# Convergence of gradient-like dynamical systems and optimization algorithms

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# Introduction

In this thesis we discuss the convergence of the trajectories of continuous dynamical systems and discrete-time optimization algorithms.

In applications one is often able to show that trajectories of a dynamical system converge to a set of equilibria. However, it is not clear if the trajectories converge to single points or show any more complicated tangential dynamics when approaching this set. This situation appears in neural network applications [88], cooperative dynamics [85, 86] and adaptive control [34, 121]. The question if a trajectory actually converges to a single state is of importance at its own right, even if it is often known that the system converges to a set of desirable states. For example, one might ask if a neural network converges to a single state, if a cooperative system converges from a initial state to a single point [86] or if an adaptive control scheme converges to a fixed controller. Here, we discuss such convergence questions for gradient-like dynamical systems.

In the discrete-time case, we focus on optimization algorithms on manifolds and their convergence properties. We consider gradient-like algorithms for several different types of optimization problems on manifolds.

Let us first discuss the continuous-time dynamical systems in more detail. We start with recalling some of the main initial approaches to prove results on the convergence of trajectories. By a slight abuse of notation, we will denote the convergence of a trajectory to single point as *pointwise convergence.*

- *Normal hyperbolicity of manifolds of equilibria.* A classical condition ensuring convergence of the integral curves to single points is normal hyperbolicity [90]. Assuming that the $\omega$-limit set is locally contained in a normally hyperbolic manifold of equilibria one can deduce that it contains at most one point. This can even be extended to the non-autonomous case [13]. We refer to the monograph of Aulbach [13] for an extensive discussion of such results.

- *Monotone dynamical systems.* The trajectories of such systems are decreasing of a given partial order on the phase space. Under some strict monotonicity conditions one can derive criteria for the convergence of trajectories. This approach goes back to Hirsch [87].

- *Gradient systems, $\dot{x} = -\operatorname{grad} f(x)$.* If the function $f$ satisfies suitable regularity conditions, one can show by more or less sophisticated methods, that the trajectories converge to single points. The classical examples for such gradient systems are Morse and Morse-Bott functions, see e.g. the monograph [77] for an exposition. A less well-known example is Łojasiewicz's convergence result for analytic functions [113].

In this thesis we will consider an extension of the gradient system approach for continuous-time dynamical systems. Let us first review the convergence properties of gradient systems. The gradient system of a real-valued function $f$ is defined by the differential equation

$$\dot{x} = -\operatorname{grad} f(x).$$

It is well known that the $\omega$-limit sets of integral curves of this system contain only critical points of the function and, in particular, if the critical points are all isolated, then bounded trajectories converge to single points [84]. This yields the convergence for Morse functions. Furthermore, gradient systems of Morse functions are generic in the class of gradient systems [129]. Hence, for generic gradient systems the trajectories converge to single points. However, one will encounter non-generic functions in some applications, for example if some additional restrictions on the class of functions are given by the application. Therefore, the behavior of trajectories of such systems is of interest, too.

It is a surprising fact, that the convergence behavior of trajectories of a non-hyperbolic gradient system can be non-trivial. A classical example by Curry [47] is the so-called *Mexican hat example*. This is a gradient system in $\mathbf{R}^2$ which has integral curves which converge to the entire unit circle. It is best visualized by a Mexican hat which has a valley on its surface circling around the center infinitely often and converging to the brim. Taking this "hat" as the graph of a function on $\mathbf{R}^2$, one sees that the gradient field has a integral curves with the unit circle as $\omega$-limit set. Exact formulas of such functions can be found in [4, 128]. Figure 1 shows a few contour lines of a function of this type.

From the Mexican hat example one can also construct gradient fields with integral curves with non-connected $\omega$-limit sets. One has just to construct a diffeomorphism[1] of an open subset $U$ of $\mathbf{R}^2$ to the whole $\mathbf{R}^2$ which maps the intersection of $U$ with the unit circle onto 2 non-trivial curves. The gradient field of the function induced by the "Mexican hat" has integral curves which contain these two curves in their $\omega$-limit set.

[1] for example $(x, y) \mapsto (x/\sqrt{1-x^2}, y/\sqrt{1-x^2})$.

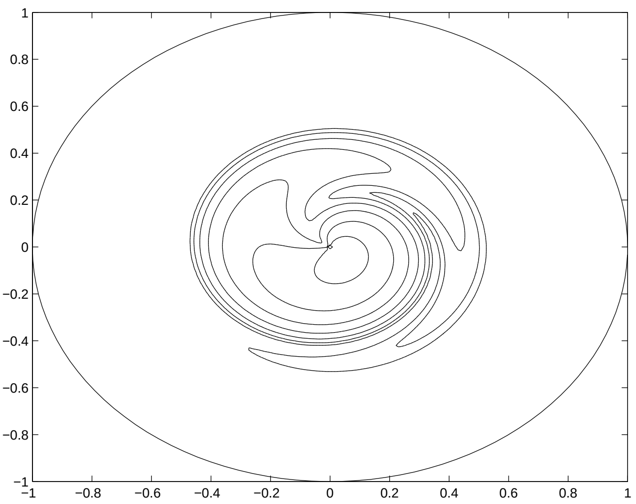


Figure 1: Contour lines for the Mexican hat-type function given by $f(r, \theta) = \exp(-1/(1-r)^2)(1 + \sin(r + \theta) + \exp(-1/(1-r)^2))$ in polar coordinates, cf. [4, 128]. The outer circle denotes the unit circle in $\mathbf{R}^2$.

However, under suitable conditions on the function it is possible to prove convergence of the integral curves even for non-Morse functions. One already mentioned example is the class of Morse-Bott functions. The convergence is based on the generalized version of the Morse lemma, see [77].

The second class of functions for which the integral curves converge are analytic functions. The convergence is a result of Łojasiewicz [113] and is stated as the following theorem.

**Theorem** *Let $M$ be an real-analytic Riemannian manifold and $f: M \to \mathbf{R}$ be real-analytic. Assume that $\gamma$ is an integral curve of* $\operatorname{grad} f$. *Then the $\omega$-limit set of $\gamma$ consists at most of one point.*

The proof is based on showing the boundedness of the length of an integral curve with the help of an estimate for the gradient of $f$ - a so-called Łojasiewicz inequality [113].

Kurdyka has extended this result to the class of functions definable in an o-minimal structure [101]. Such functions include $C^\infty$-cutoff functions like $\exp(-1/(x^2 + y^2 - 1)^2)$. He showed that a generalization of the Łojasiewicz inequality holds in the o-minimal case, which implies the convergence by analogous arguments as in the analytic case. For the integral curves of subgradient differential inclusions of non-smooth subanalytic functions Bolte at el. have proven the convergence using a version of the Łojasiewicz inequality for Clarke's generalized gradients[2] [29, 30]. Furthermore they were able to give estimates on the convergence speed of the integral curves. However, these estimates require that the Łojasiewicz exponents in the inequalities are explicitly known.

The asymptotic properties of integral curves of analytic gradient systems are even stronger. For example, Thom's gradient conjecture claims that for any integral curve $\gamma$ of an analytic gradient system, which converges to $x^*$, the limit of secants

$$\lim_{t\to\infty} \frac{\gamma(t) - x^*}{\|\gamma(t) - x^*\|} \tag{1}$$

exists. Kurdyka et al. [102] have shown that Thom's gradient conjecture holds in Euclidean space and, more generally, on Riemannian manifolds. However, some stronger conjectures of the asymptotic behavior of the integral curves are still open [102].

The various convergence results for gradient systems suggest the extension to more general, gradient-like systems. Unfortunately, there is no uniform definition of "gradient-like systems" in the literature. In this work we follow Conley [42] and call a dynamical system *gradient-like* if there exists a continuous function which is strictly decreasing on non-constant trajectories. We call this function a Lyapunov function. A simple Lyapunov argument shows that the $\omega$-limit set of an integral curve is contained in a level set of the Lyapunov function.

For stronger convergence properties of the trajectories we have to restrict the class of Lyapunov functions. Otherwise the gradient systems themselves would provide counterexamples to the convergence of trajectories to single points. Thus given the results for gradient systems above, it is naturally to require that the Lyapunov function is analytic. However, this is not sufficient for the convergence. Take for example the function $f(x, y, z) = x^2$ and the vector field

$$X(x, y, z) = \begin{pmatrix} -x^3 \\ -x^2 z \\ x^2 y \end{pmatrix}.$$

[2]Such a Łojasiewicz inequality for Clarke's generalized gradient can be extended to functions definable in an o-minimal structure, see [31].

The function $f$ is strictly increasing on the non-constant integral curves of $X$. Thus the vector field is gradient-like and $f$ is analytic. But most non-constant integral curves of $X$ converge to an entire circle in $\{0\} \times \mathbf{R}^2$.

Therefore, we need additional conditions on the vector field $X$. For differentiable Lyapunov functions $f$, the gradient-likeness of $(X, f)$ implies that $df(X(x)) \leq 0$ for the vector field $X$. In the counterexample above we have even $df(X(x)) < 0$ if $x$ is not an equilibrium. Therefore, this condition is not sufficient for convergence of the integral curves to single points. It is a natural idea to tighten this property to ensure strict convergence of the integral curves. This leads to replacing the notion of gradient-like by an angle condition on the vector field and the gradient of $f$, i.e. for any compact set $K \subset M$, there is a constant $\varepsilon_K > 0$ such that

$$- \langle \operatorname{grad} f(x), X(x) \rangle \geq \varepsilon_K \|\operatorname{grad} f(x)\| \, \|X(x)\| \,. \tag{2}$$

We will call such vector fields satisfying this condition *(AC) vector fields*. The condition (2) bounds locally the absolute value of the angle between $\operatorname{grad} f(x)$ and $X(x)$ by a constant $< \pi/2$. For an analytic $f$ this allows the use of Łojasiewicz-type arguments to show the convergence of the integral curves as in the gradient case. Note, that this angle condition requires a Riemannian metric. However, it can be easily seen that the definition of (AC) vector fields is independent of the Riemannian metric.

The extension of the convergence results for gradient systems to gradient-like ones appears in some previous works. Simon [144] considers systems $\operatorname{grad} f(x) + r(t)$, where the norm of the time-variant disturbance term $r(t)$ is bounded by $\delta \|\operatorname{grad} f(x) + r(t)\|$, $\delta \in (0, 1)$. He proves for analytic $f$ the convergence of the integral curves to single points. Note, that these systems satisfy (2) implicitly. However, Simon's condition is stronger than (2). A global version of the angle condition was first given by Andrews [9]. Andrews used it to show by a Łojasiewicz argument the convergence of the integral curves for an analytic Lyapunov function. However, his proof silently assumes that critical points of the Lyapunov function are equilibria of the vector field, which does not follow (2). In [104] it was shown that for analytic Lyapunov functions and continuous vector fields, the local version (2) is even sufficient for convergence, without any further conditions on the equilibria of the vector field and the critical points of the Lyapunov function. It was also shown that single curves with a derivative which satisfies the angle condition with respect to an analytic function, converge to a single point. In subsequent work [4], Absil et al. also consider the angle condition on single curves in $\mathbf{R}^n$. They show the weaker result that for analytic functions and under the additional condition that the curve does not meet any critical points of the function, the curve converges to a single point.

Łojasiewicz's convergence argument has also been employed for solutions of specific gradient-like second-order systems, e.g. [7, 71]. We will show later how the systems in [7] are related to (AC) vector fields. Other convergence criteria like e.g. the structure of Morse-functions, have also been applied to gradient-like second order systems [11], but we will not investigate these approaches further.

Finally, we mention some similar results on the convergence of integral curves which are not directly connected to gradient or gradient-like systems. A generalization of these convergence results is the approach of Bhat and Bernstein [22, 23]. The convergence for normally hyperbolic manifolds of equilibria is a result of a certain transversality of the vector field to the manifold of equilibria. Bhat and Bernstein define the notion of transversality to a non-smooth set of equilibria by using tangent cones and limits of the normed vector field $\|X(x)\|^{-1} X$. They show that if the vector field is transversal to a set of equilibria and a non-empty $\omega$-limit set of an integral curve is contained in this set, then it contains only one point. This is in fact again related to Łojasiewicz's convergence theorem, as an alternative proof by Hu [94] via $(a_f)$ stratifications uses an asymptotic transversality argument similar to the one in [22, 23].

A different approach from Bhat and Bernstein [21] assumes that Lyapunov function $f$ and a function $\psi : \mathbf{R} \to \mathbf{R}$ are given such that the estimate $\|X(x)\| \leq \psi'(f(x))df(X(x))$ holds. This allows to bound the length of the integral curve as in the gradient case, implying the convergence to a single point. For gradient systems this inequality follows from the Łojasiewicz inequality and is in fact the standard way for proving the convergence, cf. [101, 113].

The first chapter of this thesis deals with extensions of these convergence results to larger classes of functions and systems. We start in Section 1.1 with providing some known results on o-minimal structures and analytic-geometric categories. Furthermore we construct a special type of stratifications, which will be needed for the convergence of solutions of differential inclusions. Next, in Section 1.2, we show that the convergence results hold for Lipschitz continuous vector fields and continuous functions, which are morphisms of an analytic-geometric category. In particular, these functions need not be differentiable everywhere. Such situations have not been covered by the known results on gradient-like systems with an angle condition. Furthermore, we present some examples for (AC) vector fields with $C^2$ Lyapunov functions. We also discuss some aspects of the topology of the flow of (AC) vector fields and sketch how the results of Nowel and Szafraniec [126, 127] for analytic gradient systems can be extended to these systems. In Section 1.3 we extend the convergence results to solutions of differential inclusions. Since solutions of differential inclusions have not been yet considered in the literature in the context of gradient-like systems with angle condition, this is again an extension of the known results. As the last part of discussion of continuous-time systems, we consider the case that the Riemannian metric degenerates. This is discussed in Section 1.4 and we can develop a convergence result for this case, too. However, this convergence result does not cover the most interesting case of locally unbounded metrics, which has strong connections to the Thom conjecture and interior point methods.

The author of this thesis has submitted the convergence results from Section 1.2 and the discussion of the examples for (AC) vector fields for publication [103].

In the second part of this thesis, we discuss convergence results for discrete-time gradient-like optimization methods on manifolds.

The classical optimization theory considers optimization problems only in Euclidean spaces. However, in some applications optimization problem appear naturally on smooth manifolds. To deal with such a situation in a classical setting, the manifold has to embedded into an Euclidean space. Then a standard algorithm for constrained optimization can be applied to the optimization problem. However, this approach has several disadvantages. The dimension of the Euclidean space, in which the manifold is embedded, can be very high, leading to inefficient algorithms. Further, standard constrained optimization algorithms will in general not produce iterates on the manifold itself, thus requiring complicated projections onto the manifold.

Optimization algorithms on manifolds try to avoid these problems by using the structure of the manifold itself and not relying on any embeddings. There has been a signification interest in such optimization algorithms on manifolds in the last years, see e.g. [3,5,54,55,65,75,95,117–119,146,147,154]. So far, there are two main approaches to construct optimization algorithms on manifolds.

- The *Riemannian geometric approach* formulates the classical algorithms in the language of Riemannian geometry. This yields a direct extension of the algorithms to Riemannian manifolds. The Riemannian approach first appeared in the work of Luenberger [116]. The standard classical algorithms for unconstrained smooth optimization could be extended by this approach to Riemannian manifolds, namely gradient [65,116,146,147], conjugate gradient [146,147], Newton [65,146,147] and Quasi-Newton [65] methods. Furthermore, it is possible to extend the standard convergence results for gradient-like descent methods to this setting, see [154, 163, 164] for some results.

- The *local parameterization approach* uses local parameterizations to obtain an algorithm on the manifold. In each iteration the function is pulled back to Euclidean space by the parameterization and one step of a standard Euclidean space optimization algorithm is applied. The result is mapped back to the manifold. This yields an optimization iteration on the manifold, cf. [3, 5, 35, 37, 118–120, 143]. This approach was introduced by Shub [143] for Newton-type iterations. Shub uses a smooth retraction $\phi\colon TM \to M$, which yields local parameterizations $\phi_x\colon T_xM \to M$ from the tangent space to the manifold. This retraction type parameterizations have been studied for Newton [5,143] and trust-region [3] methods. Shub's retractions have also been used for the numerical integration on manifolds [36,37]. In the optimization context, the numerical integration of a gradient flow leads to gradient descent optimization algorithms [35, 37]. Other authors have proposed the use of parameterizations $\mathbf{R}^n \to M$ for the construction of gradient and Newton algorithms on manifolds [118–120]. These algorithms use a different parameterization to project back to the manifold. Hüper and Trumpf [95] have shown the local quadratic convergence for a class of such Newton algorithms. Unlike for the Riemannian methods, there are global convergence results only known for the trust region algorithm [2, 3].

Note, that these methods all apply only to at least continuously differentiable cost functions. For non-smooth cost functions, the theory is less known and developed. The basic tools from non-smooth analysis, subdifferentials of different types, have been extended to smooth manifolds in the last years [14, 15, 39, 107–109]. To our best knowledge, optimization methods on manifolds have only been studied for convex and quasiconvex functions, sometimes even with strong restrictions on the manifold [61, 62, 132].

In the second chapter, we consider gradient-like optimization algorithms on manifolds in different contexts. In Section 2.1 we consider gradient-like optimization algorithms using the local parameterization approach. We first introduce suitable conditions on the parameterizations of the manifold to ensure convergence of the algorithms. These conditions are much weaker than the smoothness condition of Shub [143] on the retraction $TM \to M$. We show how these conditions relate to the retractions of Shub, the exponential map and special parameterizations on homogeneous spaces. Then we give the global convergence results for gradient-like algorithms. Further, we extend a result of Absil et al. [4] on the convergence of the descent sequence to a single critical point to optimization in local parameterizations. The Section 2.2 contains an extension of the local parameterization approach to optimization of a smooth cost function over a non-smooth set. We also give a convergence result in this case, which is however much weaker than for optimization on a smooth manifold. In Section 2.3 we discuss the problem of optimizing a Lipschitz-continuous cost function over a smooth manifold. We start with a introduction of an analogue to Clarke's generalized gradient, based on the Frechét subgradient on manifolds by Ledyaev and Zhu [107–109]. Then we show the convergence of gradient-like descent algorithms, both for Riemannian algorithms and algorithms in local parameterizations. Our arguments are based on the convergence results of Teel [150] for non-smooth optimization in Euclidean space. As an application of the non-smooth optimization algorithm, we discuss in the last Section 2.4 applications to sphere packing problems, mainly on Grassmann manifolds. We start with a formulation of these problems on adjoint orbits. Then we discuss concrete examples and give explicit algorithms. In the end, numerical results for the algorithms on the real Grassmann manifold are presented.

The author of this thesis has partially presented the results on optimization of non-smooth function and sphere packing applications in the joint works with U. Helmke [105] and with G. Dirr and U. Helmke [49, 50].

# Chapter 1

# Time-continuous gradient-like systems

## 1.1 O-minimal structures and stratifications

### 1.1.1 Basic properties and definitions

In this section we recall some basic definitions and theorems of o-minimal structures on $(\mathbf{R}, +, \cdot)$ and analytic-geometric categories.

The reader is referred to [45,53,155] for a detailed discussion of o-minimal structures and analytic geometric categories. O-minimal structures on the real field $(\mathbf{R}, +, \cdot)$ are a generalization of *semialgebraic sets*, i.e. sets determined by a finite number of polynomial inequalities and equations. We recall the definition of o-minimal structures on $(\mathbf{R}, +, \cdot)$ [45, 101]:

**Definition 1.1.1** Let $\mathcal{M} = \bigcup_{n\in\mathbf{N}} \mathcal{M}_n$, where $\mathcal{M}_n$ is a family of subsets of $\mathbf{R}^n$. $\mathcal{M}$ is an *o-minimal structure on the real field* $(\mathbf{R}, +, \cdot)$ if

1. $\mathcal{M}_n$ is closed under finite set-theoretical operations,
2. $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$ implies $A \times B \in \mathcal{M}_{n+m}$,
3. for $A \in \mathcal{M}_{n+m}$ and $\pi_n \colon \mathbf{R}^{n+m} \to \mathbf{R}^m$, the projection on the first $n$ coordinates, $\pi_n(A) \in \mathcal{M}_n$ holds,
4. every semialgebraic set is contained in $\mathcal{M}$,
5. and $\mathcal{M}_1$ consists of all finite unions of points and open intervals.

Elements of $\mathcal{M}$ are said to be *definable* in $\mathcal{M}$. If the graph of a function $f \colon A \to B$ belongs to an o-minimal structure on $(\mathbf{R}, +, \cdot)$, then $f$ is called *definable in the o-minimal structure* or just *definable*.

In the last years a significant number of o-minimal structures on $(\mathbf{R}, +, \cdot)$ has been discovered. Specific examples of o-minimal structures on $(\mathbf{R}, +, \cdot)$ include, see [53, 156]:

- The class $\mathcal{R}_{\text{alg}}$ of semialgebraic sets, i.e. sets defined by polynomial inequalities and equations.
- The class $\mathcal{R}_{\text{an}}$ of restricted analytic functions, i.e. the smallest structure containing the graphs of all $f|_{[0,1]^n}$, where $f$ is an arbitrary analytic function on $\mathbf{R}^n$.
- The structure $\mathcal{R}^{\mathbf{R}}$ containing the graphs of irrational powers $x^\alpha$, $\alpha \in \mathbf{R}$. Note that $\mathcal{R}_{\text{alg}}$ contains only the graphs of rational powers.

- The structure $\mathcal{R}_{\text{exp}}$ containing the graph of the exponential function. This structure contains $C^\infty$ cut-off functions like $\exp(x^{-2})$.
- There are structures $\mathcal{R}_{\text{an}}^{\mathbf{R}}$, $\mathcal{R}_{\text{an,exp}}$ containing both $\mathcal{R}_{\text{an}}$ and $\mathcal{R}^{\mathbf{R}}$, or $\mathcal{R}_{\text{an}}$ and $\mathcal{R}_{\text{exp}}$, respectively.
- The Pfaffian closure of an o-minimal structure on $(\mathbf{R}, +, \cdot)$. This is the smallest o-minimal structure which contains the original structure as well as suitably regular solutions of definable Pfaffian equations, and therefore suitable integrals.

There are several operations available to construct new definable functions from given ones, cf. [53]. First of all, the set of functions definable in an o-minimal structure on $(\mathbf{R}, +, \cdot)$ is closed under composition. In particular any polynomial combination of definable functions is definable. Given definable functions $f_1, \ldots, f_l \colon \mathbf{R}^{n+k} \to \mathbf{R}$ the functions $x \mapsto \sup_{y \in \mathbf{R}^k} f_1(x, y)$, $z \mapsto \max\{f_1(z), \ldots, f_n(z)\}$ are definable[1]. Further, all partial derivatives of a definable function are definable. Note that compositions or other combinations of functions definable in different o-minimals structures on $(\mathbf{R}, +, \cdot)$ are not necessarily definable in an eventually larger o-minimal structure on $(\mathbf{R}, +, \cdot)$. In fact, there are known examples of different o-minimal structures on $(\mathbf{R}, +, \cdot)$ such that their union is not contained in any other o-minimal structure on $(\mathbf{R}, +, \cdot)$ cf. [136].

On analytic manifolds a counterpart of semialgebraic sets in $\mathbf{R}^n$ are the *semi- and subanalytic sets*. The semianalytic sets are locally described by a finite number of analytic equations and inequalities, while the subanalytic ones are locally projections of relatively compact semianalytic sets, see [24] for more information. The analogue generalization of semi- and subanalytic sets are the elements of analytic-geometric categories. The following definition of these categories can be found in [53].

**Definition 1.1.2** An *analytic-geometric category* $\mathcal{C}$ assigns to each real analytic manifold $M$ a collection of sets $\mathcal{C}(M)$ such that for all real analytic manifolds $M$, $N$ the following conditions hold:

1. $\mathcal{C}(M)$ is closed under finite set theoretical operations and contains $M$,
2. $A \in \mathcal{C}(M)$ implies $A \times \mathbf{R} \in \mathcal{C}(M \times \mathbf{R})$,

[1] This follows from the fact that the closure of a definable set is definable [53] and standard constructions for definable sets [53, Appendix A].

3. for proper analytic maps $f\colon M \to N$ and $A \in \mathcal{C}(M)$ the inclusion $f(A) \in \mathcal{C}(N)$ holds,

4. if $A \subset M$ and $\{U_i \mid i \in \Lambda\}$ is an open covering of $M$ then $A \in \mathcal{C}(M)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$ for all $i \in \Lambda$.

5. bounded sets $A$ in $\mathcal{C}(\mathbf{R})$ have finite boundary, i.e. the topological boundary $\partial A$ consists of a finite number of points.

Elements of $\mathcal{C}(M)$ are called $\mathcal{C}$-sets. If the graph of a continuous function $f\colon A \to B$ with $A \in \mathcal{C}(M)$, $B \in \mathcal{C}(N)$ is contained in $\mathcal{C}(M \times N)$ then $f$ is called a *morphism of* $\mathcal{C}$ or shorter a $\mathcal{C}$*-function*.

Van den Dries and Miller have shown that there is a one-to-one correspondence between o-minimal structures containing $\mathbf{R}_{\text{an}}$ and analytic-geometric categories [53, Section 3]. The following theorem recalls their results.

**Theorem 1.1.3** *For any analytic-geometric category* $\mathcal{C}$ *there is an o-minimal structure* $\mathcal{R}(\mathcal{C})$ *and for any o-minimal structure* $\mathcal{R}$ *containing* $\mathbf{R}_{an}$ *there is an analytic geometric category* $\mathcal{C}(\mathcal{R})$*, such that*

- $A \in \mathcal{C}(\mathcal{R})$ *if for all* $x \in M$ *exists an analytic chart* $\phi\colon U \to \mathbf{R}^n$*,* $x \in U$*, which maps* $A \cap U$ *onto a set definable in* $\mathcal{R}$*.*

- $A \in \mathcal{R}(\mathcal{C})$ *if it is mapped onto a bounded* $\mathcal{C}$*-set in Euclidean space by a semialgebraic bijection.*

*Furthermore, for* $\mathcal{C} = \mathcal{C}(\mathcal{R})$ *we get the back the o-minimal structure* $\mathcal{R}$ *by this correspondence, and for* $\mathcal{R} = \mathcal{R}(\mathcal{C})$ *we get again* $\mathcal{C}$*.*

**Proof:** See [53, Section 3, Appendix D]. The characterization of $\mathcal{R}(\mathcal{C})$ is slightly more general than the one in [53, Section 3], as they use a specific semialgebraic bijection. However, standard arguments show directly that both characterizations are equivalent. □

As a consequence of the correspondence between o-minimal structures and analytic-geometric categories, Theorem 1.1.3, $\mathcal{C}$-sets are locally mapped[2] to sets definable in $\mathcal{R}(\mathcal{C})$ in arbitrary analytic charts.

[2]i.e. the image of the intersection of the set and a suitably small open set is definable

**Proposition 1.1.4** *A set $A \subset M$ is in $\mathcal{C}(M)$ if and only if it is locally mapped to a set in $\mathcal{R}(\mathcal{C})$ in analytic charts, i.e. for every analytic diffeomorphism $\phi\colon U \to \mathbf{R}^n$ and every relatively compact $\mathcal{C}$-set $V$, $\overline{V} \subset U$, the set $\phi(V \cap A)$ is definable in $\mathcal{R}(\mathcal{C})$.*

**Proof:** This is shown in the argument of van den Dries and Miller for $\mathcal{C}(\mathcal{R}(\mathcal{C})) = \mathcal{C}$ [53, Proof of D.10(4)]: Since $\phi(V \cap A)$ is a bounded subset of $\mathbf{R}^n$, we see by [53, D.10 (1)] that $\phi(V \cap A)$ is a $\mathcal{C}$-set if and only if it is actually definable in $\mathcal{R}(\mathcal{C})$. □

Furthermore, $\mathcal{C}$-functions are locally mapped to definable functions by analytic charts.

**Proposition 1.1.5** *Let $f\colon M \to N$ be a $\mathcal{C}$-function and $\phi\colon U \to \mathbf{R}^m$, $U \subset M$, $\psi\colon V \to \mathbf{R}^n$, $V \subset N$ analytic local charts. Assume that we have relatively compact, open sets $U'$, $\overline{U'} \subset U$, $V'$, $\overline{V'} \subset V$ such that $f(U') \subset V'$. Then the function*

$$\psi \circ f \circ \phi^{-1}\colon \phi(U') \to \psi(V')$$

*is definable in $\mathcal{R}(\mathcal{C})$. Especially, if $f$ is a bounded $\mathcal{C}$-function $f\colon U \to \mathbf{R}$, then*

$$f \circ \phi^{-1}\colon \phi(U') \to \mathbf{R}$$

*is definable.*

**Proof:** Let $\phi\colon U \to \mathbf{R}^m$, $\psi\colon V \to \mathbf{R}^n$ analytic charts with $U \subset M$, $V \subset N$ neighborhoods of $x$ and $f(x)$. Assume that we have relatively compact subsets $U', V'$ with $\overline{U'} \subset U$, $\overline{V'} \subset V$, $x \in U'$, $f(U') \subset V'$. Since $f$ is a $\mathcal{C}$-function, the graph $\Gamma_f$ of $f$ is a $\mathcal{C}$-set. By Proposition 1.1.4 $\Gamma_f \cap (U' \times V')$ is mapped on a set $S \subset \mathbf{R}^{m+n}$ definable in $\mathcal{R}(\mathcal{C})$ by the map $(x, y) \mapsto (\phi(x), \psi(y))$. Note, that $S$ is the graph of the map $\psi \circ f \circ \phi^{-1}\colon \phi(U') \to \psi(V')$. Hence, the map $\psi \circ f \circ \phi^{-1}\colon \phi(U') \to \psi(V')$ is definable. The case $f\colon U \to \mathbf{R}$ follows directly by setting $\psi = \mathrm{Id}|_{\mathbf{R}}$. □

By Theorem 1.1.3, one can derive from o-minimal structures on $(\mathbf{R}, +, \cdot)$ the following examples for analytic geometric categories [53]:

- Subanalytic sets. While their definition originates in real analytic geometry, the class of subanalytic sets can be regarded as the analytic-geometric category derived from $\mathcal{R}_{\mathrm{an}}$.
- The analytic-geometric category derived from $\mathcal{R}_{\mathrm{an}}^{\mathbf{R}}$.

- The analytic-geometric category derived from $\mathcal{R}_{\text{an,exp}}$. Note that the class of morphisms of this category contains $C^\infty$ cut-off functions.

An analytic-geometric category $\mathcal{C}$ contains always the subanalytic sets and all subanalytic functions are $\mathcal{C}$-functions. Hence, the category of subanalytic sets is the smallest analytic-geometric category.

Similar tools for constructing new $\mathcal{C}$-functions as in the definable case are available [53]. Again we have that the composition, polynomial combinations, tangent maps and the maximum of a finite number of $\mathcal{C}$-functions of a fixed analytic-geometric category are $\mathcal{C}$-functions. The situation for the supremum is a little more subtle: given a $\mathcal{C}$-function $f : M \times N \to \mathbf{R}$ the supremum $x \to \sup_{y\in K} f(x,y)$ with $K \subset N$ compact is a $\mathcal{C}$-function. This does not hold for non-compact $K$ as the example

$$f(x,y) = \begin{cases} \sin(x^{-1}) & |x| > y^{-1} \\ 0 & |x| \leq y^{-1} \end{cases} \;;\; \sup_{y\in K} f(x,y) = \sin(x^{-1})$$

on $\mathbf{R} \times (0,\infty)$ shows[3].

It will turn out to be useful later, that the maximum of $n$ definable functions $(a,b) \to \mathbf{R}$ coincides with one of these functions on an interval $(a,\varepsilon) \subset (a,b)$.

**Lemma 1.1.6** *Let $f_i\colon (a,b) \to \mathbf{R}$, $i \in \{1,\ldots,n\}$ be a finite family of functions definable in an o-minimal structure. Then there is $\varepsilon > a$ and $j \in \{1,\ldots,n\}$ such that $f_j(x) = \max_{i=1,\ldots,n} f_i(x)$ for all $x \in (a,\varepsilon)$.*

**Proof:** As mentioned above the function $h(x) := \max_{i=1,\ldots,n} f_i(x)$ is definable. Therefore the set $A = \{(x,i) \mid x \in (a,b),\ i \in \{1,\ldots,n\}, f_i(x) = g(x)\}$ is definable, too. Thus we can define a definable function $j\colon (a,b) \to \mathbf{N}$, $j(x) := \max_{(x,i)\in A} i$. By the monotonicity theorem [53, Theorem 4.1] $j(x)$ is constant on a non-empty interval $(a,\varepsilon)$ and $f_j(x) = \max_{i=1,\ldots,n} f_i(x)$ for $j = j(y)$, $y \in (a,\varepsilon)$. $\square$

### 1.1.2 Stratifications

We discuss now some known facts on stratifications of sets in analytic-geometric category. We use the standard notions of stratifications, see [24, 53, 83, 114, 115]. Further, we refine the concept of $(a_f)$-stratifications from the literature, as we will need a stricter type of this stratifications later.

[3]Due to the accumulation of isolated critical points at 0, $\sin(x^{-1})$ cannot be a $\mathcal{C}$-function on whole $\mathbf{R}$ for any analytic-geometric category.

**Definition 1.1.7** Let $M$ be a smooth manifold.

- A *stratification* of a manifold is a locally finite, disjoint decomposition into submanifolds $S_j$, $j \in \Lambda$, the *strata*, such that $S_j \cap \overline{S_i} \neq \emptyset$, $i \neq j$, implies $S_j \subset \overline{S_i}$ and $\dim S_j < \dim S_i$. We call it a *$C^p$-stratification* if the strata are $C^p$ submanifolds.

- Given subsets $X_1, \ldots, X_k$ we call a stratification of $M$ *compatible with* $X_1, \ldots, X_k$ if each $X_j$ is the finite union of strata. If we have a set $X \subset M$ and a $C^1$-stratification $S_j$, $j \in \Lambda$ compatible with $X$, then we define the *dimension of* $X$ as

$$\dim X = \max_{j \in \Lambda} \dim S_j.$$

- A stratification $S_j$, $j \in \Lambda$ of $M$ satisfies the *Whitney condition (a)* if for strata $S_i$, $S_j$, with $S_i \subset \overline{S_j}$, and any sequence $(x_n) \subset S_j$, $x_n \to x \in S_i$, with $T_{x_n}S_j$ converging[4] to a linear space $L \subset T_xM$, we have that

$$T_xS_i \subset L.$$

Note that since $\dim M < \infty$ and stratifications are locally finite, the dimension of $X$ is well-defined and independent of a specific $C^1$-stratification[5].

Stratifications enable us to give a precise definition of a piecewise differentiable function on manifolds.

**Definition 1.1.8** Let $M$ be a manifold and $f\colon M \to \mathbf{R}$ be a continuous function. We call $f$ *piecewise $C^p$* if there is a stratification $S_j$, $j \in \Lambda$ such that $f$ is $C^p$ on the strata. We call the stratification $S_j$, $j \in \Lambda$ a *domain stratification* of $f$.

Later, we will consider functions, which are not only piecewise differentiable, but their domain stratification satisfies stronger conditions. We start with the standard notion of an $(a_f)$ or Thom-stratification, cf. [112, 115].

**Definition 1.1.9** Let $M$ be a smooth manifold and $f\colon M \to \mathbf{R}$ a continuous function. A stratification $S_j$, $j \in \Lambda$ of $M$ is an $(a_f)$ *$C^p$-stratification for $f$* if the following conditions hold:

[4]Convergence with respect to the topology of the Grassmann bundle on $M$, see [26, p. 48] for a definition of the Grassmann bundle.

[5]This follows easily from the Morse-Sard theorem [89, p.69].

- the strata are $C^p$,
- the stratification satisfies the Whitney-(a) condition
- $f$ is $C^p$ on the strata,
- the rank of $df|_{S_j}$ is constant for every stratum $S_j$ and
- the *Thom condition* holds at every point, i.e. for strata $S_j$, $S_i$, with $S_i \subset \overline{S_j}$, and any sequence $(x_n) \subset S_j$, $x_n \to x \in S_i$ with $\ker df|_{S_j}(x_n) \to L$ we have $\ker df|_{S_i}(x) \subset L$.

**Theorem 1.1.10** *Let $M$ be an analytic manifold and $f\colon M \to \mathbf{R}$ be a continuous $\mathcal{C}$-function. Then for all $p \in \mathbf{N}$ there is a $(a_f)$ $C^p$-stratification for $f$ such that the strata are $\mathcal{C}$-sets. Especially, any continuous $\mathcal{C}$-function is piecewise $C^p$.*

**Proof:** Loi proved this for functions definable in an o-minimal structure over $(\mathbf{R}, +, \cdot)$ [112]. His proof uses the standard trick from algebraic geometry of showing that the set, where the $(a_f)$-condition is violated, is definable and contains no open set. According to Theorem 1.1.3, and Propositions 1.1.4, 1.1.5 this method can be lifted to analytic-geometric categories by using local analytic charts. □

For the rest of this section, the manifold $M$ will be equipped with a Riemannian metric denoted by $\langle \cdot, \cdot \rangle$. If $f\colon M \to \mathbf{R}$ is a piecewise differentiable function and $S_j$, $j \in \Lambda$ a domain stratification of $f$, then we denote by $\operatorname{grad}_j f$ the gradient of the restriction of $f$ to $S_j$, $f|_{S_j}$, with respect to the induced metric on $S_j$.

**Lemma 1.1.11** *Let $M$ be a smooth Riemannian manifold and $f\colon M \to \mathbf{R}$ a continuous function. Assume that $S_j$, $j \in \Lambda$ is a $(a_f)$ stratification which is also a domain stratification of $f$. Let $S_i \subset \overline{S}_j$ strata and $(x_k) \subset S_j$ a sequence with $x_k \to x \in S_i$, $\operatorname{grad}_j f(x_k) \neq 0$ and*

$$\lim_{k\to\infty} \frac{\operatorname{grad}_j f(x_k)}{\|\operatorname{grad}_j f(x_k)\|} = v.$$

*Then $\pi_{T_x S_i}(v) = \lambda \operatorname{grad}_i f(x)$ with a $\lambda \in \mathbf{R}$, $\pi_{T_x S_i}$ the orthogonal projection to $T_x S_i$ with respect to the Riemannian metric.*

**Proof:** By choosing a subsequence of $(x_k)$ we can w.l.o.g. assume that $\ker df|_{S_j}(x_k)$ converges to a linear space $L \subset T_xM$. The Thom condition implies that $\ker df|_{S_i}(x) \subset L$. We denote by $v_k$ the vector

$$v_k = \frac{\operatorname{grad}_j f(x_k)}{\|\operatorname{grad}_j f(x_k)\|}.$$

Note that $v_k$ converges to a normal vector of $L$. If $\pi_{T_xS_i}(v) \neq 0$ then $T_xS_i \cap L \neq T_xS_i$. In particular, $\ker df|_{S_i}(x) \neq T_xS_i$ and $\operatorname{grad}_i f(x) \neq 0$. Since $\ker df|_{S_i}(x) \subset T_xS_i \cap L$, $\dim\ker df|_{S_i}(x) = \dim T_xS_i - 1$ and $\dim(T_xS_i \cap L) < \dim T_xS_i$, we have that $\ker df|_{S_i}(x) = T_xS_i \cap L$. Therefore $\pi_{T_xS_i}(v) = \lambda \operatorname{grad}_i f(x)$ for some $\lambda \in \mathbf{R}$. □

Unfortunately, the conditions on $(a_f)$-stratifications will not be strong enough to derive the theorems for the gradient-like systems considered later. Hence, we introduce our own notion of "strong" $(a_f)$ stratifications, which will be used later in the proofs. Note, that we call a function $f\colon M \to \mathbf{R}$ *Lipschitz continuous* at $x \in M$, if it is Lipschitz continuous at $x$ in local chart around $U$. On Riemannian manifolds this is equivalent to the existence of a neighborhood $U$ of $x$ and a constant $L > 0$ such that $|f(x) - f(y)| \leq L \operatorname{dist}(x, y)$ for all $y \in U$ with dist the Riemannian distance.

**Definition 1.1.12** Let $M$ be a Riemannian manifold and $f\colon M \to \mathbf{R}$ be a continuous function. A stratification $S_j$, $j \in \Lambda$ of $M$ is a *strong* $(a_f)$ $C^p$-*stratification for* $f$ if the following conditions hold:

1. the strata are $C^p$ submanifolds,
2. $S_j$, $j \in \Lambda$ is a Whitney $(a)$-stratification
3. $f$ is $C^p$ on the strata,
4. $\operatorname{rk} df|_{S_j}$ is constant on any stratum $S_j$,
5. if there is a $x \in S_i$ such that for all $j \in \Lambda$, and $(x_k) \subset S_j$, $x_k \to x$, the sequence $\|df|_{S_j}(x_k)\|$ is bounded, then $f$ is Lipschitz continuous in all $y \in S_i$.
6. for strata $S_i$, $S_j$, with $S_i \subset \overline{S_j}$, any sequence $(x_n) \subset S_j$, $x_n \to x \in S_i$, with $T_{x_n}S_j \to L$, $L \subset T_xM$ a linear space, and $df|_{S_j}(x_n) \to \alpha\colon L \to \mathbf{R}$ it holds that
$$\alpha(v) = df|_{S_i}(x)(v) \quad \text{for all} \quad v \in T_xS_i,$$
where $\pi_{T_xS_i}$ denotes the orthogonal projection on $T_xS_i$.

7. for strata $S_i$, $S_j$, with $S_i \subset \overline{S_j}$, any sequence $(x_n) \subset S_j$, $x_n \to x \in S_i$, with $\ker df|_{S_j}(x_n) \to L$, $L \subset T_xM$ a linear space, and $\|df|_{S_j}(x_n)\| \to \infty$ it holds that
$$T_xS_i \subset L.$$

**Remark 1.1.13** Assume that we are given a continuous function $f\colon M \to \mathbf{R}$ and a strong $(a_f)$, $C^p$-stratification $S_j$, $j \in \Lambda$, for $f$. Let $S_j$, $S_i$ strata with $S_i \subset \overline{S}_j$ and $(x_k)$ a sequence in $S_j$ with $x_k \to x \in S_i$. Furthermore, let $\ker df_{S_j}(x_k)$ converge to a linear subspace $L \subset T_xM$. By passing to a suitable subsequence of $(x_k)$ we can w.l.o.g. assume that either condition 6 or 7 hold for $(x_k)$. If condition 6 is satisfied then

$$\ker df_{S_i}(x) \subset \ker \alpha = L.$$

In the case that condition 7 holds, it yields

$$\ker df_{S_i}(x) \subset T_xS_i \subset L.$$

Hence, the conditions 6 and 7 imply the Thom condition for strong $(a_f)$ stratifications. Therefore, any strong $(a_f)$ stratification is an $(a_f)$ stratification in the sense of Definition 1.1.9.

We will show that for every $\mathcal{C}$-function a strong $(a_f)$-condition exists. To achieve this, we need some technical lemmas.

**Lemma 1.1.14** *Let $S, T \subset \mathbf{R}^n$ be $C^p$-submanifolds, $p > 1$, and definable in an o-minimal structure $\mathcal{R}$ on $(\mathbf{R}, +, \cdot)$. Assume that $S \subset \overline{T}$. Then there is a relatively open subset of $S$ such that every $C^1$ curve $\gamma\colon [0,1] \to S$ can be lifted to a family of $C^1$ curves $(\gamma_\varepsilon\colon [0,1] \to T \cup S \mid \varepsilon \in \mathbf{R}_+)$ such that*

- $\gamma_0 = \gamma$,
- *if* $\varepsilon > 0$ *then* $\gamma_\varepsilon(t) \in T$ *for all* $t \in [0,1]$,
- *the map* $(t, \varepsilon) \mapsto \gamma_\varepsilon(t)$ *is continuous,*
- *and* $\dot\gamma_\varepsilon$ *converges uniformly, to* $\dot\gamma$ *for* $\varepsilon \to 0$.

*Furthermore, if $\gamma$ is definable then the family can be chosen as a definable family, i.e. the map $(t, \varepsilon) \mapsto \gamma_\varepsilon(t)$ is definable.*

**Proof:** By curve selection with parameters [53, Theorem 4.8] there is a finite collection of $C^p$ manifolds $S_i$, with $S = \bigcup S_i$, and injective definable map $p\colon S \times (0,1) \to T$ which is $C^p$ on $S_i \times (0,1)$ and $p(x,t) \to x$ for $t \to 0$ and $x \in S$. Applying the existence of $C^p$-stratifications [53] to the image of this map, there must be a definable $C^p$ stratum $T' \subset T$ of dimension $\dim S + 1$ such that $\overline{T}' \cap S$ is open in $S$.

Recall that $T'$, $S$ satisfy the Whitney-(b) condition if for all $x \in S$ and any sequences $(x_k) \subset T$, $(y_k) \subset S$, $x_n \to x$, $y_k \to x$ with $T_{x_k}T \to L$, $L$ a linear subspace, and

$$\{r(x_k - y_k) \mid \alpha \in \mathbf{R}\} \to V,$$

$V$ a 1-dimensional linear subspace, the inclusion

$$V \subset L$$

holds [53, 115]. By the existence of Whitney-(b) stratifications [53] we can assume after eventually shrinking $S$ and $T'$ that $\overline{T}' \cap S = S$ and the Whitney-(b) condition holds for $S$ and $T'$. Note, that we can always shrink $S$ and $T'$ such that these sets are still definable. Furthermore this implies that also the Whitney-(a) condition holds in all $x \in S$, cf. [115].

Let

$$NS := \{(x,v) \in \mathbf{R}^n \times \mathbf{R}^n \mid x \in S, \langle v, w\rangle = 0 \text{ for all } w \in T_xS\}$$

be the Euclidean normal bundle of $S$. The Euclidean normal bundle is definable in $\mathcal{R}$ [45]. Consider the definable set

$$W_\varepsilon = \{x \in \mathbf{R}^n \mid x = y + v, (y,v) \in NS, \|v\| = \varepsilon\}.$$

After eventually shrinking $S$ and $T'$ there is a $\mu > 0$ such that for all $0 < \varepsilon < \mu$, $W_\varepsilon$ is a manifold [6] of codimension 1.

Assume that there is no neighborhood of $S$ on which $T'$ is transversal to the $W_\varepsilon$ for all $\varepsilon \in (0,\rho)$, $\rho > 0$. Then there exists a sequences $(x_k) \subset T'$, $(\varepsilon_k) \subset (0,\rho)$, with $x_k \to x$, $x \in S$ such that $x_k \in W_{\varepsilon_k}$ and $T_{x_k}T' \subset T_{x_k}W_{\varepsilon_k}$. We can assume that $T_{x_k}T'$ converges to a linear subspace $L$. We denote by $(y_k)$ the minimum distance projection of $x_k$ to $S$. For suitably large $k$ the $y_k$

[6]This follows from the construction of normal tubular neighborhoods for submanifolds of $\mathbf{R}^n$, see [89, Thm. 5.1 and its proof].

are well defined. The definition of the normal bundle implies that for large $k$,

$$l_k := \{r(y_k - x_k) \mid r \in \mathbf{R}\}$$

is orthogonal to $T_{x_k} W_{\varepsilon_k}$. By choosing an appropriate subsequence, we can assume that $l_k$ converges to a 1-dimensional linear subspace $V$. But as $T_{x_k} T' \subset T_{x_k} W_{\varepsilon_k}$ and $l_k$ is orthogonal to $T_{x_k} W_{\varepsilon_k}$, we see that

$$V \not\subset L.$$

This is a contradiction to the Whitney-(b) condition. Hence, the manifold $W_\varepsilon$ is transversal to $T'$ on a neighborhood of $S$.

Shrinking $S$, $T'$ and $\mu$ we can assume that $W_\varepsilon$ is transversal to $T'$ for all $\varepsilon \in (0, \mu)$. We can consider for $0 < \varepsilon < \mu$ the definable manifold

$$X_\varepsilon := W_\varepsilon \cap T'.$$

We choose a connected component $X$ of $\{(x, \varepsilon) \mid x \in X_\varepsilon, 0 < \varepsilon < \mu\}$, such that closure of $X$ contains an open set $U$ of $S$. Furthermore, we restrict the $X_\varepsilon$ to their intersection with $X$, i.e. $X \cap X_\varepsilon = X_\varepsilon$. Decreasing $\mu$ we can assume that $X_\varepsilon$ is non-empty for all $0 < \varepsilon < \mu$. We consider now the Euclidean least-distance projection $\sigma$ onto $S$. Restricting $\sigma$ to each $X_\varepsilon$ we get the family of projections $\sigma_\varepsilon \colon X_\varepsilon \to S$. Shrinking $U$, $S$ and $\mu$ we can assume that $\sigma \colon U \to S$ is smooth and $\sigma_\varepsilon$ is a $C^p$ diffeomorphism[7] for all $0 < \varepsilon < \mu$. Again, it can be ensured that the shrunken $U$, $S$ are still definable. Note, that by the smoothness of $\sigma$, $T\sigma$ is uniformly bounded on a relatively compact neighborhood of $S$. For any sequences $(\varepsilon_k) \subset \mathbf{R}_+$, $x_k \in X_{\varepsilon_k}$, with $\varepsilon_k \to 0$, $x_k \to x \in S$ the Whitney-(a) condition implies $T_{x_k} X_{\varepsilon_k} \to T_x S$. With $\sigma|_S = \mathrm{Id}_S$ we get that

$$T_{x_k} \sigma_{\varepsilon_k} \to \mathrm{Id}_{T_{\sigma(x)} S} \,. \tag{1.1}$$

As $(\sigma_\varepsilon)$ is a definable family of functions[8], there must be a relatively open subset $W$ in $S$ such that the convergence (1.1) is uniform on $\sigma^{-1}(W) \cap X$, i.e. for all $a > 0$ there exists $b > 0$ such that for all $\varepsilon > 0$, $x \in \sigma^{-1}(W) \cap X_\varepsilon$

$$\|x - \sigma(x)\| < b \text{ implies } \left\| T_x \sigma_\varepsilon - \mathrm{Id}_{T_{\sigma(x)} S} \right\| < a.$$

[7]This follows again from the construction of normal tubular neighborhoods for submanifolds of $\mathbf{R}^n$ [89, Thm. 5.1 and its proof] and the fact that straight lines are geodesics in $\mathbf{R}^n$.

[8]i.e. the map $(x, \varepsilon) \mapsto \sigma_\varepsilon(x)$ is definable.

Let $\gamma\colon [0,1] \to W$ be a $C^1$ curve in $W$. For every $X_\varepsilon$ we can choose a unique curve $\gamma_\varepsilon : [0,1] \to X_\varepsilon$ by $\gamma_\varepsilon(t) = \sigma_\varepsilon^{-1}(\gamma(t))$. By construction this gives a continuous family of curves $\gamma_\varepsilon$ such that $\gamma_0 = \gamma$. The curves $\gamma_\varepsilon$ are $C^1$ as $\sigma_\varepsilon$ is a diffeomorphism. Note that $\dot\gamma(t) = T_{\gamma_\varepsilon(t)}\sigma\dot\gamma_\varepsilon(t)$. By the uniform convergence of $T_x\sigma_\varepsilon$ to $\mathrm{Id}_{T_xS}$ the derivative $\dot\gamma_\varepsilon$ converges uniformly to $\dot\gamma$. Obviously, the family $\gamma_\varepsilon$ is definable if $\gamma$ is definable. □

**Lemma 1.1.15** *Let $M$ be an analytic manifold and $f\colon M \to \mathbf{R}$ be a continuous $\mathcal{C}$-function. Assume that we have $\mathcal{C}$-sets $S, T \subset M$ which are $C^p$-submanifolds, $p > 1$, and $S \subset \overline{T}$, $\dim S < \dim T$. Furthermore, we assume that the Thom and the Whitney-(a) condition hold for all $x \in S$ and sequences in $T$. Then the set*

$$\begin{aligned} A = \{x \in S \mid \exists (x_k) \subset T \textit{ with } x_k \to x, \\ \lim_{k\to\infty} T_{x_k}T = L, L \subset T_xM \textit{ a linear space} \\ \lim_{k\to\infty} df|_T(x_k) = \alpha\colon L \to \mathbf{R}, \alpha|_{T_xS} \neq df|_S(x)\} \\ \cap \{x \in S \mid \exists C > 0 \forall (x_k) \subset T \textit{ with } x_k \to x, \lim \|df|_T x_k\| < C\} \quad (1.2) \end{aligned}$$

*is a $\mathcal{C}$-set with* $\dim A < \dim S$.

**Remark 1.1.16** The definition of $A$ in Lemma 1.1.15 is independent of the Riemannian metric.

**Proof:** We first show that $A$ is a $\mathcal{C}$-set. By the definition of analytic-geometric categories, it is sufficient to show this locally. By Proposition 1.1.4 it is sufficient to show that $A$ is locally mapped by analytic charts to a set definable in $\mathcal{R}(\mathcal{C})$. Using analytic charts we can assume by Propositions 1.1.4,1.1.5 that $M = \mathbf{R}^n$ and $S$, $T$, $f$ are definable in $\mathcal{R}(\mathcal{C})$. Since the definition of $A$ does not depend on the Riemannian metric, we can w.l.o.g. assume that $\mathbf{R}^n$ is equipped with the Euclidean metric. Denote by $\mathrm{Grass}(n,p)$ the Grassmann manifold of $p$-dimensional linear subspaces $\mathbf{R}^n$. We use the standard identification[9] [77]

$$\mathrm{Grass}(n,p) = \{P \in \mathbf{R}^{n\times n} \mid P^2 = P, P^\top = P, \mathrm{rk}\, P = p\}.$$

Each subspace is identified with the symmetric, orthogonal projection onto itself. The manifold $\mathrm{Grass}(n,p)$ is a definable, analytic submanifold of $\mathbf{R}^{n\times n}$.

[9] This identification will also be used in later sections.

We define the subsets

$$\begin{aligned} B_1 &= \{(x,P,v) \mid && x \in T, P \in \operatorname{Grass}(n, \dim T), \forall w \in T_xT\colon Pw = w; \\ & && \forall w \in T_xT\colon \langle v, w\rangle = df|_T(x)(w)\} \\ B_2 &= \{(x,P,v) \mid && x \in S, P \in \operatorname{Grass}(n, \dim T), \\ & && \forall w \in T_xS, \langle v, w\rangle = df|_S(x)(w)\} \\ B_3 &= \{(x,P,v) \mid && x \in T, P \in \operatorname{Grass}(n, \dim T), \operatorname{Im} P = T_xT, \\ & && \forall w \in T_xT\colon \frac{\langle v, w\rangle}{\|v\|^2} = df|_T(x)(w)\} \end{aligned}$$

of $\mathbf{R}^n \times \operatorname{Grass}(n, \dim T) \times \mathbf{R}^n$. Note, that these sets are all definable in $\mathcal{R}(\mathcal{C})$. Then

$$A = \left(S \cap \pi_1\left(\overline{B_1} \setminus B_2\right)\right) \setminus \pi_1(\overline{B_3} \cap (S \times \operatorname{Grass}(n, \dim T) \times \{0\})),$$

$\pi_1$ the projection on the first component. Hence, the set $A \subset \mathbf{R}^n$ is definable[10]. Thus, we have proven that $A$ is in the general case a $\mathcal{C}$-set.

For $\dim A < \dim S$ we have to show that $A$ contains no relatively open, non-empty subset of $S$. Assume that this does not hold, w.l.o.g. $A = S$ and $S$ is relatively compact. We first show that $df|_S(x) \neq 0$ for all $x \in S$. Assume that $df|_S(x) = 0$ for a $x \in S$. Since $S = A$ there is a sequence $(x_k) \subset T$, $x_k \to x$ such that $T_{x_k}T$ converges to a linear space $L$, $\dim L = \dim T$, and there is a linear map $\alpha\colon L \to \mathbf{R}$ with

$$\lim_{k\to\infty} df|_T(x_k) = \alpha \text{ and } \alpha|_{T_xS} \neq df|_S(x) = 0.$$

By replacing $(x_k)$ with a subsequence, we can assume that $(\ker df|_T(x_k))$ converges to a linear space $L' \subset T_xM$, $\dim L' = \dim T - 1$. By the Thom condition we have $T_xS = \ker df|_S(x) \subset L'$. But since $\alpha|_{L'} = 0$, we get $\alpha|_{T_xS} = 0$. This gives a contradiction. Hence, $df|_S(x) \neq 0$ for all $x \in S$.

By Propositions 1.1.4,1.1.5 we can again assume that $M = \mathbf{R}^n$ and $S$,$T$, $f$ are definable in $\mathcal{R}(\mathcal{C})$. Again, we can assume that $M$ is equipped with the Euclidean metric. The Euclidean least distance projection $\sigma$ onto $S$ is a well defined function on a neighborhood $U$ of $S$[11]. Since the projection can be defined by

$$\sigma(x) = \{y \in S \mid \|x - y\| = \min_{z\in S} \|x - y\|\}$$

[10] The closure of a definable set is definable [53].

[11] This follows again from the construction of normal tubular neighborhoods in $\mathbf{R}^n$ [89, Thm. 5.1 and its proof] and the fact that straight lines are geodesics in $\mathbf{R}^n$.

on the set

$$U = \{x \in \mathbf{R}^n \mid \#\{y \in S \mid \|x - y\| = \min_{z \in S} \|x - y\|\} = 1\},$$

the projection is definable in $\mathcal{R}(\mathcal{C})$. We consider the function $h\colon T \cap U \to \mathbf{R}$,

$$h(x) := \frac{\langle \operatorname{grad} f|_T(x), \operatorname{grad} f|_S(\sigma(x))\rangle}{\|\operatorname{grad} f|_S(\sigma(x))\|^2}$$

where $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ denote the Euclidean scalar product and norm, grad the gradient on $S$ and $T$ with respect to the Riemannian metric induced by the Euclidean one. The function $h$ is well defined as $df|_S(x) \neq 0$ for all $x \in S$ and it is definable in $\mathcal{R}(\mathcal{C})$. Let $\Gamma_h$ be the graph of $h$. From the Thom condition follows that if for a sequence $(x_k) \subset T$, $x_k \to x \in S$ $\lim df|_T(x_k) = \alpha$ exists then $\alpha|_{T_xS} = \mu df|_S(x)$ with $\mu \in \mathbf{R}$, see Lemma 1.1.11. By (1.2) and since $A = S$, we have for any $x \in S$ a $\delta \neq 1$ with $(x, \delta) \in (\overline{\Gamma}_h \cap S \times \mathbf{R})$. By the existence of $C^p$-stratifications [53, Theorem 4.8] there is a definable open set $V$ in $S$ and a $\delta \neq 1$ such that for all $x \in V$ the set $\{x\} \times (-\infty, \delta) \cap \overline{\Gamma}_h$ is non-empty. W.l.o.g. we can assume that $S = V$. We first discuss the case $\delta < 1$. Let $R_\delta$ be the set

$$R_\delta = \{x \in T \cap U \mid h(x) < \delta\}.$$

Then $S \subset \overline{R}_\delta$. Shrinking $T$, we can w.l.o.g. assume that $h(x) < \delta$ for all $x \in T$. By Lemma 1.1.14 there is an open set $W \subset S$ such that any $C^1$ curve in $S$ can be lifted to a family of curves on $T$. Let $\gamma\colon [a, b] \to W$ be a $C^1$ integral curve of $\operatorname{grad} f|_S$. We can lift $\gamma$ to a continuous family of $C^1$ curves $\gamma_\varepsilon : [a, b] \to T$ with $\gamma_0 = \gamma$ and $\dot{\gamma}_\varepsilon$ converges uniformly to $\dot{\gamma}$. As $f$ is continuous $f_\varepsilon := f \circ \gamma_\varepsilon$ must converge uniformly to $f \circ \gamma$. Furthermore by definition of $A$, for any $y \in A$ there is a constant $C > 0$ and neighborhood $W$ of $y$ in $S \cup T$ such that for all $x \in W$: $\|df|_T(x)\| < C$. Since $\gamma([a, b])$ is compact, this yields that

$$\eta_\varepsilon(t) = \langle \operatorname{grad} f|_T(\gamma(t)), \operatorname{grad} f|_S(\gamma(t)) - \dot{\gamma}_\varepsilon(t)\rangle$$

converges uniformly to 0 for $\varepsilon \to 0$. Hence, there is a $\rho > 0$ and a continuous function $\tau\colon (0,\rho) \to \mathbf{R}_+$, with $\tau(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that

$$\begin{aligned} f_\varepsilon(b) - f_\varepsilon(a) &= \int_a^b \frac{d}{dt} f \circ \gamma_\varepsilon(t) dt = \int_a^b \langle \operatorname{grad} f|_T(\gamma_\varepsilon(t)), \dot\gamma_\varepsilon(t)\rangle \, dt \\ &\leq \int_a^b \langle \operatorname{grad} f|_T(\gamma_\varepsilon(t)), \operatorname{grad} f|_S(\gamma(t))\rangle \, dt + \tau(\varepsilon) \\ &\leq \int_a^b \delta \, \|\operatorname{grad} f|_S(\gamma(t))\|^2 \, dt + \tau(\varepsilon) \\ &= \delta(f(\gamma(b)) - f(\gamma(a))) + \tau(\varepsilon). \end{aligned}$$

Since $\delta < 1$, this gives a contradiction and $A$ contains no open set of $S$. For the case $\delta > 1$, we can use an analogous argument with a lower bound for $f_\varepsilon(b) - f_\varepsilon(a)$ which yields a contradiction. □

**Lemma 1.1.17** *Let $M$ be an analytic Riemannian manifold and $f\colon M \to \mathbf{R}$ be a continuous $\mathcal{C}$-function. Assume that we have $\mathcal{C}$-sets $S, T \subset M$ which are $C^p$-submanifolds, $p > 1$, and $S \subset \overline{T}$, $\dim S < \dim T$. Furthermore we assume that the Thom and the Whitney-(a) condition hold for all $x \in S$ and sequences in $T$. Then the set*

$$\begin{aligned} B = \{x \in S \mid \exists (x_k) \subset T \text{ with } x_k \to x, \|df|_T(x_k)\| \to \infty, \\ \lim_{k\to\infty} \ker df(x_k) = L, L \subset T_x M \text{ linear space}, T_x S \cap L \neq T_x S\}. \quad (1.3) \end{aligned}$$

*is a $\mathcal{C}$-set with $\dim B < \dim S$.*

**Remark 1.1.18** The definition of $B$ in Lemma 1.1.17 is independent of the Riemannian metric.

**Proof:** We start with showing that $B$ is a $\mathcal{C}$-set. As in the proof of Lemma 1.1.15 it is sufficient to show this locally. Thus, using local analytic charts, we can assume that $M = \mathbf{R}^n$ and $f$, $S$, $T$ definable. Since the definition of $B$ does not depend on the Riemannian metric, we equip $\mathbf{R}^n$ with the Euclidean one. We define subsets of $\mathbf{R}^n \times \mathbf{R} \times \operatorname{Grass}(n, \dim T - 1)$

$$\begin{aligned} C_1 &= \{(x, r, P) \mid x \in T, r = (1 + \|df(x)\|^2)^{-1}, df|_T(x) \circ P = 0\} \\ C_2 &= \{(x, 0, P) \mid x \in S, P \in \operatorname{Grass}(n, \dim T - 1), \exists w \in T_x S\colon Pw \neq w\}. \end{aligned}$$

These sets are definable in $\mathcal{R}(\mathcal{C})$. Then $B = \pi_1(\overline{C}_1 \cap C_2)$, $\pi_1$ the projection on the first component. Hence, $B$ is definable in $\mathcal{R}(\mathcal{C})$. Thus, the set $B$ is in the general case a $\mathcal{C}$-set.

Assume that $\dim B = \dim S$, w.l.o.g. $B = S$. Let $x \in S$ with $df|_s(x) = 0$. Since $S = B$ there is a sequence $(x_k) \subset T$, with $x_k \to x$, $\lim \ker df|_T(x_k) = L$, $L \subset T_xM$ a linear space and $T_xS \cap L \neq T_xS$. But on the other hand the Thom condition implies that $\ker df|_S(x) = T_xS \subset L$. This yields an contradiction and $df|_S(x) \neq 0$ for all $x \in S$.

Let $\sigma\colon T \to S$ the least distance projection on $S$. The map $\sigma$ is smooth and well defined after eventually shrinking $S$ and $T$. We define the function

$$h(x) = \langle \operatorname{grad} f|_T(x), \operatorname{grad} f|_S(\sigma(x)) \rangle \, .$$

Since $S = B$ and $df|_s(x) \neq 0$ for all $x \in S$, the closure of

$$X = \{x \in T \mid \ |h(x)| \geq 2 \left\| \operatorname{grad}_S f(\sigma(x)) \right\| \}$$

contains an open subset of $S$. W.l.o.g. we can assume that $X = T$ and $h(x) > 0$ for all $x \in S$.

By Lemma 1.1.14 there is a relatively open subset $W$ of $U$ such that curves in $W$ can be lifted to families of curves in $T$. We choose an integral curve $\gamma\colon [a, b] \to S$ of the vector field $\operatorname{grad} f|_S(x)$ on $S$. This curve is lifted to a continuous family of $C^1$ curves $\gamma_\varepsilon\colon [a, b] \to T$ with $\gamma_0 = \gamma$ and $\dot\gamma_\varepsilon$ converges uniformly to $\dot\gamma$. As $f$ is continuous $f_\varepsilon := f \circ \gamma_\varepsilon$ must converge uniformly to $f \circ \gamma$. On the other hand

$$\begin{aligned} f_\varepsilon(b) - f_\varepsilon(a) &= \int_a^b \frac{d}{dt} f \circ \gamma_\varepsilon(t) dt = \int_a^b \langle \operatorname{grad} f|_T(\gamma_\varepsilon(t)), \dot\gamma_\varepsilon(t) \rangle \, dt \\ &= \int_a^b \langle \operatorname{grad} f|_T(\gamma_\varepsilon(t)), \operatorname{grad} f|_S(\gamma(t)) \rangle \, dt - \tau(\varepsilon) \\ &\geq 2 \int_a^b \left\| \operatorname{grad} f|_S(\gamma(t)) \right\|^2 dt - \tau(\varepsilon) \\ &= 2(f(\gamma(b)) - f(\gamma(a))) - \tau(\varepsilon) \end{aligned}$$

with a continuous function $\tau\colon \mathbf{R}_+ \to \mathbf{R}_+$, $\tau(\varepsilon) \to 0$ for $\varepsilon \to 0$. This gives a contradiction and $B$ contains no relatively open subset of $S$. □

**Lemma 1.1.19** *Let $M$ be a smooth Riemannian manifold and $f\colon M \to \mathbf{R}$ be a continuous, piecewise differentiable function. Assume that we have a*

*$C^p$-stratification $S_j$, $j \in \Lambda$ of $M$, $p \geq 2$, such that $f$ is $C^1$ on the strata. Then $f$ is locally Lipschitz continuous in $x \in M$ if there is a neighborhood $U$ of $x$ and a constant $C > 0$ such that for all $j \in \Lambda$ and $y \in U \cap S_j$*

$$\left\| df|_{S_j}(y) \right\| < C$$

*holds.*

**Proof:** Using a local chart of a neighborhood of $U' \subset U$ of $x$ we can assume that $M = \mathbf{R}^n$ and $U \subset \mathbf{R}^n$. Let $y \in U$ and $\gamma \colon [0,1] \to U$ be the straight line between $y$ and $x$, i.e. $\gamma(t) = t(x-y) + y$. By a theorem from differential topology [89, p.78, Thm. 2.5], we can approximate $\gamma$ in the $C^2$ topology by $C^2$ curves $\gamma_k(t)$ which are transversal[12] to the strata. Then $h_k(t) = f(\gamma_k(t))$ is continuous differentiable besides a finite number of points of $[0,1]$. By the conditions on $f$ we have that $\|h_k'(t)\| \leq C\varepsilon_k \|x-y\|$ with $\varepsilon_k \to 1$. Thus $|f(y) - f(x)| \leq C \|x-y\|$. As this holds for all $y \in U$, $f$ is Lipschitz continuous in $x$. □

**Theorem 1.1.20** *Let $M$ be an analytic Riemannian manifold and $f \colon M \to \mathbf{R}$ be a continuous $\mathcal{C}$-function. Then for all $p > 1$ there is a strong $(a_f)$ $C^p$-stratification into $\mathcal{C}$-sets for $f$ and any $(a_f)$ $C^p$-stratification into $\mathcal{C}$-sets can be refined into a strong one.*

**Proof:** With Theorem 1.1.10 it is sufficient to show that any $(a_f)$ stratification can be refined into a strong one. Let $S_j$, $j \in \Lambda$ be an $(a_f)$ stratification of $f$. We define the set

$$A := \{x \in M \mid \exists i,j \in \Lambda, x \in S_i, S_i \subset \overline{S}_j, \exists (x_k) \subset S_j \colon x_k \to x, \\ \left\| df|_{S_j}(x_k) \right\| \to \infty \}.$$

Analytic charts map locally this set to sets definable in $\mathcal{R}(\mathcal{C})$, cf. Proposition 1.1.4. Thus $A$ is a $\mathcal{C}$-set. Therefore, we can refine the stratification such that $A$ is the union of strata. By abuse of notation we denote this refinement by $S_j$, $j \in \Lambda$. We chose the refinement such that $\operatorname{rk} df|_{S_j}$ is constant on the strata, cf. [53]. As $A$ is closed, a stratum $S_i$ is either contained in $A$ or all $x \in S_i$ have a neighborhood $U(x)$ such that $\left\| df|_{S_j}(y) \right\|$ is bounded for

[12] Here: if $\gamma_k(t) \in S_j$, then $\dot{\gamma}_k(t)$ and $T_{\gamma_k(t)} S_j$ span $\mathbf{R}^n$. We use later a weaker notion of transversality for differentiable curves.

all $j \in \Lambda$, $y \in U(x) \cap S_j$. Lemma 1.1.19 implies that the stratum is either contained in $A$ or $f$ is locally Lipschitz in all points of the stratum. Thus the stratification satisfies the conditions 1 - 5 of a strong $(a_f)$ stratification. For strata $S_i$, $S_j$, $S_i \subset \overline{S}_j$ let $A_{ij}$ be the set the of points, were the condition 6 fails. By the Lemmas 1.1.15 the set $A_{ij}$ has dimension $< \dim S_i$. The same holds by Lemma 1.1.17 for the set $B_{ij}$, were the condition 7 fails. A standard argument from real-algebraic geometry, cf. [53, 83, 112], implies that we can refine our stratification such that both conditions are satisfied everywhere. Note that after refinement of the stratification the conditions 1 - 4 are still satisfied. Thus our stratification is a strong $(a_f)$-stratification. □

**Remark 1.1.21** In [31] Bolte et al. show that a Whitney stratification of the graph of a function $\mathbf{R}^n \to \mathbf{R}$ with an additional regularity condition, always yields a projection formula for the Clarke generalized gradient, i.e. the Clarke generalized gradient projected to the tangent space of a stratum is the gradient of the function on the stratum. Furthermore, they show that for an arbitrary finite collection of subsets of $\mathbf{R}^n$ and a definable function, such a Whitney stratification, compatible with the subsets, always exists. While we provided a direct proof of Theorem 1.1.20, the results of Bolte et al. actually imply this theorem, too.

### 1.1.3 The Łojasiewicz gradient inequality

Our convergence theory for time-continuous gradient-like systems is based on the *Łojasiewicz gradient inequality*. This is an estimate on the gradient of a function. It was first established by Łojasiewicz for analytic functions [113], and later extended by Kurdyka to functions definable in an o-minimal structure on $(\mathbf{R}, +, \cdot)$ [101]. Bolte et al. [30, 31] have considered the Łojasiewicz gradient inequality for Clarke's generalized gradient of semi-analytic and definable functions. By Proposition 1.1.5 Kurdyka's result yields a Łojasiewicz gradient inequality for $\mathcal{C}$-functions. For our applications, we need the following version for $\mathcal{C}$-functions.

**Theorem 1.1.22** *Let $S$ be a submanifold of an analytic Riemannian manifold $M$. Furthermore let $f\colon S \to \mathbf{R}$ be a bounded, differentiable $\mathcal{C}$-function and $x^* \in M$. Assume that $S$ is equipped with a Riemannian metric $g = \langle\langle \cdot, \cdot \rangle\rangle$, such that for any compact set $K \subset M$ there is a constant $C_K$ with $\|g_x\| \leq C_K$ for all $x \in K \cap S$, $\|\cdot\|$ denoting the operator norm with respect to the Riemannian metric on $M$. Then there exist a neighborhood $U$ of $x^*$ in $M$,* *constants $C > 0$, $\rho > 0$ and a strictly increasing $C^1$-function $\psi\colon (0,\rho) \to \mathbf{R}_+$ such that for $x \in U \cap f^{-1}((0,\rho))$*

$$\left\|\operatorname{grad}_g \psi \circ f(x)\right\|_g \geq C \quad \textit{holds.} \tag{1.4}$$

*Here $\operatorname{grad}_g$ and $\|\cdot\|_g$ denote the norm and gradient with respect to $g$. Furthermore the function $\psi$ is definable in the o-minimal structure $\mathcal{R}(\mathcal{C})$ on $(\mathbf{R},+,\cdot)$.*

The theorem follows from Kurdyka's version for functions definable in an o-minimal structure on $(\mathbf{R},+,\cdot)$. However, we need the following small lemma.

**Lemma 1.1.23** *Let $M$ a Riemannian manifold and $S$ a submanifold with different Riemannian metric $g = \langle\langle\cdot,\cdot\rangle\rangle$. Assume that for any compact set $K \subset M$ there is a constant $C_K$ with $\|g_x\| \leq C_K$ for all $x \in K \cap S$, $\|\cdot\|$ denoting the operator norm with respect to the Riemannian metric on $M$. Let $f\colon S \to \mathbf{R}$ be a differentiable function. Then for any compact set $K$ there is a constant $\hat{C}_K > 0$ such that*

$$\|\operatorname{grad} f(x)\| \leq \hat{C}_K \left\|\operatorname{grad}_g f(x)\right\|_g$$

*for all $x \in S \cap K$.*

**Proof:** Note that we can write $\langle\langle v,w\rangle\rangle$ as $\langle H(x)v,w\rangle$, where $H(x)\colon T_xS \to T_xS$ is a positive definite, self adjoint linear map with respect to the Riemannian metric on $M$. Using local charts, we see that the induced vector bundle map $H\colon TS \to TS$ is continuous. Note that $\|g_x\| \leq C_K$ is equivalent to $\|H(x)\| \leq C_K$, $\|\cdot\|$ denoting the respective operator norms. As $H(x)$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, we have

$$\left\|\operatorname{grad}_g f(x)\right\|_g^2 = \left\langle \operatorname{grad} f(x), H(x)^{-1} \operatorname{grad} f(x)\right\rangle \geq C_K^{-1} \|\operatorname{grad} f(x)\|^2 .$$

This proves the lemma. □

**Łojasiewicz gradient inequality** Kurdyka [101] proved the above theorem for functions $f\colon U \to \mathbf{R}$ definable in an o-minimal structure on $(\mathbf{R},+,\cdot)$ with $U$ an bounded open subset of Euclidean $\mathbf{R}^n$. However, his proof also works for definable, bounded submanifolds of $\mathbf{R}^n$ with the Riemannian metric induced from the Euclidean one. As $f$ is a bounded $\mathcal{C}$-function the submanifold $S$ must be a $\mathcal{C}$-set. By using local charts, see Propositions 1.1.4, 1.1.5, we can assume that $f$ and $S$ are definable in $\mathcal{R}(\mathcal{C})$. Further, the theorem does not depend on the particular Riemannian metric on $M$, changing the metric just requires a change of the constant $C$ in (1.4). By Lemma 1.1.23 we change the metric to $g$ with just introducing another constant in 1.4. Thus it follows directly from the version of Kurdyka's theorem for submanifolds of Euclidean space in the o-minimal setting. $\square$

**Remark 1.1.24**

1. The bound on $\|g_x\|$ is necessary, without it the Łojasiewicz gradient inequality does not hold.

2. If the o-minimal structure $\mathcal{R}(\mathcal{C})$ is polynomially bounded, [53], we can choose $\psi(s) = s^{1-\mu}$ for a suitable $\mu \in (0,1)$, [101]. This gives the classical Łojasiewicz gradient inequality
$$\|\operatorname{grad} f(x)\| \geq C\,|f(x)|^{\mu}\,.$$

3. Unlike in Euclidean space, we do not give directly the Łojasiewicz gradient inequality on any relatively compact, open subset $U$ of $M$. This is due to the fact, that local charts will not necessarily cover a neighborhood of $\overline{U}$. This situation will be covered by the next corollary.

As a straightforward corollary we see that the Łojasiewicz gradient inequality can be extended to any compact subset of $M$ instead of single points. Our proof is similar to the argument used in [31] to show that for a domain stratification of a non-smooth, definable $f$, the functions $\psi$ can chosen uniformly for all strata near $x^*$.

**Corollary 1.1.25** *Let $K$ be a compact subset of an analytic Riemannian manifold $M$. Assume that $f\colon M \to \mathbf{R}$ is a continuously differentiable $\mathcal{C}$-function. There exist a neighborhood $U$ of $K$ in $M$, constants $C > 0$, $\rho > 0$ and a strictly increasing $C^1$ function $\psi\colon (0,\rho) \to \mathbf{R}_+$ such that for $x \in U \cap f^{-1}((0,\rho))$*
$$\|\operatorname{grad} \psi \circ f(x)\| \geq C \quad \textit{holds.} \tag{1.5}$$
*Furthermore the function $\psi$ is definable in the o-minimal structure $\mathcal{R}(\mathcal{C})$ on $(\mathbf{R}, +, \cdot)$.*

**Proof:** As $K$ is compact, we can cover it by a finite number of open set $U_i$ on which the Łojasiewicz gradient inequality holds, see Theorem 1.1.22. This gives a finite number of definable functions $\psi_i\colon (0, \rho_i) \to \mathbf{R}_+$. Let $\tilde{\rho} = \min \rho_i$. By Lemma 1.1.6 $\max \psi_i'$ coincides with one $\psi_j'$ on an interval $(0, \rho) \subset (0, \tilde{\rho})$. Thus $\psi_j\colon (0, \rho) \to \mathbf{R}_+$ is the required definable function for the neighborhood $U = \bigcup U_i$ of $K$. □

## 1.2 AC vector fields

### 1.2.1 Convergence properties of integral curves

In this section we define AC-vector fields and discuss the convergence properties of their integral curves. Here and in the sequel, we will denote the Riemannian metric on a manifold $M$ by $\langle \cdot, \cdot \rangle$.

**Definition 1.2.1** Let $M$ be a Riemannian manifold, $X$ a continuous vector field on $M$ and $f \colon M \to \mathbf{R}$ a continuous function. We call $X$ an *angle condition (AC) vector field* with associated *Lyapunov function* $f$ if the following conditions hold:

- $f$ is non-constant on open sets,
- $f$ is piecewise $C^1$ with domain stratification $S_j$, $j \in \Lambda$,
- for any compact set $K$ there is a constant $\varepsilon > 0$ such that for all $j \in \Lambda$ and all $x \in S_j \cap K$ with $X(x) \in T_x S_j$ the estimate

$$- \left\langle \operatorname{grad}_j f(x), X(x) \right\rangle \geq \varepsilon \left\| \operatorname{grad}_j f(x) \right\| \left\| X(x) \right\| \tag{AC}$$

  holds[13].

**Lemma 1.2.2** *Let $X$ be an (AC) vector field on a Riemannian manifold $M$ with Lyapunov function $f$. Assume that $f$ has the domain stratification $S_j$, $j \in \Lambda$. Then $X$ satisfies the properties of Definition 1.2.1 with respect to any refinement $\tilde{S}_l$, $l \in \tilde{\Lambda}$, i.e. any stratification such that the $S_j$ are unions of strata $\tilde{S}_l$.*

**Proof:** Let $\tilde{S}_l$, $l \in \tilde{\Lambda}$ be a refinement[14] of $S_j$, $j \in \Lambda$. Furthermore let $x \in \tilde{S}_l$ with $X(x) \in T_x \tilde{S}_l$, $X(x) \neq 0$ and $\operatorname{grad}_l f(x) \neq 0$. Then $x$ is contained in a stratum $S_j$ with $X(x) \in T_x S_j$ and $\operatorname{grad}_j f(x) \neq 0$. Note that $\pi_{T_x \tilde{S}_l}(\operatorname{grad}_j f(x)) = \operatorname{grad}_l f(x)$ and $\pi_{T_x \tilde{S}_l}(X(x)) = X(x)$, where $\pi_{T_x \tilde{S}_l}$ denotes the projection on $T_x \tilde{S}_l$ with respect to the Riemannian metric. In particular $\|\operatorname{grad}_l f(x)\| \leq \left\| \operatorname{grad}_j f(x) \right\|$. For a relatively compact neighborhood $U$ of $x$

[13] As we defined in the previous section, $\operatorname{grad}_j f$ denotes the gradient of $f$ on $S_j$ with respect to the induced Riemannian metric.

[14] We assume that $\tilde{\Lambda} \cap \Lambda = \emptyset$.

let $\varepsilon > 0$ be the constant such that the inequality (AC) is satisfied for the stratification $S_j$, $j \in \Lambda$. Since $X(x) \in T_x \tilde{S}_l$, we get

$$\begin{aligned} - \langle \operatorname{grad}_l f(x), X(x) \rangle &= - \left\langle \operatorname{grad}_j f(x), X(x) \right\rangle \\ &\geq \varepsilon \left\| \operatorname{grad}_j f(x) \right\| \|X(x)\| \geq \varepsilon \left\| \operatorname{grad}_l f(x) \right\| \|X(x)\| \end{aligned}$$

for all $x \in U$. This proves the lemma. □

**Lemma 1.2.3** *Let $X$ be a continuous vector field on an analytic Riemannian manifold $M$ and $f\colon M \to \mathbf{R}$ be a continuous $\mathcal{C}$-function, non-constant on open sets. Assume that the domain stratification of $f$ is an $(a_f)$-stratification by $\mathcal{C}$-sets and $X$ satisfies the condition of Definition 1.2.1 for all $x \in M$ which are contained in strata of dimension* $\dim M$*, i.e. for any compact set $K \subset M$ we have a constant $\varepsilon_K > 0$ such that* (AC) *holds in points of $K$ contained in strata of dimension* $\dim M$*. Then $X$ is an (AC) vector field with Lyapunov function $f$. The new $\tilde{\varepsilon}_K$ for compact sets $K$ meeting the highest dimensional strata coincides with the a priori given $\varepsilon_K$.*

**Proof:** Let $x$ be a point in a lower dimensional stratum $S_j$ with $\operatorname{grad}_j f(x) \neq 0$. We denote $\operatorname{grad}_j f(x)$ by $w$. There is a stratum $S_l$ with $\dim S_l = \dim M$ and $S_j \subset \overline{S_l}$. Since $f$ is non-constant on open sets and $\operatorname{rk} df|_{S_l}$ is constant, we have for all $y \in S_l$ that $\operatorname{grad}_l f(y) \neq 0$. Let $(x_k) \subset S_l$ be a sequence with $x_k \to x$ and $v_k := \|\operatorname{grad}_l f(x_k)\|^{-1} \operatorname{grad}_l f(x_k)$ converging to some $v \in T_x M$. By Lemma 1.1.11 we know that that $\pi_{T_x S_j}(v) = \lambda w$, where $\lambda \in \mathbf{R}$ and $\pi_{T_x S_j}$ denotes the projection on $T_x S_j$ with respect to the Riemannian metric.

By using local charts we prove now that it is possible to choose $S_l$ and a sequence $(x_k)$ with $\lambda \geq 0$. Let $\phi\colon U \to \mathbf{R}^n$ be an analytic chart around $x$. By Proposition 1.1.4 we can assume that the images of $S_i \cap U$, $i \in \Lambda$ are definable in an o-minimal structure. Denote by $\langle\langle \cdot, \cdot \rangle\rangle = \phi^{-1*} \langle \cdot, \cdot \rangle$ the pullback[15] of the Riemannian metric on $M$ and define the function $W\colon \phi(U) \to \mathbf{R}$ by $W(y) := f \circ \phi^{-1}(y)$. Note that

$$T_{x_k}\phi(v_k) = \frac{\widetilde{\operatorname{grad}}_l W(\phi(x_k))}{\left\| \widetilde{\operatorname{grad}}_l W(\phi(x_k)) \right\|}$$

[15]See [1] for a definition of pull-backs.

converges to $T_x\phi v$ where $\widetilde{\mathrm{grad}}_l$ denotes that gradient on $\phi(S_l)$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. Furthermore

$$T_x\phi(w) = \frac{\widetilde{\mathrm{grad}}_j W(\phi(x))}{\left\|\widetilde{\mathrm{grad}}_j W(\phi(x))\right\|}.$$

Thus if $\lambda < 0$ for all sequences then we have $dW(y)(\widetilde{\mathrm{grad}}_j W(\phi(x))) < C < 0$ for all $y \in \tilde{U} \cap \phi(S_l \cap U)$ with $\tilde{U}$ a suitable neighborhood of $\phi(x)$ and $C$ a constant. As we were free to choose above any maximal dimensional stratum whose closure contains $x$, we have indeed that $dW(y)(\widetilde{\mathrm{grad}}_j W(\phi(x))) < C < 0$ for all $y \in \tilde{U} \cap \phi(S_i \cap U)$, $i \in \Lambda$, $\dim S_i = \dim M$. Otherwise we could just switch the stratum and get a sequence with $\lambda \geq 0$.

Choose a $C^1$ curve $\theta$ in $\phi(S_j \cap U)$ with $\theta(0) = \phi(x)$, $\theta'(0) = \widetilde{\mathrm{grad}}_j W(\phi(x))$. The function $W \circ \theta$ is strictly increasing in an open interval $(-a, a)$ around 0. Let $t_k \in \mathbf{R}_+$ be a sequence with $t_k \to 0$. Connect $\theta(-t_k)$, $\theta(t_k)$ with a straight line $\eta_k$. Note, that the directions

$$d_k = \frac{\theta(t_k) - \theta(-t_k)}{|\theta(t_k) - \theta(-t_k)|}$$

of these lines converge to $\left\|\widetilde{\mathrm{grad}}_j W(\phi(x))\right\|^{-1} \widetilde{\mathrm{grad}}_j W(\phi(x))$. Thus for large $k$ we have after eventually shrinking $\tilde{U}$ that $dW(y)(d_k) < C < 0$ for all $y \in \tilde{U} \cap \phi(S_i \cap U)$, $i \in \Lambda$, $\dim S_i = \dim M$.

Fix a $k \in \mathbf{N}$. We consider the definable family $(l_y)$ of affine linear subspaces of $\mathbf{R}^n$ with

$$l_y = \{y + r d_k \mid r \in \mathbf{R}\}.$$

There is a $y^* \in \mathbf{R}^n$ with $\eta_k \subset l_{y^*}$. Denote by $\Sigma$ the union of the lower dimensional strata in $\tilde{U}$. Then there is a sequence $y_m \to y^*$ with $l_{y_m} \neq l_{y^*}$ such that the intersection of each $l_{y_m}$ with $\Sigma$ is finite. Otherwise, we see by using a cell decomposition [53] that the definable set

$$\mathbf{R}^n \times \Sigma \cap \{(y, z) \mid z \in l_y\}$$

would contain an open subset, which contradicts the fact that $\dim \Sigma < n$. Hence for each $\eta_k$ there is a sequence of lines $(\eta_k^m)$, $\eta_k^m \to \eta_k$ such that each $\eta_k^m$ meets $\Sigma$ in a finite number of points.

On the lines $\eta_k^m$ orientated in direction $d_k$ the function $W$ is strictly decreasing. By continuity of $W$ it must be decreasing on $\eta_k$, although not strictly. But on the other hand we have for sufficiently large $k$ that $W(\theta(-t_k)) < W(\theta(t_k))$. This gives a contradiction and there must be indeed a sequence with $\lambda \geq 0$.

If there is a sequence with $\lambda = 0$ then $v$ must be orthogonal to $T_x S_j$ and by the continuity of $X$ the (AC) inequality on $S_l$ implies that $X(x) \notin T_x S_j$. By Definition 1.2.1 the inequality (AC) does not have to be satisfied for such $x \in M$. Thus we can assume that we have a sequence $(x_k)$ with $\lambda > 0$. Due to $\|v\| = 1$ and $\|w\| = 1$ we get $\lambda \leq 1$.

Let $\varepsilon > 0$ the constant such that inequality (AC) holds on the intersection of a relatively compact neighborhood of $x$ with $S_l$. Then

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} -\left\langle v_k, \frac{X(x_k)}{\|X(x_k)\|}\right\rangle = -\left\langle v, \frac{X(x)}{\|X(x)\|}\right\rangle \\
&= -\left\langle \pi_{T_x S_j}(v), \frac{X(x)}{\|X(x)\|}\right\rangle = -\lambda\left\langle w, \frac{X(x)}{\|X(x)\|}\right\rangle \\
&\leq -\left\langle w, \frac{X(x)}{\|X(x)\|}\right\rangle .
\end{aligned}$$

Thus given a compact set $K \subset M$, we have for all $x \in K$

$$\varepsilon_K \leq -\left\langle \frac{\operatorname{grad}_j f(x)}{\|\operatorname{grad}_j f(x)\|}, \frac{X(x)}{\|X(x)\|}\right\rangle ,$$

where $\varepsilon_K > 0$ is a constant such that inequality (AC) holds on the intersection of a relatively compact neighborhood of $K$ with the strata of dimension $\dim M$. ☐

**Proposition 1.2.4** *Let $X$ be a Lipschitz continuous (AC) vector field on an analytic Riemannian manifold $M$ with an Lyapunov function $f$. Assume that $f$ is a $\mathcal{C}$-function and the stratification of $f$ consists of $\mathcal{C}$-sets. Then $X$ is gradient-like, i.e. for any non-constant integral curve $\gamma$ of $X$ the function $f \circ \gamma$ is strictly decreasing.*

**Proof:** Fix a domain stratification of $f$ and a non-constant integral curve $\gamma$. We can refine this stratification to an $(a_f)$-stratification. By Lemma 1.2.2 this refined stratification still satisfies the conditions of Definition 1.2.1. First we show that $f \circ \gamma$ is non-increasing. Take a compact interval $[a, b]$ in the domain of $\gamma$. Choose a relatively compact neighborhood $N$ of $\gamma([a,b])$ such that there is no equilibrium of $X$ on $\overline{N}$. We can approximate $X$ on $\overline{N}$ by a sequence of analytic vector fields $X_k$, i.e.

$$\lim_{k\to\infty} \sup_{x\in\overline{N}} \|X(x) - X_k(x)\| = 0.$$

As $X$ does not vanish on $\overline{N}$ the angle between $X$ and $X_k$ converges uniformly to 0 for $k \to \infty$. Hence, for sufficiently large $k$ the $X_k$ satisfy the inequality (2) in points of the highest dimensional strata with $\varepsilon_k$ depending on $\overline{N}$ and on $k$. Lemma 1.2.3 implies that the $X_k$ are indeed (AC) vector fields with Lyapunov function $f$. The integral curves of the $X_k$ are analytic and therefore belong to the analytic-geometric category. Consider the integral curves $\gamma_k$ starting at time $a$ in $\gamma(a)$. By the Lipschitz continuity of $X$, the curves $\gamma_k$ converge uniformly to $\gamma$ on the interval $[a,b]$. As they belong to the analytic-geometric category they can only leave or enter lower dimensional strata in a finite number of points in time in the compact interval $[a,b]$. Therefore, for each $k$, the interval is divided into a finite number of open subintervals on which $\gamma_k$ stays in a fixed stratum. If for a stratum $S_i$, a subinterval $(c,d) \subset (a,b)$, all $t \in (c,d)$ we have $\gamma_k(t) \in S_i$ then $f \circ \gamma_k$ is continuously differentiable on $(c,d)$ and must be decreasing by the (AC) condition. Thus by continuity of $f \circ \gamma_k$, $f \circ \gamma_k$ must be non-increasing for the whole interval $[a,b]$. As the $\gamma_k$ converge uniformly to $\gamma$, we get that $f \circ \gamma$ is non-increasing, too.

We still have to show that $f\circ\gamma$ is not constant on open intervals. Assume that $f \circ \gamma$ is constant on an interval $(a,b)$. Let $S_j$ be the highest dimensional stratum met by $\gamma|_{(a,b)}$. Note that the dimension of $S_j$ can be lower than dimension of $M$. Choose $y \in \gamma((a,b))$ and $t^* \in (a,b)$ with $y \in S_j$, and $y = \gamma(t^*)$. We have that $X(y) \neq 0$. As $S_j$ was the highest dimensional stratum met by $\gamma|_{(a,b)}$ there must be a neighborhood of $t^*$ in $(a,b)$ such that its image under $\gamma$ is contained in $S_j$. Thus $X(y) \in T_yS_j$. If $\operatorname{grad}_j f(y) \neq 0$ we get directly a contradiction from the (AC) conditions. Let us consider the other case. Since $\operatorname{rk} df|_{S_i}$ is constant on each stratum $S_i$ and $f$ is non-constant on open sets there must be a stratum $S_l$ with $\dim S_l = \dim M$, $S_j \subset \overline{S_l}$ and $\operatorname{grad}_l f(x) \neq 0$ for all $x \in S_l$. Let $(x_k)$ be a sequence in $S_l$ with $x_k \to y$ and $X(x_k) \neq 0$. W.l.o.g. let $\ker df(x_k)$ converge to a linear subspace $L \subset T_yM$. The Thom condition implies that $\ker df|_{S_j}(y) \subset L$. Because $f \circ \gamma|_{(a,b)}$ is constant, differentiable in $t^*$ and $\dot\gamma(t^*) = X(y) \in T_yS_j$, we have

$X(y) \in \ker df|_{S_j}(y) \subset T_y S_j$. This implies that[16]

$$\operatorname{dist}(X(x_k), \ker df(x_k)) \to 0.$$

which gives

$$\left\langle \frac{X(x_k)}{\|X(x_k)\|}, \frac{\operatorname{grad}_l f(x_k)}{\|\operatorname{grad}_l f(x_k)\|} \right\rangle \to 0.$$

This contradicts the definition of (AC) vector fields. Hence, $f \circ \gamma|_{(a,b)}$ cannot be constant. □

**Theorem 1.2.5** *Let $X$ be a Lipschitz continuous $AC$ vector field on an analytic Riemannian manifold $M$ with a Lyapunov function $f$. Assume that $f$ is a $\mathcal{C}$-function and its strata are $\mathcal{C}$-sets. Then the $\omega$-limit set of any integral curve of $X$ contains at most one point.*

Our proof is centered around the approach of Łojasiewicz [101, 113] to derive a bound of the length of the curve from the Łojasiewicz gradient inequality.

**Proof:** Let $\gamma$ be a non-constant integral curve of $X$ and $x^*$ be an element of the $\omega$-limit set of $\gamma$. Furthermore let $f\colon M \to \mathbf{R}$ be the associated $\mathcal{C}$-function. W.l.o.g. we can assume that $f(x^*) = 0$. Since $f \circ \gamma$ is by Proposition 1.2.4 strictly decreasing, this implies that $f \circ \gamma(t) > 0$ for all $t \in \mathbf{R}$. Refine the domain stratification of $f$ such that it is $(a_f)$ and the sign of $f$ is constant[17] on the strata. We denote this stratification by $S_j, j \in \Lambda$ and by $\tilde{\Lambda}$ the index set of the strata on which $f$ is non-constant. Applying Theorem 1.1.22 to the strata $S_j$, with $x^* \in \overline{S}_j$ and $f$ non-constant on $S_j$, we get a relatively compact neighborhood $U$ of $x^*$, constants $C_j > 0$, $\rho_j > 0$ and strictly increasing, $C^1$ functions $\psi_j\colon (0, \rho_j) \to \mathbf{R}_+$ such that for all $j \in \tilde{\Lambda}$, $x \in S_j \cap U \cap f^{-1}((0, \rho_j))$

$$\left\|\operatorname{grad}(\psi_j \circ f)|_{S_j}(x)\right\| > C_j.$$

W.l.o.g. we assume that $\psi_j(r) \to 0$ for $r \to 0$. Note that only a finite number of strata $S_j$ meet $U$, w.l.o.g. for $j \in \{1, \ldots, k\}$. Let $[t_1, t_2]$ a closed, finite interval with $\gamma([t_1, t_2]) \subset U$. Assume that we have a non-empty interval $(t_3, t_4) \subset [t_1, t_2]$ with $\gamma((t_3, t_4)) \subset S_j$. Since $\operatorname{rk} df|_{S_j}$ is constant and Proposition 1.2.4 ensures that $f \circ \gamma$ is strictly decreasing, we have $\operatorname{grad}_j f(x) \neq 0$

[16]We define for $v \in T_x M$, $L \subset T_x M$ the distance $\operatorname{dist}(v, L) = \sup_{w \in L} \|v - w\|$.

[17]i.e. either $f > 0$, $f < 0$ or $f = 0$ on each stratum

for all $x \in S_j$. Let $\varepsilon$ the (AC) constant for $\overline{U}$. For $t \in (t_3, t_4)$ we have

$$\begin{aligned} -\frac{d}{dt}(\psi_j \circ f \circ \gamma)(t) &= -\left\langle \operatorname{grad}_j(\psi_j \circ f)(\gamma(t)), \dot{\gamma}(t) \right\rangle \\ &\geq \varepsilon \psi_j'(f(\gamma(t))) \left\| \operatorname{grad}_j f(\gamma(t)) \right\| \|\dot{\gamma}(t)\| \geq \varepsilon C_j \|\dot{\gamma}(t)\| \end{aligned}$$

which gives

$$\int_{t_3}^{t_4} \|\dot{\gamma}(t)\| \, dt \leq \frac{\psi_j(f(\gamma(t_3))) - \psi_j(f(\gamma(t_4)))}{C_j \varepsilon}. \tag{1.6}$$

As in the proof of Proposition 1.2.4 we approximate $X$ on the whole manifold $M$ by analytic vector fields $X_k$. Again, in a relatively compact neighborhood $N \subset U$ of $\gamma([t_1, t_2])$ we have that $X(x) \neq 0$ for all $x \in \overline{N}$ and therefore the angle between the $X_k$ and $X$ tends uniformly to zero for large $k$. We can now apply Lemma 1.2.3 to see that the $X_k$ are indeed (AC) vector fields with Lyapunov function $f$. Let $\varepsilon_k$ be the (AC) constants for $\overline{N}$ and $X_k$. Note that $\varepsilon_k \to \varepsilon$. This is clear on the high dimensional strata as the angle between $X$ and $X_k$ tends to 0 uniformly on $N$. Lemma 1.2.3 extends this to the low dimensional strata. Let $\gamma_k$ be the integral curve of $X_k$ starting in $\gamma(t_1)$ at time $t_1$. W.l.o.g. each $\gamma_k$ is defined on $[t_1, t_2]$ and $\gamma([t_1, t_2]) \subset N$. As in the proof of Proposition 1.2.4 every $\gamma_k$ intersects, leaves or enters lower dimensional strata only finitely often on $[t_1, t_2]$ due to the analyticity of the $\gamma_k$. By Proposition 1.2.4 the $\gamma_k$ can meet the lower dimensional strata, on which $f$ is constant, only in a finite number of points for $t \in [t_1, t_2]$. Thus, these strata can be ignored when estimating the length of $\gamma_k$. Using the bound (1.6) above with $\gamma_k$ and exploiting the monotonicity of the $\psi_j$ and $f \circ \gamma_k$ we can deduce that

$$\int_{t_1}^{t_2} \|\dot{\gamma}_k(t)\| \, dt \leq \sum_{\substack{j \in \tilde{\Lambda} \\ S_j \cap U \neq \emptyset}} \frac{\psi_j(f(\gamma_k(t_1)))}{C_j \varepsilon_k}.$$

But then

$$\int_{t_1}^{t_2} \|\dot{\gamma}(t)\| \, dt \leq \sum_{\substack{j \in \tilde{\Lambda} \\ S_j \cap U \neq \emptyset}} \frac{\psi_j(f(\gamma(t_1)))}{C_j \varepsilon}.$$

Since $f(x^*) = 0$, the sum

$$\sum_{\substack{j \in \tilde{\Lambda} \\ S_j \cap U \neq \emptyset}} \frac{\psi_j(f(\gamma(t_k)))}{C_j \varepsilon}$$

converge to 0 for any sequence $t_k \to \infty$ with $\gamma(t_k) \to x^*$. Hence $\gamma(t)$ cannot leave $U$ anymore for sufficient large $t_k$ and its length is bounded. Thus $\gamma(t)$ converges to $x^*$. $\square$

### 1.2.2 Topological properties of (AC) systems

In this section we consider some topological properties of (AC) systems. We have shown in the previous section that the integral curves of an (AC) vector field converge if the Lyapunov function is from the class of $\mathcal{C}$-functions. Of course, the same holds for the integral curves of the gradient vector field of the Lyapunov function itself[18]. Hence, we can view the convergence of the integral curves of an (AC) vector field as inherited from the gradient vector field. It is natural to ask whether further topological properties of the flow are inherited from the Lyapunov function.

First we consider the question, if the linearization of the gradient field of the Lyapunov function at a non-degenerate critical point[19] and the (AC) vector field are topologically equivalent near this point. Recall, that two vector fields are called topologically equivalent, if there is a homeomorphism which maps trajectories of the flows of the first vector field onto the trajectories of the second and visa versa, and preserves the orientation of the flows on the trajectories [67, Def. 1.7.2. and 1.7.3]. From the Hartmann-Grobmann theorem [67, Thm. 1.3.1] follows that if $x \in M$ is a non-degenerate critical point of a smooth function $f\colon M \to \mathbf{R}$ then the gradient vector field is (locally) topologically equivalent to its linearization at $x$, see also [77, Proof 3.10]. One might wonder if this passes over to the (AC) vector field, too, i.e. if the (AC) vector field is topologically to the linearization of the gradient vector field in a non-degenerate critical point of the Lyapunov function. However, this is not the case. Take for example the function $f(x, y) = x^2 - y^2$ in $\mathbf{R}^2$. The point 0 is obviously a non-degenerate critical point of $f$. We choose three non-negative $C^1$-functions $\zeta_1, \zeta_2, \zeta_3\colon \mathbf{R}^2 \to \mathbf{R}_+$ on $\mathbf{R}^2$ with

$$\begin{aligned}
\zeta_1(x,y) &= \begin{cases} y^2 & \text{for } y > 0 \\ 0 & \text{for } y \le 0 \end{cases}, \\
\zeta_2(x,y) &= \zeta_1(x,-y), \\
\zeta_3(x,y) &= \begin{cases} \left(\frac{x^4}{9} - y^2\right)^2 & \text{for } |y| < \frac{x^2}{3} \\ 0 & \text{for } |y| \ge \frac{x^2}{3} \end{cases}
\end{aligned}$$

---

[18] This can be either derived from the convergence for functions definable in an o-minimal structure on $(\mathbf{R}, +, \cdot)$ [101] or as a special case of our convergence theorems.

[19] i.e. the Hessian is non-degenerate, see [77].

Further, we define the three vector fields

$$X_1(x,y) = \begin{pmatrix} -x \\ y - x^2 \end{pmatrix}, \; X_2(x,y) = \begin{pmatrix} -x \\ y + x^2 \end{pmatrix}, \; X_3(x,y) = \begin{pmatrix} -x \\ y \end{pmatrix}$$

on $\mathbf{R}^2$. Note that these three vector fields are all (AC) with the Lyapunov function $f$ given above on a suitably small neighborhood $U$ of 0. As $\zeta_1$, $\zeta_2$, $\zeta_3$ are non-negative, the sum

$$Y = \zeta_1 X_1 + \zeta_2 X_2 + \zeta_3 X_3$$

is an (AC) vector field on $U$ with Lyapunov function $f$. The figure 1.1 shows the normalized vector field, i.e. $\|Y\|^{-1} Y$, and the boundaries of the region of attraction of the point 0. The vector fields $X_1$ and $X_2$ have stable manifolds $S_1 = \{(x, x^2/3) \mid x \in \mathbf{R}\}$ and $S_2 = \{(x, -x^2/3) \mid x \in \mathbf{R}\}$ respectively [67]. Our construction implies that for $Y$ the region of attraction of the origin contains $\{(x,y) \mid |y| \leq x^2/3\}$. Further, the origin is unstable as $Y$ restricted to the $y$-axis is just an unstable linear vector field multiplied by a non-negative scalar function. Since the region of attraction of 0 contains an open set in any neighborhood of 0, but on the other hand the point 0 is not a stable equilibrium of $Y$, the vector field $Y$ cannot be topologically equivalent to any linear vector field. In particular it is not topologically equivalent near 0 to the linearization of $\operatorname{grad} f$ in 0.

However, it is possible to give a result on the topology of the set of trajectories attracted by a compact subset of a critical level set. More precisely, we can extend a theorem on Nowel and Szafraniec [126, 127] for analytic gradient vector fields to (AC) vector fields in a fairly straightforward manner. Consider an analytic function $f \colon M \to \mathbf{R}$ with gradient vector field $\operatorname{grad} f$. Nowel and Szafraniec have shown that the Čech-Alexander cohomology groups of the family of trajectories attracted by a set $K \subset f^{-1}(\{y\})$ and of the set $U \cap \{x \in M \mid f(x) > y\}$, for a suitable neighborhood $U$ of $K$, are isomorphic. For (AC) systems the same result holds under the condition that the critical points of the Lyapunov function and the vector field coincide. Thus, we have to following extension of [127, Thm. 2.25].

**Theorem 1.2.6** *Let $X$ be an (AC) vector field on an analytic Riemannian manifold $M$ with Lyapunov function $f$, which is proper, $C^1$ and a $\mathcal{C}$-function. Assume that the equilibria of $X$ and the critical points of $f$ coincide. Furthermore, let $K$ be a compact $\mathcal{C}$-set which is contained in the*

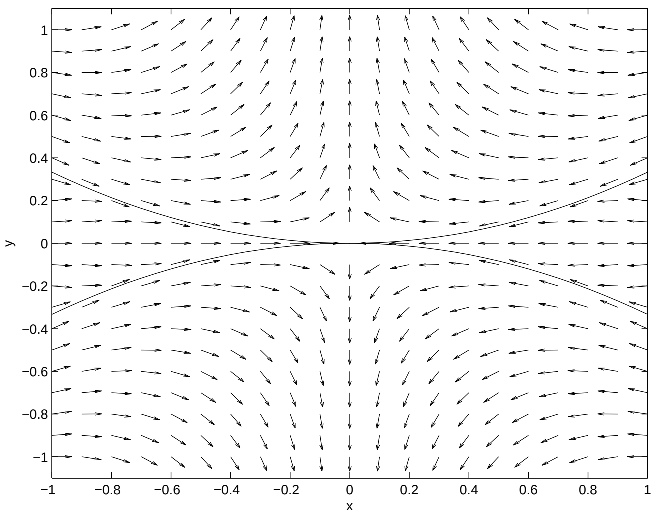


Figure 1.1: The normalized vector field $\frac{Y}{\|Y\|}$. The two solid lines denote the boundary of the region of attraction of 0.

*level set* $\{x \in M \mid f(x) = 0\}$*. We denote by* $U_\delta$ *the set* $U_\delta = \{x \in M \mid \sup_{y \in K} \operatorname{dist}(x, y) < \delta\}$*. Assume that we have an* $U_\delta$ *such that for any* $\hat{\delta} < \delta$ *the set* $U_{\hat{\delta}} \cap \{x \in M \mid f(x) > 0\}$ *is homotopy equivalent to* $U_\delta \cap \{x \in M \mid f(x) > 0\}$*. Then for sufficiently small* $y \in \mathbf{R}$*,* $y > 0$*, the Čech-Alexander cohomology groups* $\check{H}^*(\{x \in M \mid f(x) = y, \omega(x) \in K\})$ *and* $\check{H}^*(U_\delta \cap \{x \in M \mid f(x) > 0\})$ *are isomorphic.*

Note, that it is also possible to use general neighborhoods $U$ of $K$ in conjunction with the condition from [127].

**Proof:** Since the arguments of Nowel and Szafraniec with some modifications can be applied to this case, we will only sketch these changes and not provide the complete argument.

(1) First we construct a "disturbance" $h$ of $f$ as in [127]. By [53, Theorem 1.20] there is a non-negative[20] $\mathcal{C}$-function $g\colon M \to \mathbf{R}$, twice continuously differentiable, with $g(x) = 0 \Leftrightarrow x \in K$. Nowel and Szafraniec have shown, that there are constants $\rho_1 > 0$, $\rho_2 > 0$ and an odd, strictly increasing $C^1$-bijection $\phi\colon \mathbf{R} \to \mathbf{R}$ which is a $\mathcal{C}$-map and for all $x \in f^{-1}((0, \rho_1)) \cap g^{-1}((0, \rho_2))$ with $df(x) = 0$ or $dg(x) = 0$ implies $f(x) \geq 2\phi(g(x))$ [127, (2.1)]. It follows

[20] That the function is non-negative can be trivially ensured by squaring.

directly from Corollary 1.1.25 that there is a $\delta > 0$, and constants $C > 0$, $\rho_3 > 0$ and a strictly increasing $C^1$-function, definable in the o-minimal structure $\mathcal{R}(\mathcal{C})$, $\psi\colon \mathbf{R} \to \mathbf{R}_+$ such that for $x \in U_\delta \cap f^{-1}((0,\rho_3))$

$$\|\operatorname{grad} \psi \circ f(x)\| \geq C$$

holds. Further, we can assume that $\psi'(t) \to \infty$ for $t \to 0$, $\rho_3 < \min\{1, \rho_1, \rho_2\}$ and $\psi'(t)$ is $> 1$ and strictly decreasing on $(0, \rho_3)$. Note, that $\psi$ is a bijection $(0, \rho_3) \to (0, \psi(\rho_3))$. Hence, we can choose a definable, strictly increasing function $\sigma\colon (0,1) \to (0,1)$ such that $\sigma(t) \to 0$ for $t \to 0$ and for all $t \in (0,1)$

$$\psi(\sigma(t)) < t^2.$$

By the monotonicity theorem [53, Thm. 4.1], we can w.l.o.g. assume that $\sigma$ is $C^1$ on $(0,1) \cap \phi((0,\rho_3))$. Note that $t < \psi(t)$ for all $t \in (0,\rho_3)$. Hence, $\sigma(t) < t$ for $t \in (0,\rho_4) \subset \phi((0,\rho_3))$ with $\rho_4 < 1$ sufficiently small. Using the monotonicity theorem and the fact that derivatives of definable functions are definable, we can assume that $\psi'(t) > 0$ and $\sigma'(t) > 0$ for all $t \in (0,\rho_4)$. A combination of the monotonicity theorem and an integration argument shows that

$$\psi'(\sigma(t))\sigma'(t) < 1$$

for all $t$ sufficiently close to 0, i.e. for all $t \in (0,\rho_4)$ if we choose $\rho_4$ sufficiently small.

Let $U = U_\delta \cap (\phi \circ g)^{-1}((0,\rho_4)))$ We define[21] $h\colon U \to \mathbf{R}$ as

$$h(x) = f(x) - \sigma(\phi(g(x))).$$

(2) By the same argument as in [127, Lem. 2.2] we can show that $x \in U$ and $g(x)$ sufficiently small, we have $dh(x) \neq 0$

(3) Assume that we are given a $\varepsilon > 0$. Let $x \in U$ with $h(x) = 0$. Then $f(x) = \sigma(\phi(g(x)))$ and we get

$$\begin{aligned} \|\operatorname{grad}(\sigma \circ \phi \circ g)(x)\| &= \sigma'(\phi(g(x)))\, \|\operatorname{grad}(\phi \circ g)(x)\| \\ &\leq \frac{1}{\psi'(\underbrace{\sigma(\phi(g(x)))}_{=f(x)})}\, \|\operatorname{grad}(\phi \circ g)(x)\| \\ &< C^{-1}\, \|\operatorname{grad} f(x)\|\, \|\operatorname{grad}(\phi \circ g)(x)\|\,. \end{aligned}$$

[21] In fact, $h$ can be extended to a $C^1$ function on the whole manifold.

Note that $\|\text{grad}(\phi \circ g)(x)\|$ converges to 0 for $g(x) \to 0$. Thus if $g(x)$ is sufficiently small then we have that for all $x \in U$, $h(x) = 0$ implies that

$$\frac{\varepsilon}{2} \|\text{grad}\, f(x)\| > \|\text{grad}(\sigma \circ \phi \circ g)(x)\| . \tag{1.7}$$

(4) For all $x \in U$ with $h(x) = 0$, $f(x) \neq 0$ and $g(x)$ sufficiently small we have $\langle \text{grad}\, h(x), X(x)\rangle > 0$. Note that $U$ is relatively compact. Let $\varepsilon > 0$ the (AC) constant for $\overline{U}$. Then for all $x \in U$ with $h(x) = 0$ and $g(x)$ sufficiently small the estimate (1.7) holds. This gives

$$\begin{aligned} -\langle \text{grad}\, h(x), X(x)\rangle &= -\langle \text{grad}\, f(x) - \text{grad}(\sigma \circ \psi \circ g)(x), X(x)\rangle \\ &\geq \varepsilon \|X(x)\| \|\text{grad}\, f(x)\| - \|X(x)\| \|\text{grad}(\sigma \circ \psi \circ g)(x)\| \\ &\geq \frac{\varepsilon}{2} \|X(x)\| \|\text{grad}\, f(x)\| > 0 \end{aligned}$$

Since the same estimate as in the gradient case, up to a multiplicative constant, holds for the length of the integral curves of (AC) vector fields, the remaining argument is basically the same as the one for of Theorem 3.13 in [127]. Note, that since Nowel and Szafraniec consider only analytic functions $f$, we have to adapt the arguments to our case of a non-analytic $f$. But this is achieved by the use of our general disturbance term $\sigma(\phi(g(x)))$ instead of $(\phi(g(x)))^N$ in [127]. Our choice of the function $\sigma$ ensures that this does not give any problems and the proofs can be easily modified. □

We will now give some examples that the (AC) condition is crucial for the theorem above.

**Example 1.2.7** Let $M = \mathbf{R}^2 \setminus \{0\}$. In polar coordinates $(r, \theta)$, with $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ [63, p. 217] , we can define the following vector field $\tilde{X}$

$$\tilde{X}(r, \theta) = \begin{pmatrix} r(1-r)^3 \\ (1-r)^2(\sin(\theta))^2 \end{pmatrix}.$$

This yields in Euclidean coordinates the vector field $X$,

$$X(x) = X(x_1, x_2) = (1 - \|x\|)^3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1 - \|x\|)^2 x_2^2 \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

One easily checks that the whole unit circle $S^1$ is the set of equilibria of $X$ and that only the equilibria $a = (0, 1)$ and $b = (0, -1)$ have non-trivial sets of

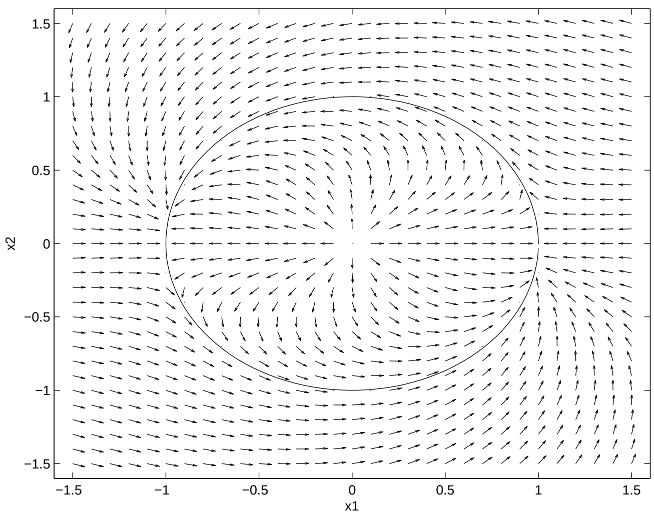


Figure 1.2: The normalized vector field $\frac{X}{\|X\|}$. For numerical reasons we plotted $\frac{X}{\|X\|}$ only if $\|X\| > 0.0001$, otherwise we plotted $X$ itself. The solid circle denotes the set of equilibria.

attraction. Figure 1.2 shows the normalized vector field, i.e. $\|X(x)\|^{-1} X(x)$. Furthermore, $X$ has the Lyapunov function $f(x) = (1 - \|x\|)^2$.
The (AC) condition is not satisfied with respect to the Lyapunov function $f$, as $\langle \|X(x)\|^{-1} X(x), x\rangle \to 0$ for $x \to S^1 \backslash \{a, b\}$. The set of non-trivial trajectories converging to $S^1$ is identical to the set of non-trivial trajectories converging to $K = \{a, b\}$. However, any neighborhood $U_\delta$ of $K$ as in Theorem 1.2.6 will have two connected components. Using this argument we see that for any $U_\delta$ for $S^1$ as in Theorem 1.2.6 and all $y \in (0, 1)$, the cohomology groups $\check{H}^*(\{x \in \mathbf{R}^2 \mid f(x) = y, \omega(x) \in K\})$ and $\check{H}^*(U_\delta \cap \{x \in \mathbf{R}^2 \mid f(x) > 0\})$ are not isomorphic.

**Example 1.2.8** Let $M = \mathbf{R}^3 \setminus \{(x_1, x_2, 0) \mid (x_1, x_2) \in \mathbf{R}^2\}$. In spherical coordinates $(r, \theta, \tau)$, with $(r, \theta, \tau) \mapsto (r\cos(\theta), r\cos(\tau)\sin(\theta), r\sin(\tau)\sin(\theta))$ for $(r, \theta, \tau) \in \mathbf{R}_+ \times (0, \pi) \times (0, 2\pi)$ [63, p. 217], we define a vector field $\tilde{Y}$ by

$$\tilde{Y}(r, \theta, \tau) = \begin{pmatrix} r(1-r) \\ \sin(2\theta) \\ 0 \end{pmatrix}.$$

Transformation to Euclidean coordinates yields a vector field $Y$ on $M$. The set of equilibria consists of the unit circle $S = \{(0, x_2, x_3) \mid x_2^2 + x_3^2 = 1\}$. The function $f(x) = (1 - \|x\|)^2$ is a Lyapunov function of $Y$. For any closed subset $K \subset S$ there is a smooth diffeomorphism $\phi$ of $M$ with the following properties:

- It leaves for any $r > 0$ the sphere $\{x \in \mathbf{R}^3 \mid \|x\| = r\}$ invariant.
- $\phi(S) \cap S = K$
- $f$ is a Lyapunov function of the induced vector field $\tilde{X} = (T_x\phi)Y(\phi(x))$.

The purpose of the diffeomorphism $\phi$ is to disturb $Y$ such that only points of $K$ in $S$ have non-trivial sets of attraction. We define the vector field $X(x) := (1 - \|x\|)^2\tilde{X}(x)$ on $M$. Then $f$ is a Lyapunov function of $X$. As in the previous case it is easy to check that $X$ does not satisfy the (AC) condition with respect to the Lyapunov function $f$. Furthermore the set of equilibria of $X$ is the whole unit sphere. Since $X$ is the scalar multiple of the pullback of $Y$, $\phi$ maps outside of the unit sphere trajectories of $Y$ onto trajectories of $X$. Hence, only subsets of $\phi(S)$ have a non-trivial region of attraction. Further, the region of attraction of $\phi(S)$ consists of $\phi(S) \cup (M \setminus S^{n-1})$. If we consider the set $S$ in $M$, then for any suitably small $\delta > 0$, the neighborhood $U_\delta = \{x \in \mathbf{R}^3 \mid \sup_{y \in S} \operatorname{dist}(x, y) < \delta\}$ satisfies the conditions of Theorem 1.2.6. However, the set of non-trivial trajectories attracted by $S$ is homeomorphic to $K \times \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \neq 0 \text{ or } (x_1, x_2) = 0\}$ and its Čech-Alexander-cohomology will in general not be isomorphic to $\check{H}^*(U_\delta \cap \{x \in M \mid f(x) > 0\})$.

### 1.2.3 Applications

In this section we consider some examples for the application of the convergence theorem for (AC) vector fields. As the first example we consider dissipative systems with a Hessian-driven damping term of the form

$$\ddot{x} = -c_1\dot{x} - c_2\,(\mathrm{Hess}_x\, f)\,\dot{x} - \mathrm{grad}\, f(x), \quad c_1, c_2 > 0. \tag{1.8}$$

Here $\mathrm{Hess}_x\, f$ denotes the Hessian of $f$. These systems were first considered by Alvarez et al. [7] for optimization purposes, albeit only on Hilbert spaces. For analytic functions $f$ they proved the convergence of the integral curves of (1.8) in Euclidean space. However, they showed this using directly the Łojasiewicz gradient inequality and not by considering any angle conditions.

**Theorem 1.2.9** *Let $M$ be a Riemannian manifold and $f : M \to \mathbf{R}$ a $C^2$ function. Then the vector field on $TM$ defined by*

$$\nabla_{\dot{x}}\dot{x} = -c_1\dot{x} - c_2\,\mathrm{Hess}_x\, f\dot{x} - \mathrm{grad}\, f(x) \tag{1.9}$$

*is an (AC) vector field. Here, $\nabla$ denotes the Riemannian connection. If $M$ is analytic with analytic Riemannian metric and $f$ is a $\mathcal{C}$-function, then $\omega$-limit set of any solution contains at most one point.*

**Proof:** Let $K\colon TTM \to TM$ denote the connection map[22], i.e. $\nabla_X Y(x) = K((T_x X)Y(x))$, and $\pi\colon TM \to M$ the tangent bundle projection, cf. [138]. We define on $TM$ the vector field $\Gamma$ as the geodesic vector field, i.e. the vector field of the geodesic flow [99, 138]. Let us recall the definition of horizontal and vertical lifts from $TM$ to $TTM$, cf. [52, 99, 138, 165]. For a map $L\colon TM \to TM$, with $\pi(L(v)) = L\pi(v)$ for all $v \in TM$, we define the horizontal and vertical lifts, $L^H$ and $L^V$, as the vector fields on $TM$ with

$$\begin{aligned} T_\eta\pi L^H(\eta) = L(\eta) \quad &, \quad KL^H(\eta) = 0, \\ T_\eta\pi L^V(\eta) = 0 \quad &, \quad KL^V(\eta) = L(\eta). \end{aligned}$$

The splitting of the tangent spaces $T_\eta TM$ into a vertical and horizontal spaces, cf. [99, 138], ensures that these lifts are well-defined. Using representations of $K$ and $T_\eta\pi$ in local coordinates [99, 138], one sees that $L^H$, $L^V$ are

[22] In some instances in literature $K$ is called the connection and $\nabla$ the covariant derivative, see [99, 106].

smooth if $L$ is a smooth function. We can now define a vector field $X$ on $TM$ by

$$X(\eta) := \Gamma(\eta) - c_1\eta^V - c_2(\mathrm{Hess}_{\pi(\eta)} f\eta)^V - (\mathrm{grad}\, f(\pi(\eta)))^V, \text{ for } \eta \in TM.$$

Let $\gamma\colon (a,b) \to TM$ be an integral curve of $X$. We define $\alpha\colon (a,b) \to M$ as the projection of $\gamma$ to $M$, i.e. $\alpha(t) = \pi(\gamma(t))$. Note, that $T_\eta\pi(\Gamma(\eta)) = \eta$ [99, Lem. 3.1.15]. Denoting $Y(\eta) = X(\eta) - \Gamma(\eta)$, we see that $T_\eta\pi Y(\eta) = 0$ and

$$\dot\alpha(t) = T_{\gamma(t)}\pi(\dot\gamma(t)) = T_{\gamma(t)}\pi\left(\Gamma(\gamma(t)) + Y(\gamma(t))\right) = T_{\gamma(t)}\pi(\Gamma(\gamma(t))) = \gamma(t).$$

Recall that

$$\nabla_{\dot\alpha(t)}\dot\alpha(t) = K\ddot\alpha(t),$$

see [99, Example 1.5.14]. Note, that this implies that $K\Gamma(\eta) = 0$, since geodesics are parallel curves. Hence,

$$\begin{aligned}
\nabla_{\dot\alpha(t)}\dot\alpha(t) &= K\dot\gamma(t) \\
&= K\Big((\Gamma(\gamma(t)) - c_1\,(\gamma(t))^V \\
&\qquad -c_2\left(\mathrm{Hess}_{\pi(\gamma(t))} f(\gamma(t))\right)^V - (\mathrm{grad}\, f(\pi(\gamma(t))))^V\Big) \\
&= -c_1\gamma(t) - c_2\,\mathrm{Hess}_{\pi(\gamma(t))} f(\gamma(t)) - \mathrm{grad}\, f(\pi(\gamma(t))) \\
&= -c_1\dot\alpha(t) - c_2\,\mathrm{Hess}_{\alpha(t)} f(\dot\alpha(t)) - \mathrm{grad}\, f(\alpha(t)).
\end{aligned}$$

Thus $\alpha(t)$ is a solution of (1.9). An analogous argument shows that the tangent curve $\dot\alpha$ of a solution $\alpha$ of (1.9) is an integral curve of $X$. Therefore, equation (1.9) yields the vector field $X$ on $TM$. We use the energy function $E\colon TM \to \mathbf{R}$,

$$E(\eta) = (1 + c_1c_2)f(\pi(\eta)) + \frac{1}{2}\left\|\eta + c_2\,\mathrm{grad}\, f(\pi(\eta))\right\|^2$$

proposed in [7] for the Hilbert space case, as Lyapunov function for $X$. Consider the function $F\colon TM \to \mathbf{R}$, $F(\eta) = \|\eta\|^2$. A straight forward calculation shows that for all $\eta \in TM$, $\zeta \in T_\eta TM$

$$dF(\eta)(\zeta) = \langle \eta, K\zeta\rangle\,,$$

see [138, p. 58]. Using $dF$, can now calculate the differential of $E$ by the chain rule. This gives for $\eta \in TM$, $\zeta \in T_\eta TM$

$$\begin{aligned} dE(\eta)(\zeta) &= (1+c_1c_2)df(\pi(\eta))(T_\eta\pi(\zeta)) \\ &\quad + \left\langle \eta + c_2 \operatorname{grad} f(\pi(\eta)), K\zeta + c_2 \underbrace{K(T_{\pi(\eta)} \operatorname{grad} f)}_{=\operatorname{Hess}_{\pi(\eta)} f}(T_\eta\pi(\zeta)) \right\rangle \\ &= (1+c_1c_2)df(\pi(\eta))(T_\eta\pi(\zeta)) \\ &\quad + \left\langle \eta + c_2 \operatorname{grad} f(\pi(\eta)), K\zeta + c_2 \operatorname{Hess}_{\pi(\eta)} f(T_\eta\pi(\zeta)) \right\rangle. \end{aligned}$$

The Riemannian metric on $M$ can be extended in a canonical way to the Sasaki metric on $TM$ [138, p. 55-56] by setting

$$\langle\langle \zeta, \xi \rangle\rangle = \langle T_\eta\pi(\zeta), T_\eta\pi(\xi)\rangle + \langle K\zeta, K\xi\rangle, \quad \zeta, \xi \in T_\eta TM.$$

By definition of the horizontal and vertical lifts, we see that the gradient of $E$ with respect to this metric is given by

$$\begin{aligned} \operatorname{grad} E(\eta) = (1+c_1c_2) \operatorname{grad} f(\pi(\eta))^H + \eta^V + c_2(\operatorname{Hess}_{\pi(\eta)} f(\eta))^H \\ + c_2(\operatorname{grad} f(\pi(\eta)))^V + c_2^2(\operatorname{Hess}_{\pi(\eta)} f(\operatorname{grad} f(\pi(\eta))))^H. \end{aligned}$$

As already shown for the Hilbert space case in [7], we get that

$$\langle\langle \operatorname{grad} E(\eta), X(\eta) \rangle\rangle = -(c_1 \|\eta\|^2 + c_2 \|\operatorname{grad} f(\pi(\eta))\|^2).$$

Note that

$$\begin{aligned} \|X(\eta)\|^2 &= \|\eta\|^2 + \left\|c_1\eta + c_2 \operatorname{Hess}_{\pi(\eta)} f(\eta) + \operatorname{grad} f(\pi(\eta))\right\|^2 \\ \|\operatorname{grad} E(\eta)\|^2 &= (1+c_1c_2)^2 \left\| \left(\operatorname{Id}_{T_{\pi(\eta)}M} + \operatorname{Hess}_{\pi(\eta)} f\right) \operatorname{grad} f(\pi(\eta)) \right\|^2 \\ &\quad + c_2^2 \|\operatorname{grad} f(\pi(\eta))\|^2 + \|\eta\|^2 + c_2^2 \left\|\operatorname{Hess}_{\pi(\eta)} f(\eta)\right\|^2. \end{aligned}$$

The continuity of $\operatorname{Hess}_x f$ yields that for any compact set $K$ we have a constant $\varepsilon_K > 0$ with

$$-\langle\langle \operatorname{grad} E(\eta), X(\eta) \rangle\rangle \geq \varepsilon_K \|\operatorname{grad} E(\eta)\| \|X(\eta)\|.$$

Hence, the vector field $X$ of (1.9) on $TM$ is an (AC) vector field. If $f$ is a $\mathcal{C}$-function and $\langle\cdot,\cdot\rangle$ is analytic then the Lyapunov function $E$ is a $\mathcal{C}$-function. Theorem 1.2.5 gives now the convergence of the integral curves. □

In the Hilbert space case, Alvarez et al. showed that the second order equation (1.8) can be transformed into a first order gradient-like system with a new Lyapunov function $W$ [7]. This first order system has the form

$$\dot{x} = -(I + J)\,\mathrm{grad}\,W(x) \tag{1.10}$$

where grad denotes the gradient with respect to the scalar product, $I$ the identity and $J$ a skew symmetric linear operator. We will now show that such first order systems are (AC) systems, too.

There are two approaches for the construction of such systems. One is to define the vector field by using a non-degenerate, non-symmetric bilinear form $b$. On a manifold $M$, we call any smooth section $b$ in the bundle of bilinear real valued maps on the tangent spaces a *bilinear form*. The form is *non-degenerate* if it is non-degenerate bilinear form on $T_xM$ for every $x \in M$. Analogous to the Riemannian metric or symplectic form case we can associate to any $C^1$ function $f\colon M \to \mathbf{R}$ a vector field $X_f$ defined by $df(x)(v) = b(X_f(x), v)$ for all $x \in M$, $v \in T_xM$. In the case of (1.10) the bilinear form would be[23] $v^\top(I + J^\top)^{-1}w$ and $X_W(x) = (I + J)\,\mathrm{grad}\,W(x)$. The other approach is to consider these vector fields as the sum of a Hamiltonian and a gradient vector field, both from the same function $f$. In the case above we have the gradient $\mathrm{grad}\,W(x)$ and the Hamiltonian vector field $J\,\mathrm{grad}\,W(x)$, if $J$ is non-degenerate. Both approaches give an (AC) vector field as the next propositions show.

**Proposition 1.2.10** *Let $M$ be a Riemannian manifold and $f\colon M \to \mathbf{R}$ be a $C^1$ function. Assume that $b$ is a non-degenerate bilinear form on $M$ such that $b_s(v,w) := 1/2(b(v,w) + b(w,v))$ defines a Riemannian metric on $M$. Then $X_f$ is an (AC) vector field.*

**Proof:** Let $x \in M$ and $U$ a relatively compact neighborhood of $x$. Since the angle condition does not depend on the metric, we can assume that $b_s$ coincides with the Riemannian metric on $M$. There exists $\varepsilon > 0$ such that

$$\|\mathrm{grad}\,f(y)\|^2 = df(\mathrm{grad}\,f(y)) = b(X_f(y), \mathrm{grad}\,f(y)) \le \varepsilon \|X_f(y)\|\,\|\mathrm{grad}\,f(y)\|$$

for all $y \in U$. Further

$$b_s(\mathrm{grad}\,f(y), X_f(y)) = df(X_f(y)) = b(X_f(y), X_f(y)) = b_s(X_f(y), X_f(y)) = \|X_f(y)\|^2\,.$$

[23]Note that $(I + J)^{-1}$ exists for all skew-symmetric $J$ since $x^\top(I + J)x = x^\top x > 0$ for all $x \in \mathbf{R}^n$, i.e. $\ker(I + J) = \{0\}$.

Thus

$$-b_s(-\operatorname{grad} f(y), X_f(y)) \geq \frac{1}{\varepsilon} \|X_f(y)\| \, \|\operatorname{grad} f(y)\| \quad \text{for all } y \in U.$$

Therefore, $X_f$ is an (AC) vector field with Lyapunov function $-f$. $\square$

**Proposition 1.2.11** *Let $M$ be a Riemannian manifold and $\Omega$ a symplectic form on $M$. If $f\colon M \to \mathbf{R}$ is a $C^1$ function then* $\operatorname{grad} f + X_f$ *is an (AC) vector field. Here $X_f$ denotes the Hamiltonian vector field of $f$ with respect to $\Omega$.*

Note that this does not follow from Proposition 1.2.10 as the sum of a Riemannian metric and a symplectic form is not necessarily non-degenerate. A simple example would be the scalar product $\langle (v_1, v_2), (w_1, w_2) \rangle = v_1 w_1 + v_2 w_1 + v_1 w_2$ and the symplectic form $\Omega((v_1, v_2), (w_1, w_2)) = v_1 w_2 - v_2 w_1$ on $\mathbf{R}^2$. Their sum is $b((v_1, v_2), (w_1, w_2)) = v_1 w_1 + 2 v_1 w_2$ which is degenerate.
**Proof:** Note that

$$\langle \operatorname{grad} f, \operatorname{grad} f + X_f \rangle = \|\operatorname{grad} f\|^2 . \tag{1.11}$$

From the equation

$$\langle \operatorname{grad} f(x), w \rangle = \Omega(X_f(x), w) \quad \text{for all } w \in T_x M,$$

we get that

$$X_f(x) = B(x) \operatorname{grad} f(x)$$

with a well-defined function $B(x)$ with values in the bundle of linear maps $T_x M \to T_x M$ over $M$. In local charts with matrix representations for $\langle \cdot, \cdot \rangle$ and $\Omega$ we see that $B$ is continuous. Thus on every relatively compact open set there is a constant $C > 0$ with

$$\|\operatorname{grad} f(x) + X_f(x)\| \leq \|\mathrm{Id}_{T_x M} + B(x)\| \, \|\operatorname{grad} f(x)\| \leq C \|\operatorname{grad} f(x)\| ,$$

where $\mathrm{Id}_{T_x M}\colon T_x M \to T_x M$ denotes the identity map and $\|I + B(x)\|$ the operator norm induced by $\langle \cdot, \cdot \rangle$ on $T_x M$. With (1.11) we get directly that on compact sets

$$-\langle -\operatorname{grad} f, \operatorname{grad} f + X_v \rangle \geq \varepsilon_K \|\operatorname{grad} f\| \, \|\operatorname{grad} f + X_f\|$$

for the constant $\varepsilon_K = C^{-1}$. $\square$

**Remark 1.2.12** One might be tempted to conjecture that the pointwise convergence also holds if the vector field has the form $X = X_f + \operatorname{grad} h$, $X_f$ the Hamiltonian vector field with respect to a function $f: M \to \mathbf{R}$ and $h$ a *different* function $M \to \mathbf{R}$. However, our approach cannot be applied to this problem. The example

$$X(x_1, x_2, x_3, x_4) = \begin{pmatrix} -2x_1 \\ x_3 \\ -x_2 \\ -2x_4 \end{pmatrix},$$

similar to the one used in the introduction, shows this. $X$ is the sum of the gradient of $f(x_1, x_2, x_3, x_4) = -x_1^2 - x_4^2$ and the Hamiltonian vector field $Y(x_1, x_2, x_3, x_4) = (0, x_3, -x_2, 0)^\top$ and has integral curves converging to the entire unit circle in $\{0\} \times \mathbf{R}^2 \times \{0\}$.

As another example we consider vector fields of the form

$$F(\operatorname{grad} h) \tag{1.12}$$

on a Riemannian manifold $M$. Here, $h$ is a real valued function on $M$ and $F: TM \to TM$ is a $C^1$ function with $F(T_xM) \subset T_xM$ for all $x \in M$, $F(0) = 0$. Furthermore, $F$ is assumed to satisfy a monotonicity condition, i.e. there is a constant $\delta > 0$ such that for all $x \in M$ and $v \in T_xM$

$$\langle DF_x(v)v, v\rangle \geq \delta \, \|v\| \, \|DF_x(v)v\| \tag{1.13}$$

holds, where $F_x = F|_{T_xM}$ and $D$ the usual derivative of functions on Hilbert spaces. Note that the inclusion $F(T_xM) \subset T_xM$ is necessary for (1.12) to be a vector field, because for each $v \in T_xM$ the value of $F(v)$ must be contained in $T_xM$. Under the conditions above the vector field (1.12) is an (AC) vector field and we get the following convergence theorem.

**Theorem 1.2.13** *Let $M$ be a Riemannian manifold and $h: M \to \mathbf{R}$ a $C^2$ function. Assume that $F: TM \to TM$ is a $C^1$ function, which satisfies for all $x \in M$: $F(T_xM) \subset T_xM$, $F(0) = 0$, and the monotonicity condition* (1.13). *Then* (1.12) *is an (AC) vector field with Lyapunov function $-h$.*

*If $M$ is an analytic manifold and $h$ is a morphism of an analytic-geometric category, then the $\omega$-limit set of an integral curve of* (1.12) *contains at most one point.*

**Proof:** We write $F_x$ for $F|_{T_xM}$. Let us define the function

$$\sigma(t) := \langle F_x(tv), v \rangle$$

for a $v \in T_xM$. Then for all $t > 0$

$$\begin{aligned}\dot{\sigma}(t) = \langle DF_x(tv)v, v \rangle &= \frac{1}{t^2} \langle DF_x(tv)tv, tv \rangle \\ &\geq \frac{\delta}{t^2} \|tv\| \|DF_x(tv)tv\| = \delta \|v\| \|DF_x(tv)v\| .\end{aligned}$$

Thus we have for all $t > 0$

$$\begin{aligned}\sigma(t) \geq \int_0^t \delta \|v\| \|DF_x(sv)v\| \, ds &= \delta \|v\| \int_0^t \left\| \frac{d}{ds} F(sv) \right\| ds \\ &\geq \delta \|v\| \left\| \int_0^t \frac{d}{ds} F(sv) ds \right\| = \delta \|v\| \|F_x(tv)\| .\end{aligned}$$

Setting $t = 1$, we get

$$\langle F(v), v \rangle \geq \delta \|v\| \|F(v)\| .$$

Thus (1.12) is an (AC) vector field. If $h$ is a $\mathcal{C}$-function we get by Theorem 1.2.5 the convergence of the integral curves. □

Similar systems of the form

$$F(\operatorname{grad} h_1(x)) - F(\operatorname{grad} h_2(x)) \tag{1.14}$$

where considered by Popov [131], although only on Hilbert spaces. Popov requires $F$ besides $C^1$ to be *$\delta$-monotone*, which can be defined in the Riemannian manifold setting as that there exists a $\delta > 0$ such that

$$\langle F(v) - F(w), v - w \rangle \geq \delta \|v - w\|^2 \quad \text{for all } x \in M, v, w \in T_xM. \tag{1.15}$$

Since $F$ is $C^1$, there is for any compact set $K$ a constant $C_k$ such that for all $v, w \in K$ $\|F(v) - F(w)\| \leq C_K \|v - w\|$. Thus (1.15) implies that (1.14) is an (AC) vector field with Lyapunov function $h_2 - h_1$. Hence, if $h_1$ and $h_2$ are $\mathcal{C}$-functions then by Theorem 1.2.5 the integral curves either escape to infinity without any $\omega$-limit points in $M$ or converge to single points. The notion of $\delta$-monotonicity is connected to condition (1.13) as there is a very similar characterization with the linearization of $F$ on $T_xM$. In fact, $\delta$-monotonicity is equivalent to *local $\delta$-monotonicity*[24], i.e.

$$\langle DF_x(v)w, w\rangle \geq \delta \left\|w\right\|^2 \quad \text{for all } x \in M, v, w \in T_xM, \tag{1.16}$$

where $F_x = F|_{T_xM}$ and $D$ the usual derivative of functions on Hilbert spaces. For completeness we give a short proof of the equivalence of (1.15) and (1.16).

**Proposition 1.2.14** *Let $M$ be a Riemannian manifold and $F\colon TM \to TM$ be a $C^1$ function with $F(T_xM) \subset T_xM$ for all $x \in M$. Then $\delta$-monotonicity of $F$ is equivalent to local $\delta$-monotonicity.*

**Proof:** If we restrict $F$ to a $T_xM$ for $x \in M$ then our definitions of local $\delta$-monotonicity and $\delta$-monotonicity on a manifold coincide with Popov's versions in [131]. With this restriction the implication of $\delta$-monotonicity from local one can be shown as in [131], Theorem 4.1. For the other implication note that

$$\begin{aligned}\langle F_x(v+tw) - F_x(v), w\rangle &= \frac{1}{t}\langle F_x(v+tw) - F_x(v), v+tw-v\rangle \\ &\geq \frac{\delta}{t}\left\|tw\right\|^2 = \delta\left\|tw\right\|\left\|w\right\|.\end{aligned}$$

Thus

$$\langle DF_x(v)w, w\rangle = \lim_{t\to 0}\frac{1}{t}\langle F_x(v+tw) - F_x(v), w\rangle \geq \lim_{t\to 0}\frac{\delta}{t}\left\|tw\right\|\left\|w\right\| = \delta\left\|w\right\|^2.$$

As the constant $\delta$ does not depend on the tangent space $T_xM$ we get the global equivalence. □

[24]This term comes from [131]. Note that it is in fact a global condition.

## 1.3 AC differential inclusions

In this section, we extend (AC) vector fields to a class of differential inclusions and prove a similar convergence theorem for their solutions.

Let $M$ be a smooth manifold and $2^{TM}$ the set of subsets of $TM$. Analogously to the Euclidean case [12], a *differential inclusion* on $M$ has the form

$$\dot{x} \in F(x)$$

where $F$ is a map $F\colon M \to 2^{TM}$ with $F(x) \subset T_xM$. To define solutions of a differential inclusion, recall that a curve $\gamma\colon [a,b] \to \mathbf{R}^n$ is called *absolutely continuous* if for any $\varepsilon > 0$ there is a $\eta > 0$ such that for any sequence $a \leq a_1 < b_1 < \ldots < a_k < b_k \leq b$ with

$$\sum_{i=1}^{k}(b_i - a_i) < \eta$$

we have

$$\sum_{i=1}^{k} \|\gamma(b_i) - \gamma(a_i)\| < \varepsilon,$$

see [12, p. 12]. We call curves defined on an open interval $(a,b)$ absolutely continuous if they are absolutely continuous on every compact subinterval of $(a,b)$. For curves on manifolds we use the standard definition of absolute continuity in local charts [111, 125], i.e. a curve $\gamma\colon [a,b] \to M$ is *absolutely continuous* if it is absolutely continuous in local charts. An absolutely continuous curve is differentiable almost everywhere [12, p. 12].

A *solution of a differential inclusion* $\dot{x} \in F(x)$ is an absolutely continuous curve $\gamma\colon (a,b) \to M$ such that

$$\dot{\gamma}(t) \in F(\gamma(t))$$

almost everywhere[25], cf. [12, p. 93/94]. Last, we recall some continuity conditions on the set-valued maps. A set-valued map $F\colon M \to 2^{TM}$ is called *upper semicontinuous* at $x \in M$, if for any neighborhood $V \subset TM$ of $F(x)$ there is a neighborhood $U$ of $x$ such that

$$\bigcup_{y \in U} F(y) \subset V,$$

[25] i.e. for all $t \in (a,b) \setminus N$, with $N$ a set of measure 0.

see [12, p.41, Def. 1]. We call $F$ simply upper semicontinuous if it is upper semicontinuous in all $x \in M$. The map $F\colon M \to 2^{TM}$ is called *lower semicontinuous* at $x \in M$, if for any sequence $(x_k) \subset M$ and $v \in F(x)$, there is a sequence $(v_k) \subset TM$ with $v_k \in F(x_k)$ and $v_k \to v$, see [12, p. 43]. Again, we call $F$ lower semicontinuous if it is lower semicontinuous in all $x \in M$.

**Definition 1.3.1** Let $M$ be a Riemannian manifold, $f\colon M \to \mathbf{R}$ a continuous function and

$$\dot{x} \in F(x),\ F\colon M \to 2^{TM}$$

a differential inclusion on $M$. We call $\dot{x} \in F(x)$ an *angle condition (AC) differential inclusion* with associated Lyapunov function $f$ if

- $f$ is non-constant on open sets,
- $f$ is piecewise $C^1$ with domain stratification $S_j$, $j \in \Lambda$,
- $F$ is lower semicontinuous,
- for any compact set $K$ there is a constant $\varepsilon_K > 0$ such that for all $j \in \Lambda$, $x \in S_j \cap K$ and $v \in F(x) \cap T_x S_j$ the estimate

$$-\left\langle \operatorname{grad}_j f(x), v \right\rangle \geq \varepsilon_K \left\| \operatorname{grad}_j f(x) \right\| \|v\| \qquad \text{(AC)}$$

holds.

**Lemma 1.3.2** *Let $M$ be a Riemannian manifold and $S \subset M$ a submanifold. Assume that we have a Lipschitz continuous function $f\colon M \to \mathbf{R}$, which is $C^1$ on $S$, and a continuous curve $\gamma\colon (-1,1) \to M$, differentiable in $0$, with $\gamma(0) \in S$ and $\gamma'(0) \in T_{\gamma(0)}S$. Then $f \circ \gamma$ is differentiable in $0$ with $(f \circ \gamma)'(0) = df|_S(\gamma(0))(\gamma'(0))$.*

**Proof:** Using a local chart we can w.l.o.g. assume that $M = \mathbf{R}^n$ equipped with the induced Riemannian metric and $\gamma(0) = 0$. Let $\alpha : (-1,1) \to S$ be a $C^1$ curve with $\alpha(0) = 0$ and $\alpha'(0) = \gamma'(0)$. Then

$$\lim_{t\to 0} \frac{\operatorname{dist}(\gamma(t), \alpha(t))}{t} = 0.$$

Let $L > 0$ be the Lipschitz constant of $f$ on a neighborhood $U$ of $0$. Then

$$\lim_{t\to 0} \frac{|f(\gamma(t)) - f(\alpha(t)|}{t} \leq \lim_{t\to 0} \frac{L \operatorname{dist}(\gamma(t), \alpha(t))}{t} = 0.$$

We have

$$\begin{aligned}
\lim_{t\to 0}\frac{f(\gamma(t))-f(0)}{t} &= \lim_{t\to 0}\frac{f(\gamma(t))-f(\alpha(t))+f(\alpha(t))-f(0)}{t}\\
&= \lim_{t\to 0}\frac{f(\alpha(t))-f(0)}{t}\\
&= df|_S(0)(\alpha'(0)) = df|_S(0)(\gamma'(0)).
\end{aligned}$$

□

**Lemma 1.3.3** *Let $M$ be Riemannian manifold with (AC) differential inclusion $\dot{x}\in F(x)$ and Lyapunov function $f\colon M\to \mathbf{R}$. Then $\dot{x}\in F(x)$ is a (AC) differential inclusion for any refinement of the stratification of $f$.*

**Proof:** The (AC) condition is preserved by the same argument as in the vector field case. The other conditions do not depend on the stratification. □

**Lemma 1.3.4** *Let $M$ be an analytic Riemannian manifold and $\dot{x}\in F(x)$ an (AC) differential inclusion. Assume that the Lyapunov function $f$ is a $\mathcal{C}$-function and the stratification is a strong $(a_f)$ stratification into $\mathcal{C}$-sets. Let $x\in S_j$. If $f$ is not Lipschitz at $x$, then $F(x)\cap T_xS_j=\{0\}$ .*

**Proof:** Let $v\in F(x)$, $v\neq 0$. Assume that $f$ is not Lipschitz at $x$. Lemma 1.1.19 implies that there is a $i\in\Lambda$ and a sequence $x_k\to x$, $x_k\in S_i$ with $\|\mathrm{grad}_i f(x_k)\|\to\infty$. Let w.l.o.g. $\ker df|_{S_i}(x_k)\to L$, $L\subset T_xM$ a linear subspace. From condition 7 of the strong $(a_f)$ stratification follows that $T_xS_j\subset L$. Since $F$ is lower semicontinuous there is a sequence $v_k\in F(x_k)$ with $v_k\to v$. By condition (AC) we have $\lim_{k\to\infty} v_k\notin L$. Thus $v\notin T_xS_j$. □

**Definition 1.3.5** Let $M$ be a manifold. A continuous curve $\gamma\colon[a,b]\to M$ is *piecewise continuously differentiable* if there is a finite number of points $a=t_1<\ldots<t_n=b$ such that $\gamma$ is $C^1$ on each interval $[t_i,t_{i+1}]$.

**Proposition 1.3.6** *Let $M$ be an analytic Riemannian manifold and $\dot{x}\in F(x)$ an (AC) inclusion. Assume that the Lyapunov function $f$ is a $\mathcal{C}$-function and the stratification consists of $\mathcal{C}$-sets. Let one of the following conditions hold:*

1. *$f$ is Lipschitz and $\gamma$ any solution of $\dot{x}\in F(x)$,*

2. *or $\gamma$ is a piecewise continuously differentiable solution of $\dot{x} \in F(x)$, with a countable number of points $t$ with $\dot{\gamma}(t) = 0$.*

*Then $f \circ \gamma$ is monotonously decreasing.*

**Proof:** By Lemma 1.3.3 we can refine the stratification into a strong $(a_f)$ stratification. Let $\gamma : (a, b) \to M$ be a solution of $\dot{x} \in F(x)$. Denote by $A$ the set of points in $(a, b)$ such that $\dot{\gamma}(t)$ is transversal[26] to the stratum of $\gamma(t)$, i.e.

$$A = \{t \in (a, b) \mid \dot{\gamma}(t) \notin T_{\gamma(t)} S_j\}.$$

For any $t \in A$, $j \in \Lambda$, $\gamma(t) \in S_j$, there is a $\delta > 0$ that for all $s \in (t - \delta, t + \delta)$, $s \neq t$, we have $\gamma(s) \notin S_j$. Hence, the intersection of $A$ with any compact subinterval of $(a, b)$ must be countable. In particular, $A$ has measure 0. Furthermore, we denote by $B$ the set of critical points of $\gamma$, i.e.

$$B = \{t \in (a, b) \mid \dot{\gamma}(t) = 0\}.$$

Case (1): $f$ is Lipschitz continuous. It is a well-known fact, that the composition of a Lipschitz continuous function and an absolutely continuous function yields an absolutely continuous function, see [16, p. 245, Ex. 14.O] for the scalar-valued case. Hence, the function $f \circ \gamma$ is absolutely continuous. Since $f$ is Lipschitz, we can apply Lemma 1.3.2 and see that for all $t \in (a, b) \setminus A$, the function $f \circ \gamma$ is differentiable and $(f \circ \gamma)'(t) = \langle \operatorname{grad}_{j(t)} f(\gamma(t)), \dot{\gamma}(t) \rangle$, with $j(t)$ defined by $\gamma(t) \in S_{j(t)}$. Thus, we have by the (AC) condition that $(f \circ \gamma)'(t) \leq 0$ for $t \in (a, b) \setminus A$. As $f \circ \gamma$ is absolutely continuous, we have that

$$(f \circ \gamma)(t) = \int_{t_0}^{t} (f \circ \gamma)'(s) ds + f(\gamma(t_0)),$$

see [12, p. 13, Thm. 1]. Since $A$ has measure 0, the function $f \circ \gamma$ is monotonously decreasing.

Case (2): $\gamma$ is piecewise continuously differentiable. Then it is sufficient to show the monotonicity of $f \circ \gamma$ on an interval $(a, b)$ where $\gamma$ is differentiable. Denote by $C$ the points $t \in (a, b)$, such that $f$ is not Lipschitz in $\gamma(t)$. If $t \in C$, then by Lemma 1.3.4 either $\dot{\gamma}(t)$ is transversal to $T_{\gamma(t)} S_j$ or $\dot{\gamma}(t) = 0$. Hence, $f$ is Lipschitz continuous in all points $\gamma(t)$ with $t \in (a, b) \setminus (A \cup B)$ and $C \subset (A \cup B)$. Since, any accumulation point of of non-Lipschitz points of $f$ must be non-Lipschitz, too, the set $C$ is closed in $(a, b)$. Let $[c, d]$ be an

[26] i.e. $\dot{\gamma}(t) \notin T_{\gamma(t)} S_i$ with $\gamma(t) \in S_i$.

arbitrary, non-trivial, compact subinterval of $(a,b)$. Then $[c,d] \cap C$ is closed and countable. Hence, the set $(c,d) \setminus C$ is a countable union of open intervals $I_k$. Note, that by the arguments for the case of Lipschitz continuous $f$, $(f \circ \gamma)'(t)$ exists for all $t \in I_k$, is $\leq 0$ and integrable[27] on $I_k$. Hence, $(f \circ \gamma)'(t)$ exists for all $t \in (c,d) \setminus C$ and is integrable on $[c,d]$. Furthermore, $f \circ \gamma$ is continuous. By a theorem from real analysis [79, p. 298/299, Ex. 18.41 d)] these conditions imply that $f \circ \gamma \colon [c,d] \to \mathbf{R}$ is absolutely continuous. Since $(f \circ \gamma)'(t) \leq 0$ almost everywhere on $[c,d]$, we get that $f \circ \gamma$ is monotonously decreasing on $[c,d]$. Since $[c,d]$ was arbitrary, this proves our claims. □

**Remark 1.3.7** The arguments in the proof above are not sufficient to cover the general case of a non-Lipschitz $f$ and arbitrary absolutely continuous solutions.

**Proposition 1.3.8** *Under the conditions of Proposition 1.3.6 the function $f \circ \gamma$ is strictly decreasing if $\gamma$ is non-constant on open sets.*

**Proof:** Refine the stratification to a strong $(a_f)$ stratification. Assume that $\gamma \colon (a,b) \to M$ is an absolutely continuous solution. By Proposition 1.3.6 $f \circ \gamma$ is monotonously decreasing. Thus, $f \circ \gamma$ is not strictly decreasing if and only if there is an non-empty interval $(c,d) \subset (a,b)$ such that $f \circ \gamma$ is constant. Assume that such an interval $(c,d)$ exists.

We consider first the case that $f$ is Lipschitz continuous. Let $S_j$ be the highest dimensional stratum met by $\gamma$ on the interval $(c,d)$. As a stratification is locally finite we can w.l.o.g. assume that $\gamma(t) \in S_j$ for all $t \in (c,d)$. If $\gamma$ in differentiable in $t$ then $\dot\gamma(t) \in T_{\gamma(t)} S_j$ and by Lemma 1.3.2 $(f \circ \gamma)'(t) = \langle \operatorname{grad}_j f(\gamma(t)), \dot\gamma(t) \rangle = 0$. But as $\gamma$ is absolutely continuous and non-constant on open sets, $\dot\gamma(t)$ cannot be zero for all $t \in (c,d)$. The (AC) condition and the fact that $\operatorname{rk} df|_{S_j}$ is constant, imply that $\operatorname{grad}_j f(x) = 0$ for all $x \in S_j$. Choose $t_0 \in (c,d)$ with $\dot\gamma(t_0) \neq 0$. Let $x = \gamma(t_0)$. There is a $i \in \Lambda$ and a sequence $(x_k) \subset S_i$ with $x_k \to x$ and $df|_{S_i}(x_k) \neq 0$. Since $F$ is lower semicontinuous, there is a sequence $v_k \in F(x_k)$ with $v_k \to \dot\gamma(t_0)$. W.l.o.g. we can assume that $v_k \neq 0$ for all $k$ and $w = \lim_{k\to\infty} \|\operatorname{grad}_i f(x_k)\|^{-1} \operatorname{grad}_i f(x_k)$

[27]By the absolute continuity of $f \circ \gamma$ on the open interval $I_k$, we see that $(f \circ \gamma)'$ is integrable on any compact subinterval of $I_k$ and the integrals are uniformly bounded. Let $J_l \subset I_k$ be a increasing sequences of compact subintervals with $\bigcup_{l=1}^{\infty} J_l = I_k$. Denote by $\psi_l$ the characteristic function of $J_l$. Since $(f \circ \gamma)'$ is non-positive, we can apply theorem on monotone sequences of integrable functions [79, p. 172, Thm. 12.22] to the sequence of functions $t \mapsto \psi_l(t)((f \circ \gamma)'(t))$. Hence, $(f \circ \gamma)'$ is integrable on $I_k$.

exists. Since $df|_{S_j}(x) = 0$, the Thom condition implies that $\langle w, \eta \rangle = 0$ for all $\eta \in T_x S_j$ and especially

$$\left\langle w, \frac{\dot{\gamma}(t_0)}{\|\dot{\gamma}(t_0)\|} \right\rangle = 0. \tag{1.17}$$

Let $K$ be a compact subset of $M$ with $x_k \in K$ for all $k \in \mathbf{N}$. By the (AC) condition, there is an $\varepsilon > 0$ such that

$$\varepsilon \leq -\left\langle \frac{v_k}{\|v_k\|}, \frac{\operatorname{grad}_i f(x_k)}{\|\operatorname{grad}_i f(x_k)\|} \right\rangle.$$

Taking the limit this gives a contradiction to equation (1.17).

Assume that $f$ is not Lipschitz continuous. In this case $\gamma$ is piecewise differentiable with countable set of critical points. We can assume that $\gamma$ is differentiable on $(c, d)$. If $f$ is not Lipschitz continuous in $\gamma(t)$, $t \in (c, d)$, then by Lemma 1.3.4 $\dot{\gamma}(t)$ is either 0 or transversal to the stratum of $\gamma(t)$. Therefore, we can find a non-trivial subinterval $[u, v]$, $u < v$ of $(c, d)$ such that $f$ is Lipschitz in all $\gamma(t)$ with $t \in [u, v]$. Hence, we apply the previous argument for Lipschitz continuous $f$ and get the desired contradiction. □

Before we prove the convergence theorem, let us recall the standard definition of the length of curve.

**Definition 1.3.9 ( cf. [26, p.158] )** Let $M$ be a Riemannian manifold and $\gamma\colon [a, b] \to M$ be an absolutely continuous curve. The *length* $L(\gamma)$ of $\gamma$ is defined by

$$L(\gamma) = \sup\{\sum_{i=1}^{k} \operatorname{dist}(\gamma(t_i), \gamma(t_{i+1})) \mid a \leq t_1 < t_2 < \ldots < t_k \leq b\}.$$

**Remark 1.3.10** On a Riemannian manifold we use the Riemannian distance to define the length of an absolutely continuous curve as above. For piecewise continuously differentiable curves $\gamma\colon [a, b] \to M$ it is well known that the above definition coincides with the arc length [26, Prop. 2, p. 158], i.e.

$$L(\gamma) = \int_a^b \|\dot{\gamma}(s)\|\, ds.$$

This also holds for absolutely continuous curves. However, the only reference in the literature of the equality for absolutely continuous curves seems to be [162, p.33].

**Theorem 1.3.11** *Let $M$ be an analytic Riemannian manifold and $\dot{x} \in F(x)$ is an (AC) inclusion. Assume that the Lyapunov function $f$ is a Lipschitz continuous $\mathcal{C}$-function and the stratification consists of $\mathcal{C}$-sets. Then any solution of $\dot{x} \in F(x)$ converges a single point or has empty $\omega$-limit set.*

Again, we derive a bound for the length of the solution from the Łojasiewicz gradient inequality.

**Proof:** Let $\gamma\colon (0,\infty) \to M$ be a solution of $\dot{x} \in F(x)$ and $x^*$ and element of the $\omega$-limit set of $\gamma$. W.l.o.g. assume that $f(x^*) = 0$ and $\gamma$ is non-constant on open intervals. In particular, $\dot{\gamma} \not\equiv 0$ on open intervals. By Proposition 1.3.8 $f \circ \gamma$ is strictly decreasing, and therefore $f \circ \gamma(t) > 0$ for all $t \in (0, \infty)$. We refine the stratification of $f$ to a strong $(a_f)$-stratification $S_j$, $j \in \Lambda$, such that the sign of $f$ is constant on the strata. Applying the Łojasiewicz gradient inequality, Theorem 1.1.22, to $f$, $x^*$ and the strata $S_j$ with $x^* \in \overline{S}_j$ and $f > 0$ on $S_j$, we get a neighborhood $U$ of $x^*$, a constant $C > 0$ and definable, strictly increasing $C^1$ functions $\psi_j\colon \mathbf{R}_+ \to \mathbf{R}_+$, $\psi_j(r) \to 0$ for $r \to 0$, such that for all $j \in \Lambda$, $x \in S_j \cap U$, with $f(x) > 0$,

$$\left\|\operatorname{grad}_j \psi_j \circ f(x)\right\| > C. \tag{1.18}$$

Let $[a,b]$ be a compact interval with $\gamma([a,b]) \subset U$. We denote by $A$ the set of points $t \in [a,b]$, where $\gamma$ is differentiable and $\dot{\gamma}(t) \in T_{\gamma(t)}S_{j(t)}$, $S_{j(t)}$ the stratum of $\gamma(t)$. Since $\gamma$ is absolutely continuous, it is non-differentiable only on a set of measure 0. Furthermore, the set of $t \in [a,b]$ such that $\dot{\gamma}(t)$ is transversal to the stratum of $\gamma(t)$ has measure 0, too. Hence, the set $[a,b] \setminus A$ has measure 0. Consider $\psi_j \circ f \circ \gamma$. By Lemma 1.3.2, the (AC) condition and (1.18) we get for all $t \in A$, $\gamma(t) \in S_j$

$$\begin{aligned} -\frac{d}{dt}(\psi_j \circ f \circ \gamma)(t) &= \left\langle \operatorname{grad}_j(\psi_j \circ f)(\gamma(t)), \dot{\gamma}(t) \right\rangle \\ &\geq \varepsilon \left\|\operatorname{grad}_j(\psi_j \circ f)(\gamma(t))\right\| \|\dot{\gamma}(t)\| \\ &\geq \varepsilon C \|\dot{\gamma}(t)\|. \end{aligned}$$

Since $f$ is Lipschitz, the function $f \circ \gamma$ is absolutely continuous on $[a,b]$. By Proposition 1.3.8 it is strictly decreasing. Since $\psi_j$ is $C^1$, the function $\psi_j \circ f \circ \gamma$ is absolutely continuous, too. Using the monotonicity of $\psi_j$ and

$f \circ \gamma$ we get the following estimate from the equations above

$$\begin{aligned}\int_{[a,b]} \|\dot\gamma(t)\|\, dt &= \int_A \|\dot\gamma(t)\|\, dt \leq \frac{1}{\varepsilon C} \sum_{S_j \cap U \neq \emptyset} \left( - \int_A \frac{d}{dt} \psi_j \circ f \circ \gamma(t) dt \right) \\ &= \frac{1}{\varepsilon C} \sum_{S_j \cap U \neq \emptyset} \psi_j \circ f \circ \gamma(a) - \psi_j \circ f \circ \gamma(b) \\ &\leq \frac{1}{\varepsilon C} \sum_{S_j \cap U \neq \emptyset} \psi_j \circ f \circ \gamma(a)\end{aligned}$$

Note that for $a \to \infty$ the right hand side becomes arbitrarily small. Therefore, $\gamma$ cannot leave $U$ and the length of $\gamma$ is finite, see Remark 1.3.10. Thus $\gamma$ converges to $x^*$. □

**Theorem 1.3.12** *Let $M$ be an analytic Riemannian manifold and $\dot x \in F(x)$ is an (AC) inclusion. Assume that the Lyapunov function $f$ is a $\mathcal{C}$-function and the stratification consists of $\mathcal{C}$-sets. Let $\gamma$ be a piecewise continuously differentiable solution of $\dot x \in F(x)$ with $\{t \mid \dot\gamma(t) = 0\}$ countable. Then $\gamma$ converges a single point or has empty $\omega$-limit set.*

**Proof:** Let $\gamma \colon (0, \infty) \to M$ be a piecewise differentiable solution. By Lemma 1.3.3 we can refine the stratification into a strong $(a_f)$-stratification. As in the proof of Proposition 1.3.6 we can argue that the set $A = \{t \in (0, \infty) \mid f \text{ is not Lipschitz in } \gamma(t)\}$ is closed and countable. Since the function $t \mapsto \int_0^t \|\dot\gamma(s)\|\, ds$ is absolutely continuous, the set $A$ does not contribute to the growth of the length of $\gamma$. For any compact interval $[c, d]$ contained in $(0, \infty) \setminus A$ the function $f \circ \gamma \colon [c, d] \to \mathbf{R}$ is absolutely continuous. We can use the same argument as in the proof of Theorem 1.3.11 to obtain bounds for the length of $\gamma$ on $[c, d]$. Using the monotonicity of $f \circ \gamma$ from Proposition 1.3.8 we get the same bounds for the length of $\gamma$ as in the $f$ Lipschitz case. This yields the convergence of $\gamma$. □

## 1.4 Degenerating Riemannian metrics

In the preceding sections we considered (AC) vector fields with piecewise differentiable Lyapunov functions. In this section we want to examine the question how this piecewise structure can be extended to the underlying space.

We will restrict ourselves to the case that we still have a smooth manifold $M$, but the Riemannian structure is only piecewise defined. To be more precise, we define a piecewise Riemannian metric as follows.

**Definition 1.4.1** Let $M$ be a smooth manifold. A *piecewise $C^p$-Riemannian metric* consists of a $C^p$ stratification $S_j$, $j \in \Lambda$ of $M$ and a family of Riemannian metrics $(g_j)$, $g_j$, on the strata.

Our convergence theorems in the preceding sections already allow non-smooth Riemannian metrics. Usually, Lipschitz or even simple continuity of the Riemannian metric is sufficient. However, with piecewise Riemannian metrics a different type of degeneration comes into play. Towards the border of a stratum $S_j$, the Riemannian metrics can degenerate, i.e. become indefinite or have a singularity.

However, the above definition of a piecewise Riemannian metric lacks any connections between the different $g_j$ on the strata. If we want to consider to asymptotic properties near the boundary of a stratum, e.g. examining the behavior of gradient curves converging to the boundary, we would need some compatibility conditions such that the metrics $g_j$ on the boundary strata govern at least some of the behavior of the metrics near these boundary strata.

**Definition 1.4.2** Let $(M, \langle\cdot,\cdot\rangle)$ be a smooth Riemannian manifold. Assume that we have a piecewise Riemannian metric $(g_j \mid j \in \Lambda)$ on $M$ such that the stratification $S_j$, $j \in \Lambda$ is Whitney-(a). We call the $g_j$ *locally bounded* if for any compact set $K$ we have $\|g_j(x)\| < C_K$ for all $j \in \Lambda$, $x \in K \cap S_j$ , where $\|\cdot\|$ denotes the operator norm with respect to $\langle\cdot,\cdot\rangle$. We call the $g_j$ *compatible* if

- they are locally bounded
- and for any $i, j \in \Lambda$, $S_i \subset \overline{S_j}$, $x \in S_i$, $(x_k) \subset S_j$, with $x_k \to x$, $\alpha = \lim_{k\to\infty} g_j(x_k)$, we have that $\alpha|_{T_xS_i} = g_i(x)$.

For (AC) vector fields the angle condition has only to be satisfied in points where the vector field is tangential to a stratum of the Lyapunov function. Therefore, the definition of (AC) vector fields extends directly to piecewise Riemannian metrics if the stratifications of the Lyapunov function and the piecewise Riemannian metric coincide. The same holds for (AC) differential inclusions. For the sake of completeness, we give the definition of (AC) inclusions in this setting.

**Definition 1.4.3** Let $M$ be a Riemannian manifold, $f\colon M \to \mathbf{R}$ a continuous function and

$$\dot{x} \in F(x),\ F\colon M \to 2^{TM}$$

a differential inclusion on $M$. Furthermore, let $(g_j)$, $g_j = \langle \cdot, \cdot \rangle_j$, be a piecewise, locally bounded, compatible Riemannian metric on $M$ with stratification $S_j$, $j \in \Lambda$. We call $\dot{x} \in F(x)$ an *angle condition (AC) differential inclusion* with associated Lyapunov function $f$ if

- $f$ is non-constant on open sets,
- $f$ is piecewise $C^1$ with domain stratification $S_j$, $j \in \Lambda$, i.e. the domain stratification coincides with the stratification for the piecewise Riemannian metric,
- $F$ is lower semicontinuous,
- for any compact set $K$ there is a constant $\varepsilon > 0$ such that for all $j \in \Lambda$, $x \in S_j \cap K$ and $v \in F(x) \cap T_x S_j$ the estimate

  $$-\left\langle \operatorname{grad}_j f(x), v \right\rangle_j \geq \varepsilon \left\| \operatorname{grad}_j f(x) \right\|_j \|v\|_j \tag{AC}$$

  holds.

With the notion of compatibility as defined above, we get the following convergence theorem.

**Theorem 1.4.4** *Let $M$ be an analytic manifold with compatible, piecewise differentiable Riemannian metric. Furthermore, let $\dot{x} \in F(x)$ be an (AC) inclusion. Assume that the Lyapunov function $f$ is a $\mathcal{C}$-function and the stratification consists of $\mathcal{C}$-sets. Let $\gamma$ be a piecewise continuously differentiable solution of $\dot{x} \in F(x)$ with $\{t \mid \dot{\gamma}(t) = 0\}$ countable. Then $\gamma$ converges a single point or has empty $\omega$-limit set. If $f$ is Lipschitz continuous, then this holds for all solutions.*

To prove the theorem we need the following lemmas.

**Lemma 1.4.5** *Let $M$ be an Riemannian manifold with compatible, piecewise differentiable Riemannian metric $(g_j \mid j \in \Lambda)$, $g_j = \langle\cdot,\cdot\rangle_j$. Furthermore, let $\dot{x} \in F(x)$ be an (AC) inclusion with Lyapunov function $f$ and stratification $S_j$, $j \in \Lambda$. For any compact set $K$ there is a $C_K > 0$ such that for all $j \in \Lambda$ and $x \in S_j \cap K$*

$$\left\|\operatorname{grad}_j f(x)\right\|_j \geq C_K \left\|\operatorname{grad} f|_{S_j}\right\|,$$

*where $\|\cdot\|_j$, $\operatorname{grad}_j$ denote norm and gradient with respect to the $g_j$ and $\|\cdot\|$, $\operatorname{grad} f|_{S_j}$ the gradient and norm with respect to the Riemannian metric on $M$.*

**Proof:** There is a positive, semidefinite map $H(x)\colon T_xM \to T_xM$ with $\langle v,w\rangle_j = \langle v, H(x)w\rangle$ for all $v,w \in T_xM$. The boundedness of the $g_j$ implies that for any $K$ there is a constant $C_K > 0$ such that for all $x \in K$

$$\|H(x)\| \leq C_K^2,$$

where $\|\cdot\|$ denotes the operator norm with respect to the Riemannian metric. Hence, for all $j \in \Lambda$, $x \in K \cap S_j$

$$C_k \left\|\operatorname{grad}_j f(x)\right\|_j \geq \left\|H(x)\operatorname{grad}_j f(x)\right\| = \left\|\operatorname{grad} f|_S(x)\right\|.$$

□

**Lemma 1.4.6** *Let $M$ be a Riemannian manifold with compatible, piecewise differentiable Riemannian metric. Furthermore, let $\dot{x} \in F(x)$ be an (AC) inclusion with Lyapunov function $f$. Assume the stratification $S_j$, $j \in \Lambda$ is strong $(a_f)$. Let $i \in \Lambda$ and $x \in S_i$ with $df|_{S_i}(x) = 0$. Then $F(x) \cap T_xS_i \subset \{0\}$.*

**Proof:** Let $v \in F(x) \cap T_xS_i$. Since the piecewise Riemannian metric is bounded, there is a smooth Riemannian metric $\langle\cdot,\cdot\rangle$ on $M$ such that $\|g_i\|$ is bounded on relatively compact sets. We choose $j \in \Lambda$ and a sequence $(x_k) \subset S_j$, $x_k \to x$ with $df|_{S_j}(x_k) \neq$ for all $k$. This is possible as $f$ is non-constant on open sets. By the lower semicontinuity of $F$ there is a sequence $v_k \in F(x_k)$ with $v_k \to v$. The (AC) condition and Lemma 1.4.5 yield constants $\varepsilon, C > 0$ such that

$$-df(x_k)(v_k) \geq \varepsilon \|v_k\|_j \left\|\operatorname{grad}_j f(x_k)\right\|_j \geq C \|v_k\|_j \left\|\operatorname{grad} f(x_k)\right\|$$

where grad, $\|\cdot\|$ denote the norm and gradient on $S_j$ with respect to $\langle \cdot, \cdot \rangle$. Thus either the (AC) condition with respect to $\langle \cdot, \cdot \rangle$ holds for the sequences $x_k$ and $v_k$ or $\|v_k\|_j \to 0$. In the first case $v = 0$ follows from the Thom condition as for smooth metrics. In the second case the compatibility and boundedness of the piecewise Riemannian metric implies that $v = 0$. □

**Proof of Theorem 1.4.4:** By the estimate in Lemma 1.4.5, the proof is nearly identical to the case of a smooth metric presented in section 1.3. The necessary lemmas and propositions also hold for piecewise, compatible Riemannian metrics. We only have to note that for proving the analogue of Lemma 1.3.4 and Proposition 1.3.8, we need the above Lemma 1.4.6. □

This raises of course the question what one can do if the Riemannian metrics on the strata are unbounded. This situation comes up in important applications like Thom's gradient conjecture or Hessian metrics from barrier functions as used in linear programming. We approach such problems by considering piecewise positive definite maps of the tangent spaces. More specifically we use the following definition.

**Definition 1.4.7** Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. A *piecewise positive definite tangent space map* of $M$ consists of a stratification $S_j$, $j \in \Lambda$, of $M$ and a family of positive definite maps $H_j(x)\colon T_xS_j \to T_xS_j$, smooth in $x$. Assume that the stratification satisfies the Whitney-(a) condition. We call the $H_j$ *locally bounded* if for any compact set $K$ in $M$ there is a constant $C_K > 0$ such that for all $j \in \Lambda$ and all $x \in K \cap S_j$

$$\|H_j(x)\| < C_k,$$

$\|\cdot\|$ denoting the operator norm with respect to $\langle \cdot, \cdot \rangle$. We call the $H_j$ *compatible* if

- the $H_j$ are locally bounded
- and for any $i, j \in \Lambda$, $S_i \subset \overline{S_j}$, $(x_k) \subset S_j$, $x \in S_i$ with $x_k \to x$ $L = \lim_{k\to\infty} H_j(x_k)$, we have that $L|_{T_xS_i} = H_i(x)$.

**Remark 1.4.8** For a function $f\colon M \to \mathbf{R}$ a piecewise positive definite tangent map $H_j$ gives a gradient-like vector field

$$X(x) = -H_j(x)\,\mathrm{grad}\, f(x).$$

We can define a piecewise Riemannian metric $g_j = \langle \cdot, \cdot \rangle_j$ by setting $\langle v, w \rangle_j = \langle H_j(x)^{-1}v, w \rangle$ where $H_j(x)^{-1}$ denotes the inverse of $H_j(x)$ on $T_xS_j$. Then

$X(x)$ can be considered as the gradient of $f$ with respect to $g_j$. The difference to the case of piecewise, compatible, locally bounded Riemannian metrics above is that here we pose compatibility and boundedness conditions on the $H_j$ instead of $H_j^{-1}$ to cover the case of unbounded Riemannian metrics.

We will now give some examples where such piecewise positive definite tangent maps show up.

**Example 1.4.9** One instance is the Thom gradient conjecture. Assume that $f\colon \mathbf{R}^n \to \mathbf{R}$ is an analytic function and $\gamma$ an integral curve of $-\operatorname{grad} f$ with $\omega(\gamma) = \{0\}$. As mentioned in the introduction the Thom gradient conjecture claims that the limit

$$\lim_{t\to 0} \frac{\gamma(t)}{\|\gamma(t)\|} \tag{1.19}$$

exists. A well-known approach, suggested by R. Thom [153], to examine this conjecture is to blow-up $\mathbf{R}^n$ at 0. This is done in the following way, cf. [148, 153]. The function $\psi(x, r) = rx$ is a blow-up of $\mathbf{R}^n$ around 0 to the cylinder $Z = S^{n-1} \times \mathbf{R} \subset \mathbf{R}^n \times \mathbf{R}$. Note that we will in the following represent the tangent spaces of $Z$ as subspaces of $\mathbf{R}^{n+1}$. Via the blow-up we can both pull back $f$ to $f \circ \psi$ and $-\operatorname{grad} f$ to

$$X(x,r) = \begin{pmatrix} r^{-1}(\operatorname{grad} f(rx) - \langle x, \operatorname{grad} f(rx)\rangle\, x) \\ \langle x, \operatorname{grad} f(rx)\rangle \end{pmatrix}$$
$$= -\begin{pmatrix} r^{-2}I & rx \\ rx^\top & 1 \end{pmatrix} \begin{pmatrix} r \operatorname{grad} f(rx) \\ \langle x, \operatorname{grad} f(rx)\rangle \end{pmatrix}$$

on $Z \setminus S^{n-1} \times \{0\}$. [28] Note, that the function $F\colon Z \to \mathbf{R}$ has the gradient

$$\operatorname{grad} F(x,r) = \begin{pmatrix} r \operatorname{grad} f(rx) \\ \langle x, \operatorname{grad} f(rx)\rangle \end{pmatrix}$$

with respect to the Riemannian metric on $Z$ induced by the Euclidean one. Hence, $X$ has the form $X = -H(x,r) \operatorname{grad} F(x,r)$ on on $Z \setminus S^{n-1} \times \{0\}$ where

$$H(x,r) := \begin{pmatrix} r^{-2}I & rx \\ rx^\top & 1 \end{pmatrix} \in \mathbf{R}^{(n+1)\times(n+1)}.$$

[28] Note, that $X$ is the gradient of $f \circ \psi$ with respect to the Riemannian metric $\left\langle \langle (v,h),(w,k)\rangle_{(x,r)} \right\rangle = r^2 \langle v, w\rangle + hk$.

As we have pulled back the vector field $-\operatorname{grad} f$ to $X$ on $Z \setminus S^{n-1} \times \{0\}$, all solutions of $-\operatorname{grad} f$ are mapped onto solutions of $X$. Thus, if all solutions $\tilde{\gamma}$ of $X$, with $\tilde{\gamma}(0) \in \mathbf{R}^n \times \mathbf{R}_+$, $\tilde{\gamma}(t) \to S^{n-1} \times \{0\}$, converge to single points, then the limit (1.19) exists and the gradient conjecture holds.

Note, that multiplying a vector field with a positive scalar function does not change the trajectories. Therefore, we can consider for convergence questions instead the vector field

$$\tilde{X}(x,r) := r^2 X(x,r).$$

The family of maps $(H(x,r) \in \mathbf{R}^{(n+1)\times(n+1)})$, induces a compatible, bounded, piecewise positive definite tangent space map on $Z$ with the stratification $S_1 = S^{n-1} \times \{0\}$, $S_2 = Z \setminus S_1$, and

$$H_1(x,0) = \operatorname{Id}_{T_{(x,0)}S_1},\ H_2(x,r) = \begin{pmatrix} I & r^2 x \\ r^2 x^\top & r^2 \end{pmatrix} \in \mathbf{R}^{(n+1)\times(n+1)}.$$

Note, that although we represent $H_2(x,r)$ as a $(n+1)\times(n+1)$ matrix, it is nevertheless easily seen that $\langle v, H_2(x,r)v\rangle > 0$ for all $(x,r) \in Z$, $r \neq 0$, $v \in T_{(x,r)}Z$. Then

$$\tilde{X}(x,r) = \begin{cases} -H_1(x,r) \operatorname{grad} F(x,r) & \text{for } r = 0 \\ -H_2(x,r) \operatorname{grad} F(x,r) & \text{for } r \neq 0 \end{cases}.$$

This shows that the Thom gradient conjecture is a special case of the question, whether integral curves of a vector field $-H_j \operatorname{grad} f$ converge, with $H_j$ a compatible, locally bounded, piecewise positive definite tangent map.

**Example 1.4.10** As a different example, we consider the logarithmic barrier function

$$h(x) = -\sum_{i=1}^{n} \log x_i,$$

on the positive orthant $\mathbf{R}^n_+ = \{(x_1,\ldots,x_n) \mid x_i > 0\}$, see [134]. This function has the Hessian

$$\operatorname{Hess}_x h = \operatorname{diag}(x_1^{-2},\ldots,x_n^{-2})$$

The inverse of the Hessian $H(x) = (\operatorname{Hess} h(x))^{-1}$ has the form $H(x) = \operatorname{diag}(x_1^2,\ldots,x_n^2)$. Its extension to the whole $\mathbf{R}^n$ yields a compatible, locally bounded piecewise positive definite tangent map $H_I$, the strata being the orthants and subspaces of the form $S_I = \{(x_1,\ldots,x_n) \in \mathbf{R}^n \mid ; \forall i \in I, x_i = 0\}$,

$I \subset \{1, \ldots, n\}$, $I \neq \emptyset$, and $H_I$ defined as the restriction of $\operatorname{diag}(x_1^2, \ldots, x_n^2)$ to $S_I$.

Differential equations of the form

$$\dot{x} = -(\operatorname{Hess}_x h)^{-1} \operatorname{grad} f(x),$$

with $h$ a barrier function, can be interpreted as continuous-time, interior point optimization methods, see [8, 10, 32]. In particular, for linear and semidefinite programming and optimization on symmetric cones, there has been a considerable interested in systems of this and similar types, see for example [17–19, 48, 58, 59, 77, 122].

The barrier function above on $\mathbf{R}_+^n$ belongs to the class of self-scaled barriers on symmetric cones [134]. However, not every self-scaled barrier on a symmetric cone has a Hessian such that its inverse can be extended to a compatible, piecewise positive definite tangent map. In fact, this can be only done if and only if the cone is isomorphic to $\mathbf{R}_+^n$. The interested reader can find this in Appendix A.1.

Unlike piecewise, compatible Riemannian metrics, we do not get the convergence even of integral curves of $H_j \operatorname{grad} f$, with $H_j$ compatible, locally bounded, piecewise positive definite tangent maps. The following example illustrates this.

**Example 1.4.11** Let $\mathcal{H} = \{(x, r) \in \mathbf{R}^{n+1} \mid r > 0\}$ a half space in $\mathbf{R}^{n+1}$ with $n \geq 2$. For any bounded vector field $X(x) \in \mathbf{R}^n$ we can define the tangent map

$$H_1(x, r) = \begin{pmatrix} r^2 & rX(x)^\top \\ rX(x) & I_n + X(x)X(x)^\top \end{pmatrix}$$

on $\mathcal{H}$. Using the Schur complement [93] it can be easily seen that $H_1(x, r)$ is positive definite. We can extend $H_1$ to a compatible, piecewise positive definite tangent map on $\mathbf{R}^{n+1}$. On $\partial\mathcal{H} = \mathbf{R}^n \times \{0\}$ we define the tangent map

$$H_2(x, 0)(v, 0) := (I_n + X(x)X(x)^\top)v, \; v \in \mathbf{R}^n.$$

On the opposite half space $-\mathcal{H} = \{(x, r) \in \mathbf{R}^{n+1} \mid r < 0\}$ the tangent map is constructed by symmetry $H_3(x, r) = H_1(x, -r)$. Now consider the linear function $f(x, r) = cr$, $c > 0$ and the vector field

$$Y(x, r) = \begin{cases} -H_1(x, r) \operatorname{grad} f(x, r) & \text{for } r > 0 \\ -H_2(x, r) \operatorname{grad} f(x, r) & \text{for } r = 0 \\ -H_3(x, r) \operatorname{grad} f(x, r) & \text{for } r < 0 \end{cases}.$$

Then

$$Y(x,r) = -c \begin{pmatrix} r^2 \\ |r|\, X(x) \end{pmatrix},$$

which is a Lipschitz continuous vector field on $\mathbf{R}^{n+1}$. Now we choose a vector field $X(x)$ which has a periodic orbit $\Theta$. Then there are solutions of $Y(x,r)$ which will converge to the entire periodic orbit for $t \to \infty$, in fact any solution in $\Theta \times (\mathbf{R} \setminus \{0\})$ shows this behavior. Therefore, for an analytic function $f$ and a locally bounded, compatible, piecewise positive definite tangent map $H_j$ the convergence theorems do not hold in general.

In dimension 2, the proof of Thom's gradient conjecture can be proven by a simple and direct argument, see [102, Prop. 2.1]. One would expect that for vector fields $-H_j(x)\,\mathrm{grad}\, f(x)$ a similar argument yields the convergence of integral curves on two dimensional manifolds. However, this is not the case, as the following example shows.

**Example 1.4.12** Let $H_1$, $H_2$, $H_3$ be the locally bounded, compatible, piecewise positive definite tangent map on $\mathbf{R}^{n+1}$ from example 1.4.11, $H_1$ defined on $\mathcal{H}$, $H_2$ on $\partial\mathcal{H}$, $H_3$ on $-\mathcal{H}$. We choose $n = 2$ and

$$X(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

As function $f$ we take $f(x,r) = cr$, $c > 0$ as in the example above. Consider the vector field

$$Y(x,r) = -H_j\, \mathrm{grad}\, f(x,r).$$

The cylinders $\mathbf{R} \times \rho S^1$, $\rho > 0$ are invariant sets of this vector field with the integral curves on the positive halfspace $\mathcal{H}$ spiralling along them while they converge to the circles $\{0\} \times \rho S^1$. If we restrict ourselves now to the cylinder $Z = \mathbf{R} \times S^1$, then we have there a vector field with integral curves spiralling along the cylinder. Since $\mathrm{grad}\, f$ is tangent to $Z$ and the restriction of $H_j$ to $Z$ gives again a locally bounded, compatible, piecewise positive definite tangent map, we have constructed the desired counterexample on a 2 dimensional manifold.

# Chapter 2

# Time-discrete gradient-like optimization methods

In this chapter, we consider discrete-time, gradient-like optimization methods on manifolds. The main goal is to develop a uniform convergence theory for gradient-like algorithms on manifolds and singular spaces. We discuss algorithms for the following three problem types:

- a smooth cost function on a manifold,
- a smooth cost function on a manifold restricted to a non-smooth constraint set,
- a non-smooth, Lipschitz continuous cost function on a manifold.

For the smooth cost functions, we use a local parameterization approach as employed in e.g. [3,5,118–120,143]. However, the families of parameterization will be subject only to weak regularity conditions, especially, substantially weaker ones than conditions on the retractions of Shub [143], For example, the families will not depend continuously on the base points. We give global convergence results for all of these algorithms. For non-smooth cost functions, we discuss both a Riemannian and a local parameterization approach. We show the convergence of gradient descent algorithms for both approaches. The descent algorithms for non-smooth cost functions are illustrated by some sphere packing problems on adjoint orbits.

# 2.1 Optimization algorithms on manifolds

## 2.1.1 Local parameterizations

We start with the discussion of families of parameterizations and suitable regularity criteria.

Without further notice, $M$ will be a smooth Riemannian manifold with Riemannian metric $g = \langle \cdot, \cdot \rangle$. The Riemannian distance between $x, y$ on $M$ is denoted by $\operatorname{dist}(x, y)$ and the exponential map at $x \in M$ by $\exp_x$. For a detailed discussion and definitions of Riemannian geometry, we refer the reader to the standard literature like e.g. [26, 51, 97, 106].

We call a function $\phi\colon M \times \mathbf{R}^n \to M$ a *family of smooth parameterizations* if for all $x \in M$ the map $y \mapsto \phi(x, y)$ is smooth. We use the notation $\phi_x$ for the function $\phi_x\colon \mathbf{R}^n \to M$, $y \mapsto \phi(x, y)$. A family of parameterizations will be denoted by $(\phi_x)$ or $(\phi_x\colon \mathbf{R}^n \to M)$. If we restrict the parameterizations to a subset $X \subset M$, we use the notations $(\phi_x \mid x \in X)$ or $(\phi_x\colon M \to \mathbf{R}^n \mid x \in X)$.

For the convergence results on descent iterations defined later, we need some regularity conditions on the family of parameterizations. First we fix some notations for operator norms of linear maps $\mathbf{R}^m \to T_xM$, $T_xM \to \mathbf{R}^m$, $m \in \mathbf{N}$. Let $A : \mathbf{R}^m \to T_xM$, $B : T_xM \to \mathbf{R}^m$, we denote by $\|A\|$, $\|B\|$ the operator norms

$$\begin{aligned}\|A\| &= \sup\{\|Av\|_g \mid v \in \mathbf{R}^m, \|v\|_e = 1\},\\ \|B\| &= \sup\{\|Bv\|_e \mid v \in T_xM, \|v\|_g = 1\},\end{aligned}$$

where $\|\cdot\|_g$ is the norm induced by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_e$ the Euclidean one on $\mathbf{R}^m$.

We now introduce some special notions of uniform continuity and equicontinuity.

**Definition 2.1.1** Let $f\colon M \to \mathbf{R}$ be a smooth function and $X \subset M$ a set. Assume that the injectivity radius[1] of all $x \in X$ is uniformly bounded below by a constant $r > 0$. We call the differential $df$ *uniformly* exp-*continuous* on $X$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$, $y \in T_xM$

$$\|y\| < \delta \text{ implies } \|d(f \circ \exp_x)(0) - d(f \circ \exp_x)(y)\| < \varepsilon.$$

Let $(\phi_x)$ a family of smooth parameterizations on $M$. We call the family $(\phi_x)$ *equicontinuous at* 0 on $X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in X$, $y \in \mathbf{R}^n$

$$\|y\| < \delta \text{ implies } \operatorname{dist}(\phi_x(y), \phi_x(0)) < \varepsilon.$$

We call the family of tangent maps $(T\phi_x\colon \mathbf{R}^n \to T_xM)$ exp-*equicontinuous at* 0 on $X$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$, $y \in \mathbf{R}^n$, with $\|y\| < \delta$,

$$\operatorname{dist}(\phi_x(y), x) < r \text{ and } \left\|T_0(\exp_x^{-1} \circ \phi_x) - T_y(\exp_x^{-1} \circ \phi_x)\right\| < \varepsilon$$

hold.

Note that $\exp_x^{-1} \circ \phi_x\colon \mathbf{R}^n \to T_xM$ is a smooth map between open sets in vector spaces. Hence, the difference $T_0(\exp_x^{-1} \circ \phi_x) - T_y(\exp_x^{-1} \circ \phi_x)$ is well-defined.

We can now give the definition of our standard assumptions on the family of parameterizations.

[1] Note, that if the injectivity radius of $x$ is $s$, then $\exp_x$ is defined for all $v \in T_xM$ with $\|v\| < s$ and a diffeomorphism of $\{v \in T_xM \mid \|v\| < s\}$ onto its image [97].

**Definition 2.1.2** Let $(\phi_x)$ be a family of parameterizations and $X \subset M$ a set. Assume that the injectivity radius of all $x \in X$ is uniformly bounded from below by a constant $r > 0$. We say that the family satisfies the *standard assumptions on $X$*, if it is a family of smooth parameterizations that satisfies the following conditions:

- $\phi_x(0) = x$ for all $x \in X$,
- there is a constant $C > 0$ such that the tangent maps $T_0\phi_x \colon \mathbf{R}^n \to T_xM$ satisfy $\|T_0\phi_x\| < C$ for all $x \in X$,
- for all $x \in X$ the tangent maps $T_0\phi_x$ are invertible, and there is a constant $C > 0$ such that for all $x \in X$: $\left\|(T_0\phi_x)^{-1}(x)\right\| < C$,
- $(T\phi_x)$ is exp-equicontinuous at $0$.

If the family of parameterizations satisfies the standard assumptions on $M$, then we say simply that it satisfies the *standard assumptions*.

**Remark 2.1.3** Since for all $x \in M$ one has $T_0 \exp_x = \mathrm{Id}_{T_xM}$, it follows that $\|T_0\phi_x\| = \|T_0(\exp_x^{-1} \circ \phi_x)\|$ and $\|(T_0\phi_x)^{-1}\| = \left\|(T_0(\exp_x^{-1} \circ \phi_x))^{-1}\right\|$. Therefore the standard assumptions can be viewed as a regularity condition with respect to the smooth map $\exp \colon TM \to M$ with $T_0 \exp|_{T_xM} = \mathrm{Id}_{T_xM}$. But then we could replace $\exp_x$ by any other smooth map $\psi \colon TM \to M$, with $T_0\psi|_{T_xM} = \mathrm{Id}_{T_xM}$, i.e. a retraction used by Shub [143]. The would yield a more general version of the standard assumptions defined above. In fact, all convergence results later can also be obtained in this setting provided that the conditions on the cost function would be suitably adapted.

**Remark 2.1.4** One might wonder, why we do not use a smooth family of parameterizations or the retractions of Shub. However, a smooth family of smooth parameterizations will not suitable under the standard assumptions for optimization purposes. The smoothness and the fact, that the parameterizations are local diffeomorphisms would imply that the tangent bundle of the manifold is trivial. Hence, it could be only applied to a limited number of applications.

The retractions of Shub [143] avoid this problem, as one has a smooth map from the tangent bundle to the manifold. However, a computer implementation requires to process the data in a coordinate form. This can achieved in two different ways: On the one hand one can use local coordinates on the tangent spaces. But then one actually works with parameterizations $\mathbf{R}^n \to M$ instead of $T_xM \to M$. Hence, there is a priori no reason to assume an underlying retraction. On the other hand, one could embed the tangent bundle in a trivial vector space bundle over the manifold, e.g. see [37, 76]. However, this increases the dimension of the problem leading to increased storage requirements and eventually reduced computational efficiency. Therefore, there is no reason to restrict oneself to the retractions proposed by Shub.

The standard assumptions imply that the family of parameterizations is equicontinuous at 0. We give a slightly more general statement of this fact, as it will be needed later when we consider optimization on non-smooth sets.

**Proposition 2.1.5** *Let $X \subset M$ be a set such that the injectivity radius of all $x \in X$ is uniformly bounded from below by a constant $r > 0$. Assume that $(\phi_x)$ is a family of parameterizations satisfying the following conditions:*

- *$\phi_x(0) = x$ for all $x \in X$.*
- *There is a constant $C > 0$ such that the tangent maps $T_0\phi_x\colon \mathbf{R}^n \to T_xM$ satisfy $\|T_0\phi_x\| < C$ for all $x \in X$.*
- *$(T\phi_x)$ is* exp*-equicontinuous at* 0.

*Then the family $(\phi_x)$ is equicontinuous at 0 on $X$. In particular, there are constants $\delta > 0$, $C > 0$ such that for all $y \in \mathbf{R}^n$ with $\|y\| < \delta$ the estimate*

$$\operatorname{dist}(\phi_x(y), x) < C\,\|y\| \quad \textit{holds.}$$

**Proof:** Let $r > 0$ denote the uniform lower bound for the injectivity radius of the $x \in X$. We consider the family of maps $(\psi_x)$

$$\psi_x := \exp_x^{-1} \circ \phi_x,$$

where $\exp_x$ is the exponential map at $x$. Note that each $\psi_x$ is well-defined and smooth on a ball $B_{R_x}(0) = \{y \in \mathbf{R}^n \mid \|y\|_e < R_x\}$, with

$$R_x := \sup\{s \in \mathbf{R}_+ \mid \forall y \in \mathbf{R}^n, \|y\|_e \leq s\colon \operatorname{dist}(\phi_x(y), x) < r\},$$

as $\exp_x$ is a diffeomorphism onto the set $\phi_x(B_{R_x}(0))$. Hence $\psi_x$ is a map $\psi_x\colon B_{R_x}(0) \to T_xM$. Since $\exp_x$ and $\phi_x$ are local diffeomorphism at 0, all $R_x$

must be positive. Fix $x \in M$ and $t > 0$, $y \in \mathbf{R}^n$ with $\|ty\|_e < R_x$. From the identity

$$\psi_x(ty) = \int_0^t D\psi_x(sy)yds,$$

we get that

$$\phi_x(ty) = \exp_x(\psi_x(ty)) = \exp_x\left(\int_0^t T_{sy}(\exp_x^{-1} \circ \phi_x)yds\right).$$

As $\operatorname{dist}(\phi_x(ty), x)$ is smaller than the injectivity radius of $x$, a standard consequence of the Gauss Lemma [97, Cor. 4.2.3, 4.2.4] shows that

$$\operatorname{dist}(\phi_x(ty), x) = \| \int_0^t T_{sy}(\exp_x^{-1} \circ \phi_x)yds\|.$$

This yields the estimates

$$\begin{aligned}\operatorname{dist}(\phi_x(ty), x) &\leq \|y\|_e \int_0^t \left\|T_{sy}(\exp_x^{-1} \circ \phi_x)\right\| ds \\ &\leq \|ty\|_e \left(\left\|T_0(\exp_x^{-1} \circ \phi_x)\right\| + \max_{0 \leq s \leq t} \left\|T_0(\exp_x^{-1} \circ \phi_x) - T_{sy}(\exp_x^{-1} \circ \phi_x)\right\|\right)\end{aligned}$$

By the assumptions on $(\phi_x)$, the family $(T\phi_x)$ is exp-equicontinuous at 0 on $X$. Moreover, there is a $C > 0$ with $\|T_0(\exp_x^{-1} \circ \phi_x)\| = \|T_0\phi_x\| < C$ for all $x \in X$. Thus for any $\varepsilon > 0$ we can choose a $\rho > 0$ independent of $x$ such that for all $\delta \in (0, \rho)$, $x \in X$ and $y \in \mathbf{R}^n$,

$$\|y\|_e < \min\{\delta, R_x\}$$

implies

$$\operatorname{dist}(\phi_x(y), x) \leq \min\{\delta, R_x\}(C + \varepsilon). \tag{2.1}$$

We have to prove that there is a uniform lower bound $R > 0$ for $R_x$. By choosing $\varepsilon = C$ in inequality (2.1), there exists a $\delta \in (0, \min\{\rho, \frac{r}{4C}\})$ such that for all $x \in X$, $y \in \mathbf{R}^n$,

$$\|y\|_e < \min\{\delta, R_x\}$$

implies

$$\operatorname{dist}(\phi_x(y), x) \leq 2C \cdot \min\{\delta, R_x\}.$$

For this $\delta$ we get that for all $x \in X$ and $y \in \mathbf{R}^n$

$$\|y\|_e < \min\{\delta, R_x\}$$

implies

$$\operatorname{dist}(\phi_x(y), x) < 2C \cdot \min\{\delta, R_x\} < \frac{r}{2}.$$

As each $\phi_x$ is smooth and defined on the whole $\mathbf{R}^n$, we can define a family of strictly increasing, continuous, surjective functions $(\sigma_x \colon (0, R_x] \to (0, r])$ by

$$\sigma_x(t) := \sup\{\operatorname{dist}(\phi_x(y), x) \mid \|y\|_e \leq t\}.$$

This yields for all $x \in M$ that

$$\sigma_x(\min\{\delta, R_x\}) < \frac{r}{2}$$

and thus

$$\min\{\delta, R_x\} < R_x.$$

Therefore, $\delta < R_x$ for all $x \in M$ and we get the uniform lower bound $R = \delta > 0$ for $R_x$.

Thus we can choose for any $\tilde{\varepsilon} > 0$ a $\delta \in (0, \min\{\rho, R, \frac{\varepsilon}{2C}\})$ such that for all $x \in X$, $y \in \mathbf{R}^n$,

$$\|y\|_e < \delta$$

implies

$$\operatorname{dist}(\phi_x(y), x) \leq 2\delta C \leq \tilde{\varepsilon}.$$

Therefore, the family $(\phi_x)$ is equicontinuous at 0 on $X$. In particular, for a $\delta \in (0, \min\{\rho, R\})$ and all $x \in X$, $y \in \mathbf{R}^n$

$$\|y\| \in (\frac{\delta}{2}, \delta) \text{ implies } \operatorname{dist}(\phi_x(y), x) \leq 2C\delta \leq 4C\|y\|.$$

Hence, for all $x \in X$ and $y \in \mathbf{R}^n$ with $\|y\|_e \in (0, \min\{\rho, R\})$ the inequality

$$\operatorname{dist}(\phi_x(y), x) < 4C\|y\|_e$$

holds. This yields the estimate on $\operatorname{dist}(\phi_x(y), x)$. □

For our convergence proofs later, we will need the following lemma on the equicontinuity of $d(f \circ \phi_x)$ for a real valued function $f \colon M \to \mathbf{R}$ and a family of parameterizations $(\phi_x : \mathbf{R}^n \to M)$.

**Lemma 2.1.6** *Let $X \subset M$ be a set such that the injectivity radius of all $x \in X$ is uniformly bounded from below by a constant $r > 0$. Further, let $(\phi_x)$ be a family of smooth parameterizations satisfying the following conditions:*

- $\phi_x(0) = x$ *for all* $x \in X$.
- *There is a constant $C > 0$ such that the tangent maps $T_0\phi_x \colon \mathbf{R}^n \to T_xM$ satisfy $\|T_0\phi_x\| < C$ for all $x \in X$.*
- $(T\phi_x)$ *is* exp*-equicontinuous at* $0$.

*Assume that $f \colon M \to \mathbf{R}$ is a smooth function with uniformly* exp*-continuous differential $df$ on $X$ and uniformly bounded $\|df(x)\|$ on $X$. Then the family of differentials $(d(f \circ \phi_x))$ is equicontinuous at $0$ on $X$, i.e. for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $y \in \mathbf{R}^n$, $x \in X$*

$$\|y\|_e \leq \delta \text{ implies } \|d(f \circ \phi_x)(y) - d(f \circ \phi_x)(0)\| < \varepsilon,$$

*$\|\cdot\|_e$ denoting the Euclidean norm on $\mathbf{R}^n$ and its operator norm for linear forms $\mathbf{R}^n \to \mathbf{R}$.*

**Proof:** Let $r > 0$ be the uniform lower bound of the injectivity radius of all $x \in X$. Fix an $x \in X$. Assume that $y \in \mathbf{R}^n$ with $\operatorname{dist}(x, \phi_x(y)) < r$. We denote by $z \in T_xM$ the vector $z := \exp_x^{-1}(\phi_x(y))$. Then

$$\begin{aligned}
&d(f \circ \phi_x)(y) - d(f \circ \phi_x)(0) \\
&\quad = df(\phi_x(y)) \circ T_y\phi_x - df(x) \circ T_0\phi_x \\
&\quad = df(\phi_x(y)) \circ T_z \exp_x \circ T_{\phi_x(y)} \exp_x^{-1} \circ T_y\phi_x - df(x) \circ T_0 \exp_x \circ T_x \exp_x^{-1} \circ T_0\phi_x \\
&\quad = (df(\phi_x(y)) \circ T_z \exp_x - df(x) \circ T_0 \exp_x) \circ T_{\phi_x(y)} \exp_x^{-1} \circ T_y\phi_x \\
&\qquad + df(x) \circ T_0 \exp_x \circ \left(T_{\phi_x(y)} \exp_x^{-1} \circ T_y\phi_x - T_x \exp_x^{-1} \circ T_0\phi_x\right) \\
&\quad = (d(f \circ \exp_x)(z) - d(f \circ \exp_x)(0)) \circ T_{\phi_x(y)} \exp_x^{-1} \circ T_y\phi_x \\
&\qquad + df(x) \circ T_0 \exp_x \circ \left(T_y(\exp_x^{-1} \circ \phi_x) - T_0(\exp_x^{-1} \circ \phi_x)\right).
\end{aligned}$$

This yields

$$\begin{aligned}
&\|d(f\circ\phi_x)(y)-d(f\circ\phi_x)(0)\| \\
&\quad\le\ \|d(f\circ\exp_x)(z)-d(f\circ\exp_x)(0)\|\,\left\|T_{\phi_x(y)}\exp_x^{-1}\circ T_y\phi_x\right\| \\
&\qquad +\|df(x)\|\,\left\|T_y(\exp_x^{-1}\circ\phi_x)-T_0(\exp_x^{-1}\circ\phi_x)\right\| \\
&\quad\le\ \|df(x)\|\,\left\|T_y(\exp_x^{-1}\circ\phi_x)-T_0(\exp_x^{-1}\circ\phi_x)\right\| \\
&\qquad +\|d(f\circ\exp_x)(z)-d(f\circ\exp_x)(0)\| \\
&\qquad\ \cdot\left(\left\|T_0\left(\exp_x^{-1}\circ\phi_x\right)\right\|+\left\|T_0\left(\exp_x^{-1}\circ\phi_x\right)-T_y\left(\exp_x^{-1}\circ\phi_x\right)\right\|\right).
\end{aligned}$$

By a standard corollary of the Gauss Lemma [97, Cor. 4.2.3, 4.2.4], the estimate $\mathrm{dist}(x,\phi_x(y))<r$ implies $\|z\|=\mathrm{dist}(x,\phi_x(y))$. Since $(\phi_x)$ is equicontinuous at 0 on $X$, see Proposition 2.1.5, we can choose for any $\varepsilon>0$ a $\delta>0$ with $\delta<r$ such that for all $x\in X$, $y\in\mathbf{R}^n$, $\|y\|_e<\delta$ implies $\|z\|=\mathrm{dist}(x,\phi_x(y))<\varepsilon$. We denote by $C_1,C_2>0$ the uniform bounds for $\|T_0\phi_x\|$ and $\|df(x)\|$ on $X$. Then uniform exp-continuity of $df$ on $X$, the exp-equicontinuity of $T\phi_x$ at 0 on $X$ and equicontinuity of $(\phi_x)$ at 0 on $X$ yield that for any $\varepsilon>0$, we can choose a $\delta>0$ such that for all $x\in X$, $y\in\mathbf{R}^n$, with $\|y\|_e<\delta$,

$$\|d(f\circ\exp_x)(z)-d(f\circ\exp_x)(0)\|<\varepsilon$$

and

$$\left\|T_0\left(\exp_x^{-1}\circ\phi_x\right)-T_y\left(\exp_x^{-1}\circ\phi_x\right)\right\|<\varepsilon$$

hold. Thus, if $\delta<r$ we get by the calculations above that for all $x\in X$, $y\in\mathbf{R}^n$, $\|y\|_e<\delta$ implies that

$$\|d(f\circ\phi_x)(y)-d(f\circ\phi_x)(0)\|<C_2\varepsilon+\varepsilon(C_1+\varepsilon).$$

This yields the equicontinuity of the family of differentials $(d(f\circ\phi_x))$ at 0.
□

**Lemma 2.1.7** *Let $X\subset M$ be a compact set. Assume that $(\phi_x)$ is a family of smooth parameterizations which satisfies the standard assumptions on $X$. Then there is a $r>0$ such that the functions $\tau_x:\mathbf{R}^n\to\mathbf{R}$*

$$\tau_x(y):=\left\|(T_y\phi_x)^{-1}\right\|$$

*are well-defined for $y \in \mathbf{R}^n$, $\|y\| < r$. Furthermore, the family $(\tau_x)$ is equicontinuous at 0, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < r$,*

$$\|y\| < \delta \text{ implies } |\tau_x(y) - \tau_x(0)| < \varepsilon.$$

**Proof:** Let us first consider a linear map $T\colon H_1 \to H_2$ between two finite dimensional Hilbert spaces $H_1, H_2$ with inner products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$. Assume that $T$ is invertible with $\|T^{-1}\| < C$ for a constant $C > 0$, where $\|T^{-1}\|$ denotes the operator norm. Let $A\colon H_2 \to H_1$ be a linear map. By standard results on Neumann series [166, p.69] the linear map $\mathrm{Id}_{H_2} - T^{-1}A$ is invertible if $\|T^{-1}A\| < 1$. Thus it is invertible if $\|A\| < \frac{1}{C}$. This implies that $T - A$ is invertible if $\|A\| < \frac{1}{C}$.

Assume that $A\colon H_2 \to H_1$ satisfies $\|A\| < \frac{1}{2C}$. Then by applying the Neumann series [166, p.69], we get

$$\begin{aligned}
\left\|T^{-1} - (T-A)^{-1}\right\| &= \left\|T^{-1}\left(\mathrm{Id}_{H_2} - (\mathrm{Id}_{H_2} - T^{-1}A)^{-1}\right)\right\| \\
&\leq \left\|T^{-1}\right\| \left\|\mathrm{Id}_{H_2} - (\mathrm{Id}_{H_2} - T^{-1}A)^{-1}\right\| \\
&= \left\|T^{-1}\right\| \left\|\sum_{k=1}^{\infty} (T^{-1}A)^k\right\| \\
&\leq C \sum_{k=1}^{\infty} \left\|T^{-1}A\right\|^k = \frac{C\left\|T^{-1}A\right\|}{1 - \underbrace{\left\|T^{-1}A\right\|}_{<1/2}} \\
&\leq 2C^2 \|A\|.
\end{aligned}$$

Now consider the family of maps $(y \mapsto T_y(\exp_x^{-1} \circ \phi_x))$ with $x \in X$. As $X$ is compact and the family $(\phi_x)$ equicontinuous at 0 on $X$ by Lemma 2.1.6, there is a $r > 0$ such that the function $y \mapsto (\exp_x^{-1} \circ \phi_x)(y)$ is well-defined and smooth for all $y \in \mathbf{R}^n$ with $\|y\| < r$. Denote by $C > 0$ a constant, such that

$$\left\|(T_0\phi_x)^{-1}\right\| < C \text{ for all } x \in X.$$

Since the family $(T\phi_x)$ is exp-equicontinuous at 0 on $X$, there is a $\delta \in (0, r)$ such that for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < \delta$,

$$\left\|T_y(\exp_x^{-1} \circ \phi_x) - T_0(\exp_x^{-1} \circ \phi_x)\right\| < \frac{1}{2C} \tag{2.2}$$

holds. Applying the considerations above with $T = T_0(\exp_x^{-1} \circ \phi_x)$, $A = T_0(\exp_x^{-1} \circ \phi_x) - T_y(\exp_x^{-1} \circ \phi_x)$, we see that $T_y(\exp_x^{-1} \circ \phi_x)$ is invertible for all $y \in \mathbf{R}^n$ with $\|y\| < \delta$ and $x \in X$. Furthermore, we get for all $x \in X$, $y \in \mathbf{R}^n$, $\|y\| < \delta$ the estimate

$$\left\| \left(T_0(\exp_x^{-1} \circ \phi_x)\right)^{-1} - \left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1} \right\| \\ \leq 2C^2 \left\| T_y(\exp_x^{-1} \circ \phi_x) - T_0(\exp_x^{-1} \circ \phi_x) \right\|.$$

This shows that $\tau_x(y)$ is well-defined for all $x \in X$ and $y \in \mathbf{R}^n$, with $\|y\| < \delta$. Applying (2.2) and the bound for $\|(T_0\phi_x)^{-1}\|$ to the estimate above, we see that for all $x \in X$, $y \in \mathbf{R}^n$, $\|y\| < \delta$,

$$\left\| \left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1} \right\| \leq 2C.$$

Since $X$ is compact, there is a $\rho > 0$ such that for all $x \in X$, $z \in M$ with $\operatorname{dist}(x, z) < \rho$, the inequality

$$\left\| T_{\exp_x^{-1}(z)} \exp_x \right\| < \frac{1}{2}$$

holds. By the equicontinuity of $\phi_x$ at 0 on $X$, we can choose a $\hat{r} > 0$ such that for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < \hat{r}$, we have

$$\operatorname{dist}(\phi_x(y), x) < \rho.$$

This yields for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < \hat{r}$ the estimate

$$\left\| T_{\exp_x^{-1}(\phi_y(x))} \exp_x \right\| < \frac{1}{2}.$$

Therefore, for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < \min\{\hat{r}, \delta\}$ we have

$$\|T_y\phi_x\| \leq \left\| \left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1} \right\| \left\| T_{\exp_x^{-1}(\phi_y(x))} \exp_x \right\| \leq C.$$

From now on, we will assume that $\delta < \hat{r}$. Let us denote for all $x, z \in M$ with $\operatorname{dist}(x, z)$ smaller than the injectivity radius of $x$, the parallel transport along the shortest geodesic between $x$ and $z$ by $\pi_{x,z} \colon T_xM \to T_zM$. Define for $x \in X$, $y \in \mathbf{R}^n$, $\|y\| < r$,

$$\mu_x(y) := \left\| T_{\exp_x^{-1}(\phi_x(y))} \exp_x - \pi_{x,\phi_x(y)} \right\|.$$

Note, that for all $x \in X$ we have $\mu_x(0) = 0$. By straightforward calculations we get for all $x \in X$ and $y \in \mathbf{R}^n$, with $\|y\| < \delta$, the estimates

$$
\begin{aligned}
&\left|\left\|(T_0\phi_x)^{-1}\right\| - \left\|(T_y\phi_x)^{-1}\right\|\right| \\
&\qquad \leq \left\|\left(T_0(\exp_x^{-1} \circ \phi_x)\right)^{-1} - \left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1}\right\| \\
&\qquad\qquad + \left|\left\|\left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1}\right\| - \left\|(T_y\phi_x)^{-1}\right\|\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\left\|\left(T_y(\exp_x^{-1} \circ \phi_x)\right)^{-1}\right\| - \left\|(T_y\phi_x)^{-1}\right\|\right| \\
&\qquad = \left|\left\|(T_y\phi_x))^{-1} T_{\exp_x^{-1}(\phi_x(y))} \exp_x\right\| - \left\|(T_y\phi_x)^{-1} \pi_{x,\phi_x(y)}\right\|\right| \\
&\qquad\qquad \leq \left\|(T_y\phi_x)^{-1}\right\| \left\|T_{\exp_x^{-1}(\phi_x(y))} \exp_x - \pi_{x,\phi_x(y)}\right\|.
\end{aligned}
$$

Combining these estimates yields

$$
\begin{aligned}
&\left|\left\|(T_0\phi_x)^{-1}\right\| - \left\|(T_y\phi_x)^{-1}\right\|\right| \\
&\qquad \leq \mu_x(y)C + 2C^2 \left\|T_y(\exp_x^{-1} \circ \phi_x) - T_0(\exp_x^{-1} \circ \phi_x)\right\|.
\end{aligned}
$$

Since $X$ is compact and $\phi_x$ equicontinuous at 0 on $X$, the family of functions $y \mapsto \mu_x(y)$ is equicontinuous at 0 on $X$. By the exp-equicontinuity of $T\phi_x$ at 0 on $X$, we can find for all $\varepsilon > 0$ a $\tilde{\delta} \in (0, \delta)$ such that for all $x \in X$, $y \in \mathbf{R}^n$, with $\|y\| < \tilde{\delta}$, we have

$$
\mu_x(y) < \varepsilon \text{ and } \left\|T_y(\exp_x^{-1} \circ \phi_x) - T_0(\exp_x^{-1} \circ \phi_x)\right\| < \varepsilon.
$$

Thus for all $x \in X$, $y \in \mathbf{R}^n$ with $\|y\| < \tilde{\delta}$ we get

$$
\left|\left\|(T_0\phi_x)^{-1}\right\| - \left\|(T_y\phi_x)^{-1}\right\|\right| \leq \varepsilon(C + 2C^2).
$$

Since we can choose $\varepsilon > 0$ arbitrary, this yields the equicontinuity of the family $(\tau_x \mid x \in X)$ at 0. □

We give now a version of the Łojasiewicz gradient inequality in a family of parameterizations.

**Theorem 2.1.8** *Let $M$ be an analytic manifold and $f \colon M \to \mathbf{R}$ a smooth $\mathcal{C}$-function. Assume that we have a family of parameterizations $(\phi_x)$ which* *satisfies the standard assumptions on a relatively compact, open set $V$. Then for any $x^* \in V$ there is a neighborhood $U \subset V$ of $x^*$, constants $r > 0$,$\rho > 0$, $C > 0$ and a strictly increasing function $\psi\colon (f(x^*), +\infty) \to \mathbf{R}_+$, definable in an o-minimal structure, with $\psi(t) \to 0$ for $t \to f(x^*)$, such that for all $y \in \mathbf{R}^n$, $x \in V$ with $\|y\| < r$, $\phi_x(y) \in f^{-1}((f(x^*), f(x^*) + \rho))$, the estimate*

$$\|\mathrm{grad}(\psi \circ f \circ \phi_x)(y)\| \geq C$$

*holds.*

**Proof:** Let $x^* \in U$. W.l.o.g. $f(x^*) = 0$. By the Łojasiewicz gradient inequality, Theorem 1.1.22, there exists a neighborhood $V \subset U$, constants $C_1 > 0$, $\rho > 0$ and a function $\psi\colon \mathbf{R}_+ \to \mathbf{R}_+$, definable in an o-minimal structure, with $\psi(t) \to 0$ for $t \to 0$, such that for all $x \in V$, with $f(x) \in (0, \rho)$,

$$\|\mathrm{grad}(\psi \circ f)(x)\| > C_1.$$

For all $x \in V$, $y, v \in \mathbf{R}^n$ with $f(\phi_x(y) \in (0, \rho)$

$$\begin{aligned} \langle \mathrm{grad}(\psi \circ f \circ \phi_x)(y), v\rangle &= d(\psi \circ f \circ \phi_x)(y)(v) \\ &= d(\psi \circ f)(x)(T_y\phi_x v) \\ &= \langle \mathrm{grad}(\psi \circ f)(x), T_y\phi_x v\rangle \\ &= \left\langle (T_y\phi_x)^\top \mathrm{grad}(\psi \circ f)(x), v\right\rangle, \end{aligned}$$

where $(T_y\phi_x)^\top\colon T_xM \to \mathbf{R}^n$ denotes the adjoint of the map $T_y\phi_x$ between the Hilbert spaces $\mathbf{R}^n$ and $T_xM$. Applying Lemma 2.1.7 we get a $r_1 > 0$ such that $(T_y\phi_x)^{-1}$ exists for all $y \in \mathbf{R}^n$, $\|y\| < r_1$ and $x \in V$. By standard arguments on adjoint operators we have for all $x \in V$ and $y \in \mathbf{R}^n$, with $\|y\| < r_1$, that

$$\left\|((T_y\phi_x)^\top)^{-1}\right\| = \left\|((T_y\phi_x)^{-1})^\top\right\| = \left\|(T_y\phi_x)^{-1}\right\|.$$

But as for all $x \in V$, $y \in \mathbf{R}^n$, with $\|y\| < r$

$$\left\|(T_y\phi_x)^\top\right\| \geq \frac{1}{\|((T_y\phi_x)^\top)^{-1}\|},$$

we get for all $x \in V$, $y \in \mathbf{R}^n$, with $\|y\| < r$ and $\phi_x(y) \in f^{-1}((0, \rho))$, that

$$\|\mathrm{grad}(\psi \circ f \circ \phi_x)(y)\| \geq \frac{C_1}{\|((T_y\phi_x)^\top)^{-1}\|}$$

holds. Let $C_2 > 0$ the uniform bound for $\|(T_0\phi_x)^{-1}\|$ on $U$. By the equicontinuity of the family of maps $(y \mapsto \left\|((T_y\phi_x)^\top)^{-1}\right\|)$ at 0 on $U$, see Lemma 2.1.7, we can find a $r_2 \in (0, r_1)$ such that for all $x \in U$, $y \in \mathbf{R}^n$, with $\|y\| < r_2$,

$$\left\|((T_y\phi_x)^\top)^{-1}\right\| < \left\|((T_0\phi_x)^\top)^{-1}\right\| + C_2 < 2C_2$$

holds. This yields the estimate

$$\|\text{grad}(\psi \circ f \circ \phi_x)(y)\| \geq \frac{C_1}{2C_2}.$$

for all $x \in V$, $y \in \mathbf{R}^n$, $\|y\| < r_2$ with $\phi_x(y) \in f^{-1}((0, \rho))$. This proves the theorem. □

In the application of Theorem 2.1.8, it would be sufficient to have an lower bound only for $\|\text{grad}(\psi \circ f \circ \phi_x)(0)\|$. This bound can be directly proven from Łojasiewicz's gradient inequality and the standard assumptions without the use of Lemma 2.1.7. However, the more general version of the Łojasiewicz's gradient inequality in local parameterizations is interesting on its own and therefore presented here.

### 2.1.2 Examples of families of parameterizations

#### Riemannian normal coordinates

As the first example, we consider the family of Riemannian normal coordinates on a complete Riemannian manifold $M$. Let us recall the definition of these coordinates, see [97, p.19]. From orthonormal bases of the tangent spaces $T_xM$ we can construct a family of linear isometries $\psi_x : \mathbf{R}^n \mapsto T_xM$. The Riemannian normal coordinates are then given by

$$\phi_x := \exp_x \circ \psi_x : \mathbf{R}^n \to M.$$

This family $(\phi_x)$ satisfies the standard assumptions.

**Proposition 2.1.9** *Assume that $M$ is a complete Riemannian manifold. Let $X \subset M$ a set and $r > 0$ such that the injectivity radius of all $x \in X$ is bounded from below by $r$. Then the Riemannian normal coordinates satisfy the standard assumptions on $X$.*

**Proof:** By definition $\phi_x(0) = x$ for all $x \in X$. It is a well-known fact from Riemannian geometry, see [51], that $\|T_0 \exp_x\| = \|T_x \exp_x^{-1}\| = 1$. As

$\|\psi_x\| = 1$ for all $x \in X$, the norms $\|T_0\phi_x\|$ and $\|(T_0\phi_x)^{-1}\|$ are uniformly bounded for all $x \in X$. The exp-equicontinuity of $T\phi_x$ at 0 follows from

$$T_y(\exp_x^{-1} \circ \phi_x) = T_{\exp(\psi_x(y))} \exp_x^{-1} \circ T_{\psi_x(y)} \exp_x \circ T_y \psi_x = T_y \psi_x = T_0 \psi_x.$$

□

**Retractions of the tangent bundle**

Our next example are the retractions of the tangent bundle proposed by Shub [143] for Newton's method on manifolds. We show how families of parameterizations satisfying the standard assumptions, can be derived from these retractions. Let us recall the definition of Shub's retractions.

**Definition 2.1.10** Let $M$ be a complete Riemannian manifold and $R\colon TM \to M$ be a smooth map. We denote by $R_x$ the restriction of $R$ to $T_xM$. The map $R$ is a *retraction*[2] if

1. $R_x(0) = x$ for all $x \in M$ and
2. $T_0R_x = \mathrm{Id}_{T_xM}$ for all $x \in M$.

Under a compactness assumption on $M$, a retraction yields a family of parameterizations which satisfies the standard assumptions.

**Proposition 2.1.11** *Let $M$ be a compact manifold and $R\colon TM \to M$ a retraction. Assume that $(\psi_x\colon \mathbf{R}^n \to T_xM)$ is a family of isometries. Then the family $(R_x \circ \psi_x)$ satisfies the standard assumptions.*

Note that we do not pose any further conditions of the family of isometries.
**Proof:** Denote by $\phi_x$ the maps $R_x \circ \psi_x\colon \mathbf{R}^n \to M$. By definition $\phi_x(0) = x$ for all $x \in M$. The condition $T_0R_x = \mathrm{Id}_{T_xM}$ yields $\|T_0\phi_x\| = \|(T_0\phi_x)^{-1}\| = 1$. By the compactness of $M$ the map $\exp^{-1} \circ R\colon TM \to TM$ is well-defined and smooth on a neighborhood of the zero-fiber $0_x$ in $TM$. As the $\psi_x$ are isometries, a compactness argument yields the exp-equicontinuity of the $T\phi_x$ at 0. □

[2]Shub [143] actually defines $R$ only on a neighborhood of the zero section $0_x$ in $TM$. However, by the conditions on $R$, we can assume w.l.o.g. that this neighborhood coincides with the whole tangent bundle. Further, our definition does not include the Lipschitz condition and boundedness condition on the first and second derivatives from [143].

**Families of parameterizations on manifolds with a Lie group action**

In [37] Celledoni and Owren propose a method for the construction of retractions for a manifold $M$ with a transitive action of a Lie group $G$. Using a special family of maps from the tangent spaces $T_xM$ to the Lie algebra $\mathfrak{g}$ of $G$, they derive specific retractions $\phi_x\colon T_xM \to M$ from a coordinate map $\mathfrak{g} \to G$ and the Lie group action on $M$. Of course, we can construct from these retractions a family of parameterizations $(\phi_x\colon \mathbf{R}^n \to M)$, satisfying the standard assumptions, as illustrated in the previous example.

However, we want that the family of parameterizations is in a certain sense invariant under the Lie group action. We use a modification of the construction of [37] to obtain such a family of parameterizations from a single local diffeomorphism $\phi\colon \mathbf{R}^n \to M$. In Proposition 2.1.12 will show that such families always satisfy the standard assumptions. We also relate this construction to the retractions of Celledoni and Owren and similar families of parameterizations $(\phi_x\colon T_xM \to M)$.

Assume that we have a smooth, transitive action $\beta\colon G \times M \to M$ of a Lie group $G$ on $M$. We use the notation $\beta_g(x) := \beta(g,x)$, for $g \in G, x \in M$. Let us recall the definitions of invariance under the group action of some differential geometric objects, see [74]. We call the Riemannian metric *invariant (under the action $\beta$)* if for all $x \in M$, $g \in G$, $v, w \in T_xM$

$$\langle v, w\rangle_x = \langle T_x\beta_g v, T_x\beta_g w\rangle_{\beta_g(x)}$$

holds. This implies that the maps $\beta_g$ are isometries. It is a well-known fact that isometries of Riemannian manifolds preserve the exponential map, see e.g. [99]. Hence, in the case of an invariant metric, the exponential map is invariant under $\beta$, i.e. for all $g \in G$: $\beta_g(\exp_x(v)) = \exp_{\beta_g(x)}(T_x\beta_g v)$. In the remaining part of this example, we will assume that a a smooth, transitive operation $\beta$ of a Lie group $G$ on $M$ is given and that the Riemannian metric $\langle\cdot,\cdot\rangle$ is invariant under $\beta$.

Using the Lie group action on $M$, we can construct a family of parameterizations satisfying the standard assumptions in the following way.

**Proposition 2.1.12** *Let $\psi\colon \mathbf{R}^n \to M$ be a local diffeomorphism and $h\colon M \to G$ a map with $\beta(h(x), \psi(0)) = x$. Then the family of smooth parameterizations*

$$\phi_x(y) := \beta(h(x), \psi(y))$$

*satisfies the standard assumptions.*

**Proof:** By construction we have that $\phi_x(0) = x$. As for all $x \in M$

$$\|T_0\phi_x\| = \left\|T_{\psi(0)}\beta_{h(x)} \circ T_0\psi\right\| = \|T_0\psi\|$$

and

$$\left\|T_0\phi_x^{-1}\right\| = \left\|(T_0\psi)^{-1} \circ \left(T_{\psi(0)}\beta_{h(x)}\right)^{-1}\right\| = \left\|(T_0\psi)^{-1}\right\|$$

hold, we get the uniform boundedness of $T_0\phi_x$ and $(T_0\phi_x)^{-1}$. The invariance of the exponential map implies that for all $x, y \in M$, $g \in G$, $v \in T_yM$, with $\mathrm{dist}(x, y)$ smaller than the injectivity radius of $x$,

$$\left\|T_{\beta_g(y)} \exp_x^{-1} v\right\| = \left\|T_y \exp^{-1}_{\beta_g(x)} \left(T_y\beta_g(v)\right)\right\|$$

holds. Hence, for any $x \in M$, $y \in \mathbf{R}^n$, $\|y\|$ sufficiently small, we have that

$$\left\|T_x \exp_x^{-1} \circ T_0\phi_x - T_{\phi_x(y)} \exp_x^{-1} \circ T_y\phi_x\right\| = \left\|T_{\psi(0)} \exp^{-1}_{\psi(0)} \circ T_0\psi - T_{\psi(y)} \exp^{-1}_{\psi(0)} \circ T_y\psi\right\|.$$

This yields the exp-equivariance of $T\phi_x$ at 0. □

The next proposition gives conditions when families of parameterizations $\phi_x\colon T_xM \to M$, like the families of retractions in [36, 37] and the parameterizations used in [76, 118], can be identified with a family from Proposition 2.1.12.

**Proposition 2.1.13** *Let $(\mu_x\colon T_xM \to M)$ be a family of local diffeomorphisms such that for all $x \in M$, $g \in G$, $w \in T_{\beta_g(x)}M$*

$$\beta_g\left(\mu_x(T_{\beta_g(x)}\beta_g^{-1}w)\right) = \mu_{\beta_g(x)}(w). \tag{2.3}$$

*Then for any $x^* \in M$ and any map $h\colon M \to G$ with $\beta(h(x), x^*) = x$, there is a family of isometries $(\psi_x\colon \mathbf{R}^n \to T_xM)$ such that for all $v \in \mathbf{R}^n$*

$$(\mu_x \circ \psi_x)(v) = \beta\left(h(x), \mu_{x^*}(\psi_{x^*}(v))\right).$$

*In particular, the family $(\mu_x \circ \psi_x)$ satisfies the standard assumptions.*

**Proof:** Let $\psi\colon \mathbf{R}^n \to T_{x^*}M$ be an isometry. We define the family of isometries $(\psi_x\colon \mathbf{R}^n \to T_xM)$ by

$$\psi_x(v) = T_{x^*}\beta_{h(x)}(\psi(v)).$$

By the invariance of the Riemannian metric we have for all $x \in M$, $v, w \in \mathbf{R}^n$

$$\begin{aligned}\langle \psi_x(v), \psi_x(w) \rangle &= \left\langle T_{x^*}\beta_{h(x)}(\psi(v)), T_{x^*}\beta_{h(x)}(\psi(w)) \right\rangle \\ &= \langle \psi(v), \psi(w) \rangle = \langle v, w \rangle .\end{aligned}$$

Thus the maps $\psi_x$ are all isometries. Let $v \in \mathbf{R}^n$ and $x \in M$. Then

$$\begin{aligned}(\mu_x \circ \psi_x)(v) &= \mu_x \left( T_{x^*}\beta_{h(x)}(\psi(v)) \right) \\ &= \beta \left( h(x), \mu_{x^*} \left( T_{\beta(h(x),x^*)}\beta^{-1}_{h(x)} \left( T_{x^*}\beta_{h(x)}(\psi(v)) \right) \right) \right) \\ &= \beta \left( h(x), \mu_{x^*}(\psi(v)) \right) = \beta \left( h(x), \mu_{x^*}(\psi_{x^*}(v)) \right) .\end{aligned}$$

Thus the family $(\psi_x)$ has the required properties. From Proposition 2.1.12 follows that the family $(\mu_x \circ \psi_x)$ satisfies the standard assumptions. □

The retractions proposed by Celledoni and Owren [37] can fit into this setting, if the maps $T_xM \to \mathfrak{g}$ and the coordinate map $\mathfrak{g} \to G$ are chosen such that the invariance condition (2.3) is satisfied. Furthermore examples for the use of families of parameterizations satisfying the invariance condition (2.3) can be found in in [5, 76, 118] for Grassmann and Stiefel manifolds. By the invariance of the exponential map under the group action, the family $(\exp_x)$ satisfies always the conditions of Proposition 2.1.13.

Manton [119] mentions also the question of the right choice of local parameterizations for Newton algorithms on homogeneous spaces. However, he suggests the use of parameterizations which preserve the local symmetries of the homogeneous space, an aspect we will not follow here any further.

### 2.1.3 Descent Iterations

We now turn to the discussion of descent iterations in a family of parameterizations.

We define the descent iterations analogous to the approaches in the literature [35, 37, 118, 120, 143].

**Definition 2.1.14** A *descent iteration* for a smooth cost function $f : M \to \mathbf{R}$ in a family of smooth parameterizations $(\phi_x : \mathbf{R}^n \to M)$ is a sequence $(x_k) \subset M$ given by

$$x_{k+1} = \phi_{x_k}(\alpha_k s_k), \quad \alpha_k \in \mathbf{R}_+, s_k \in \mathbf{R}^n$$

such that for all $k \in \mathbf{N}$

$$f(x_{k+1}) \leq f(x_k)$$

and $f(x_{k+1}) = f(x_k)$ implies $x_l = x_k$ for all $l > k$.

We call $\alpha_k$ the *step size* and $s_k$ the *descent direction*.

**Remark 2.1.15** One major argument for descent algorithms via families of smooth parameterizations instead of using geodesics, is the fact that geodesics are often too expensive to calculate [118]. The parameterizations allow to use of computationally cheaper maps. Furthermore, in many applications it is not possible to calculate the exact geodesics but only a sufficiently accurate approximation. Strictly speaking, the convergence results known for Riemannian optimization would not apply directly in these cases.

We now introduce an extension of the standard conditions for global convergence used in Euclidean optimization to our setting.

**Definition 2.1.16** A descent iteration $(x_k) \subset M$ for a smooth function $f\colon M \to \mathbf{R}$ in a family of smooth parameterizations $(\phi_x)$ satisfies the *angle condition* (AC) if there is a constant $\varepsilon > 0$ such that

$$-d(f \circ \phi_{x_k})(0)(s_k) \geq \varepsilon \, \|s_k\| \, \|\mathrm{grad}(f \circ \phi_{x_k})(0)\| \tag{AC}$$

for all $k \in \mathbf{N}$. The iteration satisfies the *first and second Wolfe-Powell conditions* (WP1), (WP2) if there are constants $\sigma$, $\rho \in (0,1)$ such that for all $k \in \mathbf{N}$

$$f(x_k) - f(x_{k+1}) \geq -\sigma d(f \circ \phi_{x_k})(0)(\alpha_k s_k) \tag{WP1}$$

$$d(f \circ \phi_{x_k})(\alpha_k s_k)(s_k) \geq \rho d(f \circ \phi_{x_k})(0)(s_k). \tag{WP2}$$

To determine a suitable step size which satisfies (WP1) and (WP2), a Wolfe-Powell line search would use an iterative sectioning algorithm as e.g. in [64, Section 2.6]. The Armijo line search takes a different approach. In particular it enforces only the first Wolfe-Powell condition.

**Definition 2.1.17** A descent iteration $(x_k) \subset M$ for $f\colon M \to \mathbf{R}$ in a smooth family of parameterizations $(\phi_x)$ determines the step size by the *Armijo line search* if there are constants $\sigma, \rho \in (0,1)$, $C > 0$ such that for all $k \in \mathbf{N}$

$$\alpha_k = C\rho^{j_k}$$

with

$$j_k := \min\left\{ j \in \mathbf{N} \mid f(x_k) - f(\phi_{x_k}(C\rho^j) \geq -\sigma d(f \circ \phi_{x_k})(0)(C\rho^j s_k) \right\}.$$

For gradient descent, i.e. $s_k = \mathrm{grad}(f \circ \phi_{x_k})(0)$, the use of the Armijo rule in local parameterizations was proposed in [118].

The convergence of descent iterations in a family of smooth parameterizations is based on a special property of the descent in Euclidean space. The standard proofs for convergence of descent iterations in $\mathbf{R}^n$, see e.g. [64,161], can be viewed from an alternative perspective: Instead of the descent sequence of points $x_k \in \mathbf{R}^n$ for a fixed cost function $f$, a sequence of functions $f_k$ evaluated only at the fixed point 0 can be considered. One can then ask whether $df_k(0)$ converges to zero in place of $df(x_k)$. The following lemma illustrates this more precisely.

**Lemma 2.1.18** *We consider the Euclidean space $\mathbf{R}^n$. Let $(f_k \colon \mathbf{R}^n \to \mathbf{R})$, $k \in \mathbf{N}$, a family of smooth functions and $(\alpha_k) \subset \mathbf{R}_+$, $(s_k) \subset \mathbf{R}^n$ sequences. Assume that the following conditions hold:*

- *$(df_k)$ is equicontinuous at 0, i.e. for all $\hat{\varepsilon} > 0$ exists a $\delta > 0$ such that*

$$\forall k \in \mathbf{N}, x \in \mathbf{R}^n \colon \ \|x\| < \delta \ \textit{implies} \ \|df_k(x) - df_k(0)\| < \hat{\varepsilon}.$$

- *for all $k \in \mathbf{N}$ we have $\|s_k\| = 1$,*
- *$f_k(0) - f_k(\alpha_k s_k) \to 0$ for $k \to \mathbf{N}$.*
- *there exists $\varepsilon > 0$ such that for all $k \in \mathbf{N}$*

$$-df_k(0)(s_k) \geq \varepsilon \left\| \mathrm{grad}\, f_k(0) \right\|,$$

- *there exist $\sigma, \rho \in (0,1)$ such that for all $k \in \mathbf{N}$*

$$\begin{aligned} f_k(0) - f_k(\alpha_k s_k) &\geq -\sigma df_k(0)(\alpha_k s_k) \\ df_k(\alpha_k s_k)(s_k) &\geq \rho df_k(0)(s_k). \end{aligned}$$

*Then $\|df_k(0)\| \to 0$ for $k \to \infty$.*

Given a descent iteration $(x_k)$ in $\mathbf{R}^n$, $x_k = \alpha_k s_k$, for a cost function $f \colon \mathbf{R}^n \to \mathbf{R}$ we can define $f_k(x) := f(x - x_k)$. W.l.o.g. we can assume that $\|s_k\| = 1$. If the descent iteration satisfies the angle and Wolfe-Powell conditions and $df$ is uniformly continuous, then the family $(f_k)$ with sequences $\alpha_k, s_k$ satisfies the conditions of Lemma 2.1.18. Furthermore, the convergence $\|df_k(0)\| \to 0$

is equivalent to $\|df(x_k)\| \to 0$. Thus the lemma above implies the standard convergence results for descent algorithms in Euclidean spaces. In fact, we have only to apply a modification of the standard arguments [64, 161] for convergence of descent iterations to prove Lemma 2.1.18.

**Proof:** The proof is a straightforward adaptation of the argument for descent iterations in Euclidean spaces, see [64, Thm 2.5.1] or [161]. The conditions on $f_k$, $\alpha_k$ and $s_k$ yield an $\varepsilon > 0$ such that for all $k \in \mathbf{N}$

$$f_k(0) - f_k(\alpha_k s_k) \geq -\sigma df_k(0)(\alpha_k s_k) \geq \varepsilon\, |\alpha_k|\, \|\mathrm{grad}\, f_k(0)\|$$

holds. As $f_k(0) - f_k(\alpha_k s_k) \to 0$, we can pass to a suitable subsequences such that $|\alpha_k| \to 0$ or $\|\mathrm{grad}\, f_k(0)\| \to 0$. Assume that $|\alpha_k| \to 0$. We define

$$r_k := df_k(\alpha_k s_k)(s_k) - df_k(0)(s_k).$$

Assume that there is a $C > 0$ with $|r_k| > C$ for all $k \in \mathbf{N}$.
Then $\|df_k(\alpha_k s_k)(s_k) - df_k(0)(s_k)\| > C$ for all $k \in \mathbf{N}$. But since $|\alpha_k| \to 0$ this would be a contradiction to the equicontinuity of the $df_k$ in 0. Thus $|r_k| \to 0$. On the other hand

$$r_k = df_k(\alpha_k s_k)(s_k) - df_k(0)(s_k) \geq (\rho - 1) df_k(0)(s_k) \geq 0.$$

Therefore $df_k(0)(s_k) \to 0$. Since $-df_k(0)(s_k) \leq \varepsilon\, \|\mathrm{grad}\, f_k(0)\|$, this yields the convergence

$$\|\mathrm{grad}\, f_k(0)\| \to 0.$$

□

Note, that for a descent iteration $(x_k) \in M$ in a family of smooth parameterizations $(\phi_x)$, we get a sequence of functions $h_k := f \circ \phi_k \colon \mathbf{R}^n \to \mathbf{R}$.

**Theorem 2.1.19** *Assume that the following conditions hold:*

- $f \colon M \to \mathbf{R}$ *is a smooth cost function,*
- $(\phi_x)$ *is a family of smooth parameterizations,*
- $(x_k)$ *is a descent iteration for* $f$ *in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *the injectivity radius of all* $x \in S = \{x \in M \mid f(x) \leq f(x_0)\}$ *is bounded from below by a constant* $r > 0$*,*

- *$df$ is uniformly* exp-*continuous on $S$,*
- *$\|df\|$ is uniformly bounded on $S$,*
- *$(\phi_x)$ satisfies the standard assumptions on $S$.*

*Then $\|df(x_k)\| \to 0$ or $f(x_k) \to -\infty$. In particular all accumulation points of $(x_k)$ are critical points of $f$.*

**Proof:** We assume that $f(x_k)$ is strictly decreasing, otherwise the claim in trivial. Define a family of smooth functions $(h_k\colon \mathbf{R}^n \to \mathbf{R})$, $k \in \mathbf{N}$ by

$$h_k(y) := f(\phi_{x_k}(y)) \text{ for all } y \in \mathbf{R}^n.$$

Let $\alpha_k \in \mathbf{R}_+$ the step sizes and $s_k \in \mathbf{R}^n$ the descent directions of the descent iteration. W.l.o.g. $\|s_k\| = 1$ for all $k \in \mathbf{N}$. As $(\phi_x)$ satisfies the standard assumptions on $S$, $df$ is uniformly exp-continuous on $S$ and $\|df\|$ is uniformly bounded on $S$, we can apply Lemma 2.1.6 to get the equicontinuity of $d(f \circ \phi_x)$ at 0 on $S$. This implies the equicontinuity of $dh_k$ at 0. Assume that $f(x_k)$ is bounded from below. Then $h_k(0) - h_k(\alpha_k s_k) = f(x_k) - f(x_{k+1}) \to 0$. As the descent iteration satisfies the angle and Wolfe-Powell conditions, we can now apply Lemma 2.1.18. Thus

$$\|d(f \circ \phi_k)(0)\| = \|dh_k(0)\| \to 0.$$

Since $\|(T_0\phi_x)^{-1}\|$ is uniformly bounded on $S$, we get that

$$\|df(x_k)\| \to 0.$$

□

**Theorem 2.1.20** *Assume that the following conditions hold:*

- *$f\colon M \to \mathbf{R}$ is a smooth cost function,*
- *$(\phi_x)$ is a family of smooth parameterizations,*
- *$(x_k)$ is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *For any compact subset $K$ of $M$, the family $(\phi_x)$ satisfies the standard assumptions on $K$, i.e. the constants used in the standard assumptions depend on $K$, and can be different for different compact sets.*

*Then all accumulation points of $(x_k)$ are critical points.*

**Proof:** We assume that $f(x_k)$ is strictly decreasing, otherwise the claim in trivial. Let $x^* \in M$ be the accumulation point of $(x_k)$. We denote by $\alpha_k \in \mathbf{R}_+$ the step sizes and by $s_k \in \mathbf{R}^n$ the descent directions of the descent iteration. W.l.o.g. $\|s_k\| = 1$ for all $k \in \mathbf{N}$. Furthermore, let $U$ be relatively compact, open neighborhood of $x^*$. There is a subsequence $(x_{k_j})$ of $(x_k)$ such that $x_{k_j} \in U$ for all $j \in \mathbf{N}$. As $f$ is smooth $\|df\|$ is bounded on $\overline{U}$ and $df$ is exp-continuous in 0 at $\overline{U}$. Furthermore, $(\phi_x)$ satisfies the standard assumptions on the compact set $\overline{U}$. Thus by Lemma 2.1.6 the family $d(f \circ \phi_x)$ is equicontinuous at 0 on $\overline{U}$. Since the descent sequence $x_k$ has an accumulation point, $f(x_k)$ is bounded from below and especially $f(x_{k_j}) - f(x_{k_j+1}) \to 0$. Define now $h_j : \mathbf{R}^n \to \mathbf{R}$ by

$$h_j(y) = f(\phi_{x_{k_j}}(y)) \text{ for all } y \in \mathbf{R}^n.$$

As the descent sequence satisfies the angle and Wolfe-Powell conditions, we can apply Lemma 2.1.18 to $h_j$, $\alpha_{k_j}$, $s_{k_j}$. This yields $\|dh_j(0)\| \to 0$. Since $\phi_x$ satisfies the standard assumptions on $\overline{U}$, this gives $\left\|df(x_{k_j})\right\| \to 0$. By the smoothness of $f$ we get $df(x^*) = 0$. □

**Corollary 2.1.21** *Assume that the following conditions hold:*

- $f : M \to \mathbf{R}$ *is a smooth cost function with compact sublevel sets*
- $(\phi_x)$ *is a family of smooth parameterizations,*
- $(x_k)$ *is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *For any compact subset $K$ of $M$, the family $(\phi_x)$ satisfies the standard assumptions on $K$, i.e. the constants in the standard assumptions depend on $K$.*

*Then the sequence $(x_k)$ converges to the set of critical points.*

**Proof:** By Theorem 2.1.20 the accumulation points of $(x_k)$ are critical points. As $f(x_k)$ is non-increasing, the sequence $(x_k)$ is contained in a compact sublevel set $S$. Therefore, $x_k$ converges to the set of critical points of $f$ in $S$. □

For $\mathcal{C}$-functions, one can show that a gradient-like descent along geodesics $(x_k)$ converges to a single point; for analytic functions and classical gradient-like descent in Euclidean spaces, see [4], and for the more general case of $\mathcal{C}$-functions with gradient-like descent along geodesics on Riemannian manifolds, see [103]. The case of analytic functions and gradient-like descent along geodesics on Riemannian manifolds has also been considered in [104]. We extend these results to descent iterations in local parameterizations.

**Theorem 2.1.22** *Assume that the following conditions hold:*

- *$M$ is an analytic manifold,*
- *$f\colon M \to \mathbf{R}$ is a smooth $\mathcal{C}$-function,*
- *$(\phi_x)$ is a family of smooth parameterizations,*
- *$(x_k)$ is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *For any compact subset $K$ of $M$, the family $(\phi_x)$ satisfies the standard assumptions on $K$, i.e. the constants in the standard assumptions depend on $K$.*

*Then the descent sequence $(x_k)$ either converges to a single point or has no accumulation points.*

**Proof:** We provide a adaptation of the convergence arguments for Euclidean and Riemannian gradient-like descent iterations [4, 103, 104] to descent iterations in a family of parameterizations. Let $x^*$ be an accumulation point of $(x_k)$. W.l.o.g. we assume $f(x^*) = 0$. By Theorem 2.1.8 there are a relatively compact neighborhood $U$ of $x^*$, constants $C_1 > 0$, $\rho > 0$ and a strictly increasing function $\psi\colon \mathbf{R}_+ \to \mathbf{R}_+$, definable in an o-minimal structure, such that for all $x \in U$, with $f(x) \in (0, \rho)$,

$$|\psi'(f(x))| \, \|\mathrm{grad}(f \circ \phi_x)(0)\| = \|\mathrm{grad}(\psi \circ f \circ \phi_x)(0)\| \geq C.$$

By the angle and Wolfe-Powell conditions, there is an $\varepsilon > 0$ such that for all $k \in \mathbf{N}$

$$f(x_k) - f(x_{k+1}) \geq \varepsilon \, \|\mathrm{grad}(f \circ \phi_{x_k})(0)\| \, \|\alpha_k s_k\| \, .$$

For $x_k \in U$ this yields

$$C_1 \varepsilon \, \|\alpha_k s_k\| \leq \psi'(f(x_k)) \, (f(x_k) - f(x_{k+1})) \, .$$

As in [103] we can deduce from the monotone increase of $\psi$, the monotone decrease of $\psi'$ and the mean value theorem that[3]

$$\psi'(f(x_k))\,(f(x_k) - f(x_{k+1})) \leq \psi(f(x_k)) - \psi(f(x_{k+1})).$$

This yields the estimate

$$C_1\varepsilon \left\|\alpha_k s_k\right\| \leq \psi(f(x_k)) - \psi(f(x_{k+1}))$$

for $x_k \in U$. By Lemma 2.1.6 there is a $C_2 > 0$ and $r > 0$ such that for all $x \in U$ and $y \in \mathbf{R}^n$ with $\|y\| < r$,

$$\operatorname{dist}(\phi_x(y), x) < C_2 \left\|y\right\|.$$

As $f(x_k) \to f(x^*)$, the difference $\psi(f(x_k)) - \psi(f(x_{k+1}))$ converges to zero. Thus, there is a $K > 0$ such that for all $k > K$ with $x_k \in U$, we have that $\|\alpha_k s_k\| < r$. This gives for $x_k \in U$, $k > K$ the estimate

$$\frac{C_1}{C_2}\varepsilon \operatorname{dist}(x_{k+1}, x_k) = \frac{C_1}{C_2}\varepsilon \operatorname{dist}(\phi_{x_k}(\alpha_k s_k), x_k) \leq \psi(f(x_k)) - \psi(f(x_{k+1})).$$

Since there is a subsequence $x_{k_j}$ converging to $x^*$ and $\psi(f(x_k)) \to 0$, we can conclude that the iterates $x_k$ cannot leave $U$ anymore for large $k$ and that

$$\sum_{k=0}^{\infty} \operatorname{dist}(x_{k+1}, x_k)$$

is bounded. Therefore, $x_k$ converges to $x^*$. □

**Example: descent on Riemannian manifolds**

As an example, we illustrate how standard gradient-like descent methods on Riemannian manifolds, see [154], can be interpreted in our setting of descent iterations in smooth parameterizations. Recall that a *Riemannian descent iteration* $(x_k) \subset M$ is given by

$$x_{k+1} = \exp_{x_k} \alpha_k s_k,$$

[3] In [4], the authors use a similar argument to get the estimate for analytic cost functions on $\mathbf{R}^n$. In this case $\psi(t)$ has the form $t^{(1-\mu)}$, $\mu \in (0,1)$ [113].

where $\alpha_k \in \mathbf{R}_+$ is the step size and $s_k \in T_{x_k}M$ the descent direction, such that for all $k \in \mathbf{N}$

$$f(x_{k+1}) \leq f(x_k)$$
$$\text{and } f(x_{k+1}) = f(x_k) \text{ implies } x_j = x_k \text{ for all } j > k.$$

Here, we assume that $M$ is a complete Riemannian manifold, otherwise the descent iteration might not be well-defined. Furthermore, we require that the injectivity radius of all $x \in M$ is uniformly bounded from below by a constant $r > 0$.

Let $(\phi_x)$ be a family Riemannian normal coordinates on $M$. By Proposition 2.1.9 the family $(\phi_x)$ satisfies the standard assumptions on $M$. Furthermore the parameterizations are given by $\phi_x = \exp_x \circ \psi_x$, where $(\psi_x : \mathbf{R}^n \to T_xM)$ is a family of linear isometries. We can define

$$\hat{s}_k := \psi_{x_k}^{-1}(s_k).$$

Then the Riemannian descent iteration $(x_k)$ can be viewed as a descent iteration in the family of smooth parameterization $(\phi_x)$ with step sizes $\alpha_k$ and descent directions $\hat{s}_k$. Recall that $(x_k)$ satisfies the angle and Wolfe-Powell conditions according to Definition 2.1.16, if there are constants $\varepsilon > 0$, $\rho, \sigma \in (0,1)$ such that for all $k \in \mathbf{N}$

$$\begin{aligned} -d(f \circ \phi_{x_k})(0)(\hat{s}_k) &\geq \varepsilon \left\|\operatorname{grad}(f \circ \phi_{x_k})(0)\right\| \left\|\hat{s}_k\right\|, \\ f(x_k) - f(x_{k+1}) &\geq -\sigma d(f \circ \phi_{x_k})(0)(\alpha_k \hat{s}_k), \\ d(f \circ \phi_{x_k})(\alpha_k \hat{s}_k)(\hat{s}_k) &\geq \rho d(f \circ \phi_{x_k})(0)(\hat{s}_k) \end{aligned}$$

hold. Since $\psi_x$ is an isometry, we have $\|\hat{s}_k\| = \|s_k\|$. Furthermore, $T_0 \exp_x$ and $T_0\psi_x$ are isometries, too. Denoting by $(T_0 \exp_x)^\top$, $(T_0\psi_x)^\top$ the adjoints of the linear maps $T_0 \exp_x \colon T_xM \to T_xM$, $T_0\psi_x \colon T_xM \to \mathbf{R}^n$ between Hilbert spaces, this yields

$$\begin{aligned} \left\|\operatorname{grad}(f \circ \phi_x)(0)\right\| &= \left\|\operatorname{grad}(f \circ \exp_{x_k} \circ \psi_{x_k})(0)\right\| \\ &= \left\|(T_0\psi_{x_k})^\top \circ (T_0 \exp_{x_k})^\top \operatorname{grad} f(x_k)\right\| = \left\|\operatorname{grad} f(x_k)\right\|. \end{aligned}$$

Furthermore, note that

$$d(f \circ \phi_{x_k})(0)(\alpha_k \hat{s}_k) = d(f \circ \exp_{x_k})(0)(\alpha_k s_k)$$

and

$$d(f \circ \phi_{x_k})(\alpha_k \hat{s}_k)(\hat{s}_k) = d(f \circ \exp_{x_k})(\psi_{x_k})(\alpha_k \hat{s}_k))(T_{\alpha_k \hat{s}_k} \psi_{x_k} \hat{s}_k) \\ = d(f \circ \exp_{x_k})(\alpha_k s_k)(s_k).$$

Therefore, the angle and Wolfe-Powell conditions are equivalent to the existence of constants $\varepsilon > 0$, $\rho, \sigma \in (0,1)$ such that

$$\begin{aligned} -d(f \circ \phi_{x_k})(0)(\hat{s}_k) &\geq \varepsilon \|\mathrm{grad}\, f(x_k)\| \, \|s_k\|, \\ f(x_k) - f(x_{k+1}) &\geq -\sigma d(f \circ \exp_{x_k})(0)(\alpha_k s_k), \\ d(f \circ \exp_{x_k})(\alpha_k s_k)(s_k) &\geq \rho df(0)(s_k). \end{aligned}$$

Hence, our convergence results imply for cost functions with compact sublevel sets the standard results on Riemannian gradient-like descent as illustrated by the following corollary, compare with [154].

**Corollary 2.1.23** *Let $M$ be a complete Riemannian manifold and $f\colon M \to \mathbf{R}$ a smooth cost function with compact sublevel sets. Assume that $x_k$ is a Riemannian descent iteration in $M$ which satisfies the following conditions*

- *There exists an $\varepsilon > 0$ such that for all $k \in \mathbf{N}$*

$$-d(f \circ \phi_{x_k})(0)(\hat{s}_k) \geq \varepsilon \|\mathrm{grad}\, f(x_k)\| \, \|s_k\|.$$

- *There exist $\rho, \sigma \in (0,1)$ such that for all $k \in \mathbf{N}$*

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq -\sigma d(f \circ \exp_{x_k})(0)(\alpha_k s_k), \\ d(f \circ \exp_{x_k})(\alpha_k s_k)(s_k) &\geq \rho df(0)(s_k). \end{aligned}$$

*Then $x_k$ converges to the set of critical points of $f$.*

**Remark 2.1.24** For functions with non-compact sublevel sets, Udriste [154] requires that the Hessian is uniformly bounded on sublevel sets. Our conditions, especially the uniform bound on $\|df\|$, are stronger. However, the conditions on $df$ could be replaced by a weak equicontinuity condition on $d(f \circ \phi_x)$. More precisely, we could assume that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in \{z \in M \mid f(x) \leq f(x_0)\}$, $v \in \mathbf{R}^n$, $\|v\| < \delta$,

$$\|d(f \circ \phi_x)(v)(v) - d(f \circ \phi_x)(0)(v)\| < \varepsilon \text{ holds.}$$

With a minor modification of Lemma 2.1.18, the convergence results hold for this weaker condition, too. This version would imply the general form of the convergence results from [154]. However, this ties the admissible functions directly to the parameterizations. While this might be viable in some cases, this can prevent a direct exchange of the parameterization or the cost function in an application.

In the same manner, we retrieve the convergence of a Riemannian descent iteration to a single point for $\mathcal{C}$-functions, cf. [4] for the Euclidean and [103] for a Riemannian version.

## 2.2 Optimization on singular sets

### 2.2.1 Motivation

We start this section be recalling the smooth constraint optimization problem.

Let $M$ be a smooth manifold, $f\colon M \to \mathbf{R}$ be a smooth function, the *cost function* and $X \subset M$ a set, the so-called *constraint set* or *set of feasible points*. The *smooth constrained optimization problem* consists of finding $x^* \in X$ such that

$$f(x^*) = \min_{x\in X} f(x).$$

The standard Euclidean space setting assumes that $M$ is the Euclidean $\mathbf{R}^n$ and $X$ is given by a number smooth equations or inequalities, i.e. $X = g^{-1}(0) \cap h^{-1}(\mathbf{R}^l_+)$ for smooth functions $g : \mathbf{R}^n \to \mathbf{R}^k$, $h : \mathbf{R}^n \to \mathbf{R}^l$.

We review some aspects of the standard Euclidean approaches. However, given the vast amount of literature in this area, it is beyond the scope of this work to give a complete overview. So we will just sketch a few parts of this area and refer the reader to the standard literature like [33, 64, 130].

A first step to solve the optimization problem is to characterize the minima of $f$ on $X$. The usual approach is to use linearization techniques or Taylor approximations for this purpose. Under suitable regularity conditions on the constraint set, the *constraint qualifications* [33], this yields first-order Lagrange-multiplier type necessary conditions for the minima of $f$. These conditions are the well-known *Karush-Kuhn-Tucker conditions.* Points satisfying these conditions are called *KKT points* and defined as follows [33, p.160]. A point $x^* \in \mathbf{R}^n$ is a KKT point if and only if there exist $\lambda \in \mathbf{R}^k$, $\mu \in \mathbf{R}^l$ such that

- $df(x^*) + \lambda^\top Dg(x^*) + \mu^\top Dh(x^*) = 0$,
- $g(x^*) = 0$, $h(x^*) \in \mathbf{R}^l_+$,
- $\mu \in \mathbf{R}^l_+$,
- $\mu^\top h(x^*) = 0$.

The Euclidean methods use Newton-type algorithms, like e.g. the SQP-methods, or penalization approaches, like penalty or barrier functions, to find a KKT point in the constraint set. However, this requires that suitable constraint qualifications hold which can impose relatively strong conditions on the geometry of the constraint set. For example, the so-called linear constraint qualifications, see [33] for a definition, imply that the constraints define locally a smooth submanifold with corners[4]. Thus these approaches are not easily applicable to e.g. arbitrary semi-algebraic sets. Furthermore, the methods often rely on a description of the constraint set by equations and inequalities. This raises problems if such a description is not easily available, would be very complicated or the description would not satisfy any type of constraint qualification. Furthermore, these approaches exploit the simple geometry of the ambient Euclidean space and are not easily extended to general Riemannian manifolds.

Here, we propose a different approach by extending the descent iterations in local parameterizations from the previous section to non-smooth sets. To motivate this approach, let us consider the case that $X$ is a algebraic subset of $\mathbf{R}^n$ or subanalytic subset of an analytic manifold $M$. Then well-known desingularization results of Hironaka [81, 82] or the uniformization theorem [24], guarantee the existence of an analytic map $\psi\colon N \to X$ with $N$ an analytic manifold of dimension $\dim X$. Note that there are constructive versions of the proof of the desingularization theorem [25,158,159], thus for analytic sets $X$ the map $\psi$ can be constructed by a finite number of blowing up operations. Assume now that we are given a smooth cost function $f\colon M \to \mathbf{R}$ and we want to optimize $f$ over $X$, i.e. find a minimum of the restriction $f|_X\colon X \to \mathbf{R}$ of $f$ to $X$. Let $(\phi_y)$ be a family of smooth parameterizations of $N$. Then we can construct a descent iteration $(y_k)$ in this family of parameterizations for the smooth function $f \circ \psi$. On the other hand we can construct a family of parameterizations $(\hat{\phi}_x : \mathbf{R}^n \to X)$ of $X$ by setting $\hat{\phi}_x := \psi \circ \phi_y$, $y \in \psi^{-1}(\{x\})$[5]. The descent iteration $(y_k)$ for $f \circ \psi$ consists of Euclidean optimization steps for the functions $f \circ \psi \circ \phi_{y_k}$, which are mapped back to $N$ by $\phi_{y_k}$. Define a sequence $(x_k) \subset X$, $x_k := \psi \circ \phi_{y_k}(0)$. This sequence is determined by Euclidean optimization steps for the function $f \circ \hat{\phi}_{x_k}$ which are mapped back to $X$ by parameterizations $\hat{\phi}_{x_k}$. Thus we can view $(x_k)$ as a descent iteration for $f$ on $X$ in the parameterizations $(\hat{\phi}_x)$. This motivates the

[4] i.e. a subset of $\mathbf{R}^n$ which is locally diffeomorphic to $\mathbf{R}^l \times \mathbf{R}^k_+ \times \{0\}^m$, $l + k + m = n$.

[5] This does not give a unique parameterization $\hat{\phi}_x$ for a $x \in X$. Therefore, we assume $\hat{\phi}_{x_k} := \psi \circ \phi_{y_k}$ for the descent iteration $(y_k)$. This assumption is feasible, as $f(\psi(y_k))$ is non-increasing, and non-decreasing only if the sequence $(y_k)$ is constant for the remainder of the sequence.

extension of the optimization on manifolds in a family of parameterizations to singular sets.

Note, that one can view the blow-up construction above as a generalization of the nonlinear coordinate transform approaches, see [56, 133], and quadratic slack variable approaches, see e.g. [96].

### 2.2.2 Parameterizations of singular sets

At first we consider families of parameterizations of a set $X \subset M$, regularity conditions on the families and necessary conditions in the parameterizations for local minima of the cost function. In the remainder of this section $M$ will again be a smooth Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$.

We define for any $X \subset M$ a *family of smooth parameterizations* as a function $\phi\colon X \times \mathbf{R}^n \to M$, such that $\operatorname{Im}\phi = X$ and $y \mapsto \phi(x, y)$ is smooth. Again, $\phi_x$ denotes the function $y \mapsto \phi(x, y)$[6].

To develop a convergence theory for the descent iterations described later, we need suitable first order condition for the minima of a smooth function $f\colon M \to \mathbf{R}$ restricted to $X$, i.e. a suitable concept of critical points on $X$. It is not obvious how a first order condition should look like. One possible approach would be to use tools from non-smooth optimization for this task. However, we assume here that the non-smooth set can be parameterized by smooth maps $\phi_x\colon \mathbf{R}^n \to M$, $\phi_x(\mathbf{R}^n) \subset X$. By considering the function in these parameterizations, i.e. $f \circ \phi_x$, we can use the well-known characterization of critical points on Euclidean $\mathbf{R}^n$, to derive a first order necessary condition for local minima. This characterization will fit seamlessly into our approach to minimization by descent iterations in the parameterizations.

**Definition 2.2.1** Let $X$ be a closed subset of $M$ with a family of smooth parameterizations $(\phi_x)$ with $\phi_x(0) = x$. Let $f : M \to \mathbf{R}$ be a smooth function. We call points $x \in X$ with $d(f \circ \phi_x)(0) = 0$ the *critical points of $f$ on $X$ induced by the family of parameterizations* or in short *critical points of $f|_X$*.

Note, that besides the smoothness and $\phi_x(0) = x$, we not impose any further restrictions on the parameterizations. It follows directly that this condition is necessary for a local minimum.

**Proposition 2.2.2** *Let $X$ be a closed subset of $M$ and $f\colon M \to \mathbf{R}$ a smooth cost function. Assume we have a family of smooth parameterizations $(\phi_x)$ of*

[6] To avoid confusion, we stress the fact that the $\phi_x$ do not have to be invertible or local diffeomorphisms.

*$X$ with $\phi_x(0) = x$. If $x^*$ is local minimum (or local maximum) of $f$ on $X$, then $d(f \circ \phi_{x^*})(0) = 0$, i.e. $x^*$ is a critical point of $f|_X$.*

**Proof:** The local minimum $x^*$ of $f$ on $X$ implies that 0 is a local minimum of the smooth function $f \circ \phi_{x^*}$. □

Furthermore, if a set $X \subset \mathbf{R}^n$ is given by smooth equations and inequalities, then all KKT points are critical points in our definition.

**Proposition 2.2.3** *Let $X$ be a closed subset of the Euclidean $\mathbf{R}^n$, which is defined by a finite number of equations and inequalities. Assume that $f : \mathbf{R}^n \to \mathbf{R}$ is a smooth cost function and $x^*$ a KKT point. Then for any family of smooth parameterizations $(\phi_x)$ of $X$ with $\phi_x(0) = x$, we get $d(f \circ \phi_{x^*})(0) = 0$.*

**Proof:** We use the same notation as in the definition of KKT points, cf. 100. Let $\phi_x : \mathbf{R}^k \to X$. Assume that $d(f \circ \phi_{x^*})(0) \neq 0$. Then there is a curve $\gamma : \mathbf{R} \to \mathbf{R}^k$ with $\gamma(0) = 0$, $(f \circ \phi_{x^*} \circ \gamma)'(0) \neq 0$. Define $\tau := \phi_{x^*} \circ \gamma$. Then by definition $\tau(0) = x^*$, $\dot{\tau}(0) \neq 0$ and $(f \circ \tau)'(0) \neq 0$. Furthermore, $\tau(t) \in X$ for all $t \in \mathbf{R}$. W.l.o.g. we can assume that $f \circ \tau(t) > 0$ for $t > 0$ and $f \circ \tau(t) < 0$ for $t < 0$. The case of different signs can be covered by replacing $\tau(t)$ with $\tau(-t)$.

Let $X = \{x \in \mathbf{R}^n \mid g(x) = 0, h(x) \geq 0\}$ for smooth functions $g\colon \mathbf{R}^n \to \mathbf{R}^l$, $h\colon \mathbf{R}^n \to \mathbf{R}^r$. As $\tau(t) \in X$ for all $t \in \mathbf{R}$, we have $g(\tau(t)) = 0$ and $h(\tau(t)) \geq 0$ for all $t \in \mathbf{R}$. Denote by $\lambda \in \mathbf{R}^l$, $\mu \in \mathbf{R}^r_+$ the Lagrange multipliers of the KKT point. Let $D$ denote the differential of maps $\mathbf{R}^n \to \mathbf{R}^m$. Then $Dg(x^*)\dot{\tau}(0) = 0$ and $\mu^\top h(\tau(t)) \geq 0$ hold for all $t \in \mathbf{R}$. By $\mu^\top h(\tau(0)) = \mu^\top h(x^*) = 0$ we get that $\mu^\top Dh(x^*)\dot{\tau}(0) = 0$. Therefore

$$df(x^*)(\dot{\tau}(0)) + \lambda^\top Dg(x^*)\dot{\tau}(0) + \mu^\top Dh(x^*)\dot{\tau}(0) = df(x^*)(\dot{\tau}(0)) \neq 0$$

which is a contradiction to the KKT conditions. Thus $d(f \circ \phi_{x^*})(0) = 0$. □

As in the smooth case, we need some regularity conditions on the family of parameterizations.

**Definition 2.2.4** Let $X$ be a subset of $M$, and $K$ a subset of $X$.[7] Assume that the injectivity radius of all $x \in K$ is bounded from below by a constant $r > 0$.

[7] As $X$ is already just a subset of $M$, the use of another subset $K \subset X$ might at a first sight not be really necessary for the problems below. One could always just try to identify $X = K$. However, in some cases the parameterizations will be actually needed on the whole space, while the conditions only holds on a subset.

A family of smooth parameterizations $(\phi_x)$ of $X$ satisfies the *weak standard assumptions* on $K$, if the following conditions hold

1. for all $x \in K\colon\ \phi_x(0) = x$,
2. there is a constant $C > 0$ such that $\|T_0\phi_x\| < C$ for all $x \in K$,
3. the family $(T\phi_x\colon \mathbf{R}^n \to T_xM)$ is exp-equicontinuous at $0$ on $K$
4. for any sequence $x_k \in K$ with $x_k \to x^* \in K$ we have that

$$\operatorname{Im} T_0\phi_{x^*} \subset \big\{w \in T_xM \mid w = \lim T_0\phi_{x_m}(v_m), (v_m) \subset \mathbf{R}^k, \\ (x_m) \text{ a subsequence of } (x_k)\big\}.$$ [8]

**Remark 2.2.5** The condition 4 in Definition 2.2.4 is equivalent to the lower semicontinuity of the set-valued map $x \mapsto \operatorname{Im} T_0\phi_x$.

**Example 2.2.6** Let $X = \{(x, y) \in \mathbf{R}^2 \mid xy = 0\}$ and

$$\phi((x, y), t) := \begin{cases} (x + t, 0) & \text{for } x \neq 0 \\ (0, y + t) & \text{otherwise} \end{cases}.$$

Then $(\phi_{(x,y)})$, with $\phi_{(x,y)}\colon t \mapsto \phi((x, y), t)$, is a family of smooth parameterizations of $X$. However, since

$$\operatorname{Im} T_0\phi_{(x,y)} = \begin{cases} \mathbf{R} \times \{0\} & \text{for } x \neq 0 \\ \{0\} \times \mathbf{R} & \text{otherwise} \end{cases}$$

condition 4 is violated and the family does not satisfy the weak standard assumptions on $X$. If we choose instead

$$\hat{\phi}((x, y), t) := \begin{cases} (x + t, 0) & \text{for } x \neq 0 \\ (0, y + t) & \text{for } y \neq 0 \\ (0, 0) & \text{for } (x, y) = (0, 0) \end{cases},$$

then the family of parameterizations $(\hat{\phi}_{(x,y)})$ of $X$, with $\phi_{(x,y)}\colon t \mapsto \phi((x, y), t)$, satisfies the weak standard assumptions on $X$.

[8] Convergence in the standard topology of the tangent bundle.

The parameterizations do not depend smoothly or even continuously on the base points. Therefore we need the following proposition to ensure that $d(f \circ \phi_{x_k})(0) \to 0$ implies that $x^*$ is a critical point in the sense of Definition 2.2.1. Note, that we use the lower semicontinuity of $x \mapsto \operatorname{Im} T_0\phi_x$ to get this result. This motivates the inclusion of this property in the definition of the weak standard assumptions.

**Proposition 2.2.7** *Let $X$ be a subset of $M$ with a family of smooth parameterizations $(\phi_x)$ which satisfies the weak standard assumptions. Assume that $K \subset X$. If $f \colon M \to \mathbf{R}$ is a smooth function and $(x_k)$ a sequence in $K$, $x_k \to x^*$, $x^* \in K$, then $d(f \circ \phi_{x_k})(0) \to 0$ implies $d(f \circ \phi_{x^*})(0) = 0$, i.e. that $x^*$ is a critical point of $f$ on $X$.*

**Proof:** If $d(f \circ \phi_{x^*})(0) \neq 0$ then there exists $v \in \operatorname{Im} T\phi_{x^*}(0)$ with $df(x^*)(v) = c \neq 0$. By the lower semicontinuity of $x \mapsto \operatorname{Im} T_0\phi_x$ and eventually passing to a subsequence of $x_k$, there exists a sequence $v_k \in \operatorname{Im} T\phi_{x_k}(0)$, $v_k \to v$. The continuity of $df$ implies that $df(x_k)(v_k) \to c$. Thus $d(f \circ \phi_{x_k})(0) \not\to 0$. □

We motivated the parameterization approach by choosing an analytic map $\psi \colon N \to X$ from a manifold to an analytic set $X$ and covering $N$ with local parameterizations $(\phi_x)$. The following proposition shows that under some regularity assumptions on $\psi$, the standard assumptions on $(\phi_x)$ imply the weak standard assumptions for the family $(\psi \circ \phi_x)$.

**Theorem 2.2.8** *Let $X$ be a subset of $M$, $K$ a subset of $X$ and $N$ another smooth Riemannian manifold. Assume that the following conditions hold:*

1. *There is a smooth map $\psi \colon N \to M$ with $\psi(N) = X$.*
2. *$\|T_y\psi\|$ is uniformly bounded on $\psi^{-1}(K)$.*
3. *For every $x \in K$, there is a $H_x \subset T_xM$ such that for all $y \in \psi^{-1}(x)$*
$$\operatorname{Im} T_y\psi = H_x.$$
4. *For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $y \in \psi^{-1}(K)$, $v \in T_yN$*
$$\|v\| < \delta \text{ implies } \left\|T_v(\exp_{\psi(y)} \circ \psi \circ \exp_y) - T_0(\exp_{\psi(y)} \circ \psi \circ \exp_y)\right\| < \varepsilon.$$
*Here the notation $\exp$ is used both for the exponential map on $M$ and on $N$.*

5. *There is a uniform lower bound $r > 0$ for the injectivity radius of all $x \in K$ and all $y \in \psi^{-1}(K)$.*

6. *There is a family of smooth parameterizations $(\phi_y \colon \mathbf{R}^n \to N)$ of $N$ which satisfies the standard assumptions on the set $\psi^{-1}(K)$.*

*Then any family of parameterizations $(\hat{\phi}_x)$ of $X$ given by*

$$\hat{\phi}_x := \psi \circ \phi_y, \text{ for a } y \in \psi^{-1}(x),$$

*satisfies the weak standard assumptions on $K$.*

**Proof:** Since $\hat{\phi}_x(0) = (\psi \circ \phi_y)(0) = \psi(y) = x$ for $y \in \psi^{-1}(x)$, we get $\hat{\phi}_x(0) = x$. Let $C$ be an uniform upper bound of $\|T_0\phi_y\|$ and $\|T_y\psi\|$ on $\psi^{-1}(K)$. Then for all $x \in K$ we have a $y \in \psi^{-1}(K)$ such that

$$\left\| T_0\hat{\phi}_x \right\| = \|T_y\psi \circ T_0\phi_y\| \leq \|T_y\psi\| \, \|T_0\phi_y\| \leq C^2.$$

Thus, $\left\| T_0\hat{\phi}_x \right\|$ is uniformly bounded on $K$. To show the exp-equicontinuity of $T\hat{\phi}_x$ at 0 on $K$, we use a similar argument as in the proof of Lemma 2.1.6. However, we have to show first that the family $(\psi \circ \exp_y : T_yN \to M)$ is equicontinuous at 0 on $\psi^{-1}(K)$. For this purpose, let $(\mu_y \colon \mathbf{R}^n \to T_yN \mid y \in N)$, $n = \dim N$ be a family of isometries. We assume that $(\hat{\psi}_x \colon \mathbf{R}^n \to M \mid x \in K)$ is a family of smooth maps such that

$$\hat{\psi}_x = \psi \circ \exp_y \circ \mu_y \quad \text{with } y \in \psi^{-1}(x).$$

Note, that this family has the following properties:

- For all $x \in K$: $\hat{\psi}_x(0) = x$.
- For all $x \in K$ there is $y \in \psi^{-1}(x)$ with $\left\| T_0\hat{\psi}_x \right\| = \|T_y\psi\| < C$, $C$ the bound defined above.
- $T\hat{\psi}_x$ is exp-equicontinuous at 0 on $K$.

Thus, we can apply Proposition 2.1.5 and get that the family $(\hat{\psi}_x)$ is equicontinuous at 0 on $K$. Since the $\mu_y$ are isometries and there are no further conditions, besides $\psi(y) = x$, on the choice of $y \in \psi^{-1}(K)$ in the definition of $\hat{\psi}_x$, this yields the equicontinuity of $(\psi \circ \exp_y : T_yN \to M)$ at 0 on $\psi^{-1}(K)$. Furthermore, by Proposition 2.1.5 the family $(\phi_y)$ is equicontinuous on $\psi^{-1}(K)$ at 0.

By the equicontinuity of $(\psi \circ \exp_y)$ and $(\phi_y)$ at 0 on $\psi^{-1}(K)$, there is a $s > 0$ such that for all $v \in \mathbf{R}^n$, $y \in \psi^{-1}(x)$, $\|v\| < s$ implies $\operatorname{dist}(x, \psi(\phi_y(0))) < r$ and $\operatorname{dist}(y, \phi_y(0)) < r$.[9] Let $x \in K$, $y \in N$ with $\psi(y) = x$ and $v \in \mathbf{R}^n$ with $\|v\| < s$. Then we get the following calculations

$$
\begin{aligned}
&T_v(\exp_x^{-1} \circ\psi \circ \phi_y) - T_0(\exp_x^{-1} \circ\psi \circ \phi_y) \\
&\quad= T_v(\exp_x^{-1} \circ\psi \circ \exp_y \circ \exp_y^{-1} \circ\phi_y) - T_0(\exp_x^{-1} \circ\psi \circ \exp_y \circ \exp_y^{-1} \circ\phi_y) \\
&\quad= T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_v\left(\exp_y^{-1} \circ\phi_y\right) \\
&\qquad -T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_0\left(\exp_y^{-1} \circ\phi_y\right) \\
&\quad= T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_v\left(\exp_y^{-1} \circ\phi_y\right) \\
&\qquad -T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_v\left(\exp_y^{-1} \circ\phi_y\right) \\
&\qquad +T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_v\left(\exp_y^{-1} \circ\phi_y\right) \\
&\qquad -T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) \circ T_0\left(\exp_y^{-1} \circ\phi_y\right) \\
&\quad= \Big(T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) - T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right)\Big) T_v\left(\exp_y^{-1} \circ\phi_y\right) \\
&\qquad +T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right)\left(T_v\left(\exp_y^{-1} \circ\phi_y\right) - T_0\left(\exp_y^{-1} \circ\phi_y\right)\right).
\end{aligned}
$$

This yields the estimate

$$
\begin{aligned}
&\left\|T_v(\exp_x^{-1} \circ\psi \circ \phi_y) - T_0(\exp_x^{-1} \circ\psi \circ \phi_y)\right\| \\
&\quad\leq \left\|T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) - T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right)\right\| \\
&\qquad\qquad \cdot \underbrace{\left\|T_v\left(\exp_y^{-1} \circ\phi_y\right)\right\|}_{=\left\|T_0\left(\exp_y^{-1} \circ\phi_y\right) - T_v\left(\exp_y^{-1} \circ\phi_y\right) - T_0\left(\exp_y^{-1} \circ\phi_y\right)\right\|} \\
&\qquad + \underbrace{\left\|T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right)\right\|}_{=\|T_y\psi\|}\left\|T_v\left(\exp_y^{-1} \circ\phi_y\right) - T_0\left(\exp_y^{-1} \circ\phi_y\right)\right\| \\
&\quad\leq \left\|T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ\psi \circ \exp_y\right) - T_0\left(\exp_x^{-1} \circ\psi \circ \exp_y\right)\right\| \\
&\qquad \cdot\left(\left\|T_0\left(\exp_y^{-1} \circ\phi_y\right)\right\| + \left\|T_0\left(\exp_y^{-1} \circ\phi_y\right) - T_v\left(\exp_y^{-1} \circ\phi_y\right)\right\|\right) \\
&\qquad + \|T_y\psi\|\left\|T_v\left(\exp_y^{-1} \circ\phi_y\right) - T_0\left(\exp_y^{-1} \circ\phi_y\right)\right\|.
\end{aligned}
$$

[9]By abuse of notation we use dist both for the Riemannian distance on $M$ and on $N$.

Note, that since $\|v\| < s$, the well-known corollary of the Gauss lemma, cf. [97, Cor. 4.2.3, 4.2.4] yields again that $\operatorname{dist}(y, \phi_y(v)) = \left\|\exp_y^{-1}(\phi_y(v))\right\|$. We denoted by $C > 0$ an uniform upper bound for $\|T_y\psi\|$ and $\|T_0\phi_y\|$ on $\psi^{-1}(K)$. The equicontinuity of $(\phi_y)$ at 0 on $\psi^{-1}(K)$, the exp-equicontinuity of $(T\phi_y)$ at 0 and condition 4 in the proposition yield for any $\varepsilon > 0$ a $\delta \in (0, s)$ such that for all $x \in K$, $y \in \psi^{-1}(K)$, $v \in \mathbf{R}^n$, with $\|v\| < \delta$, the estimates

$$\begin{aligned} \left\|T_v\left(\exp_y^{-1} \circ \phi_y\right) - T_0\left(\exp_y^{-1} \circ \phi_y\right)\right\| &< \varepsilon, \\ \left\|T_{\exp_y^{-1}(\phi_y(v))}\left(\exp_x^{-1} \circ \psi \circ \exp_y\right) - T_0\left(\exp_x^{-1} \circ \psi \circ \exp_y\right)\right\| &< \varepsilon \end{aligned}$$

hold. Combing this with the estimate above, we get for any $\varepsilon > 0$ a $\delta > 0$ such that for all $x \in K$, $y \in \psi^{-1}(K)$, $v \in \mathbf{R}^n$, with $\|v\| < \delta$:

$$\left\|T_v(\exp_x^{-1} \circ \psi \circ \phi_y) - T_0(\exp_x^{-1} \circ \psi \circ \phi_y)\right\| < \varepsilon(C + \varepsilon) + \varepsilon C.$$

Thus $(T\hat{\phi}_x)$ is exp-equicontinuous at 0 on $K$. We must still check the last condition of the weak standard assumptions. For any sequence $x_k \in K$, $x_k \to x \in K$, we can choose a $y \in \psi^{-1}(x)$ and a sequence $y_k \to y$, $y_k \in \psi^{-1}(x_k)$. Since $T\psi$ is continuous, we get that

$$\operatorname{Im} T_y\psi \subset \{w \in T_xM \mid \exists (v_k) \subset \mathbf{R}^n\colon \lim_{k\to\infty} T_{y_k}\psi(v_k) = w\}.$$

Note that $\operatorname{Im} T_0\phi_y = T_yN$ for all $y \in \psi^{-1}(K)$ and for all $x \in K$, $y \in \psi^{-1}(K)$, there is a $H_x \subset T_xM$ with $\operatorname{Im} T_y\psi = H_x$. Thus

$$\operatorname{Im} T_0(\psi \circ \phi_x) = H_x \subset \{w \in T_xM \mid \exists (v_k) \subset \mathbf{R}^n\colon \lim_{k\to\infty} T_0(\psi \circ \phi_{y_k})v_k = w\}.$$

Therefore condition 4 of the weak standard assumptions holds on $K$. □

On compact sets most of the conditions of Theorem 2.2.8 are automatically satisfied. This yields the following corollary.

**Corollary 2.2.9** *Let $X$ be a subset of $M$, $K$ a compact subset of $X$ and $N$ another smooth Riemannian manifold. Assume that the following conditions hold.*

1. *There is a smooth map $\psi\colon N \to M$ with $\psi(N) = X$.*
2. *For every $x \in K$, there is a $H_x \subset T_xM$ such that for all $y \in \psi^{-1}(x)$*

$$\operatorname{Im} T_y\psi = H_x.$$

3. *There is a smooth family of parameterizations* $(\phi_y \colon \mathbf{R}^n \to N)$ *of* $N$ *which satisfies the standard assumptions on the set* $\psi^{-1}(K)$.

*Then any family of parameterizations* $(\hat{\phi}_x)$ *of* $X$ *given by*

$$\hat{\phi}_x := \psi \circ \phi_y, \text{ for a } y \in \psi^{-1}(x),$$

*satisfies the weak standard assumptions on* $K$.

Assume that the constraint set $X$ can be stratified, i.e. there exists a stratification of $M$ compatible with $X$. If the parameterizations are then submersions onto the strata, then a point $x \in X$ is a critical point if and only if it is a critical point of the function restricted to the stratum of $x$. The following proposition illustrates this property.

**Proposition 2.2.10** *Let* $X$ *be a subset of* $M$ *and* $N$ *another smooth Riemannian manifold. Assume that the following conditions hold:*

- $S_j$, $j \in \Lambda$, *is a* $C^2$*-stratification of* $M$, *compatible with* $X$.
- $(\phi_x \colon \mathbf{R}^n \to M)$ *is a family of smooth parameterizations of* $X$.
- *For all* $j \in \Lambda$ *and* $x \in S_j$ *we have*
$$\operatorname{Im} T_0\phi_x = T_x S_j.$$
- $f : M \to \mathbf{R}$ *is a smooth function.*

*Then any* $x \in X$ *is a critical point of* $f$ *on* $X$ *induced by the parameterizations* $(\phi_x)$ *if and only if*

$$\pi_{T_x S_j}(\operatorname{grad} f(x)) = 0,$$

$\pi_{T_x S_j}$ *the orthogonal projection in* $T_x M$ *on* $T_x S_j$.

**Proof:** Let $x \in S_j$ be a point. Note that

$$\left\langle \pi_{T_x S_j}(\operatorname{grad} f(x)), v \right\rangle = \langle \operatorname{grad} f(x), v\rangle = df(x)(v) = 0$$

for all $v \in T_x S_j$. Since $\operatorname{Im} T_0\phi_x = T_x S_j$, and $d(f \circ \phi_x)(0) = df(x) \circ T_0\phi_x$ we have that

$$df(x)|_{T_x S_j} = 0 \text{ if and only if } d(f \circ \phi_x)(0) = 0.$$

Thus $\pi_{T_x S_j}(\operatorname{grad} f(x)) = 0$ if and only if $x$ is a critical point of $f|_X$. □

For parameterizations constructed from a smooth map $N \to X$ and a parameterizations of the manifold $N$, we have the following corollary of Proposition 2.2.10.

**Corollary 2.2.11** *Let $X$ be a subset of $M$ and $N$ another smooth Riemannian manifold. Assume that the following conditions hold:*

1. $\psi\colon N \to M$ *is a smooth map with* $\psi(N) = X$.
2. $S_j$, $j \in \Lambda$, *is a* $C^2$*-stratification of* $M$, *compatible with* $X$.
3. *For all* $j \in \Lambda$, $x \in S_j \cap X$ *and* $y \in \psi^{-1}(x)$ *we have*
$$\operatorname{Im} T_y\psi = T_xS_j.$$
4. $f : M \to \mathbf{R}$ *is a smooth function.*
5. $(\phi_y\colon \mathbf{R}^n \to N)$ *is a family of smooth parameterizations of* $N$ *which are all local diffeomorphisms in* $0$.

*Let* $(\hat{\phi}_x)$ *family of smooth parameterizations of* $X$ *given by*

$$\hat{\phi}_x := \psi \circ \phi_y, \text{ for a } y \in \psi^{-1}(x).$$

*Then any* $x \in X$ *is a critical point of* $f$ *on* $X$ *induced by the parameterizations* $(\hat{\phi}_x)$ *if and only if*

$$\pi_{T_xS_j}(\operatorname{grad} f(x)) = 0,$$

*where* $\pi_{T_xS_j}$ *denotes the orthogonal projection in* $T_xM$ *on* $T_xS_j$.

**Proof:** Since the $\phi_y$ are local diffeomorphisms, we see that for all $j \in \Lambda$, $x \in S_j$ and $y \in \psi^{-1}(x)$

$$\operatorname{Im} T_0\hat{\phi}_x = \operatorname{Im} T_y\psi = T_xS_j$$

holds. Thus the result follows directly from Proposition 2.2.10. □

**Remark 2.2.12** Let the assumptions of Corollary 2.2.11 hold. For strata $S_i$, $S_j$, $S_i \subset \overline{S}_j$, $S_i, S_j \subset X$ and a sequence $(x_k) \subset S_j$, $x_k \to x \in S_i$, there is a sequence $(y_k) \subset N$, $y_k \to y \in N$. Assume that $\operatorname{Im} T_{y_k}\psi$ converges to a $\tau \subset T_xM$. By smoothness of $\psi$, we have that

$$\operatorname{Im} T_y\psi \subset \tau.$$

Thus

$$T_xS_i = \operatorname{Im} T_y\psi \subset \tau = \lim \operatorname{Im} T_{y_k}\psi = \lim T_{x_k}S_j.$$

This yields that the strata $S_j \subset X$ satisfy the Whitney-(a) condition.

**Remark 2.2.13** The reader should note that a desingularization $\psi\colon N \to X$ of an analytic or even algebraic set does not necessarily satisfy the conditions of Corollary 2.2.11. Regular, self-intersecting curves in $\mathbf{R}^n$ provide already suitable counterexamples. Let $\gamma\colon \mathbf{R} \to \mathbf{R}^n$ be a regular[10], smooth curve with an $a, b \in \mathbf{R}$ such that $a \neq b$, $\gamma(a) = \gamma(b)$, and $\dot\gamma(a)$, $\dot\gamma(b)$ linearly independent. The parameterization $\gamma\colon \mathbf{R} \to X$, $X = \{\gamma(t) \mid t \in \mathbf{R}\}$ resolves the singularities of $X$. Since $\dot\gamma(a)$, $\dot\gamma(b)$ are linearly independent and $\dim X = 1$, any Whitney-(a) stratification of $X$ will have the point $\gamma(a)$ as a 0-dimensional stratum. But as $\gamma$ is regular, we have $\dim(\operatorname{Im} T_a\gamma) = \dim(\operatorname{Im} T_b\gamma) = 1$. Thus $\gamma$ does not satisfy the condition of Corollary 2.2.11. An explicit example for such a curve would be $X = \{(x, y) \in \mathbf{R}^2 \mid y^2 = x^2 - x^3\}$, see [46, p.24].

Generally, the algebraic blow-up for a singular variety does not satisfy the conditions of Corollary 2.2.11. Consider a blow-up of $\mathbf{R}^n$ with smooth center $C$, $\dim C = n - k$. We refer the reader to [141, p.71] for a detailed construction of blow-ups with a specific center. In local coordinates we can assume that $C = \{0\} \times \mathbf{R}^{n-k}$. Then the blow-up is given by

$$B = \overline{\{(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n, (x_1 : \ldots : x_k)) \mid (x_1, \ldots, x_n) \in \mathbf{R}^n\}} \subset \mathbf{R}^n \times \mathbf{P}^{k-1}.$$

where $(x_1 : \ldots, x_k)$ denotes the projective coordinates[11]. The projection $\pi$ on the first $n$-coordinates maps $B$ to $\mathbf{R}^n$. It is easily seen that $\operatorname{rk} T_y\pi = n-k+1$ for all $y \in \{0\} \times \mathbf{P}^{k-1}$. However, since $\pi$ is a diffeomorphism on $B \setminus \{0\} \times \mathbf{P}^{k-1}$ and not a local diffeomorphism in $\{0\} \times \mathbf{P}^{k-1}$, $\mathbf{R}^n$ has to be stratified into $\mathbf{R}^n \setminus C$ and $C$. But then $\operatorname{Im} T_y\pi \neq T_0C$ for all $y \in \pi^{-1}(0)$. Thus the conditions of Corollary 2.2.11 are not satisfied. This argument can be applied analogously to blow-ups of general algebraic varieties instead of $\mathbf{R}^n$.

This also illustrates that it is generally not possible to construct a family of smooth parameterizations, which satisfies the weak standard assumptions, from the algebraic blow-up construction of singular variety. In the $\mathbf{R}^n$ case above we see that for a point $x$ in the center $C$, at each point $y \in \pi^{-1}(x)$ the image of $T_y\pi$ is different. Thus constructing a family of parameterizations from parameterizations $\phi_y$ of $B$, by setting

$$\hat\phi_x = \pi \circ \phi_y, \quad y \in \pi^{-1}(x)$$

[10] i.e. $\dot\gamma(t) \neq 0$ for all $t \in \mathbf{R}$

[11] cf. [140, p. 41]

gives a family which does not satisfy the weak standard assumptions. This argument can be extended to any singular variety $X$, unless the preimages of points of the center consist of single points. But in this case $X$ would be smooth at the center anyway.

### 2.2.3 Descent iterations on singular sets

We now turn to the definition of descent iterations and the discussion of some convergence results. The definition of descent iterations in a family of parameterizations on constraint sets is analogous to the smooth case.

**Definition 2.2.14** Let $f : M \to \mathbf{R}$ be a smooth cost function and $X \subset M$ a constraint set. A *descent iteration* for $f$ in a family of smooth parameterizations $(\phi_x)$ of $X$ is a sequence $(x_k) \subset X$ given by

$$x_{k+1} = \phi_{x_k}(\alpha_k s_k), \quad \alpha_k \in \mathbf{R}_+, s_k \in \mathbf{R}^n,$$

such that for all $k \in \mathbf{N}$

$$f(x_{k+1}) \leq f(x_k)$$
$$\text{and } f(x_{k+1}) = f(x_k) \text{ implies } x_l = x_k \text{ for all } l > k.$$

We call $\alpha_k$ the *step size* and $s_k$ the *descent direction*.

For descent iterations on singular sets we define the same conditions for convergence as in the smooth case.

**Definition 2.2.15** Let $X$ be a subset of $M$ and $f \colon M \to \mathbf{R}$ a smooth cost function. A descent iteration $(x_k)$ for $f$ in a family of smooth parameterizations $(\phi_x)$ of $X$ satisfies the *angle condition* (AC) if there is a constant $\varepsilon > 0$ such that

$$-d(f \circ \phi_{x_k})(0)(s_k) \geq \varepsilon \, \|s_k\| \, \|\mathrm{grad}(f \circ \phi_{x_k})(0)\| \qquad \text{(AC)}$$

for all $k \in \mathbf{N}$. The iteration satisfies the *first and second Wolfe-Powell conditions* (WP1), (WP2) if there are constants $\sigma$, $\rho \in (0,1)$ such that for all $k \in \mathbf{N}$

$$f(x_k) - f(x_{k+1}) \geq -\sigma d(f \circ \phi_{x_k})(0)(\alpha_k s_k) \qquad \text{(WP1)}$$
$$d(f \circ \phi_{x_k})(\alpha_k s_k)(s_k) \geq \rho d(f \circ \phi_{x_k})(0)(s_k). \qquad \text{(WP2)}$$

**Remark 2.2.16** In the same way we can extend the Armijo line search to these descent iterations in a family of smooth parameterizations of $X$.

We have a similar convergence theorems as in the manifold case.

**Theorem 2.2.17** *Let $X$ be a subset of $M$ and $(x_k)$ a sequence in $X$. Assume that the following conditions hold.*

- $f\colon M \to \mathbf{R}$ *is a smooth cost function,*
- $(\phi_x)$ *is a family of smooth parameterizations of $X$,*
- $(x_k)$ *is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *the injectivity radius of all $x \in S = \{x \in X \mid f(x) \leq f(x_0)\}$ is bounded from below by a constant $r > 0$,*
- $df$ *is uniformly* exp*-continuous on $S$,*
- $\|df\|$ *is uniformly bounded on $S$,*
- $(\phi_x)$ *satisfies the weak standard assumptions on $S$.*

*Then $\|d(f \circ \phi_{x_k})(0)\| \to 0$ or $f(x_k) \to -\infty$. In particular, all accumulation points of $(x_k)$ are critical points of $f$ on $X$ induced by the family of parameterizations.*

**Proof:** We use the same argument as in the smooth case. Assume that $f(x_k)$ is strictly decreasing, otherwise the claim follows directly from the definition of descent iterations and the Wolfe-Powell conditions. Let $h_k\colon \mathbf{R}^n \to \mathbf{R}$ be defined by

$$h_k(v) := (f \circ \phi_{x_k})(v).$$

We denote by $\alpha_k \in \mathbf{R}_+$, $s_k \in \mathbf{R}^n$, the step sizes and descent directions. W.l.o.g. $\|s_k\| = 1$ for all $k \in \mathbf{N}$. Application of Lemma 2.1.6 yields the equicontinuity of $d(f \circ \phi_{x_k})$ on $S$, i.e. equicontinuity of $dh_k$ at 0. If $f(x_k)$ is bounded from below, then $h_k(0) - h_k(\alpha_k s_k) = f(x_k) - f(x_{k+1}) \to 0$. Therefore we get by Lemma 2.1.18 that

$$\|dh_k(0)\| = \|d(f \circ \phi_{x_k})(0)\| \to 0.$$

If $(x_k)$ has an accumulation point, then $f(x_k)$ is bounded from below. This proves the convergence of $\|d(f \circ \phi_{x_k})\|$ to 0. Since $\|d(f \circ \phi_{x_k})(0)\| \to 0$, Proposition 2.2.7 implies the accumulation point is a critical point of $f$ on $X$ induced by the family of parameterizations. □

**Theorem 2.2.18** *Let $X$ be a subset of $M$ and $(x_k)$ a sequence in $X$. Assume that the following conditions hold:*

- $f\colon M \to \mathbf{R}$ *is a smooth cost function,*
- $(\phi_x)$ *is a family of smooth parameterizations of $X$,*
- $(x_k)$ *is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*
- *For any compact subset $K$ of $X$, the family $(\phi_x)$ satisfies the weak standard assumptions on $K$, i.e. the constants in the weak standard assumptions depend on $K$.*

*Then all accumulation points of $(x_k)$ are critical points of $f$ on $X$ induced by the family of parameterizations.*

**Proof:** Again we can basically use the same argument as in the smooth case. Assume that $f(x_k)$ is strictly decreasing, otherwise the claim follows again directly from the Wolfe-Powell conditions. Let $x^* \in X$ be an accumulation point of the descent iteration. Note that $f(x_k)$ is bounded from below. We can choose an relatively compact, open neighborhood $U$ of $x^*$ in $X$, such that $\|df\|$ is uniformly bounded and $df$ is exp-continuous on $U$. Let $(x_{k_j}) \subset U$ be a subsequence of $(x_k)$ which converges to $x^*$. The same argument as in the proof of Theorem 2.2.17 applied to this subsequence yields

$$\left\| d(f \circ \phi_{x_{k_j}}) \right\| \to 0.$$

Thus, by Proposition 2.2.7, the accumulation point $x^*$ is a critical point of $f$ on $X$ induced by the parameterizations. $\square$

**Corollary 2.2.19** *Let $X$ be a subset of $M$ and $(x_k)$ a sequence in $X$. Assume that the following conditions hold:*

- $f\colon M \to \mathbf{R}$ *is a smooth cost function with compact sublevel sets $\{x \in X \mid f(x) \leq c\}$ on $X$.*
- $(\phi_x)$ *is a family of smooth parameterizations,*
- $(x_k)$ *is a descent iteration for $f$ in the parameterizations and satisfies the angle and Wolfe-Powell conditions,*

- *For any compact subset $K$ of $X$, the family $(\phi_x)$ satisfies the weak standard assumptions on $K$, i.e. the constants in the weak standard assumptions depend on $K$.*

*Then the sequence $(x_k)$ converges to the set of critical points of $f$ on $X$ induced by the family of parameterizations.*

**Proof:** Similar to the smooth case, this follows from fact that the iterates remain in a compact set by a straightforward application of Theorem 2.2.18.
□

**Remark 2.2.20** Assume that the parameterizations were constructed as in Theorem 2.2.8, i.e. from a smooth map $\psi\colon N \to X$ and a family of parameterizations on $N$. If the map $\psi$ is a bijection, then the descent iteration on $X$ can be lifted to a descent iteration on $N$ for the function $f \circ \psi$. Since this iteration satisfies the Wolfe-Powell and angle conditions, the convergence theorems for the smooth case yield convergence to the critical points of $f$ on $X$. However, this argument fails if $\psi$ is not bijective, since a lift $(y_k)$ of descent iteration $(x_k)$ on $X$ to $N$ will not be necessarily a descent sequence on $N$[12].

### 2.2.4 Example: Approximation by nilpotent matrices

As an illustration of our approach to optimization on singular sets, we consider the following approximation problem for a matrix $N \in \mathbf{R}^{n\times n}$: Find $\mathcal{A} \in R^{n\times n}$, with $\mathcal{A}^n = 0$, such that

$$\|N - \mathcal{A}\| = \min_{A \in \mathbf{R}^{n\times n}, A^n = 0} \|N - A\|_F\,, \tag{2.4}$$

i.e. we want to find the best approximation of $N$ in the Frobenius norm, $\|A\|_F = \sqrt{\operatorname{tr}(A^\top A)}$, by a nilpotent matrix $\mathcal{A}$. Here, the constraint set is the set of nilpotent matrices

$$X = \{A \in \mathbf{R}^{n\times n} \mid A^n = 0\}.$$

As a cost function, we choose the smooth function

$$f(A) := \|N - A\|_F^2 = \operatorname{tr}((N - A)^\top (N - A)).$$

[12]For example, $y_{k+1} \in N$ does not even have to be an element of $\operatorname{Im}\phi_{y_k}$, ($\phi_y\colon \mathbf{R}^n \to N$) the parameterizations of $N$.

Clearly, the minima of $f$ on $X$ coincide with the solutions of (2.4).

The of nilpotent matrices is a well-known singular algebraic variety. We refer the reader to [78] for more information. In particular, it is known how to construct a desingularization of $X$. Denote by $\mathrm{Flag}(n)$ the following set of $n$-tuples of subspaces of $\mathbf{R}^n$

$$\mathrm{Flag}(n) := \{(V_1, \ldots, V_n) \mid V_i \subset \mathbf{R}^n, \dim V_i = i, V_i \subset V_{i+1}\}.$$

This set is a smooth manifold, the so called *(complete) flag manifold* [77]. We can now construct the set

$$\tilde{M} := \{(A, V_1, \ldots, V_n) \mid A \in \mathbf{R}^{n\times n}, (V_1, \ldots, V_n) \in \mathrm{Flag}(n), \\ AV_i \subset V_{i-1}, AV_1 = \{0\}\}.$$

This set is a smooth manifold[13] and the projection

$$\pi((A, V_1, \ldots, V_n)) = A$$

is a smooth, surjective map onto $X$ [78]. It can even be shown that it is a desingularization in the algebraic sense, i.e. maps an open, dense, algebraic subset of $\tilde{M}$ diffeomorphically onto the set of smooth points of $X$.

To construct a family of parameterizations of $X$, we recall some facts on the flag manifold from [77, p.62]. The flag manifold is diffeomorphic to the isospectral orbit

$$F = \{\Theta \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\Theta^\top \mid \Theta \in \mathrm{SO}(n)\}$$

for any fixed sequence of eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$. The manifold $F$ is identified with $\mathrm{Flag}(n)$ by mapping an $B \in F$ to the $n$-tuple of subspaces $(V_1(B), \ldots, V_n(B))$, where $V_i(B)$ is spanned by the eigenvectors of the eigenvalues $\lambda_1, \ldots, \lambda_i$ of $B$.

This representation of the flag manifold can be used to get a simplified representation of $\tilde{M}$ and ultimately parameterizations of $X$. Clearly, $\tilde{M}$ is diffeomorphic to

$$\{(A, B) \mid A \in \mathbf{R}^{n\times n}, B \in F, AV_i(B) \subset V_{i-1}(B), AV_1(B) = \{0\}\}, \qquad (2.5)$$

since we just apply a diffeomorphism to the flag components of the tuple $(A, V_1, \ldots, V_n)$. Now, given an $(A, V_1(B), \ldots, V_n(B)) \in \tilde{M}$, $B \in F$, we want

[13] In fact, it is diffeomorphic to the cotangent bundle of $\mathrm{Flag}(n)$ [78].

to examine to relation between $A$ and $B$. Let $e_1, \dots, e_n$ the orthonormal eigenvectors of $B$ belonging to the eigenvalues $\lambda_1, \dots, \lambda_n$. Since $V_i(B)$ is spanned by $e_1, \dots, e_i$ and $AV_i(B) \subset V_{i+1}(B)$ we have that $e_1 \in \ker A$ and for all $i = 2, \dots, n$

$$Ae_i = \sum_{j=1}^{i-1} \mu_{ij} e_j$$

with $\mu_{ij} \in \mathbf{R}$. This gives the decomposition $B = \Theta \operatorname{diag}(\lambda_1, \dots, \lambda_n)\Theta^\top$ with $\Theta \in SO(n)$, $\Theta = (e_1 \dots e_n)$. Then above condition on $A$ holds if and only if $\Theta^\top A \Theta = L$, with $L \in \mathbf{R}^{n\times n}$ a lower triangular matrix. We can use this decomposition in the description of (2.5). This yields that the manifold $\tilde{M}$ is diffeomorphic to

$$M := \{(\Theta L \Theta^\top, \Theta \operatorname{diag}(\lambda_1, \dots, \lambda_n)\Theta^\top \mid \Theta \in \mathrm{SO}(n), \\ L \in \mathbf{R}^{n\times n} \text{ lower triangular}\} \subset \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n}.$$

Furthermore $X = \pi(M)$, where $\pi$ denotes the projection onto the first component.

Denote by $\mathrm{Tri}(n)$ the linear space of lower triangular $n \times n$-matrices. Let $\mu\colon \mathfrak{so}(n) \to \mathrm{SO}(n)$ be any smooth first order approximation[14] of the matrix exponential map exp. Then for a $Y = (\Theta L \Theta^\top, \Theta \operatorname{diag}(\lambda_1, \dots, \lambda_n)\Theta^\top) \in M$ we have a parameterization $\mu_Y\colon \mathrm{Tri}(n) \times \mathfrak{so}(n) \to M$

$$\hat{\phi}_Y(S, \Omega) := (\Theta\mu(\Omega)(L + S)\mu(\Omega)^\top\Theta^\top, \Theta\mu(\Omega) \operatorname{diag}(\lambda_1, \dots, \lambda_n)\mu(\Omega)^\top\Theta^\top).$$

Projecting onto the first component, we get the family of smooth parameterizations $\phi\colon \mathrm{Tri}(n) \times \mathfrak{so}(n) \to M$ of the constraint set $X$,

$$\phi_{\Theta L \Theta^\top}(S, \Omega) = \Theta\mu(\Omega)(L + S)\mu(\Omega)^\top\Theta^\top$$

for $\Theta L \Theta^\top \in X$, $\Theta \in \mathrm{SO}(n)$, $S \in \mathrm{Tri}(n)$. Note, that for a $A \in X$ the associated $\phi_A$ of the form above is not necessarily unique.

We can also derive these parameterizations from the real Schur decomposition $A = \Theta R \Theta^\top$, $\Theta \in \mathrm{SO}(n)$, $R$ upper triangular, see [66, 7.4.1]. However, our approach illustrates more clearly which role the desingularization of the set of nilpotent matrices plays.

[14] i.e. $T_0\mu = T_0 \exp$

Using this family of parameterizations, we want to obtain a gradient descent algorithm on $X$. For this task, we have to calculate the differential of $f \circ \phi_{\Theta L \Theta^\top}$. Note that

$$f \circ \phi_{\Theta L \Theta^\top}(S,\Omega) = \|N\|_F^2 - 2\,\mathrm{tr}(N^\top \phi_{\Theta L \Theta^\top}(S,\Omega)) + \|L+S\|_F^2 .$$

Hence,

$$\begin{aligned}
&d(f \circ \phi_{\Theta L \Theta^\top})(S,\Omega)(H,K) \\
&\quad= -2\,\mathrm{tr}\left(N^\top \Theta T_\Omega \mu(K)(L+S)\mu(\Omega)^\top \Theta^\top\right) + 2\,\mathrm{tr}((L+S)^\top H) \\
&\qquad -2\,\mathrm{tr}\left(N^\top \Theta \mu(\Omega)(L+S)(T_\Omega \mu(K))^\top \Theta^\top\right) - 2\,\mathrm{tr}\left(N^\top \Theta \mu(\Omega) H \mu(\Omega)^\top \Theta^\top\right) \\
&\quad= -2\,\mathrm{tr}\left(\left((L+S)\mu(\Omega)^\top \Theta^\top N^\top \Theta + (L+S)^\top \mu(\Omega)^\top \Theta^\top N \Theta\right) T_\Omega \mu(K)\right) \\
&\qquad + \mathrm{tr}\left(\left((L+S)^\top - \mu(\Omega)^\top \Theta^\top N^\top \Theta \mu(\Omega)\right) H\right),
\end{aligned}$$

for $H \in \mathrm{Tri}(n)$, $K \in \mathfrak{so}(n)$. In particular, we get with $T_0 \mu(K) = K$ that

$$\begin{aligned}
&d(f \circ \phi_{\Theta L \Theta^\top})(0,0)(H,K) \\
&\quad= -2\,\mathrm{tr}\left(\left(L\Theta^\top N^\top \Theta + L^\top \Theta^\top N \Theta\right) K\right) + 2\,\mathrm{tr}\left(\left(L^\top - \Theta^\top N^\top \Theta\right) H\right).
\end{aligned}$$

Endowing $\mathrm{Tri}(n)$ with the scalar product $\mathrm{tr}(H^\top K)$ and $\mathrm{SO}(n)$ with the Riemannian metric induced by the scalar product $-\,\mathrm{tr}(H^\top K)$ on $\mathfrak{so}(n)$, we obtain the gradient

$$\mathrm{grad}(f \circ \phi_{\Theta L \Theta^\top})(0,0) = (2(L - \pi_{\mathrm{Tri}}(\Theta^\top N \Theta)), -2\pi_{\mathfrak{so}}(\Theta^\top N \Theta L + \Theta^\top N^\top \Theta L^\top)),$$

where $\pi_{\mathrm{Tri}} \colon \mathbf{R}^{n\times n} \to \mathrm{Tri}(n)$, $\pi_{\mathfrak{so}} \colon \mathbf{R}^{n\times n} \to \mathfrak{so}(n)$ denote the orthogonal projections of the matrix space $\mathbf{R}^{n\times n}$ onto the linear spaces $\mathrm{Tri}(n)$ and $\mathfrak{so}(n)$. Note, that the decomposition $A = \Theta L \Theta^\top$ can be iteratively updated in the algorithm, since the parameterizations $\phi_{\Theta L \Theta^\top}$ give automatically a such decomposition. A similar approach of iterative updating of a singular value decomposition was also used by Helmke et al. in [75].

We get the following algorithm:

**Algorithm 2.2.21** Let $N \in \mathbf{R}^{n\times n}$. Choose an initial $A_0 = \Theta_0 L_0 \Theta_0^\top \in X$ with $L_0 \in \mathrm{Tri}(n)$, $\Theta_0 \in \mathrm{SO}(n)$.

1. Set

$$\begin{aligned}
K_k &= 2(\pi_{\mathrm{Tri}}(\Theta_k^\top N \Theta_k) - L_k) \\
H_k &= 2\pi_{\mathfrak{so}}(\Theta_k^\top N \Theta_k L_k + \Theta_k^\top N^\top \Theta_k L_k^\top).
\end{aligned}$$

2. Calculate a step size $\alpha_k$ along the curve $\alpha \mapsto \phi_{\Theta_k^\top L_k \Theta_k}(\alpha K_k, \alpha H_k)$ such that the Wolfe-Powell conditions are satisfied.

3. Set

$$\begin{aligned} L_{k+1} &= L_k + \alpha_k K_k \\ \Theta_{k+1} &= \mu(\alpha_k H_k)\Theta_k \\ A_{k+1} &= \Theta_{k+1} L_{k+1} \Theta_{k+1}^\top. \end{aligned}$$

4. Set $k = k + 1$ and go to step 1.

Unfortunately, the family of parameterizations of $X$ does not satisfy the weak standard assumptions, even on compact subsets of $X$. This can be seen by the following construction. The parameterizations have at the point $(0,0)$ the tangent map

$$T_{(0,0)}\phi_{\Theta L \Theta^\top}(H, K) = \Theta K L \Theta^\top + \Theta L K^\top \Theta^\top + \Theta H \Theta^\top,$$

$H \in \mathrm{Tri}(n)$, $K \in \mathfrak{so}(n)$. Now choose $\Theta_1, \Theta_2 \in \mathrm{SO}(n)$ such that the $n$th columns of $\Theta_1$ and $\Theta_2$ span different subspaces of $\mathbf{R}^n$. Let $(L_k) \subset \mathrm{Tri}(n)$ be a sequence with $L_k \to 0$ and $(\hat{\Theta}_k) \subset \mathrm{SO}(n)$ a sequence with $\hat{\Theta}_k \to \Theta_1$. Then

$$\lim_{k\to\infty} \operatorname{Im} T_{(0,0)}\phi_{\hat{\Theta}_k L_k \hat{\Theta}_k^\top} = \{\Theta_1 S \Theta_1^\top \mid S \in \mathrm{Tri}(n)\}$$

and

$$\operatorname{Im} T_{(0,0)}\phi_{\Theta_2 0 \Theta_2^\top} = \{\Theta_2 S \Theta_2^\top \mid S \in \mathrm{Tri}(n)\}.$$

The subspaces of $\mathbf{R}^{n\times n}$ have the same dimension. But since $\ker \Theta_i S \Theta_i^\top$, $S \in \mathrm{Tri}(n)$, is spanned by the $n$th column of $\Theta_i$, $i = 1, 2$, these subspaces do not coincide. On the other hand, for a nilpotent matrix $L$ of rank $n-1$, the associated flag of invariant subspaces is unique. Thus, no matter which parameterizations $\phi_{\Theta L \Theta^\top}$ for each $A \in X$ are chosen, there are matrices and sequences as above such that the semicontinuity condition on $\operatorname{Im} T_0\phi$ is not satisfied.

Nevertheless, we can provide a convergence result for Algorithm 2.2.21.

**Proposition 2.2.22** *Let $(A_k) \subset X$, $A_K = \Theta_k L_k \Theta_k^\top$, $(\Theta_k) \subset \mathrm{SO}(n)$, $(L_k) \subset \mathrm{Tri}(n)$ be the sequences produced by Algorithm 2.2.21. Then*

$$d(f \circ \phi_{\Theta_k L_k \Theta_k^\top})(0,0) \to 0$$

*and $\Theta_k$, $L_k$ converge.*

**Proof:** Consider the family of parameterizations ($\hat{\phi}_{(L,\Theta)}\colon \operatorname{Tri}(n) \times \mathfrak{so}(n) \to \operatorname{Tri}(n) \times \operatorname{SO}(n)$),

$$\hat{\phi}_{(L,\Theta)}(S, \Omega) = (L + S, \Theta\mu(\Omega))$$

of the smooth manifold $N = \operatorname{Tri}(n) \times \operatorname{SO}(n)$. If we identify $\operatorname{Tri}(n)$ with $\mathbf{R}^{n(n-1)/2}$, we see that the Euclidean group[15] $E(n(n-1)/2)$ acts transitively on $\operatorname{Tri}(n)$. This yields a transitive action of $G = E(n(n-1)/2) \times \operatorname{SO}(n)$ on $N$. If we equip $\operatorname{Tri}(n)$ with the scalar product $\operatorname{tr}(H^\top K)$ and $\operatorname{SO}(n)$ with a biinvariant Riemannian metric, then the product metric on $N$ is invariant under the action of $G$. It is now straightforward to verify that the conditions of Proposition 2.1.12 hold for $(\hat{\phi}_{(L,\Theta)})$. Hence, this family satisfies the standard assumptions on $N$. Instead of considering Algorithm 2.2.21 as a gradient descent algorithm on $X$, we can also consider it as a gradient-like descent algorithm on $N$ for the analytic cost function $\hat{f}\colon (L, \Theta) \mapsto f(\Theta L \Theta^\top)$ in the family of parameterizations $(\hat{\phi}_{(L,\Theta)})$. Note, that $\hat{f}$ has compact sublevel sets. The convergence results for smooth gradient-like descent, Corollary 2.1.21 and Theorem 2.1.22, yield that $(L_k, \Omega_k)$ converges to a single critical point of $\hat{f}$. □

[15] see [68] for a definition.

## 2.3 Optimization of non-smooth functions

We will now turn to the dual situation that the cost function is defined on a smooth manifold, but non-smooth itself, i.e. we consider the following problem.

Let $M \to \mathbf{R}$ be a smooth manifold and $f : M \to \mathbf{R}$ a Lipschitz continuous cost function. Find a $x^* \in M$ with

$$f(x^*) = \min_{x \in M} f(x).$$

This problem is equivalent to the previously discussed case of optimizing a smooth cost function on a non-smooth subset of a smooth manifold, i.e. to finding $(x^*, y^*) \in M \times \mathbf{R}$ such that

$$y^* = \min_{(x,y) \in G_f} y$$

with $G_f = \{(x, f(x)) \mid x \in M\} \subset M \times \mathbf{R}$ the graph of $f$. However, such reformulations of the problem can be much less accessible for optimization algorithms in practice. Therefore, it is necessary to develop separate algorithms for this problem formulation.

### 2.3.1 Generalized gradients

Our approach to non-smooth cost functions will be based on gradient descent iterations. To implement a gradient descent for non-smooth functions, we need first of all a suitable notion of a generalized gradient or subdifferential for these functions. For Euclidean or Hilbert spaces such generalized gradients and subdifferentials have been the subject of intensive studies over the last decades. We refer the reader to the monographs of Clarke [41] and Rockafellar and Wets [135] for further information. However, the case of non-smooth functions on Riemannian manifolds has received only restricted attention.

For convex functions, the classical notion of the subdifferential was extended by Ferreira and Oliveira [62] to Riemannian manifolds for the use in descent algorithms. A Frechét or viscosity subdifferential on Riemannian manifolds was examined by Ledyaev and Zhu [107–109] and Azagra et al. [15]. Both Chryssochoos and Vinter [39] and Azagra and Ferrera [14] studied proximal subdifferentials on Riemannian manifolds.

For the descent methods discussed later we will use an extension of Clarke's generalized gradient to Riemannian manifolds. It is based on dualizing and convexifying the limiting Frechét subgradient on manifolds from

Ledyaev and Zhu [108]. Most results for this generalized gradient are derived from the Euclidean counterparts in [41, 135] by applying local charts, a general technique proposed in [107, 108] for the Frechét subdifferentials on manifolds. We basically follow this approach of Ledyaev and Zhu [107, 108] and, hence, the properties of the generalized gradient presented here are very closely related to the results of Ledyaev and Zhu. However, the extension of Clarke's generalized gradient to Riemannian manifolds is not explicitly discussed in [107, 108].

We start with recalling the definition of the Frèchet subdifferential and the limiting Frèchet subdifferential from [108] on smooth manifolds. As usual, $M$ will denote a Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$.

**Definition 2.3.1 (cf. [108])** Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous function on a smooth manifold $M$. The *Frèchet subdifferential* of $f$ at $x \in M$ is the set

$$\partial_F f(x) = \{dg(x) \mid g\colon M \to \mathbf{R}, g \text{ is } C^1, \quad f - g \text{ has a local minimum at } x\}.$$

The *limiting Frèchet subdifferential* of $f$ at $x \in M$ is the set

$$\partial_L f(x) = \{\alpha \in T_x M^* \mid \exists x_n, x_n \to x, \alpha_n \in \partial_F f(x_n), \alpha_n \to \alpha\}.$$

Here the convergence $\alpha_n \to \alpha$ is defined as the convergence of the sequence $(\alpha_n)$ in the cotangent bundle of $M$.

In Euclidean or more general Banach spaces, the original definition of Clarke's generalized gradient [41] for a Lipschitz continuous function $f$ relies on the generalized directional derivative

$$f^\circ(x, v) = \limsup_{y \to x, t \to 0, t > 0} \frac{f(y + tv) - f(y)}{t}.$$

It is clearly possible to extend this definition to Riemannian manifolds. However, such an extension would have to use to parallel transport of $v \in T_x M$ to a vector $v(y) \in T_y M$ along a connecting geodesic between $x$ and $y$. A priori such a definition would depend on the Riemannian metric in a neighborhood of $x$. In contrast, the Riemannian gradient of a smooth function in a point $x \in M$ depends on the the value of the Riemannian metric at the point $x$ and not on the curvature in particular. However, the limiting Frèchet subdifferential $\partial_L f$ depends only on the topology of the cotangent bundle, i.e. only on the differentiable structure of $M$. This suggests to extend the characterization of Clarke's generalized gradient $\operatorname{grad}_C f$ in Euclidean space by the limiting Frèchet subdifferential

$$\operatorname{grad}_C f(x) = \operatorname{co}\{v \in \mathbf{R}^n \mid \langle v, \cdot \rangle \in \partial_L f(x)\}^{16},$$

see [135, Thm. 8.49], to Riemannian manifolds. Since $\partial_L f$ is independent of the Riemannian structure, such an extension will depend only on the value of the Riemannian metric in the point $x$ itself.

**Definition 2.3.2** Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous function on a Riemannian manifold. We define the generalized gradient $\operatorname{grad} f(x)$ in $x \in M$ as the set

$$\operatorname{grad} f(x) = \operatorname{co}\{v \in T_x M \mid \langle v, \cdot \rangle \in \partial_L f(x)\}.$$

This generalized gradient shares the properties of Clarke's gradient on Euclidean spaces.

**Proposition 2.3.3** *Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous function on a Riemannian manifold. Then $\operatorname{grad} f(x)$ is a non-empty, compact, convex subset of the tangent space $T_x M$. Furthermore the set-valued function $\operatorname{grad} f\colon M \to 2^{TM}$ is upper semicontinuous.*

**Proof:** Note, that $f$ is Lipschitz continuous if and only if it is Lipschitz continuous in local charts with respect to the Euclidean distance. By the transformation rules for the limiting Frèchet subdifferential [108, Cor. 1], the upper semicontinuity of $\partial_L f$ and the non-empty, compact values of $\partial_L f$ follow directly from the Euclidean results [135, Thm. 8.7, Thm. 9.13][17]. Therefore, $\operatorname{grad} f$ is upper semicontinuous with non-empty, compact values everywhere[18]. The convexity of $\operatorname{grad} f(x)$ for all $x \in M$ follows directly from the definition. □

[16]Note that Clarke actually defines his generalized gradient as a subset of the dual space [41]. However, in the Euclidean setting the dual space $(\mathbf{R}^n)^*$ is identified with $\mathbf{R}^n$ and Clarke's generalized gradient is considered as a subset of $\mathbf{R}^n$. The spaces $\mathbf{R}^n$ and $(\mathbf{R}^n)^*$ are identified via the Euclidean scalar product.

[17]The local Lipschitz continuity implies "strict continuity" in the sense of [135]

[18]The map $T_x^* M \to T_x M$, $\langle v, \cdot \rangle \mapsto v$ is linear and the convex hull of a compact set is compact.

The generalized gradient transforms under diffeomorphisms in a similar way as the gradient of a smooth function. Given a Lipschitz continuous function $f : M \to \mathbf{R}$ and a smooth diffeomorphism $h : N \to M$, it is known that

$$\partial_L(f \circ h)(x) = \{\alpha \circ T_x h \mid \alpha \in \partial_L f(h(x))\},$$

see [108, Cor. 1]. With this equivalence we see that the definition of $\operatorname{grad} f(x)$ implies

$$\operatorname{grad}(f \circ h)(x) = T_x h^\top \operatorname{grad} f(h(x)). \tag{2.6}$$

Here, $T_x h^\top$ denotes the adjoint of the tangent map $T_x h$, i.e. coincides with the linear map $T_x h^\top : T_{h(x)} M \to T_x N$, defined by

$$\langle T_x h v, w\rangle = \langle v, T_x h^\top w\rangle \text{ for all } v \in T_x N, w \in T_{h(x)} M.$$

Furthermore, we use the notation $L \operatorname{grad} f(x)$ for the set $\{Lv \mid v \in \operatorname{grad} f(x)\}$ with $L\colon T_x M \to T_y N$ a linear map.

Given any Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$, let $H(x)\colon T_x M \to T_x M$ denote a self-adjoint positive-definite linear map that depends smoothly on $x$. Then

$$\langle\langle v, w \rangle\rangle = \langle H(x) v, w\rangle$$

defines a new Riemannian metric. Let grad, $\widetilde{\operatorname{grad}}$ denote the generalized gradients, associated with $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$, respectively. A straightforward calculation shows that

$$\widetilde{\operatorname{grad}} f(x) = H(x)^{-1} \operatorname{grad} f(x), \tag{2.7}$$

which is again the analogue of the formulas for smooth functions.

**Proposition 2.3.4** *Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous function on a Riemannian manifold, $S$ an arbitrary subset of measure $0$ in $M$, which contains the points where $f$ is not differentiable. Then*

$$\operatorname{grad} f(x) = \operatorname{co}\{v \in T_x M \mid \exists x_k \to x, x_k \notin S, \lim \operatorname{grad} f(x_k) = v\}.$$

Note, that by Rademacher's theorem [135, Thm. 9.60], for any Lipschitz continuous function $f$, the set of points, where $f$ is not differentiable, has measure 0. **Proof:** Let $x \in M$, $U \subset M$ a neighborhood of $x$ and $\phi\colon U \to W \subset \mathbf{R}^n$, $\phi(x) = 0$, be a smooth local chart. We assume that $\phi$ maps $U$

diffeomorphically onto $W$. Denote by $\langle\cdot,\cdot\rangle_E$ the Euclidean scalar product on $W$ and $\langle\cdot,\cdot\rangle_R$ the Riemannian metric on $W$ induced by $\phi$, i.e. for $y \in W$

$$\langle v, w\rangle_R = \left\langle T_y\phi^{-1}v, T_y\phi^{-1}w, .\right\rangle$$

As usual we can express $\langle\cdot,\cdot\rangle_R$ as

$$\langle v, w\rangle_R = \langle v, H(y)w\rangle_E .$$

with $H\colon W \to \mathbf{R}^{n\times n}$ a smooth function. We denote by $\mathrm{grad}_R$ and $\mathrm{grad}_E$ the generalized gradient on $W$ with respect to $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_E$. By the considerations above we have that

$$\mathrm{grad}_R(f\circ\phi^{-1})(y) = H(y)^{-1}\,\mathrm{grad}_E(f\circ\phi^{-1})(y).$$

Since $\phi\colon U \to W$ is a smooth diffeomorphism, we have that for all $x \in U$.

$$\mathrm{grad}\, f(x) = T_x\phi^\top\,\mathrm{grad}_R(f\circ\phi^{-1})(\phi(x)),$$

where $T_x\phi^\top$ denotes the adjoint of the tangent map with respect to $\langle\cdot,\cdot\rangle$ on $T_xU$ and $\langle\cdot,\cdot\rangle_R$ on $T_{\phi(x)}W$. Hence, for all $x \in U$

$$\mathrm{grad}\, f(x) = T_x\phi^\top H(\phi(x))^{-1}\,\mathrm{grad}_E(f\circ\phi^{-1})(\phi(x)).$$

By [135, Thm. 8.49], the generalized gradient $\mathrm{grad}_E(f\circ\phi^{-1})(y)$ coincides with Clarke's generalized gradient. Therefore, we can apply the characterization of Clarke's generalized gradient [41, Thm. 2.5.1] by limits of the gradients of $f\circ\phi^{-1}$ and get

$$\begin{aligned}\mathrm{grad}\, f(x) = T_x\phi^\top H(\phi(x))^{-1}\,\mathrm{co}\{v \in \mathbf{R}^n \mid \exists y_k \to \phi(x), y_k \notin \phi(S),\\ \lim \mathrm{grad}_E(f\circ\phi^{-1})(y_k) = v\}.\end{aligned}$$

The transformation rules the the Riemannian gradient yield

$$\begin{aligned}&T_x\phi^\top H(\phi(x))^{-1}\,\mathrm{co}\{v \in \mathbf{R}^n \mid \exists y_k \to \phi(x), y_k \notin \phi(S),\\ &\qquad \lim \mathrm{grad}_E(f\circ\phi^{-1})(y_k) = v\}\\ &= T_x\phi^\top\,\mathrm{co}\{v \in \mathbf{R}^n \mid \exists y_k \to \phi(x), y_k \notin \phi(S),\\ &\qquad \lim \mathrm{grad}_R(f\circ\phi^{-1})(y_k) = v\}\\ &= \mathrm{co}\{v \in T_xM \mid \exists x_k \to x, x_k \notin S, \lim \mathrm{grad}\, f(x_k) = v\}.\end{aligned}$$

This proves our claim. □

**Proposition 2.3.5** *Let $x, y \in M$ be two arbitrary points such that their geodesic distance $d(x,y)$ is smaller than the injectivity radius of $x$. Assume that $f\colon M \to \mathbf{R}$ is Lipschitz and let $\gamma(t) := \exp_x tv$ be a geodesic segment between $x$ and $y$, i.e. $v \in T_xM$, $\gamma(1) = y$. Then there exists $\tau \in (0,1)$ such that*

$$f(x) - f(y) \in \{\langle w, \dot\gamma(\tau)\rangle \mid w \in \operatorname{grad} f(\gamma(\tau))\}.$$

**Proof:** We define functions $h\colon U \to \mathbf{R}$, $h(z) := (f \circ \exp_x)(z)$, $U \subset T_xM$ a suitable open set which contains $\{tv \mid t \in [0,1]\}$, and $\sigma\colon [0,1] \to T_xM$, $\sigma(t) := tv$. Since $f \circ \exp_x$ is Lipschitz continuous and $T_xM$ a finite dimensional vector space with scalar product $\langle\cdot,\cdot\rangle$, we can apply the mean value theorem for Clarke's generalized gradient [41, Thm. 2.3.7]. This yields a $\tau \in (0,1)$ such that

$$f(x) - f(y) \in \{\langle w, v\rangle \mid w \in \operatorname{grad}_C h(\sigma(\tau))\},$$

where $\operatorname{grad}_C$ denotes Clarke's generalized gradient. For the tangent map of $\exp_x$, we use the standard identification of $T_w(T_xM)$ with $T_xM$. We denote by $(T_{\tau v}\exp_x)^\top$ the adjoint of the linear map $T_{\tau v}\exp_x\colon T_xM \to T_{\exp_x(\tau v)}M$ where $T_xM$, $T_{\exp_x(\tau v)}M$ equipped with the scalar product defined by the Riemannian metric. The transformation rules for the generalized gradient yield

$$\begin{aligned}
&\{\langle w, v\rangle \mid w \in \operatorname{grad}_C h(\sigma(\tau))\}\\
&\quad= \{\langle w, v\rangle \mid w \in (T_{\tau v}\exp_x)^\top \operatorname{grad} f(\exp_x(\tau v)))\}\\
&\quad= \{\langle (T_{\tau v}\exp_x)^\top w, v\rangle \mid w \in \operatorname{grad} f(\exp_x(\tau v)))\}\\
&\quad= \{\langle w, T_{\tau v}\exp_x v\rangle \mid w \in \operatorname{grad} f(\exp_x(\tau v)))\}\\
&\quad= \{\langle w, \dot\gamma(\tau)\rangle \mid w \in \operatorname{grad} f(\exp_x(\tau v)))\}.
\end{aligned}$$

This proves the proposition. □

One important example of non-smooth cost functions are functions of the form $f(x) := \max\{f_1(x), \ldots, f_m(x)\}$. Such cost function appear in minimax problems, and in particular, in the sphere packing problems discussed later. On Banach spaces, Clarke's generalized gradient of such a maximum function $f$ can be calculated from the generalized gradients of the $f_i$ under the condition of regularity of the functions $f_i$; see [41]. Since this definition of regularity can be given by the limiting and Frechèt subdifferential [135, Cor. 8.11], it is straightforward to extend it to manifolds.

**Definition 2.3.6** A Lipschitz continuous function $f : M \to \mathbf{R}$ on a Riemannian manifold is called *regular* at $x$ if

$$\partial_F f(x) = \partial_L f(x).$$

**Remark 2.3.7** In the Euclidean case, regularity implies that $\partial_L f$ is convex valued [135, 9.16]. Using the transformation rules the Frèchet subgradient [108, Cor. 1], we see that for regular functions on a manifold the generalized gradient is just the dual of $\partial_L f$.

**Lemma 2.3.8** *A Lipschitz continuous function $f \colon M \to \mathbf{R}$ is regular if and only if it is regular in the sense of Clarke*[19] *in local charts.*

**Proof:** As stated before, the definition of regularity in Euclidean space is for Lipschitz continuous functions equivalent to Clarke's regularity of the function, cf. [135, Cor. 8.11, Thm. 9.16] and [41, Def. 2.3.4]. Thus, our claim follows directly from the transformation rules for the Frechèt and limiting Frechèt subdifferential in local charts [108, Cor. 1]. □

We can now derive a chain-rule for generalized gradients from the Euclidean counterpart.

**Proposition 2.3.9** *Let $f_1, \ldots, f_m \colon M \to \mathbf{R}$, $h \colon \mathbf{R}^m \to \mathbf{R}$ be Lipschitz continuous, regular functions and the elements of Clarke's generalized gradient $\operatorname{grad}_C h(x)$ non-negative vectors. Then*

$$g(x) = h(f_1(x), \ldots, f_m(x))$$

*is regular and*

$$\operatorname{grad} g(x) = \overline{\operatorname{co}} \left\{ \sum_{i=1}^{m} a_i v_i \mid v_i \in \operatorname{grad} f_i(x), (a_1, \ldots, a_m) \in \operatorname{grad}_C h(f_1(x), \ldots, f_m(x)) \right\}.$$

*Here, $\overline{\operatorname{co}} S$ denotes the closure of the convex hull of $S \subset T_x M$.*

Ledyaev and Zhu derive a inclusion type-chain rule for limiting Frechét subgradients [107, Thm. 4.12]. By Remark 2.3.7 we get directly an inclusion type chain-rule for regular functions from their results.

[19] i.e. [41, Def. 2.3.4]

**Proof:** By Lemma 2.3.8 the $f_i$ are Clarke regular in local charts. Hence, the regularity of $g$ follows from the Euclidean result [41, Thm. 2.3.9] and application of Lemma 2.3.8. The formula for the generalized gradient of $g$ is obtained from the Euclidean version in local charts and application of the transformation rules for the generalized gradient under diffeomorphisms and changes of the metric, compare with the proof of Proposition 2.3.4. □

Note, that if the regularity assumptions do not hold, then the equality for the generalized gradient has to be replaced by the inclusion $\operatorname{grad} g(x) \subset \overline{\mathrm{co}}\{\sum_{i=1}^m a_i v_i \mid v_i \in \operatorname{grad} f_i(x), (a_1, \ldots, a_m) \in \operatorname{grad}_C h(f_1(x), \ldots, f_m(x))\}$. From [41, Example 2.2.8] it is known that the function $g(x_1, \ldots, x_m) := \max\{x_1, \ldots, x_m\}$ is regular with generalized gradient

$$\operatorname{grad} g(x) = \{(a_1, \ldots, a_m) \mid a_i \geq 0, \sum a_i = 1, a_i \neq 0 \text{ iff } x_i = g(x)\}.$$

As in the Euclidean case, cf. [41, Prop. 2.3.12], this leads to the following result that describes the generalized gradient of $g \circ (f_1, \ldots, f_m)$.

**Corollary 2.3.10** *Let $f_1, \ldots, f_m \colon M \to \mathbf{R}$ be Lipschitz continuous, regular functions on a Riemannian manifold. Then*

$$g(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

*is regular and*

$$\operatorname{grad} g(x) = \overline{\mathrm{co}}\{\operatorname{grad} f_i(x) \mid f_i(x) = g(x)\}.$$

In [107, Thm. 4.12] an inclusion-type formula for the limiting Frechét subdifferential of $\max\{f_1(x), \ldots, f_m(x)\}$ is presented.

For the construction of an optimization algorithm, we need the standard first order necessary condition for local extrema.

**Definition 2.3.11** Let $f \colon M \to \mathbf{R}$ be a Lipschitz continuous function and $x \in M$. We call $x$ a *critical point* of $f$ if

$$0 \in \operatorname{grad} f(x).$$

**Proposition 2.3.12** *Let $f \colon M \to \mathbf{R}$ be a Lipschitz continuous function and $x \in M$ a local minimum or maximum of $f$. Then $x$ is a critical point.*

**Proof:** A point is an local minimum/maximum if and only if it is a local minimum/maximum a local chart $\phi\colon U \to \mathbf{R}^n$, $U$ a neighborhood of $x$. Hence, the proposition follows directly from the Euclidean case [41, Prop. 2.3.2] by the transformation rules for the generalized gradient. □

Note, that for the limiting Frechét subdifferential this necessary condition holds only for local minima [108]. However, if the function is regular and $x \in M$ a local maximum or minimum, then by Remark 2.3.7 we have $0 \in \partial_L f(x)$.

### 2.3.2 Riemannian gradient descent

We will now introduce by descent iterations along geodesics for optimization problems with non-smooth, Lipschitz continuous cost functions.

For convex, non-smooth functions, a subgradient descent along geodesics and its convergence under some conditions on the curvature of the manifolds has been considered by Ferreira and Oliveira [62]. We extend this to descent iterations along geodesics for non-convex, Lipschitz continuous functions on general complete Riemannian manifolds and provide a convergence result for gradient descent.

The definition of the Riemannian descent iterations is analogous to the smooth case, see [154] for an exposition of Riemannian descent methods for smooth functions. To ensure that the descent iteration is well-defined, we assume that the Riemannian manifold $M$ is complete.

**Definition 2.3.13** Let $M$ be a complete Riemannian manifold and $f\colon M \to \mathbf{R}$ be a Lipschitz continuous function. We define a *Riemannian descent iteration* as

$$x_{k+1} = \exp_{x_k}(\alpha_k s_k),$$

with $\alpha_k$ a *step size* and $s_k$ a *descent direction*. We call it *gradient descent* if $s_k \in \operatorname{grad} f(x_k)$.

Note, that we do not require the monotonicity of the sequence $(f(x_k))$. This is motivated by the fact, that we can prove a convergence result only for a type of descent iterations which are not necessarily monotone.

Unlike the smooth case, not all elements $v$ of $-\operatorname{grad} f(x_k)$ provide a direction of descent in the strict sense, i.e. such that

$$t \mapsto f(\exp_{x_k}(tv))$$

is strictly decreasing on an interval $(0, \varepsilon)$. In fact, we will see later, that such *strict descent directions* are not necessary for convergence results for this optimization approach. However, they are necessary to implement an Armijo-like step-size selection algorithm. Therefore, we give some definitions and proposition to determine such a strict descent directions. Note, that all these characterizations are analogous to their Euclidean counterparts.

**Definition 2.3.14** Let $x \in M$ and $C \subset T_xM$ compact and convex. We define[20]

$$N(C) = \{v \in T_xM \mid \forall w \in C\colon \langle v, w\rangle < 0\}.$$

**Proposition 2.3.15** *Let $f\colon M \to \mathbf{R}$ be Lipschitz continuous, $x \in M$ and $C \subset T_xM$ a compact set with* $\operatorname{grad} f(x) \subset C$. *If* $0 \notin \operatorname{co}(C)$, *then all* $v \in N(C)$ *are strict descent directions for $f$.*

**Proof:** Since for any $w_1, w_2 \in C$, $\lambda \in (0, 1)$, $v \in T_xM$, with $\langle w_1, v\rangle < 0$, $\langle w_2, v\rangle < 0$ the inequality $\langle \lambda w_1 + (1 - \lambda)w_2, v\rangle < 0$ holds, we have that $N(C) = N(\operatorname{co}(C))$. Hence, we can assume that $C$ is convex. Let $\gamma\colon \mathbf{R} \to M$ be a smooth curve with $\gamma(0) = x$, $\dot\gamma(0) = v \in N(C)$. We denote by $r > 0$ the injectivity radius of $x$. Since $\operatorname{grad} f$ is upper semicontinuous with compact values, there are a $\sigma > 0$, $\delta \in (0, r)$ such that for all $y \in M$ with $\operatorname{dist}(x, y) < \delta$ and all $w \in \operatorname{grad} f(y)$ we have

$$\langle w, \pi_{x,y}(v)\rangle < -\sigma,$$

where $\pi_{x,y}$ denotes the parallel transport along the shortest geodesic between $x$ and $y$. Since $\gamma$ is smooth, there is a $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$ we have $\operatorname{dist}(\gamma(t), x) < \delta$ and $\left\|\pi_{x,\gamma(t)}v - \dot\gamma(t)\right\| < \sigma/2$. Hence, for all $t \in (0, \varepsilon)$ and $w \in \operatorname{grad} f(\gamma(t))$ we have

$$\langle w, \dot\gamma(t)\rangle < -\frac{\sigma}{2}.$$

By mean-value Theorem 2.3.5 this implies that $f(\gamma(t)) < f(x)$ for all $t \in (0, \varepsilon)$. Setting $\gamma(t) = \exp_x(tv)$ locally around 0, we see that $v$ is a strict descent direction for $f$. □

In the Euclidean case, it is well-known that the direction of steepest descent can be obtained by the negative of the projection of 0 to the generalized gradient, i.e. $-\pi_0(\operatorname{grad} f(x))$, see e.g. [80, VIII.1] for the convex functions.

[20] To avoid confusion, we point out that the $N(C)$ used here, is not the normal cone of $C$ in $T_xM$ in the sense of the standard definition, cf. [80].

With the characterization of descent directions as above, we can apply the same arguments as in the Euclidean case and establish that $-\pi_0(\operatorname{grad} f(x))$ is a descent direction in the manifold case. In fact, by Proposition 2.3.15 we can give a more general version which enables us to use approximations of the generalized gradient.

**Proposition 2.3.16** *Let $C \subset T_xM$ be a compact, convex set. We denote by $\pi_0(C)$ the least distance projection of $0$ to $C$ with respect to the scalar product on $T_xM$ given by the Riemannian metric. Then $-\pi_0(C) \in N(C)$ or $\pi_0(C) = 0$.*

**Proof:** [21] The well-known fact that for all $w \in C$ one has $-\langle w, \pi_0(C)\rangle + \|\pi_0(C)\|^2 \leq 0$ [80, Thm. III, 3.1.1], gives directly that $\langle w, -\pi_0(C)\rangle < 0$ for all $w \in C$. □

**Corollary 2.3.17** *Let $f\colon M \to \mathbf{R}$ be Lipschitz continuous, $x \in M$ and $C \subset T_xM$ a compact, convex set with* $\operatorname{grad} f(x) \subset C$. *If $0 \notin C$, then $-\pi_0(C)$ is a strict descent direction for $f$ at $x$.*

The most striking difference between gradient descent for smooth and non-smooth functions, is that commonly used line search schemes for the step length will not provide convergence of the algorithm to critical points in the non-smooth case. In fact, there are known counterexamples for even convex, piecewise linear functions, see [80]. However, for convex functions it is known that one can choose $s_k \in \operatorname{grad} f(x_k)$ and an arbitrary sequence $\alpha_k$ with $\sum \alpha_k = \infty$, $\alpha_k \to 0$ to yield convergence of the iterates to the set of critical points [20,62]. We will reproduce this result for non-convex functions on Riemannian manifolds.

Our proof is based on the results of Teel for the optimization of non-smooth functions on Euclidean $\mathbf{R}^n$ [150]. Teel's approach is based on discretizations

$$x_{k+1} \in x_k + \alpha_k F(x_k) \tag{2.8}$$

of a differential inclusion

$$\dot{x} \in F(x). \tag{2.9}$$

Teel shows that for sufficiently small $\alpha_k$, asymptotically stable sets of (2.9) are approximated by asymptotically stable sets of (2.8). Furthermore, for

[21] We could also argue that is a byproduct from the calculation of steepest descent directions for convex functions in Euclidean spaces [80, VIII.1].

compact, asymptotically stable sets $A$ of (2.9) the existence of a function $\tau : \mathbf{R}^n \to \mathbf{R}$, such that $\alpha_k < \tau(x_k)$ implies that $A$ is asymptotically stable for (2.8) with the same basin of attraction as (2.9), is shown. For gradient or gradient-like differential inclusions the sublevel sets are asymptotically stable. Thus vanishing $\alpha_k$ imply the convergence of $(x_k)$ to a critical sublevel set, i.e. a sublevel set $\{x \in \mathbf{R}^n \mid f(x) \leq c\}$ such that $c$ is a critical value.

We extend these results to gradient descent iterations on Riemannian manifolds and prove the convergence to the set of critical points.

We start with the extension of the known results on the existence of smooth Lyapunov functions for asymptotically stable sets of differential inclusions on Euclidean space [151] to Riemannian manifolds. We recall quickly some notions of stability for differential inclusions from [150]. A compact set $A \subset M$ is called *stable* for a differential inclusion $\dot{x} \in F(x)$, if for any neighborhood $U$ of $A$ there is a neighborhood $V$ of $A$ such that all solutions starting in $V$ exist for all times $t > 0$ and remain in $U$. $A$ is called *asymptotically stable* if it is stable and there is a neighborhood $U$ of $A$ such that all solutions starting in $U$ converge to $A$. The set of points $G$ such that all solutions starting in $G$ convergence to $A$ is called the *basin of attraction*.

**Lemma 2.3.18** *Let $\dot{x} \in F(x)$ be a differential inclusion on a submanifold $M \subset \mathbf{R}^n$ with $F(x)$ a upper semicontinuous set-valued map with $F(x) \subset T_xM$, nonempty, compact, convex for all $x \in M$. Furthermore, let $F$ be uniformly bounded on $M$. Assume that $A$ is a compact, asymptotically stable set with basin of attraction $G$. Then we can extend $F$ to a upper semicontinuous, set-valued map $\hat{F}$ on $\mathbf{R}^n$ with nonempty, compact, convex values $\hat{F}(x)$, such that $A$ is asymptotically stable for the differential inclusion*

$$\dot{x} \in \hat{F}(x)$$

*with region of attraction $\hat{G} \supset G$.*

**Proof:** Let $\pi$ be the Euclidean least distance projection to $M$ and $s$ the squared distance, i.e. $s(x) = \|\pi(x) - x\|^2$, to $M$. On a suitable neighborhood $U$ of $M$ both maps are well-defined and smooth. There is a unique family of linear maps $\sigma(x)\colon T_{\pi(x)}M \to \mathbf{R}^n$, $x \in U$, defined by

$$T_x\pi\sigma(x) = \mathrm{Id}_{T_xM} .$$

Shrinking $U$, we can w.l.o.g. assume that $\|\sigma(x)\|$ is uniformly bounded. Using a smooth normal tubular neighborhood we see that $(\sigma(x))$ is a smooth family, i.e. the map $(x, v) \mapsto \sigma(x)v$ is smooth. We can extend $F$ to $U$ by setting

$$F(x) = \sigma(x)F(\pi(x)).$$

Using this extension we define

$$\hat{F}(x) = \begin{cases} F(x) + \operatorname{grad} s(x) & x \in U \\ \overline{\operatorname{co}}\,(\{v \in \mathbf{R}^n \mid x_k \to x, x_k \in U, v_k \to v, \\ \qquad v_k \in (F(x_k) + \operatorname{grad} s(x_k))\} \cup \{0\}) & x \in \partial U \\ 0 & x \in \mathbf{R}^n \setminus \overline{U} \end{cases}$$

By construction $\hat{F}$ is upper semicontinuous with closed, convex $\hat{F}(x)$. Since $F$ is uniformly bounded, the values $\hat{F}(x)$ must be compact for all $x \in \mathbf{R}^n$. Let $V$ be an open neighborhood of $A$ in $\mathbf{R}^n$. Then there is a relatively open subset $W$ of $V$, with $W \subset M$, such that any solution of $\dot{x} \in F(x)$, which starts in $W$, converges to $A$ and remains in $V$ for all $t \geq 0$. W.l.o.g. we can assume that both $V$ and $W$ are relatively compact. There is a $\delta > 0$ such that

$$\{x \in \mathbf{R}^n \mid \operatorname{dist}(x, V) < \delta\} \subset U,$$

with dist the Euclidean distance in $\mathbf{R}^n$. We define

$$\hat{V} = \{x \in U \mid \pi(x) \in V \cap M, \operatorname{dist}(x, M) < \delta\}$$

and

$$\hat{W} = \{x \in U \mid \pi(x) \in W, \operatorname{dist}(x, M) < \delta\}.$$

Let $\gamma(t)$ be a solution of

$$\dot{x} \in \hat{F}(x)$$

with $\gamma(0) \in \hat{W}$. Assume that $\gamma$ leaves $\hat{V}$ at $\tau > 0$ for the first time. By our construction of $\hat{F}$, the function $s(\gamma(t))$ is strictly decreasing on $(0, \tau)$. Thus $\pi(\gamma(\tau))$ lies in the boundary of $\hat{V} \cap M$ in $M$. By construction we have that

$$\frac{d}{dt}\pi(\gamma(t)) = T\pi(\gamma(t))\dot{\gamma}(\tau) \in T\pi(\gamma(t))\hat{F}(\gamma(t)) = F(\pi(\gamma(t)))$$

for almost all $t \in (0, \tau)$. Thus $\pi(\gamma(t))$ is a solution of $\dot{x} \in F(x)$ on $M$ for $t \in (0, \tau)$ with $\pi(\gamma(0)) \in W$. But $W$ was chosen such that all solutions starting in $W$ remain in $V \cap M$ for all $t > 0$. This gives a contradiction and $\gamma(t) \in \hat{V}$ for all $t > 0$. Arguing again that that $\pi(\gamma(t))$ is a solution of $\dot{x} \in F(x)$, we see that $\pi(\gamma(t)) \to A$. Furthermore, $s(\gamma(t))$ is strictly decreasing for all $t > 0$. Thus, $\gamma(t) \to A$. Therefore, $A$ is asymptotically stable. Let $G$ be the region of attraction of $\dot{x} \in F(x)$ in $M$. Note, that we can assume that $G \subset U$. Assume that we have a solution $\gamma(t)$ of $\dot{x} \in \hat{F}(x)$ with $\gamma(0) \in G$. There is an open interval $(0, \tau)$ such that $\gamma(t) \in U$ for all $t \in (0, \tau)$. By construction the function $s(\gamma(t))$ is absolutely continuous, non-negative and non-increasing on $(0, \tau)$. Furthermore, it is decreasing on any interval $(a, b) \subset (0, \tau)$ with $\gamma((a, b)) \cap M = \emptyset$. Hence, it must be constant 0 and $\gamma$ cannot leave $M$. Therefore, $G$ is contained in the basin of attraction of $A$ with respect to the differential inclusion $\dot{x} \in \hat{F}(x)$. $\square$

**Lemma 2.3.19** *Let $F\colon M \to 2^{TM}$ be a upper semicontinuous set-valued map with $F(x)$ a nonempty, compact, convex subset of $T_xM$. We assume that $F(x)$ is uniformly bounded on $M$. Furthermore, let $A \subset M$ be the asymptotically stable set of*

$$\dot{x} \in F(x)$$

*with basin of attraction $G$. Then there exists a smooth function $V\colon G \to \mathbf{R}_0^+$, $V(x) = 0$ on $A$, $V(x) > 0$ on $G \setminus A$, $V(x) \to \infty$ for $x \to \partial G$, such that*

$$\max_{w \in F(x)} \langle w, \operatorname{grad} V(x) \rangle \leq -V(x).$$

This is the Riemannian version of result of Teel and Praly [151] in Euclidean space.

**Proof:** By a theorem of Nash [124] we can embed $M$ isometrically in some Euclidean $\mathbf{R}^n$. Using Lemma 2.3.18, we can extend $F$ to $\mathbf{R}^n$ such that $A$ is asymptotically stable for

$$\dot{x} \in F(x)$$

with basin of attraction $\hat{G} \supset G$. The result of Teel and Praly [151] yields the existence of $\hat{V}$ with the required properties for $\hat{G}$. We get the desired $V$ on $G$ by restriction of $\hat{V}$ to $M$. $\square$

The existence of the smooth Lyapunov function allows us to use the argument of Teel for the discretization of differential inclusions [150] in the context of Riemannian manifolds. This yields the following Riemannian version of the result of Teel for Euclidean space [150, Thm. 2].

**Proposition 2.3.20** *Assume that $M$ is a complete Riemannian manifold. Let $\dot{x} \in F(x)$ be a differential inclusion with $F\colon M \to 2^{TM}$ a set-valued map, with $F(x) \subset T_xM$ compact, convex and non-empty. Assume that $A$ is compact and asymptotically stable with basin of attraction $G$. For any* *compact sets $C$, $D$, with $G \supset D \supset C \supset A$ there is a $\alpha > 0$ such that for all $\hat{\alpha} < \alpha$ we have an an $\hat{A} \subset C$ which is asymptotically stable for the iterations*

$$x_{k+1} = \exp_{x_k}(\hat{\alpha} s_k), \quad s_k \in F(x_k)$$

*and its basin of attraction contains $D$.*

**Proof:** With Lemma 2.3.19 we can extend directly the proof of Teel [150, Theorem 2] for differential inclusions on Euclidean $\mathbf{R}^n$ to Riemannian manifolds. □

Given this discretization result, we can prove the convergence to the set of critical points. However, we need first a Riemannian version of a known differentiation result for non-smooth, regular functions, cf. [142, 150].

**Lemma 2.3.21** *Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous, regular function and $\gamma\colon \mathbf{R} \to M$ an absolutely continuous curve. Then for all $a, b \in \mathbf{R}$, $a < b$, we have*

$$f(\gamma(b)) - f(\gamma(a)) = -\int_{[a,b]} \max_{v \in \operatorname{grad} f(\gamma(t))} \langle v, -\dot{\gamma}(t) \rangle \, dt,$$

*where $\int_{[a,b]}$ denotes the integral over the set of points in $[a, b]$ where $\gamma$ is differentiable.*

**Proof:** For a regular, Lipschitz continuous function $h\colon \mathbf{R}^n \to \mathbf{R}$, an absolutely continuous curve $\mu\colon \mathbf{R} \to \mathbf{R}^n$, and a $t^* \in \mathbf{R}$ such that $\mu$ and $h \circ \mu$ differentiable at $t^*$ it is known that

$$(h \circ \mu)'(t^*) = -\max_{w \in \operatorname{grad}_C h(\mu(t^*))} \langle w, -\dot{\mu}(t^*) \rangle, \tag{2.10}$$

see [142, 150]. Let $\phi\colon U \to \mathbf{R}^n$ be a local chart w.l.o.g. with $\gamma([a, b]) \subset U$. Then $h := f \circ \phi^{-1}$ is Lipschitz continuous and by Lemma 2.3.8 regular. Furthermore, $\mu := \phi \circ \gamma$ is absolutely continuous. Since $h \circ \mu$ is absolutely continuous, we get that

$$\begin{aligned}
(f \circ \gamma)(b) - (f \circ \gamma)(a) &= \int_{[a,b]} -\max_{w \in \operatorname{grad}_C h(\mu(t))} \langle w, -\dot{\mu}(t) \rangle \, dt \\
&= -\int_{[a,b]} \max_{w \in (T_{\gamma(t)}\phi)^{-\top} \operatorname{grad} f(\gamma(t))} \left\langle w, -(T_{\gamma(t)}\phi)^{\top} \dot{\gamma}(t) \right\rangle dt \\
&= -\int_{[a,b]} \max_{v \in \operatorname{grad} f(\gamma(t))} \langle v, -\dot{\gamma}(t) \rangle \, dt,
\end{aligned}$$

where $(T_{\gamma(t)}\phi)^\top$, $(T_{\gamma(t)}\phi)^{-\top}$ denote as usual the adjoint of $T_{\gamma(t)}\phi\colon T_{\gamma(t)}M \to \mathbf{R}^n$ and its inverse. □

**Theorem 2.3.22** *Assume that $M$ is a complete Riemannian manifold. Let $f\colon M \to \mathbf{R}$ be a Lipschitz continuous, regular function with bounded sublevel sets and a finite number of critical values. Assume that we have a descent iteration $(x_k)$ with*

$$x_{k+1} = \exp_{x_k}(\alpha_k s_k),$$

*such that for all $k \in \mathbf{N}$ : $s_k \in -\operatorname{grad} f(x_k)$, $\alpha_k \to 0$ and*

$$\sum_{k=1}^{\infty} \alpha_k = \infty.$$

*If $x_k$ has an accumulation point, then $x_k$ converges to the set of critical points of $f$.*

Note, that unlike Ferreira and Oliveira in their work on Riemannian gradient descent for non-smooth, convex functions [62], we do not impose any bounds on the sectional curvature of the manifold to obtain convergence to the set of critical points. However, we do not get results on the pointwise convergence of the descent sequence for convex cost functions.

**Proof:** Let $x^*$ be an accumulation point of $x_k$. We denote by $L(C)$ the sublevel set $\{x \in M \mid f(x) \leq C\}$. Let $c^* = f(x^*)$. We choose a constant $C_0 > 0$. Consider the differential inclusion

$$\dot{x} \in F(x), \quad F(x) = \begin{cases} -\operatorname{grad} f(x) & x \in L(c^* + C_0) \\ \{0\} & \text{otherwise} \end{cases}.$$

Note, that $F$ is upper semicontinuous with compact, convex values. Since the sublevel sets of $f$ are bounded, $F$ must be uniformly bounded on $M$. A straightforward Lyapunov argument, Lemma 2.3.21 and the finite number of critical values yield that the sublevel sets of $f$ with values $< c^* + C_0$ are asymptotically stable for the differential inclusion above. Since $x^*$ is an accumulation point of $(x_k)$ and $\alpha_k \to 0$, we can apply Proposition 2.3.20 to differential inclusion $\dot{x} \in F(x)$ to see that for sufficiently large $k$ the sequence $(x_k)$ remains in $L(c^*+C_0)$ and converges to the sublevel set $L(f(x^*))$. Assume now that $0 \notin \operatorname{grad} f(x^*)$. By the upper semicontinuity of $\operatorname{grad} f(x)$, there is a uniform lower bound for $\min_{v \in \operatorname{grad} f(x)} \|v\|$ on a suitable neighborhood of $x^*$. Since the map $(x, \lambda, w) \mapsto T_{\lambda w} \exp_x w$ is smooth, we can deduce that there are $r > 0$, $C_1 > 0$, $\mu > 0$ such that for all $x \in B_r(x^*)$, $w \in -\operatorname{grad} f(x)$, $\lambda \in (0, \mu)$:

$$\max_{v \in \operatorname{grad} f(\exp_x \lambda w)} \langle -T_{\lambda w} \exp_x w, v \rangle \geq C_1. \tag{2.11}$$

Here, $B_r(x^*)$ denotes the ball $\{x \in M \mid \operatorname{dist}(x, x^*) < r\}$. Let $x_{k_l} \to x^*$. As $\alpha_k \to 0$, the $\alpha_k$ are smaller than the injectivity radius of all $x \in B_r(x^*)$ for large $k$. Thus, $\sum \alpha_k = \infty$ implies that there exists a sequence $m_l \in \mathbf{N}$ such that $x_{k_l + m_l} \notin B_r(x^*)$, $x_{k_l}, \ldots, x_{k_l+m_l-1} \in B_r(x^*)$. Since $\alpha_k \to 0$ and $\operatorname{grad} f(x)$ is uniformly bounded on $B_r(x^*)$, $m_l > 2$ for all but a finite number of the $m_l$. By (2.11) and Lemma 2.3.21, we have that

$$\begin{aligned} f(x_{k_l}) - f(x_{k_l+m_l-1}) &= \sum_{i=1}^{m_l-2} f(x_{k_l}) - f(x_{k_l+i}) \\ &= \sum_{i=k_l}^{k_l+m_l-2} \int_{[0,\alpha_i]} \max_{v \in \operatorname{grad} f(\exp_x ts_i)} \langle -T_{ts_i} \exp_x s_i, v \rangle \, dt \\ &\geq C_1 \sum_{i=k_l}^{k_l+m_l-2} \alpha_i. \end{aligned}$$

Note that

$$\sum_{j=k_l}^{k_l+m_l-1} \alpha_j > r.$$

Since $\alpha_k \to 0$, this yields

$$f(x_{k_l}) - f(x_{k_l+m_l-1}) > C_2$$

with $C_2 > 0$ a constant independent of $l$. Therefore $x_{k_l+m_l-1} \in L(c^* - C_2)$. As $\alpha_k \to 0$ we can apply again Proposition 2.3.20 to choose a $C_3 > 0$ such that $(x_k)$ converges to the sublevel set $L(c^* - C_3)$. This gives a contradiction to $x_{k_l} \to x^* \notin L(c^* - C_3)$. Hence, we have proven that if an accumulation point $x^*$ of $(x_k)$ exists, then the $(x_k)$ converges to a critical sublevel set and all accumulation points of $x_k$ are critical points. Therefore, if $(x_k)$ has an accumulation point, then it converges to the set of critical points. □

For a more convenient notation, we introduce a special name for step sizes satisfying the conditions of Theorem 2.3.22.

**Definition 2.3.23** We call a sequence of step sizes $(\alpha_k)$ *harmonic* if $\alpha_k \to 0$, $\sum \alpha_k = 0$.

**Remark 2.3.24** As there is no descent condition of the form $f(x_{k+1}) < f(x_k)$ for our gradient descent, we cannot make any sensible convergence statements for iteration sequences without an accumulation point. However, this problem can only appear if $M$ is non-compact.

Instead of the fixed step sizes of Theorem 2.3.22, we can also use an *Armijo line search*, i.e. for constants $\sigma, \mu \in (0,1)$, $C > 0$, we choose

$$x_{k+1} = \exp_{x_k}\left(\mu^{-l_k} C s_k\right),$$

with $s_k \in N(\operatorname{grad} f(x_k))$, $\|s_k\| = 1$ and

$$l_k = \min\{l \in \mathbf{N} \mid f(x_k) - f\left((\exp_{x_k}\left(\mu^{-l} C s_k\right)\right) \leq \sigma C \mu^{-l} \|\pi_0(\operatorname{grad} f(x_k))\|\}.$$

However, as mentioned before, we cannot guarantee the convergence to the set of critical points for these step sizes.

### 2.3.3 Descent in local parameterizations

For smooth cost functions, one motivation for considering descent iteration in local parameterizations was the high computational costs and sometimes even the impossibility of an exact calculation of the geodesics, see Remark 2.1.15. But this argument also holds for descent iterations for non-smooth cost functions. Therefore, we will now discuss a generalized gradient descent in a family of smooth parameterizations of the manifold $M$. Again we assume, that $M$ is a complete Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$.

We use the same notations for a family of smooth parameterizations as in Section 2.1. The descent iterations are defined as for smooth cost functions.

**Definition 2.3.25** Let $f \colon M \to \mathbf{R}$ be a Lipschitz continuous function. We define a *descent iteration* in a family of smooth parameterizations $\phi_x$ of $M$ as

$$x_{k+1} = \phi_{x_k}(\alpha_k s_k).$$

We call $\alpha_k$ the step size and $s_k$ the descent direction. If $T_0\phi_{x_k}(s_k) \in \operatorname{grad} f(0)$, then we call the iteration a *gradient descent* in local parameterizations.

As for smooth cost functions, the gradient descent along geodesics is identical to the gradient descent in the family $(\exp_x)$.

The use of a family of parameterizations instead of geodesics for descent is only justified if convergence to at least the set of critical points can be shown. This needs of course again some regularity conditions of the family of parameterizations. Here, we show that our standard assumptions of Definition 2.1.2 are sufficient to extend the convergence Theorem 2.3.22 to descent iterations in a family of parameterizations.

First, we need the following generalization of Teel's discretization theorem in Euclidean space [150, Thm. 2] to discretizations by local parameterizations.

**Proposition 2.3.26** *Let $\dot{x} \in F(x)$ be a differential inclusion with $F\colon M \to 2^{TM}$ a set-valued map, with $F(x) \subset T_xM$ compact, convex and non-empty. Furthermore, let $F$ be uniformly bounded on $M$. Assume that $A$ is compact and asymptotically stable with basin of attraction $G$. Furthermore, let $(\phi_x)$ be a family of parameterizations satisfying the standard assumptions. For any compact sets $C$, $D$, with $G \supset D \supset C \supset A$, there is a $\alpha > 0$ such that for all $\hat{\alpha} < \alpha$ there is an $\hat{A} \subset C$ which is asymptotically stable for the iterations*

$$x_{k+1} = \phi_{x_k}(\hat{\alpha} s_k), \quad s_k \in T_0\phi^{-1}F(x_k) \tag{2.12}$$

*and its basin of attraction contains $D$.*

**Proof:** Again, we extend directly the argument of Teel [150] to our setting. We denote by $V\colon G \to [0,\infty)$ the smooth Lyapunov function from Lemma 2.3.19. Let $x \in M$, $y \in \mathbf{R}^n$. The standard mean-value theorem yields an $\lambda \in (0,1)$ with

$$V(\phi_x(y)) - V(x) = \langle \operatorname{grad} V(\phi_x(\lambda y)), T_{\lambda y}\phi_x(y)\rangle .$$

This gives

$$V(\phi_x(y)) - V(x) \leq \max_{w \in F(x)} \left\langle \operatorname{grad} V(\phi_x(\lambda y)), T_{\lambda y}\phi_x\left(T_x\phi_x^{-1}(w)\right)\right\rangle . \tag{2.13}$$

By the exp-equicontinuity of the family $T\phi_x$ at 0 the family $(\Omega_z \mid z \in \mathbf{R}^n)$,

$$\Omega_z = T_z\phi_x T_0\phi_x^{-1}$$

is compactly convergent at 0. Thus the family of 1-forms $\mu_z$, $z \in \mathbf{R}^n$,

$$\mu_z(v) = \langle \operatorname{grad} V(\phi_x(z)), \Omega_z v\rangle$$

is compactly convergent at 0. Therefore, for any compact subset $K \subset M$ there is a continuous function $\varepsilon\colon \mathbf{R} \to \mathbf{R}_+$, $\varepsilon(t) \to 0$ for $t \to 0$ such that for all $x \in K$, $y \in T_x\phi_x^{-1}F(x)$, $\alpha \in \mathbf{R}$, the inequality

$$V(\phi_x(\alpha y)) - V(x) \le \max_{w \in F(x)} \mu_{\alpha y}(\alpha w) \le \alpha(-V(x) + \varepsilon(\|\alpha y\|))$$

holds. Note, that the upper semicontinuity of $F$ ensures that $F$ is bounded on any compact subset of $M$. Thus using the same arguments as Teel [150] we can choose suitable sublevel sets $L(c_1)$, $L(c_2)$ of $V$ and $\alpha \in \mathbf{R}_+$ such that $D \subset L(c_1)$, $C \subset L(c_2)$, $L(c_1)$, $L(c_2)$ are invariant under the iteration 2.12 and $L(c_2)$ is reached in finite number of iterations 2.12 from each $x_0 \in L(c_1)$. This proves our claims. □

**Theorem 2.3.27** *Let $f\colon M \to \mathbf{R}$ be a Lipschitz-continuous, regular function with bounded sublevel sets and a finite number of critical values. Assume that we have a descent iteration $(x_k)$ with*

$$x_{k+1} = \phi_{x_k}(\alpha_k s_k)$$

*such that for all $k \in \mathbf{N}$ : $T_0\phi_{x_k} s_k \in -\operatorname{grad} f(x_k)$, $\alpha_k \to 0$ and*

$$\sum_{k=1}^{\infty} \alpha_k = \infty.$$

*If $x_k$ has an accumulation point, then $x_k$ converges to the set of critical points of $f$.*

**Proof:** We use the same argument as for Theorem 2.3.22. Let $x^*$ be an accumulation point of $x_k$ and $c^* = f(x^*)$. Again we choose a constant $C_0 > 0$. By Proposition 2.3.26 the descent iteration can again be viewed for large $k$ as a discretization of the differential inclusion

$$\dot{x} \in F(x), \quad F(x) = \begin{cases} -\operatorname{grad} f(x) & x \in L(c^* + C_0) \\ \{0\} & \text{otherwise} \end{cases},$$

where $F$ is upper semicontinuous with compact, convex, uniformly bounded values. This yields the convergence of $(x_k)$ to the level set $L(c^*)$. Assume that $0 \notin \operatorname{grad} f(x^*)$. As mentioned in the proof of Proposition 2.3.26, the family of maps $(\Omega_z \mid z \in \mathbf{R}^n)$

$$\Omega_z = T_z\phi_x T_0\phi_x^{-1}$$

is compact convergent for $z \to 0$. By Lemma 2.1.6 the family $(\phi_x)$ at 0 is equicontinuous at 0. These facts together with the upper-semicontinuity $\operatorname{grad} f$ yield that the family of maps $\theta_\lambda$

$$\theta_\lambda(x) := \\ \min\{\langle T_{\lambda T_0 \phi_x^{-1} w}\left(\phi_x T_0 \phi_x^{-1}(w)\right), v\rangle \mid w \in \operatorname{grad} f(x), v \in \operatorname{grad} f(\phi_x(\lambda w))\}$$

is compact convergent for $\lambda \to 0$. Note, that by definition there exists a constant $C_1 > 0$ such that for suitably small neighborhoods $U$ of $x^*$

$$\theta_0(x) > C_1, \quad x \in U.$$

Thus, we can choose an $r > 0$, $\mu > 0$ such that for all $x \in B_r(x^*)$, $w \in \operatorname{grad} f(x)$ and $\lambda \in (0, \mu)$

$$\max_{v \in \operatorname{grad} f(\phi_x(\lambda T_0 \phi^{-1}(w)))} \langle -T_{\lambda T_0 \phi_x^{-1} w} \phi_x\left(T_0 \phi_x^{-1}(w)\right), v\rangle > C_1 > 0$$

holds. We get by Lemma 2.3.21 that for all $k \in \mathbf{N}$

$$f(x_k) - f(x_{k+1}) = \int_{[0,\alpha_k]} \max_{v \in \operatorname{grad} f(\phi_x(ts_k))} \langle T_{\lambda \alpha_k s_k} \phi_x(ts_k), v\rangle \, dt$$

For all $k \in \mathbf{N}$ with $\phi_{x_k}(ts_k) \in B_r(x^*)$ for all $t \in [0, \alpha_k]$ this yields

$$f(x_k) - f(x_{k+1}) > C_1 \alpha_k.$$

The rest of the argument is identical to the Riemannian case and therefore omitted. □

### 2.3.4 Minimax problems

#### Problem formulation

We focus now on cost functions of the form

$$f(x) = \max_{i=1,\ldots,m} f_i(x)$$

with $f_i$ smooth. The *minimax problem* consists of finding the minima of $f$. We denote by $I(x)$ the set of active indices, i.e.

$$i \in I(x) \Leftrightarrow f_i(x) = f(x).$$

By Corollary 2.3.10 we have

$$\operatorname{grad} f(x) = \overline{\operatorname{co}}\{\operatorname{grad} f_i(x) \mid f_i(x) = f(x)\}.$$

Following Theorem 2.3.22 we can choose an arbitrary $s_k \in \operatorname{grad} f(x_k)$ as a direction for an optimization iteration

$$x_{k+1} = \exp_{x_k} t_k s_k.$$

However, if we use an Armijo-like descent iteration, we have to ensure that $s_k \in N(\operatorname{grad} f(x_k))$. According to Proposition 2.3.16, a sufficient choice for $s_k$ is in this case $\pi_0(\operatorname{grad} f(x_k))$. Calculating $\pi_0(\operatorname{grad} f(x_k))$ is a quadratic programming problem [20]. Denoting by $l$ the number of active indices, i.e. $l = \#I(x_k)$, this quadratic programming problem has the following formulation:

$$\begin{gathered} \text{Minimize } L^\top W L \\ L \in [0,1]^l \\ W = (\langle \operatorname{grad} f_i(x_k), \operatorname{grad} f_j(x_k)\rangle)_{i,j\in I(x)} \end{gathered}$$

and can be solved by standard algorithms.

**Smooth approximations**

A different approach to the minimax problem, is the use of a sequence $f_p$ of smooth approximations of the cost function $f = \max f_i$. A complete overview of this approach is beyond the scope of this work. Here, we discuss only two examples of smooth approximations.

One type of approximations is the use $p$-norms, an idea which apparently dates back to Pólya [38]. Assume that the functions $f_i$ are non-negative on $M$. Then we can view $f$ as the $\infty$-norm of the vector $(f_1, \ldots, f_m)$. But as the $p$-norm $\|\cdot\|_p$ converges uniformly to $\|\cdot\|_\infty$ on compact sets, we can approximate $f$ by

$$\hat{f}_p(x) := \|(f_1(x), \ldots, f_m(x)\|_p = \left(\sum_{i=1}^m (f_i(x))^p\right)^{1/p}.$$

The $\hat{f}_p$ converge uniformly to $f$ on compact sets. However, as $\hat{f}_p$ is still non-smooth, we use instead the smooth function

$$f_p(x) := (\hat{f}_p(x))^p = \sum_{i=1}^m (f_i(x))^p$$

for optimization. Obviously, the minima of $f_p$ converge to the minima of $f$ for $p \to \infty$. Basic calculus yields the gradient

$$\operatorname{grad} f_p(x) = p \sum_{i=1}^{m} (f_i(x))^{p-1} \operatorname{grad} f_i(x).$$

We can use standard gradient-like optimization methods to find the minima of $f_p$. Either increasing $p$ during the optimization or choosing a priori a sufficiently large $p$ should yield a good approximation of a minimum of $f$. This approach can also be used for $f_i$ bounded from below, by replacing $f_i(x)$ with $f_i(x) + C$ for $C$ a sufficiently large positive constant.

Another type of smooth approximation is the entropic regularization method of Li and Fang [110]. Li and Fang use the smooth approximations

$$f_p(x) = \frac{1}{p} \log \left( \sum_{i=1}^{m} \exp(p f_i(x)) \right).$$

They show that these $f_p$ converge uniformly to $f$. The gradient can be calculated as

$$\operatorname{grad} f_p(x) = \frac{\sum_{i=1}^{m} \exp(p f_i(x)) \operatorname{grad} f_i(x)}{\sum_{i=1}^{m} \exp(p f_i(x))}.$$

Again, smooth optimization algorithms can be used to achieve good approximations of the solution of the minimax problem with these $f_p$.

## 2.4 Sphere packing on adjoint orbits

### 2.4.1 General results

As an illustration of our non-smooth gradient descent algorithm, we will consider sphere packing problems on Grassmann manifolds and SVD orbits. The packing problems on Grassmann manifolds are motivated by applications in coding theory [6,69,91,92,167], where an optimal sphere packing corresponds to a code with good error correction properties. We formulate the sphere packing problem in the more general setting of adjoint orbits and reductive Lie groups. Such adjoint orbit techniques have been applied for the analysis double bracket equations and gradient flows for matrix approximation problems, e.g. [149], and for extension of Jacobi-type algorithms for singular and eigenvalue problems, e.g. [98]. Note, that there are some relations of such sphere packing problems to problems in multi-agent coordination and cooperative control, see [44, 139].

We start with some preliminaries on reductive Lie groups and adjoint orbits. For Lie groups $G, K$ we will always denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of $G$ and $K$.

Let us recall some basic notions from [100]. A Lie algebra involution $\theta$ is a Lie algebra automorphism with $\theta^2 = I$. Note, that every involution $\theta$ gives a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k}$, $\mathfrak{p}$ the $+1$, $-1$ eigenspaces of $\theta$. A Lie algebra $\mathfrak{g}$ is called reductive if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_{\mathfrak{g}}$ and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. For the classical matrix groups and matrix Lie algebras like $O(n)$, $\mathrm{GL}(n)$, $\mathfrak{so}(n)$, $\mathfrak{u}(n)$, etc., we use the same notation and definitions as [100].

We can now give the definition of a reductive Lie group from [100, 149].

**Definition 2.4.1 (cf. [100])** Let $G$ be a Lie group, $K \subset G$ a compact subgroup, $\theta$ a Lie algebra involution, $F$ a bilinear form on $\mathfrak{g}$. Assume that

- $F$ is non-degenerate, Ad-invariant and symmetric,
- $\mathfrak{g}$ is reductive,
- the Lie algebra $\mathfrak{k}$ of $K$ is the $+1$ eigenspace of $\theta$,
- the $-1$ eigenspace $\mathfrak{p}$ of $\theta$ is orthogonal to $\mathfrak{k}$ with respect to $F$,
- $(X, Y) \mapsto -F(X, \theta Y)$ is positive definite,

- the map $K \times \exp \mathfrak{p} \to G$, $(k, \exp P) = K \exp P$ is a surjective diffeomorphism,
- the automorphisms $\mathrm{Ad}_g$, $g \in G$, on the complexification $\mathfrak{g} \oplus i\mathfrak{g}$ of $\mathfrak{g}$ are inner automorphisms of $\mathfrak{g} \oplus i\mathfrak{g}$.

Then we call $(G, K, \theta, F)$ a *reductive Lie group*. Furthermore we call the decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

a *Cartan decomposition* of $\mathfrak{g}$.

Note, that for real groups, there are different definitions which do not require the last condition of Definition 2.4.1, see e.g. [160]. For example, $O(n)$ is not always a reductive group by the definition above [100, p.447, Example 3], but it would be always reductive in the sense of the definition of [160]. However, it is very easy to see that the theory and optimization methods represented here would also work for these more general definitions.

For the rest of this subsection we will fix a reductive Lie group $(G, K, \theta, F)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Note, that $K$ is always a maximal compact subgroup of $G$ [100, Prop. 7.19].

**Definition 2.4.2** For $X \in \mathfrak{g}$ we define the *adjoint orbit* or $\mathrm{Ad}_K$*-orbit* $O(X)$ as the set

$$O(X) = \{\mathrm{Ad}_k X \mid k \in K\}.$$

It is well-known that an adjoint orbit is a smooth manifold with tangent space $T_Y O(X) = \mathrm{ad}_Y \mathfrak{k}$ [149]. Let us recall the definition of the normal metric, see [27, 77, 146, 149].

**Definition 2.4.3** Let $X \in \mathfrak{p}$. Denote $(\ker \mathrm{ad}_X |_{\mathfrak{k}})^{\perp} := \{A \in \mathfrak{k} \mid \forall H \in \mathfrak{k} \cap \ker \mathrm{ad}_X \colon F(A, H) = 0\}$. Then $T_X O(X) = \mathrm{ad}_X (\ker \mathrm{ad}_X |_{\mathfrak{k}})^{\perp}$. The *normal metric* on $O(X)$ is defined by

$$\langle \mathrm{ad}_X H, \mathrm{ad}_X K \rangle = -F(K, H).$$

Several authors have shown that the gradient with respect to this metric can be computed by a double bracket relation see [27, 77, 146, 149]. The reductive case was treated by Tam [149], whose results we recall here for the convenience of the reader.

**Proposition 2.4.4** *For a smooth function $\hat{f}\colon \mathfrak{p} \to \mathbf{R}$, the gradient of $f = \hat{f}|_{O(X)}$ with respect to the normal metric is given by*

$$\operatorname{grad} f(Y) = \operatorname{ad}_Y \operatorname{ad}_Y \operatorname{grad}_F \hat{f}(Y),$$

*where* $\operatorname{grad}_F \hat{f}(Y)$ *denotes the gradient of* $\hat{f}$ *with respect to* $F$ *on* $\mathfrak{p}$.

**Proof:** See [149, Lem. 2.2, Thm 2.3, Rem. p.214]. □

If no other parameterizations are available, we will need geodesics for the generalized gradient descent algorithms. Let us recall the formula for geodesics on the adjoint orbit.

**Proposition 2.4.5** *Let $X \in \mathfrak{p}$ and $\Omega \in (\ker \operatorname{ad}_X |_{\mathfrak{k}})^{\perp}$. Then $t \mapsto \operatorname{Ad}_{\exp t\Omega} X$ is a geodesic on $O(X)$ with respect to the normal metric and all geodesics through $X$ have this form.*

**Proof:** This can be proven in greater generality for geodesics with respect to the normal metric on homogeneous spaces, see [146, p.16-19]. □

**Remark 2.4.6** While we have a simple formula for the gradient $\operatorname{grad} f(Y)$ with respect to the normal metric, we need the Lie algebra element $\Omega \in (\ker \operatorname{ad}_Y |_{\mathfrak{k}})^{\perp}$ with $\operatorname{ad}_Y \Omega = \operatorname{grad} f(Y)$ to compute the geodesic in the gradient direction. Thus, we need the isomorphism $\sigma\colon T_Y O(X) \mapsto (\ker \operatorname{ad}_Y |_{\mathfrak{k}})^{\perp}$, with $\operatorname{ad}_Y \circ \sigma = \operatorname{Id}_{T_Y O(X)}$, to implement any type of Riemannian gradient descent. Given such an isomorphism $\sigma$, we get for $Y \in O(X)$ and $\eta \in T_Y O(X)$ the formula

$$\gamma(t) = \operatorname{Ad}_{\exp(-t\sigma(\eta))} Y.$$

Note, that this construction of the exponential map on $O(X)$ is a special case of the construction of local parameterizations on homogeneous spaces proposed by Celledoni and Owren [37]. The map $\sigma$ corresponds to their inverse of the tangent map of the group action on $O(X)$, and the exponential map corresponds to their coordinate map on the Lie group.

On adjoint orbits we consider the distance induced by the bilinear form $F$ on $\mathfrak{p}$.

**Definition 2.4.7** We define the distance

$$\operatorname{dist}(P, Q) = F(P - Q, P - Q)^{1/2},$$

on an $\operatorname{Ad}_K$-orbit $O(X)$, $X \in \mathfrak{p}$.

We define now the sphere packing problem on $O(X)$ with respect to this distance.

**Problem 2.4.8** Find $m$ points $P_1, \dots, P_m$ on an $\mathrm{Ad}_K$-orbit $O(X)$, $X \in \mathfrak{p}$, such that

$$\max\{r \mid B_r(P_i) \cap B_r(P_j) = \emptyset, i < j\}$$

is maximized. Here, $B_r(P_i)$ denotes the ball

$$B_r(P_i) = \{P \in O(X) \mid \operatorname{dist}(P_i, P) < r\}.$$

However, the function above is not very accessible for optimization purposes. Instead we consider the following problem.

**Problem 2.4.9** Find $m$ points $P_1, \dots, P_m$ on an adjoint $\mathrm{Ad}_K$-orbit $O(X)$, $X \in \mathfrak{p}$, such that

$$\min \operatorname{dist}(P_i, P_j) \tag{2.14}$$

is maximized.

In fact, the solutions of these problems are equivalent.

**Proposition 2.4.10** *Let $X \in \mathfrak{p}$. A $m$-tuple of points $(P_1, \dots, P_m)$, $P_i \in O(X)$, is a solution of 2.4.8 if and only if it is a solution of 2.4.9.*

**Proof:** Let $P_1, \dots, P_m$ be points in $O(X)$. Define

$$r_1 = \max\{r \mid B_r(P_i) \cap B_r(P_j) = \emptyset, i < j\}$$

and

$$r_2 = \min \operatorname{dist}(P_i, P_j).$$

As $\operatorname{dist}(P_i, P_j) \geq 2r_1$, we have $2r_1 \leq r_2$. On the other hand for all $r < r_2/2$ and all $i < j$, $B_r(P_i) \cap B_r(P_j) = \emptyset$ holds. Thus $2r_1 = r_2$, which yields our claim. □

In (2.14) we have to maximize the minimum of a finite number of non-smooth functions. Since the non-smoothness of these function would introduce additional complications to our algorithms, we will instead minimize the maximum of a finite number of smooth functions in our sphere packing algorithms.

**Proposition 2.4.11** *Let $X \in \mathfrak{p}$. The local maxima of* (2.14) *coincide with the local minima of $f\colon (O(X))^m \to \mathbf{R}$,*

$$f(P_1, \dots, P_m) = \max_{i<j} F(P_i, P_j). \tag{2.15}$$

**Proof:** First note that the maxima of (2.14) and

$$\min_{i<j} F(P_i - P_j, P_i - P_j)$$

coincide. But as

$$\begin{aligned} F(P_i - P_j, P_i - P_j) &= F(P_i, P_i) + F(P_j, P_j) - 2F(P_i, P_j) \\ &= 2F(X, X) - 2F(P_i, P_j), \end{aligned}$$

we conclude that the local minima of $f$ are exactly the local maxima of (2.14). □

To calculate the generalized gradient, we need first the gradients of maps $(P, Q) \mapsto \operatorname{tr}(PQ)$.

**Lemma 2.4.12** *Let $X \in \mathfrak{p}$. The gradient of the map $f\colon (P, Q) \mapsto F(P, Q)$ on $(O(X))^2$ with respect to the normal metric is*

$$\operatorname{grad} f(P, Q) = (\operatorname{ad}_P \operatorname{ad}_P Q, \operatorname{ad}_Q \operatorname{ad}_Q P).$$

**Proof:** Direct consequence of 2.4.4 and that $dF(P, Q)(h, k) = F(P, k) + F(h, Q)$. □

The above lemma allows us to give the following formula for the generalized gradient. Here, and in the sequel we use the notation

$$I(P) = I(P_1, \dots, P_m) = \{(i, j) \in \mathbf{N} \times \mathbf{N} \mid F(P_i, P_j) = f(P)\}$$

for the indices of the active functions.

**Proposition 2.4.13** *The generalized gradient of $f$* (2.15) *is*

$$\operatorname{grad} f(P) = \overline{\operatorname{co}}\{(\dots, \underbrace{\operatorname{ad}_{P_i} \operatorname{ad}_{P_i} P_j}_{i\text{th entry}}, \dots, \underbrace{\operatorname{ad}_{P_j} \operatorname{ad}_{P_j} P_i}_{j\text{th entry}}, \dots) \mid (i, j) \in I(P)\}^{22}$$

[22] The "..." denote zero entries.

**Proof:** Direct consequence of Lemma 2.4.12. □

Unfortunately, generalized gradient algorithms often show bad convergence properties. Therefore, it is sometimes advisable to use one of the smooth approximations discussed in Section 2.3.4. The functions $F(P_i, P_j)$ are not necessarily non-negative. To use Pólya's $p$-norm approach we have to consider the cost function $f(P) + C_X$ with $C_X$ a suitably chosen constant. The smallest choice for $C_X$ is obviously $-\min f(P)$. However, as this value is usually not known a priori, we can also use the upper bound $C_X = F(X, X)$. In some examples, there will be better choices for $C_X$. Pólya's approach leads us to the cost function

$$f_p(P) = \frac{1}{p} \sum_{i<j} (F(P_i, P_j) + C_X)^p. \tag{2.16}$$

The calculation of the gradient of these smooth cost functions is straightforward.

**Proposition 2.4.14** *The gradient of* $f_p\colon (O(X))^m \to \mathbf{R}$, (2.16)*, with respect to the normal metric is*

$$\operatorname{grad} f_p(P) = \sum_{i<j} (F(P_i, P_j) + C_X)^{p-1} (\ldots, \operatorname{ad}_{P_i} \operatorname{ad}_{P_i} P_j, \ldots, \operatorname{ad}_{P_j} \operatorname{ad}_{P_j} P_i, \ldots).$$

**Proof:** Follows again from Lemma 2.4.12. □

### 2.4.2 Example: the real Grassmann manifold

As the first concrete example for the sphere packing on adjoint orbits we consider sphere packings on the real Grassmann manifold, i.e. the manifold of all $k$ dimensional subspaces in $\mathbf{R}^n$. It is well known that the Grassmann manifold or *Grassmannian* is a compact, smooth manifold of dimension $(n-k)k$ [77]. Denote by $\operatorname{Sym}(n)$ the set of symmetric $n \times n$ matrices. To fit the Grassmann manifold into our setting we use the identification with

$$\operatorname{Grass}(n, k, \mathbf{R}) = \left\{ P \mid P \in \operatorname{Sym}(n), P^2 = P, \operatorname{tr} P = k \right\},$$

from [77], i.e. the set of symmetric projection matrices of rank $k$. More precisely, we identify a subspace with the projection $P$ onto the subspace. In [77] it is shown that this map is a natural diffeomorphism between the Grassmann manifold and $\operatorname{Grass}(n, k, \mathbf{R})$.

Conway et al. [43] have considered sphere packings on the Grassmann manifold with respect to the so-called chordal distance. Given $k$-dimensional subspaces $\hat{P}$,$\hat{Q}$ of $\mathbf{R}^n$ with principal angles[23] $\rho_1, \ldots, \rho_k$ the *chordal distance* is defined as

$$dist(\hat{P}, \hat{Q}) = \left( \sum_{i=1}^{k} \sin^2 \rho_i \right)^{\frac{1}{2}},$$

see [43]. If $P, Q \in \operatorname{Grass}(n, k, \mathbf{R})$ are the projections onto $\hat{P}$ and $\hat{Q}$, then we have

$$\operatorname{dist}(\hat{P}, \hat{Q}) = \frac{1}{\sqrt{2}} \|P - Q\|_F,$$

where $\|A\|_F = (\operatorname{tr}(A^\top A))^{1/2}$ denotes the Frobenius norm on $\mathbf{R}^{n\times n}$ [43, Thm. 2]. Hence, the *chordal distance* is given on $\operatorname{Grass}(n, k, \mathbf{R})$ by the distance

$$\operatorname{dist}(P, Q) = \frac{1}{\sqrt{2}} \|P - Q\|_F.$$

Conway et al. [43] approached the sphere packing problem on the Grassmannian by a family of cost functions

$$f_A(P_1, \ldots, P_m) = \sum_{i<j} (\operatorname{dist}(P_i, P_j) - A)^{-1}.$$

Starting with $A = 0$, they used a Hooke-Jeeves pattern search to minimize $f_A$ while repeatedly setting $A$ to $1/2 \min \operatorname{dist}(P_i, P_j)$ after a fixed number of steps. They did not give any theoretical results on the convergence of their algorithm.

Here, we use our formalism for sphere packings on adjoint orbits and the generalized gradient descent to this problem. To apply the theory from Section 2.4.1, we need first a suitable reductive Lie group.

**Proposition 2.4.15** *Let* $\operatorname{GL}_0(n, \mathbf{R})$ *the identity component of* $\operatorname{GL}(n, \mathbf{R})$. *Then* $(\operatorname{GL}_0(n, \mathbf{R}), \operatorname{SO}(n), -X^\top, \frac{1}{2}\operatorname{tr}(XY))$ *is a reductive Lie group with Lie algebra* $\mathbf{R}^{n\times n}$ *and Cartan decomposition* $\mathbf{R}^{n\times n} = \mathfrak{so}(n) \oplus \operatorname{Sym}(n)$.

**Proof:** By [100, p. 447, Example 2] the identity component $\operatorname{SL}_0(n, \mathbf{R})$ of $\operatorname{SL}(n, \mathbf{R})$ is reductive. Thus $\operatorname{GL}_0(n, \mathbf{R})$ as the direct product of $\operatorname{SL}_0(n, \mathbf{R})$ and $\mathbf{R}_+$ is reductive, too. □

[23] See [66, 12.4.3, p. 603] for a precise definition of the principal angles.

**Proposition 2.4.16** $\mathrm{Grass}(n,k,\mathbf{R})$ *is the adjoint* $\mathrm{Ad}_{\mathrm{SO}(n)}$*-orbit of* $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

**Proof:** A matrix $P \in \mathbf{R}^{n\times n}$ is a symmetric projection matrix of rank $k$, if and only if it can be written as

$$P = \Theta^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta, \quad \Theta \in \mathrm{SO}(n).$$

Hence, the Grassmannian is the $\mathrm{Ad}_{\mathrm{SO}(n)}$-orbit of the specified matrix. □

Given the inner product $F(X,Y) = \frac{1}{2}\operatorname{tr}(XY)$ on $\mathrm{Sym}(n)$, we see that the chordal distance is given by $\operatorname{dist}(P,Q) = \sqrt{F(P-Q,P-Q)}$. Hence, we can use our machinery to solve the sphere packing problem with respect to the chordal distance on the real Grassmannian.

**Corollary 2.4.17** *Let*

$$f(P_1,\ldots,P_m) = \max_{i<j} \operatorname{tr}(P_i P_j)$$

*the cost function for the sphere packing problem on* $\mathrm{Grass}(n,k,\mathbf{R})$ *with respect to the chordal distance. The generalized gradient of* $f$ *with respect to the normal metric on* $\mathrm{Grass}(n,k,\mathbf{R})$ *is*

$$\operatorname{grad} f(P_1,\ldots,P_m) = \overline{\mathrm{co}}\{(\ldots,[P_i,[P_i,P_j]],\ldots,[P_j,[P_j,P_i]],\ldots)^{24} \mid (i,j) \in I(P)\},$$

*where* $[A,B] = AB - BA$ *denotes the usual matrix Lie bracket.*

Note, that we have scaled the cost function with a positive constant to obtain simpler formulas.

For the sphere packing problem on $\mathrm{Grass}(n,k,\mathbf{R})$ we use gradient descent in local parameterizations. As parameterizations of $\mathrm{Grass}(n,k,\mathbf{R})$, we use second order approximations of the exponential map, the QR-coordinates of Helmke et al. [76]. The QR-coordinates are maps $\phi_P\colon T_P\,\mathrm{Grass}(n,k,\mathbf{R}) \to \mathrm{Grass}(n,k,\mathbf{R})$. However, they can be fit into our setting of parameterizations $\mathbf{R}^{k(n-k)} \to \mathrm{Grass}(n,k,\mathbf{R})$. by choosing a family of isometries $(\psi_p\colon \mathbf{R}^{k(n-k)} \to T_P\,\mathrm{Grass}(n,k,\mathbf{R}))$. To give a formula for the QR-coordinates we write $P \in \mathrm{Grass}(n,k,\mathbf{R})$ as

$$P = \Theta^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta \text{ with } \Theta \in \mathrm{SO}(n). \tag{2.17}$$

For $\eta \in \mathrm{Sym}(n)$ the bracket $[\eta, P]$ has the form

$$[\eta, P] == \Theta^\top \begin{pmatrix} 0 & \xi \\ -\xi^\top & 0 \end{pmatrix} \Theta \text{ with } \eta \in \mathbf{R}^{k\times(n-k)}.$$

Then the QR-coordinates are defined as

$$\phi_P(\alpha\eta) := \Theta^\top \begin{pmatrix} (I_k + \alpha^2\xi\xi^\top)^{-1/2} & \alpha\xi(I_{n-k} + t^2\xi^\top\xi)^{-1/2} \\ -\alpha\xi^\top(I_k + \alpha^2\xi\xi^\top)^{-1/2} & (I_{n-k} + \alpha^2\xi^\top\xi)^{-1/2} \end{pmatrix}$$
$$\cdot \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (I_k + \alpha^2\xi\xi^\top)^{-1/2} & \alpha\xi(I_{n-k} + \alpha^2\xi^\top\xi)^{-1/2} \\ -\alpha\xi^\top(I_k + \alpha^2\xi\xi^\top)^{-1/2} & (I_{n-k} + \alpha^2\xi^\top\xi)^{-1/2} \end{pmatrix}^\top \Theta. \quad (2.18)$$

For a detailed discussion of the QR-coordinates, we refer the reader to [76]. Note that the terms $(I_k + \alpha^2\xi\xi^\top)^{-1/2}$, $(I_{n-k} + \alpha^2\xi^\top\xi)^{-1/2}$ can be calculated again efficiently from an singular value decomposition of $\xi$. Assuming that $\xi = U\Sigma V$ with $\Sigma \in \mathbf{R}^{k\times(n-k)}$ diagonal, $U \in SO(k)$, $V \in SO(n-k)$, we have that

$$\begin{aligned} (I_k + \alpha^2\xi\xi^\top)^{-1/2} &= U(I_k + \alpha^2\Sigma\Sigma^\top)^{-1/2}U^\top \quad \text{and} \\ (I_{n-k} + \alpha^2\xi^\top\xi)^{-1/2} &= V^\top(I_{n-k} + \alpha^2\Sigma^\top\Sigma)^{-1/2}V. \end{aligned}$$

The form of $\phi_P$ (2.18) allows us to update the decomposition (2.17) during the algorithm by setting

$$\Theta_{t+1} = \begin{pmatrix} (I_k + \alpha_t^2\xi\xi^\top)^{-1/2} & \alpha_t\xi(I_{n-k} + \alpha_t^2\xi^\top\xi)^{-1/2} \\ -\alpha_t\xi^\top(I_k + \alpha_t^2\xi\xi^\top)^{-1/2} & (I_{n-k} + \alpha_t^2\xi^\top\xi)^{-1/2} \end{pmatrix}^\top \Theta_t.$$

Thus, we avoid repeated calculation of the eigenbases of $P$. Note that a similar approach to repeated updating of a singular value decomposition was used by Helmke et al. in [75].

As an second order approximation of the exponential map, the parameterizations $\phi_P$ have the differential $T_0\phi_P = \mathrm{Id}_{T_P\,\mathrm{Grass}(n,k,\mathbf{R})}$. Therefore, we can use $-\pi_0(\mathrm{grad}\, f(P_1, \ldots, P_m))$ as the descent direction and do not have to map it onto a different tangent vector.

Note that the parameterizations $\phi_P$ use $\mathrm{ad}_P\,\eta$ for calculating $\phi_P(\eta)$. Since the map $\mathrm{ad}_P^2$ is the identity on $T_P\,\mathrm{Grass}(n,k,\mathbf{R})$, see [76, Lem. 2.2, Prop. 2.3], we have that $[[P_i,[P_i,P_j]],P_i] = [P_j,P_i]$ for all $P_i, P_j \in \mathrm{Grass}(n,k,\mathbf{R})$. Hence, at points $(P_1,\ldots,P_m) \in \mathrm{Grass}(n,k,\mathbf{R})^m$, where only one function $\mathrm{tr}(P_iP_j)$ is active, we can use

$$(\ldots, P_j, \ldots, P_i, \ldots)$$

instead of the gradient $\operatorname{grad} f(P_1,\ldots,P_m)$ for calculating the curve $\alpha \mapsto \phi_P(\alpha \operatorname{grad} f(P_1,\ldots,P_m))$.

These considerations yield the following algorithm.

**Algorithm 2.4.18** Let $(P_1^0,\ldots,P_m^0)$ be $m$ initial points in $\mathrm{Grass}(n,k,\mathbf{R})$. Calculate $\Theta_i^0 \in \mathrm{SO}(n)$ such that

$$P_i^0 = (\Theta_i^0)^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^0.$$

1. If only one $\mathrm{tr}(P_iP_j)$ is active then set

$$\eta_l^t = \begin{cases} -P_i & l = i \\ -P_j & l = j \\ 0 & \text{otherwise} \end{cases}.$$

   If more $\mathrm{tr}(P_iP_j)$ are active then set

$$(\eta_1^t,\ldots,\eta_m^t) = -\pi_0(\operatorname{grad} f(P_1,\ldots,P_m)).$$

2. Calculate $\xi_i^t$, $i = 1,\ldots,m$ by

$$\begin{pmatrix} * & \xi_i^t \\ (\xi_i^t)^\top & * \end{pmatrix} := \Theta_i^t \eta_i (\Theta_i^t)^\top.$$

3. Calculate a step size $\alpha_t$ either by a harmonic or by a Armijo step size selection[25] along the curve $\alpha \mapsto (\ldots,\phi_{P_i^t}(\alpha\eta_i^t),\ldots)$.

4. Set

$$\begin{aligned} \Theta_i^{t+1} &\\ &= \begin{pmatrix} (I_k + \alpha_t^2 \xi_i^t (\xi_i^t)^\top)^{-1/2} & \alpha_t \xi_i^t (I_{n-k} + \alpha_t^2 (\xi_i^t)^\top \xi_i^t)^{-1/2} \\ -\alpha_t (\xi_i^t)^\top (I_k + \alpha_t^2 \xi_i^t (\xi_i^t)^\top)^{-1/2} & (I_{n-k} + \alpha_t^2 (\xi_i^t)^\top \xi_i^t)^{-1/2} \end{pmatrix} \Theta_i^t, \\ P_i^{t+1} &= (\Theta_i^{t+1})^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^{t+1}. \end{aligned}$$

5. Set $t = t+1$ and go to step 1.

[25]Note, that when using the Armijo step size selection, we have to calculate the SVD of $\xi_i^t$ for the repeated calculation of $\phi_{P_i^t}(C2^{-l}\eta_i^t)$ only once.

**Corollary 2.4.19** *If the step sizes satisfy $\sum \alpha_t = \infty$, $\alpha_t \to 0$, then the iterates of Algorithm 2.4.18 converge to the set of critical points of $f$.*

**Proof:** Choose a family of isometries $(\psi_P \colon \mathbf{R}^{k(n-k)} \to T_P \operatorname{Grass}(n,k,\mathbf{R}))$. Then our claim follows from Theorem 2.3.27 with the family of parameterizations $(\phi_P \circ \psi_P)$. □

### 2.4.3 Example: the real Lagrange Grassmannian

Our next example is the real Lagrange Grassmann manifold, i.e. the manifold of Lagrangian subspaces in $\mathbf{R}^{2n}$. A $n$-dimensional subspace $V \subset \mathbf{R}^{2n}$ is Lagrangian if for all $v, w \in V$

$$v^\top J w = 0, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

see [76]. Equivalently to the Grassmann manifold, we can identify the Lagrangian subspaces with self-adjoint projection operators onto the subspaces, which satisfy $PJP = 0$. Thus one can identify the Lagrange Grassmann manifold with the *Lagrange Grassmannian* [76]

$$\operatorname{LGrass}(n) = \{P \in \operatorname{Sym}(2n) \mid P^2 = P, \operatorname{tr} P = n, PJP = 0\}.$$

A Lagrangian subspace $\hat{P}$ is identified with the $P \in \operatorname{LGrass}(n)$ such that $P$ is the projection onto $\hat{P}$.

We need now a suitable reductive group to apply our theory. For this purpose, recall that

$$\operatorname{Sp}(n) = \{X \in \operatorname{SL}(n,\mathbf{R}) \mid X^\top J X = J\}$$

is the symplectic group and

$$\mathfrak{sp}(n) = \{X \in \mathbf{R}^{n \times n} \mid X^\top J + JX = 0\}$$

its Lie algebra [100].

**Proposition 2.4.20** *Let*

$$G = \{\lambda X \in \operatorname{Sp}(n,\mathbf{R}) \mid X^\top J X = J, \lambda \in \mathbf{R}_+\}$$

*the group of symplectic $n \times n$ matrices and by $G_0$ its identity component. Furthermore, let the group $\operatorname{OSP}(n)$ be given by*

$$\operatorname{OSP}(n) = \{X \in G \mid X^\top X = I\}^{26}.$$

*Then* $(G_0, \mathrm{OSP}(n), -X^\top, \frac{1}{2}\operatorname{tr}(XY))$ *is a reductive group. The Lie algebra of* $G$ *is* $\mathfrak{g} = \mathfrak{sp}(n, \mathbf{R}) \times \mathbf{R}I_{2n}$*. The Lie algebra* $\mathfrak{g}$ *of* $G$ *has the Cartan decomposition*

$$\mathfrak{g} = \mathfrak{osp}(n) + \mathrm{PSym}(n)$$

*with* $\mathfrak{osp}(n)$[26] $= \mathfrak{so}(2n) \cap \mathfrak{sp}(n)$ *and* $\mathrm{PSym}(n) = (\mathrm{Sym}(2n) \cap \mathfrak{sp}(n)) + \mathbf{R}I_{2n}$*.*

**Proof:** The group $G$ is a closed, linear group of matrices closed under the conjugate transpose operation. Our claim follows again from [100, p. 447, Example 3]. □

**Proposition 2.4.21** $\mathrm{LGrass}(n)$ *is the* $\mathrm{Ad}_{\mathrm{OSP}(n)}$*-orbit of* $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

**Proof:** The group $\mathrm{OSP}(n)$ acts transitively on $\mathrm{LGrass}(n)$ by conjugation, see [76]. □

By our definition $\mathrm{LGrass}(n)$ is a subset of $\mathrm{Grass}(2n, n, \mathbf{R})$. Thus the chordal distance

$$\mathrm{dist}(P, Q) = \frac{1}{\sqrt{2}}\|P - Q\|_F$$

is well-defined on $\mathrm{LGrass}(n)$. In fact, as in the Grassmannian case it can also be defined via the principal angles between the Lagrangian subspaces. Given the bilinear form $F(X, Y) = \frac{1}{2}\operatorname{tr}(XY)$ on $\mathfrak{sp}(n, \mathbf{R}) \times \mathbf{R}I_{2n}$, we get again that chordal distance is $\mathrm{dist}(P, Q) = \sqrt{F(P - Q, P - Q)}$ . Thus our machinery is applicable to this example, too.

**Corollary 2.4.22** *Let*

$$f(P_1, \ldots, P_m) = \max_{i<j} \operatorname{tr}(P_i P_j)$$

*the cost function for the sphere packing problem on* $\mathrm{LGrass}(n)$ *with respect to the chordal distance. The generalized gradient of* $f$ *with respect to the normal metric on* $\mathrm{LGrass}(n)$ *is*

$$\operatorname{grad} f(P_1, \ldots, P_m) = \\ \overline{\mathrm{co}}\{(\ldots, [P_i, [P_i, P_j]], \ldots, [P_j, [P_j, P_i]], \ldots) \mid (i, j) \in I(P)\}.$$

[26] cf. [76]

As the Lagrange Grassmannian is a subset of $\text{Grass}(2n, n, \mathbf{R})$, we can consider $f$ as the restriction of the cost function given in the previous subsection to the $m$-fold product of the Lagrange Grassmannian. The generalized gradient of $f$ on the product of $\text{LGrass}(n)$ coincides with generalized gradient of $f$ on $\text{Grass}(2n, n, \mathbf{R})$ for points $P_1, \ldots, P_m \in \text{LGrass}(n) \subset \text{Grass}(2n, n, \mathbf{R})$. Furthermore, it can be shown that $\text{LGrass}(n)$ is a totally geodesic submanifold of $\text{Grass}(2n, n, \mathbf{R})$ [76]. Additionally, if we have a decomposition for $P \in \text{LGrass}(n)$

$$P = \Theta^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta, \text{ with } \Theta \in \text{OSP}(n), \tag{2.19}$$

we can define QR-coordinates $\phi_P \colon T_P \text{LGrass}(n) \to \text{LGrass}(n)$ on the Lagrange Grassmannian by the restriction of the QR-coordinates for the Grassmannian $\text{Grass}(2n, n, \mathbf{R})$ to $T_P \text{LGrass}(n)$ [76]. Hence, the Lagrange Grassmannian is an invariant set of the generalized gradient descent iteration, if the decompositions of the $P_i^t$ have all the form (2.19). This can be achieved by ensuring that in the initial decompositions

$$P_i^0 = (\Theta_i^0)^\top \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^0$$

one has $\Theta \in \text{OSP}(n)$.

Therefore, we have to following result.

**Proposition 2.4.23** *Let* $(P_1, \ldots, P_m) \in \text{LGrass}(n)^m$ *and decompositions*

$$P_i^0 = (\Theta_i^0)^\top \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^0.$$

*with* $\Theta_i^0 \in \text{OSP}(n)$. *If Algorithm 2.4.18 starts with such initial values, then it produces a generalized gradient descent on* $\text{LGrass}(n)$ *for the function* $f$. *If additionally the step sizes satisfy* $\sum \alpha_t = \infty$, $\alpha_t \to 0$, *then iterates converge to the set of critical points of* $f$.

**Proof:** It can be shown that for a decomposition (2.19) and $\eta \in \text{LGrass}(n)$ the matrices

$$\begin{pmatrix} (I_k + \xi\xi^\top)^{-1/2} & \xi(I_{n-k} + \xi^\top\xi)^{-1/2} \\ -\xi^\top(I_k + \xi\xi^\top)^{-1/2} & (I_{n-k} + \xi^\top\xi)^{-1/2} \end{pmatrix}^\top.$$

are elements of $\text{OSP}(n)$, see [76]. Thus our claims follow by induction and from Corollary 2.4.19. $\square$

### 2.4.4 Example: SVD orbit

We now demonstrate that the sphere packing problem on the set of matrices with fixed singular values with respect to the Euclidean distance is also covered by our methods. Assume we are given a $k < n$ and fixed real numbers $\lambda_1 > \ldots > \lambda_k > 0$. Let us denote $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^{k\times k}$ and $L = \begin{pmatrix} D \\ 0 \end{pmatrix} \in \mathbf{R}^{n\times k}$. As in [77, p.84] we consider the set

$$M(L) = \{ULV \mid U \in O(n), V \in O(k)\}.$$

This set is a smooth, compact manifold, see [77, p. 86, Prop. 2.2]. We call $M(L)$ the *SVD-orbit of L*. Let $\mathbf{R}^{n\times k}$ be equipped with the Euclidean scalar product, i.e. $\langle X, Y\rangle = \operatorname{tr}(X^\top Y)$. We consider the sphere packing problem on $M(L)$ with respect to the Euclidean distance

$$\operatorname{dist}_E(X, Y) = \langle X - Y, X - Y\rangle^{1/2}.$$

To embed the manifold $M(L)$ into our setting, we use the well-known trick to identify $M(L)$ with a manifold of symmetric $(n + k) \times (n + k)$ matrices, see [40, 77, 146]. The manifold $M(L)$ is identified with

$$\hat{M}(L) = \left\{ \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \middle| X \in M(L) \right\}$$

by mapping

$$X \mapsto \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix},$$

see [77, p. 90]. It is known that this manifold is an adjoint orbit [98, 149]. To give the associated reductive Lie group of $\hat{M}(L)$, we need some additional notions. Recall that $\mathrm{SO}(n, k)$ is the Lie group

$$\mathrm{SO}(n, k) = \left\{A \in \mathrm{SL}(n + k, \mathbf{R}) \mid A^\top \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix} \right\}$$

with Lie algebra

$$\mathfrak{so}(n, k) = \left\{X \in \mathfrak{sl}(n + k, \mathbf{R}) \mid X^\top \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix} + \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix} X = 0 \right\},$$

cf. [100]. For a matrix group $G$ we denote by $S(G)$ the subgroup of matrices $A \in G$ with $\det(A) = 1$. Furthermore, we use the notation $G \times K$for the set

$$G \times K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in G, B \in K \right\},$$

where $G, K$ are matrix groups or matrix lie algebras.

**Proposition 2.4.24** *We denote by* $\mathrm{SO}_0(n,k)$ *the identity component of the group* $\mathrm{SO}(n,k)$*. Then* $(\mathrm{SO}_0(n,k), S(O(n) \times O(k)), -X^\top, \mathrm{tr}(XY))$ *is a reductive Lie group. The Lie algebra of* $\mathrm{SO}_0(n,k)$ *is* $\mathfrak{so}(n,k)$ *with Cartan decomposition*

$$\mathfrak{so}(n,k) = \mathfrak{so}(n) \times \mathfrak{so}(k) + \mathrm{Sym}(n+k) \cap \mathfrak{so}(n,k).$$

*The manifold* $\hat{M}(L)$ *is the* $\mathrm{Ad}_{S(O(n)\times O(k))}$*-orbit of* $\begin{pmatrix} 0 & L \\ L^\top & 0 \end{pmatrix}$.

**Proof:** See [98, 149]. □

**Proposition 2.4.25** *Let*

$$\mathrm{dist}(A,B) = \sqrt{\mathrm{tr}((A-B)^2)}$$

*the distance on* $\mathrm{Sym}(n+k)$ *induced by* $\mathrm{tr}(AB)$*. Then for*

$$\hat{X} = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}, \ \hat{Y} = \begin{pmatrix} 0 & Y \\ Y^\top & 0 \end{pmatrix} \in \hat{M}(L)$$

*we have that* $\mathrm{dist}(\hat{X}, \hat{Y}) = \sqrt{2}\,\mathrm{dist}_E(X,Y)$.

**Proof:** This follows from

$$\mathrm{tr}\left( \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix} \right) = \mathrm{tr}\left( \begin{pmatrix} AB^\top & 0 \\ 0 & A^\top B \end{pmatrix} \right) = 2\,\mathrm{tr}(A^\top B).$$

□

Hence, we can solve the sphere packing problem on $M(L)$ by applying our machinery to the sphere packing problem on $\hat{M}(L)$.

**Corollary 2.4.26** *Let*

$$f\left( \begin{pmatrix} 0 & P_1 \\ P_1^\top & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & P_m \\ P_m^\top & 0 \end{pmatrix} \right) = \max_{i<j} 2\,\mathrm{tr}(P_i^\top P_j)$$

*the cost function for the sphere packing problem on $\hat{M}(L)$ with respect to the distance* dist*. The generalized gradient of $f$ with respect to the normal metric on $\hat{M}(L)$ is*

$$\operatorname{grad} f\left(\begin{pmatrix} 0 & P_1 \\ P_1^\top & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & P_m \\ P_m^\top & 0 \end{pmatrix}\right) = \\ \left\{ \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \;\middle|\; A \in \overline{\mathrm{co}}\{P_iP_i^\top P_j + P_jP_i^\top P_i - 2P_iP_j^\top P_i \mid (i,j) \in I(P)\} \right\}.$$

**Proof:** Straight forward calculations show that

$$\left[\begin{pmatrix} 0 & P \\ P^\top & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & P \\ P^\top & 0 \end{pmatrix}, \begin{pmatrix} 0 & Q \\ Q^\top & 0 \end{pmatrix}\right]\right] = \\ \begin{pmatrix} 0 & PP^\top Q + QP^\top P - 2PQ^\top P \\ P^\top PQ^\top + Q^\top PP^\top - 2P^\top QP^\top & 0 \end{pmatrix}.$$

for $P, Q \in \mathbf{R}^{n\times k}$. □

To determine the geodesics we have to calculate for $\eta \in T_{\hat{P}}\hat{M}(L)$ the $\Omega \in (\ker \operatorname{ad}_{\hat{P}}|_{(\mathfrak{so}(n)\times\mathfrak{so}(k))})^\perp$ with $\eta = \operatorname{ad}_{\hat{P}}\Omega$. We use the standard approach to reduce the calculations to the case that $\hat{P} \in \hat{M}(L)$ has a suitably simple structure, see e.g. [54, 76, 146]. Let $P \in M(L)$ and

$$\hat{P} = \begin{pmatrix} 0 & P \\ P^\top & 0 \end{pmatrix} \in \hat{M}(L).$$

We can write $P = ULV$, $U \in \mathrm{SO}(n)$, $V \in \mathrm{SO}(k)$. Then $\hat{P} = \hat{Q}^\top \hat{L}\hat{Q}$ with

$$\hat{Q} = \begin{pmatrix} U^\top & 0 \\ 0 & V \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & L \\ L^\top & 0 \end{pmatrix}.$$

Thus for $\Omega_1 \in \mathfrak{so}(n), \Omega_2 \in \mathfrak{so}(k)$, $\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$ we have that

$$[\hat{P}, \Omega] = \hat{Q}[\hat{L}, \hat{Q}^\top \Omega \hat{Q}]\hat{Q}^\top.$$

The following calculations are similar to the case of the compact Stiefel manifold, cf. [54]. Straightforward calculations show that

$$\ker \operatorname{ad}_{\hat{P}}|_{(\mathfrak{so}(n)\times\mathfrak{so}(k))} = \left\{ \hat{Q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix}\hat{Q}^\top \;\middle|\; S \in \mathfrak{so}(n-k) \right\}.$$

Thus

$$\left(\ker \operatorname{ad}_{\hat{P}} |_{(\mathfrak{so}(n)\times\mathfrak{so}(k))}\right)^{\perp} =$$
$$\left\{ \hat{Q} \begin{pmatrix} A & B & 0 \\ -B^\top & 0 & 0 \\ 0 & 0 & C \end{pmatrix} \hat{Q}^\top \;\middle|\; A, C \in \mathfrak{so}(k), B \in \mathbf{R}^{k\times(n-k)} \right\}.$$

Therefore, we have for any $\eta \in T_{\hat{P}}\hat{M}(L)$ the equation[27]

$$\eta = \hat{Q} \left[ \hat{L}, \begin{pmatrix} A & B & 0 \\ -B^\top & 0 & 0 \\ 0 & 0 & C \end{pmatrix} \right] \hat{Q}^\top =$$
$$\hat{Q} \begin{pmatrix} 0 & 0 & DC - AD \\ 0 & 0 & B^\top D \\ DA - CD & DB & 0 \end{pmatrix} \hat{Q}^\top, \quad A, C \in \mathfrak{so}(k), B \in \mathbf{R}^{k\times(n-k)},$$

with

$$\hat{Q} \begin{pmatrix} A & B & 0 \\ -B^\top & 0 & 0 \\ 0 & 0 & C \end{pmatrix} \hat{Q}^\top \in (\ker \operatorname{ad}_{\hat{P}} |_{(\mathfrak{so}(n)\times\mathfrak{so}(k))})^{\perp}.$$

This allows us now to calculate the geodesic in direction $\eta \in T_{\hat{P}}\hat{M}(L)$. Let

$$\hat{Q}^\top \eta \hat{Q} = \begin{pmatrix} 0 & 0 & \eta_1 \\ 0 & 0 & \eta_2 \\ \eta_1^\top & \eta_2^\top & 0 \end{pmatrix}, \eta_1 \in \mathbf{R}^{k\times k}, \eta_2 \in \mathbf{R}^{(n-k)\times k}.$$

and $A, C \in \mathfrak{so}(k)$ satisfy

$$\begin{aligned} D^{-1}\eta_1 + \eta_1^\top D^{-1} &= D^{-1}AD - DAD^{-1}, \\ \eta_1 D^{-1} + D^{-1}\eta_1^\top &= DCD^{-1} - D^{-1}CD. \end{aligned}$$

Then the geodesic $\hat{\gamma}(t)$ through $\hat{P}$ in direction $\eta$ has the form

$$\hat{\gamma}(t) = \begin{pmatrix} 0 & \gamma(t) \\ \gamma(t)^\top & 0 \end{pmatrix}$$

with

$$\gamma(t) = U \exp\left( t \begin{pmatrix} A & D^{-1}\eta_2^\top \\ -\eta_2 D^{-1} & 0 \end{pmatrix} \right) \begin{pmatrix} D\exp(-tC)V \\ 0 \end{pmatrix}.$$

[27] Recall that we defined $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$.

The formula for geodesics allows us again to update the singular value decomposition in each iteration, like Helmke et al. in [75]. Note, that it is also possible to give an explicit description of the isomorphism $\sigma\colon T_{\hat{P}}\hat{M}(L) \to \left(\ker \operatorname{ad}_{\hat{P}}|_{(\mathfrak{so}(n)\times\mathfrak{so}(k))}\right)^{\perp}$ by using Kronecker products and vec-operations[28]. We have basically to give explicit formulas for $A$ and $C$.

In an implementation of the algorithm we will of course compute only the upper triangular part of the points on $\hat{M}(L)$, i.e. our iteration will operate on the $M(L)$ itself instead of $\hat{M}(L)$. This yields the following algorithm.

**Algorithm 2.4.27** Let $(P_1^0, \ldots, P_m^0)$ be $m$ initial points in $M(L)$. Define $U_i^0 \in \mathrm{SO}(n)$, $V_i^0 \in \mathrm{SO}(k)$ by $P_i^0 = U_i^0 \begin{pmatrix} D \\ 0 \end{pmatrix} V_i^0$.

1. Set

$$\begin{aligned}
&(\zeta_1^t, \ldots, \zeta_m^t) := \\
&\qquad -\pi_0\left(\overline{\mathrm{co}}\{\left(\ldots, P_iP_i^\top P_j + P_jP_i^\top P_i - 2P_iP_j^\top P_i, \ldots,\right.\right. \\
&\qquad\qquad \left.\left. P_jP_j^\top P_i + P_iP_j^\top P_j - 2P_jP_i^\top P_j, \ldots\right) \mid (i,j) \in I(P)\}\right),
\end{aligned}$$

   where $\pi_0(C)$ denotes the least-distance projection to a convex set $C \subset \mathbf{R}^{n\times k}$ with respect to the distance $\operatorname{dist}(A,B) = \operatorname{tr}((A-B)^\top(A-B))^{1/2}$.

2. Set

$$\begin{pmatrix} H_i^t \\ K_i^t \end{pmatrix} := (U_i^t)^\top \zeta_i^t (V_i^t)^\top, \quad H_i^t \in \mathfrak{so}(k), K_i^t \in \mathbf{R}^{(n-k)\times k}.$$

   and determine $A_i^t$, $C_i^t$ by

$$\begin{aligned}
D^{-1}H_i^t + (H_i^t)^\top D^{-1} &= D^{-1}A_i^t D - DA_i^t D^{-1}, \\
H_i^t D^{-1} + D^{-1}(H_i^t)^\top &= DC_i^t D^{-1} - D^{-1}C_i^t D.
\end{aligned}$$

3. Choose a step size $\alpha_t$ by an harmonic or Armijo step size selection.

4. Set

$$\begin{aligned}
U_i^{t+1} &:= U_i^t \exp\left(\alpha_t \begin{pmatrix} A_i^t & D^{-1}(K_i^t)^\top \\ -K_i^t D^{-1} & 0 \end{pmatrix}\right) \\
V_i^{t+1} &:= \exp(-\alpha_t C_i^t) V_i^t \\
P_i^{t+1} &:= U_i^{t+1} L V_i^t
\end{aligned}$$

[28] See [93] for definition of the Kronecker product and the vec operation.

5. Set $t = t + 1$ and go to step 1.

**Corollary 2.4.28** *If the step sizes in Algorithm 2.4.27 satisfy* $\sum \alpha_t = \infty$, $\alpha_t \to 0$, *then the iterates converge to the set of critical points of* $f$.

**Proof:** This is an application of Theorem 2.3.22. □

**Remark 2.4.29** Normal metrics can be generally defined on homogeneous spaces and, in particular, on $M(L)$, too, see [77, 146]. In our case, we could construct a normal metric on $M(L)$ such that the identification $M(L) \mapsto \hat{M}(L)$ is an isometry. Hence, a generalized gradient descent for the cost function $\max_{i<j}(\mathrm{dist}_E(P_i, P_j))^2$ on $(M(L))^m$ would also yield algorithm 2.4.27.

### 2.4.5 Example: optimal unitary space-time constellations

As the next example, we consider the construction of optimal space-time constellations for multi-antenna communication channels with Rayleigh flat fading. Unitary space-time constellations for such channels were introduced by Hochwald and Marzetta [91]. The channel consists of $k$ transmitter antennas, $r$ receiver antennas and $n$ discrete time slots for transmission. The transmission model has the form [6, 69, 91]

$$R = \sqrt{\frac{\rho}{k}} SH + W,$$

where $R \in \mathbf{C}^{n\times r}$ the received signal, $S \in \mathbf{C}^{n\times k}$ the send signal, $H \in \mathbf{C}^{r\times r}$ the matrix of Rayleigh fading coefficients, $W \in \mathbf{C}^{n\times r}$ the channel noise, $\rho$ the signal to noise ratio. Furthermore, it is assumed that not the exact channel and noise coefficients $H$ and $W$, but only their distributions are known. More precisely, the fading coefficients are Gaussian distributed and the $W$ is Gaussian white noise [6, 69, 91]. A unitary space-time constellation as proposed by Hochwald and Marzetta for this channel model consists of a finite number of matrices $U_1, \ldots, U_m \in \mathbf{C}^{n\times k}$ with unitary columns, i.e. $U_i^* U_i = I_k$. Thus, a unitary space-time constellation is a finite subset of the compact, complex Stiefel manifold

$$\mathrm{St}(n, k, \mathbf{C}) = \{U \in \mathbf{C}^{n\times k} \mid U_i^* U_i = I_k\}.$$

At the receiver the signal is decoded by a maximum likelihood decoder, i.e.

$$S_R = \operatorname*{argmax}_{U=U_1,\ldots,U_m} \|R^* U\|_F,$$

with $\|A\|_F = (\mathrm{tr}(A^*A))^{1/2}$ the complex Frobenius norm, $R$ the received and $S_R$ the decoded signal [69, 91]. Note, that for each $U_i$ the value $\|R^*U_i\|_F$ only depends on the subspace of $\mathbf{C}^n$ spanned by the columns of $U_i$. Thus, the matrices $U_i$ cannot be identified directly at the receiver, but only the subspaces spanned by their columns. Therefore, to choose a unitary space-time constellations, we have to choose suitable constellations of subspaces in $\mathbf{C}^n$.

Two different design criteria for good unitary-space time constellations have been identified in the literature, see [6, 69, 92]. For a low signal-to-noise ratio it is beneficial to maximize the *diversity sum* [69]

$$\min_{i<j} \sqrt{1 - \frac{1}{k} \|U_i^* U_j\|_F}$$

or equivalently to minimize [6]

$$\max_{i<j} \|U_i^* U_j\|_F^2 . \tag{2.20}$$

If the signal-to-noise ratio is high, then maximizing the *diversity product* [69, 92]

$$\min_{i<j} \Pi_{s=1}^k \left(1 - \delta_s(U_i^* U_j)^2\right)^{1/2k} , \tag{2.21}$$

$\delta_s$ the $s$th singular value, gives constellations with good error correction properties. Both criteria are derived from Chernoff's bounds on the error probabilities, for a detailed discussion we refer the reader to [6, 69, 92].

Note, that both the diversity sum and product only depend on the spaces spanned by columns of the $U_i$, which matches the fact that we have in fact to choose subspaces of $\mathbf{C}^n$ as the points of a constellation. Thus, the choice of a good constellation with respect to one of the design criteria is an optimization problem on the complex Grassmann manifold of $k$-dimensional subspaces of $\mathbf{C}^n$. Similar to the real case, we identify the complex Grassmann manifold with the complex Grassmannian $\mathrm{Grass}(n, k, \mathbf{C})$ of hermitian projection operators onto $k$-dimensional subspaces,

$$\mathrm{Grass}(n, k, \mathbf{C}) = \left\{P \in \mathbf{C}^{n \times n} \mid P^2 = P, P^* = P, \mathrm{tr}(P) = k\right\} .$$

Here, we consider $\mathrm{Grass}(n, k, \mathbf{C})$ as a real manifold of dimension $2k(n-k)$. As in the real case, the complex Grassmannian is an adjoint orbit and fits into the reductive group setting.

**Proposition 2.4.30** *Denote by* $\mathrm{GL}_0(n, \mathbf{C})$ *the identity component of the group* $\mathrm{GL}(n, \mathbf{C})$*. Then* $(\mathrm{GL}_0(n, \mathbf{C}), U(n), -X^*, \Re \operatorname{tr}(XY))$ *is a reductive Lie group. Its Lie algebra is* $\mathbf{C}^{n \times n}$ *with Cartan decomposition*

$$\mathbf{C}^{n \times n} = \mathfrak{u}(n) + \mathrm{Herm}(n),$$

*where* $\mathrm{Herm}(n)$ *denotes the Hermitian* $n \times n$ *matrices.*

**Proof:** Analogous argument to the real case. □

**Proposition 2.4.31** *The Grassmannian* $\mathrm{Grass}(n, k, \mathbf{C})$ *is the* $\mathrm{Ad}_{U(n)}$ *orbit of* $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

**Proof:** Similar to the real case, a matrix $P \in \mathbf{C}^{n \times n}$ is a hermitian projection operator of rank $k$, if and only if it can be written as

$$P = U^* \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} U, \quad U \in U(n).$$

□

### The diversity sum

The problem of finding optimal constellations with respect to the diversity sum, can be restated as a sphere packing problem on the complex Grassmannian with respect to the distance induced by $F$. This interpretation as a sphere packing problem has already been noticed and used by Agrawal et al. in [6]. Note, that this distance is again the chordal distance, but this time between complex subspaces.

**Proposition 2.4.32** *The diversity sum* (2.20) *induces a smooth function* $f$ *on the m-fold* $\mathrm{Grass}(n, k, \mathbf{C})$ *which coincides with the minimum distance function 2.14 with respect to the distance*

$$\mathrm{dist}(P, Q) = (\Re \operatorname{tr}((P - Q)^2))^{1/2}.$$

*In particular, maximizing* (2.20) *is equivalent to the sphere packing problem on* $\mathrm{Grass}(n, k, \mathbf{C})$ *with respect to this distance.*

**Proof:** This follows directly from the fact, that for a $U \in \mathbf{C}^{n \times k}$ with $U^*U = I_k$, the projector on the column space of $U$ is $UU^*$. □

Agrawal et al. [6] approach this optimization problem by minimizing

$$\max_{i<j} \operatorname{tr}(U_j^* U_i U_i^* U_j) \tag{2.22}$$

on the $m$-fold product of the compact, complex Stiefel manifold $\operatorname{St}(n, k, \mathbf{C})$. To do so, they replace the non-smooth cost function by a family of regularizations

$$f_A = \frac{1}{A} \log \left( \sum_{i<j} \exp(A \operatorname{tr}(U_j^* U_i U_i^* U_j)) \right),$$

$U_i \in \operatorname{St}(n, k, \mathbf{C})$, $i = 1, \ldots, m$. Note, that the $f_A$ are entropic regularizations of the Minimax problem (2.22) as proposed by Li and Fang, see Section 2.3.4. Agrawal et al. use gradient descent in a overparameterization of $\operatorname{St}(n, k, \mathbf{C})$ to converge to a minimum of a regularizations. The parameter $A$ is repeatedly increased to converge to a minimum of the non-smooth cost function. However, they do not prove any theoretical results on the convergence of their algorithm. Han and Rosenthal [69, 70] use a simulated annealing algorithm to find constellations with a good diversity sum in a discrete subset on the $m$-fold product of $\operatorname{St}(n, k, \mathbf{C})$. They restrict their algorithm to the case $n = 2k$. Furthermore, they do not provide any theoretical convergence results.

We can apply our non-smooth optimization approach to this sphere packing problem.

**Corollary 2.4.33** *Let*

$$f(P_1, \ldots, P_m) = \max_{i<j} \Re \operatorname{tr}(P_i P_j)$$

*the cost function for the sphere packing problem on* $\operatorname{Grass}(n, k, \mathbf{C})$ *with respect to the distance* $(\Re \operatorname{tr}((P-Q)^2))^{1/2}$. *The generalized gradient of* $f$ *with respect to the normal metric on* $\operatorname{Grass}(n, k, \mathbf{C})$ *is*

$$\operatorname{grad} f(P_1, \ldots, P_m) = \\ \overline{\operatorname{co}}\{(\ldots, [P_i, [P_i, P_j]], \ldots, [P_j, [P_j, P_i]], \ldots) \mid (i, j) \in I(P)\}.$$

Again, we seek an efficient implementation of the generalized gradient descent by exploiting the structure of the elements of $\operatorname{Grass}(n, k, \mathbf{C})$. This can be achieved by a direct extension of the QR-coordinates to the complex

Grassmannian. We just have to replace the transpose with the complex conjugate transpose in the formulas of the real case. Thus, we can derive a generalized gradient descent algorithm on the complex Grassmannian from algorithm for the real case in a simple, straightforward manner.

**Proposition 2.4.34** *Let* $(P_1^0, \ldots, P_m^0) \in \mathrm{Grass}(n, k, \mathbf{C})^m$ *and for* $i = 1, \ldots, m$

$$P_i^0 = (\Theta_i^0)^* \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^0, \text{ with } \Theta_i^0 \in U(n).$$

*If we replace the transpose with the complex conjugate transpose operation in Algorithm 2.4.18 then it produces a generalized gradient descent on the complex Grassmannian* $\mathrm{Grass}(n, k, \mathbf{C})$ *for the function* $f$ *and the initial values above. In particular, if the step sizes satisfy* $\sum \alpha_t = \infty$, $\alpha_t \to 0$, *then the algorithm converges to the critical points of* $f$.

### The diversity product

Han and Rosenthal [69, 70] have also applied their algorithm to the problem of finding constellations with maximized diversity product. However, the same restrictions as for the diversity sum apply: they did only consider the case $n = 2k$ and optimized over a discrete subset of the $m$-fold product of the complex Stiefel manifold. Furthermore, they gave no theoretical convergence results.

Here, we will consider the application of our non-smooth optimization methods to the diversity product. As already mentioned, diversity product does only depend on the column spaces spanned by the matrices of the unitary space-time constellation and is therefore a well-defined function on the complex Grassmannian $\mathrm{Grass}(n, k, \mathbf{C})$, too. The following proposition gives an equivalent form of the diversity product on $\mathrm{Grass}(n, k, \mathbf{C})$.

**Proposition 2.4.35** *Let* $U_1, \ldots, U_m \in \mathrm{St}(n, k, \mathbf{C})$ *and* $P_1 = U_1 U_1^*, \ldots, P_m = U_m U_m^*$. *Then* $P_i \in \mathrm{Grass}(n, k, \mathbf{C})$, $i = 1, \ldots, m$ *and*

$$\min_{i<j} \Pi_{s=1}^k \left(1 - \delta_s (U_i^* U_j)^2\right)^{1/2k} = \min_{i<j} \left(\det(I_n - P_i P_j)\right)^{1/2k},$$

*where* $\delta_s(A)$ *denotes the sth singular value of* $A$.

For derivation of this formula in the case $n = 2k$ with a special representation of $\mathrm{St}(2k, k, \mathcal{C})$, see [69, 92].

**Proof:** One sees directly that $P_i \in \mathrm{Grass}(n, k, \mathcal{C})$. We show that $\Pi_{s=1}^k(1 - \delta_s(U^*V)^2) = \det(I_n - UU^*VV^*)$ for $U, V \in \mathrm{St}(n, k, \mathbf{C})$. W.l.o.g. we can assume that

$$U = \begin{pmatrix} I_k \\ 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} A \\ B \end{pmatrix},$$

$A \in \mathbf{C}^{k \times k}$, $B \in \mathbf{C}^{(n-k) \times k}$. Straightforward calculations show that

$$\begin{aligned}
\det\left(I_n - UU^*VV^*\right) &= \det\left(I_n - \begin{pmatrix} AA^* & AB^* \\ 0 & 0 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} I_k - AA^* & AB^* \\ 0 & I_{n-k} \end{pmatrix} \\
&= \det(I_k - AA^*) \\
&= \Pi_{s=1}^k(1 - \delta_s(A)^2) \\
&= \Pi_{s=1}^k(1 - \delta_s(U^*V)^2).
\end{aligned}$$

This proves our claim. □

Furthermore, we need some simple facts on $\det(I - PQ)$.

**Lemma 2.4.36** *For all positive semi-definite $P, Q \in \mathrm{Herm}(n)$ the function $\det(I - PQ)$ is real valued. For $P, Q \in \mathrm{Grass}(n, k, \mathbf{C})$ it is non-negative and bounded from above by $2^k$.*

**Proof:** First note, that for all $P, Q \in \mathbf{C}^{n \times n}$, the eigenvalues of $I - PQ$ have the form $1 - \lambda_i$ with $\lambda_i$ the eigenvalues of $PQ$. By a theorem of Horn and Johnson [93, Thm 7.6.3] for a strictly positive definite $P$, and hermitian $Q$ the eigenvalues of $PQ$ are real. A straightforward continuity argument on the eigenvalues as the zeros of the characteristic polynomial gives that $\det(I - PQ)$ is real for all positive semidefinite $P, Q$. For $P, Q \in \mathrm{Grass}(n, k, \mathbf{C})$ we have that $\|PQ\| \leq 1$, $\|\cdot\|$ denoting the operator norm. Thus $I - PQ$ has only eigenvalues $\geq 0$ and $\|I - PQ\| < 2$, i.e. the eigenvalues are bounded from above by 2. This proves our claim. □

By the above proposition and lemma we can either minimize

$$f_1(P_1, \ldots, P_m) = \max_{i<j}(2^k - \det(I - P_iP_j))$$

or

$$f_2(P_1, \ldots, P_m) = \max_{i<j} - \log(\det(I - P_iP_j))$$

on the $m$-fold product of the complex Grassmannian $\mathrm{Grass}(n, k, \mathbf{C})$ to obtain a constellation with maximized diversity product. The constant $2^k$ is added to $f_1$ to ensure that the functions are non-negative and Pólya $p$-norm approximation is applicable. The function $f_2$ is not well-defined on the whole $m$-fold product of the Grassmannian, but the singularities of $f_2$ are global minima of the diversity sum anyway and therefore the worst choice of possible constellations.

Unfortunately, neither the diversity product nor the cost functions $f_1$, $f_2$ can be regarded as cost functions for the sphere packing problem on an adjoint orbit. Nevertheless, we can use the formula for the gradients of a smooth function with respect to the normal metric, Proposition 2.4.4, to calculate the generalized gradients of $f_1$ and $f_2$.

**Proposition 2.4.37** *The generalized gradient of the cost function $f_1$ with respect to the normal metric on the m-fold Grassmannian* $\mathrm{Grass}(n, k, \mathbf{C})^m$ *is*

$$\begin{aligned}&\operatorname{grad} f_1(P_1, \ldots, P_m) = \\ &\overline{\mathrm{co}}\,\{(\ldots, [P_i, [P_j \operatorname{adj}(I - P_iP_j), P_i]], \ldots, [P_j, [\operatorname{adj}(I - P_iP_j)P_i, P_j]], \ldots) \mid \\ &\qquad\qquad 2^k - \det(I - P_iP_j) = f_1(P_1, \ldots, P_m)\big\},\end{aligned}$$

*where* $\operatorname{adj} A$ *denotes the adjoint*[29] *of $A$. The generalized gradient of $f_2$ with respect to the same metric on the m-fold Grassmannian is*

$$\begin{aligned}&\operatorname{grad} f_2(P_1, \ldots, P_m) = \\ &\overline{\mathrm{co}}\,\big\{(\ldots, [P_i, [P_i, P_j(I - P_iP_j)^{-1}]], \ldots, [P_j, [P_j, (I - P_iP_j)^{-1}P_i]], \ldots) \mid \\ &\qquad\qquad -\log(\det(I - P_iP_j)) = f_2(P_1, \ldots, P_m)\}\end{aligned}$$

**Proof:** For the space $R^{n\times n}$ it is known that $d\det(X)(H) = \operatorname{tr}(\operatorname{adj}(X)H)$, see [72, p. 304]. Considering $\mathbf{C}$ and $\mathbf{C}^{n\times n}$ as real vector spaces, one can show with the same argument as in the real case that $d\det(U)(H) = \operatorname{tr}(\operatorname{adj}(U)H)$. Hence we get for $h(P, Q) = \det(I - PQ)$, $P, Q \in \mathbf{C}^{n\times n}$, that for all $H, K \in \mathbf{C}^{n\times n}$

$$dh(P, Q)(H, K) = -\operatorname{tr}\left(\operatorname{adj}(I - PQ)HQ + \operatorname{adj}(I - PQ)PK\right).$$

By Lemma 2.4.36 we see that $h$ takes only real values on the manifold $\mathrm{Grass}(n, k, \mathbf{C}) \times \mathrm{Grass}(n, k, \mathbf{C})$. Thus the differential restricted to the tangent

[29] see [93] for a definition of the adjoint.

bundle of $\mathrm{Grass}(n, k, \mathbf{C}) \times \mathrm{Grass}(n, k, \mathbf{C})$ is real too, i.e.

$$dh(P, Q)(H, K) = -\Re \operatorname{tr}(\operatorname{adj}(I - PQ)HQ) - \Re \operatorname{tr}(\operatorname{adj}(I - PQ)PK)$$

for $P, Q \in \mathrm{Grass}(n, k, \mathbf{C})$, $H \in T_P \mathrm{Grass}(n, k, \mathbf{C})$, $K \in T_Q \mathrm{Grass}(n, k, \mathbf{C})$. By Proposition 2.4.4 we get the following gradient of $h$ with respect to the normal metric on the product $\mathrm{Grass}(n, k, \mathbf{C}) \times \mathrm{Grass}(n, k, \mathbf{C})$:

$$\operatorname{grad} h(P, Q) = ([P, [Q \operatorname{adj}(I - PQ), P]], [Q, [\operatorname{adj}(I - PQ)P, Q]]).$$

If we consider $\mathbf{C}$ and $\mathbf{C}^{n\times n}$ as real vector spaces we get for the function $k(P, Q) = -\log(\det(I - PQ))$ the differential

$$\begin{aligned} dk(P, Q)(H, K) &= \frac{1}{\det(I - PQ)} \operatorname{tr}(\operatorname{adj}(I - PQ)HQ + \operatorname{adj}(I - PQ)PK) \\ &= \operatorname{tr}((I - PQ)^{-1}HQ + (I - PQ)^{-1}PK). \end{aligned}$$

Using again that $h$ is real-valued on $\mathrm{Grass}(n, k, \mathbf{C}) \times \mathrm{Grass}(n, k, \mathbf{C})$ and Proposition 2.4.4, we get the following result for the gradient with respect to the normal metric:

$$\operatorname{grad} k(P, Q) = ([P, [P, Q(I - PQ)^{-1}]], [Q, [Q, (I - PQ)^{-1}P]]).$$

The formulas for the generalized gradient follow now directly from Corollary 2.3.10. □

We can use a generalized gradient descent on $\mathrm{Grass}(n, k, \mathbf{C})$ to search for configurations which maximize the diversity product. Again, we use the QR-coordinates to exploit the structure of the elements of $\mathrm{Grass}(n, k, \mathbf{C})$. This yields the following algorithm.

**Algorithm 2.4.38** Let $(P_1^0, \ldots, P_m^0)$ be $m$ initial points in $\mathrm{Grass}(n, k, \mathbf{C})$. Calculate $\Theta_i^0 \in U(n)$ such that

$$P_i^0 = (\Theta_i^0)^* \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^0.$$

1. Set
$$(\eta_1^t, \ldots, \eta_m^t) = -\pi_0(\operatorname{grad} f_1(P_1, \ldots, P_m)).$$
2. Calculate $\xi_i^t$, $i = 1, \ldots, m$ by
$$\begin{pmatrix} * & \xi_i^t \\ (\xi_i^t)^* & * \end{pmatrix} := \Theta_i^t \eta_i (\Theta_i^t)^*.$$

3. Calculate a step size $\alpha_t$ either by a harmonic step size selection or with the Armijo rule along the curve $(\ldots, \phi_{P_i^t}(\eta_i^t), \ldots)$.

4. Set

$$\begin{aligned}
\Theta_i^{t+1} & \\
= & \begin{pmatrix} (I_k + \alpha_t^2 \xi_i^t (\xi_i^t)^*)^{-1/2} & \alpha_t \xi_i^t (I_{n-k} + \alpha_t^2 (\xi_i^t)^* \xi_i^t)^{-1/2} \\ -\alpha_t (\xi_i^t)^* (I_k + \alpha_t^2 \xi_i^t (\xi_i^t)^*)^{-1/2} & (I_{n-k} + \alpha_t^2 (\xi_i^t)^* \xi_i^t)^{-1/2} \end{pmatrix} \Theta_i^t, \\
P_i^{t+1} & = (\Theta_i^{t+1})^* \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Theta_i^{t+1}.
\end{aligned}$$

5. Set $t = t + 1$ and go to step 1.

For the cost function $f_2$ we get the analogous algorithm.

**Corollary 2.4.39** *If the step sizes in Algorithm 2.4.38 satisfy* $\sum \alpha_t = \infty$, $\alpha_t \to 0$, *then the iterates converge to the set of critical points of* $f_1$ *(or* $f_2$ *respectively).*

**Proof:** Again, this is an application of Theorem 2.3.27 by choosing a family of isometries $(\psi_P \colon \mathbf{R}^{2k(n-k)} \to T_P \operatorname{Grass}(n, k, \mathcal{C}))$. □

### 2.4.6 Numerical results

In this section, we discuss some numerical results of our sphere packing algorithms. We consider the case of the real Grassmannian, see Section 2.4.2. Since there is a significant number of good packings known for $\mathrm{Grass}(n, k, \mathbf{R})$ with $n \leq 16$, see e.g. [43, 145], we can evaluate the results of our algorithms for this case precisely. All algorithms were implemented in MATLAB.

In Table 2.1 we display the results of our non-smooth optimization algorithm with an Armijo step size selection. Our simulations cover the case of $m$ points in $\mathrm{Grass}(n, 2, \mathbf{R})$ with $m$ ranging from 10 to 14 and $n$ from 4 to 10. The largest minimal squared distance, which was achieved by the algorithm in 80 or 200 iterations, is shown in the columns "80 steps" and "200 steps". The column "start" gives the minimal squared distance between the points of the initial configuration in $\mathrm{Grass}(n, 2, \mathbf{R})$. The results are compared with best minimal squared distances[30] of Conway et al. [43], which are displayed in the column "Conway et al.". The initial configuration was chosen in $\mathrm{Grass}(n, 2, \mathbf{R})$ by the following construction: For each point of initial configuration we chose a $A \in \mathbf{R}^{n \times n}$ with entries randomly distributed in $[-1, 1]$. We calculated a $\Theta \in \mathrm{SO}(n)$ by $\Theta = \exp(5(A - A^\top))$. Then we used the point

$$\Theta^\top \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \Theta.$$

in $\mathrm{Grass}(n, 2, \mathbf{R})$ for the initial configuration. For the Armijo step size selection we used the constants $\sigma = 10^{-4}$, $\mu = 0.5$, $C = 1$. We limited the number of line search steps in the Armijo rule to 16 and terminated the algorithm if this bound was exceeded. The algorithm achieved a significant improvement of the distances compared to the initial configuration. However, it did not reach to distances of the best known packings from Conway et al. Furthermore, we noted during the simulations that the algorithm is sensitive to the choice of the constant $\sigma$ in the Armijo step size selection. A different choice like, e.g. $\sigma = 0.1$, would lead to a significant degradation of the results. Figure 2.1 illustrates this behavior.

Table 2.2 shows the results of the non-smooth algorithm with the harmonic step size selection $\alpha_t = 1/(0.3t + 1)$. We considered the same packing problems as in Table 2.1. However, we used new initial configurations for the algorithm. Again, a significant improvement compared to the initial configuration was made, but the distances from Conway et al. were not reached.

[30] The results of Conway et al. are rounded to 4 decimal digits.

The results are similar to the Armijo version. However, this requires careful adaption of the step size formula to this specific problem and depends on the construction of the initial values. We will discuss this a little later.

As a comparison, we show in Table 2.3 the results of a gradient descent iteration for a smooth $p$-norm approximation of our non-smooth cost function. We used the $p = 6$ approximation

$$f_6(P_1, \ldots, P_m) = \frac{1}{6} \sum_{i<j} \operatorname{tr}(P_i P_j)^6.$$

A standard gradient descent in the QR-parameterization with an Armijo step size selection was applied to this function. For the Armijo rule, we used again the constants $\sigma = 10^{-4}$, $\mu = 0.5$, $C = 1$. Again, we limited to number of line search steps to 16 and terminating the algorithm if this bound was exceeded. The initial points were chosen by the same construction as above. This smooth approximation performed much better than the non-smooth algorithms, despite the fixed order $p = 6$ of the approximation. The algorithm gave results very close to the findings of Conway et al., in particular, compared to the non-smooth algorithms. Thus, it seems to be superior to the non-smooth approach. This is further illustrated in in Figure 2.2, where we compare the smooth and non-smooth algorithms for the case of packing 14 points in $\operatorname{Grass}(14, 2)$. We started with a random initial configuration. The parameter for the algorithms were chosen as above. The diagram shows the evolution of the minimal squared distance between the points during the iterations of the algorithms. Note that the minimal squared distance does not increase monotonically for the smooth algorithm, as it is a descent iteration for a different cost function - the smooth approximation $f_6$.

However, as already mentioned the similar behavior of the non-smooth algorithms with Armijo and harmonic step sizes depends on the construction of the initial points. For example by the Euler angle decomposition we can decompose each $\Theta \in \mathrm{SO}(n)$ as

$$\Theta = \begin{pmatrix} \Xi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & R(\alpha_1) \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & R(\alpha_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} R(\alpha_{n-1}) & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

with $\Xi \in \mathrm{SO}(n-1)$ and

$$R(\alpha_i) = \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ -\sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}$$

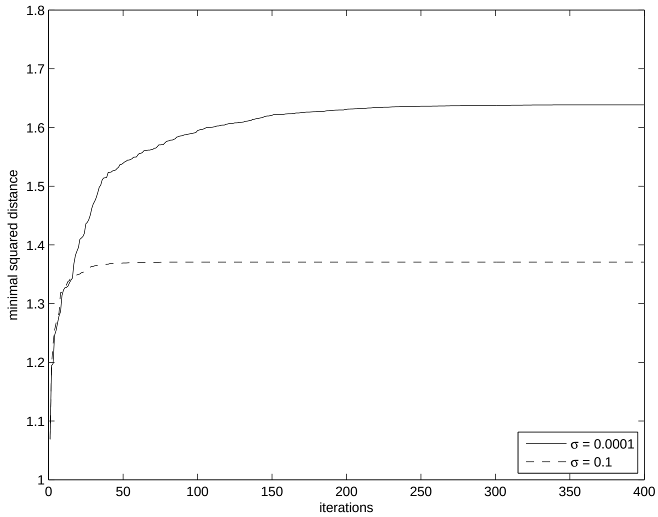


Figure 2.1: Behavior of the non-smooth algorithm with the Armijo set sizes for different values of $\sigma$, $10^{-4}$ and $10^{-1}$. We consider the problem of 14 points in Grass$(10, 2, \mathbf{R})$.

with $\alpha_i \in [0, 2\pi]$ [157]. We can choose an initial configuration by constructing recursively $\Theta$ with random $\alpha_i$ and using

$$\Theta^\top \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \Theta$$

as points of the initial configuration. In the Figures 2.3 and 2.4 we show the evolution of the algorithms starting from our standard choice of initial points and the Euler angle construction for the problem of packing 10 points on Grass$(16, 8, \mathbf{R})$. The Grassmannian Grass$(16, 8, \mathbf{R})$ was chosen because the effect is more visible than on Grass$(16, 2, \mathbf{R})$. All constants for the algorithms were chosen as in the previous simulations. In both cases the smooth approximation has the best performance. The non-smooth algorithm with Armijo step size shows similar convergence in both simulation, in particular, if we take into account that the initial configurations have different quality. However, the harmonic step size is significantly affected by the change of the initial configurations. For the Euler angle construction it has worse performance than the non-smooth algorithm with Armijo step size rule.

| n | m | start | 80 steps | 200 steps | Conway et al. |
|---|---|---|---|---|---|
| 4 | 10 | 0.42508 | 0.87146 | 0.89766 | 1.1111 |
| 4 | 11 | 0.025911 | 0.8202 | 0.83872 | 1.0000 |
| 4 | 12 | 0.14129 | 0.8801 | 0.9274 | 1.0000 |
| 4 | 13 | 0.0329 | 0.69513 | 0.76776 | 1.0000 |
| 4 | 14 | 0.19674 | 0.64854 | 0.7147 | 1.0000 |
| 5 | 10 | 0.45419 | 1.118 | 1.1305 | 1.3333 |
| 5 | 11 | 0.28611 | 1.0749 | 1.0882 | 1.3200 |
| 5 | 12 | 0.56397 | 1.0088 | 1.0383 | 1.3064 |
| 5 | 13 | 0.20946 | 1.0286 | 1.043 | 1.2942 |
| 5 | 14 | 0.37904 | 0.95747 | 1.0046 | 1.2790 |
| 6 | 10 | 0.86787 | 1.3315 | 1.4038 | 1.4815 |
| 6 | 11 | 0.70121 | 1.2454 | 1.3634 | 1.4667 |
| 6 | 12 | 0.24786 | 1.1462 | 1.1716 | 1.4545 |
| 6 | 13 | 0.61849 | 1.1475 | 1.2259 | 1.4444 |
| 6 | 14 | 0.35458 | 1.2136 | 1.2484 | 1.4359 |
| 7 | 10 | 1.0108 | 1.3621 | 1.4245 | 1.5873 |
| 7 | 11 | 0.72959 | 1.4141 | 1.4748 | 1.5714 |
| 7 | 12 | 0.73347 | 1.2922 | 1.313 | 1.5584 |
| 7 | 13 | 0.70806 | 1.3411 | 1.4235 | 1.5476 |
| 7 | 14 | 0.65897 | 1.3338 | 1.4024 | 1.5385 |
| 8 | 10 | 0.6665 | 1.4407 | 1.4828 | 1.6667 |
| 8 | 11 | 0.7486 | 1.4422 | 1.5141 | 1.6500 |
| 8 | 12 | 1.0548 | 1.486 | 1.5348 | 1.6364 |
| 8 | 13 | 0.75067 | 1.3831 | 1.4513 | 1.6250 |
| 8 | 14 | 0.76594 | 1.4244 | 1.4812 | 1.6154 |
| 9 | 10 | 1.1383 | 1.5072 | 1.6485 | 1.7284 |
| 9 | 11 | 0.90762 | 1.5139 | 1.5944 | 1.7111 |
| 9 | 12 | 0.92357 | 1.4947 | 1.5235 | 1.6970 |
| 9 | 13 | 0.90856 | 1.4811 | 1.5675 | 1.6853 |
| 9 | 14 | 0.59752 | 1.4975 | 1.5669 | 1.6752 |
| 10 | 10 | 0.95202 | 1.6428 | 1.7024 | 1.7778 |
| 10 | 11 | 1.096 | 1.624 | 1.68 | 1.7600 |
| 10 | 12 | 1.194 | 1.5693 | 1.6389 | 1.7455 |
| 10 | 13 | 1.022 | 1.5638 | 1.6346 | 1.7333 |
| 10 | 14 | 0.74117 | 1.5319 | 1.6084 | 1.7231 |

Table 2.1: **Armijo rule** Results of the sphere packing algorithm for packing $m$ points on the real Grassmannian Grass$(n, 2)$.

| n | m | start | 80 steps | 200 steps | Conway et al. |
|---|---|---|---|---|---|
| 4 | 10 | 0.32539 | 0.89374 | 0.9388 | 1.1111 |
| 4 | 11 | 0.32323 | 0.8776 | 0.92154 | 1.0000 |
| 4 | 12 | 0.19822 | 0.89603 | 0.93348 | 1.0000 |
| 4 | 13 | 0.16588 | 0.76737 | 0.82843 | 1.0000 |
| 4 | 14 | 0.080045 | 0.72348 | 0.80152 | 1.0000 |
| 5 | 10 | 0.62849 | 1.1255 | 1.1701 | 1.3333 |
| 5 | 11 | 0.33131 | 1.0557 | 1.1144 | 1.3200 |
| 5 | 12 | 0.082729 | 1.0162 | 1.0689 | 1.3064 |
| 5 | 13 | 0.41955 | 1.0006 | 1.0267 | 1.2942 |
| 5 | 14 | 0.20488 | 1.0419 | 1.0832 | 1.2790 |
| 6 | 10 | 0.78359 | 1.322 | 1.3821 | 1.4815 |
| 6 | 11 | 0.70646 | 1.2635 | 1.3107 | 1.4667 |
| 6 | 12 | 0.42362 | 1.2616 | 1.3072 | 1.4545 |
| 6 | 13 | 0.68365 | 1.2284 | 1.2835 | 1.4444 |
| 6 | 14 | 0.50561 | 1.1602 | 1.2204 | 1.4359 |
| 7 | 10 | 0.82531 | 1.4656 | 1.5207 | 1.5873 |
| 7 | 11 | 0.83157 | 1.3525 | 1.4329 | 1.5714 |
| 7 | 12 | 0.70846 | 1.41 | 1.4578 | 1.5584 |
| 7 | 13 | 0.85141 | 1.2842 | 1.3418 | 1.5476 |
| 7 | 14 | 0.66567 | 1.3387 | 1.3957 | 1.5385 |
| 8 | 10 | 1.0676 | 1.5527 | 1.6009 | 1.6667 |
| 8 | 11 | 0.76063 | 1.5078 | 1.5585 | 1.6500 |
| 8 | 12 | 0.89878 | 1.4745 | 1.5219 | 1.6364 |
| 8 | 13 | 0.70867 | 1.4867 | 1.5334 | 1.6250 |
| 8 | 14 | 0.42532 | 1.4327 | 1.4866 | 1.6154 |
| 9 | 10 | 1.0598 | 1.65 | 1.6833 | 1.7284 |
| 9 | 11 | 0.93986 | 1.5833 | 1.6243 | 1.7111 |
| 9 | 12 | 0.78188 | 1.5694 | 1.6154 | 1.6970 |
| 9 | 13 | 0.86891 | 1.5688 | 1.6041 | 1.6853 |
| 9 | 14 | 0.85208 | 1.4793 | 1.5322 | 1.6752 |
| 10 | 10 | 1.1021 | 1.6812 | 1.7186 | 1.7778 |
| 10 | 11 | 1.209 | 1.6513 | 1.6876 | 1.7600 |
| 10 | 12 | 0.81313 | 1.6027 | 1.6606 | 1.7455 |
| 10 | 13 | 1.1644 | 1.6146 | 1.6542 | 1.7333 |
| 10 | 14 | 0.79348 | 1.5657 | 1.6165 | 1.7231 |

Table 2.2: **Harmonic step size** Results of the sphere packing algorithm for packing $m$ points on the real Grassmannian $\mathrm{Grass}(n, 2)$.

| n | m | start | 80 steps | 200 steps | Conway et al. |
|---|---|---|---|---|---|
| 4 | 10 | 0.36761 | 1.1111 | 1.1111 | 1.1111 |
| 4 | 11 | 0.1865 | 0.98006 | 0.98208 | 1.0000 |
| 4 | 12 | 0.28515 | 0.99966 | 1 | 1.0000 |
| 4 | 13 | 0.21484 | 0.90421 | 0.99809 | 1.0000 |
| 4 | 14 | 0.04376 | 0.99129 | 0.99603 | 1.0000 |
| 5 | 10 | 0.54927 | 1.3119 | 1.3274 | 1.3333 |
| 5 | 11 | 0.65627 | 1.2778 | 1.2911 | 1.3200 |
| 5 | 12 | 0.38339 | 1.2639 | 1.2726 | 1.3064 |
| 5 | 13 | 0.42 | 1.258 | 1.2624 | 1.2942 |
| 5 | 14 | 0.41014 | 1.2184 | 1.2387 | 1.2790 |
| 6 | 10 | 0.76941 | 1.4784 | 1.4814 | 1.4815 |
| 6 | 11 | 0.39748 | 1.4611 | 1.4649 | 1.4667 |
| 6 | 12 | 0.29008 | 1.4273 | 1.4458 | 1.4545 |
| 6 | 13 | 0.78825 | 1.4148 | 1.4197 | 1.4444 |
| 6 | 14 | 0.53428 | 1.4092 | 1.4159 | 1.4359 |
| 7 | 10 | 0.82595 | 1.5861 | 1.5861 | 1.5873 |
| 7 | 11 | 0.74219 | 1.5638 | 1.5706 | 1.5714 |
| 7 | 12 | 0.83264 | 1.5571 | 1.5575 | 1.5584 |
| 7 | 13 | 0.74724 | 1.543 | 1.5471 | 1.5476 |
| 7 | 14 | 0.7688 | 1.5183 | 1.5313 | 1.5385 |
| 8 | 10 | 0.65001 | 1.6615 | 1.6665 | 1.6667 |
| 8 | 11 | 0.71212 | 1.649 | 1.65 | 1.6500 |
| 8 | 12 | 0.6358 | 1.6335 | 1.6362 | 1.6364 |
| 8 | 13 | 0.61274 | 1.6094 | 1.6212 | 1.6250 |
| 8 | 14 | 0.8729 | 1.6087 | 1.6148 | 1.6154 |
| 9 | 10 | 1.0903 | 1.7249 | 1.728 | 1.7284 |
| 9 | 11 | 0.93424 | 1.7074 | 1.7109 | 1.7111 |
| 9 | 12 | 1.0657 | 1.6908 | 1.6961 | 1.6970 |
| 9 | 13 | 1.038 | 1.6789 | 1.6847 | 1.6853 |
| 9 | 14 | 0.61142 | 1.6722 | 1.675 | 1.6752 |
| 10 | 10 | 1.0861 | 1.7746 | 1.7773 | 1.7778 |
| 10 | 11 | 1.045 | 1.7572 | 1.7588 | 1.7600 |
| 10 | 12 | 0.91031 | 1.7419 | 1.7452 | 1.7455 |
| 10 | 13 | 0.7923 | 1.7291 | 1.7332 | 1.7333 |
| 10 | 14 | 0.89754 | 1.7179 | 1.7223 | 1.7231 |

Table 2.3: Results of gradient descent for a smooth $p$-norm approximation of order 6 for packing problem of $m$ points on the real Grassmannian Grass$(n, 2)$.

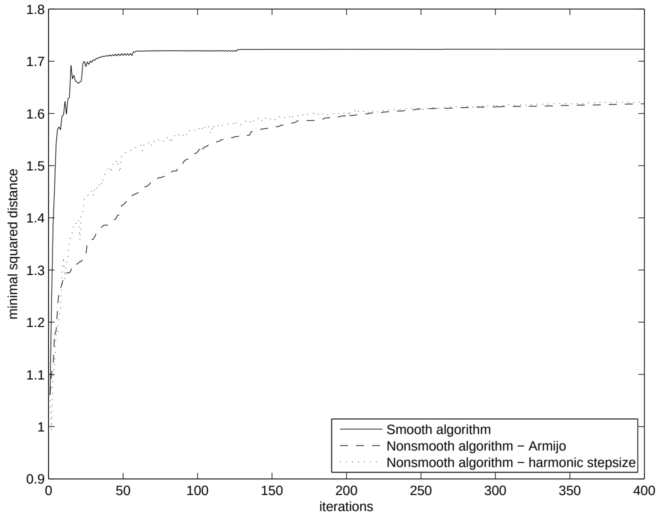


Figure 2.2: Behavior of the smooth and non-smooth algorithms for packing 14 points in Grass(10, 2).

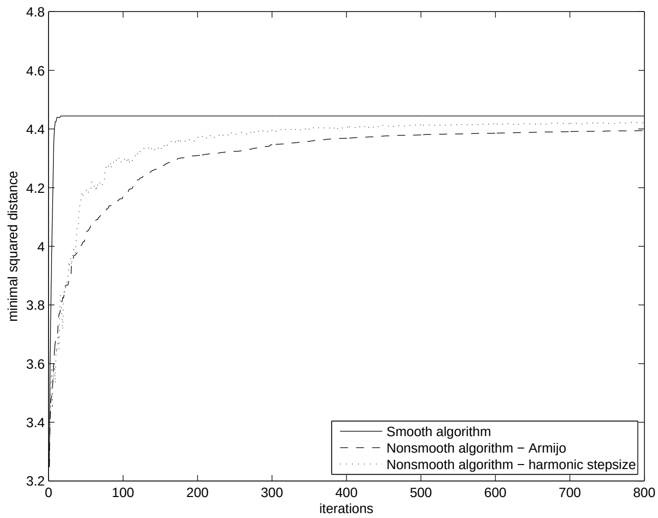


Figure 2.3: Behavior of the smooth and non-smooth algorithms for 10 points on Grass(16, 8, $\mathbf{R}$) and our standard construction of initial configurations.

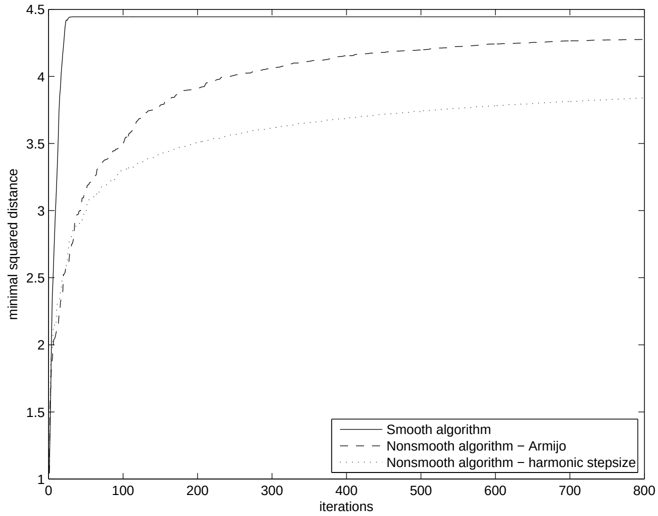


Figure 2.4: Behavior of the smooth and non-smooth algorithms for 10 points on Grass$(16, 8, \mathbf{R})$ and the Euler angle construction of initial configurations.

# Appendix A

# Additional results

## A.1 A theorem on Hessians of self-scaled barrier functions

In this section, we show that the inverse of the Hessian of self-scaled barrier function on a symmetric cone can be extended to compatible, piecewise positive definite tangent map if and only if the cone isomorphic to $\mathbf{R}^n_+$. The proof is based on the classification theorem for self-scaled barriers on symmetric cones by Güler and Hauser [73]. As a detailed introduction into the theory of Jordan algebras, symmetric cones and self-scaled barrier functions is beyond the scope of this work, we refer the reader to [57, 60, 134] for definitions and basic theorems.

**Theorem A.1.1** *Let $C$ be an open symmetric cone in an Euclidean vector space $V$ and $f$ a self-scaled barrier function on $C$. The inverse of the Hessian of $f$ can be extended to a compatible, piecewise positive definite tangent map on $V$ if and only if there is a linear isomorphism $V \to \mathbf{R}^n$ which maps $C$ onto $\mathbf{R}^n_+$.*

**Proof:** Since the cone is symmetric, we can assume by [57, Thm. III.3.1] that $V$ is an Euclidean Jordan algebra and $C$ the interior of the cone of squares in $V$. By [57, Prop. III.4.4] we can decompose $V$ into a direct sum of simple ideals $V_1,\dots,V_k$. This yields a decomposition of $C$ into a direct sum of irreducible cones $C_1, \dots, C_k$ by setting $C_i = C \cap V_i$, cf. [57, Prop. III.4.5]. Güler and Hauser [73] have shown that a self-scaled barrier function on a symmetric cone $C$ with such a decomposition has the form

$$f(x) = a_0 - \sum_{i=1}^{k} a_i \log \det_i(x),$$

with constants $a_0 \in \mathbf{R}$, $a_1, \dots, a_k \geq 1$ and $\det_i$ denoting the determinant on $V_i$. Hence, it is sufficient to assume that $V$ is simple, $C$ irreducible and $f(x) = a - \log\det(x)$ with $a \in \mathbf{R}$.[1] Let $P(x)\colon V \to V$ denote the quadratic representation[2] of $V$. For $x \in C$, the inverse of the Hessian of $f$ is $\operatorname{Hess} f(x)^{-1} = P(x)$, see [60, Prop. 6.23] or [57, Prop. II.3.2, Prop. III.4.2]. Choose a Jordan frame $c_1, \dots, c_k$. By the Peirce decomposition of

[1] Note that we can normalize $a_0 - a_1 \log(\det(x))$ to $\tilde{a}_0 - \log(\det(x))$ by multiplication with a positive constant.

[2] i.e. for $x, y \in V$, $P(x)y = 2x(xy) - (x^2)y$, see [57, p. 32]

$V$, see [57, p.62 and Prop. IV.1.1], we can define subalgebras $V_j$ in $V$ as the 1-eigenspaces of the maps $x \mapsto (c_1 + \ldots + c_j)x$. We denote by $K_j$ the interior of the cone of squares in $V_j$. By [57, Prop. IV.3.1] we can decompose $\partial C$ in the orbits $C_j = G(c_1 + \ldots + c_j)$, $j = 1, \ldots, k$, where $G$ denotes the identity component of the automorphism group of $C$. Furthermore, we have $K_j \subset C_j$ [57, Prop. IV.3.1].

Let $x \in C_j$. By [57, Prop. III.5.2] for all $g \in G$, $x \in V$ the equation $P(gx) = gP(x)g^*$, with $g^*$ the adjoint of $g$, holds. Since $C_j = G(c_1 + \ldots + c_j)$, we can therefore assume that $x = c_1 + \ldots + c_j$. By [57, Prop. IV.3.1] $\operatorname{rk} P(c_1 + \ldots + c_j) = \dim V_j$. Furthermore, $G$ is the component of closed linear group, see [57, p.4], and hence analytic [100]. Since $c_1 + \ldots + c_j \in V_j$, $K_j \subset C_j$ and $\dim K_j = \dim V_j$, we see that $\operatorname{rk} P(c_1 + \ldots + c_j) \geq \dim C_j$ if and only if $K_j$ is a relatively open subset of $C_j$.

Assume the inverse of the Hessian of $f$ can be extended to a compatible, piecewise positive definite tangent map on $V$. This implies that $\operatorname{rk} P(c_1 + \ldots + c_{k-1}) \geq \dim C_{k-1}$. But since $C_{k-1}$ is an analytic submanifold and contains an open subset of a linear subspace $V_{k-1}$ with $\dim V_{k-1} = \dim C_{k-1}$, it follows from the identity principle for analytic functions [137, Prop. 2.9, p. 11] that $C_{k-1} \subset V_{k-1}$. Since $V$ is simple, $\partial C = \overline{C}_{k-1}$ [57, p. 73]. Thus, the boundary of $C$ is contained in the linear subspace $V_{k-1}$ of $V$. Since $C$ is an open, convex subset of $V$, this implies that $C$ is a half-space. On the other hand, $C$ is self dual, i.e. $C = \{x \in V \mid \forall y \in \overline{C}, y \neq 0\colon \langle x, y \rangle > 0\}$, cf. [57, p. 4]. Therefore $C = \mathbf{R}_+$ and $V = \mathbf{R}$.

Hence we have proven that if the inverse of the Hessian can be extended to a compatible, piecewise positive definite tangent map on $V$, then $C$ must be isomorphic to $\mathbf{R}^n_+$.

On the other hand a self-scaled barrier on $\mathbf{R}^n_+$ has by the classification theorem of Güler and Hauser [73] the form[3] $f(x_1, \ldots, x_k) = a_0 - \sum_{i=1}^k a_i \log(x_i)$, with $a_0 \in \mathbf{R}$, $a_i \geq 1$. This yields the Hessian $\operatorname{diag}(a_1 x_1^{-2}, \ldots, a_k x_k^{-2})$ and its inverse can be extended to a compatible, piecewise positive definite tangent map, cf. Example 1.4.10. □

In fact, we have shown that for simple $V$ and the stratification of $\partial C$ into $C_j = G(c_1, + \ldots + c_j)$, the bilinear map $(v, w) \mapsto \langle P(x)v, w \rangle$ is a scalar product on $T_x C_j$ if and only if $C_j = \mathbf{R}_+$.

[3] For $V_i = \mathbf{R}$ we have $\det_i(x) = x$.

# Appendix B

# Notation

| | |
|---|---|
| $\mathbf{R}$ | the field of real numbers |
| $\mathbf{R}_+$ | the positive real numbers |
| $\mathbf{C}$ | the field of complex numbers |
| $\Re z$ | real part of a $z \in \mathbf{C}$ |
| $2^S$ | the power set of a set $S$ |
| $\overline{A}$ | topological closure of a set $A$ |
| $\partial A$ | topological boundary of a set $A$ |
| $\mathrm{Id}_V$ | identity mapping $V \to V$ on a vector space $V$ |
| $V^\perp$ | orthogonal complement to a subspace $V$ in a vector space $W$ with a scalar product |
| $\operatorname{co} S$ | the convex hull of a set $S$ in a vector space |
| $\overline{\mathrm{co}} S$ | the closure of $\operatorname{co} S$ for a set $S$ in a topological vector space |
| $\mathcal{R}$ | o-minimal structure, p. 12 |
| $\mathcal{C}$ | analytic geometric category, p. 13 |
| $\mathcal{C}(M)$ | the set of $\mathcal{C}$-sets on the manifold $M$, p. 13 |
| $\mathcal{R}(\mathcal{C})$ | the o-minimal structure derived from an analytic geometric category $\mathcal{C}$, p. 14 |
| $\mathcal{C}(\mathcal{R})$ | the analytic geometric category derived from an o-minimal structure $\mathcal{R}$, p. 14 |
| $\omega(\gamma)$,$\omega(x)$ | the $\omega$-limit of an integral curve $\gamma$ or the integral curve passing through $x$ |
| $f\|_T$ | restriction of a function $f\colon M \to \mathbf{R}$ to a subset $T$ |
| $T_xM$ | tangent space of a manifold $M$ |
| $T_x\phi$ | tangent map $T_xM \to T_{\phi(x)}N$ of a differentiable map $\phi\colon M \to N$ with $x \in M$ |
| $df(x)$ | differential of a function $f\colon M \to \mathbf{R}$ at $x \in M$ |
| $df\|_T(x)$ | differential at $x \in T$ of $f\|_T$ with $T \subset M$ a submanifold and $f\colon M \to \mathbf{R}$ a function |
| $\dot\gamma$, $\gamma'$ | derivative of a differentiable function $\gamma\colon \mathbf{R} \to M$ |
| $\langle\cdot,\cdot\rangle$ | Riemannian metric on a manifold $M$ |
| $\\|\cdot\\|$ | norm on a tangent space $T_xM$ induced by a Riemannian metric, also used for the induced operator norms |
| $\exp_x$ | the exponential map $T_xM \to M$ on a Riemannian manifold $M$ at the point $x \in M$ |
| $\operatorname{grad} f(x)$ | Riemannian gradient or generalized gradient, p. 123, at $x \in M$ of a function $f\colon M \to \mathbf{R}$ |

| | |
|---|---|
| $\operatorname{grad} f\|_T(x)$ | Riemannian gradient of $f\|_T$ with respect to the induced metric on a submanifold $T \subset M$ with $f\colon M \to \mathbf{R}$ a function |
| $\operatorname{grad}_j f(x)$ | Riemannian gradient of the restriction of a function $f\colon M \to \mathbf{R}$ to a stratum $S_j$ with respect to the induced metric, p. 18 |
| $\operatorname{Hess}_x f$ | Riemannian Hessian of a twice differentiable function $f\colon M \to \mathbf{R}$ at $x \in M$ |
| $\partial_F f(x)$ | Frechét subdifferential of a function $f\colon M \to \mathbf{R}$ at $x \in M$, p. 122 |
| $\partial_L f(x)$ | limiting Frechét subdifferential of a function $f\colon M \to \mathbf{R}$ at $x \in M$, p. 122 |
| $\operatorname{grad}_C f(x)$ | Clarke's generalized gradient for a function $f\colon \mathbf{R}^n \to \mathbf{R}$ |
| $I_n$ | $n \times n$ identity matrix |
| $A^\top$ | transpose of a matrix $A \in \mathbf{R}^{n\times n}$ |
| $A^*$ | complex conjugate transpose of a matrix $A \in \mathbf{C}^{n\times n}$ |
| $\operatorname{adj} A$ | adjugate of a matrix $A$ |
| $\\|A\\|_F$ | the Frobenius norm of a $n \times n$ matrix |
| $\operatorname{Flag}(n)$ | the complete flag manifold, p. 116 |
| $\operatorname{Tri}(n)$ | the linear space of lower triangular $n \times n$ matrices, p. 117 |
| $[A, B]$ | the Lie bracket, for matrices the matrix Lie bracket, i.e. $[A, B] = AB - BA$ |
| exp | the exponential map on a Lie group |
| ad | adjoint representation of a Lie algebra |
| Ad | adjoint representation of a Lie group |
| $S(G)$ | see p. 158 |
| $\mathrm{GL}(n, \mathbf{R})$, $\mathrm{GL}(n, \mathbf{C})$ | general linear group over $\mathbf{R}$ and $\mathbf{C}$ |
| $E(n)$ | Euclidean group, p. 120 |
| $\mathrm{SL}(n, \mathbf{R})$ | special linear group over $\mathbf{R}$ |
| $O(n)$ | orthogonal group |
| $\mathrm{SO}(n)$ | special orthogonal group |
| $\mathrm{Sp}(n)$ | the symplectic group, p. 154 |
| $\mathrm{OSP}(n)$ | see p. 154 |
| $\mathrm{SO}(n, k)$ | see p. 157 |
| $U(n)$ | unitary group |
| $\mathfrak{so}(n)$ | the Lie algebra of skew-symmetric $n \times n$ matrices |
| $\mathfrak{sp}(n)$ | see p. 154 |

| | |
|---|---|
| $\mathfrak{osp}(n)$ | see p. 155 |
| $\mathfrak{so}(n,k)$ | see p. 157 |
| $\mathfrak{sl}(n,\mathbf{R})$ | the Lie algebra of real $n \times n$ matrices $A$ with $\mathrm{tr}(A) = 0$ |
| $\mathfrak{u}(n)$ | the Lie algebra of skew-Hermitian $n \times n$ matrices |
| $\mathrm{Sym}(n)$ | the set of symmetric $n \times n$ matrices, p. 149 |
| $\mathrm{PSym}(n)$ | see p. 155 |
| $\mathrm{Herm}(n)$ | the set of Hermitian $n \times n$ matrices, p. 164 |
| $\mathrm{Grass}(n,k,\mathbf{R})$ | the real Grassmannian, p.149 |
| $\mathrm{LGrass}(n)$ | the Lagrange Grassmannian, p. 154 |
| $\mathrm{Grass}(n,k,\mathbf{C})$ | the complex Grassmannian, p. 163 |
| $\mathrm{St}(n,k,\mathbf{C})$ | the complex, compact Stiefel manifold, p. 162 |

# Bibliography


[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer-Verlag, New York, second edition, 1988.

[2] P.-A. Absil, C. G. Baker, and K. A. Gallivan. Convergence analysis of Riemannian trust-region methods, 2006. technical report, `http://www.optimization-online.org/DB_HTML/2006/06/1416.html`.

[3] P.-A. Absil, C. G. Baker, and K. A. Gallivan. Trust-region methods on Riemannian manifolds, 2006. Accepted for publication in Foundations of Computational Mathematics.

[4] P.-A. Absil, R. Mahony, and B. Andrews. Convergence of iterates of descent methods for analytic cost functions. *SIAM J. Optim.*, 16(2):531–547, 2005.

[5] R. L. Adler, J.-P. Dedieu, J. Y. Margulies, M. Martens, and M. Shub. Newton's method on Riemannian manifolds and a geometric model for the human spine. *IMA J. Numer. Anal.*, 22(3):359–390, 2002.

[6] D. Agrawal, T. J. Richardson, and R. L. Urbanke. Multiple-antenna signal constellations for fading channels. *IEEE Trans. on Information Theory*, 47(6):2618–2626, 2001.

[7] F. Alvarez, H. Attouch, J. Bolte, and P. Redont. A second-order gradient-like dissipative dynamical system with Hessian-driven damping. Application to optimization and mechanics. *J. Math. Pures Appl. (9)*, 81(8):747–779, 2002.

[8] F. Alvarez, J. Bolte, and O. Brahic. Hessian Riemannian gradient flows in convex programming. *SIAM J. Control Optim.*, 43(2):477–501, 2004.

[9] B. Andrews. Monotone quantities and unique limits for evolving convex hypersurfaces. *International Mathematics Research Notices*, 20:1002–1031, 1997.

[10] H. Attouch, J. Bolte, P. Redont, and M. Teboulle. Singular riemannian barrier methods and gradient-projection dynamical systems for constrained optimization. *Optimization*, 53(5-6):435–454, 2004.

[11] H. Attouch, X. Goudou, and P. Redont. The heavy ball with friction method. I: The continuous dynamical system: Global exploration of the local minima of a real-valued function by asymptotic analysis of a dissipative dynamical system. *Commun. Contemp. Math.*, 2(1):1–34, 2000.

[12] J. P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, Berlin, 1984.

[13] B. Aulbach. *Continuous and Discrete Dynamics near Manifolds of Equilibria*, volume 1058 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[14] D. Azagra and J. Ferrera. Proximal calculus on Riemannian manifolds. *Mediterranean Journal of Mathematics*, 2(4):437 – 450, 2005.

[15] D. Azagra, J. Ferrera, and F. Lpez-Mesas. Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds. *J. Funct. Anal.*, 220(2):304–361, 2005.

[16] R. G. Bartle. *A Modern Theory of Integration*. American Mathematical Society, Providence, 2001.

[17] D. Bayer and J. Lagarias. Karmarkar's linear programming algorithm and Newton's method. *Math. Program., Ser. A*, 50(3):291–330, 1991.

[18] D. Bayer and J. C. Lagarias. The nonlinear geometry of linear programming I. *Trans. Amer. Math. Soc.*, 314:499–526, 1989.

[19] D. Bayer and J. C. Lagarias. The nonlinear geometry of linear programming II. *Trans. Amer. Math. Soc.*, 314:527–581, 1989.

[20] D. P. Bertsekas. *Convex analysis and optimization*. Athema Scientific, Belmont, 2003.

[21] S. P. Bhat and D. S. Bernstein. Lyapunov analysis of semistability. In *Proceedings of the American Control Conference 1999*, pages 1608–1612, 1999.

[22] S. P. Bhat and D. S. Bernstein. Nontangency-based Lyapunov test for stability and convergence. In *Proceedings of the American Control Conference 2001*, pages 4840–4845, 2001.

[23] S. P. Bhat and D. S. Bernstein. Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibria. *SIAM J. Control Optim.*, 42(5), 2003.

[24] E. Bierstone and P. D. Milman. Semianalytic and subanalytic sets. *Publicactions Mathématiques*, 67:5–42, 1988.

[25] E. Bierstone and P. D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, 128(2):207–302, 1997.

[26] R. Bishop and R. Crittenden. *Geometry of manifolds*. Academic Press, New York, 1964.

[27] A. M. Bloch, R. W. Brockett, and T. S. Ratiu. Completely integrable gradient flows. *Commun. Math. Phys.*, 147(1):57–74, 1992.

[28] J. Bochnak, M. Coste, and M.-F. Roy. *Real Algebraic Geometry*. Springer-Verlag, Berlin, 1998.

[29] J. Bolte, A. Daniilidis, and A. S. Lewis. The Morse-Sard theorem for non-differentiable subanalytic functions. *J. Math. Anal. Appl.*, 321:729–740, 2006.

[30] J. Bolte, A. Daniilidis, and A. S. Lewis. The Łojasiewic inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. *SIAM J. Optim.*, 17:1205–1223, 2007.

[31] J. Bolte, A. Daniilidis, A. S. Lewis, and M. Shiota. Clarke subgradients of stratifiable functions, 2006. to appear in SIAM J. Optimization.

[32] J. Bolte and M. Teboulle. Barrier operators and associated gradient-like dynamical systems for constrained optimization problems. *SIAM J. Control Optim.*, 42(4):1266–1292, 2003.

[33] J. F. Bonnans, J. C. Gilbert, C. Lemaréchal, and C. Sagastizábal. *Numerical Optimization.* Springer-Verlag, Berlin, 2003.

[34] C. Byrnes, U. Helmke, and A. Morse. Necessary conditions in adaptive control. In C. Byrnes and A. Lindquist, editors, *Modelling, Identification and robust control*, pages 3–14. North Holland, 1986.

[35] E. Celledoni and S. Fiori. Descent methods for optimization on homogenous manifolds, 2007. preprint Numerics no. 1/2007, `http://www.math.ntnu.no/preprint/numerics/2007/N1-2007.pdf`.

[36] E. Celledoni and B. Owren. A class of intrinsic schemes for orthogonal integration. *SIAM J. Num. Anal.*, 40(6):2069–2084, 2002.

[37] E. Celledoni and B. Owren. On the implementation of Lie group methods on the Stiefel manifold. *Numerical Algorithms*, 32:163–183, 2003.

[38] E. Cheney. *Introduction to Approximation Theory.* McGraw-Hill, New York, 1966.

[39] I. Chryssochoos and R. Vinter. Optimal control problems on manifolds: A dynamic programming approach. *J. Math. Anal. Appl.*, 287(1):118–140, 2003.

[40] M. T. Chu and K. R. Driessel. The projected gradient method for least squares matrix approximations with spectral constraints. *SIAM J. Num. Anal.*, 27(4):1050–1060, 1990.

[41] F. H. Clarke. *Optimization and nonsmooth analysis.* Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1983.

[42] C. Conley. *Isolated invariant sets and the Morse index.* American Mathematical Society, Providence, 1978.

[43] J. H. Conway, R. H. Hardin, and N. J. A. Sloane. Packing lines, planes, etc.: Packings in Grassmannian spaces. *Experimental Mathematics*, 5:139–159, 1996.

[44] J. Cortés and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM J. Control Optim.*, 44(5):1543–1574, 2005.

[45] M. Coste. An introduction to O-minimal geometry, 2000.

[46] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms*. Spinger Science+Business Media, Inc., New York, second edition, 1997.

[47] H. B. Curry. The method of steepest descent for non-linear minimization problems. *Quarterly of Applied Mathematics*, 2:258–261, 1944.

[48] J. Dedieu and M. Shub. Newton flow and interior point methods in linear programming. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15:827–839, 2005.

[49] G. Dirr, U. Helmke, and C. Lageman. Some applications of riemannian optimization to mechanics, 2006. 3rd IFAC Worskhop on Lagrangian and Hamiltonian Methods.

[50] G. Dirr, U. Helmke, and C. Lageman. Nonsmooth riemannian optimization with applications to sphere packing and grasping, 2007. to appear in Proceedings for 3rd IFAC Worskhop on Lagrangian and Hamiltonian Methods.

[51] M. P. do Carmo. *Riemannian Geometry*. Birkhäuser, Boston, 1992.

[52] P. Dombrowski. On the geometry of the tangent bundles. *J. Reine Angew. Math.*, 210:73–88, 1962.

[53] L. Dries and C. Miller. Geometric categories and o-minimal structures. *Duke Mathematical Journal*, 84(2):497–540, 1996.

[54] A. Edelman, T. A. Arias, and S. T. Smith. The geometry of algorithms with orthogonality constraints. *SIAM J. Matrix Anal. Appl.*, 20(2):303–353 (electronic), 1999.

[55] A. Edelman and S. T. Smith. On conjugate gradient-like methods for eigen-like problems. *BIT*, 36(3):494–508, 1996.

[56] Y. Evtushenko and V. Zhadan. Stable barrier-projection and barrier-Newton methods for linear and nonlinear programming. In E. Spedicato, editor, *Algorithms for Continuous Optimization*, NATO ASI Series, pages 255–285. Kluwer Academic Publishers, 1994.

[57] J. Faraut and A. Korányi. *Analysis on Symmetric Cones.* Clarendon Press, Oxford, 1994.

[58] L. Faybusovich. Euclidean jordan algebras and affine scaling vector fields. preprint, available online at `http://www.nd.edu/~lfaybuso`.

[59] L. Faybusovich. On a matrix generalization of affine-scaling vector fields. *SIAM J. Matrix Anal. Appl.*, 16(3):886–897, 1995.

[60] L. Faybusovich. Euclidean Jordan algebras and interior-point algorithms. *Positivity*, 1:331–357, 1997.

[61] O. Ferreira and P. Oliveira. Proximal point algorithm on riemannian manifolds. *Optimization*, 51(2):257–270, 2002.

[62] O. P. Ferreira and P. R. Oliveira. Subgradient algorithm on Riemannian manifolds. *J. Optim. Theory Appl.*, 97(1):93–104, 1998.

[63] W. H. Fleming. *Functions of several variables.* Springer-Verlag, New York, second edition, 1977.

[64] R. Fletcher. *Practical Methods of Optimization.* John Wiley & Sons, Chichester, second edition, 1987.

[65] D. Gabay. Minimizing a differentiable function over a differential manifold. *J. Optim. Theory Appl.*, 37(2):177–219, 1982.

[66] G. H. Golub and C. F. Van Loan. *Matrix Computations.* The John Hopkins University Press, London, third edition, 1996.

[67] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Springer-Verlag, New York, 1983.

[68] B. C. Hall. *Lie groups, Lie algebras, and representations.* Springer-Verlag, New York, 2003.

[69] G. Han and J. Rosenthal. Geometrical and numerical design of structured unitary space time constellations. *IEEE Trans. on Information Theory*, 52(8):3722–3735, 2006.

[70] G. Han and J. Rosenthal. Good packings in the complex stiefel manifold using numerical methods. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan*, pages 1710–1719, 2006.

[71] A. Haraux and M. A. Jendoubi. Convergence of solutions of second-order gradient-like systems with analytic nonlinearities. *J. Differential Equations*, 144:313–320, 1998.

[72] D. A. Harville. *Matrix algebra from a statistician's perspective.* Springer-Verlag, New York, 1997.

[73] R. A. Hauser and O. Güler. Self-scaled barrier functions on symmetric cones and their classification. *Foundations of Computational Mathematics*, 2:121–143, 2001.

[74] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, New York, 1978.

[75] U. Helmke, K. Hüper, P. Y. Lee, and J. B. Moore. Essential matrix estimation via Newton-type methods. In *Proc. of the 16th International Symposium of Mathematical Theory of Networks and Systems, MTNS 2004, Leuven, Belgium*, 2004.

[76] U. Helmke, K. Hüper, and J. Trumpf. Newton's method on Grassmann manifolds, 2006. preprint.

[77] U. Helmke and J. B. Moore. *Optimization and Dynamical Systems.* Springer Verlag, London, 1994.

[78] U. Helmke and J. Trumpf. Conditioned invariant subspaces and the geometry of nilpotent matrices. In W. P. Dayawansa, A. Lindquist, and Y. Zhou, editors, *New Directions and Applications in Control Theory*, volume 321 of *Lecture Notes in Control and Information Sciences*, pages 123–164. Springer-Verlag, 2005.

[79] E. Hewitt and K. Stromberg. *Real and abstract analysis.* Springer-Verlag, Berlin, 1965.

[80] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms. Part 1: Fundamentals.* Springer-Verlag, Berlin, 1993.

[81] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: I. *Annals of Mathematics*, 79(1):109–203, 1964.

[82] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: II. *Annals of Mathematics*, 79(2):205–326, 1964.

[83] H. Hironaka. Subanalytic sets. In *Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki*, pages 453–493. Kinokuniya, Tokyo, 1973.

[84] J. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra.* Academic Press, New York, 1974.

[85] M. W. Hirsch. Systems of differential equations which are competitive or cooperative. I: Limit sets. *SIAM J. Math. Anal.*, 13:167–179, 1982.

[86] M. W. Hirsch. Systems of differential equations that are competitive or cooperative. II: Convergence almost everywhere. *SIAM J. Math. Anal.*, 16:423–439, 1985.

[87] M. W. Hirsch. Stability and convergence in strongly monotone dynamical systems. *J. Reine Angew. Math.*, 383:1–53, 1988.

[88] M. W. Hirsch. Convergent activation dynamics in continuous time networks. *Neural Networks*, 2(5):331 – 349, 1989.

[89] M. W. Hirsch. *Differential Topology.* Springer-Verlag, New York, 1997.

[90] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds.* Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 583.

[91] B. M. Hochwald and T. L. Marzetta. Unitary space-time modulation for multiple-antenna communications in Rayleigh flat fading. *IEEE Trans. on Information Theory*, 46(2):543–564, 2000.

[92] B. M. Hochwald and W. Sweldens. Differential unitary space-time modulation. *IEEE Trans. on Communications*, 48(12):2041–2052, 2000.

[93] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[94] X. L. Hu. *La structure des champs de gradients de fonctions analytiques réelles*. PhD thesis, Université Paris VI, Paris, 1992.

[95] K. Hüper and J. Trumpf. Newton-like methods for numerical optimization on manifolds. In *Asilomar Conference on Signals, Systems, and Computers in Pacific Grove (CA, USA)*, pages 136–139, 2004.

[96] H. T. Jongen and O. Stein. On the complexity of equalizing inequalities. *J. Glob. Optim.*, 27(4):367–374, 2003.

[97] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer-Verlag, Berlin, 2002.

[98] M. Kleinsteuber. *Jacobi-Type Methods on semisimple Lie Algebras*. Dissertation, Universität Würzburg, Würzburg, 2005.

[99] W. Klingenberg. *Riemannian Geometry*. Walter de Gruyter & Co., Berlin, 1982.

[100] A. W. Knapp. *Lie groups beyond an introduction*. Birkhäuser, Boston, second edition, 2002.

[101] K. Kurdyka. On gradients of functions definable in o-minimal structures. *Ann. Inst. Fourier*, 48(3):769–783, 1998.

[102] K. Kurdyka, T. Mostowski, and A. Parusiński. Proof of the gradient conjecture of R. Thom. *Annals of Mathematics*, 152:763–792, 2000.

[103] C. Lageman. Pointwise convergence of gradient-like systems. to appear in Mathematische Nachrichten.

[104] C. Lageman. Konvergenz reell-analytischer gradientenähnlicher Systeme. Diplomarbeit, Universität Würzburg, Würzburg, 2002.

[105] C. Lageman and U. Helmke. Optimization methods for the sphere packing problem on Grassmannians. In F. Lamnabhi-Lagarrigue, A. Loria, E. Panteley, and S. Laghrouche, editors, *Taming Heterogeneity and Complexity of Embedded Control: CTS-HYCON Workshop on Nonlinear and Hybrid Control, Paris IV Sorbonne, 10-12 July 2006*, 2007.

[106] S. Lang. *Fundamentals of Differential Geometry*. Springer-Verlag, Berlin, 1999.

[107] Y. Ledyaev and Q. J. Zhu. Nonsmooth analysis on manifolds, 2004. to appear in Transactions of AMS, available at `http://homepages.wmich.edu/~zhu/papers/man.htm`.

[108] Y. Ledyaev and Q. J. Zhu. Techniques for nonsmooth analysis on smooth manifolds: Part I :The local theory. In *Optimal control, stabilization and nonsmooth analysis*, volume 301 of *Lecture Notes in Control and Inform. Sci.*, pages 283–297. Springer-Verlag, Berlin, 2004.

[109] Y. Ledyaev and Q. J. Zhu. Techniques for nonsmooth analysis on smooth manifolds: Part II:Using deformations and flows. In *Optimal control, stabilization and nonsmooth analysis*, volume 301 of *Lecture Notes in Control and Inform. Sci.*, pages 299–311. Springer-Verlag, Berlin, 2004.

[110] S.-X. Li and S.-C. Fang. On the entropic regularization method for solving min-max problems with applications. *Mathematics of Operations Research*, 46:119–130, 1997.

[111] W. Liu and H. Sussmann. Shortest paths for sub-Riemannian metrics on rank-2 distributions. *Memoirs of the American Mathematical Society*, 118, 1995.

[112] T. L. Loi. Thom stratifications for functions definable in o-minimal structures on $(\mathbf{R}, +, \cdot)$. *C. R. Acad. Sci. Paris, Série I*, 324:1391–1394, 1997.

[113] S. Łojasiewicz. Sur les trajectoires du gradient d'une fonction analytique. In *Seminari di Geometria 1982-1983*, pages 115–117, Università di Bologna, 1983.

[114] S. Łojasiewicz. Sur la géométrie semi- et sous-analytique. *Ann. Inst. Fourier*, 43(5):1575–1595, 1993.

[115] Y.-C. Lu. *Singularity Theory and an Introduction to Catastrophe Theory.* Springer-Verlag, New York, 1976.

[116] D. G. Luenberger. The gradient projection method along geodesics. *Management Sci.*, 18:620–631, 1972.

[117] R. Mahony. *Optimization Algorithms on Homogenous Spaces.* PhD thesis, Australian National University, Canberra, 1994.

[118] J. H. Manton. Optimization algorithms exploiting unitary constraints. *IEEE Trans. on Signal Processing*, 50(3):635–650, 2002.

[119] J. H. Manton. On the various generalisations of optimisation algorithms to manifolds. In *Proc. of the 16th International Symposium of Mathematical Theory of Networks and Systems, MTNS 2004, Leuven, Belgium*, 2004.

[120] J. H. Manton, R. Mahony, and Y. Hua. The geometry of weighted low rank approximations. *IEEE Trans. on Signal Processing*, 51(2):500–514, 2003.

[121] B. Martensson. *Adaptive Stabilization.* PhD thesis, Lund Institute of Technology, Lund, 1986.

[122] N. Megiddo and M. Shub. Boundary behavior of interior point algorithms in linear programming. *Mathematics of Operations Research*, 14(1):97–146, 1989.

[123] B. S. Mordukhovich and Y. Shao. Nonsmooth sequential analysis in Asplund spaces. *Trans. Amer. Math. Soc.*, 348(4):1235–1280, 1996.

[124] J. Nash. The imbedding problem for Riemannian manifolds. *Annals of Mathematics (2)*, 63:20–63, 1956.

[125] K.-H. Neeb. Monotone functions on symmetric spaces. *Math. Ann.*, 291(2):261–273, 1991.

[126] A. Nowel and Z. Szafraniec. On trajectories of analytic gradient vector fields. *J. Differential Equations*, 184(1):215–223, 2002.

[127] A. Nowel and Z. Szafraniec. On trajectories of analytic gradient vector fields on analytic manifolds. *Topological Methods in Nonlinear Analysis*, 25:167–182, 2005.

[128] J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems.* Springer Verlag, New York, 1982.

[129] J. Palis and S. Smale. Structural stability theorems. In *Gobal Analysis*, volume 14 of *Proceedings of Symposia in Pure Mathematics*, pages 223–231, 1970.

[130] E. Polak. *Optimization.* Springer-Verlag, New York, 1997.

[131] V. M. Popov. Monotone-gradient systems. *J. Differential Equations*, 41(2):245–261, 1981.

[132] E. A. P. Quiroz and P. R. Oliveira. Proximal point methods for quasi-convex and convex functions with bregman distances, 2006. preprint, available at `http://www.optimization-online.org/DB_HTML/2006/04/1376.html`.

[133] T. Rapcsák and T. T. Thang. Nonlinear coordinate representations of smooth optimization problems. *J. Optim. Theory Appl.*, 86(2):459–489, 1995.

[134] J. Renegar. *A mathematical view of interior-point methods in convex optimization.* SIAM, Philadelphia, 2001.

[135] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis.* Springer-Verlag, Berlin, 1998.

[136] J.-P. Rolin, P. Speissegger, and A. J. Wilkie. Quasianalytic Denjoy-Carleman classes and O-minimality. *J. Amer. Math. Soc.*, 16(4):751–777, 2003.

[137] J. M. Ruiz. *The basic theory of power series.* Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, 1993.

[138] T. Sakai. *Riemannian geometry.* American Mathematical Society, Providence, RI, 1996.

[139] R. Sepulchre and A. Sarlette. Consensus optimization on manifolds, 2006. preprint.

[140] I. R. Shafarevich. *Basic Algebraic Geometry 1.* Springer-Verlag, Berlin, second edition, 1994.

[141] I. R. Shafarevich. *Basic Algebraic Geometry 2.* Springer-Verlag, Berlin, second edition, 1994.

[142] D. Shevitz and B. Paden. Lyapunov stability of nonsmooth systems. *IEEE Trans. on Automatic Control*, 39(9):1910–1914, 1994.

[143] M. Shub. Some remarks on dynamical systems and numerical analysis. In L. Lara-Carrero and J. Lewowicz, editors, *Dynamical Systems and Partial Differential Equations: Proceedings of the VII ELAM*, pages 69–92, Caracas, 1986. Equinoccio, Universidad Simon Bolivar.

[144] L. Simon. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. *Annals of Mathematics*, 118:525–571, 1983.

[145] N. J. A. Sloane. Table of best Grassmannian packings, 1997. available online at `http://www.research.att.com/~njas/grass/grassTab.html`.

[146] S. T. Smith. *Geometric Optimization Methods for Adaptive Filtering*. PhD thesis, Havard University, Cambridge, Massachusetts, 1993.

[147] S. T. Smith. Optimization techniques on Riemannian manifolds. In A. Bloch, editor, *Hamiltonian and gradient flows, algorithms and control*, volume 3 of *Fields Inst. Commun.*, pages 113–136. American Mathematical Society, Providence, 1994.

[148] F. Takens. Singularities of vector fields. *Inst. Hautes Études Sci. Publ. Math.*, 43:47–100, 1974.

[149] T.-Y. Tam. Gradient flows and double bracket equations. *Differential geometry and its Applications*, 20:209–224, 2004.

[150] A. R. Teel. Lyapunov methods in nonsmooth optimization. Part I: Quasi-Newton algorithms for Lipschitz, regular functions. In *Proceedings of the 39th IEEE Conference on Decision and Control, 2000*, volume 1, pages 112–117, 2000.

[151] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESIAM: Control, Optimization and Calculus of Variations*, 5:313–367, 2000.

[152] The MathWorks, Inc. Matlab. Software, Version 7.1.

[153] R. Thom. Problèmes rencontrés dans mon parcours mathématique: un bilan. *Publ. Math. IHES*, 70:200–214, 1989.

[154] C. Udrişte. *Convex functions and optimization methods on Riemannian manifolds.* Kluwer Academic Publishers Group, Dordrecht, 1994.

[155] L. van den Dries. *Tame topology and O-minimal structures.* Cambridge University Press, Cambridge, 1998.

[156] L. van den Dries. O-minimal structures and real analytic geometry. In *Current developments in mathematics, 1998 (Cambridge, MA)*, pages 105–152. Int. Press, Somerville, MA, 1999.

[157] N. J. Vilenkin. *Special functions and the theory of group representations.* American Mathematical Society, Providence, Rhode Island, 1968.

[158] O. Villamayor. Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup. (4)*, 22(1):1–32, 1989.

[159] O. Villamayor. Patching local uniformizations. *Ann. Sci. École Norm. Sup. (4)*, 25(6):629–677, 1992.

[160] N. R. Wallach. *Real Reductive Groups I.* Academic Press, San Diego, 1988.

[161] P. Wolfe. Convergence conditions for ascent methods. *SIAM Review*, 11:226–235, 1969.

[162] F.-E. Wolter. Interior metric, shortest paths and loops in Riemannian manifolds with not necessarily smooth boundary. Diplomarbeit, Freie Unniversitaet Berlin, Berlin, 1979. available online at `http://www.gdv.uni-hannover.de/forschung/publikationen/index.php`.

[163] Y. Yang. Optimization on Riemannian manifold. In *Proceedings of the 38th IEEE Conference on Decision and Control, 1999*, volume 1, pages 888–893, 1999.

[164] Y. Yang. Globally convergent optimization algorithms on Riemannian manifolds: Uniform framework for unconstrained and constrained optimization. *J. Optim. Theory Appl.*, 2006.

[165] K. Yano and S. Ishihara. *Tangent and cotangent bundles: differential geometry.* Marcel Dekker Inc., New York, 1973.

[166] K. Yosida. *Functional Analysis.* Spinger-Verlag, Berlin, sixth edition, 1995.

[167] L. Zheng and D. N. C. Tse. Communications on the Grassmann manifold: A geometric approach to the noncoherent multiple-antenna channel. *IEEE Trans. on Information Theory*, 48(2):359–383, 2002.