DOCTORAL THESIS

Hybrid Dynamical Systems: Modeling, Stability and Interconnection

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Abstract

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This work deals with a class of nonlinear dynamical systems exhibiting both continuous and discrete dynamics, which is called as hybrid dynamical system. We provide a broader framework of generalized hybrid dynamical systems allowing us to handle issues on modeling, stability and interconnections. Various sufficient stability conditions are proposed by extensions of direct Lyapunov method. We also explicitly show Lyapunov formulations of the nonlinear small-gain theorems for interconnected input-to-state stable hybrid dynamical systems. Applications on modeling and stability of hybrid dynamical systems are given by effective strategies of vaccination programs to control a spread of disease in epidemic systems.

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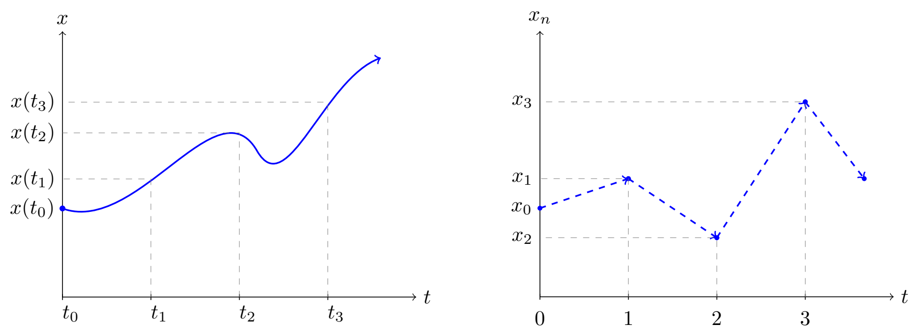
Chapter 1

Introduction

Dynamical systems are traditionally categorized as either continuous-time dynamical systems or discrete-time dynamical systems. For instances, classical mechanics like the swinging of a clock pendulum or the flow of water in a pipe can be naturally viewed as continuous-time dynamical systems, which are usually modeled by differential equations. Inventory controls, digital systems or population growth can be naturally viewed as discrete-time dynamical systems, which are usually modeled by difference equations. Figure 1.1 visualizes progresses of trajectories in continuous-time and discrete-time systems.

Numerous dynamical systems cannot be precisely placed in such categories. Particularly, they exhibit both of continuous and discrete dynamics. One of these systems is a bouncing ball. During flowing, it shows continuous dynamics described by Newton's second law of motion. In addition, it shows discrete dynamics at every jump since its velocity sign is changed instantaneously from minus to plus when the ball touches the ground, see Figure 1.2. Additional examples are provided by electronic circuits combined with analog and digital components and by mechanical systems controlled by digital computers. Such systems are called *hybrid dynamical systems* or just *hybrid systems* for short.

Basically, this work aims to provide a broader framework of hybrid systems allowing us to deal with related issues on modeling, stability and interconnections. The provided framework is based on the framework developed in [1–3], which is a solid foundations for a comprehensive theory of hybrid systems. At the same time, this work aims at providing the results from the very beginning which makes this work self-contained as suitable as possible.



(A) Continuous-time (B) Discrete-time

FIGURE 1.1: Illustrations of trajectories in continuous-time and discrete-time systems

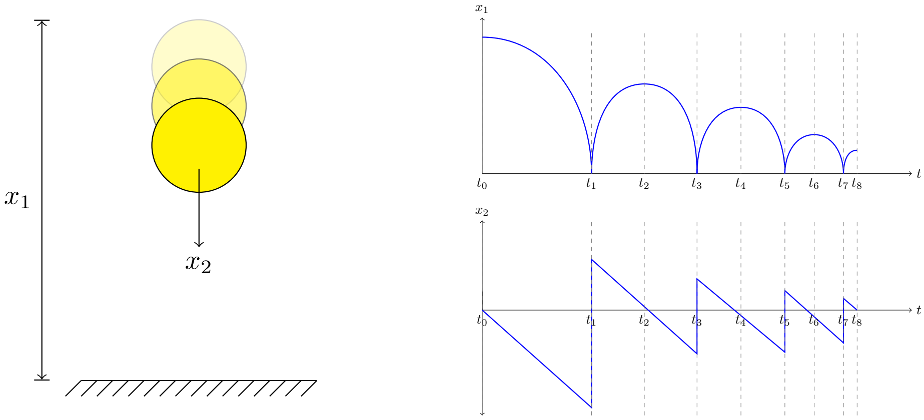


FIGURE 1.2: A bouncing ball exhibiting both continuous and discrete dynamics.

The fundamental modeling framework of hybrid systems is given in Chapter 3. In particular, a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, see (3.1), is given in the form:

$$\mathcal{H}: \quad x \in \mathcal{X} \quad \begin{cases} \dot{x} = f(x) & x \in \mathcal{C}, \\ x^+ = g(x) & x \in \mathcal{D}, \end{cases}$$

which is formulated by ordinary differential equations and difference equations. The concept of solutions is introduced in the very beginning of the chapter. Essentially, solutions always lie in the state space $\mathcal{X}$, and they are generally non-unique at points in $\mathcal{C} \cap \mathcal{D} \neq \emptyset$ since the hybrid system $\mathcal{H}$ may exhibit continuous or discrete dynamics. Moreover, we explicitly provide sufficient conditions for the existence of solutions to hybrid systems. This result arises from an application of fixed-point theory. Consequently, it can also be used to provide an iteration for finding numerical solutions to hybrid systems.

Since characterization of solutions to hybrid systems, especially in long-term trends, is significant, we therefore mainly focus on asymptotic stability of a non-empty compact set. We provide tools for stability investigation called *hybrid Lyapunov functions*. In brief, the existence of a hybrid Lyapunov function guarantees asymptotic stability of a non-empty compact set. Various types of hybrid Lyapunov candidate functions are additionally proposed with corresponding sufficient conditions to guarantee asymptotic stability of a non-empty compact set.

Additionally we proposed a notion of partial stability for hybrid dynamical systems consisting of variables like time, counters or logical parameters as a part of the state. Such variables never tend to zero, and they are required for stability of the systems. According to the provided notion, we can extend the results of stability to partial stability for hybrid systems. For instance, we can directly apply theorems like hybrid Lyapunov theorem to impulsive systems which are hybrid systems exhibiting discrete dynamics at specific impulse time sequences.

Chapter 4 deals with interconnections of hybrid dynamical systems. We firstly discuss the issues of modeling for interconnected hybrid systems in the beginning of this chapter. Motivated by a simple example of interconnected bouncing balls, the framework for interconnected hybrid systems given in the literature leads to the problem of physically meaningless solutions which implies loss of stability. We further discuss on these problems and suggest a possibility to solve them by proposing a different concept of solutions. Consequently, we introduce an extended framework for interconnected hybrid systems. Generally, an interconnected hybrid system $\mathcal{H}$ composed of n subsystem $\{{}^i\mathcal{H}\}_{i=1}^n$ with an admissible external input u is given by, see (4.7):

$$ {}^i\mathcal{H} \quad \begin{cases} {}^i\dot{x} = {}^if(x,u), & i \in I_{\mathcal{C}}(x,u), \\ {}^ix^+ = {}^ig(x,u), & i \in I_{\mathcal{D}}(x,u). \end{cases} $$

The proposed framework allows for the possibility to have continuous flows for some parts of the state also at those instants when other parts can jump. It also allows to consider one large hybrid system as an interconnection of several ones or vice versa to consider several interconnected hybrid systems as one larger hybrid system. The idea is to partition the state of a system in several parts that are allowed to jump separately while other parts are allowed to flow. In particular, the advantage of the proposed framework is that we can avoid the physically meaningless solutions.

Moreover, we investigate stability of interconnected hybrid systems. Together with extended notion of input-to-state stability (ISS) and other related notions, we provide a stability notion and results for interconnected hybrid systems. In essence, ISS for the interconnection or a subsystem is guaranteed by existence of an ISS-Lyapunov function. Moreover, existence of ISS-Lyapunov candidate function and its various corresponding conditions can also guarantee ISS property for the system. In addition, we also provide a result on the constructions of ISS-Lyapunov functions for the interconnection from ISS-Lyapunov function for subsystems.

In Chapter 5, we propose a mathematical model for a spread of disease with public vaccination programs in the framework of hybrid systems. The model is called *hybrid SIRS*, see (5.4). Unlike the epidemic models with vaccinations in [4–8] which is based on a framework of impulsive systems, hybrid SIRS model allows us to design significant effective strategies to control the spread of disease independently to predetermined vaccination time sequences. Basically, the system consists of two steady states. One is called the disease-free steady state due to no infected individuals remaining in the system at this state. Another one is called the disease steady state since at this state infected individuals remain in the system. In case of no vaccination, the disease-free steady state is globally asymptotically stable if the recovery rate is not smaller than the infection rate, i.e., the spread of the disease eventually disappears, and every individual get no more infected. Additionally, the disease-free steady state is unstable and the disease steady state is locally asymptotically stable if the infection rate is larger than the recovery rate, i.e., the infected individuals permanently remain in the population, which means the spread of disease is eventually long lasting.

Consequently, we mainly focus on the case of the infection rate being larger than the recovery rate. Further results on stability analysis of the system suggest that if vaccination programs are launched limitedly, then the epidemic eventually remains forever. Additionally, we discover that, even in the case of ideal vaccines, the vaccination can possibly fail by choosing an inappropriate strategy to limit or stop an epidemic since the number of infected individuals is permanently positive, and it is not different from the case of no vaccination or a finite number of vaccination programs. Due to this reason, we provide possible strategies to limit or stop the spread of disease. We explicitly show that the provided strategies are significantly effective to control an epidemic. Various numerical simulations are given to demonstrate a possibility to limit or stop the spread of disease.

The results on relaxed Lyapunov theorems in Chapter 3 is published in [9]. The framework for interconnected hybrid dynamical systems, generalized hybrid time domain and concept of solutions are published in [10]. Partial results on dwell-time and small-gain conditions for impulsive systems and constructions of Lyapunov functions for interconnections in Chapter 4 are published in [11] and [12] respectively.

Chapter 2

Preliminaries

In this chapter, we review some essential notions and results in control theory, dynamical systems and functional analysis which are important for this work. All of the notations and symbols are listed in Appendix A.

The set of all real numbers defines the one-dimensional *Euclidean space* denoted by $\mathbb{R}$. Denoted by $\mathbb{N}$ the set all of natural numbers including zero, $\mathbb{Z}$ the set of all integers. For a subset $R \subset \mathbb{R}$, denoted by

$$R_{>0} := \{r \in R : \ r > 0\},$$

and

$$R_{\geq 0} := R_{>0} \cup \{0\}.$$

The set of all *n-dimensional vectors* $x = (x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$, defines the n-dimensional Euclidean space denoted by $\mathbb{R}^n$. For any real number $k \in \mathbb{R}$, vectors $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define

$$k \cdot (x + y) = (k \cdot (x_1 + y_1), k \cdot (x_2 + y_2), \ldots, k \cdot (x_n + y_n)).$$

A *function* f mapping from a set A into a set B is denoted by $f : A \to B$. A function $f : A \to B$ is called a *real-valued function* if $B \subset \mathbb{R}$, and it is called a *real function* if $A, B \subset \mathbb{R}$.

The *norm* $\|x\|$ of a vector x is a real-valued function satisfying the following:

(1) For any $x \in \mathbb{R}^n$, $\|x\| \geq 0$ with $\|x\| = 0 \iff x = 0$.

(2) For any $x, y \in \mathbb{R}^n$, $\|x + y\| \leq \|x\| + \|y\|$.

(3) For any $a \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\|a \cdot x\| = |a| \cdot \|x\|$.

In this work, $\|x\|$ denotes any *L^p-norm* of a vector x, defined by

$$\|x\|_p := (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}, \quad 1 \leq p < \infty,$$

and the *L^∞-norm* of a vector x is defined by

$$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

All L^p-norms are equivalent in the sense that there exist positive real numbers c_1 and c_2 such that the inequality is satisfied:

$$c_1 \|x\|_b \leq \|x\|_a \leq c_2 \|x\|_b$$

for any $a, b \geq 1$ and $x \in \mathbb{R}^n$. For a vector x and a nonempty subset $S \subset \mathbb{R}^n$, define

$$\|x\|_S := \inf_{s \in S} |x - s| .$$

A *sequence* of vectors $x_0, x_1, x_2, \ldots, x_j, \ldots \in \mathbb{R}^n$, denoted by $\{x_j\}$, converges to a vector $x \in \mathbb{R}^n$ if $\|x_j - x\| \to 0$ as $j \to \infty$. Equivalently, a sequence $\{x_j\}$ converges to a vector x if for any $\varepsilon > 0$, there exists an integer N such that

$$\|x_j - x\| < \varepsilon \quad \forall j \geq N.$$

A sequence $\{x_j\}$ is said to be *convergent* if there exists a vector x such that $\{x_j\}$ converges to x. A sequence $\{x_j\}$ is said to be *divergent* if there exists none of vectors x such that $\{x_j\}$ converges to x. A sequence of real numbers $\{x_j\}$ is said to be:

(1) *non-decreasing* if $x_j \leq x_{j+1}$ for all $j \in \mathbb{N}$,

(2) *increasing* if $x_j < x_{j+1}$ for all $j \in \mathbb{N}$,

(3) *non-increasing* if $x_j \geq x_{j+1}$ for all $j \in \mathbb{N}$,

(4) *decreasing* if $x_j > x_{j+1}$ for all $j \in \mathbb{N}$.

A sequence x_j is said to be a *subsequence* of a sequence y_j if there exists an increasing sequence $\{n_j\}$ such that $x_j = y_{n_j}$ for all j. A vector x is an *accumulation point* of a sequence $\{x_j\}$ if there exists a subsequence of $\{x_j\}$ that converges to x. An increasing sequence of real numbers that is bounded from above converges to a real number, and a decreasing sequence that is bounded from below converges to a real number.

A subset $S \subset \mathbb{R}^n$ is said to be *open* if, for any vector $x \in S$, there exists a ε-neighborhood of x

$$B_\varepsilon(x) := \{z \in \mathbb{R}^n : \|z - x\| \leq \varepsilon\}$$

such that $B_\varepsilon(x) \subset S$. A set S is *relatively closed* in a nonempty subset $X \subset \mathbb{R}^n$ if $X \setminus S$ is open. A set S is *closed* if it is relatively closed in $\mathbb{R}^n$. Moreover, a set S is closed if and only if every convergent sequence $\{x_j\}$ with elements in S converges to a point in S. A set S is *bounded* if there exists a positive number k such that $\|x\| \leq k$ for all $x \in S$. A set S is *compact* if it is closed and bounded. A point p is a *boundary point* of a set S if every ε-neighborhood of p contains at least one point of S and one point not belonging to S. Denoted by ∂S the set of all boundary points of S. The set ∂S is called *the boundary* of a set S. An open set contains none of its boundary points, but a closed set contains all its boundary points. A point p is an *interior point* of a set S if it belongs to $S \setminus \partial S$. Denoted by int(S) the set of all interior points of a set S. The set int(S) is called *the interior* of a set S. A set S is open iff $S = \text{int}(S)$. The closure of a set S is defined by $\bar{S} := S \cup \partial S$. A set S is closed iff $S = \bar{S}$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous* at a point $x \in \mathbb{R}^n$ if $f(x_j) \to f(x)$ whenever $x_j \to x$. Equivalently, f is continuous at x if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon.$$

A function f is continuous on a set S if it is continuous at every point in S. Given a subset $X \subset \mathbb{R}^n$, a function $f : X \to \mathbb{R}^m$ is said to be continuous if f is continuous on X. If $f : A \to B$ and $g : B \to C$, the function $g \circ f : A \to C$ defined by $(g \circ f)(x) := g(f(x))$ is called *the composition of functions* f and g. Given an interval $I \subset \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be:

(1) *increasing* on I if $f(x_1) < f(x_2)$ for any $x_1 < x_2$,

(2) *decreasing* on I if $f(x_1) > f(x_2)$ for any $x_1 < x_2$,

(3) *non-decreasing* on I if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$,

(4) *non-increasing* on I if $f(x_1) \geq f(x_2)$ for any $x_1 < x_2$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable* at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The limit $f'(x)$ is called *the derivative of a function f at x*. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is increasing on an interval $I \subset \mathbb{R}$ if $f'(x) > 0$ for all $x \in I$. It is decreasing on I if $f'(x) < 0$ for all $x \in I$.

For a real function $f : A \to B$, it is said to be of class $C^0(A, B)$ if it is continuous. It is said to be of class $C^k(A, B)$ if there exists kth-derivative of f which is continuous. It is said to be of class $C^\infty(A, B)$ if there exists kth-derivative of f which is continuous for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, denote $C^k(A) := C^k(A, A)$ and $C^k := C^k(\mathbb{R})$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuously differentiable* if $f \in C^1$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the *partial derivative of f at a point $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ wrt the variable x_j* is defined as

$$\frac{\partial}{\partial x_j} f(a) := \lim_{h \to 0} \frac{f(a_1, \ldots, a_{j-1}, a_j + h, a_{j+1}, \ldots, a_n) - f(a)}{h},$$

and it is said to be *continuous differentiable* at a point $a \in \mathbb{R}^n$ if the partial derivative $\partial f_i / \partial x_j$ exists and continuous at a for $1 \leq i \leq m$ and $1 \leq j \leq n$. A function f is continuous differentiable on a set S if it is continuous differentiable at every point in S. For a continuous differentiable $f : \mathbb{R}^n \to \mathbb{R}$, *the gradient of f* is denoted by

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Consider a discrete-time dynamical system given by a system of difference equations

$$x_{j+1} = g(x_j), \tag{2.1}$$

where $j \in \mathbb{N}$ is the discrete time, $x_j \in \mathbb{R}^n$ is the state at discrete time j, $g : \mathbb{R}^n \to \mathbb{R}^n$ describes the discrete dynamics, and the initial condition is given at the discrete time $j = 0$ by $x_0 \in \mathbb{R}^n$. A sequence $\{\tau_j\}$ is a solution to the system (2.1) if it satisfies $\tau_0 = x_0$ and $\tau_{j+1} = g(\tau_j)$ for any discrete-time j. It is clear to see that there exists a non-trivial solution to the system (2.1) if g is continuous on $\mathbb{R}^n$. A point x^* is called an *equilibrium point* or a *steady state* for the system (2.1) if $g(x^*) = x^*$. The following are stability notions of discrete-time dynamical systems.

Definition 2.1 (Stability of Discrete-Time Dynamical Systems)**.** For a system in the form (2.1), a steady state x^* is called

- *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0 - x^*\| < \delta$ implies $\|x_j - x^*\| < \varepsilon$ for all $j \in \mathbb{N}$;

- *attractive* if there exists $\eta > 0$ such that $\|x_0 - x^*\| < \eta$ implies $\lim_{j\to\infty} x_j = x^*$;
- *asymptotically stable* if it is both stable and attractive.

Moreover, it is called *unstable* if it is not stable.

Consider a continuous-time dynamical system given by a system of ordinary differential equations

$$\dot{x}(t) = f\left(x\left(t\right)\right), \quad x(t_0) = x_0, \tag{2.2}$$

where t is the time, $x(t) \in \mathbb{R}^n$ is the state at time t, $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the continuous dynamics and $x(t_0)$ is the initial condition at time t_0. Each solution $x : [t_0, t^\star] \to \mathbb{R}^n$ to the differential equation in the form of (2.2) will be understood in the *Carathéodory sense* [13, 14] , i.e., function x is required to be absolutely continuous, and $\dot{x}(t) = f\left(x\left(t\right)\right)$ is required to hold for almost all $t \in [t_0, t^\star]$. For an absolutely continuous $x : [t_0, t^\star] \to \mathbb{R}^n$, the derivative $\dot{x}(t)$ exists for all $t \in [t_0, t^\star]$ except a set of measure zero.

Definition 2.2. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the *Lipschitz condition* if it holds the inequality

$$\|f(x) - f(y)\| \le L\,\|x - y\| \tag{2.3}$$

with some *Lipschitz constant* L.

Definition 2.3 (Lipschitz Continuous Functions)**.** A function f is *locally Lipschitz continuous* on a domain (open and connected set) $D \subset \mathbb{R}^n$ if each point of D has a neighborhood D_0 such that it satisfies the Lipschitz condition (2.3) for all points $x, y \in D_0$ with some Lipschitz constant Lipschitz constant L_0. A function f is *Lipschitz continuous* on a set W if it satisfies the Lipschitz condition (2.3) for all points in W with the same Lipschitz constant L. A function f is *globally Lipschitz continuous* if it is Lipschitz on $\mathbb{R}^n$.

Note that A locally Lipschitz continuous function on a domain D is not necessarily Lipschitz continuous on D. Any locally Lipschitz continuous function is differentiable almost everywhere. In case of the points where such a function is not differentiable, we use the notion of *Clarke's generalized gradient* [15] throughout this work.

The following theorems are well-known results of the existence and uniqueness of solutions to the continuous-time dynamical system (2.2), see [16–19].

Theorem 2.4 (Peano Existence Theorem [16, 17, 19])**.** *Given the continuous-time system* (2.2)*, if the function f is continuous on a neighborhood of x_0, then there exists a solution, defined in a neighborhood of t_0, to the system* (2.2)*.*

Theorem 2.5 (Picard-Linderlöf Existence and Uniqueness Theorem [18])**.** *Given the continuous-time system* (2.2)*, if the function f is globally Lipschitz continuous, then there exists one and only one solution to the system* (2.2)*.*

A point x^* is called an equilibrium point or a steady state for the continuous-time dynamical system (2.2) if $f(x^*) = 0$. The following are stability notions of continuous-time dynamical systems.

Definition 2.6 (Stability of Continuous-Time Dynamical Systems)**.** For a system in the form (2.2), a steady state x^* is called

- stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(0) - x^*\| < \delta$ implies $\|x(t) - x^*\| < \varepsilon$ for all $t \in \mathbb{R}_{\geq 0}$;
- attractive if there exists $\eta > 0$ such that $\|x(0) - x^*\| < \eta$ implies

$$\lim_{t \to \infty} x(t\,; x(0)) = x^*;$$

- asymptotically stable if it is both stable and attractive.

Moreover, it is called unstable if it is not stable.

Furthermore, the following notions are required in order to investigate stability of dynamical systems.

Definition 2.7 (Positive Definite Functions)**.** A continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *positive definite* if $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.8 (Radially Unbounded Functions)**.** A positive definite function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *radially unbounded* if $\alpha(s) \to \infty$ as $\|s\| \to \infty$.

Example 2.9. The functions

$$\begin{aligned} f_1(x) &:= (x_1 - 2x_2)^2, \\ f_2(x) &:= \frac{(x_1 + x_2)^4}{1 + (x_1 + x_2)^4} + (x_1 - 2x_2)^2 \end{aligned}$$

are not radially unbounded because along the line $x_1 = 2x_2$, the condition is not satisfied. In addition, only the function f_2 is positive definite.

Definition 2.10 (Functions of Class $\mathcal{P}$)**.** A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{P}$ if α is positive definite.

For any $\alpha_1, \alpha_2 \in \mathcal{P}$, we say $\alpha_1 < \alpha_2$ if $\alpha_1(s) < \alpha_2(s)$ for all $s > 0$. The meaning of the following: $\alpha_1 \leq \alpha_2$, $\alpha_1 > \alpha_2$, and $\alpha_1 \geq \alpha_2$ is understood in the same manner.

Definition 2.11 (Functions of Class $\mathcal{S}$)**.** A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{S}$ if it belongs to the class $\mathcal{P}$ and $\alpha < \mathrm{id}$.

Example 2.12. Consider the following:

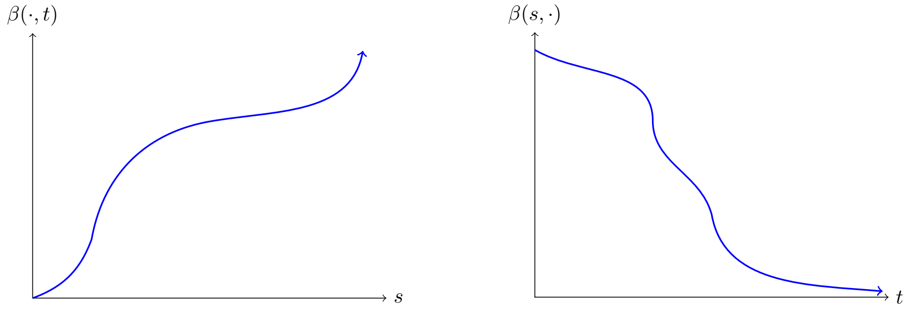


FIGURE 2.1: An illustration of a class $\mathcal{KL}$ function.

- The function $\alpha(s) := \dfrac{s}{1+|s|}$, for all $s \in \mathbb{R}_{\geq 0}$, belongs to the class $\mathcal{P}$ and $\mathcal{S}$ since $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$, and $\dfrac{s}{1+|s|} < s$ for all $s > 0$.
- The function $\alpha(s) := s^2$, for all $s \in \mathbb{R}_{\geq 0}$, belongs to the class $\mathcal{P}$ but does not belong to the class $\mathcal{S}$ since $a(s) \geq s$ for all $s \geq 1$.

Definition 2.13 (Functions of Class $\mathcal{K}$ and Class $\mathcal{K}_\infty$)**.** A function $\alpha \in \mathcal{P}$ belongs to the class $\mathcal{K}$ if it is strictly increasing, and it belongs to the class $\mathcal{K}_\infty$ if it additionally satisfies $\alpha(s) \to \infty$ as $s \to \infty$.

Example 2.14. Consider the following:

- The function $\alpha(s) := \sqrt{s}$, for any $s \geq 0$, is strictly increasing since $\alpha'(s) = \dfrac{1}{2\sqrt{s}} > 0$ for all $s > 0$. It belongs to the class $\mathcal{K}$ and also to the class $\mathcal{K}_\infty$ since it additionally satisfies $\lim_{s\to\infty} \alpha(s) = \infty$.
- The function $\alpha(s) := \tan^{-1}(s)$ is strictly increasing since $\alpha'(s) = \dfrac{1}{1+s^2} > 0$. It belongs to the class $\mathcal{K}$, but does not belong to the class $\mathcal{K}_\infty$ since $\lim_{s\to\infty} \alpha(s) = \pi/2$.
- The function $\alpha(s) := \max\left\{s, s^2\right\}$ is continuous, strictly increasing and satisfies $\lim_{s\to\infty} \alpha(s) = \infty$. Therefore, it belongs to the class $\mathcal{K}_\infty$. Note that continuous differentiability is not required for the class $\mathcal{K}$ functions.

Definition 2.15 (Functions of Class $\mathcal{L}$)**.** A continuous function $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{L}$ if it is non-increasing and satisfies $\tau(t) \to 0$ as $t \to \infty$.

Definition 2.16 (Functions of Class $\mathcal{KL}$)**.** A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each fixed non-negative t the function $\beta(\cdot, t) \in \mathcal{K}$, and for each fixed non-negative s the function $\beta(s, \cdot) \in \mathcal{L}$.

Example 2.17. Consider the following:

- The function $\beta(s,t) := \dfrac{k_1 s}{k_2 st + k_3}$, for any positive numbers k_1, k_2 and k_3, is strictly increasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{k_1 k_3}{(k_2 st + k_3)^2} > 0$$

  and decreasing in t since

$$\frac{\partial \beta}{\partial t} = -\frac{k_1 k_2 s^2}{(k_2 st + k_3)^2} < 0.$$

  Moreover, it satisfies $\beta(s,t) \to 0$ as $t \to \infty$. Hence, it belongs to class $\mathcal{KL}$.

- The function $\beta(s,t) := s^k \mathrm{e}^{-t}$, for any positive k, belongs to class $\mathcal{KL}$.

We give an illustration of a class $\mathcal{KL}$ function in Figure 2.1. The following are some important properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Theorem 2.18 ([18, 20]). *Let α_i be functions of class $\mathcal{P}$ for $i \in \mathbb{N}_4$, and β be a function of class $\mathcal{KL}$. If α_1 and α_2 belongs to class $\mathcal{K}$, α_3 and α_4 belongs to class $\mathcal{K}_\infty$, then the following is true:*

- α_3^{-1} *belongs to class $\mathcal{K}_\infty$.*
- $\alpha_1 \circ \alpha_2$ *belongs to class $\mathcal{K}$.*
- $\alpha_3 \circ \alpha_4$ *belongs to class $\mathcal{K}_\infty$.*
- $\tilde{\beta}(s,t) := \alpha_1(\beta(\alpha_2(s),t))$ *belongs to class $\mathcal{KL}$.*

Theorem 2.19 ([18]). *Let $D \subset \mathbb{R}^n$ and $B_r := \{x \in \mathbb{R}^n : \|x\| \le r\} \subset D$ for some $r > 0$. If $V : D \to \mathbb{R}_{\ge 0}$ is positive definite, then there exist class $\mathcal{K}$ functions ψ_1 and ψ_2 such that the following inequality is satisfied*

$$\psi_1(\|x\|) \le V(x) \le \psi_2(\|x\|) \quad \textit{for all } x \in B_r.$$

If the function V is radially unbounded, then the functions ψ_1 and ψ_2 can be chosen to belong to the class $\mathcal{K}_\infty$.

Some basic concepts and results of hybrid dynamical systems will be reviewed in the early beginning of the next chapter.

Chapter 3

Hybrid Dynamical Systems

A hybrid dynamical system, or hybrid system, is a dynamical system exhibiting both continuous and discrete dynamics. Hybrid dynamical modeling is widely presented in many modern real world applications such as robots controlling [21, 22], computer science [23], control systems [1, 3, 24], commercial problems [25, 26], biological and medical systems [27–30]. Moreover, hybrid phenomena have been modeled in many different frameworks since last few decades or more. Those frameworks include hybrid automata [31, 32], impulsive systems [33–36] and switched systems [37].

To work with hybrid systems, we use the framework developed in [1, 3, 24]. For the most part, there are some differences from [31, 32, 34, 38] due to not only their structure but also concept of solution to systems. The most considerable advantages of the frameworks developed in [1, 3, 24] are results on robust asymptotic stability and extended classical stability analysis tools. In addition, models such as hybrid automata, impulsive differential equations and switching systems can be translated to the framework developed in [1, 3, 24]. One of all benefits of translations is that the stability theorems can be applied to other classes of hybrid dynamical systems, e.g., the invariance principles for switching systems [39].

3.1 Modeling Framework

A hybrid system $\mathcal{H}$ is modeled in the following form

$$\mathcal{H}: \quad x \in \mathcal{X} \subset \mathbb{R}^n \quad \begin{cases} \dot{x} = f(x) & \text{if} \quad x \in \mathcal{C} \subset \mathcal{X}, \\ x^+ = g(x) & \text{if} \quad x \in \mathcal{D} \subset \mathcal{X}. \end{cases} \tag{3.1}$$

This model suggests that the state of hybrid system $\mathcal{H}$, represented by x, can change according to a differential equation $\dot{x} = f(x)$ while in the set $\mathcal{C}$, and it can change according to a difference equation $x^+ = g(x)$ while in the set $\mathcal{D}$. The notation $\dot{x}$ represents the velocity of the state x, while x^+ represents the value of the state after an instantaneous change.

The behavior of hybrid system $\mathcal{H}$ that can be described by a differential equation is referred to as *flow*, while the behavior of $\mathcal{H}$ that can be described by a difference equation is referred to as *jumps*. Consequently the following names are assigned to the four objects involved in the model (3.1): *the flow set* $\mathcal{C}$, *the flow map* f, *the jump set* $\mathcal{D}$ and *the jump map* g.

This work deals with hybrid systems in finite-dimensional spaces, i.e., the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are subsets of an n-dimensional Euclidean space $\mathcal{X} \subset \mathbb{R}^n$. For consistency reasons, we require that the flow map f be defined at least on the flow set $\mathcal{C}$, and the jump map g be defined at least on the jump set $\mathcal{D}$.

The model (3.1) can be specialized to represent the dynamics of purely continuous-time or purely discrete-time systems on $\mathbb{R}^n$, i.e., purely continuous-time systems can be captured with a flow set defined as $\mathbb{R}^n$ and an empty jump set, while the latter can be captured with an empty flow set and a jump set defined as $\mathbb{R}^n$.

For convenience reasons, we may write the the model (3.1) as $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$. In addition, these objects $\mathcal{X}, f, \mathcal{C}, g$, and $\mathcal{D}$ are called the *data of hybrid system* $\mathcal{H}$.

Note that this modeling framework, structure and concept of the model (3.1) are taken from [1, 3].

In the following examples, we show some dynamical systems from science and engineering that can be modeled with this framework. The first example is a bouncing ball. This is a classical example of hybrid dynamic phenomena.

Example 3.1 (Bouncing Ball)**.** One of classical hybrid phenomena is a bouncing ball, i.e., a ball is dropped from some height above the floor. It flows by some initial velocity and gravitation force until a collision with the floor happens. After touching the floor, the ball jumps, i.e., the velocity changes its sign instantaneously and reduces its magnitude by a restitution factor $\lambda \in [0, 1)$.

Denote the height of the bouncing ball by x_1 and its velocity by x_2. Let the state be

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X} := \mathbb{R}^2.$$

The flow map f and the flow set $\mathcal{C}$ are defined as

$$f(x) := \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} \text{ and } \mathcal{C} := \left\{ x \in \mathbb{R}^2 : x_1 \geq 0 \right\}$$

where γ represents the gravitation constant. The jump map and the jump set are respectively defined by

$$g(x) := \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} \text{ and } \mathcal{D} := \left\{ x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0 \right\}$$

where $\lambda \in [0, 1)$ denotes the restitution factor between the ball and the floor.

Example 3.2 (On/Off Switching Systems)**.** Consider a system monitoring on the value of $x_1 \in \mathbb{R}_{\geq 0}$ with an automatic on/off switch. Suppose that the positive numbers ψ_1 and ψ_2 are the lower and upper limit values of the monitored value x_1 respectively, i.e., $0 < \psi_1 \leq x_1 \leq \psi_2$. Let S_0 and S_1 be positive definite functions from $\mathbb{R}_{\geq 0}$. The value of x_1 is started at some point in the interval (ψ_1, ψ_2). It is continuously changed, and its dynamics are dependent on the switch. When the switch is turned on, x_1 increasingly evolves according to the equation $\dot{x}_1 = S_1(x_1)$. Additionally, it is decreasing according to the equation $\dot{x}_1 = -S_0(x_1)$ if the switch is turned off. Moreover, the switch is automatically turned on if x_1 reaches the lower limit value ψ_1 and automatically turned off when x_1 reaches the upper limit value ψ_2.

To model this system as $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, let us denote the state x of the system by

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X} := \mathbb{R}_{\geq 0} \times \{0, 1\}.$$

The variable x_2 indicates a stage of the switch. While the switch is turned on, we assign $x_2 = 1$. Otherwise, it is assigned by $x_2 = 0$. The flow set and the jump set are given by $\mathcal{C} := \{x \in \mathcal{X} : \ \psi_1 \leq x_1 \leq \psi_2\}$ and $\mathcal{D} := \{x \in \mathcal{X} : \ x_1 = \psi_1 \text{ or } x_1 = \psi_2\}$ respectively. The flow map f and the jump map g are defined by

$$f(x) := \begin{pmatrix} f_1(x_1) \\ 0 \end{pmatrix}, \ f_1(x_1) := \begin{cases} S_1(x_1) & \text{if } x_2 = 1, \\ -S_0(x_1) & \text{if } x_2 = 0. \end{cases}$$

$$g(x) := \begin{pmatrix} x_1 \\ g_2(x_2) \end{pmatrix}, \ g_2(x_2) := \begin{cases} 0 & \text{if } x_2 = 1, \\ 1 & \text{if } x_2 = 0. \end{cases}$$

Example 3.3 (Impulsive Systems)**.** This is a special case of hybrid systems exhibiting discrete dynamics at specific sequences of time which are given in advance. Such systems are called *impulsive systems*, see [3, 11, 34, 40–42].

Let the sequence $\{t_j\}$ be an increasing and unbounded sequence of positive numbers. Consider the differential equation

$$\dot{x} = f(x),$$

for some continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ with impulses leading to instantaneous changes at the predetermined times $t_1, t_2, \ldots$, according to

$$\Delta x(t_j) = g(x, t_j),$$

for some $g : \mathbb{R}^n \times T \to \mathbb{R}^n$ and $T = \{t_1, t_2, \ldots\}$.

To model this system as $\mathcal{H} = (\mathcal{X}, F, \mathcal{C}, G, \mathcal{D})$, firstly let us denote the state by

$$z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{X} := \mathbb{R}^{n+1}$$

where $z_1 := x$, and $z_2 := t$. The data of hybrid system $\mathcal{H}$ is given as follows. The flow set

$$\mathcal{C} := \mathbb{R}^n \times (\mathbb{R}_{\geq 0} \setminus T),$$

and the jump set

$$\mathcal{D} := \mathbb{R}^n \times T.$$

The flow map F and the jump map G are respectively defined by

$$F(z) := \begin{pmatrix} f(z_1) \\ 1 \end{pmatrix}, \quad G(z) := \begin{pmatrix} z_1 + g(z_1, t_j) \\ z_2 \end{pmatrix}.$$

3.2 Basic Assumptions

In this section, we provide some important assumptions to deal with hybrid dynamical systems.

For a hybrid dynamical system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ given by (3.1), let the following conditions be satisfied:

(BA1) $\mathcal{X}$ is open;

(BA2) $\mathcal{C}$ and $\mathcal{D}$ are relatively closed in $\mathcal{X}$;

(BA3) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous;

(BA4) $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

The above assumptions are called *basic assumptions for hybrid systems*. These are the important conditions to guarantee existence of a non-trivial solution to a hybrid system which will be presented in the upcoming sections.

Throughout this work, the basic assumptions for hybrid systems (BA1)-(BA4) are satisfied unless otherwise stated.

3.3 Concept of Solutions

The concept of a solution to a hybrid system is the topic what we are going to discuss here. It was firstly introduced in [1, 3]. Almost all of the following definitions are taken from [1]. A generalized concept of hybrid time is introduced, solutions to a hybrid system are defined and conditions for the existence of a solution to a hybrid system are addressed in this section. Moreover, we illustrate the concept of a solution by some examples from a bouncing ball, an on/off switching system and an impulsive system.

3.3.1 Hybrid Time Domains

Solutions to continuous-time systems are parameterized by time $t \in \mathbb{R}_{\geq 0}$, and solutions to discrete-time systems are parameterized by the discrete steps $j \in \mathbb{N}$, in other words, by the number of jumps. For hybrid systems, it is natural to suggest that solutions should be parameterized by both t, the amount of time passed, and j, the number of jumps that have occurred.

Definition 3.4 (Hybrid Time Domain)**.** A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\}) \tag{3.2}$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$,

$$E \cap ([0, T] \times \{0, 1, 2, \ldots, J\})$$

is a compact hybrid time domain.

Note that every hybrid time domain contains the ordered pair $(0,0)$. The last term (if existent) of a hybrid time domain is allowed to be in the form $[t_j, T) \times \{j\}$, for some $j \in \mathbb{N}$, with T finite or $T = \infty$.

Example 3.5. Consider the following:

- The set

$$\begin{aligned} E = &([0,1] \times \{0\}) \\ &\cup ([1,2] \times \{1\}) \\ &\cup ([2,2] \times \{2\}) \\ &\cup ([2,2] \times \{3\}) \\ &\cup ([2,4] \times \{4\}) \end{aligned}$$

  is a (compact) hybrid time domain. It is illustrated in Figure 3.1.

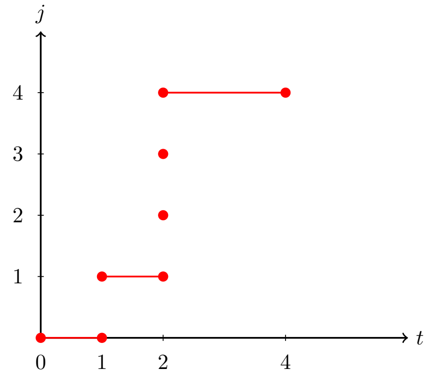


FIGURE 3.1: An example of a (compact) hybrid time domain.

- Both $[0,\infty] \times \{0\}$ and $\{0\} \times \mathbb{N}$ are hybrid time domains, but not compact.
- The ordered pairs $(0,1)$ and $(1,0)$ cannot be in the same (compact) hybrid time domain.

Definition 3.6 (The order on hybrid time domains)**.** Given E a hybrid time domain containing (t_1, j_1) and (t_2, j_2), we define

$$(t_1, j_1) \preceq (t_2, j_2) \iff t_1 + j_1 \leq t_2 + j_2,$$

and,

$$(t_1, j_1) \prec (t_2, j_2) \iff t_1 + j_1 < t_2 + j_2.$$

Note that points in two different hybrid time domain is incomparable. For instance, that is not the case either $(0,1) \preceq (1,0)$ or $(1,0) \preceq (0,1)$ since the points $(0,1)$ and $(1,0)$ cannot contain in the same hybrid time domain. They are incomparable.

Definition 3.7. Given a hybrid time domain E,

$$\sup_t E := \sup\{t \in \mathbb{R}_{\geq 0} : \exists\, j \in \mathbb{N},\ (t,j) \in E\},$$

and

$$\sup_j E := \sup\{j \in \mathbb{N} : \exists\, t \in \mathbb{R}_{\geq 0},\ (t,j) \in E\}.$$

Furthermore $\sup E := (\sup_t E, \sup_j E)$, and length $(E) := \sup_t E + \sup_j E$.

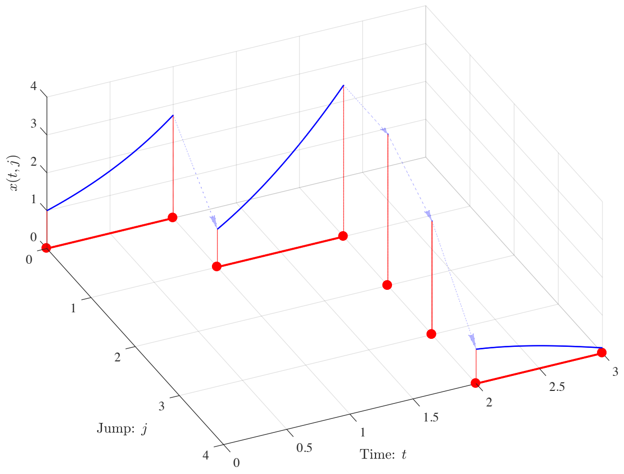


FIGURE 3.2: A hybrid arc with its corresponding hybrid time domain.

Definition 3.8 (Hybrid Arc)**.** Let E be a hybrid time domain. A function $x : E \to \mathbb{R}^n$ is a *hybrid arc* on E if for each $j \in \{0, 1, 2, \ldots, (\sup_j E - 1)\}$ the function $t \mapsto x(t, j)$ is locally absolutely continuous on the interval $[t_j, t_{j+1}]$.

Definition 3.9. Given a hybrid time domain E and a hybrid arc $x : E \to \mathbb{R}^n$, define the domain of x by

$$\text{dom } x := E,$$

and define the range of x by

$$\text{rge } x := \{y \in \mathbb{R}^n : \ \exists (t, j) \in \text{dom } x, \ x(t, j) = y\}.$$

Definition 3.10 (Types of Hybrid Arcs)**.** A hybrid arc x is said to be

(1) *nontrivial* if $\text{rge } x$ contains at least two points;

(2) *bounded* if $\sup \{\|y\| : \ y \in \text{rge } x\} < \infty$;

(3) *complete* if $\text{length } (\text{dom } x) = \infty$;

(4) *discrete* if $\sup_t \text{dom } x = 0$;

(5) *continuous* if $\sup_j \text{dom } x = 0$;

(6) *Zeno* if it is complete and $\sup_t \text{dom } x < \infty$;

(7) *eventually discrete* if $T = \sup_t \text{dom } x < \infty$ and $\text{dom } x \cap (\{T\} \times \mathbb{N})$ contains at least two points;

(8) *eventually continuous* if $J = \sup_j \text{dom } x < \infty$ and $\text{dom } x \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least two points;

(9) *absolutely hybrid* if it is neither eventually discrete nor eventually continuous.

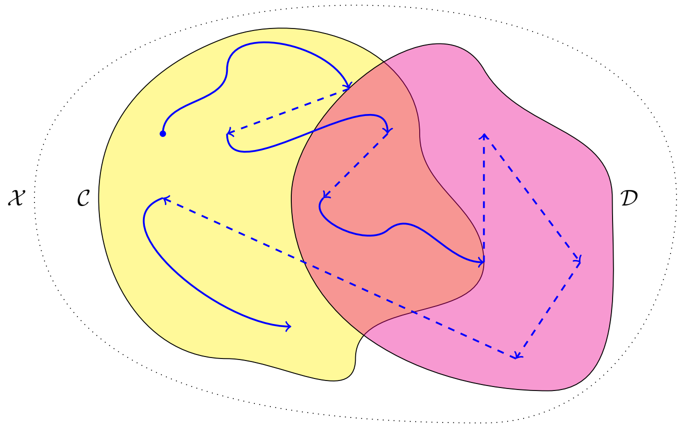


FIGURE 3.3: The behavior of solutions to hybrid systems.

Figure 3.2 illustrates a trajectory given by a hybrid arc x with its corresponding hybrid time domain drawn by solid red lines in (t, j)-plane. Each dashed line with arrow head represents a jump occurring in the system.

3.3.2 Solutions to Hybrid Systems

Given a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, its solutions are hybrid arcs x that satisfy certain conditions determined by the hybrid time domain $\operatorname{dom} x$ and the data of the hybrid system.

Definition 3.11 (Solutions to a Hybrid System)**.** A hybrid arc x is a solution to (3.1) if $x(0,0) \in \overline{\mathcal{C}} \cup \mathcal{D}$, $x(t,j) \in \mathcal{X}$ for all $(t,j) \in \operatorname{dom} x$, and

1. for all $j \in \mathbb{N}$ such that $I^j := \{t : (t,j) \in \operatorname{dom} x\}$ has nonempty interior,

$$\begin{aligned} x(t,j) &\in \mathcal{C} && \text{for all} && t \in \operatorname{int} I^j, \\ \dot{x}(t,j) &= f(x) && \text{for almost all} && t \in I^j; \end{aligned} \tag{3.3}$$

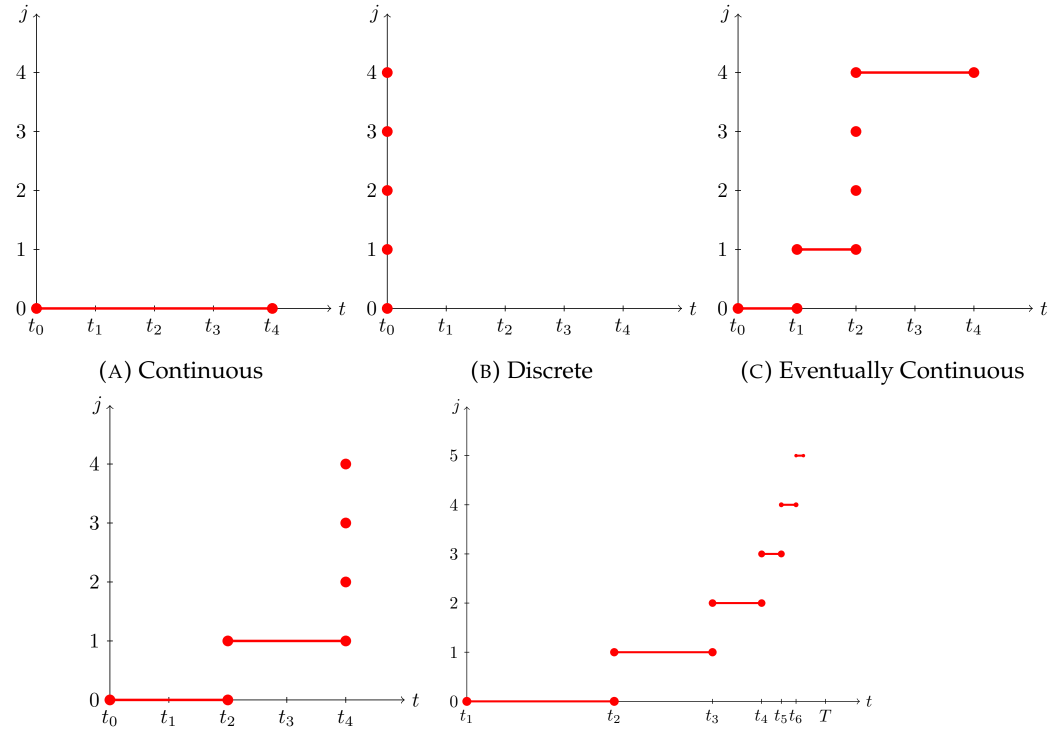
2. for all $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$\begin{aligned} x(t,j) &\in \mathcal{D}, \\ x(t,j+1) &= g(x(t,j)). \end{aligned} \tag{3.4}$$

Solutions will stay in $\mathcal{X}$. At points in $\mathcal{C} \cap \mathcal{D}$, solutions can be nonunique, i.e., they can either flow or jump. Solutions can not be continued at points in $\mathcal{C}$ where flow is not possible. At points in $\mathcal{X} \setminus (\mathcal{C} \cup \mathcal{D})$, no solution exists. In Figure 3.3, we depict the behavior of a solution to a hybrid dynamical system.

A hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ given by (3.1) with the *initial condition* $x(0,0) = \xi \in \mathcal{C} \cup \mathcal{D}$ is called an *initial value problem of the hybrid system* $\mathcal{H}$.

Definition 3.12 (Maximal Solutions)**.** A solution x to a hybrid dynamical system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ is *maximal* if there is no other solution x' to $\mathcal{H}$ such that $\operatorname{dom} x \subset \operatorname{dom} x'$ and $x(t,j) = x'(t,j)$ for all $(t,j) \in \operatorname{dom} x$. We denote by $\mathcal{S}_{\mathcal{H}}(\xi)$ the set of all maximal solutions to $\mathcal{H}$ with the initial condition $x(0,0) = \xi \in \mathcal{C} \cup \mathcal{D}$.



(A) Continuous (B) Discrete (C) Eventually Continuous

(D) Eventually Discrete (E) Zeno

FIGURE 3.4: Hybrid time domains corresponding to various types of hybrid arcs

Definition 3.13 (Classes of Hybrid Systems)**.** A hybrid system $\mathcal{H}$ is called a *complete hybrid system* if any maximal solution to $\mathcal{H}$ is complete. The system $\mathcal{H}$ is called a *Zeno hybrid system* if any maximal solution to $\mathcal{H}$ is Zeno. Moreover, a *continuous hybrid system*, a *discrete hybrid system* or other types given in Definition 3.10 are defined in the same manner.

Some examples of various types of hybrid time domains of hybrid arcs are illustrated by Figure 3.4. Note that to obtain a solution to a hybrid dynamical system, we do not know a hybrid time domain in advance. However, we collect it as simultaneous as a hybrid arc (or a solution) to the system is obtained.

The following example demonstrates how a solution to the bouncing ball system in Example 3.1 and its hybrid time domain are found.

Example 3.14 (Solution to the Bouncing Ball System)**.** Let h be a positive number. For a bouncing ball given in Example 3.1 with a given initial condition $\xi = (h, 0) \in \mathcal{C}$, say $x_1(t_0, 0) = x_1(0, 0) = h$ and $x_2(t_0, 0) = x_2(0, 0) = 0$, the hybrid time domain of the solution is written in the form

$$\bigcup_{j=0}^{\infty} \left([t_j, t_{j+1}] \times \{j\} \right),$$

and the solution is written as follows. The first arc until the first jump (continuous flow from t_0 and t_1) is given by

$$x_1(t,0) = -\frac{1}{2}\gamma t^2 + h, \tag{3.5}$$

$$x_2(t,0) = -\gamma t \tag{3.6}$$

for $t \in [t_0, t_1]$, where

$$t_1 = \sqrt{\frac{2h}{\gamma}}$$

is the first time that the ball hits the ground, i.e., $x(t_1, 0) \in \mathcal{D}$. The state after the jump at t_1 is then given by

$$x_1(t_1,1) = 0,$$
$$x_2(t_1,1) = -\lambda x_2(t_1,0) = -\lambda\left(-\gamma t_1\right)$$

Moreover, the further arcs (from t_j and t_{j+1}) are given by

$$x_1(t,j) = -\frac{1}{2}\gamma t^2 + (x_2(t_j,j) + \gamma t_j)\,t - \left(\frac{1}{2}\gamma t_j^2 + x_2(t_j,j)t_j\right), \tag{3.7}$$

$$x_2(t,j) = -\gamma t + (x_2(t_j,j) + \gamma t_j), \tag{3.8}$$

and the states after the jump at t_{j+1} are given by

$$x_1(t_{j+1}, j+1) = 0, \tag{3.9}$$

$$x_2(t_{j+1}, j+1) = -\lambda x_2(t_{j+1}, j) = -\lambda\left(-\gamma t_{j+1} + (x_2(t_j,j) + \gamma t_j)\right), \tag{3.10}$$

where

$$t_{j+1} := \frac{2x_2(t_j,j) + \gamma t_j}{2\gamma}$$

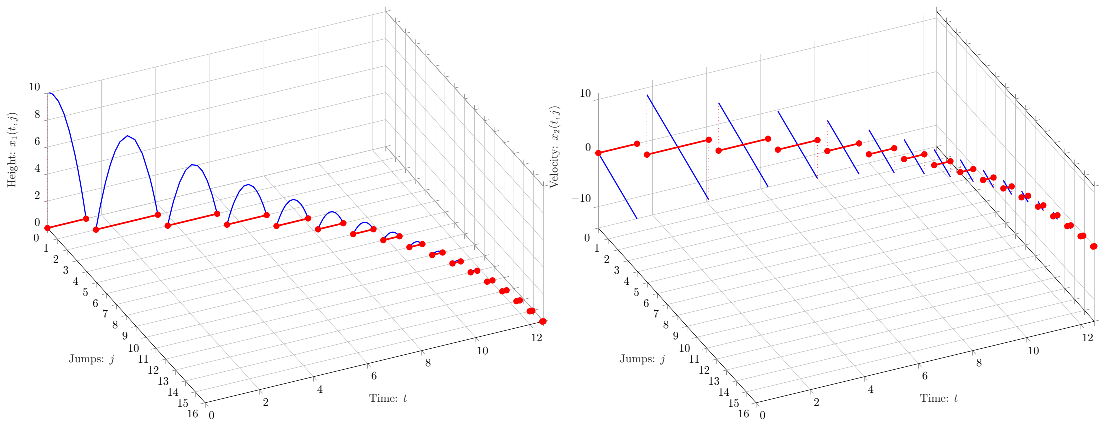
is the time at the $(j+1)$th jump. Furthermore, the total amount of time that the system spends in C is

$$\lim_{n\to\infty}\sum_{j=1}^{n}(t_{j+1} - t_j) = \sqrt{\frac{2h}{\gamma}} + \frac{2\lambda}{1-\lambda}\sqrt{\frac{2h}{\gamma}} =: t_{\max}. \tag{3.11}$$

The hybrid arc $x = (x_1, x_2)$ is a Zeno solution to the system of bouncing ball. Note that $t_{\max}$ is finite due to convergence of sequence $\{t_{j+1} - t_j\}$ and $\lambda \in (0,1)$. In addition there are infinitely many jumps until $t_{\max}$. Sometimes such $t_{\max}$ is called a *Zeno time* , see [43–46]. There is no more continuous flow after $t_{\max}$. Each interval of the hybrid time domain is of the from $[\cdot, t] \times \{\cdot\}$ such that $t < t_{\max}$, i.e., the solutions never, in mathematical sense, reach to $t_{\max}$.

In Figure 3.5, we give a numerical simulation of this system with the initial condition $x(0,0) = (10,0)$ along with the following parameters $\gamma = 9.8$, and $\lambda = 0.8$.

Example 3.15 (Solution to an On/Off Switching System)**.** Consider the on/off switching system given in Example 3.2 with the initial condition $x_1(0,0) = v$ and $x_2(0,0) = 1$, where $v \in (\psi_1, \psi_2)$. The function S_1 and S_0 are respectively given by $S_1(x_1) := kx_1$ and $S_0(x_1) := kx_1^2$, where $k > 0$. The hybrid time domain of the solution will be



(A) Height: $x_1(t,j)$.

(B) Velocity: $x_2(t,j)$.

FIGURE 3.5: To a numerical simulation of a bouncing ball.

written in the form

$$\bigcup_{j=0}^{\infty}[t_j, t_{j+1}] \times \{j\},$$

and the solution to the system is given by

$$x_1(t,0) = v\mathrm{e}^{kt}, \quad x_2(t,0) = 1 \quad \text{for } 0 \le t \le t_1 = \frac{1}{k}\ln(\frac{\psi_2}{v}).$$

Note that t_1 is the time that x_1 reaches the value of ψ_2. Therefore the switch is allowed to be turned off, i.e., the solution after t_1 is given by

$$x_1(t_1,1) = v\mathrm{e}^{kt_1}, \quad x_2(t_1,1) = 0.$$

In the next step

$$x_1(t,1) = \frac{x_1(t_1,1)}{\left(3k(t_2-t_1)x_1^3(t_1,1)+1\right)^{1/3}}, \quad x_2(t,1) = 0 \quad \text{for } t_1 \le t \le t_2$$

where

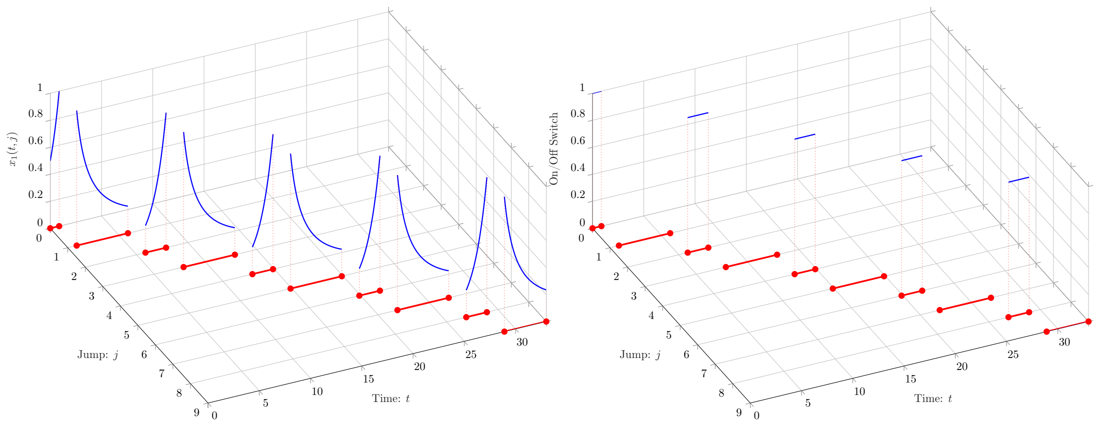
$$t_2 = t_1 + \frac{1}{3kx_1^3(t_1,1)}\left(\frac{x_1^3(t_1,1)}{\psi_1^3} - 1\right)$$

is the time at which x_1 reaches the value of ψ_1. Consequently, the switch is allowed to be turned on, and x_1 will increasingly evolve to the value of ψ_2. The solution after t_2 can be obtained by the above iteration. In Figure 3.6, we provide a numerical simulation of this system with the initial condition

$$x_1(0,0) = 0.5, \quad x_2(0,0) = 1$$

along with the following arguments: $k = 0.8$, $\psi_1 = 0.2$ and $\psi_2 = 1$. This solution is an absolutely hybrid arc.

Example 3.16 (Solution to an Impulsive System)**.** Consider a special case of hybrid systems: an impulsive system $\mathcal{H} = (\mathcal{X}, F, \mathcal{C}, G, \mathcal{D})$ given in Example 3.3 with the initial condition $z_1(0,0) = z_0 \in \mathbb{R}^n$, and $z_2(0,0) = 0$ along with an impulse time sequence $T = \{t_1, t_2, \ldots\} \subset \mathbb{R}$ such that $0 < t_1 < t_2 < \ldots < \infty$. A solution to $\mathcal{H}$ is



(A) Monitored value: $x_1(t, j)$. (B) On/Off switch: $x_2(t, j)$.

FIGURE 3.6: To a numerical simulation of an on/off switching system.

given by

$$z_1(t, 0) = z_0 + \int_0^t f(z(\tau))\, \mathrm{d}\tau,\ z_2(t, 0) = t \quad \text{for all } t \in [0, t_1],$$

$$z_1(t_1, 1) = z_1(t_1, 0) + g(z_1(t_1, 0)),\ z_2(t_1, 1) = z_2(t_1, 0) = t_1.$$

Moreover, the solution after t_1 is obtained by

$$z_1(t, j) = z_1(t_j, j) + \int_{t_j}^t f(z(\tau))\, \mathrm{d}\tau,\ z_2(t, j) = t \quad \text{for all } t \in [t_j, t_{j+1}],$$

$$z_1(t_{j+1}, j+1) = g(z_1(t_{j+1}, j)),\ z_2(t_{j+1}, j+1) = z_2(t_{j+1}, j) = t_{j+1}.$$

In Figure 3.7, we provide a numerical simulation of this system with the state $z_1 = (z_{11}, z_{12}) \in \mathcal{X} \subset \mathbb{R}^2$, the initial condition $z_1(0, 0) = (1, 0)$, $z_2(0, 0) = 0$, the functions f and g given by

$$f(z_1) = f(z_{11}, z_{12}) := \begin{pmatrix} \sin(z_{11}) + z_{12} \\ -z_{11} + \sin(z_{12}) \end{pmatrix},$$
$$g(z_1) = g(z_{11}, z_{12}) := \begin{pmatrix} z_{11} + \sin(z_{12}) \\ \sin(z_{11}) + z_{12} \end{pmatrix}$$

along with the impulse time sequence $T = \{1, 2, 3, \ldots\}$. The hybrid time domain is given by

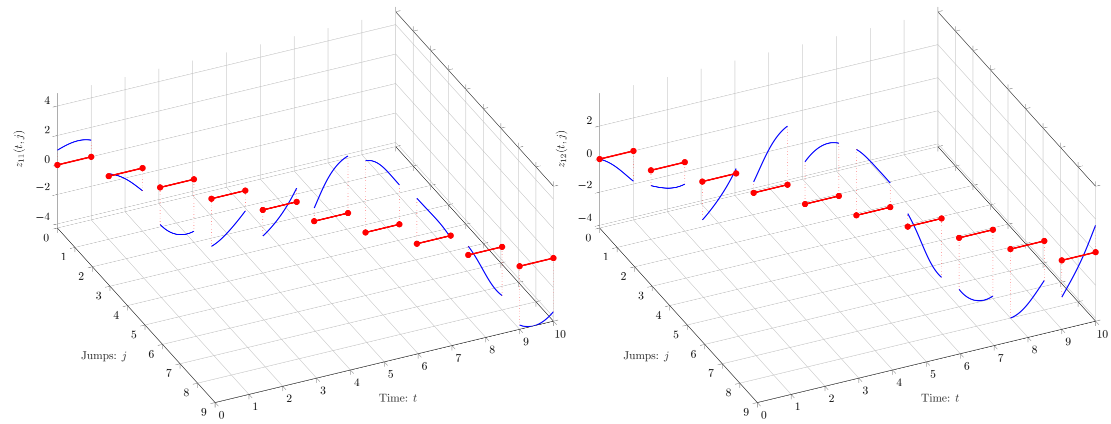
$$\bigcup_{j=1}^{\infty} ([j, j+1] \times \{j\})$$

and this solution is an absolutely hybrid arc.

3.3.3 Existence of Solutions

In this section we show sufficient conditions to guarantee existence of a nontrivial solution to a hybrid dynamical system.

Theorem 3.17 (The Existence Theorem of Hybrid Systems). *Given a hybrid system*



(A) $z_{11}(t,j)$. (B) $z_{12}(t,j)$.

FIGURE 3.7: To a numerical simulation in Example 3.16.

$\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ with the initial condition $x(0,0) = \xi \in \mathcal{C} \cup \mathcal{D}$. If the basic assumptions for hybrid systems[1] *are satisfied, then there exists a nontrivial solution to $\mathcal{H}$.*

Proof. Let the basic assumptions (BA1)-(BA4) be satisfied for the hybrid system $\mathcal{H}$. We are going to show that there exists a nontrivial hybrid arc φ such that it is a solution to $\mathcal{H}$. It is unnecessary to satisfy $\varphi \in \mathcal{S}_{\mathcal{H}}(\xi)$.

Consider the first case which $\xi \in \mathcal{D}$. Define the function $y : \{0,1\} \to \mathbb{R}^n$ by

$$y(j) := \begin{cases} \xi & \text{if } j = 0; \\ g(\xi) & \text{if } j = 1. \end{cases}$$

The function value $y(1)$ is determined due to the basic assumption (BA4), the function $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. It is clear to see that the hybrid arc φ_1 defined by

$$\varphi_1(0,j) := y(j)$$

with the hybrid time domain

$$\operatorname{dom} \varphi_1 = \{(0,0),(0,1)\}$$

is a solution to the hybrid system $\mathcal{H}$.

In the case that $\xi \in \mathcal{C} \setminus \mathcal{D}$, we show the existence of solutions to $\mathcal{H}$ as follows. By Theorem 2.4 and the basic assumption (BA3), the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, there exists a solution $z : [0,t^*] \to \mathbb{R}^n$ to the continuous-time dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = \xi.$$

Therefore the hybrid arc φ_2 defined by

$$\varphi_2(t,0) := z(t) \quad \text{for } t \in [0,t^*]$$

with the hybrid time domain $\operatorname{dom} \varphi_2 = \{(t,0) \in \mathbb{R} \times \{0\} : 0 \leq t \leq t^*\}$ is a solution to the hybrid system $\mathcal{H}$.

□

[1] See Section 3.2.

Remark 3.1. Zeno solutions are an interesting behavior in hybrid dynamical systems. Unfortunately, we cannot discover a condition for the existence of Zeno solutions. Consider the following:

(1) A hybrid arc x with $\mathrm{dom}\, x = \{0\} \times \mathbb{N}$ is discrete, complete and Zeno.

(2) A complete eventually discrete hybrid arc is Zeno.

(3) For a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$:

  (a) a solution to $\mathcal{H}$ is not Zeno if $\mathcal{D} = \emptyset$.

  (b) a solution to $\mathcal{H}$ is possibly Zeno if $\mathcal{C} \cap \mathcal{D} \neq \emptyset$ and $g(\mathcal{D}) \subset \mathcal{C}$.

  (c) if a solution to $\mathcal{H}$ is Zeno, the following condition must be satisfied:

$$\lim_{j \to \infty} (t_{j+1} - t_j) = 0 \quad \forall (t_j, j) \in \mathrm{dom}\, x.$$

3.4 Stability

In this section, we introduce notions of stability of hybrid dynamical systems. We mainly focus on asymptotic stability of a non-empty compact set. Asymptotic stability is an essential key and also fundamental property of nonlinear dynamical systems. It yields qualitative data of solutions to hybrid systems, especially long-term behaviors of the solutions. Asymptotic stability of a non-empty compact set, instead of a steady state or an equilibrium point, is important since the solutions to hybrid systems do not often settle down on an equilibrium point.

Like for other classes of nonlinear dynamical system, *Lyapunov functions* are tools for the investigation of stability of hybrid systems. Various Lyapunov theorems and the invariance principle for hybrid systems are presented in this section. The concepts of stability and invariance for hybrid systems in the form (3.1) were proposed in [1] and [47]. Lyapunov stability and hybrid invariance principles were also introduced in [1, 9, 47–49]. For simplicity, let us assume that hybrid systems are complete throughout this section, unless stated otherwise.

Definition 3.18. For a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, a non-empty compact set $\mathcal{A} \subset \mathcal{X}$ is called

- *stable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $\|x(0,0)\|_{\mathcal{A}} < \delta$ satisfies $\|x(t,j)\|_{\mathcal{A}} < \varepsilon$ for all $(t,j) \in \mathrm{dom}\, x$;
- *attractive* if any solution x to $\mathcal{H}$ satisfies x converges to $\mathcal{A}$, i.e.,

$$\|x(t,j)\|_{\mathcal{A}} \to 0 \quad \text{as } t + j \to \infty;$$

- *asymptotically stable* if it is both stable and attractive.

Note that the non-empty compact set $\mathcal{A} \subset \mathcal{X}$ in the above definition could be replaced by a single point $\varrho \in \mathcal{C} \cup \mathcal{D}$ if $\mathcal{A}$ contains only the single point ϱ, i.e., $\mathcal{A} = \{\varrho\}$.

In Figure 3.8, we depict a behavior of the trajectories in the vicinity of stable set of origin in $\mathbb{R}^2$. By choosing the initial points in the spherical neighborhood of radius δ, we can force the graph of the solution x for $(t,j) \in \mathrm{dom}\, x$ to stay entirely inside a

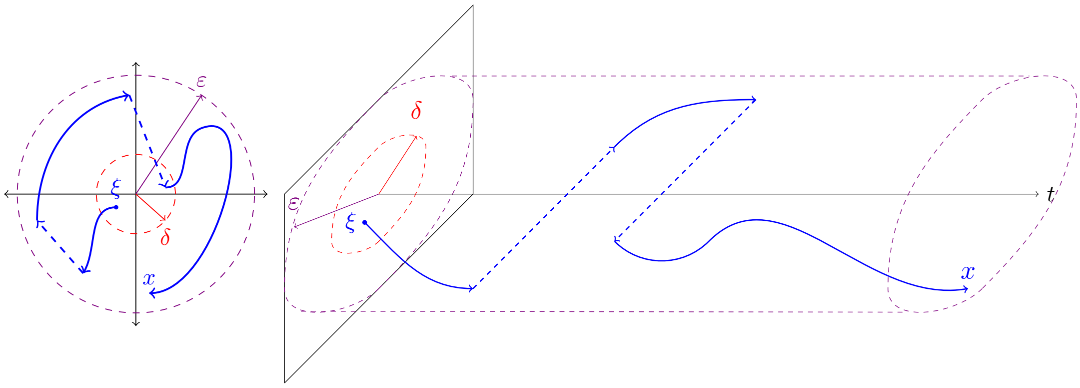


FIGURE 3.8: Visualization for stability of the set of origin in $\mathbb{R}^2$.

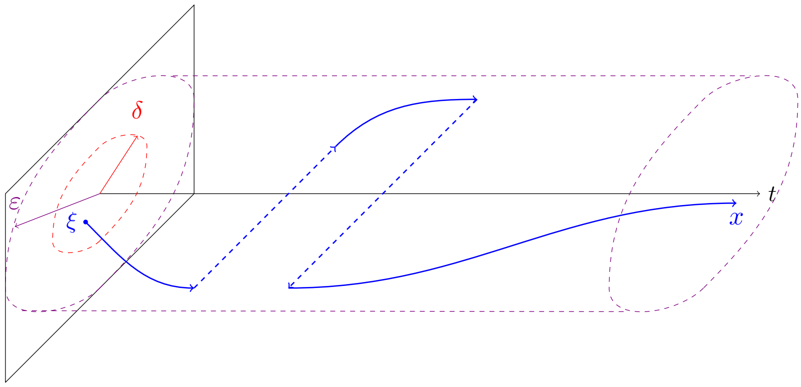


FIGURE 3.9: Visualization for asymptotic stability of the set of origin in $\mathbb{R}^2$.

given ε-tube. The behavior of the trajectories in the vicinity of asymptotically stable set of origin in $\mathbb{R}^2$ is illustrated in Figure 3.9.

3.4.1 Hybrid Lyapunov Theorem

This section addresses the hybrid Lyapunov functions as sufficient conditions for stability. The definitions below make the strict and relaxed conditions required for a function to be considered as a hybrid Lyapunov candidate function for establishing stability of a non-empty compact set for a hybrid system.

Definition 3.19 (Hybrid Lyapunov Candidate Functions)**.** Given a hybrid dynamical system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, a function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *hybrid Lyapunov candidate function* for $(\mathcal{H}, \mathcal{A})$ if it is globally Lipschitz and there exist class $\mathcal{K}_\infty$ functions ψ_1 and ψ_2 such that it satisfies

$$\psi_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \psi_2(\|x\|_{\mathcal{A}}) \quad \text{for all } x \in \mathcal{X}. \tag{3.12}$$

Definition 3.20 (Hybrid Lyapunov Functions)**.** Given a hybrid dynamical system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A})$ is called a *hybrid Lyapunov function* for $(\mathcal{H}, \mathcal{A})$ if it satisfies

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &< 0 \quad \text{for all } x \in \mathcal{C} \setminus \mathcal{A}, && (3.13)\\ V(g(x)) - V(x) &< 0 \quad \text{for all } x \in \mathcal{D} \setminus \mathcal{A}. && (3.14) \end{aligned}$$

Moreover, it is called a *relaxed hybrid Lyapunov function* for $(\mathcal{H}, \mathcal{A})$ if it satisfies

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &\leq 0 \quad \text{for all } x \in \mathcal{C} \setminus \mathcal{A}, && (3.15)\\ V(g(x)) - V(x) &\leq 0 \quad \text{for all } x \in \mathcal{D} \setminus \mathcal{A}. && (3.16) \end{aligned}$$

The following results provide sufficient conditions on a Lyapunov candidate function that guarantee stability. Even though the similar results were presented in [1, 3, 47], we intentionally propose the following theorem with an alternative proof which is possibly less complicated and difficult to understand. Various results in this chapter are also obtained from the concept of this proof. They will be presented in the forthcoming sections.

Theorem 3.21 (Hybrid Lyapunov Theorem)**.** *Given a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$,*

(L1) *if there exists a relaxed Lyapunov function for $(\mathcal{H}, \mathcal{A})$, then $\mathcal{A}$ is stable for $\mathcal{H}$.*

(L2) *if there exists a Lyapunov function for $(\mathcal{H}, \mathcal{A})$ and $\mathcal{H}$ is a complete hybrid system, then $\mathcal{A}$ is asymptotically stable for $\mathcal{H}$.*

Proof. Let V be a relaxed Lyapunov function for $(\mathcal{H}, \mathcal{A})$ satisfying the conditions (3.12), (3.13) and (3.14). Recall that the derivative of V at x

$$V'(x) = \langle \nabla V(x), f(x) \rangle .$$

For any positive number p, define

$$B_p := \{x \in \mathbb{R}^n : \|x\|_{\mathcal{A}} \leq p\} .$$

Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $\mathcal{A} \subset B_r$. Let

$$\alpha := \min_{\|x\|_{\mathcal{A}} = r} V(x),$$

then $\alpha > 0$ since $V(x) \geq \varphi_1(\|x\|_{\mathcal{A}}) > 0$ for any x with $\|x\|_{\mathcal{A}} = r$. In addition, for any $\beta \in (0, \alpha)$, let us define

$$\Omega_\beta := \{x \in B_r : V(x) \leq \beta\} .$$

Firstly, we need to show that Ω_β is in the interior of B_r. Suppose that Ω_β is not in the interior of B_r, then there is $b \in \Omega_\beta$ that lies on the boundary of B_r. Thus, $V(b) \geq \alpha > \beta$, but $V(b) \leq \beta$ for all $b \in \Omega_\beta$. From this contradiction, it follows that $\Omega_\beta \subset \operatorname{int} B_r$. Figure 3.10 illustrates the level sets used in this proof.

Secondly, we need to show that any trajectories starting in Ω_β always lie in Ω_β. Suppose that $x(0,0) \in \Omega_\beta$, $t_0 = 0$ and the hybrid time domain $\operatorname{dom} x$ is in the form of the union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$ with the last interval, if it exists, is allowed to be in the form $[t_j, T)$ with T finite of $T = \infty$.

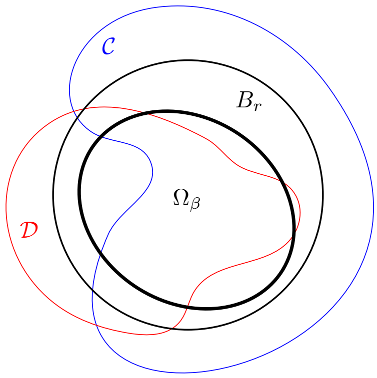


FIGURE 3.10: Level sets in the proof of Theorem 3.21.

Due to the conditions (3.13) and (3.14), it holds that

$$V(x(t_1,1)) \le V(x(t_1,0)) \le V(x(0,0)) \le \beta.$$

So we have $V(x(t_1,1)), V(x(t_1,0)) \in \Omega_\beta$. By induction, it is sufficient to conclude that $x(t,j) \in \Omega_\beta$ for any $(t,j) \in \operatorname{dom} x$.

Since V is globally Lipschitz and $V(a) = 0$ for any $a \in \mathcal{A}$, there exists $\delta > 0$ such that $\|x\|_{\mathcal{A}} < \delta$ implies $V(x) < \beta$. Then,

$$B_\delta := \{x \in \mathcal{X} : \|x\|_{\mathcal{A}} \le \delta\} \subset \Omega_\beta \subset B_r.$$

We need to show that the compact set $\mathcal{A}$ is stable. Suppose $x(0,0) \in B_\delta$. It follows that

$$x(0,0) \in B_\delta \implies x(0,0) \in \Omega_\beta \implies x(t,j) \in \Omega_\beta \implies x(t,j) \in B_r$$

for all $(t,j) \in \operatorname{dom} x$. Therefore,

$$\|x(0,0)\|_{\mathcal{A}} < \delta \implies \|x(t,j)\|_{\mathcal{A}} < r \le \varepsilon$$

for all $(t,j) \in \operatorname{dom} x$.

For (L2), let the system $\mathcal{H}$ be a complete hybrid system and V be a Lyapunov function for $(\mathcal{H}, \mathcal{A})$ satisfying the conditions (3.15) and (3.16). We have to additionally show that $\mathcal{A}$ is attractive. It is sufficient to show that $V(x(t,j)) \to 0$ as $t + j \to \infty$ since $\|x(t,j)\|_{\mathcal{A}} \le \psi_1^{-1}(V(x(t,j)))$.

Suppose a contradiction, that the trajectory x does not converge to $\mathcal{A}$, i.e., $\|x(t,j)\|_{\mathcal{A}}$ does not converge to zero as $t + j \to \infty$. Since (3.15) and (3.16) hold and V is bounded from below by zero, suppose $V(x(t,j))$ converges to some constant $c > 0$ as $t + j \to \infty$. Because V is globally Lipschitz, and it holds $V(a) = 0$ for any $a \in \mathcal{A}$, there exists a positive number q such that

$$B_q := \{x \in \mathcal{X} : \|x\|_{\mathcal{A}} \le q\} \subset \Omega_c := \{x \in B_r : V(x) \le c\}.$$

Therefore, this solution $x(t,j)$ lies outside B_q as $t + j \to \infty$ since $V(x(t,j)) \to c$. Define

$$\bar{\gamma} := \sup_{q \le \|x(t,j)\|_{\mathcal{A}} \le r} \langle \nabla V(x(t,j)), f(x(t,j)) \rangle, \quad \gamma := -\bar{\gamma},$$

$$\bar{\sigma} := \sup_{\substack{q \le \|x(t,j)\|_{\mathcal{A}} \le r \\ q \le \|x(t,j+1)\|_{\mathcal{A}} \le r}} V(x(t,j+1)) - V(x(t,j)), \quad \text{and } \sigma := -\bar{\sigma}.$$

It is clear that $\gamma > 0$ and $\sigma > 0$ due to the conditions (3.15) and (3.16) respectively. Let us denote the numbers

$$t(\eta; x) := \inf \{t \in \mathbb{R}_+ : \ (t, \eta) \in \operatorname{dom} x\},$$

and

$$j(\tau; x) := \sup \{j \in \mathbb{N} : \ (\tau, j) \in \operatorname{dom} x\}. \tag{3.17}$$

It follows that

$$\begin{aligned} V(x(t,j)) = V(x(0,0)) &+ \int_0^t V'(x(\tau, j(\tau;x)))\mathrm{d}\tau \\ &+ \sum_{\eta=1}^{j} [V(x(t(\eta;x),\eta)) - V(x(t(\eta;x),\eta-1))] \\ &\le V(x(0,0)) - \gamma t - \sigma j. \end{aligned}$$

We obtain a contradiction since $V(x(t,j))$ eventually becomes negative. □

Example 3.22 (Stability of the Bouncing Ball System)**.** Consider the bouncing ball system introduced in Example 3.1. We are going to consider stability of the compact set $\mathcal{A} = \{0\}$, i.e., the origin of $\mathbb{R}^2$.

Define a continuously differentiable function $V : \mathbb{R}^2 \to \mathbb{R}$ by

$$V(x) = \frac{1}{2}x_2^2 + \gamma x_1. \tag{3.18}$$

It follows that

$$\langle \nabla V(x), f(x) \rangle = 0 \ \text{ for all } \ x \in \mathcal{C} \setminus \mathcal{A},$$

and

$$V(g(x)) - V(x) = -\frac{1}{2}(1 - \lambda^2)x_2^2 < 0 \ \text{ for all } \ x \in \mathcal{D} \setminus \mathcal{A}.$$

Since the inequalities (3.15) and (3.16) hold, we can conclude that the origin is globally stable. However, this is not enough to guarantee asymptotic stability of the origin since the inequality (3.13) does not hold.

3.4.2 Hybrid Invariance Principle

For hybrid dynamical systems, asymptotic stability of a non-empty compact set can be assured by Theorem 3.21. In many cases, there are difficulties to find a Lyapunov function to guarantee asymptotic stability of a non-empty stable compact set for a hybrid system. However, if we can achieve a relaxed Lyapunov function, we can guarantee asymptotic stability of a nonempty compact set despite no satisfying of the strict inequalities (3.13) and (3.14) by the *invariance principles for hybrid systems*. The following definitions and theorems for the hybrid invariance principle are taken from [1, 47].

Definition 3.23. For a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, the set $\mathcal{M} \subset \mathcal{X}$ is said to be

- *weakly forward invariant* if for each $\xi \in \mathcal{M}$, there exists at least one maximal solution x with $x(t,j) \in \mathcal{M}$ for all $(t,j) \in \operatorname{dom} x$;
- *weakly backward invariant* if for each $q \in \mathcal{M}$, $N > 0$, there exist $\xi \in \mathcal{M}$ and at least one solution $x \in \mathcal{S}_{\mathcal{H}}(\xi)$ such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* \geq N$, the solution satisfies $x(t^*, j^*) = q$ and $x(t,j) \in \mathcal{M}$ for all $(t,j) \preceq (t^*, j^*)$, $(t,j) \in \operatorname{dom} x$.
- *weakly invariant* if it is both weakly forward invariant and weakly backward invariant;
- *strongly forward invariant* if for each $\xi \in \mathcal{M}$ and each $x \in \mathcal{S}_{\mathcal{H}}(\xi)$, then $x(t,j) \in \mathcal{M}$ for all $(t,j) \in \operatorname{dom} x$.

Definition 3.24. Given a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a relaxed Lyapunov function V for $(\mathcal{H}, \mathcal{A})$, define a function $u_{\mathcal{C}} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{\mathcal{C}}(x) := \langle \nabla V(x), f(x) \rangle ,$$

and a function $u_{\mathcal{D}} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{\mathcal{D}}(x) := V(g(x)) - V(x).$$

Theorem 3.25 (hybrid LaSalle's invariance principle [1])**.** *Let $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ be a complete hybrid system, $\mathcal{A}$ be a nonempty compact subset of $\mathcal{X}$ and V be a relaxed Lyapunov function for $(\mathcal{H}, \mathcal{A})$. Suppose that $\mathcal{U}$ is a neighborhood of $\mathcal{A}$, and for any maximal solution x to $\mathcal{H}$, x is bounded and $\overline{\operatorname{rge} x} \subset \mathcal{U}$. If*

$$u_{\mathcal{C}}(x) \leq 0, \quad \textit{and} \quad u_{\mathcal{D}}(x) \leq 0 \qquad \forall x \in \mathcal{U},$$

then for some constant $r \in V(\mathcal{U})$, x approaches the largest weakly invariant set in

$$V^{-1}(r) \cap \mathcal{U} \cap \left[u_{\mathcal{C}}^{-1}(0) \cup \left(u_{\mathcal{D}}^{-1}(0) \cap g(u_{\mathcal{D}}^{-1}(0)) \right) \right] \tag{3.19}$$

Theorem 3.26 (hybrid Krasovskii [1, 47])**.** *Let $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ be a complete hybrid system, $\mathcal{A}$ be a nonempty compact subset of $\mathcal{X}$ and V be a relaxed Lyapunov function for $(\mathcal{H}, \mathcal{A})$. Suppose that $\mathcal{U}$ is a neighborhood of $\mathcal{A}$, and it holds $u_{\mathcal{C}}(x) \leq 0$ and $u_{\mathcal{D}}(x) \leq 0$ for all $x \in \mathcal{U}$. If there exists $r^* > 0$ such that, for all $r \in (0, r^*)$, the largest weakly invariant subset in* (3.19) *is empty, then $\mathcal{A}$ is asymptotically stable for $\mathcal{H}$.*

The following theorem was given in [47] and provided the conditions of asymptotic stability when either $u_{\mathcal{C}}$ or $u_{\mathcal{D}}$ is strictly negative.

Theorem 3.27 ([3, 47])**.** *Let $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ be a complete hybrid system, $\mathcal{A}$ be a nonempty compact subset of $\mathcal{X}$ and V be a relaxed Lyapunov function for $(\mathcal{H}, \mathcal{A})$. Suppose that $\mathcal{U}$ is a neighborhood of $\mathcal{A}$, and it holds $u_{\mathcal{C}}(x) \leq 0$ and $u_{\mathcal{D}}(x) \leq 0$ for all $x \in \mathcal{U}$. If either*

(a1) $u_{\mathcal{C}}(x) < 0$ *for all* $x \in \mathcal{U} \setminus \mathcal{A}$,

(a2) any discrete solution x to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$ *converges to $\mathcal{A}$;*

or

(b1) $u_{\mathcal{D}}(x) < 0$ *for all* $x \in \mathcal{U} \setminus \mathcal{A}$,

(b2) any continuous solution x *to* $\mathcal{H}$ *with* $\operatorname{rge} x \subset \mathcal{U}$ *converges to* $\mathcal{A}$ *;*

is satisfied, then $\mathcal{A}$ *is asymptotically stable for* $\mathcal{H}$.

Even though Theorem 3.26 and 3.27 are consequences of the *hybrid LaSalle's invariance principle*, the similar conditions can also be obtained without any application of hybrid invariance principle, which are provided in the next subsection.

3.4.3 Relaxed Hybrid Lyapunov Theorems

We are going to provide alternative conditions being comfortable to apply, in some cases, than previous works in the literature. In term of energy functions, instead of losing energy by both of continuous and discrete dynamics, we allow energy to be lost by only one of them when all of our conditions hold for the hybrid systems. We do not require neither the strict inequalities in the hybrid Lyapunov theorems nor a consideration of hybrid invariant principles. Furthermore, existence of discrete, or continuous solutions to a hybrid system does not need to be verified in our conditions.

Theorem 3.28 (Relaxed Hybrid Lyapunov Theorem)**.** *Given a complete hybrid system* $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ *and a non-empty compact set* $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$. *Suppose that there exists a relaxed hybrid Lyapunov function* V *for* $(\mathcal{H}, \mathcal{A})$. *If the following conditions are satisfied:*

(G1) *Any maximal solution is not eventually continuous,*

(G2) $\langle \nabla V(x), f(x) \rangle \leq 0$ *for all* $x \in \mathcal{C} \setminus \mathcal{A}$,

(G3) $V(g(x)) - V(x) < 0$ *for all* $x \in \mathcal{D} \setminus \mathcal{A}$,

then $\mathcal{A}$ *is asymptotically stable for* $\mathcal{H}$.

Proof. Consider the set B_r and Ω_β defined in the proof of Theorem 3.21. Note that that $\Omega_\beta \subset \operatorname{int} B_r$.

Let us show that Ω_β is strongly invariant set. Without loss of generality, let us suppose $x(0,0) \in \Omega_\beta \cap \mathcal{C}$. Since the condition (G1) holds, there exists $t_1 > 0$ such that $x(t_1, 0) \in \mathcal{D}$. We get $x(t_1, 0) \in \Omega_\beta$ because $V(x(t_1, 0)) \leq V(x(0,0)) \leq \beta$.

If there is no $j \in \mathbb{N}$ such that $x(t_1, j) \in \mathcal{C}$, we can conclude that $x(t, j) \in \Omega_\beta$ for all $(t, j) \in \operatorname{dom} x$ because $V(x(t_1, j+1)) < V(x(t, j)) \leq V(x(t_1, 0))$ for all $j \in \mathbb{N}$. If there exists an integer j_1 such that $x(t_1, j_1)$ lies in $\mathcal{C}$, then we get $x(t_1, j_1) \in \Omega_\beta$ since $V(x(t_1, j_1)) < V(x(t_1, j_0))$ for all $j_0 \in \{0, 1, 2, \ldots, j_1 - 1\}$. Moreover, we still get $x(t, j_1) \in \Omega_\beta$ for all $t \geq t_1$ because of the condition (G2). With this procedure, we obtain that any solution starting from Ω_β always lies in Ω_β.

Since V is globally Lipschitz, and $V(a) = 0$ for all $a \in \mathcal{A}$, there exists $\delta > 0$ such that $\|x\|_{\mathcal{A}} < \delta$ implies $V(x) < \beta$. Then, $B_\delta \subset \Omega_\beta \subset B_r$. Suppose $x(0,0) \in B_\delta$. It follows that $x(0,0) \in B_\delta \implies x(0,0) \in \Omega_\beta \implies x(t,j) \in \Omega_\beta \implies x(t,j) \in B_r$ for all $(t,j) \in \operatorname{dom} x$. Therefore, $\|x(0,0)\|_{\mathcal{A}} < \delta$ implies $\|x(t,j)\|_{\mathcal{A}} < r \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$. Hence $\mathcal{A}$ is stable.

Finally, we have to show that the origin is attractive. Suppose by a contradiction that the solution x does not converge to the compact set $\mathcal{A}$. Since (G1) - (G3) hold and V

is bounded from below by zero, then V converges to some constant $c > 0$. Therefore, there exists $q > 0$ such that $B_q \subset \Omega_c$ due to the continuity of V. As $t + j \to \infty$, $x(t, j)$ lies outside B_q since $V(x(t, j)) \to c > 0$. It follows that

$$V(x(t, j)) \leq V(x(0, 0)) - \gamma t - \sigma j,$$

where γ and σ are two constants defined in the proof of Theorem 3.21.

Under the conditions (G2) and (G3), we get $\gamma \geq 0$ and $\sigma > 0$ respectively. Since $V(x(t, j))$ eventually becomes negative, we obtain a contradiction. □

Our next result is proposed in the following theorem. Instead of using the condition (F1), we can also consider complete maximal solutions to $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D}, \xi)$ which are not eventually discrete. It leads to a new result provided here.

Theorem 3.29 (Another Relaxed Hybrid Lyapunov Theorem)**.** *Given a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$. Suppose that there exists a relaxed hybrid Lyapunov function V for $(\mathcal{H}, \mathcal{A})$. If the following conditions are satisfied:*

(F1) *Any maximal solution to $\mathcal{H}$ is not eventually discrete,*

(F2) $\langle \nabla V(x), f(x) \rangle < 0$ *for all $x \in \mathcal{C} \setminus \mathcal{A}$,*

(F3) $V(g(x)) - V(x) \leq 0$ *for all $x \in \mathcal{D} \setminus \mathcal{A}$,*

then $\mathcal{A}$ is asymptotically stable for $\mathcal{H}$.

Proof. We already know from Theorem 3.28 that Ω_β is in the interior of B_r. Suppose $x(0, 0) \in \Omega_\beta$. We will show that Ω_β is strongly forward invariant when conditions (F1) - (F3) hold. Without loss of generality, let us suppose that the initial point lies in $\mathcal{D}$. Then there exists $j_1 \in \mathbb{N}$ such that $x(0, j_1) \in \mathcal{C}$ due to the condition (F1). Thus, $x(0, j_1) \in \Omega_\beta$ because $V(x(0, j_1)) \leq V(x(0, j_0)) \leq V(x(0, 0)) \leq \beta$ for all $j_0 \in \{0, 1, 2, \ldots, j_1\}$. Moreover, we can conclude that $x(t_1, j_1) \in \Omega_\beta$ for all $t_1 > 0$ because, from (F2), $V(x(t_1, j_1)) < V(x(0, j_1)) \leq \beta$ for all $t_1 > 0$ and $x(t_1, j_1) \in \mathcal{C} \setminus \mathcal{A}$. If there is no positive t such that $x(t, j_1) \in \mathcal{D}$, then $x(t, j) \in \Omega_\beta$ for all $(t, j) \in \operatorname{dom} x$.

If there exists $t_2 > t_1$ such that $x(t_2, j_1) \in \mathcal{D}$, then $x(t_2, j_1)$ will still be in Ω_β since $V(x(t_2, j_1)) < V(x(t_1, j_1))$. Since (F1) holds, there exist $j_2 \in \mathbb{N}$ such that $x(t_2, j_2) \in \mathcal{C}$. Further $x(t_2, j_2) \in \Omega_\beta$ because $V(x(t_2, j_2)) \leq V(x(t_2, i)) \leq V(x(t_2, j_1)) \leq \beta$ for all $i \in \{j_1, j_1 + 1, j_1 + 2, ..., j_2\}$. By the above procedure, we can conclude that any solution x starting form Ω_β will lie in Ω_β for all $(t, j) \in \operatorname{dom} x$.

Since V is globally Lipschitz, and $V(a) = 0$ for any $a \in \mathcal{A}$, there exists $\delta > 0$ such that $\|x\|_{\mathcal{A}} < \delta$ implies $V(x) < \beta$. Then, $B_\delta \subset \Omega_\beta \subset B_r$. For any $x(0, 0) \in B_\delta$, it follows that $x(0, 0) \in \Omega_\beta$. Since Ω_β is strongly forward invariant, then $x(t, j) \in \Omega_\beta \subset B_r$ for all $(t, j) \in \operatorname{dom} x$. Therefore, $\|x(0, 0)\|_{\mathcal{A}} < \delta$ implies $\|x(t, j)\|_{\mathcal{A}} < r \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$, i.e., the compact set $\mathcal{A}$ is stable.

Finally, we are going to show that the origin is attractive. Suppose by a contradiction that the trajectory x does not converge to the compact set $\mathcal{A}$. Since (F1) - (F3) hold and V is bounded from below by zero, then V converges to $c > 0$. Therefore, there exists $q > 0$ such that $B_q \subset \Omega_c$ due to the continuity of V. As $t + j \to \infty$, $x(t, j)$ lies outside B_q since $V(x(t,j)) \to c > 0$. It follows that

$$V(x(t,j)) \leq V(x(0,0)) - \gamma t - \sigma j,$$

where γ and σ are two constants defined in the proof of Theorem 3.21. Under the conditions (F2) and (F3), we get $\gamma > 0$ and $\sigma \geq 0$ respectively. If hybrid arc x is not Zeno, then we get a contradiction since $V(x(t,j))$ eventually becomes negative as $t + j \to \infty$. Otherwise, x is Zeno, there exists a nonnegative constant $T = \sup_t \operatorname{dom} x < \infty$. Since x is not eventually discrete, it is clear that $T > 0$. Let $x(t^*, j^*)$ be a point lying outside B_q such that $V(x(t^*, j^*)) = c$, it follows that

$$x(t^*, j^*) \in (\mathcal{C} \setminus \mathcal{A}) \cup (\mathcal{D} \setminus \mathcal{A}).$$

Suppose, without loss of generality, $x(t^*, j^*) \in \mathcal{D} \setminus \mathcal{A}$. There exists $\eta > 0$ such that $x(t^*, j^* + \eta) \in \mathcal{C} \setminus \mathcal{A}$ due to the condition (F1). Therefore there exists $\tau > 0$ such that $(t^* + \tau, j^* + \eta) \in \operatorname{dom} x$ since $x(t^*, j^* + \eta) \in \mathcal{C} \setminus \mathcal{A}$ and the hybrid arc x is allowed to flow further. Therefore, a contradiction is obtained since $t^* + \tau > T$. □

Some examples of application are illustrated by a classical hybrid phenomena and its extension as follows.

Example 3.30 (Asymptotic Stability of the Bouncing Ball System)**.** In this example, asymptotic stability of a bouncing ball system is subjected. Consider a relaxed hybrid Lyapunov function V defined by (3.18) in Example 3.22. Note that Theorem 3.21 can not be applied to verify asymptotic stability because the strict inequality (3.15) does not hold.

Since

$$\langle \nabla V(x), f(x) \rangle = 0 \ \text{ for all } \ x \in \mathcal{C} \setminus \{0\},$$

and

$$V(g(x)) - V(x) = -\frac{1}{2}(1 - \lambda^2)x_2^2 < 0 \ \text{ for all } \ x \in \mathcal{D} \setminus \{0\},$$

it follows that (G2) and (G3) are satisfied.

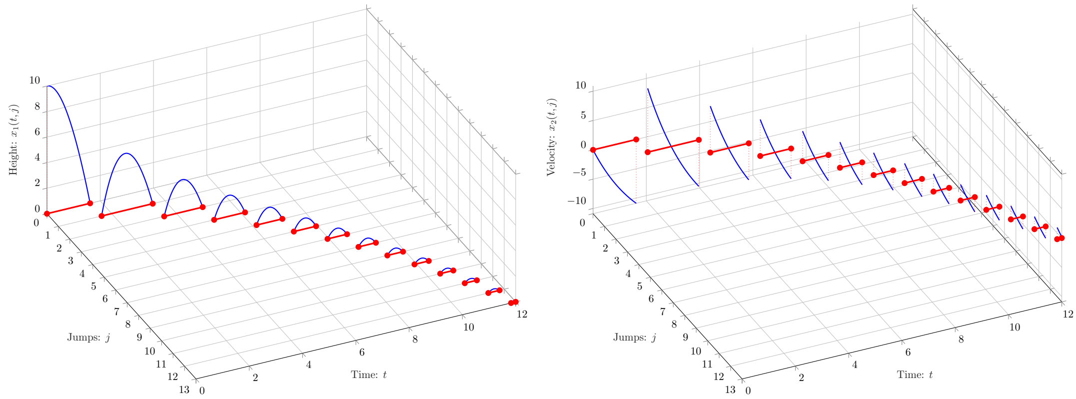
Additionally any nontrivial solution to the bouncing ball system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ is absolutely hybrid (see Definition 3.10) since, for every point $x(t,j) \in \mathcal{C} \setminus \{0\}$, there exists time t^* such that the ball reaches the floor, i.e.,

$$t^* = t + \frac{x_2(t,j) + \sqrt{(x_2(t,j))^2 + 2\gamma x_1(t,j)}}{\gamma} > t.$$

Moreover $x(t^*, j) \in \mathcal{D}$ and $g(\mathcal{D}) \subset \mathcal{C}$. Therefore, the condition (G1) automatically holds, i.e., for each nontrivial $x \in \mathcal{S}_{\mathcal{H}}(\xi)$, x is not eventually continuous. By Theorem 3.28, we can conclude that the origin is asymptotically stable.

Example 3.31 (An elastic bouncing ball with air resistance)**.** Extended Example 3.1, we put the restitution factor between the ball and the floor to be equal to one. The air resistance, or drag, is now considered in the system. Flows and jumps of the state can be described as follows:

$$f(x) := \begin{pmatrix} x_2 \\ -\gamma - kx_2 \end{pmatrix}, \quad g(x) := \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$



(A) Height (B) Velocity

FIGURE 3.11: To a numerical solution in Example 3.31.

The positive number k indicates a linear air drag constant.

To guarantee asymptotic stability of the origin, let us consider the continuously differentiable function V defined in (3.18). It follows that

$$\langle \nabla V(x), f(x) \rangle = -k x_2^2 < 0,$$

and

$$V(g(x)) - V(x) = 0.$$

The condition (F1) holds since any nontrivial solutions to the system are absolutely hybrid. The conditions (F3) is also satisfied, but (F2) is satisfied only if $x_2 \neq 0$. According to this relaxed Lyapunov function V, we cannot apply Theorem 3.29 to guarantee that the origin is asymptotically stable. In addition, we depict a numerical solution for this example in Figure 3.11.

Example 3.32. With the same flow set $\mathcal{C}$ and jump set $\mathcal{D}$ as in Example 3.31, let us redefine

$$f(x) := H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad g(x) := \begin{pmatrix} \rho_1(|x_1|) \\ \rho_2(|x_2|) \end{pmatrix},$$

where H is a Hurwitz matrix , i.e., every eigenvalue of H has negative real part, and $\rho_i \in \mathcal{K}_\infty$ with $\rho_i \leq \text{id}$ for $i = 1, 2$. There exists a positive definite symmetric matrix P [2] and a continuously differentiable function in a form [3]

$$V(x) = x^{\mathrm{T}} P x$$

such that

$$\langle \nabla V(x), f(x) \rangle = x^{\mathrm{T}} (PH + H^{\mathrm{T}} P) x < 0$$

for all $x \in \mathcal{C} \setminus \{0\}$. Furthermore, we also get

$$V(g(x)) - V(x) \leq 0 \ \text{ for all } \ x \in \mathcal{D} \setminus \{0\}.$$

We have already shown that (F2) and (F3) are satisfied. Moreover, (F1) also holds

[2] Given a positive definite symmetric matrix Q, P is the unique solution of $PH + H^{\mathrm{T}}P = -Q$.
[3] See [50], Theorem 10.1 and [18], Page 135-136.

since $g(\mathcal{D}) \subset \mathcal{C}$. Thus, we can directly conclude that the origin is globally asymptotically stable from Theorem 3.29.

Alternative conditions to guarantee asymptotic stability of hybrid dynamical systems are provided. The conditions are based on relaxed hybrid Lyapunov functions. If either (F1)–(F3) or (G1)–(G3) holds, then asymptotic stability of a non-empty compact set can be guaranteed. Although our conditions are similar to Theorem 3.27, the advantages of our results can be described as follows. Firstly, instead of an application of any invariance principles as in [1] or [47], our results are directly obtained by simple trajectory-based proofs. Secondly, we do not need to check the conditions in Theorem 3.27 which are possibly difficult to verify in some cases.

3.4.4 Dwell-Time Conditions

In the previous section, stability of hybrid dynamical systems is guaranteed by existence of hybrid Lyapunov function or relaxed Lyapunov function (with additional requirement on types of solutions). Consider a hybrid Lyapunov candidate function satisfying that it is decreasing during flow, but increasing at any jumps. Obviously, we can see that such hybrid Lyapunov candidate function grows unbounded if the system exhibits excessively discrete dynamics. However, if discrete dynamics occur not too frequently, and continuous dynamics occur long enough, then such hybrid Lyapunov candidate function may tend to decrease. The number of jumps and time during flow in the system therefore become important conditions to study stability. Such condition are called *dwell-time conditions*. In this section, we provide such conditions to guarantee the stability of hybrid dynamical systems.

To provide the results, particular hybrid dynamical systems are required. Let us introduce the special classes of hybrid dynamical systems as follow.

Definition 3.33. For a hybrid arc x, define

$$T(j,x) := \{t \in \mathbb{R}_{\geq 0} : \ (t,j) \in \operatorname{dom} x\}.$$

Definition 3.34 (Hybrid Systems of Class $L(\theta)$)**.** Let $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ be a hybrid system and θ be a positive real number.

A hybrid arc x is said to be of *class* $L(\theta)$ if either

$$J := \sup_j \operatorname{dom} x = 0$$

or the inequality

$$\inf T(j+1,x) - \inf T(j,x) \geq \theta$$

is satisfied for $j \in \{0, 1, \dots, J-1\}$.

A hybrid system $\mathcal{H}$ is called a *class* $L(\theta)$ *hybrid system* if any maximal solution to $\mathcal{H}$ is of class $L(\theta)$.

Example 3.35. Any hybrid dynamical system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D}, \xi)$ with $\mathcal{D} = \emptyset$ is a class $L(\theta)$ hybrid system for any $\theta > 0$.

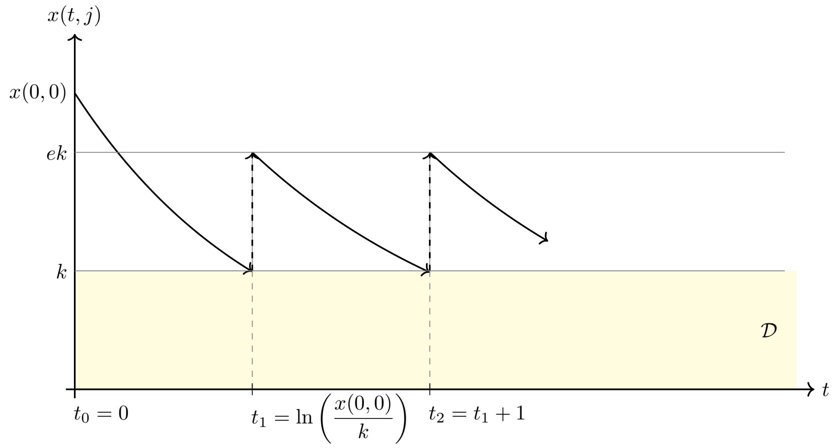


FIGURE 3.12: A solution to the system in Example 3.37.

Example 3.36. Let $\{t_j\}$ be an unbounded increasing sequence of positive real numbers. Any impulsive system with the impulse time sequence

$$T(\theta) = \{t_j \in \mathbb{R}_{>0} : \ t_{j+1} - t_j \geq \theta\}$$

is a class $L(\theta)$ hybrid system.

Example 3.37. Let k be a positive real number. Any maximal solution to the hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$, with the initial condition $x(0,0) > e \cdot k$, defined by the state $x \in \mathcal{X} \subset \mathbb{R}_{\geq 0}$,

$$f(x) := -x, \ \mathcal{C} := \{x \in \mathbb{R}_{\geq 0} : \ x > k\},$$

$$g(x) := e \cdot k, \ \text{and} \ \mathcal{D} := \{x \in \mathbb{R}_{\geq 0} : \ 0 \leq x \leq k\},$$

is of class $L(1)$. Figure 3.12 illustrates a solution to $\mathcal{H}$. The yellow region visualizes the jump set $\mathcal{D}$.

Example 3.38. The bouncing ball, proposed in Example 3.1, is not a class $L(\theta)$ hybrid system for any $\theta > 0$.

Definition 3.39 (Hybrid Systems of Class $H(\theta)$)**.** Let $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ be a hybrid system and θ be a positive real number.

A hybrid arc x is said to be of *class* $H(\theta)$ if

$$J := \sup_j \operatorname{dom} x > 0$$

and the inequality

$$\inf T(j+1, x) - \inf T(j, x) \leq \theta$$

is satisfied for $j \in \{0, 1, \ldots, J-1\}$.

A hybrid system $\mathcal{H}$ is called a *class* $H(\theta)$ *hybrid system* if any maximal solution to $\mathcal{H}$ is of class $H(\theta)$.

Example 3.40. Any hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ with $\mathcal{C} = \emptyset$ is a class $H(\theta)$ hybrid system for any $\theta > 0$.

Example 3.41. Let $\{t_j\}$ be an unbounded increasing sequence of positive real numbers. Any impulsive system with impulse time sequence

$$T(\theta) = \{t_j \in \mathbb{R}_{>0} : \ t_{j+1} - t_j \leq \theta\}$$

is a class $H(\theta)$ hybrid system.

Example 3.42. Let $\{t_j\}$ be an unbounded increasing sequence of positive real numbers. Any impulsive system with impulse time sequence

$$T(\theta) = \{t_j \in \mathbb{R}_{>0} : \ t_{j+1} - t_j = \theta\}$$

is both a class $L(\theta)$ hybrid system and a class $H(\theta)$ hybrid system.

Example 3.43. The bouncing ball system given in Example 3.1 with

$$x_1(0,0) > 0$$

is a class $H(T)$ hybrid system, where

$$T = \frac{x_2(0,0) + \sqrt{(x_2(0,0))^2 + 2\gamma x_1(0,0)}}{\gamma}.$$

For class $L(\theta)$ hybrid systems, any consecutive jumps are allowed if time has already passed by at least θ from the point of latest jump. While class $H(\theta)$ hybrid systems allow the behaviors of systems in the opposite way, i.e., any consecutive jumps must occur before time has passed by θ since the latest jump happened.

Theorem 3.44. *Let x be a maximal solution to a hybrid system $\mathcal{H}$ with its hybrid time domain*

$$\operatorname{dom} x = \bigcup_{j=0}^{\infty} [t_j, t_{j+1}] \times \{j\}.$$

(1) If the inequality

$$t_{j+1} - t_j \geq \theta \quad \forall (t_j, j) \in \operatorname{dom} x,$$

is satisfied, then x is of class $L(\theta)$.

(2) If the inequality

$$t_{j+1} - t_j \leq \theta \quad \forall (t_j, j) \in \operatorname{dom} x,$$

is satisfied, then x is of class $H(\theta)$.

Proof. For (1), it is clear to see that

$$\inf T(j+1, x) - \inf T(j, x) = t_{j+1} - t_j \geq \theta$$

for all $j \in \mathbb{N}$. The proof for (2) is omitted due to its similarity. □

The following results provide conditions to guarantee asymptotic stability hybrid systems without a hybrid Lyapunov function. The conditions required only a hybrid Lyapunov candidate function for a class $L(\theta)$ hybrid system or a class $H(\theta)$ hybrid system.

Theorem 3.45. *For a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, if there exists a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A})$ such that it satisfies for some $\varphi, \lambda \in \mathcal{P}$,*

(DL1) *$\mathcal{H}$ is a class $L(\theta)$ hybrid system for some $\theta > 0$;*

(DL2) *$\langle \nabla V(x), f(x) \rangle \leq -\varphi(V(x))$ for all $x \in \mathcal{C} \setminus \mathcal{A}$;*

(DL3) *$V(g(x)) \leq \lambda(V(x))$ for all $x \in \mathcal{D} \setminus \mathcal{A}$;*

(DL4) *The following inequality is satisfied*

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta \quad \text{for all } a > 0, \tag{3.20}$$

then $\mathcal{A}$ is stable. Additionally, if there exists $\delta > 0$ such that it satisfies

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta - \delta \quad \text{for all } a > 0, \tag{3.21}$$

then $\mathcal{A}$ is asymptotically stable.

Proof. Let us consider the first statement. It is enough to only show that any trajectory starting in Ω_β, defined in the proof of Theorem 3.21, always lies Ω_β. Suppose that $x(0,0) \in \Omega_\beta$. Since the condition (DL1) holds, the trajectory flows in $\mathcal{C}$. If there is no $T > 0$ such that $x(T, 0) \in \mathcal{D}$ then the compact set A is stable by Theorem 3.21, (L1). Note that V with (DL2) and $\mathcal{D} = \emptyset$ is a relaxed Lyapunov function. Therefore we assume that there exists $t_1 > 0$ such that $x(t_1, 0) \in \mathcal{D}$. Due to the condition (DL1), we obtain that the trajectory jumps from $x(t_1, 0) \in \mathcal{D}$ to $x(t_1, 1) = g(x(t_1, 0)) \in \mathcal{C}$. Since the condition (DL2) holds, we have for any $j \in \mathbb{N}$, $t \in [0, t_1]$,

$$V'(x(t,j)) \leq -\varphi(V(x(t,j))),$$

and so

$$-\int_0^{t_1} \frac{V'(x(t,0))\,\mathrm{d}t}{\varphi(V(x(t,0)))} \geq t_1 - 0 = t_1.$$

By substitution $V(x(t,0)) = s$ and using the condition (DL1), we therefore obtain

$$\int_{V(x(t_1,0))}^{V(x(0,0))} \frac{\mathrm{d}s}{\varphi(s)} \geq t_1 \geq \theta.$$

Note that this implies $V(x(t_1, 0)) \leq V(x(0,0))$. By replacing a by $V(x(t_1, 0))$ in the inequality (3.20) and using the condition (DL3), we then can write

$$\int_{V(x(t_1,0))}^{V(x(t_1,1))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(x(t_1,0))}^{V(g(x(t_1,0)))} \frac{\mathrm{d}s}{\varphi(s)} \leq \int_{V(x(t_1,0))}^{\lambda(V(x(t_1,0)))} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta.$$

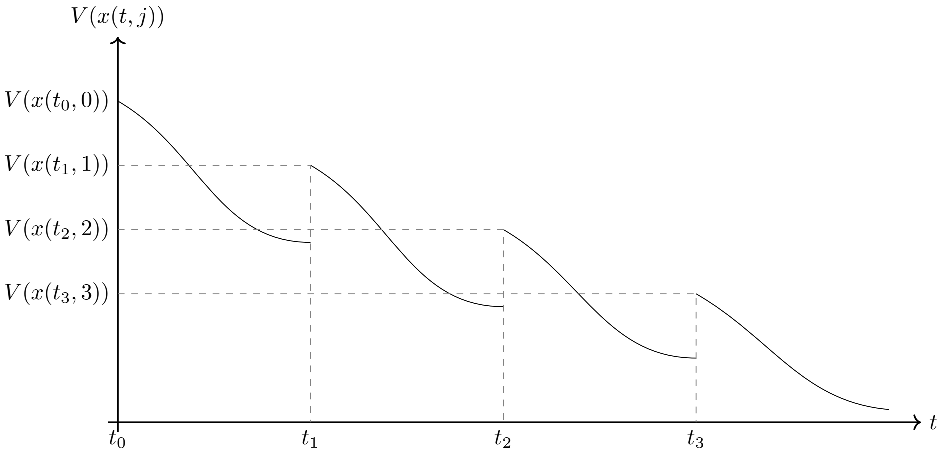


FIGURE 3.13: Illustration of hybrid Lyapunov candidate function V in Theorem 3.45.

With these two inequalities, it follows that

$$\int_{V(x(t_1,0))}^{V(x(0,0))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{V(x(t_1,0))}^{V(x(t_1,1))} \frac{\mathrm{d}s}{\varphi(s)},$$

and this implies that $V(x(t_1,1)) \leq V(x(0,0)) \leq \beta$. To guarantee that any solution starting in Ω_β always lies in Ω_β, it is sufficient to apply induction to get that

$$V(x(t_{j+1},j+1)) \leq V(x(t_j,j)) \leq V(x(0,0)) \leq \beta$$

for all $(t_j,j) \in \operatorname{dom} x$, see Figure 3.13.

For the second statement, instead of the inequality (3.20), we suppose that the inequality (3.21) holds for some $\delta > 0$. Assume that $\operatorname{dom} x$ is the union of $[t_j, t_{j+1}] \times \{j\}$ for all $j \in \mathbb{N}$. By condition (DL2), we have

$$-\int_{t_j}^{t_{j+1}} \frac{V'(x(t,j))\,\mathrm{d}t}{\varphi(V(x(t,j)))} \geq t_{j+1} - t_j \geq \theta.$$

By substitution $V(x(t,j)) = s$, we therefore get

$$\int_{V(x(t_{j+1},j))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \geq t_{j+1} - t_j \geq \theta.$$

Note that it implies $V(x(t_{j+1},j)) < V(x(t_j,j))$. Replacing $a = V(x(t_{j+1},j))$ in (3.21) and then using the condition (DL3), we see that

$$\int_{V(x(t_{j+1},j))}^{V(x(t_{j+1},j+1))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(x(t_{j+1},j))}^{V(g(x(t_{j+1},j)))} \frac{\mathrm{d}s}{\varphi(s)} \leq \int_{V(x(t_{j+1},j))}^{\lambda(V(x(t_{j+1},j+1)))} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta - \delta.$$

From two last inequalities it follows that

$$\int_{V(x(t_{j+1},j))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{V(x(t_{j+1},j))}^{V(x(t_{j+1},j+1))} \frac{\mathrm{d}s}{\varphi(s)} + \delta,$$

or it can be rewritten as

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(x(t_{j+1},j))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} - \int_{V(x(t_{j+1},j))}^{V(x(t_{j+1},j+1))} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta.$$

This implies $V(x(t_{j+1},j+1)) < V(x(t_j,j))$. So the sequence $\{V(x(t_j,j))\}$ is decreasing for $j \to \infty$, and it satisfies the inequality

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta \quad \text{for all } j \in \mathbb{N}. \tag{3.22}$$

Let us show that the sequence $\{V(x(t_j,j))\}$ converges to zero as $j \to \infty$. Suppose by contradiction that $V(x(t_j,j)) \to \alpha > 0$ as $j \to \infty$. Let

$$c := \inf_{\alpha \leq s \leq V(x(0,0))} \varphi(s),$$

then $c > 0$. From (3.22) we get

$$\delta \leq \int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \leq \frac{1}{c}\left(V(x(t_j,j)) - V(x(t_{j+1},j+1))\right),$$

That is $V(x(t_j,j)) - V(x(t_{j+1},j+1)) \geq \delta c$, but it contradicts to convergence of the sequence $\{V(x(t_j,j))\}$. So we have $V(x(t_j,j)) \to 0$ as $j \to \infty$. Recall that $V(x(t,j))$ is decreasing on every interval $[t_j, t_{j+1}]$, so

$$V(x(t_j,j)) = \sup_{(t_j,j) \preceq (t,j) \preceq (t_{j+1},j+1)} V(x(t,j)).$$

Together with the inequality $V(x(t_{j+1},j+1)) < V(x(t_j,j))$ that holds for all $j \in \mathbb{N}$. Consequently it follows, from the result $V(x(t_j,j)) \to 0$ as $j \to \infty$, that $V(x(t,j)) \to 0$ as $t + j \to \infty$, and then $\|x(t,j)\|_{\mathcal{A}} \to 0$ as $t + j \to \infty$. □

Remark 3.2. The above proof does not consider the case of finite jumps since the system is eventually continuous. Along with the condition (DL2), it is clear that $\mathcal{A}$ is asymptotically stable.

Theorem 3.46. *For a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, if there exists a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A})$ such that it satisfies for some $\varphi, \lambda \in \mathcal{P}$,*

(DH1) *$\mathcal{H}$ is a class $H(\theta)$ hybrid system for some $\theta > 0$;*

(DH2) *$\langle \nabla V(x), f(x) \rangle \leq \varphi(V(x))$ for all $x \in \mathcal{C} \cap \mathcal{U}$;*

(DH3) *$V(g(x)) \leq \lambda(V(x))$ for all $x \in \mathcal{D} \cap \mathcal{U}$;*

(DH4) *The following inequality is satisfied*

$$\int_{\lambda(a)}^{a} \frac{\mathrm{d}s}{\varphi(s)} \geq \theta \quad \textit{for all } a > 0, \tag{3.23}$$

then $\mathcal{A}$ is stable. Additionally, if there exists $\delta > 0$ such that it satisfies

$$\int_{\lambda(a)}^{a} \frac{\mathrm{d}s}{\varphi(s)} \geq \theta + \delta \quad \textit{for all } a > 0, \tag{3.24}$$

then $\mathcal{A}$ is asymptotically stable.

Remark 3.3. Either the inequality (3.23) or the inequality (3.24) implies $\lambda < \mathrm{id}$.

Proof. We are going to show stability of the system $\mathcal{H}$ of class $H(\theta)$ under the above conditions in the similar way of the proof of Theorem 3.45. For the first statement, it is enough to only show that any trajectory starting in Ω_β, defined in the proof of Theorem 3.21, always lies in Ω_β. It is easy to verify that if $\mathcal{H}$ is an eventually discrete hybrid system and the conditions (DH1)-(DH4) are satisfied, then $\mathcal{A}$ is stable for $\mathcal{H}$. Let us suppose that any maximal solution to $\mathcal{H}$ is not eventually discrete. Let $x(0,0) \in \Omega_\beta \cap \mathcal{C}$. Due to (DH1), there exists $t_1 \in (0,\theta]$ such that $x(t_1,0) \in \mathcal{D}$. Since $\varphi \in \mathcal{P}$ and (DH2) holds, we have for any $t \in [0,t_1]$,

$$V'(x(t,0)) \leq \varphi(V(x(t,0))),$$

and so

$$\int_{0}^{t_1} \frac{V'(x(t,0))\,\mathrm{d}t}{\varphi(V(x(t,0)))} \leq t_1 - 0 = t_1.$$

By substitution $V(x(t,0)) = s$, we therefore obtain

$$\int_{V(x(0,0))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)} \leq t_1 \leq \theta.$$

By replacing a by $V(x(t_1,0))$ in the inequality (3.23) and using the condition (DH3), we then can write

$$\int_{V(x(t_1,1))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(g(x(t_1,0)))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{\lambda(V(x(t_1,0)))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)} \geq \theta.$$

With these two inequalities, it follows that

$$\int_{V(x(t_1,1))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{V(x(0,0))}^{V(x(t_1,0))} \frac{\mathrm{d}s}{\varphi(s)}$$

which implies that $V(x(t_1,1)) \leq V(x(0,0)) \leq \beta$. To guarantee that any solution starting in Ω_β always lies in Ω_β, it is sufficient to apply induction to get that

$$V(x(t_{j+1},j+1)) \leq V(x(t_j,j)) \leq V(x(0,0)) \leq \beta$$

for all $(t_j,j) \in \operatorname{dom} x$.

For the second statement, instead of the inequality (3.20), we suppose that the inequality (3.21) holds for some $\delta > 0$. Assume that the hybrid time domain $\operatorname{dom} x$ is union of $[t_j,t_{j+1}] \times \{j\}$ for all $j \in \mathbb{N}$. Since $\varphi \in \mathcal{P}$, and (DH2) holds, we have

$$\int_{t_j}^{t_{j+1}} \frac{V'(x(t,j))\,\mathrm{d}t}{\varphi(V(x(t,j)))} \leq t_{j+1} - t_j.$$

By substitution $V(x(t,j)) = s$ and using (DH1), we therefore get

$$\int_{V(x(t_j,j))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} \leq t_{j+1} - t_j \leq \theta.$$

Replacing $a = V(x(t_{j+1},j))$ in (3.24) and then using (DH3), we see that

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(g(x(t_{j+1},j)))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{\lambda(V(x(t_{j+1},j)))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \theta + \delta.$$

From two last inequalities it follows that

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \int_{V(x(t_j,j))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} + \delta,$$

or it can be rewritten as

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} = \int_{V(x(t_{j+1},j+1))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} - \int_{V(x(t_j,j))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta.$$

This implies $V(x(t_{j+1},j+1)) < V(x(t_j,j))$. So the sequence $\{V(x(t_j,j))\}$ is decreasing for $j \to \infty$, and it satisfies the inequality

$$\int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta, \text{ for all } j \in \mathbb{N}. \tag{3.25}$$

Let us show that $V(x(t_j,j)) \to 0$ as $j \to \infty$. Suppose by contradiction that $V(x(t_j,j)) \to \alpha > 0$ as $j \to \infty$. Let $c = \inf_{\alpha \leq s \leq V(x(0,0))} \varphi(s)$. From (3.25) we get

$$\delta \leq \int_{V(x(t_{j+1},j+1))}^{V(x(t_j,j))} \frac{\mathrm{d}s}{\varphi(s)} \leq \frac{1}{c}\left(V(x(t_j,j)) - V(x(t_{j+1},j+1))\right),$$

That is $V(x(t_j,j)) - V(x(t_{j+1},j+1)) \geq \delta c$, but it contradicts to convergence of the sequence $\{V(x(t_j,j))\}$. So we have $V(x(t_j,j)) \to 0$ as $j \to \infty$.

However, $V(x(t,j))$ may not be decreasing on any interval $[t_j,t_{j+1}]$ since (DH2) holds. We need more investigation on this case. Without loss of generality, suppose that $V(x(t,j))$ is increasing on every interval $[t_j,t_{j+1}]$, i.e., for any interval $[t_j,t_{j+1}]$, it holds

$$\int_{V(x(t_j,j))}^{V(x(t_{j+1},j))} \frac{\mathrm{d}s}{\varphi(s)} = t_{j+1} - t_j = \theta,$$

which implies $V(x(t_j,j)) < V(x(t_{j+1},j))$, see Figure 3.14.

Therefore, it yields $V(x(t_{j+2},j+1)) < V(x(t_{j+1},j))$ for any $j \in \mathbb{N}$ since $V(x(t_{j+1},j+1)) < V(x(t_j,j))$, and it holds (DH1) and (DH2). In consequence, we obtain that the sequence $\{V(x(t_{j+1},j))\}$ is decreasing for $j \to \infty$. By way of contradiction, suppose that $V(x(t_{j+1},j)) \to \beta > 0$ as $j \to \infty$. Since each point in sequence $\{V(x(t_{j+1},j))\}$ is corresponding to tail end of j^{th}-flow, there exists $k > 0$ such that $(t_k,k) \in \operatorname{dom} x$, $x(t_k,k) \notin \mathcal{A}$, and it satisfies

$$\int_{V(x(t_k,k))}^{\beta} \frac{\mathrm{d}s}{\varphi(s)} = \theta.$$

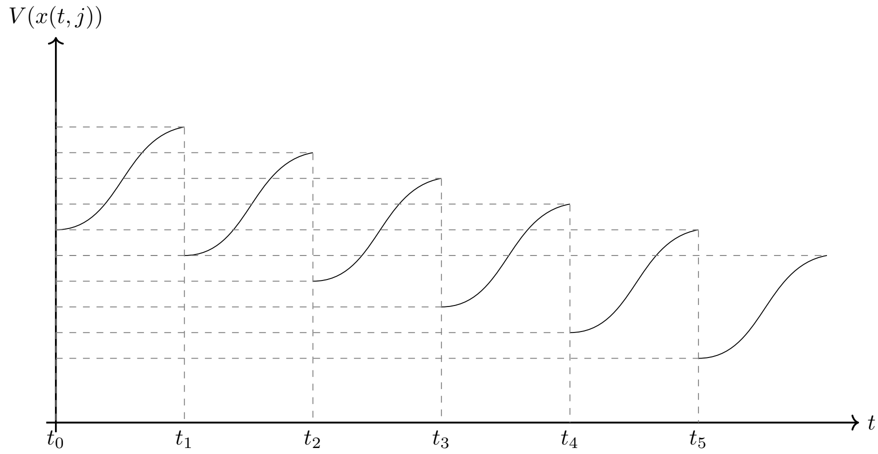


FIGURE 3.14: Illustration of hybrid Lyapunov candidate function V in Theorem 3.46.

Since V is positive definite on $\mathcal{U} \setminus \mathcal{A}$, we have $V(x(t_k, k)) > 0$, which contradicts to convergence of sequence $\{V(x(t_j, j))\}$.

Therefore, it yields $V(x(t_{j+1}, j)) \to 0$ as $j \to \infty$. Together with convergence of sequence $\{V(x(t_j, j))\}$, it follows that $V(x(t, j)) \to 0$ as $t + j \to \infty$. Therefore $\|x(t, j)\|_{\mathcal{A}} \to 0$ as $t + j \to \infty$. □

The inequalities (3.20), (3.21), (3.23) and (3.24) are called *dwell-time conditions*, which was proposed for impulsive systems in [11, 40, 51]. The following results are just special cases of hybrid Lyapunov candidate functions with their corresponding dwell-time conditions.

Corollary 3.47. *For a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, if there exists a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A})$ such that it satisfies for some $c > 0$, and $d \neq 0$:*

(1) *$\mathcal{H}$ is a class $L(\theta)$ for some $\theta > 0$;*

(2) *$\langle \nabla V(x), f(x) \rangle \leq -cV(x)$ for all $x \in \mathcal{C} \cap \mathcal{U}$;*

(3) *$V(g(x)) \leq \mathrm{e}^{-d} V(x)$ for all $x \in \mathcal{D} \cap \mathcal{U}$;*

(4) *$-d \leq c \cdot \theta$,*

then $\mathcal{A}$ is stable. Additionally, if it holds the inequality

$$-d \leq c \cdot \theta - \delta' \tag{3.26}$$

for some $\delta' > 0$, then $\mathcal{A}$ is asymptotically stable.

Proof. Consider the following inequalities

$$\langle \nabla V(x), f(x) \rangle \leq -cV(x) = -\varphi(V(x)), \; V(g(x)) \leq \mathrm{e}^{-d} V(x) = \lambda(V(x))$$

where $\varphi(s) := cs$ and $\lambda(s) := \mathrm{e}^{-d} s$ for all $s > 0$. Moreover, it holds

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} = \frac{-d}{c} \leq \theta \quad \text{for all } a > 0. \tag{3.27}$$

By Theorem 3.45, it consequently yields that $\mathcal{A}$ is stable for $\mathcal{H}$. In addition, asymptotic stability of $\mathcal{A}$ can be guaranteed by the dwell-time condition

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} = \frac{-d}{c} \leq \theta - \delta \quad \text{for all } a > 0, \text{ where } \delta = \frac{\delta'}{c}. \tag{3.28}$$

□

Remark 3.4. In case of $d > 0$ then the dwell-time condition (4) in the above corollary is automatically satisfied for any $\theta > 0$.

Remark 3.5. Various dwell-time conditions for impulsive systems given in the literature, see [36, 42], were provided by the inequality

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu, \tag{3.29}$$

or in a more general form

$$-dN(t,s) - c(t-s) \leq \ln h(t-s). \tag{3.30}$$

Note that the inequality (3.30) yields (3.29) by substitution $h(x) := \exp(\mu - \lambda x)$. Additionally, by substitution $\mu := -d$ in the inequality (3.29) and choosing time interval $[t_j, t_{j+1})$, where $\operatorname{dom} x = \cup [t_j, t_{j+1}] \times \{j\}$, and x is a solution to $\mathcal{H}$ of class $L(\theta)$, it yields $N(t, t_j) = 0$ and implies the inequality

$$\frac{c-\lambda}{-d} \geq \frac{1}{\theta}. \tag{3.31}$$

Moreover, from the dwell time condition (3.28), there exists $\lambda \in \left[\frac{c\delta}{\theta}, \infty\right)$ such that it satisfies the following inequalities

$$c\theta - \lambda\theta \leq -d \leq c\theta - c\delta,$$

which is equivalent to the inequality (3.31). We therefore summarize the relation of these dwell-time conditions as follows: the dwell-time condition (3.28) implies (3.31); the dwell-time condition (3.30) implies (3.29); and the dwell-time condition (3.29) implies (3.31).

Corollary 3.48. *For a complete hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ and a non-empty compact set $\mathcal{A} \subset \mathcal{X} \subset \mathbb{R}^n$, if there exists a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A}, \mathcal{U})$ such that it satisfies for some $c > 0$, and $d \neq 0$:*

(1) *The system $\mathcal{H}$ belongs to class $H(\theta)$ for some $\theta > 0$;*

(2) $\langle \nabla V(x), f(x) \rangle \leq cV(x)$ *for all $x \in \mathcal{C} \cap \mathcal{U}$;*

(3) $V(g(x)) \leq \mathrm{e}^{-d} V(x)$ *for all $x \in \mathcal{D} \cap \mathcal{U}$;*

(4) $d \geq c \cdot \theta$,

then $\mathcal{A}$ is stable. Additionally, if it holds the inequality

$$d \leq c \cdot \theta + \delta' \tag{3.32}$$

for some $\delta' > 0$, then $\mathcal{A}$ is asymptotically stable.

Proof. The proof is omitted due to similarity of the proof of the previous corollary. □

3.5 Partial Stability

Consider a state of hybrid systems consisting of time, counters or logical values. It is clear to see that this part of state never tends to zero. Additionally from a practical point of view, this part of state is insignificantly required for stability of the systems, see [1]. For such systems, the definitions of stability and hybrid Lyapunov function need to be modified. Suppose that the hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ can be decomposed to the following form:

$$\dot{x}^s = f^s(x^s),\ \dot{x}^c = f^c(x^s, x^c) \quad \text{if } x \in \mathcal{C}, \tag{3.33}$$

$$x^{s+} = g^s(x^s),\ x^{c+} = g^c(x^s, x^c) \quad \text{if } x \in \mathcal{D} \tag{3.34}$$

with $x = (x^s, x^c) \in \mathcal{X} \subset \mathbb{R}^n$, $x^s \in \mathcal{X}^s$, $x^c \in \mathcal{X}^c$, and $\mathcal{X} = \mathcal{X}^s \times \mathcal{X}^c$. Here x^s is the part of the state x, which we are interested in view of stability.

To provide stability notions for a part of state and related results, let us introduce the following definitions.

Definition 3.49 (Partial Stability)**.** For a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ in the form of (3.33)–(3.34) and a non-empty compact set $\mathcal{A}^s \subset \mathcal{X}^s$ is said to be

- *partially stable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that any solution $x = (x^s, x^c)$ to $\mathcal{H}$ with $\|x^s(0,0)\|_{\mathcal{A}^s} < \delta$ satisfies $\|x^s(t,j)\|_{\mathcal{A}^s} < \varepsilon$ for all $(t,j) \in \operatorname{dom} x^s$;
- *partially attractive* if any solution $x = (x^s, x^c)$ to $\mathcal{H}$ satisfies $\|x^s(t,j)\|_{\mathcal{A}^s} \to 0$ as $t + j \to \infty$;
- *partially asymptotically stable* if it is both partially stable and partially attractive.

Definition 3.50. Given a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ in the form of (3.33)–(3.34) and a non-empty compact set $\mathcal{A}^s \subset \mathcal{X}^s \subset \mathbb{R}^{n_s}$. A function $V : \mathbb{R}^{n_s} \to \mathbb{R}$ is called a *hybrid Lyapunov candidate function* for $(\mathcal{H}, \mathcal{A}^s)$ if it is globally Lipschitz, and there exist class $\mathcal{K}_\infty$ functions φ_1 and φ_2 such that it satisfies

$$\varphi_1(\|x^s\|_{\mathcal{A}^s}) \leq V(x^s) \leq \varphi_2(\|x^s\|_{\mathcal{A}^s})$$

for all $x^s \in \mathcal{X}^s$.

Definition 3.51. Given a hybrid system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ in the form of (3.33)–(3.34) and a non-empty compact set $\mathcal{A}^s \subset \mathcal{X}^s \subset \mathbb{R}^{n_s}$. A function $V : \mathbb{R}^{n_s} \to \mathbb{R}$ is called a *hybrid Lyapunov function* for $(\mathcal{H}, \mathcal{A}^s)$ if it is a hybrid Lyapunov candidate function for $(\mathcal{H}, \mathcal{A}^s)$ and satisfies

$$\langle \nabla V(x^s), f^s(x^s) \rangle < 0 \quad \text{for all } (x^s, x^c) \in \mathcal{C} \setminus (\mathcal{A}^s \times \mathcal{X}^c),$$
$$V(g^s(x^s)) - V(x^s) < 0 \quad \text{for all } (x^s, x^c) \in \mathcal{D} \setminus (\mathcal{A}^s \times \mathcal{X}^c).$$

Moreover, it is called *relaxed hybrid Lyapunov function* for $(\mathcal{H}, \mathcal{A}^s)$ if it satisfies

$$\begin{aligned} \langle \nabla V(x^s), f^s(x^s) \rangle &\leq 0 \quad \text{for all } (x^s, x^c) \in \mathcal{C} \setminus (\mathcal{A}^s \times \mathcal{X}^c), \\ V(g^s(x^s)) - V(x^s) &\leq 0 \quad \text{for all } (x^s, x^c) \in \mathcal{D} \setminus (\mathcal{A}^s \times \mathcal{X}^c). \end{aligned}$$

The above definitions only focus on the state x^s. Hybrid systems of the form (3.33)–(3.34) allow us to model and additionally investigate stability of a desired part of the state, while another part of the state is just time or other parameters usually not in point of view for stability.

The following results shows an example to guarantee stability of impulsive systems proposed in Example 3.3.

Theorem 3.52. *For a complete impulsive system $\mathcal{H} = (\mathcal{X}, F, \mathcal{C}, G, \mathcal{D})$ defined in Example 3.3 and a non-empty compact set $\mathcal{A}^s \subset \mathbb{R}^n$, if there exists a hybrid Lyapunov candidate function V for $(\mathcal{H}, \mathcal{A}^s)$ such that it satisfies for some $\varphi, \lambda \in \mathcal{P}$,*

(DL1) $T = \{t \in \mathbb{R}_{>0} : \ t_{j+1} - t_j \geq \theta\}$ *for some* $\theta > 0$;

(DL2) $\langle \nabla V(x), f(x) \rangle \leq -\varphi(V(x))$ *for all* $(x, t) \in \mathcal{C} \setminus (\mathcal{A}^s \times \mathbb{R}_{\geq 0})$;

(DL3) $V(g(x)) \leq \lambda(V(x))$ *for all* $(x, t) \in \mathcal{D} \setminus (\mathcal{A}^s \times \mathbb{R}_{\geq 0})$;

(DL4) *The following inequality holds*

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta \quad \text{for all } a > 0, \tag{3.35}$$

then $\mathcal{A}^s$ is partially stable. Additionally, if there exists $\delta > 0$ such that it satisfies

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} \leq \theta - \delta \quad \text{for all } a > 0, \tag{3.36}$$

then $\mathcal{A}^s$ is partially asymptotically stable.

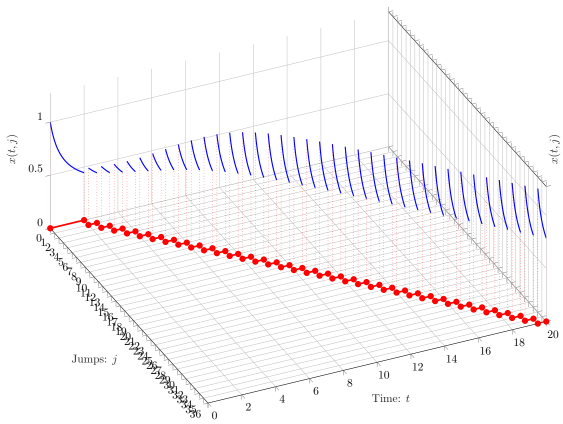
Proof. The proof is done in the same general manner as the proof of Theorem 3.45 since the time-variable t is not a part of state. □

Example 3.53. Consider the impulsive system $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ defined in Example 3.3 with $x = (x_1, x_2) \in \mathcal{X} = \mathbb{R} \times \mathbb{R}_{\geq 0}$,

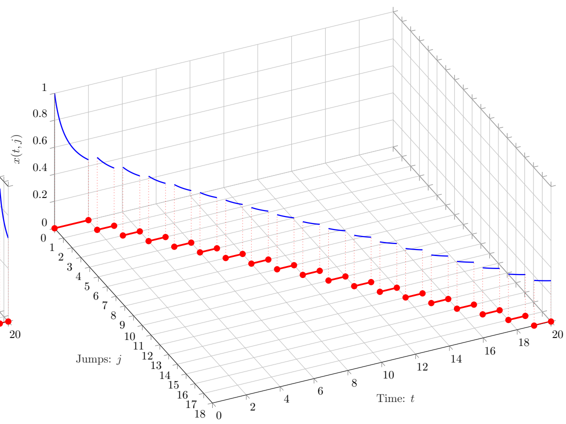
$$f(x) := \begin{pmatrix} -x_1^3 \\ 1 \end{pmatrix}, \; g(x) := \begin{pmatrix} x_1 + x_1^3 \\ 0 \end{pmatrix},$$

and $T = T_\theta := \{t \in \mathbb{R}_{>0} : \ t_{j+1} - t_j \geq \theta\}$ for some $\theta > 0$.

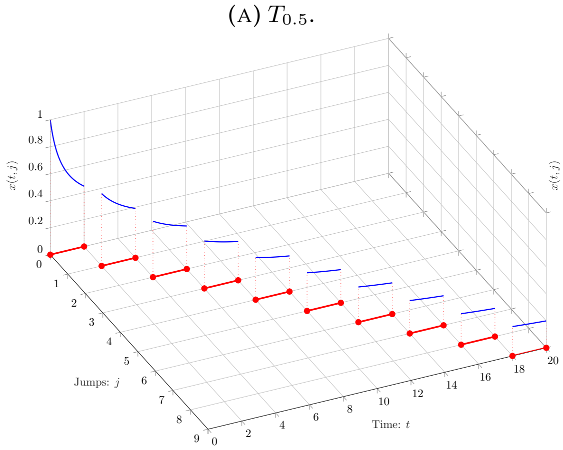
Obviously, x never tends to zero since x_2 indicated time is a part of the state. However, our focused part of state is only x_1, and it may converge to zero in long-term trends. Let us denote $x := (x^s, x^c)$, $x^c := x_2$, $x^s := x_1 \in \mathcal{X}^s := \mathbb{R}$, $f^s(x^s) := -(x^s)^3$, $g^s(x^s) := x^s + (x^s)^3$ and $\mathcal{A}^s = \{0\} \subset \mathcal{X}^s$. Define the function $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by



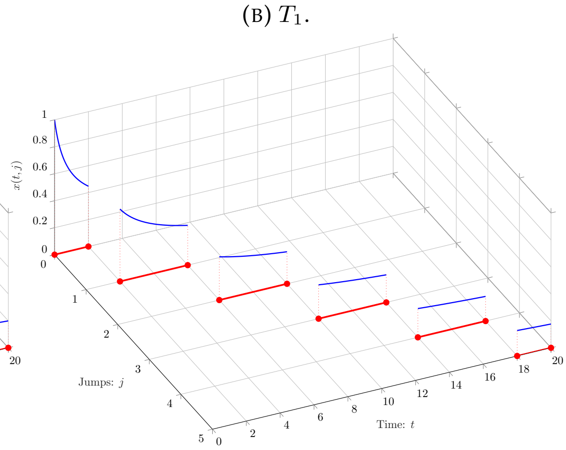
(A) $T_{0.5}$.



(B) T_1.



(C) T_2.



(D) T_4.

FIGURE 3.15: Numerical solutions to the hybrid system in Example 3.53 with various T_θ.

$V(x^s) := |x^s|$. Consider the following

$$\langle \nabla V(x^s), f^s(x^s) \rangle = \text{sign}\,(x^s) \cdot (-(x^s)^3) = -(V(x^s))^3 = -\varphi(V(x^s)),$$
$$V(g^s(x^s)) = \left|x^s + (x^s)^3\right| \leq |x^s| + |x^s|^3 = V(x^s) + (V(x^s))^3 = \lambda(V(x^s))$$

where the functions $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are defined as follows:

$$\varphi(s) := s^3, \quad \lambda(s) := s + s^3.$$

Consequently, we consider the dwell-time condition

$$\int_a^{\lambda(a)} \frac{\mathrm{d}s}{\varphi(s)} = \int_a^{a+a^3} \frac{\mathrm{d}s}{s^3} = \frac{a^2+2}{2(a^2+1)^2} < 1 \quad \text{for all } a > 0.$$

By Theorem 3.52, $\mathcal{A}^s = \{0\}$ is partially asymptotically stable for $\mathcal{H}$ if $\theta \geq 1$.

In Figure 3.15, we provide numerical simulations of this with the initial condition $x_1(0,0) = 1$ along with various impulse time sequences T_θ. It is clear to see that the frequency of discrete dynamics and period of continuous dynamics play in important role for stability in the system. In case of $T_{0.5}$, the trajectory x_1 grows unbounded while $\|x_1(t,j)\| \to 0$ as $t + j \to \infty$ in the cases of T_θ, with $\theta \geq 1$.

Chapter 4

Interconnected Hybrid Dynamical Systems

This chapter addresses the question of the composition and decomposition of hybrid dynamical systems. Motivated by the results in [10], we propose an extended framework of hybrid dynamical systems allowing us to consider an interconnection of several hybrid dynamical systems as one hybrid dynamical system and to decompose one large hybrid dynamical system into several subsystems. Results on the stability analysis of the interconnection and subsystems are also provided.

One of many currently active research field is related to interconnected large-scale systems [51–57]. Interconnections of hybrid dynamical systems were considered in [10, 58–63]. However it turns out, that the description of an interconnection in case of hybrid dynamical systems is not a trivial issue. For example, stability results for interconnections are possible only under some restrictive and physically unnatural conditions, see [61].

In particular, a natural way to consider such interconnections leads to the existence of solutions that are physically meaningless. This will be demonstrated by a simple example of an interconnection of two bouncing balls that are connected by an elastic spring. We discuss these kind of problems occurring in interconnections of hybrid dynamical systems, and we also suggest a possible way out to solve them.

4.1 Motivation

Two Bouncing Balls

Consider two bouncing balls with states ${}^1x = ({}^1x_1, {}^1x_2) \in \mathbb{R}^2$ and ${}^2x = ({}^2x_1, {}^2x_2) \in \mathbb{R}^2$ respectively. The upper-left index indicates the number of each ball. Let the balls be interconnected by an elastic spring with elastic coefficient $\mu \geq 0$, see Figure 4.1. The case $\mu = 0$ means that the balls are disconnected and move independently. The mass of the spring is ignored. The motion of balls is vertical along different lines, so that a collision is not possible. In this case there is an interaction force between the bouncing balls due to the elastic spring. By Hooke's law this force is proportional to the strain of the spring and is given by $\pm\mu({}^1x_1 - {}^2x_1)$.

Hence the dynamics of each ball is influenced by the other one, and it is given by the following equations, where again the upper index denotes the number of the ball:

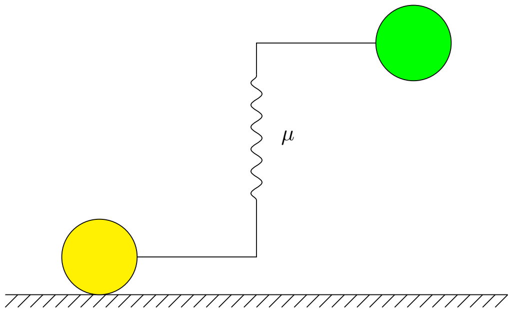


FIGURE 4.1: Two bouncing ball connected with an elastic spring.

$$
\begin{aligned}
{}^1\dot{x} &= \begin{pmatrix} {}^1x_2 \\ -\gamma - \mu({}^1x_1 - {}^2x_1) \end{pmatrix} := {}^1f({}^1x, {}^2x), ({}^1x, {}^2x) \in {}^1\mathcal{C}, \\
{}^1x^+ &= \begin{pmatrix} {}^1x_1 \\ -\lambda\, {}^1x_2 \end{pmatrix} =: {}^1g({}^1x), \quad ({}^1x, {}^2x) \in {}^1\mathcal{D},
\end{aligned}
$$

and

$$
\begin{aligned}
{}^2\dot{x} &= \begin{pmatrix} {}^2x_2 \\ -\gamma + \mu({}^1x_1 - {}^2x_1) \end{pmatrix} := {}^2f({}^1x, {}^2x), ({}^1x, {}^2x) \in {}^2\mathcal{C}, \\
{}^2x^+ &= \begin{pmatrix} {}^2x_1 \\ -\lambda\, {}^2x_2 \end{pmatrix} =: {}^2g({}^2x), \quad ({}^1x, {}^2x) \in {}^2\mathcal{D},
\end{aligned}
$$

where

$$
\begin{aligned}
{}^1\mathcal{C} &= \{({}^1x, {}^2x) \in \mathbb{R}^4 : {}^1x_1 \geq 0\}, \\
{}^2\mathcal{C} &= \{({}^1x, {}^2x) \in \mathbb{R}^4 : {}^2x_1 \geq 0\}, \\
{}^1\mathcal{D} &= \{({}^1x, {}^2x) \in \mathbb{R}^4 : {}^1x_1 = 0, {}^1x_2 \leq 0\}, \\
{}^2\mathcal{D} &= \{({}^1x, {}^2x) \in \mathbb{R}^4 : {}^2x_1 = 0, {}^2x_2 \leq 0\}.
\end{aligned}
$$

Let us now consider this interconnection of two bouncing balls as one hybrid dynamical system. For this purpose we need to define the new state and new sets $\mathcal{C}$ and $\mathcal{D}$ for the whole interconnection. It is natural to define $z := ({}^1x, {}^2x) \in \mathbb{R}^4$ as the state of the whole system.

Now we have to define $\mathcal{C}$ and $\mathcal{D}$ as well as the functions f and g describing the flow and jumps respectively for the whole system. Since it is natural to understand that if any time one of the balls jumps, the state $z \in \mathbb{R}^4$ jumps, i.e., the whole interconnection undergoes a jump, we hence define $\mathcal{D} := {}^1\mathcal{D} \cup {}^2\mathcal{D} \in \mathbb{R}^4$ and $\mathcal{C} := {}^1\mathcal{C} \cap {}^2\mathcal{C}$. A choice for f and g is as follows

$$f(z) := ({}^1f^T, {}^2f^T)^T, \tag{4.1}$$

$$g(z) := ({}^1\widetilde{g}^T, {}^2\widetilde{g}^T)^T, \tag{4.2}$$

where for $i = 1, 2$

$$
{}^i\widetilde{g}(z) = {}^i\widetilde{g}({}^1x, {}^2x) := \begin{cases} {}^ig({}^ix), & \text{if } ({}^1x, {}^2x) \in {}^i\mathcal{D}, \\ {}^ix, & \text{otherwise.} \end{cases}
$$

With this notation the interconnection can be written as one hybrid system without inputs in the form

$$\dot{z} = f(z), \quad z \in \mathcal{C}, \tag{4.3}$$

$$z^+ = g(z), \quad z \in \mathcal{D}. \tag{4.4}$$

The same approach was also used in [63, 64] to describe an interconnection of hybrid dynamical systems. Moreover we have not seen any other choice of choosing $\mathcal{C},\ \mathcal{D},\ f$ and g in the literature.

Solutions and Stability Problems

We do not see any other reasonable definition for $\mathcal{C}$, $\mathcal{D}$, f, g in the given setting written above. However, this choice leads to the following problems illustrated by the example. Consider the following initial condition ${}^1x_1(0) = {}^1x_2(0) = 0$ and ${}^2x_1(0) = h > 0$, ${}^2x_2(0) = v \in \mathbb{R}$. Then observe that the following hybrid arc

$${}^1x_1(t,j) = {}^1x_2(t,j) = 0, \quad {}^2x_1(t,j) = h, \quad {}^2x_2(t,j) = v \tag{4.5}$$

is a solution for (4.3)-(4.4) with the hybrid time domain given by $\{(0,j)\}_{j=0}^{\infty}$. This can be checked by a direct substitution of (4.5) into (4.3)-(4.4) and taking into account that in the intersection $C^i \cap D^i$ both jumps and continuous flow are allowed. In this case $t_{\max} = 0$ and the system jumps infinitely many times from a non-zero state to the same state.

This "frozen" solution appears due to the interconnection and leads to the following problems that are relevant not only for the considered example but for interconnections of other hybrid dynamical systems with a stable equilibrium point:

- The above particular solution has no physical meaning.
- This solution shows that the resting state, i.e., the origin, is not asymptotically stable any more.

This artificial loss of stability is counterintuitive. It happens due to the physically meaningless solution that needs to be ruled out by a suitable improvement of the notion of hybrid dynamical system. This is the main motivation of the extended framework of hybrid systems, and we provide such a generalization of hybrid dynamical systems below.

Moreover, there is another issue apart from the mentioned problems. In general, there is a solution corresponding to the case of one ball reaching its resting state in a finite time (after infinite number of jumps, its continuous motion stops for while) and then being pulled out from this state by the second ball (and flows again for a while after that). This problem is very interesting, but it will not be considered in this work because it appears not necessarily with an interconnection but can happen with only one ball with external input. This problem is related to the issue of extension of solutions over the Zeno behavior [43–46]. We will later give some comments about that issue.

To solve the problem of artificial solutions and related stability loss, we are going to propose an extended framework of a hybrid dynamical system in the next section.

4.2 Generalized Hybrid Dynamical Systems

Throughout this chapter we fix positive numbers n and ${}^iN \in \mathbb{N}$ such that $\sum_{i=1}^{n} {}^iN = N$. Denote $\mathbb{N}_n = \{1, 2, \ldots, n\}$. Let $x \in \mathcal{X} \subset \mathbb{R}^N$ be partitioned into n parts: $x = ({}^1x, \ldots, x^n)$ with ${}^ix \in {}^i\mathcal{X} \subset \mathbb{R}^{{}^iN}$ and $u \in U \subset \mathbb{R}^M$ be an external input. For $i \in \mathbb{N}_n$, suppose that each ${}^i\mathcal{X}$ is open. Let ${}^i\mathcal{C} \subset \mathcal{X}$ and ${}^i\mathcal{D} \subset \mathcal{X}$ be given and relatively closed in ${}^i\mathcal{X}$. Let ${}^if : \mathbb{R}^N \times U \to \mathbb{R}^{{}^iN}$ and ${}^ig : \mathbb{R}^N \times U \to \mathbb{R}^{{}^iN}$ be given and continuous. For any given $(x, u) \in \mathcal{X} \times U$, define the index sets

$$I_{\mathcal{C}}(x,u) := \{i : (x,u) \in {}^i\mathcal{C} \times U\}, \quad I_{\mathcal{D}}(x,u) := \{i : (x,u) \in {}^i\mathcal{D} \times U\}. \tag{4.6}$$

For any (x, u) such that $(x, u) \in ({}^i\mathcal{C} \cup {}^i\mathcal{D}) \times \mathcal{U}\ \forall i \in \mathbb{N}_n$, it holds that $I_C \cup I_D = \mathbb{N}_n$. Note that if ${}^i\mathcal{C} \cap {}^i\mathcal{D} \neq \emptyset$ for some i, it holds that $I_{\mathcal{C}} \cap I_{\mathcal{D}} \neq \emptyset$.

A generalized hybrid dynamical system $\mathcal{H}$ is given by

$${}^i\mathcal{H} \quad \begin{cases} {}^i\dot{x} = {}^if(x,u), & i \in I_{\mathcal{C}}(x,u); \\ {}^ix^+ = {}^ig(x,u), & i \in I_{\mathcal{D}}(x,u), \end{cases} \tag{4.7}$$

which is denoted by $\left\{{}^i\mathcal{H}\right\}_{i=1}^n$.

In case for a given x and u there are some $i \in I_{\mathcal{C}} \cap I_{\mathcal{D}} \neq \emptyset$, it is allowed for ${}^ix$ that it can flow or jump. This is similar to the case when $\mathcal{C} \cap \mathcal{D} \neq \emptyset$ for system (4.3)-(4.4), where the system may flow or jump. In the special case ${}^i\mathcal{D} = \mathcal{D}$ and ${}^i\mathcal{C} = \mathcal{C}$ for all integer $i \in [1, n]$ we arrive to the same definition of a hybrid dynamical system as in [65], whose trajectories can flow in continuous time and also jump at discrete instants.

However, our definition is more general due to the possibility to have continuous flows for some parts of the state also at those instants when other parts can jump. This definition allows to consider one large hybrid dynamical system as an interconnection of several ones or vice versa to consider several interconnected hybrid dynamical systems as one larger hybrid dynamical system. The idea is to partition the state of a system in several parts that are allowed to jump separately while other parts are allowed to flow. In this case, we have to take into account such situations when one part of the state "stops" while another part "moves".

Remark 4.1. It is obvious to see that a generalized hybrid dynamical system $\left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form (4.7) is identical to a hybrid system in the form (3.1) if $x \in \mathcal{X} \subset \mathbb{R}^N$, for some positive integer N, be partitioned into one part.

4.3 Concept of Solutions

4.3.1 Generalized Hybrid Time Domains

For a generalized hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$, we suggest that solutions is parameterized by t, the amount of time passed, and ${}^ik$, the number of jumps that have occurred in the subsystem ${}^i\mathcal{H}$. Note that the upper-left index numerates the subsystem ${}^i\mathcal{H}$ and the parts of the state x, i.e., i corresponds to ${}^ix$ and ${}^ik$ counts the jumps of this part of the state. We denote that the number k corresponds to the total number of jumps of all parts of the state.

Definition 4.1. For a generalized hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$, a point

$$\overline{k} = \left({}^1k, {}^2k, \ldots, {}^nk\right) \in \mathbb{N}^n$$

is called a *multi-index*. For a number $r \in \mathbb{R}$, multi-indices $\overline{k} = \left({}^1k, {}^2k, \ldots, {}^nk\right) \in \mathbb{N}^n$ and $\overline{s} = \left({}^1s, {}^2s, \ldots, {}^ns\right) \in \mathbb{N}^n$, define

$$r \cdot (\overline{k} + \overline{s}) := \left(r \cdot \left({}^1k + {}^1s\right), r \cdot \left({}^2k + {}^2s\right), \ldots, r \cdot \left({}^nk + {}^ns\right)\right).$$

Denote the zero multi-index by $0 := (0, \ldots, 0) \in \mathbb{N}^n$. A multi-index $\overline{p} = ({}^1p, \ldots, {}^np)$ is said to be *binary* if ${}^ip \in \{0, 1\}$ for each $i \in \{1, 2, \ldots, n\}$.

Define the function $\varsigma : \mathbb{N}^n \to \mathbb{N}$ by

$$\varsigma(\overline{k}) = \varsigma({}^1k, \ldots, {}^nk) := {}^1k + \ldots + {}^nk. \tag{4.8}$$

Definition 4.2 (Generalized Hybrid Time Domains)**.** For a generalized hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$, a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}^n$ is called a compact generalized hybrid time domain if

$$E = \bigcup_{k=0}^{K-1} \left([t_k, t_{k+1}] \times \left\{{}^1k\right\} \times \left\{{}^2k\right\} \times \ldots \times \left\{{}^nk\right\}\right) \tag{4.9}$$

with $k = \varsigma\left({}^1k, {}^2k, \ldots, {}^nk\right)$, for some finite sequence of times

$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K.$$

In addition, we assume for each $i = 1, 2, \ldots, n$ and all $k = 0, 1, \ldots, K-2$ that

$${}^ik \leq {}^i(k+1).$$

It is a *generalized hybrid time domain* if for all $(t, \overline{k}) \in E$,

$$E \cap \left([0, t] \times \left\{0, 1, 2, \ldots, \varsigma(\overline{k})\right\}^n\right)$$

is a compact generalized hybrid time domain.

According to the above definition, a generalized hybrid time domain is written by a union of finite or infinite sequence of $[t_k, t_{k+1}] \times \{{}^1k\} \times \cdots \times \{{}^nk\} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}^n$ with $k = {}^1k + \cdots + {}^nk$, where the last interval is allowed to be of the form $[t_k, T) \times \{{}^1k\} \times \{{}^2k\} \times \cdots \times \{{}^nk\}$ with T finite or $T = \infty$.

Definition 4.3 (The order on generalized hybrid time domains)**.** Given E a generalized hybrid time domain containing $(t_1, \overline{k}_1)$ and $(t_2, \overline{k}_2)$, we define

$$(t_1, \overline{k}_1) \preceq (t_2, \overline{k}_2) \iff t_1 + \varsigma(\overline{k}_1) \leq t_2 + \varsigma(\overline{k}_2),$$

and,

$$(t_1, \overline{k}_1) \prec (t_2, \overline{k}_2) \iff t_1 + \varsigma(\overline{k}_1) < t_2 + \varsigma(\overline{k}_2).$$

Definition 4.4. Given a generalized hybrid time domain E,

$$\sup_t E := \sup\left\{t \in \mathbb{R}_{\geq 0} :\ \exists\, \overline{k} \in \mathbb{N}^n,\ (t, \overline{k}) \in E\right\},$$

and

$$\sup_k E := \sup\left\{\varsigma(\overline{k}) :\ \exists t \in \mathbb{R}_{\geq 0}, (t, \overline{k}) \in E\right\}.$$

Furthermore $\sup E := (\sup_t E, \sup_k E)$, and length $(E) := \sup_t E + \sup_k E$.

Definition 4.5 (Generalized Hybrid Arc)**.** Let E be a generalized hybrid time domain. A function $x : E \to \mathbb{R}^n$ is called a *generalized hybrid arc* on E if for each $k = \varsigma(\overline{k}) \in \{0, 1, 2, \ldots, (\sup_k E - 1)\}$ the function $t \mapsto x(t, \overline{k})$ is locally absolutely continuous on the interval $[t_k, t_{k+1}]$.

Definition 4.6. Given a generalized hybrid time domain E and a generalized hybrid arc $x : E \to \mathbb{R}^n$, define the domain of x by

$$\mathrm{dom}\ x := E,$$

and define the range of x by

$$\mathrm{rge}\ x := \left\{y \in \mathbb{R}^n :\ \exists (t, \overline{k}) \in \mathrm{dom}\ x,\ x(t, \overline{k}) = y\right\}.$$

Definition 4.7 (Types of Generalized Hybrid Arcs)**.** A generalized hybrid arc x is said to be

(1) *nontrivial* if $\mathrm{rge}\ x$ contains at least two points;

(2) *bounded* if $\sup\{\|y\| :\ y \in \mathrm{rge}\ x\} < \infty$;

(3) *complete* if $\mathrm{length}\ (\mathrm{dom}\ x) = \infty$;

(4) *discrete* if $\sup_t \mathrm{dom}\ x = 0$;

(5) *continuous* if $\sup_k \mathrm{dom}\ x = 0$;

(6) *Zeno* if it is complete and $\sup_t \mathrm{dom}\ x < \infty$;

(7) *eventually discrete* if $T = \sup_t \mathrm{dom}\ x < \infty$ and $\mathrm{dom}\ x \cap (\{T\} \times \mathbb{N}^n)$ contains at least two points;

(8) *eventually continuous* if $J = \sup_k \mathrm{dom}\ x < \infty$ and $\mathrm{dom}\ x \cap (\mathbb{R}_{\geq 0} \times \{J\}^n)$ contains at least two points;

(9) *absolutely hybrid* if it is neither eventually discrete nor eventually continuous.

4.3.2 Solutions to Interconnections

This section addresses a topic of solutions to a generalized hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ and some additional discussion of artificial solutions to the interconnected bouncing balls.

Let ${}^1t_{\max}, {}^2t_{\max}, \ldots, {}^nt_{\max}$ be fixed positive real numbers and $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ be a generalized hybrid system. Throughout the rest of this chapter, these are Zeno times for ${}^1x, {}^2x, \dots, {}^nx$ respectively. For simplicity, let us assume that there exists one and only one Zeno time ${}^it_{\max}$ for ${}^ix$.

Definition 4.8 (Solutions to a generalized hybrid dynamical system)**.** For a generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ with an initial condition $x(0,0) = \xi \in \mathcal{X}$, a pair of a generalized hybrid arc x and an external input u is called a *solution to* $\mathcal{H}$ if the following is satisfied:

(i) For any i, $(\xi, u(0,0)) \in ({}^i\mathcal{C} \cup {}^i\mathcal{D}) \times U$,

(ii) For any multi-index $\overline{k} \in \mathbb{N}^n$ and almost all $t \in \mathbb{R}_{\geq 0}$ with $(t, \overline{k}) \in \operatorname{dom} x$, the following is satisfied

$${}^i\dot{x}(t,\overline{k}) = {}^if({}^1x(\min\{t, {}^1t_{\max}\}, \overline{k}), \dots, {}^nx(\min\{t, {}^nt_{\max}\}, \overline{k}), u(t,\overline{k})),$$

for any $i \in I_{\mathcal{C}}({}^1x(t,\overline{k}), \dots, {}^nx(t,\overline{k}), u(t,\overline{k}))$,

(iii) For any $(t,\overline{k}) \in \operatorname{dom} x$, binary multi-index $\overline{p} \in \mathbb{N}^n$ such that $(t, \overline{k}+\overline{p}) \in \operatorname{dom} x$ and $\varsigma(\overline{p}) \geq 1$,

$${}^ix^+(\min\{t, {}^kt_{\max}\}, \overline{k}+\overline{p}) = {}^ig({}^1x(\min\{t, {}^1t_{\max}\}, \overline{k}), \dots, {}^nx(\min\{t, {}^nt_{\max}\}, \overline{k}), u(t,\overline{k}))$$

for any $i \in I_{\mathcal{D}}({}^1x(t,\overline{k}), \dots, {}^nx(t,\overline{k}), u(t,\overline{k}))$.

Remark 4.2. The numbers ${}^it_{\max}$ are not known in advance. Each one should be found as for example in (3.11) and should be considered as a part of solution or, more precisely, of its hybrid time domain. The number ${}^it_{\max}$ is the total time during which the i-th part of the state flows.

Definition 4.9. For a generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$, a solution to $\mathcal{H}$ is said to be maximal if it cannot be extended. Denoted by $S_{\mathcal{H}}(\xi)$ the set of all maximal solutions to $\mathcal{H}$ with the initial condition $x(0,0) = \xi \in \mathcal{X}$.

Definition 4.10. A generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ is said to be complete if any maximal solution to $\mathcal{H}$ is complete. It is called a Zeno hybrid system if any of its maximal solutions is Zeno. It is called an eventually continuous (discrete) hybrid system if any of its maximal solutions is eventually continuous (discrete). It is called an absolutely hybrid system if any of its maximal solutions is absolutely hybrid.

Definition 4.11. For a generalized hybrid arc x, define

$$T(k,x) := \left\{t \in \mathbb{R}_{\geq 0} : \ (t,\overline{k}) \in \operatorname{dom} x, \varsigma(\overline{k}) = k\right\}.$$

Definition 4.12 (Generalized Hybrid Systems of Class $L(\theta)$)**.** Let $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ be a generalized hybrid system and θ be a positive real number.

A generalized hybrid arc x is said to be of *class* $L(\theta)$ if either

$$K := \sup_k \operatorname{dom} x = 0$$

or the inequality

$$\inf T(k+1, x) - \inf T(k, x) \geq \theta$$

is satisfied for $k \in \{0, 1, \ldots, K-1\}$.

A generalized hybrid system $\mathcal{H}$ is said to be *class* $L(\theta)$ if any maximal solution to $\mathcal{H}$ is of class $L(\theta)$.

Definition 4.13 (Generalized Hybrid Systems of Class $H(\theta)$)**.** Let $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ be a generalized hybrid system and θ be a positive real number.

A generalized hybrid arc x is said to be of *class* $H(\theta)$ if

$$K := \sup_k \operatorname{dom} x > 0$$

and the inequality

$$\inf T(k+1, x) - \inf T(k, x) \leq \theta$$

is satisfied for $j \in \{0, 1, \ldots, K-1\}$.

A generalized hybrid system $\mathcal{H}$ is said to be *class* $H(\theta)$ if any maximal solution to $\mathcal{H}$ is of class $H(\theta)$. .

On Additional Artificial Solutions

The advantage of Definition 4.2 and Definition 4.8 is, in particular, that we can avoid the meaningless "frozen" solutions shown in the example of two bouncing balls connected by a spring given in the section 4.1. To see this let us again consider the interconnection (4.1)-(4.1) as one hybrid dynamical system of the form (4.7) with $n = 2$, ${}^1N = {}^2N = 2$, the same sets ${}^iC$ and ${}^iD$ and $U = \emptyset$. The functions ${}^if$ and ${}^ig$ remain also unchanged. The sets $I_{\mathcal{C}}(x)$ and $I_{\mathcal{D}}(x)$ for the interconnection are given by

$$\begin{aligned}
I_{\mathcal{C}}(x) &= \{1,2\}, \quad I_{\mathcal{D}}(x) = \emptyset && \text{if } {}^1x_1 > 0,\ {}^2x_1 > 0,\\
I_{\mathcal{C}}(x) &= \{1,2\}, \quad I_{\mathcal{D}}(x) = \{1\} && \text{if } {}^1x_1 = 0,\ {}^2x_1 > 0,\\
I_{\mathcal{C}}(x) &= \{1,2\}, \quad I_{\mathcal{D}}(x) = \{2\} && \text{if } {}^1x_1 > 0,\ {}^2x_1 = 0,\\
I_{\mathcal{C}}(x) &= \{1,2\}, \quad I_{\mathcal{D}}(x) = \{1,2\} && \text{if } {}^1x_1 = 0,\ {}^2x_1 = 0.
\end{aligned}$$

Consider the same initial conditions ${}^1x_1(0) = {}^1x_2(0) = 0$, ${}^2x_1(0) = h$, ${}^2x_2(0) = v$. Now observe that the hybrid arc (4.5) is not a solution to the whole system (4.7) with these data, because it corresponds to $I_{\mathcal{C}} = \{1,2\}$, $I_{\mathcal{D}} = \{1\}$, i.e., the second subsystem is not allowed to jump. From this we see that our approach naturally avoids the additional artificial solutions discussed above.

The first arcs of solution to our example corresponding to the continuous flow up to the first jump with initial conditions ${}^1x_1(0) = {}^1x_2(0) = 0$, ${}^2x_1(0) = h$, ${}^2x_2(0) = v$ is

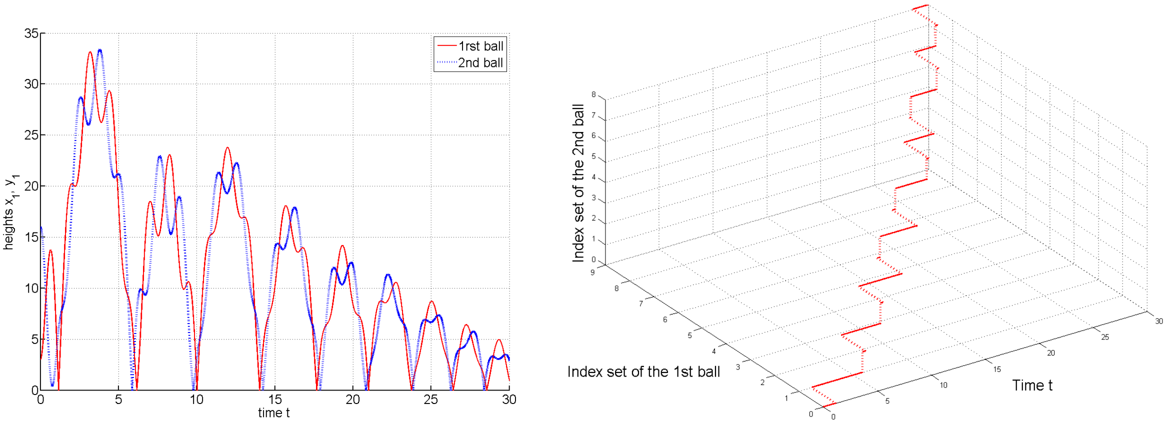


FIGURE 4.2: A solution of two bouncing ball with its corresponding generalized hybrid time domain.

given by

$$\begin{aligned}
{}^1x_1(t,0) &= -\frac{1}{2\mu}e^{-2\mu t}-\frac{1}{12}\gamma\mu t^4+\frac{1}{6}\mu v t^3-\frac{1}{2}\gamma t^2-t+\frac{1}{2\mu},\\
{}^1x_2(t,0) &= e^{-2\mu t}-\frac{1}{3}\gamma\mu t^3+\frac{1}{2}\mu v t^2-\gamma t-1,\\
{}^2x_1(t,0) &= \frac{1}{2\mu}e^{-2\mu t}+\frac{1}{12}\gamma\mu t^4-\frac{1}{6}\mu v t^3-\frac{1}{2}\gamma t^2+(v+1)t\\
&\quad +h-\frac{1}{2\mu},\\
{}^2x_2(t,0) &= -e^{-2\mu t}+\frac{1}{3}\gamma\mu t^3-\frac{1}{2}\mu v t^2-\gamma t+(v+1).
\end{aligned}$$

Further arcs can be calculated iteratively. A simulated solution and the corresponding generalized hybrid time domain are shown in Figure 4.2.

4.4 Stability

We are going to introduce different stability notions for interconnected hybrid systems and showing the relation between them. Moreover, we give a more general formulation of input-to-state stability (ISS) than those used in the literature, e.g., in [63, 66–68]. For simplicity, let us assume throughout this work that each subsystem ${}^i\mathcal{H}$ is complete, and its initial condition satisfies ${}^ix(0) = {}^i\xi \in {}^i\mathcal{C} \cup {}^i\mathcal{D}$.

Through the end of this chapter, for any $i \in \mathbb{N}_n$, let each ${}^i\mathcal{A} \subset {}^i\mathcal{X}$ be a non-empty compact set and

$$\mathcal{A} := {}^1\mathcal{A} \times \ldots \times {}^n\mathcal{A}. \tag{4.10}$$

It is obvious to see that $\mathcal{A}$ is a non-empty compact subset of $\mathcal{X}$.

Generally a hybrid system $\mathcal{H}$ of the form (4.7) also has an input u which not necessarily equal to zero. Even when we consider a decomposition of one large hybrid system without inputs, it turns that the subsystem has inputs from other subsystems. In general a decomposed system contains both internal inputs and external inputs. We are interested in a stability notion taking such inputs into account. Input-to-state stability provides a natural framework in which to formulate notions of stability with respect to input perturbations [69]. The essential supremum norm of a measurable function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is denoted by

$$\|\phi\|_\infty := \text{ess sup}\left\{\|\phi(s)\|, s \in \mathbb{R}_{\geq 0}\right\}.$$

Throughout this chapter, we require the assumption of measurable essentially bounded input u.

Definition 4.14 (Class $\mathcal{KL}^n$)**.** Let a positive $n \in \mathbb{N}$ be fixed. A function

$$\beta : \mathbb{R}_{\geq 0} \times \underbrace{\mathbb{R}_{\geq 0} \times \cdots \times \mathbb{R}_{\geq 0}}_{n} \to \mathbb{R}_{\geq 0}$$

is said to be of class $\mathcal{KL}^n$ if for fixed non-negative numbers r_i for $i = 1 \ldots n$ it satisfies $\beta(\cdot, r_1, \ldots, r_n) \in \mathcal{K}$, and it additionally holds for any $i \in \{1, 2, \ldots, n\}$ and fixed $s \geq 0$ that $\beta(s, r_1, r_2, \ldots, r_{i-1}, \cdot, r_{i+1}, \ldots, r_n) \in \mathcal{L}$.

Alternatively we can say that a function belonging to $\mathcal{KL}^n$ if it belongs to class $\mathcal{K}$ wrt the first argument while the other argument are all fixed; it belongs to class $\mathcal{L}$ wrt the second argument while the other are fixed; it belongs to class $\mathcal{L}$ wrt the third while the other are fixed; and so on.

Definition 4.15 (Input-to-State Stability)**.** A complete generalized hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ is said to be *input-to-state stable (ISS) wrt a non-empty compact set* $\mathcal{A} \subset \mathcal{X}$ if there exist $\beta \in \mathcal{KL}^{n+1}$ and $\gamma \in \mathcal{K}_\infty$ such that any solution pair (x, u) with $x(0, 0) = \xi \in \mathcal{X}$ satisfy

$$\left\|x(t, \overline{k})\right\|_{\mathcal{A}} \leq \max\left\{\beta(\|\xi\|_{\mathcal{A}}, t, \overline{k}),\ \gamma(\|u\|_\infty)\right\}, \quad \forall (t, \overline{k}) \in \text{dom } x. \tag{4.11}$$

The function γ is called an *ISS gain* for $\mathcal{H}$.

Particularly by the properties of both functions β and γ in (4.11), any solution to an ISS hybrid system wrt $\mathcal{A}$ is bounded and does not diverge from $\mathcal{A}$ in the long-run. Note that the abbreviation ISS stands for either input-to-state stability or input-to-state stable which depends on context.

Remark 4.3. By using the inequalities

$$\max_{i=1,\ldots,n}\{x_i\} \leq \sum_{i=1}^n x_i \leq n \max_{i=1,\ldots,n}\{x_i\},$$

the inequality (4.11) can be replaced by the form:

$$\left\|x(t, \overline{k})\right\|_{\mathcal{A}} \leq \tilde{\beta}(\|\xi\|_{\mathcal{A}}, t, \overline{k}) + \tilde{\gamma}(\|u\|_\infty), \quad \forall (t, \overline{k}) \in \text{dom } x,$$

where the function $\tilde{\beta} \in \mathcal{K}^{n+1}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ generally differ from β and γ in (4.11). We need both the maximization form and the summation form to deal with ISS properties.

Additional stability notions proposed in [66] are also considered in this work. Since our definition of ISS is given in a more general way, some characterizations of the following stability notions for generalized hybrid systems need further investigations.

Definition 4.16 (Global Stability)**.** A generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ is *globally stable wrt a non-empty compact set* $\mathcal{A} \subset \mathcal{X}$ if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ such that any solution pair (x, u) with $x(0,0) = \xi \in \mathcal{X}$ satisfies

$$\left\|x(t,\overline{k})\right\|_{\mathcal{A}} \leq \max\left\{\sigma(\|\xi\|_{\mathcal{A}}),\ \gamma(\|u\|_\infty)\right\}, \quad \forall (t,\overline{k}) \in \operatorname{dom} x.$$

Definition 4.17 (0-Input Stability)**.** A complete generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ is *0-input stable wrt a non-empty compact set* $\mathcal{A} \subset \mathcal{X}$ if there exists $\beta \in \mathcal{KL}^{n+1}$ such that any solution pair (x, u) with $x(0,0) = \xi \in \mathcal{X}$ satisfies

$$\left\|x(t,\overline{k})\right\|_{\mathcal{A}} \leq \beta(\|\xi\|_{\mathcal{A}}, t, \overline{k}) \quad \forall (t,\overline{k}) \in \operatorname{dom} x.$$

Theorem 4.18. *If a complete generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ is ISS wrt a non-empty compact set $\mathcal{A} \subset \mathcal{X}$, then it is globally stable and 0-input stable wrt $\mathcal{A}$.*

Proof. ISS leads to global stability by taking $\sigma(\|\xi\|_{\mathcal{A}}) := \beta(\|\xi\|_{\mathcal{A}}, 0, \ldots, 0)$. Moreover, it leads to 0-input stability under a consideration of $u = 0$. □

Definition 4.19 (Asymptotic Gain Property)**.** A hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ has *asymptotic gain property wrt a non-empty compact set* $\mathcal{A} \subset \mathcal{X}$ if there exists $\gamma \in \mathcal{K}_\infty$ such that any solution pair (x, u) with $x(0,0) = \xi \in \mathcal{X}$ is bounded and satisfies

$$\limsup_{(t_k,\overline{k}) \in \operatorname{dom} x,\ t_k + \varsigma(\overline{k}) \to \infty} \left\|x(t_k,\overline{k})\right\|_{\mathcal{A}} \leq \gamma(\|u\|_\infty).$$

Intentionally this work skips the characterizations of the extended notion of input-to-state stability for interconnected hybrid systems. However, we expect that some results like the contribution in [66] may be achieved in a similar way, which need further study.

4.5 ISS-Lyapunov Theorems

As we can see from stability conditions in the previous chapter, hybrid Lyapunov theorems are useful for the investigation of stability for hybrid dynamical systems without inputs. In this section, we provide conditions to guarantee ISS property for a generalized hybrid system $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ of the form (4.7) by using some extensions of our results in Chapter 3.

Let $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ be a complete generalized hybrid system. Due to difficulties of its generalized hybrid time domains, let us firstly start from stability for each subsystem ${}^i\mathcal{H}$ for all $i \in \mathbb{N}_n$. Suppose that each subsystem ${}^i\mathcal{H}$ is complete. To formulate stability notions for subsystem ${}^i\mathcal{H}$, we necessarily consider both external input u and additional internal inputs from other subsystems ${}^j\mathcal{H}$ for $j \in \mathbb{N}_n \backslash \{i\}$. Therefore, each subsystem ${}^i\mathcal{H}$ has the total input ${}^iv$ denoted by

$$ {}^iv := ({}^1x, \dots, {}^{i-1}x, {}^{i+1}x, \dots, {}^nx, u), \tag{4.12} $$

together with the assumption that the inequality $\left\|{}^iv\right\|_\infty < \infty$ is satisfied.

The subsystem ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$ if there exist ${}^i\beta \in \mathcal{KL}^{n+1}$, and ${}^i\gamma \in \mathcal{K}_\infty$ such that any solution pair $({}^ix, {}^iv)$ with ${}^ix(0,0) = {}^i\xi \in {}^i\mathcal{X}$ satisfies

$$ \left\|{}^ix(t,\overline{k})\right\|_{{}^i\mathcal{A}} \le \max\left\{{}^i\beta(\left\|{}^i\xi\right\|_{{}^i\mathcal{A}}, t, \overline{k}),\ {}^i\gamma(\left\|{}^iv\right\|_\infty)\right\}, \quad \forall(t,\overline{k}) \in \operatorname{dom} {}^ix, \tag{4.13} $$

Remark 4.4. Consider the case of an interconnection $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form (4.7) consisting of only one subsystem, i.e., $x \in \mathcal{X}$ is partitioned into 1 part. It is obvious to see that $\mathcal{H}$ has only one input, which is u, and the stability notion for the subsystem is identical to the stability notion of the interconnection.

For a convenience reason, let us introduce the following multi-indices.

Definition 4.20 (Multi-index Representations). A multi-index $(q_1, q_2, \dots, q_n) \in \mathbb{N}^n$ is represented by $\overline{1}_i$ if $q_i = 1$, and it is represented by $\overline{0}_i$ if $q_i = 0$. For a multi-index $\overline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, denoted by

$$ \begin{aligned} \overline{k} = \overline{1}_i &\iff k_i = 1, \\ \overline{k} = \overline{0}_i &\iff k_i = 0. \end{aligned} $$

Remark 4.5. Either $\overline{1}_i$ or $\overline{0}_i$ represents a multi-indexes, which is unnecessary unique, satisfying the certain conditions. That is not the case of comparison with $\overline{1}_i$ or $\overline{0}_i$. For instance, given η a positive integer, $\eta \cdot \overline{1}_1$ is not equal to $\underbrace{\overline{1}_1 + \ldots + \overline{1}_1}_{\eta}$ in general.

Secondly, we need to generally parameterize solutions to ${}^i\mathcal{H}$ in order to point out the time when trajectories start and stop the flows.

Without loss of generality, assume that a maximal solution $({}^ix, {}^iv)$ to the subsystem ${}^i\mathcal{H}$ starts by flow, and its corresponding generalized hybrid time domain $\operatorname{dom} {}^ix$ is a union of elements of the form $[t_k, t_{k+1}] \times \left\{{}^1k\right\} \times \ldots \times \left\{{}^nk\right\}$ where $k = {}^1k + \ldots + {}^nk$.

The hybrid arc ${}^ix$ has a first flow from the point

$$ {}^ix(t_0, 0) = {}^ix(0,0) \quad \text{to} \quad {}^ix(t_{\varsigma(\overline{0}_i)}, \overline{0}_i). $$

Note that the jump counter of ${}^ix$ is still zero since there is no jump occurring in ${}^i\mathcal{H}$, but the other jump counters are possibly equal to some positive integers.

Let $\{\eta_j\}$ be a sequence of natural numbers including zero, which $\eta_0 = 0$. Suppose additionally that, for any integer $j \ge 1$, the subsystem ${}^i\mathcal{H}$ exhibits η_j jumps between the j^{th}-flow and the $(j+1)^{\text{th}}$-flow.

Therefore we have

$$ t_{\varsigma(\overline{0}_i + \eta_1\overline{1}_i)} = t_{\varsigma(\overline{0}_i)}, $$

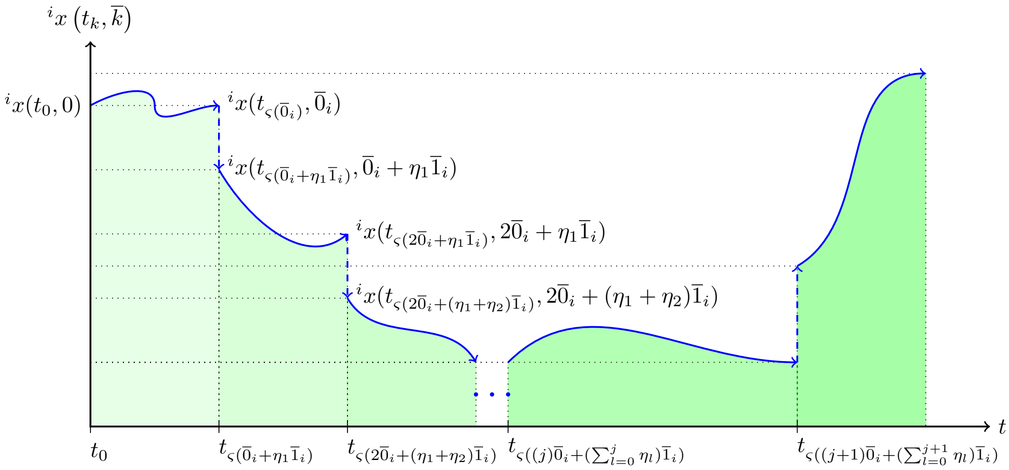


FIGURE 4.3: Trajectory ${}^i x$ with its generalized parameters.

and the hybrid arc ${}^i x$ has the second flow from the point

$$ {}^i x(t_{\varsigma(\overline{0}_i+\eta_1\overline{1}_i)}, \overline{0}_i + \eta_1\overline{1}_i) \quad \text{to} \quad {}^i x(t_{\varsigma(2\overline{0}_i+\eta_1\overline{1}_i)}, 2\overline{0}_i + \eta_1\overline{1}_i). $$

According to this manner, for any integer $j \geq 1$, the hybrid arc ${}^i x$ has the j-th flow from the point

$$ {}^i x\left(t_{\varsigma((j-1)\overline{0}_i+(\sum_{l=0}^{j-1}\eta_l)\overline{1}_i)}, (j-1)\overline{0}_i + \left(\sum_{l=0}^{j-1}\eta_l\right)\overline{1}_i \right) $$

to the point

$$ {}^i x\left(t_{\varsigma(j\overline{0}_i+(\sum_{l=0}^{j-1}\eta_l)\overline{1}_i)}, j\overline{0}_i + \left(\sum_{l=0}^{j-1}\eta_l\right)\overline{1}_i \right), $$

and

$$ t_{\varsigma(j\overline{0}_i+(\sum_{l=0}^{j-1}\eta_l)\overline{1}_i)} = t_{\varsigma(j\overline{0}_i+(\sum_{l=0}^{j}\eta_l)\overline{1}_i)}. $$

Figure 4.3 illustrates the hybrid arc ${}^i x$ with its generalized parameters.

For brevity, for any integer $j \geq 1$, let us denote

$$ \overline{\kappa}(j) := j\overline{0}_i + \left(\sum_{r=0}^{j}\eta_r\right)\overline{1}_i, \quad \overline{\nu}(j) := \overline{\kappa}(j) + \overline{0}_i, \tag{4.14} $$

$$ \kappa(j) := \varsigma(\overline{\kappa}(j)), \quad \text{and} \quad \nu(j) := \varsigma(\overline{\nu}(j)). \tag{4.15} $$

Note that the following equations

$$ t_{\kappa(0)} = t_0 = 0 \quad \text{and} \quad t_{\kappa(j)} = t_{\nu(j-1)} \tag{4.16} $$

are satisfied for any integer $j \geq 1$.

In addition, we provide Figure 4.4 as a redrawing of Figure 4.3 with the notation of κ and ν.

Thirdly, let us provide ISS-Lyapunov candidate functions, ISS-Lyapunov functions

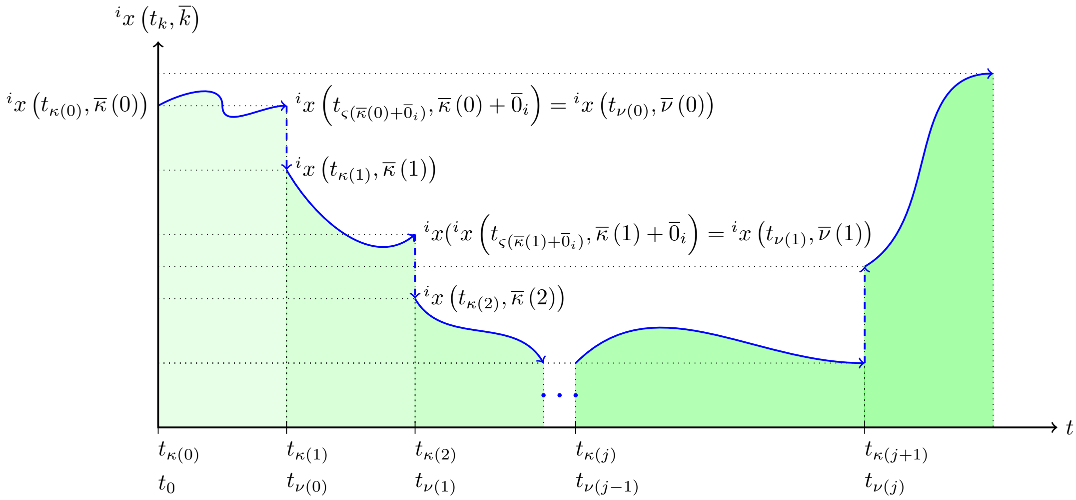


FIGURE 4.4: Trajectory ^{i}x with its generalized parameters in term of κ and ν.

and relaxed ISS-Lyapunov functions which are essentially important for stability investigation. They are an extension of hybrid Lyapunov functions from the previous chapter.

Definition 4.21 (ISS-Lyapunov Functions)**.** Given a complete generalized hybrid system $\mathcal{H}=\left\{^{i}\mathcal{H}\right\}_{i=1}^{n}$, ^{i}v an admissible input and a compact set $\mathcal{A}={}^{1}\mathcal{A}\times\ldots\times{}^{n}\mathcal{A}$ which $\emptyset\neq{}^{i}\mathcal{A}\subset{}^{i}\mathcal{X}\subset\mathbb{R}^{^{i}N}$, a function $^{i}V\;:\;\mathbb{R}^{^{i}N}\to\mathbb{R}_{\geq0}$ is called an *ISS-Lyapunov candidate function for* $(^{i}\mathcal{H},{}^{i}\mathcal{A})$ if it is globally Lipschitz and there exist $^{i}\psi_{1}$, $^{i}\psi_{2}\in\mathcal{K}_{\infty}$ such that it satisfies

$$^{i}\psi_{1}\left(\left\|^{i}x\right\|_{^{i}\mathcal{A}}\right)\leq{}^{i}V(^{i}x)\leq{}^{i}\psi_{2}\left(\left\|^{i}x\right\|_{^{i}\mathcal{A}}\right)\quad\text{for }{}^{i}x\in{}^{i}\mathcal{X}.\tag{4.17}$$

An ISS-Lyapunov candidate function ^{i}V wrt $(^{i}\mathcal{H},{}^{i}\mathcal{A})$ is called an *ISS-Lyapunov function for* $(^{i}\mathcal{H},{}^{i}\mathcal{A})$ if there exists $^{i}\vartheta\in\mathcal{K}_{\infty}$, and $^{i}\varphi\in\mathcal{P}$, such that it satisfies

$$^{i}V(^{i}x)\geq{}^{i}\vartheta(\left\|^{i}v\right\|_{\infty})\implies\begin{cases}\left\langle\nabla{}^{i}V(^{i}x),{}^{i}f(x,u)\right\rangle\leq-{}^{i}\varphi(^{i}V(^{i}x)) & \text{for }x\in{}^{i}\mathcal{C}\setminus\mathcal{A},\\ ^{i}V(^{i}g(x,u))-{}^{i}V(^{i}x)\leq-{}^{i}\varphi(^{i}V(^{i}x)) & \text{for }x\in{}^{i}\mathcal{D}\setminus\mathcal{A}.\end{cases}\tag{4.18}$$

Moreover, the function ^{i}V is called a *relaxed ISS-Lyapunov function for* $(^{i}\mathcal{H},{}^{i}\mathcal{A})$ if it additionally satisfies

$$^{i}V(^{i}x)\geq{}^{i}\vartheta(\left\|^{i}v\right\|_{\infty})\implies\begin{cases}\left\langle\nabla{}^{i}V(^{i}x),{}^{i}f(x,u)\right\rangle\leq0 & \text{for }x\in{}^{i}\mathcal{C}\setminus\mathcal{A},\\ ^{i}V(^{i}g(x,u))-{}^{i}V(^{i}x)\leq0 & \text{for }x\in{}^{i}\mathcal{D}\setminus\mathcal{A}.\end{cases}\tag{4.19}$$

In addition, a function $^{i}\vartheta$ is called an *ISS-Lyapunov gain* wrt a (relaxed) ISS-Lyapunov function ^{i}V.

Here, we are ready to provide results on stability of subsystem $^{i}\mathcal{H}$. The following theorems give sufficient conditions to guarantee ISS property of subsystem $^{i}\mathcal{H}$. The conditions are provided in such same manner as conditions in hybrid Lyapunov theorems from the previous chapter with extension for ISS.

Theorem 4.22 (ISS-Lyapunov Theorem)**.** *If there exists an ISS-Lyapunov function ${}^iV$ wrt $({}^i\mathcal{H}, {}^i\mathcal{A})$, then the system ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$.*

Proof. Suppose that ${}^iV$ is a Lyapunov function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ satisfying (4.17)–(4.18). We are going to prove ISS for the system by direct constructions of ${}^i\beta$ and ${}^i\gamma$ in the inequality (4.13). Suppose that ${}^ix(0,0) = {}^i\xi$. Define

$$R := \left\{x \in \mathbb{R}^n : \ {}^iV({}^ix) < {}^i\vartheta(\left\|{}^iv\right\|_\infty)\right\}. \tag{4.20}$$

Without loss of generality, let us suppose that $x(0,0) \in {}^i\mathcal{C}$.

Consider the first case: $x(0,0) \in R' := ({}^i\mathcal{C} \cup {}^i\mathcal{D}) \setminus R$, i.e.,

$${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) \geq {}^i\vartheta(\left\|{}^iv\right\|_\infty).$$

Due to the condition (4.18), it follows that

$$\left\langle \nabla\, {}^iV({}^ix), {}^if(x,u)\right\rangle = {}^i\dot{V}({}^ix) \leq -\,{}^i\varphi({}^iV({}^ix)) < 0 \text{ for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \tag{4.21}$$

and

$${}^iV({}^ig(x,u)) < {}^iV({}^ix) \text{ for } x \in {}^i\mathcal{D} \setminus \mathcal{A}. \tag{4.22}$$

In a similar way of the proof of Theorem 3.45, we obtain

$$\int_{{}^iV({}^ix(t_{\kappa(j)},\overline{\kappa}(j)))}^{{}^iV({}^ix(t_{\kappa(j+1)},\overline{\kappa}(j+1))} \frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\kappa(j+1)} - t_{\kappa(j)}\right) < 0.$$

Since ${}^i\varphi \in \mathcal{P}$, it follows that

$${}^iV({}^ix(t_{\kappa(j+1)}, \overline{\kappa}(j+1)) < {}^iV({}^ix(t_{\kappa(j)}, \overline{\kappa}(j))).$$

Therefore we obtain that the sequence $\left\{{}^iV({}^ix(t_{\kappa(j)}, \overline{\kappa}(j))\right\}$ and $\left\{{}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}\,(j)\right)\right)\right\}$ are decreasing as $j \to \infty$ and bounded from below by ${}^i\vartheta(\left\|{}^iv\right\|_\infty)$ while $x \in R'$. Note that the following inequalities are satisfied

$${}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}\,(j)\right)\right) < {}^iV\left({}^ix\left(t_k, \overline{k}\right)\right) < {}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\,(j)\right)\right)$$

for any

$$\left(t_k, \overline{k}\right) \in \left\{(a, \overline{a}) : \ \left(t_{\nu(j)}, \overline{\nu}\,(j)\right) \prec (a, \overline{a}) \prec \left(t_{\kappa(j)}, \overline{\kappa}\,(j)\right)\right\},$$

and there exists ${}^i\psi_1 \in \mathcal{K}_\infty$ such that

$$\left\|{}^ix\right\|_{{}^i\mathcal{A}} \leq {}^i\psi_1{}^{-1}\left({}^iV({}^ix)\right) \quad \text{for all } {}^ix \in {}^i\mathcal{X}.$$

While $x \in R'$, we are going to show that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to ${}^i\vartheta(\left\|{}^iv\right\|_\infty)$. Suppose a contradiction that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right) \to c > {}^i\vartheta(\left\|{}^iv\right\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$. Let us denote positive constants K and S corresponding to $\left(t_k, \overline{k}\right) \in \operatorname{dom}\, {}^ix$ by

$$\tilde{K} := -\tilde{K}(t_k, \overline{k}) := \sup_{\left(t_{\kappa(0)}, \overline{\kappa}(0)\right) \preceq (a, \overline{a}) \preceq (t_k, \overline{k})} {}^iV'({}^ix(a, \overline{a})),$$

$$\tilde{S} := \tilde{S}(t_k, \overline{k}) := \sup_{j \in E(t_k, \overline{k})} \left[{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\,(j)\right)\right) - {}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}\,(j)\right)\right)\right],$$

$$K := -\tilde{K}, \quad \text{and} \quad S := -\tilde{S},$$

where $E(t_k, \overline{k}) := \{j \in \mathbb{N} : (t_{\kappa(j)}, \overline{\kappa}(j)) \preceq (t_k, \overline{k})\}$. For any $(t_k, \overline{k}) \in \operatorname{dom} {}^i x$, there exists $J \in \mathbb{N}$ such that it holds

$$\begin{aligned} {}^iV\left({}^ix\left(t_k, \overline{k}\right)\right) &= {}^iV\left({}^ix\left(t_{\kappa(0)}, \overline{\kappa}(0)\right)\right) + \int_{t_{\kappa(0)}}^{t_k} {}^iV'\left({}^ix(s, \overline{k})\right)\, \mathrm{d}s \\ &\quad + \sum_{j=0}^{J} \left[{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right) - {}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}(j)\right)\right)\right] \\ &\le {}^iV\left({}^ix\left(t_{\kappa(0)}, \overline{\kappa}(0)\right)\right) - K(t_k, \overline{k}) \cdot \left(t_k - t_{\kappa(0)}\right) - S(t_k, \overline{k}) \cdot J \\ &= {}^iV\left({}^ix\left(t_{\kappa(0)}, \overline{\kappa}(0)\right)\right) + K \cdot t_{\kappa(0)} - K \cdot t_k - S \cdot J, \end{aligned}$$

which eventually becomes negative as $t_k + \varsigma(\overline{k}) \to \infty$. So we obtain a contradiction here.

Since while $x \in R'$, ${}^iV({}^ix(t_k, \overline{k})) \to {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$, let us suppose that there exists $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \in \operatorname{dom} {}^ix$ such that it satisfies

$${}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) \ge {}^i\vartheta(\|{}^iv\|_\infty),$$

and

$${}^iV({}^ix(t_k, \overline{k}))) \ge {}^i\vartheta(\|{}^iv\|_\infty) \implies (t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star).$$

Define a function $\tilde{\vartheta} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$\tilde{\vartheta}(s) := \max\left\{ \sup_{0 \le r \le {}^i\vartheta(s)} {}^iV({}^ig(r)), {}^i\vartheta(s) \right\} \qquad \text{for all } s \ge 0.$$

Together with the condition (4.17), there exists an ISS gain

$${}^i\gamma := {}^i\psi_1^{-1} \circ \tilde{\vartheta} \in \mathcal{K}_\infty$$

such that it satisfies

$$\left\|{}^ix\left(t_k, \overline{k}\right)\right\|_{\mathcal{A}} \le {}^i\psi_1^{-1}({}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)) \le {}^i\psi_1^{-1}(\tilde{\vartheta}(\|{}^iv\|_\infty)) = {}^i\gamma(\|{}^iv\|_\infty) \tag{4.23}$$

for $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \prec (t_k, \overline{k})$.

Here we are going to construct a function $\alpha : \mathbb{R}^{n+2} \to \mathbb{R}_{\ge 0}$, which provide an upper bound for ${}^iV({}^ix(t_k, \overline{k}))$ when $(t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star)$. Define

$$\alpha(s, t_0, 0) := {}^iV({}^i\xi) + {}^i\varphi({}^iV({}^i\xi)) \quad \text{for any } s > 0,$$

and $\alpha(s, t_{\nu(0)}, \overline{\nu}(0)) := y_0$ for any $s > 0$, where y_0 is a solution to the integral equation

$$\int_{{}^iV({}^i\xi)}^{y_0} \frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\nu(0)} - t_0\right).$$

For $j > 0$ and it holds the following inequality

$$t_{\nu(j-1)} < t_{\varsigma(\overline{k}^\star)}, \tag{4.24}$$

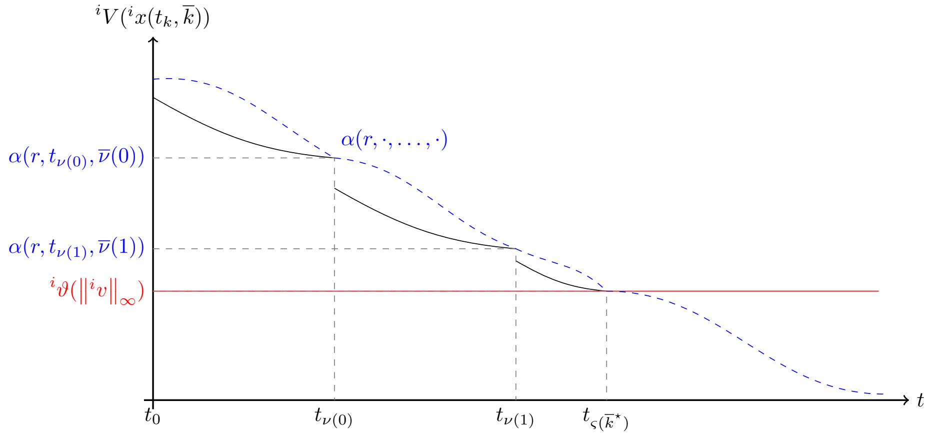


FIGURE 4.5: To the construction of function α providing an upper bound for ${}^iV$ before time reaching $t_{\varsigma(\overline{k}^\star)}$ in Theorem 4.22.

we define $\alpha(s,t_{\nu(j)},\overline{\nu}(j)) := y_j$ for any $s>0$, where y_j is a solution to the integral equation

$$\int_{a_j}^{y_j}\frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\nu(j)}-t_{\nu(j-1)}\right),$$

and

$$a_j = \alpha(s,t_{\nu(j-1)},\overline{\nu}(j-1)) - {}^i\varphi(\alpha(s,t_{\nu(j-1)},\overline{\nu}(j-1))).$$

Suppose that $j^\star$ is the greatest natural number such that it satisfies the inequality (4.24). For any positive r, we define $\alpha(r,\cdot,\dots,\cdot)$ on each interval $\left(t_{\nu(j-1)},t_{\nu(j)}\right)$ and on the interval $\left(t_{\nu(j^\star-1)},t_{\varsigma(\overline{k}^\star)}\right)$ as an arbitrary continuous decreasing function, which lies above ${}^iV$. Additionally on the interval $\left(t_{\varsigma(\overline{k}^\star)},\infty\right)$, we define $\alpha(s,\cdot,\dots,\cdot)$ as an arbitrary continuous decreasing function, which tends to zero. See Figure 4.5. By this construction, for any $(t_k,\overline{k})\in\operatorname{dom}{}^ix$ and $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, it holds that

$${}^iV({}^ix(t_k,\overline{k})) \le \alpha({}^iV({}^i\xi),t_k,\overline{k}) \le \alpha({}^i\psi_2(\|{}^i\xi\|_{\mathcal{A}}),t_k,\overline{k}),$$

where $\alpha:\mathbb{R}^{n+2}\to\mathbb{R}_{\ge 0}$ is continuous wrt the second argument, third argument, and so on; $\alpha(0,t_k,\overline{k}):=0$ for all $(t_k,\overline{k})\in\operatorname{dom}{}^ix$; $\alpha(s,\cdot,\dots,\cdot)$ is decreasing for all positive s; it holds that $\alpha(s,t_k,\overline{k})\to 0$ as $t_k+\varsigma(\overline{k})\to\infty$. So there exists ${}^i\beta\in\mathcal{KL}^{n+1}$ such that it satisfies the following inequality

$$\|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}} \le {}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right) \tag{4.25}$$

for $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, where ${}^i\beta(s,t_k,\overline{k}) = {}^i\psi_1^{-1}\left(\alpha\left({}^i\psi_2(s),t_k,\overline{k}\right)\right)$. By a combination of the inequalities (4.23) and (4.25), we finally conclude that

$$\|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}} \le \max\left\{{}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right),{}^i\gamma(\|{}^iv\|_\infty)\right\} \tag{4.26}$$

for any $(t_k,\overline{k})\in\operatorname{dom}{}^ix$.

For the case: $x(0,0)\in R$, i.e.,

$${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) < {}^i\vartheta(\|{}^iv\|_\infty).$$

If the trajectory x never leaves the set R, there exists an ISS gain

$$\gamma := {}^i\psi_1{}^{-1} \circ {}^i\vartheta \in \mathcal{K}_\infty$$

such that

$$\|{}^ix(t_k, \overline{k})\|_{{}^i\mathcal{A}} \leq {}^i\psi_1{}^{-1}({}^iV({}^ix(t_k, \overline{k}))) \leq {}^i\psi_1{}^{-1}({}^i\vartheta(\|{}^iv\|_\infty) = {}^i\gamma(\|{}^iv\|_\infty)$$

for all $(t_k, \overline{k}) \in \operatorname{dom} {}^ix$.

Otherwise, suppose that there exists $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \in \operatorname{dom} {}^ix$ such that it satisfies

$${}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) \geq {}^i\vartheta(\|{}^iv\|_\infty),$$

and

$${}^iV({}^ix(t_k, \overline{k}))) \geq {}^i\vartheta(\|{}^iv\|_\infty) \implies (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \preceq (t_k, \overline{k}).$$

A function $\alpha \in \mathcal{KL}^{n+1}$ providing an upper bound of ${}^iV({}^ix(t_k, \overline{k}))$ for any $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \preceq (t_k, \overline{k})$ may be differently defined, but it still uses the same concept of construction. For this case, the rest of proof to show existence of ${}^i\beta \in \mathcal{KL}^{n+1}$ satisfying the inequality (4.26) is intentionally omitted due to its similarity. □

Remark 4.6. According to the condition (4.18), the trajectory can leave the set R by jumping only. However, it will eventually return to the set R. In case that the initial starting point lies in the set R, the function α providing an upper bound of ${}^iV$ may be differently defined, but it still uses the same concept of construction. In addition to the proof, we assume that the trajectory reach to the set R by flowing. Note that it can also reach to the set R by jumping, but such case is skipped here since it is similar to the proof of Theorem 4.24.

Remark 4.7. In the proof of Theorem 4.22, the construction of α is given by an assumption that ${}^i\mathcal{H}$ is absolutely hybrid system, i.e., $t_{\kappa(j)}$ and $t_{\nu(j)}$ grow to infinity. The proof does not explicitly show all of the possibility. However, the concept of construction is just slightly different in case of either $t_{\kappa(j)} < \infty$ or $t_{\nu(j)} < \infty$ as $j \to \infty$. See the proof of Theorem 4.23 and Theorem 4.24 for the case of finite $t_{\kappa(j)}$ and $t_{\nu(j)}$.

Theorem 4.23 (Relaxed ISS-Lyapunov Theorem). *If there exists a relaxed Lyapunov function ${}^iV$ for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^i\vartheta \in \mathcal{K}_\infty$ and ${}^i\varphi \in \mathcal{P}$ it satisfies*

$${}^iV({}^ix) \geq {}^i\vartheta\left(\|{}^iv\|_\infty\right) \implies \begin{cases} \langle \nabla {}^iV({}^ix), {}^if(x,u) \rangle < -{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A} \\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) \leq 0 & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.27}$$

and ${}^i\mathcal{H}$ is not an eventually discrete system, then ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$.

Proof. Assume that V satisfies

$${}^iV({}^ix) \geq {}^i\vartheta\left(\|{}^iv\|_\infty\right) \implies \begin{cases} \langle \nabla {}^iV({}^ix), {}^if(x,u) \rangle < -{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A} \\ {}^iV({}^ig(x,u)) = {}^iV({}^ix) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}. \end{cases} \tag{4.28}$$

Define the set R exactly as (4.20). Let us suppose that that $x(0,0) \in R' := ({}^i\mathcal{C} \cup {}^i\mathcal{D}) \setminus R$ and ${}^ix(0,0) = {}^i\xi$. That is ${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) \geq {}^i\vartheta(\|{}^iv\|_\infty)$. With no loss of generality, we further assume that $x(0,0) \in {}^i\mathcal{C} \cap R'$. Similarly to the proof of Theorem 3.45, we obtain

$$\int_{{}^iV\left({}^ix\left(t_{\kappa(j)},\overline{\kappa}(j)\right)\right)}^{{}^iV\left({}^ix\left(t_{\kappa(j+1)},\overline{\kappa}(j+1)\right)\right)} \frac{\mathrm{d}s}{{}^i\varphi(s)} \leq -\left(t_{\kappa(j)} - t_{\kappa(j+1)}\right) < 0.$$

Since ${}^i\varphi \in \mathcal{P}$, it follows that ${}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}\left(j+1\right)\right)\right) < {}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)$. Consequently the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)\right\}$ is decreasing and bounded from below by ${}^i\vartheta\left(\|{}^iv\|_\infty\right)$ while $x \in R'$. Moreover, it holds

$${}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}\left(j\right)\right)\right) < {}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right),$$

which implies that the sequence $\left\{{}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}\left(j\right)\right)\right)\right\}$ is decreasing and bounded from below by ${}^i\vartheta\left(\|{}^iv\|_\infty\right)$ while $x \in R'$. In case that ${}^i\mathcal{H}$ is also not eventually continuous, i.e., $t_{\kappa(j)} \to \infty$ as $j \to \infty$, the proof may go along similarly to the proof of Theorem 4.22. Let us suppose that the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)\right\}$ is finite. Since the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)\right\}$ is decreasing, there exists $J \in \mathbb{N}$ such that it holds

$${}^iV\left({}^ix\left(t_{\kappa(J)}, \overline{\kappa}\left(J\right)\right)\right) = \inf\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)\right\}.$$

Note that there exists ${}^i\psi_1 \in \mathcal{K}_\infty$ such that

$$\|{}^ix\|_{{}^i\mathcal{A}} \leq {}^i\psi_1{}^{-1}\left({}^iV({}^ix)\right) \quad \text{for all } {}^ix \in {}^i\mathcal{X}.$$

While $x \in R'$, we are going to show that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to ${}^i\vartheta(\|{}^iv\|_\infty)$. Suppose for a contradiction that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to some positive number $c > {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$. Note that $t_{\kappa(j)} \to t_{\kappa(J)} < \infty$ as $j \to \infty$. Denote

$$\tilde{K} := \tilde{K}(t_k, \overline{k}) := \sup_{\left(t_{\kappa(J)},\overline{\kappa}(J)\right) \preceq (a,\overline{a}) \preceq (t_k,\overline{k})} {}^iV'({}^ix(a, \overline{a})),$$

and

$$K := -\tilde{K}.$$

For any $\left(t_k, \overline{k}\right) \succeq \left(t_{\kappa(J)}, \overline{\kappa}\left(J\right)\right)$, it holds

$$\begin{aligned}
{}^iV\left({}^ix\left(t_k, \overline{k}\right)\right) &= {}^iV\left({}^ix\left(t_{\kappa(J)}, \overline{\kappa}\left(J\right)\right)\right) + \int_{t_{\kappa(J)}}^{t_k} {}^iV'({}^ix(s, \overline{k}))\,\mathrm{d}s \\
&\leq {}^iV\left({}^ix\left(t_{\kappa(J)}, \overline{\kappa}\left(J\right)\right)\right) - K(t_k, \overline{k}) \cdot \left(t_k - t_{\kappa(J)}\right) \\
&= {}^iV\left({}^ix\left(t_{\kappa(J)}, \overline{\kappa}\left(J\right)\right)\right) + K \cdot t_{\kappa(J)} - K \cdot t_k,
\end{aligned}$$

which eventually becomes negative as $t_k + \varsigma(\overline{k}) \to \infty$. So we obtain a contradiction here.

Since while $x \in R'$, ${}^iV({}^ix(t_k, \overline{k})) \to {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$, let us suppose that there exists $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \in \operatorname{dom} {}^ix$ such that it satisfies

$${}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) \geq {}^i\vartheta(\|{}^iv\|_\infty),$$

and

$$ {}^iV({}^ix(t_k,\overline{k}))) \geq {}^i\vartheta(\|{}^iv\|_\infty) \implies (t_k,\overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)},\overline{k}^\star). $$

Define a function $\tilde{\gamma}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ by

$$ \tilde{\vartheta}(s) := \max\left\{\sup_{0\leq r\leq {}^i\vartheta(s)} {}^iV({}^ig(r)), {}^i\vartheta(s)\right\} \qquad \text{for all } s\geq 0. $$

Together with the condition (4.17), there exists an ISS gain ${}^i\gamma := {}^i\psi_1^{-1}\circ\tilde{\vartheta}\in\mathcal{K}_\infty$ such that it satisfies

$$ \left\|{}^ix\left(t_k,\overline{k}\right)\right\|_{\mathcal{A}} \leq {}^i\psi_1^{-1}({}^iV\left({}^ix\left(t_k,\overline{k}\right)\right)) \leq {}^i\psi_1^{-1}(\tilde{\vartheta}(\|{}^iv\|_\infty)) = {}^i\gamma(\|{}^iv\|_\infty) \tag{4.29} $$

for $(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)\prec(t_k,\overline{k})$.

Here we are going to construct a function $\alpha:\mathbb{R}^{n+2}\to\mathbb{R}_{\geq 0}$, which provide an upper bound for ${}^iV({}^ix(t_k,\overline{k}))$ when $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$. Define

$$ \alpha(r,t_0,0) := {}^iV({}^i\xi) + {}^i\varphi({}^iV({}^i\xi)) \quad \text{for any } r>0, $$

and $\alpha(s,t_{\nu(0)},\overline{\nu}(0)) := y_0$ for any $s>0$, where y_0 is a solution to the integral equation

$$ \int_{{}^iV({}^i\xi)}^{y_0}\frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\nu(0)}-t_0\right). $$

For $j\in\{1,2,\ldots,J-1\}$ and the following inequality is satisfied

$$ t_{\nu(j-1)} < t_{\varsigma(\overline{k}^\star)}, \tag{4.30} $$

we define $\alpha(s,t_{\nu(j)},\overline{\nu}(j)) := y_j$ for any $s>0$, where y_j is a solution to the integral equation

$$ \int_{\alpha(s,t_{\nu(j-1)},\overline{\nu}(j-1))}^{y_j}\frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\nu(j)}-t_{\nu(j-1)}\right). $$

Without loss of generality, suppose that $t_{\varsigma(\overline{k}^\star)} > t_{\nu(J-1)}$. For any positive number s, we define $\alpha(s,\cdot,\ldots,\cdot)$ on each interval $\left(t_{\nu(j-1)},t_{\nu(j)}\right)$ and the interval $\left(t_{\nu(J-1)},t_{\varsigma(\overline{k}^\star)}\right)$ as an arbitrary continuous decreasing function, which lie above ${}^iV$. Additionally on the interval $\left(t_{\varsigma(\overline{k}^\star)},\infty\right)$, we define $\alpha(s,\cdot,\ldots,\cdot)$ as an arbitrary continuous decreasing function, which tends to zero. See Figure 4.6.

By this construction, for any $(t_k,\overline{k})\in\operatorname{dom}\,{}^ix$ and $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, it holds that

$$ {}^iV({}^ix(t_k,\overline{k})) \leq \alpha({}^iV({}^i\xi),t_k,\overline{k}) \leq \alpha({}^i\psi_2(\|{}^i\xi\|_{\mathcal{A}}),t_k,\overline{k}), $$

where $\alpha:\mathbb{R}^{n+2}\to\mathbb{R}_{\geq 0}$ is continuous wrt the second argument, third argument, and so on; $\alpha(0,t_k,\overline{k}) := 0$ for all $(t_k,\overline{k})\in\operatorname{dom}\,{}^ix$; $\alpha(s,\cdot,\ldots,\cdot)$ is decreasing for all positive number s; it holds that $\alpha(s,t_k,\overline{k})\to 0$ as $t_k+\varsigma(\overline{k})\to\infty$. So there exists ${}^i\beta\in\mathcal{KL}^{n+1}$ such that it satisfies the following inequality

$$ \|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}} \leq {}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right) \tag{4.31} $$

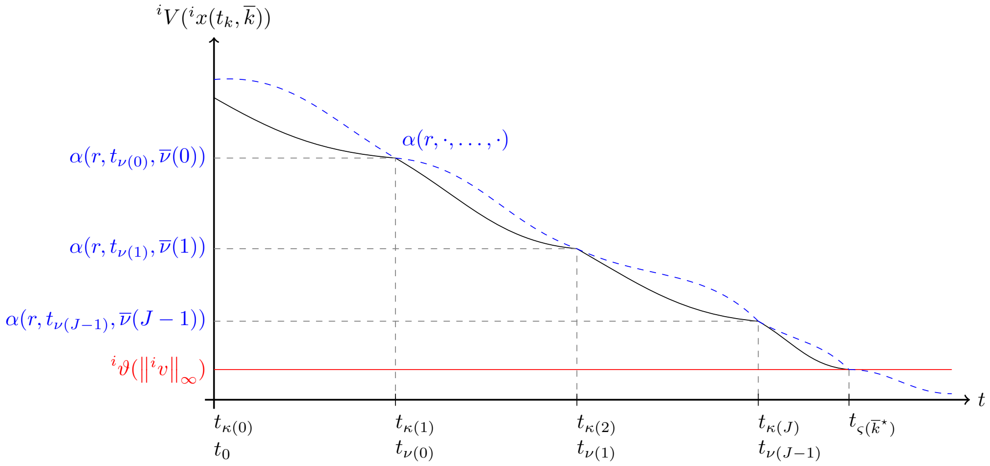


FIGURE 4.6: To the construction of function α providing an upper bound for ${}^iV$ in Theorem 4.23.

for $(t_k,\overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, where ${}^i\beta(s,t_k,\overline{k}) := {}^i\psi_1^{-1}\left(\alpha\left({}^i\psi_2(s),t_k,\overline{k}\right)\right)$. By a combination of the inequalities (4.29) and (4.31), we finally conclude that

$$\|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}} \leq \max\left\{{}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right),{}^i\gamma(\|{}^iv\|_\infty)\right\}$$

for any $(t_k,\overline{k}) \in \mathrm{dom}\ {}^ix$. □

Theorem 4.24 (Another Relaxed ISS-Lyapunov Theorem). *If there exists a relaxed Lyapunov function ${}^iV$ for $({}^i\mathcal{H},{}^i\mathcal{A})$ such that for some ${}^i\vartheta \in \mathcal{K}_\infty$ and ${}^i\varphi \in \mathcal{P}$ it satisfies*

$${}^iV({}^ix) \geq {}^i\vartheta\left(\|{}^iv\|_\infty\right) \implies \begin{cases}\left\langle \nabla\, {}^iV({}^ix),{}^if(x,u)\right\rangle \leq 0 & \text{for } x \in {}^i\mathcal{C}\setminus\mathcal{A}\\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) < -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D}\setminus\mathcal{A},\end{cases} \tag{4.32}$$

and ${}^i\mathcal{H}$ is not eventually continuous, then ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$.

Proof. Assume that V satisfies

$${}^iV({}^ix) \geq {}^i\vartheta\left(\|{}^iv\|_\infty\right) \implies \begin{cases}\left\langle \nabla\, {}^iV({}^ix),{}^if(x,u)\right\rangle = 0 & \text{for } x \in {}^i\mathcal{C}\setminus\mathcal{A}\\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) = -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D}\setminus\mathcal{A},\end{cases} \tag{4.33}$$

Define the set R exactly as (4.20). Moreover, let it hold $x(0,0) \in R' := ({}^i\mathcal{C}\cup{}^i\mathcal{D})\setminus R$ and ${}^ix(0,0) = {}^i\xi$. It follows that ${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) \geq {}^i\vartheta(\|{}^iv\|_\infty)$. Without loss of generality, we assume that $x(0,0) \in {}^i\mathcal{C}\cap R'$.

Similarly to the proofs of previous theorems, we obtain the results that the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)},\overline{\kappa}\left(j\right)\right)\right)\right\}$ and $\left\{{}^iV\left({}^ix\left(t_{\nu(j)},\overline{\nu}\left(j\right)\right)\right)\right\}$ is decreasing and bounded from below by ${}^i\vartheta(\|{}^iv\|_\infty)$ while $x \in R'$. If ${}^i\mathcal{H}$ is not eventually discrete system, this proof will go along the lines of the proof of Theorem 4.22. Let us only consider a case that ${}^i\mathcal{H}$ is eventually discrete system. In this case, there exists $J \in \mathbb{N}$ such that it satisfies $t_{\kappa(j)} \to t_{\kappa(J)} < \infty$ and $t_{\nu(j)} \to t_{\nu(J)} < \infty$ as $j \to \infty$.

While $x \in R'$, we are going to show that ${}^iV\left({}^ix\left(t_k,\overline{k}\right)\right)$ converges to ${}^i\vartheta(\|{}^iv\|_\infty)$. Suppose for a contradiction that ${}^iV\left({}^ix\left(t_k,\overline{k}\right)\right)$ converges to some positive number

$c > {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$. Denote

$$\tilde{S} := \tilde{S}(t_k, \overline{k}) := \sup_{j \in E(t_k, \overline{k})} \left[{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right) - {}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}(j)\right)\right)\right],$$

and

$$S := -\tilde{S}$$

where $E(t_k, \overline{k}) := \left\{j \in \mathbb{N} : \left(t_{\kappa(j)}, \overline{\kappa}(j)\right) \preceq (t_k, \overline{k})\right\}$. For any $\left(t_k, \overline{k}\right) \in \operatorname{dom} {}^ix$, there exists $j^* \in \mathbb{N}$ such that it holds

$$\begin{aligned}
{}^iV\left({}^ix\left(t_k, \overline{k}\right)\right) = {}^iV\left({}^ix\left(t_{\kappa(0)}, \overline{\kappa}(0)\right)\right) + \int_{t_{\kappa(0)}}^{t_k} {}^iV'\left({}^ix(s, \overline{k})\right)\, \mathrm{d}s \\
+ \sum_{j=0}^{j^*} \left[{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right) - {}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}(j)\right)\right)\right] \\
\leq {}^iV\left({}^ix\left(t_{\kappa(0)}, \overline{\kappa}(0)\right)\right) - S \cdot j^*.
\end{aligned}$$

Since ${}^i\mathcal{H}$ is an eventually discrete system, ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ eventually becomes negative as $t_k + \varsigma(\overline{k}) \to \infty$. So we obtain a contradiction here.

Since while $x \in R'$, ${}^iV({}^ix(t_k, \overline{k})) \to {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$, let us suppose that there exists $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \in \operatorname{dom} {}^ix$ such that it satisfies

$${}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) \geq {}^i\vartheta(\|{}^iv\|_\infty),$$

and

$${}^iV({}^ix(t_k, \overline{k}))) \geq {}^i\vartheta(\|{}^iv\|_\infty) \implies (t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star).$$

Without loss of generality, we additionally suppose that

$${}^iV({}^ix(t_{\nu(J)}, \overline{\nu}(J))) > {}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) > {}^i\vartheta(\|{}^iv\|_\infty).$$

Define a function $\tilde{\gamma} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\tilde{\vartheta}(s) := \max\left\{\sup_{0 \leq r \leq {}^i\vartheta(s)} {}^iV({}^ig(r)), {}^i\vartheta(s)\right\} \qquad \text{for all } s \geq 0.$$

Together with the condition (4.17), there exists an ISS gain ${}^i\gamma := {}^i\psi_1^{-1} \circ \tilde{\vartheta} \in \mathcal{K}_\infty$ such that it satisfies

$$\left\|{}^ix\left(t_k, \overline{k}\right)\right\|_{\mathcal{A}} \leq {}^i\psi_1^{-1}({}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)) \leq {}^i\psi_1^{-1}(\tilde{\vartheta}(\|{}^iv\|_\infty)) = {}^i\gamma(\|{}^iv\|_\infty) \tag{4.34}$$

for $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \prec (t_k, \overline{k})$. Note that it holds

$$t_{\nu(J)} = t_{\varsigma(\overline{k}^\star)} = t_{\varsigma(\overline{k}^\star + \overline{1}_i)}.$$

Here we are going to construct a function $\alpha : \mathbb{R}^{n+2} \to \mathbb{R}_{\geq 0}$, which provide an upper bound for ${}^iV({}^ix(t_k, \overline{k}))$ when $(t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star)$. Define

$$\alpha(s, t_0, 0) := {}^iV({}^i\xi) + {}^i\varphi({}^iV({}^i\xi)) \quad \text{for any } s > 0,$$

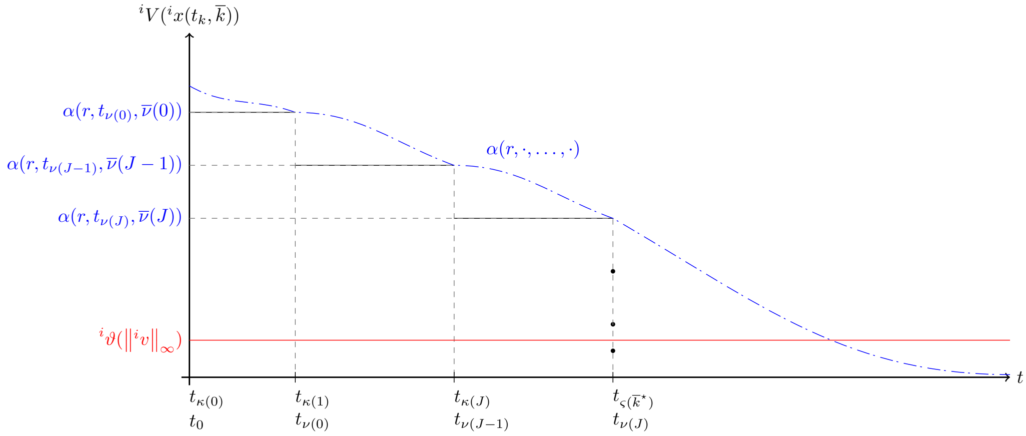


FIGURE 4.7: To the construction of function α providing an upper bound for ${}^iV$ in Theorem 4.24.

and

$$\alpha(s,t_{\nu(0)},\overline{\nu}(0)) := {}^iV({}^i\xi) \quad \text{for any } s>0.$$

For $j\in\{1,2,\dots,J\}$, we define

$$\alpha(s,t_{\nu(j)},\overline{\nu}(j)) := \alpha(s,t_{\nu(j)},\overline{\nu}(j)) - {}^i\varphi(\alpha(s,t_{\nu(j)},\overline{\nu}(j))) \quad \text{for any } s>0.$$

For any positive s, we define $\alpha(s,\cdot,\dots,\cdot)$ on each interval $\left(t_{\nu(j-1)},t_{\nu(j)}\right)$ as an arbitrary continuous decreasing function, which lies above ${}^iV$. Additionally for the interval $\left(t_{\varsigma(\overline{k}^\star)},\infty\right)$, we define $\alpha(s,\cdot,\dots,\cdot)$ as an arbitrary continuous decreasing function, which tends to zero. See Figure 4.7. By construction, for all $(t_k,\overline{k})\in\operatorname{dom}\ {}^ix$ and $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, it holds that

$${}^iV({}^ix(t_k,\overline{k}))\le\alpha({}^iV({}^i\xi),t_k,\overline{k})\le\alpha({}^i\psi_2(\|{}^i\xi\|_{\mathcal{A}}),t_k,\overline{k}),$$

where $\alpha:\mathbb{R}^{n+2}\to\mathbb{R}_{\ge0}$ is continuous wrt the second argument, third argument, and so on; $\alpha(0,t_k,\overline{k}):=0$ for all $(t_k,\overline{k})\in\operatorname{dom}\ {}^ix$; $\alpha(s,\cdot,\dots,\cdot)$ is decreasing for all positive s. Moreover, it holds that $\alpha(s,t_k,\overline{k})\to0$ as $t_k+\varsigma(\overline{k})\to\infty$. So there exists ${}^i\beta\in\mathcal{KL}^{n+1}$ such that

$$\|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}}\le{}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right), \tag{4.35}$$

for all $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, where ${}^i\beta(s,t_k,\overline{k})={}^i\psi_1^{-1}\left(\alpha\left({}^i\psi_2(s),t_k,\overline{k}\right)\right)$.

By a combination of the inequalities (4.34) and (4.35), we finally conclude that

$$\|{}^ix(t_k,\overline{k})\|_{{}^i\mathcal{A}}\le\max\left\{{}^i\beta\left(\|{}^i\xi\|_{{}^i\mathcal{A}},t_k,\overline{k}\right),{}^i\gamma(\|{}^iv\|_\infty)\right\}$$

for any $(t_k,\overline{k})\in\operatorname{dom}\ {}^ix$. □

The following results provides sufficient conditions to guarantee ISS for class $L(\theta)$ and class $H(\theta)$ generalized hybrid systems.

Theorem 4.25 (ISS-Dwell-time Condition). *If ${}^i\mathcal{H}$ is a class $L(\theta)$ generalized hybrid system for some $\theta>0$, and there exists an ISS-Lyapunov candidate function ${}^iV$ for $({}^i\mathcal{H},{}^i\mathcal{A})$*

such that for some ${}^i\vartheta \in \mathcal{K}_\infty$, ${}^i\varphi, {}^i\lambda \in \mathcal{P}$ *and* $\delta > 0$ *it satisfies*

$$ {}^iV({}^ix) \geq {}^i\vartheta\left(\left\|{}^iv\right\|_\infty\right) \implies \begin{cases} \left\langle \nabla\, {}^iV({}^ix), {}^if(x,u) \right\rangle \leq -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A} \\ {}^iV({}^ig(x,u)) \leq {}^i\lambda({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.36} $$

and

$$ \int_a^{{}^i\lambda(a)} \frac{\mathrm{d}s}{{}^i\varphi(s)} \leq \theta - \delta \quad \text{for all } a > 0, \tag{4.37} $$

then ${}^i\mathcal{H}$ *is ISS wrt* ${}^i\mathcal{A}$.

Proof. Define the set R and R' exactly as in the proof of Theorem 4.22. Suppose that that $x(0,0) \in {}^i\mathcal{C} \cap R'$ and ${}^ix(0,0) = {}^i\xi$. That is ${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) \geq {}^i\vartheta(\|{}^iv\|_\infty)$. By following the proof of Theorem 3.45, we obtain that, while $x \in R'$, the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right)\right\}$ is decreasing and bounded from below by ${}^i\vartheta(\|{}^iv\|_\infty)$, and the following inequality is satisfied

$$ \int_{{}^iV\left({}^ix\left(t_{\kappa(j+1)},\overline{\kappa}(j+1)\right)\right)}^{{}^iV\left({}^ix\left(t_{\kappa(j)},\overline{\kappa}(j)\right)\right)} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta \quad \text{for all } j \in \mathbb{N}. \tag{4.38} $$

While $x \in R'$, we are going to show that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to ${}^i\vartheta(\|{}^iv\|_\infty)$. Suppose for a contradiction that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to some positive number $c > {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$. Denote

$$ \rho := \inf_{c \leq s \leq V(x(0,0))} \varphi(s). $$

From the inequality (4.38), it follows that

$$ \delta \leq \int_{{}^iV\left({}^ix\left(t_{\kappa(j+1)},\overline{\kappa}(j+1)\right)\right)}^{{}^iV\left({}^ix\left(t_{\kappa(j)},\overline{\kappa}(j)\right)\right)} \frac{\mathrm{d}s}{\varphi(s)} \leq \frac{1}{\rho}\left({}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right) - {}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}\left(j+1\right)\right)\right)\right). $$

That is

$$ {}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}\left(j\right)\right)\right) - {}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}\left(j+1\right)\right)\right) \geq \rho\delta. $$

So a contradiction is found here.

Since while $x \in R'$, ${}^iV({}^ix(t_k, \overline{k})) \to {}^i\vartheta(\|{}^iv\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$, let us suppose that there exists $(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star) \in \operatorname{dom} {}^ix$ such that it satisfies

$$ {}^iV({}^ix(t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star))) \geq {}^i\vartheta(\|{}^iv\|_\infty), $$

and

$$ {}^iV({}^ix(t_k, \overline{k}))) \geq {}^i\vartheta(\|{}^iv\|_\infty) \implies (t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star). $$

Define a function $\tilde{\vartheta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$ \tilde{\vartheta}(s) := \max\left\{ \sup_{0 \leq r \leq {}^i\vartheta(s)} {}^iV({}^ig(r)), {}^i\vartheta(s) \right\} \qquad \text{for all } s \geq 0. $$

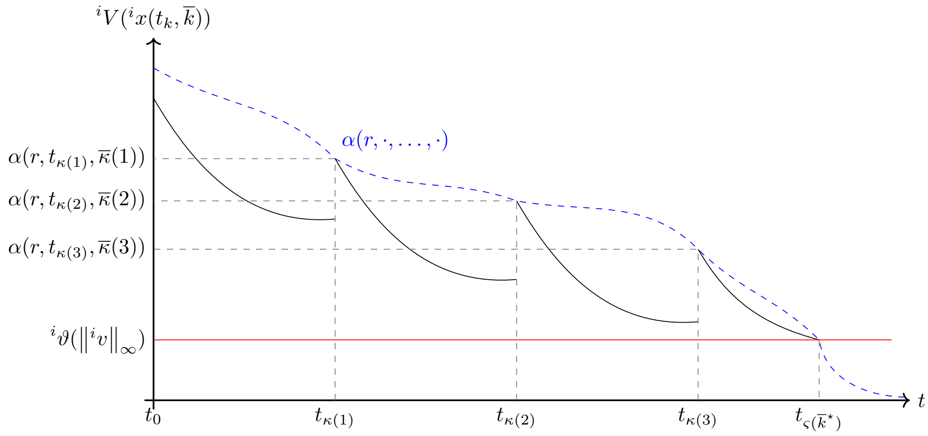


FIGURE 4.8: To the construction of function α providing an upper bound for ${}^iV$ in Theorem 4.25.

Together with the condition (4.17), there exists an ISS gain ${}^i\gamma := {}^i\psi_1^{-1} \circ \tilde{\vartheta} \in \mathcal{K}_\infty$ such that it satisfies

$$\left\|{}^ix\left(t_k,\overline{k}\right)\right\|_{\mathcal{A}} \leq {}^i\psi_1^{-1}({}^iV\left({}^ix\left(t_k,\overline{k}\right)\right)) \leq {}^i\psi_1^{-1}(\tilde{\vartheta}(\left\|{}^iv\right\|_\infty)) = {}^i\gamma(\left\|{}^iv\right\|_\infty) \tag{4.39}$$

for $(t_{\varsigma(\overline{k}^\star)},\overline{k}^\star) \prec (t_k,\overline{k})$.

Here we are going to construct a function $\alpha : \mathbb{R}^{n+2} \to \mathbb{R}_{\geq 0}$, which provide an upper bound for ${}^iV({}^ix(t_k,\overline{k}))$ when $(t_k,\overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$. Define

$$\alpha(r,t_0,0) := {}^iV({}^i\xi) + {}^i\lambda({}^iV({}^i\xi)) \quad \text{for all } r > 0,$$

and $\alpha(s,t_{\kappa(1)},\overline{\kappa}(1)) := {}^i\lambda(y_1)$ for any $s > 0$, where y_1 is a solution to the integral equation

$$\int_{{}^iV({}^i\xi)}^{y_1} \frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\kappa(1)} - t_0\right).$$

For $j > 1$ and it holds the following inequality

$$t_{\kappa(j)} < t_{\varsigma(\overline{k}^\star)}, \tag{4.40}$$

we define $\alpha(s,t_{\kappa(j)},\overline{\kappa}(j)) := {}^i\lambda(y_j)$ for any $s > 0$, where y is a solution to the integral equation

$$\int_{\alpha(s,t_{\kappa(j-1)},\overline{\kappa}(j-1))}^{y_j} \frac{\mathrm{d}s}{\varphi(s)} = -\left(t_{\kappa(j)} - t_{\kappa(j-1)}\right).$$

Suppose that $j^\star$ is the greatest natural number such that it satisfies the inequality (4.40). For any positive s, we define $\alpha(s,\cdot,\dots,\cdot)$ on each interval $\left(t_{\kappa(j-1)},t_{\kappa(j)}\right)$ and the interval $\left(t_{\kappa(j^\star)},t_{\varsigma(\overline{k}^\star)}\right)$ as an arbitrary continuous decreasing function, which lie above ${}^iV$. Additionally on the interval $\left(t_{\varsigma(\overline{k}^\star)},\infty\right)$, we define $\alpha(s,\cdot,\dots,\cdot)$ as an arbitrary continuous decreasing function, which tends to zero. See Figure 4.8. By this construction, for any $(t_k,\overline{k}) \in \operatorname{dom} {}^ix$ and $(t_k,\overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)},\overline{k}^\star)$, it holds that

$${}^iV({}^ix(t_k,\overline{k})) \leq \alpha({}^iV({}^i\xi),t_k,\overline{k}) \leq \alpha({}^i\psi_2(\left\|{}^i\xi\right\|_{\mathcal{A}}),t_k,\overline{k}),$$

where $\alpha : \mathbb{R}^{n+2} \to \mathbb{R}_{\geq 0}$ is continuous wrt the second argument, third argument, and so on; $\alpha(0, t_k, \overline{k}) := 0$ for all $(t_k, \overline{k}) \in \text{dom } {}^i x$; $\alpha(s, \cdot, \dots, \cdot)$ is decreasing for all positive number s; it holds that $\alpha(s, t_k, \overline{k}) \to 0$ as $t_k + \varsigma(\overline{k}) \to \infty$. So there exists ${}^i\beta \in \mathcal{KL}^{n+1}$ such that it satisfies the following inequality

$$\left\|{}^i x(t_k, \overline{k})\right\|_{{}^i\mathcal{A}} \leq {}^i\beta\left(\left\|{}^i\xi\right\|_{{}^i\mathcal{A}}, t_k, \overline{k}\right) \tag{4.41}$$

for $(t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star)$, where ${}^i\beta(s, t_k, \overline{k}) = {}^i\psi_1^{-1}\left(\alpha\left({}^i\psi_2(s), t_k, \overline{k}\right)\right)$. By the combination of the inequalities (4.39) and (4.41), we finally conclude that

$$\left\|{}^i x(t_k, \overline{k})\right\|_{{}^i\mathcal{A}} \leq \max\left\{{}^i\beta\left(\left\|{}^i\xi\right\|_{{}^i\mathcal{A}}, t_k, \overline{k}\right), {}^i\gamma(\left\|{}^i v\right\|_\infty)\right\}$$

for any $(t_k, \overline{k}) \in \text{dom } {}^i x$. □

Theorem 4.26 (Another ISS-Dwell-time Condition)**.** *If ${}^i\mathcal{H}$ is a class $H(\theta)$ generalized hybrid system for some $\theta > 0$, and there exists an ISS-Lyapunov candidate function ${}^iV$ for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^i\vartheta \in \mathcal{K}_\infty$, ${}^i\varphi, {}^i\lambda \in \mathcal{P}$ and $\delta > 0$ it satisfies*

$${}^iV({}^ix) \geq {}^i\vartheta\left(\left\|{}^iv\right\|_\infty\right) \implies \begin{cases} \left\langle \nabla\, {}^iV({}^ix), {}^if(x,u)\right\rangle \leq {}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A} \\ {}^iV({}^ig(x,u)) \leq {}^i\lambda({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.42}$$

and

$$\int_{{}^i\lambda(a)}^{a} \frac{\mathrm{d}s}{{}^i\varphi(s)} \geq \theta + \delta \quad \text{for all } a > 0, \tag{4.43}$$

then ${}^i\mathcal{H}$ is ISS w.r.t ${}^i\mathcal{A}$.

Proof. Define the set R and R' exactly as in the proof of Theorem 4.22. Suppose that that $x(0,0) \in {}^i\mathcal{C} \cap R'$ and ${}^ix(0,0) = {}^i\xi$. That is ${}^iV({}^ix(0,0)) = {}^iV({}^i\xi) \geq {}^i\vartheta(\left\|{}^iv\right\|_\infty)$. By following the proof of Theorem 3.46, while $x \in R'$, we obtain that the sequence $\left\{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right)\right\}$ and $\left\{{}^iV\left({}^ix\left(t_{\nu(j)}, \overline{\nu}(j)\right)\right)\right\}$ is decreasing and bounded from below by ${}^i\vartheta(\left\|{}^iv\right\|_\infty)$, and the following inequality is satisfied

$$\int_{{}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}(j+1)\right)\right)}^{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right)} \frac{\mathrm{d}s}{\varphi(s)} \geq \delta \quad \text{for all } j \in \mathbb{N}. \tag{4.44}$$

While $x \in R'$, we are going to show that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to ${}^i\vartheta(\left\|{}^iv\right\|_\infty)$. Suppose for a contradiction that ${}^iV\left({}^ix\left(t_k, \overline{k}\right)\right)$ converges to some positive number $c > {}^i\vartheta(\left\|{}^iv\right\|_\infty)$ as $t_k + \varsigma(\overline{k}) \to \infty$. Therefore, ${}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right)$ converges to $b \in \left({}^i\vartheta(\left\|{}^iv\right\|_\infty), c\right]$.

Let

$$\rho = \inf_{b \leq s \leq V(x(0,0))} \varphi(s).$$

From the inequality (4.44), we get

$$\delta \leq \int_{{}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}(j+1)\right)\right)}^{{}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right)} \frac{\mathrm{d}s}{\varphi(s)} \leq \frac{1}{\rho}\left({}^iV\left({}^ix\left(t_{\kappa(j)}, \overline{\kappa}(j)\right)\right) - {}^iV\left({}^ix\left(t_{\kappa(j+1)}, \overline{\kappa}(j+1)\right)\right)\right),$$

That is

$$^{i}V\left(^{i}x\left(t_{\kappa(j)},\overline{\kappa}\left(j\right)\right)\right)-{}^{i}V\left(^{i}x\left(t_{\kappa(j+1)},\overline{\kappa}\left(j+1\right)\right)\right)\geq\rho\delta.$$

So a contradiction is found here.

Since $^{i}V(^{i}x(t_k,\overline{k}))\rightarrow{}^{i}\vartheta(\|^{i}v\|_{\infty})$ as $t_k+\varsigma(\overline{k})\rightarrow\infty$, let us suppose that there exists $(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star})\in\operatorname{dom}{}^{i}x$ such that it satisfies

$$^{i}V(^{i}x(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star})))\geq{}^{i}\vartheta(\|^{i}v\|_{\infty}),$$

and

$$^{i}V(^{i}x(t_k,\overline{k})))\geq{}^{i}\vartheta(\|^{i}v\|_{\infty})\implies(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star}).$$

Define a function $\tilde{\gamma}:\mathbb{R}_{\geq0}\rightarrow\mathbb{R}_{\geq0}$ by

$$\tilde{\vartheta}(s):=\max\left\{\sup_{0\leq r\leq{}^{i}\vartheta(s)}{}^{i}V(^{i}g(r)),{}^{i}\vartheta(s)\right\}\qquad\text{for all } s\geq0.$$

Together with the condition (4.17), there exists an ISS gain $^{i}\gamma:={}^{i}\psi_1^{-1}\circ\tilde{\vartheta}\in\mathcal{K}_{\infty}$ such that it satisfies

$$\left\|^{i}x\left(t_k,\overline{k}\right)\right\|_{\mathcal{A}}\leq{}^{i}\psi_1^{-1}(^{i}V\left(^{i}x\left(t_k,\overline{k}\right)\right))\leq{}^{i}\psi_1^{-1}(\tilde{\vartheta}(\|^{i}v\|_{\infty}))={}^{i}\gamma(\|^{i}v\|_{\infty})\tag{4.45}$$

for $(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star})\prec(t_k,\overline{k})$.

Here we are going to construct a function $\alpha:\mathbb{R}^{n+2}\rightarrow\mathbb{R}_{\geq0}$, which provide an upper bound for $^{i}V(^{i}x(t_k,\overline{k}))$ when $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star})$. Define

$$\alpha(r,t_0,0):=y_0+{}^{i}\lambda(y_0),\text{ and }\alpha(s,t_{\nu(0)},\overline{\nu}(0)):=y_0\quad\text{for all } s>0,$$

where y_0 is a solution to the integral equation

$$\int_{^{i}V(^{i}\xi)}^{y_0}\frac{\mathrm{d}s}{\varphi(s)}=t_{\nu(0)}-t_0.$$

For $j>0$ and it holds the following inequality

$$t_{\kappa(j)}<t_{\varsigma(\overline{k}^{\star})},\tag{4.46}$$

we define $\alpha(s,t_{\nu(j)},\overline{\nu}(j)):=y_j$ for any $s>0$, where y_j is a solution to the integral equation

$$\int_{^{i}V\left(^{i}x\left(t_{\kappa(j)},\overline{\kappa}(j)\right)\right)}^{y_j}\frac{\mathrm{d}s}{\varphi(s)}=t_{\nu(j)}-t_{\kappa(j)}.$$

Suppose that $j^{\star}$ is the greatest natural number such that it satisfies the inequality (4.46). Therefore, we have $t_{\nu(j^{\star})}=t_{\varsigma(\overline{k}^{\star})}$. For any positive s, we define $\alpha(s,\cdot,\ldots,\cdot)$ on each interval $\left(t_{\kappa(j-1)},t_{\kappa(j)}\right)$ as an arbitrary continuous decreasing function, which lie above ^{i}V. Additionally on the interval $\left(t_{\varsigma(\overline{k}^{\star})},\infty\right)$, we define $\alpha(s,\cdot,\ldots,\cdot)$ as an arbitrary continuous decreasing function, which tends to zero. See Figure 4.8. By this construction, for any $(t_k,\overline{k})\in\operatorname{dom}{}^{i}x$ and $(t_k,\overline{k})\preceq(t_{\varsigma(\overline{k}^{\star})},\overline{k}^{\star})$, it holds that

$$^{i}V(^{i}x(t_k,\overline{k}))\leq\alpha(^{i}V(^{i}\xi),t_k,\overline{k})\leq\alpha(^{i}\psi_2(\|^{i}\xi\|_{\mathcal{A}}),t_k,\overline{k}),$$

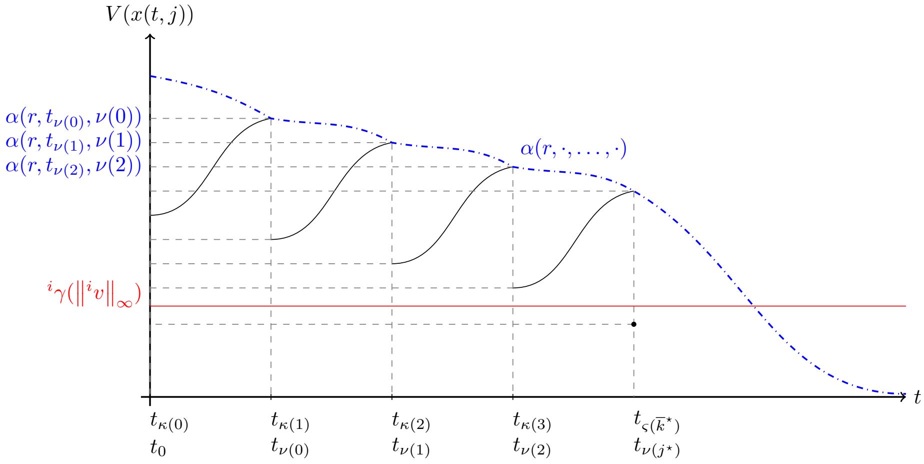


FIGURE 4.9: To the construction of function α providing an upper bound for ${}^iV$ in Theorem 4.26.

where $\alpha : \mathbb{R}^{n+2} \to \mathbb{R}_{\geq 0}$ is continuous wrt the second argument, third argument, and so on; $\alpha(0, t_k, \overline{k}) := 0$ for all $(t_k, \overline{k}) \in \text{dom } {}^ix$; $\alpha(s, \cdot, \dots, \cdot)$ is decreasing for all positive r; it holds that $\alpha(s, t_k, \overline{k}) \to 0$ as $t_k + \varsigma(\overline{k}) \to \infty$. So there exists ${}^i\beta \in \mathcal{KL}^{n+1}$ such that it satisfies the following inequality

$$\left\|{}^ix(t_k, \overline{k})\right\|_{{}^i\mathcal{A}} \leq {}^i\beta\left(\left\|{}^i\xi\right\|_{{}^i\mathcal{A}}, t_k, \overline{k}\right) \tag{4.47}$$

for $(t_k, \overline{k}) \preceq (t_{\varsigma(\overline{k}^\star)}, \overline{k}^\star)$, where ${}^i\beta(s, t_k, \overline{k}) := {}^i\psi_1^{-1}\left(\alpha\left({}^i\psi_2(s), t_k, \overline{k}\right)\right)$. By the combination of the inequalities (4.45) and (4.47), we finally conclude that

$$\left\|{}^ix(t_k, \overline{k})\right\|_{{}^i\mathcal{A}} \leq \max\left\{{}^i\beta\left(\left\|{}^i\xi\right\|_{{}^i\mathcal{A}}, t_k, \overline{k}\right), {}^i\gamma(\left\|{}^iv\right\|_\infty)\right\}$$

for any $(t_k, \overline{k}) \in \text{dom } {}^ix$. □

4.6 ISS-Lyapunov Functions for Interconnections

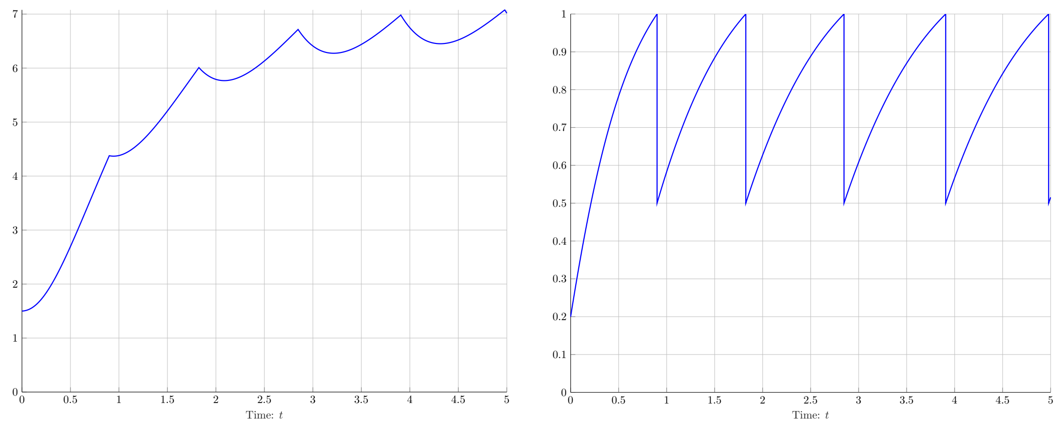
We are going to discuss on the stability of interconnected hybrid dynamical systems in this section. Consider a hybrid system $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form (4.7). Although each subsystem ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$, the interconnection $\mathcal{H}$ is not necessary to be ISS wrt $\mathcal{A}$ defined by (4.10). The following is an example to claim this fact.

Example 4.27. Consider the following hybrid system

$$\begin{aligned} \dot{x} = f(x) &:= -x + u, \quad x \in \mathcal{C} := \mathcal{X} \subset \mathbb{R}, \\ x^+ = g(x) &:= \frac{x}{2}, \quad x \in \mathcal{D} := \{x \in \mathcal{X} : x = 1\}. \end{aligned}$$

Let $V(x) = |x|$. It follows that $V(x) \geq 2\|u\|_\infty$ implies

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &= \text{sign } (x)\,(-x + u) \leq -V(x) + \|u\|_\infty \leq -\frac{V(x)}{2}, \\ V(g(x)) - V(x) &= \left|\frac{x}{2}\right| - |x| = -\frac{V(x)}{2}. \end{aligned}$$



(A) ^1x. (B) ^2x.

FIGURE 4.10: A numerical simulation in Example 4.27.

Consequently, this hybrid system is ISS wrt $\mathcal{A} := \{0\}$.

Moreover, for $i = 1, 2$, we can conclude that the system:

$$^i\mathcal{H} : {}^ix \in {}^i\mathcal{X} \subset \mathbb{R} \begin{cases} {}^i\dot{x} = {}^if({}^ix, {}^iu) := -{}^ix + {}^iu, & x \in {}^i\mathcal{C} := {}^i\mathcal{X}, \\ {}^ix^+ = {}^ig({}^ix, {}^iu) := -\dfrac{{}^ix}{2}, & x \in {}^i\mathcal{D} := \left\{x \in {}^i\mathcal{X} : \; {}^ix = 1\right\} \end{cases}$$

is ISS wrt $^i\mathcal{A} = \{0\}$.

Here, we are going to connect $^1\mathcal{H}$ and $^2\mathcal{H}$ by setting

$$^1u = \frac{16({}^2x)^2 + 1}{({}^2x)^2 + 1} + u, \quad {}^2u = \frac{{}^1x + 3}{{}^1x + 1} + u$$

where u is admissible external input. In Figure 4.10, we depict a numerical simulation of the interconnection $\mathcal{H} = \left\{{}^1\mathcal{H}, {}^2\mathcal{H}\right\}$ along with $u = 0$. It is clear to see that $\mathcal{H}$ is not 0-GAS wrt $^1\mathcal{A} \times {}^2\mathcal{A}$, which implies that $\mathcal{H}$ is not ISS.

To find the sufficient conditions to guarantee stability of interconnection consequently becomes an interesting issue. Our goal aims to provide such conditions that guarantee stability of the interconnection by investigating only on stability of subsystems.

4.6.1 Interconnections of Two Subsystems

Let us firstly consider the simplest case of interconnected ISS hybrid dynamical systems. Suppose that a system $\mathcal{H} = \left\{{}^1\mathcal{H}, {}^2\mathcal{H}\right\}$ of the form (4.7) satisfies the following requirements. For each $i, j = 1, 2$; and $i \neq j$, there exists an ISS-Lyapunov function ^iV wrt $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some $^i\psi_1, {}^i\psi_2, {}^{ij}\vartheta, {}^{iu}\vartheta \in \mathcal{K}_\infty$, and $^i\varphi \in \mathcal{P}$, it satisfies

$$^i\psi_1\left(\left\|{}^ix\right\|_{{}^i\mathcal{A}}\right) \leq {}^iV\left({}^ix\right) \leq {}^i\psi_2\left(\left\|{}^ix\right\|_{{}^i\mathcal{A}}\right) \quad \text{for all } {}^ix \in {}^i\mathcal{X},$$

and

$$^iV\left({}^ix\right) \geq {}^i\Upsilon \implies \begin{cases} \left\langle \nabla\, {}^iV({}^ix), {}^if(x, u)\right\rangle \leq -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x, u)) - {}^iV({}^ix) \leq -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.48}$$

where ${}^{i}\Upsilon = \max\left\{{}^{ij}\vartheta\left({}^{j}V\left({}^{j}x\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right)\right\}$.

Here we are going to construct an ISS-Lyapunov function to guarantee that the interconnection $\mathcal{H}$ is ISS wrt $\mathcal{A} = {}^{1}A\times{}^{2}\mathcal{A}$, i.e., a globally Lipschitz continuous $V : \mathbb{R}^n \to \mathbb{R}$ such that for some $\psi_1, \psi_2, \vartheta \in \mathcal{K}_\infty$, and $\varphi \in \mathcal{P}$, it satisfies

$$\psi_1\left(\|x\|_{\mathcal{A}}\right) \leq V\left(x\right) \leq \psi_2\left(\|x\|_{\mathcal{A}}\right) \quad \text{for all } x \in \mathcal{X},$$

and

$$V\left(x\right) \geq \vartheta\left(\|u\|_\infty\right) \implies \begin{cases} \langle \nabla V(x), f(x,u)\rangle \leq -\varphi(V(x)) & \text{for } x \in \mathcal{C}\setminus\mathcal{A}, \\ V(g(x,u)) - V(x) \leq -\varphi(V(x)) & \text{for } x \in \mathcal{D}\setminus\mathcal{A}, \end{cases}$$

where $f(x,u) := \left({}^{1}\tilde{f}^{\mathrm{T}}(x,u), {}^{2}\tilde{f}^{\mathrm{T}}(x,u)\right)^{\mathrm{T}}$, $g(x,u) := \left({}^{1}\tilde{g}^{\mathrm{T}}(x,u), {}^{2}\tilde{g}^{\mathrm{T}}(x,u)\right)^{\mathrm{T}}$,

$${}^{i}\tilde{f}(x,u) := \begin{cases} {}^{i}f(x,u) & \text{if } i \in \mathcal{I}_{\mathcal{C}}(x,u), \\ 0 & \text{otherwise.} \end{cases}, \quad {}^{i}\tilde{g}(x,u) := \begin{cases} {}^{i}g(x,u) & \text{if } i \in \mathcal{I}_{\mathcal{D}}(x,u), \\ x & \text{otherwise.} \end{cases},$$

$\mathcal{C} := {}^{1}\mathcal{C} \cup {}^{2}\mathcal{C}$ and $\mathcal{D} := {}^{1}\mathcal{D} \cup {}^{2}\mathcal{D}$.

Additionally we suppose that ${}^{12}\vartheta \circ {}^{21}\vartheta \in \mathcal{S}$. This extra condition will sufficiently guarantee that such function V exists, and the interconnection $\mathcal{H}$ is consequently ISS wrt $\mathcal{A}$. To show existence of such ISS-Lyapunov function V for $\mathcal{H}$, the following lemmas are required tools for the construction, which are motivated by the results in [52, 70].

Lemma 4.28. *For any $\rho_0 \in \mathcal{P}$, there exists $\rho \in \mathcal{S}$ such that*

- *$\rho(s) < \rho_0(s)$ for all $s > 0$,*
- *$\rho \in C^1(\mathbb{R}_{>0})$,*
- *$\rho'(s) < 1$ for all $s > 0$.*

Proof. Arbitrarily choose a constant $\lambda \in (0,1)$ and define a function $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ by $\tilde{\rho}(s) := \min\{\lambda, \rho_0(s)\}$ for all $s \geq 0$. It is clear to see that $\tilde{\rho}(s) < 1$ for all $s > 0$, and $\tilde{\rho}(s) \leq \rho_0(s)$ for all $s > 0$. Let

$$\rho_1(s) := \begin{cases} \min\limits_{a\in[s,2]} \tilde{\rho}(a) & \text{if } 0 \leq s \leq 1, \\ \min\limits_{a\in[1,s+1]} \tilde{\rho}(a) & \text{if } s > 1. \end{cases}$$

Firstly, let us show that ρ_1 is not decreasing on $(0,1)$. Suppose for the sake of contradiction that $\rho_1(s_1) > \rho_1(s_2)$ for any $0 < s_1 < s_2 < 1$. Assume $k_1 = \rho_1(s_1)$ and $k_2 = \rho_1(s_2)$. So we have $k_1 > k_2$. This contradicts the fact that

$$k_1 = \min_{a\in[s_1,2]} \tilde{\rho}(a) = \min_{a\in[s_1,s_2]\cup[s_2,2]} \tilde{\rho}(a) \leq \min_{a\in[s_2,2]} \tilde{\rho}(a) = k_2.$$

Secondly, let us show that ρ_1 is not increasing on $(1,\infty)$. We therefore suppose for a contradiction that $\rho_1(s_1) < \rho_1(s_2)$ for any $1 < s_1 < s_2$. Again, we assume that

$k_1 = \rho_1(s_1)$ and $k_2 = \rho_1(s_2)$. However, it holds

$$k_1 = \min_{a \in [1, s_1+1]} \tilde{\rho}(a) \le \min_{a \in [1, s_2+1]} \tilde{\rho}(a) = k_2.$$

So we obtain a contradiction here. Moreover, it is obvious to see that $\rho_1(s) \le \tilde{\rho}(s)$ for all $s > 0$, and $\rho_1(s-1) \le \tilde{\rho}(s)$ for all $s \ge 1$.

Let us define a function $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$\rho(s) = \begin{cases} \int_0^s \rho_1(y) \, \mathrm{d}y & \text{if } 0 \le s \le 1, \\ \int_{s-1}^s \rho_1(y) \, \mathrm{d}y & \text{if } s > 1. \end{cases}$$

Note that it yields $\rho \in C^1(\mathbb{R}_{>0})$ with

$$\rho'(s) = \begin{cases} \rho_1(s) & \text{if } 0 \le s \le 1, \\ \rho_1(s) - \rho_1(s-1) & \text{if } s > 1. \end{cases}$$

Therefore $\rho'(s) \le \rho_1(s) \le \tilde{\rho}(s) < 1$ for all $s > 0$. Furthermore it satisfies for all $s \in [0, 1]$

$$\rho(s) = \int_0^s \rho_1(y) \, \mathrm{d}y < \int_0^s \rho_1(s) \, \mathrm{d}y = s\rho_1(s) \le \rho_1(s) \le \tilde{\rho}(s) \le \rho_0(s).$$

For all $s \in (1, 2)$, it satisfies

$$\begin{aligned} \rho(s) &= \int_{s-1}^s \rho_1(y) \, \mathrm{d}y \\ &= \int_{s-1}^1 \rho_1(y) \, \mathrm{d}y + \int_1^s \rho_1(y) \, \mathrm{d}y \\ &< \int_{s-1}^1 \rho_1(1) \, \mathrm{d}y + \int_1^s \rho_1(1) \, \mathrm{d}y \\ &= \rho_1(1) \, (1 - (s-1)) + \rho_1(1)(s-1) = \rho_1(1) \le \tilde{\rho}(s) \le \rho_0(s). \end{aligned}$$

Lastly, for all $s \ge 2$, it satisfies

$$\rho(s) = \int_{s-1}^s \rho_1(y) \, \mathrm{d}y < \int_{s-1}^s \rho_1(s-1) \, \mathrm{d}y = \rho(s-1) \le \tilde{\rho}(s) \le \rho_0(s).$$

□

Lemma 4.29. *If $\sigma_1 \in \mathcal{K}$ and $\sigma_2 \in \mathcal{K}_\infty$ satisfy $\sigma_1(s) < \sigma_2(s)$ for all $s > 0$, then there exists $\sigma \in \mathcal{K}_\infty$ such that*

- *$\sigma_1(s) < \sigma(s) < \sigma_2(s)$ for all $s > 0$,*
- *$\sigma \in C^1(\mathbb{R}_{>0})$,*
- *$\sigma'(s) > 0$ for all $s > 0$.*

Proof. Define $\rho_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\rho_0(s) := \frac{1}{2}\left(s - \sigma_2^{-1} \circ \sigma_1(s)\right).$$

It is obvious to see that $\rho_0 \in \mathcal{P}$, and for all $s > 0$, it holds that

$$\begin{aligned} 2\rho_0(s) &= s - \sigma_2^{-1} \circ \sigma_1(s), \\ \sigma_2^{-1} \circ \sigma_1(s) &= s - 2\rho_0(s) < s - \rho_0(s), \\ \sigma_1(s) &< \sigma_2\left(s - \rho_0(s)\right). \end{aligned}$$

By Lemma 4.28, there exists $\rho \in \mathcal{S}$ such that $\rho(s) < \rho_0(s)$ for all $s > 0$, $\rho \in C^1(\mathbb{R}_{>0})$, and $\rho'(s) < 1$ for all $s > 0$. Therefore we have

$$\sigma_2(s - \rho_0(s)) < \sigma_2(s - \rho(s)).$$

Let us define $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\sigma(0) = 0$, and

$$\sigma(s) := \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y \quad \text{for all } s > 0.$$

We are going to show that $\sigma \in \mathcal{K}_\infty$, and it holds the desired property as follows. It is straight forward to see that for all $s > 0$,

$$\sigma(s) = \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y < \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(s)\, \mathrm{d}y = \sigma_2(s).$$

Additionally it holds

$$\sigma(s) = \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y > \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(s - \rho(s))\, \mathrm{d}y = \sigma_2(s - \rho(s)).$$

In consequence, for all $s > 0$, it yields

$$\sigma_1(s) < \sigma_2\left(s - \rho_0(s)\right) < \sigma_2(s - \rho(s)) < \sigma(s) < \sigma_2(s).$$

Consider the following derivative, for all $s > 0$,

$$\begin{aligned} \sigma'(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{\rho(s)} \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y \\ &= \frac{1}{\rho^2(s)} \left(\rho(s) \frac{\mathrm{d}}{\mathrm{d}s} \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y - \rho'(s) \int_{s-\rho(s)}^{s} \sigma_2(y)\, \mathrm{d}y \right) \\ &= \frac{1}{\rho^2(s)} \left(\rho(s) \cdot \left(\sigma_2(s) - \sigma_2(s - \rho(s)) \cdot (1 - \rho'(s)) \right) - \rho'(s) \cdot \rho(s) \cdot \sigma(s) \right) \\ &= \frac{1}{\rho(s) \cdot (\sigma(s) - \sigma_2(s - \rho(s)))} \left(\frac{\sigma_2(s) - \sigma_2(s - \rho(s))}{\sigma(s) - \sigma_2(s - \rho(s))} - \rho'(s) \right). \end{aligned}$$

Note that $\sigma(s) - \sigma_2(s - \rho(s)) > 0$, $\sigma_2(s) - \sigma_2(s - \rho(s)) > \sigma(s) - \sigma_2(s - \rho(s))$ and $\rho'(s) < 1$ for all $s > 0$. Therefore it holds $\sigma'(s) > 0$ for all $s > 0$. Since $\rho \in \mathcal{S}$ and $\sigma_2 \in \mathcal{K}_\infty$, it yields $\sigma_2(s - \rho(s)) \to \infty$ as $s \to \infty$. Together with the fact that $\sigma_2(s - \rho(s)) < \sigma(s)$ for all $s > 0$, we eventually conclude that $\sigma(s) \to \infty$ as $s \to \infty$.

□

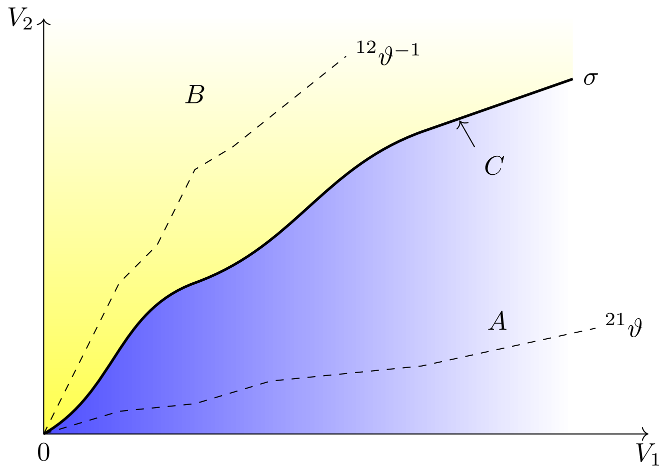


FIGURE 4.11: Level Sets: A, B and C.

Now we are ready to construct the ISS-Lyapunov function V wrt $(\mathcal{H}, \mathcal{A})$. Recall our extra condition: $^{12}\vartheta \circ {}^{21}\vartheta \in \mathcal{S}$. It follows that

$$^{21}\vartheta(s) < {}^{12}\vartheta^{-1}(s) \quad \text{for all } s > 0.$$

By Lemma 4.29, there exists $\sigma \in \mathcal{K}_\infty$ such that $\sigma \in C^1(\mathbb{R}_{>0})$, $\sigma'(s) > 0$ for all $s > 0$ and

$$^{21}\vartheta(s) < \sigma(s) < {}^{12}\vartheta^{-1}(s) \quad \text{for all } s > 0.$$

Let us define $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$V(x) := \max\left\{\sigma({}^1V({}^1x)), {}^2V({}^2x)\right\}.$$

Define the following sets, as shown in Figure 4.11.

$$\begin{aligned} A &= \left\{(x_1, x_2) \in \mathcal{X} : \ \sigma({}^1V({}^1x)) > {}^2V({}^2x)\right\}, \\ B &= \left\{(x_1, x_2) \in \mathcal{X} : \ \sigma({}^1V({}^1x)) < {}^2V({}^2x)\right\}, \\ C &= \left\{(x_1, x_2) \in \mathcal{X} : \ \sigma({}^1V({}^1x)) = {}^2V({}^2x)\right\}. \end{aligned}$$

Consider the first case: $x \in A$. In this case we have $V(x) = \sigma({}^1V({}^1x))$, and consequently

$$\langle \nabla V(x), f(x,u)\rangle = \sigma'({}^1V({}^1x))\left\langle \nabla\, {}^1V, {}^1f(x,u)\right\rangle.$$

Note that

$$^{12}\vartheta^{-1}({}^1V({}^1x)) > \sigma({}^1V({}^1x)) > {}^2V({}^2x).$$

Therefore ${}^1V({}^1x) > {}^{12}\vartheta({}^2V({}^2x))$. Define $\vartheta_1 := \sigma \circ {}^{1u}\vartheta$, which is clear that $\vartheta_1 \in \mathcal{K}_\infty$. As a consequence, $V(x) \geq \vartheta_1(\|u\|_\infty)$ implies

$$\begin{aligned} \langle \nabla V(x), f(x,u)\rangle &= \sigma'({}^1V({}^1x))\left\langle \nabla\, {}^1V, {}^1f(x,u)\right\rangle \\ &\leq -\sigma'({}^1V({}^1x))\, {}^1\varphi({}^1V({}^1x)) \\ &= -\sigma'(\sigma^{-1}(V(x)))\, {}^1\varphi(\sigma^{-1}(V(x))) \\ &= -\tilde{\varphi}_1(V(x)) \end{aligned}$$

for $x \in \mathcal{C}$, and

$$\begin{aligned} V(g(x)) - V(x) &= \sigma({}^1V({}^1g({}^1x))) - V(x) \\ &\leq \sigma({}^1V({}^1x) - {}^1\varphi({}^1V({}^1x))) - V(x) \\ &= -\left(V(x) - \sigma({}^1V({}^1x) - {}^1\varphi({}^1V({}^1x)))\right) \\ &= -\left(V(x) - \sigma(\sigma^{-1}(V(x)) - {}^1\varphi(\sigma^{-1}(V(x))))\right) \\ &= -\tilde{\varphi}_2(V(x)) \end{aligned}$$

for $x \in \mathcal{D}$, where $\varphi_1(s) := \left(\sigma' \circ \sigma^{-1}\right)(s) \cdot \left({}^1\varphi \circ \sigma^{-1}\right)(s)$, and $\varphi_2(s) := s - \sigma(\sigma^{-1}(s) - {}^1\varphi \circ \sigma^{-1}(s))$. It is clear to see that $\varphi_1 \in \mathcal{P}$. Here we further show that $\varphi_2 \in \mathcal{P}$. Suppose for a contradiction that $\varphi_2(s) = s - \sigma(\sigma^{-1}(s) - {}^1\varphi \circ \sigma^{-1}(s)) < 0$ for all $s > 0$. It follows that

$$\begin{aligned} s &< \sigma(\sigma^{-1}(s) - {}^1\varphi \circ \sigma^{-1}(s)), \\ \sigma^{-1}(s) &< \sigma^{-1}(s) - {}^1\varphi \circ \sigma^{-1}(s), \end{aligned}$$

and consequently ${}^1\varphi \circ \sigma^{-1}(s) < 0$ for all $s > 0$, which contradicts to the fact that ${}^1\varphi \circ \sigma^{-1} \in \mathcal{P}$. Therefore, in the first case there exists $\varphi_1 \in \mathcal{P}$ defined by $\varphi_1(s) = \max\{\tilde{\varphi}_1(s), \tilde{\varphi}_2(s)\}$ for all $s \geq 0$ such that $V(x) \geq \vartheta_1(\|u\|_\infty)$ implies $\langle \nabla V(x), f(x,u)\rangle \leq -\varphi_1(V(x))$ for $x \in \mathcal{C}$ and $V(g(x)) - V(x) \leq -\varphi_1(V(x))$ for $x \in \mathcal{D}$.

In the second case: $x \in B$, we have $V(x) = {}^2V({}^2x)$, and consequently

$$\langle \nabla V(x), f(x,u)\rangle = \left\langle \nabla\, {}^2V, {}^2f(x,u)\right\rangle.$$

Since it satisfies

$${}^2V({}^2x) > \sigma({}^1V({}^1x)) > {}^{21}\vartheta({}^1V({}^1x)),$$

we obtain $V(x) \geq {}^{2u}\vartheta(\|u\|_\infty)$ implies

$$\langle \nabla V(x), f(x,u)\rangle = \left\langle \nabla\, {}^2V, {}^2f(x,u)\right\rangle \leq -\,{}^2\varphi({}^2V({}^2x)) = -\,{}^2\varphi(V(x))$$

for $x \in \mathcal{C}$, and

$$V(g(x)) - V(x) = {}^2V({}^2g({}^2x)) - V(x) \leq -\,{}^2\varphi({}^2V({}^2x)) - V(x) \leq -\,{}^2\varphi(V(x))$$

for $x \in \mathcal{D}$.

Without loss of generality, we can take $V(x) = {}^2V({}^2x)$ in the last case: $x \in C$, and consequently the similar result is obtained. Combining the above results, therefore there exist $\vartheta \in \mathcal{K}_\infty$ defined by $\vartheta(s) := \max\left\{\vartheta_1(s), {}^{2u}\vartheta(s)\right\}$ for all $s \geq 0$, and $\varphi \in \mathcal{P}$ defined by $\varphi(s) := \max\left\{\varphi_1(s), {}^2\varphi(s)\right\}$ for all $s \geq 0$ such that $V(x) \geq \vartheta(\|u\|_\infty)$ implies $\langle \nabla V(x), f(x,u)\rangle \leq -\varphi(V(x))$ for $x \in \mathcal{C}$ and $V(g(x)) - V(x) \leq -\varphi(V(x))$ for $x \in \mathcal{D}$.

Note that

$$\frac{1}{2}\left(\sigma({}^1V({}^1x)) + {}^2V({}^2x)\right) \leq V(x) \leq \sigma({}^1V({}^1x)) + {}^2V({}^2x)$$

for all $x \in \mathcal{X}$. So there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, which are defined by

$$\psi_1(s) := \frac{1}{2}\left(\sigma \circ {}^1\psi_1(s_1) + {}^2\psi_1(s_2)\right), \quad \psi_2(s) := \sigma \circ {}^1\psi_2(s_1) + {}^2\psi_2(s_2)$$

for all $s = (s_1, s_2) \geq 0$ such that $\psi_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \psi_2(\|x\|_{\mathcal{A}})$.

In conclusion of stability of the interconnected ISS hybrid system $\mathcal{H} = \{{}^1\mathcal{H}, {}^2\mathcal{H}\}$, we summarize the result as follows. In order to state an assumption that subsystems ${}^1\mathcal{H}$ and ${}^2\mathcal{H}$ are ISS wrt ${}^1\mathcal{A}$ and ${}^2\mathcal{A}$ respectively, we suppose that there exist ISS-Lyapunov functions for subsystems. Along with the required condition ${}^{12}\vartheta \circ {}^{21}\vartheta \in \mathcal{S}$, we can construct an ISS-Lyapunov function for $\mathcal{H}$ as shown above. Finally, it is sufficient to conclude that $\mathcal{H} = \{{}^1\mathcal{H}, {}^2\mathcal{H}\}$ is ISS wrt $\mathcal{A}$.

4.6.2 Small-Gain Theorem

We further discuss on stability of interconnected ISS hybrid dynamical systems $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ of the form (4.7). Suppose that for each $i \in \mathbb{N}_n$, there exists an ISS-Lyapunov function ${}^iV : {}^i\mathcal{X} \to \mathbb{R}_{\geq 0}$ wrt $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^i\psi_1, {}^i\psi_2, {}^{ij}\vartheta, {}^{iu}\vartheta \in \mathcal{K}_\infty$, and ${}^i\varphi \in \mathcal{P}$, it satisfies

$$ {}^i\psi_1\left(\|{}^ix\|_{{}^i\mathcal{A}}\right) \leq {}^iV\left({}^ix\right) \leq {}^i\psi_2\left(\|{}^ix\|_{{}^i\mathcal{A}}\right) \quad \text{for all } {}^ix \in {}^i\mathcal{X}, \tag{4.49} $$

and

$$ {}^iV\left({}^ix\right) \geq {}^i\Upsilon \implies \begin{cases} \left\langle \nabla {}^iV({}^ix), {}^if(x,u)\right\rangle \leq -{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) \leq -{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.50} $$

where ${}^i\Upsilon = \max\left\{\max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta\left({}^jV\left({}^jx\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right)\right\}$.

In [54], Dashkovskiy, Rueffer and Wirth proposed sufficient conditions for the existence of an ISS Lyapunov function for a continuous-time system obtained as the interconnection of many continuous-time subsystems. The concept of Ω-path was also presented in this paper. This path plays a crucial role in the construction of a Lyapunov function for the whole network. A small gain assumption on the monotone operator induced by the gain matrix is used to constructively obtain a locally Lipschitz continuous ISS Lyapunov function for the whole network by appropriately scaling the individual Lyapunov functions for the subsystems.

The following definitions and theorems are essential tools to construct an ISS-Lyapunov function wrt $(\mathcal{H}, \mathcal{A})$ where $\mathcal{A} = {}^1\mathcal{A} \times \ldots \times {}^n\mathcal{A}$. The similar concept of the construction was also proposed in [12, 42, 53].

Definition 4.30 (Gain Operators)**.** For each $i \in \mathbb{N}_n$, let ${}^iV$ be an ISS-Lyapunov candidate function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^{ij}\vartheta, {}^{iu}\vartheta \in \mathcal{K}_\infty$, and ${}^i\varphi \in \mathcal{P}$, it satisfies

$$ {}^iV\left({}^ix\right) \geq {}^i\Upsilon \implies \begin{cases} \left\langle \nabla {}^iV({}^ix), {}^if(x,u)\right\rangle \leq {}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) \leq {}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} $$

where ${}^i\Upsilon = \max\left\{\max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta\left({}^jV\left({}^jx\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right)\right\}$.

Denote a *gain operator* $\Gamma : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ for the interconnection $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ wrt a collection $\left\{{}^iV\right\}_{i=1}^n$ by

$$\Gamma(s) := \begin{pmatrix} \max\left\{{}^{12}\vartheta(s_2), {}^{13}\vartheta(s_3), \ldots, {}^{1n}\vartheta(s_n)\right\} \\ \max\left\{{}^{21}\vartheta(s_1), {}^{23}\vartheta(s_3), \ldots, {}^{2n}\vartheta(s_n)\right\} \\ \vdots \\ \max\left\{{}^{i1}\vartheta(s_1), {}^{i2}\vartheta(s_2), \ldots, {}^{i,i-1}\vartheta(s_{i-1}), {}^{i,i+1}\vartheta(s_{i+1}), \ldots, {}^{in}\vartheta(s_n)\right\} \\ \vdots \\ \max\left\{{}^{n1}\vartheta(s_1), {}^{n2}\vartheta(s_2), \ldots, {}^{n,n-1}\vartheta(s_{n-1})\right\} \end{pmatrix}.$$

Definition 4.31 (Small-Gain Condition)**.** Let $\Gamma : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ be a gain operator for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form (4.7). We say that the small-gain condition wrt Γ is satisfied if $\Gamma \not\geq \mathrm{id}$.

Definition 4.32 (Ω-Paths)**.** A function $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{K}^n_\infty$ is called an Ω-path wrt gain operator Γ for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form (4.7) if the following is satisfied:

(Ω1) For each $i \in \mathbb{N}_n$, it holds σ_i^{-1} is locally Lipschitz continuous;

(Ω2) For any compact set $K \subset (0, \infty)$, there are finite constants $0 < c_1 < c_2$ such that for all point of differentiability of σ_i^{-1} and $i \in \mathbb{N}_n$, it holds that

$$c_1 \leq \left(\sigma_i^{-1}\right)'(s) \leq c_2 \quad \text{for all } s \in K;$$

(Ω3) For all $s > 0$, it holds that

$$\Gamma\left(\sigma(s)\right) \leq \sigma(s).$$

Theorem 4.33 (Existence of Ω-Path)**.** *An Ω-path $\sigma \in \mathcal{K}^n_\infty$ wrt gain operator Γ for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form* (4.7) *exists if and only if the small-gain condition wrt Γ is satisfied.*

Proof. ($\Rightarrow$) Since

$$\Gamma(s) = \Gamma(\sigma(\sigma^{-1}(s))) \leq \sigma(\sigma^{-1}(s)) = s$$

for all $s \geq 0$, it follows that $\Gamma \not\geq \mathrm{id}$.

($\Leftarrow$) See [54], Theorem 5.2 for $\mu = \max$ and $\Gamma_\mu \not\geq \mathrm{id}$. □

Here we are ready to provide sufficient condition to guarantee ISS for the interconnection $\mathcal{H}$. The following theorem uses the result of existence of Ω-path to construct an ISS-Lyapunov function for $\mathcal{H}$.

Theorem 4.34 (Small-Gain Theorem)**.** *Let Γ be a gain operator for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ of the form* (4.7) *wrt a collection of ISS-Lyapunov functions $\left\{{}^iV\right\}_{i=1}^n$ satisfying* (4.49)–(4.50)*. If the small-gain condition wrt Γ is satisfied, then $\mathcal{H}$ is ISS wrt $\mathcal{A}$.*

Proof. To show that $\mathcal{H}$ is ISS wrt $\mathcal{A}$, it is enough to only show existence of an ISS-Lyapunov function V wrt $(\mathcal{H}, \mathcal{A})$ such that for some $\psi_1, \psi_2, \vartheta \in \mathcal{K}_\infty$, and $\varphi \in \mathcal{P}$, it satisfies

$$\psi_1\left(\|x\|_\mathcal{A}\right) \leq V(x) \leq \psi_2\left(\|x\|_\mathcal{A}\right) \quad \text{for all } x \in \mathcal{X},$$

and

$$V(x) \geq \vartheta(\|u\|_\infty) \implies \begin{cases} \langle \nabla V(x), f(x,u)\rangle \leq -\varphi(V(x)) & \text{for } x \in \mathcal{C} \setminus \mathcal{A}, \\ V(g(x,u)) - V(x) \leq -\varphi(V(x)) & \text{for } x \in \mathcal{D} \setminus \mathcal{A}, \end{cases}$$

where $f(x,u) := \left({}^1\tilde{f}^{\mathrm{T}}(x,u), \ldots, {}^n\tilde{f}^{\mathrm{T}}(x,u)\right)^{\mathrm{T}}$, $g(x,u) := \left({}^1\tilde{g}^{\mathrm{T}}(x,u), \ldots, {}^n\tilde{g}^{\mathrm{T}}(x,u)\right)^{\mathrm{T}}$,

$${}^i\tilde{f}(x,u) := \begin{cases} {}^if(x,u) & \text{if } i \in \mathcal{I}_{\mathcal{C}}(x,u), \\ 0 & \text{otherwise.} \end{cases}, \quad {}^i\tilde{g}(x,u) := \begin{cases} {}^ig(x,u) & \text{if } i \in \mathcal{I}_{\mathcal{D}}(x,u), \\ x & \text{otherwise.} \end{cases}, \tag{4.51}$$

$\mathcal{C} := \bigcup_{i=1}^{n} {}^i\mathcal{C}$ and $\mathcal{D} := \bigcup_{i=1}^{n} {}^i\mathcal{D}$.

Since the small-gain condition wrt Γ is satisfied, there exists an Ω-path

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{K}_\infty^n.$$

Let us denote $V : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}$ by

$$V(x) = \max_{i \in \mathbb{N}_n} \sigma_i^{-1}({}^iV({}^ix)).$$

Due to a reason that

$$\frac{1}{n}\sum_{i=1}^{n} \sigma_i^{-1}({}^iV({}^ix)) \leq V(x) \leq \sum_{i=1}^{n} \sigma_i^{-1}({}^iV({}^ix)),$$

there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ defined by

$$\psi_1(s) := \frac{1}{n}\sum_{i=1}^{n} \sigma_i^{-1}({}^i\psi_1(s_i)), \text{ and } \psi_2(s) := \sum_{i=1}^{n} \sigma_i^{-1}({}^i\psi_2(s_i))$$

for all $s = (s_2, s_2) \geq 0$ such that

$$\psi_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \psi_2(\|x\|_{\mathcal{A}}).$$

Denote

$$I = \left\{ i \in \mathbb{N}_n : \ V(x) = \sigma_i^{-1}({}^iV({}^ix)) > \max_{j \in \mathbb{N}_n \setminus \{i\}} \sigma_j^{-1}({}^jV({}^jx)) \right\}.$$

Fix $i \in I$. By the property (Ω3) in Definition 4.32, it follows that

$${}^iV({}^ix) = \sigma_i(V(x)) \geq \max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta(\sigma_j(V(x))) \geq \max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta({}^jV({}^jx)).$$

By the property (Ω2) and (4.49)–(4.50), it yields

$$V(x) \geq \vartheta(\|u\|_\infty) := \max_{i \in N_n} \sigma_i^{-1}({}^{iu}\vartheta(\|u\|_\infty))$$

implies

$$\begin{aligned}\langle \nabla V(x), f(x,u)\rangle &= \left(\sigma_i^{-1}\right)'({}^iV({}^ix))\left\langle \nabla\, {}^iV, {}^if(x,u)\right\rangle \\ &\le -\left(\sigma_i^{-1}\right)'({}^iV({}^ix))\,{}^i\varphi({}^iV({}^ix)) \\ &= -\left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\,{}^i\varphi(\sigma_i(V(x))) \le -\varphi_1(V(x)),\end{aligned}$$

for $x \in \mathcal{C}$, and

$$\begin{aligned}V(g(x,u)) - V(x) &= \sigma_i^{-1}({}^iV({}^ig(x,u))) - V(x) \\ &\le \sigma_i^{-1}\left({}^iV({}^ix) - {}^i\varphi({}^iV({}^ix))\right) - V(x) \\ &= -\left(V(x) - \sigma_i^{-1}\left(\sigma_i(V(x)) - {}^i\varphi(\sigma_i(V(x)))\right)\right) \le -\varphi_2(V(x)).\end{aligned}$$

for $x \in \mathcal{D}$, where $\varphi_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\varphi_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ are defined by

$$\varphi_1(s) := \max_{i\in\mathbb{N}_n}\left\{\left((\sigma_i^{-1})' \circ \sigma_i\right)(s) \cdot \left({}^i\varphi \circ \sigma_i\right)(s)\right\},$$

and

$$\varphi_2(s) := \max_{i\in\mathbb{N}_n}\left\{s - \sigma_i^{-1}\left(\sigma_i(s) - {}^i\varphi(\sigma_i(s))\right)\right\}$$

for all $s \ge 0$. Note that $\varphi_1, \varphi_2 \in \mathcal{P}$ since for each $i \in \mathbb{N}_n$, it holds $\left(\sigma_i^{-1}\right)'(s) > 0$ and $s > \sigma_i^{-1}\left(\sigma_i(s) - {}^i\varphi(\sigma_i(s))\right)$ for all $s > 0$. In consequence, there exists $\varphi \in \mathcal{P}$ defined by $\varphi(s) := \max\{\varphi_1(s), \varphi_2(s)\}$ for all $s \ge 0$ such that it holds $V(x) \ge \vartheta(\|u\|_\infty)$ implies $\langle \nabla V(x), f(x,u)\rangle \le -\varphi(V(x))$ for all $x \in \mathcal{C}$, and $V(g(x,u)) - V(x) \le -\varphi(V(x))$ for all $x \in \mathcal{D}$. □

4.6.3 Additional Constructions

Various types of Lyapunov candidate functions can be used to provide conditions for guaranteeing stability of hybrid dynamical systems. In the foregoing subsection, we construct an ISS-Lyapunov function for $(\mathcal{H}, \mathcal{A})$ from a collection of ISS-Lyapunov functions wrt $({}^i\mathcal{H}, {}^i\mathcal{A})$. Therefore, we are going further on investigations for constructions of relaxed ISS-Lyapunov function and ISS-Lyapunov candidate function for $(\mathcal{H}, \mathcal{A})$ from a collection of relaxed ISS-Lyapunov functions or ISS-Lyapunov candidate functions wrt $({}^i\mathcal{H}, {}^i\mathcal{A})$. The results are obtained by a combination of the small-gain condition and other stability conditions guaranteeing that each subsystem ${}^i\mathcal{H}$ is ISS wrt ${}^i\mathcal{A}$.

Theorem 4.35. *For each $i \in \mathbb{N}_n$, let ${}^i\mathcal{H}$ be a non-eventually discrete hybrid system, ${}^i\mathcal{A}$ be nonempty subset of ${}^i\mathcal{X}$, and ${}^iV$ be a relaxed ISS-Lyapunov function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^i\vartheta \in \mathcal{K}_\infty$ and ${}^i\varphi \in \mathcal{P}$ it satisfies*

$${}^iV({}^ix) \ge {}^i\Upsilon \implies \begin{cases}\left\langle \nabla\, {}^iV({}^ix), {}^if(x,u)\right\rangle \le -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) \le 0 & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A},\end{cases} \tag{4.52}$$

where ${}^i\Upsilon = \max\left\{\max_{j\in\mathbb{N}_n\setminus\{i\}} {}^{ij}\vartheta\left({}^jV\left({}^jx\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right)\right\}$. Given by Γ a gain operator for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ wrt a collection $\left\{{}^iV\right\}_{i=1}^n$. If the small gain condition wrt Γ is satisfied, then $\mathcal{H}$ is ISS wrt $\mathcal{A}$.

Proof. To show that $\mathcal{H}$ is ISS wrt $\mathcal{A}$, it is enough to only show existence of a relaxed ISS-Lyapunov function V wrt $(\mathcal{H}, \mathcal{A})$ such that for some $\psi_1, \psi_2, \vartheta \in \mathcal{K}_\infty$, and $\varphi \in \mathcal{P}$, it satisfies

$$\psi_1\left(\|x\|_{\mathcal{A}}\right) \leq V(x) \leq \psi_2\left(\|x\|_{\mathcal{A}}\right) \quad \text{for all } x \in \mathcal{X},$$

and

$$V(x) \geq \vartheta\left(\|u\|_\infty\right) \implies \begin{cases} \langle \nabla V(x), f(x,u)\rangle \leq -\varphi(V(x)) & \text{for } x \in \mathcal{C} \setminus \mathcal{A}, \\ V(g(x,u)) - V(x) \leq 0 & \text{for } x \in \mathcal{D} \setminus \mathcal{A}, \end{cases}$$

where f, g, $\mathcal{C}$ and $\mathcal{D}$ are exactly defined in the proof of Theorem 4.34. The proof is going along the lines of the proof of Theorem 4.34, except $V(x) \geq \vartheta\left(\|u\|_\infty\right)$ implies

$$\begin{aligned} V(g(x,u)) - V(x) &= \sigma_i^{-1}({}^iV({}^ig(x,u))) - V(x) \\ &\leq \sigma_i^{-1}\left({}^iV({}^ix)\right) - V(x) = V(x) - V(x) = 0 \end{aligned}$$

for $x \in \mathcal{D}$. □

Theorem 4.36. *For each $i \in \mathbb{N}_n$, let ${}^i\mathcal{H}$ be a non-eventually continuous hybrid system, ${}^i\mathcal{A}$ be nonempty subset of ${}^i\mathcal{X}$, and ${}^iV$ be a relaxed ISS-Lyapunov function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some ${}^i\vartheta \in \mathcal{K}_\infty$ and ${}^i\varphi \in \mathcal{P}$ it satisfies*

$$ {}^iV({}^ix) \geq {}^i\Upsilon \implies \begin{cases} \left\langle \nabla\, {}^iV({}^ix), {}^if(x,u)\right\rangle \leq 0 & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x,u)) - {}^iV({}^ix) \leq -\,{}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.53}$$

where ${}^i\Upsilon = \max\left\{ \max_{j\in\mathbb{N}_n\setminus\{i\}} {}^{ij}\vartheta\left({}^jV\left({}^jx\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right)\right\}$. Given by Γ a gain operator for $\mathcal{H} = \left\{{}^i\mathcal{H}\right\}_{i=1}^n$ wrt a collection $\left\{{}^iV\right\}_{i=1}^n$. If the small gain condition wrt Γ is satisfied, then $\mathcal{H}$ is ISS wrt $\mathcal{A}$.

Proof. To show that $\mathcal{H}$ is ISS wrt $\mathcal{A}$, it is enough to only show existence of a relaxed ISS-Lyapunov function V wrt $(\mathcal{H}, \mathcal{A})$ such that for some $\psi_1, \psi_2, \vartheta \in \mathcal{K}_\infty$, and $\varphi \in \mathcal{P}$, it satisfies

$$\psi_1\left(\|x\|_{\mathcal{A}}\right) \leq V(x) \leq \psi_2\left(\|x\|_{\mathcal{A}}\right) \quad \text{for all } x \in \mathcal{X},$$

and

$$V(x) \geq \vartheta\left(\|u\|_\infty\right) \implies \begin{cases} \langle \nabla V(x), f(x,u)\rangle \leq 0 & \text{for } x \in \mathcal{C} \setminus \mathcal{A}, \\ V(g(x,u)) - V(x) \leq -\varphi(V(x)) & \text{for } x \in \mathcal{D} \setminus \mathcal{A}, \end{cases}$$

where f, g, $\mathcal{C}$ and $\mathcal{D}$ are exactly defined in the proof of Theorem 4.34. The proof is going along the lines of the proof of Theorem 4.34, except $V(x) \geq \vartheta\left(\|u\|_\infty\right)$ implies

$$\langle \nabla V(x), f(x,u)\rangle = \left(\sigma_i^{-1}\right)'\left({}^iV({}^ix)\right)\left\langle \nabla\, {}^iV, {}^if(x,u)\right\rangle \leq 0$$

for $x \in \mathcal{C}$. □

Theorem 4.37. *For each $i \in \mathbb{N}_n$, let ${}^i\mathcal{H}$ be a class $L(\theta)$ hybrid system, ${}^i\mathcal{A}$ be nonempty subset of ${}^i\mathcal{X}$, and ${}^iV$ be an ISS-Lyapunov candidate function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some*

${}^i\vartheta \in \mathcal{K}_\infty$, ${}^i\varphi, {}^i\lambda \in \mathcal{P}$ and a constant ${}^i\delta > 0$, it satisfies

$$ {}^iV({}^ix) \geq {}^i\Upsilon \implies \begin{cases} \langle \nabla\, {}^iV({}^ix), {}^if(x,u)\rangle \leq -\, {}^i\varphi({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{C} \setminus \mathcal{A}, \\ {}^iV({}^ig(x,u)) \leq {}^i\lambda({}^iV({}^ix)) & \text{for } x \in {}^i\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.54}$$

where ${}^i\Upsilon = \max\left\{ \max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta\left({}^jV\left({}^jx\right)\right), {}^{iu}\vartheta\left(\|u\|_\infty\right) \right\}$*, and additionally*

$$ \int_a^{{}^i\lambda(a)} \frac{\mathrm{d}s}{{}^i\varphi(s)} \leq \theta - {}^i\delta \quad \text{for all } a > 0. \tag{4.55}$$

Given by Γ a gain operator for $\mathcal{H} = \{{}^i\mathcal{H}\}_{i=1}^n$ wrt a collection $\{{}^iV\}_{i=1}^n$. If the small gain condition wrt Γ is satisfied, then there exists an ISS-Lyapunov candidate function for $(\mathcal{H}, \mathcal{A})$.

Proof. We are going to show existence of an ISS-Lyapunov candidate function V wrt $(\mathcal{H}, \mathcal{A})$ such that for some $\psi_1, \psi_2, \vartheta \in \mathcal{K}_\infty$, $\varphi, \lambda \in \mathcal{P}$ and a constant $\delta > 0$, it satisfies

$$ \psi_1\left(\|x\|_\mathcal{A}\right) \leq V(x) \leq \psi_2\left(\|x\|_\mathcal{A}\right) \qquad \text{for all } x \in \mathcal{X}, $$

$$ V(x) \geq \vartheta\left(\|u\|_\infty\right) \implies \begin{cases} \langle \nabla\, V(x), f(x,u)\rangle \leq -\, \varphi(V(x)) & \text{for } x \in \mathcal{C} \setminus \mathcal{A}, \\ V(g(x,u)) \leq \lambda(V(x)) & \text{for } x \in \mathcal{D} \setminus \mathcal{A}, \end{cases} $$

where f, g, $\mathcal{C}$ and $\mathcal{D}$ are exactly defined in the proof of Theorem 4.34. The proof is also going along the lines of the proof of Theorem 4.34, except $V(x) \geq \vartheta\left(\|u\|_\infty\right)$ implies

$$ \begin{aligned} \langle \nabla V(x), f(x,u)\rangle &= \left(\sigma_i^{-1}\right)'({}^iV({}^ix))\left\langle \nabla\, {}^iV, {}^if(x,u)\right\rangle \\ &\leq -\left(\sigma_i^{-1}\right)'({}^iV({}^ix))\, {}^i\varphi({}^iV({}^ix)) \\ &= -\left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\, {}^i\varphi(\sigma_i(V(x))) \leq -\varphi(V(x)) \end{aligned} $$

for $x \in \mathcal{C}$, , and

$$ \begin{aligned} V(g(x,u)) &= \sigma_i^{-1}({}^iV({}^ig(x,u))) \\ &\leq \sigma_i^{-1}({}^i\lambda({}^iV({}^ix))) \\ &= \sigma_i^{-1}({}^i\lambda(\sigma_i(\sigma_i^{-1}(V(x))))) \leq \lambda(V(x)) \end{aligned} $$

for $x \in \mathcal{D}$, where $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are defined by

$$ \varphi(s) := \max_{i \in \mathbb{N}_n} \left\{ \left((\sigma_i^{-1})' \circ \sigma_i\right)(s) \cdot \left({}^i\varphi \circ \sigma_i\right)(s) \right\}, $$

and

$$ \lambda(s) := \max_{i \in \mathbb{N}_n} \left\{ \left(\sigma_i^{-1} \circ {}^i\lambda \circ \sigma_i\right)(s) \right\} $$

for all $s \geq 0$. □

Theorem 4.38. *For each $i \in \mathbb{N}_n$, let ${}^i\mathcal{H}$ be a class $H(\theta)$ hybrid system, ${}^i\mathcal{A}$ be nonempty subset of ${}^i\mathcal{X}$, and ${}^iV$ be an ISS-Lyapunov candidate function for $({}^i\mathcal{H}, {}^i\mathcal{A})$ such that for some*

${}^{i}\vartheta \in \mathcal{K}_\infty$, ${}^{i}\varphi, {}^{i}\lambda \in \mathcal{P}$ and a constant ${}^{i}\delta > 0$, it satisfies

$$
{}^{i}V({}^{i}x) \geq {}^{i}\Upsilon \implies \begin{cases} \left\langle \nabla {}^{i}V({}^{i}x), {}^{i}f(x,u) \right\rangle \leq {}^{i}\varphi({}^{i}V({}^{i}x)) & \text{for } x \in {}^{i}\mathcal{C} \setminus \mathcal{A}, \\ {}^{i}V({}^{i}g(x,u)) \leq {}^{i}\lambda({}^{i}V({}^{i}x)) & \text{for } x \in {}^{i}\mathcal{D} \setminus \mathcal{A}, \end{cases} \tag{4.56}
$$

where ${}^{i}\Upsilon = \max \left\{ \max_{j \in \mathbb{N}_n \setminus \{i\}} {}^{ij}\vartheta \left({}^{j}V \left({}^{j}x \right) \right), {}^{iu}\vartheta \left(\|u\|_\infty \right) \right\}$, and additionally

$$
\int_{{}^{i}\lambda(a)}^{a} \frac{\mathrm{d}s}{{}^{i}\varphi(s)} \geq \theta + {}^{i}\delta \quad \text{for all } a > 0. \tag{4.57}
$$

Given by Γ a gain operator for $\mathcal{H} = \left\{ {}^{i}\mathcal{H} \right\}_{i=1}^{n}$ wrt a collection $\left\{ {}^{i}V \right\}_{i=1}^{n}$. If the small gain condition wrt Γ is satisfied, then there exists an ISS-Lyapunov candidate function for $(\mathcal{H}, \mathcal{A})$.

Proof. The proof is omitted due to similarity of the proof of Theorem 4.37. □

4.7 Further Problems

Here we would like to address some interesting problems that we have not discussed so far. One of them is the third kind of problems mentioned in Section 4.1. In the previous sections we have assumed for simplicity that each part ${}^{i}x$ of the whole state x can undergo at most one Zeno behavior. However, in general it may easily happen that a (sub)system undergoes several Zeno-type motions. As a simple example, we consider one bouncing ball (Example 3.1) and introduce an external input there:

$$
\dot{x} = \begin{pmatrix} x_2 \\ -\gamma + u \end{pmatrix} =: f(x,u), \quad x \in \mathcal{C},
$$

$$
x^{+} = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} =: g(x,u), \quad x \in \mathcal{D}
$$

where

$$
u(t) := \begin{cases} \gamma + 1 & \text{if } t_{\max} \leq t \leq 2t_{\max}, \\ 0 & \text{otherwise} \end{cases}
$$

and $t_{\max}$ is given by the equation (3.11). Let us take the same initial conditions as $x_1(0) = h$, $x_2 = 0$. After $t_{\max}$ the ball will be elevated to some finite height and dropped again. The second Zeno-type behavior will follow. This kind of behavior can be modeled by introducing multiple Zeno times $t_{\max,1}, t_{\max,2}, \ldots$ in an appropriate extension of the provided framework.

Moreover there is also another interesting issue that we do not consider here. A hybrid behavior can appear due to interconnection of systems that are not hybrid by their nature. For instance, a movement of a mass-point in a free space under some forces is usually modeled by ordinary differential equations and is not hybrid by its nature. However, if we consider two such mass-points in the same space, so that they can collide, then the resulting systems exhibits hybrid behavior. To model it as a hybrid systems, we will need to define the corresponding flow and jump sets due to interconnections.

Chapter 5

Hybrid Epidemic Systems

In this chapter, we propose a mathematical model for a spreading of disease with vaccination programs under a framework of hybrid dynamical system in the form of (3.1). The epidemic progress and the vaccination program are mathematically modeled by a system of differential equations and a system of difference equations respectively. Not only the effective vaccination strategies are demonstrated but the significant keys to efficiently launch the vaccination programs are also provided.

Our model is based on the works of [71–73] and additionally extended to the framework of hybrid dynamical systems [1–3, 66] which is rich in variety of tools for robust stability analysis and has useful features such as translation to other frameworks of hybrid dynamical systems including impulsive systems [33–36], switched systems [37] or hybrid automata [31, 32]. Moreover, we provide stability analysis of the new model, propose strategies to launch vaccination programs effectively and also demonstrate the provided strategies by some examples. In the very end of the chapter, we discuss some further problems which can possibly and practically be investigated in our framework.

5.1 Background

Nearly a century after the works of Kermack and McKendrick [71–73], modern mathematical models of an epidemic have been raised and allowed researchers to study an outbreak of disease by using some information of the state and the disease spreadability. As a consequence of many developments in this field, we can predict an epidemic and also enable a policy to either control or get rid of a disease. One of the difficulties to either extrapolate or model a spread of disease is to discover the factors reflecting the real-world data of the spread. Another arduousness is to deal with the dynamics of the systems because not only a spreadability of disease but also a public health policy can cause a progression of epidemic in either smooth transitions or instantaneous changes the number of infected individuals.

Numerous studies like [4–8, 74, 75] contributed mathematical models to show the effects of vaccination in epidemic system . Since the epidemic models with vaccinations proposed in [4–8] are based on impulsive systems (the launching time of vaccination is determined in advance), the systems may lack some degree of freedom to analyze a suitable launching time of vaccination program. Especially, the effect of launching time are probably difficult to be investigated due to a variety of choices to choose the impulse times in those systems. Moreover, we believe that the effectiveness and suitability of a launched vaccination program is still in question since the launching times are determined by a method of trial-and-error. Unlike [76], defining an *algebraic language* for specifying disease processes and multiple treatments from which a semantics in terms of hybrid dynamical system can be derived, we use a framework of hybrid dynamical system given in (3.1) which extends classical tools and methods to handle hybrid dynamic phenomena.

We are going to propose a mathematical model for a spread of disease in case of a vaccination program publicly launching to population. Unlike other works in this area of research, which are listed above, our vaccination program does not only depend on the states of system but also the launching time of vaccination which are not pre-determined. The vaccination is not forced to be released at the time we have no clues to launch it. This can save cost of vaccination programs and avoid inappropriate health policies.

5.2 Classic SIRS Model

Especially with large populations, continuous-time dynamical systems are used to describe a progression of epidemic. The SIR model which is firstly introduced in the works of Kermack and McKendrick [71–73] is admitted to be one of the most fundamental and significant of all epidemic mathematical models [77]. The Letters S, I and R represent different stages of disease in individuals, which stands for *susceptible*, *infected* and *recovered* respectively. According to the assumptions of SIR model, a recovered individual is assumed to be permanently immune against the disease. One of extensions of SIR model is SIRS model which allows temporarily immune. After a period of time, recovered individuals may risk to be infected again. Let us briefly overview SIRS model as follows.

5.2.1 Modeling

Let the number of population be permanently fixed and the mixing of population be homogeneous. The spread of disease is understood by means of infection getting from infected individuals. We require the assumptions as follow:

1. There are no disease deaths;
2. There are no immigrations of population;
3. The infected individuals can be recovered from the infection of disease;
4. The recovered individuals are temporarily immune against the infection of disease.

Each individual is assigned to a class representing a specific stage of the disease which is one of the following: susceptible class; infected class; and recovered class. The independent parameter of the model is time t, and we denote the part of system's state (or compartment) by S the proportion of susceptible individuals, I the proportion of infected individuals and R the proportion of recovered individuals. The state space is now ready to be defined as

$$\mathcal{X}_{SIRS} := \left\{ (S, I, R) \in [0,1]^3 : S + I + R = 1 \right\}.$$

The transfer rates between the classes are mathematically expressed as derivatives of the sizes of classes with respect to t. As a result, the dynamic is mathematically

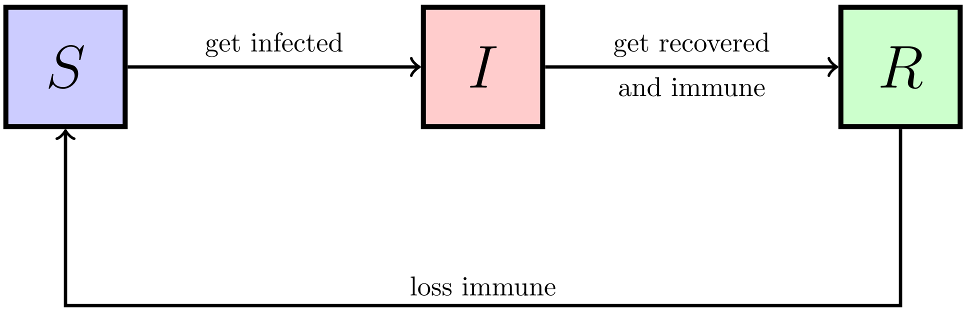


FIGURE 5.1: Transitions between compartments in SIRS model.

modeled by a system of differential equations. Let us define the epidemic system by

$$\Sigma_{SIRS} := (\mathcal{X}_{SIRS}, \Phi_{SIRS}) \tag{5.1}$$

where Φ_{SIRS} represents continuous dynamics and is given by the following ordinary differential equations:

$$\Phi_{SIRS} \begin{cases} \dot{S} &= -\beta SI + \delta R, \\ \dot{I} &= \beta SI - \gamma I, \\ \dot{R} &= \gamma I - \delta R. \end{cases}$$

The dynamics can be described as follows. Per unit time, some susceptible individuals become infected by social intercourse with any infected individual. Consequently, the number of infected individuals increasingly evolves at rate βSI, where $\beta > 0$ represents the infection rate. The infected individuals have been assumed to be recoverable at the recovery rate γ. Moreover, each of recovered individuals keeps temporary immune to the infection of disease by the infective period of $1/\delta$. With some infections of diseases like chickenpox or poliomyelitis, recovered individuals usually are practically given permanent immunity to them, δ can be assumed as zero.

5.2.2 Stability

Direct calculation shows that Σ_{SIRS} has two steady states called the disease-free steady state $E_0(\Sigma_{SIRS})$ and the disease steady state $E_+(\Sigma_{SIRS})$, where

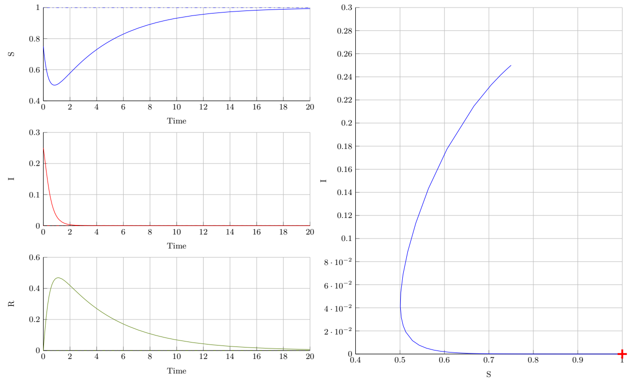
$$E_0(\Sigma_{SIRS}) = (1, 0, 0), \tag{5.2}$$

and

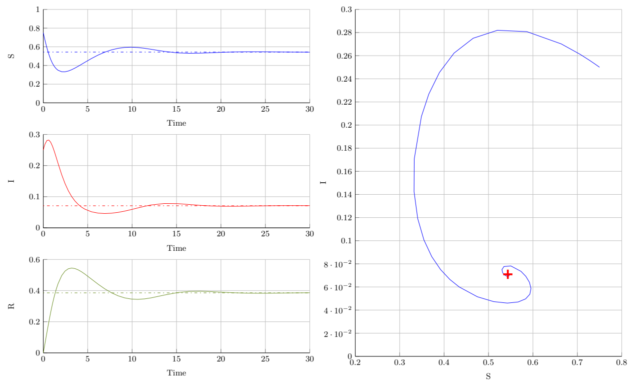
$$E_+(\Sigma_{SIRS}) = \left(\frac{\gamma}{\beta}, \frac{\delta\,(\beta - \gamma)}{\beta\,(\gamma + \delta)}, \frac{\gamma\,(\beta - \gamma)}{\beta\,(\gamma + \delta)} \right). \tag{5.3}$$

The global stability of the steady states can be acquired by means of direct Lyapunov's method. The following theorems are well-known results extracted from the literature [7, 78, 79].

Theorem 5.1. *The disease-free steady state $E_0(\Sigma_{SIRS})$ is globally asymptotically stable if $\beta \leq \gamma$.*



(A) $\beta = \gamma$.



(B) $\beta > \gamma$.

FIGURE 5.2: Simulations of the classic SIRS model.

Theorem 5.2. *The disease steady state $E_+(\Sigma_{SIRS})$ is locally asymptotically stable and the disease-free steady state $E_0(\Sigma_{SIRS})$ is unstable if $\beta > \gamma$.*

There are only two arguments affecting the behavior of solutions, which are the infection rate β and the recovery rate γ. The spread of the disease eventually disappears, and every individual get no more infected if the recovery rate is not smaller than the infection rate. In addition, the infective period does not give a huge impact to the system. In case that the infection rate β is larger than the recovery rate γ, no matter of other arguments, the infected individuals permanently remain in the population, which means the spread of disease is eventually long lasting. Figure 5.2 illustrates numerical simulations of the classic SIRS model. The simulations are given in two cases: (A) in case of $\beta = \gamma$, the trajectory tends to the disease-free steady state; (B) in case of $\beta > \gamma$, the trajectory eventually reach to the disease steady state.

5.3 Hybrid SIRS Model with Vaccination

Public health programs such as vaccination, isolation or quarantine have been applied to control a spread of disease throughout population for many decades. According to CDC[1], vaccination is injection of a killed or weakened infectious organism in order to prevent the infection of disease. This process is sometimes called immunization or inoculation. In case that infection rate β is larger than recovery rate γ, which leads to the result that an epidemic appears eternally, such a public health

[1]Centers for Disease Control and Prevention, 1600 Clifton Rd. Atlanta, GA 30333, USA

program like vaccination may help controlling an epidemical disaster. In this section, a hybrid SIRS model is introduced. A progress of epidemic is modeled by a system of differential equations based on SIRS model, and a vaccination program is applied to a class of susceptible individuals, which is modeled by a system of difference equations. Stability analysis is also provided in the very end of the section.

5.3.1 Modeling

In order to present a hybrid dynamical system for a spread of disease with vaccination, the additional assumptions for the epidemic system Σ_{SIRS} given in (5.1) are required:

1. The initial value of infected individuals is positive;
2. The epidemic appears eternally, i.e., $\beta > \gamma$;
3. No permanent immunity against disease , i.e., $\delta > 0$.

To stop or limit a spread of disease, vaccination programs can not be practically launched at the very beginning time since the vaccine against the infection of disease may not ready to be used yet. For this reason, let us introduce a *program clock* t_ν representing the realistic condition that the vaccination program will not be launched until time reaching to t_ν. Consequently, we assume that if all of the following is satisfied at time t:

1. the proportion of infected individuals I has reached to a closed set $I_v \subset [0, 1]$;
2. the proportion of susceptible individuals S has reached to a closed set $S_v \subset [0, 1]$;
3. time has already passed by the program clock $t_\nu \geq 0$,

then a vaccination program is allowed to be instantaneously launched to the class of susceptible individuals, which results that the number of vaccinated individuals becomes $\rho S(t)$, where $\rho \in [0, 1]$, since the vaccines may not be applied to all susceptible individuals. Additionally, some of the vaccinated individuals in the susceptible class are instantly moved to the recovery class by means of vaccine's performance $\nu \in [0, 1]$. It results that there are $\nu\rho S(t)$ individuals moving from the susceptible class to the recovered class, and there are $(1 - \nu)\rho S(t)$ individuals, representing vaccinated individuals who become truly infected due to the possibility of vaccine's side effects, moving from the susceptible class to the infection class.

Note that we also assume that $\min I_v$ is the lowest nonnegative number such that a proportion of infected individuals can be detected, which means no one practically knows that the infected individuals actually exist if $I(t) < \min I_v$ for any time t. Moreover, the sets S_v and I_v are called *S-detection* and *I-detection* respectively.

Here we provide a hybrid SIRS model with vaccination as follows.

$$\Sigma_{SIRS,v} := (\mathcal{X}_{SIRS,v}, \Phi_{SIRS,v}, \mathcal{C}_{SIRS,v}, \Delta_{SIRS,v}, \mathcal{D}_{SIRS,v}) \tag{5.4}$$

where

$$\mathcal{X}_{SIRS,v} := \left\{(S, I, R, t) \in [0, 1]^3 \times \mathbb{R}_{\geq 0}\right\},$$

$$\Phi_{SIRS,v}\begin{cases}\dot{S} &= -\beta SI + \delta R,\\ \dot{I} &= \beta SI - \gamma I,\\ \dot{R} &= \gamma I - \delta R,\\ \dot{t} &= 1,\end{cases}$$

if $(S, I, R, t) \in \mathcal{C}_{SIRS,v} := \mathcal{X}_{SIRS,v}$,

$$\Delta_{SIRS,v}\begin{cases}S^+ &= S - \rho S,\\ I^+ &= I + (1-\nu)\rho S,\\ R^+ &= R + \nu\rho S,\\ t^+ &= t,\end{cases}$$

if $(S, I, R, t) \in \mathcal{D}_{SIRS,v} := \{(S, I, R, t) \in \mathcal{X}_{SIRS,v} : S \in S_v, I \in I_v, t \geq t_\nu\}$.

Since the flow set $\mathcal{C}_{SIRS,v}$ and the jump set $\mathcal{D}_{SIRS,v}$ are not disjoint, the trajectories can basically either flow or jump in the overlapping of $\mathcal{C}_{SIRS,v}$ and $\mathcal{D}_{SIRS,v}$ according to $\Phi_{SIRS,v}$ and $\Delta_{SIRS,v}$ respectively. For this reason, a solution to $\Sigma_{SIRS,v}$ is generally not unique. Moreover, a vaccination program is not strictly forced to be launched even at the time that all of requirement is satisfied. It may never be launched or just temporarily delayed as long as the vaccination program's conditions are still fulfilled.

Different from [80] which propose an SIR model with state dependent vaccination, our hybrid SIRS model additionally consists of not only the extra arguments like a performance of vaccine or more flexibility to launch vaccination but also taking advantage in means of tools on stability analysis provided in the framework.

5.3.2 Stability

According to the hybrid epidemic system $\Sigma_{SIRS,v}$, time is a part of the state. We may use the notion of partial stability given in Section 3.5 to deal with the hybrid epidemic system.

However, the part of state that we focus on stability is (S, I, R). For simplicity, we are going to provide results of stability for the hybrid epidemic system with the program clock $t_\nu = 0$. Therefore, the hybrid epidemic system $\Sigma_{SIRS,v}$ is given as follows:

$$\Sigma_{SIRS,v} := (\mathcal{X}_{SIRS,v}, \Phi_{SIRS,v}, \mathcal{C}_{SIRS,v}, \Delta_{SIRS,v}, \mathcal{D}_{SIRS,v}) \tag{5.5}$$

where

$$\mathcal{X}_{SIRS,v} := \left\{(S, I, R) \in [0,1]^3\right\},$$

$$\Phi_{SIRS,v}\begin{cases}\dot{S} &= -\beta SI + \delta R,\\ \dot{I} &= \beta SI - \gamma I,\\ \dot{R} &= \gamma I - \delta R,\end{cases}$$

if $(S, I, R) \in \mathcal{C}_{SIRS,v} := \mathcal{X}_{SIRS,v}$, and

$$\Delta_{SIRS,v}\begin{cases}S^+ &= S - \rho S,\\ I^+ &= I + (1-\nu)\rho S,\\ R^+ &= R + \nu\rho S,\end{cases}$$

if $(S, I, R) \in \mathcal{D}_{SIRS,v} := \{(S, I, R) \in \mathcal{X}_{SIRS,v} : S \in S_v, I \in I_v, t \geq t_\nu\}$.

Since the system allows a chance of no vaccination, we firstly consider the trivial case where the discrete dynamics $\Delta_{x,v}$ is not exhibited. Obviously, it follows that for a hybrid epidemic system $\Sigma_{SIRS,v}$ the disease steady state $E_+(\Sigma_{SIRS,v})$ is asymptotically stable, and the disease-free steady state $E_0(\Sigma_{SIRS,v})$ is unstable.

We are now consider stability of the hybrid epidemic system $\Sigma_{SIRS,v}$ in case that vaccination program is definitely launched. Let us do the following mathematical procedure on $\Sigma_{SIRS,v}$. First of all, the system is going to be reduced by one degree of freedom, which means that R is substituted by $1 - S - I$. It follows that

$$\begin{cases} \dot{S} &= -\beta SI + \delta(1 - S - I), \\ \dot{I} &= \beta SI - \gamma I, \end{cases}$$

and

$$\begin{cases} S^+ &= S - \rho S \\ I^+ &= I + (1 - \nu)\rho S \end{cases}$$

Moreover, let us provide a translation of $\Sigma_{SIRS,v}$ to the equivalent, but lower dimension, hybrid epidemic system

$$\Sigma_{x,v} := (\mathcal{X}_{x,v}, \Phi_{x,v}, \mathcal{C}_{x,v}, \Delta_{x,v}, \mathcal{D}_{x,v}),$$

where

$$x_1 := S + \frac{\delta}{\beta}, \qquad x_2 := I,$$

$$\mathcal{X}_{x,v} := \left\{ (x_1, x_2) \in \left[\frac{\delta}{\beta}, 1 + \frac{\delta}{\beta}\right] \times [0, 1] : x_1 + x_2 \leq 1 + \frac{\delta}{\beta} \right\},$$

$$\Phi_{x,v} \begin{cases} \dot{x_1} &= -\beta x_1 x_2 - \delta x_1 + \dfrac{\delta(\beta + \delta)}{\beta}, \\ \dot{x_2} &= \beta x_1 x_2 - (\gamma + \delta) x_2, \end{cases} \quad \text{if } (x, t) \in \mathcal{C}_{x,v},$$

$$\mathcal{C}_{x,v} := \mathcal{X}_{x,v},$$

$$\Delta_{x,v} \begin{cases} x_1^+ &= (x_1 - \dfrac{\delta}{\beta}) - \rho(x_1 - \dfrac{\delta}{\beta}) + \dfrac{\delta}{\beta} \\ &= (1 - \rho) x_1 + \dfrac{\delta\rho}{\beta}, \\ x_2^+ &= x_2 + (1 - \nu)\rho(x_1 - \dfrac{\delta}{\beta}) \end{cases} \quad \text{if } (x, t) \in \mathcal{D}_{x,v},$$

$$\mathcal{D}_{x,v} := \left\{ (x, t) \in \mathcal{X}_{x,v} : x_1 \in S'_\nu, x_2 \in I_v \right\},$$

$$S'_\nu := \left\{ s' \in \left[\frac{\delta}{\beta}, 1 + \frac{\delta}{\beta}\right] : \exists\, s \in S_v,\ s' = s + \frac{\delta}{\beta} \right\},$$

Note that the disease steady state of the above system then becomes

$$E_+(\Sigma_{x,v}) = \left(\frac{\gamma + \delta}{\beta}, \frac{\delta(\beta - \gamma)}{\beta(\gamma + \delta)} \right).$$

Let us explicitly show what happens to the system if vaccinations are launched by a limited number of vaccination programs.

Theorem 5.3. *If the hybrid epidemic system $\Sigma_{x,v}$ exhibits discrete dynamics finitely, i.e.,*

$$\sup_j \text{ dom } x < \infty \tag{5.6}$$

where (x, t) is a solution to $\Sigma_{x,v}$, then the epidemic steady state $E_+(\Sigma_{x,v})$ is asymptotically stable.

Proof. Obviously, the hybrid epidemic system $\Sigma_{x,v}$ is eventually continuous system under the condition (5.6). By Theorem 5.2, we conclude that the epidemic steady state $E_+(\Sigma_{x,v})$ is asymptotically stable. □

According to the above result, we found that if vaccination programs are launched limitedly, then the epidemic eventually remains forever.

To provide additional results of stability of the disease steady state of the hybrid epidemic system $\Sigma_{x,v}$, let us provide a definition of the ideal vaccination.

Definition 5.4 (Ideal Vaccination)**.** For the hybrid epidemic system $\Sigma_{x,v}$, a vaccination program is ideal if $\nu = 1$.

In the following results, we consider stability of $\Sigma_{x,v}$ in the case of infinitely launching of ideal vaccination program, i.e., an ideal vaccine always works perfectly with no unexpected side effects. To say precisely, no increasing in the number of infected individuals is the result from an ideal vaccination program.

Theorem 5.5. *If the vaccination is ideal, and*

$$\sup S_v^{'} < \frac{\delta}{\rho\beta},$$

then the epidemic steady state $E_+(\Sigma_{x,v})$ is asymptotically stable.

Proof. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x_1, x_2) := \begin{pmatrix} (1-\rho)x_1 + (\delta\rho/\beta) \\ x_2 + (1-\nu)\rho(x_1 - \frac{\delta}{\beta}) \end{pmatrix},$$

and $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by

$$V(x_1, x_2) := x_1 - \frac{(\gamma+\delta)}{\beta}\ln(x_1) + x_2 - \frac{\delta\,(\beta-\gamma)}{\beta\,(\gamma+\delta)}\ln(x_2) - \bar{v},$$

where

$$\bar{v} := \frac{\gamma+\delta}{\beta} - \frac{(\gamma+\delta)}{\beta}\ln(\frac{\gamma+\delta}{\beta}) + \frac{\delta(\beta-\gamma)}{\beta(\gamma+\delta)} - \frac{\delta\,(\beta-\gamma)}{\beta\,(\gamma+\delta)}\ln(\frac{\delta(\beta-\gamma)}{\beta(\gamma+\delta)}).$$

It follows that

$$\begin{aligned}\dot{V}(x_1,x_2) &= \left(1-\frac{(\gamma+\delta)}{\beta x_1}\right)\left(-\beta x_1x_2-\delta x_1+\frac{\delta(\beta+\delta)}{\beta}\right)\\ &\quad+\left(1-\frac{\delta\,(\beta-\gamma)}{\beta\,(\gamma+\delta)\,x_2}\right)(\beta x_1x_2-(\gamma+\delta)x_2)\\ &= -\frac{\delta\,(-\beta x_1+\gamma+\delta)\left(-\beta\delta x_1-\beta^2x_1+\gamma\beta+\delta\beta+\gamma\delta+\delta^2\right)}{\beta^2\,(\gamma+\delta)\,x_1}\\ &= -\frac{\delta\,(\beta+\delta)\,(-\beta x_1+\gamma+\delta)^2}{\beta^2\,(\gamma+\delta)\,x_1}.\end{aligned}$$

It is clear to see that $\dot{V}(x_1,x_2)<0$ for all $(x_1,x_2,t)\in\mathcal{C}_{x,v}$. Furthermore, consider the following

$$\begin{aligned}V(g(x_1,x_2))-V(x_1,x_2) &= -\rho\left(x_1-\frac{\delta}{\beta}\right)-\left(\frac{\gamma+\delta}{\beta}\right)\ln\left(1-\rho\left(1-\frac{\delta}{\beta x_1}\right)\right)\\ &= -\rho\left(x_1-\frac{\delta}{\beta}\right)-\left(\frac{\gamma+\delta}{\beta}\right)\ln\left(1+\frac{\delta-\rho\beta x_1}{\beta x_1}\right).\end{aligned}$$

It is obvious to see that

$$\ln\left(1+\frac{\delta-\rho\beta x_1}{\beta x_1}\right)>0\quad\text{if}\quad x_1<\frac{\delta}{\rho\beta}.$$

Therefore

$$V(g(x_1,x_2))-V(x_1,x_2)<0\quad\text{for all }(x_1,x_2,t)\in\mathcal{D}_{x,v}.$$

Finally, we conclude that the epidemic steady state $E_+(\Sigma_{x,v})$ is asymptotically stable. □

Corollary 5.6. *If the vaccination is ideal, and*

$$\sup S_v<\frac{\delta(1-\rho)}{\rho\beta},$$

then the epidemic steady state $E_+(\Sigma_{SIRS,v})$ is asymptotically stable.

We eventually discover that even in the case of ideal vaccine, the vaccination can possibly fail by choosing an inappropriate strategy to limit or stop an epidemic since the number of infected population is permanently positive and not different from the case of no or a limited number of vaccinations. In the next section, we are going to discuss on public health strategies of vaccination which can decrease the number of infected individuals.

5.4 Control Plans

According to the results shown in the previous section, sufficient conditions to limit or stop a spread of disease by vaccination programs are still in question. Throughout this section, we try to deal with S-detection S_v and I-detection I_v in order to discover a sufficient condition for stopping or limiting an epidemic. The following example shows a failure of vaccination caused by inappropriate S_v.

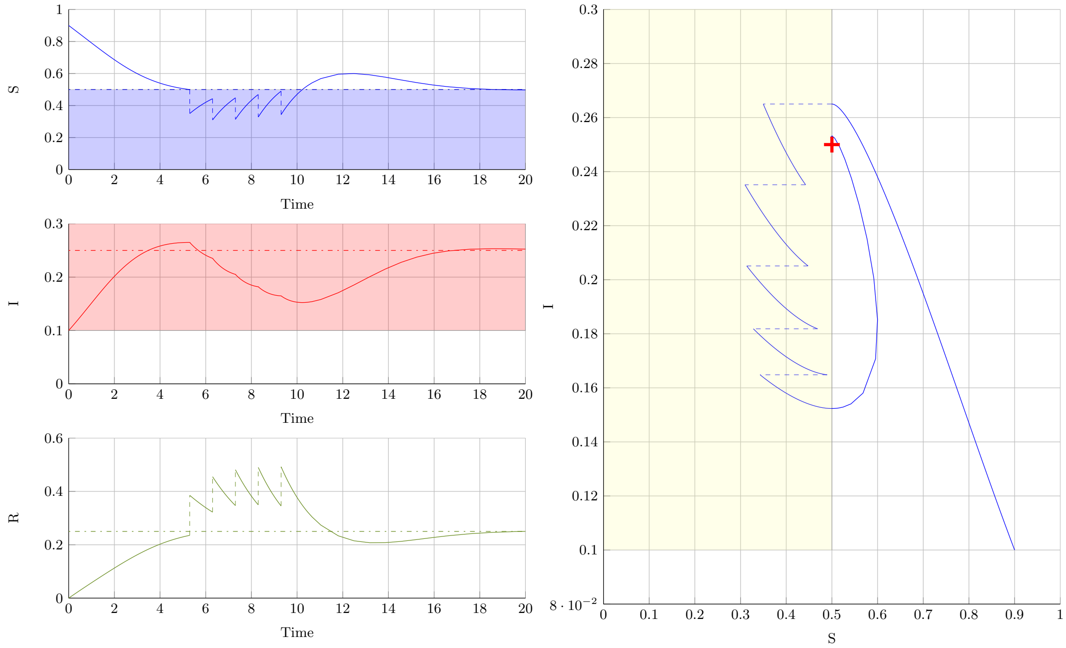


FIGURE 5.3: Solution to $\Sigma_{SIRS,v}$ in Example 5.7.

Example 5.7. Let the epidemic system $\Sigma_{SIRS,v}$ hold the arguments given in Table 5.1. The solution illustrated in Figure 5.3 shows that the vaccination is failed to control the spread of disease. Note that

$$\sup S_v = 0.5 < \frac{\delta(1-\rho)}{\rho\beta} = \frac{0.6(1-0.3)}{0.4(1.2)} = 0.75,$$

which means, by Corollary 5.6, that the disease steady state is asymptotically stable.

Remark 5.1. For any figure provided throughout the rest of this chapter: the blue, red and yellow filled regions represent the S-detection, I-detection and vaccination region respectively. Moreover, the red plus-sign indicates the epidemic steady state $E_+(\Sigma_{SIRS,v})$.

Unfortunately, a vaccination program does not always work effectively in order to limit or stop the number of infected individuals. Let us demonstrate a possibility to design a vaccination region which is exactly the jump set $\mathcal{D}_{SIRS,v}$. Basically, the number of infected individuals will be continuously decreasing in the left-hand side of the disease steady state in the SI-plane because

$$\dot{I}(t,j) = \beta S(t,j)I(t,j) - \gamma I(t,j) < 0 \quad \text{if} \quad S(t,j) < \frac{\gamma}{\beta}.$$

Therefore, the vaccination may be unnecessary to be applied on that region. Additionally, Corollary 5.6 guides us to launch vaccination appropriately, which suggest the following. We should choose the S-detection such that

$$\sup S_v \geq \frac{\delta(1-\rho)}{\rho\beta},$$

in order to avoid trajectories reaching to the disease steady state.

Captions	Arguments	Values
Infection rate	β	1.2
Recovery rate	γ	0.6
Loss of immunity rate	δ	0.6
Vaccination ratio	ρ	0.4
Vaccine performance	ν	1
S-detection	S_v	[0,0.5]
I-detection	I_v	[0.1,1]
Program clock	t_ν	0
Initial susceptible individual	$S(0,0)$	0.9
Initial infected individual	$I(0,0)$	0.1
Initial recovered individual	$R(0,0)$	0
Disease steady state	$E_+(\Sigma_{SIRS,v})$	(0.5,0.25,0.25)

TABLE 5.1: Arguments of $\Sigma_{SIRS,v}$ in Example 5.7.

In the following result, the vaccination strategies to control a spread of disease are proposed. The strategies aim to limit the spread of disease to a safely acceptable desired level $\bar{I}$ where

$$\bar{I} < I^* := \frac{\delta(\beta-\gamma)}{\beta(\gamma+\delta)}.$$

Note that the solutions to the hybrid epidemic systems $\Sigma_{SIRS,v}$ are not unique, so the time when vaccination programs are launched to the susceptible class will possibly vary according to various situations.

Strategy 5.8. *In case of a hybrid epidemic system $\Sigma_{SIRS,v}$ satisfying*

1. $S(0,0) > \dfrac{\gamma}{\beta}$*;*
2. $\beta > \gamma,\ \delta > 0,\ \rho > 0$ *and* $\nu = 1$*,*

choose the S-detection and I-detection as follows:

$$S_v := \begin{cases} \left[\dfrac{\gamma}{\beta}, \dfrac{\delta(1-\rho)}{\rho\beta}\right] & \text{if } \dfrac{\gamma}{\beta} < \dfrac{\delta(1-\rho)}{\rho\beta} < 1, \\ \left[\dfrac{\gamma}{\beta}, 1\right] & \text{otherwise}, \end{cases}$$
$$I_v := [\bar{I}, 1]$$

to control an epidemic.

In order to guarantee the efficiency of provided strategies, let us provide a definition of effective strategies as follow.

Definition 5.9 (Effective Strategies)**.** For a hybrid epidemic system $\Sigma_{SIRS,v}$, a strategy is *effective* if there exists a positive number T such that the number of infected individuals $I(t,j)$ is eventually not larger than the safely acceptable desired level

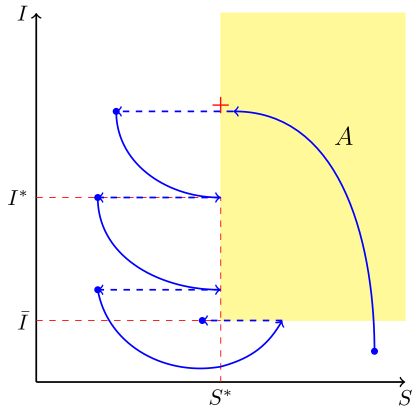


FIGURE 5.4: Graphical representation of a vaccination program in Strategy 5.8

$\bar{I} < I^*$ for any time t beyond T, where $(S^*, I^*, R^*) := E_+(\Sigma_{SIRS,v})$, i.e.,

$$\exists\, T > 0\; \forall\, t \geq T: \quad I(t,j) \leq \bar{I} < \frac{\delta(\beta-\gamma)}{\beta(\gamma+\delta)}, \quad (t,j) \in \operatorname{dom}\ I.$$

Theorem 5.10. *Strategy 5.8 is effective.*

Proof. We are going to consider the solutions on the SI-plane. Suppose that y is a solution to $\Sigma_{SIRS,v}$. Let z be a trajectory such that for all $(t,j) \in \operatorname{dom}\ y$ it satisfies

$$z(t,j) = (S(t,j), I(t,j))$$

where

$$y(t,j) = (S(t,j), I(t,j), 1 - S(t,j) - I(t,j), t)\,.$$

Define $A := S_v \times I_v$, and

$$E_+(\Sigma_{SI,\nu}) := (S^*, I^*) \in \mathbb{R}^2$$

where

$$E_+(\Sigma_{SIRS,v}) = (S^*, I^*, 1 - S^* - I^*).$$

In Figure 5.4, we depict this vaccine strategy and also visualize the vaccination region A.

According to the strategy, there exists $t_1 \geq t_\nu \geq 0$ such that the trajectories will flow to A which is assured by Corollary 5.6. The state $z(t_1, 0) \in A$ is therefore mapped by the discrete dynamics $\Delta_{SIRS,v}$. Consequently, there exists a finite natural number $j_1 > 0$ such that $z(t_1, j_1) \in [0,1]^2 \setminus A$. In case that the state satisfies $I(t_1, j_1) > \bar{I}$, then we omit the proof of this case since it is similar to the proof of Theorem 5.12.

Suppose that the state satisfies $I(t_1, j_1) = \bar{I}$. In the trivial case $\bar{I} = 0$, there are no dynamics in the system beyond this state, which is obvious to say that $I(t, j_1) = 0 < I^*$ for all $t \geq t_1$. Additionally suppose that $\bar{I} > 0$, therefore, the trajectory continuously flows to A in the direction that the proportion of infected individuals is decreasing since the state is yet in the region located on the left-hand side of $E_+(\Sigma_{SI,\nu})$ on the SI-plane. If the trajectory flows to the S-axis, then the strategy is effective. Otherwise, there exists $t_2 > t_1$ such that $z(t_2, j_1) \in A$. Indeed, $I(t, j_1) \leq \bar{I}$ for all $t \in [t_1, t_2]$.

Consequently the discrete dynamics exhibit to the system, and there exists a natural number $j_2 > j_1$ such that $z(t_2, j_2) \in [0,1]^2 \setminus A$. With this iteration, we therefore conclude that there exists $T = t_1$ such that, for any $j \geq j_1$,

$$I(t,j) \leq \bar{I} < I^* \quad \forall\, t \geq T.$$

The proof is completed since a proportion of infected individuals is eventually lower than I^*. □

Strategy 5.11. *In case a hybrid epidemic system $\Sigma_{SIRS,v}$ satisfying all of the requirements stated in Strategy 5.8 except $S(0,0) \leq \dfrac{\gamma}{\beta}$, choose I_v as given in Strategy 5.8. Moreover,*

1. *if $\bar{I} > 0$, then choose*

$$S_v := \begin{cases} \left[\dfrac{\gamma}{\beta}, \dfrac{\delta(1-\rho)}{\rho\beta}\right] & \text{if } \dfrac{\gamma}{\beta} < \dfrac{\delta(1-\rho)}{\rho\beta} < 1; \\ \left[\dfrac{\gamma}{\beta}, 1\right] & \text{otherwise}; \end{cases}$$

2. *if $\bar{I} = 0$, then pick any small number $\varepsilon > 0$ and choose*

$$S_v := \left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta} + \varepsilon\right];$$

to control an epidemic.

Remark 5.2. We can pick any small positive ε in the above strategy to control an epidemic. The smaller ε we pick results in the smaller region of vaccination programs, which practically imply to less resources of vaccination.

Theorem 5.12. *Strategy 5.11 is effective.*

Proof. Define the set A, the trajectory $z(t,j)$ and the point $E_+(\Sigma_{SI,\nu})$ as in the proof of Theorem 5.10. By Corollary 5.6, there exists $t_1 \geq t_\nu \geq 0$ such that $z(t_1, 0) \in A$ and $I(t_1, 0) \leq I^*$. Due to the discrete dynamics $\Delta_{\Sigma_{SIRS,v}}$, there exists a positive $j_1 \in \mathbb{N}$ such that $z(t_1, j_1) \in [0,1]^2 \setminus A$. Since the vaccination is ideal, $I(t_1, j_1) = I(t_1, 0) \leq I^*$. However the state $z(t_1, j_1)$ is yet in the region located on the left-hand side of $E_+(\Sigma_{SI,\nu})$. Hence the trajectories flow to A in the direction that the number of infected individuals is decreasing, which results that there exists $t_2 > t_1$ such that $z(t_2, j_1) \in A$ and $I(t_2, j_1) < I(t_1, j_1)$. As a consequence, the proportion of infected individuals gets lower on each round of the above iteration.

Since the sequence of $I(t_j, j)$ is decreasing and bounded from below by $\bar{I}$, let us consider two possible cases. The first case of consideration is $\bar{I} = 0$. We will show that the trajectory z will converge to $(S^*, \bar{I})$. Suppose by a contradiction that the proportion of infected individuals does not converge to $\bar{I}$, but it converges to $\widehat{I} > \bar{I}$, i.e., there exist $\widehat{I} > \bar{I}$, $T > 0$ and $J > 0$ such that $S(\tau, \xi) = S^*$ and $I(\tau, \xi) = \widehat{I}$ for all $\tau \geq T$ and $\xi \geq J$. However the trajectory $z(\tau, \xi) \in A$. Therefore the vaccination is launched and then $z(\tau, \xi + 1) \in [0,1]^2 \setminus A$. Since $S(\tau, \xi + 1) < S^*$, there exist $\kappa > 0$

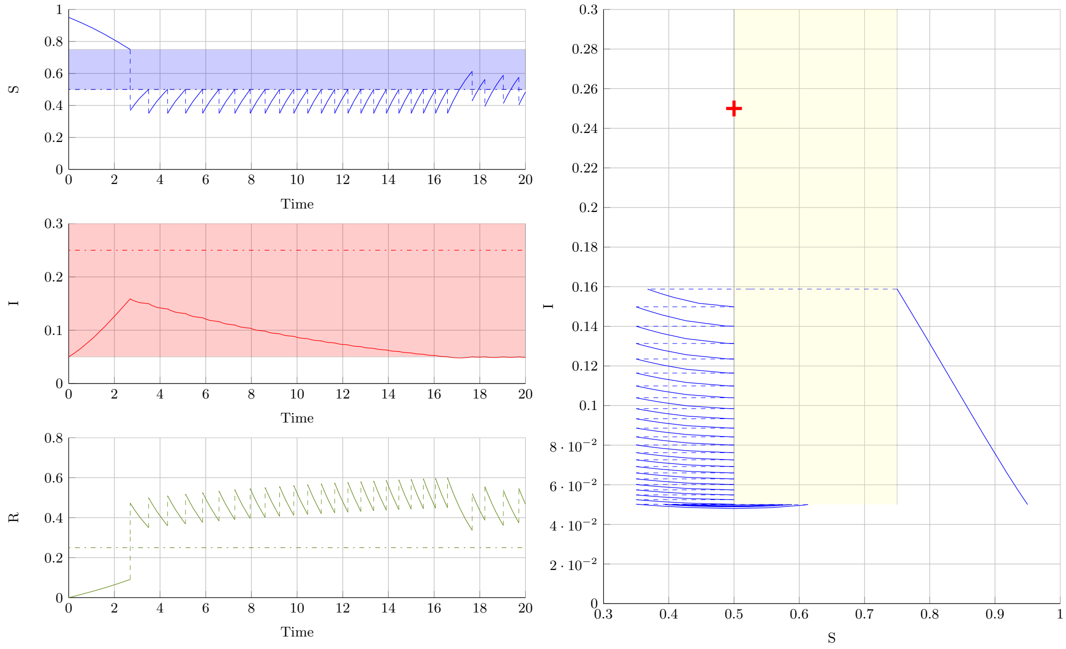


FIGURE 5.5: Solution to $\Sigma_{SIRS,v}$ in Example 5.13 with $I_v = [0.05, 1]$.

such that the state $z(\tau + \kappa, \xi + 1) \in A$ and $S(\tau + \kappa, \xi + 1) = S^*$. Note that from time $t = \tau$ to $t = \tau + \kappa$, the trajectory flows in the direction that the proportion of infected individuals decreases, i.e., $I(\tau + \kappa, \xi + 1) < \widehat{I}$. So the contradiction is obtained.

The second case is $\bar{I} \neq 0$. According to the above iteration, the trajectory will reach A and there exist $T_\nu > 0$ and $J_\nu > 0$ such that $S(T_\nu, J_\nu) > S^*$ and $I(T_\nu, J_\nu) = \bar{I}$. Therefore, the vaccinations are allowed to be launched until the trajectory is not in A. It results that there exists $k > 0$ such that $S(T_\nu, J_\nu + k) < S^*$ and $I(T_\nu, J_\nu + k) = \bar{I}$. So this iteration can be repeated infinitely many times which results that the proportion of infected individuals is limited by $\bar{I}$.

□

Example 5.13. Consider the hybrid epidemic system $\Sigma_{SIRS,v}$ along with the arguments shown in Table 5.2. In case of no vaccination, the trajectories always converge to the disease steady state $(0.5, 0.25, 0.25)$. To limit the proportion of infected individuals, Strategy 5.8 is therefore applied with $I_v = [0.05, 1]$ and $I_v = [0, 1]$. The solutions are also shown in Figure 5.5 and Figure 5.6 respectively.

Example 5.14. Consider the hybrid epidemic system $\Sigma_{SIRS,v}$ along with the arguments shown in Table 5.2 except $S(0, 0) = 0.05$ and $I(0, 0) = 0.95$. In this example, we apply Strategy 5.11. The result illustrated in Figure 5.7 shows that it can stop the spread of disease.

5.5 Discussion and Other Problems

Although the strategies proposed in the previous section are effective in our sense and can either limit or stop the spread of disease, they are practically infeasible in

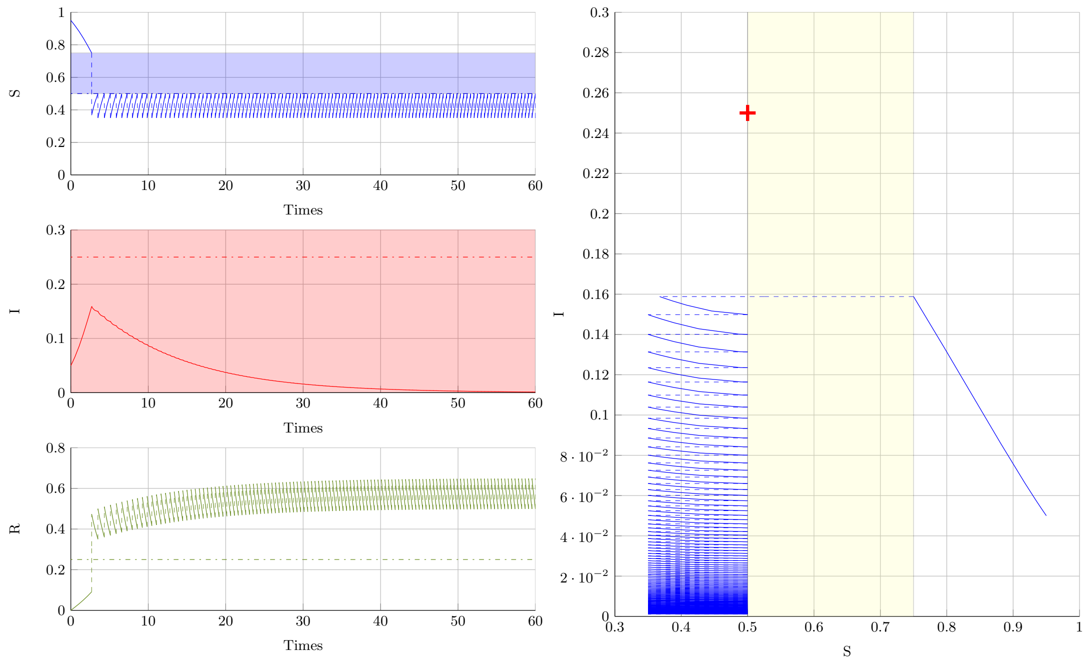


FIGURE 5.6: Solution to $\Sigma_{SIRS,v}$ in Example 5.13 with $I_v = [0, 1]$.

Captions	Arguments	Values
Infection rate	β	1.2
Recovery rate	γ	0.6
Loss of immunity rate	δ	0.6
Vaccination ratio	ρ	0.3
Vaccine performance	ν	1
S-detection	S_v	[0.5,0.75]
Program clock	t_ν	0
Initial susceptible individual	$S(0,0)$	0.95
Initial infected individual	$I(0,0)$	0.05
Initial recovered individual	$R(0,0)$	0
Disease steady state	$E_+(\Sigma_{SIRS,v})$	(0.5,0.25,0.25)

TABLE 5.2: Arguments of $\Sigma_{SIRS,v}$ in Example 5.13.

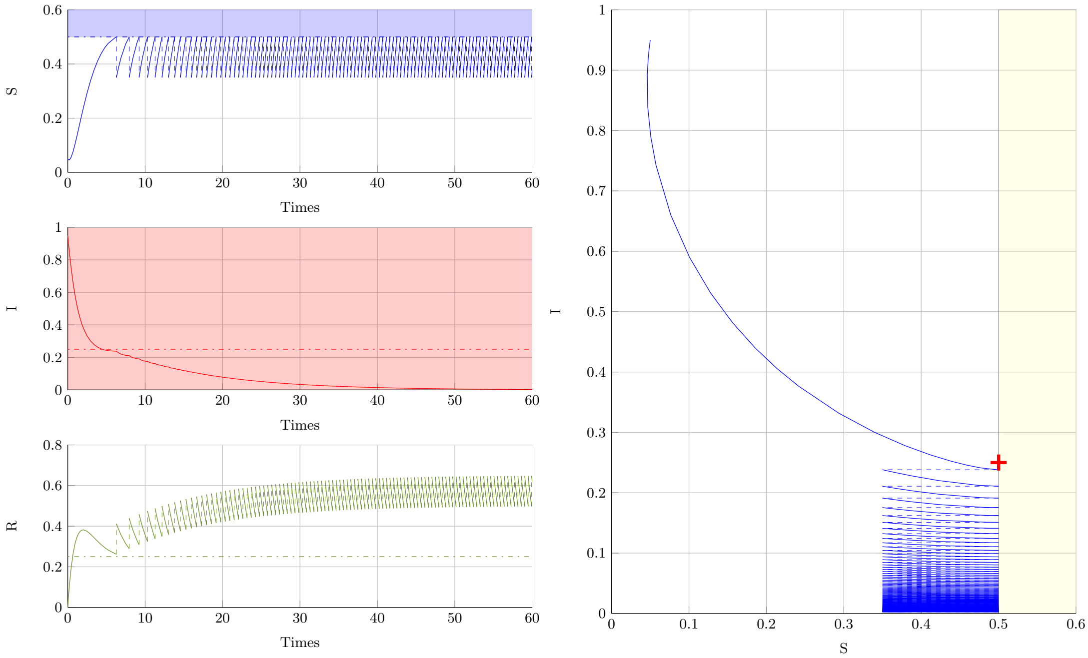


FIGURE 5.7: Solution to $\Sigma_{SIRS,v}$ in Example 5.14.

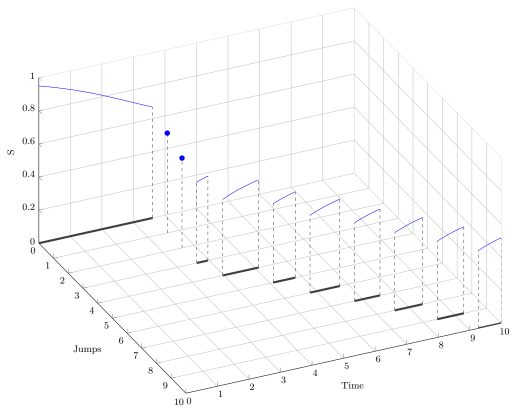


FIGURE 5.8: Trajectory of S to $\Sigma_{SIRS,v}$ in Example 5.15.

some situations. Example 5.13 and Example 5.14 show that the vaccination programs are frequently launched. This kind of situation can lead to the problems that the vaccination programs suggested by the proposed strategies can not be applied practically. Moreover, the following example shows a possibility of the strategies aiming to control the spread of disease. However, it need some cautions to apply in a real world situation.

Example 5.15. The epidemic system $\Sigma_{SIRS,v}$ holds the parameters indicated by Table 5.2 except $\rho = 0.1$ and $I_v = [0.2, 1]$. At time $t = 3.5$ approximately, multiple discrete dynamics exhibit instantly. Such cases like this need a remark since to apply vaccination many times in a row is impossible. According to its trajectory illustrated in Figure 5.8, the state needs 3 vaccinations to leave the vaccination zone. It means that, in a practical point of view, we have to apply $\rho \geq 0.3$ at a first jump. After that, the vaccination program may decrease the parameter ρ to 0.1 which can control the spread of disease to the desired level effectively.

Furthermore, the effects of vaccine performance ν and program clock t_ν are not investigated in this work. At this point we believe that they possibly give a huge impact to vaccination programs. The future study is necessary to explore these unknown effects.

Chapter 6

Conclusion

This work provides a framework for generalized hybrid dynamical systems including stability analysis and applications in epidemic systems. In Chapter 3, we introduce a fundamental framework of hybrid dynamical systems developed in [1–3]. In Theorem 3.17, sufficient conditions guaranteeing existence of a non-trivial solution to a hybrid dynamical system are provided by using Banach contraction principle and its extension. This results give us a possibility to use numerical iterations to find solutions to hybrid systems, however, it is not studied in this work. Moreover, numerous results on stability are proposed by extensions of direct Lyapunov's method. In Theorem 3.21, existence of a hybrid Lyapunov function guarantees asymptotic stability for hybrid systems. In Theorem 3.29 and Theorem 3.28, we guarantee asymptotic stability by existence of relaxed hybrid Lyapunov functions and additional conditions on characterizations of solutions. Such conditions require non-eventually discrete or continuous solutions to hybrid systems, which is easier to verify than the other conditions provided in the literature. We consequently investigate on the case of nonexistence of hybrid Lyapunov function and relaxed hybrid Lyapunov function. In such case, we consider hybrid Lyapunov candidate functions with additional conditions on characterization of solutions. We provide sufficient conditions guaranteeing asymptotic stability for a hybrid system of class $L(\theta)$ and class $H(\theta)$ in Theorem 3.45 and Theorem 3.46 respectively. Such conditions are called dwell-time conditions, which describe the frequency of discrete dynamics and duration of continuous dynamics in stable systems.

In addition, we provide a notion of partial stability for hybrid systems consisting of time, counters or logical values as a part of the state. For such systems, we modify the definitions of stability and hybrid Lyapunov functions. We assume that they can be decomposed to the form (3.33)–(3.34). The main part considered in stability is only x^s. According to the modified definitions, see Definition 3.49, Definition 3.50 and Definition 3.51, existence of hybrid Lyapunov function guarantees partial asymptotic stability for the system in the form (3.33)–(3.34). Furthermore, stability conditions provided in Theorem 3.28, Theorem 3.29, Theorem 3.45 and Theorem 3.46 can be modified to guarantee partial stability.

In Chapter 4, we firstly discuss on the issues of modeling for interconnections of hybrid systems. The issues are motivated by a simple example on interconnected bouncing balls. The framework for interconnected hybrid systems given in literature leads to the problems of physical meaningless solutions which imply to loss of stability property. We suggest a possibility to solve them by proposing a different concept of solutions. Consequently, we propose an extended framework for generalized hybrid dynamical systems allowing us to decompose one large hybrid system as an interconnection of several ones or compose several interconnected hybrid systems as one larger hybrid system, see (4.7). This framework allows the possibility to have continuous flows for some parts of the state also at those instants when other parts can jump, which avoids the mentioned issues of modeling for interconnections of hybrid systems.

According to generalize hybrid time domains in the provided framework, stability notions like input-to-state stability needs to be more generally formulated. We propose stability notions and numerous results for generalized hybrid dynamical systems. Due to the difficulties of generalized hybrid time domains, we firstly consider stability of subsystems. Note that results on stability of the interconnection are equivalent to the results on stability of subsystems if there is only one partition in the interconnection. In Theorem 4.22, existence of ISS-Lyapunov functions guarantees ISS property of subsystems. In Theorem 4.23 and Theorem 4.24, existence of relaxed ISS-Lyapunov functions with additional conditions on characterization of solutions guarantees ISS property of subsystems. In Theorem 4.25 and 4.26, existence of ISS-Lyapunov candidate functions with their corresponding dwell-time conditions guarantees ISS property of subsystems. To guarantee stability of the interconnection, we construct an ISS-Lyapunov function for the interconnection from a collection of ISS-Lyapunov functions for subsystems along with an Ω path. Basically, existence of an Ω path is guaranteed if the small-gain condition, see Definition 4.31, wrt the gain operator, see Definition 4.30, is satisfied. We therefore essentially require the satisfied small-gain condition to construct an ISS-Lyapunov function for the interconnection. Explicit constructions of ISS-Lyapunov functions for the interconnection are provided in Theorem 4.34, Theorem 4.35, Theorem 4.36, Theorem 4.37 and Theorem 4.38.

However, there are some interesting problems that we would like to address here. One of them is the problem on multiple Zeno-type motions. Consider the bouncing ball provided in Example 3.1 along with initial conditions $x_1(0) = h$, $x_2 = 0$ and the defined input u such that it takes value of $\gamma + 1$ for $t_{\max} \leq t \leq 2t_{\max}$ and is zero otherwise, where t_{max} is the corresponding Zeno time. After $t_{\max}$ this ball will be elevated to some finite height and dropped again. Consequently, the ball will tend to reach the second Zeno point, which is naturally larger than $t_{\max}$. This kind of behavior can be modeled by introducing multiple Zeno times $t_{\max,1}$ and $t_{\max,2}$ in an appropriate extension of the provided framework. Another problem is that hybrid dynamics can appear due to interconnection of systems that are not hybrid by their nature. For example, a movement of a mass-point in a free space under some forces is usually modeled by ordinary differential equations and is not hybrid by its nature. However, if we consider such two mass-points in the same space, so that they can collide, then the resulting systems exhibits hybrid phenomena. In addition, we provide Table 6.1 comparing features in our framework of generalized hybrid systems and the framework provided in the literature [63, 64]. Since we do not see any setting of interconnected hybrid systems in the literature that can handle issues on model composition or decomposition appropriately, we mark this topic with the question symbol. The check mark in the table indicates that the framework can handle the corresponding issue properly, while the cross mark indicates the opposite meaning or the framework is not ready to handle such issues.

In Chapter 5, we propose a mathematical model for a spread of disease with public vaccination programs called hybrid SIRS, see (5.4). Basically, the system consists of two steady states: the disease-free steady and the disease steady state. In case

Features	**Literature**	**Present**
Flow set	$\mathcal{C} = \cap_{i=1}^{n} {}^{i}\mathcal{C}$	$I_{\mathcal{C}}$
Jump set	$\mathcal{D} = \cup_{i=1}^{n} {}^{i}\mathcal{D}$	$I_{\mathcal{D}}$
Flow map	See (4.3)	See (4.51)
Jump map	See (4.4)	See (4.51)
Hybrid time domains	$t,\ k$	$t,\ {}^{1}k,\ {}^{2}k,\ \ldots,\ {}^{n}k,$ ${}^{1}t_{\max},\ \ldots,\ {}^{n}t_{\max}$
Inputs	✓	✓
Hybrid arcs	✓	✓
Independent dynamics of subsystems	✖	✓
Void physical-meaningless solutions	✖	✓
Global asymptotically stability	✖	✓
Composition/decomposition	?	✓
Tracking a state at $t \geq t_{\max}$	✖	✓
Multiple Zeno times in each subsystem	✖	✖
Hybrid phenomena due to interconnections	✖	✖

TABLE 6.1: Features comparison of the provided framework.

of no vaccination, the disease-free steady state is globally asymptotically stable if the recovery rate is not smaller than the infection rate, i.e., the spread of the disease eventually disappears, and every individual get no more infected. Additionally, the disease-free steady state is unstable and the disease steady state is locally asymptotically stable if the infection rate is larger than the recovery rate, i.e., the infected individuals permanently remain in the population, which means the spread of disease is eventually long lasting.

Consequently, we mainly focus on the case of the infection rate being larger than the recovery rate. The result of Theorem 5.3 suggests that if vaccination programs are launched limitedly, then the epidemic eventually remains forever. On the results in Theorem 5.5 and Corollary 5.6, we discover that even on a case of ideal vaccines, the vaccination can possibly fail by choosing an inappropriate strategy to limit or stop an epidemic since the number of infected individuals is permanently positive, and it is not different from the case of no vaccination or a finite number of vaccination programs. Furthermore, we provide Strategies 5.8 and Strategies 5.11 to limit or stop the spread of disease. We explicitly show that the provided strategies are significantly effective to control an epidemic in Theorem 5.10 and Theorem 5.12.

Even though Strategies 5.8 and Strategies 5.11 are effective in our sense and can either limit or stop the spread of disease, they are practically infeasible in some situations. We see that, in Example 5.13 and Example 5.14, the discrete dynamics exhibit very frequently in the very end of numerical simulations. Such situations can lead us to the problem of non-practical vaccination programs. Additionally, Example 5.15 shows some cautions to apply the provided strategies. Unfortunately, this work does not deal with the effects of non-ideal vaccine, i.e., $\nu < 1$, and the program clock t_{ν}. However, we believe that the further study on this model can lead us to discover some unknown and interesting results. Finally, we wish that all of the provided results would give a possibility to solve real world problems.

Appendix A

List of Symbols

$\mathcal{H}$	A hybrid system.
$\mathcal{X}$	The state space of a hybrid system.
$\mathcal{C}$	The flow set of a hybrid system.
$\mathcal{D}$	The jump set of a hybrid system.
$\dot{x},\ x'$	The derivative wrt time of the state x of a dynamical system.
x^+	The state of a hybrid dynamical system after a jump.
π	$3.1415926536\ldots$
$\mathrm{e}, \exp(1)$	$2.7182818285\ldots$
$\ln(\cdot)$	Natural logarithm function.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{\geq 0}$	$[0, \infty)$ or the set of nonnegative real numbers.
$\mathbb{R}_+,\ \mathbb{R}_{>0}$	$(0, \infty)$ or the set of positive real numbers.
$\mathbb{N}$	$\{0, 1, 2, 3, \ldots\}$ the set of natural numbers.
$\mathbb{N}_n$	$\{1, 2, 3, \ldots, n\}$.
$\mathbb{Z}$	$-\mathbb{N} \cup \mathbb{N}$ the set of integers.
$\mathbb{R}^n$	The n-dimensional Euclidean Space.
$\emptyset, \{\}$	The empty set.
$\overline{A}$	The closure of a set A.
∂A	The boundary of a set A.
$A \cup B$	The set of points belonging to set A or set B.
$A \cap B$	The set of points belonging to set A and set B.
$A \setminus B$	The set of points belonging to set A but not belonging to set B.
$A \times B$	The set of order pairs (a, b) such that $a \in A$ and $b \in B$.
max	Maximum.
min	Minimum.
sup	Supremum, the least upper bound.
inf	Infimum, the greatest lower bound.
$\lceil x \rceil$	The least integer greater than or equal to x.
x^{T}	The transpose of vector x.
(x, y)	Equivalent notation of $\left(x^{\mathrm{T}}, y^{\mathrm{T}}\right)^{\mathrm{T}}$.
$\|x\|$	The norm of a vector x.
$\lvert x \rvert$	The Euclidean norm of a vector x.
$\|x\|_p$	$(\lvert x_1\rvert^p + \lvert x_2\rvert^p + \ldots + \lvert x_n\rvert^p)^{1/p}$ the L^p-norm of vector a $x = (x_1, x_2, \ldots, x_n)$.
$\|x\|_\infty$	$\max\{\lvert x_1\rvert, \lvert x_2\rvert, \ldots, \lvert x_n\rvert\}$ the L^∞-norm of a vector $x = (x_1, x_2, \ldots, x_n)$.
$\|x\|_A$	$\inf_{a\in A} \lvert x - a\rvert$ for a non-empty $A \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$.

$f : \mathbb{R}^n \to \mathbb{R}^m$	A function or mapping f from $\mathbb{R}^n$ to $\mathbb{R}^m$.
f^{-1}	The inverse function of f.
f'	The derivative of a function f.
id	The identity mapping.
$g \circ f$	The composite function g on f.
$g^{(n)}$	The n-fold composition function on g.
$\langle x, y\rangle$	The dot product of the vectors x and y.
$\frac{\partial}{\partial x_i} f(x)$	The partial derivative of a function f at x wrt the direction x_i.
$\nabla V(x)$	The gradient of the real-valued function $V(x)$.
$\implies$	Material implication; if...then.
$\forall$	Universal quantification.
$\exists$	Existencial quantification.
C^0	The space of continuous functions.
$C^1,\ C$	The space of continuously differentiable functions.
C^k	The space of functions with k continuous derivatives.
$\mathcal{P}$	The class of positive definite functions, see Definition 2.10.
$\mathcal{S}$	The class of $\mathcal{P}$ functions, see Definition 2.11.
$\mathcal{K},\ \mathcal{K}_\infty$	The class of $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, see Definition 2.13.
$\mathcal{KL}$	The class of $\mathcal{KL}$ functions, see Definition 2.16.
$\mathcal{KL}^n$	The class of $\mathcal{KL}^n$ functions, see Definition 4.14.
$L(\theta)$	The class $L(\theta)$, see Definition 3.34 and Definition 4.12
$H(\theta)$	The class $H(\theta)$, see Definition 3.39 and Definition 4.13
$\preceq,\ \prec$	The order of hybrid time domains, see Definition 3.6 and Definition 4.3.
${}^i\mathcal{H}$	The hybrid subsystem i in an interconnection.
${}^i\mathcal{X}$	The state space of the hybrid subsystem i.
${}^i\mathcal{C}$	The flow set of the hybrid subsystem i.
${}^i\mathcal{D}$	The jump set of the hybrid subsystem i.
${}^it_{\max}$	The Zeno time of ${}^ix$, see Section 4.3.2.
$I_\mathcal{C}$	The flow index set, see 4.6.
$I_\mathcal{D}$	The jump index set, see 4.6.
$\overline{k}$	The multi-index $({}^1k, \ldots, {}^nk) \in \mathbb{N}^n$. See Definition 4.1.
$\varsigma(\overline{k})$	The summation ${}^1k + \cdots + {}^nk$, see Definition 4.1
$\overline{1}_i$	$({}^1p, \ldots, {}^{i-1}p, 1, {}^{i+1}p, {}^np) \in \mathbb{N}^n$.
$\overline{0}_i$	$({}^1p, \ldots, {}^{i-1}p, 0, {}^{i+1}p, {}^np) \in \mathbb{N}^n$.
$\overline{\kappa}(j),\ \overline{\nu}(j)$	The κ-multi-indices and a ν-multi-indices, see the equation (4.14).
$\kappa(j),\ \nu(j)$	The κ-mapping and the ν-mapping , see the equation (4.15).
$T(j,x), T(k,x)$	See Definition 3.33 or Definition 4.11.
$E_0(\Sigma_{SIRS})$	The disease-free steady state of Σ_{SIRS}, see the equation (5.2).
$E_+(\Sigma_{SIRS})$	The disease steady state of Σ_{SIRS}, see the equation (5.3).
$\bar{I}$	The safely acceptable value of infected individuals.

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