost every $t \in [0, \infty)$ and $t \mapsto p(z,t)$ is measurable for every $z \in \mathbb{D}$, see [Pom75, Th. 6.1].
Denote by $\mathcal{M}$ the set of all probability measures on $\partial\mathbb{D}$. Due to the Herglotz representation [Dur83, section 1.9], we can write $p(z,t)$ for a. e. $t \geq 0$ as

$$p(z,t) = p_{\mu_t}(z) := \int_{\partial\mathbb{D}} \frac{u+z}{u-z} \, \mu_t(du), \tag{2.1.4}$$

for some $\mu_t \in \mathcal{M}$, i.e., (2.1.3) has the form

$$\dot{f}(z,t) = -f(z,t) \int_{\partial\mathbb{D}} \frac{u+f(z,t)}{u-f(z,t)} \, \mu_t(du), \quad f(z,0) = z. \tag{2.1.5}$$

The *reachable set*

$$R := \{f_T := f(z,T) \,:\, f(z,T) \text{ is a solution to (2.1.3) and } T \geq 0\}$$

of (2.1.3) coincides with the set $\mathcal{S}^{>} := \{f : \mathbb{D} \to \mathbb{D} \,:\, f \text{ schlicht},\, f(0) = 0,\, f'(0) > 0\}$.
Note that for a solution $f(z,t)$ of 2.1.3, $f'(0,t) = e^{-t}$ holds, and thus

$$\begin{aligned} &R_T = \{f_T := f(z,T) \,:\, f(z,T) \text{ is a solution to (2.1.3)}\} = \\ &\mathcal{S}_T := \{f : \mathbb{D} \to \mathbb{D} \text{ univalent},\, f(0) = 0,\, f'(0) = e^{-T}\}, \quad T > 0. \end{aligned}$$

The sets R_T are obviously compact, but R is not. This is, however, easily remedied by considering the set $S^{\geq} := S^{>} \cup \{0\}$ instead of R: we compactify $S^{\geq}$ by adding the zero function and keep in mind, when applying the PMP, that the origin might be a boundary point of the set we want to determine. Every other boundary point will be in R and thus the PMP can be applied to the trajectory which is steered into it.

In particular, if one is interested in optimising functionals that evaluate a schlicht function (or its derivatives) in one point, the problem is equivalent to that of finding an extremal trajectory starting at the given point and being driven by a control from $\mathcal{P}$.

Roth [Rot98] proved an infinite-dimensional version of the principle, which allows to consider arbitrary complex differentiable functionals on the sets of holomorphic functions which can be expressed as reachable sets of a differential equation; however, for our purposes, a finite-dimensional version will be sufficient.

In the following section we will give the complex versions of the respective real theorems from [LM86], including a version which includes constraints on the trajectories considered.

2.2 A complex formulation of the Pontryagin principle

The so-called *fixed end time version* of the Pontryagin maximum principle on the reachable set reads as follows:

Theorem 2.1 ([LM86], Chapter 4, Th. 3)**.** *Consider the process in* $\mathbb{R}^n$

$$\dot{x} = f(x, u) \tag{2.2.1}$$

with $f(x, u)$ *and* $\frac{\partial f}{\partial x}(x, u)$ *continuous in* $\mathbb{R}^{n+m}$*. Let* $\mathcal{F}$ *be the family of all measurable controllers* $u(t)$ *on* $0 \leq t \leq T$ *that satisfy the restraint* $u(t) \subseteq \Omega \subseteq \mathbb{R}^m$ *and admit a bound for the response initiating at the point* x_0*. Let* $u^*(t)$ *have a response* $x^*(t)$ *with* $x^*(T)$ *in the boundary of the set of attainability* $R_T := \{x(T) : x \text{ is a solution to } (2.2.1)\}$*. Then there exists a nontrivial adjoint response* $\eta^*(t)$ *of*

$$\dot{\eta} = -\eta \frac{\partial f}{\partial x}(x^*(t), u^*(t))$$

such that the maximal principle obtains, that is,

$$H(\eta^*(t), x^*(t), u^*(t)) = M(\eta^*(t), x^*(t)) \text{ almost everywhere.}$$

Furthermore, if $u^*(t)$ *is bounded,* $M(\eta^*(t), x^*(t))$ *is constant everywhere.*
Here, the Hamiltonian function is

$$H(\eta, x, u) := \eta f(x, u) = \eta_1 f^1(x, u) + \cdots + \eta_n f^n(x, u)$$

and

$$M(\eta, x) := \max_{u \in \Omega} H(\eta, x, u) \quad \text{(wherever it exists).}$$

Note that this can be easily sharpened to the *free end time* version, the only difference being a stronger condition on the maximal Hamiltonian (it being zero a.e. instead of just constant):

Theorem 2.2 (cf. [Lew06], Th. 5.18)**.** *Consider the situation of Th. 2.1, but let* $u^*(t)$ *have a response* $x^*(t)$ *with* $x^*(T)$ *in the boundary of the free end time set of attainability* $R_T := \cup_{0 \leq t \leq T} R_t$*. Then there exists a nontrivial adjoint response* $\tilde{\eta}(t)$ *of*

$$\dot{\eta} = -\eta \frac{\partial f}{\partial x}(x^*(t), u^*(t))$$

such that the maximal principle obtains, that is,

$$H(\eta^*(t), x^*(t), u^*(t)) = M(\eta^*(t), x^*(t)) \text{ almost everywhere.}$$

Furthermore, if $u^*(t)$ *is bounded,* $M(\eta^*(t), x^*(t)) \equiv 0$.

We formulate the corresponding complex version.
For this purpose, we consider processes

$$\dot{w}(t) = F(w(t), \kappa(t)), \qquad w(0) = z_0 \tag{2.2.2}$$

where W is an open subset of $\mathbb{C}^n$, $z_0 \in W$, $w(t) \in W$, $\kappa(t) \in U \subseteq \mathbb{C}^m$ and $F : W \times \operatorname{cl} U \to C^n$ is continuous on $W \times U$ as well as analytic in all components $w_1, \dots, w_n$ for any $u \in \operatorname{cl} U$.
A *trajectory* of this system is the first component of the pair (w, κ) where κ is an *admissible control*, i.e., a measurable function on an interval $I \subseteq \mathbb{R}$ such that $t \mapsto F(w, \kappa(t))$ is locally integrable for any $w \in W$, and $w : I \to W$ satisfies (2.2.2). We denote by $\mathcal{X}$ the set in which admissible controls take their values, i.e. we have $\kappa(t) \in \mathcal{X}$ for every $t \in [0, \infty)$. For reasons of simplicity and because it is no restriction in our cases, we will assume that $\mathcal{X}$ is compact.
We call

$$R_T := \{z \in \mathbb{C} \,:\, z = w(T) \text{ for some trajectory } (w, \kappa) \text{ of } (2.2.2)\}$$

the *reachable set* or *set of attainability* at time T of (2.2.2).

Theorem 2.3. *If a control κ^* is optimal for the process*

$$\dot{w} = F(w, \kappa), \qquad w(0) = z_0$$

i.e., κ^* is admissible and the corresponding response w^* has the property that $w^*(T)$ is on the boundary of the reachable set R_T, then the Hamiltonian $H(\lambda, w, \kappa) := \lambda \cdot F(w, \kappa)$ fulfils

$$\max_{\kappa \in \mathcal{X}} \operatorname{Re} H(\lambda^*(t), w^*(t), \kappa(t)) = \operatorname{Re} H(\lambda^*(t), w^*(t), \kappa^*(t)) \text{ a.e. on } [0, T],$$

where λ^* is an adjoint response, i.e. a solution to

$$\dot{\lambda}_j = -\frac{\partial}{\partial w_j} H(\lambda, w^*, \kappa^*).$$

Furthermore, if κ^* is bounded, then

$$\max_{\kappa \in \mathcal{X}} \operatorname{Re} H(\lambda^*(t), w^*(t), \kappa(t)) \equiv const.$$

Proof. We write $w(t) = x(t) + iy(t)$ with $x, y : [0, T] \to \mathbb{R}^n$ and set $g := \operatorname{Re} F$, $h := \operatorname{Im} F$. Then (2.2.2) is equivalent to the following system of differential equations in $\mathbb{R}^{2n}$

$$\begin{aligned} \dot{x}_j &= g_j(w, \kappa), \\ \dot{y}_j &= h_j(w, \kappa). \end{aligned}$$

According to Th. 2.1, if κ^* is an optimal control, i.e. an admissible control such that the corresponding trajectory $w^* = (x^* + iy^*)$ fulfils $w^*(T) \in \partial R_T$, then there are adjoint responses $\eta, \mu : [0,T] \to \mathbb{R}^n$, i.e. functions that fulfil

$$\dot{\eta}_j = -\sum_{k=1}^{n} \left(\eta_k \frac{\partial g_k}{\partial x_j} + \mu_k \frac{\partial h_k}{\partial x_j} \right),$$

and

$$\dot{\mu}_j = -\sum_{k=1}^{n} \left(\eta_k \frac{\partial g_k}{\partial y_j} + \mu_k \frac{\partial h_k}{\partial y_j} \right),$$

respectively, such that the real Hamiltonian

$$\begin{aligned} H_{\mathbb{R}}(\eta^*(t), &\mu^*(t), x^*(t), y^*(t), \kappa(t)) := \\ &\eta^*(t)^T \cdot g(x^*(t) + iy^*(t), \kappa(t)) + \mu^*(t)^T \cdot h(x^*(t) + iy^*(t), \kappa(t)) = \\ &= \sum_{j=1}^{n} \eta_j^* \cdot g_j(w^*, \kappa) + \sum_{j=1}^{n} \mu_j^* \cdot h_j(w^*, \kappa) \end{aligned}$$

is optimised at κ^*.

Consider the complex functions

$$\lambda_j(t) := \eta_j(t) - i\mu_j(t), \quad j = 1, \ldots, n,$$

and the complex Hamiltonian

$$H_{\mathbb{C}}(\lambda(t), w(t), \kappa(t)) := \lambda(t) \cdot F(w(t), \kappa(t)) = \sum_{j=1}^{n} \eta_j g_j + \mu_j h_j + i\,(\eta_j h_j - \mu_j g_j)\,.$$

Firstly, we note that

$$H_{\mathbb{R}} = \operatorname{Re} H_{\mathbb{C}}.$$

Furthermore, we obviously have

$$\dot{w} = \frac{\partial}{\partial \lambda} H_{\mathbb{C}},$$

and

$$
\begin{aligned}
\dot{\lambda}_j = \dot{\eta}_j - i\dot{\mu}_j &= -\left(\frac{\partial}{\partial x_j}H_{\mathbb{R}}(\eta,\mu,x,y,\kappa) - i\frac{\partial}{\partial y_j}H_{\mathbb{R}}(\eta,\mu,x,y,\kappa)\right) = \\
&= -\left(\sum_{k=1}^{n}\left(\eta_k\frac{\partial g_k}{\partial x_j} + \mu_k\frac{\partial h_k}{\partial x_j}\right) - i\sum_{k=1}^{n}\left(\eta_k\frac{\partial g_k}{\partial y_j} + \mu_k\frac{\partial h_k}{\partial y_j}\right)\right) = \\
&\overset{*}{=} -\left(\sum_{k=1}^{n}\left(\eta_k\frac{\partial g_k}{\partial x_j} + \mu_k\frac{\partial h_k}{\partial x_j}\right) - i\sum_{k=1}^{n}\left(-\eta_k\frac{\partial h_k}{\partial x_j} + \mu_k\frac{\partial g_k}{\partial x_j}\right)\right) = \\
&= -\left(\sum_{k=1}^{n}\left(\eta_k\left(\frac{\partial g_k}{\partial x_j} + i\frac{\partial h_k}{\partial x_j}\right) + \mu_k\left(\frac{\partial h_k}{\partial x_j} - i\frac{\partial g_k}{\partial x_j}\right)\right)\right) = \\
&= -\frac{\partial}{\partial x_j}\sum_{k=1}^{n}\left(\eta_k g_k + \mu_k h_k + i\left(\eta_k h_k - \mu_k g_k\right)\right) \\
&= -\frac{\partial}{\partial x_j}H_{\mathbb{C}}(\lambda,w,\kappa) \overset{*}{=} -\frac{\partial}{\partial w_j}H_{\mathbb{C}}(\lambda,w,\kappa),
\end{aligned}
$$

where the equalities marked with $*$ follow from the Cauchy-Riemann differential equations, since each $F_k = g_k + ih_k$ is by assumption analytic in $w_j = x_j + iy_j$, $j,k = 1,\dots,n$.

If κ^* is bounded, then so are its real counterparts, and thus

$$\max_{\kappa\in\mathcal{X}} \operatorname{Re} H_{\mathbb{C}} = \max_{\kappa\in\mathcal{X}} H_{\mathbb{R}} \equiv const.$$

$\square$

Remark 2.4. *For the purposes of the Loewner equation (2.1.2), the driving term $\kappa : [0,\infty) \to \partial\mathbb{D}$ can easily be translated to a real driving term $\phi : [0,\infty) \to \mathbb{R}$ via $\kappa(t) = e^{i\phi(t)}$. Note that ϕ can always be assumed to be bounded, to be precise, $|\phi(t)| \leq \pi$. If κ is continuos, then ϕ is piece-wise continuous.*

Many interesting results in the theory of value ranges can be obtained by determining the boundary of the reachable set of a differential equation in this way. However, for some applications a version of Pontryagin's principle is needed which deals with optimising a so-called cost functional over the set of trajectories of a control system with boundary conditions:
Let $w : [0,T] \to \mathbb{C}^n$ fulfil the boundary problem

$$\dot{w} = F(w,\kappa), \quad w(0) \in S_0, \quad w(T) \in S_1, \tag{2.2.3}$$

where

$$S_j = \Phi_j^{-1}(\{0\}), \quad j = 0,1,$$

with mappings

$$\Phi_j : \mathbb{C}^n \to \mathbb{C}^{k_j},$$

which are smooth in the sense of being holomorphic in $w_1, \dots, w_n$ such that $D\Phi_j(w)$ is surjective for each $w \in \Phi_j^{-1}(\{0\})$.
We denote by Δ the set of admissible κ, with $\kappa(t) \in \mathcal{X}$ for all $t \in (0, \infty]$, which steer the trajectory w from S_0 to S_1. Note that the time T at which the trajectory ends in S_1 depends on the control κ.

We are searching for an optimal control κ^*, i.e. a driving term $\kappa^* \in \Delta$ such that

$$\max_{\kappa \in \mathcal{X}} \operatorname{Re} \int_0^T F_0(w(t), \kappa(t))\, dt$$

is taken for κ^* and the corresponding response w^*.

We introduce the so-called *augmented Hamiltonian*

$$\hat{H}_{\mathbb{C}}\left(\hat{\lambda}(t), \hat{w}(t), \kappa(t)\right) := \hat{\lambda}(t) \cdot \hat{F}(\kappa(t), \hat{w}(t)),$$

with the augmented state

$$\hat{w}(t) = (w_0(t), w(t))^T$$

which is the response to the augmented system

$$\begin{aligned} \dot{w}_0(t) &= F_0(w(t), \kappa(t)), \\ \dot{w}_j(t) &= F_j(w(t), \kappa(t)) \quad \text{for } j = 1, \dots, n. \end{aligned}$$

The function $\hat{\lambda}$ is the augmented response, i.e. a nontrivial solution to the augmented adjoint equations

$$\begin{aligned} \dot{\lambda}_0(t) &= 0, \\ \dot{\lambda}_j(t) &= -\frac{\partial}{\partial w_j} \sum_{k=1}^{n} \lambda_k(t) F_k(w(t), \kappa(t)) \quad \text{for } j = 1, \dots, n. \end{aligned}$$

The corresponding real version of the PMP (where the hat denotes the augmented real Hamiltonian, state and adjoint response) is the following:

Theorem 2.5 ([LM86], Chapter 5, Th. 1)**.** *Consider the control process in* $\mathbb{R}^n$

$$\dot{x} = f(x, u)$$

with bounded measurable controllers $u(t)$ *on various intervals* $0 \leq t \leq t_1$ *in the restraint set* $\Omega \subseteq \mathbb{R}^m$*. Let* Δ *be all admissible controllers that steer some initial* *point of X_0 to a final point in the target set X_1. For each $u(t)$ on $0 \le t \le t_1$ in Δ with response $x(t)$ let the cost functional be*

$$C(u) = \int_0^{t_1} f^0\left(x(t), u(t)\right)\, dt.$$

If $u^*(t)$ on $0 \le t \le t_1$ is minimal optimal in Δ, with augmented response $\hat{x}^*(t) = (x^{0*}(t), x^*(t))$, then there exists a nontrivial augmented adjoint response $\hat{\eta}^*(t) = (\eta_0^*, \eta^*(t))$ such that

$$\hat{H}(\hat{\eta}^*(t), \hat{x}^*(t), u^*(t)) = \max_{u \in \Omega} \hat{H}(\hat{\eta}^*(t), \hat{x}^*(t), u(t)) \text{ almost everywhere,}$$

and

$$\max_{u \in \Delta} \hat{H}(\hat{\eta}^*(t), \hat{x}^*(t), u(t))) \equiv 0 \text{ and } \eta_0^* \le 0 \text{ everywhere on } 0 \le t \le t^*.$$

Also if X_0 and X_1 (or just one of them) are manifolds with tangent spaces T_0 and T_1 at $x^*(0)$ and $x^*(t^*)$, then $\hat{\eta}^*(t)$ can be selected to satisfy the transversality conditions at both ends (or just one end)

$$\eta^*(0) \text{ orthogonal to } T_0,$$
$$\eta^*(t^*) \text{ orthogonal to } T_1.$$

Note that the real version is slightly stronger, since it allows the constraint sets X_j, $j = 0, 1$ to be arbitrary real manifolds, while, in the complex case, we restrict ourselves to a special type of manifold; this is, however, sufficient for our purposes.

We now formulate and prove the complex version:

Theorem 2.6. *If $\kappa^* : [0, T] \to \mathbb{R}$ is an optimal control for the problem*

$$\max_{\kappa \in \Delta} \operatorname{Re} \int_0^T F_0(w(t), \kappa(t))\, dt,$$

then there is a non-trivial augmented adjoint response $\hat{\lambda}^*$, i.e. a function with the following properties:

(a) $\hat{\lambda}^*$ fulfils the adjoint equation $\dfrac{d}{dt}\hat{\lambda}_j(t) = \dfrac{\partial}{\partial w_j}\hat{H}(\hat{\lambda}, \hat{w}^*, \kappa^*)$;

(b) $\lambda_0^* \ge 0$;

(c) $\operatorname{Re} \hat{H}(\hat{\lambda}^*, \hat{w}^*, \kappa^*) = \max_{\kappa \in \mathcal{X}} \operatorname{Re} \hat{H}(\hat{\lambda}^*, \hat{w}^*, \kappa)$ for almost every $t \in [0, T]$;

(d) $\max_{\kappa \in \mathcal{X}} \operatorname{Re} H(\hat{\lambda}^*, \hat{w}^*, \kappa) \equiv 0$;

(e) *For* $\xi \in \ker(D\Phi_j(w^*(t_j)))$, *we have* $\operatorname{Re}\langle \lambda^*(t_j), \overline{\xi} \rangle = 0$, *for* $j = 0, 1$, *with* $t_0 = 0$ *and* $t_1 = T$.

Proof. Again, we state the real version of this problem:
Let

$$x_j := \operatorname{Re} w_j \text{ and } y_j := \operatorname{Im} w_j.$$

Then, according to (2.2.3), $x, y : [0, T] \to \mathbb{R}^n$ are solutions to the boundary problem

$$\begin{aligned}
\dot{x} &= g(x, y, \kappa) := \operatorname{Re} F(w, \kappa), \\
\dot{y} &= h(x, y, \kappa) := \operatorname{Im} F(w, \kappa), \\
& x(0) + iy(0) \in S_0, \\
& x(T) + iy(T) \in S_1.
\end{aligned}$$

The real augmented system thus has the form

$$\begin{aligned}
\dot{x}_0 &= \operatorname{Re} F_0(x + iy, \kappa), \\
\dot{x}_j &= g_j(x, y, \kappa), \quad \dot{y}_j = h_j(x, y, \kappa), \quad j = 1, \dots, n.
\end{aligned}$$

Theorem 2.5 then states that if $\tilde{\kappa}$ is optimal for the problem

$$\min_{\kappa \in \Delta} \int_0^T g_0(w(t), \kappa(t))\, dt, \quad g_0 := -\operatorname{Re} F_0,$$

which is equivalent to solving

$$\max_{\kappa \in \Delta} \int_0^T g_0(w(t), \kappa(t))\, dt, \quad g_0 := \operatorname{Re} F_0,$$

then there are augmented responses, i.e. functions $\hat{\eta} : [0, T] \to \mathbb{R}^{n+1}$, $\mu : [0, T] \to \mathbb{R}^n$ which fulfil

$$\begin{aligned}
\dot{\eta}_0 &= 0 \text{ and } \eta_0 \leq 0, \\
\dot{\eta}_j &= -\sum_{k=0}^{n} \left(\eta_k \frac{\partial g_k}{\partial x_j} + \mu_k \frac{\partial h_k}{\partial x_j} \right), \\
\dot{\mu}_j &= -\sum_{k=0}^{n} \left(\eta_k \frac{\partial g_k}{\partial y_j} + \mu_k \frac{\partial h_k}{\partial y_j} \right), \quad j = 1, \dots, n,
\end{aligned}$$

respectively, such that

$$\hat{H}_{\mathbb{R}}(\hat{\eta}^*, \mu^*, \hat{x}^*, y^*, \kappa) = -\eta_0 g_0 + \sum_{j=1}^{n} \eta_j g_j + \mu_j h_j$$

is optimised at $\kappa = \kappa^*$.
Set

$$\lambda_0 := -\eta_0,$$
$$\lambda_j := \eta_j - i\mu_j \text{ for } j = 1, \ldots, n$$

and

$$H_{\mathbb{C}}(\lambda, w, \kappa) := \sum_{j=0}^{n} \lambda_j \cdot F_j.$$

Then we obviously have

$$\dot{w}_j = \frac{\partial}{\partial \lambda_j} H_{\mathbb{C}},$$

and

$$\dot{\lambda}_0 = \text{const}.,$$

as well as

$$\dot{\lambda}_j = -\frac{\partial}{\partial w_j} H_{\mathbb{C}},$$

as in the proof of 2.3. This proves (a) and (b).

Obviously,

$$\mathrm{Re}\, \hat{H}_{\mathbb{C}}(\lambda, w, \kappa) = \mathrm{Re}\,(\lambda_0 g_0 + H_{\mathbb{C}}(\lambda, w, \kappa)) = -\eta_0 g_0 + H_{\mathbb{R}}(\eta, \mu, x, y, \kappa),$$

and thus, Th. 2.5 immediately yields (c) and (d).

To prove (e), let $S = \Phi^{-1}\{0\}$ with a smooth mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^m$, and let

$$\phi : \mathbb{R}^{2n} \to \mathbb{R}^m, \quad \phi(x, y) = \mathrm{Re}\, \Phi(x + iy), \quad x, y \in \mathbb{R}^n,$$
$$\psi : \mathbb{R}^{2n} \to \mathbb{R}^m, \quad \psi(x, y) = \mathrm{Im}\, \Phi(x + iy), \quad x, y \in \mathbb{R}^n.$$

Then we have $x + iy \in S_0$ if and only if $\phi(x, y) = 0$ and $\psi(x, y) = 0$, i.e. if (x, y) lies in the real submanifold $S_{\mathbb{R}}$ that is defined by

$$\Phi_{\mathbb{R}}(x, y) = 0, \text{ where } \Phi_{\mathbb{R}} := \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Let $\xi = \tau + i\vartheta$, $\tau, \vartheta \in \mathbb{R}^n$, be in $\ker(D\Phi(w^*))$. This means

$$0 = D\Phi \cdot \xi = \left(\sum_{j=1}^{n} \frac{\partial \Phi_k}{\partial w_j} \cdot \xi_j\right)_{k=1,\ldots,n} \overset{*}{=} \left(\sum_{j=1}^{n} \left(\frac{\partial \phi_k}{\partial x_j} + i\frac{\partial \psi_k}{\partial x_j}\right) \cdot (\tau_j + i\vartheta_j)\right)_{k=1,\ldots,n} =$$

$$= \left(\sum_{j=1}^{n} \left(\frac{\partial \phi_k}{\partial x_j}\tau_j - \frac{\partial \psi_k}{\partial x_j}\vartheta_j + i\left(\frac{\partial \phi_k}{\partial x_j}\vartheta_j + \frac{\partial \psi_k}{\partial x_j}\tau_j\right)\right)\right)_{k=1,\ldots,n} =$$

$$\overset{*}{=} \left(\sum_{j=1}^{n} \left(\frac{\partial \phi_k}{\partial x_j}\tau_j + \frac{\partial \phi_k}{\partial y_j}\vartheta_j + i\left(\frac{\partial \psi_k}{\partial y_j}\vartheta_j + \frac{\partial \psi_k}{\partial x_j}\tau_j\right)\right)\right)_{k=1,\ldots,n},$$

where for the equations marked with $*$, we use the Riemann-Cauchy differential equations.

Thus, we have

$$\sum_{j=1}^{n} \left(\frac{\partial \phi_k}{\partial x_j}\tau_j + \frac{\partial \phi_k}{\partial y_j}\vartheta_j\right) = 0$$

and

$$\sum_{j=1}^{n} \left(\frac{\partial \psi_k}{\partial x_j}\tau_j + \frac{\partial \psi_k}{\partial y_j}\vartheta_j\right) = 0$$

for each $k = 1, \ldots, m$. This means that (τ, ϑ) lies in the tangent space to $S_{\mathbb{R}}$ at $x^* + iy^*$, and by the transversality condition of Th. 2.5, we can choose η and μ such that

$$((\eta^*)^T, (\mu^*)^T) \cdot \begin{pmatrix} \tau \\ \vartheta \end{pmatrix} = 0.$$

This is equivalent to

$$0 = \sum_{j=1}^{n} \eta_j^* \tau_j + \sum_{j=1}^{n} \mu_j^* \vartheta_j.$$

Hence, we have

$$\operatorname{Re} < \lambda^*, \bar{\xi} > = \operatorname{Re} \overline{\lambda^*}^T \cdot \bar{\xi} = \operatorname{Re} \sum_{k=1}^{n} \overline{\lambda_j^* \xi_j} = \operatorname{Re} \sum_{k=1}^{n} (\eta_j^* + i\mu_j^*)(\tau_j - i\vartheta_j) =$$

$$= \operatorname{Re} \sum_{k=1}^{n} (\eta_j \tau_j + \mu_j \vartheta_j + i(\eta_j \vartheta_j - \mu_j \tau_j)) = \sum_{k=1}^{n} (\eta_j \tau_j + \mu_j \vartheta_j) = 0.$$

$\square$

Remark 2.7. *The fact that the augmented response* $\hat{\lambda}^*$ *is nontrivial can be interpreted to mean that*

$$\lambda^*(0) \neq 0 \qquad \text{or} \qquad \lambda_0^* = 1,$$

since $\lambda^*(0) = 0$ *would imply that* $\lambda^* \equiv 0$*, and thus necessarily* $\lambda_0^* \neq 0$*, and since the adjoint equations remain valid under multiplication with a fixed positive real number, we may therefore assume that* $\lambda_0^* = 1$*.*

Chapter 3

Value sets for bounded functions on $\mathbb{D}$

The Schwarz lemma tells us that an analytic mapping $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ must map a point z_0 into the disk $\mathbb{D}_{|z_0|}$. In 1934, Rogosinki ([Rog32], see also [Dur83, p. 200]) determined the exact value range for such bounded functions with additional nomalisation for $f'(0)$ and showed that

$$\{f(z_0)\ :\ f : \mathbb{D} \to \mathbb{D} \text{ analytic},\ f(0) = 0,\ f'(0) > 0\} = \Delta(z_0),$$

where $\Delta(z_0)$ is the closed region that is bounded by the curves

$$\begin{aligned} c_1(x) &:= i|z_0|^2 e^{ix}, \quad x \in [0, \pi], \\ c_2(x) &:= \frac{x + i|z_0|}{1 + ix|z_0|} z_0, \quad x \in [0, 1], \text{ and} \\ c_3(x) &:= \frac{x - i|z_0|}{1 - ix|z_0|} z_0, \quad x \in [0, 1]. \end{aligned}$$

Roth and Schleißinger investigated the value set under the additional condition that f be schlicht, and obtained ([RS14, Th. 1.1]) that

$$\begin{aligned} &\{f(z_0)\ :\ f : \mathbb{D} \to \mathbb{D} \text{ schlicht},\ f(0) = 0\,, f'(0) > 0\} \cup \{0\} = \\ &= \{|z|e^{ix} \in \mathbb{D}\ :\ d_{\mathbb{D}}(0, z) - d_{\mathbb{D}}(0, z_0) \leq -|\arg z_0 - x|, x \in \mathbb{R}\}, \end{aligned}$$

where $d_{\mathbb{D}}$ denotes the hyperbolic distance in $\mathbb{D}$, i.e.

$$d_{\mathbb{D}}(z, w) := \log \frac{1 + \left|\frac{z-w}{1-z\overline{w}}\right|}{1 - \left|\frac{z-w}{1-z\overline{w}}\right|}.$$

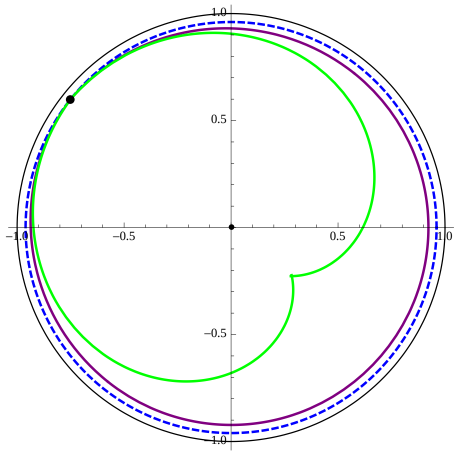

Figure 3.1: $z_0 = -0.75 + 0.6i$

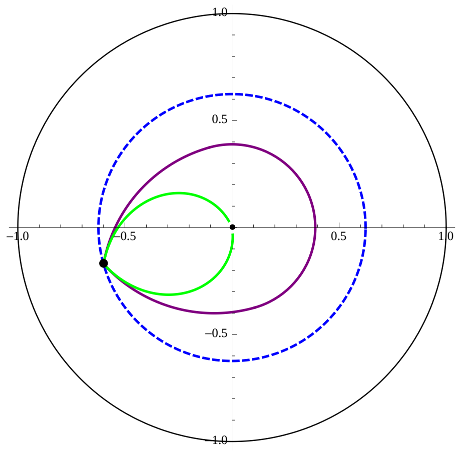

Figure 3.2: $z_0 = -0.6 - 0.17i$

The boundaries of the value sets for z_0 as determined in the lemmata by Schwarz (blue), Rogosinski (purple) and Roth/Schleißinger (green)

3.1 Schlicht functions with prescribed derivative $f'(0)$

In the next step, it is quite natural to ask what the sets look like if $f'(0)$ is fixed, i.e. $f'(0) = e^{-T}$ with $T > 0$. Hence, in the following, we will determine the value set

$$V_T(z_0) = \{f(z_0) \,:\, f \in \mathcal{S}_T\}, \quad z_0 \in \mathbb{D} \setminus \{0\},$$

where

$$\mathcal{S}_T := \{f : \mathbb{D} \to \mathbb{D} \text{ univalent}, \, f(0) = 0, \, f'(0) = e^{-T}\}, \quad T > 0.$$

The results in this and the following section are published in [KS16].

Note that Grunsky [Gru39] determined the value set $\{\log \frac{f(z)}{z} \,:\, f \in \mathcal{S}\}$ by elementary means, and in [GG76], the authors consider the set

$$\{\log(f(z_0)/z_0) \,:\, f : \mathbb{D} \to \mathbb{C} \text{ univalent}, \; f(0) = 0, \, f'(0) = 1, \, |f(z)| \leq M, \, z \in \mathbb{D}\}$$

for $M > 0$. Since $f \in \mathcal{S}_T$ if and only if $g := e^T f$ is a schlicht function $g : \mathbb{D} \to \mathbb{D}_{e^T}$ with $g(0) = 0$ and $g'(0) = 1$, see also [Rot99, p. 462], the set $V_T(z_0)$ can in principle be derived from their results.
However, we use a different and more straightforward approach to explicitly determine the set $V_T(z_0)$ by applying Pontryagin's maximum principle to the radial

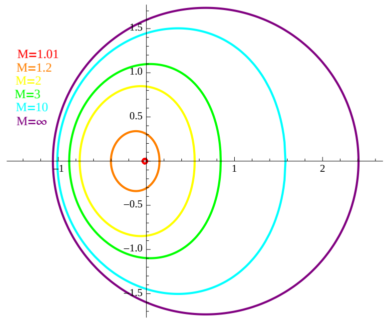

Figure 3.3: The sets $\{\log \frac{f(z)}{z} \, : \, f \in \mathcal{S}, \, f(\mathbb{D}) \subseteq \mathbb{D}_M\}$ for different values of M

Loewner equation. Finally, we mention that our results are analogous to the results of Prokhorov and Samsonova [PS15], who study univalent self-mappings of the upper half-plane having the so called hydrodynamical normalization at the boundary point ∞, see chapter 4.

For the sake of simplicity, we may assume that $z_0 \in (0,1)$; for other values of z_0, we just consider the function $z \mapsto e^{i \arg z_0} f\left(e^{-i \arg z_0} z\right)$ instead of f.

Theorem 3.1. *Let $z_0 \in (0,1)$. For $x_0 \in [-1,1]$ and $T > 0$, let $r = r(T, x_0)$ be the (unique) solution to the equation*

$$(1+x_0)(1-z_0)^2 \log(1-r) + (1-x_0)(1+z_0)^2 \log(1+r) - (1 - 2x_0 z_0 + z_0^2) \log r = (1+x_0)(1-z_0)^2 \log(1-z_0) + (1-x_0)(1+z_0)^2 \log(1+z_0) - (1 - 2x_0 z_0 + z_0^2) \log e^{-T} z_0$$

and let

$$\sigma(T, x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1 - 2x_0 z_0 + z_0^2} \left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right).$$

Furthermore, for fixed $T \geq 0$, define the two curves $C_+(z_0)$ and $C_-(z_0)$ by

$$C_\pm(z_0) := \left\{ w_\pm(x_0) := r(T, x_0) e^{\pm i \sigma(T, x_0)} \, : \, x_0 \in [-1,1] \right\}.$$

Then, if $\operatorname{arctanh} z_0 < \frac{\pi}{2}$, $V_T(z_0)$ is the closed region whose boundary consists of the two curves $C_+(z_0)$ and $C_-(z_0)$, which only intersect at $x_0 \in \{-1, 1\}$.

For $\operatorname{arctanh} z_0 \geq \frac{\pi}{2}$, there are two different cases: First assume that T is large enough that the equation

$$\frac{2(1-z_0^2)\sqrt{1-x^2}}{1 + 2x z_0 + z_0^2} \left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x)\right) = \pi \tag{3.1.1}$$

admits a solution $x \in [-1,1]$. *Then the curves* $C_+(z_0)$ *and* $C_-(z_0)$ *intersect more than twice. There is a* $\chi \in (-1,1)$ *such that* $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0)$ *is a closed Jordan curve, where*

$$\widetilde{C}_\pm(z_0) := \{w_\pm(x_0) \,:\, x_0 \in [\chi, 1]\}\,,$$

and an $\aleph \in (-1,1)$ *such that* $\widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$ *is a closed Jordan curve, where*

$$\widehat{C}_\pm(z_0) := \{w_\pm(x_0) \,:\, x_0 \in [-1, \aleph]\}\,.$$

Then $V_T(z_0)$ *is the closed region whose boundary is* $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0) \cup \widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$.
For smaller T *that do not admit a solution to (3.1.1), the set* $V_T(z_0)$ *can be described exactly as in the case of* $\operatorname{arctanh} z_0 < \frac{\pi}{2}$.

Figures 3.4 and 3.5 show the evolution of the sets $V_T(z_0)$ over time. Note that $\operatorname{arctanh} z_0 = \frac{\pi}{2} \iff z_0 = \tanh(\pi/2) \approx 0.917$.

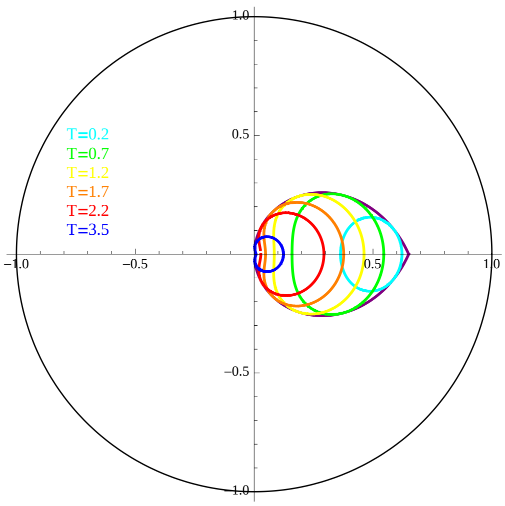

Figure 3.4: $V_T(0.65)$

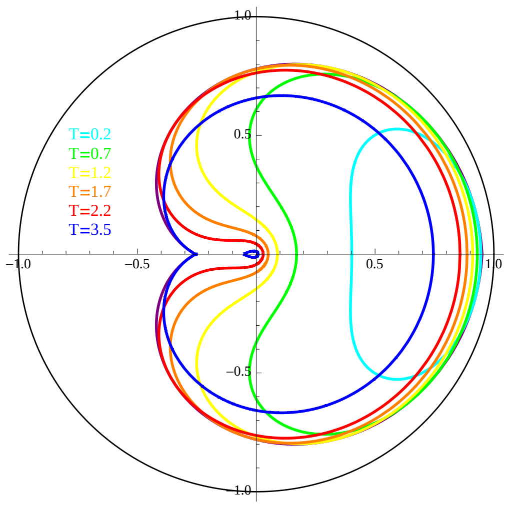

Figure 3.5: $V_T(0.95)$

The sets $V_T(z_0)$ for $z_0 = 0.65, 0.95$ and $T = 0.2 + 0.5j$, $j = 0, 1, \ldots, 4$, and $T = 3.5$. The purple curves show the boundaries of the non-fixed end time sets as determined by [RS14].

Proof. Since S_T is compact, so is V_T.
Note that for every $f \in \mathcal{S}_T$ there exists a Herglotz function $p(z,t)$ such that the solution $\{f_t\}_{t \geq 0}$ of (2.1.5) satisfies $f_T = f$; see [Pom75, Ch. 6].

Thus the description of $V_T(z_0)$ can be translated into the control theoretic problem of describing the reachable set $R_T(z_0)$ of the initial value problem

$$\dot{w}(t) = -w(t) \cdot \int_{\partial\mathbb{D}} \frac{u + w(t)}{u - w(t)} \, \mu_t(du), \qquad w(0) = z_0 \in \mathbb{D}, \tag{3.1.2}$$

where $\mu_t \in \mathcal{M}$ is a probability measure on $\partial\mathbb{D}$.
Note that the admissible right-hand sides of this equation form, for each fixed $w(t)$, a disc whose boundary corresponds exactly to the point measures in $\mathcal{M}$. Every point in this disc can thus be represented as the convex combination of two points on the circle, i.e. we can consider instead of the differential equation in (3.1.2) the Loewner equation

$$\begin{aligned}\dot{w}(t) &= -w(t) \cdot \left(s(t) \frac{\kappa_1(t) - w(t)}{\kappa_1(t) + w(t)} + (1 - s(t)) \frac{\kappa_2(t) - w(t)}{\kappa_2(t) + w(t)} \right) \\ &=: -w(t) p \left(w(t), \kappa_1(t), \kappa_2(t), s(t) \right), \end{aligned} \tag{3.1.3}$$

where $\kappa_j : [0, \infty) \to \partial\mathbb{D}$, $j = 1, 2$ and $s : [0, \infty) \to [0, 1]$ are measurable functions. We use the notation $\mathcal{X} := \partial\mathbb{D} \times \partial\mathbb{D} \times [0, 1]$.
For $\kappa_1, \kappa_2 \in \partial\mathbb{D}$, $s \in [0, 1]$, $\lambda \in \mathbb{C}$ and $w \in \mathbb{D}$ we define the Hamiltonian $H(\lambda, w, \kappa_1, \kappa_2, s)$ by

$$H(\lambda, w, \kappa_1, \kappa_2, s) = -\lambda \cdot w \cdot p(w, \kappa_1, \kappa_2, s).$$

Then (3.1.3) has the form $\dot{w}_t = \frac{\partial}{\partial\lambda} H(\lambda, w(t), \kappa_1(t), \kappa_2(t), s(t))$.

Now, if $(\kappa_1, \kappa_2, s) \in \mathcal{X}$ leads to an extremal solution $w(t)$, i.e. $w(T) \in \partial R_T(z_0)$, then $(\kappa_1(t), \kappa_2(t), s(t))$, $w(t)$ and $\lambda(t)$ satisfy Pontryagin's maximum principle (Th. 2.3): Define $\lambda(t)$ as the solution to the adjoint differential equation

$$\dot{\lambda}(t) = -\frac{\partial}{\partial w} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)), \tag{3.1.4}$$

with the initial value condition

$$\lambda(0) = e^{i\beta}, \text{ with some } \beta \in [0, 2\pi).$$

Then, for almost every $t \in [0, T]$, we have

$$\operatorname{Re} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = \max_{(k_1, k_2, \sigma) \in \mathcal{X}} \operatorname{Re} H(\lambda(t), w(t), k_1, k_2, \sigma), \tag{3.1.5}$$

and

$$\operatorname{Re} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = \mathit{const.} \quad \text{for almost all } t \in [0, T].$$

It is easy to see that $\operatorname{Re} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t))$ is maximised for triples $(\kappa(t), \kappa(t), 1)$, i.e. when

$$H(\lambda, w, \kappa_1, \kappa_2, s) = -\lambda \cdot w \cdot \frac{\kappa + w}{\kappa - w}$$

for some $\kappa \in \partial\mathbb{D}$. Thus, for almost every $t \geq 0$,

$$H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = -\lambda(t) \cdot w(t) \cdot \frac{\kappa(t) + w(t)}{\kappa(t) - w(t)},$$

where $\kappa : [0, T] \to \partial\mathbb{D}$ is measurable and (3.1.3), (3.1.4) become

$$\dot{w}(t) = -w(t) \cdot \frac{\kappa(t) + w(t)}{\kappa(t) - w(t)}, \quad w(0) = z_0 \in \mathbb{D}, \tag{3.1.6}$$

$$\dot{\lambda}(t) = -\lambda(t) \cdot \frac{w(t)^2 - 2\kappa(t)w(t) - \kappa(t)^2}{(\kappa(t) - w(t))^2}, \quad \lambda(0) = e^{i\beta}. \tag{3.1.7}$$

We now optimise the Hamiltonian by rewriting

$$\max_{\kappa \in \partial\mathbb{D}} \operatorname{Re}\left(-w\lambda \cdot \frac{\kappa + w}{\kappa - w}\right) = \max_{\phi \in \mathbb{R}} \operatorname{Re}(-\lambda w(m + re^{i\phi})) = r|\lambda w| - m \operatorname{Re}(\lambda w),$$

where

$$m = \frac{1 + |w|^2}{1 - |w|^2}, \qquad r = \frac{2|w|}{1 - |w|^2}, \qquad e^{i\phi} = \frac{w - |w|^2\kappa}{|w|\kappa - w|w|}.$$

The maximum is then obviously taken at

$$\phi = \pi - \arg(\lambda w) \quad \Leftrightarrow \quad \kappa = \frac{w}{|w|} \frac{1 + |w|e^{i\phi}}{e^{i\phi} + |w|} = w \frac{|\lambda| - \overline{\lambda w}}{|\lambda||w|^2 - \overline{\lambda w}}. \tag{3.1.8}$$

Inserting this into the phase equation (3.1.3) yields

$$\dot{w} = -w\left(m + re^{i\phi}\right),$$

or, in polar coordinates,

$$\frac{d}{dt}|w| = -|w|(m + r\cos\phi) = -|w|\left(\frac{1 + |w|^2 - 2|w|\cos(\arg\lambda + \arg w)}{1 - |w|^2}\right), \tag{3.1.9}$$

$$\frac{d}{dt}\arg w = -r\sin\phi = -\frac{2|w|\sin(\arg\lambda + \arg w)}{1 - |w|^2}, \tag{3.1.10}$$

and the costate equation (3.1.7) reads

$$\dot{\lambda} = \lambda\left(m + re^{i\phi} + 2|w|\frac{|w| + e^{i\phi}(1 + |w|^2) + |w|e^{2i\phi}}{(1 - |w|^2)^2}\right),$$

which corresponds to

$$\begin{aligned}\frac{d}{dt}|\lambda| = |\lambda|\left(m + r\cos\phi + 2|w|\frac{|w| + (1+|w|^2)\cos\phi + |w|\cos 2\phi}{(1-|w|^2)^2}\right) = \\ = |\lambda|\frac{1-|w|^4+2|w|^2-4|w|\cos(\arg\lambda+\arg w)+2|w|^2\cos(2\arg\lambda+2\arg w)}{(1-|w|^2)^2},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\arg\lambda = r\sin\phi + 2|w|\frac{|w|\sin(2\phi)+(1+|w|^2)\sin\phi}{(1-|w|^2)^2} = \\ = \frac{4|w|\sin(\arg\lambda+\arg w) - 2|w|^2\sin(2\arg\lambda+2\arg w)}{(1-|w|^2)^2}.\end{aligned} \tag{3.1.11}$$

Now we introduce the variable

$$x := \cos(\arg\lambda + \arg w),$$

which reduces our system of equations (3.1.9), (3.1.10), (3.1.11) to

$$\frac{d}{dt}|w| = -|w|\left(\frac{1+|w|^2-2|w|x}{1-|w|^2}\right) \tag{3.1.12}$$

and

$$\frac{d}{dt}x = -2|w|(1-x^2)\frac{1+|w|^2-2x|w|}{(1-|w|^2)^2} = 2\frac{1-x^2}{1-|w|^2}\frac{d|w|}{dt} \tag{3.1.13}$$

with the initial value conditions

$$|w(0)| = z_0, \qquad x(0) = x_0 := \cos\beta. \tag{3.1.14}$$

For $x_0^2 \neq 1$, separation of variables solves (3.1.13), (3.1.14) as

$$x(t) = \Phi^{-1}\left(2\operatorname{arctanh}|w(t)| - 2\operatorname{arctanh} z_0\right),$$

where

$$\Phi(y) := \operatorname{arctanh} y - \operatorname{arctanh} x_0,$$

which means

$$\begin{aligned} x(t) &= \tanh\left(2\operatorname{arctanh}|w(t)| + \operatorname{arctanh} x_0 - 2\operatorname{arctanh} z_0\right) = \\ &= \frac{(1+|w(t)|^2)(x_0-2z_0+x_0z_0^2)+2|w(t)|(1-2x_0z_0+z_0^2)}{(1+|w(t)|^2)(1-2x_0z_0+z_0^2)+2|w(t)|(x_0-2z_0+x_0z_0^2)} = \\ &= \frac{(1+|w(t)|^2)A+2|w(t)|B}{(1+|w(t)|^2)B+2|w(t)|A} \end{aligned} \tag{3.1.15}$$

with

$$A := x_0 - 2z_0 + x_0 z_0^2,$$
$$B := 1 - 2x_0 z_0 + z_0^2.$$

Note that, in fact, the denominator in (3.1.15) never equals zero for any $x_0 \in [-1, 1]$, since we have

$$\begin{aligned}(1+|w|^2)B + 2|w|A = 0 \quad \Leftrightarrow \quad |w| &= -\frac{A}{B} \pm \frac{\sqrt{A^2 - B^2}}{B} \\ &= -\frac{A}{B} \pm \frac{\sqrt{(x_0^2-1)(1-z_0^2)^2)}}{B},\end{aligned}$$

which only yields real terms for $x_0^2 = 1$, and in this case the only solution is

$$|w| = -\frac{A}{B} = \pm 1 \notin (0, 1).$$

Therefore, (3.1.15) is for all $x_0 \in [-1, 1]$ the solution to the initial value problem (3.1.13), and thus (3.1.12) can be simplified to

$$\frac{d}{dt}|w(t)| = -|w| \frac{B(1-|w|^2)}{B(1+|w|^2) + 2|w|A}, \quad |w(0)| = z_0.$$

The function

$$\Psi(y) := (A + B)\log(1 - y) - B\log(y) - (A - B)\log(1 + y)$$

is strictly monotonous on the interval $(0, 1)$, since its derivative is zero-free. Hence it is invertible, and

$$|w(t)| = \Psi^{-1}(Bt + \Psi(z_0)),$$

is the solution to the initial value problem (3.1.12), which can be verified by calculation.

To determine the value set $R_T(z_0)$, we solve the remaining initial value problem (3.1.10), which now reads

$$\frac{d}{dt} \arg w(t) = \pm \frac{2\sqrt{B^2 - A^2}}{B(1 + |w(t)|^2) + 2A|w(t)|}, \quad \arg w(0) = 0.$$

If we write

$$\arg w(t) = -G(|w(t)|),$$

where G is the solution to

$$\frac{d}{d|w|}G(|w|) = \frac{2\sqrt{B^2 - A^2}}{B(1 - |w|^2)}, \quad G(0) = 0,$$

then

$$\arg w(t) = \frac{\pm 2\sqrt{B^2 - A^2}}{B}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh}|w(t)|\right).$$

We can therefore describe candidates for the boundary points of the set $R_T(z_0)$ as follows:
For $x_0 \in [-1, 1]$, let $r = r(T, x_0)$ be the (unique) solution to the equation

$$\begin{aligned}(1 + x_0)(1 - z_0)^2 \log(1 - r) + (1 - x_0)(1 + z_0)^2 \log(1 + r) - (1 - 2x_0 z_0 + z_0^2)\log r = \\ (1 + x_0)(1 - z_0)^2 \log(1 - z_0) + (1 - x_0)(1 + z_0)^2 \log(1 + z_0) - (1 - 2x_0 z_0 + z_0^2)\log e^{-T} z_0,\end{aligned} \tag{3.1.16}$$

then $\partial R_T(z_0)$ consists of a subset of the two curves

$$C_\pm(z_0) = \left\{ w_\pm(x_0) = r(T, x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1, 1] \right\},$$

where

$$\sigma(T, x_0) = \frac{2(1 - z_0^2)\sqrt{1 - x_0^2}}{1 - 2x_0 z_0 + z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right).$$

First we consider the function $x_0 \mapsto r(T, x_0)$: By solving (3.1.16) for T and then taking the derivative with respect to x_0, we obtain

$$\frac{\partial}{\partial x_0} r(T, x_0) = -\frac{(1 - z_0)^2 r(T, x_0)(1 - r^2(T, x_0))\left(\log\left(\frac{1+r(T,x_0)}{1-r(T,x_0)}\right) - \log\left(\frac{1+z_0}{1-z_0}\right)\right)}{B(B(1 + r^2(T, x_0)) + 2A\, r(T, x_0))},$$

and since the only zeros of this term lie at $r(T, x_0) = 0$, $r(T, x_0) = \pm 1$ and $r(T, x_0) = z_0$, this immediately shows that $x_0 \mapsto r(T, x_0)$ is strictly increasing.
In particular, the curves $C_+(z_0)$ and $C_-(z_0)$ do not touch themselves.

Now we consider the first case where $z_0 < \tanh \frac{\pi}{2}$. Here, the curves never hit the negative real axis:
As the function

$$x_0 \mapsto \frac{2(1 - z_0^2)\sqrt{1 - x_0^2}}{1 - 2x_0 z_0 + z_0^2}$$

reaches its unique maximal value 2 at $x_0 = \frac{2z_0}{1+z_0^2}$, we have

$$\sigma(T, x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right) < 2 \cdot (\pi/2 - 0) = \pi.$$

Thus, they intersect only on the positive real axis and, as $\sigma(T, x_0) = 0$ if and only if $x_0 = \pm 1$, this happens exactly at $x_0 = \pm 1$. Hence, the full set $C_+(z_0) \cup C_-(z_0)$ forms the boundary of $R_T(z_0)$. Since $R_T(z_0)$ is obviously bounded, it has to consist of the bounded region enclosed by the two curves.

Next assume that $z_0 > \tanh \frac{\pi}{2}$. We have

$$\begin{aligned}\frac{\partial}{\partial x_0}\sigma(T, x_0) = &- \frac{1-z_0^2}{B\sqrt{1-x_0^2}}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right) \cdot \\ &\cdot \frac{A(1+r(T,x_0)^2) + 2Br(T,x_0)}{B(1+r(T,x_0)^2) + 2Ar(T,x_0)}.\end{aligned}$$

The zeros of this term lie clearly at the points $x_0 \neq \frac{2z_0}{1+z_0^2}$ with

$$r(T, x_0) = \frac{-B \pm \sqrt{B^2 - A^2}}{A}.$$

Since

$$\frac{-B - \sqrt{B^2 - A^2}}{A} \qquad \begin{cases} \geq 1 & \text{for } x_0 < \frac{2z_0}{1+z_0^2} \\ < 0 & \text{for } x_0 > \frac{2z_0}{1+z_0^2}, \end{cases}$$

it is clear that this term can be ignored. We focus on the equality

$$r(T, x_0) = \frac{-B + \sqrt{B^2 - A^2}}{A} \tag{3.1.17}$$

and note that here the term on the right-hand side is well-defined for all $x_0 \in [-1, 1]$, and strictly decreasing on this interval, taking values between -1 and 1. Therefore, $x_0 \mapsto h(x_0) := \frac{-B+\sqrt{B^2-A^2}}{A} - r(T, x_0)$ is continuous on $[-1, 1]$, strictly decreasing, and we have $h(-1) \geq 0$ and $h(1) \leq 0$. Thus (3.1.17) has exactly one solution $x_0 = x^*$ on $[-1, 1]$, and the function $x_0 \mapsto \sigma(T, x_0)$ increases from 0 to $\sigma(T, x^*)$ and decreases again to 0.

If T is so small that equation (3.1.1) has no solution, then we are again in the same situation: the two curves intersect only twice, namely for $x_0 = \pm 1$, and $R_T(z_0)$ is the closed region bounded by the two curves.

There is a T^* such that (3.1.1) admits a solution, but has no solution for any $T < T^*$. At this T^*, the curves $C_\pm(z_0)$ will meet for the first time, i.e. $\sigma(T^*, x^*) = \pi$. This means that at x^*, the curves both touch $\mathbb{R}^-$ at some point z^*, see Figure 3.6, and $R_T(z_0)$ (shown in green) is no longer simply connected, since the component containing the origin can obviously not be part of $R_T(z_0)$.

For slightly larger T, the curves $C_\pm(z_0)$ intersect on $\mathbb{R}^-$ twice and $\mathbb{D} \setminus (C_+(z_0) \cup C_-(z_0))$ has four components, see Figure 3.7. We denote by $K_T(z_0)$ the component (shown in orange) that arises from the intersection of the two curves near $x_0 = x^*$. Obviously, the component that contains the origin, as well as the "exterior" com-

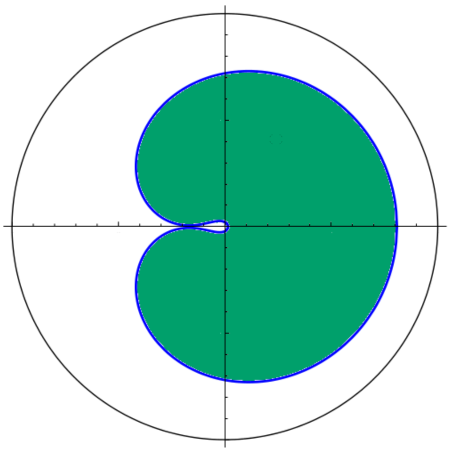

Figure 3.6: $V_{T^*}(z_0)$

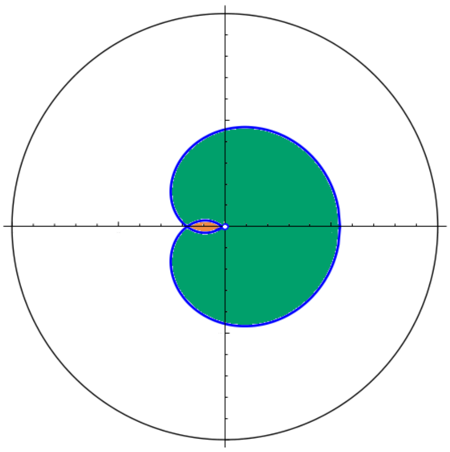

Figure 3.7: $V_{T^*+\varepsilon}(z_0)$

The evolution of the decomposition of $\mathbb{D}$ by $C_\pm(z_0)$

ponent (both shown in white) cannot be part of $R_T(z_0)$. For reasons of continuity, the "large interior" component (shown in green) must belong to $R_T(z_0)$. It remains to show that $K_T(z_0)$ also belongs to $R_T(z_0)$:
Since $z^* = w(T^*)$ for a solution $w(t)$ of the Loewner equation (3.3.10), we know that $R_T(z_0)$ contains the set $R_{T-T^*}(z^*)$, which we determined already if $T - T^*$ is small enough. In particular, $R_T(z_0)$ contains infinitely many points of $\mathbb{R}^-$. If $K_T(z_0)$ was not included in $R_T(z_0)$, then $R_T(z_0) \cap \mathbb{R}^-$ would consist of only two points, a contradiction.
For reasons of continuity, the set $R_T(z_0)$ will have the form described in the theorem for any larger T as well, and this concludes the proof. $\square$

Remark 3.2. *If $w_0 \in \partial V_T(z_0)$, then there exists exactly one control function $\kappa(t)$ such that the solution $\{f_t\}_{t\in[0,T]}$ of* (3.3.1) *with $p(z,t) = \frac{\kappa(t)+z}{\kappa(t)-z}$ satisfies $f_T(z_0) =$*

w_0. *Equation* (3.1.8) *shows that* $\kappa(t) = \exp(i\varphi(t))$ *is continuously differentiable. From [MR05], Theorem 1.1, it follows that* f *is a slit mapping in this case, i.e.* f *maps* $\mathbb{D}$ *conformally onto* $\mathbb{D} \setminus \gamma$, *where* γ *is a simple curve.*

3.2 Inverse functions

As in [PS15], it might also be of interest to determine the corresponding value sets for the inverse functions.

Firstly, in analogy to [RS14] and the set $\mathcal{V}(z_0)$, we describe the set

$$\mathcal{W}(z_0) := \{f^{-1}(z_0) \,:\, f \in \mathcal{S} \text{ with } z_0 \in f(\mathbb{D})\}.$$

In the following we write $d_{\mathbb{D}}(0, z)$, $z \in \mathbb{D}$, for the hyperbolic distance between 0 and z (using the hyperbolic metric with curvature -1), i.e. $d_{\mathbb{D}}(0, z) = 2\operatorname{arctanh}(|z|) = \log\left(\frac{1+|z|}{1-|z|}\right)$.

Theorem 3.3. *We have*

$$\begin{aligned} \mathcal{W}(z_0) &= \{f^{-1}(z_0) \,:\, f : \mathbb{D} \to \mathbb{D} \textit{ univalent, } f(0) = 0,\ f'(0) > 0 \textit{ with } z_0 \in f(\mathbb{D})\} \\ &= \{re^{i\sigma} \,:\, d_{\mathbb{D}}(0, r) \geq |\sigma| + d_{\mathbb{D}}(0, z_0),\ \sigma \in [-\pi, \pi]\}. \end{aligned}$$

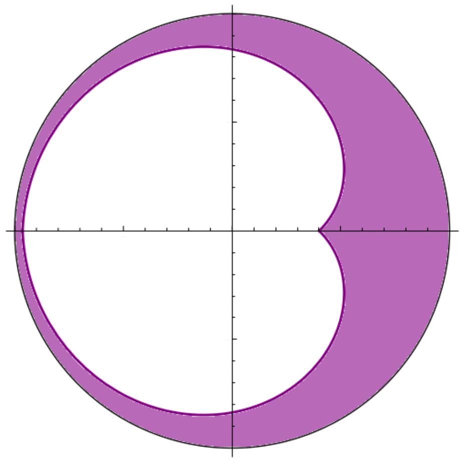

Figure 3.8: The set $\mathcal{W}(0.4)$

Furthermore, we will determine the value set

$$W_T(z_0) := \{f^{-1}(z_0) \,:\, f \in \mathcal{S}_T \text{ with } z_0 \in f(\mathbb{D})\}$$

for the inverse functions:

Theorem 3.4. *Let $z_0 \in (0,1)$. For $x_0 \in [-1,1)$ and $T>0$, let $r=r(T,x_0)$ be the (unique) positive solution to the equation*

$$(1-x_0)(1-z_0)^2 \log(1-r) + (1+x_0)(1+z_0)^2 \log(1+r) - (1+2x_0z_0+z_0^2)\log r = (1-x_0)(1-z_0)^2 \log(1-z_0) + (1+x_0)(1+z_0)^2 \log(1+z_0) - (1+2x_0z_0+z_0^2)\log e^T z_0$$

and let

$$\sigma(T,x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} r(T,x_0) - \operatorname{arctanh} z_0\right).$$

If

$$T < T^* := \log \frac{(1+z_0)^2}{4z_0},$$

then $r(T,x_0)$ can be extended continuously to $x_0=1$ and we have $W_T(z_0) = \overline{W_T(z_0)} \subset \mathbb{D}$, and $W_T(z_0)$ is the closed region bounded by the two curves

$$D_\pm(z_0) := \left\{ r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,1] \right\}.$$

Now let $T \geq T^*$ and define the two curves

$$\widetilde{D}_\pm(z_0) := \left\{ r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,1) \right\}.$$

Here we have two cases: if T is small enough that $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ intersect only at $x_0=-1$, then $\overline{W_T(z_0)}$ intersects $\partial\mathbb{D}$ and $\overline{W_T(z_0)}$ is bounded by the two curves $\widetilde{D}_\pm(z_0)$ and by the part of $\partial\mathbb{D}$ between the intersection points with the curves which includes the point 1.

Otherwise, the two curves intersect on $\mathbb{R}^-$ for the first time for some $x_0=\chi \in (-1,1)$ and $\overline{W_T(z_0)}$ is the closed region bounded by $\partial\mathbb{D}$ and the two curves

$$\widehat{D}_\pm(z_0) := \left\{ r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,\chi] \right\}.$$

In the last two cases we obtain $W_T(z_0)$ from $W_T(z_0) = \overline{W_T(z_0)} \cap \mathbb{D}$.

Proofs of Theorem 3.3 and 3.4. The proof of Theorem 3.4 is analogous to that of Theorem 3.1: we consider the inverse Loewner equation

$$\dot{w}(t) = w(t)\cdot p(w(t),t), \quad w(0) = z_0 \in \mathbb{D}, \tag{3.2.1}$$

where $p(z,t)$ is a Herglotz function.

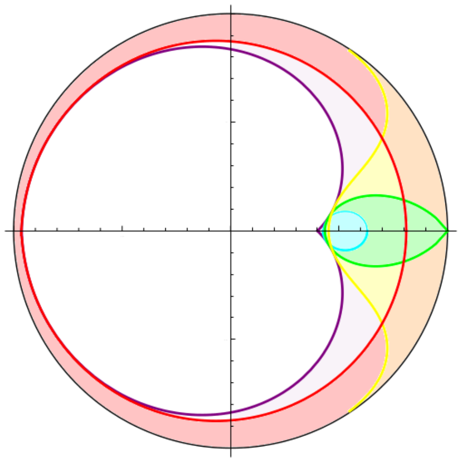

Figure 3.9: $W_T(0.4)$ for $T = 0.15, T^*, 0.3, 3$.

Here, a solution $t \mapsto w(t)$ may not exist for all time, i.e. there might be a $t_{max} > 0$ such that $w(t) \in \mathbb{D}$ for all $t < t_{max}$ but $|w(t)| \to 1$ for $t \uparrow t_{max}$. In this case, the (classical) solution to (3.2.1) ceases to exist at t_{max}. We define the reachable set

$$R'_T(z_0) = \{w(T) \,:\, w : [0,T] \to \mathbb{D} \text{ solves } (3.2.1)\}.$$

Note that we assume here that $w(t)$ exists up to $t = T$ and $w(T) \in \mathbb{D}$.
Then $W_T(z_0) = R'_T(z_0)$ is closed in the relative topology on $\mathbb{D}$, and we have

$$W_T(z_0) = \overline{W_T(z_0)} \cap \mathbb{D}.$$

Since the only difference to the case $R_T(z_0)$ consists in the sign of the left hand side of the Loewner differential equation, we can use the exact same ideas as above: Equation (3.2.1) reduces to

$$\dot{w}(t) = w(t) \cdot \left(s(t)\frac{\kappa_1(t) + w(t)}{\kappa_1(t) - w(t)} + (1 - s(t))\frac{\kappa_2(t) + w(t)}{\kappa_2(t) - w(t)} \right), \quad w(0) = z_0 \in \mathbb{D}, \tag{3.2.2}$$

where $\kappa_1, \kappa_2 : [0,T] \to \partial\mathbb{D}$ and $s : [0,T] \to [0,1]$ are measurable. We describe the boundary $\partial R'_T(z_0)$ by applying the maximum principle to (3.2.2). Again, it is obvious that the Hamiltonian

$$H'(\lambda, w, \kappa_1, \kappa_2, s) = \lambda \cdot w \cdot \left(s\frac{\kappa_1 + w}{\kappa_1 - w} + (1 - s)\frac{\kappa_2 + w}{\kappa_2 - w} \right)$$

will be optimised at triples $(\kappa, \kappa, 1)$, so we only need to consider the Hamiltonian

$$H'(\lambda, w, \kappa) = \lambda \cdot w \cdot \frac{\kappa + w}{\kappa - w}.$$

The condition (3.1.8) that is satisfied by trajectories leading to boundary points now corresponds to

$$\phi = -\arg(\lambda w),$$

which means we have to solve the system of equations

$$\frac{d}{dt}|w| = |w|\frac{1+|w|^2+2|w|x}{1-|w|^2}, \quad |w(0)| = z_0, \tag{3.2.3}$$
$$\frac{d}{dt}x = -2\frac{1-x^2}{1-|w|^2}\frac{d|w|}{dt}, \quad x(0) =: x_0 \in [-1,1].$$

We are left with

$$x(t) = \Delta^{-1}\left(2\operatorname{arctanh}|w(t)| - 2\operatorname{arctanh} z_0\right),$$

where

$$\Delta(y) = \operatorname{arctanh} x_0 - \operatorname{arctanh} y,$$

and thus

$$\begin{aligned} x(t) &= \tanh\left(\operatorname{arctanh} x_0 + 2\operatorname{arctanh} z_0 - 2\operatorname{arctanh}|w(t)|\right) = \\ &= \frac{(1+|w|^2)G - 2H|w|}{(1+|w|^2)H - 2G|w|}, \end{aligned}$$

where

$$\begin{aligned} G &:= x_0 + 2z_0 + x_0 z_0^2, \\ H &:= 1 + 2x_0 z_0 + z_0^2. \end{aligned}$$

Note that, again, this last term for x is valid for any $x_0 \in [-1,1]$.
We hence arrive at

$$\frac{d|w|}{dt} = \frac{H|w|(1-|w|^2)}{H(1+|w|^2) - 2G|w|},$$

or

$$|w(t)| = \Theta^{-1}\left(-Ht + \Theta(z_0)\right)$$

with

$$\Theta(y) = (H-G)\log(1-y) - H\log y + (G+H)\log(1+y).$$

The differential equation for the argument of the optimal trajectory w reads

$$\frac{d}{dt}\arg w(t) = \pm\frac{2|w|\sqrt{H^2-G^2}}{(1+|w|^2)H - 2G|w|},$$

which means

$$\arg w(t) = \pm\frac{2\sqrt{H^2-G^2}}{H}\left(\operatorname{arctanh}|w| - \operatorname{arctanh} z_0\right).$$

We can now describe the sets $R'_T(z_0)$:
Let $x_0 \in [-1,1)$. Then $\Theta\left((0,1)\right) = (-\infty,\infty)$ and Θ is strictly decreasing. Thus there is exactly one solution $r = r(T,x_0)$ of the equation

$$\begin{aligned}(1-x_0)(1-z_0)^2\log(1-r) + (1+x_0)(1+z_0)^2\log(1+r) - (1+2x_0z_0+z_0^2)\log r = \\ (1-x_0)(1-z_0)^2\log(1-z_0) + (1+x_0)(1+z_0)^2\log(1+z_0) - (1+2x_0z_0+z_0^2)\log e^T z_0.\end{aligned} \tag{3.2.4}$$

Furthermore we define the two curves

$$\widetilde{D}_\pm(z_0) := \left\{ r(T,x_0)e^{\pm i\sigma(T,x_0)} \,:\, x_0 \in [-1,1)\right\},$$

where

$$\sigma(T,x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1+2x_0z_0+z_0^2}\left(\operatorname{arctanh} r(T,x_0) - \operatorname{arctanh} z_0\right).$$

We take a closer look at the absolute value $r(T,x_0)$.
Firstly, the function $x_0 \mapsto r(T,x_0)$ is strictly increasing:
By solving (3.2.4) for T and then deriving with respect to x_0, we can calculate

$$\frac{\partial}{\partial x_0} r(T,x_0) = \frac{(1-z_0)^2 r(T,x_0)(1-r^2(T,x_0))\left(\log\left(\frac{1+r(T,x_0)}{1+z_0}\right) - \log\left(\frac{1-r(T,x_0)}{1-z_0}\right)\right)}{H(H(1+r^2(T,x_0)) - 2G\, r(T,x_0))},$$

and since the only zeros of this term lie at $r(T,x_0) = 0$, $r(T,x_0) = \pm 1$ and $r(T,x_0) = z_0$, this immediately shows that $x_0 \mapsto r(T,x_0)$ is strictly increasing in x_0 for $T > 0$.
Hence, we can define $r(T,x_0)$ also for $x_0 = 1$.

Note that for $x_0 = 1$, (3.2.4) simplifies to

$$2\log(1+r) - \log r = 2\log(1+z_0) - \log z_0 - T,$$

which means that the curves $D_+(z_0)$ and $D_-(z_0)$ will hit the boundary of the unit circle for the first time for

$$T = T^* := \log\frac{(1+z_0)^2}{4z_0}.$$

Next we take a closer look at the behaviour of the argument $\sigma(T, x_0)$ of the curve. We calculate

$$\frac{\partial}{\partial x_0}\sigma(T,x_0) = \frac{2(1-z_0^2)\,(\operatorname{arctanh} r(T,x_0) - \operatorname{arctanh} z_0)}{H^2}\cdot$$
$$\cdot\left(\frac{2r(T,x_0)\sqrt{1-x_0^2}(1-z_0^2)^2}{(H(1+r^2(T,x_0)) - 2G\,r(T,x_0))} - \frac{G}{\sqrt{1-x_0^2}}\right).$$

Since

$H(1+r^2(T,x_0)) - 2G\,r(T,x_0) \geq 0$ for all $x_0 \in (-1,1)$, $z_0 \in (0,1)$ and $r(T,x_0) \geq z_0$,

the term is non-negative if and only if

$$2r(T,x_0)(1-x_0^2)(1-z_0^2)^2 \geq (HG(1+r^2(T,x_0)) - 2G^2\,r(T,x_0)),$$

or

$$H(G - 2H\cdot r(T,x_0) + G\cdot r^2(T,x_0)) \leq 0,$$

which is equivalent to

$$\frac{H-\sqrt{H^2-G^2}}{G} \leq r(T,x_0) \leq \frac{H+\sqrt{H^2-G^2}}{G} \tag{3.2.5}$$

The inequality to the right always holds, since

$$\frac{H+\sqrt{H^2-G^2}}{G} \qquad \begin{cases} \leq 0 & \text{for } x_0 < -\frac{2z_0}{1+z_0^2}, \\ > 1 & \text{for } x_0 > -\frac{2z_0}{1+z_0^2}, \end{cases}$$

and of course

$$0 < r(T,x_0) \leq 1 \text{ for all } x_0 \in [-1,1).$$

The curves $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ can only intersect on $\mathbb{R}$, i.e. $\sigma(T,x_0) = k\cdot\pi$. Obviously, $\sigma(T,x_0) \geq 0$ for all x_0 so that $k \geq 0$ when the two curves intersect.

Next we show that

$$\frac{\partial}{\partial x_0}\sigma(T,x_0) > 0 \quad \text{if} \quad \sigma(T,x_0) \geq \pi. \tag{3.2.6}$$

We have

$$\log\left(1+\frac{2H-2\sqrt{H^2-G^2}}{G-H+\sqrt{H^2-G^2}}\right) \leq \frac{2H-2\sqrt{H^2-G^2}}{G-H+\sqrt{H^2-G^2}} \leq \frac{\pi H}{\sqrt{H^2-G^2}},$$

for

$$2(H\sqrt{H^2-G^2}-H^2+G^2) \leq \pi(H\sqrt{H^2-G^2}-H^2+HG),$$

and thus

$$r(T,x_0) > \tanh\left(\frac{\pi H}{2(1-z_0^2)\sqrt{1-x_0^2}}\right) \geq \frac{H-\sqrt{H^2-G^2}}{G}.$$

Hence, (3.2.5) is satisfied in this case and $\frac{\partial}{\partial x_0}\sigma(T,x_0) > 0$.

Now we consider the first case $T < T^*$:
Here, $r(T,1) < 1$ and $\sigma(T,x_0)$ is defined also for $x_0 = 1$. Furthermore, $\sigma(T,\pm 1) = 0$, i.e. the two curves $D_+(z_0)$ and $D_-(z_0)$ intersect for $x_0 = \pm 1$ on the positive real axis. Assume that the curves intersect more than twice. As $\sigma(T,x_0) > 0$ for all $x_0 \in (-1,1)$ there must be some $\rho \in (-1,1)$ with $\sigma(T,\rho) = \pi$. This is a contradiction: the function $x_0 \mapsto \sigma(T,x_0)$ is increasing for $x_0 \in [\rho,1]$ because of (3.2.6), but $\sigma(T,1) = 0$. Thus, the two curves don't intersect for $x_0 \in (-1,1)$.
Consequently, the set $R'_T(z_0)$ is the closed region enclosed by $D_+(z_0) \cup D_-(z_0)$.

Next let $T = T^*$. Then $\overline{R'_{T^*}(z_0)}$ is still the closed region bounded by $D_+(z_0) \cup D_-(z_0)$, but $R'_{T^*}(z_0) = \overline{R'_{T^*}(z_0)} \setminus \{1\}$ is not closed anymore.
In passing we note that it is not difficult to show that the solution $w(t)$ of (3.2.2) with $\kappa(t) \equiv 1$ satisfies $\lim_{t\to T^*} w(t) = 1$ and that this case corresponds to a mapping $f \in \mathcal{S}_{T^*}$ that maps $\mathbb{D}$ onto $\mathbb{D}$ minus the slit $[z_0, 1]$.

Now let $T > T^*$.
It is easy to see that the function Θ, which defines $r(T,x_0)$, is strictly decreasing, and that therefore, for fixed x_0, the term $r(T,x_0)$ is strictly increasing with growing T. Thus we know that we still have

$$r(T,x_0) \to 1 \text{ for } x_0 \to 1.$$

The driving function $\kappa(t) \equiv 1$ will now generate a mapping from $\mathbb{D}$ onto $\mathbb{D} \setminus [a,1]$ with $a < z_0$. From this it is easy to deduce that

$$L(T) := \liminf_{x_0 \to 1} \sigma(T,x_0) > 0.$$

Furthermore, $L(T)$ is increasing in $T \in [T^*,\infty)$: For a point $p = e^{i\alpha} \in \partial\mathbb{D}$ the driving function $\kappa(t) \equiv -e^{i\alpha}$ has the property that $-p \cdot \frac{\kappa(t)+p}{\kappa(t)-p} = 0$. Thus, if $e^{i\alpha} \in \overline{R'_T(z_0)}$, then also $e^{i\alpha} \in \overline{R'_S(z_0)}$ for all $S \geq T$.

If T is so small that $L(T) \leq \pi$, then the curves $\widetilde{D}_{\pm}(z_0)$ do not intersect in $\mathbb{D}$ a second time besides $x_0 = -1$ for the same reason as in the case $T < T^*$. Here, $\overline{R'_T(z_0)}$ is the closed region which is bounded by $\widetilde{D}_+(z_0)$ and $\widetilde{D}_-(z_0)$ and the part of $\partial\mathbb{D}$ which includes the point 1.

Finally, let $L(T) > \pi$. The curves $\widetilde{D}_{\pm}(z_0)$ will meet at $x_0 = -1$, and then intersect again on the negative real axis before hitting $\partial\mathbb{D}$. Because of (3.2.6) they don't intersect more than twice provided that $T > T^*$ is small enough. Hence, in this case, $\mathbb{D} \setminus (\widetilde{D}_+(z_0) \cup \widetilde{D}_-(z_0))$ has three components, see Figure 3.10.

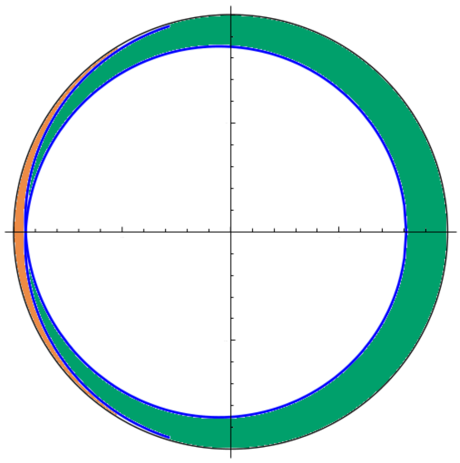

Figure 3.10: The decomposition of $\mathbb{D}$ by $\widetilde{D}_{\pm}(z_0)$

There is a simply connected component that is bounded by $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$ and does not touch $\partial\mathbb{D}$, and two simply connected components that do touch the boundary $\partial\mathbb{D}$. We denote by $W_T^{\pm}(z_0)$ the components that touch the points $+1$ (shown in green), or, respectively, -1 (shown in orange). It is clear that $\overline{R'_T(z_0)}$ has to consist of either $W_T^+(z_0)$, or $W_T^-(z_0)$, or the union of both. If it were equal to only one of the sets $W_T^{\pm}(z_0)$, this would imply that $\overline{R'_T(z_0)}$ is bounded away from parts of $\partial\mathbb{D}$, although $\overline{R'_t(z_0)}$, with some $t < T$, already touched these segments of $\partial\mathbb{D}$ – a contradiction. Thus, we must have $\overline{R'_T(z_0)} = \overline{W_T^+(z_0) \cup W_T^-(z_0)}$, and thus $\overline{R'_T(z_0)}$ is exactly the closed region bounded by $\partial\mathbb{D}$ and (in the interior) by $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$.
The same consideration applies as well for the case of more than two intersections of $\widetilde{D}_{\pm}(z_0)$ with $\mathbb{R}$, and for reasons of continuity, the inner boundary of $\overline{R'_T(z_0)}$ has to consists of $\widehat{D}_+(z_0) \cup \widehat{D}_-(z_0)$ in these cases, too.

We lastly show that the case where $\widetilde{D}_{\pm}(z_0)$ intersect for some $x_0 \in (-1, 1)$ will

actually happen:
For
$$x_0 = x^* := \frac{-2z_0}{1+z_0^2},$$
(3.2.4) reads
$$\log(1+r) + \log(1-r) - \log r = \log(1+z_0) + \log(1-z_0) - \log z_0 - T =: Y \in \mathbb{R},$$
which means
$$r = \frac{\sqrt{4+e^{2Y}} - e^Y}{2}.$$
Since $r\left(T, \frac{-2z_0}{1+z_0^2}\right)$ increases with growing T, and $r\left(T, \frac{-2z_0}{1+z_0^2}\right) \to 1$ for $T \to \infty$, it will at some point of time T become so large that
$$\operatorname{arctanh} r\left(T, \frac{-2z_0}{1+z_0^2}\right) = \frac{\pi}{2} + \operatorname{arctanh} z_0.$$
Then $\sigma(T, x^*) = 2\cdot(\operatorname{arctanh} r(T, x^*) - \operatorname{arctanh} z_0) = \pi$ and consequently the curves $\widetilde{D}_\pm(z_0)$ intersect on $\mathbb{R}^-$.

This concludes the proof of Theorem 3.4.

We finally prove Theorem 3.3 by applying the maximum principle to equation (3.2.1) in the free end time version. We have
$$\mathcal{W}(z_0) = \{w(T) \,:\, w : [0,\infty) \to \mathbb{D} \text{ solves (3.2.1)}, \, T \in [0,\infty)\}.$$
If $w(t)$ is a solution with $w(T) \in \partial\mathcal{W}(z_0)$, then we have the same setting as above and the additional information that
$$\operatorname{Re} H'(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = \max_{(k_1,k_2,\sigma)\in\mathcal{X}} \operatorname{Re} H'(\lambda(t), w(t), k_1, k_2, \sigma) = 0$$
for almost all $t \in [0,T]$, see Th. 2.2.

The optimal driving term corresponding to (3.1.8) thus has to fulfil
$$\cos\phi = -\frac{2|w|}{1+|w|^2},$$
which means
$$x = \frac{-2|w|}{1+|w|^2},$$

and thus (3.2.3) becomes

$$\frac{d}{dt}|w| = |w|\frac{1-|w|^2}{1+|w|^2}, \quad |w(0)| = z_0,$$

which is equivalent to

$$|w(t)| = \frac{-1+z_0^2+\sqrt{(1-z_0^2)^2+4z_0^2e^{2t}}}{2e^t z_0}.$$

We have

$$\frac{d}{dt}\arg w(t) = \pm\frac{2|w|}{1+|w|^2},$$

which yields

$$\frac{d}{d|w|}\arg w = \pm\frac{2}{1-|w|^2},$$

or

$$\arg w = \pm 2\left(\operatorname{arctanh}|w| - \operatorname{arctanh} z_0\right) = \pm(d_{\mathbb{D}}(0,|w|) - d_{\mathbb{D}}(0,z_0)).$$

Taking into account our results about the sets $W_T(z_0)$, we conclude that $\mathcal{W}(z_0) = \overline{\mathcal{W}(z_0)} \cap \mathbb{D}$ and that $\overline{\mathcal{W}(z_0)}$ is the closed region bounded by $\partial\mathbb{D}$ and the hyperbolic spirals

$$S_\pm(z_0) = \{re^{\pm i\sigma} \,:\, \sigma = d_{\mathbb{D}}(0,r) - d_{\mathbb{D}}(0,z_0), \sigma \in [0,\pi]\}.$$

This concludes the proof.

□

3.3 Schlicht functions with real coefficients

Another way to refine the result in [RS14] is to admit only schlicht normalised functions having only real coefficients.
We define

$$\mathcal{U} := \{f : \mathbb{D} \to \mathbb{D} \,: f \text{ schlicht},\ f(0)=0,\ f'(0)>0,$$
$$f \text{ has only real coefficients in its Taylor expansion around } 0\}.$$

Note that $\mathcal{U}$ is **not** invariant under rotation, i.e. unlike in the sections before, we must not assume that $z_0 \in (0,1)$.

The following result has been proven in [Pro92]. The proof uses Pontryagin's maximum principle, which is applied to the radial Loewner equation.
An elementary proof of the theorem is given in [Pfr16]. For the reader's convenience, we include the proof that uses Pontryagin's maximum principle.

Theorem 3.5 ([Pro92], [Pfr16]). *Let $z_0 \in \mathbb{D} \setminus \{0\}$.*
If $z_0 \in \mathbb{R}$, then $V_{\mathcal{U}}(z_0) \cup \{0\}$ is the closed interval with endpoints 0 and z_0.
Define the two curves $C_+(z_0)$ and $C_-(z_0)$ by

$$C_+(z_0) := \left\{ \frac{1}{2z_0} \left(e^t(z_0+1)^2 - 2z_0 - e^{t/2}(z_0+1)\sqrt{e^t(z_0+1)^2 - 4z_0} \right) \,:\, t \in [0,\infty] \right\},$$
$$C_-(z_0) := \left\{ \frac{1}{2z_0} \left(e^t(z_0-1)^2 + 2z_0 + e^{t/2}(z_0-1)\sqrt{e^t(z_0-1)^2 + 4z_0} \right) \,:\, t \in [0,\infty] \right\}.$$

If $z_0 \notin \mathbb{R}$, then $V_{\mathcal{U}}(z_0) \cup \{0\}$ is the closed region whose boundary consists of the two curves $C_+(z_0)$ and $C_-(z_0)$, which only intersect at $t \in \{0, \infty\}$.

Furthermore, for $z_0 \notin \mathbb{R}$, any boundary point of $V_{\mathcal{U}}(z_0)$ except 0 can be reached by only one mapping $f \in \mathcal{U}$, which is of the form

$$f_{1,t}(z) = \frac{1}{2z}(e^t(z+1)^2 - 2z - e^{t/2}(z+1)\sqrt{e^t(z+1)^2 - 4z})$$

or

$$f_{2,t}(z) = \frac{1}{2z}(e^t(z-1)^2 + 2z + e^{t/2}(z-1)\sqrt{e^t(z-1)^2 + 4z})$$

with $t \in [0, \infty)$.
The mapping $f_{1,t}$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus [2e^t - 1 - 2e^{t/2}\sqrt{e^t - 1}, 1]$ and $f_{2,t}$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus [-1, -2e^t + 1 + 2e^{t/2}\sqrt{e^t - 1}]$.

Proof of Theorem 3.5. As in the proof of Th. 3.1, we consider the radial Loewner equation

$$\dot{f}_t(z) = -f_t(z) \cdot p(f_t(z), t) \text{ for a.e. } t \geq 0, \quad f_0(z) = z \in \mathbb{D}, \tag{3.3.1}$$

with a Herglotz function $p : [0, \infty) \times \mathbb{D} \to \mathbb{C}$ and note that for almost every $t \geq 0$, we have the Herglotz representation

$$p(t, z) = \int_{\partial\mathbb{D}} \frac{u+z}{u-z}\, \nu_t(du), \tag{3.3.2}$$

for some probability measure ν_t on $\partial\mathbb{D}$.

If $f \in \mathcal{U}$, then one can find a corresponding $p(z, t)$ such that, for a.e. $t \geq 0$, the measure ν_t is symmetric with respect to the real axis and one can rewrite (3.3.2) as

$$p(z,t) = p_{\mu_t}(z) := \int_{\partial\mathbb{D}} 1/2\frac{u+z}{u-z} + 1/2\frac{\overline{u}+z}{\overline{u}-z}\, \mu_t(du) = \int_{\partial\mathbb{D}} \frac{1-z^2}{1-2z\mathrm{Re}(u)+z^2}\, \mu_t(du). \tag{3.3.3}$$

This is clear if f maps onto $\mathbb{D}$ minus two symmetric slits (μ_t is a point measure for every $t \geq 0$ in this case), and a general $f \in \mathcal{U}$ can always be approximated by such slit mappings, see Chapter I.3.8 in [Tam78].

Thus we have to determine the reachable set of the initial value problem

$$\dot{w}(t) = -w(t) \cdot p_{\mu_t}(w(t)), \quad w(0) = z_0 \in \mathbb{D}, \tag{3.3.4}$$

where $p_{\mu_t}(z)$ is a Herglotz function of the form (3.3.3) for a.e. $t \geq 0$.

For $z_0 \in \mathbb{R}$ it is easy to see that $w(t) \in \mathbb{R}$ and that the reachable set is the closed interval with endpoints 0 and z_0. In the following, assume $z_0 \notin \mathbb{R}$.

It is easy to see that the right-hand sides of (3.3.4) form the convex hull of a circle segment, and that thus any point in this set can be represented as the convex combination of two point on the circle segment, i.e. the differential equation from (3.3.4) becomes

$$\begin{aligned} \dot{w}(t) &= -w(t)\left(s(t)\frac{1-w^2(t)}{1-2w(t)\operatorname{Re}\kappa_1(t)+w^2(t)} + (1-s(t))\frac{1-w^2(t)}{1-2w(t)\operatorname{Re}\kappa_2(t)+w^2(t)}\right) \\ &=: -w(t)p(w(t),\kappa_1(t),\kappa_2(t),s(t)), \end{aligned} \tag{3.3.5}$$

where $\kappa_j : [0,\infty) \to \partial\mathbb{D}$, $j = 1,2$ and $s : [0,\infty) \to [0,1]$ are measurable functions. Again, we set $\mathcal{X} := \partial\mathbb{D} \times \partial\mathbb{D} \times [0,1]$.
For $\kappa_1, \kappa_2 \in \partial\mathbb{D}$, $s \in [0,1]$, $\lambda \in \mathbb{C}$ and $w \in \mathbb{D}$ we define the Hamiltonian $H(\lambda, w, \kappa_1, \kappa_2, s)$ by

$$H(\lambda, w, \kappa_1, \kappa_2, s) := -\lambda \cdot w \cdot p(w, \kappa_1, \kappa_2, s)).$$

Then (3.3.5) has the form $\dot{w}_t = \frac{\partial}{\partial \lambda} H(\lambda, w, \kappa_1, \kappa_2, s)$.

Now, if $(\kappa_1(t), \kappa_2(t), s(t)) \in \mathcal{X}$ leads to an extremal solution $w(t)$, i.e. $w(T) \in \partial V_{\mathcal{U}}(z_0)$, then $(\kappa_1(t), \kappa_2(t), s(t))$, $w(t)$ and $\lambda(t)$ satisfy Pontryagin's maximum principle, Th. 2.3:
Define $\lambda(t)$ as the solution to the adjoint differential equation

$$\dot{\lambda}(t) = -\frac{\partial}{\partial w} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)), \tag{3.3.6}$$

with the initial value condition

$$\lambda(0) = e^{i\beta}, \text{ with some } \beta \in [0, 2\pi).$$

Then, for almost every $t \in [0, T]$, we have

$$\operatorname{Re} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = \max_{(k_1,k_2,\sigma)\in\mathcal{X}} \operatorname{Re} H(\lambda(t), w(t), k_1, k_2, \sigma), \quad (3.3.7)$$

and

$$\operatorname{Re} H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = 0. \quad \text{for almost all } t \in [0, T]. \quad (3.3.8)$$

From (3.3.2) it is easy to see that $\operatorname{Re} H(\lambda(t), w(t), k_1, k_2, s)$ is maximised only if $k_1 = k_2$ and $s = 1$, i.e. if

$$H(\lambda, w, k_1, k_2, \sigma) = \lambda \frac{w(w^2 - 1)}{1 - 2wx + w^2} \quad (3.3.9)$$

for some $x \in [-1, 1]$. Thus, for almost every $t \geq 0$,

$$H(\lambda(t), w(t), \kappa_1(t), \kappa_2(t), s(t)) = \lambda(t) \frac{w(t)(w(t)^2 - 1)}{1 - 2w(t)\kappa(t) + w(t)^2},$$

where $\kappa : [0, T] \to [-1, 1]$ is measurable and (3.3.4) becomes

$$\dot{w}(t) = \frac{w(t)(w(t)^2 - 1)}{1 - 2w(t)\kappa(t) + w(t)^2}, \quad w(0) = z_0 \in \mathbb{D}. \quad (3.3.10)$$

For $x \in [-1, 1]$, expression (3.3.9) describes a circular arc, provided that $w \notin \mathbb{R}$. As we assumed $z_0 \notin \mathbb{R}$, we have $w(t) \notin \mathbb{R}$ for any solution of (3.3.10).
For $x \to \pm\infty$, (3.3.9) goes to 0. Hence, (3.3.7) and (3.3.8) can only be satisfied if one of the endpoints of the circular arc lies on the imaginary axis, which implies $\kappa(t) = 1$ or $\kappa(t) = -1$ for a.e. $t \geq 0$. As the circular arc varies continuously with t, it follows that $\kappa(t) = 1$ for almost all $t \geq 0$ or $\kappa(t) = -1$ for almost all $t \geq 0$. Solving (3.3.10) for these two cases leads directly to the two curves $C_\pm(z_0)$. The corresponding univalent mappings are given by $f_{1,t}$ and $f_{2,t}$.
Finally we show that the two curves intersect only for $t = s = 0$ when $z_0 \notin \mathbb{R}$:
The equality

$$f_{1,t}(z_0) = f_{2,s}(z_0)$$

for $s, t \in [0, \infty)$ gives

$$e^t = e^s \frac{1 + 2z_0 + z_0^2}{1 - 2z_0 + z_0^2} - \frac{4z_0}{1 - 2z_0 + z_0^2}.$$

Both sides of the equation describe a half-line in the complex plane which always intersect at $t = s = 0$.
It is clear that for $z_0 \in \mathbb{D} \setminus \mathbb{R}$ there are no further solutions, because $\frac{1+2z_0+z_0^2}{1-2z_0+z_0^2} \notin \mathbb{R}$

as a small computation shows:
Write $z_0 = x + iy \in \mathbb{D}$ with $y \neq 0$. Then $\operatorname{Im}\left(\frac{1+2z_0+z_0^2}{1-2z_0+z_0^2}\right) = 0$ if and only if $y - x^2y - y^3 = 0$, i.e. $x^2 = 1 - y^2$. Thus $z_0 \in \partial\mathbb{D}$, a contradiction. Hence, $V_{\mathcal{U}}(z_0) \cup \{0\}$ is the closed region whose boundary consists of the two curves $C_+(z_0)$ and $C_-(z_0)$.

Finally, if $z_0 \notin \mathbb{R}$, it follows that any boundary point of $V_{\mathcal{U}}(z_0)$ except 0 and z_0 can be reached only by the driving function $\kappa(t) \equiv 1$ or $\kappa(t) \equiv -1$, i.e. z_0 can be reached by only one mapping from $\mathcal{U}$ which is of the form $f_{1,t}$ or $f_{2,t}$ for some $t > 0$.

□

The value set for the inverse functions can be obtained quite similarly:

Theorem 3.6. *Let $z_0 \in \mathbb{D} \setminus \{0\}$ and define*

$$V_{\mathcal{U}}^*(z_0) = \{f^{-1}(z_0) \,|\, f \in \mathcal{U}, z_0 \in f(\mathbb{D})\}.$$

If $z_0 \in (0,1)$, then $V_{\mathcal{U}}^*(z_0) = [z_0, 1)$, and if $z_0 \in (-1,0)$, then $V_{\mathcal{U}}^*(z_0) = (-1, z_0]$. Define the two curves $C_+^*(z_0)$ and $C_-^*(z_0)$ by

$$C_+^*(z_0) := \left\{\frac{1}{2z_0}\left(e^t(z_0+1)^2 - 2z_0 - e^{t/2}(z_0+1)\sqrt{e^t(z_0+1)^2 - 4z_0}\right) \,:\, t \in [-\infty, 0]\right\},$$
$$C_-^*(z_0) := \left\{\frac{1}{2z_0}\left(e^t(z_0-1)^2 + 2z_0 + e^{t/2}(z_0-1)\sqrt{e^t(z_0-1)^2 + 4z_0}\right) \,:\, t \in [-\infty, 0]\right\}.$$

Now let $\operatorname{Im}(z_0) > 0$. Then $\overline{V_{\mathcal{U}}(z_0)}$ is the closed region bounded by the curves $C_+(z_0)$, $C_-(z_0)$ and $E := \partial\mathbb{D} \cap \overline{\mathbb{H}}$. The set $V_{\mathcal{U}}(z_0)$ is given by $V_{\mathcal{U}}(z_0) = \overline{V_{\mathcal{U}}(z_0)} \setminus E$.

Again, it suggests itself to ask what the sets look like if we fix the derivative $f'(0)$, i.e. for $\tau \in (0,1]$, let

$$\mathcal{U}(\tau) = \{f \in \mathcal{U} \,:\, f'(0) = \tau\}.$$

The value sets $V_{\mathcal{U}(\tau)}$ are described in [PS16].
Figure 3.11 shows the set $V_{\mathcal{U}}(z_0)$ (orange), which lies inside the heart-shaped set $V_{\mathcal{S}_>}(z_0)$ (blue) that is determined in [RS14], and the set $V_{\mathcal{U}}^*(z_0)$ (red, dashed) for $z_0 = 0.9e^{i\pi/4}$.

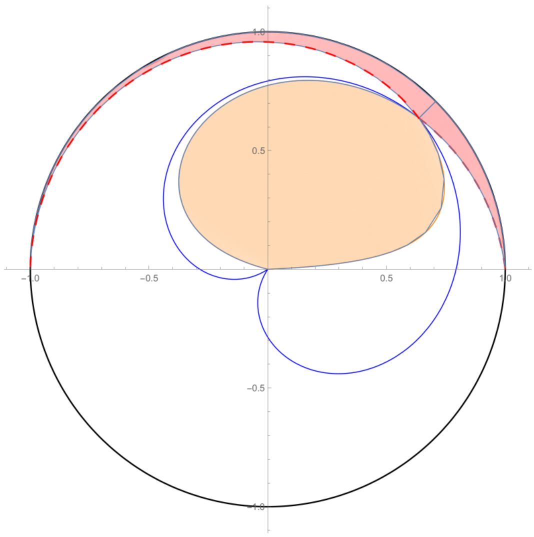

Figure 3.11: $V_{\mathcal{U}}(0.9e^{i\pi/4})$

3.4 Typically real functions

Following Rogosinski [Rog32], a holomorphic (not necessarily schlicht) mapping $f : \mathbb{D} \to \mathbb{C}$ is called *typically real* if

$$\operatorname{Im}(f(z)) \operatorname{Im}(z) \geq 0 \quad \text{for all } z \in \mathbb{D}.$$

We define the set of bounded typically real functions

$$\mathcal{T} := \{f : \mathbb{D} \to \mathbb{D} \,:\, f \text{ typically real}, \, f(0) = 0, \, f'(0) > 0\}.$$

Obviously, $\mathcal{U} \subset \mathcal{T}$.

We will determine the set

$$V_{\mathcal{T}}(z_0) := \{f(z_0) \,:\, f \in \mathcal{T}\}$$

as well as the sets

$$V_{\mathcal{T}(\tau)}(z_0) := \{f(z_0) \,:\, f \in \mathcal{T}(\tau)\}, \qquad \tau \in (0, 1],$$

where $\mathcal{T}(\tau) := \{f \in \mathcal{T} \,:\, f'(0) = \tau\}$.

The results in this and the remaining sections of chapter 3 can be found in [KS17].

From Rogosinski's work one immediately obtains an integral representation for typically real mappings, see also [Rob35], Section 2. In order to determine the value region $V_{\mathcal{T}}(z_0)$, we will need the following integral representation for bounded typically real mappings.

Theorem 3.7 ([SS82], Theorem 2.2)**.** *Let $f \in \mathcal{T}$ with $f'(0) = \tau > 0$. Then there exists a probability measure μ supported on*

$$B := \{(x, y) \in \mathbb{R}^2 \,:\, -1 \le x \le 2\tau - 1 \le y \le 1\}$$

such that

$$f(z) = \frac{\sqrt{g_{\mu,\tau}(z)} - 1}{\sqrt{g_{\mu,\tau}(z)} + 1},$$

where we take the holomorphic branch of the square root with $\sqrt{1} = 1$ and

$$g_{\mu,\tau}(z) = \int_B \frac{(1+z)^2(1 - 2(1 - 2\tau + x + y)z + z^2)}{(1 - 2xz + z^2)(1 - 2yz + z^2)}\, \mu(dxdy).$$

Remark 3.8. *In order to show $\mathcal{U} \subsetneq \mathcal{T}$ we find a function $f_0 \in \mathcal{T} \setminus \mathcal{U}$ as follows: Let $\tau = 1/2$ and let μ be the point measure in $(\tau - 1, \tau)$. Then we obtain*

$$f_0(z) = \frac{\sqrt{((1+z)^2(1+z^2))/(1+z^2+z^4)} - 1}{\sqrt{((1+z)^2(1+z^2))/(1+z^2+z^4)} + 1}.$$

The derivative $f_0'(z)$ has a zero at $z = -(1/4)+(i\sqrt{3})/4+1/2\sqrt{-(9/2) - (i\sqrt{3})/2} \in \mathbb{D}$. Hence, $f_0 \notin \mathcal{U}$.

Theorem 3.9. *Let $z_0 \in \mathbb{D} \setminus \{0\}$ and $\tau \in (0, 1]$.*
The set $V_{\mathcal{T}(\tau)}(z_0)$ is the image of the closed region bounded by the two circular arcs

$$\left\{1 + \frac{4\tau z_0}{1 - 2yz_0 + z_0^2} \,:\, y \in [2\tau - 1, 1]\right\}$$

and

$$\left\{\frac{(z_0+1)^2(1 + z_0(-4 + 4\tau - 2x + z_0))}{(z_0 - 1)^2(1 - 2xz_0 + z_0^2)} \,:\, x \in [-1, 2\tau - 1]\right\}$$

under the map $w \mapsto \frac{\sqrt{w}-1}{\sqrt{w}+1}$.

Proof. Fix some $\tau > 0$. First we show that the set

$$A(\tau) := \{g_{\mu,\tau}(z_0) \,|\, \mu \text{ is a point measure on } B\}$$

is convex. To this end, we evaluate the function

$$(x, y) \mapsto s(x, y) := \frac{(1+z)^2(1 - 2(1 - 2\tau + x + y)z + z^2)}{(1 - 2xz + z^2)(1 - 2yz + z^2)}$$

on ∂B :

(i) $s(x,y) = \frac{(1+z_0)^2}{(z_0+1)^2-4\tau z_0}$ when $x = 2\tau - 1$.

(ii) $s(x,y) = \frac{(1+z_0)^2}{(z_0+1)^2-4\tau z_0}$ when $y = 2\tau - 1$.

(iii) $s(x,y) = 1 + \frac{4\tau z_0}{1-2yz_0+z_0^2}$ when $x = -1$.

(iv) $s(x,y) = \frac{(z_0+1)^2(1+z_0(-4+4\tau-2x+z_0))}{(z_0-1)^2(1-2xz_0+z_0^2)}$ when $y = 1$.

We see that $s(\partial B)$ consists of two circular arcs connecting the points

$$P(\tau) = \frac{(1+z_0)^2}{(1+z_0)^2 - 4\tau z_0} \quad \text{and} \quad Q(\tau) = 1 + \frac{4\tau z_0}{1 - 2z_0 + z_0^2}, \tag{3.4.1}$$

and the two arcs are given by

$$\begin{aligned} s_{1,\tau} &: [2\tau - 1, 1] \to \mathbb{C}, \quad s_{1,\tau}(y) := 1 + \frac{4\tau z_0}{1 - 2yz_0 + z_0^2}, \\ s_{2,\tau} &: [-1, 2\tau - 1] \to \mathbb{C}, \quad s_{2,\tau}(x) := \frac{(z_0+1)^2(1 + z_0(-4 + 4\tau - 2x + z_0))}{(z_0 - 1)^2(1 - 2xz_0 + z_0^2)}. \end{aligned}$$

Without loss of generality, we restrict to the case $\operatorname{Im}(z_0) \geq 0$.
A short calculations shows that then

$$\begin{aligned} \frac{d}{dy} \arg \frac{d}{dy} s(x,y) &= 4 \operatorname{Im} z_0 \frac{1 - |z_0|^2}{|1 - 2yz_0 + z_0^2|^2} \geq 0, \\ \frac{d}{dx} \arg \frac{d}{dx} s(x,y) &= 4 \operatorname{Im} z_0 \frac{1 - |z_0|^2}{|1 - 2xz_0 + z_0^2|^2} \geq 0. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} s_x(x,y) &:= \frac{d}{dx} s(x,y) = \frac{4(2\tau - 1 - y) z^2 (1+z)^2}{(1 - 2xz + z^2)^2 (1 - 2yz + z^2)}, \\ s_y(x,y) &:= \frac{d}{dy} s(x,y) = \frac{4(2\tau - 1 - x) z^2 (1+z)^2}{(1 - 2yz + z^2)^2 (1 - 2xz + z^2)}, \end{aligned}$$

and thus

$$\operatorname{Im} \frac{s_x(x,y)}{s_y(x,y)} = \frac{2(y - (2\tau - 1))}{2\tau - 1 - x} \frac{1 - |z|^2}{|1 - 2xz + z^2|^2} \operatorname{Im}(z) \cdot (y - x) \geq 0 \tag{3.4.2}$$

for all $(x,y) \in B$.
This shows that any parallel to either the x- or the y-axis within B is mapped onto a convex curve, and that whenever we map a path that points inwards in B, the image also lies in the interior of $s(\partial B)$. Therefore, the convex closure of $A(\tau)$

is equal to the set $W(\tau)$ defined as the compact region bounded by the curves $s_{1,\tau}$ and $s_{2,\tau}$.
From Theorem 3.7 it follows that $V_{\mathcal{T}(\tau)}(z_0)$ is the image of

$$K(\tau) := \{g_{\mu,\tau}(z_0) \,:\, \mu \text{ is a probability measure on } B\}$$

under the map $w \mapsto \frac{\sqrt{w}-1}{\sqrt{w}+1}$. The set $K(\tau)$ is the closure of the convex hull of $A(\tau)$, i.e. $K(\tau) = W(\tau)$, which concludes the proof.

□

Figures 3.12 and 3.13 show the sets $V_{\mathcal{T}(\tau)}(z_0)$.

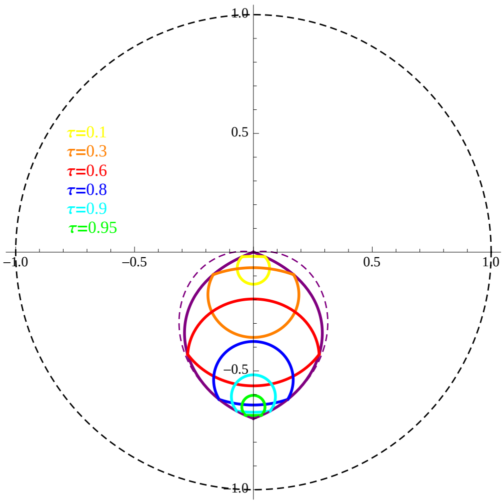

Figure 3.12: $V_{\mathcal{T}(\tau)}(-0.7i)$.

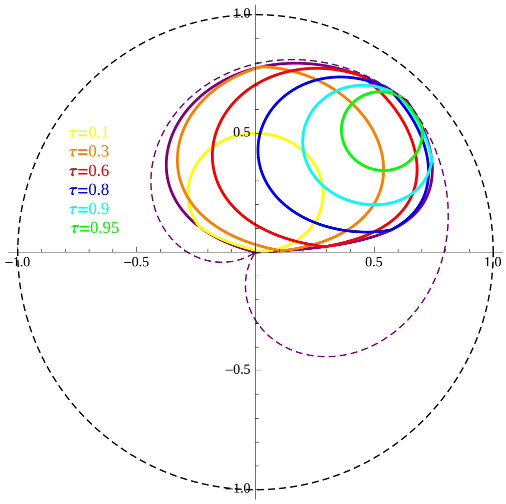

Figure 3.13: $V_{\mathcal{T}(\tau)}(0.9e^{i\frac{\pi}{4}})$.

The sets $V_{\mathcal{T}(\tau)}(z_0)$ for $z_0 = -0.7i, 0.9e^{i\frac{\pi}{4}}$ and several values of τ. The solid purple curves show the boundaries of the sets $V_{\mathcal{T}}(z_0)$; the dashed purple lines are the boundaries of the set determined by [RS14].

Corollary 3.10. *Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then $V_{\mathcal{T}}(z_0) = V_{\mathcal{U}}(z_0)$.*
Furthermore, if $z_0 \notin \mathbb{R}$, then each $w \in \partial V_{\mathcal{T}}(z_0)$ except 0 can be reached by only one mapping $f \in \mathcal{T}$, which is of the form $f_{1,t}$ or $f_{2,t}$ from Theorem 3.5.

Proof. Again, we denote by $W(\tau)$ the image of $V_{\mathcal{T}(\tau)}(z_0)$ under the map $z \mapsto (1+z)^2/(1-z)^2$, the inverse function of $w \mapsto \frac{\sqrt{w}-1}{\sqrt{w}+1}$, which is the convex region bounded by the circular arcs $s_{1,\tau}(y)$, $y \in [2\tau - 1, 1]$, and $s_{2,\tau}(x)$, $x \in [-1, 2\tau - 1]$. Consider the convex region R bounded by the circular arc

$$C := \{P(\tau) \,:\, \tau \in [0,1]\} \text{ and the line segment } L := \{Q(\tau) \,:\, \tau \in [0,1]\}\,, \quad (3.4.3)$$

where $Q(\tau)$ and $P(\tau)$ are defined as in (3.4.1).

Fix $\tau \in (0,1]$. To show that $W(\tau)$ is contained in R, assume the opposite. Then the boundary of $W(\tau)$ has to intersect either C or L in some other point besides $P(\tau)$ and $Q(\tau)$. However, it is easy to see that each of the following four equations

$$\begin{aligned} s_{1,\tau}(y) &= P(t), \quad y,t \in \mathbb{R}, \\ s_{1,\tau}(y) &= Q(t), \quad y,t \in \mathbb{R}, \\ s_{2,\tau}(x) &= P(t), \quad x,t \in \mathbb{R}, \\ s_{2,\tau}(x) &= Q(t), \quad x,t \in \mathbb{R}, \end{aligned}$$

has only one solution, namely $(y,t) = (1,\tau)$, $(y,t) = (2\tau - 1, \tau)$, $(x,t) = (-1,\tau)$, and $(x,t) = (2\tau - 1, \tau)$, respectively. (In all four cases, the second intersection point between the circles/lines is given by the limit cases $y \to \infty$ and $x \to \infty$ respectively.)

Hence, we have $W(\tau) \subset R$ for every $\tau \in (0,1]$. Finally, it is clear that every point contained in $R \setminus \{1\}$ (note that $P(0) = Q(0) = 1$) is contained in some $W(\tau)$: since every $W(\tau)$ is convex, the line segment between $P(\tau)$ and $Q(\tau)$ is always contained in $W(\tau)$.
Consequently, $\cup_{\tau\in(0,1]} W(\tau) = R \setminus \{1\}$.

Now we apply the function $z \mapsto (1+z)^2/(1-z)^2$, the inverse function of $w \mapsto \frac{\sqrt{w}-1}{\sqrt{w}+1}$, to the curves $C_+(z_0)$ and $C_-(z_0)$ from Theorem 3.5 and we obtain the curves

$$\left\{ \frac{(1+z_0)^2}{(1+z_0)^2 - 4e^{-t}z_0} \;:\; t \in [0,\infty] \right\} \text{ and } \left\{ 1 + \frac{4e^{-t}z_0}{(z_0-1)^2} \;:\; t \in [0,\infty] \right\},$$

which are the very same curves as (3.4.3). Thus, we conclude that $V_{\mathcal{T}}(z_0) = V_{\mathcal{U}}(z_0)$.

Finally, assume $z_0 \notin \mathbb{R}$ and let $w \in \partial V_{\mathcal{T}}(z_0) \setminus \{0\}$. Then $w = P(\tau)$ or $w = Q(\tau)$ for a unique $\tau \in (0,1]$ and the proof of Theorem 3.9 shows that there is only one mapping $f \in V_{\mathcal{T}(\tau)}(z_0)$ with $f(z_0) = w$. From Theorem 3.5 it follows that f is of the form $f_{1,t}$ or $f_{2,t}$. □

3.5 Functions with real coefficients

Finally, we take a look at one further value region, determined by Rogosinski in [Rog34, p. 111]: Let $\mathcal{R}$ be the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ with

$f(0) = 0$ that have only real coefficients in the power series expansion around 0. Then $V_{\mathcal{R}}(z_0)$ is the intersection of the two closed discs whose boundaries are the circles through $1, z_0, -z_0$ and through $-1, -z_0, z_0$ respectively.

Let $\mathcal{R}^{\geq}$ be the set of all holomorphic functions $f \in \mathcal{R}$ with $f'(0) \geq 0$ and $z_0 \in \mathbb{D} \setminus \{0\}$. Then we have

$$V_{\mathcal{T}}(z_0) \subset V_{\mathcal{R}^{\geq}}(z_0) \subset V_{\mathcal{R}}(z_0).$$

It is clear that $V_{\mathcal{R}^{\geq}}(z_0) \neq V_{\mathcal{R}}(z_0)$ as the point $-z_0$ belongs to $V_{\mathcal{R}}(z_0)$ and there is only one mapping $f \in \mathcal{R}(z_0)$ with $f(z_0) = -z_0$, namely $f(z) = -z$ for all $z \in \mathbb{D}$. Furthermore, if $z_0 \notin \mathbb{R}$, we have $V_{\mathcal{T}}(z_0) \subsetneq V_{\mathcal{R}^{\geq}}(z_0)$ which can be seen as follows: The boundary points of $V_{\mathcal{R}}(z_0)$ can be reached only by the functions $z \mapsto \pm z\frac{z-x}{zx-1}$, $x \in [-1, 1]$, see [Rog34, p. 111]. Hence, by Corollary 3.10 we have $\partial V_{\mathcal{T}}(z_0) \cap \partial V_{\mathcal{R}}(z_0) = \{z_0\}$. For $0 < x < 1$, the function $f(z) = z\frac{z-x}{zx-1}$ satisfies $f'(0) = x > 0$ and $f(z_0) \neq z_0$. This gives us $z_0\frac{z_0-x}{z_0x-1} \in V_{\mathcal{R}^{\geq}}(z_0) \setminus V_{\mathcal{T}}(z_0)$.

Theorem 3.11. *Let $z_0 \in \mathbb{D}\setminus\{0\}$. Then $V_{\mathcal{R}^{\geq}}(z_0)$ is the closed convex region bounded by the following three curves:*

$$\begin{aligned} A &= \left\{ z_0 \frac{z_0 - x}{z_0 x - 1} \;:\; x \in [0,1] \right\}, \quad B = \left\{ z_0 \frac{z_0 + x}{z_0 x + 1} \;:\; x \in [0,1] \right\}, \\ C &= \left\{ \frac{z_0^2(z_0 + 2x - 1)}{1 + 2xz_0 - z_0} \;:\; x \in [0,1] \right\}. \end{aligned}$$

Proof. Let $f \in \mathcal{R}^{\geq}$. Then $g(z) := (1 + f(z))/(1 - f(z))$ maps $\mathbb{D}$ into the right half-plane with $g(0) = 1$, g has only real coefficients in its power series expansion around 0 and $g'(0) = 2f'(0) \geq 0$. Due to the Herglotz representation ([Dur83], Section 1.9) we can write g as

$$g(z) = \int_{\partial\mathbb{D}} \frac{u + z}{u - z}\, \nu(du), \tag{3.5.1}$$

for some probability measure ν on $\partial\mathbb{D}$.
As g, and thus ν, is symmetric with respect to the real axis, one can rewrite (3.5.1) as

$$g(z) = \int_{\partial\mathbb{D}} 1/2\frac{u + z}{u - z} + 1/2\frac{\overline{u} + z}{\overline{u} - z}\, \nu(du) = \int_{\partial\mathbb{D}} \frac{1 - z^2}{1 - 2z\mathrm{Re}(u) + z^2}\, \nu(du), \tag{3.5.2}$$

or

$$g(z) = G_\mu(z) := \int_{[0,\pi]} \frac{1 - z^2}{1 - 2z\cos(u) + z^2}\, \mu(du), \tag{3.5.3}$$

where μ is a probability measure on $[0,\pi]$ that additionally fulfils

$$\int_{[0,\pi]} \cos(u)\,\mu(du) \geq 0, \tag{3.5.4}$$

as the last integral is equal to $g'(0)/2 = f'(0)$. Thus we need to determine the extreme points of the convex set of all measures on $[0,\pi]$ that satisfy (3.5.4). According to [Win88], Theorem 2.1, this set of extreme points is contained in the set S that consists of all point measures $\mu = \delta_\phi$ on $[0,\pi]$ satisfying (3.5.4) and of all convex combinations $\mu = \lambda\delta_\phi + (1-\lambda)\delta_\varphi$, with $\phi, \varphi \in [0,\pi]$, $\phi \neq \varphi$, $\lambda \in (0,1)$, satisfying (3.5.4).

We can now determine $V_{\mathcal{R}\geq}(z_0)$ as follows: Denote by $W_{\mathcal{R}\geq}(z_0)$ the image of $V_{\mathcal{R}\geq}(z_0)$ under the injective map $w \mapsto (1+w)/(1-w)$. Then $W_{\mathcal{R}\geq}(z_0)$ is the closure of the convex hull of the set $\{G_\mu(z_0) \,:\, \mu \in S\}$.

The point measures from S are, of course, all δ_ϕ with $\phi \in [0,\pi/2]$, and (3.5.3) gives us the curve

$$\frac{1-z_0^2}{1-2z_0x+z_0^2}, \quad x \in [0,1], \tag{3.5.5}$$

a circular arc connecting the points $\frac{1-z_0^2}{1+z_0^2}$ and $\frac{1+z_0}{1-z_0}$. This curve is the image of A under the map $w \mapsto \frac{1+w}{1-w}$. The image of B is the line segment

$$\frac{1+2xz_0+z_0^2}{1-z_0^2}, \quad x \in [0,1], \tag{3.5.6}$$

which connects $\frac{1+z_0^2}{1-z_0^2}$ to $\frac{1+z_0}{1-z_0}$.

The other measures have the form $\lambda\delta_\phi + (1-\lambda)\delta_\varphi$ with w.l.o.g. $\phi \in [0,\pi/2]$, $\varphi \in [0,\pi]$, $\phi \neq \varphi$, $\lambda \in (0,1)$ such that $\lambda\cos(\phi) + (1-\lambda)\cos(\varphi) \geq 0$. They lead to the set

$$\lambda\frac{1-z_0^2}{1-2z_0x+z_0^2} + (1-\lambda)\frac{1-z_0^2}{1-2z_0y+z_0^2}, \tag{3.5.7}$$
$$x \in [0,1],\ y \in [-1,1],\ \lambda \in (0,1),\ \lambda x + (1-\lambda)y \geq 0.$$

If we take $x = 1$, $y \in [-1,0]$ and $\lambda = \frac{y}{y-1}$ (which is equivalent to $\lambda + (1-\lambda)y = 0$), then we obtain

$$\frac{(1+z_0)(1-2(1+y)z_0+z_0^2)}{(1-z_0)(1-2yz_0+z_0^2)}. \tag{3.5.8}$$

The above expression describes a circular arc connecting the points $\frac{1+z_0^2}{1-z_0^2}$ and $\frac{1-z_0^2}{1+z_0^2}$, and this arc is the image of the curve C under the map $w \mapsto \frac{1+w}{1-w}$. Note that the point $\frac{1+z_0}{1-z_0}$ lies on the full circle, which is obtained when $y \to \infty$.
Denote by Δ the closed region bounded by the three curves (3.5.5), (3.5.6) and (3.5.8). Then Δ is convex. We are done if we can show that for all other measures from (3.5.7), $G_\mu(z_0)$ belongs to Δ. Then we can conclude that Δ is the closure of the convex hull of $\{G_\mu(z_0) \,:\, \mu \in S\}$.

First, we consider the points from (3.5.7) for $x \in [0,1]$, $y \in [-1,0]$ and again $\lambda = \frac{y}{y-x}$:

a) For $y = 1$, we obtain the curve

$$-\frac{(-1+z_0)(1+z_0(2-2x+z_0))}{(1+z_0)(1-2xz_0+z_0^2)}, \quad x \in [0,1]. \tag{3.5.9}$$

Like the curve (3.5.8), this arc also connects $\frac{1+z_0^2}{1-z_0^2}$ and $\frac{1-z_0^2}{1+z_0^2}$. For $x \to \infty$ we obtain $\frac{1-z_0}{1+z_0}$, which is the second intersection point of the full circles corresponding to (3.5.5) and (3.5.6) (note that this point is mapped onto $-z_0$ under $w \mapsto \frac{1+w}{1-w}$). We conclude that (3.5.9) is contained in Δ. The convex set bounded by (3.5.8) and (3.5.9) will be denoted by Δ_0, and is, of course, contained in Δ. (Figure 3.15 shows the image of Δ_0 under the map $w \mapsto \frac{1+w}{1-w}$.)

b) Now fix $x \in [0,1)$ and (3.5.7) becomes

$$\frac{(-1+z_0^2)(1-2(x+y)z_0+z_0^2)}{(-1+2xz_0-z_0^2)(1-2yz_0+z_0^2)}, \quad y \in [0,1]. \tag{3.5.10}$$

This set is a circular arc connecting $\frac{1-z_0^2}{1+z_0^2}$ $(y=0)$ to a point on (3.5.9) $(y=1)$. For $y \to \infty$ we obtain the point $\frac{1-z_0^2}{1-2xz_0+z_0^2}$, which lies on (3.5.5) and is not contained in Δ_0. We conclude that the arc (3.5.10) lies within Δ_0 and thus in Δ.

Finally, assume there are $x \in [0,1]$, $y \in [-1,1]$, $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y > 0$ (i.e. $G_\mu'(0) > 0$) and the corresponding point (3.5.7) lies outside Δ. Since it has nevertheless to lie in $V_{\mathcal{R}}(z_0)$, the line segment between this point and $\frac{1+z_0}{1-z_0}$ must intersect the curve (3.5.8). Thus, the set $\{f(z_0) \,:\, f \in \mathcal{R}, f'(0) > 0\}$, which does not contain the curve C, could not be convex, a contradiction.

$\square$

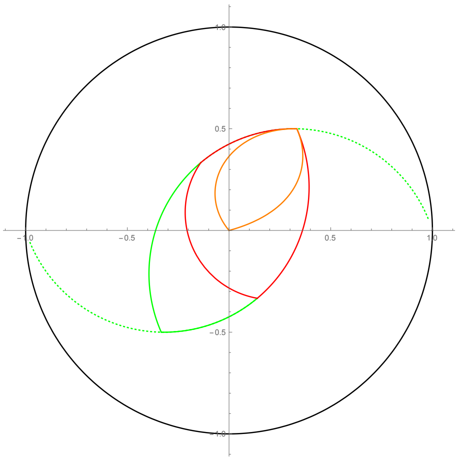

Figure 3.14: $V_{\mathcal{T}}(z_0), V_{\mathcal{R}^{\geq}}(z_0), V_{\mathcal{R}}(z_0)$.

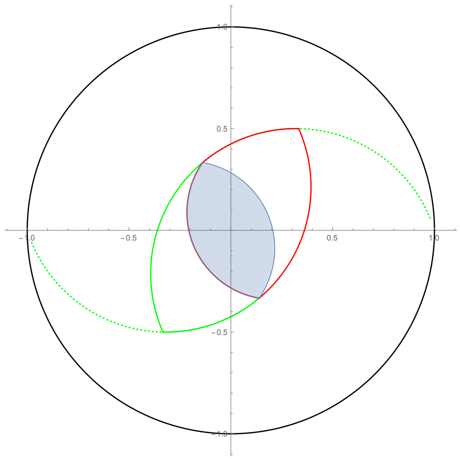

Figure 3.15: Δ_0.

The sets $V_{\mathcal{T}}(z_0)$ (orange), $V_{\mathcal{R}^{\geq}}(z_0)$ (red) and $V_{\mathcal{R}}(z_0)$ (green) are shown in Figure 3.14 for $z_0 = \frac{1}{3} + \frac{i}{2}$.

Corollary 3.12. *Let $\mathcal{R}^0 := \{f \in \mathcal{R}^{\geq} \,:\, f'(0) = 0\}$ and $z_0 \in \mathbb{D} \setminus \{0\}$. Then $V_{\mathcal{R}^0}(z_0) = \Delta_0$, i.e. $V_{\mathcal{R}^0}(z_0)$ is the closed convex set bounded by the circular arcs C and $-C$, which intersect at z_0^2 and $-z_0^2$.*

Proof. We can proceed as in the proof of Theorem 3.11. Condition (3.5.4) then has to be replaced by

$$\int_{[0,\pi]} \cos(u)\,\mu(du) = 0, \text{ which we can also write as} \tag{3.5.11}$$

$$\int_{[0,\pi]} \cos(u)\,\mu(du) \geq 0 \text{ and } \int_{[0,\pi]} \cos(u)\,\mu(du) \leq 0$$

in order to apply again [Win88], Theorem 2.1. Then we obtain that the image of $V_{\mathcal{R}^0}(z_0)$ under the map $w \mapsto (1+w)/(1-w)$ is equal to the closure of the convex hull of the set $\{G_\mu(z_0) \,:\, \mu \in S, \mu \text{ satisfies } (3.5.11)\}$.
The proof of Theorem 3.11 shows that this set is equal to Δ_0. □

Corollary 3.13. *Let $\mathcal{R}^{>} := \{f \in \mathcal{R}^{\geq} \,:\, f'(0) > 0\}$ and $z_0 \in \mathbb{D} \setminus \{0\}$. Then $V_{\mathcal{R}^{>}}(z_0) = V_{\mathcal{R}^{\geq}}(z_0) \setminus C$, where C is the curve from Theorem 3.11.*

Proof. Obviously, the curves A and B from Theorem 3.11 minus their endpoint z_0^2 and $-z_0^2$ belong to $V_{\mathcal{R}^{>}}(z_0)$. The curve C does not belong to the set $V_{\mathcal{R}^{>}}(z_0)$, but it belongs to its closure, which can be seen by approximating the curve by points from (3.5.7) for $\lambda = \frac{y-1/n}{y-x}$, $n \in \mathbb{N}$, which means the integral in (3.5.4) is equal to

$1/n$ in this case.

As $V_{\mathcal{R}^>}(z_0)$ is a convex set, we conclude that $V_{\mathcal{R}^>}(z_0)$ is equal to $V_{\mathcal{R}^\geq}(z_0) \setminus C$.

$\square$

Chapter 4

Self-mappings of the upper half-plane

4.1 Boundary normalisation and the chordal Loewner equation

The results in the previous chapter have always dealt with functions $f : \mathbb{D} \to \mathbb{D}$ with normalisation at the origin. Of course, the origin can easily be replaced by any other point $z_0 \in \mathbb{D}$ by applying a suitable automorphism of the unit disc, but the situation changes if we wish to use a boundary point $z_0 \in \partial\mathbb{D}$ as our point of normalisation: we need to assume that the function f can be continued to z_0 in the sense that $\angle \lim_{z \to z_0} f(z)$ exists. For technical reasons, one then usually considers the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$ instead of $\mathbb{D}$ (which is of course perfectly fine, since they are conformally equivalent by the Riemann mapping theorem).

Let $\mathcal{H}$ be the set of all schlicht self-mappings of $\mathbb{H}$ with *hydrodynamic normalisation* at infinity, i.e.

$$\mathcal{H} := \{f : \mathbb{H} \to \mathbb{H} \,:\, f \text{ schlicht}, \; f(z) = z - \frac{c}{z} + \gamma(z)\}, \tag{4.1.1}$$

where $\operatorname{hcap}(f) := c \geq 0$, which is usually called *half-plane capacity*, and γ satisfies $\angle \lim_{z \to \infty} z \cdot \gamma(z) = 0$.

Remark 4.1. *Let $f \in \mathcal{H}$ with $\operatorname{hcap}(f) = c$. If we transfer f to the unit disc by conjugation by the Cayley transform, then we obtain a function $\tilde{f} : \mathbb{D} \to \mathbb{D}$ having the expansion*

$$\tilde{f}(z) = z - \frac{c}{4}(z-1)^3 + \tilde{\gamma}(z),$$

where $\angle \lim_{z \to 1} \frac{\tilde{\gamma}(z)}{(z-1)^3} = 0$.

Pavel Parfenevich Kufarev ([Kuf43], [Kuf47], [KSS68]) transferred Loewner's approach to describe schlicht functions on $\mathbb{D}$ with normalisation at the interior point 0 to schlicht functions which map $\mathbb{H}$ into itself and are normalised at the boundary point ∞:
A function $f \in \mathcal{H}$ can be described as a solution to the *chordal* or *Loewner-Kufarev* equation

$$\dot{f}_t(z) = \int_{\mathbb{R}} \frac{2}{f_t(z) - u} \, \mu_t(du) \text{ for a.e. } t \geq 0, \qquad f_0(z) = z \in \mathbb{H}, \tag{4.1.2}$$

where μ_t is a Borel probability measure on $\mathbb{R}$ for every $t \geq 0$, and the function $t \mapsto \int_{\mathbb{R}} \frac{1}{u-z} \, \mu_t(du)$ is measurable for every $z \in \mathbb{H}$.

Note that a solution to (4.1.2) fulfils the hydrodynamic normalisation at ∞

$$f_t(z) = z - \frac{2t}{z} + \gamma(z), \qquad \angle \lim_{z \to \infty} z \cdot \gamma(z) = 0.$$

In a sense, the half-plane capacity can thus be seen as an analogue to the term $f'(0)$ for functions in the reachable set of the radial Loewner equation.

4.2 Schlicht functions and fixed half-plane capacities

Let $z_0 \in \mathbb{H}$. From the boundary Schwarz lemma ([Jul18], [Jul20], see also [Boa10], sec. 5) it is clear that $\text{Im}(f(z_0)) \geq \text{Im}(z_0)$ for any function $f : \mathbb{H} \to \mathbb{H}$ with hydrodynamic normalisation, and $\text{Im}(f(z_0)) = \text{Im}(z_0)$ if and only if f is the identity.

Roth and Schleißinger [RS14] proved that the additional condition of schlichtness does, in contrary to the radial case of functions $f : \mathbb{D} \to \mathbb{D}$ with interior normalisation, not change the value set:

Theorem 4.2 ([RS14] , Th. 2.4)**.**

$$V_{\mathcal{H}}(z_0) = \{w \in \mathbb{H} \,:\, \text{Im}(w) > \text{Im}(z_0)\} \cup \{z_0\}.$$

Again, an obvious next step is to ask what the sets look like if, in analogy to 3.1, we fix the half-plane capacity of a function in $\mathcal{H}$, i.e. we consider the sets

$$\mathcal{H}(T) := \{f : \mathbb{H} \to \mathbb{H} \,:\, f \text{ schlicht}, \; f(z) = z - \frac{2T}{z} + \gamma(z), \text{ where } \angle \lim_{z \to \infty} z \cdot \gamma(z) = 0\}.$$

This question was answered by Prokhorov and Samsonova [PS15] as follows:

Theorem 4.3 ([PS15], Th. 3)**.** *The domain*

$$V_{\mathcal{H}(T)}(i) := \{f(i) \,:\, f \in \mathcal{H}(T)\}, \qquad T > 0$$

is bounded by two curves L_1 and L_2 connecting the points i and $i\sqrt{1-4T}$. Denote by $K_0(\varphi, T)$, $T \geq 0$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the unique root of the equation

$$2\cos^2\varphi \log(1+\sin\varphi) + (1+\sin\varphi)^2 = 2\cos^2\varphi \log K + K^2(1-4T).$$

The curve L_1 in the complex (u, v)-plane is parameterized by the equations

$$u(T) := \frac{K_0^2(\varphi, T)(1-4T) - (1+\sin\varphi)^2}{2K_0(\varphi, T)\cos\varphi},$$
$$v(T) := \frac{1+\sin\varphi}{K_0(\varphi, T)}, \qquad \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}),$$

The curve L_2 is symmetric to L_1 with respect to the imaginary axis.

Remark 4.4 (see [PS15], p.911)**.** *Note that the sets $V_{\mathcal{H}(T)}(z_0)$ for arbitrary $z_0 \in \mathbb{H}$ can be obtained from Th. 4.3 by the observation that, for $f \in \mathcal{H}(T)$, the function $z \mapsto f(z+x) - x$, with $x \in \mathbb{R}$, lies also in $\mathcal{H}(T)$ and $z \mapsto rf\left(\frac{z}{r}\right)$ lies in $\mathcal{H}(r^2 t)$.*

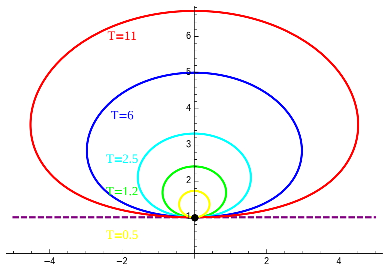

Figure 4.1: The sets $V_{\mathcal{H}(T)}(i)$ for different values of T

4.3 Symmetric mappings

In analogy to typically real functions in the radial case, we consider functions which are in a sense symmetric: Let

$$\mathcal{I} := \{f \in \mathcal{H} \,|\, f(-\overline{z}) = -\overline{f(z)} \text{ for all } z \in \mathbb{H}\}.$$

$\mathcal{I}$ consists of all $f \in \mathcal{H}$ such that the image $f(\mathbb{H})$ is symmetric with respect to the imaginary axis.
We determine

$$V_{\mathcal{I}}(z_0) := \{f(z_0) \,:\, f \in \mathcal{I}\}.$$

Theorem 4.5. *Let* $z_0 \in \mathbb{H}$. *If* $\mathrm{Re}(z_0) = 0$, *then* $V_{\mathcal{I}}(z_0) = \{z_0 + it \,:\, t \in [0, \infty)\}$.
Next, assume $\mathrm{Re}(z_0) > 0$ *and define the two curves* $C(z_0)$ *and* $D(z_0)$ *by*

$$\begin{aligned} C(z_0) &= \left\{\sqrt{z_0^2 - 4t} \,:\, t \in [0, \infty)\right\} = \\ &= \{x + iy \in \mathbb{H} \,:\, x \cdot y = \mathrm{Re}(z_0)\,\mathrm{Im}(z_0),\, x \in (0, \mathrm{Re}(z_0)]\}, \\ D(z_0) &= \left\{z_0 + e^{i\arg(z_0)} \cdot t \,:\, t \in [0, \infty)\right\}. \end{aligned}$$

Then, the set $\overline{V_{\mathcal{I}}(z_0)}$ *is the closed subset of* $\mathbb{H}$ *bounded by* $C(z_0)$ *and* $D(z_0)$, *and* $V_{\mathcal{I}}(z_0) = \{z_0\} \cup \overline{V_{\mathcal{I}}(z_0)} \setminus D(z_0)$.
The case $\mathrm{Re}(z_0) < 0$ *follows from the case* $\mathrm{Re}(z_0) > 0$ *by reflection w.r.t the imaginary axis.*

The value set $f^{-1}(z_0)$ for the inverse functions is given in a quite similar way.

Theorem 4.6. *Let* $z_0 \in \mathbb{H}$ *and define*

$$V_{\mathcal{I}}^*(z_0) = \{f^{-1}(z_0) \,:\, f \in \mathcal{I}, z_0 \in f(\mathbb{H})\}.$$

If $\mathrm{Re}(z_0) = 0$, *then* $V_{\mathcal{I}}^*(z_0) = \{z_0 - it \,:\, t \in [0, \mathrm{Im}(z_0))\}$.
Next, assume $\mathrm{Re}(z_0) > 0$ *and define the two curves* $C^*(z_0)$ *and* $D^*(z_0)$ *by*

$$\begin{aligned} C^*(z_0) &= \left\{\sqrt{z_0^2 + 4t} \,:\, t \in [0, \infty)\right\} = \\ &= \{x + iy \in \mathbb{H} \,:\, x \cdot y = \mathrm{Re}(z_0)\,\mathrm{Im}(z_0),\, x \in [\mathrm{Re}(z_0), \infty)\}, \\ D^*(z_0) &= \left\{z_0 - e^{i\arg(z_0)} \cdot t \,:\, t \in [0, |z_0|)\right\}. \end{aligned}$$

Then, the closure $\overline{V_{\mathcal{I}}^*(z_0)}$ *is the closed subset of* $\mathbb{H}$ *bounded by the curves* $C^*(z_0)$, $D^*(z_0)$ *and the positive real axis. The set* $V_{\mathcal{I}}^*(z_0)$ *is given by* $V_{\mathcal{I}}^*(z_0) = \{z_0\} \cup \overline{V_{\mathcal{I}}^*(z_0)} \setminus (D^*(z_0) \cup [0, \infty))$.
The case $\mathrm{Re}(z_0) < 0$ *follows from the case* $\mathrm{Re}(z_0) > 0$ *by reflection w.r.t the imaginary axis.*

Figure 4.2 shows the curves $C(1+i)$ and $D(1+i)$ (dashed), as well as $C^*(1+i)$ and $D^*(1+i)$.

Proof of Theorem 4.5. Without loss of generality we may assume that $z_0 \in Q_1 := \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq 0, \mathrm{Im}\, z > 0\}$.

Now consider the chordal Loewner equation

$$\dot{f}_t(z) = \int_{\mathbb{R}} \frac{2}{a - f_t(z)}\, \alpha_t(da) \text{ for a.e. } t \geq 0, \quad f_0(z) = z \in \mathbb{H}, \tag{4.3.1}$$

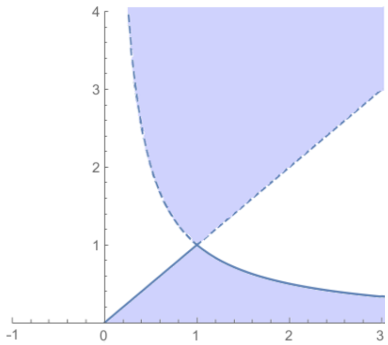

Figure 4.2: $V_{\mathcal{I}}(1+i)$ and $V_{\mathcal{I}}^*(1+i)$.

where α_t is a Borel probability measure on $\mathbb{R}$ for every $t \geq 0$, and the function $t \mapsto \int_{\mathbb{R}} \frac{1}{a-z}\, \alpha_t(da)$ is measurable for every $z \in \mathbb{H}$. For every $f \in \mathcal{H}$ there exists $T > 0$ and such a family $\{\alpha_t\}_{t\geq 0}$ of probability measures such that the solution $\{f_t\}_{t\geq 0}$ of (4.3.1) satisfies $f_T = f$; see [GB92], Theorem 5.

Now let $f_T = f \in \mathcal{I}$. Then we can find a solution $\{f_t\}_{t\in[0,T]}$ such that $f_t \in \mathcal{I}$ for all $t \in [0, T]$, which means that α_t can be written as $\alpha_t = 1/2\mu_t^* + 1/2\mu_t$, where μ_t is a probability measure supported on $[0, \infty)$ and μ_t^* is the reflection of μ_t to $(-\infty, 0]$.
This leads to the symmetric Loewner equation

$$\begin{aligned} \dot{f}_t(z) &= \int_{\mathbb{R}} \frac{1}{a - f_t(z)}\, \mu_t(da) + \int_{\mathbb{R}} \frac{1}{-a - f_t(z)}\, \mu_t(da) = \\ &= \int_{\mathbb{R}} \frac{2f_t(z)}{a^2 - f_t(z)^2}\, \mu_t(da) = \int_{\mathbb{R}} \frac{2f_t(z)}{u - f_t(z)^2}\, \nu_t(du), \end{aligned} \tag{4.3.2}$$

where we put $u = a^2 \in [0, \infty)$ and $\nu_t(A) = \mu_t(\sqrt{A})$, $\nu_t(B) = 0$ for Borel sets $A \subset [0, \infty)$ and $B \subset (-\infty, 0)$.

Thus we can consider the initial value problem

$$\dot{w}(t) = \int_{\mathbb{R}} \frac{2w(t)}{u - w(t)^2}\, \nu_t(du), \quad w(0) = z_0 \in \mathbb{H}, \tag{4.3.3}$$

and have

$$V_{\mathcal{I}}(z_0) = \{w(T) \,:\, w(t) \text{ solves } (4.3.3), T \geq 0\}. \tag{4.3.4}$$

Next, we observe that the set $\mathcal{I}_S := \{f \in \mathcal{I} \,:\, f(Q_1) = Q_1 \setminus \gamma \text{ for a simple curve } \gamma\}$ is dense in $\mathcal{I}$ by a standard argument for univalent functions; see [Dur83, Section

3.2].
Denote by δ_x the Dirac measure in $x \in \mathbb{R}$. If $f \in \mathcal{I}_S$, then we can find a continuous function $U : [0,\infty) \to [0,\infty)$ such that the measure $\nu_t = \delta_{U(t)}$ in (4.3.2) generates f, i.e. $f_T = f$; see Chapter I.3.8 in [Tam78], which considers the unit disc and the corresponding symmetric radial Loewner equation; see also [Gor86, §3], [Gor15, §5,6].

Consider the corresponding initial value problem

$$\dot{w}(t) = \frac{2w(t)}{U(t) - w(t)^2}, \quad w(0) = z_0 \in \mathbb{H}, \tag{4.3.5}$$

where $U : [0,\infty) \to [0,\infty)$ is a continuous function. Denote by $R(z_0)$ the reachable set of this equation, i.e. $R(z_0) := \{w(T) : w(t) \text{ solves } (4.3.5), T \geq 0\}$. Then $V_{\mathcal{I}_S}(z_0) \subset R(z_0)$ and because of the denseness of $\mathcal{I}_S$ in $\mathcal{I}$, we have

$$\overline{R(z_0)} = \overline{V_{\mathcal{I}}(z_0)}. \tag{4.3.6}$$

Now we determine the set $V_{\mathcal{I}}(z_0)$.

If $\mathrm{Re}(z_0) = 0$, it is clear that $w(t) \in \{z_0 + is : s \in [0,\infty)\}$ for all $t \in [0,\infty)$. The solution to (4.3.5) for $U(t) \equiv 0$ is given by $w_1(t) = \sqrt{z_0^2 - 4t}$. As $\mathrm{Im}(w_1(t)) \to \infty$ as $t \to \infty$, we conclude that $V_{\mathcal{I}}(z_0) = \{z_0 + it : t \in [0,\infty)\}$.

Now assume that $\mathrm{Re}(z_0) > 0$.

Step 1: First, we determine $R(z_0)$. We write $w(t) = \xi(t) + i\eta(t)$ and $z_0 = \xi_0 + i\eta_0$; thus (4.3.5) reads

$$\begin{aligned} \dot{\xi}(t) &= \frac{2\xi(t)(U(t) - |w(t)|^2)}{|U(t) - w(t)^2|^2}, & \dot{\eta}(t) &= \frac{2\eta(t)(U(t) + |w(t)|^2)}{|U(t) - w(t)^2|^2}, \\ \xi(0) &= \xi_0, & \eta(0) &= \eta_0. \end{aligned}$$

As $t \mapsto \eta(t)$ is strictly increasing, we can parametrize ξ and U by η and obtain

$$\frac{d\xi}{d\eta} = \frac{\xi(\eta)}{\eta} \frac{U(\eta) - |w(\eta)|^2}{U(\eta) + |w(\eta)|^2}. \tag{4.3.7}$$

Since $U(\eta) \geq 0$, we have

$$-1 \leq \frac{U(\eta) - |w(\eta)|^2}{U(\eta) + |w(\eta)|^2} < 1$$

which yields the inequalities

$$\frac{-\xi}{\eta} \le \frac{d\xi}{d\eta} < \frac{\xi}{\eta}.$$

By solving the equations $\frac{d\xi'}{d\eta} = -\frac{\xi'}{\eta}$ and $\frac{d\xi'}{d\eta} = \frac{\xi'}{\eta}$ with $\xi'(\eta_0) = \xi_0$, we arrive at

$$\frac{\eta_0}{\eta(t)} \le \frac{\xi(t)}{\xi_0} < \frac{\eta(t)}{\eta_0} \tag{4.3.8}$$

for all $t > 0$. We have equality for the left case when $U(t) \equiv 0$, which leads to the solution $w(t) = \sqrt{z_0^2 - 4t}$, i.e. the curve $C(z_0)$. The case $\frac{\xi}{\xi_0} = \frac{\eta}{\eta_0}$ corresponds to the curve $D(z_0)$, which does not belong to the set $R(z_0) \setminus \{z_0\}$.

On the Riemann sphere $\hat{\mathbb{C}}$, the two curves $\hat{C}(z_0) = C(z_0) \cup \{\infty\}$ and $\hat{D}(z_0) = D(z_0) \cup \{\infty\}$ intersect at z_0 and ∞, and form the boundary of two Jordan domains. We denote by $J(z_0)$ the closure of the one that is contained in $\mathbb{H} \cup \{\infty\}$. Note that $R(z_0) \subset J(z_0)$ by (4.3.8). We wish to show that $R(z_0) = \{z_0\} \cup J(z_0) \setminus \hat{D}(z_0)$.

To this end, consider (4.3.7) with the driving term

$$U(\eta) = \frac{1+x}{1-x}\left(\xi_0 \left(\frac{\eta}{\eta_0}\right)^{2x} + \eta^2\right), \quad -1 \le x < 1.$$

This yields

$$\frac{U(\eta) - |w(\eta)|^2}{U(\eta) + |w(\eta)|^2} \equiv x,$$

and thus

$$\xi(\eta) = \xi_0 \left(\frac{\eta}{\eta_0}\right)^x,$$

and it is easy to see that

$$J(z_0) \setminus \hat{D}(z_0) = \left\{\xi_0 \left(\frac{\eta}{\eta_0}\right)^x \,\middle|\, \eta \in [0, \infty),\ -1 \le x < 1\right\} \subseteq R(z_0).$$

Step 2: Finally, we show that $V_{\mathcal{I}}(z_0) = \{z_0\} \cup J(z_0) \setminus \hat{D}(z_0)$, which concludes the proof.
As we already know that $\overline{R(z_0)} = \overline{V_{\mathcal{I}}(z_0)}$ (equation (4.3.6)), we only need to prove that $\hat{D}(z_0) \setminus \{z_0\}$ has empty intersection with $V_{\mathcal{I}}(z_0)$.
Recall (4.3.4) and let $w(t)$ be a solution to (4.3.3). We write again $w(t) = \xi(t) + i\eta(t)$. Then

$$\dot{\xi} = 2\xi(t) \int_{\mathbb{R}} \frac{u - |w(t)|^2}{|u - w(t)^2|^2}\, \mu_t(du) < 2\xi(t) \int_{\mathbb{R}} \frac{u + |w(t)|^2}{|u - w(t)^2|^2}\, \mu_t(du),$$

and
$$\dot{\eta} = 2\eta(t) \int_{\mathbb{R}} \frac{u + |w(t)|^2}{|u - w(t)^2|^2} \, \mu_t(du).$$
Again, $t \mapsto \eta(t)$ is strictly increasing, and we parametrize ξ by η to get
$$\frac{d\xi}{d\eta} < \frac{\xi}{\eta},$$
which yields $\frac{\xi(t)}{\xi_0} < \frac{\eta(t)}{\eta_0}$ for all $t > 0$, hence $(D(z_0) \setminus \{z_0\}) \cap V_{\mathcal{I}}(z_0) = \emptyset$. $\square$

The proof of Theorem 4.6 is completely analogous.

Chapter 5

Value sets for the derivative

The Schwarz and Schwarz-Pick lemmata give estimates for $f(z)$ and $f'(z)$. The former chapters have dealt with refinements of the first estimate; this final chapter will be dedicated to the latter.

First efforts in that direction were undertaken for (unbounded) functions in the class $\mathcal{S}$:
Golusin [Gol36] used Loewner theory to find bounds for the argument of $f'(z)$, $f \in \mathcal{S}$ and showed that

$$|\arg f'(z)| \leq \begin{cases} 4\arcsin|z| & \text{for } |z| \leq \frac{1}{\sqrt{2}}, \\ \pi + \log\frac{z^2}{1-|z|^2} & \text{for } \frac{1}{\sqrt{2}} < |z| < 1. \end{cases}$$

Grad [Gra50], [SS50, Ch. XV], in a rather involved proof, determined the full value range

$$\{\log f'(\zeta) \ : \ f \in \mathcal{S}\}.$$

His method is based on Schiffer variation as well as extensive calculations[1].

In the setting of bounded functions, Dieudonné's lemma [Die31, Ch. III, p. 340] described the set

$$\{f'(\zeta) \ : \ f : \mathbb{D} \to \mathbb{D} \text{ analytic}, \ f(0) = 0, f(\zeta) = \omega\}$$

for fixed points $\zeta, \omega \in \mathbb{D} \setminus \{0\}$ to be the disc with centre $\frac{\omega}{\zeta}$ and radius $\frac{\zeta^2-|\omega|^2}{|\zeta|(1-|\zeta|^2)}$.

[1] He mentions that Grunsky told him he had found a way to determine the set by means of Loewner theory; unfortunately, he seems – to my knowledge – to never have published these results.

As in the chapters before, it is natural to ask in what way this set changes if we additionally assume the functions to be schlicht, i.e., we concern ourselves with the set

$$\mathcal{R}(\zeta,\omega) := \{f'(\zeta) \ : \ f \in S_1(\zeta,\omega)\},$$

where

$$S_1(\zeta,\omega) := \{f \in S_1 \ : \ f(\zeta) = \omega\}$$

and

$$S_1 := \{f \in \mathcal{H}(\mathbb{D}) \ : \ f(\mathbb{D}) \subseteq \mathbb{D}, f(0) = 0, f \text{ schlicht}\}.$$

Singh [Sin57] determined the minimal and maximal absolute value that $f'(\zeta)$ can take for $f \in S_1(\zeta,\omega)$, as well as the corresponding extremal functions f: he proved that the maximal value is taken at the point

$$\frac{\omega}{\zeta}\frac{(1-|\zeta|)(1+|\omega|)}{(1+|\zeta|)(1-|\omega|)},$$

and the minimal value at

$$\frac{\omega}{\zeta}\frac{(1+|\zeta|)(1-|\omega|)}{(1-|\zeta|)(1+|\omega|)}.$$

The corresponding extremal functions are the Pick functions described by

$$\frac{(1+|\omega|)^2\omega f(z)}{(\omega+|\omega|f(z))^2} = \frac{(1-|\zeta|)^2\zeta z}{(\zeta-|\zeta|z)^2}$$

and

$$\frac{(1-|\omega|)^2\omega f(z)}{(\omega-|\omega|f(z))^2} = \frac{(1+|\zeta|)^2\zeta z}{(\zeta+|\zeta|z)^2},$$

respectively.

To simplify our problem, we may assume that $\zeta,\omega \in \mathbb{R}^+$: otherwise, we consider

$$g : z \mapsto \frac{\overline{\omega}}{|\omega|} f\left(\frac{\zeta}{|\zeta|}z\right)$$

with $g(|\zeta|) = |\omega|$ and note that this transformation simply yields a rotation of the derivative: $f'(\zeta) = \frac{\omega}{|\omega|}\frac{\overline{\zeta}}{|\zeta|}g'(|\zeta|)$. Furthermore, the Schwarz lemma tells us that $\omega \le \zeta$, and $\omega = \zeta$ if and only if $f = \mathrm{id}$. Hence, in the following we assume that $\omega < \zeta$.

It is obvious that $\mathcal{R}(\zeta,\omega)$ is symmetric with respect to $\mathbb{R}$, because if $f \in S_1(\zeta,\omega)$, then so is $z \mapsto \overline{f(\overline{z})}$.

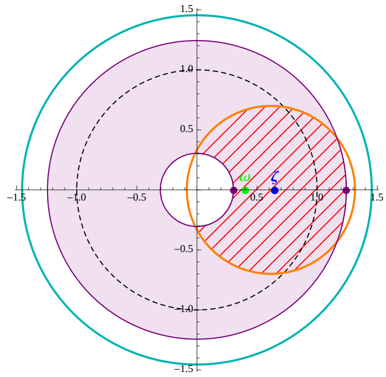

Figure 5.1: The Schwarz-Pick, Dieudonné and Singh sets for $\zeta = 0.65$ and $\omega = 0.4$

Figure 5.1 shows the sets determined by Schwarz-Pick (turquoise), Dieudonné (orange) and Singh (purple). It is immediately clear that the set we look to determine has to be a subset of the one shaded in red.

Theorem 5.1. *If $f \in S_1(\zeta,\omega)$ is extremal for this problem, i.e. the value $f'(\zeta)$ lies on the boundary of $\mathcal{R}(\zeta,\omega)$, then there is an angle $\chi \in [0, 2\pi)$ such that f fulfils the equation*

$$F^{\pm}_{\omega}(f(z)) = F^{\pm}_{\zeta}(z) \text{ for all } z \in \mathbb{D},$$

where

$$F^{\pm}_{y}(z) := 2\left(e^{i\chi} \operatorname{arctanh} \frac{\sqrt{z}}{\sqrt{y}} \pm \operatorname{arctanh}\left(\sqrt{z}\sqrt{y}\right)\right).$$

Remark 5.2. *For $\chi \in \{0, \pi\}$, this equation describes a Pick function f, cf. [Sin57], but it does not for any other value of χ.*

The proof of Th. 5.1 is based on a sort of Lagrange multiplier theorem which was proven for the class $\mathcal{S}$ by Hummel [Hum77]. Since we need families of variations for the class S_1 instead of the unbounded class $\mathcal{S}$, we need to make a few adjustments, but the general idea of the proofs in the paper can be transferred to our case one-to-one.
We will proceed in several steps: Firstly, we will introduce the variations suitable for our case. In the next step, we prove the bounded analogon to Hummel's multiplier theorem. Then, we consider the quadratic differential we obtain from this theorem and deduce the equation from Th. 5.1.

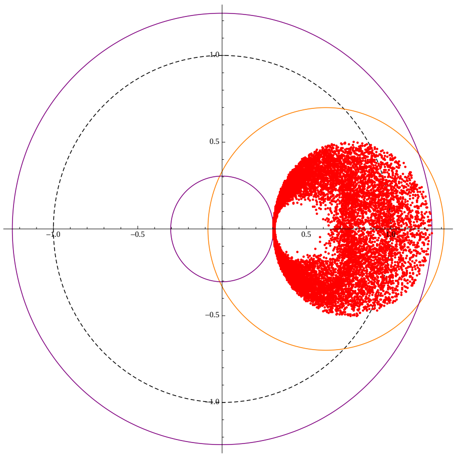

Figure 5.2: $\zeta = 0.65$, $\omega = 0.4$

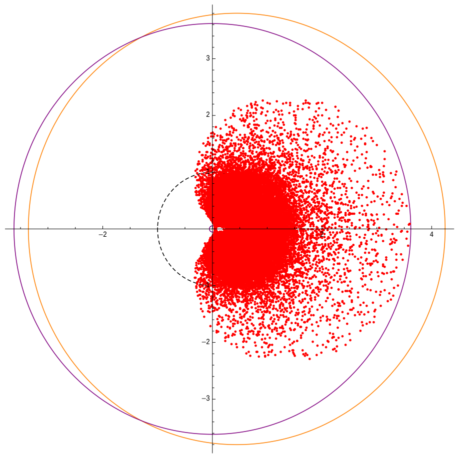

Figure 5.3: $\zeta = 0.9$, $\omega = 0.4$

Numerical evaluations of the set $\mathcal{R}(\zeta, \omega)$: every red point corresponds to the value $f'(\zeta)$ for a one-slit function $f \in S_1(\zeta, \omega)$.

5.1 Bounded variation families

We introduce the bounded equivalents of the variations used by Hummel:
Just like in his case, we firstly use a rotational variation: Let

$$^1f_\rho(z) := f\left(e^{i\rho} z\right), \quad \rho > 0.$$

Taylor expansion at $\rho = 0$ yields

$$^1f_\rho(z) = f(z) + i\rho z f'(z) + O(\rho^2),$$

where the rest term is uniform on compact subsets of S_1.
We will need a second variant, namely

$$^2f_\rho(z) := e^{i\rho} f\left(z\right), \quad \rho > 0,$$

or

$$^2f_\rho(z) = f(z) + i\rho f(z) + O(\rho^2).$$

We also use a variant of the Loewner variation, but observe that we need to drop the factor that fixes the derivative at 0 in order to preserve boundedness:
Let $\rho > 0$ and

$$g_\rho(z) := k_\rho^{-1}\left(\frac{1}{1+\rho} k_\rho(z)\right),$$

where

$$k_\rho : \mathbb{D} \to \mathbb{D},\ k_\rho(z) := \frac{z}{(1+\rho z)^2}, \quad \rho \in \partial\mathbb{D}$$

is a rotation of the Koebe function. Our third variation family then consists of functions

$${}^3 f_\rho(z) := f(g_\rho(z)) = f(z) - \rho z f'(z) \frac{\rho + z}{\rho - z} + O(\rho^2).$$

We replace the Schiffer variation by the Tammi-Schiffer variation for bounded functions[2], cf. [ST69], i.e.

$${}^4 f_\rho(z) = f(z) + \rho \left(a_0 T_0(z) + \overline{a_0} U_0(z)\right) + O(\rho^2),$$

where $a_0 \in \partial\mathbb{D}$, $z_0 \in \mathbb{D}$, and

$$\begin{aligned} T_0(z) &:= \frac{f(z)}{f(z_0) - f(z)} - \frac{q(z_0)}{2f(z_0)} \frac{z + z_0}{z - z_0} z f'(z) - \frac{z f'(z)}{2 f(z_0)}, \\ U_0(z) &:= \frac{f^2(z)}{1 - \overline{f(z_0)} f(z)} + \overline{\left(\frac{q(z_0)}{2f(z_0)}\right)} \frac{1 + \overline{z_0} z}{1 - \overline{z_0} z} z f'(z) + \frac{z f'(z)}{\overline{2 f(z_0)}}, \end{aligned}$$

and

$$q(z_0) = -\frac{f^2(z_0)}{z_0^2 f'^2(z_0)}.$$

Lastly, we replace the Marty variation by a version of the above Tammi-Schiffer variation: assume that the interior of $\mathbb{D} \backslash f(\mathbb{D})$ is non-empty. Then, for $w_0 \in (\mathbb{D} \backslash f(\mathbb{D}))^o$, the variation

$$w_\rho = w + \frac{\rho a_0 w}{w_0 - w} + \frac{\rho \overline{a_0} w^2}{1 - \overline{w_0} w}$$

has no poles on $\mathbb{D}$, so we end up with the variation family

$${}^5 f_\rho(z) = f(z) + \rho \left(a_0 \frac{f(z)}{w_0 - f(z)} + \overline{a_0} \frac{f^2(z)}{1 - \overline{w_0} f(z)} \right) + O(\rho^2), \quad a_0 \in \partial\mathbb{D}.$$

If the continuous functional J has the complex derivative Λ, i.e.

$$J(f + \varepsilon g) = J(f) + \varepsilon \Lambda(f; g) + o(\varepsilon),$$

where $\Lambda = \Lambda(f; g)$ is a continuous linear functional of g, then we get the following relations for our variation families:

[2]for a full derivation of the variation, see the appendix

In full analogy with section 3 in [Hum77], we set

$$\begin{aligned}
C :=& \Lambda\left(f; zf'(z)\right),\\
D :=& \Lambda\left(f; f(z)\right)) = J(f),\\
{}_1A(w_0) :=& \Lambda\left(f; \frac{f(z)}{w_0 - f(z)}\right),\\
{}_2A(w_0) :=& \Lambda\left(\frac{f^2(z)}{1-\overline{w_0}f(z)}\right),\\
E(z_0) :=& \Lambda\left(f; zf'(z)\frac{z+z_0}{z-z_0}\right),\\
B(z_0) :=& \frac{1}{2}\left(E(z_0) + \overline{E\left(\frac{1}{\overline{z_0}}\right)}\right).
\end{aligned}$$

and obtain

$$J\left({}^1f_\rho\right) = J(f) + i\rho C + O(\rho^2), \tag{5.1.1}$$

$$J\left({}^2f_\rho\right) = J(f) + i\rho D + O(\rho^2), \tag{5.1.2}$$

$$J\left({}^3f_\rho\right) = J(f) - \rho E(\eta) + O(\rho^2), \tag{5.1.3}$$

$$\begin{aligned}
J\left({}^4f_\rho\right) =& J(f) + \rho a_0 {}_1A\left(f(z_0)\right) - \rho\frac{a_0}{2f(z_0)}\left(q(z_0)E(z_0) + C\right)\\
&+ \rho\overline{a_0}{}_2A\left(f(z_0)\right) - \rho\frac{\overline{a_0}}{2\overline{f(z_0)}}\left(\overline{q(z_0)}E\left(\frac{1}{\overline{z_0}}\right) - C\right) + O(\rho^2),
\end{aligned} \tag{5.1.4}$$

$$J\left({}^5f_\rho\right) = J(f) + \rho\left(a_0 \cdot {}_1A\left(f(z_0)\right) + \overline{a_0}\cdot {}_2A\left(f(z_0)\right)\right) + O(\rho^2). \tag{5.1.5}$$

Taking the real part in (5.1.4), since a_0 is arbitrary and $C \in \mathbb{R}$ by (5.1.1), we obtain

$${}_1A\left(f(z_0)\right) + \overline{{}_2A\left(f(z_0)\right)} - \frac{q(z_0)}{f(z_0)}B(z_0) = 0,$$

which means we must have

$$\frac{w({}_1A(w) + \overline{{}_2A(w)})dw^2}{w^2} = -\frac{B(z)dz^2}{z^2}, \quad (w = f(z)). \tag{5.1.6}$$

Furthermore, $B(\eta) = \frac{1}{2}\left(E(\eta) + \overline{E(\eta)}\right)$ is real for $\eta \in \partial\mathbb{D}$, and since (5.1.3) yields that $\operatorname{Re} E(\eta) \geq 0$, we have

$$B(\eta) \geq 0 \quad \text{for } \eta \in \partial\mathbb{D}. \tag{5.1.7}$$

From (5.1.5) it is clear that there cannot be a $w_0 \in (\mathbb{D} \setminus f(\mathbb{D}))^o$, together with (5.1.7) this means that f has to map $\mathbb{D}$ onto $\mathbb{D}$ minus analytic slits along the trajectories of the right hand side of (5.1.6).
Note that (5.1.2) yields that $\operatorname{Im} J(f) = 0$.

5.2 A Lagrange-style multiplier theorem

In complete analogy to Th. 2 in [Hum77], we will now prove the following

Theorem 5.3. *Let J_1, J_2 be continuous linear functionals on S_1 which have complex derivatives. Let*

$$\mathcal{M} := \{f \in S_1 \,:\, J_1(f) = 0\}$$

and

$$\mathcal{T} := \{J_2(f) \,:\, f \in \mathcal{M}\}.$$

If $f \in \mathcal{M}$ is such that $J_2(f)$ is a boundary point of $\mathcal{T}$ and that $J_1(f)$ is an interior point of the set $\{J_1(g) : g \in S_1\}$ with respect to the Tammi-Schiffer variation, i.e. $J_1(f)$ is an interior point of $\{J_1(h) : h \text{ is a Tammi-Schiffer variation of } f\}$, and if all functions corresponding to J_1 and J_2 in (5.1.1) *to* (5.1.5) *are rational and non-constant, and the functions ${}_1A_1, {}_2A_1, {}_1A_2$ and ${}_2A_2$ are linearly independent, then there exist complex numbers λ_1, λ_2 such that*

(i) $\lambda_1 C_1 + \lambda_2 C_2 \in \mathbb{R}$;

(ii) $\lambda_1 D_1 + \lambda_2 D_2 \in \mathbb{R}$;

(iii) $$\lambda_1\ {}_1A_1\left(f(z)\right) + \overline{\lambda_1\ {}_2A_1\left(f(z)\right)} + \lambda_2\ {}_1A_2\left(f(z)\right) + \overline{\lambda_2\ {}_2A_2\left(f(z)\right)}$$
$$= \frac{q(z_0)}{f(z_0)}\left(\hat{B}_1(z) + \hat{B}_2(z)\right),$$

where $\hat{B}_j(z) := \frac{1}{2}\left(\lambda_j E_j(z) + \overline{\lambda_j E_j\left(\frac{1}{\bar{z}}\right)}\right)$ *for* $j = 1, 2$;

(iv) $\left(\hat{B}_1 + \hat{B}_2\right)(z) \geq 0$ *for* $z \in \partial\mathbb{D}$;

(v) the interior of $\mathbb{D} \setminus f(\mathbb{D})$ *is empty*

holds, where the terms C_j, ${}_kA_j$, and E_j, $j, k = 1, 2$, are defined in analogy with (5.1.1) *to* (5.1.5).

Proof. We closely follow the proof of Th. 2 in [Hum77]: Let $C_j, \ldots, B_j$, be the terms corresponding to those in (5.1.1) to (5.1.5) for the functionals J_j, $j = 1, 2$.

Next, we write all variations involved in real coordinates: For $j = 1, 2$, let

$$[C_j] := \begin{pmatrix} c_{j1} \\ c_{j2} \end{pmatrix}, \qquad c_{j1} + ic_{j2} = iC_j,$$
$$[D_j] := \begin{pmatrix} d_{j1} \\ d_{j2} \end{pmatrix}, \qquad d_{j1} + id_{j2} = iD_j,$$
$$[E_j(z_0)] := \begin{pmatrix} e_{j1} \\ e_{j2} \end{pmatrix}, \qquad e_{j1} + ie_{j2} = -E_j(z_0),$$

$$[S_j] := \begin{pmatrix} s_{j1} + t_{j1} & -(s_{j2} - t_{j2}) \\ s_{j2} + t_{j2} & s_{j1} - t_{j1} \end{pmatrix}, \quad s_{j1} + is_{j2} = {}_1A(w_0),\ t_{j1} + it_{j2} = {}_2A(w_0),$$
$$[Q_j] := \begin{pmatrix} q_{j1} + r_{j1} & -(q_{j2} - r_{j2}) \\ q_{j2} + r_{j2} & q_{j1} - r_{j1} \end{pmatrix},$$

where
$$q_{j1} + iq_{j2} = {}_1A_j\left(f(z_0)\right) - \frac{1}{2f(z_0)}\left(q(z_0)E_j(z_0) + C_j\right)$$
and
$$r_{j1} - ir_{j2} = {}_2A_j\left(f(z_0)\right) + \frac{1}{2\overline{f(z_0)}}\left(\overline{q(z_0)}E_j\left(\frac{1}{\overline{z_0}}\right) + C_j\right).$$
We define
$$V_j(f_\rho) := \begin{pmatrix} \operatorname{Re}\left(J_j(f_\rho) - J_j(f)\right) \\ \operatorname{Im}\left(J_j(f_\rho) - J_j(f)\right) \end{pmatrix}$$
and obtain

$$V_j({}^1f_\rho) = {}^1\rho_j[C_j] + O(\rho^2), \tag{5.2.1}$$
$$V_j({}^2f_\rho) = {}^2\rho_j[D_j] + O(\rho^2), \tag{5.2.2}$$
$$V_j({}^3f_\rho) = {}^3\rho_j[E_j(\eta)] + O(\rho^2), \tag{5.2.3}$$
$$V_j({}^4f_\rho) = [Q_j]\begin{pmatrix} x \\ y \end{pmatrix} + O(\rho^2), \tag{5.2.4}$$
$$V_j({}^5f_\rho) = [S_j]\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + O(\rho^2), \tag{5.2.5}$$

where $x + iy = \rho a_0$ in (5.1.3) and $\hat{x} + i\hat{y} = \rho a_0$ in (5.1.4). Let $\rho := \max\{{}^k\rho_j, |x + iy|, |\hat{x} + i\hat{y}|\}$.
In the next step, we combine our variations on one single vector; for example, if we choose m points $z_1, \dots, z_m \in \mathbb{D}$ and consider the corresponding x_k, y_k for

$k = 1, \ldots, m$, the complete Tammi-Schiffer variation (5.2.4) we consider reads

$$V({}^4f_\rho)(x_1, y_1, \ldots, x_m, y_m) = [Q](z_1, \ldots, z_m) \cdot X + O(\rho^2),$$

where

$$[Q](z_1, \ldots, z_m) := \begin{pmatrix} [Q_1](z_1), & \ldots, & [Q_1](z_m) \\ [Q_2](z_1), & \ldots, & [Q_2](z_m) \end{pmatrix}, \qquad X := \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_m \\ y_m \end{pmatrix}, \qquad (5.2.6)$$

and $\rho := \max\{|x_k|, |y_k|\}$.

It is easy to see that $[Q]$ is the derivative of $V({}^4f_\rho)$ at the origin and that, in fact, $V({}^4f_\rho)$ is differentiable in a neighbourhood of the origin.

If $m = 2$ and $[Q](z_1, z_2)$ is of full rank, i.e. non-singular, for some choice of z_1, z_2, we can thus use the inverse function theorem to show that $V({}^4f_\rho)$ can be inverted locally, and hence, $J_2(f)$ cannot lie on the boundary of $\mathcal{T}$.

Since $J_1(f)$ is assumed to be an interior point of $\{J_1(f) : f \in S_1\}$ with respect to the Tammi-Schiffer variation, there must be a $z_1 \in \mathbb{D}$ such that the matrix $[Q_1](z_1)$ is non-singular and therefore has rank 2. Thus, for this choice of z_1, the matrix $[Q]$ has to be of rank 2 or 3.

Case 1: If there is a $z_2 \in \mathbb{D}$ such that $[Q](z_1, z_2)$ has rank 3, then its columns span a three-dimensional subspace of $\mathbb{R}^4$, and there is a unique (up to multiplication with a scalar) non-trivial vector $v \in \mathbb{R}^4$ which is orthogonal to this subspace. Let $\lambda_1 := v_1 - iv_2$ and $\lambda_2 := v_3 - iv_4$. Then $\lambda_2 \neq 0$, or we would arrive at a contradiction to the fact that $[Q_1](z_1)$ has rank 2.

Furthermore, if we consider any matrix $[Q](z_1, z_2, z_3)$ obtained by enhancing the matrix $[Q](z_1, z_2)$ by adding a pair of columns corresponding to $[Q](z_3)$, then these new columns also have to be orthogonal to v: Else, the 4×6-Jacobian $[Q](z_1, z_2, z_3)$ of $V({}^4f_\rho) : \mathbb{R}^6 \to \mathbb{R}^4$ would contain a 4×4-matrix of full rank, and (possibly after a permutation of variables) we could apply the implicit function theorem to show that there exist a neighbourhood U of (x_1, y_2, x_2, y_2), a neighbourhood V of (x_3, y_3), and a differentiable function $f : U \to V$ such that

$$V({}^4f_\rho)\,(x_1, y_1, x_2, y_2, f(x_1, y_1, x_2, y_2)) \equiv const.$$

holds for all $(x_1, y_1, x_2, y_2) \in U$. Considering that the mapping associated with $[Q](z_1, z_2, z_3)$ corresponds to applying the Tammi-Schiffer variation three times, this means that there is a ${}^4 f_\rho$ such that $J({}^4 f_\rho)$ covers a neighbourhood of $J(f)$ – a contradiction to the assumption that $J_2(f)$ is a boundary point of $\mathcal{T}$.

In other words, for the $q_{j1}, q_{j2}, r_{j1}, r_{j2}$ associated with the 4×2-matrix $[Q](z_3)$, we must have

$$\begin{aligned}(q_{11} + r_{11})v_1 + (q_{12} + r_{12})v_2 + (q_{21} + r_{21})v_3 + (q_{22} + r_{22})v_4 = 0 \text{ and}\\ (r_{12} - q_{12})v_1 + (q_{11} - r_{11})v_2 + (r_{22} - q_{22})v_3 + (q_{21} - r_{21})v_4 = 0,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\operatorname{Re}\left(\lambda_1(q_{11} + iq_{12}) + \lambda_2(q_{21} + iq_{22}) + \overline{\lambda_1}(r_{11} + ir_{12}) + \overline{\lambda_2}(r_{22} + ir_{22})\right) = 0 \text{ and}\\ -\operatorname{Im}\left(\lambda_1(q_{11} + iq_{12}) + \lambda_2(q_{21} + iq_{22}) + \overline{\lambda_1}(r_{11} + ir_{12}) + \overline{\lambda_2}(r_{22} + ir_{22})\right) = 0.\end{aligned}$$

Since z_3 is arbitrary, we obtain (iii) by dividing by $|\lambda_2|$.

To prove (i) and (ii), we augment $[Q](z_1, z_2)$ by the columns

$$\begin{pmatrix} [C_1] \\ [C_2] \end{pmatrix} \text{ or } \begin{pmatrix} [D_1] \\ [D_2] \end{pmatrix},$$

respectively, and keep in mind that this enhanced matrix can still only have rank 3. Thus, the additional columns need to be orthogonal to v, and this yields

$$\begin{aligned}-\operatorname{Im}(\lambda_1 C_1 + \lambda_2 C_2) &= -\operatorname{Im} i\left(\lambda_1(c_{11} + ic_{12}) + \lambda_2(c_{21} + ic_{22})\right) =\\ &= v_1 c_{11} + v_2 c_{12} + v_3 c_{21} + v_4 c_{22} = \langle v, \begin{pmatrix} [C_1] \\ [C_2] \end{pmatrix} \rangle = 0,\end{aligned}$$

as well as

$$\operatorname{Im}(\lambda_1 D_1 + \lambda_2 D_2) = 0.$$

Again, division by $|\lambda_2| \neq 0$ yields (i) and (ii).

Also in complete analogy to Hummel's case, statement (iv) needs a slightly more subtle proof since we only admit $\rho > 0$. We follow his lead and consider the 4×6-matrix

$$\begin{pmatrix} Q_1(z_1) & Q_1(z_2) & [E_1](\xi_1) & [E_1](\xi_2) \\ Q_2(z_1) & Q_2(z_2) & [E_2](\xi_1) & [E_2](\xi_2) \end{pmatrix}$$

corresponding to applying two Tammi-Schiffer variations and then two Loewner variations.

The columns of this matrix span the three-dimensional subspace of $\mathbb{R}^4$ that is orthogonal to v and the positive cone generated by the four-dimensional vectors $[E](\xi_1)$ and $[E](\xi_2)$. Note that, for $\xi \in \partial\mathbb{D}$,

$$\langle v, [E](\xi)\rangle = v_1 e_{11} + v_2 e_{12} + v_3 e21 + v_4 e_{22} = \mathrm{Re}\,(\lambda_1(e_{12} + ie_{12}) + \lambda_2(e_{21} + ie_{22}) = \hat{B}(\xi).$$

If there are ξ_1, ξ_2 such that

$$\langle v, [E](\xi_1)\rangle > 0, \text{ and } \langle v, [E](\xi_2)\rangle < 0,$$

the span of the columns of the matrix in (5.2) would cover the whole of $\mathbb{R}^4$, and we could, exactly as in Hummel's proof, introduce a new real variable that depends on the restricted ones but can take any real value, and thus construct a variation that is differentiable and invertible in a neighbourhood of the origin, which would imply that $J_2(f) \notin \partial\mathcal{T}$, a contradiction. Thus, $\hat{B}(z)$ cannot change sign on $\partial\mathbb{D}$.

For the remaining statement (v), note that, since ${}_1A_1(w)$, ${}_1A_2(w)$, ${}_2A_1(w)$, and ${}_2A_2(w)$ are by assumption linearly independent and $\lambda_2 \neq 0$, the combination

$$\hat{A} := \lambda_{11}A_1 + \overline{\lambda_{12}A_1} + \lambda_{21}A_2 + \overline{\lambda_{22}A_2}$$

is a non-constant rational function. If the interior of $\mathbb{D} \setminus f(\mathbb{D})$ were non-empty, we would find a $\tilde{w} \in (\mathbb{D} \setminus f(\mathbb{D}))^o$ such that $\hat{A}(\tilde{w}) \neq 0$ – again a contradiction to $J_2(f) \in \partial\mathcal{T}$.

Case 2: If $[Q]$ has rank 2 for all possible values of z_2, the proofs apply more or less in the same way, except that there is not one, but two linearly independent vectors $v, v' \in \mathbb{R}^4$ that are both orthogonal to the two-dimensional subspace spanned by the columns of the matrix $[Q]$ in (5.2.6). This leads to two sets λ_1, λ_2 and λ_1', λ_2' defined as above; in particular, $\lambda_2 \neq 0 \neq \lambda_2'$. It is clear that (iii) and (v) then hold for both of these.

If one (or both) of the column vectors

$$\begin{pmatrix} [C_1] \\ [C_2] \end{pmatrix} \text{ and } \begin{pmatrix} [D_1] \\ [D_2] \end{pmatrix}$$

are linearly independent of the columns of the matrix $[Q]$, the augmented matrix

$$\begin{pmatrix} [Q_1](z_1) & Q_1(z_2) & [C_1] & [D_1] \\ [Q_2](z_1) & Q_2(z_2) & [C_2] & [D_2] \end{pmatrix}$$

will have rank 3 (since its rank has to be smaller than 4 to make sure that the associated variation does not admit an inverse, see above) and there is only one of the vectors v, v' left that is orthogonal to the corresponding three-dimensional subspace. The λs associated with this vector will then, by the same arguments as above, applied to the enhanced matrix instead of $[Q]$, satisfy the remaining conditions (i), (ii), and (iv).

If

$$\begin{pmatrix} [C_1] \\ [C_2] \end{pmatrix} \text{ and } \begin{pmatrix} [D_1] \\ [D_2] \end{pmatrix}$$

both lie in the span of the columns of $[Q]$, then (i) and (ii) both hold for both sets λ_1, λ_2 and $\lambda_1^{'}, \lambda_2^{'}$, and only (iv) remains to be shown for at least one of these sets of λs . To this end, assume the contrary to be true, i.e. that there are four values $\xi_1, \xi_2, \xi_3, \xi_4 \in \partial\mathbb{D}$ such that

$$\begin{aligned} \langle v, [E](\xi_1)\rangle > 0, \qquad \langle v, [E](\xi_2)\rangle < 0, \\ \langle v, [E](\xi_3)\rangle > 0, \qquad \langle v, [E](\xi_4)\rangle > 0. \end{aligned}$$

Then the positive cone generated by the vectors $[E](\xi_1)$, $[E](\xi_2)$, $[E](\xi_3)$, and $[E](\xi_4)$ covers the two-dimensional subspace of $\mathbb{R}^4$ that is orthogonal to that generated by v and v' so that the direct sum of these two is again the whole of $\mathbb{R}^4$. The rest of the argument is completely analogous to the one in case 1. □

5.3 Applying Theorem 5.3

In our case, we have $J_1(f) = f(\zeta) - \omega$ and $J_2(f) = f'(\zeta)$. Their derivatives are $\Lambda_1(g; f) = f(\zeta)$ and $\Lambda_2(g; f) = f'(\zeta)$ for any $g, f \in S_1$.
Thus, the corresponding variational terms are

$$\begin{aligned} C_1 :=& \zeta f'(\zeta), \\ D_1 :=& \omega, \\ {}_1A_1(w_0) :=& \frac{\omega}{w_0 - \omega}, \\ {}_2A_1(w_0) :=& \frac{\omega^2}{1 - \overline{w_0}\omega}, \\ E_1(z_0) :=& \zeta f'(\zeta)\frac{\zeta + z_0}{\zeta - z_0}, \\ B_1(z_0) :=& \frac{1}{2}\left(\zeta f'(\zeta)\frac{\zeta + z_0}{\zeta - z_0} - \zeta\overline{f'(\zeta)}\frac{1 + \zeta z_0}{1 - \zeta z_0}\right) \end{aligned}$$

and

$$
\begin{aligned}
C_2 :=& \left(f'(\zeta) + \zeta f''(\zeta)\right), \\
D_2 :=& f'(\zeta), \\
{}_1A_2(w_0) :=& \frac{w_0 f'(\zeta)}{(w_0 - \omega)^2}, \\
{}_2A_2(w_0) :=& \frac{\omega f'(\zeta)(2 - \omega \overline{w_0})}{(1 - \omega \overline{w_0})^2}, \\
E_2(z_0) :=& \frac{(\zeta^2 - 2\zeta z_0 - z_0^2) f'(\zeta) + (\zeta^2 - z_0^2)\zeta f''(\zeta)}{(\zeta - z_0)^2}, \\
B_2(z_0) :=& \frac{1}{2}\left(\frac{(\zeta^2 - 2\zeta z_0 - z_0^2) f'(\zeta) + (\zeta^2 - z_0^2)\zeta f''(\zeta)}{(\zeta - z_0)^2} - \right. \\
& \left. \frac{(1 + 2\zeta z_0 - \zeta^2 z_0^2)\overline{f'(\zeta)} + (1 - \zeta^2 z_0^2)\zeta \overline{f''(\zeta)}}{(1 - \zeta z_0)^2} \right)
\end{aligned}
$$

It is clear that the ${}_jA_k$, $j,k = 1,2$, are linearly independent, and that all terms are non-constant rational functions. To make sure that all conditions of Theorem 5.3 are fulfilled, we need to check whether $0 = J_1(f)$ is an interior point of the set $\{J_1(g) \, : \, g \text{ is a Tammi-Schiffer variation of } f\}$. If this is not the case, $J_1(f)$ is a boundary point and Th. 2 in the one-dimensional form (cf. [Hum77, Th. 1]) applies: By (ii), there is a $\lambda \neq 0$ such that $\lambda\omega = \lambda D \in \mathbb{R}$, and thus, since $\omega \in (0,1)$, $\lambda \in \mathbb{R}$. Hence, from (i), or $\lambda\zeta f'(\zeta) = \lambda C \in \mathbb{R}$, we immediately obtain $f'(\zeta) \in \mathbb{R}$. From (iii), we obtain that f fulfils the differential equation

$$
\lambda \frac{\omega}{f(z) - \omega} + \overline{\lambda} \frac{\omega^2}{1 - \omega f(z)} = \frac{q(z)}{2f(z)} \left(\lambda\zeta f'(\zeta) \frac{\zeta + z}{\zeta - z} - \overline{\lambda}\zeta \overline{f'(\zeta)} \frac{1 + \zeta z}{1 - \zeta z} \right),
$$

and in the light of (ii) and (i), this is equivalent to

$$
\frac{f'^2(z)}{f(z)} \frac{\omega(1 - \omega^2)}{(\omega - f(z))(1 - \omega f(z))} = f'(\zeta) \frac{\zeta(1 - \zeta^2)}{z(\zeta - z)(1 - \zeta z)}.
$$

This differential equation cannot describe a function which maps $\mathbb{D}$ onto $\mathbb{D}$ minus a slit (since we would need to have $f'(z) = 0$ at the turning point of such a slit, but the right hand side of the equation has no zeros on $\partial\mathbb{D}$).

But, by (v), the interior of $\mathbb{D} \setminus f(\mathbb{D})$ is empty – a contradiction since, as $\omega < \zeta$, f cannot be a rotation of $\mathbb{D}$. Therefore, $J_1(f)$ has to be an interior point, and all assumptions of Th. 2 are fulfilled.

We note that we may divide the terms in Th. 5.3 (i) to (v) by $|\lambda_2| > 0$ without changing the statements. We may thus assume $\lambda_2 = e^{ix}$ with some $x \in \mathbb{R}$, and relabel $\lambda := \lambda_1/|\lambda_2|$.

By condition (i), we then have

$$\lambda\zeta f'(\zeta) + e^{ix}\left(f'(\zeta) + \zeta f''(\zeta)\right) = \lambda C_1 + e^{ix} C_2 \in \mathbb{R},$$

and by (ii), we obtain

$$\lambda\omega + e^{ix} f'(\zeta) = \lambda D_1 + e^{ix} D_2 \in \mathbb{R}.$$

Thus, condition (iii) reads

$$\frac{f'^2(z)}{f(z)} \frac{P\left(f(z)\right)}{(f(z)-\omega)^2(1-\omega f(z))^2} = -\frac{Q(z)}{z(z-\zeta)^2(1-\zeta z)^2}, \tag{5.3.1}$$

where

$$\begin{aligned} P(f(z)) = & \\ & -\omega\left(\Omega(1-\omega^2) - \overline{\Theta}\right) + \\ & + \left(\Omega(1-\omega^4) - 4\omega^2 \operatorname{Re}\left(e^{ix} f'(\zeta)\right)\right) f(z) - \\ & - \omega\left(\Omega(1-\omega^2) - \Theta\right) f^2(z) \end{aligned}$$

and

$$\begin{aligned} Q(z) = & \\ & \zeta\left(\Psi(1-\zeta^2) - \overline{\Xi}\right) - \\ & - \left(\Psi(1-\zeta^4) - 4\zeta^2 \operatorname{Re}\left(e^{ix} f'(\zeta)\right)\right) z + \\ & + \zeta\left(\Psi(1-\zeta^2) - \Xi\right) z^2 \end{aligned}$$

with the notations

$$\begin{aligned} \Xi &:= e^{-ix}\overline{f'(\zeta)} + \zeta^2 e^{ix} f'(\zeta), \\ \Theta &:= e^{-ix}\overline{f'(\zeta)} + \omega^2 e^{ix} f'(\zeta), \\ \Psi &:= \lambda\zeta f'(\zeta) + e^{ix} f'(\zeta) + e^{ix}\zeta f''(\zeta), \\ \Omega &:= \lambda\omega + e^{ix} f'(\zeta). \end{aligned}$$

Lastly, Th. 5.3, (iv) means that the unit circle is a trajectory of the quadratic differential at the right hand side of (5.3.1), which, together with condition (v), implies that f maps $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ minus a slit which is a trajectory of the left hand side of (5.3.1).
In particular, the term

$$\frac{P\left(f(z)\right)}{(f(z)-\omega)^2(1-\omega f(z))^2}$$

cannot change its sign on $\partial\mathbb{D}$, which means that P has to have an even number of zeros on $f(\partial\mathbb{D})$.
The polynomial Q can be written as

$$Q(z) = \zeta\left(\Psi(1-\zeta^2) - \Xi\right)(z-b_1)(z-b_2),$$

where

$$b_1 b_2 = \frac{\Psi(1-\zeta^2) - \overline{\Xi}}{\Psi(1-\zeta^2) - \Xi}$$

and

$$-b_1 - b_2 = -\frac{\Psi(1-\zeta^4) - 4\zeta^2 \operatorname{Re} e^{ix} f'(\zeta)}{\zeta(\Psi(1-\zeta^2) - \Xi)}.$$

Since Q must have at least one zero on $\partial\mathbb{D}$, but cannot change its sign, we must hence have one double zero $b := b_1 = b_2 \in \partial\mathbb{D}$, and we can solve for Ψ and obtain

$$\Psi := \pm\frac{2\zeta|f'(\zeta)|}{1-\zeta^2}.$$

We write

$$P(w) = -\omega(\Omega(1-\omega^2) - \Theta)(w-a_1)(w-a_2).$$

Since P must also have an even number of zeros on $f(\partial\mathbb{D})$, and has to have at least one zero, we conclude that $a_1 = a_2 =: a \in \partial\mathbb{D}$.
In the same way as above, we obtain

$$\Omega := \pm\frac{2\omega|f'(\zeta)|}{1-\omega^2}.$$

With the notation

$$\chi := \arg e^{ix} f'(\zeta)$$

we can then compute that a has to take one of the values

$$a_- := \frac{e^{i\chi} + \omega}{1 + \omega e^{i\chi}} \text{ or}$$
$$a_+ := -\frac{e^{i\chi} - \omega}{1 - \omega e^{i\chi}},$$

and b one of the values

$$b_- := \frac{e^{i\chi} + \zeta}{1 + \zeta e^{i\chi}},$$
$$b_+ := -\frac{e^{i\chi} - \zeta}{1 - \zeta e^{i\chi}}.$$

It is hence clear that f has to be a one-slit function where the point a corresponds to the starting point of the slit on $\partial\mathbb{D}$, and b corresponds to the pre-image of its tip.
Thus, (5.3.1) reads

$$\frac{f'^2(z)}{f(z)}\frac{\omega\left(\omega - f(z) \pm e^{i\chi}\left(1-\omega f(z)\right)\right)^2}{(\omega - f(z))^2(1-\omega f(z))^2} = \frac{\zeta\left(\zeta - z \pm e^{i\chi}\left(1-\zeta z\right)\right)^2}{z(\zeta - z)^2(1-\zeta z)^2}, \tag{5.3.2}$$

where the signs on both sides are a priori independent of each other.

Since

$$\frac{d}{dz}\left(\pm 2\left(e^{i\chi}\operatorname{arctanh}\frac{\sqrt{z}}{\sqrt{\zeta}} \pm \operatorname{arctanh}\left(\sqrt{z}\sqrt{\zeta}\right)\right)\right) = \pm\frac{\sqrt{\zeta}(\zeta - z \pm e^{i\chi}(1-\zeta z))}{\sqrt{z}(\zeta - z)(1-\zeta z)},$$

where any combination of signs is possible, we can solve (5.3.2) by separation of variables: Let

$$F_\zeta^{\pm\pm}(z) := \pm 2\left(e^{i\chi}\operatorname{arctanh}\frac{\sqrt{z}}{\sqrt{\zeta}} \pm \operatorname{arctanh}\left(\sqrt{z}\sqrt{\zeta}\right)\right),$$

and

$$F_\omega^{\pm\pm}(z) := \pm 2\left(e^{i\chi}\operatorname{arctanh}\frac{\sqrt{z}}{\sqrt{\omega}} \pm \operatorname{arctanh}\left(\sqrt{z}\sqrt{\omega}\right)\right).$$

Then $F_\zeta^{\pm\pm}$ and $F_\omega^{\pm\pm}$ are analytic on $\mathbb{D}\setminus[0,1)$, and $F_\zeta^{\pm\pm}(0) = F_\omega^{\pm\pm}(0) = 0$. We use the principal branch of all multi-valued functions involved.
The extremal function f thus has to be a solution to

$$F_\omega^{\pm\pm}(f(z)) = F_\zeta^{\pm\pm}(z) \text{ for all } z \in \mathbb{D}\setminus\{\zeta\},$$

and it is easy to see that, in fact, the equation can be extended to $z = \zeta$ as well, as long as the first signs of $F_\zeta^{\pm\pm}$ and $F_\omega^{\pm\pm}$ coincide. This concludes the proof of Th. 5.1.

Appendix: the Tammi-Schiffer equation

We derive the Tammi-Schiffer variation by following Golusin's method [Gol46] as outlined in [Pom75, ch. 7].
For $w \in \mathbb{D}$, we consider the basic variation

$$w_\rho = w + \frac{\rho a_0 w}{w_0 - w} + \frac{\rho \overline{a_0} w^2}{1 - \overline{w_0} w}$$

with $w_0 \in \mathbb{D}$, $a_0 \in \partial\mathbb{D}$.
Obviously,

$$\operatorname{Re}\left(\frac{a_0}{w_0 - w} + \frac{\overline{a_0} w}{1 - \overline{w_0} w}\right) = 0$$

holds for $w \in \partial\mathbb{D}$, and therefore, as in [Tam78], we have

$$\begin{aligned} |w_\rho| &= |w| \left| 1 + \rho \left(\frac{a_0}{w_0 - w} + \frac{\overline{a_0} w}{1 - \overline{w_0} w}\right)\right| = \\ &= |w| \sqrt{1 + 2\rho \operatorname{Re}\left(\frac{a_0}{w_0 - w} + \frac{\overline{a_0} w}{1 - \overline{w_0} w}\right) + \rho^2 \left|\frac{a_0}{w_0 - w} + \frac{\overline{a_0} w}{1 - \overline{w_0} w}\right|^2} \\ &= |w| \left(1 + O(\rho^2)\right). \end{aligned}$$

It is hence clear that the variation preserves boundedness (as well as analyticity and standardisation at $z = 0$). Besides, the function $w \mapsto \frac{a_0 w}{w_0 - w} + \frac{\overline{a_0} w^2}{1 - \overline{w_0} w}$ is meromorphic in $\overline{\mathbb{D}}$ with a simple pole at $w = w_0$, so [Pom75, Lemma 7.3] states that this generates a univalent variation family.
We construct this family by assuming $w_0 \in f(\mathbb{D})$ and applying the procedure outlined in [Pom75, ch. 7] to

$$g_\rho(z) = f(z) + \rho \left(\frac{a_0 f(z)}{f(z_0) - f(z)} + \frac{\overline{a_0} f^2(z)}{1 - \overline{f(z_0)} f(z)}\right),$$

where $z_0 \in \mathbb{D}$ is the unique point with $f(z_0) = w_0$.
We obtain

$$\frac{d}{d\rho} g_\rho(z)\big|_{\rho=0} = \frac{a_0 f(z)}{f(z_0) - f(z)} + \frac{\overline{a_0} f^2(z)}{1 - \overline{f(z_0)} f(z)} := z f'(z) h(z)$$

and calculate for $z_0 \neq 0$

$$\operatorname{res}\left(h(z); z_0\right) = \operatorname{res}\left(a_0 \frac{f(z)}{z f'(z)} \frac{1}{f(z_0) - f(z)}; z_0\right) = -a_0 \frac{f(z_0)}{z_0 f'^2(z_0)} = a_0 q(z_0) \frac{z_0}{f(z_0)}$$

with

$$q(z_0) := -\frac{f^2(z_0)}{z_0^2 f'^2(z_0)}.$$

Note that

$$\lim_{z_0 \to 0} q(z_0) = -1,$$

and that for $z_0 = 0$, we also have $w_0 = 0$, and thus

$$\operatorname{res}(h(z); 0) = \operatorname{res}\left(a_0 \frac{f(z)}{zf'(z)} \frac{1}{(-f(z))}; 0\right) = -\frac{a_0}{f'(0)} = \lim_{z_0 \to 0} a_0 q(z_0) \frac{z_0}{f(z_0)}.$$

We can thus drop the assumption that $z_0 \neq 0$.
For $|z| > |z_0|$, we have

$$\frac{a_0}{f(z_0)} q(z_0) z_0 \frac{1}{z - z_0} = \frac{a_0}{f(z_0)} q(z_0) \sum_{k=1}^{\infty} \left(\frac{z_0}{z}\right)^k.$$

Thus, for $|z_0| < |z| < 1$,

$$h(z) = \frac{a_0}{f(z_0)} q(z_0) \sum_{k=1}^{\infty} \left(\frac{z_0}{z}\right)^k + \frac{a_0}{f(z_0)} q(z_0) + h(0) + \sum_{k=1}^{\infty} c_k z^k \text{ with some } c_k \in \mathbb{C}.$$

Note that

$$h(0) = \frac{a_0}{f(z_0)}.$$

This yields, according to [Pom75, Th. 7.3], the bounded variation family

$$\begin{aligned} f_\rho(z) = f(z) + \rho z f'(z) \Bigg(h(z) - \frac{a_0}{f(z_0)} q(z_0) z_0 \frac{1}{z - z_0} + \overline{\frac{a_0}{f(z_0)} q(z_0)} \frac{\overline{z_0} z}{1 - \overline{z_0} z} - \\ i \operatorname{Im}\left(\frac{a_0}{f(z_0)} q(z_0) + \frac{a_0}{f(z_0)}\right)\Bigg) + O(\rho^2), \end{aligned}$$

or, equivalently,

$$f_\rho(z) = f(z) + \rho\left(a_0 T_0(z) + \overline{a_0} U_0(z)\right) + O(\rho^2),$$

where $a_0 \in \partial\mathbb{D}$, $z_0 \in \mathbb{D}$, and

$$\begin{aligned} T_0(z) &:= \frac{f(z)}{f(z_0) - f(z)} - \frac{q(z_0)}{2f(z_0)} \frac{z + z_0}{z - z_0} z f'(z) - \frac{zf'(z)}{2f(z_0)}, \\ U_0(z) &:= \frac{f^2(z)}{1 - \overline{f(z_0)} f(z)} + \overline{\left(\frac{q(z_0)}{2f(z_0)}\right)} \frac{1 + \overline{z_0} z}{1 - \overline{z_0} z} z f'(z) + \frac{zf'(z)}{2f(z_0)}. \end{aligned}$$

Bibliography

[ABC10] M. Abate, F. Bracci, M. D. Contreras, and S. Díaz-Madrigal, *The evolution of Loewner's differential equations*, Eur. Math. Soc. Newsl., **78, 31-38** (2010), 1–10.

[Bie16] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. 1916, 940–955.

[Boa10] H. P. Boas, *Julius and Julia: Mastering the Art of the Schwarz Lemma*, The American Mathematical Monthly **117(9)** (2010), 770–785.

[BC14] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, and A. Vasil'ev, *Classical and stochastic Löwner-Kufarev equations*, in: A. Vasil'ev, *Harmonic and Complex Analysis and Applications*, Birkhäuser-Verlag, 2014, 39 –134.

[DeB85] L. De Branges, *A proof of the Bieberbach conjecture*, Acta Mathematica, **154(1)** (1985), 137–152.

[Die31] J. Dieudonné, *Sur les fonctions univalentes*, C. R. Acad. Sci. Paris **192** (1921), 1148–1150.

[Dur83] P. L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[FP85] C. H. FitzGerald and C. Pommerenke, *The de Branges theorem on univalent functions*, Trans. Amer. Math. Soc., **290(2)** (1985), 683–690.

[GS55] P. R. Garabedian and M. M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal. **4** (1955), 427–465.

[Gol36] G. M. Goluzin, *Sur les théorèmes de rotation dans la théorie des fonctions univalentes*, Mat. Sbornik NS, **1**(1963), 293–296.

[Gol46] G. M. Goluzin, *Variational method in conformal mapping I* (Russian), Mat. Sbornik NS **19** (1946), 203–236.

[Gor86] V. V. Goryainov, *Semigroups of conformal mappings*, Mat. Sb. (N.S.), **129(171):4** (1986), 451– 472.

[Gor15] V. V. Goryainov, *Evolution families of conformal mappings with fixed points and the Löwner-Kufarev equation*, Mat. Sb., **206:1** (2015), 39 – 68.

[GB92] V. V. Goryainov and I. Ba, *Semigroup of conformal mappings of the upper half-plane into itself with hydrodynamic normalization at infinity*, Ukraine. Mat. Zh. **44** (1992), no. 10, 1320–1329.

[GG76] V. V. Gorjaĭnov and V. Ja. Gutlyanskiĭ, *Extremal problems in the class* S_M, Mathematics collection (Russian), Izdat. "Naukova Dumka", Kiev, 1976, pp. 242–246.

[Gra50] A. Grad, A. *The region of values of the derivative of a schlicht function*, Proceedings of the National Academy of Sciences, **36(3)**(1950), 198–202.

[Gro14] T. H. Gronwall, *Some remarks on conformal representation*, Ann.of Math. **16** (1914–1915), 72–76.

[Grö30] H. Grötzsch, *Über ein Variationsproblem der konformen Abbildung*, Leipziger Berichte, **82** (1930), 251–263.

[Gru39] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45**(1939), 29–61.

[Hum77] J. A. Hummel, *Lagrange Multipliers in Variational Methods for Univalent Functions*, Journal d'Analyse Mathématique **32.1** (1977), 222–234.

[Jul18] G. Julia, *Mémoire sur l'itération des fonctions rationnelles*, J. Math. Pures Appl. (8) **1**(1918), 47–245.

[Jul20] G. Julia, *Extension nouvelle d'un lemme de Schwarz*, Acta Math. **42**(1920), 349–355.

[KS16] J. Koch and S. Schleißinger, *Value ranges of univalent self-mappings of the unit disc*, J. Math. Anal. Appl. **433** (2016), no. 2, 1772–1789.

[KS17] J. Koch and S. Schleißinger, *Three Value Ranges for Symmetric Self-mappings*, Proc. Amer. Math. Soc. **145** (2017), 1747–1761.

[Koe94] W. Koepf, *Von der Bieberbachschen Vermutung zum Satz von de Branges sowie der Beweisvariante von Weinstein*, Jahrbuch Überblicke Mathematik 1994, Vieweg, Braunschweig/Wiesbaden, 175–193.

[Kuf43] P. P. Kufarev, *On one-parameter families of analytic functions*, Rec. Math.[Mat. Sbornik] N.S. **13(55)** (1943), 87–118.

[Kuf47] P. P. Kufarev, *Theorem on solutions to some differential equation*,Tomsk. Gos. Univ. Učenye Zapiski, **5**(1947), 20–21.

[KSS68] P. P. Kufarev, V. V. Sobolev, and L. V. Sporysheva, *A certain method of investigation of extremal problems for functions that are univalent in the half-plane*, Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. **200** (1968), 142–164.

[LM86] E. B. Lee and L. Markus, *Foundations of optimal control theory*, second ed., Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1986.

[Lew06] A. Lewis, *The Maximum Principle of Pontryagin in control and in optimal control*, Lecture notes from a course held in the Department of Applied Mathematics in the Polytechnic University of Catalonia, May 2006, http://www.mast.queensu.ca/˜andrew/teaching/pdf/maximum-principle.pdf (access date: 12 October 2016)

[Löw17] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden*, Verh. Sächs. Ges. Wiss. Leipzig **69** (1917), 89–106.

[Löw23] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I*, Mathematische Annalen **89** (1923), no. 1, 103–121.

[MR05] D. E. Marshall and S. Rohde, *The Loewner differential equation and slit mappings*, J. Amer. Math. Soc. **18** (2005), no. 4, 763–778.

[Mar34] F. Marty, *Sur le module des coefficients de MacLaurin d'une fonction univalente*, CR Acad. Sci. Paris **198**(1934), 1569–1571.

[Mil71] I. M. Milin, *Univalent functions and orthonormal systems*, Moscow, 1971; English transl. Transi. Math. Monos., **50** (1977), Amer. Math. Soc, Providence, R.I., 1977

[Nev20] R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Öfvres. Finska Vet. Soc. Förh **53**(1920–1921), 1–21.

[PP09] H. J. Pesch, M. Plail, and D. Munich, *The maximum principle of optimal control: a history of ingenious ideas and missed opportunities,* Control and Cybernetics, **38(4A)** (2009), 973-995.

[Pom75] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, 1975.

[Pon78] L. S. Pontryagin, *A Short Autobiography of L. S. Pontryagin*, (in Russian), Uspekhi Matematicheskikh Nauk 33, 7–21. Russian Mathematical Surveys **33** (1978), 7–24.

[Pfr16] D. Pfrang, *Der Wertebereich typisch-reeller schlichter beschränkter Funktionen*, Master thesis, University of Würzburg, 2016.

[Pro92] D. V. Prokhorov, *The set of values of bounded univalent functions with real coefficients*, Theory of functions and approximations, Part 1 (Russian) (Saratov, 1990), Izdat. Saratov. Univ. (1992), 56–60.

[Pro02] D. V. Prokhorov, *Bounded univalent functions*, in: R. Kühnau, *Handbook of Complex Analysis, 1*, Elsevier, Amsterdam, 2002, 207–228.

[PS15] D. Prokhorov and K. Samsonova, *Value range of solutions to the chordal Loewner equation*, J. Math. Anal. Appl. **428** (2015), no. 2, 910–919.

[PS16] D. V. Prokhorov and K. Samsonova, *A Description Method in the Value Region Problem*, Complex Analysis and Operator Theory (2016).

[PV06] D. Prokhorov, and A. Vasil'ev, *Univalent Functions and Integrable Systems*, Comm. Math.Phys. 262 (2006), pp. 393-410.

[Rea55] M. O. Reade, *On close-to-convex univalent functions*, Mich. Math. J. **3** (1955), 59–62.

[Rob35] M. S. Robertson, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc., Volume 41, **8** (1935), 565–572.

[Rog32] W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen.*, Math. Z. **35** (1932), 93–121.

[Rog34] W. Rogosinski, *Zum Schwarzschen Lemma.*, Jahresber. Dtsch. Math.-Ver. **44** (1934), 258–261.

[Rot98] O. Roth, *Control Theory in* $H(\mathbb{D})$, Ph.D. thesis, University of Würzburg, 1998.

[Rot99] O. Roth, *A remark on the Loewner differential equation*, series in approximations and decompositions, **11**(1999), 461–470.

[RS14] O. Roth and S. Schleißinger, *Rogosinski's lemma for univalent functions, hyperbolic Archimedean spirals and the Loewner equation*, Bull. Lond. Math. Soc. **46** (2014), no. 5, 1099–1109.

[SS50] A. C. Schaeffer and D.C. Spencer, *Coefficient Regions for Schlicht Functions*, Colloquium publications **35**, American Mathematical Society, 1950.

[Sch38] M. M. Schiffer, *A method of variation within the family of simple functions*, Proc. London Math. Soc. (2) **44** (1928), 432–449.

[Sch39] M. M. Schiffer, *Sur la variation de la fonction de Green de domaines plans quelconques*, C.R.Acad. Sci. Paris **209**(1939), 980–982.

[Sch43] M. M. Schiffer, *Variation of the Green function and theory of the p-valued functions*, Amer. J. Math. **65**(1943), 341–360.

[Sch58] M. M. Schiffer, *Extremum problems and variational methods in conformal mapping*, Proc. International Congress of Mathematicians, Edinburgh 1958, 211–231.

[ST69] M. Schiffer and O. Tammi, *On the Coefficient Problem for Bounded Univalent Functions*, Transactions of the American Mathematical Society **140**(1969), 461–474.

[Sch00] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288.

[Sin57] V. Singh, *Interior Variations and Some Extremal Problems for Certain Classes of Univalent Functions*, Pacific J. Math. Volume 7, **3** (1957), 1485–1504.

[SS82] M. Szapiel and W. Szapiel, *Extreme points of convex sets. IV. Bounded typically real functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), no. 1-2, 49–57.

[Tam78] O. Tammi, *Extremum Problems for Bounded Univalent Functions*, Springer, Berlin-New York, 1978.

[Tei38] O. Teichmüller, *Ungleichungen zwischen den Koeffizienten schlichter Funktionen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften (1938), 363–375.

[Win88] G. Winkler, *Extreme points of moment sets*, Math. Oper. Res. **13** (1988), no. 4, 581–587.