Universal Locally Univalent Functions and Universal Conformal Metrics

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Commonly used Symbols

$\mathbb{C}$	the complex plane
$\hat{\mathbb{C}}$	the Riemann sphere $\mathbb{C} \cup \{\infty\}$
$\mathbb{D}$	the unit disc $\{z \in \mathbb{C} \,:\, \lvert z\rvert < 1\}$
$B_r(z_0)$	the open disc $\{z \in \mathbb{C} \,:\, \lvert z - z_0\rvert < r\}$
$K_r(z_0)$	the compact disc $\{z \in \mathbb{C} \,:\, \lvert z - z_0\rvert \le r\}$
$C(E)$	the set of all continuous functions $f\colon E \to \mathbb{C}$ for a subset $E \subseteq \mathbb{C}$
$\mathcal{H}(E)$	the set of all holomorphic functions defined in an open neighborhood of $E \subseteq \mathbb{C}$
$\mathcal{M}(E)$	the set of all holomorphic functions defined in an open neighborhood of $E \subseteq \mathbb{C}$
$\mathcal{B}(\Omega)$	$\{f \in \mathcal{H}(\Omega) \,:\, \sup_{z\in\Omega} \lvert f(z)\rvert \le 1\}$
$\mathcal{H}^\infty(\Omega)$	$\{f \in \mathcal{H}(\Omega) \,:\, \sup_{z\in\Omega} \lvert f(z)\rvert < \infty\}$
$\mathcal{H}_{\neq 0}(E)$	the set of all zero-free functions in $\mathcal{H}(E)$
$\mathrm{Aut}(\Omega)$	the group of all conformal automorphisms of a domain $\Omega \subseteq \mathbb{C}$
$\mathcal{G}_{lu}$	$\{f \in \mathcal{G} \,:\, f$ is locally univalent$\}$ for a set $\mathcal{G}$ of meromorphic functions
Δ	the Laplacian $\Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$
$\underline{\Delta}$	the lower generalized Laplacian
$h(E)$	the set of all harmonic functions defined in an open neighborhood of $E \subseteq \mathbb{C}$
$sh(E)$	the set of all subharmonic functions defined in an open neighborhood of $E \subseteq \mathbb{C}$
$h_\kappa(E)$	the set of all C^2 solutions of the curvature equation $\Delta u = \kappa e^{2u}$ defined in an open neighborhood of $E \subseteq \mathbb{C}$
$sh_\kappa(E)$	the set of all subsolutions of the curvature equation $\Delta u = \kappa e^{2u}$ defined in an open neighborhood of $E \subseteq \mathbb{C}$
$\Lambda_c(\Omega)$	the set of all regular conformal metrics with constant curvature $c \in \mathbb{R}$ on a domain Ω
$SK(\Omega)$	the set of all SK-metrics on a domain $\Omega \subseteq \mathbb{C}$
$SKC(\Omega)$	the set of all continuous SK-metrics on a domain $\Omega \subseteq \mathbb{C}$

κ_λ	$-\frac{\Delta \log \lambda}{\lambda^2}$, the (generalized) curvature of a conformal metric λ
$f^*\lambda$	the pullback of a conformal metric λ by a holomorphic function f
λ_Ω	the hyperbolic metric with curvature -1 on a hyperbolic domain Ω
$d_\Omega(z, w)$	the hyperbolic distance of $z, w \in \Omega$ on a hyperbolic domain Ω
$\lVert f \rVert_K$	$\max_{z \in K} \lvert f(z) \rvert$ for a compact set $K \subseteq \mathbb{C}$ and $f \in C(K)$
χ	the chordal metric on $\hat{\mathbb{C}}$
$\chi_K(f, g)$	$\max_{z \in K} \chi(f(z), g(z))$, where $K \subseteq \mathbb{C}$ is compact and $f, g \colon K \to \hat{\mathbb{C}}$ are continuous functions
$U_{K,\varepsilon}(f)$	$\{g \in \mathcal{U} \,:\, \lVert f - g \rVert_K < \varepsilon\}$, where $\mathcal{U}$ is a subset of $C(E)$ for some set $E \subseteq \mathbb{C}$, $K \subseteq \mathbb{C}$ is compact and $\varepsilon > 0$; the space $\mathcal{U}$ will always be specified by the context
$f', f^{(n)}$	the derivative, resp. the n-th derivative of a holomorphic function
$f^{[n]}$	the n-th iteration of a function f
S_f	the Schwarzian derivative $\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ of a meromorphic function f

1. Introduction

In this thesis we investigate universality problems for conformal metrics, where we understand universality with respect to families of pull-back operators. Constantly curved conformal metrics may be written with the help of locally uniformly functions, and we will also consider the existence of universal locally univalent functions. Our approach to proof universality theorems is based on a well known scheme: We want to combine Bairé's category theorem with an appropriate approximation theorem. In our case, we will first establish and then use Runge-type theorems for locally univalent functions and a local approximation result for conformal metrics.
The purpose of this chapter is to set up the stage. We summarize some central results on universal functions, Runge's theorem and conformal metrics and introduce some notation. Once this is achieved, we give a brief outline of this thesis.

1.1. Universal Functions

Let Ω be a domain in the complex plane $\mathbb{C}$ and let $\mathcal{H}(\Omega)$ be the space of all holomorphic functions on Ω. We think of $\mathcal{H}(\Omega)$ as a (closed) subspace of the Fréchet space $C(\Omega)$ of all complex-valued continuous functions on Ω equipped with the topology of locally uniform convergence. In 1929, Birkhoff [9] has shown the following:

Theorem A.
There exists a function $f \in \mathcal{H}(\mathbb{C})$ such that the set

$$\{z \mapsto f(z+n) \,:\, n \in \mathbb{N}\}$$

is dense in $\mathcal{H}(\mathbb{C})$.

In other words, there are functions in $\mathcal{H}(\mathbb{C})$ such that the sequence of translations $(f(\cdot + n))_n$ of f is *as divergent as possible*. In the context of complex analysis, this is probably the first result on universal functions. More generally a function $f \in \mathcal{H}(\Omega)$ is called *universal*, if the set $\{f \circ \phi \,:\, \phi \in \mathrm{Aut}(\Omega)\}$ is dense in $\mathcal{H}(\Omega)$, where $\mathrm{Aut}(\Omega)$ denotes the group of conformal automorphisms of Ω.

In 1941, Seidel and Walsh [50] established a related universality result for non-euclidean translations of the unit disc $\mathbb{D}$:

Theorem B.
There exists a function $f \in \mathcal{H}(\mathbb{D})$ and a sequence $(a_n) \subseteq \mathbb{D}$ such that

$$\left\{ z \mapsto f\left(\frac{z + a_n}{1 + \overline{a_n} z}\right) : n \in \mathbb{N} \right\}$$

is dense in $\mathcal{H}(\mathbb{D})$.

Theorem A and Theorem B and the Riemann mapping theorem show, that on any simply connected domain $\Omega \subseteq \mathbb{C}$ there exists a universal function. The corresponding problem for non-simply connected domains has been solved by Bernal-González and Montes-Rodríguez [8]:

Theorem C.
Let $\Omega \subseteq \mathbb{C}$ be a domain. The following assertions are equivalent:

(i) There exists a universal function $f \in \mathcal{H}(\Omega)$.

(ii) Ω is not conformally equivalent to $\mathbb{C} \setminus \{0\}$ and the group $\operatorname{Aut}(\Omega)$ is not compact.

Ruiz [16] has shown, that the set of all universal entire functions is a dense G_δ-subset of $\mathcal{H}(\Omega)$. Using a quite general setting, Gethner and Shapiro [23] and Erdmann [25] have independently shown that this is typical for universal functions.

The notion of universality has been modified for many other classes of holomorphic and meromorphic functions and beyond. We explicitly point out some results which we feel are the most relevant for this thesis. For a domain $\Omega \subseteq \mathbb{C}$ let $\mathcal{B}(\Omega) := \{f \in \mathcal{H}(\Omega) : \sup_{z \in \Omega} |f(z)| \leq 1\}$. In 1954, Heins [29] has shown that there are bounded universal functions:

Theorem D.
There exists a Blaschke product $B \colon \mathbb{D} \to \mathbb{D}$ such that the set

$$\{B \circ \phi : \phi \in \operatorname{Aut}(\mathbb{D})\}$$

is dense in $\mathcal{B}(\mathbb{D})$.

Pommerenke [46] has shown, that there are universal functions in the class S of all univalent functions $f \colon \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 0$ and $f'(0) = 1$.

Theorem E.
There exists a function $f \in S$ such that the set

$$\left\{ \frac{f \circ \phi - (f \circ \phi)(0)}{(f \circ \phi)'(0)} \,:\, \phi \in \operatorname{Aut}(\mathbb{D}) \right\}$$

is dense in S.

It is also known, that there are universal meromorphic functions. For a domain $\Omega \subseteq \mathbb{C}$ let $\mathcal{M}(\Omega)$ be the space of all meromorphic functions on Ω equipped with the topology of locally uniform convergence with respect to the chordal metric. As usual, the chordal metric on $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is defined by

$$\chi(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}$$

for $z, w \in \mathbb{C}$ and

$$\chi(z, \infty) = \chi(\infty, z) = \frac{1}{\sqrt{1 + |z|^2}}.$$

Chan [12] has shown the following:

Theorem F.
There exists a meromorphic entire function $f \in \mathcal{M}(\mathbb{C})$ with the following property: The set

$$\{z \mapsto f(z + n) \,:\, n \in \mathbb{N}\}$$

is dense in $\mathcal{M}(\Omega)$ for every domain $\Omega \subseteq \mathbb{C}$.

In recent years, universal functions for families of non-automorphisms have been studied by Bayart et al. [6] and by Große-Erdmann and Mortini [26]. We will briefly summarize their results in chapter 3, when we have introduced the necessary terminology.
Universal functions have been studied by more authors and a comprehensive overview would be too long for our purpose. Instead we refer the interested reader to the survey article [27]. However, we would like to explicitly point out one more result due to Herzog [33]. We denote the n-th derivative of a function $f \in \mathcal{H}(\Omega)$ by $f^{(n)}$.

Theorem G.
There exists a zero-free locally univalent function $f \in \mathcal{H}(\mathbb{C})$ such that the set $\{f^{(n)} : n \in \mathbb{N}\}$ is dense in $\mathcal{H}(\mathbb{C})$.

1.2. Runge's Theorem

The existence of universal functions is usually proven with the help of a suitable approximation theorem. In many cases, Runge's theorem is an adequate choice.

Definition 1.1. (a) Let $K \subseteq \mathbb{C}$ be compact. A hole of K is a bounded connected component of $\mathbb{C} \setminus K$.

(b) Let $\Omega \subseteq \mathbb{C}$ be a domain and $K \subseteq \Omega$ compact. We say that K is $\mathcal{O}$-convex in Ω, if no hole of K is relatively compact in Ω.

(c) Let $\Omega \subseteq \mathbb{C}$ be a domain and $K \subseteq \Omega$ compact. The $\mathcal{O}$-convex hull $\hat{K}$ of K is the union of K with all holes of K which are relatively compact in Ω.

Let $K \subseteq \mathbb{C}$ be compact. For $f \in C(K)$, we let $\|f\|_K = \max_{z\in K} |f(z)|$. Similarly, for continuous functions $f, g\colon K \to \hat{\mathbb{C}}$ we write $\chi_K(f, g) := \max_{z\in K} \chi(f(z), g(z))$. Finally, for a set $E \subseteq \mathbb{C}$ we write $\mathcal{H}(E)$ (resp.
$\mathcal{M}(E)$) for the set of all holomorphic (resp. meromorphic) functions, which are defined on an open neighborhood of E.

Theorem H (Runge's Theorem).
Let $\Omega \subseteq \mathbb{C}$ be a domain, $K \subseteq \Omega$ compact and $\varepsilon > 0$.

(a) Suppose that K is in addition $\mathcal{O}$-convex in Ω. Then for every $f \in \mathcal{H}(K)$ there exists $g \in \mathcal{H}(\Omega)$ such that $\|f - g\|_K < \varepsilon$. The function g can taken to be a rational function.

(b) For every $f \in \mathcal{M}(K)$ there exists a rational function $g \in \mathcal{M}(\Omega)$ such that $\chi_K(f, g) < \varepsilon$.

Functions of the form $z \mapsto \frac{1}{z-b}$ with $b \notin K$ and the maximum-principle show, that the assumption that K is $\mathcal{O}$-convex is necessary in the holomorphic case. Runge's theorem has been generalized by many authors. One of the best known of these generalizations is probably Mergelayn's theorem [40]:

Theorem I (Mergelayn's Theorem).
Let $K \subseteq \mathbb{C}$ be a compact set with connected complement, $f \in C(K) \cap \mathcal{H}(K^\circ)$ and $\varepsilon > 0$. Then there exists a polynomial $p \in \mathcal{H}(\mathbb{C})$ with $\|f - p\|_K < \varepsilon$.

There are versions of Runge's theorem for harmonic functions (see i.e. [20, Corollary 1.16]). However, we like to point out another approximation result for harmonic functions due to Keldysh [34]. For $E \subseteq \mathbb{C}$ we let E° be the interior of E and $h(E)$ be the set of functions harmonic in an open neighborhood of E.

Theorem J (Keldysh's theorem).
Let $K \subseteq \mathbb{C}$ be compact. The following are equivalent:

(i) $\mathbb{C} \setminus K$ *and* $\mathbb{C} \setminus K^\circ$ *are thin at the same points.*

(ii) $h(K)$ *is dense in* $C(K) \cap h(K^\circ)$.

1.3. Conformal Metrics

Definition 1.2 (Conformal metric). Let $\Omega \subseteq \mathbb{C}$ be a domain. A conformal pseudo-metric is a non-negative form $\lambda(z)\,|dz|$ on Ω which is not constant 0. If the form $\lambda(z)\,|dz|$ is positive, we simply call $\lambda(z)\,|dz|$ a conformal metric. We say that $\lambda(z)\,|dz|$ is regular, if its density function is C^2 in the set $\{z \in \Omega : \lambda(z) > 0\}$.

On a complex domain Ω we can think of the identity function as one globally defined coordinate. This allows us to identify a conformal metric $\lambda(z)\,|dz|$ with its density function $\lambda\colon \Omega \to [0, \infty)$ and thus we will not always make a strict distinction between a metric and its density. Note however, that the definition still makes sense on Riemann surfaces. Then of course, one has to make the distinction between the metric and its (locally defined) density.

The generalized lower Laplace-operator of a function $u\colon \Omega \to \mathbb{R}$ is defined by

$$\underline{\Delta} u(z) := \liminf_{r \to 0} \frac{4}{r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt - u(z) \right).$$

If u is C^2 in an neighborhood of z, then $\underline{\Delta} u(z) = \Delta u(z)$, where we denote, as usual, the Laplacian by $\Delta := \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$.

Definition 1.3 (Gaussian curvature). Let λ be a conformal pseudo-metric on a complex domain Ω. The (generalized) curvature of λ at a point $z \in \Omega$ with $\lambda(z) \neq 0$ is defined by

$$\kappa_\lambda(z) := -\frac{\underline{\Delta} \log \lambda(z)}{\lambda^2(z)}. \tag{1.3.1}$$

For $c \in \mathbb{R} \setminus \{0\}$, by scaling a metric λ by $|c|$, we get the metric $|c|\lambda$ with curvature

$$c^2 \kappa_{|c|\lambda} = \kappa_\lambda. \tag{1.3.2}$$

Definition 1.4 (Pullback). Let Ω_1, Ω_2 be two complex domains, λ a conformal pseudo-metric on Ω_2 and $f\colon \Omega_1 \to \Omega_2$ a non-constant holomorphic function. We then can define a pseudo-metric on Ω_1 by

$$f^*\lambda(z)\,|dz| := \lambda(f(z))|f'(z)|\,|dz|$$

and call $f^*\lambda$ the pullback of λ under f.

If λ is regular, then so is $f^*\lambda$ and a simple calculation shows that $\kappa_{f^*\lambda}(z) = \kappa_\lambda(f(z))$ wherever $f'(z) \neq 0$ and $\lambda(f(z)) \neq 0$. Thus constantly curved metrics are of particular interest in complex analysis, since then the curvature is invariant under pull-backs. For a domain $\Omega \subseteq \mathbb{C}$ and $c \in \mathbb{R}$ we let $\Lambda_c(\Omega)$ be the set of all regular conformal metrics on Ω with constant curvature c. We can think of $\Lambda_c(\Omega)$ as a subset of $C(\Omega)$ and thus $\Lambda_c(\Omega)$ is equipped with the compact-open topology. Note that (1.3.2) shows, that for constantly curved metrics it is enough to consider the three cases $c \in \{-1, 0, 1\}$.
In his seminal work, Heins [30] introduced a class of conformal metrics, which in many ways relates to the set $\Lambda_{-1}(\Omega)$ the same way as subharmonic functions relate to harmonic functions.

Definition 1.5 (SK-Metric)**.** A conformal metric $\lambda(z)\,|dz|$ on a domain $\Omega \subseteq \mathbb{C}$ is called SK-metric if the density λ is upper semi-continuous with (generalized) curvature $\kappa_\lambda \leq -1$. We let $SK(\Omega)$ be the set of all SK-metrics on a domain Ω and $SKC(\Omega) := SK(\Omega) \cap C(\Omega)$ the set of all continuous SK-metrics on Ω.

We let $\lambda_{\mathbb{D}}(z) := \frac{2}{1-|z|^2}\,|dz|$ be the hyperbolic metric on the unit disc with curvature -1. Generalizing the Schwarz-Pick Lemma, Ahlfors [3] observed that $\lambda_{\mathbb{D}}$ has an important extremal property, which often is referred to as "Ahlfors' Lemma". Heins [30] noted that Ahlfors' Lemma is true in the context of SK-metrics, and the case of equality has been independently studied by Minda [41] and Roydent [48].

Theorem K (Ahlfors' Lemma)**.**
For every $\lambda \in SK(\mathbb{D})$ we have $\lambda \leq \lambda_{\mathbb{D}}$ in $\mathbb{D}$. If equality holds for one point $z_0 \in \mathbb{D}$, then equality holds throughout $\mathbb{D}$.

Note that Schwarz-Picks' Lemma tells us that for a function $f \in \mathcal{B}(\mathbb{D})$ we have $f^*\lambda_{\mathbb{D}} = \lambda_{\mathbb{D}}$ if and only if $f \in \operatorname{Aut}(\mathbb{D})$. Thus Ahlfors' Lemma has the following useful immediate consequence, which we will also refer to as "Ahlfors' Lemma":

Proposition 1.6.
Let $\lambda \in SK(\mathbb{D})$ and $f \in \mathcal{B}(\mathbb{D})$. Suppose that there exists $z_0 \in \mathbb{D}$ such that $f^*\lambda(z_0) = \lambda_{\mathbb{D}}(z_0)$. Then $\lambda = \lambda_{\mathbb{D}}$ and $f \in \operatorname{Aut}(\mathbb{D})$.

Ahlfors' Lemma tells us, that $SK(\mathbb{D})$ has a maximal element, namely $\lambda_{\mathbb{D}}$. Using Perron's method, it can be shown that the same is true for any domain $\Omega \subseteq \mathbb{C}$, provided $SK(\Omega) \neq \emptyset$.

Definition 1.7. Let $\Omega \subseteq \mathbb{C}$ be a domain. We call a domain $\Omega \subseteq \mathbb{C}$ hyperbolic if $SK(\Omega) \neq \emptyset$. For a hyperbolic domain, we let λ_Ω be the maximal element of $SK(\Omega)$ and call λ_Ω the hyperbolic metric on Ω.

It is well-known that the following are equivalent:

(a) Ω is hyperbolic;

(b) Ω has at least two boundary points in $\mathbb{C}$;

(c) There exists a holomorphic covering mapping $f\colon \mathbb{D} \to \Omega$.

On a simply connected domain $\Omega \subsetneq \mathbb{C}$, the hyperbolic metric is given by $\lambda_\Omega = f^*\lambda_\mathbb{D}$, where $f\colon \Omega \to \mathbb{D}$ is a conformal mapping (which exists by Riemann's mapping theorem). The hyperbolic metric induces a distance d_Ω on Ω, which is given by

$$d_\Omega(u,v) = \inf_\gamma \int_\gamma \lambda_\Omega(z)\,|dz|$$

where the infimum is taken over all paths γ in Ω which connect u with v. We call d_Ω the hyperbolic distance on Ω. The hyperbolic distance is conformally invariant in the following sense: Let $f\colon \Omega_1 \to \Omega_2$ be a conformal mapping, then $d_{\Omega_1}(z,w) = d_{\Omega_2}(f(z), f(w))$ for all $z, w \in \Omega_1$.

1.4. Outline and Main Results

Now we can give a short outline of this thesis. A function $f \in \mathcal{M}(\Omega)$ is locally univalent if $f'(z) \neq 0$ for all $z \in \Omega$ with $f(z) \neq \infty$ and if all poles of f are simple. If $\mathcal{G}$ is a set of meromorphic functions, we let

$$\mathcal{G}_{lu} := \{f \in \mathcal{G} \,:\, f \text{ is locally univalent}\}.$$

For example $\mathcal{M}_{lu}(\Omega)$ is the set of all meromorphic locally univalent functions in a domain Ω, and for a compact set $K \subseteq \mathbb{C}$, $\mathcal{H}_{lu}(K)$ is the set of all functions, which are holomorphic and locally univalent in an open neighborhood of K.

In Chapter 2 we establish some auxiliary approximation results. We proof Runge-type theorems for locally univalent holomorphic and meromorphic functions and a version of Keldysh's local approximation theorem for solutions of the curvature equation $\Delta u = \kappa e^{2u}$ (we have to put some restrictions on the function κ, to ensure the existence of solutions).

In Chapter 3 we proof universality results for locally univalent functions. Among the lines, we will proof the following two theorems, which maybe are the main results of Chapter 3.

Theorem 1.8.
Let $\Omega \subseteq \mathbb{C}$ be a complex domain. Then the following assertions are equivalent:

(i) There exists $f \in \mathcal{H}_{lu}(\Omega)$ such that

$$\{f \circ \phi \,:\, \phi \in \mathrm{Aut}(\Omega)\}$$

is dense in $\mathcal{H}_{lu}(\Omega)$.

(ii) Ω is not conformally equivalent to $\mathbb{C} \setminus \{0\}$ and $\mathrm{Aut}(\Omega)$ is not compact.

Theorem 1.9.
There exists $f \in \mathcal{B}_{lu}(\mathbb{D})$ such that

$$\{f \circ \phi \,:\, \phi \in \mathrm{Aut}(\mathbb{D})\}$$

is dense in $\mathcal{B}_{lu}(\mathbb{D})$.

Note that these results may be viewed as direct analogues to the Theorems C and D for locally univalent functions. We complement our results with some (hopefully) instructive examples.
In Chapter 4 we use these theorems to obtain universality results for conformal metrics. The central result of that chapter and maybe of this thesis is the following:

Theorem 1.10.
Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $c \in \mathbb{R}$. If $c < 0$ then suppose in addition that $\Omega \neq \mathbb{C}$. Then there exists $\lambda \in \Lambda_c(\Omega)$ such that the set

$$\{\phi^* \lambda \,:\, \phi \in \mathrm{Aut}(\Omega)\}$$

is dense in $\Lambda_c(\Omega)$.

We also consider universality in the set of (continuous) SK-metrics. Many examples for universal locally univalent functions are also relevant for universal conformal metrics. However we will give additional examples which aim to highlight some differences between universal locally univalent functions and universal conformal metrics.
Throughout this thesis we will discuss numerous related open problems. Some parts of the present thesis have been published by Roth and the author [45].

1.5. A Word on Notation

We have to introduce some more basic notation, which will be used throughout this thesis. For $r > 0$ and $z_0 \in \mathbb{C}$ we let $B_r(z_0)$ resp. $K_r(z_0)$ be the open resp. closed disc with center z_0 and radius r.
For a subset E of a topological space X, we let $\overline{E}$, E° and ∂E be the closure, interior and boundary of E. If F is another subset of X, we write $E \setminus F$ for the relative complement of F in E.
A path γ in a complex domain Ω is a piecewise differentiable function $\gamma\colon [0,1] \to \Omega$. The trace of γ will be denoted by $\operatorname{tr}(\gamma)$. We call γ closed if $\gamma(0) = \gamma(1)$. For a closed path γ and $z \notin \operatorname{tr}(\gamma)$ we let $n(\gamma, z)$ be the winding number of γ around z.
Let Ω be a domain in $\mathbb{C}$, $\mathcal{U} \subseteq C(\Omega)$, $K \subseteq \Omega$ compact $f \in \mathcal{U}$ and $\varepsilon > 0$. Throughout this thesis, we denote by $U_{K,\varepsilon}(f)$ the basic K-ε-neighborhood of f in the compact open-topology on $\mathcal{U}$, that is

$$U_{K,\varepsilon}(f) \coloneqq \{g \in \mathcal{U} \,:\, \|f - g\|_K < \varepsilon\}.$$

The set $\mathcal{U}$ will always be specified by the context.

2. Approximation Results

2.1. Runge-Type Theorems for Locally Univalent Functions

Our first goal in this section is to establish "Runge-type" theorems for locally univalent functions. The strategy for the proof is the following: We first approximate the derivative of a locally univalent function $f \in \mathcal{H}_{lu}(K)$ in a way that the approximating function $g \in \mathcal{H}(\Omega)$ is zero-free and has an anti-derivative G. Then on every connected component of K, up to an additive constant, G is close to f. For different connected components this constant might be different. Our first topological proposition addresses this little complication. We let $\mathcal{H}_{\neq 0}(E)$ be the set of all zero-free functions in $\mathcal{H}(E)$.

Proposition 2.1.
Let Ω be a domain in $\mathbb{C}$, K a compact $\mathcal{O}$-convex set in Ω and $\varepsilon > 0$.

(a) Suppose that $f \in \mathcal{H}_{\neq 0}(K)$. Then there exists a connected compact $\mathcal{O}$-convex set M in Ω with piecewise differentiable boundary ∂M such that $K \subseteq M^\circ$ and a function $g \in \mathcal{H}_{\neq 0}(M)$ with $\|f - g\|_K < \varepsilon$.

(b) Suppose $f \in \mathcal{M}_{lu}(K)$. Then there exists a compact $\mathcal{O}$-convex set M in Ω with connected interior such that $K \subseteq M^\circ$ and a function $g \in \mathcal{M}_{lu}(M)$ with $\chi_K(f, g) < \varepsilon$. If $f \in \mathcal{H}_{lu}(K)$, then we can choose g so that $g \in \mathcal{H}_{lu}(M)$ with $\|f - g\|_K < \varepsilon$.

Proof. We only prove part (a); the proof of part (b) is similar. By the theorems of Runge and Hurwitz, there exists a rational function $g \in \mathcal{H}(\Omega) \cap \mathcal{H}_{\neq 0}(K)$ such that $\|f - g\|_K < \varepsilon$. Let $z_1, \ldots, z_N$ be the zeros of g in Ω. Since K is $\mathcal{O}$-convex, there exist paths $\gamma_j \colon [0, 1) \to \Omega \setminus K$ with $\gamma_j(0) = z_j$, $\gamma_j(t) \to \partial\Omega$ for $t \to 1$ and such that $W := \Omega \setminus (\mathrm{tr}\,(\gamma_1) \cup \cdots \cup \mathrm{tr}\,(\gamma_N))$ is connected. Note that W is open and $K \subseteq W$. Let (M_n) be a compact exhaustion of W with connected compact $\mathcal{O}$-convex sets in W, such that ∂M_n is piecewise differentiable for each $n \in \mathbb{N}$. Since a compact set in W is $\mathcal{O}$-convex if and only if it is $\mathcal{O}$-convex in Ω, we can take $M = M_n$ with $n \in \mathbb{N}$ sufficiently large so that $M = M_n \supset K$. ■

Theorem 2.2.
Let Ω be a domain in $\mathbb{C}$, K a compact $\mathcal{O}$-convex set in Ω and $g \in \mathcal{H}_{\neq 0}(K)$. Then there exists a sequence $(f_m) \subseteq \mathcal{H}_{\neq 0}(\Omega)$ such that $\lim_{m\to\infty} f_m = g$ uniformly on K and

$$\int_\gamma f_m(z)dz = \int_\gamma g(z)dz \tag{2.1.1}$$

for every closed curve γ in K and every $m \in \mathbb{N}$.

Proof. By Proposition 2.1 (a) we may assume that K is connected and ∂K is piecewise differentiable. Let $D_1, \ldots, D_n$ be the bounded connected components of $\mathbb{C} \setminus K$. For $j = 1, \ldots, n$ choose $z_j \in D_j \setminus \Omega$ and let γ_j be a parametrization of the positively oriented boundary ∂D_j. Then $n(\gamma_k, z_j) = \delta_{kj}$. The connectedness of K implies that $\Gamma := \bigcup_{k=1}^n \operatorname{tr}(\gamma_k)$ is a compact $\mathcal{O}$-convex set in Ω. Since every closed curve in K is homologous to a linear combination of the curves $\gamma_1, \ldots, \gamma_n$ with integer coefficients, it suffices to find a sequence $(f_m) \subseteq \mathcal{H}_{\neq 0}(\Omega)$ such that $\lim_{m\to\infty} f_m = g$ uniformly on K and equation (2.1.1) holds for $\gamma = \gamma_k$ for every $k = 1, \ldots, n$.
Now for any $j = 1, \ldots, n$ Runge's Theorem implies that there is a sequence $(w_{j,m})_m$ in $\mathcal{H}(\Omega)$ with

$$\lim_{m\to\infty} w_{j,m}(z) = \frac{1}{g(z)(z - z_j)}$$

uniformly on Γ. In particular,

$$\lim_{m\to\infty} \left(\int_{\gamma_k} w_{j,m}(z)g(z)dz \right)_{k,j=1,\ldots,n} = E_n;$$

where $E_n \in \mathbb{C}^{n\times n}$ is the identity matrix. Hence we can find a $\mu \in \mathbb{N}$ such that the matrix

$$A := \left(\int_{\gamma_k} w_{j,\mu}(z)g(z)dz \right)_{k,j=1,\ldots,n}$$

is non-singular.
We can apply a version of Runge's theorem for zero-free functions [43, Theorem 6.3.1], so that there exists a sequence (g_m) in $\mathcal{H}_{\neq 0}(\Omega)$ such that $\lim_{m\to\infty} g_m = g$ uniformly on K. Consider the functions

$$\psi_k\colon \mathbb{C}^n \to \mathbb{C}, \qquad (s_1, \ldots, s_n) \mapsto \int_{\gamma_k} \exp\left(\sum_{j=1}^n s_j w_{j,\mu}(z) \right) g(z)dz,$$

$$\psi_{k,m}\colon \mathbb{C}^n \to \mathbb{C}, \qquad (s_1, \ldots, s_n) \mapsto \int_{\gamma_k} \exp\left(\sum_{j=1}^n s_j w_{j,\mu}(z) \right) g_m(z)dz,$$

and the entire functions $\psi, \psi_m \colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\psi(s) := (\psi_1(s), \ldots, \psi_n(s)),$$
$$\psi_m(s) := (\psi_{1,m}(s), \ldots, \psi_{n,m}(s)).$$

Then $\lim_{m\to\infty} \psi_m = \psi$ locally uniformly on $\mathbb{C}^n$ and $D\psi(0) = A$ is non-singular. Hence, there exists a sequence $(s_m) = (s_{1,m}, \ldots, s_{n,m})$ in $\mathbb{C}^n$ with $\lim_{m\to\infty} s_m = 0$ and $\psi_m(s_m) = \psi(0)$ for all but finitely many $m \in \mathbb{N}$. If we put

$$f_m(z) = \exp\left(\sum_{j=1}^{n} s_{j,m} w_{j,\mu}(z)\right) g_m(z),$$

then $f_m \in \mathcal{H}_{\neq 0}(\Omega)$, $\lim_{m\to\infty} f_m = g$ uniformly on K and (2.1.1) holds for γ_k for every $k = 1, \ldots, n$. ■

Theorem 2.3.
Let Ω be a domain in $\mathbb{C}$ and let K be a compact $\mathcal{O}$-convex set. Then $\mathcal{H}_{lu}(\Omega)$ is dense in $\mathcal{H}_{lu}(K)$.

Proof. By Proposition 2.1 (b) we may assume that $f \in \mathcal{H}_{lu}(M)$ for a connected $\mathcal{O}$-convex compact set M of Ω such that $K \subseteq M^\circ$. Hence we can apply Theorem 2.2 to $f' \in \mathcal{H}_{\neq 0}(M)$, so there exists a sequence $(g_n) \subseteq \mathcal{H}_{\neq 0}(\Omega)$ with $\lim_{n\to\infty} g_n = f'$ uniformly on K and

$$\int_\gamma g_n(z)dz = \int_\gamma f'(z)dz = 0$$

for every closed curve γ in M.
Now choose a compact exhaustion $(K_k)_k$ of Ω by connected $\mathcal{O}$-convex sets in Ω with smooth boundaries and such that $K_1 = M$. Suppose we have fixed arbitrary numbers $\varepsilon > 0$, $k \in \mathbb{N}$ and a function $h \in \mathcal{H}_{\neq 0}(\Omega)$ with $\int_\gamma h(z)dz = 0$ for every closed curve γ in K_k. Then by [28, Lemma 4] there exists a function $v \in \mathcal{H}(\Omega)$ with $\|v\|_{K_k} < \varepsilon$ and $\int_\gamma e^{v(z)}h(z)dz = 0$ for every close curve γ in K_{k+1}. From this fact and an obvious induction argument, we can deduce that there exists a sequence $(v_{n,k})_k$ in $\mathcal{H}(\Omega)$ with $\|v_{n,k}\|_{K_k} < \frac{1}{2^k n}$ such that for every closed curve γ in K_k, we have

$$\int_\gamma \exp\left(\sum_{j=1}^{k} v_{n,j}(z)\right) g_n(z)dz = 0.$$

We define a holomorphic function $w_n \in \mathcal{H}(\Omega)$ by

$$w_n(z) := \sum_{j=1}^{\infty} v_{n,j}(z).$$

Clearly, we have $\|w_n\|_M < \frac{1}{n}$ and

$$\int_\gamma e^{w_n(z)} g_n(z)dz = 0$$

for every closed curve γ in Ω. This means that for fixed $z_0 \in K$ and for each $n \in \mathbb{N}$ there is an anti-derivative $G_n \in \mathcal{H}(\Omega)$ of $e^{w_n} g_n$ with $G_n(z_0) = f(z_0)$. By construction, $G_n \in \mathcal{H}_{lu}(\Omega)$. Since M is connected and $\lim_{n\to\infty} G_n' = f'$ uniformly on M we can conclude $\lim_{n\to\infty} G_n = f$ uniformly on K. ■

One might ask if other approximations theorems have locally univalent analogues. We explicitly state one associated problem:

Problem 1.
Is there a locally univalent version of Mergelyan's theorem: Let $K \subseteq \mathbb{C}$ with connected complement. For every $f \in C(K) \cap \mathcal{H}_{lu}(K^\circ)$ and every $\varepsilon > 0$, is there a function $g \in \mathcal{H}_{lu}(\mathbb{C})$ such that $\|f - g\|_K < \varepsilon$?

This problem merits some comments: First note that the strategy we used to proof Theorem 2.3 does not work for this problem any more, since a function $f \in C(K) \cap \mathcal{H}_{lu}(K^\circ)$ may have "critical points" on the boundary of K, or even worse, the derivative of f in K° might have no continuous extension to the boundary ∂K. Secondly, recently Andersson [4] has asked a similar question on zero-free approximation of functions in $C(K) \cap \mathcal{H}_{\neq 0}(K^\circ)$. The problem has been treated and partially answered by other authors, see for example in [14], [35] and [21].
We like to point out that Runge-type approximation of locally univalent functions is still possible on Riemann surfaces. Note that the definition of a compact $\mathcal{O}$-convex set still makes sense on Riemann surfaces.

Theorem 2.4.
Let R be a non-compact Riemann surface and $K \subseteq R$ compact and $\mathcal{O}$-convex. Then $\mathcal{H}_{lu}(R)$ is dense in $\mathcal{H}_{lu}(K)$.

In the following proof, convergence of a sequence of 1-forms means convergence in local coordinates.

Proof. The proof of Proposition 2.1 is valid for Riemann surfaces, thus we can assume that K is (path-)connected. Let $f \in \mathcal{H}_{lu}(K)$. Then df is a holomorphic zero-free one-form defined in a neighborhood of K and $\int_\gamma df = 0$ for every closed curve in K.
By the main result in [28] there exists a zero-free holomorphic one-form ω on R such that $\int_\gamma \omega = 0$ for every closed curve γ in R. This allows us to apply the main result of

[39] which shows, that there exists a sequence (ω_n) of holomorphic 1-forms on $\mathbb{R}$ such that $\lim_{n\to\infty} \omega_n = \omega$ on K and $\int_\gamma \omega_n = 0$ for every closed curve γ in R. Fix $z_0 \in K$. Then for every $n \in \mathbb{N}$ there exists $g_n \in \mathcal{H}_{lu}(R)$ such that $dg_n = \omega_n$ and $g_n(z_0) = f(z_0)$. Thus we have $\lim_{n\to\infty} g_n = f$ uniformly on K. ■

Remark 2.5. Theorem 2.4 shows, that every non-compact Riemann surface R carries a rich supply of locally univalent functions $f\colon R \to \mathbb{C}$. Note that it is already a deep result due to Gunning and Narasimhan [28] that every non-compact Riemann surface carries at least one locally univalent holomorphic function. Recently, Forstnerič [17] has extended the Gunning-Narasimhan theorem to Stein manifolds.
The result of Majcen [39] used in the proof above is a Runge-type theorem for holomorphic 1-forms on Stein manifolds. Its proof uses multiple-variables techniques and it might be interesting to know if there is a simpler proof for the one-dimensional case.

As a first application of Theorem 2.3 we can give a large class of domains Ω, for which $\mathcal{H}_{lu}^\infty(\Omega) := \{f \in \mathcal{H}_{lu}(\Omega) : \sup_{z\in\Omega} |f(z)| < \infty\}$ is dense in $\mathcal{H}_{lu}(\Omega)$. Domains with this property will occur in our investigation of universal locally univalent functions.

Corollary 2.6.
For $j = 1, \dots, n$ let $a_j \in \mathbb{C}$ and $r_j > \mathbb{R}$ such that the discs $K_{r_j}(a_j)$ are pairwise disjoint. Let $R > 0$ such that $K_{r_j}(a_j) \subset B_R(0)$ for every $j \in \{1, \dots, n\}$ and let

$$\Omega := B_R(0) \setminus \left(\bigcup_{j=1}^n K_{r_j}(a_j) \right).$$

Then $\mathcal{H}_{lu}^\infty(\Omega)$ is dense in $\mathcal{H}_{lu}(\Omega)$.

Proof. Let $\mathcal{U} \subseteq \mathcal{H}_{lu}(\Omega)$ open and choose $f \in \mathcal{U}$. There exists a compact $\mathcal{O}$-convex set $K \subseteq \Omega$ and $\varepsilon > 0$ such that $U_{K,\varepsilon}(f) \subseteq \mathcal{U}$. Note that K is also an $\mathcal{O}$-convex set in $\mathbb{C} \setminus \{a_1, \dots, a_n\}$. Theorem 2.3 shows us, that there exists $g \in \mathcal{H}_{lu}(\mathbb{C} \setminus \{a_1, \dots, a_n\})$ such that $\|f - g\|_K < \varepsilon$. Then obviously $g \in \mathcal{U} \cap \mathcal{H}_{lu}^\infty(\Omega)$. ■

We now consider approximation of meromorphic locally univalent functions. Our strategy is similar to the holomorphic case: We first want to use Runge's theorem to approximate *the right* derivative and then use an *integration* process. For meromorphic locally univalent functions, a suitable "derivative" is the *Schwarzian derivative*

$$S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of a function $f \in \mathcal{M}(\Omega)$. It is well known that $f \in \mathcal{M}_{lu}(\Omega)$ if and only if $S_f \in \mathcal{H}(\Omega)$. The integration process involved in the proof then is solving the differential equation $S_f = g$ for given $g \in \mathcal{H}(\Omega)$. For non-simply connected domains, this causes some troubles. As far as we know there is no (useful) way to tell for what functions g there exists a solution of the associated Schwarzian differential equation.

Theorem 2.7.
Let Ω be a simply connected domain in $\mathbb{C}$ and $K \subseteq \Omega$ compact with connected complement. Then $\mathcal{M}_{lu}(\Omega)$ is dense in $\mathcal{M}_{lu}(K)$.

Proof. By Proposition 2.1 (b) we may assume that $f \in \mathcal{M}_{lu}(M)$ for some compact $\mathcal{O}$-convex set $M \subseteq \mathbb{C}$ such that $K \subseteq M^\circ$ and such that the interior of M is connected. Since f is locally univalent in a neighborhood of M, its Schwarzian derivative S_f is holomorphic there, so $S_f \in \mathcal{H}(M)$. According to some basic facts about complex differential equations, see e.g. [37, Theorem 6.1], we can recover f from S_f by writing f as the quotient

$$f = \frac{u_1}{u_2}$$

of two linearly independent solutions $u_1, u_2 \in \mathcal{H}(M)$ of the homogeneous linear differential equation

$$w'' + \frac{1}{2} S_f \cdot w = 0. \tag{2.1.2}$$

Since $S_f \in \mathcal{H}(M)$ and $\mathbb{C} \setminus M$ has no bounded components, the classical Runge theorem shows that there exist polynomials $p_n \colon \mathbb{C} \to \mathbb{C}$ such that

$$p_n \to S_f \qquad \text{uniformly on } M.$$

We now consider the homogeneous linear differential equations corresponding to the polynomials p_n. Fix $z_0 \in M^\circ$ with $u_1(z_0) \neq 0$ and let $v_n \in \mathcal{H}(\mathbb{C})$ be the unique solution of the initial value problem

$$v_n'' + \frac{1}{2} p_n \cdot v_n = 0, \quad v_n(z_0) = u_1(z_0), \quad v_n'(z_0) = u_1'(z_0).$$

Then we clearly have

$$v_n = u_1(z_0) + u_1'(z_0)(z - z_0) - \frac{1}{2} \int_{z_0}^{z} (z - \xi) p_n(\xi) v_n(\xi) d\xi, \quad z \in \mathbb{C}.$$

Hence a standard application of Gronwall's lemma [37, Lemma 5.10] shows that the sequence (v_n) is locally bounded in G. We are therefore in a position to apply Montel's theorem and conclude that $\{v_n : n \in \mathbb{N}\}$ is a normal family. Clearly, every subsequential limit function $v \in \mathcal{H}(M)$ of (v_n) is a solution of (2.1.2) with $v(z_0) = u_1(z_0)$ and $v'(z_0) = u_1'(z_0)$. By uniqueness of this solution, we conclude $v \equiv u_1$. Consequently, we have

$$v_n \to u_1 \qquad \text{uniformly on } M.$$

For the unique solution $w_n \in \mathcal{H}(\mathbb{C})$ of the initial value problem

$$w_n'' + \frac{1}{2} p_n \cdot w_n = 0, \quad w_n(z_0) = u_2(z_0), \quad w_n'(z_0) = u_2'(z_0),$$

we arrive in a similar way at

$$w_n \to u_2 \qquad \text{uniformly on } M.$$

We claim that v_n and w_n are linearly independent for large $n \in \mathbb{N}$. For this purpose we consider the Wronskian

$$W(h,g) = hg' - h'g \in \mathcal{H}(G) \qquad \text{for } f, g \in \mathcal{H}(G).$$

Since u_1 and u_2 are solutions of the differential equation (2.1.2), there is a constant $\lambda \in \mathbb{C}$ such that $W(u_1, u_2)(z) = \lambda$ for all $z \in M$, [37, Proposition 1.4.8]. In a similar way, we see that for each $n \in \mathbb{N}$ there is $\lambda_n \in \mathbb{C}$ such that $W(v_n, w_m)(z) = \lambda_n$ for all $z \in \mathbb{C}$. By what we have already shown, $\lambda_n \to \lambda$ as $n \to \infty$. Since u_1 and u_2 are linearly independent, we have $\lambda \neq 0$, see [37, Proposition 1.4.2]. Hence $\lambda_n \neq 0$, so v_n and w_n are linearly independent for all but finitely many $n \in \mathbb{N}$.
We can therefore apply Theorem 6.1 in [37] which implies that

$$g_n := \frac{v_n}{w_n} \in \mathcal{M}_{lu}(\mathbb{C}).$$

Since $v_n \to u_1$ and $w_n \to u_2$ uniformly on M, we see that $g_n \to u_1/u_2 = f$ uniformly on M w.r.t. the chordal metric. ■

Example 2.8. Let $K := \{z \in \mathbb{C} : 1/2 \leq |z| \leq 2\}$ and $f(z) := z^2 \in \mathcal{M}_{lu}(K)$. Suppose that there is a sequence (g_n) in $\mathcal{M}_{lu}(\mathbb{C})$ which converges to f χ-uniformly on K. Then we have $S_{g_n} \to S_f$ uniformly on $\partial\mathbb{D}$. But since $S_{g_n} \in \mathcal{H}(\mathbb{C})$ for all $n \in \mathbb{N}$, the maximum principle implies $S_{g_n} \to h$ uniformly in $\mathbb{D}$ for a function $h \in \mathcal{H}(\mathbb{D})$. We have $S_f \equiv h$ on $\mathbb{D} \cap K$ and hence on $\mathbb{D} \setminus \{0\}$. This, however, contradicts the fact that 0 is a critical point of f.

Let Ω be a domain and $K \subseteq \Omega$ compact with the property that $\mathcal{M}_{lu}(\Omega)$ is dense in $\mathcal{M}_{lu}(K)$. Then example 2.8 shows that K has to be $\mathcal{O}$-convex. The obvious question is, if the reverse is true:

Problem 2.
Let Ω be a domain in $\mathbb{C}$ and let $K \subseteq \Omega$ be compact and $\mathcal{O}$-convex. Can every $f \in \mathcal{M}_{lu}(K)$ be approximated χ-uniformly on K by functions in $\mathcal{M}_{lu}(\Omega)$?

2.2. Local Approximation and the Curvature Equation

Our next goal is to establish a version of Keldysh's Theorem J for (sub-)solutions of the curvature equation

$$\Delta u = \kappa e^{2u} \tag{2.2.1}$$

when κ is a non-negative Hölder-continuous function. We refer the reader who is not familiar with potential theory to the appendix for the potential theoretic tools needed in this section.
For a nonempty set $E \subseteq \mathbb{C}$ we let $h_\kappa(E)$ be the set of all C^2-functions which satisfy (2.2.1) in an open neighborhood of E. We say that a function s is a subsolution of (2.2.1), if s is upper-semicontinuous and $\underline{\Delta} s \geq \kappa e^{2s}$. For a nonempty set $E \subseteq \mathbb{C}$ we let $sh_\kappa(E)$ be the set of all subsolutions of (2.2.1) which are defined in an open neighborhood. Let $U \subseteq \mathbb{C}$ be bounded and open, $\kappa\colon \mathbb{C} \to [0, \infty)$ a locally Hölder-continuous function and $f \in C(\partial U)$. In what follows, we let $H^f_{U,\kappa}$ be the Perron-solution of the boundary-value problem

$$\begin{cases} \Delta u = \kappa e^{2u} & \text{in } U, \\ u \equiv f & \text{on } \partial U. \end{cases}$$

If $\kappa \equiv 0$ we simply write H^f_U instead of $H^f_{U,0}$. We denote the Green-function on U with pole at $w \in U$ by $g_U(z, w)$ and extend $g_U(\cdot, w)$ to $\mathbb{C}$ by defining $g_U(z, w) = 0$ for $z \notin U$.
Our first step is to establish a Lemma on the convergence of the Perron-solutions. A similar lemma for harmonic functions can be found in [20, Lemma 1.5].

Lemma 2.9.
Let $K \subseteq \mathbb{C}$ be compact with non-empty interior. Suppose that $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K^\circ$ are thin at the same points. Further let $\kappa\colon \mathbb{C} \to [0, \infty)$ a locally Hölder-continuous function, $f \in C(\mathbb{C})$ and let (U_m) be a decreasing sequence of bounded open sets with $K = \bigcap_{m \in \mathbb{N}} U_m$. Then

$$\lim_{m \to \infty} H^f_{U_m,\kappa}(z) = H^f_{K^\circ,\kappa}(z)$$

locally uniformly in K°.

Proof. For $m \in \mathbb{N}$ let g_m be the Green-function for U_m and let g be the Green-function for K°. Further let $u_m := H^f_{U_m,\kappa}$ and $u := H^f_{K^\circ,\kappa}$. Since κ is non-negative, Riesz's decomposition theorem for subharmonic functions tells us

$$u_m(w) = H^f_{U_m}(w) - \iint_{\mathbb{C}} g_m(z,w)\kappa(z)e^{2u_m(z)}d\lambda(z) \tag{2.2.2}$$

for all $w \in U_m$ and

$$u(w) = H^f_{K^\circ}(w) - \iint_{\mathbb{C}} g_m(z,w)\kappa(z)e^{2u(z)}d\lambda(z)$$

for all $w \in K^\circ$. Since κ is locally bounded, (2.2.2) implies that the sequence (u_m) is equicontinuous. We want to apply the Arcelà-Ascoli theorem to show that (u_m) is normal in K°. Thus we show, that (u_m) is locally bounded in K°. First note that the maximum principle for subharmonic functions shows that

$$u_m(z) \leq \max_{z \in \overline{U_1}} f(z) =: c_1. \tag{2.2.3}$$

We can combine this estimate with the domain monotonicity of the Green-function to conclude

$$\iint_{\mathbb{C}} g_m(z,w)\kappa(z)e^{2u_m(z)}d\lambda(z) \leq \iint_{\mathbb{C}} g_1(z,w)c_2 d\lambda(z),$$

where $c_2 := c_1 \cdot \sup_{z \in U_1} \kappa(z)$. The function $w \mapsto \iint_{\mathbb{C}} g_1(z,w)c_2 d\lambda(z)$ is continuous on K and thus bounded there from above by a constant $c_3 \in \mathbb{R}$. We now can use the minimum principle for harmonic functions and (2.2.2) to conclude

$$u_m(w) = H^f_{U_m}(w) - \iint_{\mathbb{C}} g_m(z,w)\kappa(z)e^{2u_m(z)}d\lambda(z) \geq \min_{z \in \overline{U_1}} f(z) - c_3$$

for all $w \in K^\circ$.

We show that every accumulation point of (u_m) is $H^f_{K^\circ,\kappa}$. To this end, let (u_{m_k}) be a converging subsequence of (u_m) with limit point v. Then there exists $h \in h\,(K^\circ)$ such that

$$v(w) = h(w) - \iint_{\mathbb{C}} g(z,w)\kappa(z)e^{2v(z)}d\lambda(z).$$

First we observe that

$$\begin{aligned} u_m(w) - v(w) &= H^f_{U_m}(w) - h(w) \\ &\quad - \iint_{\mathbb{C}} \kappa(z)\left(g_m(z,w)e^{2u_m(z)} - g(z,w)e^{2v(z)}\right)d\lambda(z). \end{aligned} \tag{2.2.4}$$

Since $g_m(z,w) \to g(z,w)$ pointwise for $z \in \mathbb{C} \setminus S$, the convergence of (u_{m_k}) implies

$$\lim_{k\to\infty} \kappa(z)\left(g_m(z,w)e^{2u_m(z)} - g(z,w)e^{2v(z)}\right) = 0 \tag{2.2.5}$$

pointwise in $\mathbb{C} \setminus (S \cup \{w\})$. Since S is a polar set, we have that the set $S \cup \{w\}$ has Lebesgue-measure 0. The domain monotonicity of the Green function and (2.2.3) imply

$$\left|\kappa(z)\left(g_m(z,w)e^{2u_m(z)} - g(z,w)e^{2v(z)}\right)\right| \leq c_4 g_1(z,w), \tag{2.2.6}$$

where

$$c_4 = 2e^{2c_1} \cdot \sup_{z \in U_1} \kappa(z).$$

Thus (2.2.5), (2.2.6) and the Lebesgue-dominated convergence theorem imply

$$\lim_{k\to\infty} \iint_{\mathbb{C}} \kappa(z)\left(g_m(z,w)e^{2u_m(z)} - g(z,w)e^{2v(z)}\right) d\lambda(z) = 0 \tag{2.2.7}$$

pointwise for $w \in K^\circ$. In combination with (2.2.4) this implies

$$H^f_{U_{m_k}} - h \to 0.$$

On the other hand, we know that $H^f_{U_m}(w) \to H^f_{K^\circ}(w)$, see [20, Lemma 1.5]. Thus we can conclude $h \equiv H^f_{K^\circ}$ and so $v \equiv H^f_{K^\circ,\kappa}$. In conclusion we have shown, that (u_m) is a normal sequence in K° and $H^f_{K^\circ,\kappa}$ is the only accumulation point of (u_m). This allows us to conclude $u_m \to H^f_{K^\circ,\kappa}$ locally uniformly in K°. ■

We need to adapt a version of the "reduced function" of a harmonic function (see for example [32, Chapter 7.3]) for solutions of the curvature equation.

Definition 2.10. Let $V \subseteq \mathbb{C}$ be an open set, $\kappa\colon \mathbb{C} \to [0,\infty)$ locally Hölder-continuous and $s \in sh_\kappa(V)$ bounded from above. For $E \subseteq V$ let

$$\mathcal{P}^s_E \coloneqq \{v \in sh_\kappa(V) \,:\, v \leq s \text{ in } V \setminus E,\ v(z) \leq \sup_{w\in V} s(w) \text{ for all } z \in V\}.$$

Define

$$\hat{R}^s_E \coloneqq \sup_{v \in \mathcal{P}^s_E} v$$

and let R^s_E be the upper-semicontinuous regularization of $\hat{R}^s_E$.

Remark 2.11. In many cases there exists a maximal element in $sh_\kappa(V)$ (for example if κ is bounded form below by a positive constant). Then, if we define

$$\mathcal{P}^s_E \coloneqq \{v \in sh_\kappa(V) \,:\, v \leq s \text{ on } V \setminus E\},$$

the definition of R^s_E still makes sense. Thus in this case, we can define R^s_E even for unbounded functions $s \in sh_\kappa(V)$.

For technical reasons we define $sh_\kappa(V) \cap h_\kappa(\emptyset) = sh_\kappa(V)$ in the next theorem.

Theorem 2.12.
Let V be an open set, $\kappa\colon V \to [0,\infty)$ a locally Hölder-continuous function $E \subseteq V$ and let $s \in sh_\kappa(V)$. Then

(i) $s \leq \hat{R}^s_E \leq R^s_E$;

(ii) $\hat{R}^s_E \equiv s$ *on* $V \setminus E$ *and* $R^s_E \equiv s$ *on* $V \setminus \overline{E}$;

(iii) $R^s_E \in sh_\kappa(V) \cap h_\kappa(E^\circ)$;

(iv) There exists a polar set $S \subseteq \partial E$, *such that* $R^s_E \equiv s$ *on* $\partial E \setminus S$;

(v) If $F \subseteq E$ *then* $R^s_F \leq R^s_E$;

(vi) If (E_k) *is a decreasing sequence of open subsets ov* V *and* $E := \bigcap_{k\in\mathbb{N}} E_k$ *then*

$$\lim_{k\to\infty} R^s_{E_k} = R^s_E.$$

Proof. Note that $s \in \mathcal{P}^S_E$. With this in mind, *(i)* and *(ii)* are direct consequences of the definition R^s_E, $\hat{R}^s_E$.
The definition of R^s_E shows that $R^s_E \in sh_\kappa(V)$. On the other hand $\mathcal{P}^s_E$ is a Perron-family in E°, so $R^s_E \in sh_\kappa(E^\circ)$. Thus *(iii)* is established.
(iv): General properties of subharmonic functions show that $\hat{R}^s_E \equiv R^s_E$ except of a polar set $S \subseteq V$. By *(ii)* and *(iii)* we have $S \subseteq \partial E$.
(v) follows from the fact that $\mathcal{P}^s_E \subseteq \mathcal{P}^s_F$.
We remain to proof *(vi)* (for the *harmonic case* see e.g. [32, Theorem 8.38]). Note that by *(v)* the sequence $R^s_{E_k}$ is decreasing and bounded below by R^s_E. Hence

$$\lim_{k\to\infty} R^s_{E_k} =: v$$

exists, $v \geq R^s_E$ and $v \in sh_\kappa(V)$. By *(i)* and *(iv)* we know that $s \equiv R^s_{E_k}$ on $V \setminus E_k$ except maybe on a polar subset of ∂E_k. This implies that $v \equiv s$ on $V \setminus E$ except maybe on a polar set S. Let w be a negative subharmonic function on V with $S \subseteq w^{-1}(\{-\infty\})$. For every $\varepsilon > 0$ we then have $v + \varepsilon w \in \mathcal{P}^s_E$. The limit $\varepsilon \to 0$ implies $v \leq R^s_E$. ∎

We are now in the position to state and proof our main result of this section. Similarly as above, we define $C(K) \cap h_\kappa(\emptyset) = C(K) \cap sh_\kappa(\emptyset) = C(K)$ for technical reasons.

Theorem 2.13.
Let $K \subseteq \mathbb{C}$ be compact and let $\kappa\colon \mathbb{C} \to [0,\infty)$ be a locally Hölder-continuous function. Suppose that $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K^\circ$ are thin at the same points. Then $h_\kappa(K)$ is dense in $C(K) \cap h_\kappa(K^\circ)$.

Proof. **(1)** We first proof the case, when K has empty interior. Then the hypothesis states, that $\mathbb{C} \setminus K$ is nowhere thin.
Let $f \in C(K)$, $\varepsilon > 0$ and $R > 0$ such that $K \subseteq B_R(0)$. Let

$$m_1 \coloneqq \|f\|_K + \varepsilon, \qquad m_2 \coloneqq \|\kappa\|_{K_R(0)},$$

and let $s\colon K_R(0) \to \mathbb{R}$ be the unique solution of the boundary-value problem

$$\begin{cases} s(\xi) = 0, & \text{for } |\xi| = R; \\ \Delta s \equiv m_2 e^{2m_1}, & \text{in } B_R(0)\,. \end{cases}$$

Keldysh's theorem J tells us, that there exists an open neighborhood $U \subseteq B_R(0)$ of K and $h \in h\,(U)$ with

$$\|h - (f-s)\|_K = \|(h+s) - f\|_K < \frac{\varepsilon}{2}. \tag{2.2.8}$$

Note that then $h(z)+s(z) < f(z)+\varepsilon \leq m_1$ for all $z \in K$. By continuity, this estimate holds in an open set V with $K \subset V \subseteq U$. For every $z \in V$ we have

$$\Delta(h+s)(z) = \Delta s(z) = m_2 e^{2m_1} \geq \kappa(z) e^{2(h(z)+s(z))}.$$

Thus if we define $u \coloneqq h + s$ we have that $u \in sh_\kappa(V)$ and

$$\|f - u\|_K < \frac{\varepsilon}{2}. \tag{2.2.9}$$

Let $w \in \mathcal{P}_K^u$. Then $w \leq u$ on $V \setminus K$. Fine continuity and the hypothesis that $\mathbb{C} \setminus K$ is nowhere thin imply $w \leq u$ on K. Hence we have shown $R_K^u \leq u$. On the other hand, theorem 2.12 *(i)* tells us $R_K^u \geq u$. Thus we have established $u \equiv R_K^u$.
Now we fix a decreasing sequence (U_k) of open sets with $K = \bigcap_{k\in\mathbb{N}} U_k$. Theorem 2.12 *(v)* and *(vi)* imply that $R_{U_k}^u$ is a decreasing sequence and

$$R_{U_k}^u \to R_K^u \equiv u \qquad \text{pointwise in } V.$$

Dini's Theorem tells us, that the convergence is uniform on K. Hence there exists $N \in \mathbb{N}$ such that

$$\|R_{U_N}^u - u\|_K < \frac{\varepsilon}{2}.$$

We can combine this estimate with (2.2.9) to obtain

$$\|R^u_{U_N} - f\|_K < \varepsilon.$$

The Theorem for the case $K^\circ = \emptyset$ follows from the fact that $R^u_{U_N} \in h_\kappa(U_N) \subseteq h_\kappa(K)$.
(2) Now we proof the general case $K^\circ \neq \emptyset$. Note that in Lemma 2.9 we have already shown, that locally uniform approximation in K° is possible.
Let $f \in h_\kappa(K^\circ) \cap C(K)$ and $\varepsilon > 0$. By Tietz's extension Theorem there exists $\hat{f} \in C(\mathbb{C})$ with $\hat{f}(z) = f(z)$ for all $z \in K$. Choose a decreasing sequence (U_m) of bounded open sets with $K = \bigcap_{m\in\mathbb{N}} U_m$ and such that each U_m is regular for the Dirichlet-problem. In addition, choose an exhaustion (L_m) of K° with compact sets.
The hypothesis on K implies, that $\mathbb{C} \setminus \partial K$ is nowhere thin. It follows from what we have shown in step (1) and a continuity argument, that there exists a compact neighborhood ω of ∂K and $v \in h_\kappa(\omega)$ with

$$\|\hat{f} - v\|_\omega < \frac{\varepsilon}{3}. \tag{2.2.10}$$

There exists $N_0 \in \mathbb{N}$ such that

$$K \subseteq \omega \cup L_n \qquad \text{and} \qquad \overline{U_n} \subseteq K \cup \omega \tag{2.2.11}$$

holds for all $n \geq N_0$. Lemma 2.9 tells us

$$H^{\hat{f}}_{U_m} \to f$$

locally uniformly in K°. Thus we can choose $N \geq N_0$ such that

$$\|H^{\hat{f}}_{U_N,\kappa} - f\|_{L_N} < \frac{\varepsilon}{3}. \tag{2.2.12}$$

Let $u := H^{\hat{f}}_{U_N,\kappa}$. Since U_N is regular for the Dirichlet-problem, we have $u \equiv \hat{f}$ on ∂U_N. Thus (2.2.10) and (2.2.11) tell us

$$\|u - v\|_{\partial U_N} < \frac{\varepsilon}{3}. \tag{2.2.13}$$

On the other hand, for $\xi \in \partial\omega \cap K$ we have $\xi \in L_N$ by (2.2.11). Hence (2.2.10) and (2.2.12) imply

$$|u(\xi) - v(\xi)| < \frac{2\varepsilon}{3}. \tag{2.2.14}$$

The fact that $u, v \in h_\kappa(\omega \cap U_N)$ implies that the function $|u - v|$ is subharmonic in $\omega \cap U_N$. Note that $\partial(\omega \cap U_N) = \partial U_N \cup (\partial\omega \cap L_N)$. Equation (2.2.13) and (2.2.14) therefore show us

$$\limsup_{z\to\partial(\omega\cap U_N)} |u(z) - v(z)| < \frac{2\varepsilon}{3}.$$

The maximum-principle for subharmonic functions tells us

$$\|u - v\|_{\omega \cap U_N} < \frac{2\varepsilon}{3}.$$

We now can use (2.2.10) to conclude

$$\|u - \hat{f}\|_{\omega \cap U_N} < \varepsilon. \tag{2.2.15}$$

Note that (2.2.11) implies $K \subseteq L_N \cup (\omega \cap U_N)$. This fact, (2.2.12) and (2.2.15) allow us to conclude

$$\|u - f\|_K < \varepsilon.$$

■

Theorem 2.14.
Let K be a compact set in $\mathbb{C}$ and $\kappa\colon \mathbb{C} \to [0, \infty)$ a locally Hölder continuous function and $\varepsilon > 0$. Suppose that $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K^\circ$ are nowhere thin. Then for every $u \in sh_\kappa(K^\circ) \cap C(K)$ there exists a continuous $w \in sh_\kappa(K)$ such that $\|u - w\|_K < \varepsilon$.

Proof. Let $u \in sh_\kappa(K^\circ) \cap C(K)$. We can assume that $0 < \varepsilon < \frac{1}{2}$. The assumptions on thinness implies that $\mathbb{C} \setminus \partial K$ is nowhere thin. We can use Theorem 2.13 to find $v \in h_\kappa(\partial K)$ such that

$$u(z) + \frac{\varepsilon}{2} < v(z) < u(z) + \varepsilon \tag{2.2.16}$$

holds for all $z \in \partial K$. We can find a compact set $L \subseteq K^\circ$ such that $K^\circ \setminus L$ is regular for the Dirichlet problem and v is defined on $K^\circ \setminus L$. Let h be the unique solution of the boundary value problem

$$\begin{cases} \Delta h \equiv 0, & \text{in } K^\circ \setminus L \\ h(\xi) = 0, & \text{for } \xi \in \partial K \\ h(\xi) = -1, & \text{for } \xi \in \partial L. \end{cases}$$

Keldysh's Theorem tells us that there exists a negative function $s \in h\,(K \setminus L^\circ)$ with

$$\|s - h\|_{K \setminus L^\circ} < \varepsilon.$$

Note that then $v + s < u$ on ∂L and $v + s > u$ on ∂K. Let U be an open neighborhood of ∂K, where v and s are defined. Then the function

$$w(z) := \begin{cases} u(z), & z \in L \\ \max\{u(z),\, v(z) + s(z)\}, & z \in K \setminus L \\ v(z) + s(z), & z \in U \setminus K \end{cases}$$

is well defined and continuous on $U \cup K$. The Gluing lemma tells us $w \in sh_\kappa(K)$. Since $u \equiv w$ on L and $u \leq w \leq \max\{u, v\}$ in $K \setminus L$, it follows from (2.2.16) that $\|u - w\|_K < \varepsilon$. ■

The assumptions on K in theorem 2.14 are stronger than in theorem 2.13. This has technical reasons, but the obvious question is, if the assumption on K in theorem 2.14 can be relaxed.

Problem 3.
Let $K \subseteq \mathbb{C}$ be compact such that $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K^\circ$ are thin at the same points. Is it still true, that every function $u \in sh_\kappa(K^\circ) \cap C(K)$ can be approximated by continuous functions in $sh_\kappa(K)$?

We conclude this section with an interpretation of our results in terms of conformal metrics. We define $C(K) \cap \Lambda_{-1}(\emptyset) = C(K) \cap SK(\emptyset) = C(K)$.

Corollary 2.15.
Let $K \subseteq \mathbb{C}$ such that $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K^\circ$ are thin at the same points, and $\varepsilon > 0$.

(a) Let $\lambda \in C(K) \cap \Lambda_{-1}(K^\circ)$ be strictly positive. Then there exist an open neighborhood U of K and $\mu \in \Lambda_{-1}(U)$ with $\|\lambda - \mu\|_K < \varepsilon$.

(b) Suppose in addition that K° is regular for the Dirichlet problem. Let $\lambda \in C(K) \cap SK(K^\circ)$ be strictly positive. Then there exists an open neighborhood U of K and $\mu \in SKC(U)$ with $\|\lambda - \mu\|_K < \varepsilon$.

3. Universal Locally Univalent Functions

We have now the technology to study universal locally univalent functions. We shall work with the following definition:

Definition 3.1. Let Ω be a domain in $\mathbb{C}$, $\mathcal{G} \subseteq \mathcal{M}(\Omega)$ and Φ a family of holomorphic self-maps of Ω. A function $G \in \mathcal{G}$ is called Φ-universal in $\mathcal{G}$ if $\{G \circ \phi : \phi \in \Phi\}$ is dense in $\mathcal{G}$. If $G \in \mathcal{G}$ is $\mathrm{Aut}(\Omega)$-universal, we simply call G universal in $\mathcal{G}$.

One aim of this chapter is to provide necessary and also sufficient conditions for the existence of Φ-universal functions in the following cases for $\mathcal{G}$:

(i) $\mathcal{G} = \mathcal{H}_{lu}(\Omega)$, the set of all holomorphic locally univalent functions in Ω;

(ii) $\mathcal{G} = \mathcal{B}_{lu}(\Omega)$, the set of all bounded locally univalent functions in Ω.

Although it is not our main focus, we also proof a result for $\mathcal{G} = \mathcal{M}_{lu}(\Omega)$. We shall use the following terminology, which was introduced by Bernal and González [8] and Mortini and Grosse-Erdmann [26].

Definition 3.2. Let Ω be a domain in $\mathbb{C}$ and let (ϕ_n) be a sequence of holomorphic self-maps of Ω.

(i) We say that (ϕ_n) is run-away, if for every compact set $K \subseteq \Omega$ there exists $n \in \mathbb{N}$ with $\phi_n(K) \cap K = \emptyset$.

(ii) We say that (ϕ_n) is eventually injective, if for every compact set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that the restriction $\phi_n|_K$ is injective for all $n \geq N$.

Universality is invariant under conformal mapping, and so are eventually injective and run-away sequences:

Remark 3.3. Let Ω_1, Ω_2 be complex domains, $f \colon \Omega_1 \to \Omega_2$ a conformal mapping and let (ϕ_n) be a sequence of holomorphic self-maps of Ω_1 Define $\psi_n := f \circ \phi_n \circ f^{-1}$ Then (ψ_n) is a sequence of holomorphic self-maps of Ω_2 and the following holds:

- If (ϕ_n) is eventually injective, then so is (ψ_n);
- If (ϕ_n) is run-away, then so is (ψ_n).

On the other hand let Φ be a family of holomorphic self-maps of Ω_1, $\mathcal{G} \subseteq \mathcal{M}(\Omega_1)$ and suppose that $G \in \mathcal{G}$ is Φ-universal. Define $\Psi := \{f \circ \phi \circ f^{-1} : \phi \in \Phi\}$ and $\tilde{\mathcal{G}} := \{g \circ f^{-1} : g \in \mathcal{G}\} \subseteq \mathcal{M}(\Omega_2)$. It is straight forward that $G \circ f^{-1}$ is a Ψ-universal function in $\tilde{\mathcal{G}}$.

Example 3.4. ([8, Proposition 2.3]) A sequence $(\phi_n(z))_n := (a_n z + b_n)_n \in \text{Aut}(\mathbb{C})$ is run-away if and only if $\limsup_{n\to\infty} \min\{|b_n/a_n|, |b_n|\} = \infty$. In particular, the sequence of iterations $(\phi^{[n]})_n$ of $\phi(z) := az + b \in \text{Aut}(\mathbb{C})$ is run-away, if and only if $a = 1$ and $b \neq 0$.

The following example is a slight generalization of [8, Proposition 2.5], where only sequences in $\text{Aut}(\mathbb{D})$ are considered. It is a direct consequence of Montel's theorem and the maximum principle.

Example 3.5. A sequence (ϕ_n) of holomorphic self-maps of the unit disc is run-away if and only if $\limsup_{n\to\infty} |\phi_n(0)| = 1$.

Remark 3.6. A celebrated result due to Denjoy [15] and Wolff [51] (see also [10]) states the following: Let $\phi \in \mathcal{B}(\mathbb{D})$. Suppose that ϕ has no fixed point in $\mathbb{D}$. Then there exists a unimodular constant $\eta \in \partial\mathbb{D}$ such that $\phi^{[n]} \to \eta$ locally uniformly.
On the other hand if ϕ has a fixed point $z_0 \in \mathbb{D}$, then $(\phi^{[n]})$ can not be run-away, as for if we let $K = \{z_0\}$ then $\phi^{[n]}(K) \cap K \neq \emptyset$ for all $n \in \mathbb{N}$. Thus $(\phi^{[n]})$ is run-away if and only if ϕ has no fixed point.
Via conjugation with a conformal mapping the following statement follows: Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let ψ be a holomorphic self-map of Ω. Then $(\psi^{[n]})$ is run-away if and only if ψ has no fixed point.

Our approach to the universality result in this chapter (as well as in the next chapter) is based on a well known universality criterion.

Definition 3.7. Let X, Y be topological spaces and let $(T_j)_J$ be a family of continuous mappings $T_j \colon X \to Y$.

(i) We say that $x \in X$ is (T_j)-universal, if the orbit $\{T_j x : j \in J\}$ is dense in Y.

(ii) We say that (T_j) acts topologically transitively, if for every two non-empty open sets $U \subseteq X$, $V \subseteq Y$ there exists $j \in J$ such that $T_j(U) \cap V \neq \emptyset$.

Theorem 3.8 (Universality Criterion).
Let X be a Baire-space Y be a second countable topological space and $(T_j)_J$ a family of continuous mappings $T_j \colon X \to Y$. Suppose that (T_j) acts topologically transitively. Then the set of all (T_j)-universal elements in X is a dense G_δ-subset of X.

Note that $\mathcal{H}_{lu}(\Omega)$ is a separable Baire-space, thus the universality criterion is appropriate for our purpose.
For every holomorphic self-map ϕ of a complex domain Ω we denote with C_ϕ the associated composition operator

$$C_\phi \colon \mathcal{H}(\Omega) \to \mathcal{H}(\Omega), \quad f \mapsto f \circ \phi.$$

If ϕ is locally univalent, then C_ϕ maps $\mathcal{H}_{lu}(\Omega)$ into $\mathcal{H}_{lu}(\Omega)$. Note that Φ-universality as defined in Definition 3.1 can be viewed as universality of the family of composition operators $(C_\phi)_\Phi$. There is a well known method to show topological transitivity of a family of composition operator which relies on an application of Runge's Theorem. We already know that there is a version of Runge's Theorem for locally univalent functions, so we can adapt this method to the setting of locally univalent functions. For non-simply connected domains, this method is based on the following lemmas:

Lemma 3.9 ([26, Lemma 3.10]).
Let K and L be compact subsets of a domain $\Omega \subseteq \mathbb{C}$ with $K \subseteq L$. Let ϕ be a holomorphic self-map of Ω that is injective on some neighborhood of L. If K and $\phi(L)$ are $\mathcal{O}$-convex, then so is $\phi(K)$.

Lemma 3.10 ([26, Lemma 3.13]).
Let $\Omega \subseteq \mathbb{C}$ be a domain and let Φ be a family of holomorphic self-maps of Ω. Suppose that there exists an eventually injective sequence (ϕ_n) in Φ with the following property: For every compact $\mathcal{O}$-convex subset K of Ω and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\phi_n(K)$ is $\mathcal{O}$-convex and $\phi_n(K) \cap K = \emptyset$. Then, for every connected compact $\mathcal{O}$-convex subset K of Ω with at least two holes there exists $\phi \in \Phi$ such that $\phi(K) \cap K = \emptyset$ and $\phi(K) \cup K$ is $\mathcal{O}$-convex in Ω.

Lemma 3.11 ([8, Lemma2.12]).
Let $\Omega \subseteq \mathbb{C}$ be a domain of infinite connectivity, $(\phi_n) \subseteq \mathrm{Aut}(\Omega)$ a run-away sequence and $K, L \subseteq \Omega$ compact $\mathcal{O}$-convex sets. Then there exists $N \in \mathbb{N}$ such that $K \cap \phi_N(L) = \emptyset$ and $K \cup \phi_N(L)$ is $\mathcal{O}$-convex.

Before we start our own investigation, we briefly summarize what is known for Φ-universal functions in $\mathcal{H}(\Omega)$, if Φ is a family of holomorphic self-maps of Ω. These result give us a road map for the rest of this chapter.

- ([6, Theorem 5.2], [26, Theorem3.2], [26, Theorem 3.16]) If Ω is either simply connected or of infinite connectivity, then necessary as well es sufficient conditions for Φ are known, so that Φ-universal functions in $\mathcal{H}(\Omega)$ exist.
- ([26, Proposition 3.6], [26, Theorem 3.15]) Suppose that Ω is finitely connected, but not simply connected. Then there can be families Φ, such that there are Φ-universal functions in $\mathcal{H}(\Omega)$. However, for eventually injective sequences (ϕ_n), no (ϕ_n)-universal function exists in $\mathcal{H}(\Omega)$.
- ([6, Theorem 2.1], [26, Proposition 2.3]) There exists a Φ-universal function $f \in \mathcal{B}(\Omega)$ if and only if there exists a run-away sequence (ϕ_n) in Φ with $\phi_n^* \lambda_{\mathbb{D}} \to \lambda_{\mathbb{D}}$.

3.1. Holomorphic Locally Univalent Universal Functions

3.1.1. Necessary Conditions

We start by establishing a necessary condition for the existence of Φ-universal function in $\mathcal{H}_{lu}(\Omega)$.

Proposition 3.12.
Let Ω be a domain in $\mathbb{C}$ and let Φ be a family of locally univalent self-maps of Ω. Suppose that there is a Φ-universal function in $\mathcal{H}_{lu}(\Omega)$. Then for every relatively compact set $K \subseteq \Omega$ there exists $\phi \in \Phi$ such that $\phi(K) \cap K = \emptyset$ and the restriction $\phi|_K$ is injective. In particular there exists an eventually injective run-away sequence $(\phi_n) \subseteq \Phi$.

Proof. Suppose that $f \in \mathcal{H}_{lu}(\Omega)$ is Φ-universal in $\mathcal{H}_{lu}(\Omega)$. Choose a compact set $L \subseteq \Omega$ which contains K in its interior and which is the closure of its interior. Let

$$\delta := \frac{1}{2}\operatorname{dist}(K, \partial L) > 0, \qquad M := \sup_{z \in L} |f(z)|.$$

Then $g(z) := z + 2M + 2\delta$ belongs to $\mathcal{H}_{lu}(\Omega)$. Since f is Φ-universal in $\mathcal{H}_{lu}(\Omega)$ there exists $\phi \in \Phi$ such that

$$\|f \circ \phi - g\|_L < \delta. \tag{3.1.1}$$

This in particular implies $|f(\phi(z))| \geq |g(z)| - \delta \geq M + \delta$ for all $z \in K$, so

$$\min_{z \in K} |f(\phi(z))| \geq M + \delta > M \geq \max_{z \in K} |f(z)|$$

and thus $\phi(K) \cap K = \emptyset$. Next we fix $z_0 \in K$. Then the estimate (3.1.1) shows that for $z \in \partial L$ we have

$$|[f(\phi(z_0)) - f(\phi(z))] - [z - z_0]| < 2\delta \leq |z_0 - z|.$$

Hence, by Rouché's Theorem, $f(\phi(z_0)) - f(\phi(z))$ and $z_0 - z$ have the same numbers of zeros in L°. This implies that ϕ is injective on K.
Let (K_n) be an exhaustion of Ω with compact sets. We have already shown, that for each $n \in \mathbb{N}$ there exists $\phi_n \in \Phi$ such that $\phi_n(K_n) \cap K_n = \emptyset$ and the restriction $\phi_n|_{K_n}$ is injective. Thus (ϕ_n) is run-away and eventually injective. ■

Remark 3.13. Let ϕ be a holomorphic self-map of a domain $\Omega \subseteq \mathbb{C}$. Suppose that there exists a $(\phi^{[n]})$-universal function in $\mathcal{H}_{lu}(\Omega)$. Proposition 3.12 tells us, that $(\phi^{[n]})$ has an eventually injective subsequence. This implies that ϕ itself is injective.

The obvious question now is, to what extend the conditions in Proposition 3.12 are also sufficient. Before we discuss this question we refine the "run-away"-condition, provided that we are dealing with an eventually injective sequence in the beginning.

Theorem 3.14.
Let Ω be a domain in $\mathbb{C}$ and let (ϕ_n) be an eventually injective sequence of holomorphic self-maps of Ω. Suppose that there exists a (ϕ_n)-universal function $f \in \mathcal{H}_{lu}(\Omega)$. Then for every compact $\mathcal{O}$-convex set $K \subseteq \Omega$ and each $N \in \mathbb{N}$, there exists $n \geq N$ such that $\phi_n(K)$ is $\mathcal{O}$-convex and $\phi_n(K) \cap K = \emptyset$.

For later reference we proof a more specific statement.

Lemma 3.15.
Let Ω be a non-simply connected domain, (ϕ_n) an eventually injective sequence of locally-univalent self-maps of Ω, $f \in \mathcal{H}_{lu}(\Omega)$ a (ϕ_n)-universal function and $K \subseteq \Omega$ a compact $\mathcal{O}$-convex set. Then for every $c \in \hat{\mathbb{C}}$ there exists a subsequence (ϕ_{n_k}) of (ϕ_n) such that $f \circ \phi_{n_k} \to c$ uniformly on K and $\phi_{n_k}(K)$ is $\mathcal{O}$-convex for every $k \in \mathbb{N}$.

Proof. **(1)** Let $K \subseteq \Omega$ be $\mathcal{O}$-convex and let U be the unbounded component of $\mathbb{C} \setminus K$. Let L be a connected $\mathcal{O}$-convex set with smooth boundary and such that $K \subseteq L^\circ$. Then L has only finitely many holes, say $p \in \mathbb{N}$. For $j = 1, \ldots, p$ let O_j be the bounded connected components of $\mathbb{C} \setminus L$, let γ_j be the negative oriented boundary of O_j and let γ_0 be the positive oriented outer boundary of L. For $j = 1, \ldots, p$ choose $a_j \in O_j \setminus \Omega$. Further for $k = 0, \ldots, p$ choose pairwise distinct points $b_k \in U \cap L$. Then the following holds:

$$n(\gamma_j,\, a_k) = -\delta_{jk} \qquad \text{for } j,k \in \{1,\dots,p\},$$
$$n(\gamma_0,\, a_j) = 1 \qquad \text{for } j \in \{1,\dots,p\}$$

as well as

$$n(\gamma_j,\, b_k) = 0 \qquad \text{for } j \in \{1,\dots,p\},\, k \in \{0,\dots,p\},$$
$$n(\gamma_0,\, b_k) = 1 \qquad \text{for } k \in \{0,\dots,p\}.$$

(2) We show that there exists $g \in \mathcal{H}_{lu}(\Omega)$ with

$$0 < \operatorname{Re}\left(\frac{1}{2\pi i}\int_{\gamma_k} \frac{g'(z)}{g(z)}\,dz\right) \qquad \text{for each } k \in \{0,\dots,p\}. \tag{3.1.2}$$

Let $R > 0$ such that $L \subseteq B_R(0)$. The inverse triangle inequality and a straight forward induction argument show, that for every $j \in \{1,\dots,p\}$ we can find $c_j \in \mathbb{C}$ such that

$$\tilde{h}(z) \coloneqq \sum_{j=1}^{p} \frac{c_j}{z - a_j}$$

is a zero-free function in $\mathcal{M}_{lu}(B_R(0))$. Now, for $k \in \{0,\dots,p\}$ choose $r_k > 0$ such that the compact discs $K_{r_k}(b_k)$ are pairwise disjoint and such that $K_{r_k}(b_k) \subseteq U$. Let M be the union of the $\mathcal{O}$-convex hull of $L \setminus U$ with the discs $K_{r_k}(b_k)$ and define $h\colon M \to \mathbb{C}$ by

$$h(z) \coloneqq \begin{cases} z - b_k, & \text{if } z \in K_{r_k}(b_k) \\ \tilde{h}(z), & \text{if } z \in \widehat{L \setminus U}. \end{cases}$$

Note that for every $j \in \{1,\dots,p\}$ we have

$$\int_{\gamma_j} \frac{h'(z)}{h(z)}\,dz = 1.$$

Since U is unbounded, we have that M is $\mathcal{O}$-convex in Ω and $\operatorname{tr}(\gamma_j) \subseteq M$ for all $j \in \{1,\dots,p\}$. Fix $\varepsilon > 0$. Hurwitz's Theorem and Theorem 2.3 tell us, that there exists $g \in \mathcal{H}_{lu}(\Omega)$ such that

(i) g has a zero in $B_{r_k}(b_k)$ for each $k \in \{0,\dots,p\}$;

(ii) g is zero-free on K;

(iii) $\|h - g\|_{\operatorname{tr}(\gamma_j)} < \varepsilon$ and $\left\|\frac{h'}{h} - \frac{g'}{g}\right\|_{\operatorname{tr}(\gamma_j)} < \varepsilon$ for all $j \in \{1,\dots,p\}$.

Thus g has at least $p+1$ zeros in L° and no zeros on γ_j for $j = 1, \ldots, p$. For a suitable choice of ε, this implies

$$0 < \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_j}\frac{g'(z)}{g(z)}\,dz\right) < 1 + \frac{1}{p} \tag{3.1.3}$$

for all $j \in \{1, \ldots, p\}$. After a variation of the outer boundary of L we may assume, that g has no zero on γ_0 and still has at least $p+1$ zeros in L°. The cycle $\Gamma := \sum_{k=0}^{p}\gamma_k$ is null homologous in Ω. Thus, the argument principle tells us

$$\frac{1}{2\pi i}\sum_{k=0}^{p}\int_{\gamma_k}\frac{g'(z)}{g(z)}\,dz = \frac{1}{2\pi i}\int_{\Gamma}\frac{g'(z)}{g(z)}\,dz \geq p+1.$$

We can use this fact and (3.1.3) to conclude, that (3.1.2) holds indeed.
(3) Now fix $c \in \mathbb{C}$, we deal with the case $c = \infty$ later. For $m \in \mathbb{N}$ let $g_m = g/m + c$. Then (3.1.2) tells us

$$0 < \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_j}\frac{g_m'(z)}{g_m(z) - c}\,dz\right) \tag{3.1.4}$$

for $j \in \{0, \ldots, p\}$. The universality of f implies, that for each $m \in \mathbb{N}$ there is a subsequence $\phi_{n_k^{(m)}}$ of (ϕ_n) such that

$$\lim_{k\to\infty} f \circ \phi_{n_k^{(m)}} = g_m.$$

This implies

$$\lim_{k\to\infty}\frac{(f\circ\phi_{n_k^{(m)}})'}{f\circ\phi_{n_k^{(m)}} - c} = \frac{g_m'}{g_m - c}$$

uniformly on $\mathrm{tr}\,(\Gamma)$. On the other hand $g_m \to c$. Hence we can extract a sequence (n_m) such that

$$\begin{aligned} f\circ\phi_{n_m} - g_m &\to 0 \qquad \text{uniformly on } K\\ \frac{(f\circ\phi_{n_m})'}{f\circ\phi_{n_m} - c} - \frac{g_m'}{g_m - c} &\to 0 \qquad \text{uniformly on } \mathrm{tr}\,(\Gamma)\,. \end{aligned}$$

The first equation tells us $f \circ \phi_{n_m} \to c$ on K. We can use the second equation, (3.1.2) and the fact that (ϕ_n) is eventually injective, to find $M \in \mathbb{N}$ such that ϕ_{n_M} is injective on L for all $m \geq M$ and

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_{\phi_{n_m}(\gamma_j)}\frac{f'(z)}{f(z) - c}\,dz\right) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_j}\frac{(f\circ\phi_{n_m})'(z)}{(f\circ\phi_{n_m})(z) - c}\,dz\right) > 0. \tag{3.1.5}$$

for all $j \in \{0, \ldots, p\}$ and all $m \geq M$.

If $c = \infty$ we consider the functions $g_m := m \cdot g$ instead. Note that we then have $g_m \to \infty$ uniformly on K. As above we can find a subsequence (ϕ_{n_m}) of (ϕ_n) such that

$$\begin{aligned} f \circ \phi_{n_m} - g_m &\to 0 \qquad \text{uniformly on } K \\ \frac{(f \circ \phi_{n_m})'}{f \circ \phi_{n_m}} - \frac{g_m'}{g_m} &\to 0 \qquad \text{uniformly on } \operatorname{tr}(\Gamma). \end{aligned}$$

The first equation then tells us that $f \circ \phi_{n_m} \to \infty$ uniformly on K. We can use the same reasoning which showed (3.1.5) to show that there exists $M \in \mathbb{N}$ such that for all $m \geq M$ the function (ϕ_{n_m}) is injective on L with

$$\text{(3.1.6)} \qquad \operatorname{Re}\left(\frac{1}{2\pi i} \int_{\phi_{n_m}(\gamma_j)} \frac{f'(z)}{f(z)}\, dz \right) = \operatorname{Re}\left(\frac{1}{2\pi i} \int_{\gamma_j} \frac{(f \circ \phi_{n_m})'(z)}{(f \circ \phi_{n_m})(z)}\, dz \right) > 0.$$

(4) We show that $\phi_{n_m}(L)$ is $\mathcal{O}$-convex for every $m \in \mathbb{N}$. Then Lemma 3.9 tells us that $\phi_{n_m}(K)$ is $\mathcal{O}$-convex.
Since ϕ_{n_m} is injective on a neighborhood of L, $\phi_{n_m}(L)$ has exactly p holes. For the sake of contradiction, we assume that one of these holes, call it O, is compactly contained in Ω. Since injective holomorphic functions map boundaries to boundaries and preserve orientation, there exists $k \in \{0, \ldots, p\}$ such that the Jordan curve $\alpha := \phi_{n_m}(\gamma_k)$ is the negatively oriented boundary of O. Moreover, since O contains no point in $\mathbb{C} \setminus \Omega$, we have that

$$n(\alpha, \xi) = 0 \qquad \text{for } \xi \notin \Omega.$$

On the other hand $\overline{O} \subseteq \Omega$, hence f is holomorphic in a neighborhood of $\overline{O}$. Thus, by the argument principle, the integral

$$-\frac{1}{2\pi i} \int_\alpha \frac{f'(z)}{f(z) - c} dz \quad (c \in \mathbb{C}) \qquad \text{resp.} \qquad -\frac{1}{2\pi i} \int_\alpha \frac{f'(z)}{f(z)} dz \quad (c = \infty)$$

is the numbers of zeros of $f - c$ resp. f in O. But we have computed in (3.1.5) and (3.1.6), that this integral has negative real part, a contradiction. Thus we can conclude that $\phi_{n_m}(L)$ is $\mathcal{O}$-convex. ■

Proof of Theorem 3.14. If Ω is simply connected, then a compact set $K \subseteq \Omega$ is $\mathcal{O}$-convex if and only if it has no holes. There exists $N \in \mathbb{N}$ such that the restriction $\phi_n|_K$ is injective for every $n \geq N$. Then, by the hole invariance principle, the set $\phi_n(K)$ has no holes and thus is $\mathcal{O}$-convex. Proposition 3.12 shows, that there exists $m \geq N$ such that $\phi_m(K) \cap K = \emptyset$.

Now suppose that Ω is not simply connected, let $K \subseteq \Omega$ be a compact $\mathcal{O}$-convex set and let $f \in \mathcal{H}_{lu}(\Omega)$ be a (ϕ_n)-universal function. Lemma 3.15 (for $c = \infty$) shows, that there exists $N \in \mathbb{N}$ such that $\phi_N(K)$ is $\mathcal{O}$-convex and

$$\min_{z \in K} |f(\phi_N(z))| > \max_{z \in K} |f(z)|.$$

We can conclude $\phi_N(K) \cap K = \emptyset$. ■

Example 3.16. Let Ω be a bounded non-simply connected domain and let $R > 0$ such that $\Omega \subseteq B_R(0)$. Choose a sequence (a_n) in Ω with $a_n \to b \in \partial\Omega$ and a sequence (ρ_n) with $0 < \rho_n < \frac{1}{n}$ and $B_{\rho_n}(a_n) \subseteq \Omega$. There exist linear transformations $\phi_n(z) := \alpha_n z + \beta_n$ such that $\phi_n(B_R(0)) = B_{\rho_n}(a_n)$. Then for each $n \in \mathbb{N}$ the restriction of ϕ_n to Ω is a univalent self-map of Ω and clearly (ϕ_n) is run-away. Let $K \subseteq \Omega$ be a compact $\mathcal{O}$-convex subset with at least one hole. Then by the hole invariance principle the set $\phi_n(K)$ has at least one hole for every $n \in \mathbb{N}$. Every hole of $\phi_n(K)$ is contained in $B_{\rho_n}(a_n)$ and thus also in Ω. Hence $\phi_n(K)$ is not $\mathcal{O}$-convex for all $n \in \mathbb{N}$. Theorem 3.14 tells us, that there is no (ϕ_n)-universal function in $\mathcal{H}_{lu}(\Omega)$.

3.1.2. Simply Connected Domains

We now turn to the contrary problem: Find sufficient conditions for the existence of Φ-universal functions in $\mathcal{H}_{lu}(\Omega)$. Theorem 3.14 gives a first hint, that the geometry of Ω plays a role in this question. We consider the following three cases: Ω is simply connected, Ω is finitely but not simply connected and Ω is of infinite connectivity. For simply connected domains, the necessary condition of Proposition 3.12 is also sufficient.

Theorem 3.17.
Let Ω be a simply connected domain in $\mathbb{C}$. Suppose that Φ is a family of locally univalent self-maps of Ω which contains a run-away and eventually injective sequence. Then there exists a Φ-universal function in $\mathcal{H}_{lu}(\Omega)$ and the set of all Φ-universal functions is a dense G_δ-subset of $\mathcal{H}_{lu}(\Omega)$.

Proof. We show that the family $(C_\phi)_\Phi$ of composition operators

$$C_\phi \colon \mathcal{H}_{lu}(\Omega) \to \mathcal{H}_{lu}(\Omega)\,,\ f \mapsto f \circ \phi \qquad \phi \in \Phi$$

acts topologically transitive. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{H}_{lu}(\Omega)$ be non-empty open sets and let $f \in \mathcal{U}$, $g \in \mathcal{V}$. Then there exists a compact set $L \subseteq \Omega$ with connected complement and $\varepsilon > 0$ such that

$$U_{L,\varepsilon}(f) \subseteq \mathcal{U}, \qquad U_{L,\varepsilon}(g) \subseteq \mathcal{V}.$$

By our assumption there exist $\phi \in \Phi$ such that ϕ is injective on L and $\phi(L) \cap L = \emptyset$. Define $K := L \cup \phi(L)$ and $h \in \mathcal{H}_{lu}(K)$ by

$$h(z) := \begin{cases} f(z), & z \in L \\ g(\phi^{-1}(z)), & z \in \phi(L). \end{cases}$$

Note that K has connected complement, hence by Theorem 2.3 there exists a function $q \in \mathcal{H}_{lu}(\Omega)$ with $\|f - q\|_K < \varepsilon$. This immediately implies $q \in \mathcal{U}$ and $q \circ \phi \in \mathcal{V}$. Since $\mathcal{H}_{lu}(\Omega)$ is a Baire-space, we now can apply Theorem 3.8. ■

Corollary 3.18.
Let Ω be a simply connected domain and let ϕ be a holomorphic self-map of Ω. Then there exists a $\phi^{[n]}$-universal function $f \in \mathcal{H}_{lu}(\Omega)$ if and only if ϕ is injective and has no fixed point.

Proof. In the case $\Omega \neq \mathbb{C}$ remarks 3.6 and 3.13 show that $(\phi^{[n]})$ is eventually injective and run-away if and only if ϕ is injective and has no fixed point.

If $\Omega = \mathbb{C}$ note that if ϕ is injective, then $\phi \in \mathrm{Aut}(\mathbb{C})$. Thus remark 3.13 shows that $(\phi^{[n]})$ is eventually injective if and only if $\phi \in \mathrm{Aut}(\mathbb{C})$, i.e. $\phi(z) = az + b$. We have seen in Example 3.4 that $(\phi^{[n]})$ is run-away if and only if $a = 1$ and $b \neq 0$. A simple calculation shows, that a linear mapping $z \mapsto cz + d$ has no fixed point in $\mathbb{C}$ if and only if $c = 1$ and $d \neq 0$.

In both cases, the corollary then is a direct consequence of Proposition 3.12 and Theorem 3.17. ■

In the proof of Theorem 3.17 if the functions f, g were meromorphic one could use Theorem 2.7 instead of Theorem 2.3. This proofs the following:

Theorem 3.19.
Let Ω be a simply connected domain in $\mathbb{C}$. Suppose that Φ is a family of locally univalent self-maps of Ω which contains a run-away and eventually injective sequence. Then there exists a Φ-universal function in $\mathcal{M}_{lu}(\Omega)$ and the set of all Φ-universal functions is a dense G_δ-subset of $\mathcal{M}_{lu}(\Omega)$.

There is a version of the result of Chan, Theorem F, for locally univalent functions. For a function $f \in \mathcal{M}_{lu}(\mathbb{C})$ we let

$$T_f := \{f(\cdot + n) \,:\, n \in \mathbb{N}\}.$$

Suppose that Ω is a domain in $\mathbb{C}$ with the property, that T_f is dense in $\mathcal{M}_{lu}(\Omega)$. Then the same reasoning as in example 2.8 shows, that Ω has to be simply connected. Hence in the next corollary, we can not drop the assumption, that Ω is simply connected.

Corollary 3.20.
There exists a function $f \in \mathcal{M}_{lu}(\mathbb{C})$ with the following property: The set $T_f := \{f(\cdot + n) : n \in \mathbb{N}\}$ is dense in $\mathcal{M}_{lu}(\Omega)$ for every simply connected domain $\Omega \subseteq \mathbb{C}$.

Proof. Theorem 3.19 tells us, that there exists a universal function $f \in \mathcal{M}_{lu}(\mathbb{C})$ such that T_f is dense in $\mathcal{M}_{lu}(\mathbb{C})$. Let Ω be a simply connected domain in $\mathbb{C}$. As a consequence of theorem 2.3 $\mathcal{M}_{lu}(\mathbb{C})$ is dense in $\mathcal{M}_{lu}(\Omega)$, thus T_f is dense in $\mathcal{M}_{lu}(\Omega)$. ■

3.1.3. Finitely Connected Domains

Although in the following theorem, the domain Ω need not to be of finite connectivity, the result itself is best understood in this context. Note that Corollary 2.6 gives a large class of domains which fullfill the requirements of the theorem.

Theorem 3.21.
Let $\Omega \subseteq \mathbb{C}$ be a domain with the property, that $\mathcal{H}^{\infty}_{lu}(\Omega)$ is dense in $\mathcal{H}_{lu}(\Omega)$. Then there exist a sequence (ϕ_n) of locally univalent self-maps of Ω for which a (ϕ_n)-universal $f \in \mathcal{H}_{lu}(\Omega)$ exists.

Proof. Since $\mathcal{H}_{lu}(\Omega)$ is separable, the hypothesis implies that there exists a sequence (ψ_n) of bounded locally univalent functions on Ω which is dense in $\mathcal{H}_{lu}(\Omega)$. Let $r_n > 0$ such that the image $\psi_n(\Omega)$ is contained in the closed disc $K_{r_n}(0)$.
Now choose $b \in \partial\Omega$ and $(a_n) \subseteq \Omega$ with $\lim_{n\to\infty} a_n = b$ and a sequence (ρ_n) of positive real numbers, such that the closed discs $K_{\rho_n}(a_n)$ are pairwise disjoint and contained in Ω. For each $n \in \mathbb{N}$ let τ_n be a linear function on $\mathbb{C}$ with $\tau_n(K_{r_n}(0)) = K_{\rho_n}(a_n)$. We define ϕ_n by

$$\phi_n := \tau_n \circ \psi_n$$

and claim, that there exists a (ϕ_n)-universal function in $\mathcal{H}_{lu}(\Omega)$.
To proof the claim, we show, that the sequence (C_{ϕ_n}) acts topologically transitive on $\mathcal{H}_{lu}(\Omega)$. To this end, let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{H}_{lu}(\Omega)$ be open and choose $f \in \mathcal{U}$, $g \in \mathcal{V}$. Then there exists a compact $\mathcal{O}$-convex set $L \subseteq \Omega$ and $\varepsilon > 0$ such that

$$U_{L,\varepsilon}(f) \subseteq \mathcal{U}, \qquad U_{L,2\varepsilon}(g) \subseteq \mathcal{V}.$$

It follows from our earlier construction, that there exists $m \in \mathbb{N}$ such that $K_{\rho_m}(a_m) \cap L = \emptyset$ and $\psi_m \in U_{L,\varepsilon}(g)$. Then the set $K := L \cup K_{\rho_m}(a_m)$ is $\mathcal{O}$-convex in Ω. Define

$\tilde{h} \in \mathcal{H}_{lu}(K)$ by

$$\tilde{h}(z) := \begin{cases} f(z), & \text{for } z \in L \\ \tau_n^{-1}(z), & \text{for } z \in K_{\rho_m}(a_m)\,. \end{cases}$$

By Theorem 2.3 there exists $h \in \mathcal{H}_{lu}(\Omega)$ with $\|h - \tilde{h}\|_K < \varepsilon$. Then $h \in \mathcal{U}$ and, since $\tau_m(\psi_m(z)) \in K_{\rho_m}(a_m)$ for all $z \in \Omega$, we have

$$\begin{aligned} |h(\phi_m(z)) - g(z)| &\le |h(\phi_m(z)) - \psi_m(z)| + |\psi_m(z) - g(z)| \\ &< |h(\tau_m(\psi_m(z))) - \tau_m^{-1}(\tau_m(\psi_m(z)))| + \varepsilon < 2\varepsilon \end{aligned}$$

for all $z \in L$. This implies $h \circ \phi_m \in \mathcal{V}$. Hence we have shown, that the sequence of composition operators (C_{ϕ_n}) acts topologically transitively on $\mathcal{H}_{lu}(\Omega)$. We can now apply the universality criterion to show that there exists a (ϕ_n)-universal function $f \in \mathcal{H}_{lu}(\Omega)$. ■

In a sense, in Theorem 3.21 it is the sequence (ϕ_n) which behaves rather chaotically and not so much the universal function f. The rest of this section, we show that this is typical for finitely connected domains:

Theorem 3.22.
Let $\Omega \subseteq \mathbb{C}$ be a finitely connected domain, which is not simply connected, and let (ϕ_n) be an eventually injective sequence of locally univalent self-maps of Ω. Then no (ϕ_n)-universal function $f \in \mathcal{H}_{lu}(\Omega)$ exists.

The main difficulty in the proof is the case, when Ω is doubly connected. We need some facts about doubly connected domains, which can be found for example in [19, Chapter 6]. For $0 < r < R < \infty$ let $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$. A ring domain Ω is a doubly connected domain, such that the bounded component B_Ω of $\mathbb{C} \setminus \Omega$ is a continuum. Every ring domain Ω is conformally equivalent to some annulus $A_{r,R}$. The ratio $\frac{R}{r}$ is uniquely determined by Ω. As usual, we define the modulus of Ω by $\operatorname{mod}\Omega := \frac{1}{2\pi}\log\frac{R}{r}$. Alternatively, the modulus of Ω can be defined as the extremal length of the family of all cross-cuts in Ω. The modulus of a ring domain is conformally invariant and has the following useful monotonicity property: Let Ω_1, Ω_2 be ring domains with $\Omega_1 \subseteq \Omega_2$ and $B_{\Omega_1} \cap B_{\Omega_2} \neq \emptyset$. Then $\operatorname{mod}\Omega_1 \le \operatorname{mod}\Omega_2$ with equality if and only if $\Omega_1 = \Omega_2$.
We tactically will use the following fact: Let $\phi\colon A_{r,R} \to \Omega$ be a conformal mapping, assume $0 \notin \Omega$ and define $\gamma_\rho(t) = \phi(\rho e^{it})$ for $\rho \in (r, R)$. Then, since γ_ρ is a generator of the fundamental group of Ω, we have that $0 \in B_\Omega$ if and only if $n(\gamma_\rho, 0) \neq 0$ for one (and then for all) $\rho \in (r, R)$. We now can proof two auxiliary lemmas.

Lemma 3.23.
For $0 < r < R < \infty$ let $(\phi_n) \subseteq \mathcal{H}(A_{r,R})$ a sequence of zero-free injective mappings such that 0 lies in the bounded component of $\mathbb{C} \setminus \phi_n(A_{r,R})$ for every $n \in \mathbb{N}$. Suppose that there exists a compact set $K \subseteq A_{r,R}$ such that $\phi_n(K) \cap K \neq \emptyset$ for all $n \in \mathbb{N}$. Then (ϕ_n) has a non-constant accumulation point $\phi \in \mathcal{H}(A_{r,R})$.

Proof. Choose positive numbers $\rho_1 < \rho_2$ such that $\frac{\rho_2}{\rho_1} > \frac{R}{r}$. If for any $n \in \mathbb{N}$ we would have $\partial B_{\rho_j}(0) \subseteq \phi_n(A_{r,R})$ for $j = 1, 2$, then the assumptions on ϕ_n imply $A_{\rho_1,\rho_2} \subseteq \phi_n(A_{r,R})$. The monotonicity and the conformal invariance of the modulus of a ring domain then imply

$$\frac{1}{2\pi} \log \frac{\rho_2}{\rho_1} \leq \operatorname{mod} \phi_n(A_{r,R}) = \frac{1}{2\pi} \log \frac{R}{r}.$$

This is absurd, thus for each n, ϕ_n omits a complex number c_n with modulus either ρ_1 or ρ_2. The pigeonhole principle tells us, that at least one of this cases happens infinitely many times. Call this radius ρ.
We may without loss of generality assume, that $|c_n| = \rho$ holds for all $n \in \mathbb{N}$. If not we may instead consider an appropriate subsequence. Then the sequence $\left(\frac{\overline{c_n}}{|c_n|}\phi_n\right)$ omits the three values 0, ρ and ∞, and thus is normal. Hence, there exists a converging subsequence $\left(\frac{\overline{c_{n_k}}}{|c_{n_k}|}\phi_{n_k}\right)$ with limit $\psi \in \mathcal{H}(A_{r,R}) \cup \{\infty\}$. The assertion $\phi_{n_k}(K) \cap K \neq \emptyset$ implies $\psi \neq \infty$ and thus $\psi \in \mathcal{H}(A_{r,R})$. In addition we can assume, that $\frac{\overline{c_{n_k}}}{|c_{n_k}|}$ converges with limit $c \in \partial\mathbb{D}$. Then $\phi_{n_k} \to \phi := \overline{c} \cdot \psi$. Hurwitz Theorem tells us that ϕ is either constant or injective. If ϕ was constant, say $\phi(z) = a$ for all $z \in A_{r,R}$, then the assertion $\phi_n(K) \cap K \neq \emptyset$ and the uniform convergence of (ϕ_{n_k}) on K show $a \in K$ and in particular $|a| > r$. There exists $N \in \mathbb{N}$ such that

$$\max_{|z|=\frac{r+R}{2}} |\phi_N(z) - a| < \frac{r}{2}. \tag{3.1.7}$$

Define $\gamma(t) = \phi_N\left(\frac{r+R}{2}e^{it}\right)$ for $t \in [0, 2\pi]$. Since $|a| > r$, equation (3.1.7) then implies $n(\gamma, 0) = 0$. We can conclude that 0 lies in the unbounded component of $\phi_N(A_{r,R})$, a contradiction. ■

Lemma 3.24.
Let (ϕ_n) be an eventually injective sequence of holomorphic self-maps of a domain Ω. Suppose that for every compact set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that $K \cap \phi_n(K) = \emptyset$ for every $n \geq N$.

(i) If $\Omega = A_{r,R}$ then there exists a compact $\mathcal{O}$-convex set K and $N_1 \in \mathbb{N}$ such that $\phi_n(K)$ is not $\mathcal{O}$-convex for every $n \geq N_1$.

(ii) If $\Omega = \mathbb{D} \setminus \{0\}$ and $\phi_n(\partial B_{1/2}(0))$ is $\mathcal{O}$-convex for all $n \in \mathbb{N}$, then $\phi_n \to 0$.

Proof. (i): First let $\Omega = A_{r,R}$. Choose $r < \rho_1 < \rho_2 < R$ such that

$$\frac{\rho_1}{r} = \frac{R}{\rho_2} < \frac{\rho_2}{\rho_1}.$$

Then by the assumptions there exists $N \in \mathbb{N}$ such that ϕ_n is injective on A_{ρ_1,ρ_2} and $\overline{A_{\rho_1,\rho_2}} \cap \phi_n(\overline{A_{\rho_1,\rho_2}}) = \emptyset$ for every $n \geq N$. Assume that $\phi_n(\overline{A_{\rho_1,\rho_2}})$ is $\mathcal{O}$-convex for some $n \geq N$. Then $D := \phi_n(A_{\rho_1,\rho_2})$ would be ring domain and the bounded component of $\mathbb{C} \setminus D$ would contain 0. On the other hand, either $D \subseteq A_{r,\rho_1}$ or $D \subseteq A_{\rho_2,R}$. The monotonicity and the conformal invariance of the modulus of a ring domain then would imply

$$\frac{1}{2\pi} \log \frac{\rho_1}{r} > \operatorname{mod} D = \frac{1}{2\pi} \log \frac{\rho_2}{\rho_1}.$$

This is a contradiction, thus for $n \geq N$ the set $\phi_n(\overline{A_{\rho_1,\rho_2}})$ can not be $\mathcal{O}$-convex in Ω.
(ii): Let $K \subseteq \mathbb{D} \setminus \{0\}$ be compact, $\varepsilon_0 = \min_{z \in K} |z|$ and $\varepsilon \in (0, \varepsilon_0)$. Choose $\rho \in (1/2, 1)$ such that $\frac{\rho}{\varepsilon} > 1/\rho$ and $K \subseteq A_{\varepsilon,\rho}$.
Let $D_n = \phi_n(A_{\varepsilon,\rho})$. From the assertions on (ϕ_n) follows, that there exists $N \in \mathbb{N}$ such that D_n is ring domain with $D_n \cap A_{\varepsilon,\rho} \neq 0$ and 0 lies in the bounded component of the complement of D_n for all $n \geq N$. Then for each $n \geq N$ we have either $D_n \subseteq B_\varepsilon(0)$ or $D_n \subseteq A_{\rho,1}$. Similarly as above, we can use the conformal invariance and the monotonicity of the modulus of a ring domain to rule out the case $D_n \subseteq A_{\rho,1}$. Hence $D_n \subseteq B_\varepsilon(0)$ for all $n \geq N$, and we can conclude $\phi_n \to 0$ uniformly in K. ∎

Proof of Theorem 3.22. For the sake of contradiction assume that there exists a (ϕ_n)-universal function $f \in \mathcal{H}_{lu}(\Omega)$ and let $p \geq 1$ be the number of bounded components of $\mathbb{C} \setminus \Omega$.
(1) Suppose $2 \leq p$ and let K be a connected compact $\mathcal{O}$-convex set with exactly p holes. Then by theorem 3.14 there exists $N \in \mathbb{N}$ such that ϕ_N is injective on K, $\phi_N(K) \cap K = \emptyset$ and $\phi_N(K)$ is $\mathcal{O}$-convex. By the hole invariance principle the set $\phi_N(K)$ is connected and has exactly p holes. Since K is connected, the set $\phi_N(K)$ lies completely in one connected component of $\mathbb{C} \setminus K$. It follows, that the set $\phi_N(K) \cup K$ has $p + (p-1) = 2p - 1 > p$ holes. Thus $\phi_N(K) \cup K$ can not be $\mathcal{O}$-convex, a contradiction to Lemma 3.9
(2) We are remained to proof the case, that Ω is doubly connected. After applying a conformal mapping, we can assume that Ω is either an annulus, the punctured disc, or the punctured plane. We treat each case separately.

(i) First let $\Omega = A_{r,R}$.

By Theorem 3.14 the sequence (ϕ_n) is run-away and for every $N \in \mathbb{N}$ and every compact $\mathcal{O}$-convex set $K \subseteq \Omega$ there exists $n \geq N$ such that ϕ_n is injective on K, $\phi_n(K) \cap K = \emptyset$ and $\phi_n(K)$ is $\mathcal{O}$-convex. Lemma 3.24 shows, that such a sequence can not exist.

(ii) Next let $\Omega = \mathbb{D} \setminus \{0\}$.
By Lemma 3.15 there exists a sequence (ϕ_{n_k}) such that the set $\phi_{n_k}(\partial B_{1/2}(0))$ is $\mathcal{O}$-convex for each $k \in \mathbb{N}$ and $f \circ \phi_{n_k} \to 0$ uniformly on $\partial B_{1/2}(0)$.

Claim I: *ϕ_{n_k} is run-away.*
Otherwise, we could find $0 < r < 1/2 < R < 1$ such that $\phi_{n_k}(\overline{A_{r,R}}) \cap \overline{A_{r,R}} \neq \emptyset$ for all $k \in \mathbb{N}$. Since (ϕ_n) is eventually injective and since $\phi_{n_k}(\partial B_{1/2}(0))$ is $\mathcal{O}$-convex, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ the function ϕ_{n_k} is injective on $A_{r,R}$ and 0 lies in the bounded component of the complement of $\phi_{n_k}(A_{r,R})$. Lemma 3.23 tells us that (ϕ_{n_k}) has a non-constant accumulation point $\phi \in \mathcal{H}(A_{r,R})$. We can use that $f \circ \phi_{n_k} \to 0$ on $\partial B_{1/2}(0)$ to conclude $f \circ \phi \equiv 0$ and hence $f \equiv 0$. This is absurd.

Claim II: *0 is a removable singularity of f with $f(0) = 0$.*
Let $3 \leq N \in \mathbb{N}$. Note that Claim I and Lemma 3.24 tell us that $\phi_{n_k} \to 0$. This allows us to find $j \in \mathbb{N}$ such that $\phi_{n_j}(\partial B_{1/2}(0)) \subseteq B_{1/N}(0)$. Since $f \circ \phi_{n_k} \to 0$ we may choose j such that in addition we have $\max_{|z|=1/2} |f(\phi_{n_j}(z))| \leq 1$. Let D_N be the domain bounded by $\partial B_{1/2}(0)$ and $\phi_{n_j}(\partial B_{1/2}(0))$. Then, since $\phi_{n_k}(\partial B_{1/2}(0))$ is $\mathcal{O}$-convex, we have $D_N \subseteq \mathbb{D} \setminus \{0\}$, thus $f \in \mathcal{H}(D_N)$. On the other hand note that $A_{1/N,1/2} \subseteq D_N$ and

$$\max_{\xi \in \partial D_n} |f(\xi)| \leq M := \max\{1, \|f\|_{\partial B_{1/2}(0)}\}.$$

The maximum principle implies, that f is bounded on D_N by M. The limit $N \to \infty$ tells us, that f is bounded in $B_{1/2}(0) \setminus \{0\}$. This shows, that the singularity of f is removable. Then we have

$$f(0) = \lim_{k \to \infty} (f \circ \phi_{n_k})(1/2) = 0$$

and we have established the claim.
Now we are able to proof, that f can not be (ϕ_n)-universal. We first can apply the case $c = \infty$ of Lemma 3.15 to the set $\partial B_{1/2}(0)$. This gives us a subsequence (ϕ_{n_j}) of (ϕ_n) with $f \circ \phi_{n_j} \to \infty$ uniformly in $\partial B_{1/2}(0)$, such that $\phi_{n_j}(\partial B_{1/2}(0))$ is $\mathcal{O}$-convex for every $j \in \mathbb{N}$. Similarly as in Claim I above we can show, that $\phi_{n_j} \to 0$ uniformly on $\partial B_{1/2}(0)$. Claim II now yields $f \circ \phi_{n_j} \to f(0) \neq \infty$. This is a contradiction.

(iii) Finally we consider $\Omega = \mathbb{C} \setminus \{0\}$.
The proof is similar to the case, when $\Omega = \mathbb{D} \setminus \{0\}$; we only point out the necessary modifications. First, we can use Lemma 3.15 to obtain a subsequence (ϕ_{n_k}) such that

$\phi_{n_k}(\partial\mathbb{D})$ is $\mathcal{O}$-convex for every $k \in \mathbb{N}$ and $f \circ \phi_{n_k} \to 0$ uniformly on $\partial\mathbb{D}$. Similar arguments as in Claim I above show that (ϕ_{n_k}) is run-away. Then (ϕ_{n_k}) has a subsequence which converges either to 0 or to ∞ locally uniformly in $\mathbb{C}\setminus\{0\}$. Note that the function $f(1/z)$ is universal for the sequence $(1/\phi_n)$. This shows that, after we pass to another subsequence, there is no harm assuming that $\phi_{n_k} \to 0$ locally uniformly. As in Claim II above, it follows that 0 is a removable singularity of f with $f(0) = 0$.
By another application of Lemma 3.15 we also can find a subsequence (ϕ_{n_j}) of (ϕ_n) such that $\phi_{n_j}(\partial\mathbb{D})$ is $\mathcal{O}$-convex for every $j \in \mathbb{N}$ and $f \circ \phi_{n_j} \to 1$ uniformly on $\partial\mathbb{D}$. Again, similarly as in Claim I above one can show that (ϕ_{n_j}) is run-away. Then, since $f(0) = 0$, we must have $\phi_{n_j} \to \infty$ locally uniformly. Similarly as in Claim II above we can show, that ∞ is a removable singularity of f with $f(\infty) = 1$. However, then f would be a non-constant bounded holomorphic function in $\mathbb{C}$, what is absurd. ∎

Corollary 3.25.
Let Ω be a finitely connected domain, which is not simply connected and let ϕ be a holomorphic self-map of Ω. Then there is no $\phi^{[n]}$-universal in $\mathcal{H}_{lu}(\Omega)$.

Proof. Suppose that a $\phi^{[n]}$-universal in $\mathcal{H}_{lu}(\Omega)$ exists. Then Remark 3.13 tells us, that ϕ has to be injective. Thus the sequence $(\phi^{[n]})$ is eventually injective and Theorem 3.22 tells us, that no $\phi^{[n]}$-universal function can exists. ∎

3.1.4. Infinitely Connected Domains

Our next step is to consider domains of infinite connectivity. Note that the next theorem would be true without the assumption that Ω is of infinite connectivity. It would be enough to assume that Ω is not doubly connected. However, for simply connected domains Theorem 3.17 is stronger. For finitely domains with connectivity ≥ 3, the proof of Theorem 3.22 shows, that a family Φ of holomorphic self-maps can not match the assumptions we put on Φ in the next theorem.

Theorem 3.26.
Let Ω be a domain in $\mathbb{C}$ of infinite connectivity and let Φ be a family of locally univalent self-maps of Ω. Suppose that for every $\mathcal{O}$-convex set K in Ω there exists $\phi \in \Phi$ such that:

(i) ϕ is injective in a neighborhood of K,

(ii) $\phi(K)$ is $\mathcal{O}$-convex and $\phi(K) \cap K = \emptyset$.

Then there exists a Φ-universal function in $\mathcal{H}_{lu}(\Omega)$ and the set of all such functions is a dense G_δ-subset of $\mathcal{H}_{lu}(\Omega)$.

Proof. We show that the family $(C_\phi)_\Phi$ of composition operators act topologically transitively on $\mathcal{H}_{lu}(\Omega)$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{H}_{lu}(\Omega)$ be non-empty open sets and let $f \in \mathcal{U}, g \in \mathcal{V}$. There exists an $\mathcal{O}$-convex compact set $L \subseteq \Omega$ and $\varepsilon > 0$, such that

$$U_{L,\varepsilon}(f) \subseteq \mathcal{U}, \qquad U_{L,\varepsilon}(g) \subseteq \mathcal{V}.$$

We can enlarge L such that L is connected and has at least two holes. Then we can apply Lemma 3.10 to find $\phi \in \Phi$ such that $M := \phi(L) \cup L$ is $\mathcal{O}$-convex and $\phi(L) \cap L = \emptyset$. We now define a function $\tilde{h} \in \mathcal{H}_{lu}(M)$ by

$$\tilde{h}(z) := \begin{cases} f(z), & \text{if } z \in L \\ g(\phi^{-1}(z)), & \text{if } z \in \phi(L). \end{cases}$$

By Theorem 2.3 there exists $h \in \mathcal{H}_{lu}(\Omega)$ with $\|h - \tilde{h}\|_M < \varepsilon$. This implies

$$\|f - h\|_L < \varepsilon, \qquad \text{and} \qquad \|g - h \circ \phi\|_L < \varepsilon.$$

Hence $h \in \mathcal{U}$ and $C_\phi(h) \in \mathcal{V}$. We are now in the position to apply Theorem 3.8 and the proof is complete. ■

We can combine Theorem 3.14 and Theorem 3.26. The result gives a necessary and sufficient condition for the existence of (ϕ_n)-universal functions, provided the sequence (ϕ_n) is eventually injective. Thus the following theorem complements theorems 3.17 and 3.22.

Theorem 3.27.
Let Ω be a domain in $\mathbb{C}$ of infinite connectivity and let (ϕ_n) be an eventually injective sequence of locally univalent self-maps of Ω. Then the following assertions are equivalent:

(i) For every $\mathcal{O}$-convex compact set K in Ω and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\phi_n(K) \cap K = \emptyset$ and $\phi_n(K)$ is $\mathcal{O}$-convex.

(ii) There exists a (ϕ_n)-universal function in $\mathcal{H}_{lu}(\Omega)$.

We now are able to prove Theorem 1.8. In fact we can state a slightly more general statement. Note that the reason why we have to exclude $\mathbb{C} \setminus \{0\}$ in the following Corollary lies in Theorem 3.22.

Corollary 3.28.
Let Ω be a domain in $\mathbb{C}$ which is not conformally equivalent to $\mathbb{C} \setminus \{0\}$ and let $(\phi_n) \subseteq \operatorname{Aut}(\Omega)$. Then there exists a (ϕ_n)-universal function in $\mathcal{H}_{lu}(\Omega)$ if and only if (ϕ_n) is run-away.

Proof. First note that (ϕ_n) is eventually injective. Note that the existence of a run-away sequence in $\operatorname{Aut}(\Omega)$ implies that $\operatorname{Aut}(\Omega)$ is non-compact, so that Ω is either simply connected, conformal equivalent to $\mathbb{C} \setminus \{0\}$, or of infinite connectivity. In the first case we can use Theorem 3.17. The second case is ruled out by hypothesis. For the third case, let K be a $\mathcal{O}$-convex compact set in Ω. Then, since (ϕ_n) is run-away, we can find $N \in \mathbb{N}$ such that $\phi_N(K) \cap K = \emptyset$. Lemma 3.11 tells us, that $\phi_N(K)$ is $\mathcal{O}$-convex in Ω. We are now in the position to apply Theorem 3.27. ■

Example 3.29. ([36, Example 2.1], [44, Example 1]) Let $K := K_{1/10}(0)$,

$$\phi(z) := \frac{z - 1/2}{1 - (1/2)z} \in \operatorname{Aut}(\mathbb{D})$$

and define

$$\Omega := \mathbb{D} \setminus \left[\bigcup_{j \in \mathbb{Z}} \phi^{[j]}(K) \right].$$

Then Ω is a bounded domain of infinity connectivity and $\phi^{[j]}$ is a conformal automorphism of Ω for all $j \in \mathbb{Z}$. A straight forward computation shows $\phi^{[j]} \to -1$ locally uniformly for $j \to \infty$. Thus Ω is an example for a bounded domain of infinite connectivity with non-compact automorphism group, and $(\phi^{[j]})$ is a run-away sequence in $\operatorname{Aut}(\Omega)$.

Corollary 3.30.
Let Ω be a complex domain of infinite connectivity and let ϕ be a holomorphic self-map of Ω. Then the following are equivalent:

(i) There exists a $\phi^{[n]}$-universal function $f \in \mathcal{H}_{lu}(\Omega)$.

(ii) ϕ is injective and for every compact $\mathcal{O}$-convex set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that $\phi^{[N]}(K) \cap K = \emptyset$ and $\phi^{[N]}(K)$ is $\mathcal{O}$-convex.

The function ϕ in example 3.29 matches condition (ii) above. An example for a non-surjective function can be constructed in a similar way:

Example 3.31. ([26, Example 3.17]) Let $\psi\colon \mathbb{D} \to \mathbb{D}$ be defined by

$$\psi(z) = \frac{z}{4} + \frac{3}{4}$$

Note that $\psi(\mathbb{D}) = B_{1/4}(3/4)$. The sets

$$K_n := \psi^{[n]}(K_{1/2}(0))$$

are pairwise disjoint. Define

$$\Omega := \mathbb{D} \setminus \left(\bigcup_{n\in\mathbb{N}} K_n \right)$$

and let ϕ be the restriction of ψ to Ω. Then $\phi \notin \mathrm{Aut}(\Omega)$ and ϕ satisfies condition (ii) of Corollary 3.30.

Montes [42] has studied universal functions on Riemann surfaces. Since we have established a Runge-type theorem for locally univalent functions on Riemann surfaces, we can state and proof a version for locally univalent functions. We already have seen, that the existence of universal functions depends on the connectivity of a domain $\Omega \subseteq \mathbb{C}$. For a Riemann surfaces R, one has to find an intrinsic way in order to measure the "connectivity" of R. This can be done with the help of the Freudenthal compactification of R (see Theorem C.5).

Theorem 3.32.
Let R be a non-compact Riemann surface, which has not exactly two distinct Freudenthal ends, and Φ be a family of holomorphic self-maps of R. Suppose that for every compact $\mathcal{O}$-convex set $K \subseteq R$ there exists $\phi \in \Phi$ such that ϕ is injective on K, the set $\phi(K)$ is $\mathcal{O}$-convex and $\phi(K) \cap K = \emptyset$. Then there exists a Φ-universal function $f \in \mathcal{H}_{lu}(R)$.

The proof is similar to the proof of Theorem 3.27, and we only point out the necessary modifications: A version of Lemma 3.10 for Riemann surfaces can be found in [42, Lemma 2.17]. Instead of Theorem 2.3, one has to use Theorem 2.4.

Remark 3.33. Theorem 3.22 tells us, that we can not drop the assumption on the number of Freudenthal ends in Theorem 3.32. Also note that Theorem 3.32 would still be true, if the assumption holds only for compact $\mathcal{O}$-convex subsets of R which meet some additional regularity conditions. For more details the reader should consult [42].

Many authors have studied universal functions in several complex variables (see [13], [1], [7] and the references therein) and even beyond on Stein-manifolds [2]. To conclude this section, we formulate a related problem. Note that the answer most certainly depends on the geometry of the domain Ω.

Problem 4.
Are there universal functions in several complex variables without critical points? More precisely: Let Ω be a domain of holomorphy in $\mathbb{C}^n$ (ore even a Stein-manifold) and let $\mathcal{G}$ be the set of all function $f\colon \Omega \to \mathbb{C}^n$ without critical points. Are there universal functions in $\mathcal{G}$?

3.2. Bounded Locally Univalent Universal Functions

We now turn our attention to universal functions in $\mathcal{B}_{lu}(\mathbb{D})$. We first give a general scheme, how to obtain universal functions in $\mathcal{G}_{l.u.}$ from universal functions in $\mathcal{G} \subseteq \mathcal{M}(\mathbb{D})$. We use the following construction: For a non-constant function $f \in \mathcal{M}(\mathbb{D})$ we let $\Omega_f \subseteq \mathbb{D}$ be the set of all non-critical points of f. If $0 \in \Omega_f$, then by the uniformization theorem there exists a unique holomorphic universal covering map $\Psi_f\colon \mathbb{D} \to \Omega_f$ with $\Psi_f(0) = 0 < \Psi_f'(0)$. Further, if $\Psi\colon \mathbb{D} \to \Omega_f$ is another holomorphic covering map, then there exists $T \in \operatorname{Aut}(\mathbb{D})$ with $\Psi = \Psi_f \circ T$.

Proposition 3.34.
Let (f_n) be a sequence in $\mathcal{M}_{lu}(\mathbb{D})$ with $\lim_{n\to\infty} f_n = f \in \mathcal{M}(\mathbb{D})$ locally uniformly. Suppose that f is not constant and $0 \in \Omega_f$. Then

$$f_n \circ \Psi_{f_n} \to f \circ \Psi_f \qquad \textit{locally uniformly in } \mathbb{D}.$$

Proof. A straight forward application of Hurwitz's Theorem tells us, that

$$\Omega_{f_n} \to \Omega_f$$

in the sense of kernel convergence. This in particular implies $0 \in \Omega_n$ for all but finitely many $n \in \mathbb{N}$. A well-known Theorem of Hejhal [31] tells us $\Psi_{f_n} \to \Psi_f$ and thus $f_n \circ \Psi_{f_n} \to f \circ \Psi_f$. ■

Theorem 3.35.
Let $\mathcal{G} \subseteq \mathcal{M}(\mathbb{D})$ and let G be a non-constant universal function in $\mathcal{G}$ and let $\Psi\colon \mathbb{D} \to \Omega_G$ be a holomorphic universal covering map. Suppose that $F := G \circ \Psi \in \mathcal{G}$. Then F is universal in $\mathcal{G}_{l.u.}$.

Proof. Let $f \in \mathcal{G}_{l.u.}$. Note that $\Omega_f = \mathbb{D}$ and $\Psi_f = \mathrm{id}_{\mathbb{D}}$.
(1) We first assume $0 \in \Omega_G$ and that $\Psi = \Psi_G$.

The universality of G tells us, that there exists a sequence $(\alpha_n) \in \text{Aut}(\mathbb{D})$ with

$$G \circ \alpha_n \to f \qquad \text{locally } \chi\text{-uniformly in } \mathbb{D}. \tag{3.2.1}$$

Let $\Psi_n := \Psi_{G \circ \alpha_n}$. Proposition 3.34 and (3.2.1) allow us to conclude

$$G \circ \alpha_n \circ \Psi_n \to f \circ \Psi_f = f \tag{3.2.2}$$

locally χ-uniformly in $\mathbb{D}$. Now we observe $\alpha_n(\Omega_{G \circ \alpha_n}) = \Omega_G$, and hence $\alpha_n \circ \Psi_n$ is a universal covering map from $\mathbb{D}$ onto Ω_G. This implies that there is $\phi_n \in \text{Aut}(\mathbb{D})$ such that

$$\alpha_n \circ \Psi_n = \Psi_G \circ \phi_n.$$

Using (3.2.2) we get that

$$G \circ \Psi_G \circ \phi_n = G \circ \alpha_n \circ \Psi_n \to f$$

locally χ-uniformly in $\mathbb{D}$.
(2) We now show, that there exists a sequence (S_n) in $\text{Aut}(\mathbb{D})$ with $F \circ S_n \to f$ locally χ-uniformly.
By precomposing G with a disk automorphism we may assume 0 is not a critical point of G, and this does not change the fact, that $\{G \circ \phi \,:\, \phi \in \text{Aut}(\mathbb{D})\}$ is dense in $\mathcal{G}$. There exists $T \in \text{Aut}(\mathbb{D})$ such that $\Psi_G \circ T = \Psi$. Thus from what we have already shown, there exists a sequence $(\phi_n) \subset \text{Aut}(\mathbb{D})$ with

$$F \circ (T^{-1} \circ \phi_n) = G \circ \Psi \circ (T^{-1} \circ \phi_n) = G \circ \Psi_G \circ \phi_n \to f$$

locally χ-uniformly in $\mathbb{D}$. This shows, that we can put $S_n := T^{-1} \circ \phi_n \in \text{Aut}(\mathbb{D})$ for each $n \in \mathbb{N}$. ■

For example, we now can proof Theorem 1.9:

Corollary 3.36.
There exists a universal function in $\mathcal{B}_{lu}(\mathbb{D})$.

Proof. By Theorem D there exists a universal Blaschkeproduct $B \in \mathcal{B}(\mathbb{D})$. Clearly B is not constant, and we may assume $0 \in \Omega_B$. By Theorem 3.35 the function $F := B \circ \Psi_B \in \mathcal{B}_{lu}(\mathbb{D})$ is universal for $\mathcal{B}_{lu}(\mathbb{D})$. ■

The rest of this section we wish to establish a necessary and sufficient condition for a family $\Phi \subseteq \mathcal{B}_{lu}(\mathbb{D})$ to admit a bounded universal locally univalent function.

The first step is to establish a lemma, which will allow us to deal with run-away sequences in $\operatorname{Aut}(\mathbb{D})$. We introduce some notation. Let

$$H := \overline{\mathbb{D}} \cup [1,2] \cup K_1(3)\,, \tag{3.2.3}$$

and let Ω_ε be the ε-neighborhood of H, that is

$$\Omega_\varepsilon := \{z \in \mathbb{C} \,:\, \operatorname{dist}(z,\, H) < \varepsilon\}\,. \tag{3.2.4}$$

Lemma 3.37.
Let (T_n) be a sequence in $\operatorname{Aut}(\mathbb{D})$. Suppose that there are $\phi, \omega \in \mathbb{R}$ such that

$$\lim_{n\to\infty} T_n(0) = e^{i\varphi} \qquad \textit{and} \qquad \lim_{n\to\infty} \arg T_n'(0) = \omega.$$

Then there exists $\vartheta \in \mathbb{R}$, a sequence (ε_n) of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and a sequence of conformal mappings $f_n \colon \mathbb{D} \to \Omega_{\varepsilon_n}$, where Ω_{ε_n} is defined by (3.2.4), such that

$$f_n \to e^{-i\varphi} \operatorname{id}_{\mathbb{D}}, \qquad f_n \circ T_n \to e^{i\vartheta} \operatorname{id}_{\mathbb{D}} + 3$$

locally uniformly in $\mathbb{D}$.

Proof. Let $x_n := |T_n(0)|$, $\varphi_n := \arg T_n(0)$, $\omega_n := \arg T_n'(0)$ and $\vartheta := \omega - \varphi$. For every $\varepsilon > 0$ there exists a unique conformal mapping $f_\varepsilon \colon \mathbb{D} \to \Omega_\varepsilon$ with $f_\varepsilon(0) = 0 < f_\varepsilon'(0)$. The symmetry of Ω_ε and the normalization of f_ε imply that $f_\varepsilon^{-1}(3) \in (0,1)$ and $f_\varepsilon'(x) > 0$ for $x \in (-1,1)$.
(1) We show that there exists a sequence (ε_n) of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and $f_{\varepsilon_n}(x_n) = 3$ for every $n \in \mathbb{N}$.
Observe that Ω_ε depends continuously on ε with

$$\lim_{\varepsilon\to 0} \Omega_\varepsilon = \mathbb{D} \qquad \text{and} \qquad \lim_{\varepsilon\to\infty} \Omega_\varepsilon = \mathbb{C}$$

in the sense of kernel convergence. Recall, that $d_\Omega(\cdot,\cdot)$ is the hyperbolic distance on Ω. Consider the function

$$d \colon (0,\infty) \to (0,\infty),\ d(\varepsilon) := d_{\Omega_\varepsilon}(0,3).$$

The fact that d_Ω depends continuously on Ω and the domain monotonicity of the hyperbolic distance imply that d is a strictly decreasing continuous function with

$$\lim_{\varepsilon\to 0} d(\varepsilon) = \infty \qquad \text{and} \qquad \lim_{\varepsilon\to\infty} d(\varepsilon) = 0.$$

Thus there exists $\varepsilon_n > 0$ such that $d(\varepsilon_n) = d_{\mathbb{D}}(0, x_n)$. The fact $\lim_{n\to\infty} d_{\mathbb{D}}(0, x_n) = \infty$ allows us to conclude $\lim_{n\to\infty} \varepsilon_n = 0$. Now let $z_n = f_{\varepsilon_n}^{-1}(3)$. The conformal invariance

of the hyperbolic distance implies $d_{\mathbb{D}}(0, z_n) = d(\varepsilon_n) = d_{\mathbb{D}}(0, x_n)$. The properties of f_{ε_n} imply $z_n \in (0,1)$ and thus $z_n = x_n$.
(2) We now define

$$f_n \colon \mathbb{D} \to \Omega_{\varepsilon_n},\; f_n(z) = f_{\varepsilon_n}\left(e^{-i\varphi_n} z\right).$$

Then $f_n(0) = 0$ and $\arg f_n'(0) = -\varphi_n$. Caratheodory's kernel convergence theorem and the fact that $\lim_{n\to\infty} e^{-i\varphi_n} = e^{-i\varphi}$ imply

$$f_n \to e^{-i\varphi}\,\mathrm{id}_{\mathbb{D}}\,.$$

On the other hand note that

$$(f_n \circ T_n)'(0) = e^{-i\varphi_n} f_{\varepsilon_n}'(x_n) \cdot T_n'(x_n).$$

Thus we have

$$f_n(T_n(0)) = f_n(x_n) = 3 \qquad \text{and} \qquad \arg(f \circ T_n)'(0) = \omega_n - \varphi_n.$$

Another application of Caratheodory's kernel convergence theorem implies

$$f_n \circ T_n \to e^{i\vartheta}\,\mathrm{id}_{\mathbb{D}} + 3$$

locally uniformly in $\mathbb{D}$. ∎

We now can proof the main result of this section.

Theorem 3.38.
Let Φ be a family of locally univalent self-maps of $\mathbb{D}$. The following assertions are equivalent:

(i) There exist a run-away sequence $(\phi_n) \subseteq \Phi$ and $z_0 \in \mathbb{D}$ with

$$\lim_{n\to\infty} \phi_n^* \lambda_{\mathbb{D}}(z_0) = \lambda_{\mathbb{D}}(z_0).$$

(ii) There exists a Φ-universal function $F \in \mathcal{B}_{lu}(\mathbb{D})$.

In addition, if one of the assertion above is true, the set of all Φ-universal functions in $\mathcal{B}_{lu}(\mathbb{D})$ is a dense G_δ-subset of $\mathcal{B}_{lu}(\mathbb{D})$.

Proof. *(ii)⇒(i)*: Suppose that $F \in \mathcal{B}_{lu}(\mathbb{D})$ is Φ-universal. For $n \in \mathbb{N}$ let

$$S_n(z) := \frac{z + 1 - 1/n}{1 + (1 - 1/n)z} \in \mathrm{Aut}(\mathbb{D})\,.$$

For each $n \in \mathbb{N}$ there exists a sequence $(\phi_{m,n})_m$ with

$$F \circ \phi_{m,n} \to S_n, \qquad \text{for } m \to \infty.$$

This allows us to extract a sequence $(\phi_n) \subseteq \Phi$ such that for each $n \in \mathbb{N}$ we have

$$|F(\phi_n(0)) - S_n(0)| < \frac{1}{n}, \tag{3.2.5}$$

$$|(F \circ \phi_n)^* \lambda_{\mathbb{D}}(0) - S_n^* \lambda_{\mathbb{D}}(0)| < \frac{1}{n}. \tag{3.2.6}$$

Ahlfor's Lemma shows us

$$(F \circ \phi_n)^* \lambda_{\mathbb{D}} \le \phi_n^* \lambda_{\mathbb{D}} \le \lambda_{\mathbb{D}} = S_n^* \lambda_{\mathbb{D}}.$$

Thus (3.2.6) implies $\lim_{n\to\infty} \phi_n^* \lambda_{\mathbb{D}}(0) = \lambda_{\mathbb{D}}(0)$.
We still are remained to show that (ϕ_n) is run-away. For the sake of contradiction, assume the contrary. Then we have

$$\limsup_{n\to 0} |\phi_n(0)| < 1.$$

Montel's Theorem then tells us, that there exists a subsequence (ϕ_{n_k}) of (ϕ_n) and $\phi \in \mathcal{B}(\mathbb{D})$ with

$$\phi_{n_k} \to \phi \qquad \text{and} \qquad \phi(0) \in \mathbb{D}.$$

Then (3.2.5) tells us

$$1 = \lim_{k\to\infty} |F(\phi_{n_k}(0))| = |F(\phi(0))|.$$

The maximum principle implies, that F is constant, a contradiction to the fact that F is universal.

We split the proof of *(i)⇒(ii)* into two parts.
(1) Let (T_n) be a run-away sequence in $\mathrm{Aut}(\mathbb{D})$. We show, that there exists a (T_n)-universal function in $\mathcal{B}_{lu}(\mathbb{D})$.
We can extract a subsequence of (T_n) which fulfills the assertion of Lemma 3.37, and we may as well assume that (T_n) itself fulfills the assertions. Let φ, ϑ, (ε_n) and (f_n) be as in the conclusion of Lemma 3.37.
We show that the sequence of composition operators (C_{T_n}) acts topologically transitively on $\mathcal{B}_{lu}(\mathbb{D})$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}_{lu}(\mathbb{D})$ be non-empty and open. Then there exists $f \in \mathcal{U} \cap \mathcal{B}_{lu}(\overline{\mathbb{D}})$, $g \in \mathcal{V} \cap \mathcal{B}_{lu}(\overline{\mathbb{D}})$, a compact set $L \subseteq \mathbb{D}$ and $\delta > 0$ such that

$$U_{L,2\delta}(f) \subseteq \mathcal{U}, \qquad U_{L,2\delta}(g) \subseteq \mathcal{V}.$$

Now choose $1 < R < \frac{3}{2}$ such that $f, g \in \mathcal{B}_{lu}(K_R(0))$ and a continuous function

$$h\colon [R, 3-R] \to \mathbb{D}$$

with $h(R) = f(e^{i\varphi}R)$ and $h(3-R) = g(-e^{-i\vartheta}(3-R))$. Define

$$K \coloneqq K_R(0) \cup [R, 3-R] \cup K_R(3)$$

and

$$v\colon K \to \mathbb{D},\, v(z) \coloneqq \begin{cases} f\left(e^{i\varphi}z\right), & z \in K_R(0) \\ g\left(e^{-i\vartheta}(z-3)\right), & z \in K_R(3) \\ h(z), & z \in [R, 3-R]. \end{cases}$$

Then v is a continuous function, holomorphic in K° and $|v(z)| < 1$ for all $z \in K$. The complement of K is connected. Thus by Mergelyan's theorem there exists a polynomial q with

$$\|q - v\|_K < \min\left\{\frac{\delta}{2}, \frac{1 - \|v\|_K}{2}\right\}. \tag{3.2.7}$$

Hurwitz's theorem tells us, that we can assume that q has no critical points in an open neighborhood of $\overline{\mathbb{D}} \cup K_1(3)$. If q has critical points $w_1, \dots, w_k \in [1, 2]$ we define

$$p_n(z) \coloneqq q'(z) \prod_{j=1}^{k} \frac{\left(z - w_j - \frac{i}{n}\right)^{n_j}}{(z - w_j)^{n_j}}$$

where n_j is the multiplicity of the zero of q' at w_j. Then $\lim_{n\to\infty} p_n = q'$ locally uniformly on $\mathbb{C}$ and we can choose $N \in \mathbb{N}$ such that the function p defined by

$$p(z) \coloneqq \int_0^z p_N(w)\, dw + q(0)$$

has the property

$$\|p - q\|_K < \min\left\{\frac{\delta}{2}, \frac{1 - \|v\|_K}{2}\right\}.$$

Then equation (3.2.7) implies

$$\|p - v\|_H < \delta \qquad \text{and} \qquad \|p\|_H < 1 \tag{3.2.8}$$

where H as defined in (3.2.3). Further we have $p' = p_N \neq 0$ on H. A continuity argument now tells us that there exists $\varepsilon > 0$ such that $p \in \mathcal{B}_{lu}(\Omega_\varepsilon)$. We can assume $\varepsilon_n < \varepsilon$ for all $n \in \mathbb{N}$. Define $g_n \coloneqq p \circ f_n \in \mathcal{B}_{lu}(\mathbb{D})$. Then Lemma 3.37 implies

$$g_n \to p \circ (e^{-i\varphi}\, \mathrm{id}_{\mathbb{D}}) \qquad \text{and} \qquad g_n \circ T_n \to p \circ (e^{i\vartheta}\, \mathrm{id}_{\mathbb{D}} + 3).$$

It follows that there exists $N \in \mathbb{N}$ such that

$$|g_N(z) - p(e^{-i\varphi}z)| < \delta \tag{3.2.9}$$

$$|g_N(T_N(z)) - p(e^{i\vartheta}z + 3)| < \delta. \tag{3.2.10}$$

for all $z \in L$. On the other hand note that the definition of v shows us that

$$|p(e^{-i\varphi}z) - v(e^{-i\varphi}z)| = |p(e^{-i\varphi}z) - f(z)| < \delta \tag{3.2.11}$$

$$|p(e^{i\vartheta}z + 3) - v(e^{i\vartheta}z + 3)| = |p(e^{i\vartheta}z + 3) - g(z)| < \delta \tag{3.2.12}$$

for all $z \in \overline{\mathbb{D}}$. Combining (3.2.9), (3.2.10), (3.2.11) and (3.2.12) yields

$$\|g_N - f\|_L < 2\delta, \qquad \|g_N \circ T_N - g\|_L < 2\delta,$$

so that $g_N \in \mathcal{U}$ and $g_N \circ T_N \in \mathcal{V}$. Thus we have shown, that the sequence of composition operators (C_{T_n}) acts topologically transitively on $\mathcal{B}_{lu}(\mathbb{D})$. The universality criterion now tells us, that there exists a dense G_δ-subset of $\mathcal{B}_{lu}(\mathbb{D})$ of (T_n)-universal functions.

(2) Now let (ϕ_n) be a run-away sequence in Φ and $z_0 \in \mathbb{D}$ with

$$\lim_{n\to\infty} \phi_n^* \lambda_{\mathbb{D}}(z_0) = \lambda_{\mathbb{D}}(z_0).$$

There exists a sequence (T_n) in $\mathrm{Aut}(\mathbb{D})$ with

$$(T_n^{-1} \circ \phi_n)(0) = 0, \qquad \text{and} \qquad (T_n^{-1} \circ \phi_n)'(0) > 0.$$

By hypothesis

$$\lim_{n\to\infty} (T_n^{-1} \circ \phi_n)^* \lambda_{\mathbb{D}}(z_0) = \lambda_{\mathbb{D}}(z_0).$$

Ahlfors' lemma and a normal family argument imply that

$$T_n^{-1} \circ \phi_n \to \mathrm{id}_{\mathbb{D}} \tag{3.2.13}$$

locally uniformly in $\mathbb{D}$. Note that $T_n(0) = \phi_n(0)$. Therefore we can use that (ϕ_n) is run-away to conclude

$$\limsup_{n\to\infty} |T_n(0)| = \limsup_{n\to\infty} |\phi_n(0)| = 1.$$

Thus (T_n) is run-away. We have shown in **(1)** that the set of all (T_n)-universal functions is a dense G_δ-subset of $\mathcal{B}_{lu}(\mathbb{D})$. We complete the proof by showing, that every (T_n)-universal function $F \in \mathcal{B}_{lu}(\mathbb{D})$ is Φ-universal.

Fix $f \in \mathcal{B}_{lu}(\mathbb{D})$. The (T_n)-universality of F tells as, that there exists a subsequence (T_{n_k}) of (T_n) such that

$$F \circ T_{n_k} \to f.$$

We can use (3.2.13) to conclude

$$F \circ \phi_{n_k} = (F \circ T_{n_k}) \circ (T_{n_k}^{-1} \circ \phi_{n_k}) \to f \circ \mathrm{id}_{\mathbb{D}} = f$$

locally uniformly in $\mathbb{D}$. Thus F is Φ-universal. ■

Remark 3.39. Note that if $(\lambda_n) \subseteq \Lambda_{-1}(\mathbb{D})$ with $\lambda_n(z_0) \to \lambda_{\mathbb{D}}(z_0)$ for one $z_0 \in \mathbb{D}$ then $\lambda_n \to \lambda_{\mathbb{D}}$ locally uniformly in $\mathbb{D}$. The condition in (i) thus can be replaced with the apparently stronger statement $\phi_n^* \lambda_{\mathbb{D}} \to \lambda_{\mathbb{D}}$ locally uniformly in $\mathbb{D}$.

Example 3.40. (a similar example can be found in [6, Example 3]) Let $a_n = 1 - \frac{1}{n}$ and let $\phi_n \colon \mathbb{D} \to B_{1/n}(a_n)$ be a conformal mapping with $\phi_n(0) = a_n$. Then (ϕ_n) is a run-away sequence of injective self-maps of $\mathbb{D}$. Theorem 3.17 tells us, that there exists a (ϕ_n)-universal function in $\mathcal{H}_{lu}(\Omega)$.
On the other hand a simple computation shows

$$\limsup_{n\to\infty} \phi_n^* \lambda_{\mathbb{D}}(0) < \lambda_{\mathbb{D}}(0).$$

It follows that $\limsup \phi_n^* \lambda_{\mathbb{D}}(z) < \lambda_{\mathbb{D}}(z)$ for all $z \in \mathbb{D}$. Theorem 3.38 tells us that no (ϕ_n)-universal function exists in $\mathcal{B}_{lu}(\mathbb{D})$.

The conformal invariance of the hyperbolic metric and Remark 3.3 show, that both assertions (i) and (ii) in Theorem 3.38 are invariant under conformal mappings. Thus, using the Riemann mapping theorem, one can easily show the following version of Theorem 3.38 for proper simply connected domains Ω in $\mathbb{C}$.

Theorem 3.41.
Let Ω be a simply connected proper subdomain of $\mathbb{C}$ and let Φ be a family of locally univalent self-maps of Ω. The following assertions are equivalent:

(i) There exists a run-away sequence $(\phi_n) \subseteq \Phi$ and $z_0 \in \Omega$ with

$$\lim_{n\to\infty} \phi_n^* \lambda_\Omega(z_0) = \lambda_\Omega(z_0).$$

(ii) There exists a Φ-universal function in $\mathcal{B}_{lu}(\Omega)$.

In addition, if one of the assertions above is true, the set of all Φ-universal functions in $\mathcal{B}_{lu}(\Omega)$ is a dense G_δ-subset of $\mathcal{B}_{lu}(\Omega)$.

Corollary 3.42.
Let Ω be a simply connected proper subdomain of $\mathbb{C}$ and let (ϕ_n) be a sequence in $\mathrm{Aut}(\Omega)$. *Then there exists a (ϕ_n)-universal function in $\mathcal{B}_{lu}(\Omega)$ if and only if (ϕ_n) is run-away.*

Corollary 3.43.
Let Ω be a simply connected proper subdomain of $\mathbb{C}$ and let ϕ an holomorphic self-map of Ω. There exists a $\phi^{[n]}$-universal function $f \in \mathcal{B}_{lu}(\Omega)$ if and only if ϕ is a fix-point free conformal automorphism of Ω.

Proof. If ϕ is a fix-point free conformal automorphism of Ω, then we can use the Denjoy-Wolff theorem (see remark 3.6) to show that $(\phi^{[n]})$ is a run-away sequence in $\mathrm{Aut}(\Omega)$. Then Corollary 3.42 tells us that there exists a $\phi^{[n]}$-universal function $f \in \mathcal{B}_{lu}(\Omega)$.
To show the contrary, suppose that there exists a $\phi^{[n]}$-universal function $f \in \mathcal{B}_{lu}(\Omega)$. Then Theorem 3.41 tells us, that $\phi^{[n]}$ is run-away and that there is $z_0 \in \mathbb{D}$ such that

$$\lambda_\Omega(z_0) = \limsup_{n\to\infty} \phi_n \lambda_\Omega(z_0). \tag{3.2.14}$$

First note that the fact that $(\phi^{[n]})$ is run-away and Remark 3.6 tell us that ϕ has no fixed point.
Let $\lambda := \phi^* \lambda_\Omega$ and $\phi_n := \phi^{[n]}$. Then for each $n \in \mathbb{N}$ Ahlfors' lemma shows

$$\phi_n^* \lambda_\Omega = \phi_{n-1}^* \lambda \le \phi_{n-1}^* \lambda_\Omega.$$

Thus the sequence $\phi_n^* \lambda_\Omega$ is decreasing so that

$$\limsup_{n\to\infty} \phi_n * \lambda_\Omega \le \phi_1^* \lambda_\Omega = \lambda.$$

We now can use (3.2.14) to conclude $\lambda(z_0) = \lambda_\Omega(z_0)$. Thus Ahlfors' lemma tells us $\lambda = \lambda_\Omega$ and $\phi \in \mathrm{Aut}(\Omega)$. ∎

Example 3.44. Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be defined by

$$\phi(z) := \frac{z}{2} + \frac{1}{2}.$$

Then ϕ is injective and has no fix-point in $\mathbb{D}$. Corollary 3.18 tells us, that there exists a $\phi^{[n]}$-universal function in $\mathcal{H}_{lu}(\mathbb{D})$. On the other hand, since $\phi \notin \mathrm{Aut}(\mathbb{D})$, Corollary 3.43 tells us, that there is no $\phi^{[n]}$-universal function in $\mathcal{B}_{lu}(\Omega)$.

Theorem 3.41 is no longer true for non-simply connected domains:

Example 3.45. Let $\Omega := \mathbb{D} \setminus \{0\}$. For $n \in \mathbb{N}$ define $\phi_n\colon \Omega \to \Omega$ by $\phi_n(z) := z^n$. Note that $\phi_n^* \lambda_\Omega = \lambda_\Omega$ for every $n \in \mathbb{N}$ and $\phi_n \to 0$ locally uniformly.
Every function $F \in \mathcal{B}_{lu}(\mathbb{D} \setminus \{0\})$ can be extended to a function $F \in \mathcal{B}(\mathbb{D})$, so that $F \circ \phi_n \to F(0)$ locally uniformly in $\mathbb{D} \setminus \{0\}$. Thus no (ϕ_n)-universal function exists in $\mathcal{B}_{lu}(\Omega)$.

Universal bounded functions for sequences of automorphisms of a domain Ω have been studied in [24] (if $\Omega = \mathbb{D}$) and [44] for general domains in $\mathbb{C}$ and $\mathbb{C}^n$.

Problem 5.
Let Ω be a domain in $\mathbb{C}$ and let Φ be a family of locally univalent self-maps of Ω. Find necessary and sufficient conditions for Φ (and for Ω) for the existence of Φ-universal in $\mathcal{B}_{lu}(\Omega)$. What are necessary and sufficient conditions if $\Phi \subseteq \mathrm{Aut}(\Omega)$?

4. Universal Conformal Metrics

Definition 4.1. Let Ω be a domain in $\mathbb{C}$, let Φ be a family of locally univalent self-maps of Ω and let Λ be a family of conformal metrics on Ω. We call $\lambda \in \Lambda$ Φ-universal in Λ if the family of pullbacks $\{\phi^*\lambda \,:\, \phi \in \Phi\}$ is dense in Λ. We call λ universal in Λ if λ is $\operatorname{Aut}(\Omega)$-universal in Λ.

We are interested in the following cases for Λ:

(i) $\Lambda = \Lambda_c(\Omega)$, the set of all conformal metrics with constant curvature $c \in \mathbb{R}$.

(ii) $\Lambda = SKC(\Omega)$, the set of all continuous SK-metrics on Ω.

4.1. Constant Curvature

We can use Liouville's representation theorem and our universality results for locally univalent functions to proof the existence of universal conformal metrics with constant curvature $c \in \{-1, 0, 1\}$. We let

$$D_c := \begin{cases} \hat{\mathbb{C}}, & c > 1 \\ \mathbb{C}, & c = 0 \\ \mathbb{D}, & c < -1 \end{cases}$$

and

$$\mathcal{G}_c(\Omega) := \{f \in \mathcal{M}_{lu}(\Omega) \,:\, f(\Omega) \subseteq D_c\}.$$

The metric λ_{D_c} is the *standard* metric on D_c, that is λ_{D_c} is either the hyperbolic metric on $\mathbb{D}$ ($c = -1$), (twice) the euclidean metric on $\mathbb{C}$ ($c = 0$), or the spherical metric on $\hat{\mathbb{C}}$ ($c = 1$). We can write $\lambda_c(z) := \frac{2}{1+c|z|^2}\,|dz|$ for all $z \in D_c$ (if $c = 1$ and $z = \infty$ this has to be understood in the local coordinate $1/z$). It was Liouville [38] who has discovered the following fundamental relationship between locally univalent functions and constantly curved metrics.

Theorem 4.2.
Let $c \in \{-1, 0, 1\}$, let Ω be a simply connected domain in $\mathbb{C}$ and let $\lambda \in \Lambda_c(\Omega)$. Then

there exists a function $f \in \mathcal{G}_c(\Omega)$ such that $\lambda = f^*\lambda_c$. If $g \in \mathcal{G}_c(\Omega)$ is another function with $\lambda = g^*\lambda_c$ then there exists a holomorphic rigid motion T of D_c such that $f = T \circ g$.

Recall that the holomorphic rigid motions of D_c are

(i) the conformal automorphisms of $\mathbb{D}$ for $c = -1$;

(ii) the direct Euclidean motions of $\mathbb{C}$ for $c = 0$ (i.e. the maps $z \mapsto az + b$ with $a \in \partial\mathbb{D}$, $b \in \mathbb{C}$);

(iii) the rotations of the Riemann sphere $\hat{\mathbb{C}}$ for $c = 1$.

For simply connected domains Ω, Liouville's representation theorem gives us a continuous surjection from $\mathcal{G}_c(\Omega)$ onto the set $\Lambda_c(\Omega)$ of all conformal metrics with constant curvature c; this mapping is injective modulo the rigid motions of D_c. The next result is an immediate consequence of Liouville's theorem and shows that this map is "universality preserving":

Proposition 4.3.
Let Ω be a simply connected domain in $\mathbb{C}$, $c \in \{-1, 0, 1\}$ and let Φ be a family of locally univalent self-maps of Ω. Suppose that $f \in \mathcal{G}_c(\Omega)$ is Φ-universal in $\mathcal{G}_c(\Omega)$. Then the metric

$$\lambda := f^*\lambda_{D_c} \in \Lambda_c(\Omega)$$

is Φ-universal in $\Lambda_c(\Omega)$.

Proof. Let $\mu \in \Lambda_c(\Omega)$. By Liouville's theorem there exists a map $g \in \mathcal{G}_c(\Omega)$ such that $\mu = g^*\lambda_{D_c}$. Since f is Φ-universal in $\mathcal{G}_c(\Omega)$ there is a sequence (ϕ_n) in Φ with the property that

$$f \circ \phi_n \to g$$

locally χ-uniformly in Ω. This clearly implies

$$\phi_n^*\lambda = (f \circ \phi_n)^*\lambda_{D_c} \to g^*\lambda_{D_c} = \mu$$

locally uniformly in Ω. ∎

This gives us a geometric interpretation of theorems 3.17, 3.19 and 3.38.

Theorem 4.4.
Let Ω be a simply connected domain in $\mathbb{C}$, $c \geq 0$ and let Φ be a family of locally univalent self-maps of Ω. Suppose there exists an eventually injective run-away sequence (ϕ_n) in Φ. Then there exists a Φ-universal conformal metric $\lambda \in \Lambda_c(\Omega)$. The set of all Φ-universal conformal metrics is a dense G_δ-subset in $\Lambda_c(\Omega)$.

Theorem 4.5.
Let Ω be a simply connected proper subdomain in $\mathbb{C}$, $c < 0$ and let Φ be a family of locally univalent self-maps of Ω. Suppose that there exist a run-away sequence (ϕ_n) in Φ with

$$\lim_{n\to\infty} \phi_n^* \lambda_\Omega = \lambda_\Omega.$$

Then there exists a Φ-universal conformal metric $\lambda \in \Lambda_c(\Omega)$. The set of all Φ-universal conformal metrics is a dense G_δ-subset in $\Lambda_c(\Omega)$.

Remark 4.6. There is a reverse statement to Proposition 4.3: Let Ω be a simply connected domain, $c \in \mathbb{R}$ and let Φ be a family of locally univalent self-maps of Ω. Suppose there exists a Φ-universal metric $\lambda \in \Lambda_c(\Omega)$ (if $c < 0$ this means $\Omega \neq \mathbb{C}$) and let $f \in \mathcal{G}_c$ such that $f^* \lambda_{D_c} = \lambda$. Then f has the following universality property: The set

$$\{T \circ f \circ \phi \,:\, \phi \in \Phi,\ T \text{ is a holomorphic rigid motion of } D_c\}$$

is dense in $\mathcal{G}_c$. Indeed let $g \in \mathcal{G}_c$. Then by the universality of λ there exists a sequence (ϕ_n) in Φ such that $\phi_n^* \lambda \to g^* \lambda_{D_c}$. For each $n \in \mathbb{N}$ we can find a rigid motion of D_c with $(T_n \circ f \circ \phi_n)(0) = g(0)$ and $\arg(T_n \circ f \circ \phi_n)'(0) = \arg g'(0)$. The uniqueness assertion in Liouville's theorem tells us $T_n \circ f \circ \phi_n \to g$.

A conformal metric λ has constant curvature zero, if and only if $\log \lambda$ is harmonic. Thus we can use a well known Runge-type theorem for harmonic functions and a standard application of the universality criterion to proof a universality result for conformal metrics in $\Lambda_0(\Omega)$ even if Ω is not simply connected.

Theorem 4.7.
Let Ω be a domain of infinite connectivity and let Φ be a family of locally univalent self-maps of Ω. Suppose for every compact $\mathcal{O}$-convex set $K \subseteq \Omega$ there exists $\phi \in \Phi$ such that the restriction $\phi|_K$ is injective, $\phi(K) \cap K = \emptyset$ and $\phi(K)$ is $\mathcal{O}$-convex. Then there exists a Φ-universal conformal metric $\lambda \in \Lambda_0(\Omega)$. The set of all Φ-universal conformal metrics is a dense G_δ subset of $\Lambda_0(\Omega)$.

Proof. We show that the collection of continuous maps

$$\phi^* \colon \Lambda_0(\Omega) \to \Lambda_0(\Omega), \quad \lambda \mapsto \phi^* \lambda, \quad \phi \in \Phi,$$

acts topologically transitively on $\Lambda_0(\Omega)$ and then apply Theorem 3.8. Let $\mathcal{U}, \mathcal{V} \subseteq \Lambda_0(\Omega)$ non-empty open sets, $\lambda \in \mathcal{U}$ and $\mu \in \mathcal{V}$. There exists $\varepsilon > 0$ and a connected compact

$\mathcal{O}$-convex set $L \subseteq \Omega$ with at least two holes such that

$$U_{L,\varepsilon}(\lambda) \subseteq \mathcal{U}, \qquad U_{L,\varepsilon}(\mu) \subseteq \mathcal{V}.$$

We can use the hypothesis and Lemma 3.10 to find $\phi \in \Phi$ such that ϕ is injective on L, $\phi(L) \cap L = \emptyset$ and such that the set $K := L \cup \phi(L)$ is $\mathcal{O}$-convex. Define $h\colon K \to \mathbb{R}$ by

$$h(z) := \begin{cases} \log \lambda(z), & z \in L \\ \log[(\phi^{[-1]})^* \mu(z)], & z \in \phi(L). \end{cases}$$

Then h is harmonic in an open neighborhood of K. Hence by a well known Runge-type theorem for harmonic functions, see i.e. [20, Corollary 1.16], for every $\delta > 0$ we can find a harmonic function $u\colon \Omega \to \mathbb{R}$ such that $\|u-h\|_K < \delta$. We choose $\delta > 0$ so small, that

$$\|e^u - e^h\|_K \le \min\left\{\varepsilon, \frac{\varepsilon}{\|\phi'\|_L}\right\}.$$

Define $\nu := e^u \in \Lambda_0(\Omega)$. Then we have $\|\lambda - \nu\|_L < \varepsilon$, whence $\lambda \in \mathcal{U}$. On the other hand note that $\mu = \phi^*(e^h)$ on L, so

$$\|\phi^*\nu - \mu\|_L = \|(e^u \circ \phi - e^h \circ \phi)|\phi'|\|_L \le \|\phi'\|_L \cdot \|e^u - e^h\|_K < \varepsilon.$$

We can conclude $\phi^*\nu \in \mathcal{V}$. ∎

Let Φ be a family of locally univalent self-maps of a simply connected domain $\Omega \subseteq \mathbb{C}$ and $c \in \mathbb{R}$. Theorem 4.1 and Theorem 4.4 establish sufficient conditions for the existence of Φ-universal metrics in $\Lambda_c(\Omega)$. For universal locally univalent functions in $\mathcal{G}_c$ theorems 3.17, 3.19 and 3.38 show that these conditions are also necessary. In the setting of conformal metrics, this is not longer true. For example, the "run-away condition", which is typical for universal (locally univalent) functions, is no longer necessary.

Example 4.8. Let

$$\Phi := \{\phi \in \mathcal{B}_{lu}(\mathbb{D}) \,:\, \phi(0) = 0\}.$$

As a direct consequence of Liouville's representation theorem, the hyperbolic metric $\lambda_{\mathbb{D}}$ is Φ-universal in $\Lambda_{-1}(\mathbb{D})$. It is clear, that there can not exist a run-away sequence (ϕ_n) in Φ, since this would imply $\sup_{\phi\in\Phi} |\phi(0)| = 1$.
Similarly, if we let

$$\Psi := \{\psi \in \mathcal{H}_{lu}(\mathbb{C}) \,:\, \psi(0) = 0\}$$

then the euclidean metric is Ψ-universal in $\Lambda_0(\mathbb{C})$ and Ψ does not contain a run-away sequence.

Remark 4.9. Universal harmonic functions have been studied for example in the literature, see for example [5], [11] and the references therein. Let u be a Ψ-universal harmonic function on a domain Ω in $\mathbb{C}$. (The definition of Ψ-universal harmonic functions should be obvious.) Considering the close relation between harmonic functions and conformal metrics with constant curvature 0, one might be inclined to suspect that e^u is Ψ-universal in $\Lambda_0(\Omega)$ and vice-versa. Example 4.8 shows that this is not always the case: One can show, that if Ψ-universal functions exist, then there exists a run-away sequence in Ψ.

Proposition 4.10.
Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, $c \in \mathbb{R}$, and let Φ be a family of locally univalent functions of Ω. Suppose that there exists a Φ-universal metric $\lambda \in \Lambda_c(\Omega)$.

(a) If $c \geq 0$, then there exists an eventually injective sequence $(\phi_n) \subseteq \Phi$.

(b) If $c < 0$ (and then $\Omega \neq \mathbb{C}$), there exists a sequence $(\phi_n) \subseteq \Phi$ with $\lim_{n\to\infty} \phi_n^* \lambda_\Omega = \lambda_\Omega$. In addition, if (ϕ_n) is not run-away, then $\lambda = \lambda_\Omega$.

Proof. (a): Liouville's theorem tells us that there is $f \in \mathcal{G}_c$ with $\lambda = f^* \lambda_{D_c}$. Note that we have $\mathrm{id}_\Omega \in \mathcal{G}_c(\Omega)$.
As seen in Remark 4.6, there exists a sequence $(\phi_n) \subseteq \Phi$ and a sequence of rigid motions of D_c such that

$$T_n \circ f \circ \phi_n \to \mathrm{id}_\Omega \,.$$

Now Hurwitz's theorem shows that (ϕ_n) is eventually injective.
(b): By the universality of λ there exists $(\phi_n) \subseteq \Phi$ with

$$\phi_n^* \lambda \to \lambda_\Omega. \tag{4.1.1}$$

Ahlfors' lemma shows

$$\phi_n^* \lambda \leq \phi_n^* \lambda_\Omega \leq \lambda_\Omega$$

The limit $n \to \infty$ and (4.1.1) imply $\phi_n^* \lambda_\Omega \to \lambda_\Omega$.
If (ϕ_n) was not run-away, we could apply Montels theorem to obtain a subsequence (ϕ_{n_k}) of (ϕ_n) and a holomorphic self-map ϕ of Ω with $\phi_{n_k} \to \phi$. We can conclude $\phi^* \lambda = \lambda_\Omega$, so that $\lambda = \lambda_\Omega$ by Ahlfors' lemma. ■

Theorem 4.11.
Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let (ϕ_n) be an eventually injective sequence of locally univalent self-maps of Ω. Suppose there exists a (ϕ_n)-universal metric $\lambda \in \Lambda_{-1}(\Omega)$. Then (ϕ_n) is run-away.

Proof. We can apply a conformal mapping, so that there is no harm in assuming $\Omega = \mathbb{D}$. Choose a non-injective function $f \in \mathcal{B}_{lu}(\mathbb{D})$ (for example take $f(z) = \frac{1}{M}e^{rz}$ for suitable $r, M > 0$). For the sake of contraction suppose that (ϕ_n) is not run-away. Then $\lambda = \lambda_{\mathbb{D}}$ by Proposition 4.10. The universality of λ tells us, that there exists a subsequence (ϕ_{n_k}) of (ϕ_n) such that

$$\phi_{n_k}^* \lambda_{\mathbb{D}} = \phi_{n_k}^* \lambda \to f^* \lambda_{\mathbb{D}}.$$

A normal family argument shows, that we can in addition assume that $\phi_{n_k} \to \phi \in \mathcal{H}(\mathbb{D})$. Then, since (ϕ_n) is not run-away, we can conclude that ϕ is a self-map of $\mathbb{D}$ with $\phi^* \lambda_{\mathbb{D}} = f^* \lambda_{\mathbb{D}}$. This in particular forces ϕ to be non-constant. Since (ϕ_n) is eventually injective, Hurwitz theorem now implies that ϕ is injective. However, note that the uniqueness assertion of Liouville's theorem gives us $T \in \text{Aut}(\mathbb{D})$ with $f = T \circ \phi$. Thus f would be injective, a contradiction to the initial choice of f. ■

4.2. Universal *SK*-Metrics

We can not rely on universality theorems for locally univalent functions in order to proof universality results for conformal metrics with non-constant curvature . Instead, we use the local approximation theorems established in Chapter 2.

Theorem 4.12.
Let Φ be a family of locally univalent self-maps of $\mathbb{D}$. The following are equivalent:

(i) There exists a run-away sequence (ϕ_n) in Φ with

$$\lim_{n \to \infty} \phi_n^* \lambda_{\mathbb{D}} = \lambda_{\mathbb{D}}.$$

(ii) There exists a Φ-universal metric $\lambda \in SKC(\mathbb{D})$.

Proof. $(ii) \Rightarrow (i)$: Let $\lambda \in SKC(\mathbb{D})$ be Φ-universal. Then there exists a sequence (ϕ_n) in Φ such that

$$\lim_{n \to \infty} \phi_n^* \lambda = \lambda_{\mathbb{D}}. \tag{4.2.1}$$

Ahlfors' Lemma tells us

$$\phi_n^* \lambda \le \phi_n^* \lambda_{\mathbb{D}} \le \lambda_{\mathbb{D}}.$$

The limit $n \to \infty$ and (4.2.1) imply

$$\lim_{n \to \infty} \phi_n \lambda_{\mathbb{D}} = \lambda_{\mathbb{D}}. \tag{4.2.2}$$

If (ϕ_n) is not run-away, there exist $\phi \in \mathcal{B}(\mathbb{D})$ and a subsequence (ϕ_{n_k}) of (ϕ_n) with $\phi_{n_k} \to \phi$. We can use (4.2.1) to conclude $\phi^*\lambda = \lambda_{\mathbb{D}}$, whence $\lambda = \lambda_{\mathbb{D}}$ by Ahlfors' Lemma. Thus

$$\{\phi^*\lambda \,:\, \phi \in \Phi\} \subseteq \Lambda_{-1}(\mathbb{D}).$$

Since $\Lambda_{-1}(\mathbb{D})$ is a proper closed subset of $SK_c(\mathbb{D})$, this contradicts the universality of λ.

We split the proof that $(i) \Rightarrow (ii)$ into to steps.

(1) We show, that for every run-away sequence $(T_n) \in \operatorname{Aut}(\mathbb{D})$ there exists a (T_n)-universal element $\lambda \in SKC(\mathbb{D})$.

We may assume, that (T_n) full-fills the assertions of Lemma 3.37. Let (f_n), (ε_n), φ and ϑ as in the conclusion of Lemma 3.37.

Let $\mathcal{U}, \mathcal{V} \subseteq SKC(\mathbb{D})$ be non-empty open sets. Since $SKC(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ is dense in $SKC(\mathbb{D})$, there exist $\lambda \in \mathcal{U} \cap C(\overline{\mathbb{D}})$ and $\mu \in \mathcal{V} \cap C(\overline{\mathbb{D}})$, a compact set $L \subseteq \mathbb{D}$ and $\delta > 0$ such that

$$U_{L,2\delta}(\lambda) \subseteq \mathcal{U}, \qquad U_{L,2\delta}(\mu) \subseteq \mathcal{V}.$$

Let $H := \overline{\mathbb{D}} \cup [1,2] \cup (K_1(3))$, let $h\colon [1,2] \to (0,\infty)$ be a continuous function with $h(1) = \lambda(e^{i\varphi})$ and $h(2) = \mu(-e^{-i\vartheta})$. Define

$$\nu\colon H \to (0,\infty),\ \nu(z) := \begin{cases} \lambda(e^{i\varphi} z), & z \in \overline{\mathbb{D}} \\ \mu(e^{-i\vartheta}(z-3)), & z \in K_1(3) \\ h(z), & z \in [1,2]. \end{cases}$$

Then we have $\nu \in SK(H^\circ) \cap C(H)$. We are in the position to apply Corollary 2.15. This gives us an open set U with $H \subseteq U$ and $\zeta \in SKC(U)$ such that $\|\zeta - \nu\|_H < \delta$ Recall that in (3.2.4) we have defined Ω_ε to be the open ε-neighborhood of H and that f_n maps $\mathbb{D}$ conformally onto Ω_{ε_n}. Since $\varepsilon_n \to 0$, we may assume that $\Omega_{\varepsilon_n} \subseteq U$ for all $n \in \mathbb{N}$. Define $\eta_n := f_n^*\zeta \in SKC(\mathbb{D})$. Since $\lim_{n\to\infty} f_n = e^{-i\varphi} \operatorname{id}_{\mathbb{D}}$ locally uniformly, we can conclude

$$\eta_n \to (e^{-i\varphi} \operatorname{id}_{\mathbb{D}})^*\zeta \qquad \text{locally uniformly in } \mathbb{D}.$$

Hence we can fix $N_0 \in \mathbb{N}$ such that

$$\max_{z\in L} \left|\eta_N(z) - \zeta\left(e^{-i\varphi} z\right)\right| < \delta$$

holds for every $N \geq N_0$. Then, since $\lambda(z) = \nu(e^{-i\varphi} z)$ for $z \in \mathbb{D}$, we have

$$\max_{z\in L} |\eta_N(z) - \lambda(z)| = \max_{z\in L} \left|\eta_N(z) - \nu(e^{-i\varphi} z)\right|$$

$$\leq \max_{z \in L} \left|\eta_N(z) - \zeta(e^{-i\varphi}z)\right| + \max_{z \in \overline{\mathbb{D}}} \left|\zeta(e^{-i\varphi}z) - \nu(e^{-i\varphi}z)\right| < 2\delta.$$

The initial choice of L and δ show us $\eta_N \in \mathcal{U}$ for all $N \geq N_0$.
On the other hand we have $f_n \circ T_n \to e^{i\vartheta} \operatorname{id}_{\mathbb{D}} + 3$ locally uniformly in $\mathbb{D}$. Thus

$$T_n^* \eta_n = (f_n \circ T_n)^* \zeta \to (e^{i\vartheta} \operatorname{id}_{\mathbb{D}} + 3)^* \zeta \qquad \text{locally uniformly in } \mathbb{D}.$$

Hence there exists $N \geq N_0$ with

$$\max_{z \in L} \left|T_N^* \eta_N(z) - \zeta(e^{i\vartheta}z + 3)\right| < \delta.$$

Note that $\mu(z) = \nu(e^{i\vartheta}z + 3)$. A similar estimate as above yields

$$\max_{z \in L} |T_N^* \eta_N(z) - \mu(z)| < 2\delta,$$

whence $T_N^* \eta_N \in \mathcal{V}$. We have already established $\eta_N \in \mathcal{U}$. In conclusion, we have shown that there exist $\eta \in \mathcal{U}$ and $N \in \mathbb{N}$ with $T_N^* \eta \in \mathcal{V}$. Thus the sequence (T_n^*) of pull-back operators

$$T_n^* \colon SKC(\mathbb{D}) \to SKC(\mathbb{D}), \quad \lambda \mapsto T_n^* \lambda, \qquad n \in \mathbb{N}$$

acts topologically transitively on $SKC(\mathbb{D})$. The universality criterion shows that there exists a dense G_δ subset of $SKC(\mathbb{D})$ of Φ-universal elements.
(2) Let (ϕ_n) be a run-away sequence in Φ and $z_0 \in \mathbb{D}$ such that

$$\lim_{n \to \infty} \phi_n^* \lambda_{\mathbb{D}}(z_0) = \lambda_{\mathbb{D}}(z_0).$$

There exists a sequence $(T_n) \subseteq \operatorname{Aut}(\mathbb{D})$ with $(T_n^{-1} \circ \phi_n)(0) = 0$ and $(T_n^{-1} \circ \phi_n)'(0) > 0$. The assumption on (ϕ_n) implies

$$(T_n^{-1} \circ \phi_n)^* \lambda_{\mathbb{D}} \to \lambda_{\mathbb{D}}$$

and thus

$$T_n^{-1} \circ \phi_n \to \operatorname{id}_{\mathbb{D}} \qquad \text{locally uniformly in } \mathbb{D}. \tag{4.2.3}$$

Note that

$$\limsup_{n \to \infty} |T_n(0)| = \limsup_{n \to \infty} |\phi_n(0)| = 1,$$

whence (T_n) is run-away. We have already shown that there exist a dense G_δ-subset of $SKC(\mathbb{D})$ of (T_n)-universal elements. We proceed to show that every (T_n)-universal metric $\lambda \in SKC(\mathbb{D})$ is also Φ-universal. Note that for every $n \in \mathbb{N}$ we have

$$\phi_n^* \lambda = (T_n \circ T_n^{-1} \circ \phi_n)^* \lambda = (T_n^{-1} \circ \phi_n)^* (T_n^* \lambda). \tag{4.2.4}$$

Let $\mu \in SKC(\mathbb{D})$. Then by (T_n)-universality of λ, there exists a subsequence (T_{n_k}) of (T_n) with

$$T_{n_k} * \lambda \to \mu.$$

Now (4.2.3) and (4.2.4) imply

$$\phi_{n_k}^* \lambda \to \mathrm{id}_{\mathbb{D}}^* \, \mu = \mu.$$

Since (ϕ_{n_k}) is a sequence in Φ we have shown that λ is indeed Φ-universal. ■

Remark 4.13. The proof given above can be adapted to other situations. For example, using Corollary 2.15, this gives an alternative proof of Theorem 4.4, which does not rely on Liouville's representation theorem.

In general, a SK-metric is only upper semicontinuous. Let $UC(\mathbb{D})$ be the set of all upper semicontinuous functions $u\colon \mathbb{D} \to \mathbb{R}$. The compact-open topology is no longer the natural topology on $UC(\mathbb{D})$. Instead, it is more natural to consider the topology of decreasing convergence. This topology is defined as follows: A set $\mathcal{U} \subseteq UC(\mathbb{D})$ is open, if for every $w \in \mathcal{U}$ there exist a compact set $K \subseteq \mathbb{D}$, $\delta > 0$ and $k \in \mathbb{N}$ such that

$$\{v \in UC(\mathbb{D}) \,:\, w(z) \leq v(z) \leq \min\{k, u(z) + \delta\} \text{ for all } z \in K\} \subseteq \mathcal{U}.$$

Gauthier and Pouryayevali [22] have established the existence of Birkhoff-type universal subharmonic functions $s \in sh\,(\mathbb{C})$ if $sh\,(\mathbb{C})$ is equipped with the topology of decreasing convergence.

Let $(\lambda_n) \subseteq \Lambda_{-1}(\mathbb{D})$ be a sequence with $\lambda_n \to \lambda \in SK(\mathbb{D})$ with respect to the topology of decreasing convergence. Then in particular $\lambda_n(z) \to \lambda(z)$ pointwise. A normal family argument now shows $\lambda \in \Lambda_{-1}(\mathbb{D})$, whence $\Lambda_{-1}(\mathbb{D})$ is a closed proper subset of $SK(\mathbb{D})$.

Proposition 4.14.

Let Φ be a family of locally univalent self-maps of $\mathbb{D}$ and let $SK(\mathbb{D})$ be equipped with the topology of decreasing convergence. Then there is no Φ-universal element in $SK(\mathbb{D})$.

Proof. For the sake of contradiction, assume that $\lambda \in SK(\mathbb{D})$ is Φ-universal. Then for every open set $\mathcal{U} \subseteq SK(\mathbb{D})$ there exists $\phi \in \Phi$ such that $\phi^*\lambda \in \mathcal{U}$. Ahlfor's Lemma implies that the hyperbolic metric $\lambda_{\mathbb{D}}$ is an isolated point in $SK(\mathbb{D})$, that is $\{\lambda_{\mathbb{D}}\}$ is an open set in $SK(\mathbb{D})$. We conclude, that there exists $\phi \in \Phi$ with $\phi^*\lambda = \lambda_{\mathbb{D}}$. This implies $\lambda = \lambda_{\mathbb{D}}$, thus

$$\{\phi^*\lambda \,:\, \phi \in \Phi\} \subseteq \Lambda_{-1}(\mathbb{D}).$$

This stands in contradiction to the universality of λ, since we have seen above that $\Lambda_{-1}(\mathbb{D})$ is a proper closed subset of $SK(\mathbb{D})$. ■

We conclude this section with two related problems.

Problem 6. *(a) Let Ω be a non-simply connected hyperbolic domain in $\mathbb{C}$ and Φ a family of locally univalent self-maps of Ω. Find necessary and sufficient conditions for Φ so that a Φ-universal function in $\Lambda_{-1}(\Omega)$ resp. $SKC(\Omega)$ exists.*

(b) Do universal conformal metrics exist on non-compact (hyperbolic) Riemann surfaces?

Appendix A.

The Compact-Open topology

Definition A.1. Let $E \subseteq \mathbb{C}$, $K \subseteq E$ compact, $f \in C(E)$ and $\varepsilon > 0$. Define the set $U_{K,\varepsilon}(f)$ by

$$U_{K,\varepsilon}(f) \coloneqq \{g \in C(E) \,:\, \|f-g\|_K < \varepsilon\}.$$

The compact-open topology on $C(E)$ is the topology generated by the sets $U_{K,\varepsilon}(f)$. To be more specific, a set $U \subseteq C(E)$ is open, if for every $f \in U$ there exists a compact set $K \subseteq E$ and $\varepsilon > 0$ such that $U_{K,\varepsilon}(f) \subseteq U$.

Note that $K \subseteq L$ implies $U_{L,\varepsilon}(f) \subseteq U_{K,\varepsilon}(f)$. For a domain Ω it follows by taking the $\mathcal{O}$-convex hull, that $U \subseteq C(\Omega)$ is open if and only for every $f \in U$ there exists a compact $\mathcal{O}$-convex set $K \subseteq \Omega$ and $\varepsilon > 0$ such that $U_{K,\varepsilon}(f) \subseteq U$.
The compact-open topology is metrizable. A metric on $C(E)$ is given by

$$d(f,g) \coloneqq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}} \qquad f,g \in C(E)$$

where (K_n) is a compact exhaustion of E. Thus the compact-open topology coincides with the topology of locally uniform convergence.
Now let $\Omega \subseteq \mathbb{C}$ be a domain. We can think of $\mathcal{H}(\Omega)$ as a subspace of $C(\Omega)$. Then Weierstraß's convergence theorem tells us that $\mathcal{H}(\Omega)$ is a closed subspace of $C(\Omega)$. As a direct consequence of Runge's theorem, $\mathcal{H}(\Omega)$ is separable. In fact, the set of rational functions with rational coefficients and without poles in Ω is a countable dense subset. Note that what we have covered here is still true for meromorphic functions, if we use the chordal distance on $\hat{\mathbb{C}}$ instead of the euclidean distance.

Definition A.2. Let X be a topological space.

(a) X is called a Baire space, if a countable intersection of open dense sets is always a dense set in X.

(b) We say that $E \subseteq X$ is a G_δ-set, if E can be written as the countable intersection of open sets.

For example, Baire's category theorem [49, Theorem 2.2] states, that every complete metric space and every locally compact Hausdorff space is Baire. Every open subset of a Baire space itself is Baire. The proofs of the universality results presented in this thesis are based on the universality criterion, Theorem 3.8. The following Proposition justifies this approach.

Proposition A.3.
Let Ω be a domain in $\mathbb{C}$ and $c \in \mathbb{R}$. The following sets are second countable Baire spaces, if equipped with the compact-open topology: $\mathcal{H}_{lu}(\Omega)$; $\mathcal{M}_{lu}(\Omega)$; $\mathcal{B}_{lu}(\Omega)$; $\Lambda_c(\Omega)$; $SKC(\Omega)$.

Proof. We only proof the case $\mathcal{H}_{lu}(\Omega)$, the other cases follow in a similar fashion. Since $\mathcal{H}_{lu}(\Omega)$ is a subset of the separable metrizable space $\mathcal{H}(\Omega)$, we have that $\mathcal{H}_{lu}(\Omega)$ is second countable. Hurwitz's theorem tells us that the union of $\mathcal{H}_{lu}(\Omega)$ with all constant functions is closed in $\mathcal{H}(\Omega)$ and thus a complete metric space and $\mathcal{H}_{lu}(\Omega)$ is an open subset of this space, so $\mathcal{H}_{lu}(\Omega)$ is Baire. ■

Appendix B.

Potential Theory

We outline the Perron method for the curvature equation and some tools from potential theory, which we needed in Chapter 2. Throughout this section, we assume that Ω is a domain in $\mathbb{C}$ and $\kappa\colon \mathbb{C} \to [0, \infty)$ is a locally Hölder-continuous function.

Definition B.1 (Modification)**.** Let $u \in sh_\kappa(\Omega)$. We call $v \in sh_\kappa(\Omega)$ a modification of u, if there exists an open disc D which is compactly contained in Ω such that $v \equiv u$ in $\Omega \setminus D$ and $\Delta v = \kappa e^{2v}$ in D. If we want to be more precise, we say that v is a modification of u with respect to D.

Lemma B.2 (Gluing Lemma)**.**
If $u, v \in sh_\kappa(\Omega)$, then we also have $\max\{u, v\} \in sh_\kappa(\Omega)$.

Definition B.3 (Perron family)**.** A family $\mathcal{P} \subseteq sh_\kappa(\Omega)$ is called *Perron family*, if the following two conditions hold:

(i) If $u, v \in \mathcal{P}$, then $\max\{u, v\} \in \mathcal{P}$;

(ii) If $u \in \mathcal{P}$ and $v \in sh_\kappa(\Omega)$ is a modification of u, then $v \in \mathcal{P}$.

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f \in C(\partial B_r(z_0))$. Our assumptions on κ imply, that the boundary value problem

$$\begin{cases} \Delta u \equiv \kappa e^{2u}, & \text{in } B_r(z_0) \\ u \equiv f, & \text{on } \partial B_r(z_0) \end{cases}$$

has a unique solution. One can use this fact, the maximum principle for subharmonic functions and the Gluing lemma to show, that for each $v \in sh_\kappa(\Omega)$ and $z_0 \in \Omega$ there exists a modification u of v with respect to a disc $B_r(z_0)$ with $K_r(z_0) \subseteq \Omega$.

Theorem B.4.
Let $\mathcal{P} \subseteq sh_\kappa(\Omega)$ be a locally bounded Perron-family. Then the upper envelope $u := \sup_{v \in \mathcal{P}} v$ is a solution of the curvature equation $\Delta u = \kappa e^{2u}$.

Definition B.5 (Perron-solution of the Dirichlet Problem)**.** Suppose that Ω is bounded and let $f \in C(\partial\Omega)$. Define

$$\mathcal{P}^f_{\Omega,\kappa} := \left\{ u \in sh_\kappa(\Omega) \ : \ \limsup_{z\to\xi} u(z) \leq f(\xi) \text{ for all } \xi \in \partial\Omega \right\}$$

and

$$H^f_{f,\kappa} := \sup_{v\in\mathcal{P}_{f,\kappa}} v.$$

We call $H^f_{\Omega,\kappa}$ the Perron-solution of the Dirichlet-problem for the curvature equation. If $\kappa \equiv 0$, then we simply write H^f_Ω instead of $H^f_{\Omega,0}$.

It is easy to check, that $\mathcal{P}^f_{\Omega,\kappa}$ is a Perron-family. Thus Theorem B.4 tells us $H^f_{\Omega,\kappa} \in h_\kappa(\Omega)$.

In Chapter 2 one central concept is the concept of "thin sets":

Definition B.6. (a) A set $S \subseteq \mathbb{C}$ is called polar, if there exists an open set U with $S \subseteq U$ and $s \in sh\,(U)$ such that $S \subseteq s^{-1}(\{-\infty\})$.

(b) A set $E \subseteq \mathbb{C}$ is called thin at $z_0 \in \mathbb{C}$, if there exists $r > 0$ and a subharmonic function $s \in sh\,(B_r(z_0))$ such that

$$\limsup_{\substack{z\to z_0\\ z\in E\setminus\{z_0\}}} s(z) > s(z_0).$$

(c) A point $\xi \in \partial\Omega$ is called *regular* if the set $\mathbb{C} \setminus \Omega$ is not thin at ξ. We call Ω *regular*, if every boundary point of Ω is regular.

For every set $E \subseteq \mathbb{C}$, the set $\{z \in E \ : \ E \text{ is thin at } z\}$ is a polar set [47, Theorem 3.8.5]. We now can describe the boundary behavior of $H^f_{U,\kappa}$:

Theorem B.7.

Suppose that Ω is a bounded domain in $\mathbb{C}$. Let $f \in C(\partial\Omega)$ and let $\xi \in \partial\Omega$ be a regular boundary point of Ω. Then

$$\lim_{z\to\xi} H^f_{\Omega,\kappa}(z) = f(\xi).$$

In other words, if we define $H^f_{\Omega,\kappa}(\xi) = f(\xi)$ for $\xi \in \partial\Omega$, then there exists a polar set $E \subseteq \partial\Omega$ such that the resulting function is continuous on $\overline{\Omega} \setminus E$.

Definition B.8. Let $E \subseteq \mathbb{C}$ and $u\colon E \to [-\infty,\infty)$ be a function, which is locally bounded from above. Then the *upper semicontinuous regularization* $u^*\colon E \to [-\infty,\infty)$ of u is defined by

$$u^*(z) = \limsup_{w\to z} u(w).$$

Theorem B.9.
Let $\mathcal{V} \subseteq sh_\kappa(\Omega)$ and suppose that the upper envelope $u := \sup_{v\in\mathcal{V}} v$ is locally bounded from above. Then $u^* \in sh_\kappa(\Omega)$ and there exists a Borel measurable polar set $E \subseteq \Omega$ such that $u^* \equiv u$ holds throughout $\Omega \setminus E$.

Definition B.10 (Green function)**.** The Green-function $g_\Omega(z,w)$ of Ω with pole at $w \in \Omega$ is defined by

$$g_\Omega(z,w) = H_U^{\log|\cdot - w|}(z) - \log|z-w|, \qquad z,w \in \Omega.$$

Theorem B.11 (Riesz decomposition theorem)**.**
Let $s \in sh(\Omega)$ be integrable. Then there exists a finite Borel-measure μ on $\mathbb{C}$ such that

$$s(w) = h(w) - \iint_\Omega g_\Omega(z,w)d\mu(z)$$

where h is the least harmonic majorant of s.
If s is twice differentiable then $d\mu = \Delta s d\lambda$, where λ is the two-dimensional Lebesgue-measure.

Note that if s has a continuous extension to $\overline{\Omega}$, then the least harmonic majorant of s is H_Ω^s. In particular we have the following equation for the Perron-solution of the curvature equation: For $f \in C(\partial\Omega)$ we have

$$H_{\Omega,\kappa}^f(w) = H_\Omega^f(w) - \iint_\Omega g_\Omega(z,w)e^{2H_{\Omega,\kappa}^f(z)}d\lambda(z).$$

The first part of the following lemma can be found in [20, Lemma 1.5], the second part follows immediately by applying the first part to a suitable truncation of $\log|z-w|$. Our Lemma 2.9 can be viewed as a generalization.

Lemma B.12.
Let $K \subseteq \mathbb{C}$ be compact and suppose that $\mathbb{C}\setminus K$ and $\mathbb{C}\setminus K^\circ$ are thin at the same points and let S be the set of all points where $\mathbb{C}\setminus K$ is thin. Further let (U_m) be a decreasing sequence of bounded open sets with $K = \bigcap_{m\in\mathbb{N}} U_m$. Then

(a) $\lim_{m\to\infty} H_{U_m}^f = H_{K^\circ}^f$ *pointwise in* $\mathbb{C}\setminus S$.

(b) If $K^\circ \neq \emptyset$ *and* $w \in K^\circ$ *then* $\lim_{m\to\infty} g_{U_m}(z,w) = g_{K^\circ}(z,w)$ *pointwise in* $\mathbb{C}\setminus S$ *w.r.t.* z.

At one point in Chapter 2, we needed the fine topology:

Definition B.13 (Fine topology)**.** The fine topology on $\mathbb{C}$ is the coarsest topology on $\mathbb{C}$ in which every subharmonic function $s\colon \mathbb{C} \to \mathbb{C}$ is continuous.

Note that a set E is thin at $z_0 \in E$ if and only if z_0 is a finely isolated point of E.

Appendix C.

Riemann surfaces

Definition C.1 (Riemann surfaces)**.** A Riemann surface is a connected Hausdorff space R together with an open covering $(X_j)_{j\in J}$ of R and functions $z_j\colon X_j \to \mathbb{C}$ such that

(a) z_j is a homeomorphism of X_j onto a domain $z_j(X_j)$.

(b) each change of coordinate

$$z_k \circ z_j^{-1}\colon z_j(X_j \cap X_k) \to z_k(X_j \cap X_k)$$

is a conformal map.

We call the collection $\{(X_j, z_j)\}_J$ complex atlas on R.

Definition C.2. Let R be a Riemann surface with complex atlas $\{(X_j, z_j)\}_J$. A function $f\colon R \to \mathbb{C}$ is holomorphic if for all $j \in J$ the function $f \circ z_j^{-1}$ is holomorphic in $z_j(X_j)$.

Definition C.3. Let R be a Riemann surface with complex atlas $\{(X_j, z_j)\}_J$.

(a) For $j, k \in J$ we define $dz_k/dz_j = (z_k \circ z_j^{-1})' \circ z_j$.

(b) A holomorphic 1-form ω on R is a collection of holomorphic functions $\omega_j\colon X_j \to \mathbb{C}$ such that

$$\omega_j = \omega_k \cdot \frac{dz_k}{dz_j}$$

holds on $X_j \cap X_k$ for all $j, k \in J$.

(c) A conformal metric $\lambda(z)\,|dz|$ on R is a collection of functions $\lambda_j\colon X_j \to [0, \infty)$ such that

$$\lambda_j = \lambda_k \cdot \left|\frac{dz_k}{dz_j}\right|.$$

Example C.4. Let R be a Riemann surface with complex atlas $\{(X_j, z_j)\}_J$ and $f\colon R \to \mathbb{C}$ be holomorphic. Then we can define a holomorphic one-form df on X by $\omega_j := (f \circ z_j^{-1})' \circ z_j$. We call df the derivative of f.

At one point, we needed the *Freudenthal compactification* of a Riemann surface:

Theorem C.5 ([18]).
Let R be a Riemann surfaces. Then there exists a unique connected compact space X with $R \subseteq X$ such that

(a) R is open and dense in X;

(b) $X \setminus R$ is totally disconnected;

(c) if $e \in X \setminus R$ and U is a connected open neighborhood of e, then $U \setminus (X \setminus R)$ is connected.

Definition C.6. Let R be a Riemann surface and X as in Theorem C.5. We call X the Freudenthal compactification of R. A point $e \in X \setminus R$ is called Freudenthal end (or simply end) of R.

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