Lagrange Multiplier Methods for Constrained Optimization and Variational Problems in Banach Spaces

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DANIEL STECK
aus Chicago, IL, USA

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1. Gutachter: Prof. Dr. Christian Kanzow, Julius-Maximilians-Universität Würzburg
2. Gutachter: Prof. Dr. Michael Ulbrich, Technische Universität München
3. Gutachter: Prof. Dr. Christian Meyer, Technische Universität Dortmund

Abstract

This thesis is concerned with a class of general-purpose algorithms for constrained minimization problems, variational inequalities, and quasi-variational inequalities in Banach spaces.

A substantial amount of background material from Banach space theory, convex analysis, variational analysis, and optimization theory is presented, including some results which are refinements of those existing in the literature. This basis is used to formulate an augmented Lagrangian algorithm with multiplier safeguarding for the solution of constrained optimization problems in Banach spaces. The method is analyzed in terms of local and global convergence, and many popular problem classes such as nonlinear programming, semidefinite programming, and function space optimization are shown to be included as special cases of the general setting.

The algorithmic framework is then extended to variational and quasi-variational inequalities, which include, by extension, Nash and generalized Nash equilibrium problems. For these problem classes, the convergence is analyzed in detail. The thesis then presents a rich collection of application examples for all problem classes, including implementation details and numerical results.

Zusammenfassung

Die vorliegende Arbeit handelt von einer Klasse allgemein anwendbarer Verfahren zur Lösung restringierter Optimierungsprobleme, Variations- und Quasi-Variationsungleichungen in Banach-Räumen.

Zur Vorbereitung wird eine erhebliche Menge an Grundmaterial präsentiert. Dies beinhaltet die Theorie von Banach-Räumen, konvexe und variationelle Analysis sowie Optimierungstheorie. Manche der angegebenen Resultate sind hierbei Verfeinerungen der entsprechenden Ergebnisse aus der Literatur. Im Anschluss wird ein Augmented-Lagrange-Verfahren für restingierte Optimierungsprobleme in Banach-Räumen präsentiert. Der Algorithmus wird hinsichtlich lokaler und globaler Konvergenz untersucht, und viele typische Problemklassen wie nichtlineare Programme, semidefinite Programme oder Optimierungsprobleme in Funktionenräumen werden als Spezialfälle aufgezeigt.

Der Algorithmus wird dann auf Variations- und Quasi-Variationsungleichungen verallgemeinert, wodurch implizit auch (verallgemeinerte) Nash-Gleichgewichtsprobleme abgehandelt werden. Für diese Problemklassen werden eigene Konvergenzanalysen betrieben. Die Dissertation beinhaltet zudem eine umfangreiche Sammlung von Anwendungsbeispielen und zugehörigen numerischen Ergebnissen.

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Abbreviations and Notation

Abbreviations

a.e.	almost everywhere
e.g.	exempli gratia (for example)
etc.	et cetera (and so forth)
i.e.	id est (that is)
KKT	Karush–Kuhn–Tucker conditions
SOSC	second-order sufficient condition
RCQ	Robinson constraint qualification
ERCQ	extended Robinson constraint qualification
SRC	strict Robinson condition
LICQ	linear independence constraint qualification
MFCQ	Mangasarian–Fromovitz constraint qualification
EMFCQ	extended Mangasarian–Fromovitz constraint qualification
SMFC	strict Mangasarian–Fromovitz condition
CPLD	constant positive linear dependence condition
VI	variational inequality
QVI	quasi-variational inequality
NEP	Nash equilibrium problem
GNEP	generalized Nash equilibrium problem
PDE	partial differential equation
lsc	lower semicontinuous
usc	upper semicontinuous

Basic Sets and Relations

$\mathbb{N}$	natural numbers (without zero)
$\mathbb{N}_0$	natural numbers with zero
$\mathbb{R}$	real numbers
$\overline{\mathbb{R}}$	extended real line $\mathbb{R} \cup \{\pm\infty\}$
$\mathbb{R}_+$ ($\mathbb{R}_-$)	set of nonnegative (nonpositive) real numbers
(a, b), $[a, b]$	open and closed intervals, respectively
$\mathbb{R}^n$	space of n-dimensional real vectors
$\mathbb{R}^n_+$ ($\mathbb{R}^n_-$)	nonnegative (nonpositive) orthant in $\mathbb{R}^n$
$\|\cdot\|$	Euclidean norm on $\mathbb{R}^n$

$\leq$	partial ordering on $\mathbb{R}^n$ induced by componentwise comparison
$\leq_K$	partial ordering on a vector space induced by a convex cone K
$\mathcal{S}^n$	space of real symmetric $n \times n$-matrices
$\mathcal{S}^n_+$ ($\mathcal{S}^n_-$)	cone of positive (negative) semidefinite matrices in $\mathcal{S}^n$
$\preceq$	partial ordering on $\mathcal{S}^n$ induced by the cone $\mathcal{S}^n_+$
I_n	identity matrix of dimension $n \times n$
$\operatorname{diag}(v)$	diagonal matrix with entries $v \in \mathbb{R}^n$
A^{-1}	inverse of a nonsingular square matrix A
$A^\dagger$	Moore–Penrose pseudoinverse of a matrix A
$A^\top$	transpose of a matrix A
$\operatorname{tr}(A)$	trace of a square matrix A

Normed Spaces

X	a normed space
$\\|\cdot\\|_X$	norm on the space X
$\dim(X)$	dimension of the space X
X^*	dual space of a normed space X
$L(X,Y)$	space of bounded linear operators between normed spaces X and Y
Id_X	identity mapping on a normed space X
A^*	adjoint of an operator $A \in L(X,Y)$
$\langle\cdot,\cdot\rangle$	duality pairing of a normed space
$(\cdot,\cdot)$	scalar product of a Hilbert space
$x \perp y$	orthogonality with respect to $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$
$X \hookrightarrow Y$	embedding between normed spaces X and Y
$B_r(x)$	closed ball with radius r around some point x in a normed space
B_r^X	closed ball with radius r around zero in a normed space X
$\{x^k\} \subseteq X$	sequence of vectors in a normed space X
$\{\alpha_k\} \subseteq \mathbb{R}$	sequence of scalars
$x^k \to x$	convergence of a sequence in a topological space
$x^k \rightharpoonup x$	weak convergence of a sequence in a normed space
$\phi^k \rightharpoonup^* \phi$	weak-* convergence in the dual of a normed space
$\alpha_k \downarrow 0$	convergence to zero of a nonnegative scalar sequence $\{\alpha_k\}$
$x^k = O(\alpha_k)$	Landau symbol for $\{x^k\} \subseteq X$ satisfying $\\|x^k\\|_X \leq C\alpha_k$ with $C > 0$
$x^k = o(\alpha_k)$	Landau symbol for $\{x^k\} \subseteq X$ satisfying $\\|x^k\\|_X \leq z_k\alpha_k$ with $z_k \downarrow 0$
$\{x^k\}_{k\in I}$	subsequence of $\{x^k\}$ corresponding to $I \subseteq \mathbb{N}$
$x^k \to_I x$	convergence of the subsequence $\{x^k\}_{k\in I}$ to x

Geometry and Set Operations

$\mathcal{R}_K(y)$	radial cone of a convex set K in a point y
$\mathcal{N}_K(y)$	normal cone of a convex set K in a point y
K_∞	recession cone of a convex set K in a normed space
$\mathcal{T}_K(y)$	tangent cone of a set K in a point y
$\operatorname{cl}(A)$	closure of a set A in a topological space

$\mathrm{int}(A)$	interior of a set A in a topological space
$\mathrm{bd}(A)$	boundary of a set A in a topological space
A°	polar cone of a set A in a normed space
$A^\perp$	orthogonal complement of a set A in a Hilbert space
$A + B$	Minkowski sum of sets in a normed space
αA	α-multiple of a set in a normed space
d_C, $\mathrm{dist}(\cdot, C)$	distance function to a nonempty set C in a normed space
P_C	projection onto a nonempty closed convex set in a Hilbert space
$\sigma(\cdot, K)$	support function of a convex set K in a normed space
$S_k \xrightarrow{M} S$	set convergence in the sense of Mosco

Functions and Derivatives

$f : X \to Y$	mapping between Banach spaces X and Y
$\mathrm{epi}(f)$	epigraph of a function $f : X \to \mathbb{R}$
f', f''	first and second Fréchet-derivatives of f
∇f	transposed derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$
$\nabla^2 f$	Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$
$f_\mathcal{I}$	components $(f_i)_{i\in\mathcal{I}}$ of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$
∂f	convex subdifferential of a function $f : X \to \mathbb{R}$ or abstract generalized derivative of a function $f : X \to Y$
$\partial_B f$	Bouligand subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R}^m$
$\partial_{Cl} f$	Clarke generalized derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$
$\mathcal{W} : X \rightrightarrows Y$	set-valued mapping between Banach spaces X and Y
$\mathrm{gph}(\mathcal{W})$	graph of $\mathcal{W} : X \rightrightarrows Y$

Function Spaces

Ω	domain in $\mathbb{R}^d$
$\overline{\Omega}$	closure of $\Omega \subseteq \mathbb{R}^d$
$\partial\Omega$	boundary of $\Omega \subseteq \mathbb{R}^d$
$L^p(\Omega)$	Lebesgue space of p-integrable functions $u : \Omega \to \mathbb{R}$
$C(\overline{\Omega})$	space of continuous functions $u : \overline{\Omega} \to \mathbb{R}$
$C^k(\overline{\Omega})$	space of functions $u : \overline{\Omega} \to \mathbb{R}$ whose derivatives up to order k exist and can be extended continuously onto $\overline{\Omega}$
$C_0^\infty(\Omega)$	space of infinitely differentiable functions $u : \Omega \to \mathbb{R}$ with compact support
$W^{k,p}(\Omega)$	Sobolev space of functions $u : \Omega \to \mathbb{R}$ whose weak derivatives up to order k exist and belong to $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega)$	Sobolev space $W^{k,2}(\Omega)$
$H_0^k(\Omega)$	Sobolev space $W_0^{k,2}(\Omega)$
$H^{-1}(\Omega)$	dual space of $H_0^1(\Omega)$
∇	(weak) gradient operator

Δ	Laplace operator
Δ_p	p-Laplace operator
τu	trace of a function $u \in H^1(\Omega)$
$\partial_n u$	normal derivative of a function $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$
$H^{1/2}(\partial\Omega)$	image of $H^1(\Omega)$ under the trace operator
$H^{-1/2}(\partial\Omega)$	dual space of $H^{1/2}(\partial\Omega)$

Optimization and Related Problems

Φ	feasible set of an optimization problem or variational inequality
$\mathcal{L}$	Lagrange function of an optimization problem or variational inequality
$\mathcal{L}'$, $\mathcal{L}''$	derivatives of $\mathcal{L}$ with respect to the primal variable
$\mathcal{L}_\rho$	augmented Lagrange function of an optimization problem or variational inequality
$\mathcal{L}'_\rho$, $\mathcal{L}''_\rho$	derivatives of $\mathcal{L}_\rho$ with respect to the primal variable
$\Lambda(\bar{x})$	set of Lagrange multipliers corresponding to a stationary point of an optimization problem or variational inequality
$\mathcal{C}(\bar{x})$	critical cone corresponding to a stationary point $\bar{x}$ of an optimization problem or variational inequality
$\mathcal{C}_\eta(\bar{x})$	extended critical cone
f_ν	objective function of player ν in a Nash equilibrium problem
x^ν, $x^{-\nu}$	player variables and their complements in a Nash equilibrium problem
D_{x^ν}	partial derivative operator with respect to x^ν
∇_{x^ν}	transpose of D_{x^ν} in finite dimensions

Chapter 1

Introduction

In the last decades, optimization has emerged as one of the most fruitful branches of applied mathematics. One of the reasons for this phenomenon is that optimization lies at the intersection of many practical disciplines such as engineering, economics, and other applied sciences, and theoretical fields such as convex and variational analysis. As a result, the stellar development of optimization theory has always been motivated and driven by application contexts, whereas new theoretical findings have often resulted in new applications or different perspectives on existing ones. This research spiral is continuing even today, with new developments such as nonsmooth optimization techniques posing a variety of challenges but also resulting in an unprecedented amount of practical applications.

The technique of Lagrange multipliers is probably one of the most influential in the history of mathematical optimization. It was introduced by Joseph-Louis Lagrange in the eighteenth century for the determination of maxima and minima of functions subject to equality constraints [80]. Fast forward over two hundred years, and the Lagrange multiplier technique is at the heart of modern optimization theory and forms the basis of many algorithms for the solution of constrained optimization problems. This is epitomized by the vast amount of literature around this topic, including [24, 32, 112, 119, 163] and many more. A complete list of references is nearly impossible at this point.

In this thesis, we shall mainly be concerned with a general framework of optimization problems in Banach spaces and its extension to more sophisticated problem classes such as variational and quasi-variational inequalities, as well as (generalized) Nash equilibrium problems. The basic optimization framework we consider is a problem of the form

$$(P) \qquad \underset{x\in C}{\text{minimize}}\; f(x) \quad \text{subject to} \quad G(x) \in K, \tag{1.1}$$

where f and G are smooth functions defined on suitable Banach spaces, and C and K are convex sets. One of the aims of the thesis is to develop a class of algorithms for the solution of such problems. The methods we discuss can be classified rather broadly as *safeguarded augmented Lagrangian methods*, and they are applicable not only to optimization problems but also to variational inequalities and many more (see above). A significant emphasis is placed on a very high level of generality, and this opens up a broad spectrum of applications which will also be discussed after the theoretical investigations.

This thesis is essentially a summary of the research papers [128–136]. A substantial effort was undergone to simplify and streamline the theory, remove unnecessary assumptions, and present the results in a unified framework. In addition, a significant amount of background material is presented, including many results from the literature which are generalized, strengthened, or modified in other ways to suit the subsequent algorithmic investigations.

The following is an overview of the structure of the thesis. Chapter 2 contains a collection of fundamental results from various fields of mathematics, structured and presented in a purposeful manner to pave the way for the discussion of optimization problems. In Chapter 3, we discuss some basics of optimization theory, including first- and second-order conditions for constrained minimization problems, constraint qualifications, and similar concepts for variational inequalities. Although this section is mostly a collection of results from the literature, it also contains some developments which are new in their given form, such as the properties of second-order conditions in Section 3.1.3, the sequential optimality conditions in Section 3.2.3, and the primal-dual sensitivity analysis in Section 3.2.4.

Starting with Chapter 4, the attention is directed towards augmented Lagrangian methods for optimization and related problems. This chapter begins with a study of the algorithm for constrained minimization problems in Banach spaces; it contains a historical overview, a formal deduction of the algorithm, as well as local and global convergence analyses. The results in this chapter are largely new, and they are based on the publications [133, 135, 136], with a significant amount of improvements.

In Chapter 5, we discuss how the augmented Lagrangian algorithm can be applied to *variational inequalities* and generalized Nash equilibrium problems. This chapter is based on the papers [129, 133] and the preprint [128]. Chapter 6 then contains a discussion of *quasi-variational inequalities*, which can be seen as the most general problem class considered in the thesis. This chapter contains a theoretical investigation of such problems, a description and analysis of the augmented Lagrangian algorithm, and some additional considerations in finite dimensions, including an exact penalty method. This chapter is based on [129, 132] and the upcoming preprint [134].

In Chapter 7, we briefly discuss the concept of semismooth Newton methods and then provide a substantial collection of application examples for the proposed augmented Lagrangian algorithms, including linear and nonlinear obstacle problems, semilinear optimal control, and parameter estimation problems. We then discuss some examples of generalized Nash equilibrium problems (GNEPs), including multiobjective control problems and economic differential games, and some applications of quasi-variational inequalities (QVIs) from mechanics and superconductivity. The chapter concludes with two problem libraries for GNEPs and QVIs, respectively, in finite dimensions.

Chapter 8 contains some additional results on augmented Lagrangian techniques, including an example which demonstrates the necessity of multiplier safeguarding. Finally, in Chapter 9, some comments and future research perspectives are discussed.

Chapter 2

Background Material

This preliminary chapter establishes some fundamental notions which are indispensable for the remainder of the thesis. Most of the material presented here is simply a careful collection of results from the literature, structured and presented in a way which hopefully makes the theory as clear as possible.

The following is an outline of the structure of the chapter. In Section 2.1, we will mainly be concerned with the necessary tools from functional analysis. The results in this section can be found, for instance, in the books [38, 160, 197, 221–223]. We begin with some preliminary material on topological spaces in Section 2.1.1, where we give a brief account on different notions of compactness, and on the convergence of sequences. This section mainly serves the purpose of providing a formal basis for the topological treatment of Banach spaces. In Section 2.1.2, we give some basic results on Banach and Hilbert spaces, weak convergence, various types of continuity, and on their relationship with differentiability. Section 2.1.3 is then dedicated to the weak topology on a Banach space, its topological structure, and the resulting notions of compactness. Finally, in Section 2.1.4, we give some prominent examples of infinite-dimensional spaces, including the well-known *Lebesgue* and *Sobolev* spaces. A more comprehensive description of these spaces can be found in many textbooks, including [1, 210, 211].

In Section 2.2, the second part of the chapter, we then turn our attention to some basic concepts from convex and variational analysis. In many ways, we only scratch the surface of these enormous topics. More details on convex analysis can be found, for instance, in [13, 15], and in the famous book by Rockafellar [186]. For an overview of variational analysis, we refer the reader to the treatises [34, 163, 191], and to the book [32]. In the present thesis, we begin in Section 2.2.1 with some fundamental concepts of variational geometry such as radial, tangent, and normal cones, as well as the notion of recession cones. In Section 2.2.2, we give a fairly basic treatment of convex functions, including a variety of important examples, and the convex subdifferential.

The subsequent Sections 2.2.3 and 2.2.4 are more specialized in terms of their scope. In Section 2.2.3, we discuss a notion of convexity (and concavity) for operators with values in an arbitrary Banach space. This topic, although well known in the literature, is often covered only peripherally in textbooks on convex analysis and optimization. The history of such generalized concepts of convexity can be traced back to the doctoral thesis of J. Borwein [33], and they have since appeared most prominently in the context of vector optimization [125] and in semidefinite programming [218]. Since the material we require here is not too involved, Section 2.2.3 is self-contained and includes all the corresponding proofs (most of which are elementary).

The final part of this chapter, Section 2.2.4, deals with a class of abstract equilibrium problems and a family of existence results commonly referred to as *Ky Fan theorems*. The history of this branch of variational analysis goes back to the seminal paper [73] by Ky Fan, where he provided a geometric proof of his famous minimax inequality. This paper, in turn, has its roots in a fixed point approach developed by Knaster, Kuratowski, and Mazurkiewicz [144]. For the purposes of this thesis, we will use a slightly less well-known variant of the Ky Fan theorem which is due to Brezis, Nirenberg, and Stampacchia [39]. This result, along with a complete formal proof, will be given in Section 2.2.4, and it forms the basis of virtually all existence results for variational and quasi-variational inequalities in the subsequent parts of the thesis.

2.1 Banach Space Theory

This section is the most basic in this chapter since it establishes various notions from functional analysis. In addition, there is also a brief section on basic topology, with the aim of fixing the notation and terminology as well as providing a clear and well-founded basis for the subsequent discussions.

2.1.1 Topologies and Compactness

We begin with a discussion of some fundamental topological notions. This is necessary for the formal treatment of some aspects of optimization and variational analysis. This section is not intended as a comprehensive overview of general topology, but rather as a purposeful discussion of the particular concepts required for the subsequent chapters. A more in-depth account of general topology can be found, for instance, in [140].

Recall that a *topological space* is an arbitrary set X together with a collection $\mathcal{O}$ of subsets of X, called the *open sets*, such that

(i) arbitrary unions of open sets are again open, and

(ii) every finite intersection of open sets is open.

By convention, the trivial intersection equals the whole set X, and the trivial union is the empty set. Hence, $\mathcal{O}$ is necessarily nonempty and $\emptyset, X \in \mathcal{O}$. Prominent examples of topological spaces include metric spaces and, by extension, normed vector spaces. In those spaces, the concept of open sets can be defined via open balls, and this induces the corresponding topology.

Let X be an arbitrary topological space and $A \subseteq X$ a set. We say that A is *closed* if $X \setminus A$ is open. We write $\mathrm{cl}(A)$ for the *closure* of A, which is the intersection of all closed supersets of A, $\mathrm{int}(A)$ for the *interior* of A, which is the union of all open subsets of A, and $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(A)$ for the *boundary* of A. We call a set V a *neighborhood* of a point $x \in X$ if there is an open set $U \subseteq X$ such that $x \in U \subseteq V$. The neighborhood V itself is typically not required to be an open set, but it often suffices to consider open neighborhoods in proofs or other topological arguments.

Definition 2.1 (Hausdorff space)**.** A topological space X is called a *Hausdorff space* if, for all $x, y \in X$ with $x \neq y$, there are disjoint neighborhoods N_x of x and N_y of y.

An important notion in general topology is that of *compactness*. We say that a set $A \subseteq X$ is compact if, for every collection $\{O_i\}_{i \in I}$ of open subsets of X such that $A \subseteq \bigcup_{i \in I} O_i$, there is a finite index set $J \subseteq I$ such that $A \subseteq \bigcup_{i \in J} O_i$. One of the fundamental properties of compact sets is the following.

Lemma 2.2 (Finite intersection principle, [32, Prop. 2.4])**.** *Let X be a topological space, $A \subseteq X$ a compact set, and $\{F_i\}_{i \in I}$ a family of closed subsets of A such that, for every finite set $J \subseteq I$, the intersection $\bigcap_{i \in J} F_i$ is nonempty. Then $\bigcap_{i \in I} F_i$ is nonempty.*

In metric spaces, the notions of openness, closedness, and compactness can be fully characterized through sequences and their convergence properties. This no longer holds true for an arbitrary topological space, not even for practically relevant topologies such as the weak topology on a Banach space (see Example 2.26). Nevertheless, it will be useful to introduce and discuss appropriate notions of sequences and of convergence in topological spaces. This has two reasons. First, the treatment of sequences and their induced continuity properties is much more convenient than that of the generic (abstract) topological notions. This is particularly true when working with optimization algorithms, for which sequential continuity properties are clearly the most natural framework. The second reason why the treatment of sequences is useful is that, in some cases, the use of sequence-based continuity properties over their topological counterparts is actually *necessary*, see, for instance, Remark 2.27.

Definition 2.3 (Convergence of sequences)**.** Let X be a topological space and $\{x^k\}_{k \in \mathbb{N}}$ a sequence of points in X. We say that $\{x^k\}$ *converges to* $x \in X$, and write $x^k \to x$, if every neighborhood of x contains all but finitely many elements of $\{x^k\}$.

The above definition is obviously consistent with the standard convergence in metric spaces. If X is a Hausdorff space, then limits of sequences (if existent) are unique.

The definition of sequences and their convergence gives rise to "sequential" notions of closedness and compactness. We say that a set $A \subseteq X$ is

- *sequentially closed* if the limit of every convergent sequence from A lies in A,
- *sequentially open* if, whenever $x \in A$ and $\{x^k\} \subseteq X$, $x^k \to x$, then $x^k \in A$ for sufficiently large k, and
- *sequentially compact* if every sequence in A admits a subsequence which converges to a point in A.

It is easy to verify that every open (closed) subset of X is sequentially open (closed). Moreover, a set is sequentially open if and only if its complement is sequentially closed.

Definition 2.4 (Continuity properties)**.** Let X be an arbitrary topological space.

(a) A function $f : X \to Y$, with Y a topological space, is called *(sequentially) continuous* if $f^{-1}(O)$ is (sequentially) open in X for every open set $O \subseteq Y$.

(b) A function $f : X \to \mathbb{R}$ is called *(sequentially) lower semicontinuous* if the level sets $\{x \in X : f(x) \leq c\}$ are (sequentially) closed in X for every $c \in \mathbb{R}$.

(c) A function $f : X \to \mathbb{R}$ is called *(sequentially) upper semicontinuous* if the level sets $\{x \in X : f(x) \geq c\}$ are (sequentially) closed in X for every $c \in \mathbb{R}$.

We will often use the abbreviations *lsc* and *usc* for lower and upper semicontinuity, respectively. Clearly, f is (sequentially) lsc if and only if $-f$ is (sequentially) usc.

Note that, from a high-level perspective, all the above continuity notions can actually be recovered as special cases of plain continuity. Indeed, the sequential continuity notions are obtained if X is equipped with a slightly modified topology (see below), and the lower and upper semicontinuity of f can be recovered by equipping $\mathbb{R}$ with a suitable topology, see [140, Problem 3.F] for more details.

The definition of sequential continuity may seem rather odd at first glance. The particular definition here was chosen to closely resemble ordinary continuity and to therefore highlight the differences between the two definitions. The following result states that our notion of sequential continuity is precisely what one would intuitively expect.

Proposition 2.5. *Let X, Y be arbitrary topological spaces and $f : X \to Y$. Then the following are equivalent:*

(i) *f is sequentially continuous from X into Y.*

(ii) *Whenever $\{x^k\} \subseteq X$ and $x^k \to x$ in X, then $f(x^k) \to f(x)$ in Y.*

Proof. (i)⇒(ii): Let $\{x^k\} \subseteq X$ be an arbitrary sequence such that $x^k \to x$ in X. We need to show that $f(x^k) \to f(x)$ in Y. To this end, let $U \subseteq Y$ be an arbitrary neighborhood of $f(x)$. Without loss of generality, U is open. Then $f^{-1}(U)$ is sequentially open in X, which implies that $x^k \in f^{-1}(U)$ for k sufficiently large, and thus $f(x^k) \in U$.

(ii)⇒(i): Let $U \subseteq Y$ be an arbitrary open subset of Y. We need to show that $f^{-1}(U)$ is sequentially open in X. Let $\{x^k\} \subseteq X$ be a sequence such that $x^k \to x$ for some $x \in f^{-1}(U)$. Then $f(x^k) \to f(x) \in U$ by assumption. Hence, by the definition of convergence in Y, we obtain $f(x^k) \in U$ for sufficiently large k, and thus $x^k \in f^{-1}(U)$. □

For sequentially lsc and usc functions, it is also possible to obtain a rather intuitive characterization. This is achieved by directly using the definition of these concepts. It follows that a function $f : X \to \mathbb{R}$ is

- sequentially lsc if and only if, whenever $x \in X$ and $x^k \to x$ in X, then $f(x) \leq \liminf_{k\to\infty} f(x^k)$, and
- sequentially usc if and only if, whenever $x \in X$ and $x^k \to x$ in X, then $f(x) \geq \limsup_{k\to\infty} f(x^k)$.

Hence, these two notions are also equivalent to their conventional versions. In what follows, we shall occasionally make reference to sequential continuity properties in a single point $x \in X$. The meaning of this should be fairly obvious; for instance, we say that $f : X \to Y$, with X, Y topological spaces, is *sequentially continuous in* x if $f(x^k) \to f(x)$ in Y for every sequence $x^k \to x$ in X. Sequential lower or upper semicontinuity can be defined in an analogous manner (with $Y = \mathbb{R}$).

Proposition 2.6 (Minimization principle, [32, Thm. 2.6])**.** *Let X be an arbitrary topological space, $A \subseteq X$ a (sequentially) compact subset of X, and $f : A \to \mathbb{R}$ a (sequentially) lower semicontinuous function. Then f attains a global minimum on A.*

The above is the basic existence result for minimizers of functions on topological spaces. It is often attributed to K. Weierstrass. If X is a real Banach space endowed with the weak topology (see Section 2.1.3), then the result is sometimes called the *direct method of the calculus of variations.*

We now discuss an approach which allows us to formally interpret the sequential notions of closedness, openness, and continuity in a topological framework. The main idea is to define an auxiliary topology with the aim of representing precisely the sequential structure of X. A natural way of doing this is the following.

Definition 2.7 (Sequential topology)**.** Let X be an arbitrary topological space. Then the *sequential topology* on X is the topology given by the sequentially open subsets of X.

It is easy to see that the sequential topology is well-defined (i.e., it is always a topology). Moreover, since every closed set in a topological space is sequentially closed, it follows that the sequential topology is always stronger (finer) than the original topology of X. Finally, it is important to observe that the notion of convergence induced by the sequential topology on X is identical to the notion of convergence induced by the original topology. One direction of this equivalence is trivial (since the sequential topology is stronger than the original one), and the other direction follows from the definition of sequential openness.

It follows from Proposition 2.5 that a mapping $f : X \to Y$ from X, equipped with its sequential topology, into an arbitrary topological space Y, is continuous if and only if it maps convergent sequences to convergent sequences. The great benefit of this observation arises when applying results from general topology (which are often formulated in an abstract topological framework) to a situation where only sequential properties (such as continuity) are available. In that case, we can simply apply the desired results in the sequential topology of X, and the resulting argumentation is completely rigorous.

2.1.2 Banach and Hilbert Spaces

A *Banach space* is a normed vector space X which is complete. Throughout this thesis, we will only deal with Banach spaces where the underlying field is the real numbers, and emphasize this by calling them *real* Banach spaces.

Given a real Banach space X, we write Id_X for the identity mapping on X. For a point $x \in X$, we denote by $B_r(x) := \{y \in X : \|x - y\|_X \le r\}$ the closed r-ball around x, and we write B_r^X for the closed r-ball around $0 \in X$. Given another real Banach space Y and a mapping $T : X \to Y$, we say that T is *continuous* in $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_\delta(x_0)) \subseteq B_\varepsilon(T(x_0))$. It is easy to verify that T is continuous in every $x \in X$ if and only if it is continuous in the sense of Definition 2.4. We say that T is *Lipschitz-continuous on* $A \subseteq X$, with modulus $L \geq 0$, if $\|T(x) - T(y)\|_Y \leq L\|x - y\|_X$ for all $x, y \in A$, and *locally Lipschitz-continuous in* $x \in X$ if there exists $r > 0$ such that T is Lipschitz-continuous on $B_r(x)$. If T is Lipschitz-continuous with modulus $L = 1$, then we call T *nonexpansive*. Finally, a mapping $T : X \to Y$ is said to be *linear* if $T(\alpha x + y) = \alpha T(x) + T(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$.

Definition 2.8 (Operator and dual spaces)**.** Let X and Y be real Banach spaces. Then $L(X, Y)$ is the space of bounded linear mappings from X into Y, equipped with the norm

$$\|A\|_{L(X,Y)} := \sup_{\|x\|_X \leq 1} \|Ax\|_Y. \tag{2.1}$$

The space $L(X, \mathbb{R})$ is denoted by X^* and called the *dual space* of X.

Given $\phi \in X^*$ and $x \in X$, we will often use the *duality pairing* $\langle \phi, x\rangle := \phi(x)$ to denote the evaluation of ϕ. Note that $\langle \cdot, \cdot \rangle$ is a bilinear mapping on $X^* \times X$. Given a linear operator $T \in L(X, Y)$, we denote by $T^* \in L(Y^*, X^*)$, $\langle T^*y, x\rangle := \langle y, Tx\rangle$, the *adjoint operator* of T. We say that T is an *isomorphism* if T is bijective and its inverse lies in $L(Y, X)$. We say that T is *isometric* if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. If T satisfies both these properties, we call T an *isometric isomorphism.*

If X is not the trivial space $X = \{0\}$, then (2.1) can equivalently be written as

$$\|A\|_{L(X,Y)} = \sup_{\|x\|_X = 1} \|Ax\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X}.$$

It is well known that $L(X, Y)$ and X^* are again Banach spaces. (This even holds if X is just an arbitrary normed space.) Given a real Banach space X, we denote by $X^{**} := (X^*)^*$ the *bidual space* of X. Furthermore, we say that X is *reflexive* if the canonical embedding

$$i_X : X \to X^{**}, \quad (i_X x)(f) := f(x),$$

is surjective. Note that i_X is always injective and isometric. Hence, if X is reflexive, then i_X is an isometric isomorphism from X onto X^{**}.

We say that a real Banach space X is a *(real) Hilbert space* if the norm on X is induced by a scalar product, i.e., if there exists a symmetric bilinear mapping $(\cdot, \cdot) : X^2 \to \mathbb{R}$ such that $\|x\|_X = \sqrt{(x, x)}$ for all $x \in X$. One of the most important properties of Hilbert spaces is the following.

Theorem 2.9 (Riesz representation, [221, Section III.6])**.** *Let X be a real Hilbert space and $f \in X^*$. Then there exists a uniquely determined $x_f \in X$ such that $f = (x_f, \cdot)$. The corresponding mapping $x \mapsto (x, \cdot)$ is an isometric isomorphism from X onto X^*.*

The Riesz representation theorem has several important consequences for the analytical structure of Hilbert spaces. In particular, it implies that every real Hilbert space is reflexive. Another consequence is the following result which plays a fundamental role in the analysis of partial differential equations.

Corollary 2.10 (Lax–Milgram, [221, Section III.7]). *Let X be a real Hilbert space and $a : X^2 \to \mathbb{R}$ a bilinear form with the following properties:*

(i) *There exists $c_1 > 0$ with $|a(x,y)| \le c_1 \|x\|_X \|y\|_X$ for all $x, y \in X$.*

(ii) *There exists $c_2 > 0$ such that $a(x,x) \ge c_2 \|x\|_X^2$ for all $x \in X$.*

Then the mapping $T_a(x) := a(x,\cdot)$ is a continuous isomorphism from X onto X^* with $\|T_a\|_{L(X,X^*)} \le c_1$ and $\|T_a^{-1}\|_{L(X^*,X)} \le c_2^{-1}$.

Note that the bilinear form a in the above result is not necessarily symmetric. If a is symmetric, then it is a scalar product on X whose induced norm is equivalent to $\|\cdot\|_X$. Thus, in this case, the Lax–Milgram theorem is just the Riesz representation theorem.

Another fundamental property of Hilbert spaces is the existence of projections onto (nonempty) closed convex subsets.

Lemma 2.11 (Projection operator). *Let H be a real Hilbert space and $C \subseteq H$ a nonempty closed convex set. Then, for every $x \in H$, there is a unique point $P_C(x) \in C$ of minimal distance to x. The resulting operators $P_C : H \to H$ and $\mathrm{Id}_H - P_C$ are nonexpansive.*

In a general Banach or Hilbert space, many analytical properties of convex sets can be proved by using separation arguments. We will encounter two separation results in this thesis: the one below and Proposition 2.19 in Section 2.1.3.

Proposition 2.12 (First separation theorem, [32, Thm. 2.13]). *Let X be a real Banach space and $S, T \subseteq X$ convex sets such that S has nonempty interior and $\mathrm{int}(S) \cap T = \emptyset$. Then there exists $\phi \in X^* \setminus \{0\}$ such that $\phi(s) \ge \phi(t)$ for all $s \in S$, $t \in T$.*

If X is finite-dimensional, then the assumption on $\mathrm{int}(S)$ can be dropped. This is not the case in an arbitrary Banach space, even if the sets are closed, see [212].

Recall that the Banach open mapping theorem [197, 221] states that a surjective linear operator $A \in L(X,Y)$ between Banach spaces X and Y is *open*, in the sense that $A(U)$ is open in Y whenever U is open in X. Since A is linear, this is equivalent to the existence of an $r > 0$ such that $B_r^Y \subseteq A(B_1^X)$. Here, we state a slightly more general version of this theorem which is essentially due to Graves [92].

Theorem 2.13 (Uniform open mapping theorem). *Let X, Y be real Banach spaces and $A \in L(X,Y)$ a surjective linear operator. Then there exists $r > 0$ such that $B_r^Y \subseteq A(B_1^X)$ and, whenever $T \in L(X,Y)$ and $\delta := \|T - A\|_{L(X,Y)} < r$, then $B_{r-\delta}^Y \subseteq T(B_1^X)$.*

Proof. The first assertion is the Banach open mapping theorem. For the proof of the second assertion, we refer the reader to [58, Thm. 1.2] or [59, Thm. 5D.2]. □

We now discuss some notions of convergence on Banach spaces and their induced continuity properties. Apart from standard (strong, norm) convergence, the following are the two basic notions of sequential convergence which we will use.

Definition 2.14 (Weak and weak-* convergence)**.** Let X be a real Banach space.

(a) We say that $\{x^k\} \subseteq X$ is *weakly convergent* to $x \in X$, and write $x^k \rightharpoonup x$, if $\phi(x^k) \to \phi(x)$ for every $\phi \in X^*$.

(b) We say that $\{\phi^k\} \subseteq X^*$ is *weak-* convergent* to $\phi \in X^*$, and write $\phi^k \rightharpoonup^* \phi$, if $\phi^k(x) \to \phi(x)$ for every $x \in X$.

Given a sequence $\{x^k\} \subseteq X$, we say that x is a weak limit point of $\{x^k\}$ if there is a subsequence of $\{x^k\}$ which converges weakly to x. Note that this does not necessarily coincide with the notion of limit points in the topological sense (see Example 2.26). Weak-* limit points are defined in an analogous manner.

Definition 2.15. Let X, Y be real Banach spaces and $T : X \to Y$. We say that T is

(i) *weakly sequentially continuous* if $x^k \rightharpoonup x$ implies $T(x^k) \rightharpoonup T(x)$.

(ii) *weak-* sequentially continuous* if $Y = W^*$ for some real Banach space W, and $x^k \rightharpoonup x$ implies $T(x^k) \rightharpoonup^* T(x)$ in W^*.

(iii) *completely continuous* if $x^k \rightharpoonup x$ implies $T(x^k) \to T(x)$.

Clearly, complete continuity is the strongest notion of (sequential) continuity. In particular, it implies both ordinary and weak sequential continuity. A related notion which is frequently used in the literature is that of *compact* operators. A linear operator $T : X \to Y$ is called compact if it maps bounded sets in X to precompact sets (i.e., sets with compact closure) in Y. It is well-known and easy to verify that every compact linear operator is completely continuous, and the converse holds provided that X is reflexive.

Let X, Y be real Banach spaces. Recall that an operator $T : X \to Y$ is said to be *(Fréchet-)differentiable* in $x \in X$ if there is a bounded linear operator $T'(x) \in L(X, Y)$ such that

$$T(x + h) = T(x) + T'(x)h + o(\|h\|_X)$$

for all $h \in X$ sufficiently small. An important connection between Fréchet-differentiability and complete continuity is given by the following result.

Proposition 2.16. *Let X, Y be real Banach spaces, $T : X \to Y$ a completely continuous operator, and let T be Fréchet-differentiable in some $x \in X$. Then $T'(x) \in L(X, Y)$ is completely continuous.*

Proof. Assume that $T'(x)$ is not completely continuous. Then there is a sequence $\{w^k\} \subseteq X$ such that $w^k \rightharpoonup w \in X$, $\|w^k\|_X \leq 1$ for all k, and $\|T'(x)w^k - T'(x)w\|_Y \geq \varepsilon$ for all k and some $\varepsilon > 0$. By Fréchet-differentiability, there exists $r > 0$ such that

$$\|T(x + h) - T(x) - T'(x)h\|_Y \leq \frac{\varepsilon}{4}\|h\|_X \quad \text{for all } h \in B_r^X.$$

Observe now that $\|rw^k\|_X \le r$ for all k. It follows that $\|T(x+rw^k)-T(x)-rT'(x)w^k\|_Y \le (\varepsilon r)/4$, and the same inequality holds with w^k replaced by w. Hence,

$$\begin{aligned}\|T(x+rw^k)-T(x+rw)\|_Y &\ge \|T'(x)(rw^k-rw)\|_Y \\ &\quad - \|T(x+rw^k)-T(x)-rT'(x)w^k\|_Y \\ &\quad - \|T(x+rw)-T(x)-rT'(x)w\|_Y \\ &\ge \|T'(x)(rw^k-rw)\|_Y - \frac{\varepsilon r}{2} \ge \frac{\varepsilon r}{2},\end{aligned}$$

which contradicts the complete continuity of T. □

We now give another result which relates the complete continuity of an operator T to that of the derivative mapping $T' : X \to L(X,Y)$. For this, we need the notion of *uniform differentiability*. A differentiable operator $T : X \to Y$ is said to be uniformly differentiable on a subset $A \subseteq X$ if

$$\frac{\|T(x+h)-T(x)-T'(x)h\|_Y}{\|h\|_X} \to 0 \quad \text{as } \|h\|_X \downarrow 0,$$

uniformly for $x \in A$. This means that, for every $\varepsilon > 0$, we can choose $\delta > 0$ such that $\|T(x+h)-T(x)-T'(x)h\|_Y \le \varepsilon \|h\|_X$ whenever $x \in A$ and $\|h\|_X \le \delta$.

The following result was proved in [175]; note the different terminology in that reference.

Proposition 2.17. *Let X, Y be real Banach spaces and assume that X is reflexive. Let $T : X \to Y$ be completely continuous and uniformly differentiable on bounded subsets of X. Then $T' : X \to L(X,Y)$ is completely continuous.*

The above result admits a (partial) converse for real-valued functions. Indeed, if X is a real reflexive Banach space and $f : X \to \mathbb{R}$ a differentiable mapping, then the complete continuity of $f' : X \to X^*$ actually implies the weak sequential continuity of f. More details can be found in [222, Section 41.4].

2.1.3 The Weak Topology on a Banach Space

Throughout this section, let X be a real Banach space. A fundamental issue in infinite-dimensional spaces is the choice of topology. This is particularly critical because, in the norm topology, very few practically relevant sets are compact. Indeed, the closed unit ball B_1^X (or any other closed ball) is compact if and only if X is finite-dimensional. This underlines the necessity of a different topological approach to generic Banach spaces. The main definition in this context is the following.

Definition 2.18 (Weak topology)**.** The *weak topology* on a real Banach space X is the coarsest topology for which all $f \in X^*$ are continuous.

It is rather easy to verify that the weak topology is well-defined and a Hausdorff topology. Constructions of the above type are usually referred to as *initial topologies.* That is, one takes a family $\mathcal{F}$ of functions mapping X into arbitrary topological spaces, and then defines the initial topology with respect to that family as the coarsest topology such that all $f \in \mathcal{F}$ are continuous. The resulting topology is that generated by the preimages of open sets under the mappings in $\mathcal{F}$. In our case, the weak topology is precisely the initial topology of X with respect to the family $\mathcal{F} := X^*$.

In accordance with topological terminology, we call a set $S \subseteq X$ weakly open (closed, compact) if it is open (closed, compact) with respect to the weak topology. Since the weak topology is coarser than the strong topology, it follows that every weakly closed (open) set is strongly closed (open), and every strongly compact set is weakly compact.

Like every topology, the weak topology induces a notion of convergence, which is precisely the weak convergence defined in Definition 2.14. Thus, we call a set weakly sequentially open (closed, compact) if it is sequentially open (closed, compact) with respect to weak convergence.

Proposition 2.19 (Second separation theorem, [32, Thm. 2.14])**.** *Let $S, T \subseteq X$ be disjoint closed convex sets, and let S be weakly compact. Then there are $c_1, c_2 \in \mathbb{R}$ and $\phi \in X^*$ such that $\phi(s) \le c_1 < c_2 \le \phi(t)$ for all $s \in S$ and $t \in T$.*

The above separation theorem has several important consequences. Two particular corollaries which we need are given below.

Corollary 2.20. *Let $C \subseteq X$ be a convex set. Then the following are equivalent:* (i) *C is closed,* (ii) *C is weakly closed, and* (iii) *C is weakly sequentially closed.*

Corollary 2.21. *Let X be a real reflexive Banach space and $C \subseteq X$ a nonempty bounded closed convex set. Then C is weakly compact. Conversely, if the closed unit ball B_1^X in some real Banach space X is weakly compact, then X is reflexive.*

We now discuss the notion of weak compactness for nonconvex subsets of a real Banach space X. The main result in this direction is the following which goes back to the works of W. Eberlein and V. Šmulian.

Theorem 2.22 (Eberlein–Šmulian, [160, Thm. 2.8.6])**.** *A subset $A \subseteq X$ of a real Banach space X is weakly compact if and only if it is weakly sequentially compact.*

The following result contains a direct consequence of the Eberlein–Šmulian theorem as well as a statement which is sometimes called *Day's lemma.* A proof of this second assertion can be found in [160, Cor. 2.8.7].

Proposition 2.23. *Let $A \subseteq X$ be a weakly compact set and $S \subseteq A$. Then* (i) *S is weakly closed if and only if it is weakly sequentially closed, and* (ii) *for every point x in the weak closure of S, there is a sequence $\{x^k\} \subseteq S$ such that $x^k \rightharpoonup x$.*

Another consequence of the Eberlein–Šmulian theorem is that weak and weak sequential lower semicontinuity coincide for functions on weakly compact sets.

Corollary 2.24. *Let $A \subseteq X$ be a weakly compact set and $f : A \to \mathbb{R}$. Then f is weakly lower semicontinuous if and only if it is weakly sequentially lower semicontinuous.*

Proof. Apply Proposition 2.23 (i) to the lower level sets of f. □

We close this section with some remarks and examples.

Remark 2.25. When applying results from the literature which are formulated in a generic topological setting, it is occasionally useful to consider a slightly different topology called the *weak sequential topology* on X. This is the topology induced by weak convergence; more precisely, we call a set open in the weak sequential topology if its complement is weakly sequentially closed. This is indeed a topology (see Definition 2.7). Moreover, since every weakly closed set is weakly sequentially closed, it is stronger (finer) than the weak topology and therefore also a Hausdorff topology.

Example 2.26. Let $X := \ell^2(\mathbb{R})$ be the space of square-summable real sequences, let $\{e^k\} \subseteq X$ be the sequence of unit vectors, and consider the set $S := \{x^k\}_{k\in\mathbb{N}}$ with $x^k := \sqrt{k}e^k$. Since every weakly convergent sequence in S is necessarily bounded, every such sequence is eventually constant and its weak limit therefore lies in S. It follows that S is weakly sequentially closed and, similarly, norm closed. However, somewhat surprisingly, the set S is not weakly closed since 0 lies in the weak closure of S, see [15, Ex. 3.33].

Remark 2.27. Let X, Y be real Banach spaces and $A \in L(X, Y)$. Then the *complete continuity* of A (see Definition 2.15) is nothing but sequential continuity from the weak into the strong topology, or equivalently, continuity from the weak sequential topology (see Remark 2.25) into the strong topology. It is interesting to note that, except for trivial cases, an operator $A \in L(X, Y)$ cannot be (topologically) continuous from the weak into the strong topology. Indeed, if A has this property, then the range of A is necessarily finite-dimensional, see [38, Exercise 6.7].

2.1.4 Lebesgue, Sobolev, and Related Spaces

This section is dedicated to some prominent function spaces which will play a key role in many of our examples and applications, including the ubiquitous *Lebesgue* and *Sobolev* spaces. More details on these and related spaces can be found in many places in the literature, for instance, in [1, 62, 210, 211].

Throughout this section, we assume that $d \in \mathbb{N}$ is a natural number, $\Omega \subseteq \mathbb{R}^d$ is a bounded and sufficiently regular domain (e.g., a Lipschitz domain in the sense of [1]), and $\Gamma := \partial\Omega$ is the boundary of Ω. We write $C(\overline{\Omega})$ for the space of continuous functions $u : \overline{\Omega} \to \mathbb{R}$, equipped with the norm $\|u\|_{C(\overline{\Omega})} := \|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|$. Moreover, we write $C^k(\overline{\Omega})$ for the space of functions $u : \overline{\Omega} \to \mathbb{R}$ whose partial derivatives up to order k exist and can be extended continuously onto $\overline{\Omega}$. The norm on this space is defined as

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|s|\leq k} \|D^s u\|_\infty,$$

where the sum ranges over all multi-indices $s := (s_1, \ldots, s_d)$ with $|s| := s_1 + \cdots + s_d \leq n$, and D^s is the derivative operator $D^s := D_{x_1}^{s_1} \cdots D_{x_d}^{s_d}$. It is well-known that $C(\overline{\Omega})$ and $C^k(\overline{\Omega})$ are Banach spaces for all k. Finally, we define $C^\infty(\Omega)$ as the vector space of infinitely differentiable functions on Ω, and $C_0^\infty(\Omega)$ as the space of functions $u \in C^\infty(\Omega)$ with compact support.

Recall that a function $u : \Omega \to \mathbb{R}$ is called *measurable* if the lower level sets $\{x \in \Omega : u(x) \leq c\}$ are Lebesgue-measurable for all $c \in \mathbb{R}$. Given such a function $u : \Omega \to \mathbb{R}$, we denote by

$$\operatorname{ess\,sup}_\Omega u := \inf\{M \in \mathbb{R} : u(x) \leq M \text{ a.e. in } \Omega\}$$

the *essential supremum* of u over Ω. This allows us to define the *Lebesgue norms* $\|\cdot\|_{L^p(\Omega)}$, $1 \leq p \leq +\infty$, which are given by

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_\Omega |u|, & \text{if } p = \infty. \end{cases}$$

These norms induce the Lebesgue spaces

$$L^p(\Omega) := \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \|u\|_{L^p(\Omega)} < +\infty\big\}.$$

By a famous theorem of Fischer and Riesz, the spaces $L^p(\Omega)$, equipped with their corresponding norms, are Banach spaces. The space $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x)v(x) \,\mathrm{d}x.$$

One of the most important inequalities on Lebesgue spaces is the following. Note that we use the convention $1/\infty := 0$.

Lemma 2.28 (Hölder inequality)**.** *Let $p, q \in [1, \infty]$ and $p^{-1} + q^{-1} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $u \cdot v \in L^1(\Omega)$ and $\|u \cdot v\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}$.*

Assume now that $u \in L^p(\Omega)$ is a given function, and $s = (s_1, \ldots, s_d)$ is a given multi-index. We say that a function $v \in L^p(\Omega)$ is a *weak derivative of order s* of the function u if

$$\int_\Omega u(x) D^s\phi(x) \,\mathrm{d}x = (-1)^{|s|} \int_\Omega v(x)\phi(x) \,\mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

With this definition in place, it is common to define the *Sobolev space* $W^{k,p}(\Omega)$, where $k \in \mathbb{N}_0$ and $p \in [1, \infty]$, as

$$W^{k,p}(\Omega) := \big\{u \in L^p(\Omega) : D^s u \in L^p(\Omega) \text{ for all } s \in \mathbb{N}_0^d \text{ with } |s| \leq k\big\}.$$

These spaces become Banach spaces with the norms

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|s| \leq k} \|D^s u\|_{L^p(\Omega)}^p\right)^{1/p}, & \text{if } p < \infty, \\ \sum_{|s| \leq k} \|D^s u\|_{L^\infty(\Omega)}, & \text{if } p = \infty. \end{cases}$$

It is easy to verify that, for all $k \in \mathbb{N}_0$, the space $H^k(\Omega) := W^{k,2}(\Omega)$ is a real Hilbert space with the scalar product

$$(u, v)_{H^k(\Omega)} := \sum_{|s| \le k} \big(D^s u, D^s v\big)_{L^2(\Omega)} = \sum_{|s| \le k} \int_\Omega D^s u(x)\, D^s v(x) \, \mathrm{d}x. \tag{2.2}$$

For $k \in \mathbb{N}_0$ and $p \in [1, \infty]$, we will write $W_0^{k,p}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. By definition, this is a closed subspace of $W^{k,p}(\Omega)$ and therefore a Banach space in its own right. Similarly to above, we write $H_0^k(\Omega) := W_0^{k,2}(\Omega)$, and this is a Hilbert space with respect to the scalar product (2.2).

For the sake of convenience, we also define $L^p(\Omega, \mathbb{R}^d)$ as the space of functions $u : \Omega \to \mathbb{R}^d$ whose components belong to $L^p(\Omega)$. This space becomes a Banach space with the norm

$$\|u\|_{L^p(\Omega,\mathbb{R}^d)} := \begin{cases} \big(\int_\Omega \|u(x)\|^p\big)^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_\Omega \|u\|, & \text{if } p = \infty, \end{cases}$$

where the norm on $\mathbb{R}^d$ is the Euclidean norm. Similarly, $L^2(\Omega, \mathbb{R}^d)$ becomes a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega,\mathbb{R}^d)} := \int_\Omega u(x)^\top v(x) \, \mathrm{d}x.$$

Whenever the image space is clear from the context, we will simply write $\|u\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega,\mathbb{R}^d)}$ and $(u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega,\mathbb{R}^d)}$.

Theorem 2.29 (Poincaré inequality)**.** *Let $p \in [1, \infty)$. Then there is a constant $c > 0$, depending on Ω and p, such that $\|u\|_{L^p(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)}$ for all $u \in W_0^{1,p}(\Omega)$.*

The Poincaré inequality implies, in particular, that the space $W_0^{1,p}(\Omega)$ can equivalently be equipped with the norm $\|u\| := \|\nabla u\|_{L^p(\Omega)}$ instead of the subspace norm inherited from $W^{1,p}(\Omega)$. On $H_0^1(\Omega)$, this norm is induced by the scalar product

$$(u, v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)} = \int_\Omega \nabla u(x)^\top \nabla v(x) \, \mathrm{d}x. \tag{2.3}$$

It follows that $H_0^1(\Omega)$ is a Hilbert space with respect to this scalar product.

We now define the dual spaces $H^{-k}(\Omega) := H_0^k(\Omega)^*$, where $k \in \mathbb{N}$ is a natural number. One of the most fundamental operators on Sobolev spaces is the *Laplace operator* $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, which is defined by

$$\langle \Delta u, v\rangle := -(\nabla u, \nabla v)_{L^2(\Omega)} = -\int_\Omega \nabla u(x)^\top \nabla v(x) \, \mathrm{d}x, \quad u, v \in H_0^1(\Omega).$$

Note that $-\Delta$ is precisely the Riesz isomorphism (see Theorem 2.9) on $H_0^1(\Omega)$ if this space is equipped with the scalar product (2.3). It follows that Δ too is a continuous isomorphism. Moreover, we have $\langle -\Delta u, u\rangle = \|\nabla u\|^2_{L^2(\Omega)} \ge 0$ for all $u \in H_0^1(\Omega)$, which means that $-\Delta$ is a *positive operator*. (Since Δ is linear, this also implies that $-\Delta$ is *monotone* in the sense of Section 3.2.1.)

The Laplace operator occurs prominently in the *Poisson equation* (with Dirichlet boundary conditions). Given $u \in L^2(\Omega)$, this equation asks for the existence of $y \in H_0^1(\Omega)$ such that

$$-\Delta y = u \quad \text{a.e. in } \Omega. \tag{2.4}$$

For every $u \in H^{-1}(\Omega)$, this equation admits a so-called *weak solution* $y \in H_0^1(\Omega)$ such that the equality in (2.4) holds with respect to the space $H^{-1}(\Omega)$. This solution is simply given by $y = -\Delta^{-1}u$, and its existence follows from the Riesz representation theorem or, more generally, the Lax–Milgram theorem (Corollary 2.10).

An important concept on Sobolev-type spaces is the *trace* operator. This mapping allows us to define, in a generalized context, the notion of boundary values of Sobolev-type functions. In this thesis, we will mainly need the trace of H^1-functions. Recall that $\Gamma = \partial\Omega$ denotes the boundary of Ω.

Lemma 2.30 (Trace operator)**.** *There is a unique bounded linear operator $\tau : H^1(\Omega) \to H^{1/2}(\Gamma)$ such that $\tau u = u|_\Gamma$ for all $u \in C^1(\overline{\Omega})$.*

The trace operator is usually constructed by taking the restriction operator $|_\Gamma : C^1(\overline{\Omega}) \to L^2(\Gamma)$, where Γ is understood as a manifold (see [38]), showing that this mapping is bounded with respect to the $H^1(\Omega)$-norm on $C^1(\overline{\Omega})$, and then extending it to $\tau : H^1(\Omega) \to L^2(\Gamma)$ by means of the Hahn–Banach theorem. The extension is unique because $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$. One then defines $H^{1/2}(\Gamma)$ as the image space of τ, and it can be verified that this is indeed a Banach space, with norm given by

$$\|v\|_{H^{1/2}(\Gamma)} := \inf_{u \in H^1(\Omega),\, \tau u = v} \|u\|_{H^1(\Omega)}.$$

Note that this is nothing but the canonical norm on $H^{1/2}(\Gamma)$ induced by the isomorphy $H^1(\Omega)/\ker(\tau) \cong H^{1/2}(\Gamma)$, where the isomorphism is the mapping τ, acting on the cosets of $H^1(\Omega)$ with respect to $\ker(\tau)$, see [221, Section I.11].

Another important concept in the context of Sobolev spaces is that of *normal derivatives*. The proper definition of these requires some caution and, in particular, a different domain space. Let $H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*$ denote the dual space of $H^{1/2}(\Gamma)$.

Proposition 2.31 (Normal derivative, [210, Lem. 20.2])**.** *The space $X := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$, with norm $\|u\|_X := \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$, is a real Banach space. Moreover, the normal derivative mapping*

$$\partial_n : X \to H^{-1/2}(\Gamma), \quad \langle \partial_n u, v \rangle := \int_\Omega \Delta u(x)\, \tilde{v}(x) + \nabla u(x)^\top \nabla \tilde{v}(x)\, \mathrm{d}x,$$

where $\tilde{v} \in H^1(\Omega)$ is chosen so that $\tau\tilde{v} = v$, is well-defined and continuous.

We conclude this section by presenting a generalization of the Laplace operator which occurs in certain application contexts (see Chapter 7).

Definition 2.32 (p-Laplace operator)**.** Let $p \in [2, \infty)$. The *p-Laplace operator* or *p-Laplacian* $\Delta_p : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ is defined by

$$\langle \Delta_p u, v \rangle := -\int_\Omega \|\nabla u(x)\|^{p-2} \nabla u(x)^\top \nabla v(x) \, \mathrm{d}x, \quad u, v \in W_0^{1,p}(\Omega).$$

It is known that Δ_p is well-defined, continuous, and even continuously differentiable on $W_0^{1,p}(\Omega)$, see [104]. Moreover, $-\Delta_p$ is *monotone* (see Section 3.2.1) in the sense that

$$\langle \Delta_p u - \Delta_p v, u - v \rangle \leq 0 \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

Finally, we note that Δ_p can also be seen as the Fréchet-derivative of the function $u \mapsto p^{-1} \|\nabla u\|_{L^p(\Omega)}^p$. Recall that $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the subspace norm inherited from $W^{1,p}(\Omega)$, see the discussion after Theorem 2.29.

2.2 Elements of Variational and Convex Analysis

This section contains some basic notions of variational analysis, including various types of cones to describe the variational geometry of sets. In addition, some concepts of convex analysis are presented which will be useful in later chapters. Throughout this section, unless stated otherwise, X is always a real Banach space.

2.2.1 Tangent, Normal, and Recession Cones

This section is dedicated to the study of some basic objects which are useful when characterizing the geometric structure of sets in Banach spaces. Many aspects of the geometry of sets can be characterized through so-called cones (see below), and these play a major role in variational analysis, convex analysis, and optimization theory. The material discussed here incorporates elements from multiple books, e.g., [15, 32, 34].

Let $S \subseteq X$ be a nonempty set. We say that S is a *cone* if $\alpha S \subseteq S$ for all $\alpha > 0$. We call a cone S *pointed* if $S \cap (-S) = \{0\}$. Given an arbitrary set $S \subseteq X$, we denote by

$$S^\circ := \{\phi \in X^* : \langle \phi, s \rangle \leq 0 \text{ for every } s \in S\}$$

the *polar cone* of S. Note that $S^\circ \subseteq X^*$. If X is a real Hilbert space, we treat S° as a subset of X.

Definition 2.33 (Tangent cone)**.** Let $C \subseteq X$ be an arbitrary set and $x \in X$. Then we define the *tangent cone* $\mathcal{T}_C(x)$ as the empty set if $x \notin C$, and otherwise as

$$\mathcal{T}_C(x) := \big\{ d \in X : \exists \{x^k\} \subseteq C, \, t_k \downarrow 0 \text{ such that } x^k \to x \text{ and } (x^k - x)/t_k \to d \big\}.$$

The tangent cone plays a fundamental role in the formal description of various variational properties of sets. It is famously used in the derivation of first-order optimality conditions for constrained optimization problems, see Section 3.1.

Lemma 2.34 (Product formula)**.** *Let $X_1, \dots, X_N$ be real Banach spaces and $C_i \subseteq X_i$ closed subsets of X_i for all i. Let $C := \prod_{i=1}^N C_i$ and $x = (x_i)_{i=1}^N \in C$. Then*

$$\mathcal{T}_C(x) \subseteq \prod_{i=1}^{N} \mathcal{T}_{C_i}(x_i) \quad \text{and} \quad \mathcal{T}_C(x)^\circ = \prod_{i=1}^{N} \mathcal{T}_{C_i}(x_i)^\circ. \tag{2.5}$$

The first inclusion becomes an equality if the sets $C_1, \dots, C_N$ are convex.

Let us emphasize that the second equality in (2.5) is always satisfied, even if the sets C_i are nonconvex.

Definition 2.35 (Radial and normal cones)**.** Let $C \subseteq X$ be a convex set. We define

(a) the *radial cone* $\mathcal{R}_C(x)$ of C at $x \in X$ as $\mathcal{R}_C(x) := \emptyset$ if $x \notin C$, and otherwise

$$\mathcal{R}_C(x) := \{\alpha(c - x) : \alpha \geq 0,\, c \in C\}.$$

(b) the *normal cone* $\mathcal{N}_C(x)$ of C at $x \in X$ as $\mathcal{N}_C(x) := \emptyset$ if $x \notin C$, and otherwise

$$\mathcal{N}_C(x) := \{\phi \in X^* : \langle \phi, y - x \rangle \leq 0 \;\forall y \in C\}.$$

If X is a real Hilbert space, we treat $\mathcal{N}_C(x)$ as a subset of X instead of X^*. Both the radial and normal cones are always convex cones, and we have $\mathcal{T}_C(x) = \operatorname{cl}(\mathcal{R}_C(x))$ whenever C is convex. Moreover, the normal cone is always a closed set, and it has the representations

$$\mathcal{N}_C(x) = (C - x)^\circ = \mathcal{R}_C(x)^\circ = \mathcal{T}_C(x)^\circ.$$

The last of these formulas is sometimes taken as the general definition of the normal cone for possibly nonconvex sets C. However, it should be noted that there are a variety of different normal cones for general sets (see, for instance, [163]). Therefore, to avoid any ambiguity, we will reserve the symbol $\mathcal{N}_C$ for the case where C is convex.

The normal cone can be used to characterize the metric projection onto the underlying set (see Lemma 2.11).

Proposition 2.36. *Let H be a real Hilbert space and $C \subseteq H$ a nonempty closed convex set. Then, for $x \in H$, we have $y = P_C(x)$ if and only if $x - y \in \mathcal{N}_C(y)$.*

Conversely, for every $y \in C$, $d \in \mathcal{N}_C(y)$ if and only if $y = P_C(y + \alpha d)$ for some $\alpha > 0$. In that case, $y = P_C(y + \alpha d)$ for all $\alpha > 0$.

The following is a famous decomposition theorem involving a closed convex cone in a Hilbert space and its polar.

Lemma 2.37 (Moreau decomposition, [164])**.** *Let H be a real Hilbert space and $K \subseteq H$ a nonempty closed convex cone. Then every $y \in H$ admits a unique decomposition $y = y_1 + y_2$ with $K \ni y_1 \perp y_2 \in K^\circ$. Indeed, $y_1 = P_K(y)$ and $y_2 = P_{K^\circ}(y)$.*

We now turn to another object which describes some aspects of the geometric structure of convex sets.

Definition 2.38 (Recession cone)**.** Let $C \subseteq X$ be a nonempty convex set. Then the *recession cone* of C is the set $C_\infty := \{x \in X : x + C \subseteq C\}$.

The recession cone is always nonempty (since $0 \in C_\infty$) and a convex cone. This can be shown by first proving the convexity and then using the fact that $nx \in C_\infty$ whenever $x \in C_\infty$ and $n \in \mathbb{N}$. Finally, if C is closed, then so is C_∞.

If the set C is a convex *cone*, then it is easy to see that $C_\infty = C$. On the other hand, if C is not a cone, then the recession cone can often be used as a substitute for C in situations where a conical structure is necessary. This is the case, for instance, in the context of (partial) order relations, which closely correspond to convex cones. More details can be found in Section 2.2.3.

The following result provides some information on the polar cone $C_\infty^\circ := (C_\infty)^\circ$.

Lemma 2.39. *Let H be a real Hilbert space and $C \subseteq H$ a nonempty convex set. Then $\{y \in H : \sup_{w\in C}(w, y) < +\infty\} \subseteq C_\infty^\circ$. In particular, $\mathcal{N}_C(y) \subseteq C_\infty^\circ$ for all $y \in C$.*

Proof. Let $y \in H$ be a point with $(w, y) \le c$ for some $c \in \mathbb{R}$ and all $w \in C$. Let $x \in C_\infty$, and choose an arbitrary $x_0 \in C$. Then $x_0 + tx \in C$ for all $t > 0$, and hence $(x_0 + tx, y) \le c$. This cannot hold for all $t > 0$ if $(x, y) > 0$. Hence, $(x, y) \le 0$, and $y \in C_\infty^\circ$. □

The set $\{y \in H : \sup_{w\in C}(w, y) < +\infty\}$ in the statement of Lemma 2.39 is often called the *barrier cone* of C. Note that the inclusion stated in the lemma can be strict. In particular, there are situations where the barrier cone is not closed, and this makes it a priori impossible for it to equal C_∞°, which is always a closed cone by virtue of polarity. An example for this phenomenon can be found in [15, Exercise 6.23].

Proposition 2.40. *Let X be a real Banach space and $C \subseteq X$ a closed convex set. Let $x \in C$, $\phi \in X^*$, $\{x^k\} \subseteq C$, $\{\phi^k\} \subseteq X^*$ such that $\phi^k \in \mathcal{N}_C(x^k)$ for all k, and assume that either* (i) *$x^k \rightharpoonup x$ and $\phi^k \to \phi$, or* (ii) *$x^k \to x$ and $\phi^k \rightharpoonup^* \phi$. Then $\phi \in \mathcal{N}_C(x)$.*

Proof. Let $c \in C$. By assumption, $\langle \phi^k, c - x^k\rangle \le 0$ for all k. Under either (i) or (ii), it follows that $\langle \phi, c - x\rangle \le 0$. Since c was arbitrary, this means that $\phi \in \mathcal{N}_C(x)$. □

2.2.2 Convex Functions and Subdifferentials

The theory of convex functions is a cornerstone of modern variational analysis and optimization theory. This point is emphasized by the famous quote

> *"...the great watershed in optimization isn't between linearity and nonlinearity, but between convexity and nonconvexity."*
>
> — R. T. Rockafellar [190]

In that spirit, we shall dedicate the present and the subsequent sections to a discussion of various forms of convexity as well as their consequences. In the infinite-dimensional setting, it turns out that the distinction between convexity and nonconvexity is even more important than in finite dimensions since convex functions are much more amenable when it comes to certain minimization-related continuity properties (see below).

Definition 2.41 (Convexity)**.** Let $C \subseteq X$ be a convex set. We say that $f : C \to \mathbb{R}$ is

(i) *convex* if, for all $x, y \in C$ and $\alpha \in [0,1]$,

$$f\big((1-\alpha)x + \alpha y\big) \le (1-\alpha)f(x) + \alpha f(y).$$

(ii) *strictly convex* if, for all $x, y \in C$, $x \neq y$, and $\alpha \in (0,1)$,

$$f\big((1-\alpha)x + \alpha y\big) < (1-\alpha)f(x) + \alpha f(y).$$

(iii) *strongly convex* with modulus $c > 0$ if, for all $x, y \in C$ and $\alpha \in [0,1]$,

$$f\big((1-\alpha)x + \alpha y\big) \le (1-\alpha)f(x) + \alpha f(y) - \frac{c}{2}\alpha(1-\alpha)\|x-y\|_X^2.$$

A function $f : C \to \mathbb{R}$ is called (strictly, strongly) concave if $-f$ is (strictly, strongly) convex. It goes without saying that much of the theory of convex functions can be carried out in a similar fashion for concave functions. For the sake of simplicity, we restrict ourselves to the former class here.

One of the most fundamental examples of convex functions is the distance function $d_C : X \to \mathbb{R}$ to a convex set $C \subseteq X$. Note that the following result holds for an arbitrary Banach space X, not necessarily a Hilbert space.

Lemma 2.42 (Distance function, [15, 177])**.** *Let $C \subseteq X$ be a nonempty convex set. Then the function $d_C : X \to \mathbb{R}$, $d_C(x) := \inf_{y \in C} \|x-y\|_X$, is convex and nonexpansive.*

It is easy to see that the square of a nonnegative convex function is again convex. Thus, in the setting of Lemma 2.42, the squared distance function d_C^2 is also a convex function. If the space X is a real Hilbert space, then the squared distance function enjoys a much stronger form of regularity.

Lemma 2.43 ([15, Cor. 12.31])**.** *Let X be a real Hilbert space and $C \subseteq X$ a nonempty closed convex set. Then the squared distance function d_C^2 is convex and continuously differentiable on X with $(d_C^2)'(x) = 2(x - P_C(x))$ for all $x \in X$.*

We now discuss some continuity properties of a general convex function $f : C \to \mathbb{R}$ on a nonempty, closed, convex set $C \subseteq X$. The continuity properties of f are closely linked to the so-called *epigraph* of f, which is the set

$$\operatorname{epi}(f) := \{(x,t) \in C \times \mathbb{R} : f(x) \le t\}.$$

It is easily verified that $\operatorname{epi}(f)$ is a convex set, and that f is (weakly, weakly sequentially) lower semicontinuous if and only if $\operatorname{epi}(f)$ is (weakly, weakly sequentially) closed. The following is therefore a direct consequence of Corollary 2.20.

Proposition 2.44. *Let $C \subseteq X$ be a closed convex set and $f : C \to \mathbb{R}$ a convex function. Then the following are equivalent:* (i) *f is lower semicontinuous,* (ii) *f is weakly lower semicontinuous, and* (iii) *f is weakly sequentially lower semicontinuous.*

Note that a convex function can fail to be lower semicontinuous, even if it is defined on the whole space X. To see this, let $f : X \to \mathbb{R}$ be a discontinuous linear functional. Examples of such functionals are well-known on certain (non-complete) normed spaces X, and their existence on arbitrary infinite-dimensional spaces follows from the axiom of choice. Observe that f is convex, but if f were lower semicontinuous at some point $x \in X$, then a symmetry argument would yield the continuity of f at x and thus, by linearity, the continuity on X. This is a contradiction.

Let us now turn to the fundamental notion of generalized first-order derivatives for convex functions, the so-called convex subdifferential.

Definition 2.45 (Convex subdifferential)**.** Let $C \subseteq X$ be a convex set and $f : C \to \mathbb{R}$ a convex function. The *convex subdifferential* of f in $x \in C$ is the set

$$\partial f(x) := \big\{ d \in X^* : f(y) \geq f(x) + \langle d, y - x \rangle \ \forall y \in C \big\}.$$

The following result gives sufficient conditions for ∂f to be nonempty, and also describes the relationship between the convex subdifferential and Fréchet-derivatives. This result is a combination of various statements contained in [177].

Proposition 2.46. *Let X be a real Banach space and $f : X \to \mathbb{R}$ a continuous convex function. Then $\partial f(x)$ is nonempty at every $x \in X$. Moreover, f is Fréchet-differentiable in $x \in X$ if and only if the following two properties hold:*

(i) *$\partial f(x)$ is the singleton $\{f'(x)\}$, and*

(ii) *whenever $x^k \to x$ in X and $d^k \in \partial f(x^k)$, then $d^k \to f'(x)$.*

Property (ii) is often called norm-to-norm upper semicontinuity. The above result implies that a convex function cannot have a discontinuous Fréchet-derivative. Hence, if f is convex and differentiable, then it is continuously differentiable.

One of the key benefits of the convex subdifferential is the availability of a Fermat-type stationary result. Indeed, one of the classical assertions related to the convex subdifferential is that a point $x \in X$ minimizes a convex function $f : X \to \mathbb{R}$ if and only if $0 \in \partial f(x)$. For later reference, we state this theorem in a more general form which is essentially a combination of [34, Thm. 4.3.3] and [163, Thm. 1.88].

Theorem 2.47 (Necessary optimality condition)**.** *Let $f : X \to \mathbb{R}$ be a continuously differentiable mapping, $C \subseteq X$ a nonempty closed convex set, and $g : X \to \mathbb{R}$ a continuous convex function. If $\bar{x}$ is a local minimizer of $f + g$ on C, then*

$$0 \in f'(\bar{x}) + \partial g(\bar{x}) + \mathcal{N}_C(\bar{x}).$$

2.2.3 Concave Operators

The theory of convex functions is useful for a wide variety of application problems. There are, however, certain practical scenarios where convexity properties of nonlinear *operators*

$G : X \to Y$ are necessary, with X and Y real Banach spaces. More specifically, assume that we are dealing with an inclusion of the form

$$G(x) \in K, \quad K \subseteq Y \text{ a closed convex set.} \tag{2.6}$$

Ideally, we would like to work with a generalized notion of convexity which takes into account the mapping G and the geometry of the set K. The present section is dedicated to the analysis of such generalized convexity notions, their consequences, and their relationship with ordinary convexity.

Assume for the moment that the set K in (2.6) is a closed convex *cone*. Then K induces the order relation

$$a \le_K b \ :\Longleftrightarrow \ b - a \in K, \tag{2.7}$$

and K itself can be regarded as the nonnegative cone with respect to $\le_K$. Thus, (2.6) can be rewritten as $G(x) \ge_K 0$, which suggests that the appropriate convexity notion in this case is a generalized type of concavity with respect to the order relation $\le_K$. This property takes on the form

$$G\big((1-t)x + ty\big) \ge_K (1-t)G(x) + tG(y) \quad \text{for all } x, y \in X, \ t \in [0,1].$$

The above property is usually called *K-concavity*, and it is in fact a special case of the general concept which we define below. In the case where K is not a cone, the recession cone K_∞ turns out to be a useful substitute to define the order relation (2.7).

Definition 2.48 (Concave operator)**.** Let $G : X \to Y$ be an arbitrary mapping and $K \subseteq Y$ a closed convex set with recession cone K_∞. We say that G is *K_∞-concave* if

$$G\big((1-t)x + ty\big) \ge_K (1-t)G(x) + tG(y) \quad \text{for all } x, y \in X, \ t \in [0,1],$$

where $\le_K$ is the order relation defined by $a \le_K b :\Longleftrightarrow b - a \in K_\infty$.

Before proving that this property is in fact useful and provides some desirable properties for the constraint (2.6), we first give two important examples. These show that, for certain practically relevant cases, the notion of K_∞-concavity reduces to the corresponding "natural" convexity properties.

Example 2.49. (a) Let m, p be nonnegative integers and $Y := \mathbb{R}^{m+p}$, $K := \mathbb{R}^m_- \times \{0\}^p$. This corresponds to the case of nonlinear programming-type constraints, see Section 3.1.4. In this case, K is a closed convex cone, which implies $K_\infty = K$, and it is easy to see that G is K_∞-concave if and only if the functions G_i $(i = 1, \dots, m)$ are convex and the functions G_j $(j = m+1, \dots, m+p)$ are affine.

(b) Let X and Y be function spaces, K the negative cone in Y, and G an operator of the form $G(u)(t) := \mathcal{G}(t, u(t))$. Assume that $\mathcal{G}$ is sufficiently regular so that G is well-defined, and that $\mathcal{G}$ is convex with respect to the second variable. Then G is K_∞-concave.

Let us now discuss the analytical consequences of generalized convexity in the sense of Definition 2.48. The resulting properties can be deduced by discussing situations in which the K_∞-concavity of G yields the (ordinary) convexity of a suitable composite mapping involving G.

We say that a mapping $m : Y \to \mathbb{R}$ is K_∞-*decreasing* if it is monotonically decreasing with respect to the order $\le_K$, i.e., if $m(y_1) \le m(y_2)$ whenever $y_1 \ge_K y_2$.

Theorem 2.50. *Let X, Y be real Banach spaces, $K \subseteq Y$ a nonempty closed convex set, and $G : X \to Y$ a K_∞-concave operator. Then:*

(a) *If $m : Y \to \mathbb{R}$ is convex and K_∞-decreasing, then $m \circ G$ is convex.*

(b) *The function $d_K \circ G : X \to \mathbb{R}$ is convex.*

(c) *If $\lambda \in K_\infty^\circ$, then $x \mapsto \langle \lambda, G(x) \rangle$ is convex.*

(d) *The set $M := \{x \in X : G(x) \in K\}$ is convex.*

Proof. (a): Let $x, y \in X$ and $x_\alpha = \alpha x + (1-\alpha)y$, $\alpha \in [0,1]$. Then $G(x_\alpha) \ge_K \alpha G(x) + (1-\alpha)G(y)$ by the concavity of G. Applying m on both sides yields

$$m(G(x_\alpha)) \le m(\alpha G(x) + (1-\alpha)G(y)) \le \alpha m(G(x)) + (1-\alpha) m(G(y)),$$

where we first used the monotonicity and then the convexity of m. Hence, $m \circ G$ is convex.

(b): The function d_K is convex by Lemma 2.42. We claim that it is also K_∞-decreasing. Let $y, z \in Y$, $y \le_K z$, be arbitrary points. Then $z = y + k$ for some $k \in K_\infty$. Now, let $\varepsilon > 0$ and let $y_\varepsilon \in K$ be a point with $\|y - y_\varepsilon\|_Y \le d_K(y) + \varepsilon$. Then

$$d_K(z) = d_K(y+k) \le \|y + k - (y_\varepsilon + k)\|_Y = \|y - y_\varepsilon\|_Y \le d_K(y) + \varepsilon,$$

where the inequality in the middle uses the fact that $y_\varepsilon + k \in K$ because $k \in K_\infty$. Since $\varepsilon > 0$ was arbitrary, it follows that $d_K(z) \le d_K(y)$. Hence, d_K is K_∞-decreasing, and thus the function $d_K \circ G$ is convex by (a).

(c): The function $y \mapsto \langle \lambda, y \rangle$, with $\lambda \in K_\infty^\circ$, is obviously a convex function, and it is decreasing because $\langle \lambda, k \rangle \le 0$ for all $k \in K_\infty$. Hence, the result again follows from (a).

(d): Note that $M = \{x \in X : d_K(G(x)) \le 0\}$. Hence, M is a lower level set of the convex function $d_K \circ G$, and therefore a convex set. □

For later use, we will also need an analogue of the standard gradient inequality for convex functions. This result is contained in the following proposition.

Proposition 2.51. *Let $K \subseteq Y$ be a nonempty closed convex set and $G : X \to Y$ a differentiable K_∞-concave operator. Then $G(w) \le_K G(x) + G'(x)(w - x)$ for all $x, w \in X$.*

Proof. Let $x, w \in X$ and $d := w - x$. For all $t \in (0,1)$, the K_∞-concavity of G yields

$$G(x + td) = G\big((1-t)x + tw\big) \in (1-t)G(x) + tG(w) + K_\infty.$$

Rearranging this inclusion, dividing by t, and recalling that K_∞ is a cone yields

$$G(x) + \frac{G(x+td) - G(x)}{t} \in G(w) + K_\infty.$$

For $t \downarrow 0$, we obtain $G(x) + G'(x)d \geq_K G(w)$. Hence, the result follows. □

Let us conclude this section by mentioning an alternative, more abstract motivation for the definition of K_∞-concavity. This motivation is based on the theory of multifunctions and will play a certain role in the context of constraint qualifications, see Section 3.1.2.

Remark 2.52. Apart from the order relation induced by the recession cone K_∞, there is another way to motivate the notion of operator concavity which is closely related to the multifunction $\mathcal{W} : X \rightrightarrows Y$, $\mathcal{W}(x) := G(x) - K$. This is usually called the *feasibility mapping* of the system $G(x) \in K$, and the K_∞-concavity of G is nothing but the convexity of $\mathcal{W}$ in the multifunction sense, i.e., the convexity of the graph $\mathrm{gph}(\mathcal{W})$. More details behind this motivation can be found in Section 3.1.2 and in [32, Section 2.3.5].

2.2.4 Ky Fan's Minimax Theorem

This section is a first step towards the analysis of generic variational or equilibrium-type problems. Many such problems can be written in the general framework

$$\text{find } \hat{x} \in A: \quad \Psi(\hat{x}, y) \leq 0 \quad \forall y \in A, \tag{2.8}$$

where $A \subseteq X$ is a nonempty set and $\Psi : A \times A \to \mathbb{R}$ a scalar-valued function, usually called a *bifunction*. Problems in this abstract form are often referred to as *equilibrium problems*. More details on this problem class can be found in [30, 120, 139].

In the present section, we give two general existence theorems for equilibrium problems, with the idea of applying them to variational inequalities and related problems in later chapters. For the sake of generality, we will make use of a weakened form of concavity, called quasiconcavity.

Definition 2.53 (Quasiconcavity)**.** Let $S \subseteq X$ be a convex set. Then a function $f : S \to \mathbb{R}$ is called *quasiconcave* if

$$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\} \quad \text{for all } x, y \in S,\ \alpha \in [0,1]. \tag{2.9}$$

Clearly, every concave function is quasiconcave. Moreover, it is easy to verify that a function is quasiconcave if and only if its upper level sets $\{x \in S : f(x) \geq c\}$ are convex for all $c \in \mathbb{R}$. This, in turn, implies that the notions of upper semicontinuity, weak upper semicontinuity, and weak sequential upper semicontinuity coincide for quasiconcave functions (like they do for concave functions, see Proposition 2.44). In particular, every continuous quasiconcave function is weakly sequentially upper semicontinuous.

We now turn to the existence theory for the equilibrium problem (2.8). The most basic existence theorem is due to Ky Fan [73].

Lemma 2.54 (Ky Fan, [73]). *Let $A \subseteq X$ be a nonempty, convex, weakly compact set, and $\Psi : A \times A \to \mathbb{R}$ a mapping such that*

(i) $\Psi(x,x) \le 0$ *for all* $x \in A$,

(ii) *for every* $x \in A$, *the function* $\Psi(x,\cdot)$ *is quasiconcave, and*

(iii) *for every* $y \in A$, *the function* $\Psi(\cdot,y)$ *is weakly sequentially lsc.*

Then there exists $\hat{x} \in A$ *such that* $\Psi(\hat{x},y) \le 0$ *for all* $y \in A$.

Note that the Ky–Fan theorem is often stated in an arbitrary (Hausdorff) topological vector space, with (iii) replaced by weak lower semicontinuity. In the Banach space setting, since the set A is weakly compact, it suffices to assume weak *sequential* lower semicontinuity, see Corollary 2.24.

While the Ky–Fan theorem has manifold applications in game theory and related subjects, it turns out to be rather unwieldy when applied to the class of problems known as variational inequalities. It is in particular the continuity assumption (iii) in Lemma 2.54 which turns out to be quite unnatural in this case, see also Example 3.44 in Chapter 3. As a consequence of this fact, certain extensions of the Ky–Fan theorem have been developed, including a rather notable one due to Brezis, Nirenberg, and Stampacchia [39]. Before giving this result, we first state an auxiliary lemma, mainly for motivational purposes. Note that we call Ψ *monotone* if $\Psi(x,y) + \Psi(y,x) \ge 0$ for all $x, y \in A$.

Lemma 2.55. *Let $A \subseteq X$ be a closed convex set and $\Psi : A \times A \to \mathbb{R}$ a mapping such that either*

(a) Ψ *is weakly sequentially lsc with respect to* x, *or*

(b) Ψ *is continuous, monotone, concave with respect to* y, *and* $\Psi(x,x) \le 0$ $\forall x \in A$.

Then Ψ has the property that, whenever $x, y \in A$, $\{x^k\} \subseteq A$ converges weakly to x, and $\Psi(x^k,(1-t)x+ty) \le 0$ for all $t \in [0,1]$ and $k \in \mathbb{N}$, then $\Psi(x,y) \le 0$.

Proof. (a) If $\Psi(x^k,(1-t)x+ty) \le 0$ for all $t \in [0,1]$ and $k \in \mathbb{N}$, then we obtain, in particular, $\Psi(x^k,y) \le 0$ for all k. Taking $k \to \infty$, it follows that $\Psi(x,y) \le 0$.

(b) Let $w_t := (1-t)x+ty$. By concavity, Ψ is weakly sequentially upper semicontinuous with respect to y. Using the monotonicity of Ψ, it follows that

$$\Psi(w_t,x) \ge \limsup_{k\to\infty} \Psi(w_t,x^k) \ge \limsup_{k\to\infty} \big[-\Psi(x^k,w_t)\big] \ge 0$$

for all $t \in [0,1]$. Using again the concavity of Ψ with respect to y, we obtain

$$\Psi(w_t,x) \ge 0 \ge \Psi(w_t,w_t) \ge (1-t)\Psi(w_t,x) + t\Psi(w_t,y) \ge t\Psi(w_t,y),$$

where the last inequality uses the first one. It follows that $\Psi(w_t,y) \le 0$ for all $t > 0$, and letting $t \to 0$ yields $\Psi(x,y) \le 0$. □

Based on the result above, it now seems natural to construct an existence result for (2.8) by using the assertion of Lemma 2.55 as an abstract continuity assumption. The resulting theorem covers both the monotone and the nonmonotone case.

Theorem 2.56 (Brezis–Nirenberg–Stampacchia, [39]). *Let X be a real Banach space, $A \subseteq X$ a nonempty, convex, weakly compact set, and $\Psi : A \times A \to \mathbb{R}$ a mapping with*

(i) $\Psi(x, x) \le 0$ *for all* $x \in A$,

(ii) *for every $x \in A$, the function $\Psi(x, \cdot)$ is quasiconcave,*

(iii) *for every $y \in A$ and every finite-dimensional subspace L of X, the function $\Psi(\cdot, y)$ is lower semicontinuous on $A \cap L$, and*

(iv) *whenever $x, y \in A$, $\{x^k\} \subseteq A$ converges weakly to x, and $\Psi(x^k, (1-t)x + ty) \le 0$ for all $t \in [0, 1]$ and $k \in \mathbb{N}$, then $\Psi(x, y) \le 0$.*

Then there exists $\hat{x} \in A$ such that $\Psi(\hat{x}, y) \le 0$ for all $y \in A$.

Proof. Let $\mathcal{L}$ be the collection of all finite-dimensional subspaces L of X which intersect A. For each $L \in \mathcal{L}$, Lemma 2.54 implies the existence of an $x_L \in A \cap L$ such that $\Psi(x_L, y) \le 0$ for all $y \in A \cap L$. Now, for $L \in \mathcal{L}$, let

$$S_L := \big\{x \in A : \Psi(x, y) \le 0 \text{ for all } y \in A \cap L\big\}.$$

Then S_L is nonempty for all $L \in \mathcal{L}$. Observe furthermore that the family $\{S_L\}_{L \in \mathcal{L}}$ has the following finite intersection property: whenever $L_1, \ldots, L_n \in \mathcal{L}$, then the intersection $S_{L_1} \cap \ldots \cap S_{L_n}$ is nonempty since it contains the set $S_{L'}$ corresponding to the linear hull L' of $L_1 \cup \ldots \cup L_n$. This implies that the intersection $\mathrm{cl}_w(S_{L_1}) \cap \ldots \cap \mathrm{cl}_w(S_{L_n})$ is also nonempty, where cl_w denotes the weak closure. Since A is weakly compact, it follows from Lemma 2.2 that there is a point $\hat{x} \in A$ with $\hat{x} \in \mathrm{cl}_w(S_L)$ for all $L \in \mathcal{L}$.

We claim that $\hat{x}$ has the desired property. To this end, let $y \in A$ be an arbitrary point, and let L denote the linear hull of $\hat{x}$ and y. Then $L \in \mathcal{L}$ and thus $\hat{x} \in \mathrm{cl}_w(S_L)$. By Proposition 2.23, there is a sequence $\{x^k\} \subseteq S_L$ such that $x^k \rightharpoonup \hat{x}$. The definition of S_L now implies that

$$\Psi(x^k, w) \le 0 \quad \text{for all } w \in A \cap L$$

for all $k \in \mathbb{N}$. Since $A \cap L$ contains the line segment connecting $\hat{x}$ and y, it follows from property (iv) that $\Psi(\hat{x}, y) \le 0$, and the proof is complete. □

The importance of the above theorem can hardly be overstated. It plays a fundamental role in the existence theory for variational inequalities and even quasi-variational inequalities, see Chapters 3 and 6.

Note that, in their original paper [39], Brezis, Nirenberg, and Stampacchia formulated the above theorem for an arbitrary Hausdorff topological vector space and used the continuity property (iii) with respect to nets (or filters) instead of sequences. In the Banach space setting, we can dispense with these notions due to Proposition 2.23.

Chapter 3

Theory of Optimization and Variational Problems

This chapter focuses on the theoretical background of constrained optimization and variational inequalities in general Banach spaces. Most of the theory is inspired by the book [32], although the results we present are often reformulated, extended, or modified in other ways to suit the algorithmic applications we will develop later on.

The literature on optimization theory is enormous, especially when finite-dimensional nonlinear programming is taken into account. For the general theory, we refer the reader to [13, 112, 222], and quite notably [32]. These books also include a bibliography of various milestone publications in optimization theory over the last decades, something which is outside the scope of this thesis. Some information on Banach space optimization, albeit in a more specialized context, can also be found in [211]. For nonlinear programming, much of the state-of-the-art theory can be found in parts of the aforementioned references, in the monographs [25, 172], in the encyclopedia-style book [80], and of course in the references therein.

This chapter also deals in some detail with variational inequalities (VIs). Indeed, we will treat these problems in a slightly more general framework designed to accommodate nonconvex optimization problems. The resulting problem class will be referred to as *variational problems*, but we will often use this term and "variational inequality" interchangeably. The theory of variational inequalities is fairly well-known in the literature. It can be found, for instance, in the monographs [70, 143] and, to a lesser extent, the book [32], as well as the more application-oriented treatises [12, 88–90]. In this thesis, we develop the corresponding theory in tandem with constrained optimization, either by reducing the VI to an optimization problem or by proving the corresponding results directly for VIs, with optimization to be seen as a special case.

The following is an overview of the structure of this chapter. Section 3.1 is dedicated to constrained optimization problems in Banach spaces. We discuss the well-known Karush–Kuhn–Tucker (KKT) conditions in Section 3.1.1, and give some results on constraint qualifications and their consequences in Section 3.1.2. This section also contains a brief discussion of multifunctions and metric regularity. More details on these topics can be found, for instance, in [11,32,163,191]. Section 3.1.3 is dedicated to second-order sufficient optimality conditions and their consequences. Finally, in Section 3.1.4, we specialize some of the concepts and results from the preceding sections for the case of finite-dimensional nonlinear programming.

In Section 3.2, we deal with variational problems (inequalities) in their full generality. Section 3.2.1 is dedicated to the notion of *pseudomonotonicity*, a fundamental concept of continuity for VIs which can also be seen as a generalization of monotonicity. This notion can be traced back to the work of Brezis [37], and it is indispensable for the subsequent chapters since it provides a unified framework for the analysis of monotone and nonmonotone VIs (and, a fortiori, convex and nonconvex optimization problems). We continue with a discussion of the Karush–Kuhn–Tucker (KKT) conditions for variational inequalities in Section 3.2.2, where we essentially demonstrate how these can be extracted from the corresponding conditions for optimization problems. In Section 3.2.3, we then deal with an asymptotic analogue of the KKT conditions, designed to facilitate optimality assertions about limit points of sequences generated by numerical algorithms. The developments in this section are in the spirit of related works in nonlinear programming [7,8,28]. Finally, in Section 3.2.4, we provide a quantitative stability analysis for variational inequalities in terms of so-called *error bounds*. The theory in this section forms the basis for the rate-of-convergence analysis of many optimization algorithms, including the augmented Lagrangian methods discussed in Chapters 4 and 5.

3.1 Constrained Optimization

Throughout this section, we consider a generic nonlinear optimization problem of the form

$$(P) \qquad \underset{x\in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x)\in K, \tag{3.1}$$

where X, Y are real Banach spaces, $f : X \to \mathbb{R}$ and $G : X \to Y$ are continuously differentiable functions, and $K \subseteq Y$ is a nonempty closed convex set. We say that a point $x \in X$ is *feasible* if $x \in C$ and $G(x) \in K$, and denote by

$$\Phi := C \cap G^{-1}(K) = \{x \in C : G(x) \in K\}$$

the *feasible set* of (P). Note that K is assumed to be a convex set, but we have made no convexity or concavity assumptions on the mapping G. In particular, the feasible set Φ may not be convex. The idea behind the above formulation is that G models the possible “nonlinearity” (or nonconvexity) of the problem.

3.1.1 First-Order Optimality Conditions

One of the most fundamental concepts in the study of (differentiable) optimization problems is that of first-order necessary conditions. These are conditions which involve the first derivatives of the functions f and G and which have to be satisfied by local or global solutions of (P). The most prominent form of such conditions is the so-called KKT system which we will define below. Before doing so, it will be useful to first state a general first-order condition which involves the tangent cone to the feasible set.

Lemma 3.1. *If $\bar{x}$ is a local minimizer of (P), then $f'(\bar{x})d \geq 0$ for all $d \in \mathcal{T}_\Phi(\bar{x})$.*

The above is always a necessary optimality condition for $\bar{x}$ to be a local minimizer. Note that, if Φ is convex, then we can restate the condition as $f'(\bar{x})(x - \bar{x}) \geq 0$ for all $x \in \Phi$. If, in addition, f is a convex function, then any point $\bar{x}$ satisfying this inequality also satisfies

$$f(x) - f(\bar{x}) \geq f'(\bar{x})(x - \bar{x}) \geq 0 \quad \text{for all } x \in \Phi.$$

Hence, in that case, every point $\bar{x}$ which is stationary in the sense of Lemma 3.1 is a global minimizer of (P).

Let us now return to the general case. Note that we can equivalently state the assertion of Lemma 3.1 as

$$-f'(x) \in \mathcal{T}_\Phi(\bar{x})^\circ, \tag{3.2}$$

where $^\circ$ is the polar cone from Section 2.2.1. This condition forms the basis of the theory of KKT conditions. Indeed, under suitable regularity assumptions on the mapping G, the cone $\mathcal{T}_\Phi(\bar{x})^\circ$ can be represented analytically, and this yields a more concrete and thus more convenient form of first-order optimality conditions.

A central role will be played by the *Lagrange function* or *Lagrangian* of (P), which is the function

$$\mathcal{L} : X \times Y^* \to \mathbb{R}, \quad \mathcal{L}(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle. \tag{3.3}$$

This allows us to state the so-called Karush–Kuhn–Tucker or KKT conditions of (P) as follows. Note that we use the notation $\mathcal{L}'$ for the derivative of $\mathcal{L}$ with respect to x (the primal variable).

Definition 3.2 (KKT point)**.** A pair $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ is a *KKT point* of (P) if

$$-\mathcal{L}'(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})).$$

We say that $\bar{x} \in X$ is a *stationary point* of (P) if $(\bar{x}, \bar{\lambda})$ is a KKT point for some multiplier $\bar{\lambda} \in Y^*$, and denote by $\Lambda(\bar{x})$ the set of such multipliers.

Note that the above inclusions imply that $\bar{x} \in C$ and $G(\bar{x}) \in K$, since otherwise at least one of the corresponding normal cones would be empty. It follows that every stationary point of (P) is necessarily feasible. Let us also remark that we always have $\bar{\lambda} \in K_\infty^\circ$, where K_∞ is the recession cone of K, see Lemma 2.39. This can be interpreted as a sign property of the Lagrange multiplier.

Example 3.3 (Cone constraints)**.** Assume that $K \subseteq Y$ is a closed convex *cone*. Then the inclusion $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x}))$ in the KKT conditions can equivalently be stated as

$$G(\bar{x}) \in K, \quad \bar{\lambda} \in K^\circ, \quad \text{and} \quad \left\langle \bar{\lambda}, G(\bar{x}) \right\rangle = 0.$$

These three conditions are often referred to as *complementarity conditions*. Recalling that any closed convex cone induces a (partial) order relation, we can interpret these conditions as $G(\bar{x})$ being nonnegative, $\bar{\lambda}$ being nonpositive (in the dual sense), and their product being equal to zero.

We have already alluded to the fact that certain regularity properties are needed for the KKT conditions to be necessary optimality conditions for (P). Such properties are usually called *constraint qualifications*; they ensure that the feasible set is well-behaved and that, roughly speaking, the reconstruction of its geometry from first-order information is possible. We will discuss the rich theory behind constraint qualifications and their consequences in some more detail in Section 3.1.2. For the present section, we focus on the connection between these conditions and the KKT system. The main constraint qualification we use is the following.

Definition 3.4 (Robinson constraint qualification)**.** Let $x \in X$ be a feasible point for (P). We say that the *Robinson constraint qualification (RCQ)* holds in x if

$$0 \in \operatorname{int}\big[G(x) + G'(x)(C - x) - K\big].$$

The above condition was introduced by Robinson in [184] in the context of certain stability properties of nonlinear inclusions. A more detailed study of RCQ, its consequences, and some related conditions will be conducted in Section 3.1.2.

Theorem 3.5 (KKT conditions under RCQ, [32, Thm. 3.9])**.** *Let $\bar{x}$ be a local minimizer of (P) and assume that RCQ holds in $\bar{x}$. Then the set of Lagrange multipliers $\Lambda(\bar{x})$ is nonempty, closed, convex, and bounded in Y^*.*

It is possible to show that, under RCQ, the set $\Lambda(\bar{x})$ is indeed *weak-* compact*, where the weak-* topology on Y^* is defined similarly to the weak topology from Definition 2.18. The further study of this topology is not necessary for our purposes, and thus we will not go beyond this parenthetical remark.

Let us now turn to stronger constraint qualification-type conditions. In particular, we will make use of a strict version of RCQ which guarantees the uniqueness of the Lagrange multiplier. This condition will also play a certain role in the primal-dual stability analysis of optimization problems and variational inequalities, see Section 3.2.4.

Definition 3.6 (Strict Robinson condition)**.** Let $\bar{x} \in X$ be a feasible point. We say that the *strict Robinson condition (SRC)* holds in $\bar{x}$ if there exists $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$0 \in \operatorname{int}\big[G(\bar{x}) + G'(\bar{x})(C_0 - \bar{x}) - K_0\big],$$

where $C_0 := \{x \in C : \mathcal{L}'(\bar{x}, \bar{\lambda})(x - \bar{x}) = 0\}$ and $K_0 := \{y \in K : \langle \bar{\lambda}, y - G(\bar{x}) \rangle = 0\}$.

Clearly, the strict Robinson condition implies RCQ. However, it should be emphasized that SRC is not a constraint qualification in the conventional sense since it presupposes the existence of $\bar{\lambda}$ and therefore depends not only on the constraint system but rather on the problem as a whole. Indeed, there may be different objective functions which attain a local or global minimizer at the same point in Φ, but SRC may only hold for some of them. A related discussion can be found in Section 3.1.4 and in [215].

Proposition 3.7 (KKT conditions under SRC)**.** *Let $\bar{x} \in X$ be a stationary point such that SRC holds in $\bar{x}$. Then the corresponding Lagrange multiplier is unique, i.e., $\Lambda(\bar{x})$ is a singleton.*

Proof. Let $\bar{\lambda} \in \Lambda(\bar{x})$ be the multiplier satisfying SRC. By assumption, there exists $r > 0$ such that $B_r^Y \subseteq G(\bar{x}) + G'(\bar{x})(C_0 - \bar{x}) - K_0$. Now, let $\lambda \in \Lambda(\bar{x})$ be arbitrary, and let $y \in B_r^Y$. Then $y = G(\bar{x}) + G'(\bar{x})(c - \bar{x}) - k$ with $c \in C_0$ and $k \in K_0$. It follows that

$$\begin{aligned} \langle \lambda - \bar{\lambda}, y \rangle &= \big\langle G'(\bar{x})^*(\lambda - \bar{\lambda}), c - \bar{x} \big\rangle - \langle \lambda - \bar{\lambda}, k - G(\bar{x}) \rangle \\ &= \big\langle f'(\bar{x}) + G'(\bar{x})^* \lambda, c - \bar{x} \big\rangle - \langle \lambda, k - G(\bar{x}) \rangle \geq 0, \end{aligned}$$

where the second equality uses the definitions of C_0 and K_0, and the final inequality uses the fact that $\lambda \in \Lambda(\bar{x})$. Since this holds for all $y \in B_r^Y$, we conclude that $\langle \lambda - \bar{\lambda}, y \rangle = 0$ for all $y \in B_r^Y$, which is equivalent to $\lambda - \bar{\lambda} = 0$. □

The following remark contains an important sufficient condition for the Robinson constraint qualification and its strict counterpart.

Remark 3.8. If $C = X$, then the surjectivity of $G'(\bar{x})$ implies the Robinson constraint qualification, and it implies the strict Robinson condition if $\bar{x}$ is a stationary point. Therefore, the surjectivity of $G'(\bar{x})$ can be seen as the strongest constraint qualification. In finite-dimensional nonlinear programming, it is even stronger than the linear independence constraint qualification (LICQ, see Section 3.1.4), and it is therefore almost never needed in this case. However, in the infinite-dimensional case, the distinction and isolation of active and inactive parts of constraints is not so easy, and the surjectivity of $G'(\bar{x})$, which does not depend on these concepts, sometimes allows us to prove stronger optimality-related statements. An example of this phenomenon can be found in Section 3.2.3.

Assume now that we have a point $\hat{x}$ which is "almost" a solution of (P). A popular definition in this context is that of *ε-minimizers*: given $\varepsilon > 0$, we say that $\hat{x} \in \Phi$ is an ε-minimizer of (P) if $f(\hat{x}) \leq f(x) + \varepsilon$ for all $x \in \Phi$. For such approximate minimizers, it is indeed possible to obtain an inexact analogue of the KKT conditions. This result is usually called Ekeland's variational principle.

Proposition 3.9 (Ekeland's variational principle, [32, Thm. 3.23])**.** *Let $\bar{x} \in \Phi$ be an ε-minimizer of (P), let $\delta := \varepsilon^{1/2}$, and assume that RCQ holds at every $x \in B_\delta(\bar{x}) \cap \Phi$. Then there exist another ε-minimizer $\hat{x}$ of (P) and $\lambda \in Y^*$ such that $\|\hat{x} - \bar{x}\|_X \leq \delta$,*

$$\operatorname{dist}\big(-\mathcal{L}'(\hat{x}, \lambda), \mathcal{N}_C(\hat{x})\big) \leq \delta, \quad \textit{and} \quad \lambda \in \mathcal{N}_K(G(\hat{x})).$$

In the case where $C = X$, it follows that $\mathcal{N}_C(\hat{x}) = \{0\}$, and thus the first condition in the above equation reduces to $\|\mathcal{L}'(\hat{x}, \lambda)\|_{X^*} \leq \delta$.

Note that, as we shall see later, the Robinson constraint qualification remains invariant under small perturbations of the constraint system. In particular, if $\bar{x}$ is a feasible point satisfying RCQ, then there always exists $\delta > 0$ such that RCQ holds at every $x \in B_\delta(\bar{x}) \cap \Phi$. The assumption made in Ekeland's variational principle requires that we can choose $\delta := \varepsilon^{1/2}$, where $\varepsilon > 0$ is the constant from the definition of ε-optimality.

Let us close this section with a general remark on the feasible set Φ. The observation below is useful and should be kept in mind when dealing with the Robinson constraint qualification and its strict counterpart.

Remark 3.10. The analytical representation of the feasible set of (P) is in general not unique. In particular, we can always re-write the constraint system as $(G(x), x) \in K \times C$, $x \in X$, which essentially amounts to replacing G by the mapping $x \mapsto (G(x), x)$, K by $K \times C$, and C by X. In this formulation, the KKT conditions take on the form

$$f'(\bar{x}) + G'(\bar{x})^* \bar{\lambda} + \bar{\mu} = 0, \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})), \quad \bar{\mu} \in \mathcal{N}_C(\bar{x}),$$

and they are therefore equivalent to the KKT system given in Definition 3.2. It is also interesting to note that the Robinson constraint qualification remains invariant under this transformation of the constraint system, see [32, Lem. 2.100]. The same holds for the strict Robinson condition, which can be seen as RCQ for the sets C_0 and K_0.

3.1.2 Constraint Qualifications and Regularity

We have already seen in the previous section that certain regularity properties are necessary to ensure that the KKT conditions are necessary optimality conditions. Such properties are usually called *constraint qualifications.* The present section is now dedicated to a more detailed study of these conditions, the relationships between them, and their consequences. The analysis here is based on the theoretical framework established in [32], with some slight modifications and extensions.

The Robinson constraint qualification is closely linked to stability properties of certain multifunctions, in particular the so-called *feasibility mapping* $\mathcal{F}_G : X \rightrightarrows Y$, $\mathcal{F}_G(x) := G(x) - K$. Therefore, it is necessary to first discuss some elements of multifunction theory. Since this is not the primary subject of this thesis, we will keep the discussion fairly superficial and only mention the key results. More details can be found in [32].

A multifunction $\mathcal{W} : X \rightrightarrows Y$ is a function mapping each point $x \in X$ to a subset $\mathcal{W}(x)$ of Y. Occasionally, a multifunction of this form is interpreted as an (ordinary) mapping into the power set of Y, but it will be convenient to treat multifunctions distinctly from ordinary functions in order to facilitate the use of certain multifunction-tailored notation and terminology. For instance, we define the *graph* of $\mathcal{W}$ as

$$\operatorname{gph}(\mathcal{W}) := \{(x, y) \in X \times Y : y \in \mathcal{W}(x)\}.$$

Moreover, given arbitrary subsets S_1 of X and S_2 of Y, we define the *image* of S_1 and the *preimage* of S_2 under $\mathcal{W}$ as

$$\mathcal{W}(S_1) := \bigcup_{s \in S_1} \mathcal{W}(s) \quad \text{and} \quad \mathcal{W}^{-1}(S_2) := \{x \in X : \mathcal{W}(x) \cap S_2 \neq \emptyset\}.$$

The set $\mathcal{W}(X)$ is sometimes called the *range* of $\mathcal{W}$, and $\mathcal{W}^{-1}(Y) = \{x \in X : \mathcal{W}(x) \neq \emptyset\}$ the *domain* of $\mathcal{W}$. Note that, for the treatment of multifunctions, it is no restriction to always consider mappings defined on the whole space X. This is because a multifunction

$\mathcal{W} : S \rightrightarrows Y$ defined on a subset $S \subseteq X$ can trivially be extended to X by setting $\mathcal{W}(x) := \emptyset$ whenever $x \notin S$.

Given a multifunction $\mathcal{W} : X \rightrightarrows Y$, we say that $\mathcal{W}$ is *closed* (resp. *convex*) if its graph $\operatorname{gph}(\mathcal{W})$ is a closed (resp. convex) subset of $X \times Y$. One of the most fundamental results on multifunctions is the following generalized open mapping theorem due to Robinson and Ursescu.

Theorem 3.11 (Generalized open mapping theorem, [32, Thm. 2.70])**.** *Let $\mathcal{W} : X \rightrightarrows Y$ be a closed convex multifunction and $y \in \operatorname{int} \mathcal{W}(X)$. Then $y \in \operatorname{int} \mathcal{W}(B_r(x))$ for all $x \in \mathcal{W}^{-1}(y)$ and all $r > 0$.*

Note that the above result generalizes the ordinary (Banach) open mapping theorem (see Theorem 2.13). Indeed, if $T \in L(X, Y)$ is a surjective linear operator, then its graph is closed by continuity, and convex by linearity. Since $0 \in \operatorname{int} T(X)$, it follows that $0 \in \operatorname{int} T(B_1^X)$, which means that $B_r^Y \subseteq T(B_1^X)$ for some $r > 0$. Hence, Theorem 3.11 implies (the first part of) Theorem 2.13.

We now turn to a fundamental property of multifunctions which also plays a crucial role in the analysis of constraint systems for optimization problems. In what follows, we say that a property holds *near* a point if it holds in a neighborhood of that point.

Definition 3.12 (Metric regularity)**.** A multifunction $\mathcal{W} : X \rightrightarrows Y$ is said to be *metrically regular* at $(\bar{x}, \bar{y}) \in \operatorname{gph}(\mathcal{W})$ if there exists $c > 0$ such that, for all (x, y) near $(\bar{x}, \bar{y})$,

$$\operatorname{dist}\bigl(x, \mathcal{W}^{-1}(y)\bigr) \leq c \operatorname{dist}\bigl(y, \mathcal{W}(x)\bigr). \tag{3.4}$$

Note that we did not assume $\mathcal{W}$ to be a closed convex multifunction in Definition 3.12. If these conditions hold, then it is possible to give a full characterization of metric regularity.

Proposition 3.13 (Robinson–Ursescu, [32, Thm. 2.83])**.** *Let $\mathcal{W} : X \rightrightarrows Y$ be a closed convex multifunction. Then $\mathcal{W}$ is metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph}(\mathcal{W})$ if and only if $\bar{y} \in \operatorname{int} \mathcal{W}(X)$.*

Let us now discuss how the concept of metric regularity can be applied to the optimization problem (P). We first consider the case where $C = X$. The main approach is to linearize the mapping G in the neighborhood of a point x, thus obtaining a convex constraint, and to apply Proposition 3.13. In this context, the interior condition from Proposition 3.13 becomes precisely the Robinson constraint qualification.

Theorem 3.14 (Stability theorem, [32, Prop. 2.89])**.** *Let $C = X$ and let $\bar{x} \in \Phi$ be a feasible point. Then RCQ holds in $\bar{x}$ if and only if the multifunction $\mathcal{F}_G : X \rightrightarrows Y$, $\mathcal{F}_G(x) := G(x) - K$, is metrically regular at the point $(\bar{x}, 0) \in X \times Y$.*

We now give two corollaries of this result. First, we give a consequence of the theorem in the general case where $C \neq X$. The main idea here is to use Remark 3.10.

Corollary 3.15. *Let RCQ hold in a point $\bar{x} \in \Phi$. Then there exists $c > 0$ such that* $\operatorname{dist}(x, \Phi) \leq c \operatorname{dist}(G(x), K)$ *for all $x \in C$ near $\bar{x}$.*

Proof. Let $G_2 : X \to X \times Y$, $G_2(x) := (x, G(x))$, and $K_2 := C \times K$. By Remark 3.10, RCQ holds for the constraint $G_2(x) \in K_2$ in $\bar{x}$. Thus, by Theorem 3.14, the mapping $\mathcal{F}_{G_2}(x) := G_2(x) - K_2$ is metrically regular at the point $(\bar{x}, 0, 0) \in X^2 \times Y$. Hence, there exists $c > 0$ such that, whenever $x \in C$ is sufficiently close to $\bar{x}$, then

$$\operatorname{dist}(x, \Phi) \le c \operatorname{dist}\bigl(G_2(x), K_2\bigr) = c \operatorname{dist}\bigl((x, G(x)), C \times K\bigr) = c \operatorname{dist}\bigl(G(x), K\bigr). \qquad \square$$

The above property is often called *metric subregularity*, *calmness*, or simply an *error bound to the feasible set*. This should not be confused with error bounds to solution sets, which also play a prominent role in optimization theory, see Section 3.2.4.

The distance estimate provided by Corollary 3.15 can be used to obtain an analytical representation of the tangent cone $\mathcal{T}_\Phi(x)$. The following result is precisely the geometric property which lies at the heart of the KKT conditions.

Corollary 3.16 ([32, Cor. 2.91])**.** *Let $x \in \Phi$ be a feasible point and assume that RCQ holds in x. Then $\mathcal{T}_\Phi(x) = \{d \in \mathcal{T}_C(x) : G'(x)d \in \mathcal{T}_K(G(x))\}$.*

We now discuss multiple conditions which are related to the Robinson constraint qualification (RCQ). In this context, it will be convenient to define an analogue of RCQ which is not restricted to feasible points. To keep a clear distinction, we call the resulting condition the *extended* Robinson constraint qualification.

Definition 3.17 (Extended Robinson constraint qualification)**.** Let $x \in X$ be an arbitrary, not necessarily feasible point. We say that the *extended Robinson constraint qualification (extended RCQ, ERCQ)* holds in x if

$$0 \in \operatorname{int}\bigl[G(x) + G'(x)(C - x) - K\bigr].$$

Note that the condition defining ERCQ is the same as for the standard Robinson constraint qualification. The only difference is that, for the latter, the point x has to be feasible, whereas ERCQ is defined for arbitrary points.

An important property of (E)RCQ is its invariance under small perturbations of the constraint system. Various forms of this statement can be found in the literature, e.g., in [32, 227]. A particular form of this invariance arises if only the base point x is perturbed. We then end up with the conclusion that, if RCQ holds in a point x, then it holds in a neighborhood of x. The following result shows that RCQ actually holds "uniformly" in a neighborhood of x, where uniformness is understood in the radius of a ball around zero in Y, and in certain uniform bounds on the sets C and K.

Proposition 3.18 (Local invariance of RCQ)**.** *Let RCQ hold in some $\bar{x} \in \Phi$. Then there are $r, \delta > 0$ such that, for all $x \in B_\delta(\bar{x})$, ERCQ holds in x with*

$$B_r^Y \subseteq G'(x)\bigl[(C - x) \cap B_1^X\bigr] - \bigl[(K - G(x)) \cap B_1^Y\bigr]. \tag{3.5}$$

Proof. By the generalized open mapping theorem (Theorem 3.11), RCQ also holds in $\bar{x}$ with respect to the "localized" sets $C_\ell := B_{1/2}(\bar{x}) \cap C$ and $K_\ell := B_{1/2}(G(\bar{x})) \cap K$. Taking into account Remark 3.10, it follows that

$$0 \in \operatorname{int}\left[\begin{pmatrix} G(\bar{x}) \\ \bar{x} \end{pmatrix} + \begin{pmatrix} G'(\bar{x}) \\ \mathrm{Id}_X \end{pmatrix} X - K_\ell \times C_\ell\right].$$

By [32, Remark 2.88], this implies the existence of $\delta > 0$ such that

$$0 \in \operatorname{int}\left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} A \\ \mathrm{Id}_X \end{pmatrix} X - K_\ell \times C_\ell\right]$$

for all $(a,b) \in Y \times X$ and $A \in L(X,Y)$ with $\|(a,b) - (G(\bar{x}), \bar{x})\|_{Y\times X} \leq \delta$ and $\|A - G'(\bar{x})\|_{L(X,Y)} \leq \delta$. Shrinking δ if necessary, this yields the existence of $r > 0$ such that

$$B_r^{Y\times X} \subseteq \begin{pmatrix} G(x) \\ x \end{pmatrix} + \begin{pmatrix} G'(x) \\ \mathrm{Id}_X \end{pmatrix} X - K_\ell \times C_\ell \tag{3.6}$$

for all $x \in B_\delta(\bar{x})$. Without loss of generality, let $\delta < 1/2$, and let $G(x) \in B_{1/2}(G(\bar{x}))$ for all $x \in B_\delta(\bar{x})$.

We now claim that (3.5) holds with the given r and δ. To see this, let $x \in B_\delta(\bar{x})$, and let $y \in B_r^Y$ be an arbitrary point. Then $(y, 0) \in B_r^{Y\times X}$ and, by (3.6), there exist $d \in X$, $k \in K_\ell$, and $c \in C_\ell$ such that $y = G(x) + G'(x)d - k$ and $0 = x + d - c$. Observe now that

$$C_\ell - x \subseteq (C - x) \cap B_1^X \quad \text{and} \quad K_\ell - G(x) \subseteq (K - G(x)) \cap B_1^Y$$

since $\|x - \bar{x}\|_X \leq 1/2$ and $\|G(x) - G(\bar{x})\|_Y \leq 1/2$. This finally yields

$$\begin{aligned} y = G(x) + G'(x)(c - x) - k &\in G(x) + G'(x)(C_\ell - x) - K_\ell \\ &\subseteq G'(x)\big[(C - x) \cap B_1^X\big] - \big[(K - G(x)) \cap B_1^Y\big]. \end{aligned}$$

The proof is complete. □

We now discuss some conditions which are related or equivalent to the Robinson constraint qualification.

Proposition 3.19 ([32, Prop. 2.97]). *For $x \in \Phi$, consider the following assertions:*

(a) *The Robinson constraint qualification holds in x.*

(b) *We have $G'(x)\mathcal{R}_C(x) - \mathcal{R}_K(G(x)) = Y$.*

(c) *We have $G'(x)\mathcal{T}_C(x) - \mathcal{T}_K(G(x)) = Y$.*

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). *If either Y is finite-dimensional or K has nonempty interior, then* (a)-(c) *are equivalent.*

Condition (b) in the above result is often called the *Zowe–Kurcyusz constraint qualification.* It was introduced in [227] for the study of Lagrange multipliers. As stated by the result, this condition is equivalent to RCQ.

An important sufficient condition for RCQ and its extended version is the so-called *linearized Slater condition.* Indeed, this condition is equivalent to ERCQ under certain assumptions. The details can be found in the following result which is an adaptation of [32, Lem. 2.99].

Proposition 3.20. *Let $x \in X$ be an arbitrary point and assume that there exists $\hat{x} \in C$ such that $G(x) + G'(x)(\hat{x} - x) \in \operatorname{int}(K)$. Then ERCQ holds in x. The converse is true provided that* $\operatorname{int}(K)$ *is nonempty.*

Proof. If $G(x)+G'(x)(\hat{x}-x) \in \text{int}(K)$, then $0 \in \text{int}[G(x)+G'(x)(\hat{x}-x)-K]$, and ERCQ holds. Conversely, assume that ERCQ holds and that $\text{int}(K)$ is nonempty. Assume, by contradiction, that the convex sets

$$A_1 := G(x) + G'(x)(C - x), \quad A_2 := \text{int}(K)$$

have empty intersection. By the first separation theorem (Proposition 2.12), there exists a nonzero $\lambda \in Y^*$ such that $\langle \lambda, G(x) + G'(x)(c - x)\rangle \geq \langle \lambda, k\rangle$ for all $c \in C$, $k \in K$. Now, let $y \in Y$ be an arbitrary vector such that $\langle \lambda, y\rangle < 0$. Then, for $t > 0$, the vector ty does not belong to $G(x) + G'(x)(C - x) - K$. It follows that the latter cannot contain a ball around zero, and this is the desired contradiction. □

Let us now assume that the operator G is K_∞-concave in the sense of Definition 2.48, where K_∞ the recession cone of K. In this case, it is possible to considerably strengthen the connection between the Robinson and Slater-type conditions.

Proposition 3.21. *Let $G : X \to Y$ be K_∞-concave and assume that the feasible set $\Phi = C \cap G^{-1}(K)$ is nonempty. Consider the following assertions:*

(a) *There is a feasible point x such that RCQ holds in x.*

(b) *For every point $x \in X$, ERCQ holds in x.*

(c) *We have $0 \in \text{int}[G(C) - K]$.*

(d) *There is a point $\hat{x} \in C$ such that $G(\hat{x}) \in \text{int}(K)$.*

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftarrow$ (d)*. If* $\text{int}(K)$ *is nonempty, then* (a)-(d) *are equivalent.*

Proof. (d) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are clear. To prove (c) $\Rightarrow$ (b), let $x \in X$ be an arbitrary point. By Proposition 2.51, we have

$$G(w) - K \subseteq G(x) + G'(x)(w - x) - K$$

for all $w \in X$. Since the union of the left-hand side over $w \in C$ contains a ball around zero, so does the union of the right-hand side over $w \in C$, and this is precisely ERCQ.

For (a) $\Rightarrow$ (c), let $G_2(x) := (x, G(x))$, $K_2 := C \times K$, and consider the multifunction $\mathcal{W}(x) := G_2(x) - K_2$. By Remark 2.52, $\mathcal{W}$ is a closed convex multifunction, and by Theorem 3.14 it is metrically regular at the point $(x, 0, 0) \in \text{gph}(\mathcal{W})$. From Proposition 3.13, it follows that $(0, 0)$ lies in the interior of the range of $\mathcal{W}$. Now, let $y \in Y$ be sufficiently close to zero. Then $(0, y)$ lies in the range of $\mathcal{W}$. Hence, there exists $x \in X$ such that $(0, y) \in \mathcal{W}(x) = (x - C) \times (G(x) - K)$. It follows that $x \in C$ and thus $y \in G(C) - K$. We have shown that $G(C) - K$ contains a ball around zero in Y.

Finally, if $\text{int}(K)$ is nonempty, then (c) $\Leftrightarrow$ (d) follows from [32, Prop. 2.106]. □

An important property of ERCQ is that it guarantees that, whenever x is a stationary point of a certain measure of infeasibility, then x is actually a feasible point. For the sake of later reference, we formulate this result in a slightly more general framework.

Proposition 3.22. *Let $i : Y \hookrightarrow H$ densely for some real Hilbert space H, and let $\mathcal{K} \subseteq H$ be a closed convex set with $i^{-1}(\mathcal{K}) = K$. Let $\bar{x} \in X$ be a stationary point of the problem $\min_{x\in C} d_{\mathcal{K}}^2(G(x))$, and assume that ERCQ holds in $\bar{x}$ with respect to the constraint system of (P). Then $G(\bar{x}) \in K$.*

Proof. Let $r > 0$ be such that $B_r^Y \subseteq G(\bar{x}) + G'(\bar{x})(C - \bar{x}) - K$. Then, for any $y \in B_r^Y$, there are $z \in C$ and $w \in K$ such that $y = G(\bar{x}) + G'(\bar{x})(z - \bar{x}) - w$. Thus, we have

$$\begin{aligned}\langle G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x})), y\rangle = \big\langle G'(\bar{x})^*\big[G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x}))\big], z - \bar{x}\big\rangle \\ + \langle G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x})), G(\bar{x}) - w\rangle.\end{aligned}$$

Observe that $G'(\bar{x})^*[G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x}))]$ is just the derivative of $\frac{1}{2}d_{\mathcal{K}}^2 \circ G$ in $\bar{x}$. Hence, the first term above is nonnegative by the minimizing property of $\bar{x}$, and so is the second term by standard projection inequalities. Thus, $\langle G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x})), y\rangle \geq 0$ for all $y \in B_r^Y$, which implies $\langle G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x})), y\rangle = 0$ for all $y \in B_r^Y$ and, since Y is dense in H, it follows that $G(\bar{x}) - P_{\mathcal{K}}(G(\bar{x})) = 0$. This completes the proof. □

We conclude this section by giving an example for a specific constraint system which occurs frequently in practical applications. The discussion of this example once again highlights the fact that the analytical representation of the feasible set can have a significant impact on the fulfillment of constraint qualifications.

Example 3.23 (Box constraints in Lebesgue spaces)**.** Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain, and $X := L^2(\Omega)$. Let the feasible set be given by box constraints, i.e.,

$$\Phi = \{u \in X : u_a \leq u \leq u_b \text{ almost everywhere in } \Omega\},$$

where $u_a, u_b \in X$ and $u_a \leq u_b$. In practice, such constraints are considered "simple", and they are therefore often included in the set C of implicit constraints. Nevertheless, let us discuss here how these constraints can be formulated analytically through the mapping G. There are two canonical possibilities of doing so: on the one hand, we can simply define $G(u) := u$ and $K := \Phi$ (since this is a convex set). This formulation satisfies all constraint qualifications since $G'(u) = \mathrm{Id}_X$ is surjective for all $u \in X$. On the other hand, we can represent the feasible set through the definitions

$$\hat{G}(u) := (u - u_a, u_b - u), \quad \hat{K} := \{(v, w) \in X^2 : v, w \geq 0 \text{ almost everywhere}\}. \tag{3.7}$$

The latter formulation has the advantage that $\hat{K}$ is a closed convex *cone*, whereas K is not. Despite this, (3.7) has the severe disadvantage that the Robinson constraint qualification typically does not hold at points $u \in \Phi$, see [211] for more details. An intuitive way to verify this irregularity is to note that if RCQ holds, then it remains stable under small perturbations of the constraint system (see [32]). However, even if u_a and u_b are "well separated", it is fairly easy to construct small perturbations (in the sense of L^2) which make the lower and upper bounds coincide on some set of positive measure. If this happens, then the set of Lagrange multipliers corresponding to a stationary point becomes unbounded, and RCQ is violated.

3.1.3 Second-Order Sufficient Conditions

In this section, we deal with sufficient optimality conditions based on second-order derivatives. Since we are dealing with a generic Banach space setting, the formulation of such conditions will be slightly more complicated than in finite-dimensional nonlinear programming. We essentially follow the approach conducted in [32], with some modifications to allow for slightly more general and practical results.

Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (P). Throughout this section, we assume that f and G are continuously differentiable in a neighborhood of $\bar{x}$, and twice differentiable in $\bar{x}$. Consider, for $\eta > 0$, the *extended critical cone*

$$\mathcal{C}_\eta(\bar{x}) := \left\{ d \in \mathcal{T}_C(\bar{x}) : \begin{array}{l} f'(\bar{x})d \le \eta \|d\|_X, \\ \operatorname{dist}\big(G'(\bar{x})d, \mathcal{T}_K(G(\bar{x}))\big) \le \eta \|d\|_X \end{array} \right\}. \tag{3.8}$$

Note that $\mathcal{C}_\eta$ depends on $\bar{x}$ only. The following is the general form of second-order sufficient conditions which we will use throughout this section.

Definition 3.24 (Second-order sufficient condition)**.** We say that the *second-order sufficient condition (SOSC)* holds in a KKT point $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ of (P) if there are $\eta, c > 0$ such that

$$\mathcal{L}''(\bar{x}, \bar{\lambda})(d, d) \ge c\|d\|_X^2 \quad \text{for all } d \in \mathcal{C}_\eta(\bar{x}).$$

The above should be considered the "basic" second order condition which can be stated without any assumptions on the specific structure of (P). For many problem classes, it is possible to state more refined second-order conditions which are either equivalent to Definition 3.24 or turn out to have similar implications.

One of the most important consequences of second-order conditions is the local quadratic growth of the objective function on the feasible set, i.e., the existence of $c > 0$ such that $f(x) \ge f(\bar{x}) + c\|x - \bar{x}\|_X^2$ for all $x \in \Phi$ near $\bar{x}$, see, for instance, [32, Thm. 3.63]. Here, we will prove a slightly stronger version of this statement with the aim of applying it to the augmented Lagrangian method in Chapter 4. In this context, it will be essential to discuss the impact of SOSC on sequences of points $\{x^k\}$ which are not necessarily feasible but satisfy some kind of asymptotic feasibility, e.g., of the form $d_K(G(x^k)) \to 0$. It turns out that the quadratic growth condition can be extended to such points.

For the statement of this result, we use the Landau symbol $a_k = o(b_k)$ for nonnegative real sequences $\{a_k\}$ and $\{b_k\}$, which means that $a_k \le z_k b_k$ for some null sequence $\{z_k\}$. The sequences $\{a_k\}$ and $\{b_k\}$ themselves are not required to converge to zero.

Theorem 3.25 (Extended quadratic growth)**.** *Let SOSC hold in a KKT point $(\bar{x}, \bar{\lambda})$ of (P). Then there are $r, c > 0$ such that, for every sequence $\{x^k\} \subseteq B_r(\bar{x}) \cap C$ with $d_K(G(x^k)) = o(\|x^k - \bar{x}\|_X)$, we have*

$$\liminf_{k \to \infty} \left[f(x^k) - f(\bar{x}) - c\|x^k - \bar{x}\|_X^2 \right] \ge 0. \tag{3.9}$$

Proof. Let $\eta, \bar{c} > 0$ be the constants from SOSC and choose r small enough so that

$$\left|f(\bar{x}+d) - f(\bar{x}) - f'(\bar{x})d\right| \le \frac{\eta}{2}\|d\|_X, \tag{3.10}$$

$$\left\|G(\bar{x}+d) - G(\bar{x}) - G'(\bar{x})d\right\|_Y \le \frac{\eta}{2}\|d\|_X, \tag{3.11}$$

and $$\left|\mathcal{L}(\bar{x}+d,\bar{\lambda}) - \mathcal{L}(\bar{x},\bar{\lambda}) - \mathcal{L}'(\bar{x},\bar{\lambda})d - \frac{1}{2}\mathcal{L}''(\bar{x},\bar{\lambda})(d,d)\right| \le \frac{\bar{c}}{4}\|d\|_X^2 \tag{3.12}$$

for all $d \in X$ with $\|d\|_X \le r$. Furthermore, set

$$c := \min\left\{\frac{\eta}{2r}, \frac{\bar{c}}{4}\right\}. \tag{3.13}$$

Now, let $\{x^k\} \subseteq B_r(\bar{x}) \cap C$ be a sequence with $d_K(G(x^k)) = o(\|x^k - \bar{x}\|_X)$, and set $d^k := x^k - \bar{x}$. Without loss of generality, we assume that $\{x^k\}$ realizes the lim inf in (3.9). If $f'(\bar{x})d^k > \eta\|d^k\|_X$ for infinitely many k, then by (3.10) and (3.13) we obtain

$$f(x^k) - f(\bar{x}) \ge f'(\bar{x})d^k - \frac{\eta}{2}\|d^k\|_X \ge \frac{\eta}{2}\|d^k\|_X \ge c\|d^k\|_X^2$$

for all these k, which implies (3.9). We now consider the case where $f'(\bar{x})d^k \le \eta\|d^k\|_X$ for all but finitely many k. From (3.11), the fact that $K - G(\bar{x}) \subseteq \mathcal{T}_K(G(\bar{x}))$, and $d_K(G(x^k)) = o(\|d^k\|_X)$, it is easy to deduce that

$$\operatorname{dist}\big(G'(\bar{x})d^k, \mathcal{T}_K(G(\bar{x}))\big) \le \operatorname{dist}\big(G(\bar{x}) + G'(\bar{x})d^k, K\big) \le \frac{\eta}{2}\|d^k\|_X + o(\|d^k\|_X).$$

Hence, $d^k \in \mathcal{C}_\eta(\bar{x})$ for sufficiently large k. Applying (3.12), (3.13) and SOSC yields

$$\mathcal{L}(x^k,\bar{\lambda}) - \mathcal{L}(\bar{x},\bar{\lambda}) - \mathcal{L}'(\bar{x},\bar{\lambda})d^k \ge \frac{\bar{c}}{2}\|d^k\|_X^2 - \frac{\bar{c}}{4}\|d^k\|_X^2 \ge c\|d^k\|_X^2. \tag{3.14}$$

Observe now that $-\mathcal{L}'(\bar{x},\bar{\lambda})d^k \le 0$ since $(\bar{x},\bar{\lambda})$ is a KKT point and $d^k \in C - \bar{x}$. Moreover,

$$\mathcal{L}(x^k,\bar{\lambda}) - \mathcal{L}(\bar{x},\bar{\lambda}) = f(x^k) - f(\bar{x}) + \big\langle \bar{\lambda}, G(x^k) - G(\bar{x})\big\rangle,$$

and the last term is asymptotically nonpositive since $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x}))$. Inserting this into (3.14), we obtain $f(x^k) - f(\bar{x}) \ge c\|d^k\|_X^2 + o(1)$, and the result follows. ☐

The ordinary quadratic growth condition follows easily as a corollary of the above theorem.

Corollary 3.26 (Quadratic growth)**.** *Let $(\bar{x},\bar{\lambda})$ be a KKT point of (P) satisfying SOSC. Then there are $r, c > 0$ such that $f(x) \ge f(\bar{x}) + c\|x - \bar{x}\|_X^2$ for all $x \in B_r(\bar{x}) \cap \Phi$. In particular, $\bar{x}$ is a strict local minimizer of (P).*

Note that a stationary point $\bar{x}$ satisfying the second-order sufficient condition is necessarily a strict local minimizer but, in general, not an *isolated* local minimizer. To see this, consider the following example, due to Robinson [185]:

$$\underset{x\in\mathbb{R}}{\text{minimize}}\ x^2 \quad \text{subject to} \quad x^6 \sin(1/x) = 0,$$

where the constraint function is understood to be zero for $x = 0$. It is easy to see that $\bar{x} := 0$ is the unique global solution of this minimization problem, and SOSC is satisfied for any $\bar{\lambda} \in \Lambda(\bar{x})$. However, every point of the form $x := (k\pi)^{-1}$, $k \in \mathbb{N}$, is a local minimizer, and thus $\bar{x}$ is not an isolated local minimizer.

We now give a second corollary of Theorem 3.25 which will be particularly useful for later results. The main idea is that we can use the theorem to give a sufficient condition for a sequence of asymptotically feasible points to converge to $\bar{x}$.

Corollary 3.27. *Let $(\bar{x}, \bar{\lambda})$ be a KKT point of (P) satisfying SOSC. Then there exists $r > 0$ such that, whenever $\{x^k\} \subseteq B_r(\bar{x}) \cap C$ is a sequence with $d_K(G(x^k)) \to 0$ and $\limsup_{k\to\infty} f(x^k) \le f(\bar{x})$, then $x^k \to \bar{x}$ (strongly) in X.*

Proof. Let $r, c > 0$ be as in Theorem 3.25 and $\{x^k\} \subseteq B_r(\bar{x}) \cap C$ a sequence with the stated properties. Assume that $\{x^k\}$ does not converge to $\bar{x}$. Passing onto a subsequence if necessary, we may assume that $\|x^k - \bar{x}\|_X \ge \varepsilon$ for all k and some $\varepsilon > 0$. Then $d_K(G(x^k)) = o(\|x^k - \bar{x}\|_X)$ holds trivially; hence, by Theorem 3.25, we obtain

$$0 \le \liminf_{k\to\infty} \left[f(x^k) - f(\bar{x}) - c\|x^k - \bar{x}\|_X^2 \right] \le -c \limsup_{k\to\infty} \|x^k - \bar{x}\|_X^2,$$

where we used the fact that $\limsup_{k\to\infty} f(x^k) \le f(\bar{x})$ by assumption. It follows that $\|x^k - \bar{x}\|_X \to 0$, which is the desired contradiction. □

Let us close this section by mentioning two general situations in which the second-order sufficient condition from Definition 3.24 can be simplified.

Remark 3.28. (a) If RCQ holds in $\bar{x}$, we can make the extended critical cone slightly smaller by replacing the estimate $\operatorname{dist}\big(G'(\bar{x})d, \mathcal{T}_K(G(\bar{x}))\big) \le \eta\|d\|_X$ with the simple inclusion $G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x}))$. The resulting second-order condition is equivalent to Definition 3.24, see [32, Remark 3.68].

(b) If X is finite-dimensional, then we can replace the extended critical cone by the *(ordinary) critical cone*

$$\mathcal{C}(\bar{x}) := \big\{ d \in \mathcal{T}_C(\bar{x}) : f'(\bar{x})d \le 0,\ G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x})) \big\}.$$

In that case, SOSC can equivalently be stated as $\mathcal{L}''(\bar{x}, \bar{\lambda})(d, d) > 0$ for all $d \in \mathcal{C}(\bar{x}) \setminus \{0\}$, see [32, Thm. 3.63] and its proof. We will use this simpler form of second-order conditions in later sections.

3.1.4 Nonlinear Programming

In this section, we briefly outline how some of the conditions of the preceding sections can be specialized for the important case of *nonlinear programming (NLP)*. A more comprehensive treatment of this subject can be found in many textbooks on optimization, for instance, in the references [16, 25, 48, 172].

To fix the problem setting, let $m, p \in \mathbb{N}_0$ be given numbers, and consider the constrained minimization problem

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad g(x) \le 0,\ e(x) = 0, \tag{3.15}$$

where $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, and $e : X \to \mathbb{R}^p$ are continuously differentiable. The inequality constraint $g(x) \le 0$ is understood componentwise. If either m or p is equal to zero, we treat the corresponding constraint as nonexistent.

Note that we have made no assumptions on the particular structure of the space X. In traditional NLP, this space is often assumed to be of the form $X = \mathbb{R}^n$, $n \in \mathbb{N}$, and we will indeed use this setting in the context of second-order conditions below. However, the remaining concepts in this section do not require X to be of this form, or to even be finite-dimensional, and thus we will work with a general Banach space X.

Clearly, the nonlinear program (3.15) can be cast into the framework of our general constrained problem (P) with $C := X$, $Y := \mathbb{R}^{m+p}$, $G := (g, e)$, and $K := \mathbb{R}^m_- \times \{0\}^p$. Thus, we can readily apply the concepts of the previous sections such as constraint qualifications or second-order sufficient conditions to (3.15). However, in the NLP setting, many of these conditions can be reformulated in a much more elementary manner, and they also often have different names since, from a historical perspective, they were developed independently (usually much earlier).

We begin by discussing some of the common constraint qualifications. For a point $x \in \mathbb{R}^n$ and subsets $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, p\}$, we say that the set of gradients $\{\nabla g_i(x)\}_{i \in I} \cup \{\nabla e_j(x)\}_{j \in J}$ is *positively linearly dependent* if there are nontrivial coefficients (i.e., not all equal to zero) $\lambda_i \ge 0$, $i \in I$, and $\mu_j \in \mathbb{R}$, $j \in J$, such that

$$\sum_{i \in I} \lambda_i \nabla g_i(x) + \sum_{j \in J} \mu_j \nabla e_j(x) = 0.$$

Note that the coefficients λ_i corresponding to the inequality constraints are required to be nonnegative, whereas the remaining coefficients μ_j are arbitrary real numbers. From a formal point of view, this is slightly imprecise since we attribute the positive linear dependence to the union $\{\nabla g_i(x)\}_{i \in I} \cup \{\nabla e_j(x)\}_{j \in J}$, but impose special conditions on the coefficients of the gradients $\{\nabla g_i(x)\}_{i \in I}$. However, this mild inconsistency should not introduce any confusion, and it simplifies the terminology in what follows.

If there is no nontrivial linear combination of the above form, i.e., if the vectors $\{\nabla g_i(x)\}_{i \in I} \cup \{\nabla e_j(x)\}_{j \in J}$ are not positively linearly dependent, then we call them *positively linearly independent.*

Definition 3.29 (Constraint qualifications for NLP)**.** Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary point and let $\mathcal{I} := \{i = 1, \dots, m : g_i(\bar{x}) = 0\}$. We say that

(a) the *linear independence constraint qualification (LICQ)* holds in $\bar{x}$ if the set of gradients $\{\nabla g_i(\bar{x})\}_{i \in \mathcal{I}} \cup \{\nabla e_j(\bar{x})\}_{j=1}^p$ is linearly independent.

(b) the *Mangasarian–Fromovitz constraint qualification (MFCQ)* holds in $\bar{x}$ if the set of gradients $\{\nabla g_i(\bar{x})\}_{i \in \mathcal{I}} \cup \{\nabla e_j(\bar{x})\}_{j=1}^p$ is positively linearly independent.

(c) the *extended MFCQ (EMFCQ)* holds in $\bar{x}$ if the gradients $\{\nabla g_i(\bar{x})\}_{i\in\mathcal{I}'}\cup\{\nabla e_j(\bar{x})\}_{j=1}^p$ with $\mathcal{I}' := \{i = 1,\dots,m : g_i(\bar{x}) \geq 0\}$ are positively linearly independent.

(d) the *constant positive linear dependence condition (CPLD)* holds in $\bar{x}$ if, whenever $I \subseteq \mathcal{I}$ and $J \subseteq \{1,\dots,p\}$ are subsets such that the gradients $\{\nabla g_i(x)\}_{i\in I}\cup\{\nabla e_j(x)\}_{j\in J}$ are positively linearly dependent in $x := \bar{x}$, then they are linearly dependent for all x in a neighborhood of $\bar{x}$.

Note that some of these conditions are usually defined for feasible points only. However, it will occasionally be convenient to deal with, say, LICQ, for arbitrary points which are not necessarily feasible (see, for instance, Section 6.3.2).

Clearly, LICQ implies MFCQ, and MFCQ implies CPLD. Moreover, if the point x is feasible, then MFCQ and EMFCQ coincide. Note also that MFCQ is precisely the Robinson constraint qualification for the problem (3.15), see [32]. Interestingly, the connection between EMFCQ and ERCQ is a little more nuanced. Using Proposition 3.20, it is easy to show that the latter implies the former if only inequality constraints are present. In the general case, however, the two conditions are not related.

Example 3.30 (EMFCQ versus ERCQ)**.** (a) Consider the constraint function $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) := (x, -x)^\top$, at $\bar{x} := 1$. Then $g(\bar{x}) = g'(\bar{x}) = (1, -1)^\top$, and thus it is easy to see that $g(\bar{x}) + g'(\bar{x})X - K$ cannot contain a ball around zero since every point y in that set satisfies $y_1 + y_2 \geq 0$. On the other hand, EMFCQ is satisfied since g_1 is the only active or violated constraint at $\bar{x}$, and $\nabla g_1(\bar{x}) \neq 0$.

(b) Let $g(x) := 1 + 2x$, $e(x) := 1 + x$, and $\bar{x} := 0$. Then $g(\bar{x}) = e(\bar{x}) = 1$, $\nabla g(\bar{x}) = 2$, and $\nabla e(\bar{x}) = 1$. Hence, EMFCQ is violated in $\bar{x}$ since $\nabla g(\bar{x}) - 2\nabla e(\bar{x}) = 0$. On the other hand, an easy calculation shows that

$$\begin{pmatrix} g(\bar{x}) \\ e(\bar{x}) \end{pmatrix} + \begin{pmatrix} g'(\bar{x}) \\ e'(\bar{x}) \end{pmatrix} X - K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : 2y \leq x + 1 \right\}.$$

This set contains a ball around zero, and thus ERCQ is satisfied in $\bar{x}$.

Assume now that $\bar{x}$ is a local minimizer of (3.15), and that any of the constraint qualifications from Definition 3.29 is satisfied in $\bar{x}$. Then $\bar{x}$ together with a suitable vector of Lagrange multipliers satisfies the KKT conditions. In the present setting, the multiplier vector takes on the form $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{m+p}$, and the KKT conditions can be written as

$$\mathcal{L}'(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0, \quad 0 \leq \bar{\lambda} \perp g(\bar{x}) \leq 0, \quad e(\bar{x}) = 0, \tag{3.16}$$

where $\mathcal{L}(x, \lambda, \mu) := f(x) + \lambda^\top g(x) + \mu^\top e(x)$ is the Lagrange function (see also (3.3)), $\mathcal{L}'$ is its derivative with respect to x, and the condition $\bar{\lambda} \perp g(\bar{x})$ is shorthand for $\bar{\lambda}^\top g(\bar{x}) = 0$. Moreover, the set of Lagrange multipliers $\Lambda(\bar{x}) \subseteq \mathbb{R}^{m+p}$ is bounded if MFCQ holds in $\bar{x}$ (see also Theorem 3.5), and it is a singleton if LICQ holds in $\bar{x}$.

Similarly to ERCQ, the extended MFCQ implies that any stationary point of a certain measure of infeasibility is a feasible point. This result is contained in the following proposition.

Proposition 3.31. *Let $\bar{x} \in X$ be a stationary point of the function $m(x) := \|g_+(x)\|^2 + \|e(x)\|^2$, and let EMFCQ hold in $\bar{x}$. Then $\bar{x}$ is feasible, i.e., $g(\bar{x}) \le 0$ and $e(\bar{x}) = 0$.*

Proof. By assumption, the derivative of m must vanish in $\bar{x}$. This implies that

$$0 = \nabla g(\bar{x}) g_+(\bar{x}) + \nabla e(\bar{x}) e(\bar{x}) = \sum_{g_i(\bar{x}) \ge 0} g_i(\bar{x}) \nabla g_i(\bar{x}) + \sum_{j=1}^{p} e_j(\bar{x}) \nabla e_j(\bar{x}).$$

It then follows from EMFCQ that all the coefficients in the above linear combination must be zero. Hence, $\bar{x}$ is a feasible point. □

We now return to the KKT system of (3.15) and define a strict version of the Mangasarian–Fromovitz constraint qualification.

Definition 3.32 (Strict Mangasarian–Fromovitz condition)**.** Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be a KKT point of (3.15), let $\mathcal{I} := \{i = 1, \dots, m : g_i(\bar{x}) = 0\}$, and $\mathcal{I}_+ := \{i \in \mathcal{I} : \bar{\lambda}_i > 0\}$. We say that the *strict Mangasarian–Fromovitz condition (SMFC)* holds in $(\bar{x}, \bar{\lambda}, \bar{\mu})$ if

(i) the vectors $\{\nabla g_i(\bar{x})\}_{i \in \mathcal{I}_+} \cup \{\nabla e_j(\bar{x})\}_{j=1}^{p}$ are linearly independent, and

(ii) there exists $d \in \mathbb{R}^n$ such that $\nabla g_i(\bar{x})^\top d = 0$ for all $i \in \mathcal{I}_+$, $\nabla g_i(\bar{x})^\top d < 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}_+$, and $\nabla e_j(\bar{x})^\top d = 0$ for all $j = 1, \dots, p$.

We say that SMFC holds in $\bar{x}$ if there exists $(\bar{\lambda}, \bar{\mu}) \in \Lambda(\bar{x})$ such that SMFC holds in $(\bar{x}, \bar{\lambda}, \bar{\mu})$.

The above condition can be seen as a special case of the strict Robinson condition from Definition 3.6, see [32, Remark 4.49]. Hence, by Proposition 3.7, it follows that SMFC implies the uniqueness of the Lagrange multiplier $(\bar{\lambda}, \bar{\mu})$. In fact, these conditions are equivalent.

Proposition 3.33 ([154])**.** *Let $\bar{x}$ be a stationary point of* (3.15)*. Then $\Lambda(\bar{x})$ is a singleton if and only if SMFC holds in $\bar{x}$.*

Let us stress that SMFC implicitly assumes the existence of $(\bar{\lambda}, \bar{\mu})$ and therefore depends on the whole problem (3.15), not only the constraint functions. In this context, it is interesting to observe the following: given g and e, if the set $\Lambda(\bar{x})$ is a singleton for *every* objective function f such that $\bar{x}$ is a local minimizer of (3.15), then indeed LICQ must hold in $\bar{x}$. This was observed in [215] and underlines the fact that SMFC is not a constraint qualification.

We now turn to a second-order sufficient condition for the nonlinear program (3.15). For the remainder of this section, we assume that $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$. In this case, we can dispense with the extended critical cone from Section 3.1.3 and simply use the (ordinary) critical cone, which takes on the form

$$\mathcal{C}(\bar{x}) := \big\{ d \in \mathbb{R}^n : f'(\bar{x}) d \le 0,\ g'_{\mathcal{I}}(\bar{x}) d \le 0,\ e'(\bar{x}) d = 0 \big\},$$

where $\mathcal{I} := \{i = 1, \dots, m : g_i(\bar{x}) = 0\}$ is again the set of active indices. The resulting second-order condition can be stated as follows.

Definition 3.34 (Second-order sufficient condition for NLP)**.** Let $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{n+m+p}$ satisfy the KKT conditions (3.16). We say that the *second-order sufficient condition (SOSC)* holds in $(\bar{x}, \bar{\lambda}, \bar{\mu})$ if

$$d^\top [\nabla^2_{xx} \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})] d > 0 \quad \text{for all } d \in \mathcal{C}(\bar{x}) \setminus \{0\}.$$

This form of the second-order sufficient condition is slightly simpler than Definition 3.24, but the two conditions are equivalent since $X = \mathbb{R}^n$ is finite-dimensional (see Remark 3.28). In particular, the above condition implies the local quadratic growth of f on Φ, and it implies that $\bar{x}$ is a strict local minimizer of (3.15).

3.2 Variational Inequalities

We now turn to a class of variational problems, or variational inequalities (VIs), which can be seen as a generalization of nonlinear optimization. Let X be a real Banach space, Φ a nonempty closed subset of X, and $F : X \to X^*$ a (nonlinear) mapping. The *variational inequality* corresponding to F and Φ, occasionally denoted by $\mathrm{VI}(F, \Phi)$, is the problem of finding $x \in X$ such that

$$(V) \qquad x \in \Phi, \quad \langle F(x), d \rangle \geq 0 \quad \forall d \in \mathcal{T}_\Phi(x). \tag{3.17}$$

This condition is heavily inspired by the first-order necessary conditions for constrained minimization problems from Lemma 3.1. Indeed, if $F = f'$ for some differentiable function $f : X \to \mathbb{R}$, then (V) represents a first-order necessary condition for the optimization problem

$$\underset{x \in X}{\text{minimize}} \ f(x) \quad \text{subject to} \quad x \in \Phi.$$

It should be noted, however, that the scope of VIs far exceeds that of optimization problems. Indeed, some of the most prominent applications of VIs are (generalized) Nash equilibrium problems, or (G)NEPs, which are optimization-type problems involving multiple players and corresponding functions. We will discuss NEPs and GNEPs in some more detail in Section 5.3. The introduction to Chapter 5 also contains a more comprehensive literature review of variational inequalities and their applications in economics, mechanics, and many related fields. In the present section, we shall mainly be concerned with a general analysis of variational inequalities in the form (3.17), with the understanding that many more tangible problem classes can be reformulated as VIs and are therefore implicitly subsumed by our approach.

Observe that, if the feasible set Φ is convex, then (V) can equivalently be stated as

$$x \in \Phi, \quad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \Phi. \tag{3.18}$$

This is because, in the convex case, the polar cone of $\mathcal{T}_\Phi(x)$ coincides with the normal cone in the sense of convex analysis, see the discussion after Definition 2.35. The above is often taken to be the canonical form of variational inequalities. For our purposes, however, it will be convenient to deal with the more general framework (V) since this allows us to implicitly treat nonconvex minimization and related problems.

In many cases, the feasible set Φ has an analytical representation of the form

$$\Phi = \{x \in C : G(x) \in K\}, \tag{3.19}$$

where $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, Y is a real Banach space, and $G : X \to Y$ a smooth mapping. For the sake of generality, we will begin this section by working with a general set Φ, without assuming any representation of the above form. Starting with Section 3.2.2, we will work with a feasible set as in (3.19).

3.2.1 Pseudomonotone Operators

This section is dedicated to a discussion of various continuity and monotonicity properties for the operator $F : X \to X^*$. The basic definitions related to monotonicity are summarized below.

Definition 3.35 (Monotonicity properties)**.** Let $F : X \to X^*$. We say that F is

(i) *monotone* if $\langle F(x) - F(y), x - y\rangle \geq 0$ for all $x, y \in X$.

(ii) *strictly monotone* if $\langle F(x) - F(y), x - y\rangle > 0$ for all $x, y \in X$ with $x \neq y$.

(iii) *strongly monotone* with modulus $c > 0$ if $\langle F(x) - F(y), x - y\rangle \geq c\|x - y\|_X^2$ for all $x, y \in X$.

The monotonicity notions above are closely linked to the convexity concepts from Section 2.2.2. Indeed, if $f : X \to \mathbb{R}$ is a differentiable function, then f is (strictly, strongly) convex if and only if $f' : X \to X^*$ is (strictly, strongly) monotone. Moreover, in the case of strong convexity and monotonicity, the corresponding constants in the definitions can be chosen equivalently.

The aforementioned notions of monotonicity play a fundamental role in the analysis of variational inequalities. If the feasible set Φ is convex and the operator F is monotone, then it follows that the solution set of (V) is always a convex set (possibly empty). If F is furthermore *strictly* monotone, then the solution of (V), if it exists, is unique.

Despite this, it turns out that monotonicity is too restrictive an assumption for many practical variational problems. In particular, the restriction of our analysis to monotone VIs would rule out nonconvex optimization and Nash equilibrium problems. Therefore, a more general approach is necessary. To this end, we use the following notion of *pseudomonotonicity*, due to Brezis [37].

Definition 3.36 (Pseudomonotonicity)**.** We say that an operator $F : X \to X^*$ is *pseudomonotone* if, whenever

$$\{x^k\} \subseteq X, \quad x^k \rightharpoonup x, \quad \text{and} \quad \limsup_{k\to\infty} \left\langle F(x^k), x^k - x\right\rangle \leq 0,$$

then

$$\langle F(x), x - y\rangle \leq \liminf_{k\to\infty} \left\langle F(x^k), x^k - y\right\rangle \quad \text{for all } y \in X.$$

Despite its somewhat peculiar appearance, the notion of pseudomonotonicity will play a fundamental role in the subsequent theory of VIs. Some sufficient conditions for pseudomonotone operators are summarized in the following lemma.

Lemma 3.37 (Sufficient conditions for pseudomonotonicity)**.** *Let X be a real Banach space and $T, U : X \to X^*$ given operators. Then:*

(a) *If T is monotone and continuous, then T is pseudomonotone.*

(b) *If, for every $y \in X$, the mapping $x \mapsto \langle T(x), x - y\rangle$ is weakly sequentially lsc, then T is pseudomonotone.*

(c) *If T is completely continuous, then T is pseudomonotone.*

(d) *If T is continuous and* $\dim(X) < +\infty$*, then T is pseudomonotone.*

(e) *If T and U are pseudomonotone, then $T + U$ is pseudomonotone.*

Proof. (b) is obvious. The remaining assertions can be found in [223, Prop. 27.6]. □

It follows from the above observations that the concept of pseudomonotone operators provides a unified approach to different classes of operators, including monotone and completely continuous ones. Property (b) in the above lemma is occasionally referred to as *(Ky–)Fan-hemicontinuity* as it is closely related to the assumptions of the Ky Fan theorem (Lemma 2.54). At the end of this section, we will present an example which shows that this property is strictly stronger than pseudomonotonicity.

Let us now give a simple class of pseudomonotone operators which arises frequently in practical scenarios.

Example 3.38 (Derivative mappings)**.** In many applications, the operator F is the derivative mapping of a functional $f : X \to \mathbb{R}$. Assume that we are in this scenario, that X is reflexive, and that $f = f_1 + f_2$ with f_1 a smooth convex function and f_2 nonconvex, but weakly sequentially continuous and uniformly differentiable on bounded subsets of X. Then f_1' is monotone and continuous, and f_2' is completely continuous by Proposition 2.17. It follows that $F = f_1' + f_2'$ is pseudomonotone by Lemma 3.37.

Recall that an operator is said to be bounded if it maps bounded sets to bounded sets. The following lemma gives an important property of bounded pseudomonotone operators and generalizes a result from [223] since we do not assume the space X to be reflexive.

Lemma 3.39. *Let $F : X \to X^*$ be a bounded pseudomonotone operator. Then F is* demicontinuous*, i.e., it maps strongly convergent sequences to weak-* convergent sequences. In particular, if* $\dim(X) < +\infty$*, then F is continuous.*

Proof. Let $\{x^k\} \subseteq X$ be a sequence with $x^k \to x$ for some $x \in X$. Observe that $\{F(x^k)\}$ is bounded in X^* and hence

$$\left|\left\langle F(x^k), x^k - x\right\rangle\right| \le \|F(x^k)\|_{X^*}\|x^k - x\|_X \to 0.$$

Thus, by pseudomonotonicity, we obtain

$$\langle F(x), x - y\rangle \leq \liminf_{k\to\infty} \langle F(x^k), x^k - y\rangle = \liminf_{k\to\infty} \langle F(x^k), x - y\rangle \tag{3.20}$$

for all $y \in X$, where we used the boundedness of $\{F(x^k)\}$ and the fact that $x^k \to x$. Inserting $\tilde{y} := 2x - y$ for an arbitrary $y \in X$, we also obtain

$$\langle F(x), y - x\rangle = \langle F(x), x - \tilde{y}\rangle \leq \liminf_{k\to\infty} \langle F(x^k), x^k - \tilde{y}\rangle = \liminf_{k\to\infty} \langle F(x^k), y - x\rangle, \tag{3.21}$$

where the last equality uses the fact that $\{F(x^k)\}$ is bounded and that $x^k - \tilde{y} = y - x + o(1)$. Putting (3.20) and (3.21) together, it follows that $\langle F(x^k), x - y\rangle \to \langle F(x), x - y\rangle$ for all $y \in X$. This implies $F(x^k) \rightharpoonup^* F(x)$, and the proof is done. □

It follows from Lemmas 3.37 and 3.39 that, for a finite-dimensional space X (without loss of generality, a Hilbert space), an operator $F : X \to X$ is bounded and pseudomonotone if and only if it is continuous.

We now present two existence results for variational inequalities. For the sake of generality and since we will need this result later, we first prove an existence theorem for generalized VIs involving, in addition to the mapping F, a lower semicontinuous convex function φ. Such problems are usually called variational inequalities *of the second kind.* The main tool in the following proof is the Brezis–Nirenberg–Stampacchia theorem discussed in Section 2.2.4.

Theorem 3.40. *Let $\Phi \subseteq X$ be a nonempty, convex, weakly compact set, $F : X \to X^*$ a bounded pseudomonotone operator, and $\varphi : X \to \mathbb{R}$ a convex, lower semicontinuous function. Then there exists $\bar{x} \in \Phi$ such that*

$$\langle F(\bar{x}), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \leq 0 \quad \text{for all } y \in \Phi.$$

Proof. We claim that the mapping $\Psi : \Phi^2 \to \mathbb{R}$, $\Psi(x, y) := \langle F(x), x - y\rangle + \varphi(x) - \varphi(y)$, satisfies the assumption of the Brezis–Nirenberg–Stampacchia theorem (Theorem 2.56). Clearly, $\Psi(x, x) \leq 0$ for every $x \in \Phi$, and Ψ is (quasi-)concave with respect to the second argument. Moreover, by Lemma 3.39, Ψ is lower semicontinuous with respect to the first argument on $\Phi \cap L$ for any finite-dimensional subspace L of X. Finally, let $x, y \in \Phi$, let $\{x^k\} \subseteq \Phi$ be a sequence converging weakly to x, and assume that

$$\Psi(x^k, (1 - t)x + ty) \leq 0 \quad \forall t \in [0, 1],\ \forall k \in \mathbb{N}. \tag{3.22}$$

We need to show that $\Psi(x, y) \leq 0$. By (3.22), we have in particular that $\Psi(x^k, x) \leq 0$ and $\Psi(x^k, y) \leq 0$ for all k. The first of these conditions implies that

$$\begin{aligned} 0 \geq \limsup_{k\to\infty} \Psi(x^k, x) &\geq \limsup_{k\to\infty} \langle F(x^k), x^k - x\rangle + \liminf_{k\to\infty} \left[\varphi(x^k) - \varphi(x)\right] \\ &\geq \limsup_{k\to\infty} \langle F(x^k), x^k - x\rangle, \end{aligned}$$

where we used the weak sequential lower semicontinuity of φ. Hence, by the pseudomonotonicity of F, we obtain

$$\begin{aligned}\Psi(x,y) &= \langle F(x), x-y\rangle + \varphi(x) - \varphi(y) \\ &\le \liminf_{k\to\infty} \left[\left\langle F(x^k), x^k - y\right\rangle + \varphi(x^k) - \varphi(y)\right] = \liminf_{k\to\infty} \Psi(x^k, y) \le 0.\end{aligned}$$

Therefore, Ψ satisfies all the requirements of Theorem 2.56, and the result follows. □

Clearly, we can recover the variational inequality (V) by setting $\varphi \equiv 0$. This immediately yields the following existence result.

Corollary 3.41. *Let $\Phi \subseteq X$ be a nonempty, convex, weakly compact set, and $F : X \to X^*$ a bounded pseudomonotone operator. Then (V) admits a solution.*

In many situations, the weak compactness of C can be replaced by an appropriate kind of radial unboundedness.

Remark 3.42 (Coercivity)**.** Assume that X is reflexive and F is *coercive* in the sense that, for all $y \in X$,

$$\frac{\langle F(x), x-y\rangle}{\|x-y\|_X} \to +\infty \quad \text{as } \|x\|_X \to +\infty. \tag{3.23}$$

Then the weak compactness of Φ in Theorem 3.40 and Corollary 3.41 can be replaced by closedness, see [39, Thm. 1]. Note that every strongly monotone operator satisfies (3.23).

The importance of pseudomonotonicity for VIs goes beyond the existence theory. Indeed, a rather important consequence of this property is the stability of solutions and approximate solutions of the VI under weak convergence.

Proposition 3.43. *Let $\Phi \subseteq X$ be a nonempty closed convex set and $F : X \to X^*$ a pseudomonotone operator. Assume that $\{x^k\} \subseteq X$ converges weakly to a point $\bar{x} \in \Phi$ and that there are null sequences $\{\delta_k\}, \{\varepsilon_k\} \subseteq \mathbb{R}$ (possibly negative) such that*

$$\left\langle F(x^k), y - x^k\right\rangle \ge \delta_k + \varepsilon_k \|y - x^k\|_X \quad \forall y \in \Phi$$

for all k. Then $\bar{x}$ is a solution of the VI.

Proof. Since $\bar{x} \in \Phi$, we obtain in particular that $\liminf_{k\to\infty}\langle F(x^k), \bar{x} - x^k\rangle \ge 0$. The pseudomonotonicity of F therefore implies that

$$\langle F(\bar{x}), y - \bar{x}\rangle \ge \limsup_{k\to\infty} \left\langle F(x^k), y - x^k\right\rangle \ge 0 \quad \text{for all } y \in \Phi.$$

Hence, $\bar{x}$ is a solution of the VI. □

Under the assumptions of Proposition 3.43, it follows in particular that weak limit points of a sequence of (exact) solutions of the VI are again solutions of the VI. This means that the solution set of (V) is weakly sequentially closed.

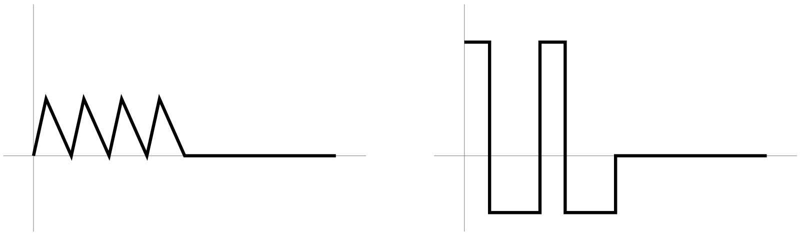


Figure 3.1: The sequence from Example 3.44 for $k = 4$: u_k (left) and ∇u_k (right).

We conclude this section by presenting an example of a bounded pseudomonotone operator which satisfies neither (b) nor (c) from Lemma 3.37. In particular, the example shows that the continuity from part (b), also called Ky–Fan-hemicontinuity, is strictly stronger than pseudomonotonicity. This implies that a recent publication claiming the equivalence of the two properties, see [199], is erroneous.

Example 3.44. Let $X := W_0^{1,3}(0,1)$ be one of the Sobolev spaces from Section 2.1.4, and let $F : X \to X^*$ be the (negative) p-Laplacian defined by

$$\langle F(u), v\rangle := \int_0^1 |\nabla u(t)|\nabla u(t)\nabla v(t)\,\mathrm{d}t.$$

Then F is monotone and continuous, hence pseudomonotone. Now, for each $k \in \mathbb{N}$, let $u_k : [0,1] \to \mathbb{R}$ be the piecewise linear function with value $1/k$ at $t = (3i+1)/(6k)$ for $i = 0, \ldots, k-1$, and value zero at $t = i/(2k)$ for $i = 0, \ldots, k$, and on $[1/2, 1]$. Clearly, $u_k \to 0$ in $L^3(0,1)$. Moreover, the weak derivative of u_k is (almost everywhere) given by

$$\nabla u_k(t) = \begin{cases} 6, & \text{if } t \in \left(\frac{i}{2k}, \frac{3i+1}{6k}\right) \text{ with } i = 0, \ldots, k-1, \\ -3, & \text{if } t \in \left(\frac{3i-2}{6k}, \frac{i}{2k}\right) \text{ with } i = 1, \ldots, k, \\ 0, & \text{if } t \in \left(\frac{1}{2}, 1\right), \end{cases}$$

see Figure 3.1. It follows from standard arguments (e.g., [4, Exercise 8.7, p. 254]) that $\nabla u_k \rightharpoonup 0$ in $L^3(0,1)$. Hence, $u_k \rightharpoonup u := 0$ in $X = W_0^{1,3}(0,1)$. Now, let $v(t) := \alpha \min\{t, 1-t\}$ with $\alpha \gg 0$, and observe that $\|\nabla u_k\|^3_{L^3(0,1)} = 45$ for all k. Finally, we have $\langle F(u), u - v\rangle = 0$, but an elementary calculation shows that

$$\langle F(u_k), u_k - v\rangle = \|\nabla u_k\|^3_{L^3(0,1)} - \int_0^1 |\nabla u_k(t)|\nabla u_k(t)\nabla v(t)\,\mathrm{d}t = 45 - 3\alpha.$$

Thus, if α is large enough, it follows that $\langle F(u_k), u_k - v\rangle$ is a negative constant for all k, and thus property (b) from Lemma 3.37 is violated.

3.2.2 KKT-Type Conditions

Throughout the remainder of this chapter, we assume that the feasible set Φ of the VI has an analytical representation of the form

$$\Phi = \{x \in C : G(x) \in K\}, \tag{3.24}$$

where $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, Y is a real Banach space, and $G : X \to Y$ a continuously differentiable mapping. It follows that, if $F = f'$ for some differentiable function $f : X \to \mathbb{R}$, then (V) can be seen as a first-order necessary condition (in the sense of Lemma 3.1) of the constrained optimization problem

$$\underset{x \in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in K.$$

Let us now discuss a general VI of the form (V), with an arbitrary operator $F : X \to X^*$, and with the feasible set given by (3.24). It turns out that such VIs admit a similar theory of first-order necessary (KKT) conditions and Lagrange multipliers as constrained optimization problems. To see this, let $\bar{x}$ be a solution of (V), and assume that the constraint system of (V) satisfies the Robinson constraint qualification (RCQ, see Definition 3.4) in $\bar{x}$. Then Corollary 3.16 implies that

$$\mathcal{T}_\Phi(\bar{x}) = \{d \in \mathcal{T}_C(\bar{x}) : G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x}))\}.$$

In particular, the definition of the VI implies that $\bar{d} := 0$ is a solution of the constrained minimization problem

$$\underset{d \in X}{\text{minimize}}\ \langle F(\bar{x}), d\rangle \quad \text{subject to} \quad d \in \mathcal{T}_C(\bar{x}),\ G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x})). \tag{3.25}$$

Observe now that RCQ for this transformed problem in $\bar{d} = 0$ takes on the form

$$0 \in \operatorname{int}\big[G'(\bar{x})\mathcal{T}_C(\bar{x}) - \mathcal{T}_K(G(\bar{x}))\big].$$

Since $C - \bar{x} \subseteq \mathcal{T}_C(\bar{x})$ and $K - G(\bar{x}) \subseteq \mathcal{T}_K(G(\bar{x}))$, this condition is implied by RCQ for the original constraint system of (V). Thus, we obtain from Theorem 3.5 that there is a nonempty, bounded, and convex set $\Lambda(\bar{x}) \subseteq Y^*$ of Lagrange multipliers for (3.25). The KKT conditions of this problem in $\bar{d} = 0$ take on the form

$$-F(\bar{x}) - G'(\bar{x})^*\bar{\lambda} \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})). \tag{3.26}$$

This prompts us to define the *Lagrange function* or *Lagrangian* of (V), in the variational inequality sense, as the mapping

$$\mathcal{L} : X \times Y^* \to X^*, \quad \mathcal{L}(x, \lambda) := F(x) + G'(x)^*\lambda. \tag{3.27}$$

The KKT system of the VI is then nothing but the KKT system (3.26) of the transformed problem (3.25).

Definition 3.45 (KKT point)**.** A point $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ is a *KKT point* of (V) if

$$-\mathcal{L}(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})).$$

We say that $\bar{x} \in X$ is a *stationary point* of (V) if $(\bar{x}, \bar{\lambda})$ is a KKT point for some multiplier $\bar{\lambda} \in Y^*$, and denote by $\Lambda(\bar{x})$ the set of such multipliers.

If the VI originates from a constrained optimization problem, then the Lagrangian (3.27) in the variational sense is the derivative of the Lagrange function (3.3) of the underlying optimization problem. In this case, the KKT system from Definition 3.45 is consistent with its optimization counterpart from Definition 3.2.

The preceding discussion implies that Theorem 3.5 on the necessity of the KKT system under RCQ can directly be carried over to the VI setting. Interestingly, it turns out that the converse implication is much stronger in the present case. More precisely, the KKT conditions are *always* sufficient for "optimality", where optimality has to be understood in the variational sense. The reason why the problem (V) admits this strong connection is that its definition uses the tangent cone to Φ and not the set Φ itself.

Proposition 3.46. *If $(\bar{x}, \bar{\lambda})$ is a KKT point, then $\bar{x}$ is a solution of (V). Conversely, if $\bar{x}$ is a solution of (V) and RCQ holds in $\bar{x}$, then $\Lambda(\bar{x})$ is nonempty and bounded in Y^*.*

Proof. The converse part follows from Theorem 3.5 and the arguments preceding Definition 3.45. To prove that the KKT conditions are always sufficient, let $(\bar{x}, \bar{\lambda})$ be a KKT point of (V), and let $d \in \mathcal{T}_\Phi(\bar{x})$. Then $d = \lim_{k\to\infty}(x^k - \bar{x})/t_k$ with $\{x^k\} \subseteq \Phi$, $x^k \to \bar{x}$, and $t_k \downarrow 0$. In particular, we have $d \in \mathcal{T}_C(\bar{x})$, and it follows from the KKT conditions that $\langle \mathcal{L}(\bar{x}, \bar{\lambda}), d\rangle \geq 0$. This yields

$$\langle F(\bar{x}), d\rangle \geq \langle -G'(\bar{x})^*\bar{\lambda}, d\rangle = \left\langle -G'(\bar{x})^*\bar{\lambda}, \lim_{k\to\infty} \frac{x^k - \bar{x}}{t_k} \right\rangle = -\lim_{k\to\infty} \frac{1}{t_k} \big\langle \bar{\lambda}, G'(\bar{x})(x^k - \bar{x})\big\rangle.$$

But $G'(\bar{x})(x^k - \bar{x}) = G(x^k) - G(\bar{x}) + o(t_k)$ and therefore

$$\langle F(\bar{x}), d\rangle \geq -\lim_{k\to\infty} \frac{1}{t_k} \big\langle \bar{\lambda}, G(x^k) - G(\bar{x})\big\rangle \geq 0,$$

where we used $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x}))$ and $G(x^k) \in K$ for all k. □

It is also possible to extend the strict Robinson condition from constrained optimization (SRC, Definition 3.6) to variational inequalities. This has to be done explicitly because SRC is not a constraint qualification, i.e., it cannot be attributed solely to the constraint system of the underlying problem. In any case, the natural extension of Definition 3.6 is the following.

Definition 3.47 (Strict Robinson condition)**.** We say that the *strict Robinson condition (SRC)* for the VI (V) holds in $\bar{x} \in \Phi$ if there exists $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$0 \in \operatorname{int}\big[G(\bar{x}) + G'(\bar{x})(C_0 - \bar{x}) - K_0\big],$$

where $C_0 := \{x \in C : \mathcal{L}(\bar{x}, \bar{\lambda})(x - \bar{x}) = 0\}$ and $K_0 := \{y \in K : \langle \bar{\lambda}, y - G(\bar{x})\rangle = 0\}$.

Arguing as in Proposition 3.7, it follows that the SRC for VIs implies the uniqueness of the corresponding Lagrange multiplier.

Due to the strong connection between the KKT conditions and the VI, it is not necessary to formulate second-order sufficient conditions for variational problems. Nevertheless, for theoretical considerations, it will be convenient to define an analogue of the second-order sufficient condition from constrained optimization. Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (V) and define, for $\eta > 0$, the *extended critical cone*

$$\mathcal{C}_\eta(\bar{x}) := \left\{ d \in \mathcal{T}_C(\bar{x}) : \begin{array}{l} \langle F(\bar{x}), d \rangle \leq \eta \|d\|_X, \\ \operatorname{dist}\bigl(G'(\bar{x})d, \mathcal{T}_K(G(\bar{x}))\bigr) \leq \eta \|d\|_X \end{array} \right\}. \tag{3.28}$$

The following is the basic second-order condition for variational inequalities. We assume that F is continuously differentiable and G twice continuously differentiable near $\bar{x}$.

Definition 3.48 (Second-order sufficient condition)**.** Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (V). We say that the *second-order sufficient condition (SOSC)* holds in $(\bar{x}, \bar{\lambda})$ if there are $\eta, c > 0$ such that

$$\bigl\langle \mathcal{L}'(\bar{x}, \bar{\lambda})d, d \bigr\rangle \geq c\|d\|_X^2 \quad \text{for all } d \in \mathcal{C}_\eta(\bar{x}).$$

Note that we use the terminology "second-order sufficient condition" mainly for the sake of consistency with the corresponding condition from constrained optimization (Definition 3.24). For variational problems such as (V), there is actually no need for sufficiency conditions to complement the KKT system because of Proposition 3.46.

3.2.3 Sequential KKT Conditions

This section is dedicated to a rather pragmatic concept of asymptotic optimality. Many practical algorithms for constrained optimization, variational inequalities, etc., iteratively construct a primal-dual sequence $\{(x^k, \lambda^k)\}$ which satisfies the KKT conditions in an asymptotic sense. Therefore, it makes sense to analyze such "sequential" analogues of the KKT conditions in more detail. The discussion in this section is also motivated by similar approaches in finite dimensions, see [7, 8, 28].

Recall that the KKT conditions of (V) are given by

$$-\mathcal{L}(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})).$$

It is particularly the second condition which requires special care when trying to formulate a sequential analogue of the KKT conditions. The main definition we will use is the following.

Definition 3.49 (Asymptotic KKT sequence)**.** We say that a sequence $\{(x^k, \lambda^k)\} \subseteq C \times Y^*$ is an *asymptotic KKT sequence* for (V) if there exist null sequences $\{\varepsilon^k\} \subseteq X^*$ and $\{r_k\} \subseteq \mathbb{R}$ such that, for all k,

$$\varepsilon^k - \mathcal{L}(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \bigl\langle \lambda^k, y - G(x^k) \bigr\rangle \leq r_k \ \forall y \in K. \tag{3.29}$$

Our main aim in this section is to give sufficient conditions which guarantee that, if $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence and $\bar{x}$ is a (possibly weak) limit point of $\{x^k\}$, then $\bar{x}$ is a stationary point of (V). In this context, it is worth mentioning that Definition 3.49 imposes no conditions on the attainment of feasibility. This aspect is left unspecified for the sake of flexibility; indeed, we will mainly be concerned with scenarios where $\bar{x}$ is some kind of limit point of $\{x^k\}$ and we already know from a preliminary analysis that $\bar{x}$ is a feasible point.

Note that, while the conditions posed in Definition 3.49 seem reasonably weak, it is possible to generalize the asymptotic KKT concept even further. In particular, in our formulation, the second inequality in (3.29) is assumed to hold *uniformly* on K. If K is unbounded, then it may be more natural to require some kind of uniformness of the inequality on bounded subsets of K. In any case, however, the augmented Lagrangian method which we will discuss in later chapters satisfies the uniform bound from (3.29), and a more general analysis is therefore not necessary for our purposes.

Before analyzing the optimality properties of limit points, we first give a property of asymptotic KKT sequences which is interesting in its own right. Indeed, it turns out that the existence of an asymptotic KKT sequence is a necessary optimality condition for constrained optimization problems, even in the absence of constraint qualifications. For the formulation of this result, consider a problem of the form (3.1), that is

$$\underset{x\in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x)\in K, \tag{3.30}$$

where $f: X\to\mathbb{R}$ is a continuously differentiable function. This corresponds to the variational setting (V) with $F := f'$. The proof of the following result is inspired by [28, Thm. 3.1].

Proposition 3.50. *Let $\bar{x}$ be a local minimizer of* (3.30). *Assume that X is reflexive, Y is a real Hilbert space, and that f and $d_K\circ G$ are weakly sequentially lsc in $\bar{x}$. Then there is an asymptotic KKT sequence $\{(x^k,\lambda^k)\}\subseteq C\times Y$ such that $x^k\to\bar{x}$.*

Proof. Let $r>0$ be such that $\bar{x}$ minimizes f on $B_r(\bar{x})\cap\Phi$. For $k\in\mathbb{N}$, consider the problem

$$\underset{x\in X}{\text{minimize}}\ f(x)+\|x-\bar{x}\|_X^2+kd_K^2(G(x)) \quad \text{subject to} \quad x\in B_r(\bar{x})\cap C. \tag{3.31}$$

Since X is reflexive and the objective function in the above problem is weakly sequentially lsc, the problem admits a minimizer $x^k\in B_r(\bar{x})\cap C$. Passing to a subsequence if necessary, we may assume that $x^k\rightharpoonup\hat{x}$ for some $\hat{x}\in B_r(\bar{x})\cap C$. Observe now that

$$f(x^k)+\|x^k-\bar{x}\|_X^2+kd_K^2(G(x^k))\le f(\bar{x}) \tag{3.32}$$

for all k by the minimizing property of x^k. Dividing by k and taking the limit $k\to\infty$, it follows that $d_K(G(\hat{x}))=0$, i.e., $\hat{x}$ is feasible. By (3.32), we also obtain $f(\hat{x})+\|\hat{x}-\bar{x}\|_X^2\le f(\bar{x})$. But $f(\bar{x})\le f(\hat{x})$, hence $\hat{x}=\bar{x}$ and (3.32) implies that $x^k\to\bar{x}$. In particular, we have $\|x^k-\bar{x}\|_X<r$ for sufficiently large k, and from (3.31) we obtain the existence of a sequence $\{\varepsilon^k\}\subseteq X^*$ such that $-\varepsilon^k\in\partial\|x^k-\bar{x}\|_X^2$ and

$$0\in f'(x^k)-\varepsilon^k+2kG'(x^k)^*\big[G(x^k)-P_K(G(x^k))\big]+\mathcal{N}_C(x^k).$$

Observe now that $\|\cdot\|_X^2$ is Fréchet-differentiable in the point $0\in X$ with derivative $0\in X^*$. Hence, by Proposition 2.46, we obtain $\varepsilon^k\to 0$ in X^*. Moreover, the sequence $\lambda^k := 2k[G(x^k)-P_K(G(x^k))]$ satisfies $\langle\lambda^k, y-G(x^k)\rangle\le 0$ for all $y\in K$ by Proposition 2.36. □

Our subsequent efforts in this section are devoted to the analysis of conditions which guarantee that, if $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence and $\bar{x}$ is a limit point of $\{x^k\}$, then there exists $\bar{\lambda} \in Y^*$ such that $(\bar{x}, \bar{\lambda})$ is a KKT point. We will mainly be concerned with the analysis of *weak* limit points, which implies that, occasionally, we will need complete continuity properties of the mapping G or its derivative. Of course, if $\bar{x}$ is a strong limit point of $\{x^k\}$, then many of the subsequent assumptions can be simplified considerably.

Theorem 3.51. *Let $\{(x^k, \lambda^k)\} \subseteq C \times Y^*$ be an asymptotic KKT sequence with $x^k \rightharpoonup \bar{x}$ for some $\bar{x} \in \Phi$. Assume that F is bounded and pseudomonotone, that G and G' are completely continuous, and that RCQ holds in $\bar{x}$. Then the sequence $\{\lambda^k\}$ is bounded in Y^*, and every weak-* limit point of $\{\lambda^k\}$ belongs to $\Lambda(\bar{x})$.*

Proof. We first prove the boundedness of $\{\lambda^k\}$ in Y^*. Applying the generalized open mapping theorem (Theorem 3.11) to the multifunction $\mathcal{W}(u) := G(\bar{x}) + G'(\bar{x})u - K$ on the domain $C - \bar{x}$, we obtain the existence of $r > 0$ such that

$$B_r^Y \subseteq G(\bar{x}) + G'(\bar{x})\big[(C - \bar{x}) \cap B_1^X\big] - K.$$

Now, let $\{y^k\} \subseteq Y$ be a sequence of unit vectors such that $\langle \lambda^k, y^k \rangle \geq \frac{1}{2}\|\lambda^k\|_{Y^*}$. Then

$$-ry^k = G(\bar{x}) + G'(\bar{x})(v^k - \bar{x}) - z^k$$

with $\{v^k\} \subseteq C$ a bounded sequence and $\{z^k\} \subseteq K$. It follows that $ry^k = z^k - G(x^k) - G'(x^k)(v^k - x^k) + \delta^k$ with $\delta^k \to 0$ as $k \to \infty$. Let k be large enough so that $\|\delta^k\|_Y \leq r/4$. Then, by the asymptotic KKT conditions (3.29), we obtain

$$\begin{aligned}\frac{r}{2}\|\lambda^k\|_{Y^*} \leq \big\langle \lambda^k, ry^k \big\rangle &\leq \big\langle \lambda^k, z^k - G(x^k) \big\rangle - \big\langle \lambda^k, G'(x^k)(v^k - x^k) \big\rangle + \frac{r}{4}\|\lambda^k\|_{Y^*} \\ &\leq \big\langle \lambda^k, z^k - G(x^k) \big\rangle + \big\langle F(x^k) - \varepsilon^k, v^k - x^k \big\rangle + \frac{r}{4}\|\lambda^k\|_{Y^*}.\end{aligned}$$

Now, using again (3.29) and the boundedness of F, it follows that the first two terms are bounded from above by some constant $c > 0$. Hence, $\frac{r}{4}\|\lambda^k\|_{Y^*} \leq c$.

We now show the second assertion. Let $I \subseteq \mathbb{N}$ be an (infinite) subset such that $\lambda^k \rightharpoonup^*_I \bar{\lambda}$ in Y^*. By (3.29) and Proposition 2.40, we have $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x}))$. Now, let $y \in C$ be arbitrary. Then, by (3.29),

$$\big\langle \varepsilon^k, y - x^k \big\rangle \leq \big\langle F(x^k), y - x^k \big\rangle + \big\langle \lambda^k, G'(x^k)(y - x^k) \big\rangle. \tag{3.33}$$

By complete continuity, we have $G'(x^k) \to G'(\bar{x})$ and $G'(\bar{x})(y - x^k) \to G'(\bar{x})(y - \bar{x})$, see Proposition 2.16. Hence, $G'(x^k)(y - x^k) \to G'(\bar{x})(y - \bar{x})$. We now argue as in Proposition 3.43. Inserting $y := \bar{x}$ into (3.33) yields $\liminf_{k\to\infty}\langle F(x^k), \bar{x} - x^k \rangle \geq 0$. Hence, by pseudomonotonicity, we obtain that, for all $y \in C$,

$$\langle F(\bar{x}), y - \bar{x} \rangle + \langle \bar{\lambda}, G'(\bar{x})(y - \bar{x}) \rangle \geq \limsup_{k\to\infty}\big[\big\langle F(x^k), y - x^k \big\rangle + \big\langle \lambda^k, G'(x^k)(y - x^k) \big\rangle\big] \geq 0.$$

But this means that $-\mathcal{L}(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x})$. Hence, $(\bar{x}, \bar{\lambda})$ is a KKT point of (V). □

We now consider the case where $C = X$, the primal sequence $\{x^k\}$ converges weakly to a point $\bar{x}$, and the derivative operator $G'(\bar{x})$ is surjective. This is perhaps the strongest possible constraint qualification (see Remark 3.8) and it has the great benefit that, unlike conditions such as RCQ, the surjectivity of $G'(\bar{x})$ does not depend on the function value $G(\bar{x})$. Therefore, it is possible to obtain a convergence result for asymptotic KKT sequences under only the convergence $G'(x^k) \to G'(\bar{x})$, with no convergence of the values $G(x^k)$. For later reference, we state this result in a slightly more general framework.

Proposition 3.52. *Let $\{x^k\} \subseteq X$, $\{T_k\} \subseteq L(X,Y)$, and $\{\lambda^k\} \subseteq Y^*$ be sequences such that $F(x^k) + T_k^*\lambda^k \rightharpoonup^* 0$. Assume that $x^k \rightharpoonup \bar{x}$ for some $\bar{x} \in X$, $F(x^k) \rightharpoonup^* F(\bar{x})$, $T_k \to T$ for some $T \in L(X,Y)$, and that T is surjective. Then $\{\lambda^k\}$ converges weak-* in Y^* to the unique solution of $F(\bar{x}) + T^*\lambda = 0$.*

Proof. We first show that $\{\lambda^k\}$ is weak-* convergent. Let $\hat{y} \in Y$ be an arbitrary point. It suffices to show that $\langle \lambda^k, \hat{y}\rangle$ is convergent. Let $r > 0$ be as in the uniform version of the Banach open mapping theorem (Theorem 2.13), so that $B_r^Y \subseteq T(B_1^X)$. Assume, without loss of generality, that $\hat{y} \in B_r^Y$, and let $\hat{w} \in B_1^X$ be a point such that $T\hat{w} = \hat{y}$. Set $\delta_k := \|T_k - T\|_{L(X,Y)}$, and let k be sufficiently large so that $\delta_k < r$. Then $\|\hat{y} - T_k\hat{w}\|_Y \le \delta_k$ and, by Theorem 2.13, there are points $d^k \in X$ such that $T_k d^k = \hat{y} - T_k\hat{w}$ and

$$\|d^k\|_X \le \frac{\|\hat{y} - T_k\hat{w}\|_Y}{r - \delta_k} \le \frac{\delta_k}{r - \delta_k}.$$

Define $w^k := \hat{w} + d^k$. Then $w^k \to \hat{w}$ and $T_k w^k = \hat{y}$ by definition. Hence,

$$0 \leftarrow \left\langle F(x^k) + T_k^*\lambda^k, w^k \right\rangle = \langle F(\bar{x}), \hat{w}\rangle + o(1) + \left\langle \lambda^k, \hat{y} \right\rangle.$$

Thus, we obtain $\langle \lambda^k, \hat{y}\rangle \to -\langle F(\bar{x}), \hat{w}\rangle$. Since $\hat{y} \in Y$ was arbitrary, this implies that $\{\lambda^k\}$ is weak-* convergent in Y^*.

Let $\bar{\lambda}$ denote the weak-* limit of $\{\lambda^k\}$. Using $F(x^k) + T_k^*\lambda^k \rightharpoonup^* 0$, it follows that $F(\bar{x}) + T^*\bar{\lambda} = 0$, and $\bar{\lambda}$ is unique since T^* is injective. □

We now briefly turn to the case of nonlinear programming (NLP) type constraints. Here, RCQ boils down to the Mangasarian–Fromovitz constraint qualification (MFCQ, see Definition 3.29), and thus the results of Theorem 3.51 can readily be applied to the NLP setting. However, a more detailed analysis using the specific structure of NLP constraints allows us to prove a similar assertion under the CPLD constraint qualification (see Definition 3.29). For this result, we need the following Carathéodory-type lemma.

Lemma 3.53 (Carathéodory, [28, Lem. 3.1])**.** *Let $u \in \mathbb{R}^n$ be a vector of the form*

$$u = \sum_{i=1}^m \lambda_i v^i + \sum_{j=1}^p \mu_j w^j,$$

where $\lambda_i \ge 0$, $v^i \in \mathbb{R}^n$ for $i = 1,\dots,m$, and $\mu_j \in \mathbb{R}$, $w^j \in \mathbb{R}^n$ for $j = 1,\dots,p$. Then there exist subsets $I \subseteq \{1,\dots,m\}$, $J \subseteq \{1,\dots,p\}$ and coefficients $\lambda_i' \ge 0$, $i \in I$, and $\mu_j' \in \mathbb{R}$, $j \in J$, such that the vectors $\{v^i\}_{i\in I} \cup \{w^j\}_{j\in J}$ are linearly independent and

$$u = \sum_{i\in I} \lambda_i' v^i + \sum_{j\in J} \mu_j' w^j.$$

Let us now consider a variational inequality (or optimization problem) where the constraints are of nonlinear programming form, i.e.,

$$\Phi = \{x \in \mathbb{R}^n : g(x) \leq 0,\ e(x) = 0\}, \tag{3.34}$$

where $n \in \mathbb{N}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $e : \mathbb{R}^n \to \mathbb{R}^p$, and $m, p \in \mathbb{N}_0$. In this case, it is possible to use a slightly different asymptotic KKT concept, see the theorem below. For the sake of completeness, we include assertions for all of LICQ, MFCQ, and CPLD.

Theorem 3.54. *Let the feasible set of* (V) *be given in the form* (3.34). *Assume that* $\{(x^k, \lambda^k, \mu^k)\} \subseteq \mathbb{R}^{n+m+p}$ *is a sequence such that* $x^k \to \bar{x}$ *for some* $\bar{x} \in \Phi$ *and*

$$F(x^k) + \nabla g(x^k)\lambda^k + \nabla e(x^k)\mu^k \to 0, \quad \min\{-g(x^k), \lambda^k\} \to 0, \quad \text{as } k \to \infty. \tag{3.35}$$

Then the following assertions hold:

(a) *If CPLD holds in* $\bar{x}$*, then* $\bar{x}$ *is a stationary point (and a solution) of the VI.*

(b) *If MFCQ holds in* $\bar{x}$*, then* $\{(\lambda^k, \mu^k)\}$ *is bounded, and its limit points belong to* $\Lambda(\bar{x})$*.*

(c) *If LICQ holds in* $\bar{x}$*, then* $\{(\lambda^k, \mu^k)\}$ *converges to the unique element in* $\Lambda(\bar{x})$*.*

Proof. (a) Since (3.35) remains true if we replace λ^k by $\max\{\lambda^k, 0\}$, we may assume, without loss of generality, that $\lambda^k \geq 0$ for all k. Observe furthermore that $\lambda_i^k \to 0$ whenever $g_i(\bar{x}) < 0$. Thus, we obtain from (3.29) that

$$F(x^k) + \sum_{g_i(\bar{x})=0} \lambda_i^k \nabla g_i(x^k) + \sum_{j=1}^{p} \mu_j^k \nabla e_j(x^k) \to 0.$$

By Lemma 3.53, there are subsets $I_k \subseteq \{i : g_i(\bar{x}) = 0\}$ and $J_k \subseteq \{1, \ldots, p\}$ such that, for all k, the gradients $\{\nabla g_i(x^k)\}_{i \in I_k} \cup \{\nabla e_j(x^k)\}_{j \in J_k}$ are linearly independent and

$$F(x^k) + \sum_{i \in I_k} \hat{\lambda}_i^k \nabla g_i(x^k) + \sum_{j \in J_k} \hat{\mu}_j^k \nabla e_j(x^k) \to 0 \tag{3.36}$$

with suitable coefficients $\hat{\lambda}_i^k \geq 0$, $i \in I_k$, and $\hat{\mu}_j^k \in \mathbb{R}$, $j \in J_k$. Passing onto a subsequence if necessary, we may assume that $I_k = I$ and $J_k = J$ for all k with some subsets $I \subseteq \{i : g_i(\bar{x}) = 0\}$ and $J \subseteq \{1, \ldots, p\}$. To conclude the proof, it suffices to show that the sequence $\{(\hat{\lambda}^k, \hat{\mu}^k)\}$ is bounded. If this were not the case, then we could divide (3.36) by $\|\hat{\lambda}^k\| + \|\hat{\mu}^k\|$, take the limit $k \to \infty$ on a suitable subsequence and obtain nontrivial coefficients $\alpha_i \geq 0$, $i \in I$, and $\beta_j \in \mathbb{R}$, $j \in J$, such that

$$\sum_{i \in I} \alpha_i \nabla g_i(\bar{x}) + \sum_{j \in J} \beta_j \nabla e_j(\bar{x}) = 0.$$

Hence, by CPLD, the gradients $\{\nabla g_i(x)\}_{i \in I} \cup \{\nabla e_j(x)\}_{j \in J}$ should be linearly independent in a neighborhood of x, which is the desired contradiction.

(b) Assume that $\{(\lambda^k, \mu^k)\}$ is unbounded. Arguing as in the proof of (a), we can divide the first limit in (3.35) by $\|\lambda^k\| + \|\mu^k\|$, take the limit $k \to \infty$ on a suitable subsequence, and obtain the positive linear dependence (in the sense of Section 3.1.4) of the gradients $\{\nabla g_i(\bar{x})\}_{i\in\mathcal{I}} \cup \{\nabla e_j(\bar{x})\}_{j=1}^p$, where $\mathcal{I} = \{i : g_i(\bar{x}) = 0\}$ is the set of active indices. This contradicts MFCQ.

(c) It is well-known that LICQ implies MFCQ and the uniqueness of $(\bar{\lambda}, \bar{\mu}) \in \Lambda(\bar{x})$. From (b), it follows that $\{(\lambda^k, \mu^k)\}$ is bounded, and its limit points are equal to $(\bar{\lambda}, \bar{\mu})$. Thus, $(\lambda^k, \mu^k) \to (\bar{\lambda}, \bar{\mu})$, and the proof is complete. □

3.2.4 Error Bounds and Lipschitz Stability

As a second essential ingredient for our algorithmic approach in later chapters, we now analyze quantitative stability properties of the KKT system of the variational inequality (V). By extension, the analysis of course also applies to optimization problems. The stability theory is closely linked to so-called error bounds, i.e., computable upper bounds on the distance of certain points to the (primal-dual) solution set of (V). Such error bounds are essential for rate-of-convergence analyses of optimization algorithms.

We begin our investigations by considering the special case where $C = X$ (the general case will then be deduced by arguing as in Remark 3.10). In what follows, we equip the product of two Banach spaces with the norm induced by the sum of the component norms. Recall also that $\Lambda(\bar{x})$ denotes the set of Lagrange multipliers in a stationary point $\bar{x}$. The basic result on the stability of the KKT system of (V) is the following, which is a slightly simplified version of [32, Thm. 5.9].

Lemma 3.55. *Let $C = X$ and let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (V) satisfying SOSC and SRC. Then there exists $c > 0$ such that, for all $\theta = (\alpha, \beta) \in X^* \times Y$ sufficiently close to $(0, 0)$, if $(x_\theta, \lambda_\theta)$ satisfies the perturbed KKT system*

$$\mathcal{L}(x, \lambda) = \alpha, \quad \lambda \in \mathcal{N}_K(G(x) - \beta), \tag{3.37}$$

and x_θ is sufficiently close to $\bar{x}$, then $\|x_\theta - \bar{x}\|_X + \|\lambda_\theta - \bar{\lambda}\|_{Y^*} \le c\|\theta\|_{X^*\times Y}$.

Perturbations of the form (3.37) are usually called *canonical* perturbations. If the parameter β is omitted and only the Lagrange function is perturbed, then one speaks of *tilt* perturbations.

We now direct our efforts towards a more convenient form of stability which does not require us to explicitly specify the perturbation parameters in (3.37), but allows us to estimate the distance to the primal-dual solution set for arbitrary points (x, λ). Such estimates are usually called *error bounds*. For the precise formulation of these, we will assume that the constraint function G maps into a Hilbert space instead of the Banach space Y. To emphasize the modified problem setting, let H be a real Hilbert space, $\mathcal{K} \subseteq H$ a nonempty closed convex set, and $G : X \to H$ a given mapping. With an obvious change of notation, the feasible set of (V) is now

$$\Phi := \{x \in X : G(x) \in \mathcal{K}\}, \tag{3.38}$$

and the KKT system takes on the form of the generalized equations

$$\mathcal{L}(x,\lambda) = F(x) + G'(x)^*\lambda = 0 \quad \text{and} \quad \lambda \in \mathcal{N}_{\mathcal{K}}(G(x)). \tag{3.39}$$

Formally, one can think of H and $\mathcal{K}$ as being nothing but placeholders for Y and K. However, we will later encounter situations where multiple spaces (as well as embeddings) are involved, in which case it becomes crucial to distinguish Banach and Hilbert spaces. The present change of notation is chosen to be consistent with these structures which we will encounter, for instance, in Chapters 4 and 5.

Observe that, due to the Hilbert space setting, the normal cone inclusion in (3.39) can be reformulated through the projection onto $\mathcal{K}$. Indeed, by Proposition 2.36, we have $\lambda \in \mathcal{N}_{\mathcal{K}}(G(x))$ if and only if $G(x) = P_{\mathcal{K}}(G(x)+\lambda)$. Thus, in order to quantify the violation of (3.39), it is rather intuitive to define the residual mapping

$$\Theta(x,\lambda) := \|\mathcal{L}(x,\lambda)\|_{X^*} + \|G(x) - P_{\mathcal{K}}(G(x)+\lambda)\|_H. \tag{3.40}$$

Clearly, the generalized equations (3.39) are equivalent to $\Theta(x,\lambda) = 0$. Indeed, we shall now see that, under certain assumptions, the function Θ allows us to quantify not only the violation of (3.39) but also the distance of (x,λ) to the primal-dual solution set. The main tool in this direction is a characterization of error bounds in terms of certain Lipschitz-type properties such as those given in Lemma 3.55. This equivalence has appeared in various forms in the literature [57, 76, 122], albeit mostly in a finite-dimensional setting. The following result shows that the equivalence holds for general variational problems of the form (V).

Theorem 3.56 (Characterization of local error bounds)**.** *Let $(\bar{x},\bar{\lambda}) \in X \times H$ be a KKT point of (V), with the feasible set given by* (3.38)*. Then the following are equivalent:*

(a) *There are a neighborhood U of $\bar{x}$ and $c > 0$ such that, for all $\theta = (\alpha,\beta) \in X^* \times H$ close to* $(0,0)$*, any solution $(x_\theta,\lambda_\theta) \in U \times H$ of the perturbed KKT system*

$$\mathcal{L}(x,\lambda) = \alpha, \quad \lambda \in \mathcal{N}_{\mathcal{K}}(G(x) - \beta) \tag{3.41}$$

satisfies the estimate $\|x_\theta - \bar{x}\|_X + \operatorname{dist}\big(\lambda_\theta, \Lambda(\bar{x})\big) \le c\|\theta\|_{X^*\times H}$.

(b) *There are a neighborhood U of $\bar{x}$ and $c > 0$ such that, for all $(x,\lambda) \in U \times H$ with $\Theta(x,\lambda)$ sufficiently small, we have the error bound*

$$\|x - \bar{x}\|_X + \operatorname{dist}\big(\lambda, \Lambda(\bar{x})\big) \le c\Theta(x,\lambda).$$

Proof. (b) $\Rightarrow$ (a): Let $\theta = (\alpha,\beta) \in X^* \times H$. It is an easy consequence of Lemma 2.11 that the mapping $y \mapsto y - P_{\mathcal{K}}(y + \lambda_\theta)$ is nonexpansive. Hence, we obtain

$$\|G(x_\theta) - P_{\mathcal{K}}(G(x_\theta)+\lambda_\theta)\|_H \le \|\beta\|_H + \|G(x_\theta) - \beta - P_{\mathcal{K}}(G(x_\theta) - \beta + \lambda_\theta)\|_H.$$

Since $\lambda_\theta \in \mathcal{N}_{\mathcal{K}}(G(x_\theta) - \beta)$, the last term is equal to zero, and we obtain $\Theta(x_\theta,\lambda_\theta) \le \|\alpha\|_{X^*} + \|\beta\|_H = \|\theta\|_{X^*\times H}$. Choosing $\theta = (\alpha,\beta)$ sufficiently close to 0, we see that $\Theta(x_\theta,\lambda_\theta)$ becomes arbitrarily small. Hence, we can apply (b) and obtain

$$\|x_\theta - \bar{x}\|_X + \operatorname{dist}\big(\lambda_\theta, \Lambda(\bar{x})\big) \le c\Theta(x_\theta,\lambda_\theta) \le c\|\theta\|_{X^*\times H}.$$

(a) ⇒ (b): Shrinking U if necessary, we may assume that $\|G'(x)^*\|_{L(H,X^*)} \le c_1$ for all $x \in U$ with some constant $c_1 \ge 0$. Now, let $(x, \lambda) \in U \times H$. We will use (x, λ) to construct a solution of the perturbed KKT system (3.41). Set $\delta := \Theta(x, \lambda)$ and define

$$y_\theta := P_{\mathcal{K}}(G(x) + \lambda), \quad \lambda_\theta := G(x) + \lambda - y_\theta.$$

Let $\alpha := \mathcal{L}(x, \lambda_\theta)$ and $\beta := G(x) - y_\theta$. Then $\lambda_\theta \in \mathcal{N}_{\mathcal{K}}(y_\theta)$ and, hence, (x, λ_θ) solves the perturbed KKT system corresponding to $\theta := (\alpha, \beta)$. Moreover, we have $\|\beta\|_H = \|y_\theta - G(x)\|_H = \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H \le \delta$ and $\|\lambda_\theta - \lambda\|_H = \|\beta\|_H \le \delta$. This implies

$$\|\theta\|_{X^* \times H} = \|\mathcal{L}(x, \lambda_\theta)\|_{X^*} + \|\beta\|_H \le \|\mathcal{L}(x, \lambda)\|_{X^*} + (c_1 + 1)\|\beta\|_H \le (c_1 + 2)\delta.$$

Hence, if $\delta = \Theta(x, \lambda)$ is small enough, then θ becomes arbitrarily close to 0. We can therefore apply (a) to (x, λ_θ) and obtain $\|x - \bar{x}\|_X + \operatorname{dist}\big(\lambda_\theta, \Lambda(\bar{x})\big) \le c\|\theta\|_{X^* \times H} \le c(c_1 + 2)\delta$. But $\|\lambda_\theta - \lambda\|_H \le \delta$ and, hence, $\operatorname{dist}\big(\lambda_\theta, \Lambda(\bar{x})\big) \ge \operatorname{dist}\big(\lambda, \Lambda(\bar{x})\big) - \delta$ by the nonexpansiveness of the distance function. This finally yields

$$\|x - \bar{x}\|_X + \operatorname{dist}\big(\lambda, \Lambda(\bar{x})\big) \le \big[c(c_1 + 2) + 1\big]\delta,$$

and the proof is complete. □

Let us stress that the distance estimate provided by the above theorem holds if x is close to $\bar{x}$; in particular, no assumption on the proximity of λ to $\Lambda(\bar{x})$ is necessary. We also remark that (a) does not make any assertion about the existence of solutions to the perturbed KKT conditions (3.41). These may have solutions for some but not all θ.

Before we give some corollaries of the above theorem, let us remark that the function Θ is locally Lipschitz-continuous with respect to x, and globally so with respect to λ. Hence, if the error bound from Theorem 3.56 holds, then we actually have the "double" error bound

$$c_1\Theta(x, \lambda) \le \|x - \bar{x}\|_X + \operatorname{dist}\big(\lambda, \Lambda(\bar{x})\big) \le c_2\Theta(x, \lambda) \tag{3.42}$$

with suitable constants $c_1, c_2 > 0$. As before, this holds for all $(x, \lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small.

Let us now give a direct corollary of Theorem 3.56 and Lemma 3.55.

Corollary 3.57. *Let $(\bar{x}, \bar{\lambda})$ be a KKT point of (V), with the feasible set given by* (3.38), *such that SOSC and SRC hold in $(\bar{x}, \bar{\lambda})$. Then $\Lambda(\bar{x}) = \{\bar{\lambda}\}$ and there are $c_1, c_2 > 0$ such that, for all $(x, \lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small, we have*

$$c_1\Theta(x, \lambda) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H \le c_2\Theta(x, \lambda),$$

where $\Theta : X \times H \to \mathbb{R}$ is the residual function given by (3.40).

Proof. The uniqueness follows from Proposition 3.7. The error bound follows from Theorem 3.56, Lemma 3.55, and the arguments before (3.42). □

Let us now turn to the case where the feasible set has the form

$$\Phi := \{x \in X : x \in C,\ G(x) \in \mathcal{K}\}, \tag{3.43}$$

with $C \subseteq X$ a nonempty closed convex set. As before, $G : X \to H$, H is a real Hilbert space, and $\mathcal{K} \subseteq H$ a nonempty closed convex set. In this case, the residual function Θ takes on a slightly more general form which includes (3.40) as a special case.

Corollary 3.58. *Let $(\bar{x}, \bar{\lambda})$ be a KKT point of (V), with X a real Hilbert space and the feasible set given by* (3.43)*, such that SOSC and SRC hold in $(\bar{x}, \bar{\lambda})$. Then $\Lambda(\bar{x}) = \{\bar{\lambda}\}$ and there are $c_1, c_2 > 0$ such that, for all $(x, \lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small, we have*

$$c_1\Theta(x, \lambda) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H \le c_2\Theta(x, \lambda), \tag{3.44}$$

where $\Theta(x, \lambda) := \|x - P_C(x - \mathcal{L}(x, \lambda))\|_X + \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H$.

Proof. Taking into account Remarks 3.10 and 3.28, it follows that SOSC and SRC hold for the constraint system $(G(x), x) \in \mathcal{K} \times C$ in the point $\bar{x} \in X$, with the Lagrange multiplier pair $(\bar{\lambda}, \bar{\mu}) \in H \times X$, where $\bar{\mu} := -F(\bar{x}) - G'(\bar{x})^*\bar{\lambda}$. Thus, by Corollary 3.57, we obtain an error bound of the form

$$c_1\hat{\Theta}(x, \lambda, \mu) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H + \|\mu - \bar{\mu}\|_X \le c_2\hat{\Theta}(x, \lambda, \mu)$$

for all $(x, \lambda, \mu) \in X \times H \times X$ with x sufficiently close to $\bar{x}$ and $\hat{\Theta}(x, \lambda, \mu)$ sufficiently small, where

$$\hat{\Theta}(x, \lambda, \mu) := \|F(x) + G'(x)^*\lambda + \mu\|_X + \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H + \|x - P_C(x + \mu)\|_X. \tag{3.45}$$

To deduce the desired form of the error bound (3.44), let $(x, \lambda) \in X \times H$ be a point with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small. Define $\mu := -\mathcal{L}(x, \lambda) \in X$. Inserting the triple (x, λ, μ) into (3.45), the first term vanishes, and we are left with $\hat{\Theta}(x, \lambda, \mu) = \Theta(x, \lambda)$. This implies that

$$c_1\Theta(x, \lambda) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H + \|\mu - \bar{\mu}\|_X \le c_2\Theta(x, \lambda).$$

It remains to show that $\|\mu - \bar{\mu}\|_X = O(\|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H)$ if x is close to $\bar{x}$, where $\mu = -\mathcal{L}(x, \lambda)$. But this follows from the local Lipschitz-continuity of F and G'. Hence, the proof is complete. □

The above is the main practical result of this section and will be instrumental in proving asymptotic convergence results for the augmented Lagrangian method in Chapters 4 and 5.

We conclude this section by briefly discussing the case where the set $\mathcal{K}$ is *polyhedral*, i.e., it can be represented by finitely many linear equalities and inequalities.

Remark 3.59. For certain problem classes, it is possible to establish error bounds under weaker assumptions than those given above. An important example arises if the set $\mathcal{K}$ is (generalized) polyhedral, e.g., in nonlinear programming. Roughly speaking, one can use Hoffman's lemma [32, Thm. 2.200] to get the "dual part" of the error bound for free, while the primal part again follows from SOSC. As a result, one obtains a primal-dual error bound under SOSC alone, see [122], with the restriction that the Lagrange multiplier is not necessarily unique. Unsurprisingly, this result does not extend to the non-polyhedral case, which shows that additional assumptions such as SRC are inevitable.

Example 3.60. Let $X := H := \ell^2(\mathbb{R})$ be the space of square-summable real sequences. Consider the variational inequality arising from the optimization problem (P) with $f(x) := \|x\|_X^2/2$ and the constraint $G(x) \in \mathcal{K}$, where $G : X \to H$, $G(x) := (x_i/i)_{i=1}^\infty$, and $\mathcal{K}$ is the nonnegative cone in X. It is easy to see that $(\bar{x}, \bar{\lambda}) := (0, 0)$ is the unique KKT point of this problem, and that SOSC holds. Now, let $x^k := e^k/k$ and $\lambda^k := -e^k$, where $\{e^k\}$ is the sequence of unit vectors. Then

$$\Theta(x^k, \lambda^k) = \|\mathcal{L}(x^k, \lambda^k)\|_{X^*} + \|G(x^k) - P_{\mathcal{K}}(G(x^k) + \lambda^k)\|_H = k^{-2}$$

for all k, where $\mathcal{L}$ is the Lagrangian in the variational inequality sense, see (3.27). Moreover, $x^k \to \bar{x}$, but $\lambda^k \not\to \bar{\lambda}$. Hence, the local error bound (3.42) does not hold. (In particular, SRC cannot hold, even though the Lagrange multiplier is actually unique.) A slightly different example is obtained by setting $\hat{x}^k := e^k/k^2$ and $\hat{\lambda}^k := -e^k/k$. In this case, $(\hat{x}^k, \hat{\lambda}^k) \to (\bar{x}, \bar{\lambda})$, but an easy calculation shows that

$$\Theta(\hat{x}^k, \hat{\lambda}^k) = k^{-3} \quad \text{and} \quad \|\hat{x}^k - \bar{x}\|_X + \|\hat{\lambda}^k - \bar{\lambda}\|_H = k^{-2} + k^{-1}.$$

In particular, the error bound is violated even if the multiplier is close to $\bar{\lambda}$.

Chapter 4

Augmented Lagrangian Methods in Constrained Optimization

This chapter is dedicated to a thorough discussion of the augmented Lagrangian method (ALM) for constrained minimization problems of the form discussed in Chapter 3. More specifically, we deal with a problem of the form

$$(P) \qquad \underset{x \in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in K, \tag{4.1}$$

where, as before, X, Y are real Banach spaces, $f : X \to \mathbb{R}$ and $G : X \to Y$ are continuously differentiable functions, and $C \subseteq X$ as well as $K \subseteq Y$ are nonempty closed convex sets. To facilitate the application of the augmented Lagrangian technique, we assume that $i : Y \hookrightarrow H$ densely for some real Hilbert space H. This implies that we are working in the *Gel'fand triple* framework

$$Y \overset{i}{\hookrightarrow} H \cong H^* \overset{i^*}{\hookrightarrow} Y^*. \tag{4.2}$$

Furthermore, we assume that there is a closed convex set $\mathcal{K} \subseteq H$ such that $i^{-1}(\mathcal{K}) = K$. This allows us to interpret the constraint $G(x) \in K$ equivalently as $G(x) \in \mathcal{K}$. Note that we will usually suppress the embedding for the sake of brevity.

It should be stressed that the above framework is extremely general, and the resulting augmented Lagrangian method therefore covers a very broad spectrum of applications. Moreover, many prominent problem classes can be recovered as special cases of (P), and they are thus implicitly covered by our analysis. For many of these problem classes, there is existing literature on augmented Lagrangian techniques, and the analysis in this chapter subsumes and generalizes most of these approaches:

- **Nonlinear programming.** This is the historical origin of the augmented Lagrangian technique. Indeed, the algorithm goes back to the seminal works by Hestenes [100] and Powell [178], and in its early days it was commonly referred to as the *method of multipliers*. The technique was further developed by many authors in the later parts of the 20th century, including Rockafellar [187–189], Bertsekas [24], and Conn, Gould, and Toint

[45–47], who created the LANCELOT software package. The algorithm was rediscovered by Andreani, Birgin, Martínez, and co-authors in [5, 6, 26, 27], a series of publications which culminated in the book [28] and the ALGENCAN software package.

In today's nonlinear programming landscape, algorithms such as interior point methods [81, 91] or sequential quadratic programming [91, 122] are often preferred to methods of augmented Lagrangian type, mainly due to their fast local convergence characteristics. In contrast, the augmented Lagrangian method possesses very strong global convergence properties, and it has been found to work rather well on degenerate problem classes such as problems with complementarity constraints [124]. A state-of-the-art local convergence analysis of the ALM for nonlinear programming is given in [74]. More discussion on nonlinear programming in general, and on the corresponding algorithms, can be found in [24, 25, 48, 172], and in the encyclopedia [80].

• **Function space optimization.** One of the main motivations for the generalization of augmented Lagrangian methods to the level of generality represented by (P) is the advent of function space optimization problems. Some early references in this context include [20, 22, 114–117, 217], and the book [82]. Most of these publications are restricted to very specific problem settings such as convex optimization problems or finite-dimensional constraints. In [23, 118], an augmented Lagrangian-type penalty scheme was proposed, in combination with a semismooth Newton method, for the solution of state-constrained optimal control problems. The resulting method came to be known as *Moreau–Yosida regularization*; it was further developed in [101, 103], and it is today considered a standard approach for state-constrained optimal control [109, 119, 214]. Some other techniques for such problems include Lavrentiev regularization [108, 161], interior point methods [151, 200], and the so-called virtual control approach [150], which is related to the augmented Lagrangian technique [149].

• **Semidefinite programming.** Another notable problem class which occurs as a byproduct of the general convergence theory in this chapter is the special case of semidefinite programming or, more generally, C^2-cone reducible programming (see Section 4.3.3). Methods of augmented Lagrangian type are quite popular for these problems [145–147, 208, 219] and for related problem classes such as second-order cone programming [156, 157]. The theoretical framework in this chapter, in combination with a recent stability analysis of the aforementioned problem classes [57], allows us to strengthen the known local convergence results for the augmented Lagrangian method which can be found in the literature.

The purpose of this chapter is to develop the augmented Lagrangian method for a general problem of the form (P), thereby subsuming the above problem classes. The core of the theory is tailored towards function space optimization, and the prototypical applications to be kept in mind are state-constrained optimal control, obstacle problems [192], the Signorini problem [12, 88], and similar examples. More details along with numerical implementations will be described in Chapter 7. The theory presented below can also be seen as a precursor to the augmented Lagrangian method for variational and quasi-variational inequalities, see Chapters 5 and 6.

The results in this chapter are essentially based on the publications [133,135,136], with a substantial amount of modifications aimed at making the theory simpler, more general, and more readily applicable. The structure of the chapter is as follows. In Section 4.1, we provide some background on the augmented Lagrangian method. Section 4.1.1 is dedicated the original method of multipliers by Hestenes and Powell. In Section 4.1.2, we demonstrate how a slack variable approach can be used to formally deduce the augmented Lagrangian method for a general problem of the form (P), and in Section 4.1.3 we give the resulting algorithm along with some basic properties.

Section 4.2 contains a thorough convergence analysis for the ALM from a global point of view. In Section 4.2.1, we begin by giving some sufficient conditions for the existence of penalized solutions, and in Section 4.2.2 we establish some rather simple convergence results under the assumption that the penalized subproblems are solved in an (essentially) global sense. In Section 4.2.3, we state convergence results in terms of the first-order necessary (KKT) conditions. The results in Section 4.2 can be seen as generalizations of various works in the literature, including [103, 114–117, 119]. Some related results can also be found in [137, 138].

Finally, Section 4.3 is dedicated to the local convergence of the augmented Lagrangian algorithm. In Section 4.3.1, we analyze the existence and behavior of local minimizers of the augmented Lagrange function, and in Section 4.3.2 we provide a quantitative analysis which yields primal-dual rate of convergence results as well as the boundedness of the sequence of penalty parameters. We conclude in Section 4.3.3 by demonstrating how the results can be specialized for the class of C^2-cone reducible optimization problems, which encompasses semidefinite and second-order cone programming. The results in Section 4.3 can be seen as generalizations of various findings contained in [26, 156, 157, 208]. Some related results can also be found in [137].

4.1 Motivation and Statement of the Algorithm

This preliminary section provides some background on the augmented Lagrangian method, including a historical overview and a formal statement of the method for a general problem of the form (P).

4.1.1 The Original Method of Multipliers

In its initial form, the method of multipliers is an algorithm for the solution of equality-constrained minimization problems in finite dimensions. Here, we present this original method in a slightly more general framework. Consider an equality-constrained optimization problem of the form

$$\underset{x\in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad h(x) = 0, \tag{4.3}$$

where $f : X \to \mathbb{R}$, $C \subseteq X$ is a closed convex set, and $h : X \to H$. We assume that X is a real Banach space and H is a real Hilbert space. In the special case of the original method of multipliers, we have $X := \mathbb{R}^n$, $H := \mathbb{R}^m$ with $m, n \in \mathbb{N}$, and $C := X$.

The basic idea is to tackle (4.3) by combining elements of Lagrangian theory with a penalty-type scheme. Recall that the Lagrangian of the problem takes on the form $\mathcal{L}(x,\lambda) = f(x) + \big(\lambda, h(x)\big)$. By adding a positive multiple of $\|h(x)\|_H^2$, we penalize the violation of the equality constraint, thus ending up with the *augmented Lagrangian*

$$\mathcal{L}_\rho(x,\lambda) := f(x) + \big(\lambda, h(x)\big) + \frac{\rho}{2}\|h(x)\|_H^2. \tag{4.4}$$

From an algorithmic perspective, we now proceed as follows. Given a penalty parameter ρ_k and a current estimate λ^k of the Lagrange multiplier, we compute x^{k+1} as a minimizer (or approximate minimizer) of (4.4) on C so that, ideally, x^{k+1} is close to feasibility (if ρ_k is large) and close to being a minimizer of the Lagrangian $\mathcal{L}(\cdot,\lambda^k)$. Let us assume, for the moment, that the functions f and h are continuously differentiable, and that x^{k+1} is an exact minimizer of $\mathcal{L}_{\rho_k}(\cdot,\lambda^k)$ on C. Then Lemma 3.1 yields the inclusion

$$\mathcal{N}_C(x^{k+1}) \ni -\mathcal{L}'_{\rho_k}(x^{k+1},\lambda^k) = -f'(x^{k+1}) - h'(x^{k+1})^*(\lambda^k + \rho_k h(x^{k+1})).$$

This immediately suggests $\lambda^{k+1} := \lambda^k + \rho_k h(x^{k+1})$ as the new estimate of the Lagrange multiplier, which is often called the *Hestenes–Powell multiplier update.*

After the above procedure is completed, the penalty parameter is updated based on a heuristic test. The most common option is to keep ρ_k if the constraint violation has decreased sufficiently, and to increase it otherwise. We thus end up with the following overall algorithm.

Algorithm 4.1 (Original method of multipliers)**.** Let $(x^0,\lambda^0) \in X \times H$, $\rho_0 > 0$, let $\gamma > 1$, $\tau \in (0,1)$, and set $k := 0$.

Step 1. If (x^k,λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Compute an approximate solution x^{k+1} of the problem

$$\underset{x\in X}{\text{minimize}}\ \mathcal{L}_{\rho_k}(x,\lambda^k). \tag{4.5}$$

Step 3. Update the vector of multipliers to $\lambda^{k+1} := \lambda^k + \rho_k h(x^{k+1})$.

Step 4. If $\|h(x^{k+1})\|_H \le \tau\|h(x^k)\|_H$ holds, set $\rho_{k+1} := \rho_k$. Otherwise, set $\rho_{k+1} := \gamma\rho_k$.

Step 5. Set $k \leftarrow k+1$ and go to Step 1.

4.1.2 Inequality Constraints and Slack Variables

Having established the classical multiplier method for equality-constrained problems, we now outline how the algorithm can be extended to the inequality-constrained case. To this end, we consider an optimization problem of the form (P), that is,

$$\underset{x\in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in K,$$

where, as before, $f : X \to \mathbb{R}$ and $G : X \to Y$ are given mappings, and $C \subseteq X$ and $K \subseteq Y$ nonempty closed convex sets. Moreover, H is a real Hilbert space with $i : Y \hookrightarrow H$ densely, and $\mathcal{K} \subseteq H$ is a closed convex set with $i^{-1}(\mathcal{K}) = K$. In this setting, we can restate (P) as the problem

$$(P_H) \qquad \underset{x \in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in \mathcal{K}. \tag{4.6}$$

We can transform this problem into an equality-constrained problem by adding an artificial variable $s \in \mathcal{K}$, also called a *slack variable*. This results in the equality-constrained problem

$$\underset{(x,s) \in C \times \mathcal{K}}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) - s = 0.$$

In the context of the equality-constrained framework (4.3) from the previous section, this essentially amounts to defining the mapping $h : X \times H \to H$, $h(x, s) := G(x) - s$. The new problem is now an equality-constrained optimization problem on the space $X \times H$, and its augmented Lagrangian in the sense of (4.4) is given by

$$\mathcal{L}_\rho^s(x, s, \lambda) = f(x) + \big(\lambda, h(x, s)\big) + \frac{\rho}{2}\|h(x, s)\|_H^2.$$

In order to transform the augmented Lagrangian into a form where s is eliminated, observe that we can rewrite $\mathcal{L}_\rho^s$ as

$$\mathcal{L}_\rho^s(x, s, \lambda) = f(x) + \frac{\rho}{2}\left\| G(x) + \frac{\lambda}{\rho} - s \right\|_H^2 - \frac{\|\lambda\|_H^2}{2\rho}. \tag{4.7}$$

Taking into account the constraint $s \in \mathcal{K}$, we can now minimize this formula with respect to s for each fixed $x \in X$. Since s occurs only in the middle term, the result involves, by definition, the squared distance function $d_{\mathcal{K}}^2$.

Definition 4.2 (Augmented Lagrange function)**.** For $\rho > 0$, the *augmented Lagrange function* or *augmented Lagrangian* of (P) is the function

$$\mathcal{L}_\rho : X \times H \to \mathbb{R}, \quad \mathcal{L}_\rho(x, \lambda) := f(x) + \frac{\rho}{2} d_{\mathcal{K}}^2\left(G(x) + \frac{\lambda}{\rho} \right) - \frac{\|\lambda\|_H^2}{2\rho}. \tag{4.8}$$

Before discussing some other observations and consequences of the slack variable approach, we first give some general properties of the augmented Lagrangian.

Proposition 4.3. *Let $\mathcal{L}_\rho : X \times H \to \mathbb{R}$ be the augmented Lagrangian* (4.8)*. Then:*

(a) *$\mathcal{L}_\rho$ is concave and continuously differentiable with respect to λ.*

(b) *If f is convex and G is $\mathcal{K}_\infty$-concave, then $\mathcal{L}_\rho$ is convex with respect to x.*

(c) *If f and G are continuously differentiable, then $\mathcal{L}_\rho$ is so with respect to x.*

(d) *If $x \in X$ is a feasible point, then $\mathcal{L}_\rho(x, \lambda) \le f(x)$ for all $x \in X$ and $\lambda \in H$.*

Proof. (a): The concavity follows from the fact that $\mathcal{L}_\rho(x, \cdot)$ is an infimum of affine functions, and the continuous differentiability follows from that of $d_{\mathcal{K}}^2$.

(b): This follows from Theorem 2.50.

(c): This follows again from the continuous differentiability of $d_{\mathcal{K}}^2$.

(d): If $G(x) \in \mathcal{K}$, then $d_{\mathcal{K}}(G(x) + \lambda/\rho) \le \|\lambda\|_H/\rho$ by the nonexpansiveness of the distance function. Hence, $\mathcal{L}_\rho(x, \lambda) \le f(x) + (\rho/2)\|\lambda\|_H^2/\rho^2 - \|\lambda\|_H^2/(2\rho) = f(x)$. □

Let us close this section by mentioning some byproducts of the slack variable approach. For fixed λ and ρ, the minimizing value of s in (4.7) is given by $\bar{s}(x) := P_{\mathcal{K}}(G(x) + \lambda/\rho)$. It follows that

$$h(x, \bar{s}(x)) = G(x) - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho}\right). \tag{4.9}$$

The above expression will play a certain role later on. Recall that, in the original method of multipliers (Algorithm 4.1), the norm of the equality constraint was used to determine whether the penalty parameter ρ_k should be increased after a given iteration. The above calculations suggest that (4.9) should be used to control ρ_k in the general case.

Another byproduct of the slack variable technique is a natural candidate for the Lagrange multiplier update. Assume that $\lambda^k \in H$ is a given estimate of the Lagrange multiplier of (P_H), that $\rho_k > 0$, and x^{k+1} is the next primal iterate (typically, some kind of minimizer of $\mathcal{L}_{\rho_k}(\cdot, \lambda^k)$). Taking into account the update rule in Algorithm 4.1, the next dual iterate is given by

$$\lambda^{k+1} = \lambda^k + \rho_k h(x^{k+1}, \bar{s}(x^{k+1})) = \rho_k \left[G(x^{k+1}) + \frac{\lambda^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{\lambda^k}{\rho_k}\right)\right].$$

This formula will play a fundamental role in the subsequent algorithms. Note that the above updating scheme can also be motivated (in the differentiable case) by looking at the stationarity condition of $\mathcal{L}_{\rho_k}(\cdot, \lambda^k)$, evaluated in x^{k+1}.

4.1.3 The Algorithm and Basic Properties

This section presents the main algorithmic framework for the remainder of the chapter. It is based on the method of multipliers from Section 4.1.1 and the slack variable transformation from Section 4.1.2, but it differs from the original multiplier method in one key aspect: the use of a safeguarded multiplier sequence. This will be the main tool to obtain much sharper convergence assertions than those which are possible for the traditional algorithm. A more detailed discussion can be found after the method below, and in Section 8.3, where we demonstrate the necessity of multiplier safeguarding.

Recall that we are dealing with a problem of the form (P), that we are working in the Gel'fand triple framework (4.2), and that $\mathcal{K} \subseteq H$ is a nonempty closed convex set with $i^{-1}(\mathcal{K}) = K$. The algorithm now proceeds by augmenting the constraint $G(x) \in \mathcal{K}$ in the space H. This means that, in a sense, we are not really attempting to solve (P) but the transformed problem (P_H). Nevertheless, we will see that many convergence properties of the augmented Lagrangian method can be stated accurately in terms of (P) (using, for instance, constraint qualifications for that problem).

For the precise specification of the method below, we will need a means of controlling the penalty parameter ρ. Motivated by (4.9), it is natural to use the function

$$V(x, \lambda, \rho) = \left\| G(x) - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho}\right) \right\|_H, \tag{4.10}$$

which can be seen as a composite measure of feasibility and complementarity at the current iterates. Using this function, the augmented Lagrangian method can be given as follows.

Algorithm 4.4 (ALM for constrained optimization)**.** Let $(x^0, \lambda^0) \in X \times H$, $\rho_0 > 0$, let $B \subseteq H$ be a nonempty bounded set, $\gamma > 1$, $\tau \in (0, 1)$, and set $k := 0$.

Step 1. If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in B$ and compute an approximate solution x^{k+1} of the problem

$$\underset{x \in C}{\text{minimize}} \ \mathcal{L}_{\rho_k}(x, w^k). \tag{4.11}$$

Step 3. Update the vector of multipliers to

$$\lambda^{k+1} := \rho_k \left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right]. \tag{4.12}$$

Step 4. Let $V_{k+1} := V(x^{k+1}, w^k, \rho_k)$ and set

$$\rho_{k+1} := \begin{cases} \rho_k, & \text{if } k = 0 \text{ or } V_{k+1} \le \tau V_k, \\ \gamma \rho_k, & \text{otherwise.} \end{cases} \tag{4.13}$$

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

Some remarks are in order. First among them is the fact that we have not specified what constitutes an "approximate solution" in Step 2. There are multiple options in this regard. For instance, we could require that x^{k+1} is an (approximate) global minimizer of $\mathcal{L}_{\rho_k}(\cdot, w^k)$. This is probably the simplest assumption from a theoretical point of view, but it is effectively restricted to problems where some form of convexity is present. On the other hand, we could also require that x^{k+1} is some kind of approximate stationary point of (4.11). This is more realistic in the nonconvex case, but it is also more intricate to deal with in theoretical terms. We will analyze both these approaches individually in the subsequent sections.

In practical terms, the augmented subproblems are typically solved by applying an appropriate generalized Newton method. The necessity for such methods stems from the fact that the augmented Lagrangian is once but in general not twice continuously differentiable with respect to x. A more detailed discussion of this problem and of the resulting Newton-type methods will be given in Chapter 7.

The second remark pertains to the sequence $\{w^k\}$, which will occasionally be referred to as the *safeguarded (Lagrange) multiplier sequence*. The presence of w^k can be seen as the distinctive feature of the algorithm, and it separates the method from traditional augmented Lagrangian schemes. Indeed, in Algorithm 4.4, we use w^k in certain places where conventional algorithms simply use λ^k. The main motivation is that w^k is always a bounded sequence (it is specifically required to be so), and this is the main ingredient to obtain sharper global convergence results. As a consequence, the above algorithm has strictly stronger convergence properties than its traditional counterpart. An actual example demonstrating this fact is somewhat involved and will be given in Section 8.3. Note that, despite the boundedness of $\{w^k\}$, the sequence $\{\lambda^k\}$ in Algorithm 4.4 can still be unbounded. The actual choice of w^k allows for a certain degree of freedom. For instance, we could always choose $w^k := 0$, thus obtaining an algorithm which is essentially a quadratic penalty method. In practice, it is usually advantageous to keep w^k as close as possible to λ^k, for instance, by choosing the set B as a simple but large bounded set, and taking

$$w^k := P_B(\lambda^k)$$

for all k. This choice has the advantage that, if the sequence $\{\lambda^k\}$ is indeed bounded and the set B is large enough, then we can expect to have $w^k = \lambda^k$ for all k. On the other hand, if $\{\lambda^k\}$ is unbounded, then the safeguarding scheme will prevent w^k from escaping to infinity.

Finally, let us remark that the penalty updating scheme in (4.13) makes a distinction between the cases $k = 0$ and $k \geq 1$. This is because the value V_0 is formally undefined since we do not have w^{-1} and ρ_{-1}. In practice, it is often beneficial to treat this initial step differently, for instance, by simply setting $w^{-1} := w^0$, $\rho_{-1} := \rho_0$, and performing the penalty update in the same way as for $k \geq 1$. In any case, the treatment of this initial step has no impact on the convergence theory.

We now prove some basic properties for the iterates generated by Algorithm 4.4. Note that the choice of x^{k+1} in Step 2 is still unspecified. Despite this, the nature of the multiplier update (4.12) allows us to prove two assertions which hold completely independently of x^{k+1}.

Lemma 4.5. *We have $\lambda^k \in \mathcal{K}_\infty^\circ$ for all k. Moreover, there is a null sequence $\{r_k\} \subseteq \mathbb{R}_+$ such that $\big(\lambda^k, y - G(x^k)\big) \leq r_k$ for all $y \in \mathcal{K}$ and $k \in \mathbb{N}$.*

Proof. Let $s^{k+1} := P_\mathcal{K}(G(x^{k+1}) + w^k/\rho_k)$. Then $\lambda^{k+1} \in \mathcal{N}_\mathcal{K}(s^{k+1})$ by Proposition 2.36, and thus $\lambda^{k+1} \in \mathcal{K}_\infty^\circ$ by Lemma 2.39. For the second assertion, observe first that

$$G(x^{k+1}) = \frac{\lambda^{k+1} - w^k}{\rho_k} + s^{k+1}. \tag{4.14}$$

Using the fact that $\lambda^{k+1} \in \mathcal{N}_\mathcal{K}(s^{k+1})$, we obtain

$$\begin{aligned}\big(\lambda^{k+1}, y - G(x^{k+1})\big) &= \left(\lambda^{k+1}, y - \frac{1}{\rho_k}(\lambda^{k+1} - w^k) - s^{k+1}\right) \\ &\leq \frac{1}{\rho_k}\Big[\big(\lambda^{k+1}, w^k\big) - \|\lambda^{k+1}\|_H^2\Big] =: r_{k+1}. \end{aligned}\tag{4.15}$$

We claim that this sequence $\{r_{k+1}\}$ satisfies $\limsup_{k\to\infty} r_{k+1} \leq 0$. This yields the desired result (by replacing r_k with $\max\{0, r_k\}$). If $\{\rho_k\}$ is bounded, then (4.13) and (4.14) imply $\|\lambda^{k+1} - w^k\|_H/\rho_k \to 0$ and therefore $\|\lambda^{k+1} - w^k\|_H \to 0$. This yields the boundedness of $\{\lambda^{k+1}\}$ in H as well as $(\lambda^{k+1}, w^k) - \|\lambda^{k+1}\|_H^2 = (\lambda^{k+1}, w^k - \lambda^{k+1}) \to 0$. Hence, $r_k \to 0$. Assume now that $\rho_k \to \infty$. Note that (4.15) is a quadratic function in λ. A simple calculation therefore shows that $r_{k+1} \leq \|w^k\|_H^2/(4\rho_k)$ and, hence, $\limsup_{k\to\infty} r_k \leq 0$. □

The first assertion of the above lemma can be interpreted as a sign property of the multiplier sequence, and the second assertion can be described roughly as a kind of "asymptotic normality" between λ^k and $G(x^k)$. Note that, by virtue of the Gel'fand triple $Y \hookrightarrow H \hookrightarrow Y^*$, this inequality also holds if we replace $\mathcal{K}$ by K and the scalar product by the duality pairing on $Y^* \times Y$. Recall furthermore that the KKT conditions of (P) postulate the existence of a Lagrange multiplier $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x})) \subseteq Y^*$. The second assertion of Lemma 4.5 is essentially an asymptotic analogue of this condition, and it will prove useful in the convergence analysis later on (see Section 4.2.3).

Example 4.6 (Cone constraints)**.** If the set $\mathcal{K}$ is a closed convex *cone*, then some parts of Algorithm 4.4 and Lemma 4.5 can be simplified. In this case, we can use the Moreau decomposition (Lemma 2.39) to restate the multiplier update (4.12) as $\lambda^{k+1} = P_{\mathcal{K}^\circ}(w^k + \rho_k G(x^{k+1}))$. Moreover, the first assertion of Lemma 4.5 simply becomes $\lambda^k \in \mathcal{K}^\circ$ for all k. This implies that $(\lambda^k, y) \leq 0$ for all $y \in \mathcal{K}$, and it is easy to see that the second assertion is then equivalent to

$$\liminf_{k\to\infty} \left\langle \lambda^k, G(x^k) \right\rangle \geq 0.$$

4.2 Global Convergence Theory

In this section, we analyze the convergence characteristics of Algorithm 4.4 from a global point of view. The main aim is to impose reasonable assumptions on the sequence $\{x^k\}$ and to then state results on weak limit points of this sequence. For the sake of generality, we will conduct dedicated analyses under varying assumptions (mainly pertaining to the manner in which the augmented subproblems are solved). In addition, many special cases such as finite-dimensional problems are discussed.

4.2.1 Existence of Penalized Solutions

In most situations, the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C. This is satisfied, in particular, if f itself is already bounded from below on C, or if, roughly speaking, the penalty parameter is sufficiently large to make $\mathcal{L}_\rho$ coercive on the infeasible set. In any case, if $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C, then the augmented subproblems necessarily admit approximate minimizers. Recall (see Section 3.1.1) that $\hat{x} \in C$ is called an ε-minimizer of a function $L : X \to \mathbb{R}$ on C if $L(\hat{x}) \leq L(x) + \varepsilon$ for all $x \in C$.

Proposition 4.7. *Let $w \in H$, $\rho > 0$, and assume that the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C. Then the following assertions hold:*

(a) *For any $\varepsilon > 0$, there is an ε-minimizer $x_\varepsilon \in C$ of $\mathcal{L}_\rho(\cdot, w)$ on C.*

(b) *If the functions f and G are continuously differentiable, then we can choose x_ε so that it additionally satisfies* $\operatorname{dist}(-\mathcal{L}'_\rho(x_\varepsilon, w), \mathcal{N}_C(x_\varepsilon)) \leq \varepsilon^{1/2}$.

Proof. The first assertion follows from the lower boundedness assumption. The second property is a consequence of Ekeland's variational principle (Proposition 3.9). □

We now discuss the existence of exact minimizers. The main proof technique is the direct method of the calculus of variations (see Proposition 2.6). For this, we need an appropriate kind of lower semicontinuity of the augmented Lagrangian. The following lemma provides two sufficient conditions for this property.

Lemma 4.8. *Assume that f is weakly sequentially lsc and G is either* (i) *continuous and $\mathcal{K}_\infty$-concave, or* (ii) *weakly sequentially continuous. Then, for each $\rho > 0$ and $w \in H$, the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is weakly sequentially lsc on X.*

Proof. Let $w \in H$ and $\rho > 0$. It suffices to verify the weak sequential lower semicontinuity of the function $h(x) := d_{\mathcal{K}}^2(G(x) + w/\rho)$. Observe that $d_{\mathcal{K}}$ is weakly sequentially lsc by Proposition 2.44. Hence, under (ii), we immediately obtain the same for h.

Consider now (i). In that case, the function h is convex (by Theorem 2.50) and continuous, thus again weakly sequentially lsc by Proposition 2.44. □

The weak sequential lower semicontinuity of the augmented Lagrangian yields the existence of penalized solutions if we assume either the weak compactness of the set C or an appropriate growth condition. We say that a function $J : X \to \mathbb{R}$ is *coercive* if $J(x^k) \to +\infty$ whenever $\{x^k\} \subseteq X$ and $\|x^k\|_X \to +\infty$.

Corollary 4.9. *Let $w \in H$, $\rho > 0$, and let one of the conditions in Lemma 4.8 be satisfied. If either* (i) *C is weakly compact, or* (ii) *X is reflexive and $\mathcal{L}_\rho(\cdot, w)$ is coercive, then the problem $\min_{x \in C} \mathcal{L}_\rho(x, w)$ admits a global minimizer.*

Clearly, a sufficient condition for the coercivity of the augmented Lagrangian is that of the objective function f. Even if this property does not hold, then it is common for $\mathcal{L}_\rho(\cdot, w)$ to be coercive if, roughly speaking, the objective function is coercive on the feasible set Φ and not too badly behaved outside of it. In that case, the penalty term in (4.8) yields the coercivity of $\mathcal{L}_\rho(\cdot, w)$ on the complement of Φ.

4.2.2 Convergence to Global Minimizers

In this section, we analyze the convergence properties of Algorithm 4.4 under the assumption that we can solve the subproblems in an (essentially) global sense. This is of course a rather restrictive requirement and can, in general, only be expected under certain convexity assumptions. However, the resulting theory is still appealing due to its simplicity. Indeed, the results below merely require some rather mild form of continuity (no differentiability), and can easily be extended to the case where the function f is extended-valued, i.e., it is allowed to take on the value $+\infty$.

Assumption 4.10 (Global minimization)**.** We assume that f and $d_{\mathcal{K}} \circ G$ are weakly sequentially lsc on C and that $x^k \in C$ for all k. Moreover, for every $x \in C$, there is a null sequence $\{\varepsilon_k\} \subseteq \mathbb{R}$ such that $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) \le \mathcal{L}_{\rho_k}(x, w^k) + \varepsilon_{k+1}$ for all k.

Some remarks are due. Recall that, for convex functions, weak sequential lower semicontinuity is implied by ordinary continuity (see Proposition 2.44). Thus, if f is a continuous convex function, then f is weakly sequentially lsc.

A similar comment applies to the weak sequential lower semicontinuity of the function $d_{\mathcal{K}} \circ G$. Indeed, there are two rather general situations in which this condition is satisfied: if G is weakly sequentially continuous, then $d_{\mathcal{K}} \circ G$ is weakly sequentially lsc since $d_{\mathcal{K}}$ is so by Proposition 2.44. On the other hand, if G is continuous and $\mathcal{K}_\infty$-concave in the sense of Definition 2.48, then $d_{\mathcal{K}} \circ G$ is a continuous convex function (by Theorem 2.50) and thus again weakly sequentially lsc. Let us also remark that, if G is continuous and affine, then both the above cases apply.

Finally, another salient feature of Assumption 4.10 is the dependence of the sequence $\{\varepsilon_k\}$ on the comparison point $x \in C$. The motivation behind this is that, if (P) is a smooth convex problem and the point x^{k+1} is "nearly stationary" in the sense that $\operatorname{dist}(-\mathcal{L}'_{\rho_k}(x^{k+1}, w^k), \mathcal{N}_C(x^{k+1})) \le \delta$ for some (small) $\delta > 0$, then, by convexity, we obtain an estimate of the form

$$\begin{aligned} \mathcal{L}_{\rho_k}(x, w^k) &\ge \mathcal{L}_{\rho_k}(x^{k+1}, w^k) + \mathcal{L}'_{\rho_k}(x^{k+1}, w^k)(x - x^{k+1}) \\ &\ge \mathcal{L}_{\rho_k}(x^{k+1}, w^k) - \delta \|x^{k+1} - x\|_X. \end{aligned}$$

This suggests that we should allow the sequence $\{\varepsilon_k\}$ in Assumption 4.10 to depend on the point x. In any case, the stated assumption is satisfied automatically if x^{k+1} is a global ε_{k+1}-minimizer of $\mathcal{L}_{\rho_k}(\cdot, w^k)$ for some null sequence $\{\varepsilon_k\}$.

We now turn to the convergence analysis of Algorithm 4.4 under Assumption 4.10. The theory is divided into separate analyses of feasibility and optimality. Since the augmented Lagrangian method is, at its heart, a penalty-type algorithm, the attainment of feasibility is particularly important for the success of the algorithm. A closer look at the definition of the augmented Lagrangian suggests that, if ρ is large, then the minimization of $\mathcal{L}_\rho$ essentially reduces to that of the infeasibility measure $d_{\mathcal{K}}^2(G(x))$. Hence, we can expect (weak) limit points of the sequence $\{x^k\}$ to be minimizers of this auxiliary function, which means that, roughly speaking, these points are "as feasible as possible." A precise statement of this assertion can be found in the following lemma.

Lemma 4.11. *Let $\{x^k\}$ be generated by Algorithm 4.4, let Assumption 4.10 hold, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x}$ is a global minimizer of the function $d_{\mathcal{K}} \circ G$ on C. In particular, if the feasible set of (P) is nonempty, then $\bar{x}$ is feasible.*

Proof. Note that C is weakly sequentially closed by Corollary 2.20, hence $\bar{x} \in C$. To show the desired minimization property, we first consider the case where $\{\rho_k\}$ remains bounded. Then (4.13) and the definition of V yield

$$d_{\mathcal{K}}(G(x^{k+1})) \le \left\| G(x^{k+1}) - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right\|_H \to 0.$$

The weak sequential lower semicontinuity of $d_{\mathcal{K}} \circ G$ therefore implies that $d_{\mathcal{K}}(G(\bar{x})) = 0$. Hence, $\bar{x}$ is feasible and there is nothing to prove.

Assume now that $\rho_k \to \infty$, and let $x^{k+1} \rightharpoonup_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$. Let $x \in C$ be an arbitrary point and let $\{\varepsilon_k\}$ be the corresponding null sequence from Assumption 4.10.

Then $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) \le \mathcal{L}_{\rho_k}(x, w^k) + \varepsilon_{k+1}$ for all k, which implies

$$f(x^{k+1}) + \frac{\rho_k}{2} d_{\mathcal{K}}^2\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) \le f(x) + \frac{\rho_k}{2} d_{\mathcal{K}}^2\left(G(x) + \frac{w^k}{\rho_k}\right) + \varepsilon_{k+1}. \tag{4.16}$$

Observe now that $\{f(x^{k+1})\}_{k\in I}$ is bounded from below since f is weakly sequentially lsc. Hence, dividing both sides by ρ_k and taking the lim inf for $k \in I$, we obtain

$$\liminf_{k\in I} d_{\mathcal{K}}^2\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) \le \liminf_{k\in I} d_{\mathcal{K}}^2\left(G(x) + \frac{w^k}{\rho_k}\right) = d_{\mathcal{K}}^2(G(x)).$$

Using the fact that $w^k/\rho_k \to 0$ and that $d_{\mathcal{K}} \circ G$ is weakly sequentially lsc, it follows that the left-hand side is greater than or equal to $d_{\mathcal{K}}^2(G(\bar{x}))$. This completes the proof. □

The idea to link the feasibility properties of the iterates $\{x^k\}$ to the minimization of the infeasibility measure $d_{\mathcal{K}}^2 \circ G$ is a recurring theme in the convergence theory of augmented Lagrangian methods. In fact, we will encounter similar statements in the context of stationary points and (quasi-)variational inequalities.

Let us now turn to the optimality part.

Theorem 4.12. *Let $\{x^k\}$ be generated by Algorithm 4.4, let Assumption 4.10 hold, and assume that the feasible set of (P) is nonempty. Then $\limsup_{k\to\infty} f(x^{k+1}) \le f(x)$ for every $x \in \Phi$. Moreover, every weak limit point of $\{x^k\}$ is a global solution of (P).*

Proof. Let $x \in X$ be an arbitrary feasible point, and let $\{\varepsilon_k\}$ be the sequence from Assumption 4.10. By Proposition 4.3 (d), we have

$$f(x^{k+1}) + \frac{\rho_k}{2} d_{\mathcal{K}}^2\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) - \frac{\|w^k\|_H^2}{2\rho_k} \le \mathcal{L}_{\rho_k}(x, w^k) + \varepsilon_{k+1} \le f(x) + \varepsilon_{k+1}. \tag{4.17}$$

Clearly, if $\rho_k \to \infty$, then $\|w^k\|_H^2/(2\rho_k) \to 0$. In this case, the nonnegativity of $d_{\mathcal{K}}$ and the fact that $\varepsilon_k \to 0$ imply $\limsup_{k\to\infty} f(x^{k+1}) \le f(x)$.

Consider now the case where $\{\rho_k\}$ remains bounded. The triangle inequality yields

$$d_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) \le \left\|\frac{w^k}{\rho_k}\right\|_H + \left\|G(x^{k+1}) - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right)\right\|_H.$$

The last term converges to zero by the penalty updating scheme (4.13). Using the boundedness of $\{w^k\}$ and squaring on both sides, it is easy to deduce that

$$\limsup_{k\to\infty}\left[d_{\mathcal{K}}^2\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) - \left\|\frac{w^k}{\rho_k}\right\|_H^2\right] \le 0.$$

Since $\{\rho_k\}$ is bounded, it follows again from (4.17) that $\limsup_{k\to\infty} f(x^{k+1}) \le f(x)$.

Finally, let $x^{k+1} \rightharpoonup_I \bar{x}$ for some (infinite) subset $I \subseteq \mathbb{N}$. Then $\bar{x}$ is feasible by Lemma 4.11, and the weak sequential lower semicontinuity of f implies that $f(\bar{x}) \le \liminf_{k\in I} f(x^{k+1}) \le f(x)$ for every $x \in \Phi$. Hence, $\bar{x}$ is a global solution of (P). □

If the problem is convex with strongly convex objective, then it is possible to considerably strengthen the results of the previous theorem. Recall that, in this case, the weak sequential lower semicontinuity of f from Assumption 4.10 is implied by (ordinary) continuity. Recall also that a sufficient condition for the convexity of the feasible set Φ is the $\mathcal{K}_\infty$-concavity of G, see Section 2.2.3. Moreover, if G is $\mathcal{K}_\infty$-concave, then the distance function $d_\mathcal{K} \circ G$ is convex, and thus the weak sequential lower semicontinuity from Assumption 4.10 is implied by (ordinary) continuity of G.

Corollary 4.13. *Let $\{x^k\}$ be generated by Algorithm 4.4 and let Assumption 4.10 hold. Assume that X is reflexive, f is strongly convex on C, and the feasible set of (P) is nonempty and convex. Then $\{x^k\}$ converges strongly to the unique solution of (P).*

Proof. Note that Assumption 4.10 implies that the feasible set Φ is closed. Since f is strongly convex, the existence and uniqueness of the solution $\bar{x}$ follows from standard arguments. Now, denoting by $c > 0$ the modulus of convexity of f, it follows that

$$\frac{c}{8}\|x^{k+1} - \bar{x}\|_X^2 \le \frac{f(x^{k+1}) + f(\bar{x})}{2} - f\left(\frac{x^{k+1} + \bar{x}}{2}\right) \tag{4.18}$$

for all k. Moreover, by Theorem 4.12, we have $\limsup_{k\to\infty} f(x^{k+1}) \le f(\bar{x})$. Taking into account that f is bounded from below, it follows from (4.18) that $\{x^k\}$ is bounded. Since X is reflexive and every weak limit point of $\{x^k\}$ is a solution of (P) by Theorem 4.12, it follows that $x^k \rightharpoonup \bar{x}$. Since f is weakly sequentially lsc and $\limsup_{k\to\infty} f(x^{k+1}) \le f(\bar{x})$, we conclude that $f(x^{k+1}) \to f(\bar{x})$. Moreover, since $(x^k + \bar{x})/2 \rightharpoonup \bar{x}$, we also have $f(\bar{x}) \le \liminf_{k\to\infty} f\big((x^k + \bar{x})/2\big)$. Hence, (4.18) implies that $\|x^{k+1} - \bar{x}\|_X \to 0$. □

4.2.3 Stationarity of Limit Points

The theory on global minimization which we have developed in the preceding section is certainly appealing from a theoretical point of view. However, the practical relevance of the corresponding results is essentially limited to problems where some form of convexity is present. It therefore seems natural to conduct a dedicated analysis for the augmented Lagrangian method which, instead of global minimization, takes into account the optimality and stationary concepts from Section 3.1.

The present section is dedicated to precisely this approach. To that end, we assume that the functions defining the optimization problem are continuously differentiable and that we are able to compute local minimizers or stationary points of the subproblems (4.11) which occur in the algorithm. Note that the first-order optimality conditions of these problems (compare with Lemma 3.1) are given by

$$-\mathcal{L}'_{\rho_k}(x, w^k) \in \mathcal{N}_C(x).$$

Similarly to the previous section, we will allow for certain inexactness terms. A natural way of doing this is by considering the inexact first-order optimality condition

$$\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x, w^k) \in \mathcal{N}_C(x),$$

where $\varepsilon^{k+1} \in X^*$ is an error term. For $k \to \infty$, the degree of inexactness should vanish in the sense that $\varepsilon^k \to 0$. Hence, we arrive at the following assumption.

Assumption 4.14 (Convergence to KKT points)**.** We assume that

(i) f and G are continuously differentiable on X,

(ii) the derivative f' is bounded and pseudomonotone,

(iii) G and G' are completely continuous on C, and

(iv) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Recall that $\mathcal{L}_{\rho_k}$ is continuously differentiable by Proposition 4.3. The derivative $\mathcal{L}'_{\rho_k}$ (with respect to x) can be calculated by applying the chain rule together with a standard projection theorem such as Lemma 2.43. One obtains

$$\mathcal{L}'_{\rho_k}(x, w^k) = f'(x) + \rho_k G'(x)^* \left[G(x) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x) + \frac{w^k}{\rho_k} \right) \right] \tag{4.19}$$

and, in particular, $\mathcal{L}'_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}'(x^{k+1}, \lambda^{k+1})$.

As in the previous section, we treat the questions of feasibility and optimality in a separate manner. For the feasibility part, we relate the augmented Lagrangian to the infeasibility measure $d_{\mathcal{K}}^2 \circ G$.

Lemma 4.15. *Let $\{x^k\}$ be generated by Algorithm 4.4 under Assumption 4.14, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x}$ is a stationary point of the problem* $\min_{x \in C} d_{\mathcal{K}}^2(G(x))$.

Proof. Let $x^{k+1} \rightharpoonup_I \bar{x}$ for some index set $I \subseteq \mathbb{N}$. Observe that $\bar{x} \in C$ by Corollary 2.20. If $\{\rho_k\}$ is bounded, then we can argue as in Lemma 4.11 to see that $\bar{x}$ is feasible, and there is nothing to prove. If $\rho_k \to \infty$, then Assumption 4.14 implies that

$$\varepsilon^{k+1} - f'(x^{k+1}) - G'(x^{k+1})^* \lambda^{k+1} \in \mathcal{N}_C(x^{k+1})$$

for all $k \in \mathbb{N}$. We now divide this inclusion by ρ_k, use the definition of λ^{k+1} and the fact that $\mathcal{N}_C(x^{k+1})$ is a cone. It follows that

$$\frac{\varepsilon^{k+1} - f'(x^{k+1})}{\rho_k} - G'(x^{k+1})^* \left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right] \in \mathcal{N}_C(x^{k+1}).$$

We now take the limit $k \to_I \infty$, use the boundedness of $\{f'(x^{k+1})\}$ (by Assumption 4.14), and Proposition 2.40. This yields $G'(\bar{x})^*[P_{\mathcal{K}}(G(\bar{x})) - G(\bar{x})] \in \mathcal{N}_C(\bar{x})$, which is precisely the first-order optimality condition of $\min_{x \in C} d_{\mathcal{K}}^2(G(x))$. □

The above lemma indicates that weak limit points of the sequence $\{x^k\}$ have a strong tendency to be feasible points. Apart from the heuristic appeal of the result, there are several nontrivial cases where Lemma 4.15 automatically implies the feasibility of the limit point $\bar{x}$. Here, two cases in particular deserve a special mention: first, let us assume that the mapping G is $\mathcal{K}_\infty$-concave in the sense of Definition 2.48 (for instance, G could be affine). In this case, the function $d_{\mathcal{K}}^2 \circ G$ is convex by Theorem 2.50, and it follows that $\bar{x}$ is a global minimizer of this function. Hence, if the feasible set Φ is nonempty, then $\bar{x} \in \Phi$. The second interesting case arises if the point $\bar{x}$ satisfies the extended Robinson constraint qualification from Definition 3.17. In this case, the feasibility of $\bar{x}$ follows from Proposition 3.22.

We now analyze the optimality properties of limit points. Recall that $\mathcal{L}'_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}'(x^{k+1}, \lambda^{k+1})$ for all k. Hence, combining Assumption 4.14 and Lemma 4.5, we obtain the asymptotic conditions (for $k \geq 1$)

$$\varepsilon^k - \mathcal{L}'(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \langle \lambda^k, y - G(x^k) \rangle \leq r_k \ \forall y \in K. \tag{4.20}$$

Note that the second inequality also holds with K replaced by $\mathcal{K}$. This means that the primal-dual sequence $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence in the sense of Section 3.2.3. Hence, our main approach is to employ the results from that section to obtain the optimality of weak limit points of $\{x^k\}$. The main result in this direction is the following.

Theorem 4.16. *Let $\{(x^k, \lambda^k)\}$ be generated by Algorithm 4.4 under Assumption 4.14, let $x^{k+1} \rightharpoonup_I \bar{x}$ for some index set $I \subseteq \mathbb{N}$, and let $\bar{x}$ satisfy ERCQ with respect to the constraint system of (P). Then $\bar{x}$ is a stationary point of (P), the sequence $\{\lambda^{k+1}\}_{k \in I}$ is bounded in Y^*, and each of its weak-* limit points belongs to $\Lambda(\bar{x})$.*

Proof. Note that $\bar{x}$ is feasible by Lemma 4.15 and Proposition 3.22. By (4.20), the sequence $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence in the sense of Definition 3.49. The result now follows by applying Theorem 3.51, with $F := f'$. □

Observe that the sequence $\{\lambda^k\}$ is only bounded in Y^* and not necessarily in H. If the extended RCQ holds with respect to the transformed constraint $G(x) \in \mathcal{K}$ (instead of the original condition $G(x) \in K$), then the result remains true with Y^* replaced by H. However, this assumption is too restrictive for many applications, in particular those where (P) is regular (in the constraint qualification sense) with respect to the original space Y, but not with respect to the larger space H. Some examples demonstrating this fact can be found in Chapter 7.

In the context of optimality properties, it is worthwhile to briefly discuss the case of bounded penalty parameters. This is particularly interesting because any assertion made under this assumption is a *necessary* condition for the boundedness of $\{\rho_k\}$. It turns out that no constraint qualifications are needed in the bounded case, and the algorithm produces a Lagrange multiplier in H.

Corollary 4.17. *Let $\{(x^k, \lambda^k)\}$ be generated by Algorithm 4.4, let Assumption 4.14 hold, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. If $\{\rho_k\}$ remains bounded, then $\{\lambda^k\}$ has a bounded subsequence in H, and $\bar{x}$ satisfies the KKT conditions of (P) with a multiplier in H.*

Proof. By (4.20), the sequence $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence for (P). Now, let $x^{k+1} \rightharpoonup_I \bar{x}$ on some subset $I \subseteq \mathbb{N}$, and assume that $\{\rho_k\}$ remains bounded. By arguing as in the proof of Lemma 4.15, it follows that $\bar{x} \in \Phi$. Moreover, by the definition

$$\lambda^{k+1} = \rho_k \left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right]$$

of λ^{k+1}, and the boundedness of all the involved quantities, the sequence $\{\lambda^{k+1}\}_{k \in I}$ is bounded in H. Thus, this sequence admits a weak limit point in H, and this point is a Lagrange multiplier in $\bar{x}$ by (4.20). □

The above result implies that $\{\rho_k\}$ can only remain bounded if (P) admits a multiplier in H. We will revisit the boundedness of $\{\rho_k\}$ at the end of Section 4.3.2.

If the optimization problem in question is a finite-dimensional nonlinear program, then the assumptions required for convergence can be simplified considerably. In this case, items (ii) and (iii) from Assumption 4.14 are satisfied trivially, and we can use the constraint qualifications from Section 3.1.4 instead of the general conditions from Banach space optimization. In particular, ERCQ can be replaced by EMFCQ, and it is also possible to prove an analogue of Theorem 4.16 under the assumptions that the limit point $\bar{x}$ is feasible and satisfies the CPLD constraint qualification.

Theorem 4.18. *Let $X := C := \mathbb{R}^n$, $Y := H := \mathbb{R}^m$, and $K := \mathcal{K} := \mathbb{R}^m_-$ for some $m \in \mathbb{N}$. Let $\{x^k\}$ be generated by Algorithm 4.4 under Assumption 4.14, and $\bar{x}$ a limit point of $\{x^k\}$. Then the following assertions hold:*

(a) *If $\bar{x}$ is feasible and CPLD holds in $\bar{x}$, then $\bar{x}$ is a stationary point of (P).*

(b) *If EMFCQ holds in $\bar{x}$, then $\bar{x}$ is feasible and a stationary point of (P).*

Proof. By Lemma 4.15 and Proposition 3.31, the assumptions of (b) imply those of (a). Hence, we only need to prove (a). Let $I \subseteq \mathbb{N}$ be an index set with $x^{k+1} \to_I \bar{x}$. By Theorem 3.54, it suffices to show that

$$\mathcal{L}'(x^{k+1}, \lambda^{k+1}) \to 0 \quad \text{and} \quad \min\{-G(x^{k+1}), \lambda^{k+1}\} \to 0 \tag{4.21}$$

as $k \to_I \infty$. The first of these assertions follows from (4.20). For the second, note that $\lambda^{k+1} \geq 0$ for all k by Lemma 4.5, and that $G(x^{k+1}) \to_I G(\bar{x}) \leq 0$. Hence, we need to show that $\lambda_i^{k+1} \to 0$ whenever $G_i(\bar{x}) < 0$ for some i. Let i be an index with $G_i(\bar{x}) < 0$. Then $G_i(x^{k+1}) < 0$ for $k \in I$ sufficiently large. We now distinguish two cases. If $\{\rho_k\}$ remains bounded, then $\min\{-G_i(x^{k+1}), w_i^k/\rho_k\} \to 0$ by (4.13), hence $w_i^k \to 0$, and thus

$$\lambda_i^{k+1} = \max\{0, w_i^k + \rho_k G_i(x^{k+1})\} = 0 \tag{4.22}$$

for $k \in I$ sufficiently large. On the other hand, if $\rho_k \to \infty$, then (4.22) also holds eventually since $\{w_i^k\}$ is bounded and $G_i(x^{k+1}) \to_I G_i(\bar{x}) < 0$. Thus, in either case, we have $\lambda_i^{k+1} = 0$ for sufficiently large k whenever $G_i(\bar{x}) < 0$. This shows that $\min\{-G(x^{k+1}), \lambda^{k+1}\} \to 0$, and the result follows from (4.21) and Theorem 3.54. □

We now return to the general case and provide two additional results which can be useful to obtain convergence in certain special cases. First, let us consider the case of convex constraints. In this case, we can treat (P) as a variational inequality and apply the convergence theory for variational inequalities which we will deduce in Chapter 5. The resulting theorem requires neither the complete continuity of G or G' nor any constraint qualification. For the proof, the reader is referred to Theorem 5.8.

Proposition 4.19. *Let $\{x^k\}$ be generated by Algorithm 4.4, let Assumption 4.14* (i), (ii), (iv) *hold, let G be $\mathcal{K}_\infty$-concave on C, and assume that Φ is nonempty. Then every weak limit point $\bar{x}$ of $\{x^k\}$ satisfies $\bar{x} \in \Phi$ and $f'(\bar{x})d \geq 0$ for all $d \in \mathcal{T}_\Phi(\bar{x})$.*

Another special case arises if $C = X$ and the operator $G'(\bar{x})$ is surjective, where $\bar{x}$ is again a weak limit point of the sequence $\{x^k\}$. If we already know (e.g., by Proposition 4.19) that $\bar{x}$ is a stationary point of (P), then it is possible to prove the weak-* convergence of a subsequence of $\{\lambda^k\}$ under weaker assumptions than those in Theorem 4.16.

Proposition 4.20. *Let $\{x^k\}$ be generated by Algorithm 4.4 and let $x^{k+1} \rightharpoonup_I \bar{x}$ for some $I \subseteq \mathbb{N}$ and $\bar{x} \in X$. Assume that $\bar{x}$ is a stationary point of (P), that $C = X$, f' is weak-* sequentially continuous, G' is completely continuous, and that $G'(\bar{x})$ is surjective. Then $\{\lambda^{k+1}\}_{k \in I}$ converges weak-* to the unique element in $\Lambda(\bar{x})$.*

Proof. By (4.20), the sequence $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence for (P). Hence, the result follows from Proposition 3.52. □

Remark 4.21. If we know from the specific problem structure or from some other convergence result (e.g., Corollary 4.13) that the sequence $\{x^k\}$ or one of its subsequences is *strongly* convergent, then we can dispense with the pseudomonotonicity and complete continuity assumptions. In this case, the assertions of Lemma 4.15 and Theorem 4.16 remain true under Assumption 4.14 (i) and (iv) only.

4.3 Local Convergence Theory

This section is dedicated to a local convergence analysis of Algorithm 4.4. The basic situation we will consider is that (P) admits a local solution $\bar{x}$, and we will analyze conditions which provide some information on the behavior of the augmented Lagrange function in a vicinity of $\bar{x}$. In addition, we will give quantitative results on the rate of convergence of the iterates.

4.3.1 Existence of Local Minimizers

As a first step in the local convergence analysis, we consider a local minimizer of (P) and ask whether the augmented Lagrangian admits local minimizers near this point. As we shall see, the answer to this question is closely linked to the fulfillment of second-order sufficient conditions (SOSC) of the form given in Definition 3.24.

When using the second-order condition, special care needs to be taken because the embedding $Y \hookrightarrow H$ allows us to interpret the constraint in (P) either in Y or in H.

We have already seen that this makes a strong difference for constraint qualifications, and the situation for SOSC is quite similar. The second-order condition in H, for instance, requires the existence of Lagrange multipliers in H, which in itself is already a restriction. Nevertheless, this is in a sense the more "natural" second-order condition for the augmented Lagrangian method since the augmentation is performed in H. Thus, for the most part of this section (with the exception of Proposition 4.25), we will make the following assumption.

Assumption 4.22 (Local convergence)**.** There is a KKT point $(\bar{x}, \bar{\lambda}) \in X \times H$ of (P) which satisfies the SOSC from Definition 3.24 with respect to the space H.

The basic approach to the existence of local minimizers is the following. Let $r > 0$ be a sufficiently small radius, $B \subseteq H$ a bounded set, and consider, for $\rho > 0$ and $w \in B$, the "localized" problem

$$\underset{x \in X}{\text{minimize}} \; \mathcal{L}_\rho(x, w) \quad \text{subject to} \quad x \in B_r(\bar{x}) \cap C. \tag{4.23}$$

Under suitable assumptions, this problem admits minimizers (or approximate minimizers) in $B_r(\bar{x}) \cap C$. If we can now show that, for sufficiently large ρ, these minimizers actually lie in the interior of $B_r(\bar{x})$, then the spherical constraint in (4.23) is superfluous and we obtain local minimizers of $\mathcal{L}_\rho(\cdot, w)$ subject to $x \in C$.

The above property can equivalently, and more conveniently, be stated in terms of sequences. We need to show that, whenever $\{w^k\} \subseteq B$ is an arbitrary (bounded) sequence, $\rho_k \to \infty$, and, for all k, y^{k+1} is an (approximate) solution of

$$\underset{x \in X}{\text{minimize}} \; \mathcal{L}_{\rho_k}(x, w^k) \quad \text{subject to} \quad x \in B_r(\bar{x}) \cap C, \tag{4.24}$$

then $\|y^{k+1} - \bar{x}\|_X < r$ for all k sufficiently large. Indeed, we will show that any such sequence converges to $\bar{x}$, and the existence of local minimizers of the augmented Lagrangian subproblems follows directly by the above arguments.

To prove the convergence of minimizers of (4.24) to $\bar{x}$, we will make use of Corollary 3.27, which is a consequence of the second-order sufficient condition. This result guarantees the convergence $y^{k+1} \to \bar{x}$ if we are able to show that $d_{\mathcal{K}}(G(y^{k+1})) \to 0$ and $\limsup_{k \to \infty} f(y^{k+1}) \le f(\bar{x})$ as $k \to \infty$.

Lemma 4.23. *Let Assumption 4.22 hold. Then there is a radius $r > 0$ such that the following holds: whenever $\{w^k\} \subseteq H$ is a bounded sequence, $\rho_k \to \infty$, $\varepsilon_k \to 0$, and, for all k, y^{k+1} is an ε_{k+1}-minimizer of (4.24), then $y^{k+1} \to \bar{x}$.*

Proof. Let $r > 0$ be as in Corollary 3.27. Shrinking r if necessary, we may assume that f is bounded on $B_r(\bar{x})$. In particular, it follows that $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on $B_r(\bar{x})$ for all $\rho > 0$ and $w \in B$.

Now, let $\{y^{k+1}\}$ be as specified. Then the ε_{k+1}-minimality of y^{k+1} and Proposition 4.3 yield

$$f(y^{k+1}) + \frac{\rho_k}{2} d_{\mathcal{K}}^2\left(G(y^{k+1}) + \frac{w^k}{\rho_k}\right) - \frac{\|w^k\|_H^2}{2\rho_k} \le \mathcal{L}_{\rho_k}(\bar{x}, w^k) + \varepsilon_{k+1} \le f(\bar{x}) + \varepsilon_{k+1} \tag{4.25}$$

for all k. Dividing by ρ_k and using the boundedness of $\{w^k\}$ and $\{f(y^{k+1})\}$, it follows that $d_{\mathcal{K}}(G(y^{k+1}) + w^k/\rho_k) \to 0$ and thus $d_{\mathcal{K}}(G(y^{k+1})) \to 0$. Moreover, since $\rho_k \to \infty$, we also obtain from (4.25) that $\limsup_{k\to\infty} f(y^{k+1}) \le f(\bar{x})$. Hence, the desired convergence follows from Corollary 3.27. □

The above result implies that the augmented Lagrangian subproblem admits approximate local minimizers in a neighborhood of $\bar{x}$, provided that ρ is large enough and ε is sufficiently small. By using Ekeland's variational principle (Proposition 3.9), we can extend this statement to additionally obtain some kind of approximate stationarity.

Theorem 4.24. *Let Assumption 4.22 hold and let $B \subseteq H$ be a bounded set. Then there are $\bar{\rho}, \bar{\varepsilon}, r > 0$ such that, for all $w \in B$, $\rho \ge \bar{\rho}$, and $\varepsilon \in (0, \bar{\varepsilon})$, there is a point $x = x_{\rho,\varepsilon}(w) \in C$ with $\|x - \bar{x}\|_X < r$ and the following properties:*

(i) *x is an ε-minimizer of $\mathcal{L}_\rho(\cdot, w)$ on $B_r(\bar{x}) \cap C$,*

(ii) *x satisfies $\operatorname{dist}\big(-\mathcal{L}'_\rho(x, w), \mathcal{N}_C(x)\big) \le \varepsilon^{1/2}$, and*

(iii) *$x = x_{\rho,\varepsilon}(w) \to \bar{x}$ uniformly on B as $\rho \to \infty$ and $\varepsilon \to 0$.*

Proof. Let $r > 0$ be as in Lemma 4.23. For $\rho > 0$ and $w \in B$, consider the problem

$$\underset{x\in X}{\text{minimize}}\ \mathcal{L}_\rho(x, w) \quad \text{subject to} \quad x \in C_r := B_r(\bar{x}) \cap C.$$

Observe that the constraint $x \in C_r$ trivially satisfies the Robinson constraint qualification. Hence, by Ekeland's variational principle (Proposition 3.9), there are points $x = x_{\rho,\varepsilon}(w)$ such that x satisfies (i) and, in addition, $\operatorname{dist}\big(-\mathcal{L}'_\rho(x, w), \mathcal{N}_{C_r}(x)\big) \le \varepsilon^{1/2}$. By Lemma 4.23, it follows that $x_{\rho,\varepsilon} \to \bar{x}$ uniformly on B as $\rho \to \infty$ and $\varepsilon \to 0$. Hence, there are $\bar{\rho}, \bar{\varepsilon} > 0$ such that $\|x_{\rho,\varepsilon}(w) - \bar{x}\|_X < r$ for all $\rho \ge \bar{\rho}$, $\varepsilon \in (0, \bar{\varepsilon})$, and $w \in B$. But $\mathcal{N}_{C_r}(x) = \mathcal{N}_C(x)$ whenever $x \in C$ and $\|x - \bar{x}\|_X < r$. Hence, the result follows. □

If X is reflexive and the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is weakly sequentially lsc, then the assertions of the above theorem remain valid if we replace the ε-minimizers by exact minimizers. In this case, we obtain points $x = x_\rho(w)$ which satisfy (i) and (ii) with $\varepsilon := 0$ and which converge to $\bar{x}$ uniformly on B as $\rho \to \infty$. Sufficient conditions for the weak sequential lower semicontinuity of $\mathcal{L}_\rho(\cdot, w)$ were given in Lemma 4.8.

If the mapping G is completely continuous, then it is possible to prove a similar result under the second-order sufficient condition with respect to the space Y. This result is a generalization of a theorem from [137].

Proposition 4.25. *Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (P) which satisfies SOSC with respect to the space Y, and $B \subseteq H$ a bounded set. Assume that*

(i) *the space X is reflexive,*

(ii) *f is weakly sequentially lsc on X, and*

(iii) *G is completely continuous from X into Y.*

Then there are $\bar{\rho}, r > 0$ such that, for every $w \in B$ and $\rho \ge \bar{\rho}$, the problem $\min_{x\in C} \mathcal{L}_\rho(x, w)$ admits a local minimizer $x = x_\rho(w)$ in $B_r(\bar{x}) \cap C$, and $x_\rho \to \bar{x}$ uniformly on B as $\rho \to \infty$.

Proof. We argue similarly to the proof of Lemma 4.23 and Theorem 4.24. Let $r > 0$ be small enough so that $\bar{x}$ is a strict local minimizer of f on $B_r(\bar{x}) \cap \Phi$, and such that the assertions of Corollary 3.27 hold. Let $\{w^k\} \subseteq B$ and $\rho_k \to \infty$ be arbitrary sequences and, for each k, let $y^{k+1} \in B_r(\bar{x}) \cap C$ be a minimizer of (4.24). Note that y^{k+1} exists since $B_r(\bar{x})$ is weakly compact.

We need to show that $y^{k+1} \to \bar{x}$ as $k \to \infty$. By Theorem 4.12 and the reflexivity of X, it follows that $y^{k+1} \rightharpoonup \bar{x}$ and $f(y^{k+1}) \to f(\bar{x})$. Since G is completely continuous, we obtain $G(y^{k+1}) \to G(\bar{x})$, which implies that $d_K(G(y^{k+1})) \to 0$ (note the K instead of $\mathcal{K}$). Thus, Corollary 3.27 yields the convergence $y^{k+1} \to \bar{x}$, and the proof is complete. □

We now prove a result which is very similar to Theorem 4.24 but explicitly deals with the case where X is finite-dimensional. In this case, the assumptions required for the existence of local minimizers can be simplified considerably. In fact, we only need that $\bar{x}$ is a strict local minimizer, which is weaker than SOSC. We do not even need the existence of Lagrange multipliers.

Proposition 4.26. *Assume that X is finite-dimensional, that $B \subseteq H$ is a bounded set, and $\bar{x}$ is a strict local minimizer of (P). Then there are $\bar{\rho}, r > 0$ such that, for every $w \in B$ and $\rho \geq \bar{\rho}$, the problem $\min_{x \in C} \mathcal{L}_\rho(x, w)$ admits a local minimizer $x = x_\rho(w)$ which lies in $B_r(\bar{x}) \cap C$. Moreover, $x_\rho \to \bar{x}$ uniformly on B as $\rho \to \infty$.*

Proof. We again argue similarly to the proof of Lemma 4.23 and Theorem 4.24. Let $r > 0$ be small enough so that $\bar{x}$ is a strict local minimizer of f on $B_r(\bar{x}) \cap \Phi$. Let $\{w^k\} \subseteq B$ and $\rho_k \to \infty$ be arbitrary sequences and, for each k, let $y^{k+1} \in B_r(\bar{x}) \cap C$ be a minimizer of (4.24). Note that y^{k+1} exists by compactness.

We need to show that $y^{k+1} \to \bar{x}$ as $k \to \infty$. By Theorem 4.12, it follows that every limit point of $\{y^k\}$ is a (global) minimizer of f on $B_r(\bar{x}) \cap \Phi$. The uniqueness of $\bar{x}$ therefore implies that $y^{k+1} \to \bar{x}$, and the proof is complete. □

Let us close this section by showing that the local minimizers obtained in the above results are in general not unique. It is possible to obtain uniqueness under significantly stronger assumptions, namely the so-called *strong regularity* of the KKT system. In finite-dimensional nonlinear programming, this property is equivalent to a stronger version of SOSC together with the linear independence constraint qualification. More details can be found in [24, 32]. If these assumptions are weakened, then we cannot expect unique local minimizers of the augmented Lagrangian, as illustrated by the following example.

Example 4.27. Consider the following quadratic program, due to Kyparisis [154]:

$$\underset{x \in \mathbb{R}^3}{\text{minimize}}\ x_1^2 - \frac{1}{2} x_2^2 \quad \text{subject to} \quad G(x) := \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_1 \end{pmatrix} \leq 0.$$

An easy calculation shows that $\bar{x} := (0, 0)$ is the unique solution of this problem, and $\bar{\lambda} := (0, 0, 0)$ the unique Lagrange multiplier. In particular, by Proposition 3.33, the strict

Mangasarian–Fromovitz condition (SMFC) holds in $\bar{x}$, which in this case is equivalent to the strict Robinson condition. Moreover, the critical cone at $\bar{x}$ is given by

$$\mathcal{C}(\bar{x}) = \{d \in \mathbb{R}^2 : d_1 \le 0,\ d_2 \in [d_1, -d_1]\}.$$

This makes it easy to verify that SOSC (in the sense of Definition 3.34) is satisfied. On the other hand, the linear independence constraint qualification (LICQ) does not hold in $\bar{x}$ since $\nabla G_1(\bar{x}) + \nabla G_2(\bar{x}) - 2\nabla G_3(\bar{x}) = 0$.

For the analysis of the augmented Lagrangian, assume now that $\rho > 1$ is arbitrarily large and $\varepsilon > 0$ arbitrarily small. Define

$$\lambda := \left(\frac{\varepsilon}{\rho}, \frac{\varepsilon}{\rho}, \varepsilon + \frac{\varepsilon}{\rho}\right).$$

Then λ is arbitrarily close to $\bar{\lambda}$, but the augmented Lagrangian $\mathcal{L}_\rho(\cdot, \lambda)$ has the stationary points

$$x^{(1)} := \left(\frac{-\varepsilon}{\rho}, \frac{\varepsilon}{\rho}\right) \quad \text{and} \quad x^{(2)} := \left(\frac{-\varepsilon}{\rho}, \frac{-\varepsilon}{\rho}\right).$$

4.3.2 Rate of Convergence Analysis

We are now in a position to discuss the convergence of Algorithm 4.4 from a quantitative point of view. Throughout this section, we assume that the space X is a real Hilbert space, that there is a local minimizer $\bar{x} \in X$ of (P) with a unique Lagrange multiplier $\bar{\lambda} \in H$, and that the local error bound from Section 3.2.4 holds in $(\bar{x}, \bar{\lambda})$. In our setting, this condition takes on the form (compare with Corollary 3.58)

$$c_1 \Theta(x, \lambda) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H \le c_2 \Theta(x, \lambda) \tag{4.26}$$

for all $(x, \lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small, where Θ is the residual

$$\Theta(x, \lambda) := \|x - P_C(x - \mathcal{L}'(x, \lambda))\|_X + \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H.$$

The regularity assumptions mentioned above may seem rather stringent in view of the Gel'fand triple framework $Y \hookrightarrow H \hookrightarrow Y^*$. Indeed, a sufficient condition for the local error bound is a combination of the second-order sufficient condition (SOSC, see Definition 3.24) and the strict Robinson condition (SRC, see Definition 3.6), both with respect to the space H. This effectively rules out certain applications where the embedding $Y \hookrightarrow H$ is too weak, but the underlying issue is that we simply cannot expect the results in this section to hold if the constraint system of (P) is only regular with respect to the space Y. This is also evidenced by the fact that the rate-of-convergence analysis will enable us to prove the boundedness of the penalty sequence $\{\rho_k\}$, and this actually *implies* the existence of a Lagrange multiplier in H under certain assumptions, see Corollary 4.17 and the discussion after Corollary 4.32 below.

Despite these restrictions, the theory we develop here is still applicable to a fair amount of nontrivial problems such as control-constrained optimal control, elliptic parameter estimation problems, and of course optimization in finite dimensions (see also Section 4.3.3). For more details, we refer the reader to Chapter 7.

Assumption 4.28 (Rate of convergence)**.** We assume that

(i) X is a real Hilbert space with f and G continuously differentiable on X,

(ii) $(\bar{x}, \bar{\lambda}) \in X \times H$ is a KKT point of (P) which satisfies the error bound (4.26),

(iii) the primal-dual sequence $\{(x^k, \lambda^k)\}$ converges strongly to $(\bar{x}, \bar{\lambda})$ in $X \times H$,

(iv) the safeguarded multiplier sequence satisfies $w^k := \lambda^k$ for k sufficiently large, and

(v) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Two assumptions which may require some elaboration are (iii) and (iv). Note that we already know, by Theorem 4.24, that the augmented Lagrangian admits approximate local minimizers and stationary points in a neighborhood of $\bar{x}$. We shall now see that, if the algorithm chooses these local minimizers (or any other points sufficiently close to $\bar{x}$), then we automatically obtain the convergence $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ in $X \times H$. In this case, the sequence $\{\lambda^k\}$ is necessarily bounded in H, so it is reasonable to assume that the safeguarded multipliers are eventually chosen as $w^k = \lambda^k$. The following result can therefore be considered as (retrospective) justification for Assumption 4.28.

Proposition 4.29. *Let Assumption 4.28* (i), (ii), (v) *hold, and let RCQ hold in $\bar{x}$ with respect to the space H. Then there exists $r > 0$ such that, if $x^k \in B_r(\bar{x})$ for sufficiently large k, then $\Theta(x^k, \lambda^k) \to 0$ and $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ strongly in $X \times H$.*

Proof. Let $r > 0$ be small enough so that the error bound (4.26) holds for all $(x, \lambda) \in X \times H$ with $x \in B_r(\bar{x})$ and $\Theta(x, \lambda)$ sufficiently small. Shrinking r if necessary, we may also assume that f' and G' are bounded on $B_r(\bar{x})$ and, by Proposition 3.18, that there exists $s > 0$ such that

$$B_s^H \subseteq G(x) + G'(x)\big[(C - x) \cap B_1^X\big] - \mathcal{K} \tag{4.27}$$

for all $x \in B_r(\bar{x})$. Assume now that $x^k \in B_r(\bar{x})$ for all k sufficiently large. The proof is divided into multiple steps.

Step 1. We first show that $d_{\mathcal{K}}(G(x^{k+1})) \to 0$ as $k \to \infty$. If $\{\rho_k\}$ remains bounded, then this readily follows from the penalty updating scheme (4.13). On the other hand, if $\rho_k \to \infty$, then we can argue as in the proof of Lemma 4.15 to obtain that

$$\delta^{k+1} - G'(x^{k+1})^*\big[G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}))\big] \in \mathcal{N}_C(x^{k+1}) \tag{4.28}$$

for some null sequence $\{\delta^k\} \subseteq X^*$. (Note that this step uses the boundedness of f' and G' on $B_r(\bar{x})$.) We claim that this implies $d_{\mathcal{K}}(G(x^{k+1})) \to 0$. Let $y \in B_s^H$ be an arbitrary vector. By (4.27), there exist sequences $\{c^k\} \subseteq C$ and $\{z^k\} \subseteq \mathcal{K}$ such that $c^k \in B_1(x^k)$ and $y = G(x^{k+1}) + G'(x^{k+1})(c^{k+1} - x^{k+1}) - z^{k+1}$ for all k. Hence,

$$\begin{aligned} \big(G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1})), y\big) &= \big\langle G'(x^{k+1})^*\big[G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}))\big], c^{k+1} - x^{k+1}\big\rangle \\ &\quad + \big(G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1})), G(x^{k+1}) - z^{k+1}\big) \\ &\geq \langle \delta^{k+1}, c^{k+1} - x^{k+1}\rangle \geq -\|\delta^{k+1}\|_{X^*} \end{aligned}$$

for all k, where we used (4.28) and standard projection inequalities. Since this lower bound is uniform with respect to $y \in B_s^H$, it is easy to infer that $G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1})) \to 0$ in H, which yields $d_{\mathcal{K}}(G(x^{k+1})) \to 0$. This concludes the proof of Step 1.

Step 2. We now demonstrate that $\Theta(x^k, \lambda^k) \to 0$. Observe that $\mathcal{L}'(x^{k+1}, \lambda^{k+1}) = \mathcal{L}'_{\rho_k}(x^{k+1}, w^k)$ and $\phi^{k+1} := \varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k. Using Proposition 2.36 and the nonexpansiveness of the projection P_C, it follows that the first term in the definition of $\Theta(x^{k+1}, \lambda^{k+1})$ satisfies

$$\begin{aligned} &\|x^{k+1} - P_C(x^{k+1} - \mathcal{L}'(x^{k+1}, \lambda^{k+1}))\|_X \\ &\le \|x^{k+1} - P_C(x^{k+1} + \phi^{k+1})\|_X + \|\varepsilon^{k+1}\|_X = \|\varepsilon^{k+1}\|_X, \end{aligned} \tag{4.29}$$

which converges to zero. Hence, it remains to show that $G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}) + \lambda^{k+1}) \to 0$ as $k \to \infty$. To this end, define the sequence $s^{k+1} := P_{\mathcal{K}}(G(x^{k+1}) + w^k/\rho_k)$. Then $s^{k+1} \in \mathcal{K}$ and $\lambda^{k+1} \in \mathcal{N}_{\mathcal{K}}(s^{k+1})$ for all k (by Proposition 2.36 and the definition of λ^{k+1}). We now use the fact that $y \mapsto y - P_{\mathcal{K}}(y + \lambda^{k+1})$ is nonexpansive, which is an easy consequence of Lemma 2.11. Therefore, the inverse triangle inequality yields

$$\begin{aligned} &\|G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}) + \lambda^{k+1})\|_H \\ &\le \|G(x^{k+1}) - s^{k+1}\|_H + \|s^{k+1} - P_{\mathcal{K}}(s^{k+1} + \lambda^{k+1})\|_H. \end{aligned} \tag{4.30}$$

The last term is equal to zero since $\lambda^{k+1} \in \mathcal{N}_{\mathcal{K}}(s^{k+1})$, see again Proposition 2.36. Hence, to prove the claim, it remains to show that $\|s^{k+1} - G(x^{k+1})\|_H \to 0$. The proof of this assertion is divided into two cases. If $\{\rho_k\}$ is bounded, then it readily follows from the penalty updating scheme (4.13). On the other hand, if $\rho_k \to \infty$, then we see that

$$\|s^{k+1} - G(x^{k+1})\|_H \le \|s^{k+1} - P_{\mathcal{K}}(G(x^{k+1}))\|_H + d_{\mathcal{K}}(G(x^{k+1})) \to 0,$$

where we used the boundedness of $\{w^k\}$ and Step 1. This concludes the proof of Step 2.

Step 3. We finally deduce that $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ in $X \times H$. Recall that $x^k \in B_r(\bar{x})$ for all k and that $\Theta(x^k, \lambda^k) \to 0$ by Step 2. Hence, the claim is an immediate consequence of the error bound (4.26). □

We will now prove convergence rates for the primal-dual sequence $\{(x^k, \lambda^k)\}$. Since the distance of (x^k, λ^k) to $(\bar{x}, \bar{\lambda})$ admits both upper and lower estimates relative to the residual terms $\theta_k := \Theta(x^k, \lambda^k)$ by (4.26), we will largely base our analysis on the sequence $\{\theta_k\}$, and the results on the primal-dual sequence $\{(x^k, \lambda^k)\}$ will follow directly.

Lemma 4.30. *Let Assumption 4.28 hold, and let $\theta_k := \Theta(x^k, \lambda^k)$. Then*

$$\left(1 - \frac{c_2}{\rho_k}\right)\theta_{k+1} \le \|\varepsilon^{k+1}\|_X + \frac{c_2}{\rho_k}\theta_k$$

for all $k \in \mathbb{N}$ sufficiently large.

Proof. Using the definition of θ_{k+1} and (4.29), we have

$$\theta_{k+1} \leq \|\varepsilon^{k+1}\|_X + \|G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}) + \lambda^{k+1})\|_H. \tag{4.31}$$

Now, let $k \in \mathbb{N}$ be large enough so that $w^k = \lambda^k$. Consider again the sequence $s^{k+1} := P_{\mathcal{K}}(G(x^{k+1}) + \lambda^k/\rho_k)$. Using (4.30) and the definition of λ^{k+1}, we see that

$$\|G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}) + \lambda^{k+1})\|_H \leq \|G(x^{k+1}) - s^{k+1}\|_H = \frac{\|\lambda^{k+1} - \lambda^k\|_H}{\rho_k}. \tag{4.32}$$

Inserting this into (4.31) and using the triangle inequality yields

$$\theta_{k+1} \leq \|\varepsilon^{k+1}\|_X + \frac{1}{\rho_k}\big(\|\lambda^{k+1} - \bar{\lambda}\|_H + \|\lambda^k - \bar{\lambda}\|_H\big).$$

Now, by Assumption 4.28 and since $x^k \to \bar{x}$, there is a $c_2 > 0$ such that $\|\lambda^k - \bar{\lambda}\|_H \leq c_2\theta_k$ for all $k \in \mathbb{N}$ sufficiently large. Hence,

$$\theta_{k+1} \leq \|\varepsilon^{k+1}\|_X + \frac{c_2}{\rho_k}\theta_{k+1} + \frac{c_2}{\rho_k}\theta_k,$$

again for $k \in \mathbb{N}$ sufficiently large. Reordering gives the desired result. □

With the above lemma, it is easy to deduce convergence rates for the primal-dual sequence $\{(x^k, \lambda^k)\}$.

Theorem 4.31. *Let Assumption 4.28 hold and assume that $\varepsilon^{k+1} = o(\theta_k)$. Then:*

(a) *For every $q \in (0,1)$, there exists $\bar{\rho}_q > 0$ such that, if $\rho_k \geq \bar{\rho}_q$ for sufficiently large k, then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ Q-linearly in $X \times H$ with rate q.*
(b) *If $\rho_k \to \infty$, then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ Q-superlinearly in $X \times H$.*

Proof. Let $k \in \mathbb{N}$ be large enough so that $w^k = \lambda^k$. By Lemma 4.30, if ρ_k is large enough so that $1 - c_2/\rho_k > 0$, then

$$\frac{\theta_{k+1}}{\theta_k} \leq \frac{c_2}{\rho_k - c_2} + o(1). \tag{4.33}$$

This implies the desired Q-rates for the sequence $\{\theta_k\}$. The corresponding rates for $\{(x^k, \lambda^k)\}$ are then an easy consequence of the error bound (4.26). □

The assumption $\varepsilon^{k+1} = o(\theta_k)$ in the above theorem says that, roughly speaking, the degree of inexactness should be small enough to not affect the rate of convergence. Note that we are comparing ε^{k+1} to the optimality measure θ_k of the previous iterates (x^k, λ^k). Hence, it is easy to ensure this condition in practice, for instance, by always computing the next iterate x^{k+1} with a precision $\|\varepsilon^{k+1}\|_X \leq z_k\theta_k$ for some fixed null sequence z_k.

Corollary 4.32. *Let Assumption 4.28 hold and assume that the subproblems occurring in Algorithm 4.4 are solved exactly, i.e., that $\varepsilon^k = 0$ for all k. Then $\{\rho_k\}$ remains bounded.*

Proof. Let $k \in \mathbb{N}$ be sufficiently large so that $w^k = \lambda^k$, let $s^{k+1} = P_{\mathcal{K}}(G(x^{k+1}) + \lambda^k/\rho_k)$, and define $V_{k+1} := V(x^{k+1}, w^k, \rho_k) = \|G(x^{k+1}) - s^{k+1}\|_H$. To prove the boundedness of $\{\rho_k\}$, we need to show that $V_{k+1} \le \tau V_k$ for sufficiently large k, where $\tau \in (0,1)$ is the constant from Algorithm 4.4. Using (4.32) and the fact that $-\mathcal{L}'(x^{k+1}, \lambda^{k+1}) = -\mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, we obtain

$$V_{k+1} \ge \|G(x^{k+1}) - P_{\mathcal{K}}(G(x^{k+1}) + \lambda^{k+1})\|_H = \theta_{k+1}$$

for all $k \in \mathbb{N}$. Using again (4.32) and the error bound (4.26), it follows that

$$V_{k+1} = \frac{\|\lambda^{k+1} - \lambda^k\|_H}{\rho_k} \le \frac{\|\lambda^{k+1} - \bar{\lambda}\|_H + \|\lambda^k - \bar{\lambda}\|_H}{\rho_k} \le \frac{c_2}{\rho_k}(\theta_{k+1} + \theta_k)$$

for $k \in \mathbb{N}$ sufficiently large (recall that $x^k \to \bar{x}$). Putting these inequalities together yields

$$\frac{V_{k+1}}{V_k} \le \frac{c_2}{\rho_k} \frac{\theta_{k+1} + \theta_k}{\theta_k} = \frac{c_2}{\rho_k}\left(1 + \frac{\theta_{k+1}}{\theta_k}\right).$$

If we now assume that $\rho_k \to \infty$, then it is easy to deduce from (4.33) that $V_{k+1}/V_k \to 0$. Hence, $V_{k+1}/V_k \le \tau$ for all k sufficiently large, which contradicts $\rho_k \to \infty$. □

The boundedness of $\{\rho_k\}$ obviously rules out the Q-superlinear convergence of Theorem 4.31 (b). However, the former is usually considered more significant in practice since it prevents the subproblems from becoming excessively ill-conditioned.

Remark 4.33. If inexact solutions are allowed for the augmented Lagrangian subproblems, then the boundedness of $\{\rho_k\}$ requires a slightly modified updating rule for the penalty parameter since the one used in Algorithm 4.4 does not take into account the current measure of optimality. Indeed, if we replace the function V from (4.10) by

$$\tilde{V}(x, \lambda, \rho) := V(x, \lambda, \rho) + \|x - P_C(x - \mathcal{L}'(x, \lambda))\|_X,$$

then it is possible to show that $\{\rho_k\}$ remains bounded under the assumptions of Theorem 4.31. A proof for the case $C = X$ can be found in [133], and the extension to the general case is straightforward (see also [26, 28]). It is furthermore worth noting that the global and local convergence results from Sections 4.2.3 and 4.3.1 remain valid with this "modified" penalty updating scheme. This is because the proofs of these results only use the fact that $V(x^{k+1}, w^k, \rho_k) \to 0$ if $\{\rho_k\}$ remains bounded, which clearly still holds if the penalty updating scheme uses $\tilde{V}$ instead of V.

Remark 4.34. In the case of finite-dimensional nonlinear programming, it is possible to obtain similar rate of convergence results to those above under the second-order sufficient condition only. In this case, one obtains that $(x^k, \lambda^k) \to (\bar{x}, \lambda)$ Q-linearly for some $\lambda \in \Lambda(\bar{x})$ which is not necessarily equal to $\bar{\lambda}$. This result can be found in [74]. The reason why this is possible is that, for nonlinear programming, the set $\mathcal{K}$ is polyhedral and therefore, as mentioned in Section 3.2.4, the second-order condition implies a local primal-dual error bound without any constraint qualification. This approach is not possible if $\mathcal{K}$ is not polyhedral, as evidenced by Example 3.60.

4.3.3 C^2-Cone Reducible Programming

In this section, we consider a special case of (P) which arises if the constraint set has a certain geometric structure. The prototypical applications we have in mind are semidefinite programming, second-order cone programming, and related problems. More details will be given further below. To avoid overburdening the presentation, we leave out some of the results and proofs in this section and instead make frequent references to the literature. A more comprehensive exposition of the theory can be found in [57, 135, 203] and in the book [32].

Throughout this section, we assume that the spaces X, Y, H defining (P) are finite-dimensional. In this case, it is no restriction to assume that all these spaces are Hilbert spaces; in the finite-dimensional context, these are often referred to as *Euclidean spaces*. Moreover, the dense embedding $Y \hookrightarrow H$ is necessarily the identity mapping, and thus we can simply drop the space Y and directly consider the constrained optimization problem

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in \mathcal{K}, \tag{4.34}$$

where $G : X \to H$ and $\mathcal{K} \subseteq H$ is a nonempty closed convex set. Note that we do not include an additional constraint set $C \subseteq X$ for the sake of simplicity.

We assume throughout that the functions f and G are twice continuously differentiable. In this situation, we already know that Algorithm 4.4 possesses good local convergence properties if the problem satisfies a suitable second-order sufficient condition together with an appropriate constraint qualification, see Section 4.3.2.

The approach we consider here depends on a local reduction property of the set $\mathcal{K}$ to a pointed closed convex cone. This allows us to locally transform the constraint system into a simpler one, and we can then apply properties such as second-order conditions or constraint qualifications to the reduced problem, yielding sharper optimality results.

Definition 4.35 (C^2-cone reducibility)**.** We say that $\mathcal{K}$ is *C^2-cone reducible at $y_0 \in \mathcal{K}$* if there exist a pointed closed convex cone $D \subseteq Z$ in some finite-dimensional space Z, a neighborhood N of y_0, and a twice continuously differentiable mapping $\Xi : N \to Z$ such that $\Xi(y_0) = 0$, $\Xi'(y_0)$ is onto, and $\mathcal{K} \cap N = \Xi^{-1}(D) \cap N$. We say that $\mathcal{K}$ is *C^2-cone reducible* if the above holds at every $y_0 \in \mathcal{K}$.

Some examples where the set $\mathcal{K}$ is C^2-cone reducible include nonlinear programming, semidefinite programming, second-order cone programming, and any combination thereof. More details will be given further below.

Assume now that $\bar{x} \in X$ is a local minimizer of (4.34), and that the set $\mathcal{K}$ is C^2-cone reducible at $y_0 := G(\bar{x})$. This implies that $\bar{x}$ is also a local minimizer of the *reduced problem*

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad \Xi(G(x)) \in D. \tag{4.35}$$

This problem possesses a key theoretical advantage over (4.34) in that the new constraint mapping satisfies $z_0 := \Xi(G(\bar{x})) = 0$. This implies that, when forming variational objects such as radial, tangent, or normal cones (see Section 2.2.1), we simply obtain

$$\mathcal{T}_D(z_0) = \mathcal{R}_D(z_0) = D \quad \text{and} \quad \mathcal{N}_D(z_0) = D^\circ.$$

In particular, there is no "gap" between the radial and tangent cones, and the local geometry of D is completely described by the conical structure of D itself.

We now apply the Robinson constraint qualification (RCQ) and its strict counterpart (SRC) to the reduced problem (4.35). Taking into account Proposition 3.19 and the fact that $\mathcal{T}_{\mathcal{K}}(G(\bar{x})) = \mathcal{T}_D(z_0)$, RCQ for the reduced problem takes on the form

$$G'(\bar{x})X + \mathcal{T}_{\mathcal{K}}(G(\bar{x})) = H$$

and therefore coincides with RCQ for the problem (4.34). For the strict Robinson condition, the situation is slightly different, and we obtain the following condition.

Definition 4.36 (Reduced SRC)**.** We say that the *reduced strict Robinson condition (reduced SRC, R-SRC)* holds in $\bar{x}$ if there exists $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$G'(\bar{x})X + \mathcal{T}_{\mathcal{K}}(G(\bar{x})) \cap \bar{\lambda}^{\perp} = H.$$

The reduced SRC yields the uniqueness of $\bar{\lambda}$, and it is weaker than the ordinary SRC (Definition 3.6) for the problem (4.34), see [32, p. 299]. At the end of this section, we will present two examples which show that this implication is strict.

The second-order sufficient condition from Definition 3.24 can also be applied to the reduced problem (4.35). In fact, this can be done in a slightly more general context by using a form of SOSC which takes into account the whole multiplier set (see [32]). The resulting condition can then be reformulated using only the problem primitives f, G, and $\mathcal{K}$. This process involves the so-called *second-order tangent set* to $\mathcal{K}$ in a point $y \in \mathcal{K}$ and a direction $h \in H$, which is given by

$$\mathcal{T}_{\mathcal{K}}^2(y, h) := \left\{ w \in H : \operatorname{dist}(y + th + \tfrac{1}{2}t^2 w, \mathcal{K}) = o(t^2),\ t \geq 0 \right\}.$$

Now, let $\sigma(y, S) := \sup_{z \in S}(y, z)$ be the support function of a closed convex set $S \subseteq H$. Then the second-order condition takes on the following form.

Definition 4.37 (Reduced SOSC)**.** Let $(\bar{x}, \bar{\lambda}) \in X \times H$ be a KKT point of (4.34). We say that the *reduced second-order sufficient condition (reduced SOSC, R-SOSC)* holds in $(\bar{x}, \bar{\lambda})$ if the set $\mathcal{K}$ is C^2-cone reducible at $G(\bar{x})$ and

$$\sup_{\lambda \in \Lambda(\bar{x})} \left\{ \mathcal{L}''(\bar{x}, \lambda)(d, d) - \sigma\big(\lambda, \mathcal{T}_{\mathcal{K}}^2(G(\bar{x}), G'(\bar{x})d)\big) \right\} > 0 \tag{4.36}$$

for all $d \in \mathcal{C}(\bar{x}) \setminus \{0\}$, where $\mathcal{C}(\bar{x}) := \{d \in X : f'(\bar{x})d \leq 0,\ G'(\bar{x})d \in \mathcal{T}_{\mathcal{K}}(G(\bar{x}))\}$.

Similarly to the general form of SOSC from Definition 3.24, the reduced SOSC implies the local quadratic growth of the objective function, i.e., the existence of $c > 0$ such that $f(x) \geq f(\bar{x}) + c\|x - \bar{x}\|_X^2$ for all feasible points x near $\bar{x}$. In particular, it follows that $\bar{x}$ is a strict local minimizer of the problem. What sets R-SOSC apart from the general SOSC is that, under certain assumptions, it is actually *equivalent* to the quadratic growth condition. More details can be found in [32, Theorems 3.86 and 3.137].

The reduced versions of SOSC and SRC imply a local primal-dual error bound in the sense of Section 3.2.4, and they are in fact equivalent to the error bound property under certain assumptions. As in Section 3.2.4, let $\Theta : X \times H \to \mathbb{R}$ be the residual mapping

$$\Theta(x, \lambda) := \|\mathcal{L}'(x, \lambda)\|_X + \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H.$$

The following result is a consequence of the theory in [57, 133, 135].

Proposition 4.38. *Assume that the problem* (4.34) *admits a KKT point* $(\bar{x}, \bar{\lambda})$ *which satisfies the reduced SOSC and reduced SRC. Then* $\Lambda(\bar{x}) = \{\bar{\lambda}\}$ *and there are* $c_1, c_2 > 0$ *such that, for all* $(x, \lambda) \in X \times H$ *with* x *near* $\bar{x}$ *and* $\Theta(x, \lambda)$ *sufficiently small,*

$$c_1 \Theta(x, \lambda) \le \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H \le c_2 \Theta(x, \lambda). \tag{4.37}$$

Conversely, if $\Lambda(\bar{x})$ *is a singleton, the error bound* (4.37) *is satisfied, and RCQ holds in* $\bar{x}$*, then both the reduced SOSC and reduced SRC hold in* $(\bar{x}, \bar{\lambda})$*.*

The error bound property implies a local convergence and rate of convergence result for the augmented Lagrangian method (Algorithm 4.4). For the sake of clarity, we restate our assumptions in the present setting.

Assumption 4.39 (Local convergence for C^2-cone reducible problems)**.** Assume that

(i) $(\bar{x}, \bar{\lambda}) \in X \times H$ is a KKT point of (4.34) which satisfies the error bound (4.37),
(ii) the primal-dual sequence $\{(x^k, \lambda^k)\}$ converges to $(\bar{x}, \bar{\lambda})$,
(iii) the safeguarded multiplier sequence satisfies $w^k := \lambda^k$ for k sufficiently large, and
(iv) there is a null sequence $\{\varepsilon_k\} \subseteq \mathbb{R}_+$ such that $\|\mathcal{L}'_{\rho_k}(x^{k+1}, w^k)\|_X \le \varepsilon_{k+1}$ for all k.

Note that the above is basically a reformulation of Assumption 4.28. For the sake of simplicity, we have replaced the vectorial sequence $\{\varepsilon^k\} \subseteq X$ in the latter by a scalar null sequence $\{\varepsilon_k\}$. A sufficient condition for assumption (ii) was given in Proposition 4.29, where it was shown that $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ if the error bound (4.37) holds, RCQ is satisfied in $\bar{x}$, and the primal iterates $\{x^k\}$ eventually lie in a sufficiently small neighborhood of $\bar{x}$.

Clearly, in the case where $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$, it is reasonable to assume that the safeguarded multipliers are eventually chosen as $w^k := \lambda^k$. Thus, the conditions in Assumption 4.39 are realistic, and we obtain the following result which is essentially a restatement of Theorem 4.31. Recall that $\theta_k := \Theta(x^k, \lambda^k)$.

Theorem 4.40. *Let* $(\bar{x}, \bar{\lambda})$ *be a KKT point of the problem* (4.34)*, and let Assumption 4.39 hold. Then there exists* $\bar{\rho} > 0$ *such that, if* $\rho_k \ge \bar{\rho}$ *for* k *sufficiently large and* $\varepsilon_{k+1} = o(\theta_k)$*, then* $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ *Q-linearly with rate proportional to* $1/\rho_k$*.*

This constitutes our main local convergence result for C^2-cone reducible programs. We now turn to two problem classes which are arguably the most prominent applications of the reduction approach.

Semidefinite Programming

Semidefinite programming (SDP), linear or nonlinear, revolves around constraints which impose semidefiniteness of certain matrices. We write $\mathcal{S}^n$ for the space of symmetric $n \times n$-matrices, equipped with the scalar product $(A, B) := \mathrm{tr}(A^\top B)$, $\mathcal{S}^n_+$ ($\mathcal{S}^n_-$) for the subsets of positive (negative) semidefinite matrices, and $A \succeq 0$ ($A \preceq 0$) for positive (negative) semidefiniteness. With these definitions, a typical SDP is given by

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad g(x) \leq 0,\ e(x) = 0,\ \mathcal{G}(x) \preceq 0, \tag{4.38}$$

where X is a finite-dimensional space and $g : X \to \mathbb{R}^m$, $e : X \to \mathbb{R}^p$, $\mathcal{G} : X \to \mathcal{S}^n$ are given mappings. This problem corresponds to the general setting (4.34) with

$$G(x) := (g(x), e(x), \mathcal{G}(x)), \qquad \begin{array}{l} H := \mathbb{R}^m \times \mathbb{R}^p \times \mathcal{S}^n, \\ \mathcal{K} := \mathbb{R}^m_- \times \{0\}^p \times \mathcal{S}^n_-. \end{array}$$

Note that $\mathcal{K}$ is C^2-cone reducible because it is a Cartesian product of C^2-cone reducible sets. Indeed, given a point $(y_0, z_0, A_0) \in \mathcal{K}$, the local reduction of $\mathcal{K}$ takes on the form

$$\Xi(y, z, A) := ((y - y_0)_{\mathcal{I}}, z, \Xi_{\mathcal{S}^n_-}(A)), \quad D := \mathbb{R}^{|\mathcal{I}|}_- \times \{0\}^p \times D_{\mathcal{S}^n_-},$$

where $\mathcal{I} := \{i = 1, \ldots, m : (y_0)_i = 0\}$ is the index set of active inequality constraints in y_0, and $\Xi_{\mathcal{S}^n_-}$ and $D_{\mathcal{S}^n_-}$ constitute the reduction of the negative semidefinite cone $\mathcal{S}^n_-$ at A_0. More details can be found in [203].

For semidefinite programming, the Lagrange multiplier occurring in the KKT conditions can be split as $\bar{\lambda} = (\bar{\mu}, \bar{\nu}, \bar{\Gamma})$ with $\bar{\mu} \in \mathbb{R}^m$, $\bar{\nu} \in \mathbb{R}^p$, and $\bar{\Gamma} \in \mathcal{S}^n$. With an obvious change of notation, the Lagrange function now becomes

$$\mathcal{L}(x, \mu, \nu, \Gamma) := f(x) + \mu^\top g(x) + \nu^\top e(x) + (\Gamma, \mathcal{G}(x)),$$

and the KKT system takes on the form

$$\mathcal{L}'(\bar{x}, \bar{\mu}, \bar{\nu}, \bar{\Gamma}) = 0, \quad 0 \leq \bar{\mu} \perp g(\bar{x}) \leq 0, \quad e(\bar{x}) = 0, \quad 0 \preceq \bar{\Gamma} \perp \mathcal{G}(\bar{x}) \preceq 0,$$

where $\perp$ denotes orthogonality with respect to the corresponding scalar products. The reduced SRC and SOSC conditions can be reformulated more explicitly in the case of SDP. A characterization of the former can be stated in terms of certain linear independences and the existence of a Mangasarian–Fromovitz type vector. The resulting condition is fairly involved and can be found in [202, 224]. As for reduced SOSC, the σ-term occurring in (4.36) can be calculated explicitly by taking into account the geometric structure of $\mathcal{S}^n_+$, and the condition can therefore be rewritten as

$$\sup_{(\mu,\nu,\Gamma) \in \Lambda(\bar{x})} \big\{ \mathcal{L}''(\bar{x}, \mu, \nu, \Gamma)(d, d) - 2\big(\Gamma, (\mathcal{G}'(\bar{x})d)\mathcal{G}(\bar{x})^\dagger(\mathcal{G}'(\bar{x})d)\big) \big\} > 0 \tag{4.39}$$

for all $d \in \mathcal{C}(\bar{x}) \setminus \{0\}$, where † denotes the Moore–Penrose pseudoinverse, see [32, 203, 224]. Note that the functions g and e provide no contribution to the σ-term since they represent constraints for which the corresponding factor in the set $\mathcal{K}$ is polyhedral.

Corollary 4.41. *Let $(\bar{x}, \bar{\mu}, \bar{\nu}, \bar{\Gamma})$ be a KKT point of* (4.38)*, and let Assumption 4.39 hold. Then there exists $\bar{\rho} > 0$ such that, if $\rho_k \geq \bar{\rho}$ for k sufficiently large and $\varepsilon_{k+1} = o(\theta_k)$, then $(x^k, \mu^k, \nu^k, \Gamma^k) \to (\bar{x}, \bar{\mu}, \bar{\nu}, \bar{\Gamma})$ Q-linearly with rate proportional to $1/\rho_k$.*

Let us briefly discuss the case of a *linear* semidefinite program. Given a problem of the form

$$\underset{x \in \mathcal{S}^n}{\text{minimize}}\ (c, x) \quad \text{subject to} \quad Ax = b,\ x \succeq 0,$$

where $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$, and $A : \mathcal{S}^n \to \mathbb{R}^m$ is a linear operator, it is customary to apply the augmented Lagrangian method to the *dual* problem

$$\underset{y \in \mathbb{R}^m}{\text{maximize}}\ b^\top y \quad \text{subject to} \quad A^* y - c \succeq 0,$$

since this yields subproblems which are smooth, unconstrained minimization problems on $\mathbb{R}^m$. It turns out that R-SOSC and R-SRC for the dual problem are closely related to the corresponding primal properties. In fact, assuming that the problem admits a unique primal-dual solution pair, it can be shown that R-SOSC for the primal problem is equivalent to R-SRC for the dual problem. By duality, this also holds with R-SOSC and R-SRC interchanged. Hence, if both conditions hold for the primal problem, then they also hold for the dual problem (primal-dual uniqueness follows automatically in this case). The corresponding investigations can be found in [226].

Second-Order Cone Programming

For second-order cone programs (SOCP), the theoretical analysis is very similar to semidefinite programming. Throughout this section, we write $w := (w_0, \bar{w})$ for a generic element in $\mathbb{R}^{1+m}$. Let $\mathcal{K} \subseteq \mathbb{R}^{1+m}$ be the *second-order (Lorentz, ice-cream) cone*

$$\mathcal{K} := \{(w_0, \bar{w}) \in \mathbb{R}^{1+m} : w_0 \geq \|\bar{w}\|_2\},$$

where $\|\cdot\|_2$ is the Euclidean norm. The analysis below can easily be extended to the case where additional inequality, equality, or multiple second-order cone constraints are present. In any case, the resulting set $\mathcal{K}$ is C^2-cone reducible [203].

As in the case of semidefinite programming, the reduced SOSC from Definition 4.37 can be reformulated to take into account the particular geometry of the problem. The resulting condition is given by

$$\sup_{\lambda \in \Lambda(\bar{x})} \left\{ \mathcal{L}''(\bar{x}, \lambda)(d, d) + d^\top \mathcal{H}(\bar{x}, \lambda) d \right\} > 0 \tag{4.40}$$

for all $d \in \mathcal{C}(\bar{x}) \setminus \{0\}$, where

$$\mathcal{H}(\bar{x}, \lambda) := -\frac{\lambda_0}{G_0(\bar{x})} G'(\bar{x})^\top \begin{pmatrix} 1 & 0 \\ 0 & -I_m \end{pmatrix} G'(\bar{x})$$

if $G(\bar{x}) \in \mathrm{bd}(\mathcal{K}) \setminus \{0\}$, and $\mathcal{H}(\bar{x}, \lambda) := 0$ otherwise, see [31, 157, 225]. Similar to before, G_0 denotes the first component of G.

Corollary 4.42. *Let $(\bar{x}, \bar{\lambda})$ be a KKT point of a nonlinear SOCP, and let Assumption 4.39 hold. Then there exists $\bar{\rho} > 0$ such that, if $\rho_k \geq \bar{\rho}$ for k sufficiently large and $\varepsilon_{k+1} = o(\theta_k)$, then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ Q-linearly with rate proportional to $1/\rho_k$.*

We close this section with two examples which demonstrate that the reduced version of SRC is, in general, strictly weaker than the ordinary SRC from Definition 3.6.

Example 4.43. (a) Let $X := \mathbb{R}$, $H := \mathbb{R}^2$, and consider the optimization problem (4.34) with $f(x) := x$, $G(x) := (x, 0)$, and $\mathcal{K}$ the closed unit ball in H. Clearly, $\bar{x} := -1$ is the global minimizer of this problem, and it is easy to see that $\bar{\lambda} := (-1, 0)$ is the corresponding (unique) Lagrange multiplier. Moreover, the set $\mathcal{K}$ is C^2-cone reducible in $\bar{y} := G(\bar{x}) = (-1, 0)$ to the cone $D := [0, +\infty)$ by means of the mapping $\Xi(x) := 1 - x_1^2 - x_2^2$. A straightforward calculation shows that $\mathcal{T}_{\mathcal{K}}(\bar{y}) \cap \bar{\lambda}^{\perp} = \bar{\lambda}^{\perp}$; on the other hand, the set $\mathcal{K}_0$ from Definition 3.6 is given by $\mathcal{K}_0 = \{\bar{y}\}$, and it follows that $\mathcal{T}_{\mathcal{K}_0}(\bar{y}) = \{0\}$, see Figure 4.1. We conclude that $G'(\bar{x})X + \mathcal{T}_{\mathcal{K}}(\bar{y}) \cap \bar{\lambda}^{\perp} = H$ and $G'(\bar{x})X + \mathcal{T}_{\mathcal{K}_0}(\bar{y}) \neq H$.

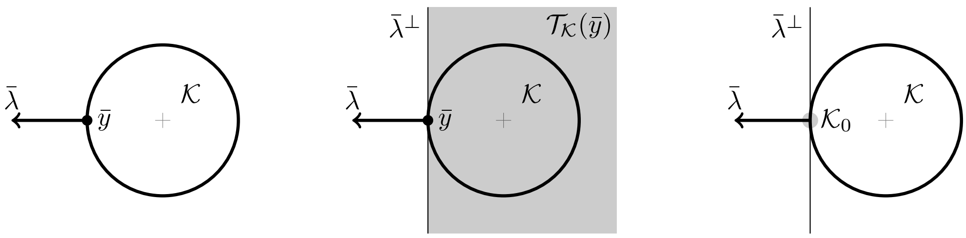


Figure 4.1: The setting of Example 4.43 (a), the tangent cone to $\mathcal{K}$, and the set $\mathcal{K}_0$.

(b) This example is a second-order cone program. Let $X := \mathbb{R}$, $H := \mathbb{R}^3$, $f(x) := -2x$, $G(x) := (x, 0, 2 - x)$, and $\mathcal{K} := \{y \in \mathbb{R}^3 : y_3 \geq \sqrt{y_1^2 + y_2^2}\}$. An easy calculation shows that $\bar{x} := 1$ is the global minimizer of the problem, and $\bar{\lambda} := (1, 0, -1)$ is the corresponding (unique) Lagrange multiplier. Moreover, with $\bar{y} := G(\bar{x}) = (1, 0, 1)$, we have

$$\mathcal{N}_{\mathcal{K}}(\bar{y}) = \{\alpha\bar{\lambda} : \alpha \geq 0\} \quad \text{and} \quad \mathcal{T}_{\mathcal{K}}(\bar{y}) = \mathcal{N}_{\mathcal{K}}(\bar{y})^{\circ} = \{\bar{\lambda}\}^{\circ}.$$

Hence, $\mathcal{T}_{\mathcal{K}}(\bar{y}) \cap \bar{\lambda}^{\perp} = \bar{\lambda}^{\perp}$. On the other hand, the set $\mathcal{K}_0$ is the intersection of $\mathcal{K}$ with the plane $\bar{\lambda}^{\perp}$, which is given by $\mathcal{K}_0 = \{\alpha\bar{y} : \alpha \geq 0\}$. Therefore, $\mathcal{T}_{\mathcal{K}_0}(\bar{y}) = \operatorname{span}(\bar{y})$, and it follows that $G'(\bar{x})X + \mathcal{T}_{\mathcal{K}}(\bar{y}) \cap \bar{\lambda}^{\perp} = H$ but $G'(\bar{x})X + \mathcal{T}_{\mathcal{K}_0}(\bar{y}) \neq H$.

Chapter 5

Augmented Lagrangian Methods for Variational Inequalities

In this chapter, we present a generalization of the augmented Lagrangian method (ALM) to variational inequalities (VIs) or, more generally, variational problems (in the sense of Section 3.2). The algorithm can be seen as a generalization of the ALM for constrained optimization, but it should be kept in mind that some arguments from optimization theory are simply not possible for variational problems. This is because many of these arguments rely on descent properties of the function f or, more generally, the ability to compare function values in order to get an indicator of optimality. This is not possible for general VIs. In turn, the variational framework has the significant advantage that we are now able to model more general optimization-related problems such as Nash and generalized Nash equilibrium problems.

The main framework we consider throughout this chapter is a VI of the form

$$(V) \qquad x \in \Phi, \quad \langle F(x), d\rangle \geq 0 \quad \forall d \in \mathcal{T}_\Phi(x), \tag{5.1}$$

where X is a real Banach space, $\Phi \subseteq X$ a nonempty closed set, and $F : X \to X^*$ a given mapping. Recall that, if $F = f'$ for some differentiable function $f : X \to \mathbb{R}$, then (V) represents the first-order necessary conditions (in the sense of Lemma 3.1) of the optimization problem

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad x \in \Phi.$$

Observe also that, if Φ is a convex set, then (V) can be restated as

$$x \in \Phi, \quad \langle F(x), y - x\rangle \geq 0 \quad \forall y \in \Phi. \tag{5.2}$$

The notion of variational inequalities is ubiquitous in modern optimization theory, and there is a variety of monographs specifically targeting this problem class, including [12, 70, 88–90, 143]. The VI has also become a standard tool for the modeling of various problems in the applied sciences [50, 87], in particular mechanics [40, 148, 174, 205]. In addition, the VI is often treated as part of many textbooks on constrained optimization, see, for instance, [13, 32].

One of the most important special cases (in a sense to be specified later) of the VI is the (generalized) Nash equilibrium problem, or (G)NEP for short. This is a class of optimization-type problems where multiple agents are involved, each with their own objective function and constraint set (see Section 5.3). The history of these problems can be traced back to the works of Nash [168, 169], to Arrow and Debreu [9, 53], and to Rosen [194]. A more detailed account of GNEPs, their history, and the surrounding theory, can be found in the books [14, 111] and the contemporary survey papers [65, 77]. The practical scope of GNEPs is enormous, with applications including economics, network design, electromagnetics, aerodynamics, and many more. In this regard, the reader is again referred to the survey papers [65, 77]. Due to its modeling power, the GNEP is a particular class of problems which has also enjoyed a substantial amount of applications involving infinite-dimensional spaces. In this context, a rather popular class of examples is that of *differential games*, which are multiobjective problems related to the evolution of dynamical systems involving ordinary differential equations [83, 84, 99, 181, 196]. A related but different problem class is concerned with the (multiobjective) optimal control of partial differential equations [35, 61, 106, 107, 182]. More applications in the infinite-dimensional context can be found in [41, 54, 183, 195, 209], and in the references of all these publications.

A survey of some standard algorithms for the solution of VIs can be found in [71]. For GNEPs, some notable references include [60, 64, 66, 86] and the survey papers [65, 77].

In this chapter, we present and discuss the augmented Lagrangian method for a general problem of the form (V). Similarly to the previous chapter, we assume that the feasible set Φ has a representation of the form

$$\Phi = \{x \in C : G(x) \in K\}, \tag{5.3}$$

where X, Y are real Banach spaces, $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, and $G : X \to Y$ is a continuously differentiable mapping. To facilitate the application of the augmented Lagrangian technique, we again assume that $i : Y \hookrightarrow H$ densely for some real Hilbert space H, and that $\mathcal{K} \subseteq H$ is a closed convex set satisfying $i^{-1}(\mathcal{K}) = K$. Hence, we are once more working in the *Gel'fand triple* framework

$$Y \overset{i}{\hookrightarrow} H \cong H^* \overset{i^*}{\hookrightarrow} Y^*.$$

The problem setting (5.3) is extremely general and encompasses a variety of constraint mappings (see Chapter 4 for a related discussion). In particular, the augmented Lagrangian method which we will present below can be seen as a generalization of the algorithm from [6, 121, 176] for VIs with nonlinear programming constraints.

The results in this chapter are based on the publications [129, 133], the preprint [128], and the arguments and proofs from Chapter 4. The following is an outline of the structure of the chapter. In Section 5.1, we discuss the augmented Lagrangian method from a general point of view, demonstrate how the algorithm can be motivated, and analyze its relationship to the corresponding method for constrained optimization problems (Algorithm 4.4).

Section 5.2 is dedicated to a comprehensive convergence analysis of the augmented Lagrangian method for VIs, including the existence of penalized solutions in Section 5.2.1, the global convergence properties for VIs with convex constraints in Section 5.2.2, the primal-dual convergence characteristics for nonconvex problems in Section 5.2.3, and the rate of convergence in Section 5.2.4. Some of the arguments and proofs are straightforward adaptations of their optimization counterparts, but this chapter also includes many results which are new or different from those in Chapter 4.

In Section 5.3, we provide a more detailed account of generalized Nash equilibrium problems. In particular, we show how these problems are related to variational inequalities, indicating that many of the convergence results from Section 5.2 can readily be applied to the GNEP setting. In addition, we give a slightly different convergence analysis which takes into account the specific structure of GNEPs, both in the infinite-dimensional (Section 5.3.2) and the finite-dimensional case (Section 5.3.3).

5.1 Discussion and Statement of the Algorithm

This section provides a brief discussion of the augmented Lagrangian method from a motivational point of view. In particular, we analyze how the method is related to its counterpart from constrained optimization, state the main algorithmic framework for the chapter, and give some basic properties.

5.1.1 Relationship with Constrained Optimization

We shall now outline how the augmented Lagrangian method for variational inequalities can be deduced from that for constrained optimization problems (Algorithm 4.4). Since variational problems of the form (V) contain minimization problems as a special case, it is natural to construct the augmented Lagrangian in a manner such that the definitions for VIs and optimization problems are consistent.

Assume, for the moment, that the VI in question originates from a minimization problem of the form

$$\underset{x\in C}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x)\in K \tag{5.4}$$

with $f : X \to \mathbb{R}$ a continuously differentiable function. Thus, we have $F := f'$. The augmented Lagrangian of (5.4) in the optimization sense takes on the form

$$\mathcal{L}_\rho^{\mathrm{Opt}}(x,\lambda) = f(x) + \frac{\rho}{2} d_{\mathcal{K}}^2\left(G(x)+\frac{\lambda}{\rho}\right) - \frac{\|\lambda\|_H^2}{2\rho}, \tag{5.5}$$

see Definition 4.2, where the superscript $^{\mathrm{Opt}}$ emphasizes the fact that this is the augmented Lagrangian corresponding to the optimization problem (5.4).

The augmented Lagrangian method for (5.4) now generates a sequence of constrained minimization problems of the form $\min_{x\in C} \mathcal{L}_\rho^{\mathrm{Opt}}(x,\lambda)$. Since C is a convex set, these problems correspond to the variational inequalities

$$x\in C, \quad \left\langle D_x\mathcal{L}_\rho^{\mathrm{Opt}}(x,\lambda), y-x\right\rangle \ge 0 \quad \forall y\in C,$$

and the derivative $D_x \mathcal{L}_\rho^{\mathrm{Opt}}(x,\lambda)$ can be written as

$$D_x \mathcal{L}_\rho^{\mathrm{Opt}}(x,\lambda) = f'(x) + \rho G'(x)^* \left[G(x) + \frac{\lambda}{\rho} - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho} \right) \right].$$

This immediately suggests an appropriate definition of the augmented Lagrangian for a general VI of the form (V).

Definition 5.1 (Augmented Lagrangian)**.** For $\rho > 0$, the *augmented Lagrange function* or *augmented Lagrangian* of (V) is the function $\mathcal{L}_\rho : X \times H \to X^*$,

$$\mathcal{L}_\rho(x,\lambda) := F(x) + \rho G'(x)^* \left[G(x) + \frac{\lambda}{\rho} - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho} \right) \right]. \tag{5.6}$$

In view of the above motivation, this definition is consistent with the corresponding one for constrained optimization problems (Definition 4.2). Note that, if $\mathcal{K}$ is a closed convex *cone*, then we can simplify the above formula to $\mathcal{L}_\rho(x,\lambda) = F(x) + G'(x)^* P_{\mathcal{K}^\circ}(\lambda + \rho G(x))$ by using the Moreau decomposition (Lemma 2.37).

5.1.2 Statement of the Method

We now present the augmented Lagrangian method for the variational inequality (V). For the construction of our algorithm, we will need a means of controlling the penalty parameters. To this end, we define the utility function

$$V(x,\lambda,\rho) := \left\| G(x) - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho} \right) \right\|_H. \tag{5.7}$$

This function is carried over from the augmented Lagrangian method for constrained optimization problems, where it arises from the slack variable transformation discussed in Section 4.1.2. The function (5.7) can be seen as a composite measure of feasibility and complementarity.

Algorithm 5.2 (ALM for variational inequalities)**.** Let $(x^0,\lambda^0) \in X \times H$, $\rho_0 > 0$, let $B \subseteq H$ be a nonempty bounded set, $\gamma > 1$, $\tau \in (0,1)$, and set $k := 0$.

Step 1. If (x^k,λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in B$ and compute an approximate solution x^{k+1} of the VI

$$x \in C, \quad \left\langle \mathcal{L}_{\rho_k}(x,w^k), y - x \right\rangle \geq 0 \quad \forall y \in C. \tag{5.8}$$

Step 3. Update the vector of multipliers to

$$\lambda^{k+1} := \rho_k \left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right]. \tag{5.9}$$

Step 4. Let $V_{k+1} := V(x^{k+1}, w^k, \rho_k)$ and set

$$\rho_{k+1} := \begin{cases} \rho_k, & \text{if } k = 0 \text{ or } V_{k+1} \leq \tau V_k, \\ \gamma \rho_k, & \text{otherwise.} \end{cases} \tag{5.10}$$

Step 5. Set $k \leftarrow k+1$ and go to Step 1.

The above algorithm can be seen as a natural extension of the augmented Lagrangian method for constrained optimization problems (Algorithm 4.4). Indeed, if $F = f'$ for some differentiable function $f : X \to \mathbb{R}$, then (V) represents a first-order necessary condition of the minimization of f on Φ, and the augmented subproblems generated by Algorithm 5.2 can be viewed as first-order necessary conditions of the augmented subproblems generated by Algorithm 4.4. In particular, the KKT systems of these problems are equivalent; thus, if one decides to apply the augmented Lagrangian scheme solely in terms of the KKT conditions of the original problem and the augmented subproblems, then the two algorithms coincide.

Apart from the above remark, we shall not discuss Algorithm 5.2 in too much detail since the discussion is essentially the same as that of Algorithm 4.4 in Chapter 4. In particular, the boundedness of w^k is crucial to the algorithm, and it is natural to choose w^k as

$$w^k := P_B(\lambda^k) \quad \text{for all } k,$$

where B is a simple but large bounded subset of H. Another property which deserves to be highlighted at this point is the following. The definition of the augmented Lagrangian (in the VI sense) and of λ^{k+1} in Step 3 implies that

$$\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}(x^{k+1}, \lambda^{k+1}) \quad \text{for all } k \in \mathbb{N}. \tag{5.11}$$

This property will be crucial in the subsequent discussion since it allows us to analyze the primal-dual convergence properties of Algorithm 5.2 in terms of asymptotic KKT-type conditions.

The following result contains some basic properties of the algorithm which hold regardless of the choice of x^{k+1} in Step 2. Its proof is identical to that of Lemma 4.5 and therefore omitted.

Lemma 5.3. *We have $\lambda^k \in \mathcal{K}_\infty^\circ$ for all k. Moreover, there is a null sequence $\{r_k\} \subseteq \mathbb{R}_+$ such that $\left(\lambda^k, y - G(x^k)\right) \le r_k$ for all $y \in \mathcal{K}$ and $k \in \mathbb{N}$.*

As in the optimization case, the assertions of the above lemma and the formulation of Algorithm 5.2 can be simplified if $\mathcal{K}$ is a closed convex *cone* (see Example 4.6).

Let us close this section by remarking that the augmented subproblems (5.8) can also be interpreted as variational inequalities of the second kind. Indeed, let $w^k \in H$ and $\rho_k > 0$ be given, and consider the variational problem

$$x \in C, \quad \langle F(x), y - x\rangle + \frac{\rho_k}{2}\left[P_{\rho_k}(y, w^k) - P_{\rho_k}(x, w^k)\right] \ge 0 \quad \forall y \in C, \tag{5.12}$$

where $P_\rho(x, w) := d_{\mathcal{K}}^2(G(x) + w/\rho)$ is the penalization term which forms the basis of the augmented Lagrangian approach. Observe that (5.12) is well-defined even if G is nonsmooth. If G is continuously differentiable, then a directional derivative argument implies that any solution of (5.12) necessarily satisfies the variational inequality of the first kind given in (5.8). The converse holds provided that $P_\rho(\cdot, w)$ is convex, which is the case if G is $\mathcal{K}_\infty$-concave in the sense of Section 2.2.3.

5.2 Convergence Theory

This section provides a systematic convergence analysis of Algorithm 5.2. As we shall see, it is often possible to adapt the arguments and proof techniques used in constrained optimization (see Chapter 4) in order to obtain convergence results for VIs. Whenever the adaptation is straightforward (e.g., it merely amounts to replacing the derivative f' from the optimization context by the operator F), we will either omit or shorten the corresponding proofs.

On the other hand, there are some scenarios where different arguments are necessary. This is because variational problems do not admit the use of descent properties or, more generally, they do not allow us to compare function values in order to get an indicator of optimality. This is in stark contrast to the optimization case, see, for instance, Sections 4.2.2 and 4.3.1.

5.2.1 Existence of Penalized Solutions

As a first step in the convergence analysis, we analyze situations in which the augmented subproblems (5.8) are guaranteed to admit solutions for all k. Since we are not necessarily dealing with a constrained minimization problem, we cannot invoke the arguments from Section 4.2.1 to obtain the existence of approximate solutions. Instead, we have to assume either some form of compactness or coercivity and apply the general existence theorem for VIs (Theorem 3.40).

Proposition 5.4. *Assume that C is weakly compact, F is pseudomonotone, and either*

(i) *G is $\mathcal{K}_\infty$-concave on C, or*

(ii) *G and G' are completely continuous on C.*

Then the augmented Lagrangian subproblems (5.8) *admit solutions for all k.*

Proof. (i) For $k \in \mathbb{N}$, let $h_k(x) := d_{\mathcal{K}}^2(G(x) + w^k/\rho_k)$. Then h_k is convex, continuously differentiable, and $\mathcal{L}_{\rho_k}(x, w^k) = F(x) + (\rho_k/2)h_k'(x)$. Consider the mapping

$$\Psi_k(x, y) := \langle F(x), x - y\rangle + \frac{\rho_k}{2}\big[h_k(x) - h_k(y)\big].$$

By Theorem 3.40, there exists a point $\hat{x} \in C$ such that $\Psi_k(\hat{x}, y) \le 0$ for all $y \in C$. Thus, the point $\hat{x}$ is a maximizer of $\Psi_k(\hat{x}, \cdot)$, with maximum value equal to zero. By Lemma 3.1, this implies $D_y\Psi_k(\hat{x}, \hat{x})(y - \hat{x}) \le 0$ for all $y \in C$, and it is easy to check that this is precisely the desired variational inequality.

(ii) In this case, it follows from Lemma 3.37 that the augmented Lagrangian $\mathcal{L}_{\rho_k}(\cdot, w^k)$ is pseudomonotone for every k. Hence, the claim follows from Corollary 3.41. □

In many cases, the weak compactness of C can be substituted by an appropriate form of coercivity. The basic result in this direction is the following.

Proposition 5.5. *Assume that X is reflexive, F is strongly monotone, and $G : X \to Y$ is $\mathcal{K}_\infty$-concave. Then the augmented subproblems* (5.8) *admit solutions for all k.*

Proof. Note that the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is the sum of F and the derivative of the convex function $x \mapsto (\rho/2)d_{\mathcal{K}}^2(G(x) + w/\rho)$. This implies that $\mathcal{L}_\rho(\cdot, w)$ is strongly monotone for all $\rho > 0$ and $w \in H$. The claim is therefore a consequence of Theorem 3.40 and Remark 3.42. □

5.2.2 Convergence for Convex Constraints

We now analyze the global convergence characteristics for the augmented Lagrangian method. In this section, we are mainly concerned with the case of convex constraints. As discussed in Section 2.2.3, the natural analytic notion of convexity is the $\mathcal{K}_\infty$-concavity of the constraint mapping G.

In addition to this property, we will also need a certain assumption on the manner in which the augmented subproblems (5.8) are solved. Observe that these problems can be written as $-\mathcal{L}_{\rho_k}(x, w^k) \in \mathcal{N}_C(x)$. Thus, a natural assumption is

$$\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$$

for some null sequence $\{\varepsilon^k\} \subseteq X^*$. This is consistent with a similar assumption made in Section 4.2.3. To obtain the optimality of limit points, we will also need an appropriate continuity property of the mapping F. The following is a summary of the assumptions we will use in this section.

Assumption 5.6 (Convex constraints)**.** We assume that

(i) the mapping F is bounded and pseudomonotone,

(ii) G is continuously differentiable on X and $\mathcal{K}_\infty$-concave on C, and

(iii) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Similar to before, our analysis first deals with the attainment of feasibility and then with optimality. Since we are working in the setting of convex constraints, we can expect the iterates to be asymptotically feasible. This is indeed the case.

Lemma 5.7. *Let Assumption 5.6 hold, and let $\bar{x}$ be a weak limit point of the sequence $\{x^k\}$. Then $\bar{x}$ is a minimizer of the convex function $d_{\mathcal{K}} \circ G$ on C. In particular, if the feasible set Φ of (V) is nonempty, then $\bar{x} \in \Phi$.*

Proof. Note that the function $d_{\mathcal{K}} \circ G$ is convex by Theorem 2.50 and continuous, hence weakly sequentially lower semicontinuous (by Proposition 2.44). If $\{\rho_k\}$ remains bounded, then the penalty updating scheme (5.10) implies that

$$d_{\mathcal{K}}(G(x^{k+1})) \le \left\| G(x^{k+1}) - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right\|_H \to 0$$

and therefore $d_{\mathcal{K}}(G(\bar{x})) = 0$. We now assume that $\rho_k \to \infty$ and define the auxiliary functions $h_k(x) = d_{\mathcal{K}}^2(G(x) + w^k/\rho_k)$. Note that h_k is continuously differentiable by Lemma 2.43. Let $x^{k+1} \rightharpoonup_I \bar{x}$ for some (infinite) subset $I \subseteq \mathbb{N}$ and assume that there is a point $y \in X$ with $d_{\mathcal{K}}(G(y)) < d_{\mathcal{K}}(G(\bar{x}))$. The weak sequential lower semicontinuity of $d_{\mathcal{K}} \circ G$ and the boundedness of $\{w^k\}$ imply that

$$\liminf_{k\in I} h_k(x^{k+1}) = \liminf_{k\in I} d_{\mathcal{K}}^2\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right) \geq d_{\mathcal{K}}^2(G(\bar{x}))$$

and $h_k(y) \to d_{\mathcal{K}}^2(G(y))$. Hence, there is a constant $c_1 > 0$ such that $h_k(x^{k+1}) - h_k(y) \geq c_1$ for all $k \in I$ sufficiently large. Since h_k is convex by Theorem 2.50, it follows that

$$\left\langle h_k'(x^{k+1}), y - x^{k+1}\right\rangle \leq h_k(y) - h_k(x^{k+1}) \leq -c_1 \tag{5.13}$$

for all $k \in I$ sufficiently large. Now, let $\{\varepsilon^k\}$ be the sequence from Assumption 5.6. Using Lemma 2.43 for the derivative of h_k, we obtain

$$\begin{aligned}\left\langle \varepsilon^{k+1}, y - x^{k+1}\right\rangle &\leq \left\langle \mathcal{L}_{\rho_k}(x^{k+1}, w^k), y - x^{k+1}\right\rangle \\ &= \left\langle F(x^{k+1}), y - x^{k+1}\right\rangle + \frac{\rho_k}{2}\left\langle h_k'(x^{k+1}), y - x^{k+1}\right\rangle.\end{aligned}$$

Since F is a bounded operator by Assumption 5.6, there is a constant $c_2 \in \mathbb{R}$ such that $\langle F(x^{k+1}), y - x^{k+1}\rangle \leq c_2$ for all $k \in I$. This together with (5.13) implies

$$\left\langle \varepsilon^{k+1}, y - x^{k+1}\right\rangle \leq c_2 - \frac{\rho_k c_1}{2} \to -\infty.$$

Since $\{x^{k+1}\}_{k\in I}$ is bounded and $\varepsilon^k \to 0$, this is a contradiction. □

The above result guarantees that every weak limit point $\bar{x}$ automatically minimizes the constraint violation even if the feasible set Φ is empty. This is not unlike similar results which we have already discovered for constrained optimization problems, see, for instance, Lemma 4.11 or Lemma 4.15.

We now turn to the main global convergence result.

Theorem 5.8. *Let Assumption 5.6 hold, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. If the feasible set Φ of (V) is nonempty, then $\bar{x}$ is feasible and a solution of (V).*

Proof. Let $x^{k+1} \rightharpoonup_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$. The feasibility of $\bar{x}$ follows from Lemma 5.7. For the optimality, let $y \in \Phi$ be any feasible point. Then $\langle \mathcal{L}_{\rho_k}(x^{k+1}, w^k), y - x^{k+1}\rangle \geq \langle \varepsilon^{k+1}, y - x^{k+1}\rangle$ by Assumption 5.6 and, using (5.11), we get

$$\begin{aligned}\left\langle \varepsilon^{k+1}, y - x^{k+1}\right\rangle &\leq \left\langle F(x^{k+1}) + G'(x^{k+1})^*\lambda^{k+1}, y - x^{k+1}\right\rangle \\ &= \left\langle F(x^{k+1}), y - x^{k+1}\right\rangle + \left(\lambda^{k+1}, G'(x^{k+1})(y - x^{k+1})\right) \\ &\leq \left\langle F(x^{k+1}), y - x^{k+1}\right\rangle + \left(\lambda^{k+1}, G(y) - G(x^{k+1})\right),\end{aligned}$$

where we used the fact that $x \mapsto (\lambda^{k+1}, G(x))$ is convex by Theorem 2.50 and Lemma 5.3. Using again Lemma 5.3, we now obtain $\langle F(x^{k+1}), y - x^{k+1}\rangle \geq \langle \varepsilon^{k+1}, y - x^{k+1}\rangle + r_{k+1}$ with a null sequence $\{r_k\} \subseteq \mathbb{R}$. Since y is arbitrary and F is pseudomonotone, it follows from Proposition 3.43 that $\langle F(\bar{x}), y - \bar{x}\rangle \geq 0$ for all $y \in \Phi$, and the proof is complete. □

If we assume the strong monotonicity of F, then (V) admits a unique solution $\bar{x} \in \Phi$, and the iterates generated by Algorithm 5.2 converge strongly to $\bar{x}$.

Corollary 5.9. *Let Assumption 5.6 hold, let X be reflexive, and F strongly monotone on C. Then $\{x^k\}$ converges strongly to the unique solution of (V).*

Proof. Existence and uniqueness of the solution follow from standard arguments, see Section 3.2.1. Let $\bar{x} \in C$ be the solution. For the proof of convergence, observe that

$$c\|x^{k+1} - \bar{x}\|_X^2 \le \langle F(x^{k+1}) - F(\bar{x}), x^{k+1} - \bar{x}\rangle \tag{5.14}$$

for all k and some $c > 0$. We first show that $\{x^k\}$ is bounded. The proof of Theorem 5.8 shows that $\langle F(x^{k+1}), y - x^{k+1}\rangle \ge \langle \varepsilon^{k+1}, y - x^{k+1}\rangle + r_{k+1}$ for all $y \in \Phi$ and $k \in \mathbb{N}$, where $\{\varepsilon^k\} \subseteq X^*$ and $\{r_k\} \subseteq \mathbb{R}$ are the null sequences from Assumption 5.6 and Lemma 5.3, respectively. Inserting $y := \bar{x}$ and applying this inequality to (5.14), it follows that

$$c\|x^{k+1} - \bar{x}\|_X^2 \le \left\langle \varepsilon^{k+1} - F(\bar{x}), x^{k+1} - \bar{x}\right\rangle - r_{k+1}. \tag{5.15}$$

This implies the existence of an $M > 0$ such that $c\|x^{k+1} - \bar{x}\|_X^2 \le M\|x^{k+1} - \bar{x}\|_X - r_{k+1}$ for all k, which yields the boundedness of $\{x^k\}$. Since X is reflexive and $\bar{x}$ is the unique solution of (V), it now follows from Theorem 5.8 that $x^{k+1} \rightharpoonup \bar{x}$. Thus, $\langle F(\bar{x}), x^{k+1} - \bar{x}\rangle \to 0$, and (5.15) finally yields $\|x^{k+1} - \bar{x}\|_X \to 0$. □

5.2.3 Primal-Dual Convergence

We now state convergence theorems based on the KKT conditions of (V). The results and proofs in this section are basically identical to those from Section 4.2.3, the only modification being that we now deal with a general mapping $F : X \to X^*$ instead of the derivative f' which occurs in the optimization context. As a consequence, we omit the proofs of the subsequent results. More details can be found in Section 4.2.3.

Assumption 5.10 (Convergence to KKT points)**.** We assume that

(i) F is bounded and pseudomonotone,

(ii) G is continuously differentiable on X,

(iii) G and G' are completely continuous on C, and

(iv) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

The above is essentially an adaptation of Assumption 4.14. Note that (iv) is an inexact version of the VI subproblem (5.8). Indeed, this problem can be written as

$$-\mathcal{L}_{\rho_k}(x, w^k) \in \mathcal{N}_C(x),$$

and condition (iv) in Assumption 5.10 states that x^{k+1} satisfies this condition up to an error term $\varepsilon^{k+1} \in X^*$ which vanishes asymptotically.

We now turn to the convergence analysis of Algorithm 5.2 under Assumption 5.10. As always, we begin by considering the asymptotic feasibility of the iterates. The proof of the following result is identical to that of Lemma 4.15.

Lemma 5.11. *Let $\{x^k\}$ be generated by Algorithm 5.2 under Assumption 5.10, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x}$ is a stationary point of the problem* $\min_{x \in C} d_{\mathcal{K}}^2(G(x))$.

As in the optimization case, there are multiple special cases where Lemma 5.11 guarantees the actual feasibility of the point $\bar{x}$. First, if the mapping G is $\mathcal{K}_\infty$-concave (see Section 2.2.3), then $d_{\mathcal{K}}^2 \circ G$ is a convex function. Hence, in this case, it follows that $\bar{x}$ is a global minimizer of this function, and if the feasible set is nonempty, then $\bar{x}$ is a feasible point. The second special case arises if the extended RCQ is satisfied in $\bar{x}$. In that case, the feasibility of $\bar{x}$ follows from Proposition 3.22.

We now turn to the optimality of limit points. As in the setting of constrained optimization, the main tool is the asymptotic KKT property of the primal-dual sequence $\{(x^k, \lambda^k)\}$, which in this case takes on the form (for $k \geq 1$)

$$\varepsilon^k - \mathcal{L}'(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \langle \lambda^k, y - G(x^k) \rangle \leq r_k \ \forall y \in K \tag{5.16}$$

with a null sequence $\{r_k\} \subseteq \mathbb{R}$. Note that (5.16) follows from Assumption 5.10 (iv), the fact that $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}(x^{k+1}, \lambda^{k+1})$ for all k, and Lemma 5.3.

Theorem 5.12. *Let $\{(x^k, \lambda^k)\}$ be generated by Algorithm 5.2 under Assumption 5.10, let $x^{k+1} \rightharpoonup_I \bar{x}$ for some index set $I \subseteq \mathbb{N}$, and let $\bar{x}$ satisfy ERCQ with respect to the constraint system of (V). Then $\bar{x}$ is a stationary point (and a solution) of (V), the sequence $\{\lambda^{k+1}\}_{k \in I}$ is bounded in Y^*, and its weak-* limit points belong to $\Lambda(\bar{x})$.*

We now turn to the special case of nonlinear programming-type VIs. Recall that, if X is finite-dimensional, then property (i) from Assumption 5.10 is equivalent to the continuity of F. Moreover, due to the special structure of the constraints, we can use the CPLD constraint qualification to obtain the optimality of limit points.

Theorem 5.13. *Let $X := C := \mathbb{R}^n$, $Y := H := \mathbb{R}^m$, and $K := \mathcal{K} := \mathbb{R}^m_-$ for some $m, n \in \mathbb{N}$. Let $\{x^k\}$ be generated by Algorithm 5.2 under Assumption 5.10, and $\bar{x}$ a limit point of $\{x^k\}$. If $\bar{x}$ is feasible and CPLD holds in $\bar{x}$, then $\bar{x}$ is a stationary point and a solution of (V).*

As in the optimization case, it is possible to prove a stronger assertion for the dual sequence under the assumption that $\bar{x}$ is a solution of (V) and $G'(\bar{x})$ is surjective. The proof of this result is identical to that of Proposition 4.20.

Proposition 5.14. *Let $\{x^k\}$ be generated by Algorithm 5.2 and let $x^{k+1} \rightharpoonup_I \bar{x}$ for some $I \subseteq \mathbb{N}$ and $\bar{x} \in X$. Assume that $\bar{x}$ is a solution of (V), that $C = X$, F is weak-* sequentially continuous, G' is completely continuous, and that $G'(\bar{x})$ is surjective. Then $\{\lambda^{k+1}\}_{k \in I}$ converges weak-* to the unique element in $\Lambda(\bar{x})$.*

5.2.4 Rate of Convergence

We now analyze the convergence of the augmented Lagrangian method from a quantitative point of view. The theory below is essentially identical to that of Section 4.3.2 for optimization problems, and it crucially depends on the primal-dual error bound from Section 3.2.4, which in the present case takes on the form

$$c_1\Theta(x,\lambda) \le \|x-\bar{x}\|_X + \|\lambda-\bar{\lambda}\|_H \le c_2\Theta(x,\lambda) \tag{5.17}$$

for all $(x,\lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x,\lambda)$ sufficiently small, where Θ is the residual

$$\Theta(x,\lambda) := \|x - P_C(x - \mathcal{L}(x,\lambda))\|_X + \|G(x) - P_{\mathcal{K}}(G(x)+\lambda)\|_H.$$

Here, $\mathcal{L}$ stands for the Lagrangian in the variational inequality sense (see Section 3.2). This assumption along with some other properties are collected below.

Assumption 5.15 (Rate of convergence)**.** We assume that

(i) X is a real Hilbert space, F is continuous, and G continuously differentiable on X,

(ii) $(\bar{x},\bar{\lambda}) \in X \times H$ is a KKT point of (V) which satisfies the error bound (5.17),

(iii) the primal-dual sequence $\{(x^k,\lambda^k)\}$ converges strongly to $(\bar{x},\bar{\lambda})$ in $X \times H$,

(iv) the safeguarded multiplier sequence satisfies $w^k := \lambda^k$ for k sufficiently large, and

(v) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1},w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

One assumption which might merit some discussion is the convergence $(x^k,\lambda^k) \to (\bar{x},\bar{\lambda})$ in $X \times H$. This assumption was also used for optimization problems in Chapter 4. Sufficient conditions for the convergence $x^k \to \bar{x}$ can be found in that chapter (for constrained optimization), and in Corollary 5.9 for the VI case. In addition, the following result provides some useful information in this regard and also shows that the primal convergence $x^k \to \bar{x}$ implies the convergence of $\{\lambda^k\}$ to $\bar{\lambda}$ in H. The proof of this result is identical to that of Proposition 4.29.

Proposition 5.16. *Let Assumption 5.15* (i), (ii), (v) *hold, and let RCQ hold in $\bar{x}$ with respect to the space H. Then there exists $r > 0$ such that, if $x^k \in B_r(\bar{x})$ for sufficiently large k, then $\Theta(x^k,\lambda^k) \to 0$ and $(x^k,\lambda^k) \to (\bar{x},\bar{\lambda})$ strongly in $X \times H$.*

The basic approach to deduce convergence rates is to first consider the sequence $\theta_k := \Theta(x^k,\lambda^k)$ which, due to the error bound (5.17), converges to zero with the same order as the distance of (x^k,λ^k) to $(\bar{x},\bar{\lambda})$. The main convergence result is the following, which is a simple adaptation of Theorem 4.31.

Theorem 5.17. *Let Assumption 5.15 hold and assume that $\varepsilon^{k+1} = o(\theta_k)$. Then:*

(a) *For every $q \in (0,1)$, there exists $\bar{\rho}_q > 0$ such that, if $\rho_k \ge \bar{\rho}_q$ for sufficiently large k, then $(x^k,\lambda^k) \to (\bar{x},\bar{\lambda})$ Q-linearly in $X \times H$ with rate q.*

(b) *If $\rho_k \to \infty$, then $(x^k,\lambda^k) \to (\bar{x},\bar{\lambda})$ Q-superlinearly in $X \times H$.*

The assumption $\varepsilon^{k+1} = o(\theta_k)$ in the above theorem says that, roughly speaking, the degree of inexactness should be small enough to not affect the rate of convergence. Note that we are comparing ε^{k+1} to the optimality measure θ_k of the previous iterates (x^k,λ^k). Hence, it is easy to ensure this condition in practice, for instance, by always computing the next iterate x^{k+1} with a precision $\|\varepsilon^{k+1}\|_X \le z_k\theta_k$ for some fixed null sequence z_k.

The following result can be shown in the same manner as Corollary 4.32.

Corollary 5.18. *Let Assumption 5.15 hold and assume that the subproblems occurring in Algorithm 5.2 are solved exactly, i.e., that* $\varepsilon^k = 0$ *for all* k*. Then* $\{\rho_k\}$ *remains bounded.*

The boundedness of $\{\rho_k\}$ obviously rules out the Q-superlinear convergence of Theorem 4.31 (b). However, the former is usually considered more significant in practice since it prevents the subproblems from becoming excessively ill-conditioned.

If inexact solutions are allowed for the augmented Lagrangian subproblems, then it is possible to guarantee the boundedness of $\{\rho_k\}$ by using the same modified penalty updating scheme as in Remark 4.33, see [133] for more details.

5.3 Generalized Nash Equilibrium Problems

We now turn our attention to an important class of variational inequalities, the so-called *generalized Nash equilibrium problems (GNEPs).* Let $N \in \mathbb{N}$ be a natural number (the number of players), let each player $\nu = 1, \dots, N$ be given a real Banach space X_ν, and let $X := X_1 \times \dots \times X_N$. We write $x = (x^1, \dots, x^N)$ for a generic element in X, and will often use the notation $x = (x^\nu, x^{-\nu})$ to emphasize the role of player ν's variable in the vector x. In this notation, we have

$$x^\nu \in X_\nu \quad \text{and} \quad x^{-\nu} \in X_{-\nu} := \prod_{\mu \neq \nu} X_\mu.$$

Note that this is merely a matter of notational convenience and clarity. In particular, it does not entail any kind of reordering of the spaces X_ν or the components x^ν which constitute x.

Assume now that each player ν is given a continuously differentiable objective function $f_\nu : X \to \mathbb{R}$, and that $\Phi \subseteq X$ is a nonempty closed set. The GNEP we consider takes on the form

$$\underset{x^\nu \in X_\nu}{\text{minimize}}\ f_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad (x^\nu, x^{-\nu}) \in \Phi. \tag{5.18}$$

Observe that f_ν depends on the whole variable x, but player ν attempts to minimize f_ν with respect to x^ν only. The constraint $x = (x^\nu, x^{-\nu}) \in \Phi$ is often called a *joint* or *shared constraint* since it is the same for every player. If Φ is a convex set and the functions f_ν are convex with respect to x^ν for all ν, then we call the GNEP *jointly convex.*

Note that, if the set Φ has a product representation of the form $\Phi = \Phi_1 \times \dots \times \Phi_N$, with $\Phi_\nu \subseteq X_\nu$ for all ν, then we can rewrite the GNEP (5.18) as

$$\underset{x^\nu \in X_\nu}{\text{minimize}}\ f_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad x^\nu \in \Phi_\nu, \tag{5.19}$$

which is called a *(standard) Nash equilibrium problem (NEP).*

For the most part of this section, we assume that the feasible set has an analytical representation of the form

$$\Phi = \{x \in C : G(x) \in K\}, \tag{5.20}$$

where Y is a real Banach space, $G : X \to Y$ a continuously differentiable mapping, and $K \subseteq Y$ a nonempty closed convex set. For the sake of simplicity, the set C is assumed to be of the form $C := \prod_{\nu=1}^{N} C_\nu$ with nonempty closed convex sets $C_\nu \subseteq X_\nu$. We will use this representation to derive KKT-type conditions for GNEPs and to apply the augmented Lagrangian method for their solution.

This section is divided into three parts. We begin by establishing the theoretical foundations of GNEPs in Banach spaces and the relationship between jointly constrained GNEPs and variational inequalities. After that, we present the augmented Lagrangian method in a general Banach space setting and give a brief convergence analysis. Special emphasis will be placed on a comparison to the corresponding algorithm for variational inequalities (Algorithm 5.2). Finally, in Section 5.3.3, we specialize some of the results and assumptions for finite-dimensional problems, thus paving the way for practical applications in a finite-dimensional framework.

5.3.1 Theoretical Background

We begin with some general discussions and definitions for the GNEP (5.18). The concepts below do not depend on the specific form (5.20) of the feasible set but hold if Φ is an arbitrary nonempty subset of X.

For a given player ν and a point $x^{-\nu} \in X_{-\nu}$, let

$$\Phi_\nu(x^{-\nu}) := \{x^\nu \in X_\nu : (x^\nu, x^{-\nu}) \in \Phi\}$$

be the set of feasible points for player ν (with respect to $x^{-\nu}$). Note that, if the GNEP is a NEP, then Φ_ν is independent of $x^{-\nu}$ for all ν.

Definition 5.19. Let $\bar{x} \in \Phi$ be a feasible point. We say that $\bar{x}$ is a

(a) *generalized Nash equilibrium* or simply a *solution* of the GNEP if, for every ν,

$$f_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \le f_\nu(y^\nu, \bar{x}^{-\nu}) \quad \text{for all } y^\nu \in \Phi_\nu(\bar{x}^{-\nu}). \tag{5.21}$$

(b) *normalized (Nash) equilibrium* of the GNEP if

$$\sum_{\nu=1}^{N} f_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \le \sum_{\nu=1}^{N} f_\nu(y^\nu, \bar{x}^{-\nu}) \quad \text{for all } y \in \Phi. \tag{5.22}$$

Note that every normalized equilibrium is also a generalized Nash equilibrium, which can be seen by inserting points of the form $y := (y^\nu, \bar{x}^{-\nu})$ into (5.22). The converse however is not true in general. For NEPs, it is easy to see that both concepts are equivalent.

The notion of normalized equilibria is closely linked to the *Nikaido–Isoda* function

$$\Psi(x, y) := \sum_{\nu=1}^{N} \left[f_\nu(x^\nu, x^{-\nu}) - f_\nu(y^\nu, x^{-\nu}) \right]. \tag{5.23}$$

It is evident that a point $\bar{x} \in \Phi$ is a normalized equilibrium if and only if

$$\Psi(\bar{x}, y) \le 0 \quad \forall y \in \Phi. \tag{5.24}$$

This is an equilibrium problem in the sense of Section 2.2.4. Observe that $\Psi(x,x) = 0$ for all $x \in X$. Hence, if $\bar{x}$ satisfies (5.24), then it is a maximizer of $\Psi(\bar{x},\cdot)$ on Φ, with optimal value equal to zero. It therefore follows from standard first-order necessary conditions (e.g., Lemma 3.1) that $D_y\Psi(\bar{x},\bar{x})d \le 0$ for all $d \in \mathcal{T}_\Phi(\bar{x})$. This condition can be rewritten as the variational inequality (VI)

$$\bar{x} \in \Phi, \quad \langle F(\bar{x}), d\rangle \ge 0 \quad \forall d \in \mathcal{T}_\Phi(\bar{x}), \tag{5.25}$$

where $F(x) := (D_{x^\nu} f_\nu(x))_{\nu=1}^N$. It follows that we can tackle the GNEP (5.18) by solving the VI (5.25) instead. Note that, by the preceding arguments, we have that (5.25) is always a necessary condition for $\bar{x}$ to be a normalized equilibrium of the GNEP. Moreover, it is a sufficient condition if the GNEP is jointly convex, i.e., if Φ is a convex set and the functions f_ν are convex with respect to x^ν for all ν.

Let us now discuss the existence of solutions to the GNEP (5.18). Taking into account the above arguments, it seems natural to construct an existence result by analyzing the variational inequality (5.25) and applying the basic existence result for VIs from Section 3.2.1. Assume that the GNEP is jointly convex, so that the VI (5.25) is sufficient for $\bar{x}$ to be a normalized equilibrium of the GNEP. We are then faced with the following problem: the existence result for VIs (Corollary 3.41) requires that F is a pseudomonotone operator, and it is not immediately clear how this property relates to the structure of the GNEP and the objective functions f_ν. Recall that sufficient conditions for pseudomonotonicity include either complete continuity or ordinary continuity together with monotonicity. Clearly, the former is a very restrictive property since it would require the complete continuity of all the derivatives $D_{x^\nu} f_\nu$.

On the other hand, the monotonicity of F is also a somewhat restrictive property for general GNEPs. Note that the convexity of f_ν with respect to x^ν does not imply the monotonicity of F, since the latter property would require some knowledge about the dependence of $D_{x^\nu} f_\nu$ on the whole vector x. Indeed, assuming for the moment that the functions f_ν are twice continuously differentiable, the derivative F' takes on the form

$$F'(x) = \begin{pmatrix} D^2_{x^1x^1} f_1(x) & \cdots & D^2_{x^Nx^1} f_1(x) \\ \vdots & \ddots & \vdots \\ D^2_{x^1x^N} f_N(x) & \cdots & D^2_{x^Nx^N} f_N(x) \end{pmatrix}, \tag{5.26}$$

and the monotonicity of F would be equivalent to the positivity of this block operator for all x. The convexity of f_ν with respect to x^ν yields the positivity of the diagonal blocks $D^2_{x^\nu x^\nu} f_\nu(x)$ for all x, but it does not entail any information on the off-diagonal parts of $F'(x)$. Vaguely speaking, one could expect (5.26) to be positive if, in addition to the positivity of the diagonal blocks, the operator exhibits some form of "diagonal dominance". Returning to the GNEP, this could be interpreted as the fact that player ν has more influence on his own objective than the other players do. This appears reasonable and can be expected to hold in certain applications, but the above arguments are still very vague and do not cover the GNEP in its full generality.

It follows that caution must be exercised when applying existence results for variational inequalities to the GNEP setting. A notable exception is the finite-dimensional case, where the (bounded) pseudomonotonicity of F is equivalent to simple continuity, and thus we can expect this property to hold regardless of whether F is monotone or not. In any case, we shall now present a more general existence result which covers both the finite- and infinite-dimensional case. The main idea is to directly apply the Ky–Fan minimax theorem (Lemma 2.54) to the characterization (5.24) of normalized equilibria, without appealing to the variational inequality (5.25).

Proposition 5.20. *Let $\Phi \subseteq X$ be nonempty, convex, weakly compact, and let Ψ be weakly sequentially lsc with respect to x. Then the GNEP admits a normalized equilibrium.*

The assumption that Ψ is weakly sequentially lsc with respect to x arises naturally from the Ky–Fan theorem. However, unless X is finite-dimensional (in which case it is implied by ordinary continuity), this is a nontrivial requirement due to the minus sign in the Nikaido–Isoda function (5.23). Clearly, a sufficient condition is the weak sequential lower semicontinuity of the functions

$$x \mapsto f_\nu(x^\nu, x^{-\nu}) - f_\nu(y^\nu, x^{-\nu})$$

for all ν and fixed y^ν, which can be expected to hold in certain applications. In fact, a rather common situation is $f_\nu(x) = f_\nu^1(x) + f_\nu^2(x^\nu)$, where f_ν^1 is weakly sequentially continuous (e.g., if it involves operators which are compact or completely continuous), and f_ν^2 is weakly sequentially lsc (e.g., it is convex and continuous). This setting encompasses various potential-type games as well as multiobjective optimal control problems (see Section 7.3).

We now discuss KKT-type optimality conditions for the GNEP (5.18). To that end, we assume that the constraint set Φ has an analytical representation of the form (5.20), i.e., we have

$$\Phi = \{x \in C : G(x) \in K\},$$

where $C := \prod_{\nu=1}^{N} C_\nu$ is a set of player-specific constraints, $C_\nu \subseteq X_\nu$ and $K \subseteq Y$ are nonempty closed convex sets, Y is a real Banach space, and $G : X \to Y$ a continuously differentiable mapping. Moreover, we define the *Lagrangian of player ν* as

$$\mathcal{L}^\nu : X \times Y^* \to \mathbb{R}, \quad \mathcal{L}^\nu(x, \lambda) := f_\nu(x) + \langle \lambda, G(x) \rangle. \tag{5.27}$$

Given a Nash equilibrium $\bar{x} \in \Phi$, a rather natural way to construct first-order necessary conditions is to form the KKT system for every player ν and to then concatenate these systems for all ν. This results in the following overall system which we call the KKT system of the GNEP.

Definition 5.21 (KKT point)**.** A tuple $(\bar{x}, \bar{\lambda}^1, \ldots, \bar{\lambda}^N) \in X \times (Y^*)^N$ is a *KKT point* of the GNEP if

$$-D_{x^\nu} \mathcal{L}^\nu(\bar{x}, \bar{\lambda}^\nu) \in \mathcal{N}_{C_\nu}(\bar{x}^\nu) \quad \text{and} \quad \bar{\lambda}^\nu \in \mathcal{N}_K(G(\bar{x})) \quad \text{for all } \nu.$$

We call $\bar{x} \in \Phi$ a *stationary point* of the GNEP if there exist $\bar{\lambda}^1, \ldots, \bar{\lambda}^N \in Y^*$ such that $(\bar{x}, \bar{\lambda}^1, \ldots, \bar{\lambda}^N)$ is a KKT point of the GNEP.

Since the above is just the collection of the KKT systems of all players, it follows that the relationship between the GNEP and its KKT conditions is essentially the same as for constrained optimization problems (see Section 3.1.1). In particular, if $\bar{x}$ is a generalized Nash equilibrium and a suitable constraint qualification holds for each player's optimization problem, then there exist multipliers $\bar{\lambda}^1, \ldots, \bar{\lambda}^N \in Y^*$ such that $(\bar{x}, \bar{\lambda}^1, \ldots, \bar{\lambda}^N)$ is a KKT point of the GNEP. Conversely, if $\bar{x}$ is a stationary point of the GNEP, the functions f_ν are convex with respect to x^ν for all ν, and the feasible sets $\Phi_\nu(\bar{x}^{-\nu}) \subseteq X_\nu$ are convex, then $\bar{x}$ is a solution of the GNEP.

When dealing with normalized equilibria, it is possible to give a more refined KKT system which takes advantage of the joint structure of the constraint set. Indeed, if $\bar{x}$ is a normalized equilibrium, then it is necessarily a solution of the variational inequality (5.25), and the KKT conditions of this problem take on the form (compare with Definition 3.45)

$$-F(\bar{x}) - G'(\bar{x})^* \bar{\lambda} \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})).$$

By Lemma 2.34, we have $\mathcal{N}_C(\bar{x}) = \mathcal{N}_{C_1}(\bar{x}^1) \times \cdots \times \mathcal{N}_{C_N}(\bar{x}^N)$. Thus, we arrive at the following condition.

Definition 5.22 (Normalized KKT point)**.** A point $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ is a *normalized KKT point* of the GNEP if

$$-D_{x^\nu} \mathcal{L}^\nu(\bar{x}, \bar{\lambda}) \in \mathcal{N}_{C_\nu}(\bar{x}^\nu) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})) \quad \text{for all } \nu. \tag{5.28}$$

The distinctive feature of normalized KKT points is the fact that the multiplier $\bar{\lambda}$ is the same for every player. It follows that every normalized KKT point of the GNEP is an (ordinary) KKT point of the GNEP with $\bar{\lambda}^\nu := \bar{\lambda}$ for all ν.

The connection between normalized KKT points and normalized equilibria follows from the connection between the latter and the corresponding variational inequality (5.25). Indeed, if $\bar{x} \in \Phi$ is a normalized equilibrium of the GNEP and the constraint $G(x) \in K$ satisfies a suitable constraint qualification in $\bar{x}$, then $\bar{x}$ is a normalized stationary point. The converse holds provided that Φ is a convex set and the functions f_ν are convex with respect to x^ν for all ν.

5.3.2 Problems in Banach Spaces

This subsection is dedicated to the augmented Lagrangian method (ALM) for a jointly convex GNEP of the form (5.18). On the following pages, we work with a problem whose feasible set Φ has the form (5.20), i.e.,

$$\Phi = \{x \in C : G(x) \in K\},$$

where $C := \prod_{\nu=1}^N C_\nu$ is a set of player-specific constraints, $C_\nu \subseteq X_\nu$ and $K \subseteq Y$ are nonempty closed convex sets, Y is a real Banach space, and $G : X \to Y$ a continuously differentiable mapping. The algorithm is constructed similarly to that in Section 5.1.2: we assume that there is a real Hilbert space H together with a dense embedding $i : Y \hookrightarrow H$, and that $\mathcal{K} \subseteq H$ is a closed convex set with $i^{-1}(\mathcal{K}) = K$. Thus, given $\rho > 0$, we can define the *augmented Lagrangian of player* ν as the function (compare with Definition 4.2)

$$\mathcal{L}_\rho^\nu : X \times H \to \mathbb{R}, \quad \mathcal{L}_\rho^\nu(x,\lambda) := f_\nu(x) + \frac{\rho}{2} d_\mathcal{K}^2\left(G(x) + \frac{\lambda}{\rho}\right) - \frac{\|\lambda\|_H^2}{2\rho}.$$

As in the optimization case, we note that the last term can be omitted since it plays no role in the minimization of $\mathcal{L}_\rho^\nu$ with respect to x^ν.

For the definition of our penalty updating scheme, we also define the auxiliary function

$$V(x,\lambda,\rho) = \left\| G(x) - P_\mathcal{K}\left(G(x) + \frac{\lambda}{\rho}\right)\right\|_H. \tag{5.29}$$

This enables us to formulate our algorithm as follows.

Algorithm 5.23 (ALM for jointly convex GNEPs)**.** Let $(x^0,\lambda^0) \in X \times H$, $\rho_0 > 0$, let $B \subseteq H$ be a nonempty bounded set, $\gamma > 1$, $\tau \in (0,1)$, and set $k := 0$.

Step 1. If (x^k,λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in B$ and compute an approximate solution x^{k+1} of the NEP consisting of the minimization problems

$$\underset{x^\nu \in C_\nu}{\text{minimize}} \; \mathcal{L}_{\rho_k}^\nu(x^\nu, x^{-\nu}, w^k). \tag{5.30}$$

Step 3. Update the vector of multipliers to

$$\lambda^{k+1} := \rho_k\left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_\mathcal{K}\left(G(x^{k+1}) + \frac{w^k}{\rho_k}\right)\right]. \tag{5.31}$$

Step 4. Let $V_{k+1} := V(x^{k+1}, w^k, \rho_k)$ and set

$$\rho_{k+1} := \begin{cases} \rho_k, & \text{if } k = 0 \text{ or } V_{k+1} \le \tau V_k, \\ \gamma\rho_k, & \text{otherwise.} \end{cases} \tag{5.32}$$

Step 5. Set $k \leftarrow k+1$ and go to Step 1.

Let us stress that, for all intents and purposes, Algorithm 5.23 can be seen as a special case of Algorithm 5.2. The only difference is that the augmented subproblems now take on the form of Nash equilibrium problems. These can be rewritten as variational inequalities, in which case they take on the form (5.8), but the Nash equilibrium framework may facilitate the use of subproblem solution methods which take into account the specific Nash structure.

The convergence of Algorithm 5.23 can be shown in different ways, in particular by simply appealing to the results in Section 5.2. Here, we present a slightly different analysis which takes into account the Nikaido–Isoda function (5.23) and can therefore be considered GNEP-specific. The following are the assumptions which we use. For the sake of brevity, we write

$$\mathcal{L}_k^\nu(x) := \mathcal{L}_{\rho_k}^\nu(x, w^k). \tag{5.33}$$

Assumption 5.24 (Global convergence for GNEPs)**.** We assume that

(i) for every $x \in X$, the functions $f_\nu(\cdot, x^{-\nu})$ are convex and differentiable,

(ii) the operator G is $\mathcal{K}_\infty$-concave and differentiable,

(iii) the Nikaido–Isoda function Ψ is weakly sequentially lsc with respect to x, and

(iv) there is a null sequence $\{\varepsilon^k\} \subseteq X^* = X_1^* \times \cdots \times X_N^*$ such that, for all ν and k,

$$x^{\nu,k+1} \in C_\nu \quad \text{and} \quad \varepsilon^{\nu,k+1} - D_{x^\nu} \mathcal{L}_k^\nu(x^{k+1}) \in \mathcal{N}_{C_\nu}(x^{\nu,k+1}).$$

In many ways, the above assumptions are similar to those used in Section 5.2.2. The main difference is that we use the weak sequential lower semicontinuity of the Nikaido–Isoda function instead of the pseudomonotonicity of the mapping F from (5.25).

Before diving into the convergence analysis, let us present a simple result which guarantees the existence of penalized solutions if the set C is weakly compact.

Lemma 5.25. *Let Assumption 5.24* (i)-(iii) *be satisfied and let C be weakly compact. Then the augmented NEPs* (5.30) *admit solutions for all k.*

Proof. Let $k \in \mathbb{N}$ and $h_k(x) := d_{\mathcal{K}}^2(G(x) + w^k/\rho_k)$. Observe that h_k is convex and continuous, hence weakly sequentially lsc by Proposition 2.44. Now, let

$$\Psi_k(x, y) := \Psi(x, y) + \frac{\rho_k}{2}\Big[h_k(x) - h_k(y)\Big].$$

Then Ψ_k is weakly sequentially lsc with respect to x. Hence, by the Ky–Fan theorem (Lemma 2.54), there exists $\hat{x} \in C$ such that $\Psi_k(\hat{x}, y) \le 0$ for all $y \in C$. We claim that $\hat{x}$ solves the augmented NEP (5.30). To this end, let μ be an arbitrary player index and let $y^\mu \in C_\mu$. With $y := (y^\mu, \hat{x}^{-\mu}) \in C$ we obtain

$$\begin{aligned} 0 \ge \Psi_k(\hat{x}, y) &= \sum_{\nu=1}^N \Big[f_\nu(\hat{x}^\nu, \hat{x}^{-\nu}) - f_\nu(y^\nu, \hat{x}^{-\nu})\Big] + \frac{\rho_k}{2}\Big[h_k(\hat{x}) - h_k(y)\Big] \\ &= f_\mu(\hat{x}^\mu, \hat{x}^{-\mu}) - f_\mu(y^\mu, \hat{x}^{-\mu}) + \frac{\rho_k}{2}\Big[h_k(\hat{x}) - h_k(y)\Big] = \mathcal{L}_k^\mu(\hat{x}) - \mathcal{L}_k^\mu(y). \end{aligned}$$

This shows that $\hat{x}$ is a Nash equilibrium of (5.30). $\square$

We note in passing that the existence of penalized solutions can also be shown by rewriting the augmented subproblems as variational inequalities. Indeed, these problems then take on the form

$$x \in C, \quad \langle \mathcal{L}_{\rho_k}^V(x, w^k), y - x\rangle \ge 0 \quad \forall y \in C,$$

where $\mathcal{L}_\rho^V$ is the augmented Lagrangian in the variational inequality sense, see (5.6). More details on these subproblems can be found in Section 5.2.1.

We now turn to the convergence analysis of Algorithm 5.23 and begin by discussing the attainment of feasibility.

Lemma 5.26. *Let Assumption 5.24 hold, and let $\bar{x}$ be a weak limit point of the sequence $\{x^k\}$. Then $\bar{x}$ is a minimizer of the convex function $d_{\mathcal{K}} \circ G$ on C. In particular, if the feasible set Φ of the GNEP is nonempty, then $\bar{x} \in \Phi$.*

Proof. Note that $d_{\mathcal{K}} \circ G$ is convex by Theorem 2.50 and continuous, hence weakly sequentially lsc. Let us first consider the case where $\{\rho_k\}$ remains bounded. Then the penalty updating scheme (5.32) together with $\tau \in (0,1)$ yields

$$d_{\mathcal{K}}(G(x^{k+1})) \leq \left\| G(x^{k+1}) - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right\|_H \to 0.$$

Hence, $d_{\mathcal{K}}(G(\bar{x})) = 0$, and the claim follows. We now assume that $\rho_k \to \infty$, and define again the functions $h_k(x) := d_{\mathcal{K}}^2(G(x) + w^k/\rho_k)$. As in the previous proof, the functions h_k are convex and continuous, thus weakly sequentially lsc. Now, let $x^{k+1} \rightharpoonup_I \bar{x}$ for some index set I, and assume that there exists $y \in C$ with $d_{\mathcal{K}}(G(y)) < d_{\mathcal{K}}(G(\bar{x}))$. Then

$$\liminf_{k \in I} \left[h_k(x^{k+1}) - h_k(y) \right] = \liminf_{k \in I} \left[d_{\mathcal{K}}^2(G(x^{k+1})) - d_{\mathcal{K}}^2(G(y)) \right] > 0.$$

Hence, there is a constant $c_1 > 0$ such that $h_k(x^{k+1}) - h_k(y) \geq c_1$ for all $k \in I$ sufficiently large. Since h_k is continuously differentiable by Lemma 2.43, it follows that

$$h_k'(x^{k+1})(y - x^{k+1}) \leq h_k(y) - h_k(x^{k+1}) \leq -c_1 \tag{5.34}$$

for all $k \in I$ sufficiently large. Now, let $\{\varepsilon^k\}$ be as in Assumption 5.24. Then

$$\begin{aligned}
\left\langle \varepsilon^{k+1}, y - x^{k+1} \right\rangle &\leq \sum_{\nu=1}^N \left\langle D_{x^\nu} \mathcal{L}_k^\nu(x^{k+1}), y^\nu - x^{\nu,k+1} \right\rangle \\
&= \sum_{\nu=1}^N \left[D_{x^\nu} f_\nu(x^{k+1})(y^\nu - x^{\nu,k+1}) \right] + \frac{\rho_k}{2} h_k'(x^{k+1})(y - x^{k+1}) \\
&\leq \sum_{\nu=1}^N \left[f_\nu(y^\nu, x^{-\nu,k+1}) - f_\nu(x^{k+1}) \right] + \frac{\rho_k}{2} h_k'(x^{k+1})(y - x^{k+1}) \\
&= \frac{\rho_k}{2} h_k'(x^{k+1})(y - x^{k+1}) - \Psi(x^{k+1}, y),
\end{aligned}$$

where Ψ is the Nikaido–Isoda function from (5.23). By Assumption 5.24, Ψ is weakly sequentially lsc with respect to the first argument; hence, there is a constant $c_2 \in \mathbb{R}$ such that $\Psi(x^{k+1}, y) \geq c_2$ for all $k \in I$. This together with (5.34) implies

$$\left\langle \varepsilon^{k+1}, y - x^{k+1} \right\rangle \leq -\frac{\rho_k c_1}{2} - c_2 \to -\infty$$

and therefore contradicts $\varepsilon^k \to 0$. □

Having established the feasibility of weak limit points, we now turn to the optimality part. Since we augmented the constraint $G(x) \in K$ in a joint manner, we can expect convergence to *normalized* Nash equilibria, and this is precisely the assertion of the following theorem.

Theorem 5.27. *Let Assumption 5.24 hold, and let the feasible set of* (5.18) *be nonempty. Then every weak limit point of* $\{x^k\}$ *is a normalized equilibrium of the GNEP.*

Proof. Let $x^{k+1} \rightharpoonup_I \bar{x}$ for some $I \subseteq \mathbb{N}$. Note that $\bar{x}$ is feasible by Lemma 5.26. Let $y \in \Phi$ be an arbitrary point. An easy calculation shows that $D_{x^\nu}\mathcal{L}_k^\nu(x^{k+1}) = D_{x^\nu}\mathcal{L}^\nu(x^{k+1}, \lambda^{k+1})$ for all k. Since $y^\nu \in C_\nu$ for all ν, Assumption 5.24 implies that

$$\begin{aligned}\left\langle \varepsilon^{\nu,k+1}, y^\nu - x^{\nu,k+1}\right\rangle &\le D_{x^\nu}\mathcal{L}_k^\nu(x^{k+1})(y^\nu - x^{\nu,k+1})\\ &\le f_\nu(y^\nu, x^{-\nu,k+1}) - f_\nu(x^{k+1}) + \left\langle \lambda^{k+1}, D_{x^\nu}G(x^{k+1})(y^\nu - x^{\nu,k+1})\right\rangle,\end{aligned}$$

where we used the convexity of f_ν with respect to x^ν in the last estimate. Summing this inequality over all ν and using the convexity of $x \mapsto \langle \lambda^{k+1}, G(x)\rangle$ (by Theorem 2.50) yields

$$\begin{aligned}\left\langle \varepsilon^{k+1}, y - x^{k+1}\right\rangle &\le -\Psi(x^{k+1}, y) + \left\langle \lambda^{k+1}, G'(x^{k+1})(y - x^{k+1})\right\rangle\\ &\le -\Psi(x^{k+1}, y) + \left\langle \lambda^{k+1}, G(y) - G(x^{k+1})\right\rangle.\end{aligned}$$

Taking the limit $k \to_I \infty$ on both sides, using Lemma 5.3, $\varepsilon^k \to 0$, and the weak sequential lower semicontinuity of Ψ with respect to x, we obtain $\Psi(\bar{x}, y) \le 0$. Since $y \in \Phi$ was arbitrary, it follows that $\bar{x}$ is a normalized equilibrium. □

5.3.3 The Finite-Dimensional Case

We now present a variant of the augmented Lagrangian method for the solution of jointly convex GNEPs in finite dimensions. Some parts of the algorithm and its convergence analysis are essentially special cases of the general algorithmic framework from the preceding section (or from Section 5.1.2). However, the finite-dimensional case allows for a more concrete discussion of certain algorithmic aspects, such as the solution of the augmented subproblems, and allows us to simplify many assumptions such as constraint qualifications. The study of the finite-dimensional algorithm is also motivated by the fact that many applications are naturally given in this setting. For more details, we refer the reader to the references [65, 66], and to the examples in Section 7.5. Thus, it makes sense to formulate the corresponding algorithm explicitly instead of treating it as a special instance of the abstract high-level methods from the previous sections.

The general framework we consider throughout this section is a jointly constrained GNEP with $N \in \mathbb{N}$ players, where player ν attempts to solve the problem

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}}\ f_\nu(x) \quad \text{subject to} \quad g(x) \le 0,\ h(x) \le 0, \tag{5.35}$$

with smooth functions $f_\nu : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$. For the sake of simplicity, we assume that the functions f_ν are convex with respect to x^ν, and that g and h are convex with respect to x. Hence, the GNEP is again jointly convex. Much of the theory below can be carried out in a similar fashion for nonconvex problems.

The purpose of the two constraint functions g and h in (5.35) is to account for the possibility of *partial penalization*: the constraints defined by g will be penalized, whereas h

is an (optional) constraint which will remain in the penalized subproblems. This approach allows for a certain degree of flexibility: for $p = 0$, we obtain the case of *full penalization*, where the subproblems become unconstrained. On the other hand, one might use h to model only the player-specific (non-coupling) constraints of the problems, so that the penalized subproblems become standard NEPs.

Our aim will be to compute a normalized KKT point of (5.35), that is, a triple $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{n+m+p}$ such that

$$D_{x^\nu} \mathcal{L}^\nu(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0, \quad 0 \le \bar{\lambda} \perp g(\bar{x}) \le 0, \quad 0 \le \bar{\mu} \perp h(\bar{x}) \le 0, \tag{5.36}$$

for all ν, where $\mathcal{L}^\nu(x, \lambda, \mu) := f_\nu(x) + \lambda^\top g(x) + \mu^\top h(x)$ is the Lagrange function of player ν. Note that we can rewrite the last two conditions in (5.36) as $\min\{-g(\bar{x}), \bar{\lambda}\} = 0$ and $\min\{-h(\bar{x}), \bar{\mu}\} = 0$, where min is understood componentwise.

We have already observed in the preceding section that a (smooth) jointly convex GNEP can be rewritten as a variational inequality. To this end, let

$$F(x) := (\nabla_{x^\nu} f_\nu(x))_{\nu=1}^N, \quad \Phi := \{x \in \mathbb{R}^n : g(x) \le 0,\ h(x) \le 0\}.$$

Note that Φ is just the feasible set of the GNEP, and that Φ is convex. As in the previous section, it follows that a point $\bar{x}$ is a normalized equilibrium of (5.35) if and only if it satisfies the variational inequality

$$\bar{x} \in \Phi, \quad F(\bar{x})^\top (y - \bar{x}) \ge 0 \quad \forall y \in \Phi. \tag{5.37}$$

The KKT conditions of this VI are given by

$$\mathcal{L}^V(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0, \quad \min\{-g(\bar{x}), \bar{\lambda}\} = 0, \quad \min\{-h(\bar{x}), \bar{\mu}\} = 0, \tag{5.38}$$

where $\mathcal{L}^V(x, \lambda, \mu) := F(x) + \nabla g(x)\lambda + \nabla h(x)\mu$ is the Lagrangian of (5.37) in the variational inequality sense. Note that (5.38) is just a condensed version of (5.36).

We now apply an augmented Lagrangian scheme to the GNEP. Note that we could equivalently construct the algorithm for a general VI of the form (5.37) which need not originate from a jointly constrained GNEP. The two constructions are equivalent, but for the sake of later applications we focus on the GNEP setting. It should also be remarked that, for the GNEP case, the augmented Lagrangian method can be interpreted a little more naturally since we can consider it as a penalization scheme applied to each player's objective function.

Let $\lambda \in \mathbb{R}^m$, $\rho > 0$, and consider the (partially) augmented Lagrangian of player ν, that is,

$$\mathcal{L}_\rho^\nu(x, \lambda) := f_\nu(x) + \frac{\rho}{2} \left\| \left(g(x) + \frac{\lambda}{\rho} \right)_+ \right\|^2 - \frac{\|\lambda\|^2}{2\rho}. \tag{5.39}$$

The following is the basic algorithm which we consider throughout this section.

Algorithm 5.28 (ALM for jointly convex GNEPs in $\mathbb{R}^n$)**.** Let $(x^0, \lambda^0, \mu^0) \in \mathbb{R}^{n+m+p}$, $\rho_0 > 0$, $w^{\max} \ge 0$, $\gamma > 1$, $\tau \in (0, 1)$, and set $k := 0$.

Step 1. If (x^k, λ^k, μ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in [0, w^{\max}]^m$ and compute an approximate KKT point (x^{k+1}, μ^{k+1}) of the NEP consisting of the minimization problems

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}} \; \mathcal{L}^\nu_{\rho_k}(x, w^k) \quad \text{subject to} \quad h(x) \le 0. \tag{5.40}$$

Step 3. Update the vector of multipliers to $\lambda^{k+1} := \max\{0, w^k + \rho_k g(x^{k+1})\}$.

Step 4. If

$$\big\|\min\{-g(x^{k+1}), \lambda^{k+1}\}\big\| \le \tau \big\|\min\{-g(x^k), \lambda^k\}\big\|, \tag{5.41}$$

then set $\rho_{k+1} := \rho_k$. Otherwise, set $\rho_{k+1} := \gamma \rho_k$.

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

The above algorithm is fundamentally similar to Algorithm 5.23, with three minor differences. First, the boundedness of w^k is now specified more explicitly through a threshold parameter $w^{\max}$, and secondly, the nonpenalized constraints are given analytically through the function h. Finally, Algorithm 5.28 uses a slightly different updating rule for the penalty parameter, where the quantity $\big\|\min\{-g(x^k), \lambda^k\}\big\|$ is used as an indicator of feasibility and complementarity at the current iterates. This differs slightly from the update suggested by the slack variable approach (see Section 4.1.2), but the two updating schemes can be used in a similar manner when proving convergence to KKT points. As a side effect, the updating rule (5.41) has the advantage that it is naturally defined even for $k = 0$, which is not the case for the previously used one.

The definition of λ^{k+1} in Step 3 implies that, for all k, we have

$$\nabla_{x^\nu} \mathcal{L}^\nu_k(x^{k+1}) = \nabla_{x^\nu} f_\nu(x^{k+1}) + \nabla_{x^\nu} g(x^{k+1}) \lambda^{k+1} \tag{5.42}$$

where, similarly to before, we use the notation $\mathcal{L}^\nu_k(x) := \mathcal{L}^\nu_{\rho_k}(x, w^k)$. The above equality can be seen as the main motivation for the definition of λ^{k+1}.

Assumption 5.29 (Convergence to KKT points)**.** We assume that

(i) the functions f_ν are differentiable and convex with respect to x^ν, and the partial derivatives $\nabla_{x^\nu} f_\nu$ are continuous with respect to the whole vector x,

(ii) the mappings g and h are convex (i.e., they have convex component functions) and continuously differentiable, and

(iii) at Step 2 of Algorithm 5.28, we choose (x^{k+1}, μ^{k+1}) such that, for all ν,

$$\nabla_{x^\nu} \mathcal{L}^\nu_k(x^{k+1}) + \nabla_{x^\nu} h(x^{k+1}) \mu^{k+1} \to 0 \quad \text{and} \quad \min\{-h(x^{k+1}), \mu^{k+1}\} \to 0.$$

In the present situation, the above assumptions are natural and do not need much motivation. One detail which may warrant some discussion is the solution criterion of the augmented subproblems. Note that, in our framework (5.35), the nonpenalized constraints are given through the analytical representation $h(x) \le 0$ instead of the abstract set $C \subseteq X$ which we used in the previous section. This makes it more natural to state the (approximate) optimality conditions of the augmented problems in terms of Lagrange multipliers instead of using the normal cone to the set C. This is precisely what we did in Assumption 5.29.

Let us now begin the convergence analysis. As usual, we start by addressing the feasibility of limit points. The main result in this direction is the following.

Lemma 5.30. *Let $\{x^k\}$ be generated by Algorithm 5.28 under Assumption 5.29, let $\bar{x}$ be a limit point of $\{x^k\}$, and assume that the function h satisfies CPLD in $\bar{x}$. Then $\bar{x}$ is a global solution of*

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \|g_+(x)\|^2 \quad \text{subject to} \quad h(x) \le 0. \tag{5.43}$$

In particular, if there are feasible points, then $\bar{x}$ is feasible.

Proof. Let $x^{k+1} \to_I \bar{x}$ for some $I \subseteq \mathbb{N}$. If the sequence $\{\rho_k\}$ remains bounded, then (5.41) implies $\min\{-g(x^{k+1}), \lambda^{k+1}\} \to 0$, which yields $g(\bar{x}) \le 0$. Hence, $\bar{x}$ is feasible and there is nothing to prove. Assume now that $\rho_k \to \infty$. By Assumption 5.29, we have

$$\nabla_{x^\nu} f_\nu(x^{k+1}) + \nabla_{x^\nu} g(x^{k+1})(w^k + \rho_k g(x^{k+1}))_+ + \nabla_{x^\nu} h(x^{k+1})\mu^{k+1} \to 0$$

for all ν. Dividing this equation by ρ_k and omitting some vanishing terms, we obtain

$$\nabla_{x^\nu} g(x^{k+1}) g_+(x^{k+1}) + \nabla_{x^\nu} h(x^{k+1})\hat{\mu}^{k+1} \to_I 0 \tag{5.44}$$

for all ν, where $\hat{\mu}^{k+1} := \mu^{k+1}/\rho_k$. We now claim that $\min\{-h(x^{k+1}), \hat{\mu}^{k+1}\} \to_I 0$ as $k \to \infty$. To see this, note that $\min\{-h(x^{k+1}), \mu^{k+1}\} \to 0$ by Assumption 5.29; hence, $\liminf_{k\to\infty} \mu_j^{k+1} \ge 0$ for all j, which implies $\liminf_{k\to\infty} \hat{\mu}_j^{k+1} \ge 0$. Moreover, $h(x^{k+1}) \to_I h(\bar{x}) \le 0$ and, if $h_j(\bar{x}) < 0$ for some $j = 1, \ldots, p$, then $\mu_j^{k+1} \to 0$ and thus also $\hat{\mu}_j^{k+1} \to 0$. This shows that $\min\{-h(x^{k+1}), \hat{\mu}^{k+1}\} \to_I 0$.

Collecting the equations (5.44) for all ν and using the fact that $\nabla g(x) g_+(x) = \nabla \frac{1}{2}\|g_+(x)\|^2$, we therefore have

$$\nabla \frac{1}{2}\|g_+(x^{k+1})\|^2 + \nabla h(x^{k+1})\hat{\mu}^{k+1} \to_I 0 \quad \text{and} \quad \min\{-h(x^{k+1}), \hat{\mu}^{k+1}\} \to_I 0$$

as $k \to \infty$. It then follows from CPLD (see Theorem 3.54) that the limit point $\bar{x}$ is a stationary point of the optimization problem (5.43). Since this is a convex problem, $\bar{x}$ is a global minimizer, and the proof is complete. □

The above result shows that the augmented Lagrangian method has no trouble achieving feasibility for jointly convex GNEPs in finite dimensions. This is not surprising due to the convex structure of the constraints.

We now prove the optimality of limit points. This result is again based on the CPLD constraint qualification, this time applied to the pair $(g, h) : \mathbb{R}^n \to \mathbb{R}^{m+p}$.

Theorem 5.31. *Let $\{x^k\}$ be generated by Algorithm 5.28 under Assumption 5.29, and let $\bar{x}$ be a limit point of $\{x^k\}$. If the feasible set is nonempty and the function (g, h) satisfies CPLD in $\bar{x}$, then $\bar{x}$ is feasible and a normalized equilibrium of the GNEP.*

Proof. Under the given assumptions, the function h itself also satisfies CPLD. Hence, the feasibility of $\bar{x}$ follows from Lemma 5.30. To obtain the optimality, let $x^k \to_I \bar{x}$ for some $I \subseteq \mathbb{N}$, and observe that $g(x^k) \to_I g(\bar{x}) \leq 0$. We first claim that $\min\{-g(x^k), \lambda^k\} \to 0$ as $k \to_I \infty$. This is clear if $\{\rho_k\}$ is bounded, see (5.41). If $\rho_k \to \infty$ and $g_i(\bar{x}) < 0$ for some i, then $\lambda_i^k = \max\{0, w_i^{k-1} + \rho_{k-1} g_i(x^k)\} = 0$ for sufficiently large $k \in I$. Hence, in either case, we have $\min\{-g(x^k), \lambda^k\} \to_I 0$. Now, define the mappings

$$F(x) := (\nabla_{x^\nu} f_\nu(x))_{\nu=1}^N \quad \text{and} \quad \mathcal{L}^V(x, \lambda, \mu) := F(x) + \nabla g(x)\lambda + \nabla h(x)\mu.$$

Taking into account (5.42) and Assumption 5.29, an elementary calculation shows that

$$\mathcal{L}^V(x^k, \lambda^k, \mu^k) \to 0, \quad \min\{-g(x^k), \lambda^k\} \to 0, \quad \text{and} \quad \min\{-h(x^k), \mu^k\} \to 0,$$

as $k \to_I \infty$. The result now follows from Theorem 3.54. □

The above constitutes our main global convergence result for GNEPs of the form (5.35). Note that, due to the structure of the algorithm, we always obtain a *normalized* equilibrium.

Let us close this section with a remark. It is worth observing that the study of jointly constrained GNEPs does not require any analytical tools which exceed the theory of optimization problems or variational inequalities. The reason for this is that normalized equilibria correspond to VIs, which implies that we can use many of the same arguments and constraint qualifications from optimization theory. This is in stark contrast to the case of a "fully" generalized Nash equilibrium problem, i.e., a GNEP where the constraints are different for each player but still depend on the whole vector x. Indeed, we will see in Section 6.3.3 that the GNEP in its full generality requires the study of GNEP-tailored constraint qualifications, and these add an extra layer of complexity to the problem and the discussion of practical algorithms.

Chapter 6

Quasi-Variational Inequalities

This chapter is dedicated to the study of variational inequalities (VIs) with implicit constraints, usually called *quasi-variational inequalities (QVIs)*. More specifically, let X be a real Banach space, $\Phi : X \rightrightarrows X$ a set-valued mapping, and $F : X \to X^*$ a nonlinear operator. The problem we consider consists of finding $x \in X$ such that

$$(Q) \qquad x \in \Phi(x), \quad \langle F(x), d\rangle \geq 0 \quad \forall d \in \mathcal{T}_{\Phi(x)}(x). \tag{6.1}$$

The above problem is obviously inspired by the variational inequality framework from Chapter 5. Indeed, if the set-valued mapping Φ is constant, i.e., $\Phi(x)$ is independent of x, then (Q) reduces to the variational inequality (5.1). Let us furthermore observe that, if Φ is *convex-valued*, i.e., if $\Phi(x)$ is a convex set for all x, then (Q) can equivalently be stated as

$$x \in \Phi(x), \quad \langle F(x), y - x\rangle \geq 0 \quad \forall y \in \Phi(x), \tag{6.2}$$

which is a generalization of the variational inequality (5.2).

We say that a point $x \in X$ is *feasible* for the QVI if $x \in \Phi(x)$. Note that, in the QVI context, we do not use the symbol Φ to denote the set of feasible points. Instead, Φ is the parametric set from the definition of (Q).

The notion of QVIs was introduced by Bensoussan and Lions [17] in the context of impulse control problems. The QVI has since emerged as a universal tool for the modeling of various equilibrium-type scenarios in the natural sciences. Its applications include game theory [95], solid and continuum mechanics [19, 98, 148, 174], economics [111,126], probability theory [141], transportation [29,52,201], biology [94], and stationary problems in superconductivity, thermoplasticity, or electrostatics [3, 104, 105, 153, 193]. For further information, we refer the reader to the corresponding papers, the monographs [12,18,148,167], and the references therein. An important feature of QVIs is that they can be used to model generalized Nash equilibrium problems (GNEPs) in their full generality (whereas ordinary VIs correspond to jointly constrained problems, see Section 5.3). This opens up a broad spectrum of further applications in economics and game theory, see [14, 53, 65, 77, 95, 111] and the references therein.

In comparison to ordinary variational inequalities, the treatment of QVIs turns out to be substantially more difficult. This is because many pathological situations can occur: for instance, the set $\Phi(x)$ can be empty or nonconvex for certain x. Moreover, it is clear that some kind of continuous dependence of Φ on x will be necessary for a tractable analysis of the problem. The most prominent notion in this regard is that of Mosco-convergence (see [166, 167] and Section 6.1.1), and this property will play a fundamental role for the theory in this chapter.

The literature on algorithms for the solution of QVIs is quite diverse, and many methods are designed for specific problem classes. For GNEPs, penalty methods are some of the most successful techniques [66, 69, 86], along with interior point [60] and Newton-type methods [64, 123]. Another popular class of QVIs is the *moving set case*, which arises if $\Phi(x) = c(x) + \Phi_0$ for some fixed set $\Phi_0 \subseteq X$ and a single-valued mapping $c : X \to X$. For these problems, there is a large amount of literature revolving around fixed point approaches [42, 171, 173, 198]. Another algorithmic technique for QVIs is based on gap functions [85, 96, 97]. Finally, in [127, 176], an augmented Lagrangian algorithm was proposed for the solution of QVIs in finite dimensions. This method can be seen as a special case of the algorithm we will present below.

In the present thesis, we will mainly be concerned with QVIs where the feasible set mapping Φ has an analytical representation of the form

$$\Phi(x) = \{y \in C : G(x, y) \in K\}, \tag{6.3}$$

where X, Y are real Banach spaces, $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, and $G : X^2 \to Y$ is a possibly nonlinear mapping. Note that C is independent of x. Hence, the parametric part of the constraints is completely modeled by the mapping G. This framework allows for a very high degree of flexibility and encompasses many of the aforementioned application cases. More details will be given in Chapter 7.

This chapter is based on the theory in [129, 132] and the upcoming preprint [134]. The material is structured as follows. In Section 6.1, we begin by analyzing in some detail the theoretical background of QVIs. Section 6.1.1 in particular deals with the important notion of Mosco-convergence, and also contains a prototypical existence result for general quasi-variational inequalities. In Section 6.1.2, we analyze constraint qualifications and Karush–Kuhn–Tucker (KKT) conditions for QVIs. Finally, in Section 6.1.3, we specialize some of these concepts for QVIs with nonlinear programming constraints.

Starting with Section 6.2, we turn our attention to the augmented Lagrangian method for a general QVI of the form (Q). In Section 6.2.1, we discuss and state the corresponding algorithm, and in Section 6.2.2 we give some sufficient conditions for the existence of penalized solutions. Section 6.2.3 then deals with convergence results for QVIs where the feasible set mapping is convex-valued, and in Section 6.2.4 we analyze the convergence for general QVIs using a primal-dual approach.

In Section 6.3, we show how some of the results surrounding the augmented Lagrangian method can be specialized for QVIs in finite dimensions. In Sections 6.3.1 and 6.3.3, we give some improved convergence results for QVIs and GNEPs, respectively. Finally, in Section 6.3.2, we demonstrate how the augmented Lagrangian technique can be used to construct an exact penalty method for finite-dimensional QVIs. The resulting algorithm can be seen as a generalization of the methods from [51, 55, 56, 78].

6.1 Theory of Quasi-Variational Inequalities

This section contains a discussion of various theoretical properties of QVIs. At this point, it is also appropriate to state one of the most important special cases of QVIs, the *generalized Nash equilibrium problems (GNEPs).*

Example 6.1 (Generalized Nash equilibrium problems)**.** Let $N \in \mathbb{N}$ be a natural number and X_ν, $\nu = 1, \dots, N$, a collection of Banach spaces. We define $X := \prod_{\nu=1}^N X_\nu$ and write $x = (x^\nu, x^{-\nu})$ for a generic element in X, where $x^\nu \in X_\nu$ and $x^{-\nu} \in X_{-\nu} := \prod_{\mu \neq \nu} X_\mu$. Consider the GNEP where player ν attempts to solve

$$\underset{x^\nu \in X_\nu}{\text{minimize}}\ f_\nu(x) \quad \text{subject to} \quad x^\nu \in \Phi_\nu(x^{-\nu}). \tag{6.4}$$

Here, $f_\nu : X \to \mathbb{R}$ is a continuously differentiable function and $\Phi_\nu : X_{-\nu} \rightrightarrows X_\nu$ a set-valued mapping. A point $\bar{x}$ is said to be a solution of the GNEP if, for all ν, $\bar{x}^\nu$ is a solution of (6.4) for fixed $\bar{x}^{-\nu}$. Combining Lemmas 2.34 and 3.1, it is easy to see that any solution $\bar{x}$ of the GNEP is necessarily a solution of the QVI (Q) corresponding to the mappings

$$F(x) := \big(D_{x^\nu} f_\nu(x)\big)_{\nu=1}^N \quad \text{and} \quad \Phi(x) := \prod_{\nu=1}^N \Phi_\nu(x^{-\nu}).$$

The converse holds whenever the first-order conditions given in Lemma 3.1 are sufficient for optimality in the players' optimization problems. This is the case, for instance, if f_ν is convex with respect to x^ν and $\Phi_\nu(x^{-\nu})$ is convex for all $x^{-\nu} \in X_{-\nu}$.

6.1.1 Mosco-Convergence and Existence Results

This section is dedicated to one of the most fundamental properties of QVIs, the Mosco-convergence of sets, and the resulting continuity property of the feasible set mapping Φ. In what follows, we will define and analyze these properties and their consequences. In particular, the Mosco-continuity of Φ can be used to state an existence theorem for solutions of QVIs which generalizes some related results from [2, 79, 95, 165, 176].

Definition 6.2 (Mosco-convergence)**.** Let S and S_k, $k \in \mathbb{N}$, be subsets of X. We say that $\{S_k\}$ *Mosco-converges* to S, and write $S_k \xrightarrow{M} S$, if

(i) for every $y \in S$, there is a sequence $y^k \in S_k$ such that $y^k \to y$, and

(ii) whenever $y^k \in S_k$ for all k and y is a weak limit point of $\{y^k\}$, then $y \in S$.

The concept of Mosco-convergence plays a key role in multiple aspects of the analysis of QVIs such as existence [153, 167], approximation [155], or the convergence of algorithms [104]. It is typically used as part of a continuity property of the mapping Φ.

Definition 6.3 (Weak Mosco-continuity)**.** Let $\Phi : X \rightrightarrows X$ be a set-valued mapping and $x \in X$. We say that Φ is *weakly Mosco-continuous in* x if $x^k \rightharpoonup x$ implies $\Phi(x^k) \xrightarrow{M} \Phi(x)$. If this holds for every $x \in X$, we simply say that Φ is *weakly Mosco-continuous.*

The two conditions defining weak Mosco-continuity are often referred to as complete (or strong) inner and weak outer semicontinuity. Observe moreover that, if Φ is weakly Mosco-continuous, then $\Phi(x)$ is weakly closed for all $x \in X$.

In many practical application examples, the weak Mosco-continuity can be verified by using the specific structure of the underlying feasible set mapping. Moreover, we will later give a generic sufficient condition for the weak Mosco-continuity in terms of certain constraint qualifications, see Section 6.1.2.

One of the most important consequences of the Mosco-continuity of Φ is that, under a suitable assumption on the mapping F, every weak limit point of a sequence of approximate solutions of the QVI is an exact solution. This result is a generalization of Proposition 3.43 for the variational inequality case.

Proposition 6.4. *Let F be a bounded pseudomonotone operator and let Φ be weakly Mosco-continuous. Assume that $\{x^k\} \subseteq X$ converges weakly to $\bar{x}$, that $\bar{x} \in \Phi(\bar{x})$, and that there are null sequences $\{\delta_k\}, \{\varepsilon_k\} \subseteq \mathbb{R}$ (possibly negative) such that*

$$\left\langle F(x^k), y - x^k \right\rangle \geq \delta_k + \varepsilon_k \|y - x^k\|_X \quad \forall y \in \Phi(x^k) \tag{6.5}$$

for all k. Then $\bar{x}$ is a solution of the QVI.

Proof. By Mosco-continuity, there is a sequence $\bar{x}^k \in \Phi(x^k)$ such that $\bar{x}^k \to \bar{x}$. Inserting $\bar{x}^k$ into (6.5) yields $\liminf_{k\to\infty}\langle F(x^k), \bar{x}^k - x^k\rangle \geq 0$ and, since $\{F(x^k)\}$ is bounded, $\liminf_{k\to\infty}\langle F(x^k), \bar{x} - x^k\rangle \geq 0$. The pseudomonotonicity of F therefore implies that

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq \limsup_{k\to\infty} \left\langle F(x^k), y - x^k \right\rangle \quad \text{for all } y \in X. \tag{6.6}$$

To show that $\bar{x}$ solves the QVI, let $y \in \Phi(\bar{x})$. Using the Mosco-continuity of Φ, we obtain a sequence $y^k \in \Phi(x^k)$ such that $y^k \to y$. By (6.5), we have $\liminf_{k\to\infty}\langle F(x^k), y^k - x^k\rangle \geq 0$, hence $\liminf_{k\to\infty}\langle F(x^k), y - x^k\rangle \geq 0$, and (6.6) implies that $\langle F(\bar{x}), y - \bar{x}\rangle \geq 0$. □

Note that Proposition 6.4 implies, in particular, that the solution set of (Q) is weakly sequentially closed (under the given assumptions).

The existence of solutions to QVIs is a rather delicate topic. One of the reasons for this is the rather involved nature of the problem, especially the mapping Φ. Indeed, even the task of finding a feasible point, i.e., a point satisfying $x \in \Phi(x)$, is a nonlinear fixed point problem which needs to be solvable in order to even have a chance of solving the QVI. As a result, QVIs are seldom tackled in their full generality, but often in the context of specific problem settings or under rather strong structural assumptions, see [2, 153]. The few "generic" existence results which do exist often work with a substantial amount of technical assumptions, see [12, 79, 142, 167, 204], or pertain to the finite-dimensional case only [176].

In what follows, we shall develop a prototypical existence theorem which is relatively simple and which can be seen as a natural extension of the basic results for VIs (see Section 3.2.1), and of the finite-dimensional existence result from [176].

A rather intuitive approach to the existence of solutions is given by Proposition 6.4. This result suggests that we can tackle the QVI by solving a sequence of approximating problems and then using a limiting argument to obtain a solution of (Q). For the precise implementation of this idea, we need two auxiliary results. The first is a consequence of the Brezis–Nirenberg–Stampacchia theorem (Theorem 2.56) and guarantees the existence of solutions for the approximating problems. This result will also be useful later on.

Lemma 6.5. *Let $A \subseteq X$ be a nonempty, convex, weakly compact set, $F : X \to X^*$ a bounded pseudomonotone operator, and $\varphi : A^2 \to \mathbb{R}$ a mapping such that*

(i) *$\varphi(x, x) = 0$ for all $x \in A$,*

(ii) *φ is concave with respect to y, and*

(iii) *φ is weakly sequentially lsc with respect to x.*

Then there exists $\hat{x} \in A$ such that $\langle F(\hat{x}), \hat{x} - y\rangle + \varphi(\hat{x}, y) \le 0$ for all $y \in A$.

Proof. The proof is similar to that of Theorem 3.40. We claim that the mapping $\Psi : A^2 \to \mathbb{R}$, $\Psi(x, y) := \langle F(x), x - y\rangle + \varphi(x, y)$, satisfies the assumption of the Brezis–Nirenberg–Stampacchia theorem (Theorem 2.56). Clearly, $\Psi(x, x) \le 0$ for every $x \in A$, and Ψ is (quasi-)concave with respect to the second argument. Moreover, by the properties of pseudomonotone operators (Lemma 3.39), Ψ is lower semicontinuous with respect to the first argument on $A \cap L$ for any finite-dimensional subspace L of X. Finally, let $x, y \in A$, let $\{x^k\} \subseteq A$ be a sequence converging weakly to x, and assume that

$$\Psi(x^k, (1-t)x + ty) \le 0 \quad \forall t \in [0, 1],\ \forall k \in \mathbb{N}. \tag{6.7}$$

We need to show that $\Psi(x, y) \le 0$. By (6.7), we have in particular that $\Psi(x^k, x) \le 0$ and $\Psi(x^k, y) \le 0$ for all k. The first of these conditions implies that

$$\begin{aligned} 0 \ge \limsup_{k\to\infty} \Psi(x^k, x) &\ge \limsup_{k\to\infty} \big\langle F(x^k), x^k - x\big\rangle + \liminf_{k\to\infty} \varphi(x^k, x) \\ &\ge \limsup_{k\to\infty} \big\langle F(x^k), x^k - x\big\rangle, \end{aligned}$$

where we used the weak sequential lower semicontinuity of φ with respect to x and the fact that $\varphi(x, x) = 0$. Hence, by the pseudomonotonicity of F, we obtain

$$\begin{aligned} \Psi(x, y) &= \langle F(x), x - y\rangle + \varphi(x, y) \\ &\le \liminf_{k\to\infty} \big[\big\langle F(x^k), x^k - y\big\rangle + \varphi(x^k, y)\big] = \liminf_{k\to\infty} \Psi(x^k, y) \le 0. \end{aligned}$$

Therefore, Ψ satisfies all the requirements of Theorem 2.56, and the result follows. □

Apart from the above result, we will also need some information on the behavior of the "parametric" distance function $(x, y) \mapsto d_{\Phi(x)}(y)$. Here, the weak Mosco-continuity of Φ plays a key role and allows us to prove the following lemma.

Lemma 6.6. *Let $\Phi : X \rightrightarrows X$ be weakly Mosco-continuous. Then, for every $y \in X$, the distance function $x \mapsto d_{\Phi(x)}(y)$ is weakly sequentially upper semicontinuous on X.*

If, in addition, there are nonempty subsets $A, B \subseteq X$ such that $\Phi(A) \subseteq B$ and B is weakly compact, then the function $x \mapsto d_{\Phi(x)}(x)$ is weakly sequentially lsc on A.

Proof. Let $y \in X$ be a fixed point and let $\{x^k\} \subseteq X$, $x^k \rightharpoonup x \in X$. If $w \in \Phi(x)$ is an arbitrary point, then there is a sequence $w^k \in \Phi(x^k)$ such that $w^k \to w$. It follows that

$$\|y - w\|_X = \lim_{k\to\infty} \|y - w^k\|_X \geq \limsup_{k\to\infty} d_{\Phi(x^k)}(y).$$

Since $w \in \Phi(x)$ was arbitrary, this implies that $d_{\Phi(x)}(y) \geq \limsup_{k\to\infty} d_{\Phi(x^k)}(y)$.

We now prove the second assertion. Let $\{x^k\} \subseteq A$ be a sequence with $x^k \rightharpoonup x \in A$. Without loss of generality, let $d_{\Phi(x^k)}(x^k) \to \liminf_{k\to\infty} d_{\Phi(x^k)}(x^k)$, and let $\Phi(x^k)$ be nonempty for all k. Choose points $w^k \in \Phi(x^k)$ such that $\|x^k - w^k\|_X \leq d_{\Phi(x^k)}(x^k) + 1/k$. By assumption, the sequence $\{w^k\}$ is contained in the weakly compact set B, and thus there is an index set $I \subseteq \mathbb{N}$ such that $w^k \rightharpoonup_I w$ for some $w \in B$. Since $w^k \in \Phi(x^k)$ for all k, the weak Mosco-continuity of Φ implies $w \in \Phi(x)$. It follows that

$$d_{\Phi(x)}(x) \leq \|x - w\|_X \leq \liminf_{k\in I} \|x^k - w^k\|_X = \liminf_{k\to\infty} d_{\Phi(x^k)}(x^k).$$

This completes the proof. □

We now turn to the main existence result for QVIs.

Theorem 6.7. *Consider a QVI of the form* (Q)*. Assume that* (i) *F is bounded and pseudomonotone,* (ii) *Φ is weakly Mosco-continuous, and* (iii) *there is a nonempty, convex, weakly compact set $A \subseteq X$ such that, for all $x \in A$, $\Phi(x)$ is nonempty, closed, convex, and contained in A. Then the QVI admits a solution $\bar{x} \in A$.*

Proof. For $k \in \mathbb{N}$, let $\Psi_k : A^2 \to \mathbb{R}$ be the bifunction

$$\Psi_k(x, y) := \langle F(x), x - y\rangle + k\big[d_{\Phi(x)}(x) - d_{\Phi(x)}(y)\big].$$

By Lemmas 6.5 and 6.6, there exist points $x^k \in A$ such that $\Psi_k(x^k, y) \leq 0$ for all $y \in A$. Since A is weakly compact, the sequence $\{x^k\}$ admits a weak limit point $\bar{x} \in A$. Moreover, by assumption, there are points $y^k \in \Phi(x^k) \subseteq A$ for all k. For these points, we obtain

$$0 \geq \Psi_k(x^k, y^k) = \big\langle F(x^k), x^k - y^k\big\rangle + k d_{\Phi(x^k)}(x^k).$$

By the boundedness of A and F, the first term is bounded. Hence, dividing by k, we obtain $d_{\Phi(x^k)}(x^k) \to 0$, thus $d_{\Phi(\bar{x})}(\bar{x}) = 0$ by Lemma 6.6, and hence $\bar{x} \in \Phi(\bar{x})$.

Finally, we claim that $\bar{x}$ solves the QVI. Observe that, for all k and $y \in \Phi(x^k)$,

$$0 \geq \Psi_k(x^k, y) = \big\langle F(x^k), x^k - y\big\rangle + k d_{\Phi(x^k)}(x^k) \geq \big\langle F(x^k), x^k - y\big\rangle.$$

Thus, by Proposition 6.4, it follows that $\bar{x}$ is a solution of the QVI. □

The applicability of the above theorem depends most crucially on the weak Mosco-continuity of Φ and the existence of the weakly compact set A. It is possible to modify the theorem by requiring some form of coercivity instead.

Example 6.8. This example is based on [153]. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded domain, let $X := H_0^1(\Omega)$, and consider the QVI given by the functions

$$F(u) := -\Delta u - f, \qquad \Phi(u) := \{v \in H_0^1(\Omega) : \|\nabla v\| \leq \psi(u)\},$$

where $\|\cdot\|$ is the Euclidean norm, $f \in H^{-1}(\Omega)$, and $\psi : H_0^1(\Omega) \to L^\infty(\Omega)$ is completely continuous. Observe that F is monotone and continuous, hence pseudomonotone by Lemma 3.37. Assume now that $c_1 \leq \psi(u) \leq c_2$ for all u and some $c_1, c_2 > 0$. Then Φ is weakly Mosco-continuous by [153, Lem. 1]. Moreover, $0 \in \Phi(u)$ for all $u \in H_0^1(\Omega)$, and the Poincaré inequality (Theorem 2.29) implies the existence of $R > 0$ such that $\Phi(u) \subseteq B_R^X$ for all $u \in H_0^1(\Omega)$. We conclude that all the requirements of Theorem 6.7 are satisfied; hence, the QVI admits a solution.

6.1.2 KKT Conditions and Constraint Qualifications

The purpose of this section is to discuss first-order optimality conditions involving Lagrange multipliers for the QVI (Q). To that end, we assume that the feasible set mapping Φ has the analytical representation

$$\Phi(x) := \{y \in C : G(x, y) \in K\}, \tag{6.8}$$

where $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, Y is a real Banach space, and $G : X^2 \to Y$. A rather intuitive approach to such conditions is to observe that, if $\bar{x}$ is a solution of (Q), then it is also a solution of the (ordinary) VI

$$x \in A, \quad \langle F(x), d\rangle \geq 0 \quad \forall d \in \mathcal{T}_A(x), \tag{6.9}$$

where $A := \Phi(\bar{x})$ is considered fixed. The constraint set of this problem can be written as

$$A = \{y \in C : G(\bar{x}, y) \in K\}. \tag{6.10}$$

In other words, the parametric nature of the problem is resolved, and the constraint system can be written in such a manner that it is consistent with the theory of variational inequalities (Section 3.2).

The above observation implies two things. First, leaving some technical details aside, it is reasonable to expect that the KKT conditions of QVIs are not substantially more difficult than those for standard VIs. Secondly, when defining constraint qualifications for QVIs, these should only depend on the partial derivative of G with respect to y and not on the behavior of G with respect to the parametric variable x.

Concerning the first of the above remarks, it is natural to define the *Lagrange function* or *Lagrangian* of (Q) as the mapping

$$\mathcal{L} : X \times Y^* \to X^*, \quad \mathcal{L}(x, \lambda) := F(x) + D_y G(x, x)^* \lambda. \tag{6.11}$$

The corresponding first-order optimality conditions are given as follows.

Definition 6.9 (KKT point)**.** A tuple $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ is a *KKT point* of (Q) if

$$-\mathcal{L}(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x}, \bar{x})).$$

We call $\bar{x}$ a *stationary point* if $(\bar{x}, \bar{\lambda})$ is a KKT point for some multiplier $\bar{\lambda} \in Y^*$, and denote by $\Lambda(\bar{x})$ the set of such multipliers.

The specific form of the constraint set (6.10) immediately suggests an appropriate way to define constraint qualifications for QVIs. The most notable one is the Robinson constraint qualification, which is contained in the following definition. For the sake of completeness, we also define an extended version of this condition.

Definition 6.10 (Robinson constraint qualification for QVIs)**.** Let $x \in X$ be an arbitrary point. We say that x satisfies

(i) the *extended Robinson constraint qualification for QVIs (QVI-ERCQ)* if

$$0 \in \operatorname{int}\big[G(x,x) + D_y G(x,x)(C - x) - K\big]. \tag{6.12}$$

(ii) the *Robinson constraint qualification for QVIs (QVI-RCQ)* if x is feasible and satisfies QVI-ERCQ, i.e., equation (6.12).

It follows from the definition that QVI-RCQ is nothing but RCQ for the constraint system $y \in C$, $G(x, y) \in K$, with respect to y, applied in the point $y := x$. This is consistent with the observations before Definition 6.9.

The following result is an immediate consequence of Proposition 3.46.

Proposition 6.11. *If $(\bar{x}, \bar{\lambda})$ is a KKT point, then $\bar{x}$ is a solution of (Q). Conversely, if $\bar{x}$ is a solution of (Q) and QVI-RCQ holds in $\bar{x}$, then the set of Lagrange multipliers $\Lambda(\bar{x})$ is nonempty and bounded in Y^*.*

We now turn to another consequence of QVI-RCQ which is a certain metric regularity of the feasible set(s). This property can also be used to deduce the weak Mosco-continuity of Φ (see Corollary 6.13). The main idea is that, given some reference point $\bar{x}$, we can view the parameter x in the feasible set mapping $\Phi(x)$ as a perturbation parameter and apply the sensitivity framework from [32] and Section 3.1.2. Recall that metric regularity is defined as a distance estimate to the set $\Phi(x)$ for all x in a neighborhood of the reference point $\bar{x}$, see Definition 3.12. An important observation in this context is that, given suitable continuity properties of the mappings G and $D_y G$, we can actually take the neighborhood around $\bar{x}$ with respect to the *weak (sequential) topology* of X. A precise statement of this fact can be found in the following lemma. For the sake of clarity, we state this result in a slightly different setting and assume that G is defined on $U \times X$, where U is an arbitrary topological space (e.g., the space X endowed with the weak or weak sequential topology).

Lemma 6.12. *Let $\bar{x} \in X$ and $\bar{y} \in \Phi(\bar{x})$. Assume that G and $D_y G$ are continuous on $U \times X$, where U is an arbitrary topological space, and that*

$$0 \in \operatorname{int}\big[G(\bar{x}, \bar{y}) + D_y G(\bar{x}, \bar{y})(C - \bar{y}) - K\big].$$

Then there are $c > 0$ *and a neighborhood* N *of* $(\bar{x}, \bar{y})$ *in* $U \times X$ *such that* $\operatorname{dist}(y, \Phi(x)) \leq c \operatorname{dist}(G(x, y), K)$ *for all* $(x, y) \in N$ *with* $y \in C$.

Proof. Let $G_2 : U \times X \to X \times Y$ be the mapping $G_2(x, y) := (y, G(x, y))$, and define the set $K_2 := C \times K$. Observe that G_2 is Fréchet-differentiable with respect to y, that both G_2 and $D_y G_2$ are continuous on $U \times X$, and that $\Phi(x) = \{y \in X : G_2(x, y) \in K_2\}$. By Remark 3.10, we have $0 \in \operatorname{int}[G_2(\bar{x}, \bar{y}) + D_y G_2(\bar{x}, \bar{y})X - K_2]$. Hence, by [32, Thm. 2.87], there are $c > 0$ and a neighborhood N of $(\bar{x}, \bar{y})$ in $U \times X$ such that

$$\operatorname{dist}\big(y, \Phi(x)\big) \leq c \operatorname{dist}\big(G_2(x, y), K_2\big)$$

for all $(x, y) \in N$. Clearly, if $y \in C$, then $\operatorname{dist}(G_2(x, y), K_2) = \operatorname{dist}(G(x, y), K)$. □

As mentioned before, we can use Lemma 6.12 to prove the weak Mosco-continuity of Φ. To this end, we only need to apply the lemma in the special case where U is the space X equipped with the weak sequential topology (see Definition 2.7 and Remark 2.25).

Corollary 6.13 (Sufficient condition for weak Mosco-continuity)**.** *Let* $\bar{x}$ *be a feasible point for (Q). Assume that* G *is* K_∞*-concave with respect to* y*, that QVI-RCQ holds in* $\bar{x}$*, and that* G *satisfies the continuity property*

$$x^k \rightharpoonup x, \quad y^k \to y \quad \Longrightarrow \quad \begin{array}{c} G(x^k, y^k) \to G(x, y), \\ D_y G(x^k, y^k) \to D_y G(x, y) \end{array} \tag{6.13}$$

for all $x, y \in X$. *Then* Φ *is weakly Mosco-continuous in* $\bar{x}$.

Proof. We first show the weak outer semicontinuity of Φ. Let $x^k \rightharpoonup \bar{x}$ and $y^k \in \Phi(x^k)$, $y^k \rightharpoonup \bar{y}$. Then $\{y^k\} \subseteq C$, which implies $\bar{y} \in C$. Furthermore, Proposition 2.51 yields

$$G(x^k, \bar{y}) + D_y G(x^k, \bar{y})(y^k - \bar{y}) \in G(x^k, y^k) + K_\infty \subseteq K$$

for all k. Taking the limit $k \to \infty$ and using (6.13), it follows that the left-hand side converges weakly to $G(\bar{x}, \bar{y})$. Hence, $G(\bar{x}, \bar{y}) \in K$, and $\bar{y} \in \Phi(\bar{x})$.

For the inner semicontinuity, let $x^k \rightharpoonup \bar{x}$ and $\bar{y} \in \Phi(\bar{x})$. By assumption, the constraint system $y \in C$, $G(\bar{x}, y) \in K$, satisfies the Robinson constraint qualification in $\bar{x}$; thus, by Proposition 3.21, it satisfies RCQ in every $y \in \Phi(\bar{x})$, in particular for $y := \bar{y}$. Now, let U denote the space X equipped with the weak sequential topology (see Remark 2.25), so that G and $D_y G$ are continuous on $U \times X$. By Lemma 6.12, there exists $c > 0$ such that

$$\operatorname{dist}\big(\bar{y}, \Phi(x^k)\big) \leq c \operatorname{dist}\big(G(x^k, \bar{y}), K\big)$$

for $k \in \mathbb{N}$ sufficiently large. Since the right-hand side converges to zero as $k \to \infty$, we can choose points $y^k \in \Phi(x^k)$ with $\|y^k - \bar{y}\|_X \to 0$. This completes the proof. □

6.1.3 Nonlinear Programming Constraints

In this section, we briefly touch upon the case of a QVI in finite dimensions where the feasible set has the form

$$\Phi(x) := \big\{y \in \mathbb{R}^n : g(x,y) \le 0,\ e(x,y) = 0\big\} \tag{6.14}$$

for continuously differentiable mappings $g : \mathbb{R}^{2n} \to \mathbb{R}^m$, $e : \mathbb{R}^{2n} \to \mathbb{R}^p$, and $m, p \in \mathbb{N}_0$. This problem can obviously be cast into the Banach space framework of the previous sections by setting $Y := \mathbb{R}^{m+p}$ and defining the mapping $G := (g, e)$. However, as in the case of constrained optimization, it is possible to give more refined constraint qualifications for QVIs by taking into account the specific structure of (6.14). The resulting conditions can be seen as natural extensions of those from Section 3.1.4.

As in the optimization case, a key concept for the following definition is that of *positive linear independence*. Given a point $x \in \mathbb{R}^n$ and subsets $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, p\}$, we say that the set of gradients $\{\nabla_y g_i(x,x)\}_{i\in I} \cup \{\nabla_y e_j(x,x)\}_{j\in J}$ is positively linearly dependent if there are nontrivial coefficients $\lambda_i \ge 0$, $i \in I$, and $\mu_j \in \mathbb{R}$, $j \in J$, such that

$$\sum_{i\in I} \lambda_i \nabla_y g_i(x,x) + \sum_{j\in J} \mu_j \nabla_y e_j(x,x) = 0.$$

Note that the coefficients λ_i corresponding to the inequality constraints are required to be nonnegative, whereas the remaining coefficients μ_j are arbitrary real numbers. If such a vanishing linear combination does not exist, we call the vectors positively linearly independent.

The constraint qualifications defined below are adaptations of the conditions from Section 3.1.4 for nonlinear programming. These include, in particular, the linear independence constraint qualification (LICQ), the Mangasarian–Fromovitz constraint qualification (MFCQ), the extended MFCQ (EMFCQ), and the constant positive linear dependence condition (CPLD), see Definition 3.29.

Definition 6.14 (Constraint qualifications for QVIs)**.** Let $\bar{x} \in X$ be an arbitrary point and let $\mathcal{I} := \{i = 1, \dots, m : g_i(\bar{x}, \bar{x}) = 0\}$. We say that

(a) *QVI-LICQ* holds in $\bar{x}$ if the set of gradients $\{\nabla_y g_i(\bar{x},\bar{x})\}_{i\in\mathcal{I}} \cup \{\nabla_y e_j(\bar{x},\bar{x})\}_{j=1}^p$ is linearly independent.

(b) *QVI-MFCQ* holds in $\bar{x}$ if the set of gradients $\{\nabla_y g_i(\bar{x},\bar{x})\}_{i\in\mathcal{I}} \cup \{\nabla_y e_j(\bar{x},\bar{x})\}_{j=1}^p$ is positively linearly independent.

(c) *QVI-EMFCQ* holds in $\bar{x}$ if the gradients $\{\nabla_y g_i(\bar{x},\bar{x})\}_{i\in\mathcal{I}'} \cup \{\nabla_y e_j(\bar{x},\bar{x})\}_{j=1}^p$ with $\mathcal{I}' := \{i = 1, \dots, m : g_i(\bar{x},\bar{x}) \ge 0\}$ are positively linearly independent.

(d) *QVI-CPLD* holds in $\bar{x}$ if, whenever $I \subseteq \mathcal{I}$ and $J \subseteq \{1, \dots, p\}$ are subsets such that the gradients $\{\nabla_y g_i(x,x)\}_{i\in I} \cup \{\nabla_y e_j(x,x)\}_{j\in J}$ are positively linearly dependent in $x := \bar{x}$, then they are linearly dependent for all x in a neighborhood of $\bar{x}$.

As in the optimization case, QVI-MFCQ can be seen as a special case of the Robinson constraint qualification for QVIs, applied to the constraint system (6.14). This follows from the derivation of the first-order optimality conditions in Section 6.1.2. In particular, QVI-MFCQ and QVI-RCQ are equivalent to (standard) MFCQ and RCQ, respectively, for the reduced problem (6.9). Hence, they are mutually equivalent by the discussion in Section 3.1.4.

With an obvious change of notation, we denote by $\mathcal{L}(x,\lambda,\mu) := F(x) + \nabla_y g(x,x)\lambda + \nabla_y e(x,x)\mu$ the Lagrange function of (Q) with the constraint set (6.14), where $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$. Then the KKT conditions of (Q) can be written as

$$\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu}) = 0, \quad 0 \le \bar{\lambda} \perp g(\bar{x},\bar{x}) \le 0, \quad e(\bar{x},\bar{x}) = 0.$$

The following result can be shown using the same arguments as in Theorem 3.54.

Proposition 6.15. *Let $\{(x^k,\lambda^k,\mu^k)\} \subseteq \mathbb{R}^{n+m+p}$ be a sequence such that $x^k \to \bar{x}$ for some $\bar{x} \in \Phi(\bar{x})$, $\mathcal{L}(x^k,\lambda^k,\mu^k) \to 0$, and $\min\{-g(x^k,x^k),\lambda^k\} \to 0$ as $k \to \infty$. Then the following assertions hold.*

(a) *If QVI-CPLD holds in $\bar{x}$, then $\bar{x}$ is a stationary point (and a solution) of the QVI.*

(b) *If QVI-MFCQ holds in $\bar{x}$, then $\{(\lambda^k,\mu^k)\}$ is bounded and its limit points lie in $\Lambda(\bar{x})$.*

(c) *If QVI-LICQ holds in $\bar{x}$, then $\{(\lambda^k,\mu^k)\}$ converges to the unique element in $\Lambda(\bar{x})$.*

6.2 The Augmented Lagrangian Method

We now formulate and discuss the augmented Lagrangian method for a QVI of the form (Q). Throughout this section, we again assume that the feasible set mapping has the representation

$$\Phi(x) := \{y \in C : G(x,y) \in K\}, \tag{6.15}$$

where X, Y are real Banach spaces, $C \subseteq X$ and $K \subseteq Y$ are nonempty closed convex sets, and $G : X^2 \to Y$ is a continuously differentiable mapping.

6.2.1 Discussion and Statement of the Algorithm

We now present the augmented Lagrangian method for the QVI (Q). The main approach is to penalize the function G and therefore reduce the QVI to a sequence of standard VIs. Throughout the remainder of this section, we assume that $i : Y \hookrightarrow H$ densely for some real Hilbert space H, and that $\mathcal{K}$ is a closed convex subset of H with $i^{-1}(\mathcal{K}) = K$.

For $\rho > 0$, consider the augmented Lagrangian $\mathcal{L}_\rho : X \times H \to X^*$ given by

$$\mathcal{L}_\rho(x,\lambda) := F(x) + \rho D_y G(x,x)^* \left[G(x,x) + \frac{\lambda}{\rho} - P_{\mathcal{K}}\left(G(x,x) + \frac{\lambda}{\rho} \right) \right]. \tag{6.16}$$

Note that, if $\mathcal{K}$ is a cone, then we can simplify the above formula to $\mathcal{L}_\rho(x,\lambda) = F(x) + D_y G(x,x)^* P_{\mathcal{K}^\circ}(\lambda + \rho G(x,x))$ by using Moreau's decomposition (Lemma 2.37).

For the construction of our algorithm, we will need a means of controlling the penalty parameters. To this end, we define the utility function

$$V(x,\lambda,\rho) := \left\| G(x,x) - P_{\mathcal{K}}\left(G(x,x) + \frac{\lambda}{\rho} \right) \right\|_H. \tag{6.17}$$

The function V is a composite measure of feasibility and complementarity; it arises from an inherent slack variable transformation which is often used to define the augmented Lagrangian for inequality or cone constraints.

Algorithm 6.16 (ALM for quasi-variational inequalities)**.** Let $(x^0, \lambda^0) \in X \times H$, $\rho_0 > 0$, let $B \subseteq H$ be a nonempty bounded set, $\gamma > 1$, $\tau \in (0,1)$, and set $k := 0$.

Step 1. If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in B$ and compute an approximate solution x^{k+1} of the VI

$$x \in C, \quad \big\langle \mathcal{L}_{\rho_k}(x, w^k), y - x \big\rangle \geq 0 \quad \forall y \in C. \tag{6.18}$$

Step 3. Update the vector of multipliers to

$$\lambda^{k+1} := \rho_k \left[G(x^{k+1}, x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}, x^{k+1}) + \frac{w^k}{\rho_k} \right) \right]. \tag{6.19}$$

Step 4. If $k = 0$ or

$$V(x^{k+1}, w^k, \rho_k) \leq \tau V(x^k, w^{k-1}, \rho_{k-1}) \tag{6.20}$$

holds, set $\rho_{k+1} := \rho_k$; otherwise, set $\rho_{k+1} := \gamma \rho_k$.

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

Let us make some simple observations. As in the previous chapters, one of the central features of the above algorithm is the use of the bounded sequence $\{w^k\}$ in certain places where traditional algorithms might use the sequence $\{\lambda^k\}$. This boundedness property is crucial for the subsequent convergence analysis. In practice, one usually tries to keep w^k as "close" as possible to λ^k, e.g., by defining $w^k := P_B(\lambda^k)$, where B (the bounded set from the algorithm) is chosen suitably to allow cheap projections.

The definition of λ^{k+1} implies that, regardless of the manner in which x^{k+1} is computed (exactly or inexactly), we always have the equality

$$\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}(x^{k+1}, \lambda^{k+1}) \quad \text{for all } k \in \mathbb{N}, \tag{6.21}$$

which follows directly from the definition of $\mathcal{L}_\rho$ and the multiplier updating scheme (6.19). This equality is the main motivation for the definition of λ^{k+1}.

Lemma 6.17. *We have $\lambda^k \in \mathcal{K}_\infty^\circ$ for all k. Moreover, there is a null sequence $\{r_k\} \subseteq \mathbb{R}_+$ such that $\big(\lambda^k, y - G(x^k, x^k)\big) \leq r_k$ for all $y \in \mathcal{K}$ and $k \in \mathbb{N}$.*

Proof. The proof is analogous to that of Lemma 4.5. Let $s^{k+1} := P_{\mathcal{K}}(G(x^{k+1}, x^{k+1}) + w^k/\rho_k)$. Then $\lambda^{k+1} \in \mathcal{N}_{\mathcal{K}}(s^{k+1})$ by Proposition 2.36, and thus $\lambda^{k+1} \in \mathcal{K}_\infty^\circ$ by Lemma 2.39. For the second assertion, observe that

$$G(x^{k+1}, x^{k+1}) = \frac{\lambda^{k+1} - w^k}{\rho_k} + s^{k+1}. \tag{6.22}$$

Using the fact that $\lambda^{k+1} \in \mathcal{N}_{\mathcal{K}}(s^{k+1})$, we obtain

$$\begin{aligned}\left(\lambda^{k+1}, y - G(x^{k+1}, x^{k+1})\right) &= \left(\lambda^{k+1}, y - \frac{1}{\rho_k}(\lambda^{k+1} - w^k) - s^{k+1}\right) \\ &\le \frac{1}{\rho_k}\Big[\left(\lambda^{k+1}, w^k\right) - \|\lambda^{k+1}\|_H^2\Big] =: r_{k+1}. \end{aligned} \tag{6.23}$$

We claim that this sequence $\{r_{k+1}\}$ satisfies $\limsup_{k\to\infty} r_{k+1} \le 0$. This yields the desired result (by replacing r_k with $\max\{0, r_k\}$). If $\{\rho_k\}$ is bounded, then (6.20) and (6.22) imply $\|\lambda^{k+1} - w^k\|_H/\rho_k \to 0$ and therefore $\|\lambda^{k+1} - w^k\|_H \to 0$. This yields the boundedness of $\{\lambda^{k+1}\}$ in H as well as $(\lambda^{k+1}, w^k) - \|\lambda^{k+1}\|_H^2 = (\lambda^{k+1}, w^k - \lambda^{k+1}) \to 0$. Hence, $r_k \to 0$. Assume now that $\rho_k \to \infty$. Note that (6.23) is a quadratic function in λ. A simple calculation therefore shows that $r_{k+1} \le \|w^k\|_H^2/(4\rho_k)$ and, hence, $\limsup_{k\to\infty} r_k \le 0$. □

As in the optimization case, we note that the assertion of Lemma 6.17 can be simplified if $\mathcal{K}$ is a closed convex *cone*. In this case, the first assertion becomes $\lambda^k \in \mathcal{K}^\circ$ for all k, and the second assertion is equivalent to

$$\liminf_{k\to\infty} \left\langle \lambda^k, G(x^k, x^k)\right\rangle \ge 0.$$

6.2.2 Existence of Penalized Solutions

We first discuss some sufficient conditions for the existence of solutions to the augmented subproblems (6.18). In the quasi-variational context, this question turns out to be substantially more complicated than for constrained optimization or variational inequalities, mainly due to the additional dependence of the constraint mapping G on the point x. As a result, only parts of the arguments from Proposition 5.4 (for the VI case) can be carried over to the present situation.

Proposition 6.18. *Assume that C is weakly compact, F is bounded and pseudomonotone, and one of the following conditions is satisfied:*

(a) *G is completely continuous with respect to x, weakly sequentially continuous with respect to (x, y), and $\mathcal{K}_\infty$-concave with respect to y, or*

(b) *G and $D_y G$ are completely continuous with respect to (x, y).*

Then the augmented Lagrangian subproblems (6.18) *admit solutions for all k.*

Proof. (a) For $k \in \mathbb{N}$, let $h_k(x, y) := d_{\mathcal{K}}^2(G(x, y) + w^k/\rho_k)$. Observe that h_k is convex and continuously differentiable with respect to y, weakly sequentially continuous with respect to x, and weakly sequentially lsc with respect to (x, y). Moreover, we have $\mathcal{L}_{\rho_k}(x, w^k) = F(x) + (\rho_k/2) D_y h_k(x, x)$ for all k. Consider the mapping

$$\Psi_k(x, y) := \langle F(x), x - y\rangle + \frac{\rho_k}{2}\big[h_k(x, x) - h_k(x, y)\big].$$

By Lemma 6.5, there exists a point $\hat{x} \in C$ such that $\Psi_k(\hat{x}, y) \le 0$ for all $y \in C$. Thus, the point $\hat{x}$ is a maximizer of $\Psi_k(\hat{x}, \cdot)$, with maximum value equal to zero. By Lemma 3.1, this implies $D_y\Psi_k(\hat{x},\hat{x})(y-\hat{x}) \le 0$ for all $y \in C$, and it is easy to check that this is precisely the desired variational inequality.

(b) In this case, it follows from Lemma 3.37 that the augmented Lagrangian $\mathcal{L}_{\rho_k}(\cdot, w^k)$ is pseudomonotone for every k. Hence, the claim follows from Corollary 3.41. □

6.2.3 Convergence for Convex Constraints

In this section, we analyze the global convergence properties of Algorithm 6.16 for QVIs where the set $\Phi(x)$ is convex for all x. In the situation where $\Phi(x) = \{y \in C : G(x,y) \in K\}$ as in (6.15), the natural analytic notion of convexity is the $\mathcal{K}_\infty$-concavity of G with respect to y, see Section 2.2.3. To obtain the stationarity of weak limit points, we will employ Proposition 6.4. Thus, we will need the pseudomonotonicity of F and the weak Mosco-continuity of Φ.

The above discussion is reflected in the following set of assumptions.

Assumption 6.19 (Primal convergence for QVIs)**.** We assume that

(i) the mapping F is bounded and pseudomonotone,

(ii) the multifunction Φ is weakly Mosco-continuous,

(iii) the operator G is $\mathcal{K}_\infty$-concave with respect to y,

(iv) $d_{\mathcal{K}} \circ G$ is weakly sequentially lower semicontinuous, and

(v) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Note that (i) and (ii) were also used in Theorem 6.7 on the existence of solutions. The weak sequential lower semicontinuity of $d_{\mathcal{K}} \circ G$ holds, for instance, if G is weakly sequentially continuous with respect to (x,y). Finally, the assumption (v) on the sequence $\{x^k\}$ is just an inexact version of the VI subproblem (6.18).

Similarly to before, the convergence analysis is split into separate discussions of feasibility and optimality. We begin with the feasibility part.

Lemma 6.20. *Let Assumption 6.19 hold and let $\bar{x}$ be a weak limit point of $\{x^k\}$. If $\Phi(\bar{x})$ is nonempty, then $\bar{x}$ is feasible.*

Proof. Let $I \subseteq \mathbb{N}$ be an index set such that $x^{k+1} \rightharpoonup_I \bar{x}$. Observe first that $\bar{x} \in C$ since C is closed and convex, hence weakly sequentially closed. It therefore remains to show that $G(\bar{x},\bar{x}) \in K$. If $\{\rho_k\}$ remains bounded, then the penalty updating scheme (6.20) implies

$$d_{\mathcal{K}}(G(x^{k+1},x^{k+1})) \le \left\| G(x^{k+1},x^{k+1}) - P_{\mathcal{K}}\left(G(x^{k+1},x^{k+1}) + \frac{w^k}{\rho_k} \right) \right\|_H \to 0.$$

Since $d_{\mathcal{K}} \circ G$ is weakly sequentially lsc, we obtain $d_{\mathcal{K}}(G(\bar{x},\bar{x})) = 0$ and thus $G(\bar{x},\bar{x}) \in K$. Assume now that $\rho_k \to \infty$ and that $G(\bar{x},\bar{x}) \notin K$, or equivalently $d_{\mathcal{K}}^2(G(\bar{x},\bar{x})) > 0$. Since $\Phi(\bar{x})$ is nonempty, we can choose an $y \in \Phi(\bar{x})$, and by inner semicontinuity there exists a sequence $y^{k+1} \in \Phi(x^{k+1})$ such that $y^{k+1} \to_I y$. Now, let $h_k(x,y) := d_{\mathcal{K}}(G(x,y) + w^k/\rho_k)$. Observe that h_k is convex with respect to y by Theorem 2.50, and that h_k^2 is continuously differentiable with respect to y by Lemma 2.43. Moreover, since the distance function is nonexpansive, it follows that $h_k(x^{k+1}, y^{k+1}) \le \|w^k\|_H/\rho_k$ for all k, and thus

$$\liminf_{k\to\infty} \left[h_k(x^{k+1}, x^{k+1}) - h_k(x^{k+1}, y^{k+1})\right] = \liminf_{k\to\infty} h_k(x^{k+1}, x^{k+1}) \ge d_{\mathcal{K}}(G(\bar{x}, \bar{x})) > 0.$$

Hence, there is a constant $c_1 > 0$ such that $h_k^2(x^{k+1}, x^{k+1}) - h_k^2(x^{k+1}, y^{k+1}) \ge c_1$ for all $k \in I$ sufficiently large. The convexity of h_k (and of h_k^2) now yields

$$\left\langle D_y(h_k^2)(x^{k+1}, x^{k+1}), y^{k+1} - x^{k+1}\right\rangle \le h_k^2(x^{k+1}, y^{k+1}) - h_k^2(x^{k+1}, x^{k+1}) \le -c_1.$$

By the boundedness of F, there is a constant $c_2 \in \mathbb{R}$ such that $\langle F(x^{k+1}), y^{k+1} - x^{k+1}\rangle \le c_2$ for all $k \in I$. Now, let $\{\varepsilon^k\}$ be the sequence from Assumption 6.19. Observe that $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = F(x^{k+1}) + (\rho_k/2) D_y(h_k^2)(x^{k+1}, x^{k+1})$ for all k. Therefore,

$$\left\langle \varepsilon^{k+1}, y^{k+1} - x^{k+1}\right\rangle \le \left\langle \mathcal{L}_{\rho_k}(x^{k+1}, w^k), y^{k+1} - x^{k+1}\right\rangle \le c_2 - \frac{\rho_k c_1}{2} \to -\infty,$$

which contradicts $\varepsilon^{k+1} \to 0$. □

The above lemma guarantees that the weak limit point $\bar{x}$ is "as feasible as possible" in the sense that it is feasible if (and only if) the set $\Phi(\bar{x})$ is nonempty. In many particular examples of QVIs (e.g., for moving-set problems, see below), we know a priori that $\Phi(x)$ is nonempty for all $x \in X$, and this directly yields the feasibility of $\bar{x}$.

Theorem 6.21. *Let Assumption 6.19 hold and let $\bar{x}$ be a weak limit point of $\{x^k\}$. If $\Phi(\bar{x})$ is nonempty, then $\bar{x}$ is feasible and a solution of the QVI.*

Proof. The feasibility follows from Lemma 6.20. For the optimality part, we apply Proposition 6.4. To this end, let $y^k \in \Phi(x^k)$ be arbitrary, let $\{\varepsilon^k\}$ be the sequence from Assumption 6.19, and recall that $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}(x^{k+1}, \lambda^{k+1})$ by (6.21). Thus,

$$\begin{aligned}\left\langle \varepsilon^{k+1}, y^{k+1} - x^{k+1}\right\rangle &\le \left\langle F(x^{k+1}) + D_y G(x^{k+1}, x^{k+1})^* \lambda^{k+1}, y^{k+1} - x^{k+1}\right\rangle \\ &= \left\langle F(x^{k+1}), y^{k+1} - x^{k+1}\right\rangle + \left(\lambda^{k+1}, D_y G(x^{k+1}, x^{k+1})(y^{k+1} - x^{k+1})\right) \\ &\le \left\langle F(x^{k+1}), y^{k+1} - x^{k+1}\right\rangle + \left(\lambda^{k+1}, G(x^{k+1}, y^{k+1}) - G(x^{k+1}, x^{k+1})\right)\end{aligned}$$

for all k, where we used the convexity of $y \mapsto (\lambda^{k+1}, G(x^{k+1}, y))$, see Theorem 2.50. Since $G(x^{k+1}, y^{k+1}) \in \mathcal{K}$, the last term is bounded from above by r_{k+1}, where $\{r_k\}$ is the null sequence from Lemma 6.17. Taking into account that $\varepsilon^k \to 0$ by Assumption 6.19, the result is now a consequence of Proposition 6.4. □

The above theorem guarantees the optimality of any weak limit point of the sequence generated by Algorithm 6.16. Despite this, it should be pointed out that the result is purely "primal" in the sense that no assertions are made for the multiplier sequence. We will investigate the dual (or, more precisely, primal-dual) behavior of the augmented Lagrangian method in more detail in Section 6.2.4.

If the mapping F is strongly monotone, then it is possible to obtain strong convergence of the iterates.

Corollary 6.22. *Let Assumption 6.19 hold and let F be strongly monotone on C. If $x^k \rightharpoonup_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$ and $\bar{x}$ is a solution of the QVI, then $x^k \to_I \bar{x}$.*

Proof. By the weak Mosco-continuity of Φ, there is a sequence $\bar{x}^k \in \Phi(x^k)$ such that $\bar{x}^k \to \bar{x}$. By the proof of Theorem 6.21, we have $\liminf_{k\in I}\langle F(x^k), \bar{x}^k - x^k\rangle \geq 0$ and therefore $\liminf_{k\in I}\langle F(x^k), \bar{x} - x^k\rangle \geq 0$. The strong monotonicity of F yields

$$c\|x^k - \bar{x}\|_X^2 \leq \big\langle F(x^k) - F(\bar{x}), x^k - \bar{x}\big\rangle = \big\langle F(x^k), x^k - \bar{x}\big\rangle - \big\langle F(\bar{x}), x^k - \bar{x}\big\rangle.$$

But the lim sup of the first term is less than or equal to zero, and the second term converges to zero since $x^k \rightharpoonup_I \bar{x}$. Hence, $\|x^k - \bar{x}\|_X \to 0$, and the proof is complete. □

The following is one of the most prominent classes of QVIs, see the discussion at the beginning of the chapter.

Example 6.23 (Moving set case)**.** A common class of QVIs arises if the feasible set is given by $\Phi(x) = c(x) + \Phi_0$ for some mapping $c : X \to X$ and a nonempty closed convex set $\Phi_0 \subseteq X$. This is usually called the *moving set case*, and it can be modeled by defining

$$G(x, y) := y - c(x) \quad \text{and} \quad K := \Phi_0.$$

(If the set Φ_0 is given through analytical constraints, these may also be incorporated into the function G.) In the above formulation, the mapping G is linear and thus $\mathcal{K}_\infty$-concave with respect to y (for any choice of $\mathcal{K}$). Assume now that F is bounded and pseudomonotone, and c is completely continuous. Then it is easy to see that Φ is weakly Mosco-continuous and $d_\mathcal{K} \circ G$ is weakly sequentially lsc. Thus, it follows from Theorem 6.21 that every weak limit point of the sequence $\{x^k\}$ is a solution of the QVI.

6.2.4 Primal-Dual Convergence Analysis

The purpose of this section is to consider the behavior of the augmented Lagrangian method in the absence of convexity. The analysis below is based on the KKT system of the QVI and can therefore be characterized as *primal-dual.* In particular, we will present results which guarantee the boundedness or convergence of the dual sequence $\{\lambda^k\}$. These results are also useful in the case of convex constraints. Hence, the present section is not intended to supersede the preceding one but rather to complement it.

Assumption 6.24 (Primal-dual convergence)**.** We assume that

(i) the operator F is bounded and pseudomonotone,

(ii) the mappings G and $D_y G$ are completely continuous, and

(iii) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

As in Chapters 4 and 5, one of the main ingredients in the primal-dual convergence analysis is an asymptotic analogue of the KKT conditions. In the present context, this condition takes on the form

$$\varepsilon^k - \mathcal{L}(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \big\langle \lambda^k, y - G(x^k, x^k)\big\rangle \leq r_k \ \forall y \in K \tag{6.24}$$

for all $k \geq 1$, where $\varepsilon^k \to 0$ in X^* and $r_k \to 0$ in $\mathbb{R}$. The above conditions can be verified by using Lemma 6.17 and observing that $\mathcal{L}(x^{k+1}, \lambda^{k+1}) = \mathcal{L}_{\rho_k}(x^{k+1}, w^k)$ for all k, see (6.21). This suggests that we can obtain the stationarity of limit points by using a suitable constraint qualification. Before doing so, however, we first have to deal with the attainment of feasibility.

Lemma 6.25. *Let Assumption 6.24 hold and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x} \in C$ and $-D_y(d_{\mathcal{K}}^2 \circ G)(\bar{x}, \bar{x}) \in \mathcal{N}_C(\bar{x})$. If $\bar{x}$ satisfies QVI-ERCQ, then $\bar{x}$ is feasible.*

Proof. Note that $\bar{x} \in C$ since C is weakly sequentially closed. If $\{\rho_k\}$ is bounded, then (6.20) implies that $d_{\mathcal{K}}(G(x^{k+1}, x^{k+1})) \to 0$, which yields $G(\bar{x}, \bar{x}) \in \mathcal{K}$. Hence, there is nothing to prove. Now, let $\rho_k \to \infty$, let $x^{k+1} \rightharpoonup_I \bar{x}$ on some subset $I \subseteq \mathbb{N}$, and let $\{\varepsilon^k\} \subseteq X^*$ be the sequence from Assumption 6.24. Then

$$\varepsilon^{k+1} - F(x^{k+1}) - D_y G(x^{k+1}, x^{k+1})^* \lambda^{k+1} \in \mathcal{N}_C(x^{k+1})$$

for all $k \in \mathbb{N}$. We now divide this inclusion by ρ_k, use the definition of λ^{k+1}, and the fact that $\mathcal{N}_C(x^{k+1})$ is a cone. It follows that

$$-D_y G(x^{k+1}, x^{k+1})^* \left[G(x^{k+1}, x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}, x^{k+1}) + \frac{w^k}{\rho_k} \right) \right] + \frac{\varepsilon^{k+1} - F(x^{k+1})}{\rho_k} \in \mathcal{N}_C(x^{k+1}).$$

Taking the limit $k \to_I \infty$ and using Proposition 2.40 yields

$$- D_y G(\bar{x}, \bar{x})^* [G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x}))] \in \mathcal{N}_C(\bar{x}), \tag{6.25}$$

which is the first claim. Assume now that ERCQ holds in $\bar{x}$, and let $r > 0$ be such that $B_r^Y \subseteq G(\bar{x}, \bar{x}) + D_y G(\bar{x}, \bar{x})(C - \bar{x}) - K$. Then, for any $y \in B_r^Y$, there are $z \in C$ and $w \in K$ such that $y = G(\bar{x}, \bar{x}) + D_y G(\bar{x}, \bar{x})(z - \bar{x}) - w$. In particular, we have

$$\begin{aligned} \langle G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})), y \rangle &= \left\langle D_y G(\bar{x}, \bar{x})^* \left[G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})) \right], z - \bar{x} \right\rangle \\ &\quad + \langle G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})), G(\bar{x}, \bar{x}) - w \rangle. \end{aligned}$$

The first term is nonnegative by (6.25), and so is the second term by standard projection inequalities. Hence, $\langle G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})), y \rangle \geq 0$ for all $y \in B_r^Y$, which implies $\langle G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})), y \rangle = 0$ for all $y \in B_r^Y$ and, since Y is dense in H, it follows that $G(\bar{x}, \bar{x}) - P_{\mathcal{K}}(G(\bar{x}, \bar{x})) = 0$. This completes the proof. □

The above is our main feasibility result for this section. Note that the assertion $-D_y(d_{\mathcal{K}}^2 \circ G)(\bar{x}, \bar{x}) \in \mathcal{N}_C(\bar{x})$ has a rather natural interpretation: the function $d_{\mathcal{K}}^2 \circ G$ is a measure of infeasibility, and the lemma above states that any weak limit point $\bar{x}$ of $\{x^k\}$ has a (partial) minimization property for this function on the set C. The observation that an extended form of RCQ yields the actual feasibility of the point $\bar{x}$ is motivated by similar arguments from optimization theory (see Section 3.1.2).

Having dealt with feasibility, we now turn to the main primal-dual convergence result. Recall that $\Lambda(\bar{x}) \subseteq Y^*$ denotes the set of Lagrange multipliers in a given point $\bar{x}$.

Theorem 6.26. *Let Assumption 6.24 hold, let $x^k \rightharpoonup_I \bar{x}$ on some subset $I \subseteq \mathbb{N}$, and assume that $\bar{x}$ satisfies QVI-ERCQ. Then $\bar{x}$ is feasible, the sequence $\{\lambda^k\}_{k\in I}$ is bounded in Y^*, and each of its weak-* accumulation points belongs to $\Lambda(\bar{x})$.*

Proof. The feasibility of $\bar{x}$ follows from Lemma 6.25. Observe that $G(x^k, x^k) \to_I G(\bar{x}, \bar{x})$ and $D_y G(x^k, x^k) \to_I D_y G(\bar{x}, \bar{x})$. Using the sequences $\{r_k\}$ and $\{\varepsilon^k\}$ Lemma 6.17 and Assumption 6.24, respectively, we have

$$\varepsilon^k - \mathcal{L}(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \big\langle \lambda^k, y - G(x^k, x^k) \big\rangle \leq r_k \ \forall y \in K \tag{6.26}$$

for all $k \geq 1$. To prove the boundedness of $\{\lambda^k\}_{k\in I}$, we now proceed as in Theorem 3.51. Applying the generalized open mapping theorem (Theorem 3.11) to the multifunction $\mathcal{W}(u) := G(\bar{x}, \bar{x}) + D_y G(\bar{x}, \bar{x})u - K$ on the domain $C - \bar{x}$, we obtain $r > 0$ such that

$$B_r^Y \subseteq G(\bar{x}, \bar{x}) + D_y G(\bar{x}, \bar{x})\big[(C - \bar{x}) \cap B_1^X\big] - K.$$

By the definition of the dual norm, we can choose a sequence $\{y^k\} \subseteq Y$ of unit vectors such that $\langle \lambda^k, y^k \rangle \geq \frac{1}{2}\|\lambda^k\|_{Y^*}$. For every k, we can write

$$-ry^k = G(\bar{x}, \bar{x}) + D_y G(\bar{x}, \bar{x})(v^k - \bar{x}) - z^k$$

with $\{v^k\} \subseteq C$ a bounded sequence and $\{z^k\} \subseteq K$. Since $D_y G(\bar{x}, \bar{x})$ is completely continuous (by Proposition 2.16), it follows that $ry^k = z^k - G(x^k, x^k) - D_y G(x^k, x^k)(v^k - x^k) + \delta^k$ with $\delta^k \to 0$ as $k \to_I \infty$. Assume now that k is large enough so that $\|\delta^k\|_Y \leq r/4$. Then, by (6.26),

$$\begin{aligned}\frac{r}{2}\|\lambda^k\|_{Y^*} \leq \big\langle \lambda^k, ry^k \big\rangle &\leq \big\langle \lambda^k, z^k - G(x^k, x^k) \big\rangle - \big\langle \lambda^k, D_y G(x^k, x^k)(v^k - x^k) \big\rangle + \frac{r}{4}\|\lambda^k\|_{Y^*} \\ &\leq \big\langle \lambda^k, z^k - G(x^k, x^k) \big\rangle + \big\langle F(x^k) - \varepsilon^k, v^k - x^k \big\rangle + \frac{r}{4}\|\lambda^k\|_{Y^*}.\end{aligned}$$

By (6.26), the first two terms on the right-hand side are bounded from above by some $c > 0$. Reordering the inequality yields $\frac{r}{4}\|\lambda^k\|_{Y^*} \leq c$, and the result follows.

Finally, let us show that every weak-* limit point of $\{\lambda^k\}$ is a Lagrange multiplier. Without loss of generality, we assume that $\lambda^k \rightharpoonup_I^* \bar{\lambda}$ for some $\bar{\lambda} \in Y^*$ on the same subset $I \subseteq \mathbb{N}$ where $\{x^k\}$ converges. By the second inequality in (6.26) and the fact that $G(x^k, x^k) \to_I G(\bar{x}, \bar{x})$, we obtain $\bar{\lambda} \in \mathcal{N}_K(G(\bar{x}, \bar{x}))$. Now, let $y \in C$. Then, by (6.26),

$$\big\langle \varepsilon^k, y - x^k \big\rangle \leq \big\langle F(x^k), y - x^k \big\rangle + \big\langle \lambda^k, D_y G(x^k, x^k)(y - x^k) \big\rangle.$$

By complete continuity, we have $D_y G(x^k, x^k) \to_I D_y G(\bar{x}, \bar{x})$ and $D_y G(\bar{x}, \bar{x})(y - x^k) \to_I D_y G(\bar{x}, \bar{x})(y - \bar{x})$, see Proposition 2.16. Hence, $D_y G(x^k, x^k)(y - x^k) \to_I D_y G(\bar{x}, \bar{x})(y - \bar{x})$. The result now follows by arguing as in Theorem 3.51. □

Similarly to the optimization case, we now give a final result under the assumption that the derivative $D_y G(\bar{x}, \bar{x}) \in L(X, Y)$ is a surjective operator. In this case, it is possible to obtain the weak-* convergence of the corresponding dual sequence to the unique Lagrange multiplier in $\bar{x}$. The proof is essentially identical to that of Proposition 4.20 and therefore an application of Proposition 3.52.

Proposition 6.27. *Let $\{x^k\}$ be generated by Algorithm 5.2 and let $x^k \rightharpoonup_I \bar{x}$ for some $I \subseteq \mathbb{N}$ and $\bar{x} \in X$. Assume that $\bar{x}$ is a solution of (Q), that $C = X$, F is weak-* sequentially continuous, $D_y G$ is completely continuous, and that $D_y G(\bar{x}, \bar{x})$ is surjective. Then $\{\lambda^k\}_{k \in I}$ converges weak-* to the unique element in $\Lambda(\bar{x})$.*

It follows that a QVI of the moving set type, if modeled as above, will always satisfy the surjectivity assumption from Proposition 6.27, and it satisfies QVI-RCQ in any feasible point. This has strong consequences for the convergence properties of the augmented Lagrangian method for such problems. We collect these observations in the following example.

Example 6.28 (Moving set case, continued)**.** Assume that we are in the moving set case from Example 6.23, i.e., that $\Phi(x) = c(x) + \Phi_0$ with $c : X \to X$ and $\Phi_0 \subseteq X$ a nonempty closed convex set. Let $G(x, y) := y - c(x)$ and $K := \Phi_0$. Assume that F is bounded and pseudomonotone, and that c is completely continuous. Then we know from Example 6.23 that every weak limit point $\bar{x}$ of $\{x^k\}$ is a solution of the QVI. Since $D_y G(x, x)$ is the identity mapping on X for all $x \in X$, we then obtain from Proposition 6.27 that the corresponding subsequence of $\{\lambda^k\}$ converges weak-* to the unique element in $\Lambda(\bar{x})$.

6.3 The Algorithm in Finite Dimensions

Having established the basic theory of the augmented Lagrangian method, we now specialize some of the preceding results for the finite-dimensional case. In addition, this section also contains an overview of the algorithm for finite-dimensional generalized Nash equilibrium problems, and an exact penalty scheme based on the augmented Lagrangian technique. The findings in this section are essentially a summary of the papers [127, 132].

6.3.1 Quasi-Variational Inequalities

Throughout this section, we consider a QVI where the feasible set mapping has the form

$$\Phi(x) = \{ y \in \mathbb{R}^n : g(x, y) \leq 0, \ h(x, y) \leq 0 \} \tag{6.27}$$

with two continuously differentiable functions $g : \mathbb{R}^{2n} \to \mathbb{R}^m$ and $h : \mathbb{R}^{2n} \to \mathbb{R}^p$. The purpose of this approach is to account for the possibility of partial penalization: the constraints defined by g will be penalized, whereas h is an (optional) constraint function which will stay as a constraint in the penalized subproblems. We stress that this framework is very general and gives us some flexibility to deal with different situations. The most natural choices are probably the following ones:

1. Penalize all contraints. This full penalization approach is the simplest and most straightforward approach where, formally, we set $p = 0$. The resulting subproblems are unconstrained and therefore become nonlinear equations.
2. Another natural splitting is the case where h covers the non-parametric constraints (i.e., those which do not depend on x), whereas g subsumes the remaining constraints.

The resulting penalized problems then become standard VIs and are therefore easier to solve than the original QVI since the (presumably) difficult constraints are moved into the objective function.

3. Finally, for certain problems, it might make sense to include some of the parametric constraints into h. In this case, the subproblems themselves are QVIs, but might still be easier to solve than the original QVI, e.g., if they belong to a particular subclass of QVIs or the remaining constraints satisfy a certain structure.

In the present setting, the Lagrange function of the QVI takes on the form $\mathcal{L}(x,\lambda,\mu) = F(x) + \nabla_y g(x,x)\lambda + \nabla_y h(x,x)\mu$, where $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$. The corresponding KKT conditions are given by

$$\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu}) = 0, \quad \min\{-g(\bar{x},\bar{x}),\bar{\lambda}\} = 0, \quad \text{and} \quad \min\{-h(\bar{x},\bar{x}),\bar{\mu}\} = 0.$$

If $p = 0$, then we can simply discard μ and state the above system with λ alone.

In order to formally describe the partial penalization scheme, we now consider the set-valued mapping $\Phi_h : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\Phi_h(x) = \{y \in \mathbb{R}^n : h(x,y) \le 0\}, \tag{6.28}$$

and the *(partial) augmented Lagrangian*

$$\mathcal{L}_\rho(x,\lambda) = F(x) + \nabla_y g(x,x)(\lambda + \rho g(x,x))_+, \tag{6.29}$$

where $\rho > 0$ and $\lambda \in \mathbb{R}^m$. The following is the algorithmic framework we consider in this section.

Algorithm 6.29 (ALM for quasi-variational inequalities in $\mathbb{R}^n$)**.** Let $(x^0,\lambda^0,\mu^0) \in \mathbb{R}^{n+m+p}$, $\rho_0 > 0$, $w^{\max} \ge 0$, $\gamma > 1$, $\tau \in (0,1)$, and set $k := 0$.

Step 1. If (x^k,λ^k,μ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in [0,w^{\max}]^m$ and compute an approximate KKT point (x^{k+1},μ^{k+1}) of the penalized QVI

$$x \in \Phi_h(x), \quad \mathcal{L}_{\rho_k}(x,w^k)^\top d \ge 0 \quad \forall d \in \mathcal{T}_{\Phi_h(x)}(x). \tag{6.30}$$

Step 3. Update the vector of multipliers to $\lambda^{k+1} := \max\{0, w^k + \rho_k g(x^{k+1},x^{k+1})\}$.

Step 4. If

$$\big\|\min\{-g(x^{k+1},x^{k+1}),\lambda^{k+1}\}\big\| \le \tau \big\|\min\{-g(x^k,x^k),\lambda^k\}\big\|, \tag{6.31}$$

then set $\rho_{k+1} := \rho_k$. Otherwise, set $\rho_{k+1} := \gamma\rho_k$.

Step 5. Set $k \leftarrow k+1$ and go to Step 1.

The above method is similar to Algorithm 6.16 but differs in the level of generality which is allowed for the nonpenalized constraints $h(x,y) \le 0$. In the present case, these constraints are allowed to be nonconvex and parametric, whereas in Section 6.2 the nonpenalized constraints were assumed to be of a simpler form. Moreover, the penalty parameter is increased using a slightly different updating rule, compare (6.31) and (6.20). Both schemes can be used in a similar manner when proving convergence to stationary points. Note also that a similarly modified updating scheme was already employed in Algorithm 5.28 for finite-dimensional GNEPs.

As in the general case, let us stress once again that the above method uses a bounded sequence $\{w^k\}$ as a partial replacement for the multiplier sequence $\{\lambda^k\}$. In the above algorithm, the boundedness is enforced by confining w^k to the m-dimensional hypercube $[0, w^{\max}]^m$, and it is indeed common to simply define w^k as the projection onto this set, i.e., $w^k := \min\{\lambda^k, w^{\max}\}$, where the minimum is understood componentwise.

Assumption 6.30 (Global convergence)**.** At Step 2 of Algorithm 6.29, we obtain x^{k+1} and μ^{k+1} such that, for $k \to \infty$, we have

$$\mathcal{L}_{\rho_k}(x^{k+1}, w^k) + \nabla_y h(x^{k+1}, x^{k+1})\mu^{k+1} \to 0,$$
$$\min\{-h(x^{k+1}, x^{k+1}), \mu^{k+1}\} \to 0.$$

This is a very natural assumption which asserts that the pair (x^{k+1}, μ^{k+1}) satisfies an approximate KKT condition for the augmented subproblems, and the degree of inexactness converges to zero for $k \to \infty$. The actual solvability of the subproblems obviously depends on the structure of the underlying QVI; for certain classes of QVIs, this topic is discussed in more detail in [127].

The definition of λ^{k+1} implies that

$$\mathcal{L}_{\rho_k}(x^{k+1}, w^k) = F(x^{k+1}) + \nabla_y g(x^{k+1}, x^{k+1})\lambda^{k+1} \tag{6.32}$$

for all k. This equality will be useful in the convergence analysis.

Let us now turn to a convergence analysis of Algorithm 6.29 under Assumption 6.30. As usual, we begin by considering the feasibility properties of the iterates $\{x^k\}$. In this context, the following observation is helpful: if ρ is large, then the augmented Lagrangian (6.29) is dominated by the term $\nabla_y \frac{1}{2}\|g_+(x,x)\|^2 = \nabla_y g(x,x) g_+(x,x)$. This means that we can expect the iterates to exhibit some kind of asymptotic stationarity of the function $\|g_+(x,y)\|^2$ with respect to the second variable. One has to take into account that the additional h-constraints may be of parametric nature as well. Thus, it is intuitive to model the feasibility properties by considering the auxiliary QVI

$$x \in \Phi_h(x), \quad (\nabla_y \|g_+(x,x)\|^2)^\top d \geq 0 \quad \forall d \in \mathcal{T}_{\Phi_h(x)}(x). \tag{6.33}$$

We will call this problem the *Feasibility QVI.* Note that (6.33) represents the first-order necessary conditions (in the tangent cone sense) of the minimization of $y \mapsto \|g_+(x,y)\|^2$ subject to $y \in \Phi_h(x)$, stated in the point $y := x$. Thus, the Feasibility QVI encodes the intuitive minimization property, discussed above, in a quasi-variational framework. We can therefore expect the iterates $\{x^k\}$ to eventually become approximate solutions of (6.33). A precise statement of this assertion is contained in the following lemma.

Lemma 6.31. *Let Assumption 6.30 hold, and let $\bar{x}$ be a limit point of $\{x^k\}$. Then:*

(a) *If h satisfies QVI-CPLD in $\bar{x}$, then $\bar{x}$ is a stationary point of the Feasibility QVI.*

(b) *If the function (g,h) satisfies QVI-EMFCQ in $\bar{x}$, then $\bar{x}$ is feasible.*

Proof. (a): Let $x^{k+1} \to_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$. If $\{\rho_k\}$ is bounded, then $\bar{x}$ is feasible by (6.31), and there is nothing to prove. Assume now that $\rho_k \to \infty$. By Assumption 6.30 and the definition of $\mathcal{L}_\rho$, we have

$$F(x^{k+1}) + \nabla_y g(x^{k+1}, x^{k+1})\big(w^k + \rho_k g(x^{k+1}, x^{k+1})\big)_+ + \nabla_y h(x^{k+1}, x^{k+1})\mu^{k+1} \to 0$$

and $\min\{-h(x^{k+1}, x^{k+1}), \mu^{k+1}\} \to 0$ as $k \to \infty$. Dividing by ρ_k and omitting some vanishing terms, we obtain

$$\nabla_y g(x^{k+1}, x^{k+1}) g_+(x^{k+1}, x^{k+1}) + \nabla_y h(x^{k+1}, x^{k+1})\hat{\mu}^{k+1} \to_I 0,$$

where $\hat{\mu}^{k+1} := \mu^{k+1}/\rho_k$. It is easy to verify that $\min\{-g(x^{k+1}, x^{k+1}), \hat{\mu}^{k+1}\} \to_I 0$ (see the proof of Lemma 5.30). The result therefore follows from Proposition 6.15.

(b): By (a), there exists $\hat{\mu} \in \mathbb{R}^p$ such that

$$\nabla_y \|g_+(\bar{x}, \bar{x})\|^2 + \nabla_y h(\bar{x}, \bar{x})\hat{\mu} = 0 \quad \text{and} \quad \min\{-h(\bar{x}, \bar{x}), \hat{\mu}\} = 0.$$

Expanding the sums and omitting some vanishing terms, we obtain

$$2 \sum_{g_i(\bar{x},\bar{x})>0} \nabla_y g_i(\bar{x}, \bar{x}) g_i(\bar{x}, \bar{x}) + \sum_{h_j(\bar{x},\bar{x})=0} \nabla_y h_j(\bar{x}, \bar{x})\hat{\mu}_j = 0.$$

By QVI-EMFCQ, it follows that the first sum must be empty, and $\bar{x}$ is feasible. □

One of the advantages of the Feasibility QVI is that it lends itself to a problem-specific analysis which often guarantees the feasibility of $\bar{x}$ without any additional assumptions. To see this, observe the following: if the functions g and h are convex with respect to y, then every solution $\bar{x}$ of the Feasibility QVI is necessarily a global solution of

$$\underset{y \in \mathbb{R}^n}{\text{minimize}} \ \|g_+(\bar{x}, y)\|^2 \quad \text{subject to} \quad h(\bar{x}, y) \le 0. \tag{6.34}$$

That is, $\bar{x}$ is a (partial) minimizer of the infeasibility measure $\|g_+(x,y)\|^2$. This argument has an important consequence: if, in addition to the convexity of g and h, the feasible set mapping Φ is nonempty-valued, then the minimal value in (6.34) is necessarily zero (attained at any $y \in \Phi(\bar{x})$), and thus we obtain from the minimality of $\bar{x}$ that $\bar{x}$ is a feasible point.

Using the above argument, we obtain the feasibility of limit points of Algorithm 6.29 for large classes of QVIs, including the ubiquitous moving-set case (see the introduction of Section 6.3). Interestingly, the Feasibility QVI admits even more practical applications and interpretations. For instance, it yields a very intuitive feasibility framework for generalized Nash equilibrium problems (see Section 6.3.3), and it can also be used to guarantee the feasibility of limit points for QVIs with bilinear constraints. For more details, we refer the reader to [132].

We now turn to the optimality of limit points. As usual, the main approach is to state an asymptotic KKT-type condition for the sequence $\{(x^k, \lambda^k, \mu^k)\}$, and then use Proposition 6.15 to obtain the optimality of limit points of $\{x^k\}$. Observe that, by Assumption 6.30 and (6.32), we have

$$\mathcal{L}(x^k, \lambda^k, \mu^k) \to 0 \quad \text{and} \quad \min\{-h(x^k, x^k), \mu^k\} \to 0. \tag{6.35}$$

Thus, the only missing condition for Proposition 6.15 is $\min\{-g(x^k, x^k), \lambda^k\} \to 0$. If this condition is verified, then Proposition 6.15 together with a suitable constraint qualification yields the optimality of limit points.

Theorem 6.32. *Let Assumption 6.30 hold and let $\bar{x}$ be a limit point of $\{x^k\}$. Then $\bar{x}$ is a stationary point of the QVI provided that one of the following holds:*

(a) *$\bar{x}$ is feasible and the function (g, h) satisfies QVI-CPLD in $\bar{x}$.*

(b) *The function (g, h) satisfies QVI-EMFCQ in $\bar{x}$.*

Proof. Note that QVI-CPLD for the function (g, h) implies QVI-CPLD for the function h. Hence, by Lemma 6.31, the assumptions of (b) imply those of (a), and it therefore suffices to prove (a). Let $x^{k+1} \to_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$. By (6.35) and Proposition 6.15, the claim follows if we are able to prove that

$$\min\{-g(x^{k+1}, x^{k+1}), \lambda^{k+1}\} \to_I 0. \tag{6.36}$$

By (6.31), this holds if $\{\rho_k\}$ is bounded. Now, let $\rho_k \to \infty$. Observe that $\lambda^{k+1} \geq 0$ for all k and $g(x^{k+1}, x^{k+1}) \to_I g(\bar{x}, \bar{x}) \leq 0$. Hence, we need to show that $\lambda_i^{k+1} \to_I 0$ whenever $g_i(\bar{x}, \bar{x}) < 0$ for some i. If i is such an index, then $g_i(x^{k+1}, x^{k+1}) < 0$ for $k \in I$ sufficiently large. Since $\{w_i^k\}$ is bounded and $g_i(x^{k+1}, x^{k+1}) \to_I g_i(\bar{x}, \bar{x}) < 0$, it follows that

$$\lambda_i^{k+1} = \max\{0, w_i^k + \rho_k g_i(x^{k+1}, x^{k+1})\} = 0 \tag{6.37}$$

for $k \in I$ sufficiently large. This shows that (6.36) holds in either case, and the result follows from Proposition 6.15. □

Part (b) of the above theorem is more convenient to state since QVI-EMFCQ directly implies the feasibility of $\bar{x}$. However, in many practical scenarios, assertion (a) is actually the sharper one. This is because, for many problem classes, the fulfillment of the Feasibility QVI (6.33) yields the feasibility of $\bar{x}$. In these cases, we obtain the feasibility from Lemma 6.31, without QVI-EMFCQ.

6.3.2 An Exact Penalty Method

We now consider a modification of the augmented Lagrangian method for QVIs (Algorithm 6.40) which, under suitable conditions, is an exact penalty method. Throughout this section, we assume that the constraint system of (Q) is given by

$$\Phi(x) := \{y \in \mathbb{R}^n : g(x, y) \leq 0\},$$

and that $F : \mathbb{R}^n \to \mathbb{R}^n$ is at least continuously differentiable and g twice continuously differentiable. The exact penalty approach only works if we are performing a full penalization of the constraints. Hence, we omit the function h from Section 6.3.1.

The basic approach is to remove the explicit multipliers in the augmented Lagrangian and replace them by a multiplier function which is dependent on x. More precisely, for a given x, we compute λ as a solution of the minimization problem

$$\underset{\lambda \in \mathbb{R}^m}{\text{minimize}} \ \|F(x) + \nabla_y g(x,x)\lambda\|^2 + \|\mathcal{D}_g(x)\lambda\|^2, \tag{6.38}$$

where $\mathcal{D}_g(x) := \operatorname{diag}(g_1(x,x), \ldots, g_m(x,x))$. This is a linear least-squares problem, and the following lemma states precisely when it has a unique solution.

Lemma 6.33 ([55, Prop. 2])**.** *The multiplier problem* (6.38) *has a unique solution whenever g satisfies QVI-LICQ in x. In this case, the solution vector $\lambda(x)$ is given by*

$$\lambda(x) = -M_*(x)^{-1}\nabla_y g(x,x)^\top F(x), \tag{6.39}$$

where $M_*(x)$ is the positive definite matrix $M_*(x) = \nabla_y g(x,x)^\top \nabla_y g(x,x) + \mathcal{D}_g(x)^2$.

As a logical consequence of the above lemma, we make the blanket assumption that g satisfies QVI-LICQ at every point $x \in \mathbb{R}^n$. This implies that $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a well-defined and continuously differentiable function. Note that one might ask whether, in the absence of QVI-LICQ, a particular solution of (6.38) could be used as a substitute, e.g., the minimum-norm solution. However, as fairly simple examples show, the resulting function $\lambda(x)$ might fail to even be continuous.

With QVI-LICQ and the smoothness of the multiplier function in mind, we consider the following basic algorithm for the realization of the exact penalty approach. Here and below, $\mathcal{L}_\rho$ is the augmented Lagrange function

$$\mathcal{L}_\rho(x,\lambda) = F(x) + \nabla_y g(x,x)(\lambda + \rho g(x,x))_+.$$

Algorithm 6.34 (Exact penalty method for QVIs in $\mathbb{R}^n$)**.** Choose $x^0 \in \mathbb{R}^n$, $\rho_0 > 0$, $\gamma > 1$, and set $k := 0$.

Step 1. Compute $\lambda^k := \lambda(x^k)$ as a solution to (6.38).

Step 2. If (x^k, λ^k) is a KKT point of the QVI: STOP.

Step 3. Compute x^{k+1} as an (exact) zero of the function $x \mapsto \mathcal{L}_{\rho_k}(x, \lambda(x))$.

Step 4. Set $\rho_{k+1} := \gamma\rho_k$, set $k \leftarrow k+1$, and go to Step 1.

Note that the exact penalty concept requires that we solve both the least-squares problem (6.38) and the nonlinear subproblem in Step 3 exactly. The latter always possesses solutions; for instance, it is easy to see that $\mathcal{L}_\rho(x, \lambda(x)) = 0$ for all $\rho > 0$ if x is a KKT point of the QVI, since in this case the least-squares problem (6.38) yields the corresponding (unique) Lagrange multiplier.

In general, the function $x \mapsto \mathcal{L}_\rho(x, \lambda(x))$ is not continuously differentiable, although it is semismooth in a sense to be specified later (see Chapter 7). If $(\bar{x}, \bar{\lambda})$ is a KKT point of the QVI and the *strict complementarity* condition holds, i.e., if $\bar{\lambda}_i > 0$ for all i with $g_i(\bar{x}, \bar{x}) = 0$, then it is easy to see that the function $x \mapsto \mathcal{L}_\rho(x, \lambda(x))$ is continuously differentiable in a neighborhood of $\bar{x}$.

It turns out that the exact penalty method is theoretically very similar to the augmented Lagrangian method for QVIs (Algorithm 6.29). To see this, note that, if x^{k+1} is a zero of $\mathcal{L}_{\rho_k}(x, \lambda(x))$, then it is also a zero of the function

$$\mathcal{L}_{\rho_k}(x, w^k), \quad \text{with} \quad w^k := \lambda(x^{k+1}).$$

Furthermore, if $\{x^{k+1}\}$ converges to $\bar{x}$ on some subsequence $I \subseteq \mathbb{N}$, then $\{w^k\}_{k \in I}$ is bounded. This implies that Algorithm 6.34 inherits many convergence properties from Algorithm 6.29, although it is, strictly speaking, not a special case of the latter, since the penalty parameter is handled differently and $\lambda(x)$ is not necessarily positive. In any case, the following result can be shown similarly to Lemma 6.31.

Lemma 6.35. *Assume that Algorithm 6.34 does not terminate finitely, and let $\bar{x}$ be a limit point of the sequence $\{x^k\}$. Then*

$$\nabla_y \|g_+(\bar{x}, \bar{x})\|^2 = 0. \tag{6.40}$$

It should be noted that (6.40) is nothing but the Feasibility QVI from Section 6.3.1, which takes on the above form in the case of full penalization.

We now turn to the central property of Algorithm 6.34, which is the exactness property. To this end, we first prove a technical lemma.

Lemma 6.36. *Let x be a zero of $\mathcal{L}_\rho(x, \lambda(x))$, and let*

$$s := \min\left\{-g(x, x), \frac{\lambda(x)}{\rho}\right\}, \quad t := \max\left\{-g(x, x), \frac{\lambda(x)}{\rho}\right\}.$$

Then $Ms = 0$, where $M := \nabla_y g(x, x)^\top \nabla_y g(x, x) - G(x) \operatorname{diag}(t)$. Furthermore, if $s = 0$, then $(x, \lambda(x))$ is a KKT point of the QVI.

Proof. By definition, $\lambda(x)$ is a solution of the least-squares problem (6.38). The first-order necessary conditions of this problem can be written as

$$\nabla_y g(x, x)^\top \mathcal{L}(x, \lambda(x)) = -\mathcal{D}_g(x)^2 \lambda(x).$$

The definition of s and t implies that $\rho s_i t_i = -g_i(x, x)\lambda_i(x)$ for all $i = 1, \dots, m$, and hence $\nabla_y g(x, x)^\top \mathcal{L}(x, \lambda(x)) = \rho \mathcal{D}_g(x) \operatorname{diag}(t) s$. Observe now that

$$0 = \mathcal{L}_\rho(x, \lambda(x)) = \mathcal{L}(x, \lambda(x)) - \rho \nabla_y g(x, x) s. \tag{6.41}$$

We therefore obtain

$$0 = \nabla_y g(x, x)^\top \mathcal{L}_\rho(x, \lambda(x)) = \rho\big[\mathcal{D}_g(x) \operatorname{diag}(t) - \nabla_y g(x, x)^\top \nabla_y g(x, x)\big] s.$$

This proves the first part. If $s = 0$, then $\min\{-g(x, x), \lambda(x)/\rho\} = 0$, which implies $g(x, x) \le 0$, $\lambda(x) \ge 0$, and $\lambda(x)^\top g(x, x) = 0$. It then follows from (6.41) that $(x, \lambda(x))$ is a KKT point of the QVI. $\square$

The following is the central result of this section. It shows that Algorithm 6.34 terminates after finitely many outer iterations, and the corresponding iterates x^k and λ^k constitute a KKT point of the underlying QVI.

Theorem 6.37. *Assume that the iterates* $\{x^k\}$ *generated by Algorithm 6.34 remain bounded and that every solution of* (6.40) *is a feasible point. Then the algorithm terminates finitely and produces a KKT point of the QVI.*

Proof. Assume that the method does not terminate finitely, i.e., we obtain sequences $\{x^k\}$ and $\{\rho_k\}$ with $\rho_k \to \infty$. Since $\{x^k\}$ remains bounded, we can choose a subset $I \subset \mathbb{N}$ such that $x^{k+1} \to_I \bar{x}$. By Lemma 6.35 and our assumptions, it follows that $\bar{x}$ is feasible. Using the notation from Lemma 6.36 and the fact that x^{k+1} is a zero of $\mathcal{L}_{\rho_k}(x, \lambda(x))$ for all k, we now obtain a sequence of matrices M_k with $M_k s^k = 0$, where

$$M_k = \nabla_y g(x^{k+1}, x^{k+1})^\top \nabla_y g(x^{k+1}, x^{k+1}) - \mathcal{D}_g(x^{k+1}) \operatorname{diag}(t^k).$$

But $t^k = \max\{-g(x^{k+1}, x^{k+1}), \lambda(x^{k+1})/\rho_k\} \to_I -g(\bar{x}, \bar{x})$ and, hence, $M_k \to_I M_*(\bar{x})$ with M_* from Lemma 6.33. It follows that M_k is nonsingular for sufficiently large $k \in I$. This implies $s^k = 0$ and the result follows from Lemma 6.36. □

Note that the above theorem uses two central assumptions: first, we need the sequence $\{x^k\}$ to remain bounded. This is a rather standard condition in the context of similar exact penalty methods [51, 55]. Secondly, we require that every solution of the Feasibility QVI (which, in this case, takes on the form (6.40)) is a feasible point. This is a similar condition to those discussed in Section 6.3.1 (see also [132]). Furthermore, it is essentially equivalent to Assumption B from [55].

We close this section by noting that, for the special case of generalized Nash equilibrium problems, there already exist exact penalty methods, see [66, 69, 86]. In the terminology of optimization problems, however, these methods correspond to the nonsmooth exact ℓ_1 penalty function, whereas the approach here is motivated by the differentiable exact penalty function from [55]. Consequently, even in the context of generalized Nash equilibrium problems, the present approach has better smoothness properties than existing (exact) penalty schemes which implies that the resulting subproblems are usually easier to solve.

Remark 6.38. Similar to the approach presented here, it is also possible to construct exact penalty methods for constrained optimization problems. This can be achieved by applying the above framework to the VI reformulation of the problem, but it can also be achieved by directly using the augmented Lagrange function $\mathcal{L}_\rho^{\mathrm{Opt}}$ from constrained optimization (see Chapter 4) and inserting the multiplier function $\lambda(x)$, see [55]. Recall that $\mathcal{L}_\rho(x, \lambda) = \nabla_x \mathcal{L}_\rho^{\mathrm{Opt}}(x, \lambda)$. However, despite this, the two exact penalty approaches are *not* identical, since the optimization variant results in a derivative of the form

$$\nabla\big[\mathcal{L}_\rho^{\mathrm{Opt}}(x, \lambda(x))\big] = \nabla_x \mathcal{L}_\rho^{\mathrm{Opt}}(x, \lambda(x)) + \nabla\lambda(x) \nabla_\lambda \mathcal{L}_\rho^{\mathrm{Opt}}(x, \lambda(x)),$$

whereas the exact penalty function $\mathcal{L}_\rho(x, \lambda(x))$ (in the VI sense) does not depend on the derivative $\nabla\lambda(x)$. This has the implication that, if one wishes to use second-order methods for the minimization of $\mathcal{L}_\rho^{\mathrm{Opt}}(x, \lambda(x))$, one actually needs the second-order differentiability of $\lambda(x)$, which in turn requires the *three times* differentiability of the functions defining the optimization problem, as opposed to the (Q)VI technique which would only require twice differentiability. For more details, the reader is referred to [51, 55].

6.3.3 Generalized Nash Equilibrium Problems

We conclude this chapter by showing how the augmented Lagrangian method can be specialized for generalized Nash equilibrium problems (GNEPs) in finite dimensions. The resulting algorithm is essentially equivalent to Algorithm 6.29, but it can be stated more naturally in the GNEP setting and some of the theoretical aspects have a more natural interpretation.

The basic framework we consider throughout this section is a GNEP with $N \in \mathbb{N}$ players, each in control of a variable $x^\nu \in \mathbb{R}^{n_\nu}$. As in Section 5.3, we write $x = (x^\nu, x^{-\nu}) \in \mathbb{R}^n$, and consider the GNEP where player ν attempts to solve the problem

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}}\ f_\nu(x) \quad \text{subject to} \quad g^\nu(x) \le 0,\ h^\nu(x) \le 0. \tag{6.42}$$

Here, $f_\nu : \mathbb{R}^n \to \mathbb{R}$, $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$, $h^\nu : \mathbb{R}^n \to \mathbb{R}^{p_\nu}$ are continuously differentiable functions, and

$$n := \sum_{\nu=1}^N n_\nu, \quad m := \sum_{\nu=1}^N m_\nu, \quad p := \sum_{\nu=1}^N p_\nu, \quad \text{with } n_\nu \in \mathbb{N} \text{ and } m_\nu, p_\nu \in \mathbb{N}_0.$$

Here and throughout, we assume that the functions f_ν, g_ν, h_ν are convex with respect to x^ν, so that (6.42) is a convex minimization problem for all ν. In this situation, the GNEP is typically called *player-convex*.

Similarly to Section 6.3.1, the purpose of the above splitting scheme is to account for the possibility of partial penalization: the constraints g^ν will be penalized by the augmented Lagrangian approach, whereas the functions h^ν stay as constraints in the corresponding subproblems. This is also the reason why p_ν is allowed to be zero, since this corresponds to the case where no additional (nonpenalized) constraints are given for player ν. We also allow some of the numbers m_ν to vanish, although m should be strictly positive since otherwise there would be no constraints to penalize.

The notions of generalized Nash equilibrium and of KKT points can be defined similarly to Section 5.3.2. For the sake of brevity, let $G^\nu := (g^\nu, h^\nu)$ denote the full constraint function of player ν. We say that $\bar{x} \in \mathbb{R}^n$ is a *generalized Nash equilibrium* or simply a *solution* of (6.42) if, for all ν, we have $G^\nu(\bar{x}) \le 0$ and

$$f_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \le f_\nu(x^\nu, \bar{x}^{-\nu}) \quad \forall x^\nu \in \mathbb{R}^{n_\nu} : G^\nu(x^\nu, \bar{x}^{-\nu}) \le 0.$$

Moreover, a tuple $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is a *KKT point* of (6.42) if, for all ν, we have

$$\begin{gathered} \nabla_{x^\nu} f_\nu(\bar{x}) + \nabla_{x^\nu} g^\nu(\bar{x}) \bar{\lambda}^\nu + \nabla_{x^\nu} h^\nu(\bar{x}) \bar{\mu}^\nu = 0, \\ \min\{-g^\nu(\bar{x}), \bar{\lambda}^\nu\} = 0, \quad \text{and} \quad \min\{-h^\nu(\bar{x}), \bar{\mu}^\nu\} = 0. \end{gathered}$$

For the subsequent analysis, we will also need certain constraint qualifications for GNEPs.

Definition 6.39 (Constraint qualifications for GNEPs)**.** Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary point and let $\mathcal{I}_\nu := \{i = 1, \dots, m_\nu + p_\nu : G_i^\nu(\bar{x}) = 0\}$ for all ν. We say that

(a) *GNEP-LICQ* holds in $\bar{x}$ if, for all ν, the set of gradients $\{\nabla_{x^\nu} G_i^\nu(\bar{x})\}_{i \in \mathcal{I}_\nu}$ is linearly independent.

(b) *GNEP-MFCQ* holds in $\bar{x}$ if, for all ν, the set of gradients $\{\nabla_{x^\nu} G_i^\nu(\bar{x})\}_{i \in \mathcal{I}_\nu}$ is positively linearly independent.

(c) *GNEP-EMFCQ* holds in $\bar{x}$ if, for all ν, the set of gradients $\{\nabla_{x^\nu} G_i^\nu(\bar{x})\}_{i \in \mathcal{I}'_\nu}$ with $\mathcal{I}'_\nu := \{i = 1, \dots, m_\nu + p_\nu : G_i^\nu(\bar{x}) \geq 0\}$ is positively linearly independent.

(d) *GNEP-CPLD* holds in $\bar{x}$ if, for all ν, whenever $I_\nu \subseteq \mathcal{I}_\nu$ is a subset such that the gradients $\{\nabla_{x^\nu} G_i^\nu(x)\}_{i \in I_\nu}$ are positively linearly dependent in $x := \bar{x}$, then they are linearly dependent for all x in a neighborhood of $\bar{x}$.

The GNEP (6.42) is strongly connected to a quasi-variational inequality of the form discussed in Section 6.3.1. Indeed, assume that $\bar{x}$ is a solution of the GNEP, and define the functions

$$F(x) := (\nabla_{x^\nu} f_\nu(x))_{\nu=1}^N, \qquad \begin{array}{l} g(x,y) := (g^\nu(y^\nu, x^{-\nu}))_{\nu=1}^N, \\ h(x,y) := (h^\nu(y^\nu, x^{-\nu}))_{\nu=1}^N. \end{array}$$

Using Lemma 3.1, it is easy to see that $\bar{x}$ is then a solution of the QVI defined by the operator F and the feasible set (6.27). Moreover, we have

$$\nabla_y g(x,x) = \begin{pmatrix} \nabla_{x^1} g^1(x) & & \\ & \ddots & \\ & & \nabla_{x^N} g^N(x) \end{pmatrix},$$

where unspecified blocks are understood to be zero, and similarly for h. This implies that the GNEP constraint qualifications above coincide with their QVI counterparts from Definition 6.14.

The following is a description of the algorithm we will use throughout this section. Given $\rho_\nu > 0$, the *(partial) augmented Lagrangian* of player ν takes on the form

$$\mathcal{L}_{\rho_\nu}^\nu(x, \lambda^\nu) := f_\nu(x) + \frac{\rho_\nu}{2} \left\| \left(g^\nu(x) + \frac{\lambda^\nu}{\rho_\nu} \right)_+ \right\|^2 - \frac{\|\lambda^\nu\|^2}{2\rho_\nu}, \tag{6.43}$$

where $\lambda^\nu \in \mathbb{R}^{m_\nu}$.

Algorithm 6.40 (Augmented Lagrangian method for GNEPs in $\mathbb{R}^n$)**.** Let $(x^0, \lambda^0, \mu^0) \in \mathbb{R}^{n+m+p}$. Let $w^{\max} \geq 0$, $\rho_{\nu,0} > 0$, $\gamma_\nu > 1$, $\tau_\nu \in (0,1)$ for all ν, and set $k := 0$.

Step 1. If (x^k, λ^k, μ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in [0, w^{\max}]^m$ and compute an approximate KKT point (x^{k+1}, μ^{k+1}) of the GNEP where player ν attempts to solve

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}}\ \mathcal{L}_{\rho_{\nu,k}}^\nu(x, w^{\nu,k}) \quad \text{subject to} \quad h^\nu(x) \leq 0. \tag{6.44}$$

Step 3. For every $\nu = 1, \dots, N$, update the multiplier vector to

$$\lambda^{\nu,k+1} := \max\{0, w^{\nu,k} + \rho_{\nu,k} g^\nu(x^{k+1})\}. \tag{6.45}$$

Step 4. For all $\nu = 1, \dots, N$, if

$$\big\|\min\{-g^\nu(x^{k+1}), \lambda^{\nu,k+1}\}\big\| \le \tau_\nu \big\|\min\{-g^\nu(x^k), \lambda^{\nu,k}\}\big\|, \tag{6.46}$$

then set $\rho_{\nu,k+1} := \rho_{\nu,k}$. Otherwise, set $\rho_{\nu,k+1} := \gamma_\nu \rho_{\nu,k}$.

Step 5. Set $k \leftarrow k+1$ and go to Step 1.

Up to some minor technical details, Algorithm 6.40 is essentially a special case of Algorithm 6.29. One difference lies in the formulation of the subproblems, which in the present case are given as GNEPs, whereas in Section 6.3.1 they are formulated as quasi-variational inequalities. This distinction plays no role if the subproblems are solved in a KKT sense. Another technical difference between the two algorithms is that, in Algorithm 6.40, we allow distinct sequences $\{\rho_{\nu,k}\}$ for every player, and they are updated with possibly different parameters γ_ν and τ_ν.

It follows that the convergence theory for Algorithm 6.40 can be carried out in a similar fashion to the previous section. We therefore skip many of the proofs in this section, instead pointing out and discussing the different interpretations and consequences of some of the results in the GNEP framework. A more detailed account of the proofs can also be found in [129].

The following is the basic convergence assumption for Algorithm 6.40. As in Section 5.3, we write $\mathcal{L}_k^\nu(x) := \mathcal{L}_{\rho_{\nu,k}}^\nu(x, w^{\nu,k})$ for the augmented Lagrangian of player ν at iteration k.

Assumption 6.41 (Global convergence for GNEPs in $\mathbb{R}^n$)**.** At Step 2 of Algorithm 6.40, we obtain (x^{k+1}, μ^{k+1}) such that, for all ν, we have

$$\nabla_{x^\nu} \mathcal{L}_k^\nu(x^{k+1}) + \nabla_{x^\nu} h^\nu(x^{k+1}) \mu^{\nu,k+1} \to 0 \quad \text{and} \quad \min\{-h^\nu(x^{k+1}), \mu^{\nu,k+1}\} \to 0$$

We once again begin by addressing the feasibility of limit points of $\{x^k\}$. We have seen in the previous section that the feasibility properties can be modeled by considering an auxiliary problem. In the present setting, this can be motivated as follows. Due to the structure of the augmented Lagrangian (6.43) and since we solve the subproblems (6.44) in a Nash sense, we can expect some kind of stationarity of the function $\|g_+^\nu(x)\|^2$ for all ν, but with respect to x^ν only. Taking into account the additional h-constraints, this leads to the GNEP where player ν attempts to solve

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}} \ \|g_+^\nu(x)\|^2 \quad \text{subject to} \quad h^\nu(x) \le 0. \tag{6.47}$$

We will refer to this problem as the *game of infeasibility* or *Feasibility GNEP* since it describes the best we can expect regarding the feasibility of the limit points: player ν minimizes the violation of the constraint g^ν with respect to his own variable x^ν, subject to the nonpenalized constraints described by h^ν.

Note that the Feasibility GNEP is a special case of the Feasibility QVI from Section 6.3.1. However, the former has a more tangible interpretation and therefore merits a dedicated discussion. As in the QVI case, it turns out that limit points of Algorithm 6.40 are stationary points of the Feasibility GNEP. The proof of this result is basically identical to Lemma 6.31.

Lemma 6.42. *Let Assumption 6.41 hold and let $\bar{x}$ be a limit point of $\{x^k\}$. Then:*

(a) *If the functions h^ν satisfy GNEP-CPLD in $\bar{x}$, then $\bar{x}$ is a stationary point of the Feasibility GNEP* (6.47).

(b) *If the functions (g^ν, h^ν) satisfy GNEP-EMFCQ in $\bar{x}$, then $\bar{x}$ is feasible.*

As indicated by the above theorem, the Feasibility GNEP plays a fundamental role in the analysis of penalty-type algorithms such as the augmented Lagrangian method. An interesting case in which the Feasibility GNEP has some structural properties is the following, which covers, as a special case, the jointly convex GNEP. Assume that the functions g^ν describe a shared constraint (which we denote by g) and that h^ν is a function of x^ν only. Furthermore, assume that both g and h^ν are convex. Hence, player ν's optimization problem takes the form

$$\underset{x^\nu \in \mathbb{R}^{n_\nu}}{\text{minimize}} \; f_\nu(x) \quad \text{subject to} \quad g(x) \le 0, \; h^\nu(x^\nu) \le 0. \tag{6.48}$$

For such GNEPs, we can prove the following theorem which makes the same assertion as Lemma 6.42 (b). Note, however, that we do not require any constraint qualification for the function g.

Corollary 6.43. *Consider a GNEP of the form* (6.48), *with g, h^ν convex functions, and assume that the feasible set of the problem is nonempty. Then every KKT point of the corresponding Feasibility GNEP* (6.47) *is a feasible point for* (6.48).

Proof. By assumption, there are multipliers $\hat{\mu}^\nu \in \mathbb{R}^{p_\nu}$ such that

$$\nabla_{x^\nu} \|g_+(\bar{x})\|^2 + \nabla h^\nu(\bar{x}^\nu)\hat{\mu}^\nu = 0 \quad \text{and} \quad \min\{-h^\nu(\bar{x}^\nu), \hat{\mu}^\nu\} = 0$$

for all ν. Hence, $\bar{x}$ together with $\hat{\mu} := (\hat{\mu}^1, \dots, \hat{\mu}^N)$ is a KKT point of the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; \|g_+(x)\|^2 \quad \text{subject to} \quad h^\nu(x^\nu) \le 0 \text{ for all } \nu. \tag{6.49}$$

Since this is a convex optimization problem, it follows that $\bar{x}$ is a global minimizer. By assumption, the optimal value of (6.49) is zero, and thus $\bar{x}$ is feasible. □

We now prove the optimality of limit points of Algorithm 6.40. The definition of $\lambda^{\nu,k+1}$ in the algorithm implies that

$$\nabla_{x^\nu} \mathcal{L}_k^\nu(x^{k+1}) = \nabla_{x^\nu} f_\nu(x^{k+1}) + \nabla_{x^\nu} g^\nu(x^{k+1})\lambda^{\nu,k+1}$$

for all k. Together with Assumption 6.41, this suggests that (x^k, λ^k, μ^k) is some kind of asymptotic KKT sequence for the GNEP (6.42). (This concept was formally defined in Section 3.2.3, but only for variational inequalities.) This property implies the following result whose proof is basically identical to that of Theorem 6.32.

Theorem 6.44. *Let Assumption 6.41 hold and let $\bar{x}$ be a limit point of $\{x^k\}$. Then $\bar{x}$ is a stationary point of the GNEP provided that one of the following holds:*

(a) *$\bar{x}$ is feasible and the functions (g^ν, h^ν) satisfies GNEP-CPLD in $\bar{x}$.*

(b) *The functions (g^ν, h^ν) satisfies GNEP-EMFCQ in $\bar{x}$.*

We close this section with a remark on the relationship between Algorithm 6.40 and Algorithm 5.28 for jointly convex GNEPs.

Remark 6.45. The method discussed in this section can also be seen as a generalization of Algorithm 5.28 for jointly convex GNEPs in finite dimensions. Indeed, that algorithm can be recovered as a special case of the present one by simply choosing the parameters $\lambda^{\nu,0}$, $\mu^{\nu,0}$, $\rho_{\nu,0}$, γ_ν, and τ_ν from Algorithm 6.40 in a manner which is independent of ν.

Chapter 7

Applications

This chapter presents a variety of applications of the theoretical framework from Chapters 4 to 6. In addition, we discuss a general type of semismooth Newton methods which can be used to solve the subproblems occurring in the augmented Lagrangian method.

Recall that the optimization framework from Chapter 4 encompasses multiple problem types such as nonlinear programming, semidefinite and second-order cone programming, and function space optimization. For nonlinear programming, there exists a plethora of literature on the numerical behavior of augmented Lagrangian methods, see, for instance, [5, 26, 124]. For semidefinite programming, many implementation details and applications can be found in the works of Kočvara and Stingl [145–147]. As a result, the main focus in this chapter will be on function space optimization problems.

One of the most fundamental ingredients for a practical application of the augmented Lagrangian scheme is a suitable generalized Newton method for the solution of the augmented subproblems. Speaking in the optimization case, the distinctive feature of these problems is that they are once but not twice continuously differentiable. Hence, an indispensable tool for their solution is a sufficiently general theory of *semismooth functions* and corresponding Newton-type methods. Historically, the concept of semismoothness was introduced by Mifflin [162], and the semismooth Newton method goes back to the works of Qi and Sun [180] and Kummer [152] in finite dimensions, see also [179]. The method was extended to the general Banach space setting in [43, 213]; for more details, see [109, 214]. For certain problem types, the semismooth Newton algorithm coincides with a primal-dual active set strategy, see [21, 23, 102].

This chapter is structured as follows. Section 7.1 contains some preliminary thoughts and considerations for the practical application of the augmented Lagrangian scheme. In particular, this section contains a brief discussion of semismooth functions and the resulting generalized Newton methods which play a fundamental role in the solution of the augmented subproblems. In Section 7.1.2, we present a semismooth Levenberg–Marquardt type algorithm which can be seen as a globalization or regularization of the semismooth Newton method. Section 7.1.3 then contains some remarks and a discussion of the discretization techniques employed in the numerical experiments.

The subsequent sections then deal with various types of problem classes and corresponding applications. Most of the examples can be found in similar forms in the publications [129, 132, 133, 136] or the preprints [128, 134]. Section 7.2 is dedicated to constrained optimization problems in function spaces, including convex and nonconvex obstacle-type problems as well as state-constrained optimal control. Section 7.3 contains generalized Nash equilibrium problems (GNEPs) in a Banach space setting, including environmental differential games and multiobjective control problems involving partial differential equations. In Section 7.4, we present some applications of the quasi-variational inequality (QVI) framework from Chapter 6. This section contains an implicit version of the well-known Signorini problem [18, 167] and a QVI involving parametric gradient constraints which arises, for instance, in superconductivity [104, 193]. Finally, in Sections 7.5 and 7.6, we give some applications of GNEPs and QVIs, respectively, in finite dimensions.

7.1 Preliminary Considerations

This section contains some preliminary discussion for the implementation of the augmented Lagrangian method. In particular, we will analyze the concept of semismoothness and present two algorithms for the solution of the augmented subproblems.

7.1.1 Semismooth Newton-type Methods

The practical implementation of the augmented Lagrangian technique crucially depends on the fulfillment of generalized smoothness properties. Recall that the augmented Lagrange function $\mathcal{L}_\rho$ (in the optimization sense) is once but not twice continuously differentiable with respect to x since the derivative $\mathcal{L}_\rho'$ with respect to x involves the projection operator $P_\mathcal{K}$, see the discussion in Section 4.1.3. If $\mathcal{K}$ is not a subspace of H, then this projection is in general a nonsmooth function.

The present section is therefore dedicated to a brief description of generalized smoothness concepts, in particular the so-called *semismoothness*. In many applications, it is possible to show that the projection operator $P_\mathcal{K}$ is indeed semismooth, and this facilitates the application of Newton-type methods for the solution of the augmented subproblems.

For the sake of generality, we conduct the subsequent analysis in the framework of a general nonlinear operator

$$T : X \to Y, \quad \text{where } X \text{ and } Y \text{ are real Banach spaces.}$$

The main concept of generalized smoothness we will consider is the following.

Definition 7.1 (Semismooth mapping)**.** Let X, Y be real Banach spaces and $T : X \to Y$ a continuous mapping. We say that T is *semismooth* on X if there exists a set-valued mapping $\partial T : X \rightrightarrows L(X, Y)$ with nonempty images such that, for all $x \in X$,

$$\sup_{M \in \partial T(x+s)} \|T(x+s) - T(x) - Ms\|_Y = o(\|s\|_X) \quad \text{as } \|s\|_X \to 0. \tag{7.1}$$

In this case, we also say that T is *∂T-semismooth*.

It is possible to restrict the definition of semismoothness to subsets of X. This modification is fairly straightforward and not necessary for our purposes.

The notion of semismoothness is a fundamental concept of generalized smoothness which will allow us to formulate Newton-type algorithms with fast local convergence properties. Clearly, if T is continuously differentiable, then it is also semismooth, and we can choose $\partial T(x) := \{T'(x)\}$ for all x. However, it is interesting to note that, even in the differentiable case, the generalized derivative ∂T is not unique. Indeed, given any finite set $S \subseteq X$, we can always change the values $\partial T(x)$, $x \in S$, in an essentially arbitrary manner, and the resulting set-valued mapping will still be a generalized derivative of T. This is because, for any $x \in X$, we have $x + s \notin S$ for all $s \neq 0$ sufficiently small, regardless of whether $x \in S$ or not.

If the spaces X, Y are finite-dimensional and the function T is locally Lipschitz-continuous, then there are two common candidates for generalized derivatives. In this case, a famous theorem of Rademacher [191] asserts that T is Fréchet-differentiable on a dense subset $D_T \subseteq X$, and this motivates the *Bouligand subdifferential*

$$\partial_B T(x) := \bigl\{M \in L(X,Y) : \exists x^k \to x,\ \{x^k\} \subseteq D_T,\ \text{such that } T'(x^k) \to M\bigr\},$$

as well as the *Clarke subdifferential* $\partial_{Cl} T(x)$, which is the convex hull of $\partial_B T(x)$. Many common nonsmooth functions T on finite-dimensional spaces can be shown to be semismooth with respect to ∂_B or ∂_{Cl}; indeed, a common definition in the finite-dimensional context is to say that T is semismooth if it is $\partial_{Cl} T$-semismooth in the sense of Definition 7.1. Some examples of $\partial_{Cl} T$-semismooth functions include the absolute value function, the positive (negative) part mapping on $\mathbb{R}^n$ (see below), and the projection onto the cone of positive semidefinite matrices [207].

An important property of semismoothness is its propagation under various mathematical operations. Note that we call a set-valued mapping $\mathcal{W} : X \rightrightarrows Y$ locally bounded if, for every $x \in X$, there exists $r > 0$ such that $\mathcal{W}(B_r(x))$ is bounded in Y.

Proposition 7.2 ([109, Thm. 2.10])**.** *Let X, Y, Z, X_i, Y_i be real Banach spaces.*

(a) *If the operators $T_i : X \to Y_i$ are ∂T_i-semismooth for $i = 1, 2$, then (T_1, T_2) is $(\partial T_1 \times \partial T_2)$-semismooth.*

(b) *If the operators $T_i : X \to Y$ are ∂T_i-semismooth for $i = 1, 2$, then $T_1 + T_2$ is $(\partial T_1 + \partial T_2)$-semismooth.*

(c) *If $T_1 : Y \to Z$ and $T_2 : X \to Y$ are ∂T_i-semismooth, the mapping ∂T_1 is locally bounded, and T_2 is locally Lipschitz-continuous, then $T_1 \circ T_2$ is semismooth with*

$$\partial(T_1 \circ T_2)(x) = \{M_1 M_2 : M_1 \in \partial T_1(T_2(x)),\ M_2 \in \partial T_2(x)\}.$$

We now present the basic semismooth Newton algorithm for the solution of the nonlinear equation

$$T(x) = 0, \tag{7.2}$$

where $T : X \to Y$ is a semismooth operator and X, Y are real Banach spaces.

Algorithm 7.3 (Basic semismooth Newton method)**.** Let $x^0 \in X$ and $k := 0$.

Step 1. If x^k is a sufficiently accurate zero of T: STOP.

Step 2. Choose $M_k \in \partial T(x^k)$ and compute d^k as a solution of the Newton equation

$$M_k d^k = -T(x^k). \tag{7.3}$$

Step 3. Set $x^{k+1} := x^k + d^k$, set $k \leftarrow k+1$, and go to Step 1.

Note that the choice of M_k in Step 2 is essentially arbitrary. Furthermore, the Newton equation (7.3) may have none or multiple solutions depending on M_k. In an ideal scenario, the operators M_k are invertible, and thus this equation admits a unique solution d^k.

Theorem 7.4 ([109, Thm. 2.12])**.** *Let $T : X \to Y$ be a semismooth operator, $\bar{x} \in X$ a point with $T(\bar{x}) = 0$, and assume that there exist $r, c > 0$ such that all elements $M \in \partial T(x)$, $x \in B_r(\bar{x})$, admit bounded inverses $M^{-1} : Y \to X$ with*

$$\|M^{-1}\|_{L(Y,X)} \le c \quad \text{for all} \quad M \in \partial T(x),\ x \in B_r(\bar{x}).$$

Then, if x^0 is chosen sufficiently close to $\bar{x}$, the sequence generated by Algorithm 7.3 converges Q-superlinearly to $\bar{x}$.

Let us now discuss how the above algorithm and convergence result can be applied in practice. To this end, assume that the nonlinear mapping T is the derivative of the augmented Lagrangian $\mathcal{L}_\rho$ from Chapter 4, which takes on the form

$$\mathcal{L}_\rho'(x, w) = f'(x) + \rho G'(x)^* \left[G(x) + \frac{w}{\rho} - P_{\mathcal{K}}\left(G(x) + \frac{w}{\rho} \right) \right]. \tag{7.4}$$

For the sake of simplicity, we assume that no additional constraints are present, so that the augmented subproblems arising in Algorithm 4.4 can be solved by means of the nonlinear equation $\mathcal{L}_\rho'(x, w) = 0$.

To make the discussion more focused and more concrete, we restrict ourselves to two typical examples. We first begin by considering the case where Y is finite-dimensional. This simple base case will allow us to highlight the key ideas and concepts of generalized smoothness. Assume that $Y := H := \mathbb{R}^m$ and $K := \mathcal{K} := \mathbb{R}^m_-$ for some $m \in \mathbb{N}$, i.e., we are dealing with the nonlinear program

$$\underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \le 0. \tag{7.5}$$

In this case, the projection $P_{\mathcal{K}}$ takes on the simple form $P_{\mathcal{K}}(y) = \min\{y, 0\}$, where the minimum is understood componentwise. Moreover, passing to the gradient mappings, (7.4) can be rewritten as

$$\nabla \mathcal{L}_\rho(x, w) = \nabla f(x) + \nabla G(x) \max\{w + \rho G(x), 0\},$$

where ∇ is understood with respect to x. Thus, the question of semismoothness reduces to that of the maximum function on $\mathbb{R}^m$. It turns out that this function is indeed semismooth, and its generalized derivatives can be constructed by means of suitable diagonal matrices.

Lemma 7.5. *The mapping* $m : \mathbb{R}^m \to \mathbb{R}^m$, $m(x) := \max\{x, 0\}$, *is semismooth with generalized derivatives given by*

$$V \in \partial m(x) \iff V = \operatorname{diag}(v),\ v_i \begin{cases} = 1, & \text{if } x_i > 0, \\ \in [0,1], & \text{if } x_i = 0, \\ = 0, & \text{if } x_i < 0. \end{cases}$$

The proof of this lemma is elementary and therefore omitted. It follows from the above result and Proposition 7.2 that the gradient of $\mathcal{L}_\rho$ is semismooth, and thus Algorithm 7.3 can be employed for the solution of the subproblems arising in the augmented Lagrangian method.

Let us now consider a more general framework which is prototypical for function space problems. Let Y be a suitable function space densely embedded into $H := L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain. This covers many spaces of practical interest such as $H^1(\Omega)$, $H_0^1(\Omega)$, or $C(\overline{\Omega})$. Assume furthermore that $K = Y_-$ and $\mathcal{K} = H_-$, so that we are dealing with an inequality constraint of the form

$$G(u) \leq 0 \quad \text{a.e. in } \Omega.$$

In this case, the derivative of the augmented Lagrangian (7.4) takes on the form

$$\mathcal{L}_\rho'(u, w) = f'(u) + G'(u)^* \max\{w + \rho G(u), 0\}, \tag{7.6}$$

where we write u instead of x to emphasize the function space setting. Thus, the semismoothness of $\mathcal{L}_\rho'$ once again depends on that of the maximum function.

Assume now that $Y \hookrightarrow L^p(\Omega)$ for some $p > 2$. It is well-known that $\max\{\cdot, 0\} : L^p(\Omega) \to L^2(\Omega)$ is semismooth in this case, see, for instance, [214]. In the present situation, one further has to take into account that, according to the specification of the augmented Lagrangian method, the "shift" vector w is in general an element of $L^2(\Omega)$. Thus, the maximum function in (7.6) cannot be interpreted as a mapping from $L^p(\Omega)$ to $L^2(\Omega)$, but it is easy to see that the semismoothness of the mapping $\max\{\cdot, 0\}$ can be extended to mappings of the form $\max\{\cdot, w\}$, where $w \in L^2(\Omega)$. The generalized derivatives of these mappings can be constructed by considering suitable elements in $L(L^p(\Omega), L^2(\Omega))$ induced by pointwise multiplication with a function in $L^s(\Omega)$, where $s \in [2, \infty)$ is chosen so that $1/2 = 1/s + 1/p$.

Proposition 7.6. *Let* $\Omega \subseteq \mathbb{R}^d$ *be a bounded domain,* $1 \leq q < p \leq +\infty$ *given numbers, and* $w \in L^q(\Omega)$. *Then the mapping* $m(u) := \max\{u, w\}$ *is semismooth from* $L^p(\Omega)$ *into* $L^q(\Omega)$ *with generalized derivatives almost everywhere given by*

$$z \in \partial m(u) \iff z \in L^s(\Omega),\ z(x) \begin{cases} = 1, & \text{if } u(x) > w(x), \\ \in [0,1], & \text{if } u(x) = w(x), \\ = 0, & \text{if } u(x) < w(x), \end{cases} \tag{7.7}$$

where $s \in [1, \infty)$ *is chosen so that* $1/q = 1/s + 1/p$.

Proof. The proof is an adaptation of [119, Example 8.12]. We need to show that (7.7) is a generalized derivative of m in the sense of Definition 7.1. Let $u \in L^p(\Omega)$ be an arbitrary point, $\{s_k\} \subseteq L^p(\Omega)$ a (strong) null sequence, $u_k := u + s_k$, and $d_k \in \partial m(u_k)$ for all k. Then there is a subsequence $\{s_k\}_{k\in I}$ of $\{s_k\}$ such that $s_k(t) \to_I 0$ for almost every $t \in \Omega$. Writing $v := m(u)$, $v_k := m(u_k)$, and using the semismoothness of $\max\{\cdot, w(t)\}$ in $u(t)$ for all $t \in \Omega$, we obtain

$$s_k(t)^{-1}\big|v_k(t) - v(t) - d_k(t)s_k(t)\big| \to_I 0 \quad \text{for a.e. } t \in \Omega,$$

where the quotient on the left is understood to be zero whenever $s_k(t) = 0$. Observe now that $|v_k(t) - v(t) - d_k(t)s_k(t)| \le 2|s_k(t)|$ for almost all t by the nonexpansiveness of the max function. Thus, we can apply Lebesgue's dominated convergence theorem and deduce that $s_k^{-1}(v_k - v - d_k s_k) \to_I 0$ in $L^r(\Omega)$ for all $r \in [1,\infty)$. We can then apply Hölder's inequality to conclude that

$$\frac{\|v_k - v - d_k s_k\|_{L^q(\Omega)}}{\|s_k\|_{L^p(\Omega)}} \le \|s_k^{-1}(v_k - v - d_k s_k)\|_{L^s(\Omega)} \to_I 0. \tag{7.8}$$

Since the above argument can be repeated for any subsequence of $\{s_k\}$, it follows that the limit in (7.8) holds with I replaced by $\mathbb{N}$, and the proof is complete. □

We now state a practical corollary of the above result in terms of the function $u \mapsto \|(u+w)_+\|_{L^2(\Omega)}^2$, where $w \in L^2(\Omega)$. This mapping arises naturally as part of the augmented Lagrangian approach, and Proposition 7.6 yields a generalized second-order derivative of this function. Moreover, it is interesting to note that we can choose the generalized derivatives to be *nonnegative*, where a bilinear mapping $a : X^2 \to \mathbb{R}$ is called nonnegative if $a(x,x) \ge 0$ for all $x \in X$. If a is given through an operator $M : X \to X^*$, then this can equivalently be stated as $\langle Mx, x\rangle \ge 0$ for all $x \in X$.

Corollary 7.7. *Let $p > 2$ and $w \in L^2(\Omega)$. Then the function $P : L^p(\Omega) \to \mathbb{R}$, $P(u) := \|(u+w)_+\|_{L^2(\Omega)}^2$, is continuously differentiable on $L^p(\Omega)$. Its derivative P' is semismooth, and $\partial P'$ can be chosen such that all elements $M \in \partial P'(u)$, $u \in L^p(\Omega)$, are nonnegative.*

Proof. By Lemma 2.43, the function P is even continuously differentiable on $L^2(\Omega)$. Its derivative is given by $P'(u) = 2(u+w)_+$ for all $u \in L^p(\Omega)$. By Proposition 7.6, this is a semismooth function with generalized derivatives almost everywhere given by

$$z \in \partial P'(u) \iff z \in L^s(\Omega),\ z(x) \begin{cases} = 1, & \text{if } u(x) + w(x) > 0, \\ \in [0,1], & \text{if } u(x) + w(x) = 0, \\ = 0, & \text{if } u(x) + w(x) < 0, \end{cases}$$

where s is chosen so that $1/p + 1/s = 1/2$. It remains to show that all elements $z \in \partial P'(u)$ are nonnegative operators, in the sense that $\langle v, zv\rangle \ge 0$ for all $v \in L^p(\Omega)$. But this is clear from the definition of z. □

The above results imply that the semismooth Newton method can be used to compute stationary points of the augmented Lagrange function, provided that $Y \hookrightarrow L^p(\Omega)$ for some $p > 2$.

Let us close this section by noting that, while the semismooth Newton method provides a powerful framework for the solution of nonlinear equations in Banach spaces, it has a definite drawback in that it is a local method only. For Theorem 7.4, we need the starting point x^0 to be sufficiently close to the solution $\bar{x}$, and the method may fail to converge if x^0 is far from $\bar{x}$ unless extremely strong assumptions are made. Indeed, a rather pathological example in this direction is the function

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) := \begin{cases} x - 2, & \text{if } x < -1, \\ 3x, & \text{if } x \in [-1, 1], \\ x + 2, & \text{if } x > 1. \end{cases}$$

This function is strongly monotone and semismooth but, for any starting point $x^0 \notin [-1, 1]$, the semismooth Newton method eventually oscillates between the points -2 and 2. Note that the example is nonsmooth, but it is easy to construct a smooth example where Newton's method exhibits the same pathological behavior.

7.1.2 A Levenberg–Marquardt Algorithm

We now present a more sophisticated algorithm, one of Levenberg–Marquardt type, which is based on the semismooth Newton method but aims to overcome some of its drawbacks. In particular, the algorithm has significantly better global convergence characteristics, and it can achieve fast local convergence even for nonisolated solutions. The analysis of Levenberg–Marquardt methods has been an active research subject in the past decades, see [63, 72, 220]. The algorithm and results below will be presented in a finite-dimensional framework, although there is no apparent reason why the method could not be extended to a suitable infinite-dimensional setting. A vaguely related algorithm using trust-region techniques can be found in [214].

Throughout this section, we consider the nonlinear equation

$$T(x) = 0, \tag{7.9}$$

where $T : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz-continuous semismooth mapping. Here and throughout, since the involved spaces are finite-dimensional, semismoothness is understood in terms of the Clarke subdifferential ∂_{Cl}.

Algorithm 7.8 (Semismooth Levenberg–Marquardt algorithm)**.** Let $x^0 \in \mathbb{R}^n$ and $k := 0$.

Step 1. If x^k is a sufficiently accurate zero of T: STOP.

Step 2. Choose $M_k \in \partial T(x^k)$, $\alpha_k \geq 0$, and compute d^k as a solution of the equation

$$(M_k^\top M_k + \alpha_k I_n) d^k = -T(x^k). \tag{7.10}$$

Step 3. Set $x^{k+1} := x^k + d^k$, set $k \leftarrow k + 1$, and go to Step 1.

The equation (7.10) is also called *damped Newton equation.* In practice, it is seldom solved by forming the product $M_k^\top M_k$. Instead, one solves the equivalent linear least squares problem

$$\underset{d\in\mathbb{R}^n}{\text{minimize}}\ \|T(x^k)+M_kd\|^2+\alpha_k\|d\|^2. \tag{7.11}$$

Observe that this is just the least squares problem corresponding to the overdetermined linear equation

$$\begin{pmatrix} M_k \\ \sqrt{\alpha_k}I_n \end{pmatrix} d \overset{!}{=} \begin{pmatrix} -T(x^k) \\ 0 \end{pmatrix}. \tag{7.12}$$

This equation reduces to the standard Newton equation (7.3) if $m = n$ and $\alpha_k = 0$. On the other hand, if $\alpha_k > 0$, then the matrix in (7.11) has full column rank and the corresponding least squares problem admits a unique solution. This solution can be determined efficiently by computing, for instance, a QR decomposition of the matrix (with column pivoting).

In practice, the damping factor α_k is often used to provide some kind of globalization. Indeed, if $\alpha_k > 0$ is large, then the minimization problem (7.11) will eventually force d^k to become small. Assuming for the moment that T is continuously differentiable, it follows that d^k will eventually lie in a region where the first-order approximation

$$T(x^k+d)\approx T(x^k)+T'(x^k)d=T(x^k)+M_kd$$

is sufficiently accurate. Hence, given the nature of the minimization problem (7.11), it is reasonable to expect that $\|T(x^k+d^k)\|<\|T(x^k)\|$. This observation can be used to construct a heuristic globalization scheme which is often sufficient in practice: given α_k, we compute d^k as a solution of (7.10). If $\|T(x^k+d^k)\|<\|T(x^k)\|$, then we proceed as in Algorithm 7.8 and choose the next regularization parameter as $\alpha_{k+1} := c_1\alpha_k$ with $c_1 \in (0,1)$. Otherwise, we repeatedly multiply α_k by some factor $c_2 > 1$ and recompute d^k until the descent condition $\|T(x^k+d^k)\|<\|T(x^k)\|$ is satisfied.

We now discuss the local convergence characteristics of Algorithm 7.8. To this end, let $\mathcal{S} := T^{-1}(0)$ denote the solution set of (7.9), and let $\bar{x} \in \mathcal{S}$ be a fixed point. Note that $\mathcal{S}$ is not assumed to consist of isolated points. In the nonisolated case, the role of the regularity assumption from Theorem 7.4 is taken on by a so-called *error bound condition*, which postulates the existence of constants $r, c > 0$ such that

$$\operatorname{dist}(x,\mathcal{S})\le c\|T(x)\| \quad \text{for all } x\in B_r(\bar{x}). \tag{7.13}$$

Under this and a few other technical assumptions, it is possible to show that, given x^0 sufficiently close to $\bar{x}$, and choosing $\alpha_k := \|T(x^k)\|^2$ for all k, the Levenberg–Marquardt algorithm converges Q-superlinearly (or even Q-quadratically) to a point in $\mathcal{S}$, see [63]. We will not state this result here explicitly due to the overhead induced by the required technical assumptions. We will however state the following related result from [72] on the differentiable case.

Theorem 7.9. *Let T be continuously differentiable with locally Lipschitz-continuous derivative on $\mathbb{R}^n$, let $\mathcal{S} := T^{-1}(0)$ denote the solution set of* (7.9)*, and let $\bar{x} \in \mathcal{S}$. Assume* *that there exist $r, c > 0$ such that the error bound* (7.13) *holds. Then, if x^0 is sufficiently close to $\bar{x}$ and $\alpha_k = \|T(x^k)\|^\delta$ for all k with $\delta \in [1, 2]$, it follows that $\{x^k\}$ converges Q-quadratically to an element in $\mathcal{S}$.*

Let us close this section by noting that Levenberg–Marquardt type regularization schemes can also be applied in a slightly different manner when dealing with minimization problems. Consider, for the sake of simplicity, an unconstrained problem of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with semismooth derivative (also called an SC^1-function). Given a current iterate $x^k \in \mathbb{R}^n$ and $M_k \in \partial(\nabla f)(x^k)$, a natural regularization technique is to choose $\alpha_k \geq 0$ and solve the modified Newton equation

$$(M_k + \alpha_k I_n) d^k = -\nabla f(x^k),$$

the vague idea being that α_k dampens the impact of negative eigenvalues of M_k on the search direction. Note that d^k becomes approximately parallel to the negative gradient direction $-\nabla f(x^k)$ if α_k is large. The above regularization scheme is closely related to trust-region methods [48].

7.1.3 Some Remarks on Discretization

In the subsequent sections, a significant emphasis will be placed on problems in infinite dimensions (function spaces). In practice, these problems are then solved by using a suitable discretization scheme and solving the resulting finite-dimensional discretized problems. A common procedure is to solve the problems with increasingly fine levels of discretization and to use the resulting observations as an indicator of how well the theoretical background of the numerical algorithm works on the continuous (infinite-dimensional) level.

In what follows, we will mainly employ simple discretization schemes using finite differences. This has the advantage that the resulting discretized problems are fairly simple and closely resemble the original problem in their structure. Moreover, the use of simple discretization techniques allows us to focus on the theoretical properties of the augmented Lagrangian method, their consequences, and their verification in practice, without becoming too distracted by implementation details.

It is also possible to solve many of the problems below by using a finite element type discretization (see, e.g., [36]). We will implicitly use this approach for some selected examples which we solve by using the FEniCS software package [158, 159]. In this case, however, the finite element discretization is not performed explicitly on the problem level but implicitly by the program.

7.2 Constrained Optimization in Banach Spaces

We now turn to an array of practical applications for the augmented Lagrangian methods presented in Chapters 4 to 6. The present section deals with constrained optimization problems in a function-space setting. The basic framework we consider is that of Chapter 4, i.e., an optimization problem of the form

$$(P) \qquad \underset{x\in C}{\text{minimize}}\; f(x) \quad \text{subject to} \quad G(x) \in K, \tag{7.14}$$

where X, Y are real Banach spaces, $f : X \to \mathbb{R}$ and $G : X \to Y$ are sufficiently smooth functions, and $C \subseteq X$ as well as $K \subseteq Y$ are nonempty closed convex sets. Moreover, H is a real Hilbert space, $i : Y \to H$ a dense embedding, and $\mathcal{K} \subseteq H$ a closed convex set with $i^{-1}(\mathcal{K}) = K$.

Since the examples below will be in a function space context, we will often write u instead of x to denote the optimization variable, whereas x will occasionally denote the underlying function parameter, i.e., x lies in some finite-dimensional domain. This change of notation should be rather clear from the context and hopefully does not cause any confusion. Moreover, for a function space Y, we will denote by Y_- (Y_+) the nonpositive (nonnegative) cone in Y.

In the subsequent discussion, a significant emphasis will be placed on the applicability of the theoretical convergence results in practice. These include, in particular, the global convergence properties from Sections 4.2.2 and 4.2.3, and the local results from Sections 4.3.1 and 4.3.2. As discussed in Section 7.1.3, the practical solution of the problems requires a suitable discretization approach, and the algorithm is therefore formally applied at the discrete level. The apparent change in behavior of the algorithm with increasingly fine levels of discretization is then taken as an indicator of the validity of the theory on the infinite-dimensional level.

In practice, due to the discretization, the augmented Lagrangian method (Algorithm 4.4) is applied to a finite-dimensional approximation of the corresponding problem. The implementation was done in MATLAB and uses the algorithmic parameters

$$\lambda^0 := 0, \quad \rho_0 := 1, \quad B := [-10^6, 10^6] \quad \gamma := 10, \quad \tau := 0.1$$

(where B is understood in a suitable function space, usually an L^2-space), together with a problem-dependent starting point u^0. The sequence $\{w^k\}$ is chosen as $w^k := P_B(\lambda^k)$, i.e., it is a safeguarded analogue of the multiplier sequence.

Recall that our algorithmic framework contains the quadratic penalty or Moreau–Yosida regularization technique as a special case (for $w^k \equiv 0$). Since this method is rather popular for function space problems (see the discussion at the beginning of Chapter 4), the present section also contains a numerical comparison of that method to the augmented Lagrangian scheme. In order to make the comparison fair, two modifications were incorporated into the methods. For the Moreau–Yosida scheme, it does not make sense to update the penalty parameter conditionally, and it is therefore increased in every iteration. On the other hand, for the augmented Lagrangian method, the penalty updating scheme was modified slightly in order to be well-defined for $k = 0$. This is achieved by formally setting $w^{-1} := w^0$ and $\rho_{-1} := \rho_0$, see the discussion in Section 4.1.3.

7.2.1 The Obstacle Problem

The obstacle problem is one of the basic and most prominent examples of optimization problems in Banach spaces, with many applications in the engineering sciences and in mathematical physics [192].

For the formal description of the problem, let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, and consider the minimization problem

$$\underset{u \in H_0^1(\Omega)}{\text{minimize}}\ J(u) \quad \text{subject to} \quad u \geq \psi, \tag{7.15}$$

where $J(u) := \|\nabla u\|^2_{L^2(\Omega)}$ and $\psi \in H_0^1(\Omega)$ is a fixed obstacle. In the context of our general framework (P), we have

$$X := Y := H_0^1(\Omega), \quad C := X, \quad G(u) := u - \psi, \quad K := Y_+,$$
$$H := L^2(\Omega), \quad \mathcal{K} := H_+.$$

Note that the objective function J is strongly convex by the Poincaré inequality (Theorem 2.29), and that the feasible set Φ is closed and convex. It follows from standard arguments that the obstacle problem admits a unique solution $\bar{u} \in X$. Moreover, the derivative $G'(\bar{u}) = \mathrm{Id}_X$ is surjective from X onto Y, and thus there exists a uniquely determined Lagrange multiplier $\bar{\lambda} \in Y^* = H^{-1}(\Omega)$, see Section 3.1.1. The KKT system takes on the form

$$J'(\bar{u}) + \bar{\lambda} = 0 \quad \text{and} \quad 0 \leq \bar{u} - \psi \perp \bar{\lambda} \leq 0,$$

where the negativity of $\bar{\lambda}$ is understood in the sense of $H^{-1}(\Omega)$, i.e., $\langle \bar{\lambda}, u \rangle \leq 0$ for all $u \in H_0^1(\Omega)$ with $u \geq 0$.

We now apply the augmented Lagrangian method to the problem. The subproblems are solved by computing stationary points (which, due to convexity, are also global minimizers). Note that these problems always admit solutions since J is strongly convex and G is linear, which implies that the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is also strongly convex for all $\rho > 0$ and $w \in H$. In fact, by Corollary 7.7, the generalized second order derivatives of the augmented Lagrangian are uniformly positive, and this implies that the semismooth Newton method converges superlinearly by Theorem 7.4.

Due to the strong convexity of J and the surjectivity of $G'(u)$, it follows that the augmented Lagrangian method enjoys powerful global convergence properties. Indeed, the primal sequence $\{u^k\}$ converges strongly to $\bar{u}$ in X by Corollary 4.13 (or Corollary 5.9), and the dual sequence $\{\lambda^k\}$ converges weak-* by Proposition 4.20. Actually, we have from the definition of the augmented Lagrangian and λ^{k+1} that

$$J'(u^{k+1}) + \lambda^{k+1} = \mathcal{L}'_{\rho_k}(u^{k+1}, w^k) \to 0 \quad \text{as } k \to \infty,$$

which implies $\lambda^k \to -J'(\bar{u}) = \bar{\lambda}$ strongly in Y^*.

We now present some numerical results for $\Omega := (0,1)^2$ and the obstacle

$$\psi(x,y) := \max\left\{0.1 - 0.5 \left\| \begin{pmatrix} x - 0.5 \\ y - 0.5 \end{pmatrix} \right\|, 0\right\},$$

see Figure 7.1. For the solution process, we choose $n \in \mathbb{N}$ and discretize Ω by means of a standard grid which consists of n (interior) points per row or column, i.e., n^2 interior points in total. Furthermore, we use

$$J(u) = \|\nabla u\|_{L^2(\Omega)}^2 = -\langle \Delta u, u \rangle_X \quad \text{for all } u \in X$$

and approximate the Laplace operator by a standard five-point finite difference scheme. The subproblems occurring in the algorithm are unconstrained minimization problems which we solve by applying the semismooth Newton method from Section 7.1.1.

	Augmented Lagrangian			Moreau–Yosida		
n	outer	inner	final ρ_k	outer	inner	final ρ_k
16	6	9	10^4	7	11	10^7
32	7	13	10^5	7	15	10^7
64	7	17	10^5	7	18	10^7
128	7	22	10^6	8	22	10^8
256	8	25	10^7	8	27	10^8

Table 7.1: Numerical results for the obstacle problem from Section 7.2.1.

Table 7.1 contains the inner and outer iteration numbers together with the final penalty parameters for different values of the discretization parameter n. Both the augmented Lagrangian and Moreau–Yosida methods scale rather well with increasing dimension; in particular, the outer iteration numbers remain nearly constant. Performance-wise, the two methods perform very similarly, with the augmented Lagrangian method holding a slight advantage in terms of iteration numbers and penalty parameters.

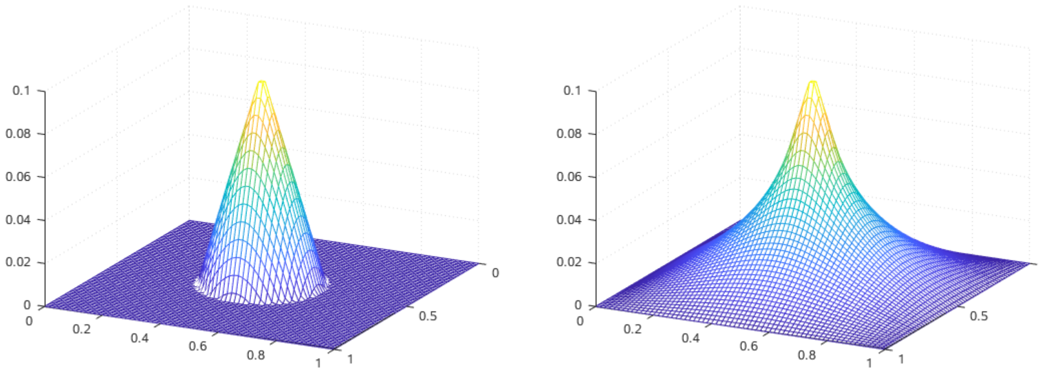
The fact that the outer iteration numbers of the augmented Lagrangian method remain nearly constant with increasing n is a good indicator that the theory behind the algorithm works in this case, and that the method achieves convergence on the continuous (infinite-dimensional) level.

Remark 7.10. The obstacle problem always admits a unique Lagrange multiplier $\bar{\lambda} \in H^{-1}(\Omega)$. Under suitable regularity assumptions on the obstacle ψ, it is possible to show that actually $\bar{\lambda} \in L^2(\Omega)$. In that case, the solution $\bar{u}$ satisfies the second-order sufficient condition with respect to the space $H = L^2(\Omega)$. However, in any case, one cannot expect the problem to be regular (in the constraint qualification sense) with respect to H since the range of the constraint mapping is completely contained in $H_0^1(\Omega)$.

7.2.2 Bratu's Obstacle Problem

This section contains a nonlinear variation of the obstacle problem which can be used to model nonlinear diffusion phenomena occurring, for instance, in combustion and in semiconductors. More details on this problem can be found in [70, 110].

The minimization problem presented here is a non-quadratic and nonconvex problem which differs from that of the previous section in the choice of objective function. To this



(a) Constraint function ψ (b) Solution $\bar{u}$

Figure 7.1: Numerical results for the obstacle problem with $n = 64$.

end, let

$$J(u) := \|\nabla u\|_{L^2(\Omega)}^2 - \alpha \int_\Omega e^{-u(x)} \, \mathrm{d}x \tag{7.16}$$

for some fixed $\alpha > 0$. Similarly to before, we consider the minimization problem

$$\underset{u \in H_0^1(\Omega)}{\text{minimize}} \ J(u) \quad \text{subject to} \quad u \geq \psi \tag{7.17}$$

with a fixed obstacle $\psi \in X$. In the context of our general framework (P), we have

$$X := Y := H_0^1(\Omega), \quad C := X, \quad G(u) := u - \psi, \quad K := Y_+,$$
$$H := L^2(\Omega), \quad \mathcal{K} := H_+.$$

To ensure well-definedness of the objective J, we require that $\Omega \subseteq \mathbb{R}^2$. This allows us to prove the following result.

Lemma 7.11. *If $\Omega \subseteq \mathbb{R}^2$, then the functional J given by* (7.16) *is well-defined, continuously Fréchet-differentiable, and weakly sequentially lsc from $H_0^1(\Omega)$ into $\mathbb{R}$. Moreover, the derivative $J' : H_0^1(\Omega) \to H^{-1}(\Omega)$ is bounded and pseudomonotone.*

Proof. By [136, Lem. 7.1], the function J is continuously differentiable and weakly sequentially lsc on $H_0^1(\Omega)$. The proof in that reference also shows that the integral term J_2 in the definition of J is weakly sequentially continuous, uniformly differentiable on bounded subsets of $H_0^1(\Omega)$, and J_2' is bounded on bounded subsets of $H_0^1(\Omega)$. It follows that J' is also a bounded operator. Moreover, J_2' is completely continuous by Proposition 2.17, and J' is pseudomonotone by Example 3.38. □

Due to the constraint $u \geq \psi$, the functional J is coercive on the feasible set of (7.17). The above lemma therefore yields the existence of a solution $\bar{u} \in X$. As with the standard obstacle problem, the derivative $G'(\bar{u})$ is surjective and therefore we obtain the existence of a unique Lagrange multiplier $\bar{\lambda} \in Y^* = H^{-1}(\Omega)$.

Let us now discuss the convergence properties of the augmented Lagrangian method applied to the Bratu problem. Note that, as opposed to the obstacle problem, the objective function (7.16) is no longer convex. Hence, we can only expect to compute stationary points of the augmented subproblems which are not necessarily local or global minimizers. In this scenario, it follows from Propositions 4.19 and 4.20 that every weak limit point $\hat{u}$ of the primal sequence $\{u^k\}$ is a stationary point of the problem, and the corresponding subsequence of $\{\lambda^k\}$ converges weak-* in $H^{-1}(\Omega)$ to the unique Lagrange multiplier in $\hat{u}$.

To analyze how the method behaves in practice, we again considered $\Omega := (0,1)^2$ and implemented the Bratu problem using the same obstacle and a similar implementation as for the standard obstacle problem. The resulting images are given in Figure 7.2, and some iteration numbers are given in Table 7.2. As with the obstacle problem, we observe

	Augmented Lagrangian			Moreau–Yosida		
n	outer	inner	final ρ_k	outer	inner	final ρ_k
16	6	13	10^4	7	15	10^7
32	7	17	10^5	7	17	10^7
64	7	19	10^5	7	19	10^7
128	8	24	10^6	8	23	10^8
256	8	24	10^6	8	28	10^8

Table 7.2: Numerical results for the Bratu problem from Section 7.2.2.

that both the augmented Lagrangian and Moreau–Yosida regularization methods scale well with increasing dimension, and the augmented Lagrangian method once again holds a certain advantage in terms of iteration numbers and penalty parameters. In fact, the gap between the two methods is slightly bigger than for the standard obstacle problem.

7.2.3 State-Constrained Optimal Control Problems

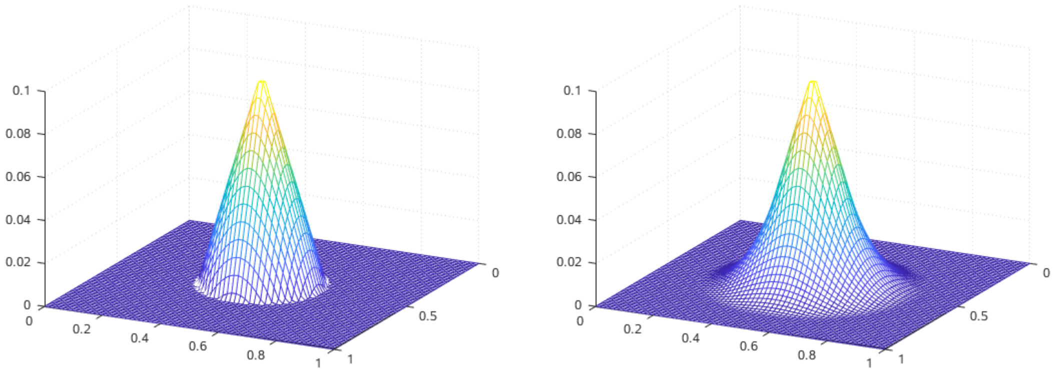
We now turn to a rather prominent class of optimization problems with partial differential equation (PDE) constraints. Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain. We consider the optimal control problem given by the functional

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}, \tag{7.18}$$

where $y \in H^1_0(\Omega) \cap C(\overline{\Omega})$ and $u \in L^2(\Omega)$, together with the PDE and state constraints

$$-\Delta y + d(y) = u \quad \text{in } H^{-1}(\Omega), \quad \text{and} \quad y \geq \psi \quad \text{in } \Omega. \tag{7.19}$$

Here, $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is the Laplace operator, $\alpha > 0$ is a regularization parameter, and $y_d \in L^2(\Omega)$, $\psi \in C(\overline{\Omega})$, $\psi \leq 0$ on $\partial\Omega$, are given functions. The nonlinearity d in the elliptic equation is induced by a function $d : \mathbb{R} \to \mathbb{R}$, which is assumed to be sufficiently regular and monotonically increasing.



(a) Constraint function ψ (b) Solution $\bar{u}$

Figure 7.2: Numerical results for the Bratu problem with $\alpha = 1$ and $n = 64$.

We now present a standard technique in PDE-constrained optimization which consists of eliminating the variable y. By elliptic regularity results, the PDE in (7.19) admits, for any given $u \in L^2(\Omega)$, a unique solution $y = S(u) \in H_0^1(\Omega) \cap C(\overline{\Omega})$. The resulting mapping $S : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ is completely continuous and continuously differentiable, see [211], and thus we can restate (7.16) as the smooth minimization problem

$$\underset{u \in L^2(\Omega)}{\text{minimize}}\ \bar{J}(u) := J(S(u), u) \quad \text{subject to} \quad S(u) \geq \psi. \tag{7.20}$$

This is usually called the *reduced form* of the problem. In the context of our general framework (7.14), we have the data

$$X := L^2(\Omega), \quad C := X, \quad Y := C(\overline{\Omega}), \quad G(u) := S(u) - \psi, \quad K := Y_+,$$
$$H := L^2(\Omega), \quad \mathcal{K} := H_+.$$

We can now apply the augmented Lagrangian method to eliminate the constraint $S(u) \geq \psi$, thus obtaining a sequence of penalized problems. For efficiency reasons, we tackle these problems by reintroducing the state variable y and writing the problems as

$$\begin{aligned} &\underset{y,u}{\text{minimize}} \quad J(y,u) + \frac{\rho_k}{2} \left\| \left(y - \psi + \frac{w^k}{\rho_k} \right)_- \right\|_H^2 - \frac{\|w^k\|_H^2}{2\rho_k} \\ &\text{subject to} \quad -\Delta y + d(y) = u \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{7.21}$$

We now discuss the applicability of the convergence results from Chapter 4. Due to the nonconvexity of the problem, we can only expect to compute stationary points of the augmented subproblems. This makes the theory from Section 4.2.3 a natural candidate for the present situation. To apply the main results from that section, we need to verify the following properties:

	Augmented Lagrangian			Moreau–Yosida		
n	outer	inner	final ρ_k	outer	inner	final ρ_k
16	6	16	10^4	6	19	10^6
32	7	21	10^5	7	22	10^7
64	7	23	10^6	7	25	10^7
128	7	26	10^6	8	30	10^8
256	8	31	10^7	9	37	10^9

Table 7.3: Numerical results for the optimal control problem from Section 7.2.3.

- The mapping $G' : X \to L(X, Y)$ is completely continuous. In the present setting, since $X = L^2(\Omega)$ is reflexive and $G'(u) \in L(X, Y)$ is completely continuous for all u (by Proposition 2.16), this is equivalent to the following property: whenever $u^k \rightharpoonup u$ and $h^k \rightharpoonup h$ in X, then $G'(u^k)h^k \to G'(u)h$ strongly in Y. A proof of this statement (for the Neumann case) can be found in [137, Lem. 4.7].
- The mapping $\bar{J}' : X \to X^*$ is bounded and pseudomonotone. Note that $\bar{J}'(u) = S'(u)^*(S(u) - y_d) + \alpha u$ for all $u \in X$. As seen above, S and S' are completely continuous, hence bounded (since X is reflexive). This implies the boundedness of $\bar{J}'$. The pseudomonotonicity follows from the fact that the first term in $\bar{J}'$ is completely continuous and the second term is monotone (see Lemma 3.37).

Moreover, we need the Robinson constraint qualification (RCQ) to hold at feasible points of (7.20). For this, the following observation is helpful. Since the set K has a nonempty interior, RCQ is equivalent to the linearized Slater condition

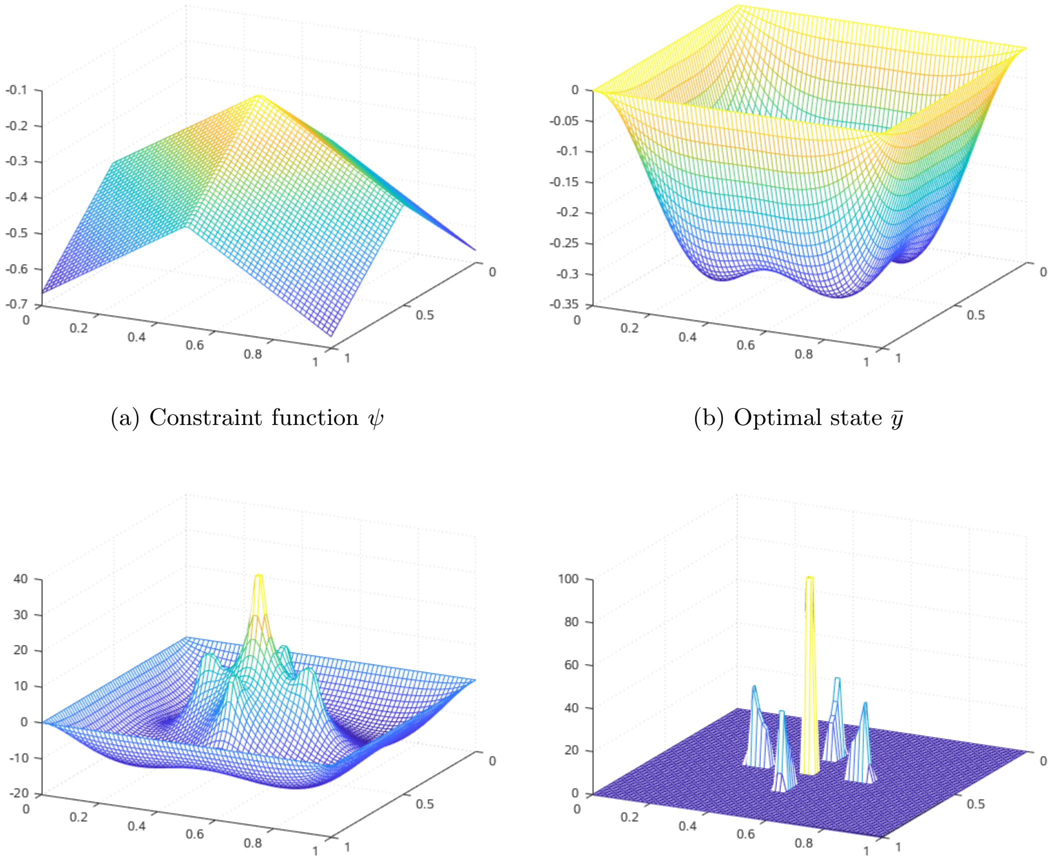
$$\exists \hat{u} \in X : \quad G(u) + G'(u)(\hat{u} - u) \in \operatorname{int}(K),$$

which is a standard assumption in the optimal control context. If the linearized Slater condition holds, then we obtain RCQ (and its extended version from Section 3.1.2). This implies that the main results from Section 4.2.3 are applicable. In particular, we obtain from Theorem 4.16 that every weak limit point $\bar{u}$ of the sequence $\{u^k\}$ is a stationary point of (7.20), the corresponding subsequence of $\{\lambda^k\}$ is bounded in $C(\overline{\Omega})^*$, and its weak-* limit points are Lagrange multipliers in $\bar{u}$.

We now turn to numerical results. The following test problem is similar to the example presented in [170]. Let $\Omega := (0, 1)^2$, $d(y) := y^3$, $\alpha := 10^{-3}$, and

$$\psi(x) := -\frac{2}{3} + \frac{1}{2} \min\{x_1 + x_2, 1 + x_1 - x_2, 1 - x_1 + x_2, 2 - x_1 - x_2\}.$$

Clearly, in this setting, (7.18) and its reformulation (7.20) are nonconvex problems. The augmented subproblems are solved by applying the MATLAB function `fmincon`, where the Hessian of the objective is approximated by a generalized second-order derivative in the sense of Section 7.1.1. Table 7.3 contains the resulting iteration numbers and final penalty parameters for both the augmented Lagrangian and Moreau–Yosida regularization methods. As with the previous examples, both methods scale well with increasing dimension, and the augmented Lagrangian method is more efficient in terms of iteration numbers and penalty parameters.



(a) Constraint function ψ

(b) Optimal state $\bar{y}$

(c) Optimal control $\bar{u}$

(d) Lagrange multiplier $\bar{\lambda}$ (negative)

Figure 7.3: Numerical results for the optimal control problem from Section 7.2.3 ($n = 64$).

The state constraint ψ and the results of our method are given in Figure 7.3. It is interesting to note that the multiplier $\bar{\lambda}$ appears to be much less regular than the optimal control $\bar{u}$ and state $\bar{y}$. This is not surprising because, due to our construction, we have

$$\bar{u} \in L^2(\Omega), \quad \bar{y} \in C(\overline{\Omega}), \quad \text{and} \quad \bar{\lambda} \in C(\overline{\Omega})^*.$$

The latter is well-known to be the space of Radon measures on $\overline{\Omega}$, which is a superset of $L^2(\Omega)$. In fact, the convergence data shows that the (discrete) L^2-norm of $\bar{\lambda}$ grows approximately linearly as n increases, possibly even diverging to $+\infty$, which suggests that the underlying (infinite-dimensional) problem (7.20) does not admit a multiplier in $L^2(\Omega)$ but only in $C(\overline{\Omega})^*$.

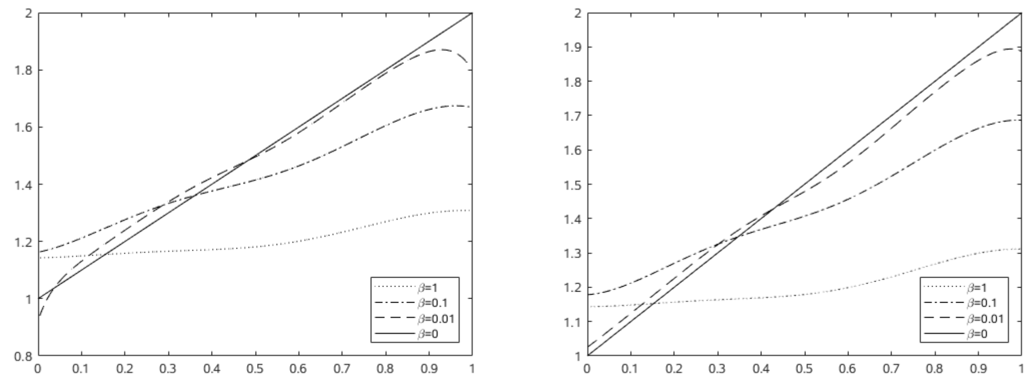


Figure 7.4: Computed solutions q of the parameter estimation problem from Section 7.2.4, with $n = 256$ (left) and $n = 1024$ (right).

7.2.4 Parameter Estimation in Elliptic Systems

The example presented here is a constrained optimization problem, based on [113, 115], which aims at estimating an unknown (functional) parameter in an elliptic differential system. For more details, we refer the reader to [113, 115, 119].

The problem presented here is interesting because it involves function spaces but allows for an application of the augmented Lagrangian method without resorting to embedding techniques (as in the previous examples). This yields linear convergence of the algorithm and boundedness of the sequence of penalty parameters, both of which are confirmed by numerical experiments.

For the sake of simplicity, we restrict ourselves to the one-dimensional case. Let $\Omega \subseteq \mathbb{R}$ be a bounded interval and consider the elliptic differential equation

$$-\nabla(q\nabla u) = f, \quad u \in H_0^1(\Omega), \tag{7.22}$$

where $q \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$. The parameter estimation problem now consists of minimizing the tracking-type functional

$$J(q,u) := \frac{1}{2}\|u - z\|^2_{H_0^1(\Omega)} + \frac{\beta}{2}\|q\|^2_{H^1(\Omega)} \tag{7.23}$$

subject to (7.22) and $q \geq \alpha$, where $z \in H_0^1(\Omega)$ and $\alpha, \beta > 0$. To formulate this problem in our framework, let

$$X := H^1(\Omega) \times H_0^1(\Omega), \quad C := \{(q,u) \in X : q \geq \alpha\}, \quad Y := H := H_0^1(\Omega),$$
$$K := \mathcal{K} := \{0\}, \quad G(q,u) := -\Delta^{-1}\big(\nabla(q\nabla u) + f\big).$$

Note that G is essentially the differential equation (7.22), but premultiplied with $-\Delta^{-1}$ to map the result back into $H_0^1(\Omega)$.

	$n=256,\ \beta=1$		$n=256,\ \beta=0.01$		$n=1024,\ \beta=1$		$n=1024,\ \beta=0.01$	
k	ρ_k	θ_k	ρ_k	θ_k	ρ_k	θ_k	ρ_k	θ_k
0	1	2.54e+04	1	2.54e+04	1	2.05e+05	1	2.05e+05
1	1	4.64e-01	1	1.21e-01	1	4.44e-01	1	6.59e-02
2	10	7.48e-02	10	5.01e-02	10	5.83e-02	10	2.52e-02
3	10	4.35e-03	10	4.71e-03	10	2.99e-03	10	2.07e-03
4	10	2.86e-04	10	4.68e-04	10	1.90e-04	10	1.83e-04
5	10	1.95e-05	10	4.68e-05	10	1.72e-05	10	2.15e-05

Table 7.4: Iteration progress for the parameter estimation problem from Section 7.2.4.

The existence of minimizers of (7.23) can be shown by eliminating u in (7.22) and using the coercivity of J (see [115] for more details). Let $(\bar{q}, \bar{u})$ be a solution of the problem. Observe that we can rewrite the constraints as $\tilde{G}(q,u) := (q, G(q,u)) \in C \times K$ (see Remark 3.10), and the derivative of this constraint takes on the form

$$\tilde{G}'(\bar{q},\bar{u}) = \begin{pmatrix} \mathrm{Id}_{H^1(\Omega)} & 0 \\ T_{\bar{u}} & T_{\bar{q}} \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} T_{\bar{u}}(q) := -\Delta^{-1}(\nabla(q\nabla\bar{u})), \\ T_{\bar{q}}(u) := -\Delta^{-1}(\nabla(\bar{q}\nabla u)). \end{array}$$

Observe now that $T_{\bar{q}} : H_0^1(\Omega) \to H_0^1(\Omega)$ is surjective. This follows from the fact that $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism and that $u \mapsto \nabla(\bar{q}\nabla u)$ is surjective onto $H^{-1}(\Omega)$ by the Lax–Milgram theorem (since $\bar{q} \geq \alpha > 0$). It therefore follows that the whole operator $\tilde{G}'(\bar{q},\bar{u})$ is surjective, which implies the existence and uniqueness of Lagrange multipliers $\bar{\lambda} \in H^{-1}(\Omega)$, corresponding to the constraint $G(q,u) \in K$, and $\bar{\mu} \in H^1(\Omega)^*$, corresponding to the constraint $q \geq \alpha$.

Let us furthermore assume that the second-order sufficient condition holds in $(\bar{q},\bar{u})$. The precise verification of this condition would require the knowledge of the solution, but the second-order condition is very plausible since the objective in (7.23) is strongly convex (by virtue of the H^1-regularization term). Under the present assumptions, the problem admits the local primal-dual error bound from Corollary 3.58. The corresponding residual function $\Theta : X \times H \to \mathbb{R}$ takes on the form

$$\Theta(q,u,\lambda) = \|(q,u) - P_C((q,u) - \mathcal{L}'(q,u,\lambda))\|_X + \|G(q,u)\|_H.$$

It follows that we can expect strong convergence of the primal-dual iterates generated by the augmented Lagrangian method, with linear rate of convergence proportional to the inverse penalty parameter (see Section 4.3.2).

The example we present is [113, Ex. 6]. The domain $\Omega := (0,1)$ is discretized by means of $n \in \mathbb{N}$ points, including boundary points, and the derivative operators are approximated by forward differences. The problem is constructed by setting

$$q_0(x) := 1 + x, \quad z(x) := u_0(x) := \sin(\pi x), \quad f(x) := (1+x)\pi^2 \sin(\pi x) - \pi\cos(\pi x),$$

so that $-\nabla(q_0 \nabla u_0) = f$. Since $z = u_0$, an exact solution of (7.23) for $\beta = 0$ is simply given by (q_0, u_0). For $\beta > 0$, which is the preferable case from a numerical perspective, the solutions are different in general. The initial values for the algorithm are chosen as $(q^0, u^0, \lambda^0) := (1, 0, 0)$, together with $w^k := P_B(\lambda^k)$ and B the closed ball with radius 10^6 around zero in $H_0^1(\Omega)$. The termination criteria for the outer and inner iterations are $\Theta(q, u, \lambda) \leq 10^{-4}$ and $\|\mathcal{L}'_{\rho_k}(q, u, w^k) + (\mu^k, 0)\|_{X^*} \leq 10^{-6}$, respectively, where μ^k is the Lagrange multiplier corresponding to the constraint $q \geq \alpha$. Finally, the augmented subproblems were solved by the MATLAB routine `fmincon` which takes into account the lower bound constraint.

Table 7.4 contains the corresponding iteration numbers for different values of n and β, as well as the optimality measures $\theta_k := \Theta(q^k, u^k, \lambda^k)$. As suggested by the theory, we observe linear convergence of θ_k, with rate proportional to $1/\rho_k$, and the sequences of penalty parameters remain bounded. Finally, Figure 7.4 compares the computed solutions q for different n and β to the exact solution q_0 for $\beta = 0$.

7.3 Generalized Nash Equilibrium Problems in Banach Spaces

This section is dedicated to various examples of Nash and generalized Nash equilibrium problems (NEPs and GNEPs, respectively) in a Banach space setting. The basic framework we consider is that of Section 5.3, i.e., we have $N \in \mathbb{N}$ players, and each player attempts to solve the optimization problem

$$\underset{x^\nu \in C_\nu}{\text{minimize}} \; f_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad G(x^\nu, x^{-\nu}) \in K, \tag{7.24}$$

where $X := X_1 \times \cdots \times X_N$ and Y are real Banach spaces, $f_\nu : X \to \mathbb{R}$ and $G : X \to Y$ are continuously differentiable functions, and $C_\nu \subseteq X_\nu$ and $K \subseteq Y$ are nonempty closed convex sets. We define $C := C_1 \times \cdots \times C_N$ and denote by $\Phi := C \cap G^{-1}(K)$ the feasible set of (7.24). Moreover, there is a real Hilbert space H together with a dense embedding $i : Y \to H$ such that

$$i^{-1}(\mathcal{K}) = K, \quad \text{with } \mathcal{K} \subseteq H \text{ a nonempty closed convex set.}$$

The applications below will all have the property that G is a linear operator and f_ν is convex with respect to x^ν for all ν. Hence, in the terminology of Section 5.3, the resulting GNEPs are *jointly convex*. Some applications of the augmented Lagrangian method to more general problems will be given in Section 7.5, albeit in a finite-dimensional context.

In the present section, we will mainly be concerned with function space related problems. The notation we adopt is similar to that of the previous sections, i.e., we write $u = (u^\nu, u^{-\nu})$ for the optimization variable, whereas x will occasionally denote a point from some finite-dimensional domain Ω. Moreover, for a given function space Y, we denote by Y_- (Y_+) the nonpositive (nonnegative) cone in Y.

Recall that a jointly convex GNEP can be rewritten as the equivalent variational inequality (VI)

$$u \in \Phi, \quad \langle F(u), v - u \rangle \geq 0 \quad \forall v \in \Phi,$$

where Φ is the feasible set of (7.24) and $F(u) := (D_{u^\nu} f_\nu(u))_{\nu=1}^N$, see the discussion in Section 5.3.1. It follows that the GNEP is covered by a wide range of convergence theorems from Chapter 5, in particular those pertaining to VIs (Sections 5.2.2 to 5.2.4) and those specifically tailored to GNEPs (Section 5.3.2).

As in the optimization case, a significant emphasis will be placed on the verification of the theoretical requirements of the convergence theorems in practice. For the solution process, the problems are then discretized by means of finite differences or finite elements. The implementation details are different for each problem and will therefore be explained directly in the corresponding sections.

7.3.1 Multiobjective Optimal Control

We begin with a basic optimal control problem with multiple objectives, based on [35, 61]. Some of the corresponding theory is similar to the single-objective case discussed in Section 7.2.3. The analysis below will also form the basis for the state-constrained multiobjective problems we will consider in Section 7.3.2.

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain, and $N \in \mathbb{N}$ a natural number. The problem we consider here is a standard Nash equilibrium problem (NEP) with N players, where each player ν attempts to minimize the objective function

$$J_\nu(y, u^\nu) := \frac{1}{2}\|y - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\|u^\nu\|_{L^2(\Omega)}^2 \tag{7.25}$$

with respect to $u^\nu \in L^2(\Omega)$, subject to the partial differential equation and pointwise control constraints

$$-\Delta y = \sum_{\nu=1}^N u^\nu + f \quad \text{and} \quad u^\nu \in U_{\text{ad}}^\nu, \tag{7.26}$$

where

$$U_{\text{ad}}^\nu := \{u^\nu \in L^2(\Omega) : u_a^\nu \le u^\nu \le u_b^\nu \text{ a.e. in } \Omega\}$$

and $u_a^\nu, u_b^\nu \in L^2(\Omega)$, $u_a^\nu \le u_b^\nu$. The remaining problem parameters satisfy $\alpha_\nu > 0$ and $y_d^\nu \in L^2(\Omega)$ for all ν. Similarly to Section 7.2.3, we can use the fact that the PDE in (7.26) admits a unique solution for each right-hand side. The resulting solution operator $S : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ is linear and compact. Defining the *control-to-state* mapping

$$y(u) := S\left(\sum_{\nu=1}^N u^\nu + f\right),$$

we can now restate player ν's optimization problem as

$$\underset{u^\nu \in L^2(\Omega)}{\text{minimize}}\ \bar{J}_\nu(u) := J_\nu(y(u), u^\nu) \quad \text{subject to} \quad u^\nu \in U_{\text{ad}}^\nu. \tag{7.27}$$

We now discuss how the augmented Lagrangian method can be applied to this problem. Recall that Nash equilibrium problems can be reformulated as variational inequalities (VIs), see Section 5.3. Hence, we can tackle (7.27) by either treating it as a NEP and applying Algorithm 5.23, or by treating it as a VI and using Algorithm 5.2. These approaches are essentially equivalent, which means that we can choose which convergence results to apply to this scenario. Observe that the VI reformulation of (7.27) takes on the form

$$u \in U_{\mathrm{ad}}, \quad \langle F(u), v-u \rangle \geq 0 \quad \forall v \in U_{\mathrm{ad}},$$

where $U_{\mathrm{ad}} := U^1_{\mathrm{ad}} \times \cdots \times U^N_{\mathrm{ad}}$ and $F(u) := (D_{u^\nu} \bar{J}_\nu(u))_{\nu=1}^N$. An elementary calculation shows that derivative $F'(u)$ can be written as the block operator

$$F'(u) = \begin{pmatrix} S^*S + \alpha_1 \operatorname{Id}_{L^2(\Omega)} & \cdots & S^*S \\ \vdots & \ddots & \vdots \\ S^*S & \cdots & S^*S + \alpha_N \operatorname{Id}_{L^2(\Omega)} \end{pmatrix},$$

where all off-diagonal blocks are equal to S^*S. This implies that F is strongly monotone, in the sense that $\langle F(u) - F(v), u - v \rangle \geq c\|u-v\|_X^2$ for all $u, v \in X$, where the constant c can be chosen as $c := \min\{\alpha_1, \ldots, \alpha_N\}$. We now set

$$X := Y := L^2(\Omega)^N, \quad C := X, \quad G(u) := u, \quad K := U_{\mathrm{ad}},$$
$$H := L^2(\Omega)^N, \quad \mathcal{K} := U_{\mathrm{ad}}.$$

Note that $G'(u)$ is the identity mapping for all u, and therefore surjective. It follows from Corollary 5.9 that $\{u^k\}$ converges strongly to the unique solution of the NEP. Indeed, by Theorem 5.17, the primal-dual sequence $\{(u^k, \lambda^k)\}$ is strongly convergent, with Q-rate proportional to $1/\rho_k$.

We now present a numerical example which is based on [35]. The Nash equilibrium problem has $N = 2$ players and is constructed in such a way that the optimal solution is known analytically. Let $\Omega := (0,1)^2$ be the unit square and define $\alpha_\nu := 1$, $u_a^\nu := -0.5$, and $u_b^\nu := 0.5$ for all ν. Consider the functions

$$\bar{y}(x) := \sin(\pi x_1)\sin(\pi x_2), \qquad \begin{aligned} \bar{p}^1(x) &:= -\sin(2\pi x_1)\sin(2\pi x_2), \\ \bar{p}^2(x) &:= -\sin(3\pi x_1)\sin(3\pi x_2), \end{aligned}$$

as well as $y_d^\nu := \bar{y} + \Delta \bar{p}^\nu$ and $\bar{u}^\nu := P_{[u_a^\nu, u_b^\nu]}(-\bar{p}^\nu/\alpha_\nu)$ for all ν, and finally $f := -\Delta\bar{y} - \bar{u}^1 - \bar{u}^2$. Then it is easy to see that $\bar{u}$ is a Nash equilibrium. The corresponding state is given by $\bar{y}$, the variables $\bar{p}^\nu$ are the so-called adjoint states of the players, and the Lagrange multiplier is given by $\bar{\lambda} := (-\bar{p}^1 - \alpha_1 \bar{u}^1, -\bar{p}^2 - \alpha_2 \bar{u}^2)$.

The implementation of the augmented Lagrangian method for the above problem is done in MATLAB and uses the algorithmic parameters

$$\lambda^0 := 0, \quad \rho_0 := 1, \quad B := [-10^6, 10^6]^N \subseteq L^2(\Omega)^N, \quad \gamma := 10, \quad \tau := 0.5.$$

The augmented subproblems are solved by applying the semismooth Newton algorithm from Section 7.1.1. (Strictly speaking, the derivative of $\mathcal{L}_\rho$ is only semismooth on $L^p(\Omega)$ for $p > 2$ in this case, but the Newton method turns out to work rather well nonetheless.) The corresponding numerical results are given in Table 7.5, where each line contains the

	$n = 64$			$n = 256$			$n = 1024$		
k	ρ_k	θ_k	dist_k	ρ_k	θ_k	dist_k	ρ_k	θ_k	dist_k
0	1	5.08e-01	5.43e-01	1	5.02e-01	5.37e-01	1	5.01e-01	5.35e-01
1	1	8.59e-02	1.71e-01	1	8.47e-02	1.69e-01	1	8.44e-02	1.69e-01
2	1	4.30e-02	8.54e-02	1	4.23e-02	8.46e-02	1	4.22e-02	8.44e-02
3	10	2.15e-02	4.24e-02	10	2.12e-02	4.23e-02	10	2.11e-02	4.22e-02
4	10	1.95e-03	3.41e-03	10	1.92e-03	3.81e-03	10	1.92e-03	3.83e-03
5	10	1.78e-04	8.13e-04	10	1.75e-04	3.17e-04	10	1.74e-04	3.47e-04
6	10	1.61e-05	8.95e-04	10	1.59e-05	5.08e-05	10	1.59e-05	2.96e-05
7	10	1.47e-06	9.08e-04	10	1.45e-06	5.63e-05	10	1.44e-06	3.23e-06
8	10	1.33e-07	9.09e-04	10	1.31e-07	5.74e-05	10	1.31e-07	3.50e-06
9	10	1.21e-08	9.09e-04	10	1.20e-08	5.75e-05	10	1.19e-08	3.59e-06
10	10	1.10e-09	9.09e-04	10	1.09e-09	5.75e-05	10	1.08e-09	3.60e-06

Table 7.5: Numerical results for the optimal control Nash equilibrium problem from Section 7.3.1.

values of the penalty parameter ρ_k, the optimality measure θ_k, and the distance dist_k of (u^k, λ^k) to $(\bar{u}, \bar{\lambda})$. We observe good consistency of the results with our established theory; in particular, the rate of convergence is roughly proportional to $1/\rho_k$. Note that the distances dist_k stop decreasing after a certain point because of the inexactness induced by the discretization; in particular, if we discretize the (known) optimal solution pair $(\bar{u}, \bar{\lambda})$, we do not obtain an *exact* solution of the discretized problem. This phenomenon is also evidenced by the fact that the "limit" value of dist_k decreases as n increases.

Remark 7.12. It is extremely important that we define the constraint system with G and K as above. Indeed, if we rewrite the inclusion $u^\nu \in U_{\text{ad}}^\nu$ as the two constraints $u^\nu \geq u_a^\nu$ and $u^\nu \leq u_b^\nu$, then we cannot apply the local convergence analysis from Section 5.2.4 since the strict Robinson condition is violated. Indeed, even the Robinson constraint qualification is violated, see Example 3.23.

7.3.2 Multiobjective Optimal Control with State Constraints

This section is dedicated to a multiobjective optimal control problem which is similar to the framework of the previous section but includes additional state constraints. Problems of this type arise, for instance, in the optimization of aerodynamic designs if multiple (conflicting) objectives are taken into account [182, 209], or in spot-market models [106]. For further reading about this problem class, the reader is referred to [61, 106].

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain, and $N \in \mathbb{N}$ a natural number. As in the previous section, we consider a problem where each player ν attempts to minimize

$$J_\nu(y, u^\nu) := \frac{1}{2}\|y - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\|u^\nu\|_{L^2(\Omega)}^2 \tag{7.28}$$

over all $u^\nu \in L^2(\Omega)$, subject to the partial differential equation and pointwise control and state constraints

$$-\Delta y = \sum_{\nu=1}^{N} \mathcal{X}_{\Omega_\nu} u^\nu + f, \quad u^\nu \in U_{\text{ad}}^\nu, \quad \text{and} \quad y \geq \psi \quad \text{a.e. in } \Omega, \tag{7.29}$$

where $y_d^\nu \in L^2(\Omega)$, $\alpha_\nu > 0$, $f \in L^2(\Omega)$, and $\psi \in C(\overline{\Omega})$. Moreover, $\mathcal{X}_{\Omega_\nu} : \mathbb{R}^d \to \{0,1\}$ denotes the characteristic function of a suitable player-specific domain $\Omega_\nu \subseteq \Omega$. Similarly to before, the admissible control sets U_{ad}^ν are given by

$$U_{\text{ad}}^\nu = \{u^\nu \in L^2(\Omega) : u_a^\nu \leq u^\nu \leq u_b^\nu \text{ a.e. in } \Omega\}$$

with $u_a^\nu, u_b^\nu \in L^2(\Omega)$ and $u_a^\nu \leq u_b^\nu$. The set $U_{\text{ad}} := U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N$ is the set of admissible controls for all players. We once again use the fact that the PDE in (7.29) admits a unique solution y for any right-hand side in $L^2(\Omega)$. The resulting solution operator $S : L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ is linear and compact. Thus, we can define the control-to-state mapping via

$$y(u) := S\left(\sum_{\nu=1}^{N} \mathcal{X}_{\Omega_\nu} u^\nu + f\right).$$

This allows us to state the *reduced form* of player ν's optimization problem as

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad \bar{J}_\nu(u) := \frac{1}{2}\|y(u) - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\|u^\nu\|_{L^2(\Omega)}^2 \\ &\text{subject to} \quad u^\nu \in U_{\text{ad}}^\nu, \quad y(u) \geq \psi \text{ a.e. in } \Omega. \end{aligned} \tag{7.30}$$

In the notation of our abstract setting (7.24), we have

$$X := L^2(\Omega)^N, \quad C := U_{\text{ad}}, \quad G(u) := y(u) - \psi, \quad Y := C(\overline{\Omega}), \quad K := Y_+,$$
$$H := L^2(\Omega), \quad \mathcal{K} := H_+.$$

The Nikaido–Isoda function of the GNEP is given by

$$\Psi(u,v) = \sum_{\nu=1}^{N} \Big[\bar{J}_\nu(u^\nu, u^{-\nu}) - \bar{J}_\nu(v^\nu, u^{-\nu})\Big], \tag{7.31}$$

where $v = (v^\nu, v^{-\nu})$ and $u, v \in L^2(\Omega)^N$. Observe that, for all ν, the objective function $\bar{J}_\nu$ is weakly sequentially lsc with respect to u^ν, and weakly sequentially *continuous* with respect to $u^{-\nu}$ (since $u \mapsto y(u)$ is completely continuous). Thus, it is easy to see that the Nikaido–Isoda function Ψ is weakly sequentially lsc with respect to u. It follows that we can apply the theoretical framework of Section 5.3.2. In particular, due to the weak compactness of the set U_{ad}, the GNEP admits a normalized equilibrium, and every weak limit point of the sequence $\{u^k\}$ generated by the augmented Lagrangian method is such a normalized solution.

In fact, due to the analytical structure of the problem, it is possible to obtain much sharper convergence results. To this end, we once again use the fact that the GNEP can be rewritten as a variational inequality. Using the arguments in the previous section, the resulting mapping $F(u) := (D_{u^\nu} \bar{J}_\nu(u))_{\nu=1}^N$ is strongly monotone on X. Since the feasible set Φ is a convex set, we obtain that the GNEP admits a *unique* normalized equilibrium $\bar{u}$, and the sequence $\{u^k\}$ generated by the algorithm converges strongly to $\bar{u}$.

Remark 7.13. In practical applications, one often needs to restrict the observation of the state y. In this scenario, the first part of the cost functional could take on the form $J_\nu^1(u) := \frac{1}{2}\|T_\nu y(u) - y_d^\nu\|_{W_\nu}^2$ with real Hilbert spaces W_ν, continuously differentiable (possibly nonlinear) operators $T_\nu : H_0^1(\Omega) \cap C(\overline{\Omega}) \to W_\nu$, and desired states $y_d^\nu \in W_\nu$. In this setting, the Nikaido–Isoda function is still weakly sequentially lsc with respect to u, but the mapping F will in general not be strongly monotone.

We now discuss the convergence properties of the dual variables. Let $\bar{u}$ be the unique solution of the problem, and $\bar{y} := y(\bar{u})$ the corresponding optimal state. A standard regularity assumption in the state-constrained context is the existence of a Slater point, i.e., of a point $\hat{u} \in U_{\mathrm{ad}}$ and $\sigma > 0$ such that

$$y(\hat{u}) \geq \psi + \sigma \quad \text{in } \overline{\Omega}. \tag{7.32}$$

This means that $G(\hat{u}) = y(\hat{u}) - \psi$ lies in the interior of $K = C_+(\overline{\Omega})$. Hence, this condition is a special case of the Slater condition discussed in Section 3.1.2, applied in the space $Y = C(\overline{\Omega})$. Moreover, in the present situation, it is equivalent to the Robinson constraint qualification by Proposition 3.21.

If the Slater condition (7.32) is satisfied, then it follows that the problem admits a Lagrange multiplier in $\bar{u}$. The resulting first-order (KKT) system can be stated as

$$\begin{gathered} \left\langle D_{u^\nu} \bar{J}_\nu(\bar{u}) + \mathcal{X}_{\Omega_\nu}^* S^* \bar{\lambda}, u - \bar{u} \right\rangle \geq 0 \quad \text{for all } u \in U_{\mathrm{ad}}, \\ \bar{\lambda} \leq 0, \quad \bar{y} \geq \psi, \quad \text{and} \quad \langle \bar{\lambda}, \bar{y} - \psi \rangle = 0, \end{gathered}$$

where the last three conditions are equivalent to $\bar{\lambda} \in \mathcal{N}_K(\bar{y} - \psi)$. Note that the inequality $\bar{\lambda} \leq 0$ has to be understood in the dual sense, i.e., $\langle \bar{\lambda}, \varphi \rangle_Y \leq 0$ for all $\varphi \in K$. In other words, $\bar{\lambda}$ belongs to the polar cone K° of K.

The second implication of the Slater condition is that the multiplier sequence $\{\lambda^k\}$ generated by the augmented Lagrangian method is bounded in $C(\overline{\Omega})^*$, and each of its weak-* limit points is a Lagrange multiplier in $\bar{u}$. This is a consequence of Theorem 5.12.

Now, let us briefly consider the augmented subproblems which occur in every iteration of the algorithm. Since the set $C = U_{\mathrm{ad}}$ is weakly compact, the existence of solutions to these problems can be argued as for the original problem itself. Introducing adjoint state variables $p^\nu \in L^2(\Omega)$ for every player, we can characterize the subproblems by means of the first-order systems

$$-\Delta y = \sum_{\nu=1}^N \mathcal{X}_{\Omega_\nu} u^\nu + f, \tag{7.33a}$$

$$-\Delta p^\nu = y - y_d^\nu + \big(w^k + \rho_k(y - \psi)\big)_-, \tag{7.33b}$$

$$(\mathcal{X}_{\Omega_\nu} p^\nu + \alpha_\nu u^\nu, v - u^\nu) \geq 0 \quad \forall v \in U_{\mathrm{ad}}^\nu \tag{7.33c}$$

for all ν.

We now discuss a numerical example based on a four-player game from [106], where $f \equiv 1$, $u_a^\nu := -12$, $u_b^\nu := 12$ for all ν, and $\alpha := (2.8859, 4.3374, 2.5921, 3.9481)$. The state constraint ψ is given by

$$\psi(x_1, x_2) := \cos\big(5\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}\big) + 0.1,$$

and the desired states y_d^ν are defined as $y_d^\nu := \xi_\nu - \xi_{5-\nu}$, where

$$\xi_\nu(x_1, x_2) = 10^3 \max\big\{0, 1 - 4\max\{|x_1 - z_\nu^1|, |x_2 - z_\nu^2|\}\big\}$$

and $z^1 := (0.25, 0.75, 0.25, 0.75)$ as well as $z^2 := (0.25, 0.25, 0.75, 0.75)$.

The subproblems arising within the computation are solved exactly by applying an active set method [21, 23] to the corresponding KKT conditions (7.33). The algorithm was stopped as soon as the quantities $\|(\psi - y^k)_+\|_{C(\overline{\Omega})}$ and $|(\lambda^k, y^k - \psi)|$ drop below 10^{-6}. Since the NEPs in the inner iterations are solved exactly, these values represent the residual of the first-order system (the stationarity part is always zero.) The algorithmic parameters are given by

$$\lambda^0 := 0, \quad \rho_0 := 1, \quad B := [-10^6, 10^6] \subseteq L^2(\Omega), \quad \gamma := 10, \quad \tau := 0.1.$$

The table below displays some outer iteration numbers, accumulated inner iteration numbers, and final penalty parameters $\rho_{\max}$ for different levels of discretization.

n	16	32	64	128	256
outer it.	12	12	13	14	14
inner it.	24	28	34	43	50
$\rho_{\max}$	10^7	10^9	10^{10}	10^{12}	10^{12}

It is worth noting that the outer iteration numbers remain nearly constant as n increases, and the penalty parameters increase only moderately. This suggests that the algorithm works quite well for the multiobjective optimal control problem.

To improve the computational efficiency and reduce the number of iterations on the finest mesh, a common technique is to apply a nested grid strategy. The approach here is adapted from [106]; the mesh is refined whenever $\rho \geq 1000n^2$ is satisfied, with $n = 512$ the final sample size. The next table displays the outer and inner iteration numbers with the corresponding maximum of the penalty parameter using this nested grid approach.

n	4	8	16	32	64	128	256	512	Σ
outer it.	6	1	1	1	1	1	1	4	16
inner it.	8	2	4	4	5	5	5	14	47
$\rho_{\max}$	10^5	10^6	10^7	10^8	10^9	10^{10}	10^{11}	10^{15}	

Figures 7.5 to 7.7 depict the numerical solutions of the problem, computed on a triangular mesh with $n = 128$ grid points.

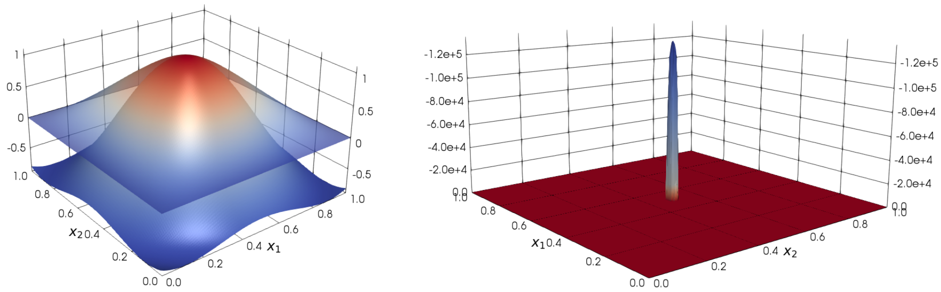


Figure 7.5: Computed solutions for the multiobjective control problem from Section 7.3.2. Left: optimal state y (top) and state constraint ψ, Right: Lagrange multiplier λ.

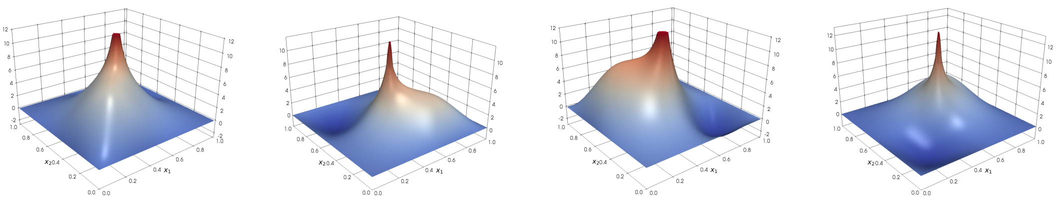


Figure 7.6: Computed solutions for the multiobjective control problem from Section 7.3.2. Control variables $\bar{u} = (\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4)$.

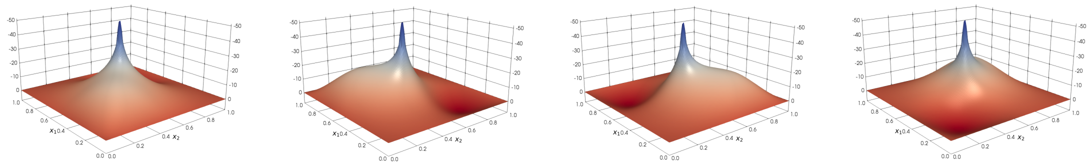


Figure 7.7: Computed solutions for the multiobjective control problem from Section 7.3.2. Adjoint state variables $\bar{p} = (\bar{p}^1, \bar{p}^2, \bar{p}^3, \bar{p}^4)$.

7.3.3 Differential Economic Games

We end with a problem from the class of N-person differential games. Problems of this type are rather popular in the literature [83, 84, 181]. They arise, for instance, in economic simulations if the underlying model is not limited to a fixed point in time but takes into account a whole time interval. In such cases, infinite-dimensional spaces arise naturally, and the problems can therefore be tackled by our algorithmic framework.

The particular example we present here is an environmental management problem based on the framework in [99]. The N players are given by N companies that compete on a common market. Let the strategy space of all players be given by $X := L^2(0,T)^N$. Let $u^\nu(t) \in \mathbb{R}$ denote the investment (control) of company ν at time t and $y^\nu(t) \in \mathbb{R}$ the production capacity. Then u^ν and y^ν are coupled through the differential equation

$$\dot{y}^\nu(t) + b_\nu y^\nu(t) = u^\nu(t), \qquad y^\nu(0) = y_0^\nu, \tag{7.34}$$

where $b_\nu \in \mathbb{R}$. Further, at each time, the investments $u^\nu(t)$ are bounded in the sense that

$$u^\nu \in U_{\text{ad}}^\nu := \{u \in L^2(0,T) : 0 \le u \le u_{\max} \text{ a.e. in } (0,T)\}, \tag{7.35}$$

with $u_{\max} \in L_+^2(0,T)$. We set $U_{\text{ad}} := U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N$. The production y^ν induces a certain environmental pollution. For the sake of pollution control, the companies have to comply with legal requirements. A global constraint of pollution is given by

$$(Ey)(t) \le \psi(t) \quad \text{for all } t \in [0,T], \tag{7.36}$$

where $\psi \in C[0,T]$, $y := (y^1, \dots, y^N)$, and $E : C[0,T]^N \to C[0,T]$ is a bounded linear operator modeling the emission rate in terms of the installed capacity. Each company's production and adjustment costs are given by the function $q_\nu(y^\nu(t), u^\nu(t))$. The market price $r_\nu(y(t))$ of the observed product is associated with the total supply $\sum_{\nu=1}^N y^\nu(t)$. Hence, the revenue of the ν-th company can be modeled via $r_\nu(y(t)) \cdot y^\nu(t)$. Since each company aims for maximal profit, the resulting problem is a GNEP where each player attempts to minimize the objective function

$$J_\nu(y, u^\nu) := \int_0^T q_\nu(y^\nu(t), u^\nu(t)) - r_\nu(y(t)) y^\nu(t) \, \mathrm{d}t$$

subject to the constraints imposed by Equations (7.34) to (7.36). Here and below, we tacitly assume that q_ν and r_ν are sufficiently regular so that J_ν is well-defined on the space $C[0,T]^N \times L^2(0,T)$.

By a well-known theorem of Carathéodory, the differential equation (7.34) admits, for each $u^\nu \in L^2(0,T)$, a unique solution which can be written explicitly as

$$y^\nu(t) = e^t y_0^\nu + \int_0^t e^{t-s} u^\nu(s) \, \mathrm{d}s, \tag{7.37}$$

see [93, p. 30] and [206, p. 488]. Using this expression and the fact that $H^1(0,T)$ is compactly embedded in $C[0,T]$, one can easily infer that $y^\nu \in H^1(0,T)$ and that the control-to-state mappings $y^\nu(\cdot) : L^2(0,T) \to C[0,T]$ are completely continuous (compact). Similarly to the previous examples, we can now pass to the reduced form by inserting $y := y(u) = (y^\nu(u^\nu))_{\nu=1}^N$ into the objective functions J_ν. This results in the GNEP where player ν attempts to solve

$$\begin{aligned} &\underset{u^\nu \in L^2(0,T)}{\text{minimize}} \quad \bar{J}_\nu(u) := J_\nu(y(u), u^\nu) \\ &\text{subject to} \quad u^\nu \in U_{\text{ad}}^\nu, \quad E(y(u)) \le \psi \text{ in } [0,T]. \end{aligned} \tag{7.38}$$

Let us now assume that q_ν and r_ν are sufficiently well behaved so that, for all ν, the functional $\bar{J}_\nu$ is both convex and continuously differentiable with respect to u^ν. Arguing as in Section 7.3.2, it is easy to see that the Nikaido–Isoda function of the reduced GNEP is weakly sequentially lsc with respect to the first variable. By Proposition 5.20, it follows that the GNEP admits a normalized equilibrium $\bar{u} \in X$. Despite this, the problem is more complex than those in the previous sections since the operator $F := (D_{u^\nu}\bar{J}_\nu)_{\nu=1}^N$ may fail to be strongly monotone (depending on the functions q_ν and r_ν).

We now discuss how the augmented Lagrangian method (Algorithm 5.23) can be applied to problem (7.38). In the notation of our general framework (7.24), we have

$$X := L^2(0,T)^N, \quad C := U_{\text{ad}}, \quad G(u) := E(y(u)) - \psi, \quad Y := C[0,T], \quad K := Y_-,$$
$$H := L^2(0,T), \quad \mathcal{K} := H_-.$$

It follows from the theory in Section 5.3.2 that, provided the feasible set of (7.38) is nonempty, then every weak limit point of the sequence $\{u^k\}$ is a normalized equilibrium of the problem. For the convergence of the dual iterates, we again assume the existence of a Slater point, i.e., of a point $\hat{u} \in U_{\text{ad}}$ and $\sigma > 0$ such that

$$E(y(\hat{u}))(t) \le \psi(t) - \sigma \quad \text{for all } t \in [0,T].$$

This is equivalent to $G(\hat{u})$ being an interior point of K. Since the interior of K is nonempty, the above assumption is equivalent to the Robinson constraint qualification (see Proposition 3.21). Hence, the Slater condition implies the existence of a Lagrange multiplier $\bar{\lambda} \in C[0,T]^*$ corresponding to the pollution constraint (7.36).

Numerical Results

The following is a practical example which models the development of a market involving two companies, with one holding a monopoly in the beginning. We set $N := 2$, $T := 3$, and use the functions

$$q_\nu(y^\nu(t), u^\nu(t)) := \frac{a_1}{2} y^\nu(t)^2 + \frac{a_2}{2} u^\nu(t)^2, \qquad r_\nu(y(t)) := \frac{c}{\varepsilon + \sum_{\nu=1}^N y^\nu(t)},$$
$$(Ey)(t) := e_1 y^1(t) + e_2 y^2(t),$$

where $\varepsilon > 0$. The implementation for this example was done in MATLAB. We discretize the appearing time derivative by finite differences and solve the augmented subproblems

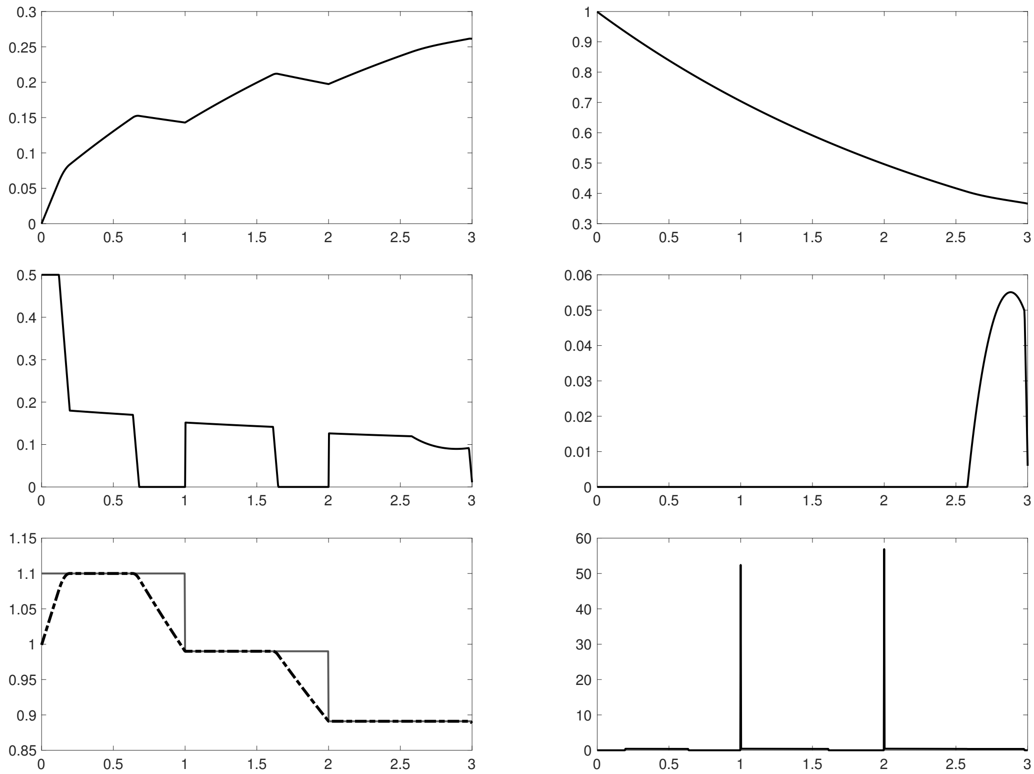


Figure 7.8: (Example 3) From top to bottom, from left to right: Computed states y_h^1, y_h^2, controls u_h^1, u_h^2, state constraint ψ (gray) and $E(y_h)$ (dotted black), multiplier λ_h.

by applying a semismooth Newton method with a desired accuracy of 10^{-6}. The overall algorithm is stopped as soon as the residual of the corresponding KKT system drops below 10^{-6}. The algorithmic parameters are chosen as

$$\lambda^0 := 0, \quad \rho_0 := 1, \quad B := [-100, 100] \subseteq L^2(0,T), \quad \gamma := 10, \quad \tau := 0.1.$$

The parameters of the model are given by $u_{\max} := 0.5$, $a_1 := 0.7$, $a_2 := 0.2$, $b_1 = 0.2$, $b_2 := 0.35$, $c := 1$, $e_1 := 2$, $e_2 := 1$, and $\varepsilon = 10^{-9}$. The initial values of the state are given by $y^1(0) := 0$, $y^2(0) := 1$. The state constraint is defined by

$$\psi(t) = \begin{cases} 1.1, & \text{for } t \in [0,1], \\ 0.99, & \text{for } t \in (1,2], \\ 0.891, & \text{for } t \in (2,3] \end{cases}$$

which is a decrease of ten percent after every third of the time interval. The following table shows some iteration numbers for the given parameters, and Figure 7.8 depicts the computed solutions for $n = 10^3$ grid points.

n	16	32	64	128	256	512	1024
outer it.	8	8	9	9	9	10	10
inner it.	100	146	175	128	108	109	113
$\rho_{\max}$	10^2	10^3	10^4	10^5	10^5	10^5	10^6

7.4 Quasi-Variational Inequalities in Banach Spaces

This section now deals with applications of the augmented Lagrangian technique in the quasi-variational inequality (QVI) context. The basic framework we consider is that of Chapter 6, i.e., the problem of finding $x \in X$ such that

$$(Q) \qquad x \in \Phi(x), \quad \langle F(x), d\rangle \geq 0 \quad \forall d \in \mathcal{T}_{\Phi(x)}(x), \tag{7.39}$$

where X is a real Banach space, $F : X \to X^*$ a suitable operator, and $\Phi : X \rightrightarrows X$ a set-valued mapping of the form

$$\Phi(x) = \{y \in C : G(x, y) \in K\},$$

with Y another real Banach space, $G : X^2 \to Y$ a continuously differentiable mapping, and $C \subseteq X$ and $K \subseteq Y$ nonempty closed convex sets, respectively. As per usual, the space Y is assumed to be densely embedded into a real Hilbert space H via a mapping $i : Y \to H$, and there is a closed convex set $\mathcal{K} \subseteq H$ such that $i^{-1}(\mathcal{K}) = K$.

The above is a fairly general framework (see the discussion at the beginning of Chapter 6), and the resulting applications cover a broad spectrum of special cases. In particular, the above framework can be used to model QVIs or generalized Nash equilibrium problems (GNEPs) in finite dimensions. These two important application cases will be discussed in Sections 7.5 and 7.6. Here, we will mainly focus on applications in the function space context, and demonstrate how these can be solved with the augmented Lagrangian method. Similarly to before, the underlying problems are analyzed on the continuous (infinite-dimensional) level, with a special emphasis on the discussion of the assumptions used in the convergence theory (Sections 6.2.3 and 6.2.4). The problems are then solved numerically using a suitable discretization approach. The implementation was done in MATLAB and uses the algorithmic parameters

$$\lambda^0 := 0, \quad \rho_0 := 1, \quad B := [-10^6, 10^6] \quad \gamma := 10, \quad \tau := 0.1$$

(where B is understood in a suitable function space, usually an L^2-space), together with a problem-dependent starting point u^0. The sequence $\{w^k\}$ is chosen as $w^k := P_B(\lambda^k)$, i.e., it is a safeguarded analogue of the multiplier sequence.

Let us also remark that there is a certain overlap between the QVI setting considered here and the GNEP setting from Section 7.3. In particular, the augmented Lagrangian method for QVIs could also be used to compute generalized Nash equilibria, including *non-normalized* equilibria of, for instance, the multiobjective control problem in Section 7.3.2. To avoid redundancy, this application is not included in the presentation here again. More details can be found in [134].

Since our examples are defined in function spaces, we will typically use the notation (u, v) instead of (x, y) for the variable pairs in the space X^2. This should be rather clear from the context and hopefully does not introduce any confusion. Moreover, similarly to before, Y_+ (Y_-) denotes the nonnegative (nonpositive) cone in some function space Y.

7.4.1 An Implicit Signorini Problem

The application presented here is an implicit version of the celebrated Signorini problem [12, 70, 88]. In comparison to the traditional problem, which can be seen as a variational inequality, the implicit problem involves a compliant (i.e., deformable) obstacle on the boundary of the domain, which makes the problem a quasi-variational inequality. More details on the implicit Signorini problem can be found in [18, 167].

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary Γ, and let X denote the space

$$X := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}, \quad \text{with } \|u\|_X := \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}.$$

Recall that the trace operator τ maps $H^1(\Omega)$ into $H^{1/2}(\Gamma)$, that $H^{1/2}(\Gamma)^* =: H^{-1/2}(\Gamma)$, and that the normal derivative $\partial_n : X \to H^{-1/2}(\Gamma)$ is well-defined and continuous. For fixed elements $h_0, \phi \in H^{1/2}(\Gamma)$ with $\phi \geq 0$, consider the set-valued mapping

$$\Phi(u) := \{v \in X : \tau v \geq h(u) \text{ on } \Gamma\}, \quad h(u) := h_0 - \langle \phi, \partial_n u \rangle,$$

where $h : H^1(\Omega) \to H^{1/2}(\Gamma)$ and the duality pairing is understood between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. The problem in question now is the QVI

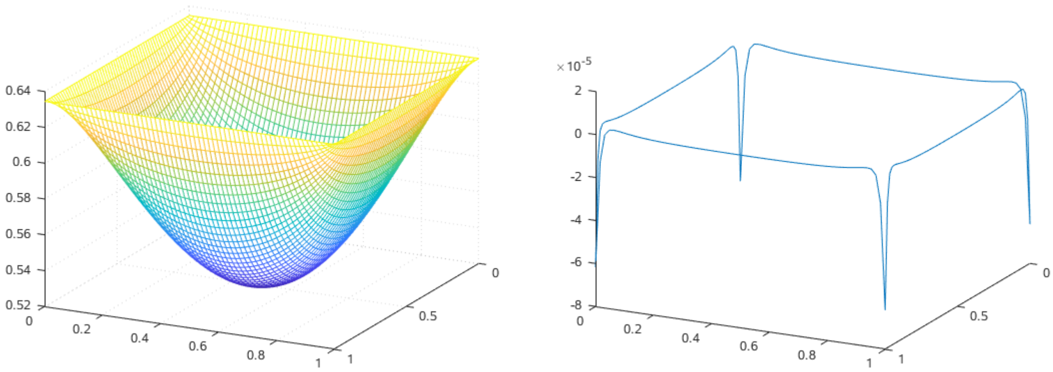
$$u \in \Phi(u), \quad \langle Au - f, v - u \rangle \geq 0 \quad \forall v \in \Phi(u),$$

where $A : X \to X^*$ is a monotone differential operator and $f \in H^{-1}(\Omega)$. This problem can be cast into our general framework by choosing

$$X = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}, \quad C := X, \quad F(u) := Au - f,$$
$$Y := H^{1/2}(\Gamma), \quad G(u, v) := \tau v - h(u), \quad K := Y_+, \quad H := L^2(\Gamma), \quad \mathcal{K} := H_+.$$

Observe now that the Signorini problem satisfies the assumptions of Section 6.2.3. The mapping F is bounded (since it is affine) and pseudomonotone (since it is monotone and continuous, see Lemma 3.37). Note also that G is linear (hence $\mathcal{K}$-concave) with respect to v. Moreover, G is weakly sequentially continuous, which implies that $d_{\mathcal{K}} \circ G$ is weakly sequentially lsc. The final property which remains to be verified is the weak Mosco-continuity of Φ. But this property is immediate from the fact that $u \mapsto \langle \phi, \partial_n u \rangle$ maps into a one-dimensional subspace of Y.

It follows from the theory in Section 6.2.3 that every weak limit point of the sequence $\{u^k\}$ generated by the augmented Lagrangian method (Algorithm 6.16) is a solution of the QVI. (Note that $\Phi(u)$ is nonempty for all $u \in X$, as was assumed in Lemma 6.20.) For the convergence of the dual sequence, the following observation is helpful: since



(a) Solution $\bar{u}$ (b) Constraint value $G(\bar{u}, \bar{u})$

Figure 7.9: Numerical solution of the implicit Signorini problem from Section 7.4.1 ($n = 64$).

$C = X$, the sequences $\{u^k\}$ and $\{\lambda^k\}$ generated by the algorithm satisfy $\mathcal{L}(u^{k+1}, \lambda^{k+1}) = \mathcal{L}_{\rho_k}(u^{k+1}, w^k) \to 0$ as $k \to \infty$ (see (6.21)). This implies that

$$0 \leftarrow \langle F(u^k), h \rangle + \langle \tau^* \lambda^k, h \rangle = \langle F(u^k), h \rangle + \langle \lambda^k, \tau h \rangle$$

for all $h \in X$. Since the range of the trace operator contains the fractional Sobolev space $H^{3/2}(\Gamma)$, see [1], it follows that a subsequence of $\{\lambda^k\}$ converges weak-* in $H^{3/2}(\Gamma)^*$.

We now present some numerical results for the domain $\Omega := (0,1)^2$ and the differential operator $Au := u - \Delta u$. For the implementation of the method, we set $H := L^2(\Gamma)$, $\mathcal{K} := H_+$, and discretize the domain Ω by means of a uniform grid with $n \in \mathbb{N}$ points per row or column (including boundary points), i.e., n^2 points in total. The remaining problem parameters are given by $f \equiv -1$ and $\phi = h_0 \equiv 1$.

n	16	32	64	128	256
outer it.	9	9	9	10	10
inner it.	42	41	42	47	52
$\rho_{\max}$	10^4	10^4	10^5	10^5	10^5

We observe that the method scales rather well with increasing dimension n. In particular, the outer iteration numbers and final penalty parameters remain nearly constant, and the increase in terms of inner iteration numbers is very moderate.

7.4.2 Parametric Gradient Constraints

The example in this section is a QVI with pointwise constraints on the gradients of the involved function. Such problems arise, for instance, in the magnetization of superconductors, in certain elastoplastic torsion problems, and in electrostatics. More details can be found in [104, 153].

Let $d, p \geq 2$ and $X := W_0^{1,p}(\Omega)$ for some bounded domain $\Omega \subseteq \mathbb{R}^d$. Consider the set-valued mapping

$$\Phi(u) := \{v \in X : \|\nabla v\| \leq \psi(u)\}, \tag{7.40}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$ and $\psi : X \to L^\infty(\Omega)$, as well as the resulting QVI

$$u \in \Phi(u), \quad \langle -\Delta_p u - f, v - u \rangle \geq 0 \quad \forall v \in \Phi(u),$$

where $f \in X^*$ and $\Delta_p : X \to X^*$ is the p-Laplacian defined by

$$\langle \Delta_p u, v \rangle := -\int_\Omega \|\nabla u(x)\|^{p-2} \nabla u(x)^\top \nabla v(x) \, \mathrm{d}x.$$

This problem can be cast into our general framework (Q) by defining

$$X = W_0^{1,p}(\Omega), \quad C := X, \quad F(u) := -\Delta_p u - f, \quad Y := L^2(\Omega),$$
$$G(u, v) := \|\nabla v\| - \psi(u), \quad K := Y_-.$$

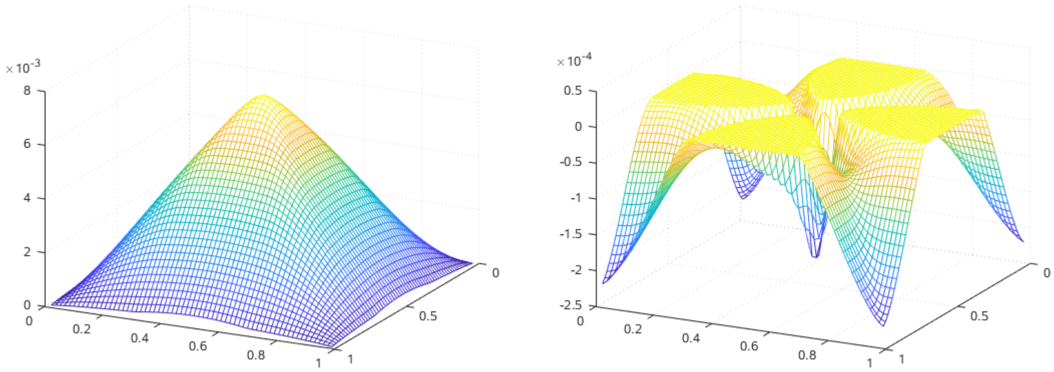
Observe that F is monotone, bounded, and continuous (see Section 2.1.4 and [104]), hence pseudomonotone by Lemma 3.37. Assume now that ψ is completely continuous and satisfies $\psi(u) \geq c_1$ for all u and some $c_1 > 0$. Then Φ can be shown to be weakly Mosco-continuous, see [153, Lem. 1].

When applying the augmented Lagrangian method to the above problem, a significant challenge lies in the analytical formulation of the feasible set. Observe that the original formulation in (7.40) is nonsmooth. This issue is probably not critical since the nonsmoothness is a rather "mild" one, but nevertheless an alternative formulation is necessary to formally apply our algorithmic framework. To this end, we first discretize the problem and then reformulate the (finite-dimensional) gradient constraint as $G(u, v) \leq 0$ with $G(u, v) = \|\nabla v\|^2 - \psi(u)^2$.

For the discretized problems, all the continuity assumptions on F and G from Sections 6.2.3 and 6.2.4 are satisfied trivially. Moreover, the feasible sets $\Phi(u)$ are nonempty for all $u \in X$, and QVI-ERCQ (or QVI-EMFCQ) holds everywhere since the point zero is a Slater point of the mapping $v \mapsto G(u, v)$ for all $u \in X$ (see the discussions in Sections 3.1.2 and 3.1.4). It follows from Theorem 6.26 (or Theorem 6.32) that every limit point $\bar{u}$ of $\{u^k\}$ is a solution of the QVI, the corresponding dual subsequence is bounded, and its limit points are (discretized) Lagrange multipliers in $\bar{u}$.

As a numerical application, we consider a slightly modified version of [104, Example 3] on the unit square $\Omega := (0, 1)^2$. Note that the original example in that reference is actually solved by the solution of the p-Laplace equation $-\Delta_p u - f = 0$, $u \in W_0^{1,p}(\Omega)$. Hence, we have modified the example to use the same function f as in [104] but with the constraint function replaced by $\Psi(u) := 0.01 + 2|\int_\Omega u(x) \, \mathrm{d}x|$. We discretize the problem with $n \in \mathbb{N}$ interior points per row or column, and use backward differences to approximate the gradient and p-Laplace operators.

n	16	32	64	128	256
outer it.	8	7	7	7	7
inner it.	63	52	63	73	103
$\rho_{\max}$	10^4	10^4	10^4	10^4	10^5



(a) Solution $\bar{u}$ (b) Constraint value $G(\bar{u}, \bar{u})$

Figure 7.10: Numerical results for the gradient-constrained QVI with $n = 64$.

As before, the method scales rather well with increasing dimension n, and the outer iteration numbers and final penalty parameters remain nearly constant.

7.5 Generalized Nash Equilibrium Problems in $\mathbb{R}^n$

The present section is the penultimate of this chapter and contains applications of the augmented Lagrangian method for *finite-dimensional* generalized Nash equilibrium problems (GNEPs). Our main objective is to apply the algorithmic framework to the problem collection used in [66], see also the preprint version of that paper. The problems in this collection are GNEPs consisting of player problems of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\ f_\nu(x) \quad \text{subject to} \quad g^\nu(x) \le 0.$$

Here and throughout, $f_\nu : \mathbb{R}^n \to \mathbb{R}$ and $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$ are convex with respect to x^ν and continuously differentiable. The problems are tackled by using the augmented Lagrangian method from Section 6.3.3 or, in the case of joint convexity, Algorithm 5.28 from Section 5.3.3. (For the set of parameters we will choose below, the algorithms coincide in the jointly convex case.) For the sake of simplicity, all the problems are tackled by full penalization (i.e., all constraints are penalized and the resulting subproblems are unconstrained). The implementation was done in MATLAB and uses the parameters

$$\rho_{\nu,0} := 1, \quad w^{\max} := 10^6, \quad \text{and} \quad \begin{cases} \tau_\nu := 0.1,\ \gamma_\nu := 10, & \text{if } n \le 100, \\ \tau_\nu := 0.5,\ \gamma_\nu := 2, & \text{if } n > 100. \end{cases}$$

This represents a quite aggressive penalization scheme for small problems and a more cautious scheme for large problems. This distinction has turned out to be very efficient for our problem set. For the computation of the initial multipliers $\lambda^{\nu,0}$, we consider the KKT conditions of player ν in x^0, which can be stated as

$$\nabla_{x^\nu} f_\nu(x^0) + \nabla_{x^\nu} g^\nu(x^0)\lambda^{\nu,0} = 0 \quad \text{and} \quad \min\{-g^\nu(x^0), \lambda^{\nu,0}\} = 0. \tag{7.41}$$

We now choose $\lambda^{\nu,0}$ in a least-squares sense by setting $\lambda_i^{\nu,0} := 0$ for every i with $g_i^\nu(x^0) < 0$ and using the MATLAB function `lsqnonneg` to compute the remaining components of $\lambda_i^{\nu,0}$ as approximate solutions of the first equality in (7.41).

The augmented subproblems are solved by reformulating them as the semismooth equations

$$F_k(x) := \begin{pmatrix} \nabla_{x^1} \mathcal{L}^1_{\rho_{1,k}}(x, w^{1,k}) \\ \vdots \\ \nabla_{x^N} \mathcal{L}^N_{\rho_{N,k}}(x, w^{N,k}) \end{pmatrix} \overset{!}{=} 0$$

and then applying a semismooth Levenberg–Marquardt algorithm similar to that presented in Section 7.1.2 (see [129] for more details). These equations are solved up to a residual of $\varepsilon > 0$ in the infinity norm. Finally, the overall stopping criterion we use is

$$\|\nabla_{x^\nu} \theta_\nu(x) + \nabla_{x^\nu} g^\nu(x)\lambda^\nu\|_\infty \le \varepsilon, \quad \|g_+^\nu(x)\|_\infty \le \varepsilon, \quad \text{and} \quad |g^\nu(x)^\top \lambda^\nu| \le \varepsilon$$

for every ν. Here, ε is some prescribed stopping tolerance which we set to 10^{-8}.

The results are presented in Table 7.6. For a given problem, N denotes the number of players, n is the total number of variables, and x^0 is the initial point. If only a number is reported here, this means that all variables were initiated with that value. Moreover, k is the number of outer iterations, i_{total} is the accumulated number of inner iterations, and "F" denotes a failure. We also include certain values which measure the feasibility, optimality and complementarity at the computed solution. These are denoted R_f, R_o and R_c, respectively. The values are calculated as follows:

$$\begin{aligned} R_f &:= \max_{\nu=1,\dots,N} \|g_+^\nu(x)\|_\infty, \\ R_o &:= \max_{\nu=1,\dots,N} \|\nabla_{x^\nu} f_\nu(x) + \nabla_{x^\nu} g^\nu(x)\lambda^\nu\|_\infty, \\ R_c &:= \max_{\nu=1,\dots,N} |g^\nu(x)^\top \lambda^\nu|. \end{aligned}$$

Clearly, some remarks are in order:

1. With the exception of problem A.8, the augmented Lagrangian method was able to solve every problem quite efficiently. It is particularly noteworthy that the method achieves a very high accuracy, typically in the region of 10^{-10}.
2. The algorithm was also tested with different choices of $w^{\max}$. Recall that, for $w^{\max} = 0$, the algorithm is essentially a quadratic penalty method. The following table lists some values for $w^{\max}$ and corresponding failure numbers.

$w^{\max}$	0	10	10^2	10^4	10^6
failures	29	18	10	1	1

Example	N	n	x^0	k	i_{total}	R_f	R_o	R_c	$\rho_{\max}$
A.1	10	10	0.01	7	20	1.5e-10	8.9e-16	4.2e-11	100
			0.1	6	13	8e-09	5.9e-13	2.1e-09	100
			1	7	19	1.5e-10	2.9e-16	4.2e-11	100
A.2	10	10	0.01	9	108	4.7e-09	2.3e-09	1.4e-09	1000
			0.1	8	70	2.9e-09	4.1e-14	2.9e-11	100
			1	10	192	4.9e-10	5.6e-14	1.5e-10	1e+05
A.3	3	7	0	1	4	0	1e-09	0	1
			1	1	5	0	3.6e-15	0	1
			10	1	5	0	1.7e-10	0	1
A.4	3	7	0	12	63	2.6e-11	1.5e-09	2.6e-09	1e+04
			1	0	0	0	0	0	1
			10	11	202	2.5e-12	4.6e-10	7.4e-10	1e+04
A.5	3	7	0	8	20	2e-10	1.7e-13	4.8e-10	1000
			1	8	20	3.5e-10	4.9e-13	8.3e-10	1000
			10	10	27	6.9e-09	1e-13	6.2e-09	1000
A.6	3	7	0	14	68	1.9e-11	6.6e-10	4.2e-09	1e+04
			1	11	92	9.8e-12	4.5e-09	5.1e-09	1e+04
			10	14	82	1.9e-11	6.6e-10	4.2e-09	1e+04
A.7	4	20	0	13	35	6.6e-12	1.7e-11	2.3e-09	1e+04
			1	12	39	1.1e-11	1.4e-11	3.8e-09	1e+04
			10	12	52	4.1e-12	1.2e-11	1.7e-09	1e+05
A.8	3	3	0	F					
			1	1	4	4.9e-11	4.9e-11	4.9e-11	1
			10	3	14	4.5e-12	4.9e-12	4.5e-12	10
A.9a	7	56	0	9	46	2.3e-09	8e-15	7.6e-09	10
A.9b	7	112	0	26	75	2.8e-10	1e-14	2.7e-09	16
A.10a	8	24	see [66]	11	243	9.8e-13	4.5e-11	4.5e-12	1e+05
A.10b	25	125	see [66]	19	2519	6.7e-10	1.8e-11	4.9e-09	64
A.10c	37	222	see [66]	40	3658	7.2e-13	9.3e-12	1.6e-09	5e+05
A.10d	37	370	see [66]	19	2527	2.9e-11	2.3e-12	3.1e-10	256
A.10e	48	576	see [66]	18	4048	1.2e-10	7.1e-12	1.5e-09	256
A.11	2	2	0	9	17	6.4e-09	2.9e-15	3.2e-09	10
A.12	2	2	(2,0)	1	5	0	8.9e-16	0	1
A.13	3	3	0	4	20	3.3e-09	7.6e-12	1.9e-09	1
A.14	10	10	0.01	1	8	0	8.2e-14	0	1
A.15	3	6	0	1	7	0	2.8e-14	0	1
A.16a	5	5	10	10	26	1.3e-10	6e-14	3.7e-09	10
A.16b	5	5	10	9	26	6.1e-11	3.6e-15	1.1e-09	10
A.16c	5	5	10	7	23	9e-10	1.5e-13	6.4e-09	10
A.16d	5	5	10	9	24	4e-09	2.1e-14	1.9e-09	1
A.17	2	3	0	8	20	4.5e-11	3.4e-13	1.1e-10	100
A.18	2	12	0	9	34	1.3e-11	1.1e-11	2.4e-10	1000
			1	9	34	1.3e-11	1.2e-11	2.4e-10	1000
			10	9	32	1.3e-11	1.8e-11	2.4e-10	1000

Table 7.6: Numerical results of the augmented Lagrangian method, applied to a collection of finite-dimensional generalized Nash equilibrium problems.

3. Clearly, the overall speed of the algorithm crucially depends on how quickly the subproblems are solved. In this regard, the semismooth Levenberg–Marquardt algorithm seems to be a very strong choice. Some of the problems were investigated on a sample basis, and the Levenberg–Marquardt method appears to be superlinearly convergent for all of them.
4. Another factor which greatly affects the performance of the algorithm is the choice of algorithmic parameters. The presented values are quite simple and straightforward. However, for some problems, fine-tuning the parameters can yield a significant speed improvement.
5. For problem A.8 with the starting point $x^0 = 0$, the subproblem algorithm is unable to compute a solution and, hence, the overall iteration breaks down. Another peculiarity of problem A.8 is that, for a suitable choice of parameters, one can get the algorithm to converge to the infeasible point $\bar{x} = (1.5, 0, 2)^\top$. This point (together with its corresponding multipliers) satisfies the stationarity part of the KKT conditions, but (due to the infeasibility) is not a solution of the GNEP. Furthermore, one can easily verify that $\bar{x}$ is a solution of the "Feasibility GNEP" from Section 6.3.3, as suggested by Lemma 6.42, but the GNEP-EMFCQ does not hold in $\bar{x}$. This corroborates the assertion of that result.

7.6 Quasi-Variational Inequalities in $\mathbb{R}^n$

The purpose of this section is to give detailed practical results on the augmented Lagrangian method for QVIs with nonlinear programming constraints. The results are obtained for both the basic method (Algorithm 6.29) and its exact penalty modification (Algorithm 6.34). A suitable collection of test problems is the QVILIB library [68]. The QVIs in this collection follow a simple structure: for each problem, the constraint set $\Phi(x)$ is given by

$$\Phi(x) = \{y \in \mathbb{R}^n : g^I(y) \leq 0 \text{ and } g^P(x, y) \leq 0\},$$

where g^I and g^P describe the *independent* and *parametrized* constraints, respectively. This structure lends itself to both partial and full penalization (recall that the exact penalty method always uses full penalization). The resulting subproblems then become either standard VIs or, in the latter case, nonlinear equations. For the solution of these problems, we decided to employ a semismooth Levenberg–Marquardt type method (see Section 7.1.2) together with the well-known Fischer–Burmeister complementarity function

$$\phi_{\mathrm{FB}}(a, b) := \sqrt{a^2 + b^2} - a - b,$$

see [67, 75], which allows us to transform a VI into a nonlinear equation.

After the above discussion, we are left with four methods:

- the semismooth Levenberg–Marquardt method, applied directly to the KKT system of the QVI by use of the Fischer–Burmeister function. This method will be denoted by **Semi**.

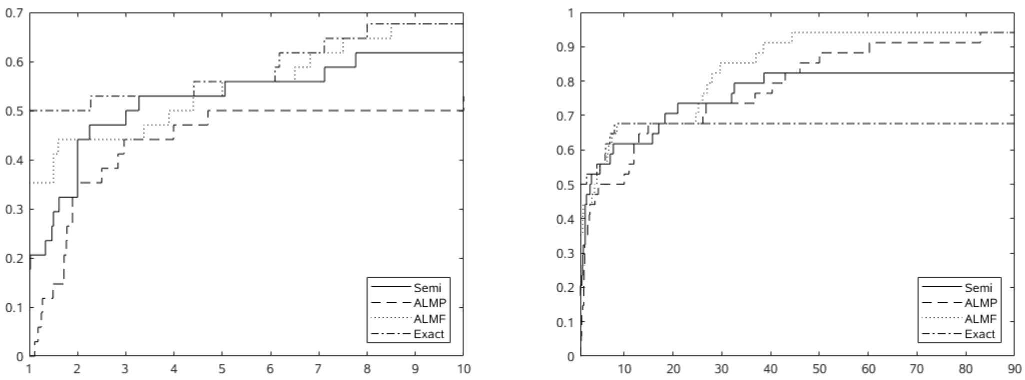


Figure 7.11: Performance profiles (based on CPU times) of the four algorithms from Section 7.6.

- the augmented Lagrangian method (Algorithm 6.29) using partial penalization and the formula $w^k = \min\{\lambda^k, w^{\max}\}$. This method will be denoted by **ALMP**.
- the augmented Lagrangian method like above, but with full penalization, denoted **ALMF**.
- the exact penalty method (Algorithm 6.34), denoted **Exact**.

We now describe some implementation details for our methods, using the notation of Section 6.3.1. The starting points x^0 are given by the test library, and the subproblems occurring within the penalty methods are solved with the termination criterion

$$\left\| \begin{pmatrix} \mathcal{L}_{\rho_k}(x, w^k) + \nabla_y h(x,x)\mu \\ \phi_{\text{FB}}(-h(x,x), \mu) \end{pmatrix} \right\|_\infty \le 10^{-8},$$

where ϕ_{FB} is the (vectorized) Fischer–Burmeister function, see above. For the outer algorithm, we use the stopping criterion

$$\left\| \begin{pmatrix} F(x) + \nabla_y g(x,x)\lambda + \nabla_y h(x,x)\mu \\ \min\{-g(x,x), \lambda\} \\ \min\{-h(x,x), \mu\} \end{pmatrix} \right\|_\infty \le \varepsilon := 10^{-4}.$$

Both these inequalities can, of course, be written more tersely for the methods which do not use h. Finally, we use the initial Lagrange multipliers $(\lambda^0, \mu^0) := (0,0)$ and the algorithmic parameters

$$\rho_0 := 1, \quad w^{\max} := 10^{10}, \quad \gamma := 5, \quad \text{and} \quad \tau := 0.9,$$

which are chosen to favor robustness over efficiency. Note that some algorithms, such as the exact penalty method, only use a subset of the above parameters.

The results are presented in Table 7.7, which contains problem data and iteration numbers, and Figure 7.11, which contains the performance profiles of the four methods

Name	n	m	p	**Semi**	**ALMP**	**ALMF**	**Exact**
BiLin1A	5	3	10	57	23	22	2
BiLin1B	5	3	10	27	49	49	1
Box1A	5	10	0	6	38	38	1
Box1B	5	10	0	–	–	–	1
Box2A	500	1000	0	146	13	13	1
Box2B	500	1000	0	74	16	16	1
Box3A	500	1000	0	177	12	12	1
Box3B	500	1000	0	774	38	38	2
KunR11	2500	2500	0	504	12	12	$*$
KunR12	4900	4900	0	–	11	11	$*$
KunR21	2500	2500	0	277	1	1	$*$
KunR22	4900	4900	0	269	1	1	$*$
KunR31	2500	2500	0	340	29	29	$*$
KunR32	4900	4900	0	621	35	35	$*$
MovSet1A	5	1	0	8	36	36	1
MovSet1B	5	1	0	–	–	–	2
MovSet2A	5	1	0	9	42	42	1
MovSet2B	5	1	0	–	44	44	1
MovSet3A1	1000	1	0	45	3	3	1
MovSet3A2	2000	1	0	96	3	3	1
MovSet3B1	1000	1	0	42	3	3	4
OutKZ31	62	62	62	7	16	19	–
OutKZ41	82	82	82	12	22	10	–
OutZ40	2	2	4	5	1	1	1
OutZ41	2	2	4	6	1	1	1
OutZ42	4	4	4	14	6	6	1
OutZ43	4	4	0	5	8	8	1
OutZ44	4	4	0	5	7	7	1
RHS1A1	200	199	0	110	1	1	1
RHS1B1	200	199	0	388	1	1	1
RHS2A1	200	199	0	116	1	1	1
RHS2B1	200	199	0	114	1	1	1
Scrim21	2400	2400	2400	846	28	30	$*$
Scrim22	4800	4800	4800	–	28	30	$*$

Table 7.7: Numerical results of four algorithms for a library of quasi-variational inequality problems. Note: $*$ denotes a problem where QVI-LICQ is violated.

(using CPU time as the underlying metric). The table is structured as follows: each row represents a problem from the QVILIB library. The name of the problem is given in the first column, followed by the dimensions n, m and p. The final four columns list the iteration numbers for each of our four methods, where "–" denotes a failure. In view of the results, some remarks are in order:

1. The augmented Lagrangian methods (**ALMP** and **ALMF**) are able to solve most problems, the only exceptions being `Box1B` and `MovSet1B`. A quick analysis shows that the failure for these problems is due to the inability of the Newton method to solve the subproblems at certain iterations. This possibly could have been avoided with a different choice of sub-algorithm.
2. The Newton method has 5 failures, which shows that the implementation (although fairly simple) is quite robust (in particular, more robust than the standard semismooth Newton method from [67, 127]), but not as robust as the augmented Lagrangian methods.
3. The exact penalty method is able to solve most of the smaller problems extremely quickly, usually requiring only 1 or 2 iterations (recall that 1 iteration means that the penalty parameter $\rho_0 = 1$ is already exact). However, the algorithm exhibits failures for some of the larger problems. A quick analysis shows that, in particular, the `Kun*` and `Scrim*` examples seem to violate QVI-LICQ (at least numerically), which makes the exact penalty method entirely unsuited for these problems.
4. Regarding the actual performance of the methods, note that the iteration numbers in Table 7.7 do not show the whole picture since the cost of a single step differs greatly between the methods (for instance, the Newton method only solves a linear equation per step). A more realistic performance comparison is given by the performance profiles in Figure 7.11. We observe that the augmented Lagrangian method performs significantly better when using full penalization, even rivaling the semismooth Newton method in terms of overall efficiency. The exact penalty method is extremely efficient but suffers from a lack of robustness due to the strong theoretical requirements of the method.
5. All four algorithms were also tested with a desired accuracy of $\varepsilon = 10^{-8}$. For most problems, this did not cause any difficulties. The failure numbers in this setting are given by 7 (**Semi**), 4 (**ALMP** and **ALMF**), and 3 (**Exact**, excluding the examples where LICQ is violated).
6. For some problem classes, the algorithms exhibit completely different behaviour. For instance, the `RHS*` examples turn out to be extremely easy for the augmented Lagrangian methods and quite hard for the semismooth Newton method.
7. The two problems `Box1B` and `MovSet1B` appear to be very hard for the first three methods, which agrees with the numerical results from [127]. Interestingly, though, the exact penalty method is able to solve these problems very easily, requiring only 1 and 2 iterations, respectively.

To conclude, the augmented Lagrangian approach appears to be a very robust and efficient solver for QVIs under fairly mild assumptions. On the other hand, the exact penalty method, which is basically just a by-product of our analysis, requires stronger assumptions (i.e., QVI-LICQ) to even be well-defined, but seems to work extremely well if these assumptions are satisfied. In addition, it is able to solve certain problems which pose difficulties to both the augmented Lagrangian and semismooth Newton methods.

Chapter 8

Additional Results

This chapter collects some supplementary results whose purpose is to enrich the theoretical framework of the augmented Lagrangian technique. For the sake of simplicity and clarity, the results in this section will be restricted to the case of a constrained minimization problem of the form

$$(P) \qquad \underset{x \in X}{\text{minimize}}\ f(x) \quad \text{subject to} \quad G(x) \in \mathcal{K}, \tag{8.1}$$

where X and H respectively denote real Banach and Hilbert spaces, $f : X \to \mathbb{R}$ and $G : X \to H$ are suitable mappings, and $\mathcal{K} \subseteq H$ is a nonempty closed convex set. Note that, in comparison to the framework from Chapter 4, we do not use an intermediate space Y together with an embedding $Y \hookrightarrow H$. The reason is that the results below can be stated most accurately in terms of the space H itself.

For $w \in H$ and $\rho > 0$, the augmented Lagrangian of (P) takes on the form

$$\mathcal{L}_\rho : X \times H \to \mathbb{R}, \quad \mathcal{L}_\rho(x, \lambda) := f(x) + \frac{\rho}{2} d_{\mathcal{K}}^2\left(G(x) + \frac{\lambda}{\rho}\right) - \frac{\|\lambda\|_H^2}{2\rho}. \tag{8.2}$$

This chapter is organized as follows. In Section 8.1, we give a general result characterizing the KKT points of (P) as primal-dual saddle points of the function $\mathcal{L}_\rho$. This property was observed by Rockafellar in [187] for convex minimization problems. Here, we show that it is valid for arbitrary optimization problems. This allows us to interpret the augmented Lagrangian algorithm as a gradient ascent method for the dual variable.

In Section 8.2, we demonstrate a connection between the augmented Lagrangian and the concept of *epigraphical convergence* (also called Γ-*convergence*). This is a notion of functional convergence designed to facilitate limit processes in minimization problems, see [10, 49, 191] for more details.

Section 8.3 is based on the publication [130] and contains an explicit example demonstrating the necessity of multiplier safeguarding in augmented Lagrangian methods. The existence of such examples can be seen as the most tangible practical motivation for the use of safeguarded algorithms such as those in Chapters 4 to 6.

Finally, Section 8.4 is based on the paper [131]. Here, we show how the augmented Lagrangian method is related to the famous *proximal point algorithm* from convex analysis (see [15]). This connection also goes back to the work of Rockafellar [187,189] for nonlinear programming, and in the present chapter we show that the connection can be extended to a general problem of the form (P). This provides another perspective on the convergence properties of the augmented Lagrangian algorithm.

8.1 Saddle Points of the Augmented Lagrangian

In this section, we shall see that KKT points of constrained optimization problems can always be characterized by means of primal-dual stationary points of the augmented Lagrange function. We consider a problem of the form (P), where f and G are assumed to be continuously differentiable. Recall that the KKT conditions of the problem take on the form

$$\mathcal{L}'(\bar{x},\bar{\lambda}) = 0 \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_{\mathcal{K}}(G(\bar{x})),$$

and that the augmented Lagrangian of (P), for $\rho > 0$, is given by

$$\mathcal{L}_\rho(x,\lambda) = f(x) + \frac{\rho}{2} d_{\mathcal{K}}^2\left(G(x) + \frac{\lambda}{\rho}\right) - \frac{\|\lambda\|_H^2}{2\rho}. \tag{8.3}$$

The main definition we will use below is the following.

Definition 8.1 (Primal-dual stationary point)**.** We say that a pair $(\bar{x},\bar{\lambda}) \in X \times H$ is a *primal-dual stationary point* of $\mathcal{L}_\rho$ if $D_x\mathcal{L}_\rho(\bar{x},\bar{\lambda}) = 0$ and $D_\lambda\mathcal{L}_\rho(\bar{x},\bar{\lambda}) = 0$.

Note that we write $D_x\mathcal{L}_\rho$ instead of $\mathcal{L}_\rho'$ for the sake of clarity. The following is the main result of this section.

Theorem 8.2. *Let $(\bar{x},\bar{\lambda}) \in X \times H$ be an arbitrary point. The following are equivalent:*

(i) *$\bar{x}$ is feasible and $(\bar{x},\bar{\lambda})$ is a KKT point of the problem (P).*

(ii) *$(\bar{x},\bar{\lambda})$ is a primal-dual stationary point of $\mathcal{L}_\rho$ for some $\rho > 0$.*

(iii) *$(\bar{x},\bar{\lambda})$ is a primal-dual stationary point of $\mathcal{L}_\rho$ for all $\rho > 0$.*

Proof. The partial derivatives of the augmented Lagrangian take on the form

$$D_x\mathcal{L}_\rho(x,\lambda) = f'(x) + \rho G'(x)^*\left[G(x) + \frac{\lambda}{\rho} - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho}\right)\right], \tag{8.4}$$

$$D_\lambda\mathcal{L}_\rho(x,\lambda) = G(x) - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho}\right). \tag{8.5}$$

We now prove the desired equivalences. Note that (iii)⇒(ii) is clear.

(ii)⇒(i): If $(\bar{x},\bar{\lambda})$ is a primal-dual stationary point for some $\rho > 0$, then (8.5) implies $G(\bar{x}) \in \mathcal{K}$ and $\bar{\lambda}/\rho \in \mathcal{N}_{\mathcal{K}}(G(\bar{x}))$, which yields $\bar{\lambda} \in \mathcal{N}_{\mathcal{K}}(G(\bar{x}))$. Moreover, inserting (8.5) into (8.4), we obtain $f'(\bar{x}) + G'(\bar{x})^*\bar{\lambda} = 0$, and thus $(\bar{x},\bar{\lambda})$ is a KKT point of the optimization problem.

(i)⇒(iii): Let $\rho > 0$ be arbitrary. Then $\bar{\lambda}/\rho \in \mathcal{N}_{\mathcal{K}}(G(\bar{x}))$ since the latter is a cone. Hence, (8.5) implies that $D_\lambda\mathcal{L}_\rho(\bar{x},\bar{\lambda}) = 0$. Inserting this into (8.4), we see that $D_x\mathcal{L}_\rho(\bar{x},\bar{\lambda}) = D_x\mathcal{L}(\bar{x},\bar{\lambda}) = 0$. Thus, $(\bar{x},\bar{\lambda})$ is a primal-dual stationary point of $\mathcal{L}_\rho$. □

The above result provides a different perspective on the augmented Lagrangian method. To this end, assume that we are in the notation of Algorithm 4.4, so that x^{k+1} is an approximate minimizer of $\mathcal{L}_{\rho_k}(\cdot, w^k)$ for certain $\rho_k > 0$ and $w^k \in H$. This can be viewed as the "primal" step of the algorithm. An elementary calculation shows that the next dual iterate λ^{k+1} is given by

$$\lambda^{k+1} = w^k + \rho_k D_\lambda \mathcal{L}_{\rho_k}(x^{k+1}, w^k). \tag{8.6}$$

It follows that we can interpret the multiplier update in the augmented Lagrangian method as a gradient ascent step for the dual variable, with step size equal to ρ_k. The overall algorithm can therefore be viewed as repeatedly

(i) minimizing $\mathcal{L}_{\rho_k}(\cdot, w^k)$ to obtain x^{k+1}, and then

(ii) performing a gradient ascent step for $\mathcal{L}_{\rho_k}(x^{k+1}, \cdot)$ to obtain λ^{k+1} from w^k.

Observe also that the derivative $D_\lambda \mathcal{L}_{\rho_k}$ is globally Lipschitz-continuous with respect to λ, with Lipschitz constant given by $1/\rho_k$, see (8.5). Hence, the step size in (8.6) is equal to the inverse Lipschitz constant of $D_\lambda \mathcal{L}_{\rho_k}$, which is a standard step size in the context of gradient-type descent (or ascent) schemes [16].

8.2 Epigraphical Convergence

The purpose of this section is to analyze the augmented Lagrange function in the context of a certain notion of functional convergence, the so-called *epigraphical convergence* (also known as Γ-*convergence*). This property is related to the behavior of approximate or asymptotic minimizers of the augmented Lagrangian (see Section 4.2.2), but it may also be of independent theoretical interest and could lead to subsequent developments. The main definition is the following. Note that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ denotes the extended real line.

Definition 8.3 (Epigraphical convergence)**.** Let X be a real Banach space and $f_k, f : X \to \overline{\mathbb{R}}$ given functions. We say that $\{f_k\}$ *converges epigraphically* or *epiconverges* to f if the following two conditions are satisfied:

(i) whenever $\{x^k\} \subseteq X$ and $x^k \to x \in X$, then $f(x) \le \liminf_{k\to\infty} f_k(x^k)$.

(ii) for every $x \in X$, there is a sequence $\{x^k\} \subseteq X$ such that $x^k \to x$ and $f(x) \ge \limsup_{k\to\infty} f_k(x^k)$.

The main result in this section is the following. Note that we write δ_Φ for the indicator function of the feasible set $\Phi = G^{-1}(\mathcal{K})$, i.e., for the function defined as zero on Φ and $+\infty$ elsewhere. (This is the indicator function from convex analysis, but in the present case, the set Φ is not assumed to be convex.)

Theorem 8.4. *Let $\{w^k\} \subseteq H$ be a bounded sequence, $\rho_k \to \infty$, and $\mathcal{L}_k : X \to \mathbb{R}$, $\mathcal{L}_k(x) := \mathcal{L}_{\rho_k}(x, w^k)$. If f and $d_{\mathcal{K}} \circ G$ are lower semicontinuous, then the sequence $\{\mathcal{L}_k\}$ converges epigraphically to $f + \delta_\Phi$.*

Proof. To prove property (i) from Definition 8.3, let $x^k \to x$ for some $x \in X$. We distinguish two cases. if $x \notin \Phi$, then $d_{\mathcal{K}}(G(x)) > 0$, and it is easy to infer that $d_{\mathcal{K}}(G(x^k) + w^k/\rho_k)$ is asymptotically positive. Moreover, the lower semicontinuity of f implies that $\{f(x^k)\}$ is bounded from below. Hence, in this case, it follows that

$$\begin{aligned}\mathcal{L}_k(x^k) &= f(x^k) + \frac{\rho_k}{2} d_{\mathcal{K}}^2\left(G(x^k) + \frac{w^k}{\rho_k}\right) - \frac{\|w^k\|_H^2}{2\rho_k} \\ &\to +\infty = (f + \delta_\Phi)(x),\end{aligned}$$

which implies the claim. Assume now that $x \in \Phi$. From the definition of $\mathcal{L}_\rho$, we have that

$$\liminf_{k\to\infty} \mathcal{L}_k(x^k) \geq \liminf_{k\to\infty} \left[f(x^k) - \frac{\|w^k\|_H^2}{2\rho_k} \right] \geq f(x) = (f + \delta_\Phi)(x).$$

Hence, the claim also follows in this case.

We now prove property (ii) from Definition 8.3. Let $x \in X$ be an arbitrary point and set $x^k := x$ for all k. If x is infeasible, then we can argue as above to see that $\mathcal{L}_k(x^k) \to +\infty = (f + \delta_\Phi)(x)$, so that the property is satisfied. If x is feasible, then we have from Proposition 4.3 that $\mathcal{L}_k(x^k) \leq f(x)$ for all k, and this immediately implies

$$(f + \delta_\Phi)(x) = f(x) \geq \limsup_{k\to\infty} \mathcal{L}_k(x^k).$$

The proof is complete. □

The above proof technique can also be used to deduce a property which is stronger than epiconvergence. Indeed, if the functions f and $d_{\mathcal{K}} \circ G$ are *weakly* sequentially lsc, then we obtain that $\{\mathcal{L}_k\}$ converges epigraphically to $f + \delta_\Phi$, and property (i) from Definition 8.3 actually holds for all *weakly* convergent sequences $\{x^k\}$. This type of convergence is often called *Mosco-convergence*, see [49]. (It corresponds to its namesake set convergence from Section 6.1.1, applied to the epigraphs of the functions.)

The epiconvergence (and Mosco-convergence) of the augmented Lagrangians can be used to obtain certain results on approximate or asymptotic minimizers which closely resemble those in Section 4.2.2, see [49] for more details. However, the results in that section are actually somewhat stronger and simpler than those which follows from the general epiconvergence theory, and the proofs in Section 4.2.2 are much more elementary than an application of this abstract theoretical framework.

8.3 The Necessity of Multiplier Safeguarding

This section contains an explicit example which demonstrates the necessity of multiplier safeguarding in augmented Lagrangian methods. This can be regarded as the most tangible motivation for the use of safeguarded multiplier sequences. Throughout this section, we will deal with a nonlinear programming problem of the form

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(x) \quad \text{subject to} \quad g(x) \leq 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, $m, n \in \mathbb{N}$, are continuously differentiable mappings. For the sake of clarity, the following is essentially a restatement of the augmented Lagrangian method (Algorithm 4.4) for such problems. Here and throughout, we write $x_+ := \max\{0, x\}$ for a vector in $\mathbb{R}^n$, where max is understood componentwise.

Algorithm 8.5 (ALM for nonlinear programming)**.** Let $(x^0, \lambda^0) \in \mathbb{R}^{n+m}$, $\rho_0 > 0$, let $w^{\max} \geq 0$, $\gamma > 1$, $\tau \in (0, 1)$, and set $k := 0$.

Step 1. If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Let $w^k := \min\{\lambda^k, w^{\max}\}$ and compute an approximate solution x^{k+1} of the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; \mathcal{L}_{\rho_k}(x, w^k). \tag{8.7}$$

Step 3. Update the vector of multipliers to $\lambda^{k+1} := (w^k + \rho_k g(x^{k+1}))_+$.

Step 4. Let $V_{k+1} := \left\| \min\{-g(x^{k+1}), w^k/\rho_k\} \right\|$ and set

$$\rho_{k+1} := \begin{cases} \rho_k, & \text{if } k = 0 \text{ or } V_{k+1} \leq \tau V_k, \\ \gamma \rho_k, & \text{otherwise.} \end{cases} \tag{8.8}$$

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

We will refer to the above algorithm as the *safeguarded* augmented Lagrangian method (ALM). Throughout this section, we will also make reference to the *traditional* or *classical* ALM, which is understood to be the same method as Algorithm 8.5, with the exception that w^k is always chosen as λ^k in Step 2 (regardless of boundedness).

We have already seen that the safeguarded ALM allows for a very rich global convergence theory, and we shall now see that a similar theory is not possible for the classical ALM. Indeed, we will construct an example where both the traditional and modified ALMs generate sequences of stationary points (in fact, local minimizers) of the corresponding augmented subproblems in such a way that the sequences each have two accumulation points, one of which is infeasible and violates basically any constraint qualification, whereas the other accumulation point is feasible (though different for both methods) and satisfies essentially all constraint qualifications, and is therefore necessarily a KKT point of (8.9) for the safeguarded ALM (in view of Section 4.2.3), whereas it does not correspond to a stationary point for the classical ALM.

The problem we consider is the simple nonlinear programming problem

$$\underset{x \in \mathbb{R}}{\text{minimize}} \; f(x) := x \quad \text{subject to} \quad g(x) := 1 - x^3 \leq 0. \tag{8.9}$$

Note that $\bar{x} := 1$ is the unique solution of this problem; moreover, an easy calculation shows that $(\bar{x}, \bar{\lambda}) := (1, 1/3)$ is the only KKT point. A key issue in our subsequent analysis is the fact that g has a stationary point at $x = 0$, and that this point is not feasible. Observe that (8.9) is easy in the sense that both the objective function and the feasible set are convex, although the representation of the feasible set is nonconvex.

The Classical Method

Let us consider the traditional augmented Lagrangian method applied to problem (8.9). The subsequent analysis is fairly general and only assumes (mainly for the sake of convenience) that $\rho_0 > 1/3$ and $\lambda^0 \le 1/3$.

It is easily seen that, for all $\lambda \ge 0$ and $\rho > 0$, the function $\mathcal{L}_\rho(\cdot, \lambda)$ is coercive on $\mathbb{R}$. Moreover, using the formula

$$\mathcal{L}_\rho'(x, \lambda) = f'(x) + g'(x)\big(\lambda + \rho g(x)\big)_+$$

for the derivative of $\mathcal{L}_\rho$, we obtain $\mathcal{L}_\rho'(0, \lambda) = 1$ and $\mathcal{L}_\rho'(1, \lambda) = 1 - 3\lambda$. It follows that $\mathcal{L}_\rho$ always attains a local minimum in $(-\infty, 0)$ and, if $\lambda > 1/3$, it attains another local minimum in $(1, +\infty)$. Let $\{x^k\}_{k\ge1}$ be a sequence of such local minimizers such that

- for k odd, x^k is the largest local minimizer in $(-\infty, 0)$,
- for k even, x^k is the smallest local minimizer in $(1, +\infty)$.

If k is odd, then we have $x^k < 0$ and $g(x^k) > 1$. It follows that

$$\lambda^k = \big(\lambda^{k-1} + \rho_{k-1} g(x^k)\big)_+ \ge \rho_{k-1}. \tag{8.10}$$

Since $\rho_{k-1} > 1/3$, we conclude that x^{k+1} is well-defined. Another property of the sequence $\{x^k\}$ is boundedness.

Lemma 8.6. *For the classical ALM, we have* $x^k \in [-1, 2]$ *for all* $k \ge 1$.

Proof. If k is odd, then $x^k < 0$. Moreover, $\mathcal{L}_{\rho_{k-1}}'(-1, \lambda^{k-1}) = 1 - 3(\lambda^{k-1} + 2\rho_{k-1})_+ < 0$, which implies $x^k > -1$ since x^k is defined as the largest local minimizer in $(-\infty, 0)$.

Before showing that $x^k \le 2$ for k even, we need some information about the multiplier sequence $\{\lambda^k\}$. First, if $k > 1$ is even, then $0 = \mathcal{L}_{\rho_{k-1}}'(x^k, \lambda^{k-1}) = 1 - 3(x^k)^2\lambda^k$, which implies $\lambda^k = 1/(3(x^k)^2) \le 1/3$. By our assumption on λ^0, this assertion also holds for $k = 0$. Hence, using $\rho_0 > 1/3$ and $x^k \ge -1$ for k odd, it follows that

$$\lambda^k = \big(\lambda^{k-1} + \rho_{k-1} g(x^k)\big)_+ \le \frac{1}{3} + 2\rho_{k-1} \le 3\rho_k,$$

again for k odd. We now use this inequality to prove $x^k \le 2$ for k even. To this end, let $k > 1$ be even, and note that

$$\mathcal{L}_{\rho_{k-1}}'(2, \lambda^{k-1}) = 1 - 12(\lambda^{k-1} - 7\rho_{k-1})_+ = 1,$$

since $k - 1$ is odd. Hence, the definition of x^k as the smallest local minimizer in $(1, +\infty)$ implies $x^k < 2$. $\square$

The boundedness of $\{x^k\}$ implies that the sequence has at least one limit point in $[-1, 0]$ and one in $[1, 2]$. In particular, we have $\rho_k \to \infty$, for otherwise every limit point of $\{x^k\}$ would have to be feasible (because of the penalty updating scheme).

On the other hand, (8.10) implies that $\lambda^k/\rho_k \geq \gamma^{-1}$ for odd k (where γ is defined in Algorithm 8.5). Let

$$\hat{x} := \left(1 + \frac{1}{2\gamma}\right)^{1/3},$$

so that $g(\hat{x}) = -1/(2\gamma)$. It follows that, for all $x \in [1, \hat{x}]$ and k even,

$$\begin{aligned}\mathcal{L}'_{\rho_{k-1}}(x, \lambda^{k-1}) &= 1 - 3\rho_{k-1}x^2 \left(g(x) + \frac{\lambda^{k-1}}{\rho_{k-1}}\right)_+ \\ &\leq 1 - 3\rho_{k-1}\left(\frac{1}{\gamma} - \frac{1}{2\gamma}\right)_+ = 1 - \frac{3}{2\gamma}\rho_{k-1} < 0\end{aligned}$$

for sufficiently large values of ρ_{k-1}. Since $\rho_k \to \infty$, we conclude that $x^k > \hat{x}$ for sufficiently large (even) k. In particular, any accumulation point of $\{x^k\}$ in $[1, 2]$ is strictly greater than 1. But none of these points correspond to a KKT point of the problem (8.9).

The Safeguarded Method

We now consider the modified method applied to problem (8.9). For the sake of convenience, we will again make certain assumptions on the algorithmic parameters. That is, we assume $\rho_0 > 1/3$, $\lambda^0 \leq 1/3$, and $w^{\max} > 1/3$.

These assumptions allow us to compare the algorithm fairly easily to the classical ALM. In particular, we can choose $\{x^k\}$ as before, and the proof of Lemma 8.6 can be carried over as well.

Lemma 8.7. *For the safeguarded ALM, we have* $x^k \in [-1, 2]$ *for all* $k \geq 1$.

Proof. The proof is virtually identical to that of Lemma 8.6. Note that $w^k \leq \lambda^k$ for all k; hence, any upper bound for λ^k automatically translates to one for w^k. □

As with the classical ALM, it follows that the sequence generated by the modified ALM has at least two limit points, one in $[-1, 0]$ and one in $[1, 2]$. Using standard convergence theorems (e.g., Lemma 4.15 and Theorem 4.18), we know that

- every limit point of $\{x^k\}$ is a stationary point of $\|g_+(x)\|^2$, and
- every feasible limit point of $\{x^k\}$ where g satisfies CPLD is a KKT point.

Clearly, the interval $[-1, 0]$ consists only of infeasible points. However, the point $x = 0$ is the only point in this interval which is a stationary point of $\|g_+\|^2$. Hence, the subsequence of $\{x^k\}$ consisting of odd k must converge to $x = 0$. On the other hand, the interval $[1, 2]$ consists entirely of feasible points, and CPLD (in fact, LICQ) holds at every one of these points. Hence, the subsequence of $\{x^k\}$ consisting of even k converges to $x = 1$ which is the solution (and only stationary point) of the optimization problem (8.9).

Numerical Results

This part contains some numerical results illustrating the practical behavior of the two methods. The algorithmic parameters were chosen as

$$x^0 := -1, \quad \lambda^0 := 0, \quad \rho_0 := 1, \quad \tau := 0.1, \quad \gamma := 2, \quad w^{\max} = 10^4.$$

The subproblems are solved with the MATLAB function `fminunc` and a tolerance of 10^{-12}. The overall stopping criterion is

$$|f'(x) + \lambda g'(x)| \leq 10^{-4} \quad \text{and} \quad |\min\{-g(x), \lambda\}| \leq 10^{-4}.$$

Table 8.1 shows the iterates generated by both algorithms.

	Classical algorithm			Safeguarded algorithm		
k	x^k	λ^k	ρ_k	x^k	λ^k	ρ_k
0	-1.000000	0.0000e+00	2^0	-1.000000	0.0000e+00	2^0
1	-0.537207	1.1550e+00	2^0	-0.537207	1.1550e+00	2^0
2	1.247365	2.1424e-01	2^0	1.247365	2.1424e-01	2^0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	-0.009021	4.0963e+03	2^{13}	-0.009021	4.0963e+03	2^{13}
16	1.144714	2.5438e-01	2^{14}	1.144714	2.5438e-01	2^{14}
17	-0.004511	1.6384e+04	2^{15}	-0.004511	1.6384e+04	2^{15}
18	1.144714	2.5438e-01	2^{16}	1.092837	2.7910e-01	2^{16}
19	-0.002255	6.5536e+04	2^{17}	-0.002255	6.5536e+04	2^{17}
20	1.144714	2.5438e-01	2^{18}	1.024810	3.1739e-01	2^{17}
21	-0.001128	2.6214e+05	2^{19}	-0.001595	1.3107e+05	2^{18}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	-0.000002	1.7180e+10	2^{35}	-0.000100	3.3554e+07	2^{26}
38	1.144714	2.5438e-01	2^{36}	1.000050	3.3330e-01	2^{26}
39	0.000000	6.8719e+10	2^{37}	-0.000084	6.7109e+07	2^{27}
40	1.144714	2.5438e-01	2^{38}	1.000025	3.3332e-01	2^{27}

Table 8.1: Comparison of the classical and safeguarded augmented Lagrangian methods.

As suggested by the theory, both algorithms generate sequences with two accumulation points. Up to iteration 17, the methods perform identically. This is because the bound $w^{\max} = 10^4$ only becomes active when $|\lambda^k| > w^{\max}$, which first occurs for $k = 17$. Starting with $k = 18$, the classical ALM essentially alternates between 1.144714 and (almost) zero. Recall that the latter is a stationary point of the constraint function $g(x) = 1 - x^3$, which is one of the key reasons for this behavior. However, the former limit point is strictly feasible and perfectly regular (it satisfies LICQ), but is neither a stationary point of the optimization problem nor of the constraint. Note that the classical ALM keeps exhibiting this pathological behavior as k increases, but Table 8.1 stops at $k = 40$ for which the safeguarded ALM terminates successfully.

As with the standard method, the safeguarded ALM generates a sequence $\{x^k\}$ with two accumulation points, one of which is again the (irregular) point zero. Moreover, up to $k = 17$, the iterates are identical to those generated by the classical ALM, but as soon as the safeguarding becomes active (at $k = 18$), the subsequence of $\{x^k\}$ with even k starts to converge to the point $\bar{x} = 1$, which is the unique solution and stationary point of the problem. As a result, the method terminates successfully at $k = 40$. It is also worth noting that the penalty parameter remains much smaller for the modified method since the algorithm is "making progress" during the iterations 18–40.

The table also shows that both methods generate multiplier sequences which are unbounded (at least on a subsequence). A closer look at the ratio w^k/ρ_k shows that this sequence converges to zero for the safeguarded ALM (which is clear since $\{w^k\}$ is bounded), whereas the corresponding sequence λ^k/ρ_k (for k odd) from the classical ALM appears to converge to 1/2. This agrees with our theoretical analysis.

8.4 Relationship with Proximal Point Methods

We now turn to a brief discussion of proximal point methods and their relationship to the augmented Lagrangian algorithm. Consider an optimization problem of the form (P) where, for the sake of simplicity, the set $\mathcal{K}$ is a closed convex *cone*. Assume furthermore that f is convex and that G is $\mathcal{K}_\infty$-concave and continuously differentiable. Let

$$q(\lambda) := \inf_{x \in X} \mathcal{L}(x, \lambda).$$

Then the *dual problem* of (P) is given by

$$\underset{\lambda \in \mathcal{K}^\circ}{\text{maximize}} \; q(\lambda). \tag{8.11}$$

Note that q is a concave function since it is an infimum of affine functions. Since $\mathcal{K}^\circ$ is a convex set, it follows that the above is a concave maximization problem. For $c > 0$, let

$$\operatorname{prox}_{cq}(w) := \underset{\lambda \in \mathcal{K}^\circ}{\operatorname{argmax}} \left\{ q(\lambda) - \frac{1}{2c} \|\lambda - w\|_H^2 \right\}$$

denote the *proximal mapping* corresponding to (8.11). Note that the function occurring inside the argmax above is strongly concave. Since H is a Hilbert space, this function admits a unique maximizer, and thus the proximal mapping is well-defined.

The famous proximal point algorithm [15] is defined by the recursion

$$c_k > 0, \quad \lambda^{k+1} := \operatorname{prox}_{c_k q}(\lambda^k), \quad k \in \mathbb{N}_0.$$

We will now see that this iterative procedure is strongly related to the augmented Lagrangian method (Algorithm 4.4). Note that, since $\mathcal{K}$ is assumed to be a cone, the latter algorithm consists of the basic iteration

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} \, \mathcal{L}_{\rho_k}(x, w^k) \quad \text{and} \quad \lambda^{k+1} := P_{\mathcal{K}^\circ}(w^k + \rho_k G(x^{k+1})).$$

The main result in this section is the following.

Theorem 8.8. *Let $w \in H$ and $\rho > 0$. Let $\hat{x}$ be a minimizer of $\mathcal{L}_\rho(\cdot, w)$ on X, let $\hat{\lambda} := P_{\mathcal{K}^\circ}(w + \rho G(\hat{x}))$, and $\hat{\mu} = \operatorname{prox}_{\rho q}(w)$. Then $\hat{\mu} = \hat{\lambda}$, and $\hat{x}$ is a point where the infimum in $q(\hat{\lambda})$ is attained.*

Proof. We first claim that $(\hat{x}, \hat{\lambda})$ is a saddle point of the convex-concave function

$$h : X \times \mathcal{K}^\circ \to \mathbb{R}, \quad h(x, \lambda) = \mathcal{L}(x, \lambda) - \frac{1}{2\rho}\|\lambda - w\|_H^2.$$

To verify this saddle-point property, note that the definition of $\hat{x}$ implies that

$$0 \in \partial_x \mathcal{L}_\rho(\hat{x}, w) = \partial f(\hat{x}) + G'(\hat{x})^* \hat{\lambda} = \partial_x h(\hat{x}, \hat{\lambda}).$$

Hence, $\hat{x}$ is a minimizer of the convex function $h(\cdot, \hat{\lambda})$. On the other hand, $h(\hat{x}, \cdot)$ is a quadratic function of the form

$$h(\hat{x}, \lambda) = \langle \lambda, G(\hat{x}) \rangle - \frac{1}{2\rho}\|\lambda - w\|_H^2 + c = -\frac{1}{2\rho}\|\lambda - w - \rho G(\hat{x})\|_H^2 + \tilde{c},$$

where $c, \tilde{c}$ are constants independent of λ. Therefore, the (unique) maximizer of $h(\hat{x}, \cdot)$ on $\mathcal{K}^\circ$ is $\hat{\lambda} = P_{\mathcal{K}^\circ}(w + \rho G(\hat{x}))$. This proves the saddle-point property of $(\hat{x}, \hat{\lambda})$.

A standard saddle point theorem (e.g., [13, Prop. 2.105]) now implies that $(\hat{x}, \hat{\lambda})$ satisfies

$$h(\hat{x}, \hat{\lambda}) = \max_{\lambda \in \mathcal{K}^\circ} h(\hat{x}, \lambda) = \max_{\lambda \in \mathcal{K}^\circ} \min_{x \in X} h(x, \lambda). \tag{8.12}$$

On the other hand, $\hat{\mu}$ is characterized by

$$\begin{aligned}
\hat{\mu} = \operatorname{prox}_{\rho q}(w) &= \operatorname*{argmax}_{\lambda \in \mathcal{K}^\circ} \left\{ q(\lambda) - \frac{1}{2\rho}\|\lambda - w\|_H^2 \right\} \\
&= \operatorname*{argmax}_{\lambda \in \mathcal{K}^\circ} \left\{ \inf_{x \in X} \mathcal{L}(x, \lambda) - \frac{1}{2\rho}\|\lambda - w\|_H^2 \right\} \\
&= \operatorname*{argmax}_{\lambda \in \mathcal{K}^\circ} \left\{ \inf_{x \in X} h(x, \lambda) \right\}.
\end{aligned}$$

Using (8.12), the uniqueness of $\hat{\mu}$ implies $\hat{\mu} = \hat{\lambda}$, and the statement follows. □

It follows from the above theorem that the safeguarded augmented Lagrangian method can be viewed as a proximal-point type algorithm, applied to the dual problem (8.11), where $\lambda^{k+1} = \operatorname{prox}_{\rho_k q}(w^k)$ for all k. This is actually a slightly modified proximal point method (with w^k instead of λ^k). For more details, the reader is referred to [131].

Chapter 9

Comments and Outlook

The results in the preceding chapters provide a fairly comprehensive picture of augmented Lagrangian methods for constrained optimization, variational inequalities, generalized Nash equilibrium problems, and quasi-variational inequalities in Banach spaces. In this chapter, we conclude the thesis by summarizing the main results, highlighting some essential assumptions, and discussing possible topics of future research.

Constrained Optimization

The formal study of constrained optimization problems began in Chapter 3, where a substantial amount of background material was presented. Two aspects which deserve a special mention are the consequences of second-order sufficient conditions in Section 3.1.3 and the analysis of asymptotic KKT-type conditions in Section 3.2.3. The discussions in these sections are clearly motivated by the augmented Lagrangian method, but they may be applicable to other optimization algorithms as well.

In Chapter 4, a significant effort was dedicated to a thorough description of augmented Lagrangian methods for constrained minimization problems in Banach spaces. This chapter contains a formal deduction of the algorithm (Section 4.1) as well as global (Section 4.2) and local (Section 4.3) convergence analyses. The main results in this chapter are probably Theorem 4.12 on global optimality, Theorem 4.16 on stationary points under the Robinson constraint qualification, Theorem 4.24 on the existence of local minimizers of the augmented problems, and Theorem 4.31 on the rate of convergence of the algorithm. A special mention should go to the analysis of C^2-cone reducible programs in Section 4.3.3, which has led to strengthened local convergence results as compared to those in the literature.

There are multiple aspects of the above theory which could lead to interesting developments or future research topics. One of the most obvious ideas would be an extension of the algorithmic framework to nonsmooth problems. The field of nonsmooth optimization has experienced a steady growth in the last decades, mainly due to the fact that nonsmooth structures arise naturally in many application contexts. These include, for instance, optimal control problems with nonsmooth differential equations [44]. Another class of examples arises in mathematical programming with complementarity constraints (MPCC) if the constraints are reformulated through the use of nonsmooth complementarity functions (see [70, 75, 214]). This problem class includes, by extension, bilevel optimization problems if the lower level problem is reformulated through first-order optimality (KKT) conditions. For such applications and for nonsmooth optimization in general, the availability of a general-purpose algorithm seems like a worthwhile goal which might be the subject of future research.

The aforementioned MPCCs can also be considered as a possible research avenue in their own right. It has already been found that augmented Lagrangian methods possess certain properties for these problems in finite dimensions [124]. It could be interesting to analyze the applicability of the algorithm for MPCCs in function spaces (see [216]), or for other problems with disjunctive constraints such as vanishing or switching constraints. This can also be motivated by the fact that some results in this thesis may be directly applicable to such problems, e.g., the results on global minimization in Section 4.2.2 and those using second-order conditions in Section 4.3.1.

Finally, another possible research topic are composite minimization problems with an objective of the form $f(x) + \varphi(G(x))$, where φ is a convex but nonsmooth function which is then regularized by an augmented Lagrangian-type approach, see [117, 119]. Problems of this form occur quite frequently in sparse optimization and image processing.

Variational Inequalities

The theory of variational inequalities was presented in tandem with constrained optimization in Chapter 3. Many concepts and properties presented there are either well-known or simple extensions of those from constrained optimization. Two aspects which should be mentioned specifically are the asymptotic KKT concepts from Section 3.2.3, which may lead to interesting developments for other optimization algorithms, and the primal-dual stability analysis from Section 3.2.4. For the latter, there are in particular the error bound equivalence (Theorem 3.56) and the resulting primal-dual error bounds (Corollaries 3.57 and 3.58) which are probably new in their given form.

A fairly comprehensive analysis of augmented Lagrangian methods for variational inequalities was conducted in Chapter 5. Many findings in this chapter are extensions of those for constrained optimization, but there are also some approaches tailored specifically to VIs. The main results are Theorem 5.8 for VIs with convex constraints, Theorem 5.12 on stationary points, and Theorem 5.17 on the rate of convergence. Moreover, the chapter also contains a section on generalized Nash equilibrium problems, their connection to VIs, and the optimality properties of augmented Lagrangian algorithms for these problems. The main results in this direction are Theorem 5.27 for Banach space problems and Theorem 5.31 for GNEPs in finite dimensions.

Two possible research topics in this context are variational inequalities of the second kind or, even more generally, equilibrium problems in the sense of Section 2.2.4. These problem types are strongly related to nonsmooth optimization (see above) since, in the smooth case, VIs of the second kind and equilibrium problems can be reformulated as ordinary VIs. It follows that an analysis of these problems would be useful, in particular, for nondifferentiable (generalized) Nash equilibrium problems.

Quasi-Variational Inequalities

A detailed study of quasi-variational inequalities was conducted in Chapter 6. Due to the parametric nature of the constraint set, the theory of QVIs is substantially more difficult than for standard VIs, and a continuity property involving Mosco-convergence was shown to be of fundamental importance for the existence of solutions and their stability under weak convergence. In addition, a formal approach to the first-order optimality (KKT) conditions of QVIs was presented.

The augmented Lagrangian method for QVIs was discussed in Section 6.2, with convergence results similar to those for standard VIs (albeit under different assumptions). The main results for QVIs in Banach spaces are Theorem 6.21 for QVIs with convex constraints and Theorem 6.26 on the primal-dual convergence to stationary points. In the finite-dimensional case, several refined results were presented in Section 6.3, in particular Theorem 6.32 for QVIs, Theorem 6.44 for GNEPs, and Theorem 6.37 on an exact penalty modification of the augmented Lagrangian technique.

Some natural research topics for QVIs are given by the numerous extensions of this problem class which exist in the literature, including set-valued mappings $F : X \rightrightarrows X^*$ (see [204]), QVIs of the second kind, or quasi-equilibrium problems (see [79, 155]). As hinted in the previous discussion of VIs, these classes are intimately related to nonsmooth optimization or Nash equilibrium problems, which is why they occur quite frequently in practical applications and corresponding models.

Final Comments

The study of practical algorithms for optimization-related problems is intimately related to the study of their theoretical properties. This fact is widely acknowledged in the optimization community, and it was one of the driving factors which eventually led to the development of this thesis. The material in the preceding chapters underlines the fact that the interplay between theory and practice is particularly important when dealing with optimization problems (variational inequalities, etc.) in general Banach spaces, and shows that a sound development of functional analysis and optimization theory is indispensable for a tractable approach to many practical problems. In particular, in order to attain the achieved level of generality, the combination of various ingredients from different branches of convex, functional, and variational analysis was necessary. With this in mind, it is the author's hope that the theory, practical results, and the numerous remarks and observations presented throughout this thesis will prove useful to other researchers.

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