# Robust Stability of Differential Equations with Maximum

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# Contents

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# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of natural numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}_+$ | the set of nonnegative real numbers |
| $\mathbb{R}^n$ | the set of $n$-dimensional vectors |
| $x^\top$ | transposition of a vector $x \in \mathbb{R}^n$ |
| $\|x\|$ | $:= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$ |
| $C(J;\mathbb{R}^n)$ | the set of continuous functions $f : J \to \mathbb{R}^n$ defined on the open set $J \subseteq \mathbb{R}$ |
| $\|\|f\|\|_J$ | $:= \sup\limits_{t\in J} \|f(t)\|$, norm of $f : J \to \mathbb{R}$ (sometimes, we write $\|\|f\|\|$ if the set $J$ is clear from the context) |
| $PC(J;\mathbb{R}^n)$ | the set of piecewise continuous (right continuous) functions $f : J \to \mathbb{R}^n$ defined on the open set $J \subseteq \mathbb{R}$ |
| $AC(J;\mathbb{R}^n)$ | the set of absolutely continuous functions $f : J \to \mathbb{R}^n$ defined on the open set $J$ |
| $L_\infty(J;\mathbb{R}^n)$ | the set of essentially bounded measurable functions $f : J \to \mathbb{R}^n$ defined on the open set $J \subseteq \mathbb{R}$ |
| $\|\|A\|\|$ | $:= \max\limits_{i=1,\cdots,n} \sum\limits_{j=1}^{n} \|a_{i,j}\|$, matrix norm of $A \in \mathbb{R}^{n\times n}$ |
| $\mathcal{P}$ | $:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous},\ \gamma(0) = 0,\ \text{and } \gamma(s) > 0 \text{ for } s > 0\}$ |
| $\mathcal{K}$ | $:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\}$ |
| $\mathcal{K}_\infty$ | $:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}$ |
| $\mathcal{L}$ | $:= \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim\limits_{t\to\infty} \gamma(t) = 0\right\}$ |
| $\mathcal{KL}$ | $:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta(\cdot, t) \in \mathcal{K}\ \forall t \geq 0,\ \beta(s, \cdot) \in \mathcal{L}\ \forall s > 0\}$ |

For any $x, y \in \mathbb{R}^n$ we define $x \geq y :\Leftrightarrow\ x_i \geq y_i, \quad \text{for all } i = 1, \cdots, n.$

# List of Abbreviations

| | |
|---|---|
| **ODE** | **O**rdinary **D**ifferential **E**quation |
| **FDE** | **F**unctional **D**ifferential **E**quation |
| **RFDE** | **R**etarded **F**unctional **D**ifferential **E**quation |
| **SD-DDE** | **S**tate **D**ependent-**D**elay **D**ifferential **E**quation |
| **DDE** | **D**elay **D**ifferential **E**quation |
| **GAS** | **G**lobally **A**symptotically **S**table |
| **0-GAS** | zero **G**lobally **A**symptotically **S**table |
| **GES** | **G**lobally **E**xponentially **S**table |
| **0-GES** | zero **G**lobally **E**xponentially **S**table |
| **(e)($\delta$)ISS** | (**e**xponentially) (**i**ncrementally) **I**nput-to-**S**tate **S**table |
| **(e)ISpS** | (**e**xponentially) **I**nput-to-**S**tate **p**ractical **S**table |
| **KBM** | **K**rylov, **B**ogolyubov, **M**itropolskiy |

# Introduction

In many applications from areas such as opinion dynamics [68], biosciences [34, 42, 82], economics [6, 59, Chapter 2], ecology [46], one assumes, that rate of change of the system's state at current time $t$ is determined not only by the present state, but also by the past state of the system. Therefore one deals with model with time-delays. General class of such problems is called *retarded functional differential equations (RFDE)* and they have the form

$$\dot{x}(t) = F(t, x_t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $h \in \mathbb{R}_+$, and $x_t \in C([-h, 0]; \mathbb{R}^n)$ is defined by

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-h, 0].$$

By $\dot{x}(t)$ we mean the right-hand derivative of $x$ at $t$.
Equation (1) is a general type of differential equation which includes the following classes of equations.

- *Ordinary differential equations (ODE)* if

$$F(t, x_t) := f(t, x(t)), \quad t \geq 0.$$

- *Functional integro-differential equations*, for example if

$$F(t, x_t) := \int_{-h}^{0} g(t, \theta, x(t + \theta)) d\theta, \quad t \geq 0.$$

- *Pantograph equation* if

$$F(t, x_t) := ax(t) + bx(\lambda t), \quad t \geq 0,$$

  where $0 < \lambda < 1$. It describes the mechanical properties of a current collection of a locomotive [75, 55]. And also arises in application to bioscience [42].

- *State dependent-delay differential equations (SD-DDE)*

$$F(t, x_t) := f(t, x(t - h(t, x))), \quad h : [0, \infty) \times [0, \infty) \to \mathbb{R}_+.$$

  Sometimes it also called *autoregulative functional differential equations.*
  Particular cases of SD-DDE are

– *delay differential equations (DDE) with constant delay* if

$$F(t, x_t) := f(t, x(t-h)), \quad h \in \mathbb{R}_+.$$

Note that sometimes such equation is also called *differential difference equation* [38, p.37];

– *delay differential equations with time varying delay*

$$F(t, x_t) := f(t, x(t-h(t))), \quad h : [0, \infty) \to \mathbb{R}_+;$$

– *differential equations with maximum*

$$F(t, x_t) := f(t, \max_{s\in[t-h,t]} x(s)), \quad t \geq 0.$$

Function $h(\cdot)$ is called *delay function* or *lag.*

The systematic study of RFDE started at the 1950s with contributions from Myshkis's work [20] (see also its translation into German [72]) in USSR and from works of Bellman and Danskin [16] and Bellman and Cooke [15] in the USA. We mentioned about the monography of J.Hale [38] and its further edition with S.Verduyn Lunel [40], which present the basic theory of equation (1). Up to now, this is the most cited book about RFDE.

## Differential equations with maximum

In this thesis we focus on SD-DDE perturbed by an input, where the dynamics depends on the maximum of the solution taken over a past time interval

$$\dot{x}(t) = f(t, x(t), \max_{s\in[t-h,t]} x(s), u(t)), \qquad t \in [0, \infty), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $h > 0$ and

$$\max_{s\in[t-h,t]} x(s) = \Big( \max_{s\in[t-h,t]} x_1(s), \cdots, \max_{s\in[t-h,t]} x_n(s) \Big)^\top,$$

and the input function $u \in L_\infty([0, \infty); \mathbb{R}^m)$, $m \in \mathbb{N}$. With initial condition

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

where $\varphi \in C([-h, 0]; \mathbb{R}^n)$ is the initial function. The state-space of the dynamical system (2) is the Banach space $C([-h, 0]; \mathbb{R}^n)$ therefore (2) is an infinite-dimensional dynamical system. Such kind of RFDE is called *differential equation with maximum* [11]. Observe that the max-operator is nonlinear. Indeed, for any $h > 0$

$$\sqrt{2} = \max_{s\in[-h,0]} \Big( \sin\frac{2\pi t}{h} + \cos\frac{2\pi t}{h} \Big) \neq \max_{s\in[-h,0]} \sin\frac{2\pi t}{h} + \max_{s\in[-h,0]} \cos\frac{2\pi t}{h} = 2,$$

therefore if $f$ is a linear function, the equation (2) does not possess properties of linear equation i.e., it may happen that the sum of two solutions of the equation (2) is not a solution to (2). We observe in Chapter 2 that system (2) is a system of state dependent, piecewise continuous delay differential equations. Qualitative theory and some methods for analysis of unperturbed differential equations with maximum are presented in [11].

Systems of type (2) appear in modeling of various practical processes.

**Financial market model.** Stochastic model of financial market, proposed in [6], is based on three main principles:

- comparison current price of stock with reference level;
- comparison local maximum of over past time interval with the current price;
- trading on latest news (which is treated as a noise in a model).

Let $r(t)$ be an instantaneous return at time $t$ of the stock price $S(t)$ at time $t$ then

$$S(t) = S(0)e^{\int_0^t r(u)du}, \quad t \geq 0.$$

It is assumed that market consists of $M_1$ *reference* and $M_2$ *technical level traders*. Furthermore, it is supposed that none of the traders change their trading strategies and they are endowed with infinite lives.

*Reference traders* act according to the comparison current return with some reference level $r_l \in \mathbb{R}$, $l = 1, \cdots, M_1$, $M_1 \in \mathbb{N}$. In particular, if return is higher (lower) then $r_l$ (level $\tau_l$, in general, different for different traders) this is a signal of future increasing (decreasing) return and therefore traders buy (sell) stocks. The planned instantaneous excess demand of all reference traders over time interval $(t, t + dt)$ is $\sum_{i=1}^{M_1} \alpha_l(r(t) - r_l)dt$, where $\alpha_l \in \mathbb{R}$, $l = 1, \cdots, M_1$. The sign of $\alpha_l$ depends on strategy of $l$-th trader.

*Technical traders* possess memory. They compare maximum instantaneous return over the past $h > 0$ time periods with some tolerant level $\tau_j \in \mathbb{R}$, $j = 1, \cdots, M_2$, $M_2 \in \mathbb{N}$. In particular, if maximum of instantaneous return is lower then $\tau_j$ then the stock's price will increase. In this case the planned instantaneous excess demand of all technical traders over the time interval $(t, t+dt)$ is $\sum_{i=1}^{M_2} \beta_j(\max_{s\in[t-h,t]} r(s) - r(t) - \tau_j)dt$, where $\beta_j > 0$, $j = 1, \cdots, M_2$. Which means, that if $(\max_{s\in[t-h,t]} r(s) - r(t) - \tau_j)dt$ is positive (negative) then traders buy (sell) stocks.

Both groups of traders react on the news which is independent of the past return. Let $B$ be one-dimensional Brownian motion, $\xi_i > 0$. Therefore the stocks' price is given by

$$dr(t) = \left( \sum_{l=1}^{M_1} \alpha_l(r(t) - r_l) + \sum_{l=1}^{M_1} \beta_j(\max_{s\in[t-h,t]} r(s) - r(t) - \tau_j) \right) dt + \xi dB(t), \quad t \geq 0.$$

The discrete model of financial market with maximum is discussed in details in [88, Chapter 5].

**Bioscience.** Differential equations with maximum are used to describe the vision process in the compound eye of a horseshoe crab [34], where the presence of max in a model is required by biological features of such animals. Chemostat model, which involves maximum and minimum of the state is proposed in [52] (see Example 1.2.2, pp. 7-8). Application of difference equations with maximum to medicine are available in [19].

**Photovoltaic model.** In [9, 8] an application of equations with supremum to the maximum power point tracking control of solar energy plant is discussed.

Further applications are available in [2, p.166] and [81]. Moreover, scalar differential inequality with maximum in the linear form, well known as a Halanay inequality, is a helpful tool for the stability analysis of systems with delays [36, §4.5], [18, 66], to name just a few.

Observe that differential equations with maximum possess different properties than constant DDE.

**Example 1.** *Compare*

$$\dot{x}(t) = \max_{s\in[t-h,t]} x(s), \quad t \geq 0, \ h > 0, \tag{3}$$

*with*

$$\dot{x}(t) = x(t-h), \quad t \geq 0, \ h > 0. \tag{4}$$

*For any nondecreasing initial condition, maximum in equation* (3) *is attained in the right end of the interval* $[t-h, t]$ *for all* $t \in [0, \infty)$ *therefore equation* (3) *reduces to the ODE* $\dot{x}(t) = x(t)$ *for all* $t \in [0, \infty)$*. However, equation* (4) *cannot be reduced to an ODE for any initial condition and for any* $h > 0$*.*

The next example illustrates the influence of maximum presence on stability property.

**Example 2.** *Consider the following delay differential equation*

$$\dot{y}(t) = -y(t-2), \quad t \geq 0, \tag{5}$$

*where* $y(t) \in \mathbb{R}$*. With the initial condition*

$$\varphi(t) = 1, \ t \in [-2, 0]. \tag{6}$$

*Its solution is given by (see [72, p.8], [20, p.21])*

$$y(t) = \sum_{j=0}^{n} \frac{(-1)^j [t-2(j-1)]^j}{j!}, \tag{7}$$

*for* $2(n-1) \le t \le 2n; \quad n = 1, 2, \cdots$. *Using (7) one can write down the solution to the problem (5),(6) on* $[0, 6]$, *that is*

$$y(t) = \begin{cases} 1 - t, & t \in [0, 2], \\ \frac{1}{2}(t-2)^2 - t + 1, & t \in [2, 4], \\ -\frac{1}{6}(t-4)^3 + \frac{1}{2}(t-2)^2 - t + 1, & t \in [4, 6]. \end{cases}$$

*Now consider differential equation with maximum*

$$\dot{x}(t) = - \max_{s \in [t-2, t]} x(s), \quad t \ge 0, \tag{8}$$

*where* $x(t) \in \mathbb{R}$. *With the same initial condition (6). On* $[0, t^*]$ *solutions to problems (5),(6) and (8),(6) coincide, where* $t^* = 6 - 2\sqrt{\frac{2}{3}}$ *satisfies* $\max_{s \in [t^*-2, t^*]} x(s) = x(t^*)$ *and* $x(t^* - 2) = x(t^*)$. *Then for all* $t \in [t^*, \infty)$ *the solution to problem (8),(6) coincides with the solution to the problem of the ODE* $\dot{x}(t) = -x(t)$ *with* $x(t^*) = x_0 \in (-\infty, 0)$. *Observe, that the solution to equation (5) oscillates while for the same value of delay* $h = 2$ *the trajectory of equation (8) does not and, moreover, approaches zero (see Figure 1) (in section 3.4, it is proved that it approach zero as an exponential function). These examples illustrate that behavior of the equations with constant delays and the ones of equations with maximum are different. So, the independent study of stability properties of the equations with maximum is absolutely necessarily.*

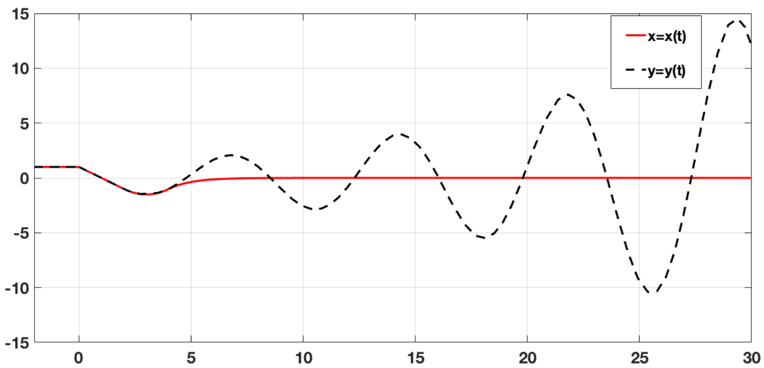


FIGURE 1: *Graphs of solutions to the problem with delay (5),(6) and to the problem with maximum (8),(6).*

Stability theory of differential equations with maximum with usage of Razumikhin function is given in [11, Chapter 4]. Where authors proposed the extension of Razumikhin method, in order to generalize considerations, two different measures for initial functions and for solutions are used and fundamental results for different types of stability are obtained.

Some earliest results dedicated to the stability of equations with maximum by trajectory approach are available in [93, 10]. Where authors compare asymptotic stability of linear constant DDE and differential equations with maximum in a linear form, and proved that in spite of the complicated structure, equations with maximum possess better stability property then constant DDE. This result is recalled in the section 3.6. of this thesis. We refer to [7] where convergence of solution to differential equations with max-operator in a linear form is studied and to [48, 13, 79] for some further stability results without usage of any methods of Lyapunov function.

Very little is known about systems with maximum perturbed by an external input. For instance, observation and stabilization problems for systems involving supremum are studied in [3]. Optimal control associated with class of system with supremum in the linear form is considered in [9] where for such problem the Pontryagin-like Minimum Principle is proved. See also [91] for advances in optimal control problem of differential equations with supremum. Observe that stability of systems with maximum with respect to perturbing signal has not been studied yet.

## Input-to-state stability

Robust stability of infinite-dimensional dynamical systems plays important role in applications. The input-to-state stability (ISS) framework, originally introduced for finite-dimensional systems in [83], describes global stability properties of an equilibrium in case of perturbations and seems to be promising for infinite-dimensional systems as well. This property is invariant under nonlinear coordinate transformations and extends the classical global asymptotic stability of an equilibrium to the case of systems that have external perturbing signals. If an unperturbed system possesses a global asymptotically stable equilibrium point (which is by definition invariant and attracting set) then, in case of perturbation, the ISS property of the system means that instead of the equilibrium point there is an invariant and globally attracting domain. The size of this domain is a nonlinear function of the norm of input function. This framework was very successful in studying nonlinear interconnected systems and construction of ISS-Lyapunov functions for them [24].

During the last five years the ISS framework was actively developed for infinite-dimensional systems [69, 23, 71, 70, 53]. The reason of these activities is the success of this framework in many applications to finite-dimensional case. There are some works devoted to ISS property of systems with delays, where the Lyapunov-Krasovskiy functional [78] or the Razumikhin function [89] is applied. However, none of them provides explicit expressions for the ISS gain functions. Note that for applying Lyapunov method one should guess a Lyapunov function, which is, sometimes, matter not only of experience, but also of luck. The alternative way can be a trajectory approach, which is, to the best of author's knowledge, is not available for the ISS analysis of SD-DDE systems. In Chapter 3 we study ISS of dynamical system governed by differential equations with maximum and derive an explicit expression of the ISS estimate. Neither Lyapunov-Krasovskiy technique nor Razumikhin technique is used for this purpose.

## Approximation methods

Differential equations with maximum in the linear form do not possess properties of a linear equations due to nonlinearity of the max-operator. Analytic solutions of such equations can hardly be obtained, therefore we have to address to approximations methods. In [44, 28, 45] approximation methods for differential equations with maximum, which are based on the method of successive approximation, are proposed. The most general of these results is given in [44], where authors study behavior of lower and upper solutions with different initial time intervals and applied Euler method for the suggested scheme to calculate numerical solution. See also [1] for method of successive approximation for difference equations with maximum. To the best of author's knowledge there are no ready-made functions in computational systems, like for example Matlab, to solve numerically differential equations with maximum. There exist a lot of works devoted to methods of numerical approximations of SD-DDE [14, 54, 29, 33, 26, 47, 43, Chapter 9]), just named a few. Nevertheless, cited methods require smoothness of delay function, which due to properties of max-operator obtained in Chapter 1, cannot be satisfied for differential equations with maximum. In this work, we suggest a numerical method for computing solutions to differential equations with maximum. Our method is based on the rectangle method that requires only continuity of the first derivative of the solution, no additional assumptions about delay function are assumed. Recall that for others methods, for example trapezoid or Simpson method more regularity of solution is required.

Another type of approximation methods is the averaging method.

## Averaging method for ISS analysis

Initially, averaging method appeared with need to solve the problems of celestial mechanics. The idea of averaging method is, that the right hand side of a time varying system of ODE, which describes, for example oscillations, is replaced by an averaged one, i.e. without explicit time dependence. To the best of author's knowledge, the first averaging schemes were considered by Gauss [27] and Fatou [25]. However, they proposed particular schemes and the general method was developed by Bogolybov and Mitropolskiy in 1934 [60]. They showed that under certain change of coordinates, which allows us to exclude $t$ from right hand side of equation, solution of time-varying (original) system is approximated by a solution of time-invariant (averaged) system.

The first works devoted to the averaging method for RFDE appeared in 1960s [35, 37, 39]. Within cited papers the most general results are obtained by Hale [39], where author showed that time-varying system of RFDE with small parameter may be approximated by a time-invariant ODE. Later Lehman and Weibel [65] proposed another approach. Using local averaged, authors showed, that approximation of RFDE by an averaged RFDE leads, compare with Hale's approach, to the more accurate error of approximation (see Ex.5.2 [65]).

The general method for infinite-dimensional systems is available in [41]. Some averaging schemes for differential equations with maximum are given in [80, 11, chapter 7], where authors used techniques of proofs that differ from the idea of the proof which is applied in [65].

Mentioned above literature about averaging is devoted to results of approximation on an finite time interval. This method can also be extended on an infinite time interval [12] and, furthermore, it is used for stability analysis of ODE [32], of RFDE [64].

The idea to apply averaging method in order to analyze the stability of dynamical system with input is initially used in [73], where authors introduced the notation of averaged system of ODE with input (strong averaged and weak averaged, which differ by the roles played by an input function) and shown that the existence of ISS-Lyapunov function

- for strong averaged system implies uniform semi-global practical stability of the original system;
- for weak averaged system implies weaker, so called, 'ISS like' property.

The second results is weaker, nevertheless, the advantage to use weak averaged is, that it exists for wider then the strong average class of functions, and in addition, 'ISS like' result is useful for analysis of singular perturbed systems [73].

Such framework has been successfully used for different classes of problems [96, 98, 97, 74, 90, 63, 95, 94]. Yang and Wang [98] considered time-varying RFDE with input and proved that if a strong averaged system of RFDE admitted an ISS-Razumikhin function, then time-varying (original) system is semi-globally input-to-state practically stable.

Observe that averaging is applied for ISS analysis only under assumption of existence of a Lyapunov function (for ODE) or a Razumikhin function (for RFDE), justification of applying averaging method for RFDE with input are not available in the literature. In chapter 3 of this thesis we fill this gab by providing, without usage of any Lyapunov techniques, the justification of averaging method for controlled systems with maximum and apply these results to investigate ISS properties of time-varying systems with maximum.

## Contribution of this Thesis

The main achievements of this dissertation are the following:

1. It is proved that system with maximum and input may change infinite-dimension on an one-dimension along its solution.
2. Comparison lemma for differential equations with maximum is obtained.
3. The ISS of system in the linear form is proved, the explicit expression of the ISS estimate is derived. Neither Lyapunov-Krasovskiy method nor Razumikhin method is used for this purpose.
4. The classical averaging method is extended to a system of differential equations with maximum affected by an input. ISS property of nonlinear system with maximum is studied without usage of any Lyapunov techniques in the context of the averaging method.
5. A numerical method of the first order for differential equations with maximum is developed.

# Outline of this Thesis

This thesis contains four main chapters, one introduction chapter and one concluding chapter. At the beginning of chapters 1, 2 we introduce all necessary basic notions and notations. Each chapter ends with specific comments to the results from the chapter.

**Chapter 1.** We introduce max and arg max operators and study several of their properties, which are used in next chapters. In particular, we prove that max-operator is sublinear, periodic, monotonic, nondifferentiable and Lipschitz continuous; arg max is nondecreasing, piecewise continuous operator.

**Chapter 2.** In preliminaries of Chapter 2 we remark some properties of systems of differential equations with maximum, which follow immediately from results of Chapter 1. In section 2.2 comparison lemma for differential equations with maximum is proved. Next, in section 2.3 we show that perturbed system with maximum in the linear form may change its infinite dimension to a finite one along its solution, i.e. infinite-dimensional system reduces to an ODE. We deal with so called Multi-Mode Multi Dimensional systems (see [49]). In section 2.4, we compare stability properties of scalar differential equations with maximum with constant delay equation. Using the result from section 2.3 we prove that global exponential stability of equation with maximum does not depend on the length of the past time interval, which is not true for DDE with constant delay. In section 2.5, using neither Lyapunov-Krasovskiy method nor Razumikhin method, we provide the input-to-state stability results for time-varying systems of differential equation with maximum in the linear form. Moreover, the explicit expressions for the ISS gain function is obtained. Several examples are considered in subsection 2.5.1.

Some results from this Chapter were presented on the 20th IFAC World Congress 2017, [22].

**Chapter 3** is devoted to the ISS analysis of systems of differential equations with maximum via averaging method. In section 3.1 we briefly recall the classical averaging method for ODE system. The notations of weak averaged and strong averaged systems are introduced in section 3.2. The difference between these definitions is discussed and an illustrative example is considered. In section 3.3, we prove that the solutions to strong averaged and weak averaged systems approximate a solution to original time-varying system on a finite time interval. In the next section we extend the results from the section 3.3 to infinite time interval and, moreover, without usage of any Laypunov techniques, prove that exponential incremental ISS (e$\delta$ISS) of strong averaged and weak averaged systems implies exponential input-to-state practical stability (eISpS) of the original system.

**Chapter 4.** A numerical approximation method of solutions to differential equations with maximum is suggested in Chapter 4. The proposed method is based on the left rectangle method that requires only continuity of the first derivative of the solution, no additional conditions about delay function are assumed. The method is illustrated by an example. Observe that the suggested method is an extension of the one from [21], where only constant initial functions were considered.

**Chapter 5.** In the last chapter, results of the whole thesis are summarized and possible directions for future research are provided.

# Chapter 1

# Properties of max-operator

In this chapter max-operator and arg max are introduced, some of their properties are studied. In particular, it is shown that max is sublinear, periodic, monotonic, nondifferentiable, Lipschitz continuous operator; arg max is nondecreasing, piecewise continuous operator. Several examples are provided to illustrate some properties.

## 1.1 Preliminaries

Let $h > 0$, $t_0 \in \mathbb{R}$ be given and $[t_0 - h, T) \subset \mathbb{R}$, $t_0 < T \leq \infty$ be a time interval. For a scalar valued function $g \in C([t_0 - h, T); \mathbb{R})$, we introduce the map

$$\max : C([t_0 - h, T); \mathbb{R}) \to C([t_0, T); \mathbb{R}), \quad g \overset{\max}{\longmapsto} g_h^\vee, \tag{1.1}$$

where

$$\forall t \in [t_0, T) : \quad g_h^\vee(t) := \max_{s \in [t-h, t]} g(s) \in \mathbb{R}. \tag{1.2}$$

Similarly for a vector valued function $g \in C([t_0 - h, T); \mathbb{R}^n)$, $n \in \mathbb{N}$ we define component-wise

$$\forall t \in [t_0, T) : \quad g_h^\vee(t) := \left(g_{1,h}^\vee(t), g_{2,h}^\vee(t), \ldots, g_{n,h}^\vee(t)\right)^\top \in \mathbb{R}^n. \tag{1.3}$$

Sometimes, we write $g^\vee$ instead of $g_h^\vee$ to simplify notation, if the value $h$ is clear from the context.

For $\alpha \in C([t_0 - h, T); \mathbb{R})$ we introduce the map

$$\arg\max : C([t_0 - h, T); \mathbb{R}) \to PC([t_0, T); \mathbb{R}), \quad \alpha \overset{\arg\max}{\longmapsto} \arg\max \alpha,$$

$$\forall t \in [t_0, T) : \quad (\arg\max \alpha)(t) := \sup\left\{\tau \in [t_0 - h, T) : \max_{s \in [t-h, t]} \alpha(s) = \alpha(\tau)\right\} \in \mathbb{R}, \tag{1.4}$$

similarly, for a vector valued function $\alpha \in C([t_0 - h, T); \mathbb{R}^n)$ we define

$$\forall t \in [t_0, T) : \quad (\arg\max \alpha)(t) := \Big((\arg\max \alpha_1)(t), \cdots, (\arg\max \alpha_n)(t)\Big)^\top \in \mathbb{R}^n. \tag{1.5}$$

Using the definition (1.5) we can rewrite the definition (1.3) as follows

$$g_h^{\vee}(t) = \Big(g_1\Big((\arg\max\, g_1)(t)\Big), \cdots, g_n\Big((\arg\max\, g_n)(t)\Big)\Big)^{\top}. \tag{1.6}$$

From (1.4) it follows that $\arg\max\alpha$ is piecewise right continuous function satisfying

$$t - h \le (\arg\max\alpha)(t) \le t, \quad t \ge t_0.$$

To illustrate the above definitions, consider the following example.

**Example 3.** *Let $g(t) = \sin t$ on $[-1, 6]$ and $h = 1$. Then we have*

$$g_h^{\vee}(t) = \begin{cases} \sin t, & t \in [0, \frac{\pi}{2}], \\ 1, & t \in [\frac{\pi}{2}, \frac{\pi}{2}+1], \\ \sin(t-1), & t \in [\frac{\pi}{2}+1, t^*), \\ \sin t, & t \in [t^*, 6] \end{cases} \qquad (\arg\max\,\sin)(t) = \begin{cases} t, & t \in [0, \frac{\pi}{2}], \\ \frac{\pi}{2}, & t \in [\frac{\pi}{2}, \frac{\pi}{2}+1], \\ t-1, & t \in [\frac{\pi}{2}+1, t^*), \\ t, & t \in [t^*, 6], \end{cases}$$

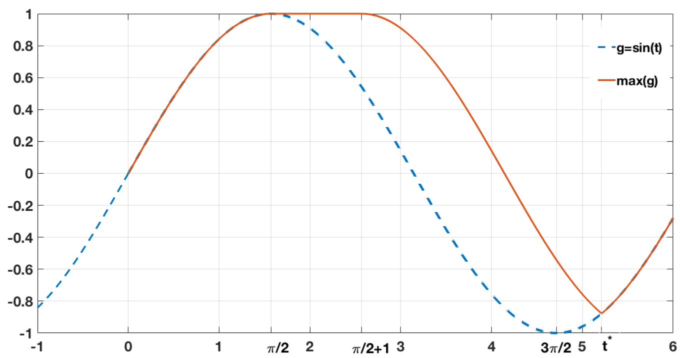


FIGURE 1.1: *The graphs of $g(t) = \sin t$, $g_1^{\vee}(t) = \max_{s\in[t-1,t]} g(s)$.*

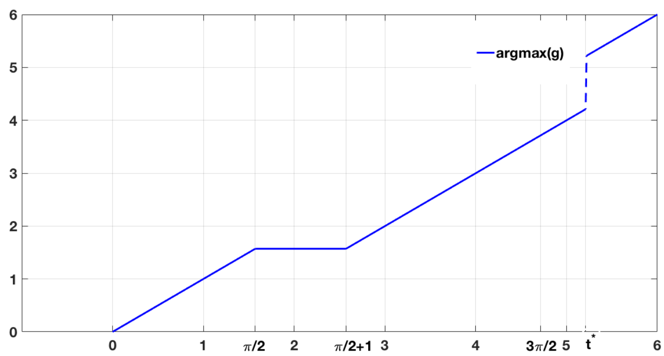


FIGURE 1.2: *The graph of $(\arg\max\,\sin)(t)$.*

*where $t^* := \frac{1}{2}(3\pi+1)$ is the unique solution of $\sin t = \sin(t-1)$ on $[\frac{\pi}{2}+1, 6]$ (Figures 1.1,1.2).*

**Proposition 1.** *Let* $h > 0,\ t_0 \in \mathbb{R},\ T \in (t_0, \infty)$ *and* $g \in C([t_0 - h, T); \mathbb{R})$*. Then* $\arg\max\, g : [t_0, T) \mapsto \mathbb{R}$ *is nondecreasing function on* $[t_0, T)$*.*

*Proof.* Let $t_1, t_2 \in [t_0, T)$ be such that $t_1 \le t_2$. Set

$$\tau_1 := (\arg\max\, g)(t_1), \quad \tau_2 := (\arg\max\, g)(t_2).$$

Consider possible cases
<u>Case 1.</u> Let $h > 0$ be such that $t_1 - h < t_2 - h < t_1 < t_2$. Then if

1. $\tau_1 \in [t_1 - h, t_2 - h]$ it follows $\tau_1 \le t_2 - h \le \tau_2$;
2. $\tau_2, \tau_2 \in [t_2 - h, t_1]$, by (1.4) it follows $\tau_1 = \tau_2$;
3. $\tau_2 \in [t_1, t_2]$ it follows $\tau_1 \le t_1 \le \tau_2$.

<u>Case 2.</u> Let $h > 0$ be such that $t_1 - h < t_1 < t_2 - h < t_2$. Then it follows $\tau_1 < \tau_2$.
<u>Case 3.</u> Let $t_1 = t_2$. Then by (1.4) it follows $\tau_1 = \tau_2$.
Hence $\tau_1 \le \tau_2$. The proposition is proved. □

## 1.2 Properties of max-operator

Let $h > 0,\ t_0 \in \mathbb{R}$. $Max$-operator possesses the following properties:

1. *Sublinearity.*
   For continuous functions $g, \eta : [t_0 - h, T) \to \mathbb{R}^n$, $t_0 < T \le \infty$ the following holds
   $$\forall t \in [t_0, T) : (g + \eta)^\vee(t) \le g^\vee(t) + \eta^\vee(t). \tag{1.7}$$
   Indeed, fix any $t \in [t_0, T)$. Fix any $i \in \{1, \cdots, n\}$. Let $t_1, t_2, t_3 \in [t_0, T)$ be such that
   $$g_i(t_1) + \eta_i(t_1) = (g_i + \eta_i)^\vee(t), \quad g_i(t_2) = {g_i}^\vee(t), \quad \eta_i(t_3) = {\eta_i}^\vee(t).$$
   Then we have
   $$(g_i + \eta_i)^\vee(t) = g_i(t_1) + \eta_i(t_1) \le {g_i}^\vee(t) + {\eta_i}^\vee(t) = g_i(t_2) + \eta_i(t_3). \tag{1.8}$$
   Inequality (1.8) holds for any $i \in \{1, \cdots, n\}$, hence (1.7) is proved.
   Moreover, for any $\alpha \in \mathbb{R}_+$ the map defined by (1.1) satisfies
   $$(\alpha + g)_h^\vee = \alpha + g_h^\vee, \qquad (\alpha g)_h^\vee = \alpha g_h^\vee.$$
2. *Monotonicity.*
   Let $\xi, g \in C([t_0 - h, T); \mathbb{R}^n)$, $t_0 < T \le \infty$. If $\xi(t) \ge g(t)$ for all $t \in [t_0 - h, T)$, then $\xi_h^\vee(t) \ge g_h^\vee(t)$ for all $t \in [t_0, T)$. Moreover, if for any $i = 1, 2, \cdots, n$ function $g_i$ is nondecreasing (nonincreasing) for all $[t_0 - h, T)$, then function $g_i^\vee$ is nondecreasing (nonincreasing) for all $t \in [t_0, T)$. In this case for all $t_1 \ge t_2$ it follows that $g^\vee(t_1) \ge g^\vee(t_2)$ (resp. $g^\vee(t_1) \le g^\vee(t_2)$).

3. *Periodicity*
   Let $g \in C([t_0 - h, \infty); \mathbb{R})$ be periodic with period $T' > 0$. Then for any $h > 0$, the function $g_h^\vee$ is periodic on $[t_0, \infty)$ with the same period $T'$. Indeed, for all $t \in [t_0, \infty)$

$$g_h^\vee(t) := \max_{s\in[t-h,t]} g(s) = \max_{s\in[t-h,t]} g(s+T') = \max_{s\in[t-h+T',t+T']} g(s) =: g_h^\vee(t+T).$$

4. *Nondifferentiability.*
   $Max$-operator does not preserves smoothness. Indeed, consider function $g(t) = sin\, t$, $t \in [-\frac{3\pi}{2}, \frac{\pi}{2}]$. Take $h = \pi$, then for all $t \in [-\frac{3\pi}{2}, \frac{\pi}{2}]$ we obtain

$$g_\pi^\vee(t) = \begin{cases} 1, & t \in [-\frac{3\pi}{2}, -\frac{\pi}{2}], \\ \sin(t-\pi), & t \in [\frac{\pi}{2}, 0], \\ \sin\, t, & t \in [0, \frac{\pi}{2}]. \end{cases}$$

One can observe on Figure 1.3 that $g_h^\vee(\cdot)$ is not differentiable at point $t = 0$,

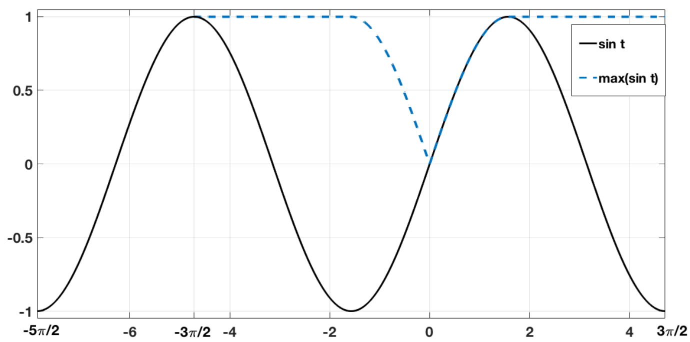


FIGURE 1.3: *The graph of $g(t) = \sin\, t$ and its maximum for $h = \pi$.*

where the one side derivatives are

$$g_\pi^{\vee+}(0) = \lim_{t\to 0+} \frac{g_\pi^\vee(t) - g_\pi^\vee(0)}{t} = \lim_{t\to 0+} \frac{\sin t}{t} = 1,$$

$$g_\pi^{\vee-}(0) = \lim_{t\to 0-} \frac{\sin(t-\pi)}{t} = -\lim_{t\to 0-} \frac{\sin t}{t} = -1.$$

5. The next Lemma provides useful, for the work with max-operator estimate.

**Lemma 1.** *Let $h > 0$, $t_0 \in \mathbb{R}$, $t_0 < T \leq +\infty$, $n \in \mathbb{N}$ and $g \in C([t_0-h, T); \mathbb{R}^n)$. Then for all $t \in [t_0, T)$*

$$n \max_{s\in[t-h,t]} |g(s)| =: n|g^\vee|(t) \geq |g^\vee(t)| \geq |g^\vee|(t). \tag{1.9}$$

*Proof.* For any fixed $t \in [t_0, T)$ we have

$$
\begin{aligned}
|g^\vee(t)| &= |(g_1(\arg\max_{s\in[t-h,t]} g_1(s)), \dots, g_n(\arg\max_{s\in[t-h,t]} g_n(s)))^\top| \\
&= \sqrt{\sum_{i=1}^n g_i^2(\arg\max_{s\in[t-h,t]} g_i(s))} \geq \sqrt{\sum_{i=1}^n g_i^2(\arg\max_{s\in[t-h,t]} |g(s)|)} \\
&= |g(\arg\max_{s\in[t-h,t]} |g(s)|)| = |g^\vee|(t).
\end{aligned}
$$

This implies the second inequality in (1.9). To prove the first one observe that for any $i = 1, \dots, n$ the following holds

$$
\max_{s\in[t-h,t]} |g(s)| = \max_{s\in[t-h,t]} \sqrt{\sum_{k=1}^n g_k^2(s)} \geq \max_{s\in[t-h,t]} |g_i(s)|.
$$

Taking the sum left and right for all $i = 1, \dots, n$, and using that $\sum_{i=1}^n c_i \geq \sqrt{\sum_{i=1}^n c_i^2}$ holds for any $c_i \geq 0, i = 1, \dots, n$, and $|g_i(s)| \geq g_i(s)$ for any $s$, we obtain

$$
\begin{aligned}
n \max_{s\in[t-h,t]} |g(s)| &\geq \sum_{i=1}^n \max_{s\in[t-h,t]} |g_i(s)| \geq \sqrt{\sum_{i=1}^n (\max_{s\in[t-h,t]} |g_i(s)|)^2} \\
&\geq \sqrt{\sum_{i=1}^n (\max_{s\in[t-h,t]} g_i(s))^2} = \left| \max_{s\in[t-h,t]} g(s) \right| = \left| g^\vee(t) \right|.
\end{aligned}
\tag{1.10}
$$

□

6. *Lipschitz continuity.*

**Definition 1.** Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be two normed spaces. A map $\chi : X \mapsto Y$ is called *Lipschitz continuous* if there exists a Lipschitz constant $L > 0$ such that for all $x_1, x_2 \in X$ the following holds

$$
||\chi(x_1) - \chi(x_2)||_Y \leq L||x_1 - x_2||_X.
$$

**Lemma 2.** *Let $h > 0$, $t_0 \in \mathbb{R}$, $t_0 < T \leq +\infty$, $n \in \mathbb{N}$. Max-operator is Lipschitz continuous with Lipschitz constant $L = n$ i.e.*

$$
\forall v, w \in C([t_0 - h, T); \mathbb{R}^n) : \quad ||v_h^\vee - w_h^\vee||_{[t_0,T)} \leq n||v - w||_{[t_0-h,T)}. \tag{1.11}
$$

*Proof.* For short we denote $v_h^\vee =: v^\vee$. We obtain

$$
\begin{aligned}
&||v^\vee - w^\vee||_{[t_0,T)} := \sup_{t\in[t_0,T)} |v^\vee(t) - w^\vee(t)| = \sup_{t\in[t_0,T)} |(v - w + w)^\vee(t) - w^\vee(t)| \\
&\overset{(1.7)}{\leq} \sup_{t\in[t_0,T)} \left| \max_{s\in[t-h,t]} (v(s) - w(s)) \right| \overset{(1.10)}{\leq} n \sup_{t\in[t_0-h,T)} \max_{s\in[t-h,t]} |v(s) - w(s)| \\
&= n \sup_{t\in[t_0-h,T)} |v(t) - w(t)| =: n||v - w||_{[t_0-h,T)}.
\end{aligned}
$$

Lemma is proved. □

## 1.3 Concluding remarks

In this chapter some properties of max and arg max operators are studied. In the next chapters we use them to investigate behavior of systems with maximum.

# Chapter 2

# Stability of the systems of differential equations with maximum in the linear form

In this chapter we study stability properties of dynamical systems governed by differential equations with maximum and, in particular robustness with respect to perturbation. Such systems are nonlinear and infinite-dimensional. Stability of systems with maximum without input is studied in [11, Chapter 4], [93, 10]. There are a few works [2, 3, 9, 22, 91, 8] studying such systems perturbed by an input, and none of cited works provides input-to-state stability (ISS) analysis. Although there are results devoted to the ISS study of systems with delay where the Lyapunov-Krasovskiy functional [78] or the Razumikhin function [89] is applied, nevertheless, neither of these works provides an explicit expressions for the ISS gain function.

This chapter is organized as follows: in the next section we introduce all necessary basic notions and notations and prove that the equation with maximum does not possess backward uniqueness property. In section 2.2, we review comparison results for constant and time-varying DDE, and prove comparison lemma for equation with maximum. Next, in section 2.3 we study the behavior of scalar differential equations with maximum and perturbing signal. It is proved that infinite dimension of such system may change to a finite along its solution i.e., we deal with Multi-Mode Multi Dimensional systems [49]. In section 2.4, stability properties of scalar differential equations with maximum and constant delay are compared. Using results from the previous section, it is proved that for equation with maximum global exponential stability does not depend on the length $h$ of the past time interval, which is not true for the constant delay equations. In section 2.5, we study ISS properties of dynamical systems with maximum in the linear form. Using neither method of Lyapunov-Krasovskiy functional nor Razumikhin approach the exponential ISS of such systems is proved, the explicit expression for ISS gain function is derived. Furthermore, we prove that zero-exponential global asymptotic stability of such systems implies exponential ISS. Several examples are discussed in subsection 2.5.1. In the last section we conclude the results of this chapter and sketch possible directions for future research.

## 2.1 Preliminaries

Let $h > 0$. Consider the system of differential equations with *max*-operator and input $u \in L_\infty([0,\infty);\mathbb{R}^m)$, $m \in \mathbb{N}$ of the form

$$\dot{x}(t) = f(t, x(t), x_h^\vee(t), u(t)), \quad t \geq 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}$, $f \in C([0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ with initial condition

$$x(t) = \varphi(t), \quad t \in [-h, 0], \tag{2.2}$$

where $\varphi \in C([-h,0];\mathbb{R}^n)$. By $\dot{x}(t)$ we mean the right-hand side derivative of $x$ at $t$.

*Remark* 1. Note that (2.1) is a system of retarded functional differential equations with state-dependent piecewise continuous delay. The state-space of this system is $C([-h,0];\mathbb{R}^n)$, therefore we deal with an infinite-dimensional dynamical system.

*Remark* 2. Since max is sublinear operator, system (2.1) is, in general, nonlinear i.e., it can happen that the sum of two solutions of the problem (2.1),(2.2) is not a solution to (2.1),(2.2).

**Definition 2.** For every input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ and for every initial function $\varphi \in C([-h,0];\mathbb{R}^n)$, a function $x(\cdot) = x(\cdot\,;\varphi,h,u)$ is called *a solution* to the problem (2.1), (2.2) if there is a $T_f \in (0,\infty]$ such that $x \in C([-h,T_f);\mathbb{R}^n)$, $x(t) = \varphi(t)$ for all $t \in [-h,0]$, $x$ is absolutely continuous on $[0,T_f)$ and satisfies differential equation (2.1) almost everywhere on $[0,T_f)$; $x$ is a *maximal solution* if it has no right extension that is also a solution.

For each input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ the function $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by $F(t, x(t), x_h^\vee(t)) := f(t, x(t), x_h^\vee(t), u(t))$.

Suppose that $\Omega$ is an open subset of $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. The function $F : \Omega \to \mathbb{R}^n$ is said to satisfy the *Carathéodory conditions* on $\Omega$ if

(i) for each fixed $t \in \mathbb{R}_+$, function $F(t,\cdot\,,\cdot)$ is continuous;

(ii) for each fixed $\xi_1, \xi_2 \in \mathbb{R}^n$ function $F(\cdot\,,\xi_1,\xi_2)$ is measurable;

(iii) for any fixed $(t,\xi_1,\xi_2) \in \Omega$ there is a neighborhood $W(t,\xi_1,\xi_2)$ and a Lebesgue integrable function $\zeta : [0,\infty) \to \mathbb{R}_+$ such that

$$|F(t,\xi_1',\xi_2')| \leq \zeta(t) \quad \text{for all} \;\; (t,\xi_1',\xi_2') \in W(t,\xi_1,\xi_2);$$

(iv) for each compact subset $\overline{\Omega} \subseteq \Omega$ there is a locally integrable function $l : [0,\infty) \to \mathbb{R}_+$ such that

$$|F(t,\xi_1,\xi_2) - F(t,\overline{\xi_1},\overline{\xi_2})| \leq l(t)\Big(||\xi_1 - \xi_2|| + ||\overline{\xi_1} - \overline{\xi_2}||\Big),$$

for all $\xi_1, \xi_2, \overline{\xi_1}, \overline{\xi_2} \in \overline{\Omega}$, and for almost all $t \in [0,\infty)$.

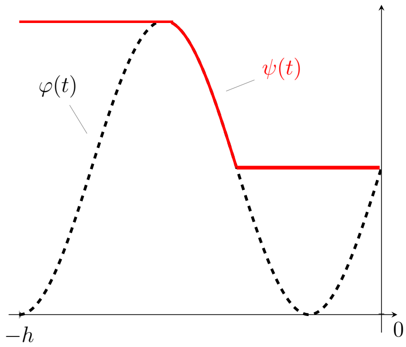


FIGURE 2.1: Graphs of a function $\psi$ and a function $\varphi$.

Existence of unique maximal solution to RFDE (1) under Carathéodory conditions is given in [40] (see Theorems 2.1, 2.3, 3.1, 3.2). Recall that differential equation with maximum

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), x_h^{\vee}(t)), & t &\geq 0, \\ x(t) &= \varphi(t), & t &\in [-h, 0], \end{aligned} \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$, $h > 0$, $\varphi \in C([-h, 0]; \mathbb{R}^n)$, $F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, is a particular case of RFDE considered in [40] with $x_t := x_h^{\vee}$, moreover, *max*-operator is Lipschitz continuous, by Lemma 2, therefore under Carathéodory conditions imposed on function $F$, existence of solution $x(\cdot) = x(\cdot\,; \varphi, h)$ of the problem (2.3) follows from Theorem 2.1 [40], its uniqueness from Theorem 2.3 [40], its extension to maximal solution from Theorems 3.1, 3.2 [40]. Therefore we can formulate the following theorem.

**Theorem 2.** *Let $h > 0$, $\varphi \in C([-h, 0]; \mathbb{R}^n)$. Assume $F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions. Then there exists a unique solution $x : [-h, T_f) \to \mathbb{R}^n$, $T_f \in (0, \infty]$ of the problem (2.3), and every solution can be extended to a maximal solution.*

For any function $\varphi \in C([-h, 0]; \mathbb{R}^n)$ we define the function $\psi \in C([-h, 0]; \mathbb{R}^n)$ by

$$\psi(t) := \max_{s \in [t, 0]} \varphi(s), \quad t \in [-h, 0]. \tag{2.4}$$

Here *max* is taken component-wise as in (1.3). Note that each component of the function $\psi(\,\cdot\,)$ defined by (2.4) is a nonincreasing function (see Figure 2.1).

Indeed, let $t_1, t_2 \in [-h, 0]$ be such that $t_1 < t_2$. Then we have

$$\psi(t_1) = \max_{s \in [t_1, 0]} \varphi(s) \geq \max_{s \in [t_2, 0]} \varphi(s) = \psi(t_2),$$

and

$$\forall t \in [-h, 0] : \quad \max_{s \in [t, 0]} \varphi(s) = \psi(t) = \max_{s \in [t, 0]} \psi(s), \tag{2.5}$$

holds. Also,

$$\forall t \in [-h, 0] : \quad \psi(t) \geq \varphi(t),$$

and

$$\psi(0) = \varphi(0).$$

Consider the equation with maximum (2.1) with the initial condition

$$x(t) = \psi(t), \quad t \in [-h, 0], \tag{2.6}$$

where the function $\psi$ is defined by (2.5).

**Assumption 1.** *For any initial function $\varphi \in C([-h,0];\mathbb{R}^n)$, $h > 0$ and any input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ there exists unique solution $x(\cdot) = x(\cdot\,;\varphi,h,u)$ of the problem (2.1), (2.2) defined on $[0,\infty)$.*

**Lemma 3.** *Let $h > 0$ and Assumption 1 be satisfied. Then for any $\varphi \in C([-h,0];\mathbb{R}^n)$ and any input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ the solution of the problem* (2.1), (2.2) *coincides with the solution of the problem* (2.1), (2.6) *on* $[0,\infty)$.

*Proof.* Let $h > 0$, $\varphi \in C([-h,0];\mathbb{R}^n)$ and $u \in L_\infty([0,\infty);\mathbb{R}^m)$. By the Assumption 1 for any initial function there exists solution to the problem (2.1),(2.2) on $[0,\infty)$, therefore denote by $x_1(\cdot) = x(\cdot\,;\varphi,h,u)$ and $x_2(\cdot) = x(\cdot\,;\psi,h,u)$ the solutions to the Cauchy problems (2.1), (2.2) and (2.1),(2.6) on $[0,\infty)$, resp. Let $t \in [0,h]$. Then we have

$$\begin{aligned}
\dot{x}_1(t) &= f(t, x_1(t), x^{\vee}_{1,h}(t), u) = f(t, x_1(t), \max\{\max_{s\in[t-h,0]} \varphi(s), \max_{s\in[0,t]} x_1(s)\}, u) \\
&\overset{(2.5)}{=} f(t, x_1(t), \max\{\max_{s\in[t-h,0]} \psi(s), \max_{s\in[0,t]} x_1(s)\}, u).
\end{aligned}$$

Therefore, $x_1(\cdot) = x(\cdot\,;\varphi,h,u)$ is the solution to the Cauchy problem (2.1),(2.6) on $[0,h]$. By the Assumption 1 there exists unique solution of (2.1) with initial condition (2.6), hence

$$\forall t \in [0,h]: \quad x_1(t) = x_2(t). \tag{2.7}$$

Let $t \geq h$. Consider equation (2.1) for all $t \geq h$ with the initial condition

$$\forall t \in [0,h]: \quad x(t) = x_1(t). \tag{2.8}$$

By the Assumption 1 there exists unique solution $x_1(\cdot)$ to the problem (2.1), (2.8) and in conjunction with (2.7) we conclude that $x_1(\cdot) = x_2(\cdot)$ on $[h,\infty)$, where $x_2(\cdot)$ is the solution to (2.1) with initial condition $x(t) = x_2(t)$ for all $t \in [0,h]$. Therefore, $x_1(t) = x_2(t)$ on $[0,\infty)$. This proves the lemma. $\square$

*Remark* 3. Lemma 3 shows that, without loss of generality we can restrict our consideration to the case of nonincreasing initial functions.

The following notion of stability, originally introduced for ODEs in [83], is used in this chapter:

**Definition 3.** System (2.1) is called *input-to-state stable from $u$ to $x$* if there exist a function $\gamma$ of class $\mathcal{K}_\infty$ and a function $\beta$ of class $\mathcal{KL}$ such that for each input $u \in L_\infty([0,\infty);\mathbb{R}^m)$ and each initial function $\varphi \in C([-h,0];\mathbb{R}^n)$, $h > 0$ the unique solution $x(\cdot) = x(\cdot\,;\varphi,h,u)$ to (2.1),(2.2) exists for all $t \in [0,\infty)$ and furthermore it satisfies

$$|x(t)| \leq \beta(||\varphi||,t) + \gamma(||u||), \quad t \geq 0. \tag{2.9}$$

The function $\gamma$ is called *ISS-gain*.

**Definition 4.** If in Definition 3 function $\beta(s,t) = Me^{-\lambda t}s$, for any $s,t \in \mathbb{R}_+$ and some $M > 0, \lambda > 0$, then system (2.1) is called *exponentially ISS (eISS)*.

Throughout this Chapter we assume, that $f(t,0,0,0) = 0$, for all $t \geq 0$ so that, $x(t) \equiv 0$ is an equilibrium of (2.1).

**Definition 5.** The solution $x \equiv 0$ of the system (2.1) with $u \equiv 0$, is said to be *zero stable* if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $||\varphi|| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \in [0,\infty)$. It is said to be *zero asymptotically stable* if it is zero stable, and for some $\delta > 0$ $||\varphi|| < \delta$ implies $\lim\limits_{t\to\infty} x(t) = 0$. It is *zero globally asymptotically stable (0-GAS)* if it is zero asymptotically stable with $\delta = \infty$. It is *zero globally exponentially stable (0-GES)* if there exist $c > 0$, $k > 0$ such that for any $\varphi \in C([-h,0];\mathbb{R}^n)$ a solution satisfies $|x(t)| \leq ce^{-kt}||\varphi||$, for all $t \in [0,\infty)$.

In the special case when $f$ has no input signal

$$\dot{x}(t) = f(t, x(t), x_h^\vee(t)), \quad t \geq 0, \tag{2.10}$$

instead of 0-GAS (0-GES) of $x \equiv 0$ we say that $x \equiv 0$ is globally asymptotically stable (GAS) (globally exponentially stable (GES)).

## 2.2 Comparison lemmas

In many problems of mathematical control theory one needs to compute bounds of a solution $x(\cdot\,;u)$ to equation $\dot{x} = f(t,x,u)$ without knowing the solution itself. One of the powerful tools, which is widely used for this purpose, is a comparison lemma (see [56, Lemma 3.4, p.102] and e.g. [85, 84] for its applications).

In the literature there are some comparison results for delay equation in the form

$$\dot{x}(t) = -\sum_{i=1}^{n} q_i(t)x(t-\tau_i(t)), \quad t \geq 0, \tag{2.11}$$

with initial condition

$$x(t) = \varphi(t), \quad t \in [\min_{i=1,\cdots,n}\inf_{s\geq 0}(s-\tau_i(s)), 0]. \tag{2.12}$$

where $\tau_i \in C([0,\infty);\mathbb{R}_+)$, $i = 1,\cdots,n$, $\varphi \in C([\min\limits_{i=1,\cdots,n}\inf\limits_{s\geq 0}(s-\tau_i(s)),0];\mathbb{R}_+)$. For example, in [62] the equation (2.11) in case $\tau_i(t) = \tau \in \mathbb{R}$ is studied. The authors consider two different initial conditions and compare corresponding solutions. In [61] the case of continuous variable delays is considered and the following comparison result is obtained:

**Theorem 3.** *Let* $\tau_i \in C([0,\infty);\mathbb{R}_+)$, $i = 1,\cdots,n$, $\varphi \in C([\min_{i=1,\cdots,n} \inf_{s\geq 0}(s-\tau_i(s)),0];\mathbb{R}_+)$, $L \in (0,\infty]$. *Assume* $x(t) > 0$ *for all* $t \in [0,L)$ *satisfies (2.11) and (2.12) on* $[0,L)$. *Suppose* $y \in C([\min_{i=1,\cdots,n} \inf_{s\geq 0}(s-\tau_i(s));\mathbb{R})$ *and* $y \in C^1([0,\infty);\mathbb{R})$ *satisfies*

$$\begin{array}{ll} \dot{y}(t) \geq -\sum\limits_{i=1}^{n} q_i(t)y(t-\tau_i(t)), & t \geq 0, \\ y(t) \leq \varphi(t), & t \in [\min\limits_{i=1,\cdots,n} \inf\limits_{s\geq 0}(s-\tau_i(s)),0], \end{array}$$

*with* $y(0) = \varphi(0) > 0$. *Then* $y(t) \geq x(t)$ *for all* $t \in [0,L)$, *where the interval can be closed if* $L$ *is finite.*

To the best of our knowledge, comparison results for general state-dependent delay differential equation are not available in the literature.

Here we propose a comparison lemma for differential equations with maximum in the form

$$\begin{array}{ll} \dot{x}(t) = f(t,x(t),x_h^{\vee}(t)), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-h,0], \end{array} \tag{2.13}$$

where $x(t) \in \mathbb{R}$, $h > 0$, $\varphi \in C([-h,0];\mathbb{R})$.

**Assumption 2.** $f \in C([0,\infty)\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ *satisfies the implication*

$$y_1 \leq y_2 \;\Rightarrow\; f(t,x,y_1) \leq f(t,x,y_2) \;\; \textit{for all} \;\; t \geq 0, x \in \mathbb{R}.$$

**Lemma 4.** *Let* $h > 0$, $\varphi \in C([-h,0];\mathbb{R})$. *Suppose*

- *the Assumption 2 holds;*
- *there exists* $x(\cdot) = x(\cdot\,;\varphi,h)$ *solution to (2.13) on* $[-h,T), 0 < T \leq \infty$;
- *there is* $y \in C([-h,\infty);\mathbb{R})$ *such that* $y$ *is right differentiable on* $[0,\infty)$, *satisfies*

$$\dot{y}(t) < f(t,y(t),y_h^{\vee}(t)), \quad t \in [0,T), \tag{2.14}$$

*and* $y(t) < \varphi(t)$ *for all* $t \in [-h,0]$. *Then* $y(t) < x(t)$ *for all* $[0,T)$.

*Proof.* For $t = 0$ we have $y(0) < x(0)$ and $y_h^{\vee}(0) < x_h^{\vee}(0)$. Consider the function $r(t) := x(t) - y(t)$ on $[0,T)$. Assume there exists $t^* \in (0,T)$ such that $r(t^*) = 0$ and $r(t) > 0$ for all $t \in [0,t^*)$. Note that $r(0) > 0$. Then there exists $\varepsilon > 0$ such that $r$ is nonincreasing on $(t^*-\varepsilon,t^*]$, therefore $\dot{r}(t^*) \leq 0$. On the other hand, by Assumption 2 we have

$$\dot{y}(t^*) < f(t^*,y(t^*),y_h^{\vee}(t^*)) \leq f(t^*,x(t^*),x_h^{\vee}(t^*)) = \dot{x}(t^*). \tag{2.15}$$

It follows from (2.15) that $\dot{r}(t^*) > 0$ which contradicts $\dot{r}(t^*) \leq 0$. Thus, such $t^*$ does not exist. This proves the lemma. □

## 2.3 Scalar differential equations with maximum and input

Consider the scalar problem with *max*-operator and input in the linear form

$$\begin{aligned} \dot{x}(t) &= ax(t) + bx_h^{\vee}(t) + u(t), \quad && t \geq 0, \\ x(t) &= \varphi(t), && t \in [-h, 0], \end{aligned} \tag{2.16}$$

where $x(t) \in \mathbb{R}$, $h > 0$, $a, b \in \mathbb{R}$, $u \in L_\infty([0,\infty);\mathbb{R})$, $\varphi \in C([-h,0];\mathbb{R})$. Recall that system (2.16) defines an infinite-dimensional dynamical system. Since the right-hand side of the problem (2.16) satisfies Carathéodory conditions, Theorem 2 assures that for every input $u \in L_\infty([0,\infty);\mathbb{R})$ and every initial function $\varphi \in C([-h,0];\mathbb{R})$ there exists a unique maximal solution $x(\cdot) := x(\cdot\,;\varphi, h, u)$ on $[0, T_f)$, $T_f > 0$. Remark that the the right-hand side of (2.16) is sublinear hence $T_f = +\infty$ (see §2.2 [58]). We define, for this $x(\cdot)$

$$\begin{aligned} D_{x(\cdot)} &= \left\{ t \in [0,\infty) \mid x(t) < x_h^{\vee}(t) \right\}, \\ G_{x(\cdot)} &= \left\{ t \in [0,\infty) \mid x(t) = x_h^{\vee}(t) \leq 0 \right\}. \end{aligned}$$

Note that by these definitions for any solution $x(\cdot)$ to the problem (2.16) it holds $D_{x(\cdot)} \cap G_{x(\cdot)} = \varnothing$.

**Theorem 4.** *Let $h > 0$, $u \in L_\infty([0,\infty);\mathbb{R})$ with $u(t) \leq 0$ for all $t \in [0,\infty)$ and $a, b \in \mathbb{R}$ be such that*

$$a + b < 0. \tag{2.17}$$

*Let $x(\cdot)$ be unique solution of the problem (2.16) on $[0,\infty)$. Then*

$$[0,\infty) = D_{x(\cdot)} \cup G_{x(\cdot)}. \tag{2.18}$$

*Proof.* By the definition of sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ the following holds

$$[0,\infty) \supset D_{x(\cdot)} \cup G_{x(\cdot)}.$$

We will prove that

$$[0,\infty) \subset D_{x(\cdot)} \cup G_{x(\cdot)}. \tag{2.19}$$

Assume there exists $t_1 \in [0,\infty)$ such that $t_1 \notin D_{x(\cdot)} \cup G_{x(\cdot)}$ which indicates that

$$x(t_1) = x_h^{\vee}(t_1) > 0. \tag{2.20}$$

Then

$$\dot{x}(t_1) = ax(t_1) + bx_h^{\vee}(t_1) + u(t_1) = x(t_1)(a+b) + u(t_1).$$

Since $u(t) \leq 0$ for all $t \geq 0$ and taking into account (2.17) it follows that there exists $\varepsilon > 0$ such that $\dot{x}(t) < 0$ almost everywhere on $(t_1 - \varepsilon, t_1 + \varepsilon)$. Assume $\tilde{a}, \tilde{b} \in (t_1 - \varepsilon, t_1 + \varepsilon)$ such that

$$\tilde{a} < \tilde{b} \Longrightarrow x(\tilde{a}) \leq x(\tilde{b}), \tag{2.21}$$

then

$$x(\tilde{b}) - x(\tilde{a}) = \int_{\tilde{a}}^{\tilde{b}} \dot{x}(t)dt < 0,$$

which contradict to (2.21). Therefore, $x$ is strictly decreasing on $(t_1 - \varepsilon, t_1 + \varepsilon) \Rightarrow x(t_1) < x_h^\vee(t_1)$. This contradicts (2.20). Hence inclusion (2.19) is obtained. $\square$

*Remark* 4. From the proof of Theorem 4 it follows that if $D_{x(\cdot)}$ is bounded then there exists $t^* \in [0, \infty)$ such that the solution $x(\cdot)$ to the problem (2.16) satisfies the ODE $\dot{x}(t) = (a+b)x(t) + u(t)$ for all $t \geq t^*$. In this sense the infinite-dimensional system (2.16) reduces to one-dimensional after $t^*$.

**Example 4.** *Consider the following problem*

$$\begin{array}{ll} \dot{x}(t) = \frac{1}{2}x(t) - 2x_h^\vee(t) + u(t), & t \geq 0, \\ x(t) = 0, & t \in [-h, 0], \end{array} \tag{2.22}$$

*with $h = 2$, $u \in L_\infty([0,\infty);\mathbb{R})$. The condition (2.17) holds and let $u(t) := -e^{-t} - 1 < 0$ for all $t \in [0, \infty)$. The analytic solution to the problem (2.22) is*

$$x(t) = \begin{cases} -\frac{8}{3}e^{\frac{1}{2}t} + \frac{2}{3}e^{-t} + 2, & t \in [0,2], \\ (-e^2 - 2\frac{59}{125})e^{\frac{1}{2}t} + 5\frac{1}{3}e^{\frac{1}{2}t-1} + 7\frac{23}{98}e^{-t} + 10, & t \in [2, 3\frac{97}{100}], \\ -1727\frac{1}{10}e^{-\frac{3t}{2}} - 2e^{-t} - \frac{2}{3}, & t \in [3\frac{97}{100}, \infty). \end{cases}$$

*The point $t^* = 3\frac{97}{100}$ satisfies $x(t^*) = x(t^* - h)$ and $x(t^*) = x_h^\vee(t^*)$ i.e., $x(t) < x_h^\vee(t)$ for all $t < t^*$, which indicates that the equation (2.22) reduces to ODE for all $t^* \geq 3\frac{97}{100}$ i.e., we have $D_{x(\cdot)} = [0, 3\frac{97}{100})$ and $G_{x(\cdot)} = [3\frac{97}{100}, \infty)$ (see Figure 2.2).*

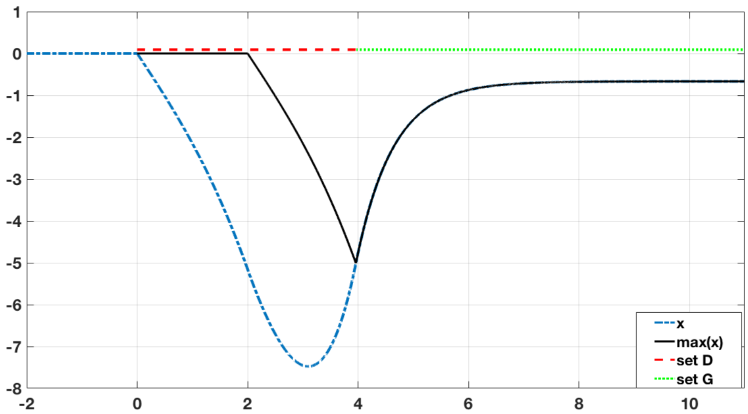


FIGURE 2.2: *Sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ for the solution to (2.22).*

In the considered example differential equation reduces to an ODE at $t^*$ and remains ODE for all $t \geq t^*$. However, in general, dimension of system with maximum and input may change back from one to infinity. For example, let us consider the following Cauchy problem

$$\begin{aligned} &\dot{x}(t) = -x_2^{\vee}(t) - u(t), && t \in [0,\infty), \\ &x(t) = 0 && t \in [-2,0]. \end{aligned} \tag{2.23}$$

Choose $u(t) = -|\sin t| \leq 0$ for all $t \geq 0$. By Theorem 2 there exists unique maximal solution $x(\cdot)$ on $[0,\infty)$ of (2.23) for which, by Theorem 4, we have

$$[0,\infty) = D_{x(\cdot)} \cup G_{x(\cdot)}.$$

The sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ are shown in the Figure 2.3.

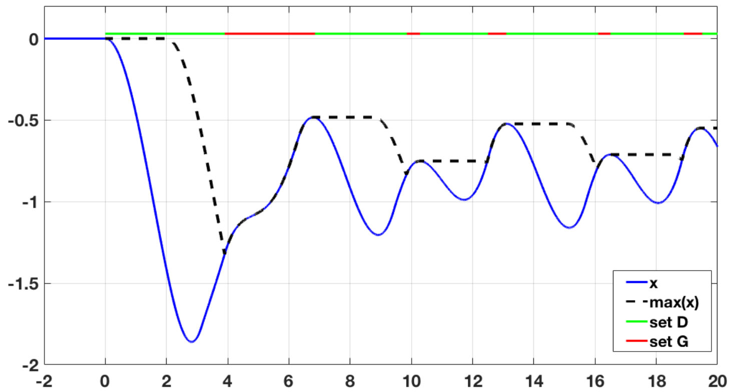


FIGURE 2.3: *Sets $D_{x(\cdot)}$ and $G_{x(\cdot)}$ for the solution to the problem (2.23).*

**Lemma 5.** *Let $a,b \in \mathbb{R}$, $t^* \geq 0$, $h > 0$, $\varphi \in C([t^*-h,t^*];\mathbb{R})$ be given and function $x \in C([t^*-h,\infty);\mathbb{R})$ be such that*

$$\begin{aligned} &\dot{x}(t) = ax(t) + bx_h^{\vee}(t), && t \geq t^*, \\ &x(t) = \varphi(t), && t \in [t^*-h,t^*]. \end{aligned} \tag{2.24}$$

*Assume (2.17) holds and*

$$\varphi_h^{\vee}(t^*) = \varphi(t^*) \leq 0. \tag{2.25}$$

*Then for all $t \in [t^*,\infty)$ the solution to the problem (2.24) coincides with the solution of*

$$\begin{aligned} &\dot{x}(t) = (a+b)x(t), && t \geq t^*, \\ &x(t^*) = \varphi(t^*) \leq 0. \end{aligned} \tag{2.26}$$

*Proof.* It follows from (2.25) that there exists $\varepsilon > 0$ such that the solution to the problem (2.24) coincides with the solution to the problem (2.26) for all $t \in [t^*,t^*+\varepsilon]$. By (2.17) and continuity of $\dot{x}$ we obtain $\dot{x}(t) \geq 0$ for all $t \in [t^*,t^*+\varepsilon]$ then $x$ is nondecreasing on $[t^*,t^*+\varepsilon]$ and, therefore, maximum is attained at the right end of $[t-h,t]$ for all $t \in [t^*,t^*+\varepsilon]$. Since the solution to (2.26), $x(t) = \varphi(t^*)e^{(a+b)(t-t^*)}$ is nondecreasing function for all $t \geq t^*$, max is taken at the right end of $[t-h,t]$ for all $t \geq t^*$, thus the solution to the problem (2.24) coincides with the solution of the problem (2.26) for all $t \geq t^*$. $\square$

*Remark* 5. Lemma 5 shows that the problem with maximum (2.24) reduces to an ODE for all $t \geq t^*$.

**Proposition 5.** *Assume $t_0 \geq 0$, $h > 0$, $b > 0$ and function $x \in C([t_0 - h, \infty); \mathbb{R})$ is such that*

$$\begin{aligned} \dot{x}(t) &= -b x_h^{\vee}(t), \quad t \in [t_0, \infty), \\ x_h^{\vee}(t_0) &=: K < 0. \end{aligned} \tag{2.27}$$

*Then $x$ is increasing function for all $t \in [t_0, \infty)$ and $x(t) < 0$ for all $t \in [t_0, \infty)$.*

*Proof.* Let $t_1 = \sup\{t > t_0 : x(s) < 0 \text{ for all } s \in [t_0 - h, t]\}$. Then $x_h^{\vee}(t) < 0$ for all $t \in [t_0, t_1)$, $\dot{x}(t) > 0$ for all $t \in [t_0, t_1)$ therefore $x$ is increasing function on $[t_0, t_1)$. Assume $t_1 < \infty$, then $x(t_1) = 0$ and there exists $t_2 \in [t_0, t_1]$ such that $x(t_2) = K$. Then $x_h^{\vee}(t) = x(t)$ for all $t \in [t_2, t_1]$ and by Lemma 5 it follows $x(t) = Ke^{-b(t_2 - t)} \Longrightarrow x(t_1) = Ke^{-b(t_2 - t_1)} \neq 0$. Hence $x(t) < 0$ for all $t \in [t_0, \infty)$ and $x$ is increasing function on $[t_0, \infty)$. □

## 2.4 Stability for one dimensional differential equation with max-operator

Here we consider scalar equations with maximum without input. In [99] it is proved that the trivial solution to the constant delay equation

$$\dot{x}(t) = -b x(t - h), \quad t \in [0, \infty), \ h > 0, \ b > 0,$$

is GES if and only if $bh \in [0, \frac{\pi}{2})$. The following theorem shows that GES of the trivial solution to the equation with maximum does not depend on the value of $h$.

**Theorem 6.** *Let $h > 0$, $b > 0$, $\varphi \in C([-h, 0]; \mathbb{R})$. Suppose $x(\cdot) = x(\cdot\,; \varphi, h)$ is the solution to the Cauchy problem*

$$\begin{aligned} \dot{x}(t) &= -b x_h^{\vee}(t), \quad t \geq 0 \\ x(t) &= \varphi(t), \qquad [-h, 0], \end{aligned} \tag{2.28}$$

*Then the trivial solution to the problem (2.28) is GES.*

*Proof.* By Theorem 3.1 [93] it follows that the trivial solution of the problem (2.28) is asymptotically stable. Notice, that system (2.28) is a particular case of (2.16) for which Theorem 4 holds. Assume that $D_{x(\cdot)}$ is bounded, then for some $t^* \geq 0$ by Remark 4 and Lemma 5, it follows that the solution to the problem (2.28) coincides with the solution to

$$\begin{aligned} \dot{x}(t) &= -b x(t), \quad t \geq t^*, \\ x(t^*) &= x_h^{\vee}(t^*). \end{aligned}$$

Then $|x(t)| \leq |x(t^*)| e^{-b(t - t^*)}$ for all $t \geq t^*$ and therefore, there is $c = c(\varphi, h, t^*) > 0$ such that $|x(t)| \leq ce^{-bt} ||\varphi||$ for all $t \geq 0$. Next, assume $D_{x(\cdot)}$ is unbounded, then by definition of $D_{x(\cdot)}$ for all $t \geq 0$ the solution for problem (2.28) coincides with the solution to the delay differential equation

$$\begin{aligned} &\dot{x}(t) = -bx(t-h), && t \geq 0 \\ &x(t) = \varphi(t), && t \in [-h, 0]. \end{aligned} \tag{2.29}$$

By Remark 2.2 [57] asymptotic stability of DDE (2.29) is equivalent to GES of (2.29). Thus theorem is proved. □

**Proposition 7.** *Let $c \in \mathbb{R}$, $h > 0$ and $\varphi \in C([-h, 0]; \mathbb{R})$. Then the trivial solution to*

$$\begin{aligned} &\dot{x}(t) = -bx_h^{\vee}(t) + c, && t \geq 0, \\ &x(t) = \varphi(t), && t \in [-h, 0], \end{aligned} \tag{2.30}$$

*is GES.*

*Proof.* Denote $z(t) := x(t) - \frac{c}{b}$. Then

$$\begin{aligned} &\dot{z}(t) = -bz_h^{\vee}(t), && t \geq 0, \\ &z(t) = \varphi(t) - \tfrac{c}{b}, && t \in [-h, 0]. \end{aligned} \tag{2.31}$$

Applying Theorem 6 to the problem (2.31) we obtain the GES of the system (2.31) and, therefore, of (2.30).

□

Stability of the problem

$$\begin{aligned} &\dot{x}(t) = -ax(t) - bx_h^{\vee}(t), && t \geq 0, \\ &x(t) = \varphi(t), && t \in [-h, 0], \end{aligned} \tag{2.32}$$

where $a, b \in \mathbb{R}$, $h > 0$, $\varphi \in C([-h, 0]; \mathbb{R})$ is studied in [93]. There it is shown that the trivial solution of the system (2.32) is GAS if and only if $a, b$ satisfy

$$\begin{aligned} &b > -a, && \text{for} \quad a \geq -1/h, \\ &b > \tfrac{1}{h} e^{-ah-1}, && \text{for} \quad a \leq -1/h. \end{aligned} \tag{2.33}$$

See Figure 2.4.

Recall that the trivial solution to the delay differential equation in the form

$$\begin{aligned} &\dot{x}(t) = -ax(t) - bx(t-h), && t \geq 0, \\ &x(t) = \varphi(t), && t \in [-h, 0], \end{aligned} \tag{2.34}$$

where $a, b \in \mathbb{R}$, $h > 0$, $\varphi \in C([-h, 0]; \mathbb{R})$, is GAS if and only if $a, b$ satisfy

$$a > -\frac{1}{h}, \quad -a < b < r(a),$$

where $r(a) = \lambda(h \sin \lambda)^{-1}$, $\lambda$ is the unique root of $a = -\lambda h^{-1} cotg \lambda$ in the interval $(0, \pi)$ [5], [40, pp.134-135]. See Figure 2.5.

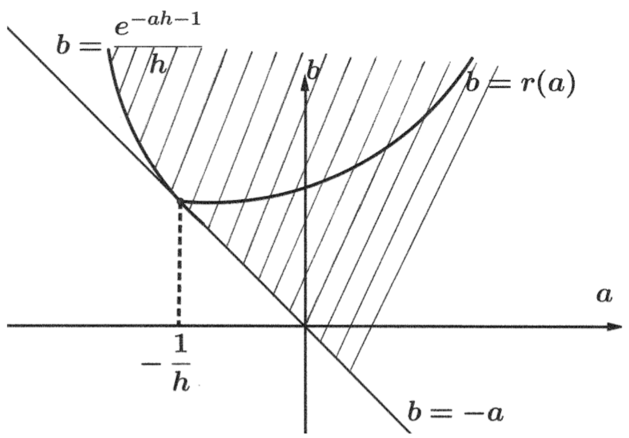


FIGURE 2.4: *Shaded region stands for the set of parameters for which the trivial solution to the problem (2.32) is GAS.*

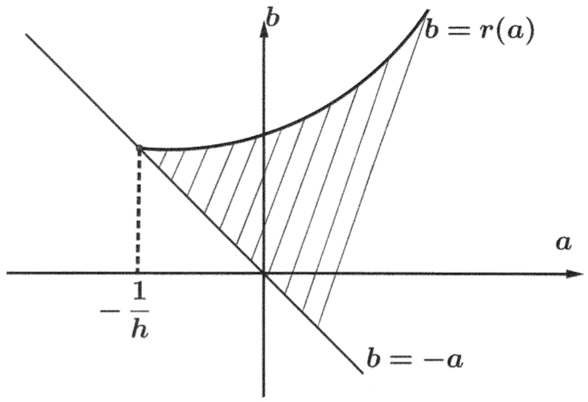


FIGURE 2.5: *Shaded region stands for the set of parameters for which the trivial solution to the system (2.34) is GAS.*

Compare shaded regions on the Figure 2.4 and Figure 2.5 one observes that the set of parameters $a, b, h$ for which the trivial solution to the problem with maximum (2.32) is GAS includes the one to the problem with delay (2.34).

## 2.5 Input-to-state stability of the systems with maximum in the linear form

Consider the following system of differential equations with maximum in the linear form

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)x_h^{\vee}(t) + u(t), && t \geq 0, \\ &x(t) = \varphi(t), && t \in [-h, 0], \end{aligned} \tag{2.35}$$

where $x(t) \in \mathbb{R}^n$, $A, B \in C([0,\infty); \mathbb{R}^{n\times n})$, initial function $\varphi \in C([-h,0]; \mathbb{R}^n)$, input $u \in L_\infty([0,\infty); \mathbb{R}^n)$.

The right-hand side of (2.35) satisfies Carathéodory conditions thus by Theorem 2, for every input $u \in L_\infty([0,\infty);\mathbb{R}^n)$ and every initial function $\varphi \in C([-h,0];\mathbb{R}^n)$ there exists a unique maximal solution $x(\cdot) := x(\cdot\,;\varphi,h,u)$ to the problem (2.35) on $[0,T_f)$, $T_f > 0$. Since the right-hand side of (2.35) is sublinear, $T_f = +\infty$ (see §2.2 [58]).

The scalar case of the system (2.35) with a periodic input $u(\cdot)$ is studied in [48, 13] where several stability properties are obtained.

Now we investigate whether the system (2.35) is ISS and what is the corresponding gain function.

First, derive the following auxiliary result for the scalar case.
Consider the differential inequality

$$\dot{x}(t) \leq -a(t)x(t) + b(t)x_h^\vee(t) + w(t), \quad t \geq 0, \tag{2.36}$$

with a non-negative initial function

$$x(t) = \varphi(t), \quad t \in [-h,0], \tag{2.37}$$

where $x(t) \in \mathbb{R}$; $\varphi \in C([-h,0];\mathbb{R}_+)$; $h > 0$, $a,b \in C([0,\infty);(0,\infty))$, input $w \in L_\infty([0,\infty);\mathbb{R})$.

**Lemma 6.** *Let $x \in AC([0,\infty);\mathbb{R}_+)$ be a solution to (2.36),(2.37). Assume $a,b \in C([0,\infty);(0,\infty))$ are such that*

$$\forall t \in [0,\infty): \quad a(t) - b(t) \geq \delta > 0,$$

*where $\delta := \inf\limits_{t\in[0,\infty)} (a(t)-b(t))$. Then there exists $\lambda > 0$ such that*

$$x(t) \leq \max\{1,h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{\|w\|}{\delta}, \quad t \geq 0. \tag{2.38}$$

*Proof.* Define the function $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$H(t,\lambda^*) := \lambda^* - a(t) + b(t)e^{\lambda^* h}, \quad \lambda^* \in \mathbb{R}_+, \quad t \in \mathbb{R}_+. \tag{2.39}$$

Set

$$F(\lambda^*) := \sup_{t\in[0,\infty)} H(t,\lambda^*), \quad \lambda^* \in \mathbb{R}_+. \tag{2.40}$$

Then

$$\begin{aligned} F(0) &= \sup_{t\in[0,\infty)} H(t,0) = \sup_{t\in[0,\infty)} (-a(t)+b(t)) \\ &= -(-\sup_{t\in[0,\infty)} (-[a(t)-b(t)])) = -\inf_{t\in[0,\infty)} (a(t)-b(t)) = -\delta < 0. \end{aligned} \tag{2.41}$$

Write

$$\begin{aligned} \underline{a} &:= \inf_{t\in[0,\infty)} a(t), \quad \underline{b} := \inf_{t\in[0,\infty)} b(t), \\ \overline{a} &:= \sup_{t\in[0,\infty)} a(t), \quad \overline{b} := \sup_{t\in[0,\infty)} b(t), \end{aligned}$$

since functions $a, b$ are bounded on $[0, \infty)$ by assumption $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in \mathbb{R}_+$ are finite numbers. Set $\overline{F}(\lambda^*) := \lambda^* - \underline{a} + \overline{b}e^{\lambda^* h}$, $\underline{F}(\lambda^*) := \lambda^* - \overline{a} + \underline{b}e^{\lambda^* h}$ then

$$\underline{F}(\lambda^*) \leq F(\lambda^*) \leq \overline{F}(\lambda^*), \quad \lambda^* \in \mathbb{R}_+.$$

Since $\underline{F}(\lambda^*) \to \infty$, $\overline{F}(\lambda^*) \to \infty$, when $\lambda^* \to \infty$, one obtains

$$F(\lambda^*) \to \infty, \quad \text{as } \lambda^* \to \infty. \tag{2.42}$$

Let $\delta^*$ be such that $0 < \delta^* < \delta$. Then from (2.41), (2.42) and by the continuity of $F$ it follows that there exists $\lambda > 0$ such that,

$$F(\lambda) = \sup_{t \in [0,\infty)} H(t, \lambda) = \sup_{t \in [0,\infty)} (\lambda - a(t) + b(t)e^{\lambda h}) = -\delta^* < 0.$$

Hence

$$\forall t \in [0, \infty): \quad \lambda - a(t) + b(t)e^{\lambda h} \leq -\delta^* < 0. \tag{2.43}$$

For any $\varepsilon > 0$ define the function $q \in C([-h, \infty); \mathbb{R}_+)$ by

$$q(t) := [\max\{1, h\} \|\varphi\| + \varepsilon] \, e^{-\lambda t} + \frac{\|w\|}{\delta}.$$

Observe that $q(t) > x(t)$ for all $t \in [-h, 0]$ by this definition. We prove that $q(t) > x(t)$ for all $t \in (0, \infty)$. Suppose this is not true, i.e., there exists $t_1 > 0$ such that $q(t_1) = x(t_1)$ and $q(t) > x(t)$ for all $t \in (0, t_1)$. Therefore, for the left side derivative the following holds

$$\dot{q}(t_1-) \leq \dot{x}(t_1-).$$

At the same time

$$\begin{aligned}
\dot{q}(t_1-) &= -\lambda e^{-\lambda t_1} [\max\{1, h\} \|\varphi\| + \varepsilon] > (-a(t_1) + b(t_1)e^{\lambda h})e^{-\lambda t_1} [\max\{1, h\} \|\varphi\| + \varepsilon] \\
&= -a(t_1)e^{-\lambda t_1} [\max\{1, h\} \|\varphi\| + \varepsilon] + b(t_1)e^{\lambda h}e^{-\lambda t_1} [\max\{1, h\} \|\varphi\| + \varepsilon] \\
&= -a(t_1) \left[q(t_1) - \frac{\|w\|}{\delta}\right] b(t_1) \left[q(t_1 - h) - \frac{\|w\|}{\delta}\right] \\
&= -a(t_1)x(t_1) + b(t_1)q(t_1 - h) - (-a(t_1) + b(t_1))\frac{\|w\|}{\delta}.
\end{aligned}$$

Let $t' \in [t_1 - h, t_1]$ be such that $x_h^\vee(t_1) = x(t')$.

In case $t_1 \geq h$ we have $t' > 0$ and from the choice of the point $t_1$ and monotonicity (decreasing) of the function $q$ on $[0, \infty)$ we obtain $x(t') \leq q(t') < q(t_1 - h)$.

In case $t_1 < h$ if $t' < 0$ then $x(t') \leq \|\varphi\| < q(t_1 - h)$. Therefore,

$$\dot{q}(t_1-) > -a(t_1)x(t_1) + b(t_1)x_h^\vee(t_1) + \|w\| > \dot{x}(t_1-).$$

The obtained contradiction proves the inequality $q(t) > x(t)$ for all $t \in (0, \infty)$ and the claim of lemma. $\square$

*Remark* 6. Let $a, b \in (0, \infty)$ with $a - b > 0$, $h \in \mathbb{R}_+$, $w \in L_\infty([0,\infty);\mathbb{R})$, and $\varphi \in C([-h,0];\mathbb{R})$. Then Lemma 6 is also true for the following integral inequality

$$|x(t)| \leq ||\varphi||e^{-at} + \int_0^t e^{-a(t-\tau)}\Big(b|x_h^\vee|(\tau) + |w(\tau)|\Big)d\tau, \quad t \in [0,\infty). \tag{2.44}$$

To see this observe, that for the constant coefficients $a, b$ differential inequality (2.36) is equivalent to

$$x(t) \leq \varphi(0)e^{-at} + \int_0^t e^{-a(t-\tau)}\Big(bx_h^\vee(\tau) + w(\tau)\Big)d\tau, \quad t \in [0,\infty).$$

Then for all $t \in [0,\infty)$ we obtain

$$|x(t)| \quad \leq \quad ||\varphi||e^{-at} + \int_0^t e^{-a(t-\tau)}\Big(b|x_h^\vee(\tau)| + |w(\tau)|\Big)d\tau \tag{2.45}$$

$$\overset{(1.10)}{\leq} \quad ||\varphi||e^{-at} + \int_0^t e^{-a(t-\tau)}\Big(b|x_h^\vee|(\tau) + |w(\tau)|\Big)d\tau. \tag{2.46}$$

Assume that the following assumption hold for the system (2.35):

**Assumption 3.**

*(i) Let $\tilde{b} \in \mathbb{R}_+$, $B \in C([0,\infty);\mathbb{R}^{n\times n})$ in (2.35) be such that $\|B(t)\| \leq \tilde{b}$ for all $t \in [0,\infty)$.*

*(ii) Let there exist a function*

$$V : \mathbb{R}^m \to \mathbb{R}, \quad y \mapsto V(y) = y^T P y,$$

*where $P \in \mathbb{R}^{n\times n}$ is a positive definite, symmetric matrix $P \in \mathbb{R}^{n\times n}$, i.e. for some constants $\alpha_2 \geq \alpha_1 > 0$*

$$\alpha_1|y|^2 \leq V(y) \leq \alpha_2|y|^2 \quad \textit{for all } y \in \mathbb{R}^n. \tag{2.47}$$

*(iii) The inequality*

$$y^T(PA(t) + A^T(t)P)y \leq -\alpha_3|y|^2 \textit{ for all } t \geq 0, \quad y \in \mathbb{R}^n,$$

*holds with*

$$\alpha_3 > 2n\tilde{b}m\frac{\alpha_2}{\alpha}, \tag{2.48}$$

*where $\alpha = \min\{1, \alpha_1\}$, $m = ||P||$.*

**Theorem 8.** *Let $\tilde{b} \in \mathbb{R}_+, n \in \mathbb{N}$, $h, \lambda, \alpha, m, \alpha_2, \alpha_3 > 0$, Assumption 3 hold, $u \in L_\infty([0,\infty);\mathbb{R}^n)$ and $\varphi \in C([-h,0];\mathbb{R}^n)$. Then the following estimate for the solution to the problem (2.35) holds*

$$|x(t)| \le \frac{1}{\sqrt{\alpha_1}} \max\{1,h\} \|\varphi\| e^{-\lambda t} + \frac{2\alpha_2 m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m} \|u\|, \ t \ge 0.$$

*Proof.* Compute the derivative $\dot{V}(x(t))$ along the solution $x$

$$\begin{aligned} \dot{V}(x(t)) &= x^T(t) \left(PA(t) + A^T(t)P\right) x(t) \\ &+ \left(B^T(t)x_h^{\vee T}(t) + u^T(t)\right) Px(t) + x^T(t)P\left(B(t)x_h^\vee(t) + u(t)\right) \\ &\le -\alpha_3 |x(t)|^2 + 2|x(t)| \|B(t)P\| \left(|x_h^\vee(t)|\right) + 2|x(t)| \|P\| \|u\|. \end{aligned} \tag{2.49}$$

Let $\varepsilon > 0$ be an arbitrary small number. Define the function $R : [-h,\infty) \to (0,\infty)$ such that

$$R^2(t) := \begin{cases} V(x(t)) + \varepsilon, & t \ge 0, \\ |\varphi(t)|^2 + \varepsilon, & t \in [-h,0). \end{cases} \tag{2.50}$$

From (2.47) it follows that for all $t \in [0,\infty)$

$$|x(t)| \le \sqrt{\frac{V(x(t))}{\alpha_1}} < \sqrt{\frac{V(x(t)) + \varepsilon}{\alpha_1}} = \sqrt{\frac{R^2(t)}{\alpha_1}} = \frac{R(t)}{\sqrt{\alpha_1}} \le \frac{R(t)}{\sqrt{\alpha}}.$$

Let $\xi \in [-h,\infty)$ be such that $\max\limits_{s\in[t-h,t]} |x(s)| = |x(\xi)|$. Assume $t \in (h,\infty)$. Then $\xi > 0$ and by (1.9) we obtain

$$\begin{aligned} |x_h^\vee(t)| &\le n \max_{s\in[t-h,t]} |x(s)| = n|x(\xi)| \le n\frac{\sqrt{V(x(\xi))}}{\sqrt{\alpha_1}} \\ &< n\frac{\sqrt{V(x(\xi)) + \varepsilon}}{\sqrt{\alpha_1}} = n\frac{R(\xi)}{\sqrt{\alpha_1}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha_1}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha}}. \end{aligned} \tag{2.51}$$

Now, let $t \in [0,h]$. If $\xi \in (0,\infty)$ then inequality (2.51) holds. If $\xi \in [-h,0]$ then

$$|x_h^\vee(t)| \le n \max_{s\in[t-h,t]} |x(s)| = n|x(\xi)| = n|\varphi(\xi)| < nR(\xi) \le n\frac{R(\xi)}{\sqrt{\alpha}} \le n\frac{R_h^\vee(t)}{\sqrt{\alpha}}.$$

Then for all $t \in [0,\infty)$ we obtain

$$2R(t)\dot{R}(t) = \dot{V}(x(t)) \le -\alpha_3\frac{R^2(t)}{\alpha_2} + 2R(t)\frac{n\tilde{b}m}{\alpha}R_h^\vee(t) + \frac{2R(t)}{\sqrt{\alpha}}m\|u\|,$$

or

$$\dot{R}(t) \le -\alpha_3\frac{R(t)}{2\alpha_2} + \frac{n\tilde{b}m}{\alpha}R_h^\vee(t) + \frac{m\|u\|}{\sqrt{\alpha_1}}.$$

By Lemma 6 with

$$a(t) = \frac{\alpha_3}{2\alpha_2}, \qquad b(t) = \frac{n\tilde{b}m}{\alpha}, \qquad w(t) = \frac{m\,\|u\|}{\sqrt{\alpha_1}}, \tag{2.52}$$

there exists $\lambda > 0$ such that

$$R(t) \leq \max\{1, h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{2\alpha_2\sqrt{\alpha_1}m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m}\,\|u\|\,.$$

Since $\varepsilon > 0$ is an arbitrary number we can take the limit for $\varepsilon \to 0$ and obtain

$$\sqrt{V(x(t))} \leq \max\{1, h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{2\alpha_2\sqrt{\alpha_1}m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m}\,\|u\|\,.$$

Therefore from $|x(t)| \leq \frac{\sqrt{V(x(t))}}{\sqrt{\alpha_1}}$ for all $t \in [0, \infty)$ we obtain

$$|x(t)| \leq \frac{1}{\sqrt{\alpha_1}}\max\{1, h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{2\alpha_2 m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m}\,\|u\|\,.$$

□

*Remark* 7. Note that from the proof of the Theorem 8 it follows that system (2.35) is not only ISS but even eISS.

**Corollary 9.** *The inequality (2.48) implies* $\frac{\alpha_3\alpha}{2nm\alpha} > \tilde{b} \geq ||B(t)||$ *for all* $t \in [0, \infty)$*. Therefore Theorem 8 provides upper bound for* $||B(t)||$ *such that GES of* $\dot{x}(t) = A(t)x(t)$ *implies eISS of the problem with maximum (2.35).*

The next statement follows immediately from the Theorem 8.

**Corollary 10.** *Let problem (2.35) be eISS. Then problem (2.35) is 0-GES.*

In case of constant coefficients the proof of the Theorem 8 can be sufficiently simplified. Consider system

$$\dot{x}(t) = Ax(t) + Bx_h^{\vee}(t) + u(t), \quad t \geq 0, \tag{2.53}$$

where $h > 0, A, B \in \mathbb{R}^{n\times n}$, $u \in L_\infty([0, \infty); \mathbb{R}^n)$. With initial condition

$$x(t) = \varphi(t), \qquad t \in [-h, 0],$$

where $\varphi \in C([-h, 0]; \mathbb{R}^n)$.

**Theorem 11.** *Let* $h > 0, n \in \mathbb{N}, A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n}$*. Assume there exist* $k > 0, \mu > 0$ *such that*

$$||e^{At}|| \leq ke^{-\mu t} \quad \textit{for all} \;\; t \in [0, \infty), \tag{2.54}$$

*and*

$$||B|| < \frac{\mu}{kn}. \tag{2.55}$$

*Then system (2.53) is eISS.*

*Proof.* Multiplying equation (2.35) from the both sides by $e^{-At}$ we obtain

$$\forall t \geq 0: \quad e^{-At}\dot{x}(t) = Ae^{-At}x(t) + e^{-At}(B\check{x}_h(t) + u(t)),$$

and, hence

$$\forall t \geq 0: \quad (e^{-At}x(t))'_t = e^{-At}(Bx_h^{\vee}(t) + u(t)). \tag{2.56}$$

Take the integral on $[0, t]$ from the both sides of (2.56) we have

$$\forall t \geq 0: \quad x(t) = \varphi(0)e^{At} + \int_0^t e^{A(t-s)}(Bx_h^{\vee}(s) + u(s))ds. \tag{2.57}$$

Take norm from (2.57), for all $t \in [0, \infty)$

$$|x(t)| \leq ||\varphi||\,||e^{At}|| + \int_0^t ||e^{A(t-s)}||(||B||\,|x_h^{\vee}(s)| + |u(s)|)ds \tag{2.58}$$

$$\overset{(2.54),(1.10)}{\leq} k||\varphi||e^{-\mu t} + k\int_0^t e^{-\mu(t-s)}(n||B||\,|x_h^{\vee}|(s) + |u(s)|)ds. \tag{2.59}$$

Under the assumptions $\mu, k > 0$ and (2.55), the assertions of the Lemma 6 are satisfied for inequality (2.58), therefore from the Remark 6 and Lemma 6 it follows that there exists $\lambda > 0$, such that for all $t \in [0, \infty)$ the following holds

$$|x(t)| \leq k \max\{1, h\}\,||\varphi||e^{-\lambda t} + \frac{||u||}{\mu - nk||B||}.$$

Which proofs the theorem. □

### 2.5.1 Examples

**Example 5.** *Let $h > 0$, $a > b > 0$, $u \in L_\infty([0, \infty); \mathbb{R})$, $\varphi \in C([-h, 0]; \mathbb{R})$. Consider a scalar system with input*

$$\begin{array}{ll} \dot{x}(t) = -ax(t) + bx_h^{\vee}(t) + u(t), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-h, 0], \end{array}$$

*Applying Theorem 8 with $A(t) = -a$, $B(t) = b$, $P = m = 1$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2a > 2b$ we get $\gamma = \frac{1}{a-b}$ and*

$$|x(t)| \leq \max\{1, h\}\,\|\varphi\|\,e^{-\lambda t} + \frac{1}{a-b}\,\|u\|\,, \quad t \geq 0,$$

*for some $\lambda > 0$ i.e., the considered problem is eISS.*

**Example 6.** *Consider*

$$\begin{aligned} \dot{x}(t) &= -2x(t) + \sin t\, x_h^{\vee}(t) + u(t), && t \geq 0, \\ x(t) &= 1, && t \in [-h, 0]. \end{aligned} \tag{2.60}$$

*for $h = 2$. Check the eISS property. Apply Theorem 8 in case $A(t) = -2$, $B(t) = \sin t$, $|\sin t| \leq 1 =: \tilde{b}$ for all $t \in [0, \infty)$ and check the conditions of the Theorem 8:*

*(i) choose $V(x) = x^2$ then $P = 1 = m$ and $\alpha_1 = \alpha_2 = \alpha = 1$;*

*(ii) $A(t) = -2$ then $2x(-2x) = -4x^2$ and the inequality $\alpha_3 = 4 > 2$ holds.*

*Since the conditions of Theorem 8 hold then by Theorem 8 there exists $\lambda > 0$ such that*

$$\beta(\|\varphi\|, t) = \frac{1}{\sqrt{\alpha_1}} \max\{1, h\} \|\varphi\| e^{-\lambda t} = 2e^{-\lambda t}, \quad t \geq 0.$$

$$\gamma(\|u\|) = \frac{2\alpha_2 m}{\alpha\alpha_3 - 2\alpha_2 n\tilde{b}m} \|u\| = \|u\|,$$

*and, therefore*

$$|x(t)| \leq 2e^{-\lambda t} + ||u||, \quad t \geq 0.$$

*One can check that $\lambda = 0.27$ satisfies $\lambda - 2 + e^{2\lambda} < 0$. The Figure 2.6 illustrates the behavior of solution to the problem (2.60) and its input-to-state estimate in case $u(t) = \cos t$ for all $t \in [0, \infty)$.*

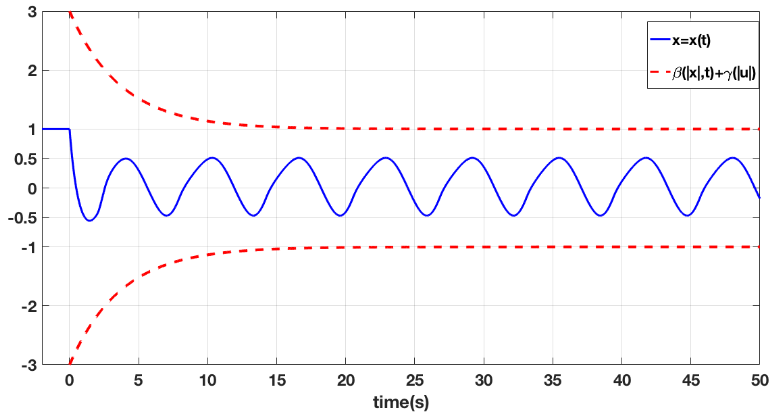


FIGURE 2.6: *Graph of the solution to the problem (2.60) and its boundedness.*

**Example 7.** *Consider the following system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(t)x_h^{\vee}(t) + u(t), && t \geq 0 \\ x(t) &= \varphi(t), && t \in [-h, 0]. \end{aligned} \tag{2.61}$$

*where*

$$A = \begin{pmatrix} -3 & 1 \\ -\frac{1}{4} & -2 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{3}\cos t & 0 \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

$h = 1$, $\varphi \in C([-1, 0]; \mathbb{R}^2)$, $u \in L_\infty([0, \infty); \mathbb{R}^2)$ *Find the ISS estimate for the solution of the problem (2.61) using Theorem 8. Let* $\tilde{b} := \frac{1}{3} = ||B(t)||$ *for all* $t \in [0, \infty)$. *Check Assumption 3 (ii),(iii). Define*

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

*then* $||P|| = m = 2$ *and*

$$V(x) = x^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = 2x_1^2 + 2x_2^2 \quad \text{for all } x \in \mathbb{R}^2.$$

*Then the estimate (2.47) holds for* $\alpha_1 = \alpha_2 = 2$. *Find* $\alpha_3$ *such that the following inequality holds*

$$\begin{aligned} x^T(PA + A^TP)x &= x^T \left[ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -\frac{1}{4} & -2 \end{pmatrix} + \begin{pmatrix} -3 & -\frac{1}{4} \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] x \\ &= x^T \begin{pmatrix} -12 & \frac{3}{2} \\ \frac{3}{2} & -8 \end{pmatrix} x = -x^TQx \leq -\tilde{\lambda}_{min}(Q)|x|^2 = -\alpha_3|x|^2, \end{aligned} \tag{2.62}$$

*where*

$$Q = \begin{pmatrix} 12 & -\frac{3}{2} \\ -\frac{3}{2} & 8 \end{pmatrix},$$

*and* $\alpha_3 = \tilde{\lambda}_{min}(Q) = \min\{\tilde{\lambda}_1, \tilde{\lambda}_1\}$. *Here* $\tilde{\lambda}_1 = \frac{25}{2}$, $\tilde{\lambda}_2 = \frac{15}{2}$ *are eigenvalues of* $Q$. *Notice that Assumption 3 (ii),(iii) holds with* $\alpha_3 = \frac{15}{2} > \frac{16}{3}$. *Hence, the conditions of the Theorem 8 hold and one obtains the following estimate for the solution of the problem (2.61)*

$$|x(t)| \leq \frac{1}{\sqrt{2}}||\varphi||e^{-\lambda t} + \frac{48}{13}||u||, \quad t \geq 0,$$

*where* $\lambda > 0$ *satisfies* $\lambda - \frac{15}{8} + \frac{4}{3}e^\lambda < 0$.

## 2.6 Concluding remarks and open problems

In this chapter we have provided analysis of robustness properties of dynamical systems with maximum using neither Lyapunov-Krasovskiy approach nor Razumikhin approach. Nevertheless, many questions in the context of perturbed dynamical systems with maximum remain to be solved. Just a few are named.

1. In section 2.3 we have proved that infinite-dimensional systems in the linear form may reduce to an one-dimensional system along its solution. It is likely that the time instance $t^*$ of changing dimension depends on parameters $a, b, h$ and input function, to find this dependence is of interest.

2. Extension of the result of Theorem 4 to the case of the system $\dot{x}(t) = f(t, x(t), x_h^\vee(t), u(t))$ is of avenue for future research.

3. Under Assumption 3 the trivial solution to the ODE $\dot{x}(t) = A(t)x(t)$ is asymptotically stable and therefore $A(t) \neq 0$ for all $t \in [0, \infty)$. Hence Theorem 8 does not cover the case of system $\dot{x}(t) = B(t)x_h^\vee(t) + u(t)$.

Some results from this chapter were presented on the 20th IFAC World Congress [22].

# Chapter 3

# Input-to-state stability for differential equations with maximum via averaging method

Averaging method appeared with need to solve the problems of celestial mechanics. The idea of the averaging method is, that the right-hand side of a time varying system of differential equations is replaced by averaged one, i.e. without explicit time dependence. The justification of averaging method was proposed in works [17, 60]. There authors shown that under certain conditions a solution of a time-varying (original) system is approximated by a solution of time-invariant (averaged) system.

Observe that averaging method is an effective tool to study the stability of time-varying systems [31, 32, 30, 4]. Its application to the stability analysis of ODE systems under the action of some perturbations is introduced in [73]. Where for nonautonomous (original) systems with input, definitions of strong averaged and weak averaged systems are introduced. It is shown that existence of ISS-Lyapunov function of weak averaged and strong averaged systems provides some ISS properties of original system. Extension of these results to RFDE systems is available in [98, 97]. In particular, in [98] it is shown that existence of ISS-Razumikhin function for strong averaged system of RFDE implies ISpS of original system.

To the best of our knowledge, either the proof of closeness solutions of original and averaged systems or ISS analysis of RFDE system via averaging is available in the literature only with usage of Lyapunov technique. In this chapter, on the base of trajectory approach, we obtain the justification of averaging method for system of differential equations with maximum and input, and apply averaging for the ISS analysis.

This chapter is organized as follows: the classical averaging method for ODEs system is recalled in section 3.1. In section 3.2 definitions of strong and weak averaged systems are introduced, the difference between them is discussed. Next, in section 3.3, without usage of any Lyapunov techniques, we obtain the justification of averaging method for differential equations with maximum and input on a finite time interval. Both cases (the existence of strong and weak averages) are considered. An illustrative example is provided. In section 3.4 the results from the previous section are extended to an infinite time interval, and by the trajectory estimate it is proved that exponential incremental input-to-state stability (e$\delta$ISS) of averaged (strong averaged and weak averaged) system implies exponential input-to-state practical stability (eISpS) of the original system. In the last section we conclude results of this chapter and provide possible direction for future research.

## 3.1 Averaging method for ODE

The classical averaging method is applied to the system of ODE in the standard (by Bogolyubov) form

$$\begin{aligned} &\dot{x}(t) = \varepsilon F(t, x(t)), \quad t \geq 0, \\ &x(0) = x_0, \end{aligned} \tag{3.1}$$

where $\varepsilon > 0$ is a parameter, $x_0 \in \mathbb{R}^n$, $F : \mathbb{R}_+ \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$.

**Definition 6.** [50, p.36] Suppose that $F : \mathbb{R}_+ \times D \to \mathbb{R}^n$ is continues and uniformly bounded such that $|F(t,x)| \leq K$ for all $(t,x) \in \mathbb{R}_+ \times D$, $F$ is Lipschitz continuous i.e, for all $(t,x^1) \in \mathbb{R}_+ \times D$ and for all $(t,x^2) \in \mathbb{R}_+ \times D$ there exists a $Q > 0$ such that $|F(t,x^1) - F(t,x^2)| \leq Q||x^1 - x^2||$. Furthermore, suppose that the average

$$\overline{F}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T F(\tau, x) d\tau, \quad T > 0, \tag{3.2}$$

exists uniformly for all $x \in D$. Then $F$ is called *KBM-vectorfield* (KBM stands for Krylov, Bogolyubov, Mitropolskiy).

The following time-invariant system

$$\begin{aligned} &\dot{y}(t) = \varepsilon \overline{F}(y(t)), \quad t \geq 0, \\ &y(0) = x_0. \end{aligned} \tag{3.3}$$

is called *averaged* system.

**Theorem 12.** *[17, §26]. Assume*

1. *$F$ is KBM-vectorfield with average $\overline{F}$.*
2. *The solution $y(\cdot)$ of the problem (3.3) belongs to an interior subset of $D$ on $[t_0, \infty)$.*

*Then for arbitrary small $\eta > 0$ and arbitrary large $L > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, and for all $t \in [0, \frac{L}{\varepsilon}]$ the estimate $|x(t) - y(t)| \leq \eta$ holds.*

Theorem 12 is known as Bogolyubov's theorem.

**Theorem 13.** *[12] Let conditions 1,2 of Theorem 12 hold. Assume that there exists a trivial solution to the problem (3.3) and it is asymptotically stable. Then for arbitrary small $\eta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, and for all $t \in [0, \infty)$ the estimate $|x(t) - y(t)| \leq \eta$ holds.*

*Remark* 8. [17, §26]. If domain $D$ is bounded then the Bogolyubov's theorem remains valid in case uniform existence of the limit (3.2) is relaxed to the existence of it for every fixed $x \in D$.

First extension of averaging method to RFDE system is based on coordinate transformation and assumption that value of delay $h$ in averaged system can be neglected [39] (see also [35] for earliest results). Later in [65] it is shown that, in some cases, approach from [39] leads to a big error (see Ex.5.2 in [65]) and an alternative result is proved. Some averaging results for differential equations with maximum are available in the literature [80, 11, chapter 7].

## 3.2 Weak and strong averages

In this section we introduce the definitions of weak and strong averages originally discussed for ODEs system with input [73].
Let $t_0 \in \mathbb{R}$, $h > 0$. Consider the dynamical system defined as

$$\begin{array}{ll} \dot{x}(t) = \varepsilon f(t, x(t), x_h^{\vee}(t), u(t)), & t \in [t_0, \infty), \\ x(t) = \varphi(t), & t \in [t_0 - h, t_0], \end{array} \tag{3.4}$$

where parameter $\varepsilon$ is positive, state variable $x(\cdot)$ takes value in $D \subset \mathbb{R}^n$ ($D$ is any domain), $u \in L_\infty([t_0, \infty); U)$, $U \subset \mathbb{R}^m$. Let $\varphi \in C([t_0-h, t_0]; D)$ be an initial function. Suppose $f$ satisfies:

**Assumption 4.** *$f \in C([t_0, \infty) \times D \times D \times U; \mathbb{R}^n)$ is uniformly bounded that is there exists $\eta \geq 0$ such that $|f(t, x, y, u)| \leq \eta$ for all $(t, x, y, u) \in [t_0, \infty) \times D \times D \times U$.*

**Assumption 5.** *$f : [t_0, \infty) \times D \times D \times U \to \mathbb{R}^n$ is Lipschitz continuous that is, for all $(x, y, u) \in D \times D \times U$ and for all $(x, y', u') \in D \times D \times U$ there exists $M > 0$ such that*

$$|f(t, x, y, u) - f(t, x', y', u')| \leq M \left( ||x - x'|| + ||y - y'|| + ||u - u'|| \right).$$

Under Assumptions 4, 5 imposed on $f$, by Theorem 2 for every $u \in L_\infty([t_0, \infty); U)$, and for every $\varphi \in C([t_0 - h, t_0]; D)$, $h > 0$ there exists a unique solution

$$x(\cdot; t_0, \varphi, h, u) : [t_0, \infty) \mapsto D,$$

to the problem (3.4).

**Definition 7.** (**Weak average**) A Lipschitz function $f_{wa} : D \times D \times U \to \mathbb{R}^n$ is said to be a *weak average* of map $t \mapsto f(t, x, y, u)$ if there exist function $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that, for all $T \geq T^*$ and for all $t \geq 0$ the following holds:

$$\begin{aligned} \forall x, y \in D, \forall u \in U : \left| f_{wa}(x, y, u) - \frac{1}{T} \int_t^{t+T} f(\tau, x, y, u) d\tau \right| & \\ \leq \beta_{av}(\max\{|x|, |y|, |u|, 1\}, T). & \end{aligned} \tag{3.5}$$

In case the map $f$ in (3.4) admits weak average the system

$$\begin{array}{ll} \dot{y}(t) = \varepsilon f_{wa}(y(t), y_h^{\vee}(t), u(t)), & t \in [t_0, \infty), \\ y(t) = \varphi(t), & t \in [t_0 - h, t_0], \end{array} \tag{3.6}$$

is called *weak averaged* system to the system (3.4).

**Definition 8.** (**Strong average**) A Lipschitz function $f_{sa} : D \times D \times U \to \mathbb{R}^n$ is said to be a *strong average* of map $t \mapsto f(t, x, y, u)$ if there exist function $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that for all $u \in L_\infty([t_0, \infty); U)$, for all $T \geq T^*$ and for all $t \geq 0$ the following holds:

$$\forall x, y \in D, \forall u \in L_\infty([t_0, \infty); U) : \left| \frac{1}{T} \int_t^{t+T} \Big( f_{sa}(x, y, u(\tau)) - f(\tau, x, y, u(\tau)) \Big) d\tau \right| \leq \beta_{av}(\max\{|x|, |y|, ||u||, 1\}, T). \tag{3.7}$$

In case the map $f$ admits strong average the following system is called *strong averaged* system to the system (3.4)

$$\begin{aligned} \dot{y}(t) &= \varepsilon f_{sa}(y(t), y_h^\vee(t), u(t)), \quad && t \in [t_0, \infty), \\ y(t) &= \varphi(t), && t \in [t_0 - h, t_0]. \end{aligned} \tag{3.8}$$

The difference between these two definitions of average is that in (3.5) input $u$ is treated as a constant wheres in (3.7) $u$ is a function from $L_\infty([t_0, \infty); U)$.

Observe that in case $u = 0$ strong and weak averages coincide.
Consider the following example

**Example 8.** *Consider the nonautonomous scalar differential equation*

$$\dot{x}(t) = \left( -x(t) + \frac{2+t}{1+t} u(t) \right), \quad t \geq 0, \tag{3.9}$$

*where $h > 0$, $x(t) \in \mathbb{R}$, $u \in L_\infty([0, \infty); U)$. We show that*

$$f_{wa}(x, u) = -x + u, \tag{3.10}$$

*is the weak average of $f(t, x, u) = -x + \frac{2+t}{1+t} u$ as it is defined in the Definition 7. Indeed, for all $T > 0$ and for all $t \geq 0$ the following holds*

$$\begin{aligned} &\left| -x + u - \frac{1}{T} \int_t^{t+T} \left( -x + \frac{2+\tau}{1+\tau} u \right) d\tau \right| = \left| u - \frac{u}{T} \int_t^{t+T} \left( \frac{2+\tau}{1+\tau} \right) d\tau \right| \\ &= \left| u - \frac{u}{T}(T - \ln(t+T+1) + \ln(t+1)) \right| = \left| \frac{u}{T} \ln\left( \frac{t+1}{t+T+1} \right) \right| \\ &\leq \frac{|u|}{T} \left| \ln(t+1) \right| = \beta_{av}(\max\{|u|, 1\}, T). \end{aligned}$$

*Now we show that (3.10) also is the strong average for $f(t, x, u) = -x + \frac{2+t}{1+t} u$ as it is defined in the Definition 8. To see that, note that for all $T > 0$, for all $u \in$*

$L_\infty([0,\infty);U)$ *and all* $t \geq 0$ *the following holds*

$$\left| \frac{1}{T} \int\limits_{t}^{t+T} \left( -x + u(\tau) + x - \frac{2+\tau}{1+\tau} u(\tau) \right) d\tau \right| = \left| \frac{1}{T} \int\limits_{t}^{t+T} u(\tau) \left( -\frac{1}{1+\tau} \right) d\tau \right|$$
$$\leq \frac{||u||}{T} \int\limits_{t}^{t+T} \frac{d\tau}{1+\tau} = \frac{||u||}{T} \left( \ln(t+T+1) - \ln(t+1) \right) = \frac{||u||}{T} \ln\left( 1 + \frac{T}{t+1} \right)$$
$$\leq \frac{||u||}{T} \ln(1+T) = \beta_{av}(\max\{||u||, 1\}, T).$$

In the considered example the strong average coincides with the weak one, but this is not always the case. Moreover, the existence of the strong average implies the existence of the weak one, however the opposite is not true (see Example 1 in [73]). Some properties of weak and strong averages are considered in [73].

## 3.3 Averaging method for differential equations with maximum and input on a finite time interval

In this section we consider system (3.4) and we show that under certain conditions imposed on $f$ solutions to averaged systems (weak averaged (3.6) and strong averaged (3.8)) approximate solution of the problem (3.4) on a finite time interval.

**Definition 9.** Let $\mathcal{U} \subset L_\infty([t_0,\infty);U)$. The family of functions $\mathcal{U}$ is called *uniformly equicontinuous* if for every $\lambda > 0$ there exists $\delta > 0$ such that $|u(\tau) - u(s)| < \lambda$ for every $u \in \mathcal{U}$ and for all points $\tau, s$ with $|\tau - s| < \delta$.

**Theorem 14.** *Let* $t_0 \in \mathbb{R}$, $h > 0$. *Consider system (3.4). Suppose*

*1) Assumptions 4 and 5 hold;*

*2)* $u \in \mathcal{U}$*;*

*3) there exists a weak average* $f_{wa}$ *of* $f$*;*

*4) for every input signal* $u \in \mathcal{U}$*, for every* $\varphi \in C([t_0 - h, t_0]; D)$ *the solution* $y_{wa}(\cdot; t_0, \varphi, h, u)$ *to the problem (3.6) belongs to an interior subset of* $D$ *on* $[t_0, \infty)$.

*Then for any* $c > 0$*, and any* $L > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(c, L) > 0$*, such that for all* $\varepsilon \in (0, \varepsilon_0]$*, and all* $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$*:*

$$\sup_{t \in [t_0, t_0 + \frac{L}{\varepsilon}]} |x(t; t_0, \varphi, h, u) - y_{wa}(t; t_0, \varphi, h, u)| \leq c. \tag{3.11}$$

*Proof.* For simplicity, set $x(\cdot) := x(\cdot; t_0, \varphi, h, u)$, $y(\cdot) := y_{wa}(\cdot; t_0, \varphi, h, u)$. By Assumptions 4,5 solutions $x(\cdot)$ and $y(\cdot)$ to problems (3.4) and (3.6) resp. exist for all $t \geq t_0$.

Write (3.4) and (3.6) in integral forms

$$\forall t \geq t_0: \ x(t) = \varphi(t_0) + \varepsilon \int\limits_{t_0}^{t} f(\tau, x(\tau), x_h^{\vee}(\tau), u(\tau))d\tau,$$

$$\forall t \geq t_0: \ y(t) = \varphi(t_0) + \varepsilon \int\limits_{t_0}^{t} f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau))d\tau.$$

Subtract $y(\cdot)$ from $x(\cdot)$, add $f(t, y(t), y_h^{\vee}(t), u(t))$ and $-f(t, y(t), y_h^{\vee}(t), u(t))$ to the right part of equation. Then, for all $t \geq t_0$,

$$\begin{aligned} x(t) - y(t) = \varepsilon &\int\limits_{t_0}^{t} \Big( f(\tau, x(\tau), x_h^{\vee}(\tau), u(\tau)) - f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau \\ &+ \varepsilon \int\limits_{t_0}^{t} \Big( f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau. \end{aligned}$$

By the Assumption 5

$$\begin{aligned} \forall t \in [t_0, \infty): \ |x(t) - y(t)| \leq \varepsilon M &\int\limits_{t_0}^{t} \Big( |x(\tau) - y(\tau)| + |x_h^{\vee}(\tau) - y_h^{\vee}(\tau)| \Big) d\tau \\ &+ \varepsilon \left| \int\limits_{t_0}^{t} \Big( f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau \right|. \end{aligned} \tag{3.12}$$

Notice that

$$\begin{aligned} \forall t \in [t_0, \infty): \quad |x_h^{\vee}(t) - y_h^{\vee}(t)| = |(x - y + y)_h^{\vee}(t) - y_h^{\vee}(t)| &\overset{(1.7)}{\leq} |(x - y)_h^{\vee}(t)| \\ &\overset{(1.10)}{\leq} n \sup_{t \in [t_0, \infty)} |x(t) - y(t)|. \end{aligned} \tag{3.13}$$

Then for inequality (3.12) we obtain

$$\forall t \in [t_0, \infty): \quad |x(t) - y(t)| \leq \varepsilon M(1+n) \int\limits_{t_0}^{t} \sup_{s \in [t_0, \tau]} |x(s) - y(s)| d\tau + I(t), \tag{3.14}$$

where

$$\forall t \in [t_0, \infty): \quad I(t) := \varepsilon \left| \int\limits_{t_0}^{t} \Big( f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau \right|.$$

Find an approximation of $I(t)$ for all $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$. Divide $[t_0, t_0 + \frac{L}{\varepsilon}]$ into $p$ equal parts such that $t_i = t_0 + \frac{iL}{\varepsilon p}$, $i = 0, \cdots, p$. Let $t \in [t_k, t_{k+1})$ for all $k = 0, \cdots, p-1$, then

$$
\begin{aligned}
I(t) = {}& \underbrace{\varepsilon \left| \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \Big( f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau \right|}_{(a)} \\
& \underbrace{+ \, \varepsilon \int_{t_k}^{t} \Big| f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big| d\tau}_{(b)} \, .
\end{aligned}
\tag{3.15}
$$

Observe that, the length of interval $[t_k, t]$ may be smaller then $\frac{L}{\varepsilon p}$.

(a): For simplicity, set $y(t_i) =: y_i$, $y_h^{\vee}(t_i) =: y_i^{\vee}$. Add $\pm f(t, y_i, y_i^{\vee}, u_i)$, and $\pm f_{wa}(y_i, y_i^{\vee}, u_i)$, then

$$
\begin{aligned}
& \varepsilon \left| \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \Big( f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big) d\tau \right| \\
& \le \underbrace{\varepsilon \left| \int_{t_i}^{t_{i+1}} \Big( f(\tau, y_i, y_i^{\vee}, u_i) - f_{wa}(y_i, y_i^{\vee}, u_i) \Big) d\tau \right|}_{(a.1)} \\
& \underbrace{+ \, \varepsilon \int_{t_i}^{t_{i+1}} \Big| f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f(\tau, y_i, y_i^{\vee}, u_i) \Big| d\tau}_{(a.2)} \\
& \underbrace{+ \, \varepsilon \int_{t_i}^{t_{i+1}} \Big| f_{wa}(y_i, y_i^{\vee}, u_i) - f_{av}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big| d\tau}_{(a.3)} \, .
\end{aligned}
\tag{3.16}
$$

(a.1): Split integral into two integrals and by the Definition 7 we have

$$\varepsilon\left|\int_{t_i}^{t_{i+1}}\Big(f(\tau,y_i,y_i^\vee,u_i)-f_{wa}(y_i,y_i^\vee,u_i)\Big)d\tau\right|$$
$$=\varepsilon\left|\int_{t_0}^{t_{i+1}}\Big(f(\tau,y_i,y_i^\vee,u_i)-f_{wa}(y_i,y_i^\vee,u_i)\Big)d\tau-\int_{t_0}^{t_i}\Big(f(\tau,y_i,y_i^\vee,u_i)-f_{wa}(y_i,y_i^\vee,u_i)\Big)d\tau\right|$$
$$\leq\varepsilon\left|\int_{t_0}^{t_{i+1}}\Big(f(\tau,y_i,y_i^\vee,u_i)-f_{wa}(y_i,y_i^\vee,u_i)\Big)d\tau\right|+\varepsilon\left|\int_{t_0}^{t_i}\Big(f(\tau,y_i,y_i^\vee,u_i)-f_{wa}(y_i,y_i^\vee,u_i)\Big)d\tau\right|$$
$$\leq 2\varepsilon t_i\beta_{av}(\max\{|y_h^\vee|,|u|,1\},t_i)\overset{\varsigma:=\varepsilon t_i}{=}2\varsigma\beta_{av}(\max\{|y_h^\vee|,|u|,1\},\frac{\varsigma}{\varepsilon})$$
$$\leq 2\sup_{\varsigma\in[t_0,t_0+L]}\Big(\varsigma\beta_{av}(\max\{|y_h^\vee|,|u|,1\},\frac{\varsigma}{\varepsilon}\Big).$$

Function $\sup\limits_{\varsigma\in[t_0,t_0+L]}\big(\varsigma\beta_{av}(\max\{|y_h^\vee|,|u|,1\},\frac{\varsigma}{\varepsilon}\big)\to 0$ as $\varepsilon\to 0$.

(a.2): By the Assumption 5, using equicontinuity of inputs and by the inequality (3.13) we obtain

$$\varepsilon\int_{t_i}^{t_{i+1}}\Big|f(\tau,y(\tau),y_h^\vee(\tau),u(\tau))-f(\tau,y_i,y_i^\vee,u_i)\Big|d\tau$$
$$\leq\varepsilon M\left(\int_{t_i}^{t_{i+1}}\Big(|y(\tau)-y_i|+|y_h^\vee(\tau)-y_i^\vee|+|u(\tau)-u_i|\Big)d\tau\right)$$
$$\leq\varepsilon M\left(\frac{\lambda L}{\varepsilon p}+\int_{t_i}^{t_{i+1}}\Big(|y(\tau)-y_i|+|y_h^\vee(\tau)-y_i^\vee|\Big)d\tau\right)$$
$$\leq\varepsilon M\left(\lambda\frac{L}{\varepsilon p}+(n+1)\int_{t_i}^{t_{i+1}}\sup_{s\in[t_0,\tau]}|y(s)-y_i|d\tau\right)$$
$$\leq\varepsilon M\left(\lambda\frac{L}{\varepsilon p}+(n+1)\int_{t_i}^{t_{i+1}}\sup_{s\in[t_0,\tau]}\left|y_i-\varepsilon\int_{t_i}^{s}f_{wa}(y(\theta),y_h^\vee(\theta),u(\theta))d\theta-y_i\right|d\tau\right)$$
$$\overset{Assum.4,}{\leq}\varepsilon M\left(\lambda\frac{L}{\varepsilon p}+\varepsilon\eta(n+1)\int_{t_i}^{t_{i+1}}\int_{t_i}^{\tau}d\theta\,d\tau\right)=\varepsilon M\left(\lambda\frac{L}{\varepsilon p}+\varepsilon\eta(n+1)\int_{t_i}^{t_{i+1}}(\tau-t_i)d\tau\right)$$
$$=\varepsilon M\left(\frac{\lambda L}{\varepsilon p}+\frac{(n+1)\varepsilon\eta}{2}(t_{i+1}-t_1)^2\right)=\frac{LM}{p}\left(\lambda+\frac{(n+1)\eta L}{2p}\right).\tag{3.17}$$

(a.3): By the approach used for integral (a.2) we obtain

$$\varepsilon \int\limits_{t_i}^{t_{i+1}} \left| f_{wa}(y_i, y_i^{\vee}, u_i) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \right| d\tau \le \frac{LM}{p}\left(\lambda + \frac{(n+1)\eta L}{2p}\right).$$

(b): Using inequality $|\nu - \varrho| \le |\nu| + |\varrho|$ for all $\nu, \varrho$, and taking into account that for all
$t \in [t_0, t_0 + \frac{L}{\varepsilon}]$, $t_k - t \le \frac{L}{\varepsilon p}$ we have

$$\forall t \in [t_0, t_0 + \frac{L}{\varepsilon}]: \quad \varepsilon \int\limits_{t_k}^{t} \left| f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{wa}(y(\tau), y_h^{\vee}(\tau), u(\tau) \right| d\tau \le 2\eta \frac{L}{p}.$$

Therefore we obtain

$$I(t) \le 2p \sup_{\varsigma \in [t_0, t_0+L]} \left( \varsigma \beta_{av}(\max\left\{|y_h^{\vee}|, |u|, 1\right\}, \frac{\varsigma}{\varepsilon} \right) + 2LM\left(\lambda + \frac{(n+1)\eta L}{2m}\right) + 2\eta\frac{L}{m} =: \tilde{d}.$$

Inequality (3.14) holds for all $t \in [t_0, \infty)$, then for all $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$

$$\sup_{s \in [t_0, t]} |x(s) - y(s)| \le (1+n)\varepsilon M \int\limits_{t_0}^{t} \sup_{s \in [t_0, \tau]} |x(s) - y(s)| d\tau + \tilde{d}. \tag{3.18}$$

It is easy to check that assumptions of the Gronwall's lemma hold for the inequality (3.18), therefore

$$\sup_{t \in [t_0, t_0 + \frac{L}{\varepsilon}]} |x(t) - y(t)| \le \tilde{d} e^{\varepsilon M \int_{t_0}^{t} d\tau} \le \tilde{d} e^{ML}.$$

Choose $p$, such that

$$e^{ML}\left(2LM\left(\lambda + \frac{(n+1)\eta L}{2p}\right) + 2\eta\frac{L}{p}\right) \le \frac{c}{2}.$$

Then fix $p$ and choose $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$2pe^{ML}\varsigma\beta_{av}(\max\left\{|y_h^{\vee}|, |u|, 1\right\}, \frac{\varsigma}{\varepsilon}) \le \frac{c}{2}.$$

Therefore $\sup_{s \in [t_0, t_0 + \frac{L}{\varepsilon}]} |x(t) - y(t)| \le c$. Theorem is proved. □

The next theorem proves the closeness of solutions to original (3.4) and strong averaged (3.8) systems. Observe that under assumption of existence of strong average $f_{sa}$ one can remove the assumption of uniformly equicontinuous of set $\mathcal{U}$.

**Theorem 15.** *Let $t_0 \in \mathbb{R}$, $h > 0$, $U \subset \mathbb{R}^m$, $D \subset \mathbb{R}^n$. Consider system (3.8) and suppose*

1. *Assumptions 4 and 5 hold;*
2. *there exists strong average $f_{sa}$ of $f$ ;*
3. *for every $u \in L_\infty([t_0,\infty);U)$ and every $\varphi \in C([t_0-h,t_0];D)$ the solution $y_{sa}(\cdot;t_0,\varphi,h,u)$ to the problem (3.8) belongs to an interior subset of $D$ on $[t_0,\infty)$.*

*Then for any $\tilde{c}>0$, and any $L>0$ there exists $\varepsilon_0=\varepsilon_0(\tilde{c},L)>0$, such that for all $\varepsilon\in(0,\varepsilon_0]$, and all $t\in[t_0,t_0+\frac{L}{\varepsilon}]$:*

$$\sup_{t\in[t_0,t_0+\frac{L}{\varepsilon}]} |x(t;t_0,\varphi,h,u)-y_{sa}(t;t_0,\varphi,h,u)| \le \tilde{c}. \tag{3.19}$$

*Proof.* Set $x(\cdot):=x(\cdot;t_0,\varphi,h,u)$, $y(\cdot):=y_{sa}(\cdot;t_0,\varphi,h,u)$. The proof is similar to the proof of the Theorem 14 with the following changes: instead of (3.16) for all $t\in[t_0,t_0+\frac{L}{\varepsilon}]$ we have:

$$\begin{aligned}
R(t) &:= \varepsilon\left|\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\Big(f(\tau,y(\tau),y_h^\vee(\tau),u(\tau))-f_{sa}(y(\tau),y_h^\vee(\tau),u(\tau))\Big)d\tau\right| \\
&\le \underbrace{\varepsilon\left|\int_{t_i}^{t_{i+1}}\Big(f(\tau,y_i,y_i^\vee,u(\tau))-f_{sa}(y_i,y_i^\vee,u(\tau))\Big)d\tau\right|}_{(a)} \\
&\quad \underbrace{+\,\varepsilon\int_{t_i}^{t_{i+1}}\Big|f(\tau,y(\tau),y_h^\vee(\tau),u(\tau))-f(\tau,y_i,y_i^\vee,u(\tau))\Big|d\tau}_{(b)} \\
&\quad \underbrace{+\,\varepsilon\int_{t_i}^{t_{i+1}}\Big|f_{sa}(y_i,y_i^\vee,u(\tau))-f_{sa}(y(\tau),y_h^\vee(\tau),u(\tau))\Big|d\tau}_{(c)} \\
&\quad \underbrace{+\,\varepsilon\int_{t_k}^{t}\Big|f(\tau,y(\tau),y_h^\vee(\tau),u(\tau))-f_{sa}(y(\tau),y_h^\vee(\tau),u(\tau))\Big|d\tau}_{(d)}.
\end{aligned}$$

(a): Then by the Definition 8,

$$\varepsilon\left|\int_{t_i}^{t_{i+1}}\Big(f(\tau,y_i,y_i^\vee,u(\tau))-f_{sa}(y_i,y_i^\vee,u(\tau))\Big)d\tau\right| \le 2\sup_{\varsigma\in[t_0,t_0+L]}(\varsigma\beta_{sa}(\max\{|y_h^\vee|,||u||,1\},\frac{\varsigma}{\varepsilon}).$$

(b): Similarly to (3.17), we obtain

$$\varepsilon \int_{t_i}^{t_{i+1}} \Big| f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f(\tau, y_i, y_i^{\vee}, u(\tau)) \Big| d\tau \le \varepsilon M \int_{t_i}^{t_{i+1}} \Big( |y(\tau) - y_i| + |y^{\vee}(\tau) - y_i^{\vee}| \Big) d\tau$$

$$\le \varepsilon^2 M(1+n)\eta \int_{t_i}^{t_{i+1}} (\tau - t_i) d\tau = \frac{M(1+n)\eta L^2}{2p^2}.$$

For (c) and (d)

$$\varepsilon \int_{t_i}^{t_{i+1}} \Big| f_{sa}(y_i, y_i^{\vee}, u(\tau)) - f_{sa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big| d\tau \le \frac{M(1+n)\eta L^2}{2p^2},$$

and

$$\varepsilon \int_{t_k}^{t} \Big| f(\tau, y(\tau), y_h^{\vee}(\tau), u(\tau)) - f_{sa}(y(\tau), y_h^{\vee}(\tau), u(\tau)) \Big| d\tau \le 2\eta \frac{L}{p},$$

resp. Hence, for all $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$ we obtain

$$R(t) \le p \sup_{\varsigma \in [t_0, t_0 + L]} (\varsigma \beta_{sa}(\max\{|y_h^{\vee}|, ||u||, 1\}, \frac{\varsigma}{\varepsilon}) + \frac{L\eta}{p}\Big(M(1+n)L + 2\Big).$$

Similar to the Theorem 14, choose $p$, such that $e^{ML}\left(\frac{L\eta}{p}\Big(M(1+n)L+2\Big)\right) \le \frac{\tilde{c}}{2}$. Then fix $p$ and choose $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ we obtain $pe^{ML} \sup_{\varsigma \in [t_0, t_0+L]} (\varsigma \beta_{sa}(\max\{|y_h^{\vee}|, ||u||, 1\}, \frac{\varsigma}{\varepsilon}) \le \frac{\tilde{c}}{2}$. Therefore estimate (3.19) is obtained.

□

**Example 9.** *Consider differential equation with maximum*

$$\begin{aligned} \dot{x}(t) &= \varepsilon(-x(t)\sin t - x_h^{\vee}(t) - u(t)), & t \in [0, \infty), \\ x(t) &= 0, & t \in [-1, 0], \end{aligned} \tag{3.20}$$

*where $x(t) \in \mathbb{R}$, $h > 0$, $\varepsilon > 0$. Consider continuous inputs $u : [0, \infty) \mapsto \mathbb{R}$ such that, $|u(t)| \le 1$ for all $t \in [0, \infty)$, denote such space by $\mathcal{U}$. Notice, that r.h.s of (3.20) is continuous function on each argument and it is Lipschitz continuous with constant $M = 1$, indeed for all $t \in [0, \infty)$, for all $x, y, x_h^{\vee}, y_h^{\vee}, u, v \in \mathbb{R}$*

$$\begin{aligned} |f(t, x, x_h^{\vee}, u) - f(t, y, y_h^{\vee}, v)| &= |-x\sin t - x_h^{\vee} - u - (-y\sin t - y_h^{\vee} - v)| \\ &\le |x - y| + |y_h^{\vee} - x_h^{\vee}| + |u - v|. \end{aligned}$$

*Then by the Theorem 2 there exists unique maximal solution $x(\cdot) := x(\cdot; h, u)$ to (3.20) for all $t \ge 0$. Observe that $f_{sa}(y_h^{\vee}, u) = -y_h^{\vee} - u$ is a strong average for $f(t, x, x_h^{\vee}, u) = -x\sin t - x_h^{\vee} - u$. Indeed, for all $T > 0$, for every $u \in \mathcal{U}$, and for all*

$t \geq 0$

$$\left|\frac{1}{T}\int\limits_t^{t+T}\Big(f(\tau,x,x_h^\vee,u(\tau)) - f(\tau,y,y_h^\vee,u(\tau))\Big)d\tau\right| = \left|\frac{x}{T}\int\limits_t^{t+T}\sin\tau d\tau\right| \leq \frac{2|x|}{T} =: \beta_{av}(|x|,T).$$

*Hence, the strong averaged system to the system (3.20) is given by*

$$\begin{array}{ll} \dot{y}(t) = \varepsilon(-y_h^\vee(t) - u(t)), & t \in [0,\infty), \\ y(t) = 0, & t \in [-1,0]. \end{array} \tag{3.21}$$

*Let* $\tilde{c} = \frac{1}{2}$, $L = 10$ *then by the proof of the Theorem 15 we may choose* $p$ *such that,*

$$e^{ML}\left(\frac{L}{p}\left(M(1+n)\eta L + 2\eta\right)\right) \leq \frac{1}{4}. \tag{3.22}$$

*Constant* $M = 1$ *is already found. In order to find* $\eta$ *consider (3.20) in the integral form*

$$\forall t \in [0,\infty): \quad x(t) = -\varepsilon\int\limits_0^t \Big(x(\tau)\sin\tau + x_h^\vee(\tau) + u(\tau)\Big)d\tau.$$

*For all* $t \in [0, \frac{L}{\varepsilon}]$*:*

$$|x(t)| \leq \varepsilon\left(\int\limits_0^t\Big(|x(\tau)| + |x_h^\vee(\tau)|\Big)d\tau + \int\limits_0^t |u(\tau)|d\tau\right) \leq L + 2\varepsilon\int\limits_0^t |x_h^\vee|(\tau)d\tau. \tag{3.23}$$

*Take supremum from the both sides of (3.23) on* $[0,t]$

$$\textit{for all } t \in [0,\frac{L}{\varepsilon}]: \sup_{s\in[0,t]}|x(s)| \leq L + 2\varepsilon\int\limits_0^t \sup_{\xi\in[0,\tau]}|x_h^\vee|(\xi)d\tau. \tag{3.24}$$

*For inequality (3.24) conditions of Gronwall's lemma hold therefore,*

$$\sup_{t\in[0,\frac{L}{\varepsilon}]}|x(t)| \leq Le^{2\varepsilon\int\limits_0^t d\tau} \leq Le^{2L}. \tag{3.25}$$

*Then for all* $t \in [0, \frac{L}{\varepsilon}]$*:*

$$|f(t,x,x_h^\vee,u)| = |-x(t)\sin t - x_h^\vee(t) - u(t)| \leq |x(t)| + |x_h^\vee|(t) + |u(t)| \leq 2Le^{2L} + 1 =: \eta.$$

*Hence from (3.22)*

$$p \leq 4e^{10}(440e^{20} + 21).$$

*Fix* $p = 4e^{10}(440e^{20}+21)$. *Since* $\beta_{av}(|x|, T) := \frac{2|x|}{T} \overset{(3.25)}{\leq} \frac{20e^{20}}{T}$ *we have* $20e^{30}p \sup_{\xi \in [0,10]} \frac{\xi^2}{\varepsilon} \leq \frac{1}{4} \Rightarrow \varepsilon_0 \leq 8000e^{30}p$. *Then for any* $\varepsilon \in (0, 8000e^{30}p]$

$$\sup_{t \in [0, \frac{10}{\varepsilon}]} |x(t) - y(t)| \leq \frac{1}{2}.$$

*See Figure 3.1 for the trajectory of (3.20) and (3.21) in case* $u(t) = \cos t$, $\varepsilon = \frac{1}{2}$, $h = 1, \varphi(t) = 1$ *for all* $t \in [-h, 0]$.

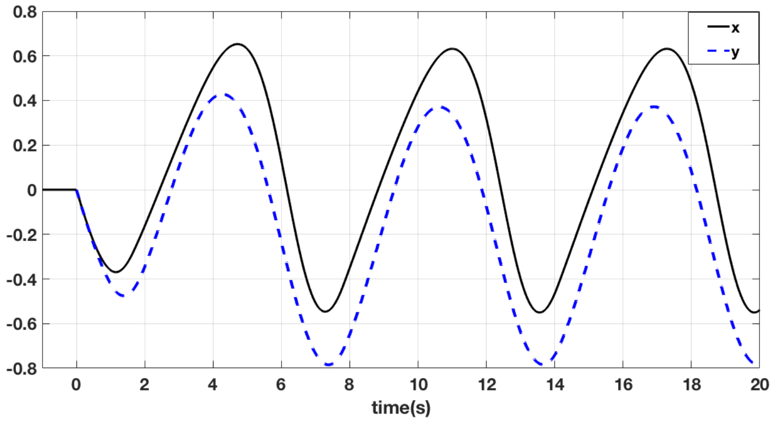


FIGURE 3.1: *The solutions of original (3.20) and averaged (3.21) system.*

## 3.4 Input-to-state stability via averaging method

Here we extend the results from the previous section. Consider system (3.4). We prove the closeness of solutions to original and averaged (weak and strong) systems on infinite time interval and, moreover, we show that e$\delta$ISS of averaged system (weak and strong) implies eISpS of the original system.

Assume that

$$f(t, 0, 0, 0) = 0, \quad \text{for all} \quad t \geq t_0, \tag{3.26}$$

so that $x(t) \equiv 0$ is an equilibrium of he system (3.4).

**Definition 10.** The system (3.4) is *exponentially input-to-state practically stable (eISpS)* if there exist $Q > 0$, $\mu > 0$, and $\mathcal{K}_\infty$-function $\gamma$, and nonnegative constant $p$ such that for any initial function $\varphi \in C([t_0 - h, t_0]; \mathbb{R}^n)$ and any input $u \in L_\infty([t_0, \infty); \mathbb{R}^m)$ the unique solution $x(\cdot; t_0, \varphi, h, u)$ to (3.4) exists for all $t \geq t_0$, and furthermore it satisfies

$$|x(t; t_0, \varphi, h, u)| \leq Qe^{-\mu(t-t_0)}||\varphi|| + \gamma(||u||) + p, \quad t \in [t_0, \infty).$$

**Definition 11.** The system (3.4) is *exponentially incrementally input-to-state stable (e$\delta$ISS)* if there exist $Q > 0$, $\mu > 0$, and $\mathcal{K}_\infty$-function $\gamma$, such that for any initial functions $\varphi, \tilde{\varphi} \in C([t_0 - h, t_0]; \mathbb{R}^n)$ and any inputs $u, \tilde{u} \in L_\infty([t_0, \infty); \mathbb{R}^m)$ the unique solution $y(\cdot; t_0, \varphi, h, u)$ to (3.6) exists for all $t \geq t_0$, and furthermore it satisfies

$$|y_{wa}(t; t_0, \varphi, h, u) - y_{wa}(t; t_0, \tilde{\varphi}, h, u)| \leq Qe^{-\mu(t-t_0)}||\varphi - \tilde{\varphi}|| + \gamma(||u - \tilde{u}||), \quad t \in [t_0, \infty).$$

Since (3.26) is assumed, from the Definition 11 it follows that e$\delta$ISS implies eISS just comparing an arbitrary solution with the trivial one.

**Theorem 16.** *Assume conditions 1-4 of Theorem 14 hold. Let (3.26) hold and system (3.6) be e$\delta$ISS. Then for any $\theta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\theta) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following holds*

$$\sup_{t \geq t_0} |x(t; t_0, \varphi, h, u) - y_{wa}(t; t_0, \varphi, h, u)| \leq \theta. \tag{3.27}$$

*Moreover, the original system (3.4) is eISpS with respect to $\mathcal{U}$.*

*Proof.* Set $x(\cdot) := x(t; t_0, \varphi, h, u)$, $y_{wa}(\cdot) := y_{wa}(t; t_0, \varphi, h, u)$. Consider the partition of the time axis

$$[t_0, t_0 + \frac{L}{\varepsilon}] \cup [t_0 + \frac{L}{\varepsilon}, t_0 + \frac{2L}{\varepsilon}] \cup \cdots \cup [t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}] \cup \cdots, \quad k = 1, 2, \cdots,$$

where constant $L > 0$ is determined. On each interval $[t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}]$, $k = 1, 2, \cdots$ we define $y_{wa,k}(\cdot)$ as a solution to

$$\begin{array}{ll} \dot{y}(t) = \varepsilon f_{wa}(y(t), y_h^\vee(t), u(t)), & t \in [t_0 + \frac{kL}{\varepsilon}, \infty), \\ y_{wa,k}(t) = x(t), & t \in [t_0 + \frac{kL}{\varepsilon} - h, t_0 + \frac{kL}{\varepsilon}], \end{array} \tag{3.28}$$

see Figure 3.2. Then

$$\forall t \in [t_0, \infty): \quad |x(t) - y_{wa}(t)| \leq |x(t) - y_{wa,k}(t)| + |y_{wa,k}(t) - y_{wa}(t)|. \tag{3.29}$$

By Theorem 14, for any fixed $L > 0$,

$$\forall t \in [t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}] \; \forall k = 1, 2, \cdots: \quad |x(t) - y_{wa,k}(t)| \leq c.$$

If $k = 0$, then

$$\forall t \in [t_0, t_0 + \frac{L}{\varepsilon}]: \quad y_{wa,0}(t) = y_{wa}(t).$$

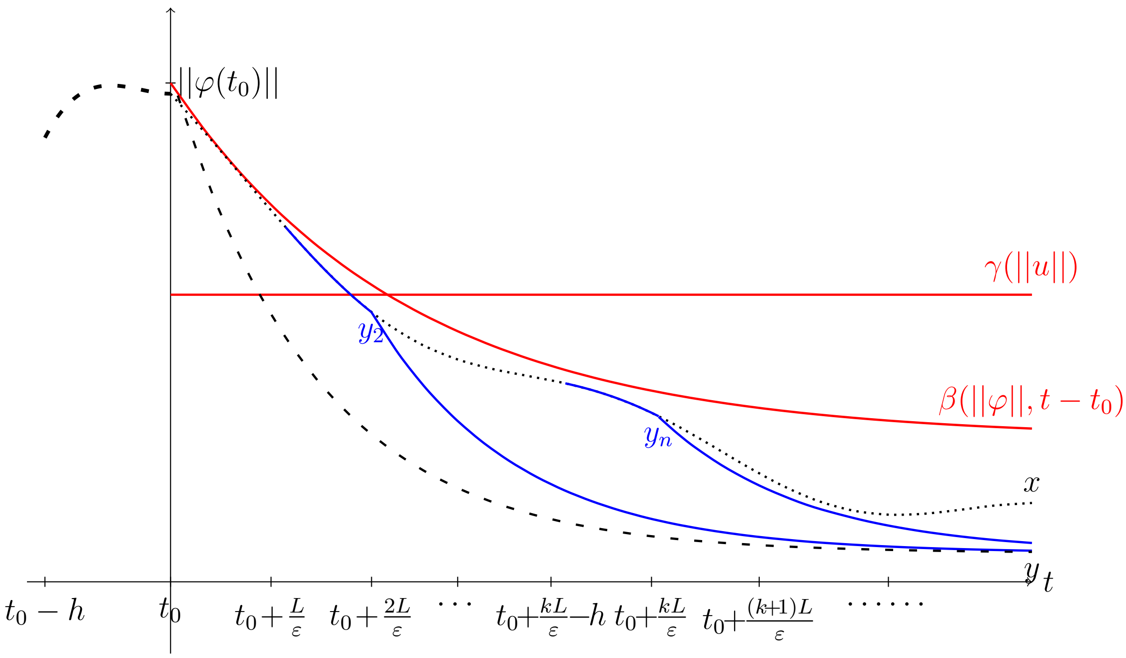


FIGURE 3.2: Solutions to original, strong averaged system and to the system (3.28).

Without lost of generality, assume $\frac{L}{\varepsilon} \geq h$. Since (3.6) is e$\delta$ISS, for $k \geq 1$, for all $t \in [t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}]$ :

$$\begin{aligned} |y_{wa,k}(t) - y_{wa}(t)| &\leq \tilde{Q}e^{-\mu(t-(t_0+k\frac{L}{\varepsilon}))} \sup_{s\in[t_0+\frac{kL}{\varepsilon}-h, t_0+\frac{kL}{\varepsilon}]} |y_{wa,k}(s) - y_{wa}(s)| \\ &\leq \tilde{Q}e^{-\mu(t-(t_0+k\frac{L}{\varepsilon}))} \sup_{s\in[t_0+\frac{kL}{\varepsilon}-h, t_0+\frac{kL}{\varepsilon}]} \left(|y_{wa,k}(s) - y_{wa,k-1}(s)| + |y_{wa}(s) - y_{wa,k-1}(s)|\right), \end{aligned} \tag{3.30}$$

where $\tilde{Q} > 0$ and $\mu > 0$. Then by the definition of $y_{wa,k}(\cdot)$,

$$\sup_{s\in[t_0+\frac{kL}{\varepsilon}-h, t_0+\frac{kL}{\varepsilon}]} |y_{wa,k}(s) - y_{wa,k-1}(s)| = \sup_{s\in[t_0+\frac{kL}{\varepsilon}-h, t_0+\frac{kL}{\varepsilon}]} |x(s) - y_{wa,k-1}(s)| \overset{\text{Th. 14}}{\leq} c.$$

For simplicity define $\delta_{k-1} := \sup_{s\in[t_0+\frac{kL}{\varepsilon}-h, t_0+\frac{kL}{\varepsilon}]} |y_{wa}(s) - y_{wa,k-1}(s)|$, $k = 1, 2, \cdots$ . Then for (3.30), for all $t \in [t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}]$:

$$|y_{wa,k}(t) - y_{wa}(t)| \leq \tilde{Q}e^{-\mu(t-(t_0+k\frac{L}{\varepsilon}))}(c + \delta_{k-1}). \tag{3.31}$$

Taking sup on an interval $[t_0 + \frac{(n+1)L}{\varepsilon} - h, t_0 + \frac{(n+1)L}{\varepsilon}]$ from both sides of inequality (3.31) we obtain

$$\begin{aligned}\delta_k &= \sup_{s\in[t_0+\frac{(k+1)L}{\varepsilon}-h, t_0+\frac{(k+1)L}{\varepsilon}]} |y_{wa,k}(s) - y_{wa}(s)| \\ &\leq \tilde{Q} \sup_{s\in[t_0+\frac{(k+1)L}{\varepsilon}-h, t_0+\frac{(k+1)L}{\varepsilon}]} e^{-\mu(s-(t_0+k\frac{L}{\varepsilon}))}(c + \delta_{k-1}) \\ &= \tilde{Q}e^{-\mu(\frac{L}{\varepsilon}-h)}(c + \delta_{k-1}).\end{aligned} \tag{3.32}$$

Set $q := \tilde{Q}e^{-\mu(\frac{L}{\varepsilon}-h)}$ and notice that $\delta_0 = 0$ hence

$$\begin{aligned}&\delta_1 \leq qc, \\ &\delta_2 \leq qc(q+1), \\ &\delta_3 \leq qc(q^2+q+1), \\ &\cdots, \\ &\delta_n \leq qc(q^k + q^{k-1} + \cdots + 1). \\ &\cdots\end{aligned}$$

Choose $L$ sufficiently large so that $q < 1$, then $\delta_k \leq \frac{cq}{1-q}$, as $k \to \infty$. Thus for (3.30)

$$\forall t \in [t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}]\ \forall k = 1, 2, \cdots:\ |y_{wa,k}(t) - y_{wa}(t)| \leq q\left(c + \frac{cq}{1-q}\right) = \frac{qc}{1-q}.$$

For any $k = 1, 2, \cdots$, on each interval $[t_0 + \frac{kL}{\varepsilon}, t_0 + \frac{(k+1)L}{\varepsilon}]$ we have

$$|x(t) - y_{wa}(t)| \leq c(1 + \frac{q}{1-q}).$$

Then

$$\forall t \in [t_0, \infty):\quad |y_{wa}(t) - x(t)| \leq c(1 + \frac{q}{1-q}) =: \theta. \tag{3.33}$$

Therefore estimate (3.27) holds. Moreover, observe since $\delta$eISS of (3.6) implies ISS of (3.6) and in conjunction with estimate (3.33) we obtain for all $t \geq t_0$

$$|x(t)| \leq |y_{wa}(t)| + |x(t) - y_{wa}(t)| \leq \beta(||\varphi||, t - t_0) + \gamma(||u||) + \theta,$$

which proofs eISpS of the system (3.4). □

**Theorem 17.** *Assume conditions 1-3 of Theorem 15 hold. Let (3.26) hold and system (3.8) be e$\delta$ISS. Then for any $\tilde{\theta} > 0$ there exists $\varepsilon_0 = \varepsilon_0(\tilde{\theta}) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the following hold*

$$\sup_{t\geq t_0} |x(t; t_0, \varphi, h, u) - y_{sa}(t; t_0, \varphi, h, u)| \leq \tilde{\theta}.$$

*Moreover, the original system (3.4) is eISpS.*

The proof is similar to the proof of the Theorem 16.

## 3.5 Concluding remarks

In this chapter we have proposed extension of averaging method to the perturbed system with maximum and we have applied it to ISS study of differential equations with maximum affected by an input. In particular, without usage of any Lyapunov techniques it has been shown that e$\delta$ISS of averaged systems implies eISpS of original system. Remark that compare with eISS, e$\delta$ISS is more restrictive assumption. Observe Theorems 14, 15 are interesting not only for the ISS analysis, they itself provide valuable results regarding to averaging method for differential equations with maximum and input. It is likely that the results from this chapter can be extended on general RFDE systems.

# Chapter 4

# Numerical method for differential equation with max-operator

Numerical methods for SD-DDE are available in many papers and monographs (we cited only some of them [14, 26, 29, 33, 47, 54, 43, Chapter 9]). For applying these methods one should assume that delay function is smooth. Since the delay function in problem

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), x_h^{\vee}(t)), && t \in [0, T), \\ &x(t) = \varphi(t), && t \in [-h, 0], \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}$, $h > 0$, $T \in (0, \infty]$, is piecewise continuous (see Remark 1) cited methods are not applicable to the calculation a solution of the problem (4.1) and therefore it is necessary to develop numerical method for differential equations with maximum.

In this chapter, we propose a numerical method for computing the solution to (4.1) which is based on the left rectangle method that requires only continuity of the first derivative of the solution, no additional assumptions about a delay function are assumed. Recall that for trapezoid or Simpson method more regularity of $x$ is used. Suggested in this chapter method is an extension of the one from [21], where only constant initial functions were considered. We refer to [28] for a numerical method for calculation a solution to the problem (4.1), which is based on construction of approximation by lower and upper solutions to (4.1).

## 4.1 Equivalent problem to a problem of differential equations with maximum

Let $h > 0$, $T > 0$. Consider Cauchy problem in the form (4.1). Assume the initial function $\varphi : [-h, 0] \mapsto \mathbb{R}$ is continuous, function $f : [0, T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from (4.1) satisfies the following

**Assumption 6.** *(i)* $f : (t, x, y) \mapsto f(t, x, y)$ *is continuous*

*(ii)* $f : (t, x, y) \mapsto f(t, x, y)$ *is Lipschitz continuous in* $(x, y) \in \mathbb{R} \times \mathbb{R}$ *for all* $t \in [0, T)$ *with Lipschitz constant* $L \geq 0$.

**Theorem 18.** *Consider problem (4.1). Let Assumption 6 (i) hold. Then the problem (4.1) is equivalent to the following problem*

$$v(t) = \begin{cases} \varphi(t), & t \in [-h, 0] \\ f\Big(t, \int_0^t v(\tau)d\tau + \varphi(0), \Big(\int_0^s v(\tau)d\tau\Big)_h^{\vee}(t) + \varphi(0)\Big), & t \in (0, T), \end{cases} \tag{4.2}$$

*where* $\max_{s\in[t-h,t]} \int_0^s v(\tau)d\tau =: \left(\int_0^s v(\tau)d\tau\right)_h^{\vee}(t)$. *If* $v \in C((0,T);\mathbb{R})$ *is a solution to the problem (4.2) then*

$$x(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ \int_0^t v(\tau)d\tau + \varphi(0), & t \in (0, T), \end{cases}$$

*solves the problem (4.1).*

*Proof.* Let $x \in AC([0,T);\mathbb{R})$ be a solution to the problem (4.1). Set $\dot{x}(\cdot) =: v(\cdot)$ on $(0,T)$ and $\varphi(\cdot) = v(\cdot)$ on $[-h, 0]$. Since $x$, as a solution to (4.1), is an absolutely continuous function, its derivative is defined almost everywhere, therefore $x(t) - \varphi(0) = \int_0^t v(\tau)d\tau$, and $x_h^{\vee}(t) = \left(\int_0^s v(\tau)d\tau\right)_h^{\vee}(t) + \varphi(0)$ for all $t \in (0,T)$. Since $x(\cdot)$ satisfies (4.1) it follows that $v(\cdot)$ solves problem (4.2). Conversely, let $v \in C((0,T);\mathbb{R})$ be a solution to (4.2). Define

$$x(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ \int_0^t v(\tau)d\tau + \varphi(0), & t \in (0, T). \end{cases}$$

Taking the right time derivative of $x(\cdot)$ on $(0,T)$, $\dot{x}(t) = v(t)$ for all $t \in (0,T)$, that satisfies (4.1) due to (4.2). Thus, the theorem is proved. □

Let $B > 0$ be such that

$$\sup_{t\in[0,T)} |f(t, 0, 0)| \le B \tag{4.3}$$

and

$$\sup_{t\in[-h,0]} |\varphi(t)| \le B. \tag{4.4}$$

Consider the following integral equation

$$\forall t \in [0, T): \quad u(t) = L \int_0^t u(\tau)d\tau + \tilde{B}, \tag{4.5}$$

where $\tilde{B} \geq 0$. It is easy to check that

$$\forall t \in [0, T): \quad u(t) = \tilde{B} e^{Lt}. \tag{4.6}$$

is a solution to (4.5). Define the sequences $\{z_n(\cdot)\}_{n=0}^{\infty}$ as

$$\forall t \in [0, T) \;\; \forall n = 0, 1, \cdots : \quad z_{n+1}(t) = L \int\limits_0^t z_n(\tau) d\tau,$$

where $z_0(t) := u(t)$ for all $t \in [0, T)$. Show that

$$\forall t \in [0, T): \quad 0 \leq ... \leq z_{n+1} \leq z_n(t) \leq ... \leq z_0(t).$$

Indeed

$$\forall t \in [0, T): \quad z_1(t) = L \int\limits_0^t z_0(\tau) d\tau \leq L \int\limits_0^t u(\tau) d\tau + \tilde{B} = u(t) = z_0(t).$$

Thus, $z_1(t) \leq z_0(t)$ for all $t \in [0, T)$. Now, let $0 \leq z_n(t) \leq z_{n-1}(t)$ for all $t \in [0, T)$, then

$$\forall t \in [0, T): \quad z_{n+1}(t) = L \int\limits_0^t z_n(s) ds \leq L \int\limits_0^t z_{n-1}(\tau) d\tau = z_n(t),$$

i.e, for all $t \in [0, T)$ we obtain $z_{n+1}(t) \leq z_n(t)$. Notice, $z_n \in C([0, T); \mathbb{R})$, $n = 0, 1, \cdots$. For all $t \in [0, T]$ the sequence $\{z_n(\cdot)\}_{n=0}^{\infty}$ is nonincreasing and bounded from below. For this reason there exists measurable function $\overline{z} : [0, T) \mapsto \mathbb{R}$ such that $\lim\limits_{n \to \infty} z_n(t) = \overline{z}(t)$ for all $t \in [0, T)$.
By Lebesgue's dominated convergence theorem

$$\forall t \in [0, T): \quad \overline{z}(t) = L \int\limits_0^t \overline{z}(\tau) d\tau. \tag{4.7}$$

Compare (4.7) with (4.5) and conclude that there exists unique solution $\overline{z}(t) = 0$ to the integral equation (4.7) on $[0, T)$. By the Dini theorem [51, Th.12.1, p.157], the sequence $\{z_n(\cdot)\}_{n=0}^{\infty}$ uniformly converges to $\overline{z}(\cdot)$ on $[0, T)$. Consider the sequence $\{v_n(\cdot)\}_{n=0}^{\infty}$ on $[0, T)$ such that

$$v_{n+1}(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ f(t, \int\limits_0^t v_n(\tau) d\tau + \varphi(0), \left( \int\limits_0^s v_n(\tau) d\tau \right)_h^{\vee} (t) + \varphi(0)), & t \in [0, T), \end{cases}$$

where $v_0(t) := 0$ for all $t \in [-h, T)$. Let

$$\overline{u}(t) = \begin{cases} \tilde{B}, & t \in [-h, 0], \\ u(t), & t \in [0, T). \end{cases}$$

and

$$\overline{z}_n(t) = \begin{cases} 0, & t \in [-h, 0], \\ z_n(t), & t \in [0, T). \end{cases}$$

**Theorem 19.** *Consider (4.1). Let $f$ satisfy Assumption 6. Then there exists continuous solution $v : [0, \infty) \mapsto \mathbb{R}^n$ to the problem (4.2). Moreover,*

$$\forall t \in [-h, T): \quad |v(t)| \leq \overline{u}(t), \tag{4.8}$$

*and*

$$\forall t \in [-h, T) \;\; \forall n = 0, 1, 2, \cdots : \quad |v(t) - v_n(t)| \leq \overline{z}_n(t). \tag{4.9}$$

*Proof.* First let us show $|v_n(t)| \leq \overline{u}(t)$ on $[-h, T)$ for all $n = 0, 1, 2, \ldots$. Note that for all $t \in [-h, T)$, $0 = |v_0(t)| \leq \overline{u}(t)$, and let by induction

$$\forall t \in [0, T) \;\; \forall n = 0, 1, 2, \cdots : \quad |v_n(t)| \leq \overline{u}(t). \tag{4.10}$$

Proof that $|v_{n+1}(t)| \leq \overline{u}(t)$ for all $t \in [0, T)$, for any $n = 0, 1, 2, \ldots$. Indeed, for all $t \geq 0$:

$$\begin{aligned} |v_{n+1}(t)| \quad & \leq \quad \left| f\left(t, \int_0^t v_n(\tau) d\tau + \varphi(0), \left(\int_0^s v_n(\tau) d\tau\right)_h^{\vee} (t) + \varphi(0)\right) - f(t, 0, 0)\right| \\ & + \quad |f(t, 0, 0)| \\ & \overset{Assum.6(ii)}{\leq} \quad L\left(\int_0^t |v_n(\tau)|\, d\tau + \left(\int_0^s |v_n(\tau)| d\tau\right)_h^{\vee} (t)\right) + 2L|\varphi(0)| + |f(t, 0, 0)| \\ & \overset{(4.3),(4.4)}{\leq} \quad L\left(\int_0^t |v_n(\tau)|\, d\tau\right) + B(2L+1) \overset{(4.10)}{\leq} L \int_0^t u(\tau) d\tau + \tilde{B} \overset{(4.5)}{=} u(t) = \overline{u}(t), \end{aligned}$$

where $\tilde{B} := B(2L + 1)$. Next, using induction we show that

$$\forall t \in [0, T) \;\; \forall p = 0, 1, 2, \cdots : \quad |v_{n+p}(t) - v_n(t)| \leq z_n(t). \tag{4.11}$$

If $n = 0$ then

$$\forall t \in [0, T): \quad |v_p(t) - v_0(t)| = |v_p(t)| \leq u(t) =: z_0(t).$$

Let $n = k$ then

$$\forall t \in [0, T): \quad |v_{k+p}(t) - v_k(t)| \leq z_k(t). \tag{4.12}$$

Thus for all $t \in [0, T)$ :

$$
\begin{aligned}
|v_{k+1+p}(t) - v_{k+1}(t)| = & \Bigg| f\left(t, \int_0^t v_{k+p}(\tau)d\tau + \varphi(0), \left(\int_0^s v_{k+p}(\tau)d\tau\right)_h^{\vee}(t) + \varphi(0)\right) \\
& - f\left(t, \int_0^t v_k(\tau)d\tau + \varphi(0), \left(\int_0^s v_k(\tau)d\tau\right)_h^{\vee}(t) + \varphi(0)\right)\Bigg| \\
& \overset{Assum.6(ii),(1.11)}{\leq} L\int_0^t |v_{k+p}(\tau) - v_k(\tau)|\, d\tau \overset{(4.12)}{\leq} L\int_0^t z_k(\tau)d\tau =: z_{k+1}(t).
\end{aligned}
$$

Thus we obtain (4.11). The sequence $\{z_n(\cdot)\}_{n=0}^{\infty}$ uniformly converges to zero on $[0, T)$, hence according to (4.11) the sequence $\{v_n(\cdot)\}_{n=0}^{\infty}$ uniformly converge to $v(\cdot)$ on $[0, T)$, where $v \in C([0, T); \mathbb{R}^n)$ which proves (4.8). Let $p \to \infty$ in (4.11) then we have (4.9).
*Uniqueness.* Assume the opposite. Let besides a solution $v(\cdot) = v(\cdot; \varphi, h)$ to equation (4.2) there exist another solution such that

$$
\overline{w}(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ w(t), & t \in [0, T), \end{cases}
$$

where $w \in C([0, T); \mathbb{R})$, and by (4.8) $|\overline{w}(t)| \leq \overline{u}(t)$ on $[-h, T)$. Proof that

$$
\forall t \in [-h, T) \;\; \forall n = 0, 1, \cdots : \quad |\overline{w}(t) - v_n(t)| \leq \overline{z}_n(t). \tag{4.13}
$$

It is sufficient to show that (4.13) holds for all $t \in [0, T)$. Using induction, for all $t \in [0, T)$ with $n = 0$ we have $|w(t) - v_0(t)| \leq u(t) = z_0(t)$. Let for all $t \in [0, T)$ for any $n = 0, 1, \cdots$, $|w(t) - v_n(t)| \leq z_n(t)$, and show that $|w(t) - v_{n+1}(t)| \leq z_{n+1}(t)$ on $[0, T)$. Indeed, for all $t \in [0, T)$

$$
\begin{aligned}
|w(t) - v_{n+1}(t)| = & \Bigg| f\left(t, \int_0^t w(\tau)d\tau + \varphi(0), \left(\int_0^s w(\tau)d\tau\right)_h^{\vee}(t) + \varphi(0)\right) \\
& - f\left(t, \int_0^t v_n(\tau)d\tau + \varphi(0), \left(\int_0^s v_n(\tau)d\tau\right)_h^{\vee}(t) + \varphi(0)\right)\Bigg| \\
& \overset{Assum.6(ii)}{\leq} L\int_0^t |w(\tau) - v_n(\tau)|\, d\tau \overset{(4.13)}{\leq} L\int_0^t z_n(\tau)d\tau = z_{n+1}(t).
\end{aligned}
$$

Thus $\lim\limits_{n\to\infty} \overline{v}_n(t) = \overline{w}(t)$ converges uniformly for all $t \in [0, T)$. Therefore $\overline{v}(t) = \overline{w}(t)$ for all $t \in [0, T)$.
Theorem 19 is proved. $\square$

## 4.2 Numerical method

The following definition is used to derive an error estimate of numerical method.

**Definition 12.** [100]. The continuity modulus of a function $\mu : [0, T] \to \mathbb{R}$, $0 < T < \infty$ with step $l \in [0, T]$ is called a function $\omega_\mu(\cdot)$ defined as follows

$$\omega_\mu(l) = \sup \{ |\mu(t_1) - \mu(t_2)| : \ t_1, t_2 \in [0, T]; \ |t_1 - t_2| \le l \} .$$

*Remark* 9. From the definition 12 it follows that $\lim\limits_{l \to 0.} \omega_\mu(l) = 0$ if and only if $\mu \in C([0, T]; \mathbb{R})$.

Let $l$ be the step of integration, $\Delta = \{t_0, t_1, \cdots, t_n, \cdots, t_N = T = Nl\}$, $N \in \mathbb{N}$ be a mesh of points, $\mu_k$ is approximation to $\mu(t_k)$. Then by the formula of left rectangles

$$\int_0^T \mu(t) dt = l \sum_{k=0}^{n-1} \mu_k + r_n(\mu).$$

From above equality

$$\begin{aligned} \forall t \in [0, T] : \quad |r_n(\mu)| &= \left| \int_0^T \mu(t) dt - l \sum_{k=0}^{n-1} \mu_k \right| = \left| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mu(t) dt - \sum_{k=0}^{n-1} \mu_k (t_{k+1} - t_k) \right| \\ &= \left| \sum_{k=0}^{n-1} \int_{t_k}^{t_k+1} \mu(t) dt - \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mu_k dt \right| = \left| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\mu(t) - \mu_k) \, dt \right| \\ &\le \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\mu(t) - \mu_k| \, dt. \end{aligned}$$

For all $t \in [t_k, t_{k+1}]$ we have $|\mu(t) - \mu_k| \le |\mu(t) - \mu(t_k)| + |\mu(t_k) - \mu_k| \le \omega_\mu(l) + \delta_k$ with $\delta_k := |\mu(t_k) - \mu_k|$ . Hence

$$|r_n(\mu)| \le l\omega_\mu(l) + l \sum_{k=0}^{n-1} \delta_k. \tag{4.14}$$

This proves the following

**Lemma 7.** *Let $T \in (0, \infty)$ and $\mu \in C([0, T]; \mathbb{R})$. Then for the error of left rectangle method estimate (4.14) is valid.*

In spite of wide usage of rectangle method the proof of the Lemma 7 has not been found in the literature devoted to numerical methods.

By $x_k$ we denote the approximation of solution $x(t_k)$ to the problem (4.1) at $t = t_k$, also by $v_k$ we denote the approximation of solution $v(t_k)$ of (4.2). Assume, that $x_k$

and $v_k, k = 0, \cdots, n$ are already found. Show how $x_{n+1}, v_{n+1}$ can be calculated. Note, that $x(t) - \varphi(0) = \int\limits_0^t v(\tau)d\tau$ for all $t \in [0, T)$. By the left rectangle formula

$$x(t_{n+1}) = \int\limits_0^{t_{n+1}} v(\tau)d\tau + \varphi(0) = \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} v(\tau)d\tau + \varphi(0) \approx l \sum_{k=0}^{n} v_k + \varphi(0).$$

Hence $x_{n+1} = l \sum\limits_{k=0}^{n} v_k + \varphi(0)$. Then

$$\begin{aligned} |x(t_{n+1}) - x_{n+1}| &= \left| \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} v(\tau)d\tau - l \sum_{k=0}^{n} v_k \right| = \left| \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} v(\tau)d\tau - \sum_{k=0}^{n} v_k(t_{k+1} - t_k) \right| \\ &= \left| \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} v(\tau)d\tau - \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} v_k d\tau \right| \le \sum_{k=0}^{n} \int\limits_{t_k}^{t_{k+1}} |v(\tau) - v_k| \, d\tau. \end{aligned} \tag{4.15}$$

Denote $\delta_k := |v(t_k) - v_k|$ then

$$\forall t \in [t_k, t_{k+1}] : \quad |v(t) - v_k| \le |v(t) - v(t_k)| + |v(t_k) - v_k| \le \omega_v(l) + \delta_k. \tag{4.16}$$

From (4.15), (4.16) and Lemma 7 it follows that

$$|x(t_{n+1}) - x_{n+1}| \le l \sum_{k=0}^{n} \delta_k + T\omega_v(l). \tag{4.17}$$

By (4.2) for all $t \ge 0$, $v(t) = f\left(t, \int\limits_0^t v(\tau)d\tau + \varphi(0), \left(\int\limits_0^s v(\tau)d\tau\right)_h^\vee (t) + \varphi(0)\right)$. Then $v(t_{n+1}) = f\left(t_{n+1}, x(t_{n+1}), x_h^\vee(t_{n+1})\right)$. Let $\overline{s}$ be such that $x(\overline{s}) = x_h^\vee(t_{n+1})$. Calculate an approximation $x_{n+1}^\vee$ of $x(\overline{s})$. If $\overline{s} \in [-h, 0] \cap [t_{n+1} - h, t_{n+1}]$, then $x(\overline{s}) = \overline{x}_{n+1} = \varphi(t)$ since $v(t) = \varphi(t)$ for all $t \in [-h, 0]$. Let $\overline{s}$ belong to one of the intervals $[t_{n+1-k}, t_{n+2-k}], 1 \le k \le \frac{h}{l}$ and $t_{n+1-k} \ge 0$. If $\overline{s} = t_{n+1-k}$, then by the formula of left rectangles

$$\begin{aligned} x(\overline{s}) = x(t_{n+1-k}) = \int\limits_0^{t_{n+1-k}} v(\tau)d\tau + \varphi(0) &= \sum_{i=0}^{n-k} \int\limits_{t_i}^{t_{i+1}} v(\tau)d\tau + \varphi(0) \\ &\approx l \sum_{i=0}^{n-k} v(t_i) + \varphi(0) \approx l \sum_{i=0}^{n-k} v_i + \varphi(0) = x_{n+1}^\vee. \end{aligned}$$

From here and till the end of this chapter for simplicity set $x^{\vee}_{h,n+1} =: x^{\vee}_{n+1}$, and $x^{\vee}_h =: x^{\vee}$. From (4.17)

$$\left|x(\overline{s}) - x^{\vee}_{n+1}\right| \leq l \sum_{i=0}^{n-k} \delta_i + T\omega_v(l). \tag{4.18}$$

Now consider the case $t_{n+1-k} < \overline{s} < t_{n+2-k}$. Then

$$\begin{aligned} x(\overline{s}) &= \int\limits_0^{t_{n+1-k}} v(\tau)d\tau + \int\limits_{t_{n+1-k}}^{\overline{s}} v(\tau)d\tau + \varphi(0) \\ &\approx l \sum_{i=0}^{n-k} v(t_i) + v(t_{n+1-k})(\overline{s} - t_{n+1-k}) + \varphi(0) \approx l \sum_{i=0}^{n+1-k} v(t_i) + \varphi(0) \\ &\approx l \sum_{i=0}^{n+1-k} v_i + \varphi(0) = x^{\vee}_{n+1}. \end{aligned}$$

The following estimate holds

$$\left|x(\overline{s}) - x^{\vee}_{n+1}\right| \leq l \sum_{i=0}^{n+1-k} \delta_i + T\omega_v(l). \tag{4.19}$$

If $\overline{s} = t_{n+2-k}$, then similarly it follows that $x^{\vee}_{n+1} = l \sum\limits_{i=0}^{n+1-k} v_i + \varphi(0)$ and estimate (4.19) is valid.

Let now $k = 1, \overline{s} \in [t_n, t_{n+1}]$. In this case $x^{\vee}_{n+1} = l \sum\limits_{i=0}^{n} v_i + \varphi(0)$,

$$\left|x(\overline{s}) - x^{\vee}_{n+1}\right| \leq l \sum_{i=0}^{n} \delta_i + T\omega_v(l). \tag{4.20}$$

Therefore, in all considered cases $x^{\vee}_{n+1}$ approximates $\overline{x} = x^{\vee}_h(t_{n+1})$. For $\overline{x}_{n+1}$ the estimate (4.20) is valid. Hence,

$$v_{n+1} = f(t_{n+1}, x_{n+1}, x^{\vee}_{n+1}). \tag{4.21}$$

In order to prove the convergence of numerical method we need the following auxiliary result.

**Lemma 8.** *Let $l > 0$, $T \in (0, \infty), N \in \mathbb{N}$. Consider the uniform mesh $\Delta = \{t_k = kl, k = 0, 1, \cdots, N, Nl = T\}$. Assume functions $q_l : \Delta \to \mathbb{R}$, $\alpha_l : \Delta \to \mathbb{R}_+$, are such that*

$$|q_l(t_{n+1})| \leq Ll \sum_{k=0}^{n-1} |q_l(t_k)| + \tilde{B}, \tag{4.22}$$

$$\alpha_l(t_{n+1}) \geq Ll \sum_{k=0}^{n-1} \alpha_l(t_k) + \tilde{B}, n = 0, 1, \cdots, N-1. \tag{4.23}$$

*where* $q_l(t_0) = 0, \alpha_l(t_0) = \tilde{B} \geq 0, L > 0$. *Then* $|q_l(t_n)| \leq \alpha_l(t_n)$. *Moreover, (4.23) is satisfied by* $\alpha_l(t_n) = u(t_n)$, *where* $u(\cdot)$ *is a solution to (4.5).*

*Proof.* By assumption of Lemma $|q_l(t_0)| \leq \alpha_l(t_0)$. Let $|q_l(t_k)| \leq \alpha_l(t_k)$, $k = 0, 1, \cdots, n$. Then $|q_l(t_{n+1})| \leq \alpha_l(t_{n+1})$ follows from (4.22) and (4.23).
Next, we show that $\alpha_l(t_n) = u(t_n)$, $n = 0, 1, \cdots, N$ satisfies (4.23). From (4.6) it follows that $u(t) > 0$ for all $t \in [0, T]$ and it is increasing for all $t \in [0, T]$. Then from (4.5) we obtain

$$\begin{aligned} u(t_{n+1}) &= L \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} u(\tau) d\tau + \tilde{B} \geq L \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} u(t_k) d\tau + \tilde{B} \\ &= L \sum_{k=0}^{n} u(t_k)(t_{k+1} - t_k) + \tilde{B} = Ll \sum_{k=0}^{n} u(t_k) + \tilde{B}. \end{aligned}$$

Hence, Lemma is proved. □

A similar result to the Lemma 8 was obtained in [92] in order to show the convergence of numerical method for differential equations of fractional order.
Now we can prove the following theorem.

**Theorem 20.** *Let $f$ satisfy Assumption 6. Then*

$$\lim_{N \to \infty} \max_{k=0,\cdots,n} |v(t_k) - v_k| = 0, \tag{4.24}$$

$$\lim_{N \to \infty} \max_{k=0,\cdots,n} |x(t_k) - x_k| = 0, \tag{4.25}$$

*hold for the approximation of solutions to (4.1) and (4.2) resp.*

*Proof.* Using our previous notation and from (4.17), (4.20) we have

$$\begin{aligned} \delta_{n+1} &:= |v(t_{n+1}) - v_{n+1}| = |f(t_{n+1}, x(t_{n+1}), x^{\vee}(t_{n+1})) - f(t_{n+1}, x_{n+1}, x^{\vee}_{n+1})| \\ &\overset{Assum.6(ii)}{\leq} L\left(|x(t_{n+1}) - x_{n+1}| + \left|x^{\vee}(t_{n+1}) - x^{\vee}_{n+1}\right|\right) \\ &= L\left(l \sum_{k=0}^{n} \delta_k + T\omega_v(l)\right). \end{aligned} \tag{4.26}$$

Applying Lemma 8 to (4.26) in case $\tilde{B} = LT\omega_v(l)$ we obtain

$$\delta_n \leq u(t_n) = \tilde{B} e^{Lt_n} \leq \tilde{B} e^{LT}. \tag{4.27}$$

If $l = \frac{T}{N} \to 0$, then by Remark 9 $\omega_v(l) \to 0$, then $\lim_{N\to\infty} \max_{n=0,1,\cdots,N-1} \delta_n = 0$. From (4.17) and (4.27) follows that

$$|x(t_{n+1}) - x_{n+1})| \le l \sum_{k=0}^{n} \tilde{B}e^{LT} + T\omega_v(l) = T(\tilde{B}e^{LT} + \omega_v(l)),$$

and hence $\lim_{N\to\infty} \max_{n=0,1,\cdots,N-1} |x(t_n) - x_n| = 0$. □

## 4.3 Results of numerical experiments

To illustrate the proposed method we consider scalar differential equation with maximum. The analytic solution is derived and it is compared with numerical approximation for different values of step integration.

**Example 10.** *Consider the following differential equations with maximum*

$$\begin{array}{ll} \dot{x}(t) = -x_1^{\vee}(t), & t \in [0, \infty), \\ x(t) = -\sin t + 1, & t \in [-1, 0]. \end{array} \tag{4.28}$$

*The analytic solution to the problem (4.28) can be written as follows*

$$x(t) = \begin{cases} -t - \cos(1-t) + \cos 1 + 1, & t \in [0,1], \\ \frac{t^2}{2} - (2+\cos 1)t - \sin(2-t) + \frac{1}{2} + 2\cos 1 + \sin 1, & t \in [1,2], \\ -\frac{t^3}{6} + \frac{(3+\cos 1)t^2}{2} - 3t - (t-3)\sin 1 + \cos(3-t) - 3\cos 1(t-1) - \frac{1}{6}, & t \in [2, 1+\frac{3}{e}] \\ -\frac{7}{30}e^{-t+2+\frac{3}{e}}, & t \in [1+\frac{3}{e}, \infty]. \end{cases} \tag{4.29}$$

*Figure 4.1 illustrates analytic solution and numerical solution with step integration* $l = 0.1$ *to the problem (4.28) on* $[-1, 5]$ *. Table 4.1 shows the values of* $x_k$ *-numerical solution by the proposed method and* $x(t_k)$ *- the analytic solution calculated at points* $t_k$*. In the same table the difference between analytic and numerical solutions for steps* $l = 0.5, l = 0.1, l = 0.05, l = 0025$ *are presented. In particular, for the step* $l = 0.5$ *we get maximum value of* $|x(t_k) - x_k|$ *on an interval* $[0, 5]$ *which is 0.3211, for the step* $l = 0.1$ *the maximum value is 0.0515, for* $l = 0.05$ *is 0.0237, and for* $l = 0.0025$ *is 0.0030. Thus for step l we can point numbers* $\varepsilon_l$ *such that* $|x(t_k) - x_k| \le \varepsilon_l$*, and* $\varepsilon_{0.5} = 0.3211$, $\varepsilon_{0.1} = 0.0515, \varepsilon_{0.05} = 0.0237, \varepsilon_{0.0025} = 0.0030$.

## 4.4 Concluding remarks and open problems

In this chapter we have extended method proposed in [21] to the case of nonconstant initial conditions. The approximation methods of higher order are of interest.

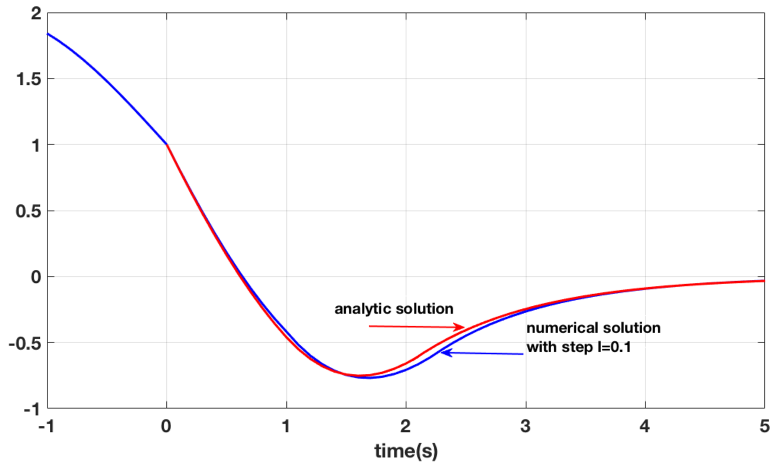


FIGURE 4.1: Plots of analytic solution and numerical solution with step $l = 0.1$ of the problem (4.28).

TABLE 4.1: Estimate of the difference between analytic and numerical solutions of the problem (4.28).

| $t_k$ | $x(t_k)$ | $l = 0.5,\ x_k,$ $\lvert x(t_k) - x_k\rvert$ | $l = 0.1,\ x_k,$ $\lvert x(t_k) - x_k\rvert$ | $l = 0.05,\ x_k,$ $\lvert x(t_k) - x_k\rvert$ | $l = 0.0025,\ x_k,$ $\lvert x(t_k) - x_k\rvert$ |
|---|---|---|---|---|---|
| 0.5 | 0.1627 | 0.2603 0.0976 | 0.1811 0.0184 | 0.1718 0.0091 | 0.1632 0.0005 |
| 1.0 | -0.4597 | -0.3997 0.0600 | -0.4172 0.0425 | -0.4386 0.0211 | -0.4586 0.0011 |
| 1.5 | -0.7328 | -0.8359 0.1031 | -0.7843 0.0515 | -0.7429 0.0101 | -0.7337 0.0009 |
| 2.0 | -0.6802 | -0.8699 0.1897 | -0.7070 0.0268 | -0.6598 0.0204 | 0.6805 0.0003 |
| 2.5 | -0.4721 | -0.7500 0.2779 | -0.4273 0.0448 | -0.4484 0.0237 | -0.4727 0.0006 |
| 3.0 | -0.2863 | -0.3801 0.0938 | -0.3268 0.0405 | -0.2648 0.0215 | -0.2893 0.0030 |
| 3.5 | -0.1737 | -0.3277 0.1540 | -0.1532 0.0205 | -0.1901 0.0164 | -0.1760 0.0023 |
| 4.0 | -0.1053 | -0.3184 0.2131 | -0.0917 0.0139 | -0.0950 0.0103 | -0.1068 0.0014 |
| 4.5 | -0.0639 | -0.3850 0.3211 | -0.0545 0.0094 | -0.0549 0.0090 | -0.0655 0.0016 |
| 5.0 | -0.0388 | -0.0238 0.0150 | -0.0329 0.0059 | -0.0335 0.0053 | -0.0393 0.0005 |

# Chapter 5

# Conclusion

In this thesis we have provided analysis of robustness properties of dynamical systems governed by differential equations with maximum of solution taken over past time interval. We have shown that such systems are particular case of SD-DDE with piecewise continuous delay function, they are infinite-dimensional and nonlinear. We have obtained comparison lemma (Lemma 4) for differential equations with maximum and we have proved that infinite-dimensional system with maximum may reduce to one-dimensional (Theorem 4, Lemma 5). By these results we have shown that global exponential stability of scalar equation with maximum is independent on $h$ (Theorem 6, Propositions 7).

We have accomplished ISS analysis of systems with maximum with usage neither Lyapunov-Krasovskiy technique nor Razumikhin technique but on the base of trajectory estimate. Particularly, for system in the linear form we proved ISS theorem (Theorem 8), which provides an explicit formula of the ISS gain function. To obtain this, we have proved the auxiliary lemma (Lemma 6), which gives an estimate from above of solution to differential inequality with maximum. For nonlinear system ISS study is achieved via averaging method (Theorems 16,17). Justification of averaging method has been also obtained (Theorems 14,15). Furthermore, we have proposed the numerical method for differential equations with maximum which is based on the rectangle method integration.

### Open problems

A lot of open problems in context of differential equations with maximum, especially in nonzero input case, remain to be solved. In addition to remarks in the end of each chapter we listed other open questions.

1. It is likely that the results from Chapter 3 can be extended to general SD-DDE. Also, observe that in [98] ISS analysis is provided via averaging method with usage ISS-Razumikhin approach. Application ISS-Lyapunov-Krasovskiy method is of interest.

2. In future one may study other robustness properties of differential equations with maximum, such that asymptotic gain property (every trajectory must ultimately stay not far from zero) and global stability (small initial states and controls produce uniformly small trajectory).

3. In this thesis we have studied RFDE in the form (2), however in the literature differential equation with maximum in the following form are considered

$$\dot{x}(t) = f(t, x(t), \max_{s\in[0,t]} x(s), u(t)), \quad t \geq 0, \tag{5.1}$$

(see [76, 77] for zero input case), which, together with equation (2), is a particular case of

$$\dot{x}(t) = f(t, x(t), \max_{s\in[g(t),\gamma(t)]} x(s), u(t)), \quad t \geq 0, \tag{5.2}$$

where $g(t) \leq \gamma(t) \leq t$, $t \in [0, \infty)$, $u \in L_\infty([0, \infty); \mathbb{R}^m)$. Qualitative analysis of system (5.2) without input is provided in [11], stability analysis is available in [10, 7, 11], however ISS of system (5.2) are of avenue for future research.

4. One may consider problem (5.2) in case functions $g, \tilde{\gamma}$ depend on the state of the system. To the best of author's knowledge such problem has not been studied yet.

5. In this work we have considered only continuous time systems. There are some works devoted to the stability of difference equations with maximum [87, 86, 67], nevertheless, robust stability is completely unstudied topic.

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