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Augmented Lagrangian Methods for State Constrained Optimal Control Problems

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VERONIKA KARL
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1. Gutachter:	Prof. Dr. Daniel Wachsmuth
	Julius-Maximilians-Universität Würzburg
2. Gutachter:	Prof. Dr. Ira Neitzel
	Rheinische Friedrich-Wilhelms-Universität Bonn

Abstract

This thesis is concerned with the solution of control and state constrained optimal control problems, which are governed by elliptic partial differential equations. Problems of this type are challenging since they suffer from the low regularity of the multiplier corresponding to the state constraint. Applying an augmented Lagrangian method we overcome these difficulties by working with multiplier approximations in $L^2(\Omega)$. For each problem class, we introduce the solution algorithm, carry out a thoroughly convergence analysis and illustrate our theoretical findings with numerical examples.
The thesis is divided into two parts. The first part focuses on classical PDE constrained optimal control problems. We start by studying linear-quadratic objective functionals, which include the standard tracking type term and an additional regularization term as well as the case, where the regularization term is replaced by an $L^1(\Omega)$-norm term, which makes the problem ill-posed. We deepen our study of the augmented Lagrangian algorithm by examining the more complicated class of optimal control problems that are governed by a semilinear partial differential equation.
The second part investigates the broader class of multi-player control problems. While the examination of jointly convex generalized Nash equilibrium problems (GNEP) is a simple extension of the linear elliptic optimal control case, the complexity is increased significantly for pure GNEPs. The existence of solutions of jointly convex GNEPs is well-studied. However, solution algorithms may suffer from non-uniqueness of solutions. Therefore, the last part of this thesis is devoted to the analysis of the uniqueness of normalized equilibria.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Lösung von kontroll- und zustandsbeschränkten Optimalsteuerungsproblemen mit elliptischen partiellen Differentialgleichungen als Nebenbedingungen. Da die zur Zustandsbeschränkung zugehörigen Multiplikatoren nur eine niedrige Regularität aufweisen, sind Probleme dieses Typs besonders anspruchsvoll. Zur Lösung dieser Problemklasse wird ein augmentiertes Lagrange-Verfahren angewandt, das Annäherungen der Multiplikatoren in $L^2(\Omega)$ verwendet. Für jede Problemklasse erfolgt eine Präsentation des Lösungsalgorithmus, eine sorgfältige Konvergenzanalysis sowie eine Veranschaulichung der theoretischen Ergebnisse durch numerische Beispiele.
Die Arbeit ist in zwei verschiedene Themenbereiche gegliedert. Der erste Teil widmet sich klassischen Optimalsteuerungsproblemen. Dabei wird zuerst der linear-quadratische und somit konvexe Fall untersucht. Hier setzt sich das Kostenfunktional aus einem Tracking-Type Term sowie einem $L^2(\Omega)$-Regularisierungsterm oder einem $L^1(\Omega)$-Term zusammen. Wir erweitern unsere Analysis auf nichtkonvexe Probleme. In diesem Fall erschwert die Nichtlinearität der zugrundeliegenden partiellen Differentialgleichung die Konvergenzanalysis des zugehörigen Optimalsteuerungsproblems maßgeblich.
Der zweite Teil der Arbeit nutzt die Grundlagen, die im ersten Teil erarbeitet wurden und untersucht die allgemeiner gehaltene Problemklasse der Nash-Mehrspielerprobleme. Während die Untersuchung von konvexen verallgemeinerten Nash-Gleichsgewichtsproblemen (engl.: Generalized Nash Equilibrium Problem, kurz: GNEP) mit einer für alle Spieler identischen Restriktion eine einfache Erweiterung von linear elliptischen Optimalsteuerungsproblemen darstellt, erhöht sich der Schwierigkeitsgrad für Mehrspielerprobleme ohne gemeinsame Restriktion drastisch. Die Eindeutigkeit von normalisierten Nash-Gleichgewichten ist, im Gegensatz zu deren Existenz, nicht ausreichend erforscht, was insbesondere eine Schwierigkeit für Lösungsalgorithmen darstellt. Aus diesem Grund wird im letzten Teil dieser Arbeit die Eindeutigkeit von Lösungen gesondert betrachtet.

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师傅领进门，修行在个人。

Teachers open the door. You enter by yourself. (Chinese proverb)

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List of Symbols and Abbreviations

Abbreviations

a.e.	almost everywhere
e.g.	exempli gratia (for example)
PDE	partial differential equation
KKT	Karush–Kuhn–Tucker (conditions)
SSC	second-order sufficient condition
VI	variational inequality
QVI	quasi-variational inequality
NEP	Nash equilibrium problem
GNEP	generalized Nash equilibrium problem
NI	Nikaido-Isoda
WCCQ	weak convergence constraint qualification

General Notation

$\exists$	there exists
$\forall$	for all
$\mathbb{N}$	natural numbers with zero
$\mathbb{N}_0$	natural numbers without zero
$\mathbb{R}$	real numbers
$\mathbb{R}^n$	n-dimensional vector space of real numbers
$\overline{\mathbb{R}}$	extended real numbers $\mathbb{R} \cup \{+\infty\}$
$\mathrm{meas}(A)$	Lebesgue measure of a measurable set A
$(\cdot)_+$	$\max(0, \cdot)$ in a pointwise almost everywhere sense
$(a, b)_+$	$(\int_\Omega ab \, \mathrm{d}x)_+$

Normed Spaces

U	normed space
$\|\cdot\|_U$	norm on the space U
U^*	dual space of the normed space U
$L(U, W)$	space of bounded linear operators from the normed space U to W
$(\cdot, \cdot)_H$	inner product in a Hilbert space H
$(\cdot, \cdot)$	inner product in $L^2(\Omega)$
$\langle \cdot, \cdot \rangle_{U^*, U}$	duality pairing in a normed space U
$U \hookrightarrow W$	embedding between normed spaces U and W
$U \overset{c}{\hookrightarrow} W$	compact embedding between normed spaces U and W
$B_r(u)$	closed ball with radius r around u in a normed space

$(u_k)_k \subseteq U$	sequence of vectors in a normed space U
$(u_{k'})_{k'} \subseteq U$	subsequence of $(u_k)_k$ in a normed space U
$(\rho_k)_k \subseteq \mathbb{R}$	sequence of scalars
u^*	(weak) limit point of a sequence
$u_k \to u^*$	strong convergence of a sequence in a normed space to u^*
$u_k \rightharpoonup u^*$	weak convergence of a sequence in a normed space to u^*
$\varphi_k \rightharpoonup^* \varphi$	weak-* convergence of a sequence in the dual of a normed space to φ

Set Operations

$\mathcal{N}_C(u)$	Fréchet normal cone of a convex set C in a point u
$\mathcal{T}_C(u)$	tangent cone of a convex set C in a point u
C°	polar cone of a set C in a normed space
C^*	dual cone of a set C in a normed space
$\mathrm{int}(C)$	interior of a set C
$\mathrm{dist}(\cdot, C)$	distance function to a nonempty set C
P_C	projection onto a nonempty closed convex set in a Hilbert space
$C_k \xrightarrow{M} C$	convergence of sets in the sense of Mosco

Functions and Derivatives

$f\colon U \to W$	a mapping between Banach spaces U and W
$\mathrm{dom}(f)$	effective domain of a function f
$D_G f$	Gâteaux derivative of f
f'	Fréchet derivative of f
$D_u f$	partial Fréchet derivative with respect to u
∂f	convex subdifferential of a function $f\colon U \to \mathbb{R}$
$\Phi\colon U \rightrightarrows W$	a set-valued mapping between Banach spaces U and W
$\mathrm{gph}(\Phi)$	graph of Φ

Function Spaces

Ω	domain in $\mathbb{R}^d$
$\overline{\Omega}$	closure of Ω in $\mathbb{R}^d$
Γ	boundary of Ω in $\mathbb{R}^d$
$L^p(\Omega)$	Lebesgue space of p-times Lebesgue-integrable functions
$L^\infty(\Omega)$	Banach space of essentially bounded functions
$W^{k,p}(\Omega)$	Sobolev space of functions $u\colon \Omega \to \mathbb{R}$ whose weak derivatives up to order k exist and belong to $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega)$	Sobolev space $W^{k,2}(\Omega)$
$H_0^k(\Omega)$	Sobolev space $W_0^{k,2}(\Omega)$
$C(\overline{\Omega})$	space of uniformly bounded continuous functions $u\colon \overline{\Omega} \to \mathbb{R}$
$\mathcal{M}(\overline{\Omega})$	dual space $C(\overline{\Omega})^*$ of $C(\overline{\Omega})$, i.e., space of regular Borel measures on $\overline{\Omega}$
D^α	differential operator with respect to the multi-index $\alpha \in \mathbb{N}_0^n$
∇	gradient operator
Δ	Laplacian

Optimization

U_{ad}	admissible set of an optimization problem
F_{ad}	feasible set of an optimization problem
$\mathcal{L}$	Lagrangian function of an optimization problem
$\mathcal{L}', \mathcal{L}''$	derivatives of the Lagrangian with respect to the primal variable
f	objective function of an optimization problem
f_{AL}	objective function of the augmented Lagrange subproblem corresponding to f
f^{ν}	objective function of the ν-th player in a Nash equilibrium problem
f_{AL}^{ν}	objective function of the augmented Lagrange subproblem of the ν-th player
$u^{\nu}, u^{-\nu}$	player variable and their complements

CHAPTER 1

INTRODUCTION

In this work we study optimal control problems of distributed type, which are governed by elliptic partial differential equations with homogeneous Dirichlet or Neumann boundary conditions. Moreover, we imply inequality constraints on the variables of the optimal control problem. Single player, as well as multi-player optimal control problems are investigated.

1.1 Motivation

The aim of this section is to give a short introduction into optimal control theory and its applications.

Many processes in nature and technology can be described by partial differential equations (PDE). Among them can be found processes like heat distribution, diffusion, wave propagation, fluid flows, elastic deformation and option pricing. With the help of a *control* variable u, we want to influence the solution or *state* y of a partial differential equation:

Control u → | Partial Differential Equation | → State y.

This process should take place in such a way that a certain objective functional $J(y,u)$, which depends on y and u, will be minimized. Various applications in the industrial, medical and economical context are covered by this setting. However, lots of real world applications require additional constraints on the control u and the state y. Let us illustrate this on the following example from the medical field: In cancer therapy [40], one wants to approximate a desired heat distribution y_d in a domain Ω in order to fight cancer cells. In this case, the control u represents a heat source, which acts inside the domain. However, the control u only possesses bounded heating or cooling capacities. Moreover, due to health reasons, the patient's body temperature should not exceed a certain maximum. In this way, additional constraints on the state and control are arising. Another application for pointwise state constrained optimal control problems can be found in the production process for bulk single crystals [92].

Let the control space U and the state space Y be some Banach spaces to be specified. Then, a general optimal control problem for a control $u \in U$ and a state $y \in Y$ takes the form:

$$\underset{y\in Y,\, u\in U}{\text{minimize}}\ J(y,u) \quad \text{subject to} \quad y = Su, \quad u \in U_{\text{ad}}, \quad y \in Y_{\text{ad}}.$$

Here, $J\colon Y \times U \to \mathbb{R}$ denotes the objective functional, while $S\colon U \to Y$ is the solution operator of an underlying partial differential equation. The sets U_{ad} and Y_{ad} represent the previously mentioned constraints on the control and the state. If every control $u \in U$ admits a unique state

y, we can eliminate the state by setting $y := Su$. Then the objective functional is only dependent on u, i.e., $f(u) := J(Su, u)$, the so-called *reduced formulation*. Thus, the abstract problem turns into the following formulation:

$$\underset{u\in U}{\text{minimize}}\ f(u) \quad \text{subject to} \quad u \in F_{\text{ad}} := \{u \in U_{\text{ad}},\ Su \in Y_{\text{ad}}\}. \tag{1.1}$$

The proceeding in optimal control theory usually is as follows: The discussion of PDE constrained optimal control problems requires the analysis of the underlying PDE concerning solvability, uniqueness and regularity of solutions. Moreover, existence results concerning optimal controls and optimality conditions have to be established. These results can be used to develop suitable solution algorithms.

1.2 Formulation of the Problem

In this thesis, we are mainly concerned with the following PDE constrained optimal control problem. Let $\Omega \subseteq \mathbb{R}^d$ denote an open, bounded domain and let U and Y be given by the function spaces $U := L^2(\Omega)$ and $Y := Y_{PDE} \cap C(\overline{\Omega})$. The space Y_{PDE} depends on the type of underlying partial differential equations. For our purposes suitable candidates will be the spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. We are searching for optimal controls $u \in L^2(\Omega)$ that solve the following optimization problem

$$\begin{aligned} \underset{u\in L^2(\Omega)}{\text{minimize}}\ & f(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to}\ & u_a(x) \le u(x) \le u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \le \psi(x) \qquad \text{in } \overline{\Omega}, \end{aligned} \tag{1.2}$$

where $y_d \in L^2(\Omega)$, $\psi \in C(\overline{\Omega})$ and $\alpha > 0$. The control u is constrained by lower and upper bounds u_a, u_b which are assumed to be elements of $L^2(\Omega)$ with $u_a(x) \le u_b(x)$. The solution operator $S\colon L^2(\Omega) \to Y_{PDE} \cap C(\overline{\Omega})$ maps the control u to the solution y of an associated, possibly nonlinear, PDE. In this thesis, S is always assumed to be continuously Fréchet differentiable and completely continuous. For linear S, problem (1.2) is a strictly convex optimization problem. However, if S is nonlinear, for instance if S is the solution operator of a semilinear partial differential equation, problem (1.2) turns into a non-convex problem.

1.2.1 Linear Optimal Control Problems

Let us first consider the case that S is linear. It is well known that under suitable constraint qualifications first-order necessary optimality conditions for problem (1.2) can be established. In general, however, the Lagrange multiplier associated to the state constraint $y \le \psi$ is only a measure in $C(\overline{\Omega})^* = \mathcal{M}(\overline{\Omega})$, see, e.g., [26]. Under additional assumptions it has been proven in [32] that the multiplier is an element of the more restrictive space $H^{-1}(\Omega)$. These assumptions are satisfied, e.g., for ψ being constant. This low regularity of the Lagrange multiplier makes the numerical solution of state constrained optimal control problems challenging. Thus, in recent years different approaches were studied to overcome this problem. These approaches have in common that the state constraint is relaxed in a suitable way. Let us mention Lavrentiev-regularization [61, 93], which turns the control problem into a problem with mixed control-state constraints. Penalization-based approaches were studied in [53, 57, 58, 65], their combination with a path-following strategy was investigated in [55, 56]. Both types of methods are obtained as special cases of the so-called virtual control regularization approach developed in [79, 80]. Interior point methods, which generate feasible iterates are considered for instance in [82, 112], where in the recent work [82] complexity estimates are provided.

Augmented Lagrangian methods are well-known in optimization. However, there is only a limited number of publications dedicated to the application of such methods to optimal control problems with state constraints. In [14, 15] the state equation is augmented, but the inequality constraints on the state are still present in the augmented Lagrangian subproblem. In [17, 63] problems with constraints in a Hilbert space are studied. However, the convergence analysis needs that the constraints are in a finite-dimensional space. Nevertheless, the natural choice for the state constraint function space is $C(\overline{\Omega})$, which is not a Hilbert space. This limits the applicability of the above mentioned results. The goal of this work is therefore to analyze the classical augmented Lagrangian method in the general setting of problems with state constraints: state constraints in $C(\overline{\Omega})$ (not in a – possibly finite-dimensional – Hilbert space) with multipliers in $C(\overline{\Omega})^*$.

In the first part of this thesis, we present an augmented Lagrangian algorithm that solves a sequence of subproblems that are control constrained only, i.e.,

$$\begin{aligned} &\underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad f_{AL}(u_\rho, \mu, \rho) := f(u_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(Su_\rho - \psi))_+ \right\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u_a(x) \le u_\rho(x) \le u_b(x) \quad \text{a.e. in } \Omega, \end{aligned} \tag{1.3}$$

where S denotes the same solution operator as for the unregularized problem (1.2), $\rho > 0$ and $\mu \in L^2(\Omega)$. Moreover, $(\cdot)_+ := \max(0, \cdot)$ in the pointwise everywhere sense. Compared to the unregularized problem, the occurring subproblems can be solved by efficient optimization algorithms. We establish a special update rule that performs the classical augmented Lagrange update only if a sufficient decrease of the maximal constraint violation and the violation of the complementarity condition is achieved. This type of update rule has its predecessors in finite dimensional nonlinear optimization [36, 37, 84]. Further, this update allows us to guarantee the $L^1(\Omega)$-boundedness of generated multiplier approximations, which is crucial for the convergence analysis, since it is necessary for obtaining a weak-* convergent subsequence in $\mathcal{M}(\overline{\Omega})$. While penalty methods suffer from the fact that the penalty parameter tends to infinity, augmented Lagrangian methods for finite-dimensional optimization problems do not require this property. Surprisingly, such a result is not available for the augmented Lagrangian method studied in this thesis. However, for the case that S is linear, we will prove, see Theorem 3.18, that the penalty parameters are bounded only if there is a multiplier to the state constraint in $L^2(\Omega)$, which is not the case in general. Such an observation was also made in the contribution [73]. There, a modified augmented Lagrangian method is investigated, which is in the spirit of recent developments for finite-dimensional optimization problems [18] and allows for a simpler convergence analysis.

1.2.2 Semilinear Optimal Control Problems

If the solution operator S is nonlinear, problem (1.2) turns into a non-convex optimization problem. The convergence analysis of solution algorithms of non-convex optimal control problems suffers significantly from non-uniqueness of local and global solutions and only few contributions can be found in the literature. Let us mention the so-called virtual control approach [80], Lavrentiev regularization [97], and Moreau-Yosida regularization [94]. All of these publications discuss under which conditions local solutions of the unregularized problem can be approximated by sequences of local solutions of the regularized problems, but do not provide convergence results for the overall iterative solution method. The convergence analysis of safe-guarded augmented Lagrangian methods has been considered in [20, 73].

Our goal is to extend the augmented Lagrangian method presented in Chapter 3 and to provide the corresponding convergence analysis in order to solve (1.2) for nonlinear S. By penalizing the state constraint, one has to solve subproblems of the type (1.3). Given penalty parameters ρ_k and multiplier estimates μ_k, new iterates (y_{k+1}, u_{k+1}) of the algorithm are computed as *stationary* points of (1.3) for $(\rho, \mu) := (\rho_k, \mu_k)$.

The question of convergence of the algorithm is linked to the question of feasibility of limit points of iterates that are only stationary points of the augmented Lagrangian subproblem. In particular, the subproblem may have stationary points that are located arbitrarily far from the feasible set and there is no rule to determine which stationary points have to be chosen in the solution process of the subproblem in order to guarantee convergence. Specifically for augmented Lagrangian methods, feasibility of limit points is not guaranteed, see for instance [71]. Consequently, feasibility is either imposed as an additional assumption [36, 37, 73] or is an implication of a constraint qualification [20, 73]. Let us mention that the classical quadratic penalty method is contained in [20, 73] as a special case, and the comments regarding feasibility of limit points apply equally to this method.

The crucial point of augmented Lagrangian methods is the questions when and how to update the penalty parameter and multiplier. As for the linear case, we use an update rule that performs the classical augmented Lagrangian update only if a sufficient decrease of the maximal constraint violation and the violation of the complementarity condition is achieved. Accordingly, during all other steps the penalty parameter is increased, but the multiplier remains unchanged. This allows us to conclude feasibility of a weak limit point if and only if an infinite number of multiplier updates is executed, see Theorem 5.9. It would be favourable if the penalty parameter is increased only finitely many times. In this case, the penalty parameter is only bounded in exceptional situations, i.e., if the multiplier is a function in $L^2(\Omega)$, see Theorem 5.28. In practice, solutions of the augmented Lagrangian subproblems are obtained by iterative methods, which naturally use the previous iterate as starting point. Thus, it is realistic to expect that the iterates stay in a neighbourhood of a local solution of the original problem. One main result of this section is to prove that such a situation can occur, i.e., for each iteration, we provide existence of a stationary point of the subproblem in exactly this neighborhood. Therefore, we investigate the auxiliary problem

$$\begin{aligned} \underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad & f_{AL}(u_\rho, \mu, \rho) := f(u_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(Su_\rho - \psi))_+ \right\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_a(x) \le u_\rho(x) \le u_b(x) \quad \text{a.e. in } \Omega, \\ & \left\| \bar{u} - u_\rho \right\|_{L^2(\Omega)} \le r, \end{aligned} \tag{1.4}$$

that possesses solutions that are close enough to a local solution $\bar{u}$ of (1.2). We will prove under a quadratic growth condition that for ρ large enough global solutions of this auxiliary problem are local solutions of the augmented Lagrangian subproblem. Moreover, if we assume that the algorithm chooses the global solutions of the auxiliary problem as KKT points of the augmented Lagrangian subproblem and the penalty parameter remains bounded, then the multiplier is a function in $L^2(\Omega)$.

1.2.3 Ill-Posed Optimal Control Problems with Sparse Controls

Another challenging problem is to replace the regularizing Tikhonov term in (1.2) by an $L^1(\Omega)$-norm term, which results in the optimal control problem

$$\begin{aligned} \underset{u \in L^2(\Omega)}{\text{minimize}} \quad & f(u) := \frac{1}{2} \left\| Su - y_d \right\|^2_{L^2(\Omega)} + \beta \left\| u \right\|_{L^1(\Omega)} \\ \text{subject to} \quad & u_a(x) \le u(x) \le u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \le \psi(x) \qquad \text{in } \overline{\Omega}, \end{aligned} \tag{1.5}$$

where $\beta > 0$ is a fixed parameter. The motivation for the $L^1(\Omega)$-term in the cost functional is the following: The optimal solution $\bar{u}$ of this optimal control problem is sparse, i.e., the control is zero on large parts of the domain if β is large enough. This can be used in the optimal placement of controllers, especially in situations where it is not desirable to control the system from the whole domain Ω, see [113]. Such sparsity promoting optimal control problems without state constraints have been studied in e.g. [118–120] for optimal control of linear partial differential equations and in [28, 30] for the optimal control of semilinear equations. For sufficient second-order conditions for the state constrained sparsity promoting optimal control problem with a semilinear partial differential equation we refer to [34].

Our aim is to modify and extend the method that will be used for solving (1.2). The main idea is the following: To deal with the ill-posedness of (1.5), we add a Tikhonov regularization. To overcome the problems, that arise due to the pointwise state constraints, we apply the augmented Lagrangian method. Thus, in every iteration we examine the optimal control problem

$$\begin{aligned} &\underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad f_{AL}^{\alpha}(u_\rho, \mu, \rho) := f(u_\rho) + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\left\|(\mu + \rho(Su_\rho - \psi))_+\right\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u_a(x) \le u_\rho(x) \le u_b(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{1.6}$$

Both variables, the regularization parameter α and the penalization parameter ρ are coupled in our method. During the algorithm we decrease the regularization parameter $\alpha \to 0$ while increasing the penalization parameter $\rho \to \infty$. Since the decrease of α is a classical Tikhonov regularization approach, we aim to achieve strong convergence against the solution of (1.5). Apart from the augmented Lagrangian method there exist some other different approaches to deal with state constraints. We want to mention [89], in which a simultaneous Tikhonov and Lavrentiev regularization had been applied for (1.5) without an additional $L^1(\Omega)$-norm therm. There, the motivation was to derive error estimates under a source condition and the assumption that the state constraints are not active for solutions of (1.5). Furthermore, the authors assumed that for the lower bound on the control it holds $u_a = 0$. In our approach, we do not assume any of the above, which allows us to apply our method to a larger class of problems.

1.2.4 Jointly Convex Multi-Player Optimal Control Problems

The optimal control problems (1.2) and (1.3) can basically be taken as single player optimal control problems that can be customized to a larger problem class: the *multi-player optimal control problems* or *generalized Nash equilibrium problems* (GNEP). Let $N \in \mathbb{N}$ denote the number of players. The player $\nu \in \{1, ..., N\}$ is in control of the variable $u^\nu \in L^2(\Omega)$. The strategies of all players, except the ν-th player are denoted by $u^{-\nu} \in L^2(\Omega)^{N-1}$. To emphasize the role of player ν's variable, we use the notation $u := (u^\nu, u^{-\nu})$. We are searching for a control $u \in L^2(\Omega)^N$ such that for all ν the control u^ν solves the following associated PDE constrained optimal control problem:

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad f^\nu(u) := \frac{1}{2}\|Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u^\nu \in U_{\text{ad}}^\nu := \{u^\nu \in L^2(\Omega) \mid u_a^\nu(x) \le u^\nu(x) \le u_b^\nu(x) \ \text{ a.e. in } \Omega\}, \\ &\qquad\qquad\quad Su(x) \le \psi(x) \quad \text{in } \overline{\Omega}, \end{aligned} \tag{1.7}$$

where $S\colon L^2(\Omega)^N \to Y_{PDE} \cap C(\overline{\Omega})$ denotes the solution operator of the underlying partial differential equation. Since the state constraint $Su \le \psi$ is the same for all players, it is commonly referred to as *joint constraint*, turning problem (1.7) into a so-called *jointly convex generalized Nash equilibrium problem*. Defining the set

$$F_{\text{ad}} := \left\{ u \in U_{\text{ad}} := U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N \mid Su \le \psi \right\},$$

the generalized Nash equilibrium problem (1.7) can be expressed as a *variational inequality* (VI) problem, or simply as the problem of finding a point $\bar{u} \in F_{\text{ad}}$ such that

$$(F(\bar{u}), v - \bar{u}) \geq 0 \quad \forall v \in F_{\text{ad}},$$

where $F(u) := \big(D_{u^1} f^1(u), ..., D_{u^N} f^N(u)\big)$. Problems of this type have widely been studied in finite dimensions, see for instance the survey papers [44,48]. For GNEPs in Banach spaces Carlson [24] extended the work of Rosen [108] and provided conditions for the existence and uniqueness of so-called normalized Nash equilibria. Most other papers only deal with specific problem classes, e.g. [21, 102–104, 109, 110, 115] for standard NEPs and [42, 51, 59, 60] for GNEPs. In [59, 60] GNEPs in the convex optimal control setting with pointwise state and control constraints have been studied. Here, a Moreau-Yosida type penalty approach has been made to overcome the problems that are arising due to the state constraints. We also want to mention the contribution [69], where an augmented Lagrangian method with safeguarded multiplier has been applied to GNEPs in a setting that includes also the jointly convex optimal control case. Jointly convex GNEPs exhibit better solution properties, i.e., normalized equilibria of the GNEPs as well as solutions of the arising subproblems are unique in our optimal control setting.

We will solve (1.7) by applying a simple extension of the augmented Lagrangian method developed for (1.2). Including the state constraints into the objective functional leads to a system of problems where each player's minimization problem is given by

$$\begin{aligned} \underset{u_\rho^\nu \in L^2(\Omega)}{\text{minimize}} \quad & f^\nu_{AL}(u_\rho, \mu, \rho) := f^\nu(u^\nu_\rho, u^{-\nu}_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(S u_\rho - \psi))_+ \right\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u^\nu_\rho \in U^\nu_{\text{ad}}. \end{aligned}$$

In this case, the remaining constraints do not depend on the other players' controls. Problems of this type are called *Nash equilibrium problems* (NEP).

1.2.5 Generalized Multi-Player Optimal Control Problems

The investigation of multi-player optimal control problems complicates if the control-to-state mapping and the state constraint ψ_ν differs for each player. In this case, the control u^ν solves the following associated PDE constrained optimal control problem:

$$\begin{aligned} \underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad & f^\nu(u^\nu, u^{-\nu}) := \frac{1}{2} \|S_\nu u - y^\nu_d\|^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2} \|u^\nu\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & U^\nu_{\text{ad}} := \{u^\nu \in L^2(\Omega) \mid u^\nu_a(x) \leq u^\nu(x) \leq u^\nu_b(x) \ \text{ a.e. in } \Omega\}, \\ & S_\nu u(x) \leq \psi_\nu(x) \quad \text{in } \overline{\Omega}, \end{aligned} \tag{1.8}$$

where $S_\nu \colon L^2(\Omega)^N \to Y_{PDE} \cap C(\overline{\Omega})$ is the solution operator of the respective linear elliptic PDE. The bounds u^ν_a, u^ν_b are assumed to be $L^2(\Omega)$ functions and $\psi_\nu \in C(\overline{\Omega})$ for all ν. The N different state constraints may lead to N different multipliers. Let us emphasize that each player's state, i.e., the solution $S_\nu u$ of each players PDE is affected by the other players' strategies $u^{-\nu}$, leading to a coupled system of optimal control problems. Defining the multifunction $F_{\text{ad}}(u) \colon L^2(\Omega)^N \rightrightarrows L^2(\Omega)^N$

$$F_{\text{ad}}(u) := \left\{ v \in L^2(\Omega)^N \mid \forall \nu = 1, \ldots, N \; v^\nu \in U^\nu_{\text{ad}} \ \text{ and } \ S_\nu(v^\nu, u^{-\nu}) \leq \psi_\nu \right\},$$

the generalized Nash equilibrium problem (1.8) can be expressed as a *quasi-variational inequality* (QVI) problem, i.e., as the problem of finding a point $\bar{u} \in F_{\text{ad}}(\bar{u})$ such that

$$(F(\bar{u}), v - \bar{u}) \geq 0 \quad \forall v \in F_{\text{ad}}(\bar{u}),$$

where $F(u) := \left(D_{u^1} f^1(u), ..., D_{u^N} f^N(u)\right)$. QVIs have been introduced in the context of stochastic impulse control in the paper [13] by Bensoussan and Lions, who also recognized the connection between generalized Nash games and quasi-variational inequalities [12]. QVIs became a powerful modelling tool for various application areas, for instance mechanics, economics and biology. Only few approaches have been made to solve finite-dimensional QVIs numerically. The first globally convergent algorithm using a fixed point approach has been made in [35]. In [45] an interior point method has been applied to the arising KKT conditions. Our approach is based on the work of Pang and Fukushima [98], see also the extensions [68, 70], where a sequential penalty approach has been proposed to general QVIs in finite dimensions. In the recent years the interest in QVIs in infinite dimensions strongly increased since they permit to model various physical phenomena. Here, Barett and Prigozhin [7–10] did lots of research concerning formation and growth of sand piles, determination of lakes and superconductivity, see also [83, 105]. Concerning existence results and convergence theory, the concept of weak Mosco-continuity [96] plays a fundamental role. In order to prove existence of solution of the corresponding QVI, we will work with a Slater-type constraint qualification, which implies weak Mosco-continuity on the admissible set U_{ad}.

We solve (1.8) by applying the augmented Lagrangian method that will be developed in Chapter 3 and 5. We include the state constraints into the objective functional, while the control constraints are treated directly. This leads to a Nash equilibrium problem, where each player's minimization problem is given by

$$\begin{aligned}\underset{u_\rho^\nu \in L^2(\Omega)}{\text{minimize}} \quad & f_{AL}^\nu(u_\rho, \mu^\nu, \rho_\nu) := f^\nu(u_\rho^\nu, u_\rho^{-\nu}) + \frac{1}{2\rho_\nu} \left\|(\mu^\nu + \rho_\nu(S_\nu u_\rho - \psi_\nu))_+\right\|_{L^2(\Omega)}^2 \\ \text{subject to} \quad & u_\rho^\nu \in U_{\text{ad}}^\nu.\end{aligned}$$

In this way, the subproblem has been simplified to a standard variational inequality, which can be equivalently reformulated in form of the GNEP's optimality conditions, that has to be solved during the solution process.

In [60] a GNEP in the optimal control setting with pointwise state and control constraints was investigated. The authors provided a result that establishes the associated optimality system supposing that the state equation is of the form $Ay = \sum_{\nu=1}^N u^\nu$ and the single states y^ν are given via $y^\nu := K^\nu y$, where $K^\nu \in L(Y_{PDE}, L^2(\Omega))$. However, in the following, this work concentrated on the special class of jointly convex GNEPs that admit normalized equilibria. Moreover, in [72] a Lagrange multiplier method for QVIs in a rather general setting has been developed that includes the optimal control case.

1.2.6 Uniqueness of Non-Reducible Multi-Player Control Problems

As already mentioned, the investigation of multi-player control problems in the function space setting often considers problems of the type where each player aims at minimizing the optimal control problem

$$\begin{aligned}\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad & \frac{1}{2}\|CSu - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\|u^\nu\|_{L^2(\Omega)}^2 \\ \text{subject to} \quad & u^\nu \in U_{\text{ad}}^\nu, \quad g(u) \in K,\end{aligned} \tag{1.9}$$

where the operator $S\colon L^2(\Omega) \to Y_{PDE}$ denotes the solution operator of an underlying linear elliptic partial differential equation with a suitable Banach space Y_{PDE}. Further, $C \in L(Y_{PDE}, L^2(\Omega))$ and $g \in L(L^2(\Omega)^N, Y_{PDE})$ are given linear and continuous mappings, $y_d^\nu \in L^2(\Omega)$, and α_ν is a non-negative regularization parameter. Moreover, the set $U_{\text{ad}}^\nu \subset L^2(\Omega)$ is bounded, closed, convex and $K \subseteq Y_{PDE}$ is a closed, convex cone. The joint constraint $g(u) \in K$ coincides for each player, which makes the problem a jointly convex GNEP. Problems of this type have first been investigated in [59]. Here, $C := \mathrm{Id}$, $g(u) := Su - \psi$, $\psi \in C(\overline{\Omega})$, and K is the cone of non-negative continuous functions. The authors extended their investigation in [60]. In [42, 69], general settings of jointly convex GNEPs, which include the optimal control case, have been investigated. Some of the literature above investigates the solution of (1.9) via the application of a Moreau-Yosida regularization or a Lagrange multiplier method. Supposing K is given as the cone of non-positive continuous function, this treatment requires for each player the solution of the following subproblem:

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \ \frac{1}{2}\|CSu - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\|(\mu + \rho g(u))_+\|^2_{L^2(\Omega)} \\ &\text{subject to} \ \ u^\nu \in U^\nu_{\text{ad}}, \end{aligned} \tag{1.10}$$

where $\rho > 0$ is a positive penalization parameter and $\mu \in L^2(\Omega)$. In this case, each player's constraint does not depend on the opponents' strategies $u^{-\nu}$, which makes (1.10) a simpler Nash equilibrium problem. For the Moreau-Yosida regularization, one typically chooses μ equal to zero, while the augmented Lagrangian approach uses an adaptive update of μ, which is dependent on the previous iterates. In general, this process yields fewer outer iterations for the augmented Lagrangian method. However, solving (1.10) with a semi-smooth Newton method, the different choice of μ does in general hardly affect the number of inner iterations. The existence of (generalized) Nash equilibria is well studied for both problems. A common approach is to apply the Kakutani-Fan-Glicksberg Theorem [49]. Moreover, solutions of problem (1.9) as well as solutions of (1.10) can be characterized via solutions of strongly monotone variational inequalities. Applying the theory of VIs [78, Chap. III] directly yields existence of solutions. Finally, we want to mention that both problems fall into the category of potential games in the sense of Monderer [95]. In our setting, this means that the problem can be reduced to a single convex PDE constrained optimization problem [60, Prop. 3.10]. Existence of unique solutions of this problems can be deduced by standard arguments from optimization theory.

However, the situation becomes considerably more complicated if we generalize (1.9) in the following way: Now, each player aims at minimizing

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \ \frac{1}{2}\|C_\nu Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u^\nu\|^2_{L^2(\Omega)} \\ &\text{subject to} \ \ u^\nu \in U^\nu_{\text{ad}}, \quad g(u) \in K. \end{aligned} \tag{1.11}$$

The setting is the same as for (1.9) but now the linear mapping $C_\nu \in L(Y_{PDE}, L^2(\Omega))$ may differ for each player. The corresponding augmented Nash equilibrium problem is given by

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \ \frac{1}{2}\|C_\nu Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u^\nu\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\|(\mu + \rho g(u))_+\|^2_{L^2(\Omega)} \\ &\text{subject to} \ \ u^\nu \in U^\nu_{\text{ad}}. \end{aligned} \tag{1.12}$$

Existence of solutions for these problems can again be proven by a fixed point approach [60, Thm. 3.4]. However, problems of this type cannot be reduced to a single control problem and we can not expect in general that the resulting first-order optimality system is a (strongly) monotone VI. Thus, uniqueness of normalized solutions is not clear. The first idea concerning the assurance of a unique Nash equilibrium has been investigated in finite dimensions by Rosen [108], who defined the notion of *strict diagonal convexity*. This notion was extended to the infinite dimensional setting by Carlson [22–25] and requires that the combined objectives are strictly diagonal convex. This basically coincides with the property that the resulting first-order optimality system is a strongly monotone VI. A sufficient condition is given in [108] by a certain kind of definiteness of the second derivative of the combined objective functionals. However, it is not clear if this condition is satisfied in our case. In our approach we will show existence and uniqueness of variational equilibria of the GNEP (1.11) and the NEP (1.12) by imposing an assumption on the regularization parameter $\alpha > 0$.

Let us motivate the investigation of problems of the type (1.11) and (1.12). In multicriterion optimization, interaction between several criteria must be considered. Here, Tang, Désidéri and Periaux [115] investigated an airfoil design optimization problem, where they combined the so-called adjoint method with a formulation from game theory. The authors considered different design targets as objectives, which correspond to minimization problems. The design variable is, by physical considerations, split into several subsets corresponding to the design targets, which results in the choice $C_\nu := \chi_\nu$, where χ_ν denotes the standard characteristic function of a subset Ω_ν of Ω. However, each design variable effects the same physical system, which is described by partial differential equations. Hence, the objectives are mutually in conflict and we are in the situation of a non-reducible NEP.

It is a common choice to solve the arising subproblems (1.12) with a semi-smooth Newton method. We will show that the method can be expected to converge superlinear only if α is sufficiently large, see Theorem 8.11. For values of α, which do not satisfy this condition, it might be possible to show at least in an experimental way that no other equilibrium exists. Therefore, one might use a simple path-following on α with an initial value, which is greater than the critical value. As soon as α drops below this critical value, one could apply a deflation technique for semi-smooth equations [47] in order to search for distinct solutions and provide experimental evidence that no other equilibrium exists.

1.3 Structure of the Thesis

The outline of this thesis is as follows.

Chapter 2 starts with recalling all necessary notations and results from functional analysis, PDE and optimization theory. In particular, we clarify existence and uniqueness of solutions of the underlying PDE. We introduce the control-to-state mapping and derive a characterization of its adjoint operator. Moreover, partial differential equations that incorporate measures on its right side are briefly discussed. We continue with some existence results concerning general optimization problems. After that, we derive the corresponding necessary optimality conditions. First, we limit ourselves to multiplierfree formulations. Introducing suitable constraint qualifications, we end up with optimality conditions involving Lagrange multipliers. Moreover, we introduce the reader to the augmented Lagrangian method, which is investigated throughout this thesis. Finally, we collect results concerning solvability and optimality conditions of the problems (1.2) and (1.3).

In the first part of the thesis we will focus on the investigation of a solution algorithm for the optimal control problem (1.2).

Chapter 3 concerns the investigation of an augmented Lagrangian algorithm for problem (1.2), as already sketched in Section 1.2.1. We restrict ourselves to the case that S is linear. We prove strong convergence of the primal variables as well as weak convergence of the adjoint states and weak-* convergence of the multipliers associated to the state constraint. In addition, we show that the sequence of generated penalty parameters is bounded only in exceptional situations, which is different from classical results in finite-dimensional optimization.

Chapter 4 deals with the ill-posed optimal control problem (1.5). We couple the augmented Lagrangian method from Chapter 3 with a Tikhonov regularization. The coupling between the regularization parameter introduced by the Tikhonov regularization and the penalty parameter from the augmented Lagrangian method allows us to prove strong convergence of the controls and their corresponding states. Moreover, convergence results proving the weak convergence of the adjoint state and weak-* convergence of the multiplier are provided.

Chapter 5 aims at extending the results from Chapter 3 to a larger class of optimal control problems in order to solve non-convex elliptic problems. We show strong convergence of subsequences of the primal variables to a local solution of the original problem as well as weak convergence of the adjoint states and weak-* convergence of the multipliers associated to the state constraint. We use an auxiliary function and prove the existence of a KKT point of the subproblem in arbitrary small neighborhoods of a local solution of the original problem under a quadratic growth condition.

The second part of this thesis is devoted to generalized Nash equilibrium problems in the optimal control setting.

Chapter 6 investigates an augmented Lagrangian algorithm for jointly convex multi-player optimal control problems. We adapt the augmented Lagrangian method from Chapter 3 and show strong convergence of the primal variables to the unique normalized equilibrium as well as weak convergence of the adjoint states and weak-* convergence of the multipliers associated to the joint constraint.

Chapter 7 contains an extension of Chapter 6, which considers generalized Nash equilibrium problems. Under a Slater-type constraint qualification, which implies weak Mosco-continuity of the feasible set, we prove an existence result. Further, we prove convergence of the applied method.

In Chapter 8 we investigate a special class of Nash equilibrium problems that cannot be reduced to single player optimal control problems. We derive a sufficient condition, that proves the existence and uniqueness of normalized solutions. Problems of this type can be solved by a semi-smooth Newton method. Applying the same condition as needed for the uniqueness of solutions, we derive superlinear convergence for the associated Newton method and the equivalent active-set method. We also provide detailed finite element discretizations for both methods.

CHAPTER 2

BACKGROUND

This chapter aims at collecting all necessary notations and results from functional analysis, PDE and optimization theory. In the first sections, we focus on functional analysis, introduce function spaces and the differentiation in normed spaces. We will skip the proofs and refer the reader for instance to the books [5, 38, 41, 116, 121].

2.1 Basics from Functional Analysis

A normed vector space is called a *Banach space* if it is complete with respect to its norm. Let U and W denote Banach spaces that are endowed with the norms $\|\cdot\|_U$ and $\|\cdot\|_W$, respectively. An operator $A\colon U \to W$ is called *continuous* if $\lim_{n\to\infty} u_n = u$ in U implies $\lim_{n\to\infty} Au_n = Au$ in W. Further, if A is linear, it is called *bounded* if there exists a constant $c > 0$ independent of u such that $\|Au\|_W \leq c\,\|u\|_U$. For linear operators boundedness and continuity coincide. Let $L(U, W)$ denote the space of all linear and continuous operators from U to W and assume $A \in L(U, W)$. Then, the quantity

$$\|A\|_{L(U,W)} = \sup_{u\in U,\ \|u\|_U=1} \|Au\|_W$$

is finite and called *operator norm*. Endowed with this norm, $L(U, W)$ is a Banach space itself. The space $U^* := L(U, \mathbb{R})$ is called the dual space of U. For elements $u^* \in U^*$ and $u \in U$ we define the *duality paring*

$$u^*(u) =: \langle u^*, u\rangle_{U^*,U}.$$

The *canonical embedding* of U in the *bidual space* $U^{**} := (U^*)^*$

$$i_U\colon U \to U^{**}, \quad u \mapsto [u^* \in U^* \mapsto u^*(u)]$$

defines a linear, continuous isometry. If i_U is surjective, the Banach space U is called *reflexive*. In this case we identify U^{**} with U. Let H denote a complete Banach space, whose norm is induced by an inner product, i.e.,

$$\|u\|_H := \sqrt{(u, u)_H}.$$

Then H is called a *Hilbert space*. The following theorem characterizes the dual space H^* of a Hilbert space H.

Theorem 2.1 (Fréchet-Riesz [5, Thm. 6.1][38, §3 Thm. 3.4]). *Let H denote a Hilbert space with inner product $(\cdot,\cdot)_H$. Then the mapping $R_H\colon H \to H^*$, $(R_H u)(f) = (f, u)_H$ is bijective, isometric and conjugate linear. With other words, for every linear and continuous functional $F \in H^*$ there exists an element $f \in H$ such that $F(u) = (f, u)_H$ and $\|f\|_H = \|F\|_{H^*}$.*

Due to the Riesz-Theorem, we can identify H^* with H. Consequently, every Hilbert space is reflexive.
For a linear operator $A\colon U \to W$ and an arbitrary element $w^* \in W^*$ the *adjoint operator* $A^\star\colon W^* \to U^*$ is defined by $(A^\star w^*)u := w^*(Au)$ and we can write

$$\langle Au, w^* \rangle_{W,W^*} = \langle u, A^\star w^* \rangle_{U,U^*} \quad \forall w^* \in W^*,\ u \in U.$$

Let H_1, H_2 denote Hilbert spaces, $A \in L(H_1, H_2)$ and let $R_{H_i}\colon H_i \to H_i^*$ denote the Riesz isometries from Theorem 2.1. Then the adjoint operator (in the Hilbert space sense) A^* is given by $R_{H_1}^{-1} A^\star R_{H_2}$ and it is characterized by the relationship

$$(Au, w)_{H_2} = (u, A^* w)_{H_1} \quad \forall u \in H_1,\ w \in H_2.$$

For $H_1 = H_2$ we say that A is self-adjoint if $A^* = A$. In the following, we will make no difference between the adjoint operator $A^\star$ and the adjoint operator in Hilbert spaces A^* and use only the notation A^*.

We distinguish between different kinds of convergence:

Definition 2.2. Let U be a real Banach space. We say that a sequence

a) $(u_n)_n \subseteq U$ *converges strongly* to $u \in U$ and write $u_n \to u$ if $\|u_n - u\|_U \to 0$.
b) $(u_n)_n \subseteq U$ *converges weakly* to $u \in U$ and write $u_n \rightharpoonup u$ if $f(u_n) \to f(u)$ for every $f \in U^*$.
c) $(f_n)_n \subseteq U^*$ *converges weak-** to $f \in U^*$ and write $f_n \rightharpoonup^* f$ if $f_n(u) \to f^*(u)$ for every $u \in U$.

We refer to u^* as a (weak) limit point of a sequence $(u_n)_n$ if there exists a subsequence $(u_{n_k})_{n_k}$ such that $u_{n_k} \rightharpoonup u^*$, $u_{n_k} \to u^*$, respectively. If u^* is *the* (weak) limit of $(u_n)_n$, then the whole sequence converges (weakly). We extend the well-known concepts of closed and compact sets:

Definition 2.3. Let U denote a normed vector space and $M \subseteq U$. We say that M is

a) *closed* if for every sequence $(u_n)_n \subseteq M$ with $u_n \to u$ it holds $u \in M$.
b) *weakly sequentially closed* if for every sequence $(u_n)_n \subseteq M$ with $u_n \rightharpoonup u$ it holds $u \in M$.
c) *(weakly) sequentially compact* if every sequence $(u_n)_n \subseteq M$ contains a (weak) convergent subsequence with (weak) limit in M.
d) Let $M^* \subseteq U^*$. Then M^* is *weak-* sequentially compact* if every sequence $(u_n^*)_n \subseteq M^*$ contains a weak-* convergent subsequence with weak-* limit in M^*.

Given a Banach space U and a bounded sequence $(u_n)_n \subseteq U$ the following results are helpful to determine if U contains a convergent subsequence in the weak or weak-* sense.

Theorem 2.4 (Banach-Alaoglu [5, Thm. 8.5]). *Let U be a separable normed linear space. Then the closed unit ball in U^* is weak-* sequentially compact.*

Theorem 2.5 (Eberlein-Šmulyan [121, Thm. VIII.3.18, Thm. VIII.6.1]). *Let U be a normed space. Then, the closed unit ball is weakly sequentially compact if and only if U is reflexive.*

In particular, Theorem 2.5 implies that for a reflexive Banach space U, every non-empty, bounded, closed, convex set is weakly sequentially compact. Moreover, [121, Thm. VIII.6.1] implies that these sets are also weakly compact. Let us define some different kinds of continuity for a given function $A\colon U \to W$.

Definition 2.6. Let U, V be real Banach spaces and $A\colon U \to V$. We say that the function A is in a point $u \in U$

a) *continuous* if $u_n \to u$ implies $Au_n \to Au$.
b) *weakly continuous* if $u_n \rightharpoonup u$ implies $Au_n \rightharpoonup Au$.
c) *completely continuous* if $u_n \rightharpoonup u$ implies $Au_n \to Au$.

We say that a linear operator $A\colon U \to W$ is *compact* if it maps bounded sets in U into precompact sets in W, i.e., sets with compact closure. This definition immediately yields the complete continuity of compact operators.

Theorem 2.7 ([5, Lem. 20.2]). *Let U, W be real Banach spaces and $A\colon U \to W$ be compact. Then A is completely continuous. The converse holds true if U is reflexive.*

2.2 Function Spaces

Let us introduce some standard function spaces. Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ denote an open, bounded domain with boundary $\Gamma := \partial\Omega$. The *space of bounded continuous functions* $u\colon \overline{\Omega} \to \mathbb{R}$, endowed with the norm $\|u(x)\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|$, is denoted by $C(\overline{\Omega})$. It is well known, that $C(\overline{\Omega})$ is a separable Banach space. We define $C^\infty(\Omega)$ as the vector *space of infinitely differentiable functions* on Ω, and $C_0^\infty(\Omega)$ as the space of functions $u \in C^\infty(\Omega)$ with compact support. Moreover, $C^{0,1}(\overline{\Omega})$ denotes the space of *Lipschitz-continuous functions*

$$C^{0,1}(\overline{\Omega}) := \left\{ f \in C(\overline{\Omega}) \mid L := \sup \left\{ \frac{|f(u) - f(v)|}{|u - v|} < \infty, u \neq v \in \overline{\Omega} \right\} \right\}.$$

With the norm $\|u\|_{C^{0,1}(\overline{\Omega})} := \|u\|_\infty + L$, this is a Banach space. According to the Riesz-Radon theorem, the *space of regular Borel measures* $\mathcal{M}(\overline{\Omega})$ on $\overline{\Omega}$ is the dual space of $C(\overline{\Omega})$. The Riesz-Markow representation theorem [111, Theorem 6.19] yields the representation

$$\langle \mu, y \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = \int_{\overline{\Omega}} y \,\mathrm{d}\mu.$$

We identify a function $\tilde{\mu} \in L^2(\Omega)$ with an element $\mu \in \mathcal{M}(\overline{\Omega})$ via

$$\langle \mu, y \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = \int_{\overline{\Omega}} y \,\mathrm{d}\mu := \int_\Omega y\tilde{\mu} \,\mathrm{d}x = (y, \tilde{\mu})_{L^2(\Omega)} \quad \forall y \in C(\overline{\Omega}).$$

Thus, the definition of the norm yields the basic estimate

$$\|\mu\|_{\mathcal{M}(\overline{\Omega})} = \sup_{\substack{y\in C(\overline{\Omega}),\\ \|y\|_{C(\overline{\Omega})}=1}} \left| \int_{\overline{\Omega}} y \,\mathrm{d}\mu \right| = \sup_{\substack{y\in C(\overline{\Omega}),\\ \|y\|_{C(\overline{\Omega})}=1}} \left| \int_\Omega y\tilde{\mu} \,\mathrm{d}x \right| \le \|\tilde{\mu}\|_{L^1(\Omega)}.$$

Since $\mathcal{M}(\overline{\Omega})$ is the dual of the separable space $C(\overline{\Omega})$, the Banach-Alaoglu theorem (Theorem 2.4) yields, that the closed unit ball in $\mathcal{M}(\overline{\Omega})$ is weak-* sequentially compact. Thus, every bounded sequence $(u_n)_n \subseteq \mathcal{M}(\overline{\Omega})$ contains a weak-* convergent subsequence with weak limit in $\mathcal{M}(\overline{\Omega})$.

Further, for a measurable function $u\colon \Omega \to \mathbb{R}$ the *essential supremum* of u over Ω is denoted by $\operatorname{ess\,sup}_\Omega u := \inf\{M \in \mathbb{R} \mid u(x) \le M \text{ a.e. in } \Omega\}$. We define the norms

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \,\mathrm{d}x \right)^{1/p} \text{ if } 1 \le p < \infty \quad \text{and} \quad \|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_\Omega |u|.$$

It is well known that the *Lebesgue spaces*

$$L^p(\Omega) := \{u\colon \Omega \to \mathbb{R} \text{ measurable and } \|u\|_{L^p(\Omega)} < \infty\},$$

equipped with the corresponding norms, are Banach spaces. In particular, for $1 < p < \infty$, they are reflexive. For $p = 2$, the corresponding norm is induced by the scalar product

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\,\mathrm{d}x,$$

via $\|u\|^2_{L^2(\Omega)} = (u,u)_{L^2(\Omega)}$, which makes the space $L^2(\Omega)$ a Hilbert space. Consequently, by the Eberlein-Šmulyan theorem (Theorem 2.5), every bounded sequence $(u_n)_n \subseteq L^p(\Omega), 1 < p < \infty$ contains a weak convergent subsequence with weak limit in $L^p(\Omega)$.

Lemma 2.8 (Hölder inequality [5, Lem. 3.18]). *Let $p,q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, where $\frac{1}{\infty} := 0$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and $\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$.*

For the special case $p = q = 2$, the Hölder inequality

$$\int_\Omega uv\,\mathrm{d}x \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

is known as *Cauchy-Schwarz inequality*.

The last important class of function spaces we are concerned with are the *Sobolev spaces* $W^{k,p}(\Omega)$, where $k \in \mathbb{N}_0$ and $p \in [1,\infty]$, i.e.,

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k\}.$$

Here, $\alpha = (\alpha_1, ..., \alpha_d)$ is a given multi-index and $D^\alpha u$ the associated weak derivative. Equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)}\right)^{1/p}, \quad \text{if } 1 \leq p < \infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} := \max_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)},$$

these spaces become Banach spaces, which are reflexive for $1 < p < \infty$. For $p = 2$ we set $H^k(\Omega) := W^{k,2}(\Omega)$. The scalar product

$$(u,v)_{H^k(\Omega)} = \sum_{0\leq|\alpha|\leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

induces the norm $\|u\|^2_{H^k(\Omega)} = (u,u)_{H^k(\Omega)}$. This makes $H^k(\Omega)$ a Hilbert space. Finally, for $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ we define the space $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. In particular, we set $H_0^k(\Omega) := W_0^{k,2}(\Omega)$, which is again a Hilbert space. As a closed subspace of the Banach space $W^{k,p}(\Omega)$ the space $W_0^{k,p}(\Omega)$ is a Banach space itself. Next, for $k \in \mathbb{N}$ we define $H^{-k}(\Omega)$ as the dual space of $H_0^k(\Omega)$. A prominent candidate, which involves this function space, is the Laplace operator $\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ which is an isometric isomorphism between these spaces. If the boundary Γ satisfies a certain regularity, the *trace operator* allows us to interpret the functions in $W_0^{1,p}(\Omega)$ as functions of $W^{1,p}(\Omega)$ with zero boundary values.

Theorem 2.9 (Trace theorem [5, A 8.6]). *Let Ω be a bounded domain with Lipschitz boundary Γ. Let $1 \leq p \leq \infty$. Then there exists a linear and continuous operator $\tau\colon W^{1,p}(\Omega) \to L^p(\Gamma)$ such that for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ it holds $\tau u := u\big|_\Gamma$ almost everywhere on Γ.*

Thus, for bounded domains Ω with Lipschitz boundary Γ we define

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) \mid u\big|_\Gamma = 0\}.$$

Sobolev spaces enjoy a certain regularity that can be characterized with the help of embedding theorems.

Theorem 2.10 (Sobolev embedding theorem [1, Thm. 5.4]). *Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be an open bounded domain with Lipschitz boundary. Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Then the following embeddings are continuous:*

a) for $kp < d$: $W^{k,p} \hookrightarrow L^q(\Omega)$ for all $1 \leq q \leq \frac{dp}{d-kp}$,

b) for $kp = d$: $W^{k,p} \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$,

c) for $kp > d$: $W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.

All of the embeddings above are compact if q is strictly smaller than the corresponding upper bound. Further, for arbitrary open, bounded domains $\Omega \subset \mathbb{R}^d$ all assertions hold for $W^{k,p}(\Omega)$ replaced by $W_0^{k,p}(\Omega)$.

In particular, the Sobolev embedding theorem implies the following frequently used embeddings:

$$\begin{aligned} H^1(\Omega) &\hookrightarrow C(\overline{\Omega}), && \text{for } \Omega \subseteq \mathbb{R}, \\ H^1(\Omega) &\hookrightarrow L^q(\Omega),\ 1 \leq q < \infty, && \text{for } \Omega \subseteq \mathbb{R}^2, \\ H^1(\Omega) &\hookrightarrow L^6(\Omega), && \text{for } \Omega \subseteq \mathbb{R}^3, \\ W^{1,s}(\Omega) &\overset{c}{\hookrightarrow} L^2(\Omega),\ s \in (1, d/(d-1)) && \text{for } \Omega \subseteq \mathbb{R}^d,\ d \in \{2,3\}. \end{aligned}$$

2.3 Differentiation in Normed Spaces

In this section, we introduce the basic concepts of differentiability in infinite dimensional normed spaces. The concept of derivatives is needed for the characterization of minimizers of an optimization problem, see Section 2.5.3. We start with the notion of the convex subdifferential.

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ denote the extended values of the real numbers. The *effective domain* of a function $f\colon U \to \overline{\mathbb{R}}$ is defined by $\operatorname{dom}(f) := \{u \in U \mid f(u) < \infty\}$. If $\operatorname{dom}(f) \neq \emptyset$, the functional f is called *proper*.

Definition 2.11. Let U be a normed space, $f\colon U \to \overline{\mathbb{R}}$ proper and $u \in \operatorname{dom}(f)$. Then

$$\partial f(u) := \{u^* \in U^* \mid f(v) - f(u) \geq \langle u^*, v-u\rangle_{U^*,U} \quad \forall v \in U\}$$

denotes the *(convex) subdifferential* of f in u. Each element $u^* \in \partial f(u)$ is called *subgradient*.

Let us collect some properties of the subdifferential.

Theorem 2.12 ([43, Prop. I.5.2, I.5.5, I.5.6, I.5.7]). *Let $f, g\colon U \to \overline{\mathbb{R}}$ be proper and convex.*

a) The subdifferential ∂f is monotone.

b) Let f be continuous at a point $u \in \operatorname{dom}(f)$. Then $\partial f(u) \neq \emptyset$. Moreover, $\partial f(v) \neq \emptyset$ for all $v \in \operatorname{int}(\operatorname{dom}(f))$.

c) Let $u \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$*. Then* $\partial f(u) + \partial g(u) \subset \partial(f+g)(u)$*. Equality holds if there exists an* $\hat{u} \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$ *with* f *continuous in* $\hat{u}$*.*

d) Let $A \in L(W,U)$*. Then for each* $u \in \operatorname{dom}(f \circ A)$ *holds the inclusion* $\partial(f \circ A)(u) \supseteq A^*\partial f(u) := \{A^*v^* \mid v^* \in \partial f(Au)\}$*. Equality holds if there exists a* $\hat{w} \in W$ *such that* f *is continuous in* $A\hat{w}$*.*

Let us give two examples.

Example 2.13. For a convex set C with corresponding indicator function δ_C, the subdifferential for all $u \in C$ is given by the *(convex) normal cone* of the set C at the point u, i.e.,

$$\partial\delta_C(u) = \{u^* \in U^* \mid \langle u^*, v-u\rangle_{U^*,U} \leq 0 \quad \forall v \in C\} =: \mathcal{N}_C(u).$$

Example 2.14. Let us consider $j(u) := \|u\|_{L^1(\Omega)}$. By definition every $\lambda \in \partial j(u)$ satisfies

$$\int_\Omega \lambda(v-u)\,\mathrm{d}x \leq \|v\|_{L^1(\Omega)} - \|u\|_{L^1(\Omega)} \quad \forall v \in L^1(\Omega). \tag{2.1}$$

Since j is a convex function with $\operatorname{dom}(j) = L^1(\Omega)$ the subdifferential is always non-empty. Moreover, $\lambda \in \partial j(u) \subseteq L^\infty(\Omega)$ if and only if

$$\lambda \begin{cases} = 1 & \text{if } u(x) > 0, \\ = -1 & \text{if } u(x) < 0, \\ \in [-1,1] & \text{if } u(x) = 0, \end{cases}$$

see [28]. We will need the subdifferential of $j(u)$ to establish first-order conditions for an ill-posed state constrained optimal control problem with sparse controls in Chapter 4.

Note, that the calculus rules for the convex subdifferential are only valid for convex functions. Thus, in order to derive optimality conditions for non-convex optimality problems, it is necessary to establish a different notion of differentiability. Let $f\colon U \to W$ with U, W normed spaces. If for $u \in U$ and $h \in U$ the limit

$$\delta f(u;h) := \lim_{t \to 0^+} \frac{f(u+th) - f(u)}{t} \tag{2.2}$$

in W exists, then it is called the *directional derivative* of f at u in direction h and f is called *directional differentiable* in u in direction h.

Definition 2.15 (Gâteaux derivative). If the directional derivative (2.2) exists in u for all directions h and there exists a linear and continuous operator $A \in L(U,W)$ such that

$$\delta f(u;h) = Ah,$$

then we call A the *Gâteaux derivative* of f in u and use the notation $D_G f := A$.

For $f\colon U \to \overline{\mathbb{R}}$ proper, convex and Gâteaux differentiable, it holds $\partial f(u) = \{D_G f(u)\}$, i.e., the subdifferential is a singleton [43, Prop. I.5.3]. The following characterization of convexity for Gâteaux differentiable functions is well known.

Lemma 2.16 ([43, Prop. I.5.4]). *Let* $f\colon U \to \mathbb{R}$ *be Gâteaux differentiable on the convex set* $C \subseteq U$*. Then* f *is convex on* C *if and only if*

$$f(v) - f(u) \geq D_G f(u)(v-u) \quad \forall u, v \in C.$$

Moreover, the function is strictly convex if the inequality is strict for all $u \neq v \in C$*.*

Definition 2.17 (Fréchet derivative). We say that the function f is *Fréchet differentiable* in a point u if there exists $A \in L(U,W)$ such that

$$\lim_{\|h\|_U \to 0} \frac{\|f(u+h) - f(u) - Ah\|_W}{\|h\|_U} = 0.$$

We write $f'(u) := A$ for the Fréchet derivative of f at the point u.

Every Fréchet differentiable function is Gâteaux differentiable and the derivatives coincide. Further, in contrast to Gâteaux differentiability, Fréchet differentiability implies continuity of a function.

Lemma 2.18 (Chain rule [116, Thm. 2.20]). *Let U, V, W be normed linear spaces. Let $g\colon U \to V$ and $f\colon V \to W$ be Fréchet differentiable at $u \in U$ and $g(u)$, respectively. Then $f \circ g$ is Fréchet differentiable at u and it holds*

$$(f \circ g)'(u) = f'(g(u))g'(u).$$

2.4 PDE Theory

Throughout this thesis we will encounter partial differential equations with homogeneous Dirichlet boundary conditions as well as homogeneous Neumann boundary conditions. However, in this chapter we will treat the Neumann case only. Nevertheless, we want to point out that similar results also hold true for the Dirichlet case. For further details we refer to [26, 29, 50, 116].

Let Ω be a bounded Lipschitz domain. Throughout this thesis we will assume that the second-order elliptic operator A satisfies the following properties:

Assumption 2.19. Let A denote the following operator

$$(Ay)(x) := -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij}(x)\partial_{x_i}y(x)) + a_0(x)y(x),$$

where the coefficients a_{ij} with, $1 \le i,j \le d$ and $a_0 \ge 0$ a.e. in Ω are given as functions in $L^\infty(\Omega)$ and satisfy $a_{ij} = a_{ji}$. In addition, we assume that A is a uniform elliptic operator, i.e., there is $\delta > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \delta|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. in } \Omega$$

is satisfied. Furthermore, in the case of Neumann boundary conditions we assume $a_0 \not\equiv 0$ and define the co-normal derivative

$$\partial_{\nu_A} y = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}y(x)\nu_j(x),$$

where ν denotes the outward unit normal vector on Γ.

2.4.1 Elliptic Partial Differential Equations

We start with the investigation of the partial differential equation

$$\begin{aligned} Ay &= u \quad \text{in } \Omega, \\ \partial_{\nu_A} y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.3}$$

Throughout this chapter, we assume the linear elliptic operator A and the co-normal derivative ∂_{ν_A} to satisfy Assumption 2.19. We are searching for a solution of (2.3) in the solution space $Y := H^1(\Omega)$. Later on, we will refer to the solution of this equation as the *state* y. The equation itself will therefore be called *state equation*. First, we introduce the corresponding bilinear form $a : Y \times Y \to \mathbb{R}$ by

$$a(y,v) := \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i} y(x)\partial_{x_j} + a_0(x)y(x) \right) v(x)\,\mathrm{d}x.$$

Then, we define a linear and continuous functional on Y

$$F(v) := \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

A function $y \in Y$ is called a weak solution of (2.3) if it holds

$$a(y,v) = F(v) \quad \forall v \in Y.$$

Based on the Lax-Milgram Lemma, one can deduce existence and uniqueness of weak solutions.

Lemma 2.20 (Lax-Milgram [116, Lem. 2.2]). *Let Y be a real Hilbert space and let $a\colon Y \times Y \to \mathbb{R}$ be a bilinear form. If there exist constants c_1 and c_2 such that a satisfies the following conditions:*

a) Boundedness: $|a(y,v)| \le c_1 \|y\|_Y \|v\|_Y \quad \forall y,v \in Y$,
b) Coercivity: $a(y,y) \ge c_2 \|y\|_Y^2 \quad \forall y \in Y$,

then for every $F \in Y^*$ the equation

$$a(y,v) = F(v) \quad \forall v \in Y$$

admits a unique solution $y \in Y$. Furthermore, there exists a constant c independent of F such that

$$\|y\|_Y \le c\,\|F\|_{Y^*}.$$

Applying Lemma 2.20 we can deduce the following theorem:

Theorem 2.21. *For every $u \in L^2(\Omega)$ the elliptic partial differential equation* (2.3) *admits a unique weak solution $y \in H^1(\Omega)$. Moreover, there exists a constant c independent of u such that the following estimate is satisfied*

$$\|y\|_{H^1(\Omega)} \le c\,\|u\|_{L^2(\Omega)}.$$

However, treating pointwise state constraints requires the continuity of the state y. This is due to the fact that the forthcoming optimality theory is in the need of a convex cone K with non-empty interior. This is satisfied for $Y = C(\overline{\Omega})$. We will give more details in Section 2.5.3. The following theorem yields the desired higher regularity of the state y.

Theorem 2.22. *The elliptic partial differential equation* (2.3) *admits a unique weak solution in $y \in H^1(\Omega) \cap C(\overline{\Omega})$. Moreover, there exists a constant c independent of u such that*

$$\|y\|_{H^1(\Omega)} + \|y\|_{C(\overline{\Omega})} \le c\,\|u\|_{L^2(\Omega)}.$$

If in addition $(u_n)_n$ is such that $u_n \rightharpoonup u$ in $L^2(\Omega)$, then the corresponding solutions $(y_n)_n$ of (2.3) *converge strongly in $H^1(\Omega)$ and $C(\overline{\Omega})$ to the solution y of* (2.3) *to data u.*

Proof. The corresponding proof can be found in Casas [27, Theorem 3.1]. The Dirichlet case is treated in [29, Theorem 2.1]. □

We introduce the *control-to-state mapping* or the *solution operator* $S := A^{-1}$ that maps every $u \in L^2(\Omega)$ to the unique weak solution of (2.3).

$$S\colon L^2(\Omega) \to (H^1(\Omega) \cap C(\overline{\Omega})), \quad u \mapsto y, \quad y = Su. \tag{2.4}$$

The previous theorem allows to derive the following properties of the solution operator.

Theorem 2.23. *The control-to-state mapping* $S\colon L^2(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega})$, $u \mapsto y$ *is a linear, continuous, hence, Fréchet differentiable operator. Moreover, S is compact.*

Proof. The operator S is linear, continuous and completely continuous, see Theorem 2.22. Since $L^2(\Omega)$ is a reflexive Banach space, we can conclude the compactness of S, see [38, Proposition VI.3.3]. □

The complete continuity of the solution operator S is a crucial property. It will be essentially needed for carrying out our convergence analysis. Due to the embedding $\iota\colon H^1(\Omega) \hookrightarrow L^2(\Omega)$ we can consider the solution operator S as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$. However, for the sake of simplicity, we will neglect an explicit specification of the embedding operator ι.

The adjoint operator of the solution operator S from the previous theorem is the solution operator of a PDE itself, the so-called *adjoint equation*.

Lemma 2.24. *Let y be the weak solution of* (2.3) *with associated solution operator S. Then for $z \in L^2(\Omega)$ the function $H^1(\Omega) \ni p := S^*z$ is the solution of*

$$\begin{aligned} A^*p &= z \quad \textit{in}\ \Omega, \\ \partial_{\nu_{A^*}} p &= 0 \quad \textit{on}\ \Gamma. \end{aligned}$$

Proof. We know from Theorem 2.22 that the solution operator $S = A^{-1}$ is linear and continuous. Hence, $(A^*)^{-1} = (A^{-1})^*$ is a linear and continuous operator and we can conclude existence and uniqueness of a solution $p \in H^1(\Omega)$. For Neumann boundary conditions the characterization of p follows directly from [116, Lem. 2.31]. The Dirichlet case can be treated in the same way. □

2.4.2 Elliptic Equations with Measures

Let us now investigate partial differential equations where elements of $\mathcal{M}(\overline{\Omega})$ appear on the right hand side of the equation. We will encounter this type of PDEs during the derivation of optimality conditions of optimal control problems. Let $z \in L^2(\Omega)$, $\mu \in \mathcal{M}(\overline{\Omega})$ be a regular Borel measure that can be split up as $\mu = \mu_\Omega + \mu_\Gamma$ where μ_Ω denotes the restriction of μ on Ω, i.e., $\mu|_\Omega$ and μ_Γ the restriction on Γ. We investigate the partial differential equation

$$\begin{aligned} A^*p &= z + \mu_\Omega \quad \text{in}\ \Omega, \\ \partial_{\nu_{A^*}} p &= \mu_\Gamma \quad \text{on}\ \Gamma. \end{aligned} \tag{2.5}$$

Following Tröltzsch [116, Sec. 7.2.3] and Casas [27], a function $p \in W^{1,s}(\Omega)$, $s \in (1, d/(d-1))$ is called a *very weak solution* of (2.5) if it satisfies

$$\begin{aligned} \int_\Omega A^*v(x)p(x)\,\mathrm{d}x &+ \int_\Gamma \partial_{\nu_{A^*}} v(x)p(x)\,\mathrm{d}s(x) \\ &= \int_\Omega z(x)v(x)\,\mathrm{d}x + \int_\Omega v(x)\,\mathrm{d}\mu_\Omega(x) + \int_\Gamma v(x)\,\mathrm{d}\mu_\Gamma(x), \end{aligned}$$

for all $v \in V^{r,s} := \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid A^*v \in L^r(\Omega), \partial_{\nu_{A^*}} \in L^s(\Gamma)\}$.
The following theorem states existence and uniqueness of the very weak solution of (2.5). This result is due to [27, Thm. 4.3], see also [2]. The Dirichlet case is treated in [26, Thm. 4].

Theorem 2.25. *Let S denote the solution operator* (2.4) *and set $p := S^*(z + \mu) \in L^2(\Omega)$. Then p is the unique very weak solution of* (2.5) *that satisfies $p \in W^{1,s}(\Omega), s \in (1, d/(d-1))$. Moreover, there exists a constant $c > 0$ independent of μ such that*

$$\|p\|_{W^{1,s}(\Omega)} \leq c \left(\|z\|_{L^2(\Omega)} + \|\mu\|_{\mathcal{M}(\overline{\Omega})} \right).$$

2.5 Optimization Theory

The aim of this section is to give a short introduction into optimization theory. We will gather well-established results from literature that are necessary for the reader to follow the subsequent investigations of this thesis. This includes existence of solutions as well as primal and dual optimality conditions. Note that we will just give a brief overview, which is far from being complete. For more details, we refer the reader to the books [6, 19, 39, 43, 62, 90, 116].

We first focus on the minimization problem of the general type

$$\underset{u \in U}{\text{minimize}} \ f(u) \quad \text{subject to} \quad u \in \mathcal{F} \subseteq U. \tag{2.6}$$

Here, U is a Banach space and $f \colon U \to \overline{\mathbb{R}}$ a given mapping. The set $\mathcal{F}$ is called the *feasible set*. A point $u \in \mathcal{F}$ is a *feasible point*. Later on, we will focus on feasible sets given by

$$\mathcal{F} := \{u \in U \mid u \in C,\ g(u) \in K\},$$

where $C \subseteq U$ and $K \subseteq Y$ are non-empty, closed, convex sets and Y is a Banach space. Moreover, $g \colon U \to Y$ is continuously Fréchet differentiable.

2.5.1 Cones

In this section we will introduce some basic concepts concerning various types of cones that are needed for our optimization theory. Let $K \subset U$ be a non-empty set. We call the set K a *cone* if $y \in K$ implies $cy \in K$ for all positive $c \in \mathbb{R}$. Moreover, for an arbitrary set K we define the *dual* and the *polar cone* via

$$K^* := \{\phi \in U^* \mid \langle \phi, u \rangle \geq 0,\ \forall u \in K\} \qquad \text{(Dual cone)}$$
$$K^\circ := \{\phi \in U^* \mid \langle \phi, u \rangle \leq 0,\ \forall u \in K\} \qquad \text{(Polar one).}$$

Obviously, it holds $K^* = -K^\circ$.

Definition 2.26 (Tangent cone, normal cone and radial cone). Let $C \subseteq U$ denote an arbitrary set.

a) For $u \in U$ the *tangent cone* of C at u is defined by

$$\mathcal{T}_C(u) := \left\{ d \in U \mid \exists (u_k)_k \subseteq U, \exists (t_k)_k \searrow 0 : u_k \to u, \frac{u_k - u}{t_k} \to d \right\}.$$

Let $C \subseteq U$ be a convex set. We define

b) the *normal cone* $\mathcal{N}_C(u)$ of C at u as $\mathcal{N}_C(u) := \{\phi \in U^* \mid \langle \phi, v - u \rangle \leq 0\ \forall v \in C\}$.

c) the *radial cone* $\mathcal{R}_C(u)$ of C at u as $\mathcal{R}_C(u) := \{d \in C \mid d = \alpha(v - u)\ \forall v \in C, \alpha > 0\}$.

Whenever C is convex we have $\mathcal{T}_C(u) = \overline{\mathcal{R}_C(u)}$. Using this representation we obtain

$$\mathcal{N}_C(u) = \mathcal{T}_C(u)^\circ. \tag{2.7}$$

We will use this representation to reformulate the optimality conditions in a more suitable way, see Theorem 2.39. The following lemma gives a characterization of the inclusion $\mu \in \mathcal{T}_K(y)^\circ$, $y \in K$ for a convex cone K.

Lemma 2.27 ([11, Ex. 6.39]). *Let $K \subseteq Y$ be a non-empty convex cone. Further, let $y \in K$ and $\mu \in Y^*$. Then it holds, that $\mu \in \mathcal{T}_K(y)^\circ$ if and only if $\mu \in K^\circ$, $\langle \mu, y \rangle_{Y^*,Y} = 0$.*

A convex cone K induces the order relation

$$y \leq_K v \quad \Leftrightarrow \quad y - v \in K.$$

We say that a function g is *convex with respect to* K if

$$g(\lambda u_1 + (1 - \lambda u_2)) \leq_K \lambda g(u_1) + (1 - \lambda) g(u_2) \quad \forall u_1, u_2 \in U, \lambda \in [0, 1].$$

A class of functions that satisfy this *K-convexity* are affine linear functions. K-convexity coincides with the standard definition of convexity if K is the convex cone of non-positive real numbers.

Lemma 2.28 ([90, 8.2 Prop. 2]). *Let $g \colon U \to Y$ be convex with respect to the convex cone $K \subseteq Y$. Then the set $\{u \in U \mid g(u) \in K\}$ is convex.*

2.5.2 Existence of Solutions

In this section we will collect existence results for the solution of the minimization problem (2.6).

Definition 2.29 (Global and local solution). A function $\bar{u} \in \mathcal{F}$ is called

a) *global solution* or *optimal solution* of (2.6) if it satisfies $f(\bar{u}) \leq f(u)$ for all $u \in \mathcal{F}$.
b) *local solution* or *local optimal solution* of (2.6) if there exists a $\delta > 0$ such that for all $u \in \mathcal{F}$ with $\|u - \bar{u}\|_U \leq \delta$ it holds $f(\bar{u}) \leq f(u)$.

Weak lower semicontinuity and weak coercivity are crucial properties for proving existence of solutions.

Definition 2.30 (Weak lower semicontinuity). We say that a function $f \colon U \to \overline{\mathbb{R}}$ is *weakly lower semicontinuous* (w.l.s.c.) in $u \in U$ if for all weak convergent sequences $u_n \rightharpoonup u$ it holds $f(u) \leq \liminf_{n\to\infty} f(u_n)$. If f is w.l.s.c. in every $u \in U$, we say that f is w.l.s.c. in U.

Definition 2.31 (Weak coercivity). A function $f \colon U \to \overline{\mathbb{R}}$ is called *weakly coercive* if for every sequence $(u_n)_n \subset U$ with $\|u_n\| \to \infty$ it holds $\lim_{\|u_n\|_U \to \infty} f(u_n) = \infty$.

Theorem 2.32 (Weierstrass [11, Thm. 1.28][43, II Prop. 1.2]). *Let U be a normed space and $\mathcal{F} \subseteq U$ a non-empty, closed, convex set. Further, let $f \colon U \to \overline{\mathbb{R}}$ be proper and weakly lower semicontinuous. If either (i) $\mathcal{F}$ is weakly compact or (ii) U is a reflexive Banach space and f is weakly coercive on $\mathcal{F}$, then* (2.6) *has a global solution.*

Since every convex and continuous function in a Banach space is weakly lower semicontinuous, we arrive directly at the following corollary:

Corollary 2.33 ([43, Prop. 1.2]). *The weak lower semicontinuity of f in Theorem 2.32 can be replaced by convexity combined with continuity. Moreover, if f is strictly convex and continuous, then the solution is unique.*

2.5.3 Optimality Conditions

In this section we will derive necessary and sufficient optimality conditions.

Primal Optimality Conditions

The tangent cone can be used to establish first-order optimality conditions for constrained optimization problems.

Lemma 2.34 ([62, Thm. 1.5.2]). *Let U be a normed space, $\mathcal{F} \subseteq U$ be a closed, convex set and $f\colon U \to \mathbb{R}$ a proper, Fréchet differentiable function. If $\bar{u}$ is a local solution of* (2.6)*, then it holds*

$$\langle f'(\bar{u}), d\rangle_{U^*,U} \geq 0 \quad \forall d \in \mathcal{T}_{\mathcal{F}}(\bar{u}). \tag{2.8}$$

Condition (2.8) can be reformulated in $-f'(\bar{u}) \in \mathcal{T}_{\mathcal{F}}(\bar{u})^\circ$. However, this condition is rather abstract and hard to analyze. If f is a proper, Gâteaux differentiable function and $\mathcal{F}$ a convex set, minima can be characterized in a more concrete way.

Lemma 2.35 ([116, Lem. 2.21]). *Let U be a normed space, $f\colon U \to \overline{\mathbb{R}}$ a proper, Gâteaux differentiable function and $\mathcal{F} \subseteq U$ a convex set.*

a) If $\bar{u}$ is a local minimum of problem (2.6)*, then it holds*

$$D_G f(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in \mathcal{F}. \tag{2.9}$$

b) If f is additionally convex, then $\bar{u}$ is a global solution of problem (2.6) *if and only if* (2.9) *is satisfied.*

To end this section, let us briefly consider problems that are not even Gâteaux differentiable. If $f\colon U \to \overline{\mathbb{R}}$ is a convex function, minima can be characterized with the help of the (convex) subdifferential, see Definition 2.11. Reformulating the constrained minimization problem (2.6), we aim at minimizing the function $\tilde{f}(u) := f(u) + \delta_{\mathcal{F}}$ over $u \in U$. Hence, (2.6) turned into a (convex), unconstrained minimization problem whose minima can be characterized with Fermat's Theorem, which is a direct consequence of the definition of the subdifferential.

Theorem 2.36 (Fermat's Theorem). *Let $f\colon U \to \overline{\mathbb{R}}$ be proper. Then $\bar{u}$ is a minimizer of f if and only if $0 \in \partial f(\bar{u})$.*

A popular example is the optimal control problem with sparse controls

$$\underset{u \in L^2(\Omega)}{\text{minimize}} \; f(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \beta\|u\|_{L^1(\Omega)} \quad \text{subject to} \quad u \in U_{\text{ad}}, \quad Su \leq \psi,$$

where $\beta \geq 0$. This problem will be treated in Chapter 4. We can apply Fermat's Theorem 2.36 to the minimization problem (2.6) and derive directly first-order necessary optimality conditions.

Lemma 2.37. *Let $f\colon U \to \overline{\mathbb{R}}$ be a proper, convex function and $\mathcal{F}$ a convex set. Assume that there exists $\hat{u} \in \operatorname{int}(\mathcal{F}) \cap \operatorname{dom}(f)$. Then, $\bar{u}$ is a global solution of* (2.6) *if and only if for all $g \in \partial f(\bar{u})$*

$$\langle g, v - \bar{u}\rangle_{U^*,U} \geq 0 \quad \forall v \in \mathcal{F}.$$

Since $\partial f = \{D_G f\}$ for Gâteaux differentiable functions, the optimality conditions from Lemma 2.37 coincide with those from Lemma 2.35. However, the existence of an inner point of the feasible set F_{ad} is crucial and not trivial. We will see in the next section that this assumption, the so-called *Slater condition*, is also a suitable regularity condition for deriving existence of Lagrange multipliers .

Primal-Dual Optimality Conditions

In this section we will apply the Karush-Kuhn-Tucker theory in order to derive first-order optimality conditions. Under a suitable constraint qualification we will show existence of Lagrange multipliers. Let us recall the minimization problem

$$\text{minimize } f(u) \quad \text{subject to} \quad u \in \mathcal{F} = \{u \in U \mid u \in C,\, g(u) \in K\}. \tag{2.10}$$

Throughout this section we assume that U and Y are Banach spaces. Further, f and g are Fréchtet differentiable functions and the sets $C \subseteq U$ and $K \subseteq Y$ are non-empty, closed, convex sets.

The optimality conditions from the previous section only consider the primal variables. Incorporating the special structure of the set $\mathcal{F}$ we arrive at the following definition:

Definition 2.38 (KKT-Point). A tuple $(\bar{u}, \bar{\mu}, \bar{v}) \in U \times Y^* \times U^*$ is called *Karush-Kuhn-Tucker* (KKT) *point* of (2.10) if it satisfies

$$\bar{u} \in \mathcal{F}, \qquad f'(\bar{u}) + g'(\bar{u})^*\bar{\mu} + \bar{v} = 0, \qquad \bar{\mu} \in \mathcal{T}_K(g(\bar{u}))^\circ, \qquad \bar{v} \in \mathcal{T}_C(\bar{u})^\circ. \tag{2.11}$$

The elements $\bar{\mu}$ and $\bar{v}$ are called *Lagrange multipliers*.

If K is a convex cone, the characterization (2.7) and Lemma 2.27 make it possible to reformulate the KKT conditions.

Theorem 2.39. *Let $K \subseteq Y$ be a non-empty convex cone. Then for any feasible point $\bar{u} \in \mathcal{F}$ the KKT conditions* (2.11) *are equivalent to*

$$\begin{aligned} \langle f'(\bar{u}) + g'(\bar{u})^*\bar{\mu}, u - \bar{u})\rangle_{U^*,U} \geq 0 \quad \forall u \in C,\\ \langle \bar{\mu}, g(\bar{u})\rangle_{Y^*,Y} = 0, \quad \bar{\mu} \in K^\circ. \end{aligned}$$

Definition 2.40 (Zowe-Kurcyusz constraint qualification). Let $\bar{u} \in \mathcal{F}$ be a feasible point. We say that

a) the *Zowe-Kurcyusz constraint qualification* (ZKCQ) is satisfied in $\bar{u}$ if

$$Y = g'(\bar{u})\mathcal{R}_C(\bar{u}) - \mathcal{R}_K(g(\bar{u})).$$

b) the *Robinson constraint qualification* (RCQ) is satisfied in $\bar{u}$ if

$$0 \in \operatorname{int}\left(g(\bar{u}) + g'(\bar{u})(C - \bar{u}) - K\right).$$

In 1979 Zowe and Kurcyusz proved the generalized open mapping theorem [122, Theorem 2.1]. Applying this result, it can be shown that these two constraint qualifications are equivalent.

Theorem 2.41 ([122]). *For a feasible point $\bar{u} \in \mathcal{F}$, the Zowe-Kurcyusz condition and the Robinson condition are equivalent.*

These constraint qualifications allow us to achieve existence of Lagrange multipliers.

Theorem 2.42 ([122, Thm. 4.1]). *Let $\bar{u}$ be a local solution of the minimization problem* (2.10) *and ZKCQ be satisfied. Then there exist Lagrange multipliers $(\bar{\mu}, \bar{v}) \in Y^* \times U$ such that* (2.11) *is satisfied.*

The Zowe-Kurcyusz condition is a very abstract condition. The following constraint qualification is often easier to verify. From now on we assume K to be a convex cone.

Definition 2.43 (Linearized Slater condition). A feasible point $\bar{u} \in \mathcal{F}$ satisfies the *linearized Slater condition* if there exists $\hat{u} \in C$ with

$$g(\bar{u}) + g'(\bar{u})(\hat{u} - \bar{u}) \in \operatorname{int}(K).$$

Lemma 2.44 ([19, Lem. 2.99]). *If $\bar{u} \in \mathcal{F}$ satisfies the linearized Slater condition, then ZKCQ is satisfied in $\bar{u}$. The converse is true, if $\operatorname{int}(K)$ is non-empty.*

For the remaining part of this section, we will assume that f is a convex function and g is convex with respect to K.

Theorem 2.45 ([62, Thm. 1.55]). *Let g be convex with respect to K and assume that there exists $\hat{u} \in C$ such that*

$$g(\hat{u}) \in \operatorname{int}(K). \tag{2.12}$$

Then the linearized Slater condition is satisfied in every feasible point $\bar{u} \in \mathcal{F}$. In particular, ZKCQ is satisfied for every feasible point.

Condition (2.12) is called *Slater condition*. In opposite to all other constraint qualification the Slater condition does not depend on the optimal solution $\bar{u}$, which is a huge advantage. However, it still requires necessarily that the interior of the convex cone K is non-empty. This assumption is crucial and by far not trivial, since it is often not satisfied. A popular example is the cone

$$K := L^2(\Omega)_- := \{y \in L^2(\Omega) \mid y \leq 0 \text{ a.e. in } \Omega\},$$

which does not contain an inner point, see [116, Chap. 6]. However, the space of non-positive continuous functions $C(\overline{\Omega})$ is an appropriate candidate instead. This makes it reasonable to consider pointwise inequality constraints of the type $y \leq \psi$, with $y, \psi \in C(\overline{\Omega})$. Finally, the following result states that, for convex functions, the KKT conditions are not only necessary but also sufficient.

Theorem 2.46. *Let $f\colon U \to \mathbb{R}$ be convex and $g\colon U \to Y$ be convex with respect to K. Let $(\bar{u}, \bar{\mu}, \bar{v})$ denote a KKT point of (2.10). Then $\bar{u}$ is a global solution of (2.10).*

Proof. We chose an arbitrary point $u \in \mathcal{F}$. Since f is convex we arrive with Lemma 2.16 at

$$f(u) - f(\bar{u}) \geq f'(\bar{u})(u - \bar{u}) = -\langle \bar{\mu}, g'(\bar{u})(u - \bar{u})\rangle - \langle \bar{v}, u - \bar{u}\rangle.$$

We know that $u - \bar{u} \in \mathcal{R}_C(\bar{u}) \subset \mathcal{T}_C(\bar{u})$ and $\bar{v} \in \mathcal{T}_C(\bar{u})^\circ$. Hence, we obtain $\langle \bar{v}, u - \bar{u}\rangle \leq 0$. The convexity of g implies $g(\bar{u} + t(u - \bar{u})) \in K, \forall t \in [0,1]$. The Fréchet differentiability of g implies $\lim_{t\to 0} \frac{1}{t}\left(g(\bar{u} + t(u - \bar{u})) - g(\bar{u})\right) = g'(\bar{u})(u - \bar{u})$ and by definition of the tangent cone $g'(\bar{u})(u - \bar{u}) \in \mathcal{T}_K(g(\bar{u}))$. Together with $\bar{\mu} \in \mathcal{T}_K(\bar{u})^\circ$ this shows $\langle \bar{\mu}, g'(\bar{u})(u - \bar{u})\rangle \leq 0$. Putting all together we obtain $f(u) \geq f(\bar{u})$ for all feasible u. Thus, $\bar{u}$ is a global solution of (2.10). □

2.6 The Augmented Lagrangian Method

Following [114] this section aims at giving a short introduction in augmented Lagrangian methods in Banach spaces.

The Method of Multipliers for Equality Constraints

In 1969 Hestenes [52] and Powell [101] provided an efficient method, the *method of multipliers*, to solve finite-dimensional minimization problems, which include equality constraints. Adapting the general framework from finite dimensions, we consider a Banach space U, a Hilbert space H and the minimization problem

$$\underset{u\in C}{\text{minimize}} \;\; f(u) \quad \text{subject to} \quad h(u) = 0. \tag{2.13}$$

Further, $f\colon U \to \mathbb{R}$, $h\colon U \to H$ are given mappings and $C \subseteq U$ is a non-empty closed convex set. Penalizing the violation of the equality constraint, we investigate the *augmented Lagrangian*

$$\mathcal{L}_{AL}(u,\mu,\rho) := \underbrace{f(u) + (\mu, h(u))}_{=:\mathcal{L}(u,\mu)} + \frac{\rho}{2}\, \|h(u)\|_H^2 ,$$

where $\mathcal{L}(u,\mu)$ denotes the Lagrangian function of (2.13) and $\rho \in \mathbb{R}$ a positive parameter, the so-called *penalty parameter*. Assuming that the functions f and h are continuously differentiable and μ and ρ are fixed, we obtain from Lemma 2.35 that a solution $\bar{u}$ of $\min_{u\in C} \mathcal{L}_{AL}(u,\mu,\rho)$ has to satisfy

$$\begin{aligned} & (\mathcal{L}'_{AL}(\bar{u},\mu,\rho), v-\bar{u}) \geq 0 \quad \forall v \in C \\ \Leftrightarrow \quad & (f'(\bar{u}) + h'(\bar{u})^*(\mu + \rho h(\bar{u})), v - \bar{u}) \geq 0 \quad \forall v \in C. \end{aligned}$$

Comparing this inequality with the first-order conditions for the Lagrangian function of (2.13) given by $(f'(\bar{u}) + h'(\bar{u})^*\mu, v-\bar{u}) \geq 0 \;\forall v \in C$, this immediately suggests to consider

$$\bar{\mu} := \mu + \rho h(u) \tag{2.14}$$

as a reasonable estimate for the Lagrange multiplier. Choosing an initial value for μ_1 and ρ_1, the method of multipliers now basically consists of the following two steps: For given μ_k and ρ_k one computes the minimizer of the corresponding augmented Lagrangian. After that, the multiplier is updated according the *Hestenes-Powell multiplier update* from (2.14). Hence, we obtain

$$\begin{aligned} u_k &= \arg\min_{u\in C} \mathcal{L}_{AL,k}(u,\mu_k,\rho_k), \\ \mu_{k+1} &= \mu_k + \rho_k h(u_k). \end{aligned}$$

It remains to fix an update rule for the penalty parameter. Usually, if the violation of the equality constraints shows sufficient decrease, it is reasonable to keep the current value of ρ. Otherwise the penalty parameter will be increased by a factor $\theta > 1$.

The Method of Multipliers for Inequality Constraints

Let us now adapt the concept for equality constraints to problems of the type

$$\text{minimize} \;\; f(u) \quad \text{subject to} \quad u \in \mathcal{F} = \{u \in U \mid u \in C,\; g(u) \in K\},$$

where $f\colon U \to \mathbb{R}$ and $g\colon U \to Y$ are given mappings. Further, $C \subseteq U$ is a non-empty closed convex set and $K \subseteq Y$ is a closed convex cone. Assuming that $E\colon Y \hookrightarrow H$ densely and $\mathcal{K} \subseteq H$ is a closed convex set with $E^{-1}(\mathcal{K}) = K$, we are allowed to equivalently investigate the problem

$$\text{minimize} \;\; f(u) \quad \text{subject to} \quad u \in \mathcal{F} = \{u \in U \mid u \in C,\; g(u) \in \mathcal{K}\}. \tag{2.15}$$

In order to apply the theory of the original method of multipliers, we need to reformulate (2.15) as an equality constrained problem. To do so, we introduce the *slack variable* $s \in \mathcal{K}$ and consider the equality constrained problem

$$\underset{(u,s)\in C\times\mathcal{K}}{\text{minimize}}\ f(u) \quad \text{subject to} \quad g(u) - s = 0.$$

Defining $h\colon U \times H \to H,\ h(u,s) := g(u) - s$, the corresponding augmented Lagrangian on the space $U \times H$ is given by

$$\begin{aligned}\mathcal{L}^s_{AL}(u,s,\mu,\rho) &= f(u) + (\mu, h(u,s)) + \frac{\rho}{2}\left\|h(u,s)\right\|_H^2 \\ &= f(u) + \frac{\rho}{2}\left\|\frac{\mu}{\rho} + g(u) - s\right\|_H^2 - \frac{\|\mu\|_H^2}{2\rho}.\end{aligned}$$

Minimizing the last formula for each fixed $u \in U$ with respect to $s \in \mathcal{K}$, results in the following augmented Lagrangian function, which is only dependent on u and μ:

$$\mathcal{L}_{AL}\colon U \times H \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{L}_{AL}(u,\mu,\rho) := f(u) + \frac{\rho}{2}\mathrm{dist}^2\left(\frac{\mu}{\rho} + g(u), \mathcal{K}\right) - \frac{\|\mu\|_H^2}{2\rho},$$

where $\mathrm{dist}(\cdot,\mathcal{K}) = \inf_{s\in\mathcal{K}}\|\cdot - s\|_H$. Moreover, for fixed multiplier μ and penalty parameter ρ, the element $\bar{s}(u) := P_{\mathcal{K}}(g(u) + \frac{\mu}{\rho})$ is the minimal value of s. Thus,

$$h(u,\bar{s}(u)) = g(u) - P_{\mathcal{K}}\left(g(u) + \frac{\mu}{\rho}\right)$$

and for the current iterates u_{k+1}, μ_k and ρ_k the multiplier update corresponding to (2.14) yields

$$\mu_{k+1} = \mu_k + \rho_k h(u_{k+1},\bar{s}(u_{k+1})) = \rho_k\left(g(u_{k+1}) + \frac{\mu_k}{\rho_k} - P_{\mathcal{K}}\left(g(u_{k+1}) + \frac{\mu_k}{\rho_k}\right)\right). \tag{2.16}$$

Applying the classical method of multipliers results in the following general algorithm.

Algorithm 2.1 Original Augmented Lagrangian Method for Cone Constraints

Let $(u_0,\mu_1) \in U \times H$ and $\rho_1 > 0$ be given. Choose $\theta > 1, \tau \in (0,1)$ and set $k := 1$.

1: Compute a solution u_k of the problem $\min\limits_{u\in U} \mathcal{L}_{AL,k}(u,\mu_k,\rho_k)$.
2: Update the multiplier μ_{k+1} according to (2.16).
3: If it holds $\|h(u_k,\bar{s}(u_k))\|_H \le \tau\|h(u_{k-1},\bar{s}(u_{k-1}))\|_H$, set $\rho_{k+1} := \rho_k$. Otherwise, increase the penalty parameter $\rho_{k+1} := \theta\rho_k$.
4: If the stopping criterion is not satisfied, set $k := k+1$ and go to step 1.

However, for infinite dimensional problems Algorithm 2.1 is not directly applicable. First, the convergence analysis of the algorithm requires a certain boundedness property of the Lagrange multiplier, which is not guaranteed by the basic multiplier method. Moreover, at least in finite dimensions, augmented Lagrangian methods possess the advantageous property that the penalty remains bounded, i.e., the penalty parameter has to be increased only finitely many times. Since a penalty parameter tending to infinity may cause heavy numerical instabilities during numerical computations it is favourable to transfer this property to the infinite dimensional setting.

Conn et al. [36, 37] introduced an algorithm for finite-dimensional nonlinear optimization problems, which is constructed in such a way that the penalty parameter remains bounded. This algorithm differs from the original method of multipliers by distinguishing after each iteration whether the step has been successful or not. If the step has been successful, the multiplier is updated, while the penalty parameter remains unchanged. Otherwise, if the step has been not successful, the multiplier is not updated, but the penalty parameter is increased. Based on this finite-dimensional investigations, we adapted this procedure to the infinite dimensional case and end up with the following general algorithm for inequality constrained optimization problems.

Algorithm 2.2 Augmented Lagrangian Method for Cone Constraints

Let $(u_0, \mu_1) \in U \times H$ and $\rho_1 > 0$ be given. Choose $\theta > 1, \tau \in (0, 1)$ and set $k := 1$.

1: Compute a solution u_k of the problem $\min_{u \in C} \mathcal{L}_{AL,k}(u, \mu_k, \rho_k)$.

2: If the step is successful, update the multiplier $\mu_{k+1} := \rho_k \left(g(u_k) + \frac{\mu_k}{\rho_k} - P_{\mathcal{K}} \left(g(u_k) + \frac{\mu_k}{\rho_k} \right) \right)$ and set $\rho_{k+1} := \rho_k$.

3: Otherwise, increase the penalty parameter $\rho_{k+1} := \theta \rho_k$.

4: If a certain stopping criterion is not satisfied, set $k := k + 1$ and go to step 1.

Let us point out the main difference between Algorithm 2.2 and the original method from Algorithm 2.1. The Lagrange multiplier is not updated in every iteration of the algorithm, but only if a step is considered to be successful. To determine if this is the case, we will basically check on a reasonable measure of feasibility and complementarity. Moreover, we aim to choose our update rule from Step 2 and Step 3 in such a way that the sequence of multipliers $(\mu_k)_k$ is bounded in $L^1(\Omega)$. This is crucial to obtain a weak-* convergent subsequence of multipliers in $\mathcal{M}(\overline{\Omega})$. We will specify the algorithm applied to pointwise state constrained optimal control problems in Section 2.7.3.

Another way to handle this challenge has been proposed in [73], see also [114]. Here, the authors apply the multiplier update rule from the original method of multipliers. Thus, in every iteration of the algorithm the subproblem consists in solving

$$\underset{u \in C}{\text{minimize}}\ \mathcal{L}_{AL,k}(u, w_k, \rho_k),$$

and updating the Lagrange multiplier via

$$\mu_{k+1} := \rho_k \left(g(u_k) + \frac{w_k}{\rho_k} - P_{\mathcal{K}} \left(g(u_k) + \frac{w_k}{\rho_k} \right) \right).$$

Here, $w_k \in H$ is an element of a bounded set $B \subset H$ instead of the multiplier μ_k. In practice, the authors choose $w_k := P_B(\mu_k)$ as the so-called *safeguarded multiplier sequence*. In this way, $(w_k)_k$ is a bounded sequence, which is, in this case, the main ingredient to obtain suitable convergence results.

2.7 The Optimal Control Problems – Basic Results

This section aims at collecting basic results that are needed for the discussion of PDE constrained optimal control problems with pointwise state constraints. Hereby, we restrict ourselves to the case that the solution operator S is linear, i.e., the underlying partial differential equation is linear. For results concerning nonlinear PDEs we refer the reader to Chapter 5. For more details about optimal control theory we refer the reader to the books [39, 62, 116]. Moreover, we will specify the augmented Lagrangian algorithm that will be used to solve state constrained optimal control problems.

2.7.1 The State Constrained Optimal Control Problem

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$ denote an open bounded domain with Lipschitz boundary. We start by introducing the optimal control problem given in its reduced formulation

$$\begin{aligned} \underset{u \in L^2(\Omega)}{\text{minimize}} \quad & f(u) := \frac{1}{2} \|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_a(x) \le u(x) \le u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \le \psi(x) \qquad \text{in } \overline{\Omega}, \end{aligned} \tag{P}$$

where $y_d \in L^2(\Omega)$ and α is a positive regularization parameter. The upper and lower constraints on the control u satisfy $u_a(x) \le u_b(x)$ and are elements of $L^2(\Omega)$, while ψ is given in $C(\overline{\Omega})$. The operator S denotes the control-to-state mapping of the linear elliptic partial differential equation (2.3). Thus, by Theorem 2.23, $S \colon L^2(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega})$ is linear, continuous and compact. In particular, it is completely continuous and Fréchet differentiable. We are searching for a control $u \in U := L^2(\Omega)$ with associated state $y \in Y := H^1(\Omega) \cap C(\overline{\Omega})$. We define the *admissible* and *feasible* set

$$\begin{aligned} U_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u_a(x) \le u(x) \le u_b(x) \text{ a.e. in } \Omega\}, \\ F_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u \in U_{\text{ad}},\ Su(x) \le \psi \text{ in } \overline{\Omega}\}. \end{aligned}$$

It is easy to see, that the set U_{ad} is non-empty, bounded, closed and convex, hence weakly compact. For S linear, it is straight forward to prove that the feasible set is also weakly compact.

Theorem 2.47. *Assume that the set F_{ad} is non-empty. Then, F_{ad} is weakly compact and* (*P*) *has a unique global solution.*

Proof. The boundedness of U_{ad} implies boundedness of F_{ad}. Moreover, the complete continuity of S allows us to conclude closedness of F_{ad}. The linearity and continuity of S implies that $g(u) := Su - \psi$ is convex with respect to the closed convex cone of non-positive functions. Thus, Lemma 2.28 yields convexity of F_{ad} and we obtain its weak compactness. Due to the linearity of S, the objective function f is continuous and strictly convex ($\alpha > 0$), hence, weakly lower semicontinuous and we can apply Corollary 2.33 to obtain existence of a unique solution. □

However, if S is a nonlinear operator, for instance the solution operator of a semilinear partial differential equation, then the feasible set F_{ad} is still closed, but not convex. Nevertheless, one can conclude its weak compactness and apply Theorem 2.32.

Let us establish first-order necessary optimality conditions for the convex problem (*P*). Due to convexity, these conditions are also sufficient.

Theorem 2.48 (Primal first-order optimality conditions). *The control $\bar{u}$ is a minimizer of* (*P*) *if and only if*

$$(S^*(S\bar{u} - y_d) + \alpha\bar{u}, u - \bar{u}) \ge 0 \quad \forall u \in F_{\text{ad}}.$$

Proof. We know from the proof of Theorem 2.47 that F_{ad} is convex. Thus, we can apply Lemma 2.35. □

However, this formulation does not reveal the difficulties that are arising from the pure state constraints, since the corresponding constraints are hidden in the feasible set F_{ad}. This is why a further investigation of the optimality conditions with the help of Lagrange multipliers is helpful. For this purpose, the fulfilment of a suitable constraint qualification is required. For problem (*P*) the Slater condition (2.12) is a convenient choice.

Theorem 2.49 (First-order optimality conditions with multipliers). *Assume that there exists $\hat{u} \in U_{\text{ad}}$ and $\sigma > 0$ such that*

$$S\hat{u}(x) \leq \psi(x) - \sigma \quad \text{in } \overline{\Omega} \tag{2.17}$$

is satisfied. Then, $\bar{u}$ is a solution of (P) if and only if there exists an adjoint state $\bar{p} \in W^{1,s}(\Omega), s \in (1, d/(d-1))$ and a Lagrange multiplier $\bar{\mu} \in \mathcal{M}(\overline{\Omega})$ such that the following optimality system is satisfied:

$$\begin{aligned} \bar{y} = S\bar{u}, \qquad \bar{p} =& S^*(S\bar{u} - y_d + \bar{\mu}), \\ (\bar{p} + \alpha\bar{u}, u - \bar{u})) \geq& 0 \quad \forall u \in U_{\text{ad}}, \\ \langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} =& 0, \quad \bar{\mu} \geq 0, \quad \bar{y} \leq \psi. \end{aligned}$$

Here, the property $\bar{\mu} \geq 0$ means that $\langle \bar{\mu}, \xi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \geq 0$ for all $\xi \in C(\overline{\Omega})$ with $\xi \geq 0$, i.e., $\bar{\mu}$ lies in the dual of the non-negative cone $C(\overline{\Omega})_+$.

Proof. We set $U := L^2(\Omega)$, $Y := H^1(\Omega) \cap C(\overline{\Omega})$, $C := U_{\text{ad}}$, $g(u) := Su - \psi$, and $K := C(\overline{\Omega})_- := \{u \in C(\overline{\Omega}) \mid u(x) \leq 0 \text{ in } \overline{\Omega}\}$. Let $\bar{u}$ denote a solution of (P). Since we assumed that the Slater condition is satisfied, we can conclude from Theorem 2.45 and Lemma 2.44 that ZKCQ is satisfied in every feasible point. Hence, existence of a Lagrange multiplier $\bar{\mu} \in Y^*$ follows with Theorem 2.42. Since K is a non-empty convex cone we arrive with Theorem 2.39 at

$$\begin{aligned} (S^*(S\bar{u} - y_d + \bar{\mu} + \alpha\bar{u}, u - \bar{u}) \geq 0 \quad \forall u \in U_{\text{ad}}, \\ \langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{\mu} \in C(\overline{\Omega})_-^\circ. \end{aligned}$$

The definition of the polar cone yields

$$\begin{aligned} C(\overline{\Omega})_-^\circ &= \{\phi \in Y^* \mid \langle \phi, \xi \rangle \leq 0, \forall \xi \in C(\overline{\Omega})_-\} \\ &= \{\phi \in Y^* \mid \langle \phi, \xi \rangle \geq 0, \forall \xi \in C(\overline{\Omega})_+\} = C(\overline{\Omega})_+^*. \end{aligned}$$

Thus, $\mu \in C(\overline{\Omega})_+^*$. Introducing the adjoint states $\bar{p} := S^*(S\bar{u} - y_d + \bar{\mu})$, we end up with the desired optimality system. Casas [27] showed that the adjoint state is an element of the space $W^{1,s}$, $s \in (1, d/(d-1))$. Theorem 2.46 shows that the KKT conditions are also sufficient. □

Remark 2.50. By Theorem 2.25 we know that the adjoint state $\bar{p}$ from Theorem 2.49 is the very weak solution of the adjoint equation

$$\begin{aligned} A^*\bar{p} &= \bar{y} - y_d + \bar{\mu}_\Omega && \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p} &= \bar{\mu}_\Gamma && \text{on } \Gamma. \end{aligned}$$

In general, the Lagrange multiplier $\bar{\mu}$ and thus the adjoint state $\bar{p}$ from Theorem 2.49 need not to be unique. A sufficient condition for uniqueness of the adjoint state $\bar{p}$ and the Lagrange multiplier $\bar{\mu}$ is given by a certain separation condition on the active sets, see [80, Lemma 1]. There the following result was proven:

Lemma 2.51. *Let $\bar{u}$ be an optimal control of (P) and let (2.17) be satisfied. Moreover, suppose there exists $\delta > 0$ such that the active sets*

$$\begin{aligned} A_y &= \{x \in \overline{\Omega} \mid \bar{y}(x) = \psi(x)\} \\ A_u &= \{x \in \overline{\Omega} \mid \bar{u}(x) = u_a(x) \ \vee \ \bar{u}(x) = u_b(x)\} \end{aligned}$$

satisfy $\operatorname{dist}(A_y, A_u) \geq \delta$, i.e., the active sets are well separated. Then, the corresponding adjoint state $\bar{p}$ and the Lagrange multiplier $\bar{\mu}$ are uniquely determined.

2.7.2 The Augmented Lagrangian Subproblem

We consider the augmented Lagrangian subproblem

$$\begin{aligned} \underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad & f_{AL}(u_\rho,\mu,\rho) := f(u_\rho) + \frac{1}{2\rho}\left\|(\mu + \rho(Su_\rho - \psi))_+\right\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_\rho \in U_{\text{ad}}, \end{aligned} \tag{P_{AL}}$$

where S denotes the same solution operator as given in the unregularized problem (P). Further, $\alpha > 0$, ρ is a positive penalization parameter and $\mu \in L^2(\Omega)$.

Theorem 2.52 (Existence of an optimal control). *The optimization problem* (P_{AL}) *admits a unique global solution.*

Proof. The set U_{ad} is weakly compact. Since S is linear and continuous, the cost functional f_{AL} is strictly convex ($\alpha > 0$) and continuous in u_ρ. Hence, the claim follows by Corollary 2.33. □

First-order optimality conditions can now directly be established by Lemma 2.35. Thanks to the convexity of (P_{AL}), these conditions are also sufficient.

Theorem 2.53 (First-order optimality conditions). *The control* $\bar{u}_\rho$ *is a global solution of* (P_{AL}) *if and only if*

$$(S^*(S\bar{u}_\rho - y_d + (\mu + \rho(S\bar{u}_\rho - \psi))_+) + \alpha\bar{u}_\rho, u - \bar{u}_\rho) \geq 0 \quad \forall u \in U_{\text{ad}},$$

which is equivalent to the existence of $\bar{p}_\rho \in H^1(\Omega)$ *and* $\bar{\mu}_\rho \in L^2(\Omega)$ *such that*

$$\begin{aligned} \bar{y}_\rho = S\bar{u}_\rho, \qquad \bar{p}_\rho = S^*(S\bar{u}_\rho - y_d + \bar{\mu}_\rho), \\ (\bar{p}_\rho + \alpha\bar{u}_\rho, u - \bar{u}_\rho)) \geq 0 \quad \forall u \in U_{\text{ad}}, \\ \bar{\mu}_\rho = (\mu + \rho(S\bar{u}_\rho - \psi))_+ \end{aligned}$$

is satisfied.

Remark 2.54. By Lemma 2.24 we know that the adjoint state $\bar{p}_\rho \in H^1(\Omega)$ from Theorem 2.53 is the weak solution of the adjoint equation

$$\begin{aligned} A^*\bar{p}_\rho &= \bar{y}_\rho - y_d + \bar{\mu}_\rho && \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p}_\rho &= 0 && \text{on } \Gamma. \end{aligned}$$

Remark 2.55. In contrast to the original problem (P), the multipliers $\bar{p}_\rho$ and $\bar{\mu}_\rho$ from (P_{AL}) are unique. This is due to the explicit construction of $\bar{\mu}_\rho$.

2.7.3 The Augmented Lagrangian Algorithm

We aim at solving the pointwise state constrained optimal control problem (P) with the augmented Lagrangian method that has been introduced in Section 2.6. To satisfy the therein used general framework we set

$$\begin{gathered} U := L^2(\Omega), \quad Y := H^1(\Omega) \cap C(\overline{\Omega}), \quad C := U_{\text{ad}}, \quad g(u) := Su - \psi, \quad K := C(\overline{\Omega})_- \\ H := L^2(\Omega), \quad \mathcal{K} := L^2(\Omega)_-. \end{gathered}$$

The corresponding augmented Lagrangian is given by $\mathcal{L}_{AL}\colon L^2(\Omega) \times L^2(\Omega) \times \mathbb{R} \to \mathbb{R}$

$$\mathcal{L}_{AL}(u,\mu,\rho) := f(u) + \frac{\rho}{2}\left\|(\frac{\mu}{\rho} + Su - \psi)_+\right\|^2_{L^2(\Omega)} - \frac{\|\mu\|^2_{L^2(\Omega)}}{2\rho}.$$

Rearranging the second term and noting that the last term can be neglected (for minimization with respect to u), we intend to solve

$$\underset{u \in U_{\text{ad}}}{\text{minimize}} \ \mathcal{L}_{AL}(u, \mu, \rho) := f(u) + \frac{1}{2\rho} \|(\mu + \rho(Su - \psi))_+\|^2_{L^2(\Omega)}, \tag{2.18}$$

which is exactly the augmented Lagrangian subproblem (P_{AL}). Let $\bar{u}_k$ denote a solution of (2.18) for given $\rho_k > 0$ and $\mu_k \in L^2(\Omega)$. Then an update candidate for the Lagrange multiplier is according to (2.16) given by

$$\mu_{k+1} = (\mu_k + \rho_k(S\bar{u}_k - \psi))_+ . \tag{2.19}$$

Formula (2.19) will play an important role in the convergence analysis. Let $\bar{u}$ denote a solution of (P) and $\bar{u}_k$ a solution of (P_{AL}) for given $\rho_k > 0$ and $\mu_k \in L^2(\Omega)$. We define

$$R_k := \|(S\bar{u}_k - \psi)_+\|_{C(\bar{\Omega})} + |(\bar{\mu}_k, \psi - S\bar{u}_k)|.$$

As already pointed out in Section 2.6, the boundedness of the multiplier estimates in $L^1(\Omega)$ is a crucial issue for the convergence analysis of augmented Lagrangian methods in this setting. This boundedness property cannot be guaranteed for the sequence $(\bar{u}_k)_k$, which denotes the sequence of solutions of the augmented Lagrangian subproblem $\min_{u \in U_{\text{ad}}} \mathcal{L}_{AL}(u, \mu_k)$ for given μ_k and ρ_k. Inspired by [36, 37], we will therefore investigate the sequence of so-called *successful* iterates $(u_n^+)_n$ with corresponding multiplier approximations μ_n^+. Here, we consider a step to be successful if the quantity R_k shows, compared to the last successful step, sufficient decrease, i.e., $R_k \le \tau R_{n-1}^+ := \|(Su_{n-1}^+ - \psi)_+\|_{C(\bar{\Omega})} + |(\mu_{n-1}^+, \psi - Su_{n-1}^+)|$. During all successful steps the multiplier update as given in (2.19) is carried out, while the penalty parameter remains unchanged. If the step is not successful, the penalty parameter is increased.

Applying Algorithm 2.2 to the optimal control problem (P), we arrive at the following algorithm:

Algorithm 2.3 Modified Augmented Lagrangian Algorithm for Problem (P)

Let $(\bar{u}_0, \mu_1) \in L^2(\Omega) \times L^2(\Omega)$ and $\rho_1 > 0$ be given with $\mu_1 \ge 0$. Choose $\theta > 1$, $\tau \in (0, 1)$, $R_0^+ \gg 0$ and set $k := 1, n := 1$.

1: Compute a solution $\bar{u}_k$ of $\min_{u \in U_{\text{ad}}} \mathcal{L}_{AL,k}(u, \mu_k)$.

2: Compute $\bar{\mu}_k := (\mu_k + \rho_k(S\bar{u}_k - \psi))_+$.

3: Compute a measure for feasibility and complementarity R_k.

4: If $R_k \le \tau R_{n-1}^+$, then the step k is **successful**: We set

$$\mu_{k+1} = \bar{\mu}_k, \qquad \rho_{k+1} := \rho_k$$

and define

$$u_n^+ := \bar{u}_k, \quad \mu_n^+ := \mu_{k+1}, \quad R_n^+ := R_k.$$

Finally, we set $n := n + 1$.

5: Otherwise, the step k is **not successful**: We set $\mu_{k+1} := \mu_k$ and increase the penalty parameter $\rho_{k+1} := \theta \rho_k$.

6: If $R_{n-1}^+ \le \epsilon$ then stop, otherwise set $k := k + 1$ and go to step 1.

Let us comment on the choice of R_k. This quantity depends on the sufficient decrease of feasibility and complementarity. Moreover, R_k can be used as termination criterion in Algorithm 2.3, where the algorithm is stopped if this quantity is small enough.

The algorithm is well-defined. However, one of the main tasks is to prove that infinitely many steps are successful. Let us point out the importance of this issue. Assume that only a finite number of steps are successful and let m denote the index of the last successful step. Then all steps k with $k > m$ are not successful. According to Algorithm 2.3, step 5, we do not gain any new iterates u_n^+. Moreover there is no update on the Lagrange multiplier, while the penalty parameter ρ tends to infinity. Hence, the algorithm is caught in an infinite loop between steps 1, 2, 3, and 5. In order to prove that only a finite numbers of steps are not successful, we will investigate the solutions of the augmented subproblem (P_{AL}) with fixed multiplier approximation μ and penalization parameter ρ tending to infinity. This choice reduces the method to the classical quadratic penalty approach with additional shift parameter μ, which is also known under the name *Moreau-Yosida regularization* [53, 55, 58].

2.8 Multi-Player Optimization Problems

2.8.1 Generalized Nash Equilibrium Problems

In this section, we will introduce the reader to the different solution concepts for (generalized) Nash equilibrium problems. Let $1 < N \in \mathbb{N}$ denote the number of players. Each player $\nu \in \{1, ..., N\}$ is in control of the variable $u^\nu \in U^\nu$, where U^ν is a real Banach space. The strategies of all players, except the ν-th player are denoted by $u^{-\nu} \in U^{-\nu}$, leading to the notation $u := (u^\nu, u^{-\nu})$. The strategy space of all players is given by $U := U^1 \times \cdots \times U^N$. Each player aims at minimizing

$$\underset{u^\nu \in U^\nu}{\text{minimize}} \; f^\nu(u^\nu, u^{-\nu}) \quad \text{subject to} \quad u^\nu \in \mathcal{F}^\nu(u^{-\nu}), \tag{2.20}$$

where

$$\mathcal{F}^\nu(u^{-\nu}) := \{v^\nu \in C^\nu \mid g_\nu(v^\nu, u^{-\nu}) \in K\} \subseteq U.$$

Here, $f^\nu \colon U \to \mathbb{R}$ is a convex and continuous objective functional, $C^\nu \subset U^\nu$ is a non-empty, bounded, closed, convex set, and $K \subseteq Y$ a closed, convex cone, where Y is a Banach space. Further, $g_\nu \colon U \to Y$ is a convex, continuously Fréchet differentiable mapping. Throughout this section, we assume $f^\nu(\cdot, u^{-\nu})$ to be convex and continuously Fréchet differentiable for any given $u^{-\nu}$. A point $\bar{u} \in U$ is called *feasible*, if $\bar{u} \in \mathcal{F}(\bar{u}) := \mathcal{F}^1(\bar{u}^{-1}) \times \cdots \times \mathcal{F}^N(\bar{u}^{-N})$. Solutions of the GNEP are defined as follows.

Definition 2.56 (Generalized Nash equilibrium). A feasible point $\bar{u}$ is a *generalized Nash equilibrium* or a *solution of the GNEP* (2.20) if and only if for every ν it holds

$$f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \leq f^\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v^\nu \in \mathcal{F}^\nu(\bar{u}^{-\nu}). \tag{2.21}$$

From now on, we use the notation

$$F \colon U \to U^*, \quad F(\bar{u}) := (D_{u^1} f^1(\bar{u}), ..., D_{u^N} f^N(\bar{u})).$$

Since the objective functional f^ν is convex in u^ν for given $u^{-\nu}$, condition (2.21) can for each player be equivalently expressed as the ν-th players optimality condition, see Lemma 2.35. Thus, concatenating each player's optimality condition yields another characterisation of a generalized Nash equilibrium.

Lemma 2.57 (First-order optimality condition). *A feasible point $\bar{u} \in \mathcal{F}(\bar{u})$ is a solution of the GNEP* (2.20) *if and only if*

$$(D_{u^\nu} f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}), v^\nu - \bar{u}^\nu) \geq 0 \quad \forall v^\nu \in \mathcal{F}^\nu(\bar{u}^{-\nu}), \tag{2.22}$$

is satisfied for all ν.

Moreover, due to the convexity of the objective functional the generalized Nash equilibrium problem can be reformulated as a *quasi-variational inequality (QVI)*.

Lemma 2.58. *The GNEP* (2.20) *can be reformulated as the quasi variational inequality*

$$\bar{u} \in \mathcal{F}(\bar{u}), \quad (F(\bar{u}), v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{F}(\bar{u}) := \mathcal{F}^1(\bar{u}^{-1}) \times \cdots \times \mathcal{F}^N(\bar{u}^{-N}). \tag{QVI}$$

Proof. Concatenating each player's optimality conditions (2.22) it follows immediately that any solution $\bar{u}$ of the GNEP is a solution of (QVI). For the converse, since $f^\nu(\cdot, u^{-\nu})$ is convex, we obtain for all $v \in \mathcal{F}(\bar{u})$ with Lemma 2.16

$$\sum_{\nu=1}^{N} \left(f^\nu(v^\nu, \bar{u}^{-\nu}) - f^\nu(\bar{u}^\nu, \bar{u}^{-\nu})\right) \geq \sum_{\nu=1}^{N} (D_{u^\nu} f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}), v^\nu - \bar{u}^\nu) = (F(\bar{u}), v - \bar{u}) \geq 0.$$

Fixing ν and inserting the points $v := (v^\nu, \bar{u}^{-\nu}) \in \mathcal{F}(\bar{u})$, where $v^\nu \in \mathcal{F}^\nu(\bar{u}^{-\nu})$ is chosen arbitrary, we arrive at

$$f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \leq f^\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v^\nu \in \mathcal{F}^\nu(\bar{u}^{-\nu}),$$

which implies that any solution $\bar{u}$ of the QVI is a solution of the GNEP. □

Let Φ denote the solution operator of the variational inequality associated to F and $\mathcal{F}(u)$ for some fixed u:

$$w = \Phi(u) \quad \Leftrightarrow \quad w \in \mathcal{F}(u), \quad (F(w), v - w) \geq 0 \quad \forall v \in \mathcal{F}(u).$$

Then u is a solution of (QVI) if and only if

$$u = \Phi(u). \tag{2.23}$$

Thus, searching for a generalized Nash equilibrium of (2.20) results in solving the fixed point equation (2.23). Kakutani's fixed point theorem [67] is an important tool for proving existence of solutions of generalized Nash equilibria in finite dimensions. In 1952, Glicksberg [49] showed that Kakutani's theorem can be extended from the Euclidean space to convex linear topological spaces which implies the minimax theorem of Ky Fan [46] for continuous games with continuous payoff.

Theorem 2.59 (Kakutani-Fan-Glicksberg [3, Cor. 17.55]). *Let $\mathcal{K}$ be a non-empty, compact, and convex subset of a locally convex Hausdorff space X. Further let $\Phi : \mathcal{K} \rightrightarrows \mathcal{K}$ have a closed graph and non-empty convex values. Then the set of fixed points of Φ is compact and non-empty.*

In order to apply Theorem 2.59 we consider the setting

$$U^\nu := (L^2(\Omega), \tau_{weak}), \qquad X := \prod_{\nu=1}^{N} U^\nu, \qquad \mathcal{K} := C := C^1 \times ... \times C^N,$$

where τ_{weak} denotes the weak topology on $L^2(\Omega)$. The sets $C^\nu \in L^2(\Omega)$ are non-empty, closed, bounded and convex, hence weakly sequentially compact. Moreover, the Eberlein-Šmulian Theorem (Theorem 2.5) yields weak compactness. We recall that graph(Φ) is weakly sequentially closed if

$$x_k \rightharpoonup x^*,\ y_k \in \Phi(x_k),\ y_k \rightharpoonup y^* \quad \text{implies} \quad y^* \in \Phi(x^*).$$

The following result helps us now to check the assumptions of Theorem 2.59.

Lemma 2.60. *Let $\mathcal{K}$ denote a non-empty, weakly compact, and convex subset $\mathcal{K} \subset X$ of a Banach space X and $\Phi\colon \mathcal{K} \rightrightarrows \mathcal{K}$. If graph$(\Phi)$ is weakly sequentially closed, Φ has a closed graph in the weak topology.*

Proof. For the Graph of Φ we have graph$(\Phi) \subseteq \mathcal{K}^2$, with $\mathcal{K}^2$ weakly compact. Let us assume that graph(Φ) is weakly sequentially closed. Then graph(Φ) is also weakly sequentially compact. By a result from Eberlein and Šmulian, we obtain that graph(Φ) is weakly compact, hence weakly closed. This coincides with Φ having a closed graph in the weak topology. □

Thus, in order to apply Theorem 2.59, it is enough to check if graph(Φ) is weakly sequentially closed and Φ has non-empty convex values.

2.8.2 Jointly Convex GNEPs

Let us investigate a slight modification of the GNEP (2.20). From now on the functions g_ν coincide for each player. Thus, each player attempts to solve

$$\underset{u^\nu \in U^\nu}{\text{minimize}}\ f^\nu(u^\nu, u^{-\nu}) \quad \text{subject to} \quad (u^\nu, u^{-\nu}) \in \mathcal{F}, \tag{2.24}$$

where

$$\mathcal{F} := \{u \in C \mid g(u) \in K\} \subseteq U.$$

denotes the *feasible set*. Here, $f^\nu\colon U \to \mathbb{R}$ is a convex and continuous objective functional, $C := C^1 \times \cdots \times C^N$ where $C^\nu \subset U^\nu$ are non-empty, closed, convex sets, and $K \subseteq Y$ a closed, convex cone, where Y is a Banach space. Further, $g\colon U \to Y$ is a convex, continuously Fréchet differentiable mapping. Since in this case the constraint $g(u) \in K$ is the same for each player, it is commonly called *joint constraint*. Problems with this particular structure are called *jointly convex generalized Nash equilibrium problems*. The solution concept of jointly convex GNEPs involves another characterization of solutions which is based on the *Nikaido-Isoda (NI) function*

$$\Psi(u, v) := \sum_{\nu=1}^{N} f^\nu(u^\nu, u^{-\nu}) - \sum_{\nu=1}^{N} f^\nu(v^\nu, u^{-\nu}).$$

Definition 2.61 (Normalized equilibrium/ Variational equilibrium). Let $\bar{u} \in \mathcal{F}$ be a feasible point. Then $\bar{u}$ is called a *normalized Nash equilibrium (NE)* or *variational equilibrium* if

$$\sum_{\nu=1}^{N} f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \le \sum_{\nu=1}^{N} f^\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v \in \mathcal{F}. \tag{2.25}$$

Note, that the above characterization of a normalized equilibrium can be equivalently expressed as

$$\Psi(\bar{u}, v) \le 0 \quad \forall v \in \mathcal{F}. \tag{2.26}$$

Every variational equilibrium is a generalized Nash equilibrium. This can be seen by inserting the points $v := (v^\nu, \bar{u}^{-\nu})$, with $v^\nu \in \mathcal{F}(\bar{u}^{-\nu})$, into the Definition 2.61. Note, that the characterization (2.26) is equivalent to $\bar{u}$ being a solution of the concave maximization problem

$$\max_{v \in \mathcal{F}} \Psi(\bar{u}, v). \tag{2.27}$$

Establishing first-order necessary optimality conditions of the concave maximization problem (2.27), we obtain that a normalized Nash equilibrium $\bar{u}$ can equivalently be characterized as a point $\bar{u} \in \mathcal{F}$ that solves the following variational inequality (VI):

$$\bar{u} \in \mathcal{F}, \quad \langle F(\bar{u}), v - \bar{u}\rangle \ge 0 \quad \forall v \in \mathcal{F}. \tag{2.28}$$

If f^ν is for instance given by a tracking-type objective functional and the set C is additionally bounded, it is possible to prove existence of solutions by applying the Kakutani-Fan-Glicksberg-Theorem 2.59. However, this proof does not imply uniqueness of solutions. Nevertheless, existence of a unique normalized equilibrium can be shown via the standard solution theory of variational inequalities, assumed F is strongly monotone.

Definition 2.62 (Monotone operators). Let U denote a real Banach space with $C \subseteq U$. We say that an operator $A\colon U \to U^*$ is

a) *monotone* on C if $\langle Au - Aw, u - w\rangle_{U^*,U} \geq 0$ for all $u, w \in C$.

b) *strongly monotone* on C if there exists a constant $c > 0$ such that

$$\langle Au - Aw, u - w\rangle_{U^*,U} \geq c\,\|u - w\|_U^2 \quad \forall u, w \in C.$$

Theorem 2.63 (Existence of a normalized Nash equilibrium [78, Thm. III.1.4]). *If the sets C^ν are non-empty, closed, convex, and bounded and F is monotone, then there exists a normalized solution of the GNEP* (2.24)*. Moreover, if the sets C^ν are non-empty, closed and convex and F is strongly monotone, then the solution is unique.*

Let us emphasize that unique equilibria are often crucial for numerical implementations and throughout the execution of convergence analysis. If the problem under consideration falls into the category of potential games in the sense of Monderer [95], it is possible to reduce the GNEP (2.20) to a single convex optimization problem. Existence of unique solutions of this problems can then be deduced by standard arguments from optimization theory, see for instance [60, Prop 3.10]. However, there are special types of GNEPs that cannot be reduced to a single optimization problem and it cannot be expected in general that the resulting first-order optimality system is a (strongly) monotone VI. We will treat an example of this case in Chapter 8.

2.8.3 Nash Equilibrium Problems

Let us now take a closer look at the case that each player's constraints are independent of the other players' strategies. In this situation each player aims at minimizing

$$\underset{u^\nu \in U^\nu}{\text{minimize}}\ f^\nu(u^\nu, u^{-\nu}) \quad \text{subject to} \quad u^\nu \in \mathcal{F}^\nu \subseteq U^\nu. \tag{2.29}$$

Here, $f^\nu\colon U \to \mathbb{R}$ is a linear and continuous objective functional and $\mathcal{F}^\nu \subseteq U^\nu$ is a non-empty, closed, convex set. Problems of this type are called *Nash equilibrium problems (NEP)*. In an analogue way to GNEPs, we define the feasible set $\mathcal{F} := \mathcal{F}^1 \times \cdots \times \mathcal{F}^N$ and a solution of the NEP:

Definition 2.64. Let $\bar{u} \in \mathcal{F}$. Then $\bar{u}$ is a *Nash equilibrium* or a *solution of the NEP* (2.29) if and only if for every ν it holds

$$f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \leq f^\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v^\nu \in \mathcal{F}^\nu.$$

Since the set of constraints does not depend on u, reformulating the NEP (2.29) results in a variational inequality instead of a QVI (see Lemma 2.58).

Lemma 2.65. *The NEP* (2.29) *can be equivalently reformulated as the variational inequality*

$$\bar{u} \in \mathcal{F}, \quad (F(\bar{u}), v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{F}. \tag{VI}$$

Proof. Concatenating the first-order necessary optimality conditions for the ν-th problem for any fixed $\bar{u}^{-\nu}$ results in

$$(D_{u^\nu} f_\nu(\bar{u}^\nu, \bar{u}^{-\nu}), v^\nu - \bar{u}^\nu) \geq 0 \quad \forall v^\nu \in \mathcal{F}^\nu$$

and we arrive directly at the variational inequality (VI). The converse can be shown like for the QVI case, see Lemma 2.58. □

Applying the solution theory of (strongly) monotone variational inequalities yields the following existence result.

Theorem 2.66 (Existence of a Nash equilibrium [78, Cor. III.1.8]). *If the sets $\mathcal{F}^\nu$ are non-empty, closed, convex and bounded and F is monotone, then there exists a solution of the NEP* (2.29)*. Moreover, if the sets $\mathcal{F}^\nu$ are non-empty, closed and convex and F is strongly monotone, then the solution is unique.*

Part I

Optimal Control Problems

CHAPTER 3

LINEAR OPTIMAL CONTROL PROBLEMS

This chapter deals with the solution of a convex optimal control problem governed by an elliptic partial differential equation with homogeneous Neumann boundary conditions and pointwise control and state constraints. The problem is given by

$$\text{minimize} \ \ J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \tag{3.1}$$

subject to

$$\begin{aligned} Ay &= u && \text{in } \Omega, \\ \partial_{\nu_A} y &= 0 && \text{on } \partial\Omega, \\ y &\leq \psi && \text{in } \overline{\Omega}, \\ u_a \leq u &\leq u_b && \text{a.e. in } \Omega. \end{aligned}$$

Here, A denotes a linear elliptic operator of second-order. For instance, we can choose $A := -\Delta + \mathrm{Id}$. As already mentioned in the introduction (Section 1.2.1), the Lagrange multiplier associated to the state constraint $y \leq \psi$ is only a measure in $C(\overline{\Omega})^* = \mathcal{M}(\overline{\Omega})$. By applying an augmented Lagrangian method, the state constraints are replaced by a penalization term, which is augmenting the inequality constraint in the objective functional. In this way, we have to solve a sequence of optimal control problems that are only control constrained. We establish an update rule that performs the classical augmented Lagrangian update (see Section 2.6) only if a sufficient decrease of the maximal constraint violation and the violation of the complementarity condition is achieved. This is crucial for our convergence analysis, since it enables us to conclude boundedness of the multiplier sequence in $L^1(\Omega)$.

We start this chapter by giving a detailed formulation, including optimality conditions, of the problem to solve in Section 3.1. We develop the augmented Lagrangian method in Section 3.2. In order to guarantee the boundedness of generated multiplier approximations, we investigate a special multiplier update rule: the classical multiplier update is performed only if a certain measure of feasibility and violation of complementarity shows sufficient decrease, see Section 3.2.2. The convergence of the method is studied in Section 3.3. The main results of this section are boundedness of iterates (Lemma 3.14) and their convergence (Theorem 3.15) to the solution of the original problem. In addition, we show that the sequence of generated penalty parameters is bounded only in exceptional situations, which is different from classical results in finite-dimensional optimization (Theorem 3.18). We demonstrate the performance of the method for selected problems in Section 3.4. The results of this chapter can be found in the publication [77].

3.1 The Optimal Control Problem

This section concerns a detailed introduction of the optimal control problem under investigation. We make the concrete problem setting clear, collect results about existence and uniqueness of solutions and derive optimality conditions.

3.1.1 Problem Setting

Denote by $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded Lipschitz domain with boundary Γ. We aim at solving the following optimal control problem:

Minimize

$$J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}$$

over all $(y,u) \in (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega)$ subject to the elliptic equation

$$\begin{aligned} (Ay)(x) &= u(x) && \text{in } \Omega, \\ (\partial_{\nu_A} y)(x) &= 0 && \text{on } \Gamma, \end{aligned}$$

and subject to the pointwise state and control constraints

$$\begin{aligned} y(x) &\le \psi(x) && \text{in } \overline{\Omega}, \\ u_a(x) \le u(x) &\le u_b(x) && \text{a.e. in } \Omega. \end{aligned}$$

In the sequel, we will work with the following set of standing assumptions.

Assumption 3.1. Let Assumption 2.19 be satisfied. Further, the given data satisfy $y_d \in L^2(\Omega)$, $\alpha > 0$, $u_a, u_b \in L^2(\Omega)$ with $u_a \le u_b$ and $\psi \in C(\overline{\Omega})$.

Due to the assumptions above, we know from Theorem 2.22 that for every $u \in L^2(\Omega)$ there exists a uniquely determined weak solution $y \in H^1(\Omega) \cap C(\overline{\Omega})$ of the state equation. The control-to-state mapping $S\colon u \mapsto y$ is linear and continuous from $L^2(\Omega)$ to $H^1(\Omega) \cap C(\overline{\Omega})$. Further, by Theorem 2.23, S is compact. We introduce the reduced formulation of (3.1)

$$\begin{aligned} &\underset{u \in L^2(\Omega)}{\text{minimize}} && f(u) := \frac{1}{2}\,\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\,\|u\|^2_{L^2(\Omega)} \\ &\text{subject to} && u_a(x) \le u(x) \le u_b(x) \quad \text{a.e. in } \Omega, \\ & && Su(x) \le \psi(x) \quad \text{in } \overline{\Omega}, \end{aligned} \tag{P}$$

and recall the *admissible* and *feasible* sets

$$\begin{aligned} U_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u_a(x) \le u(x) \le u_b(x) \text{ a.e. in } \Omega\}, \\ F_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u \in U_{\text{ad}},\ Su(x) \le \psi(x) \text{ in } \overline{\Omega}\}. \end{aligned}$$

Let us mention that the analysis below does not rely on the particular structure of U_{ad}. In fact, all the results are valid if U_{ad} is assumed to be non-empty, convex, and closed in $L^2(\Omega)$. Theorem 2.47 yields existence of a unique solution.

Theorem 3.2 (Existence of solutions of the optimal control problem). *Let Assumption 3.1 be satisfied. Assume that the feasible set F_{ad} is non-empty. Then there exists a uniquely determined global solution $\bar{u} \in L^2(\Omega)$ of (P).*

3.1.2 Optimality Conditions

Existence of Lagrange multipliers to state constrained optimal control problems is not guaranteed without any regularity assumptions. In the sequel, we will work with the following Slater condition.

Assumption 3.3 (Slater condition). We assume that there exists $\hat{u} \in U_{\text{ad}}$ and $\sigma > 0$ such that for $\hat{y} = S\hat{u}$ it holds

$$\hat{y}(x) \leq \psi(x) - \sigma \quad \forall x \in \overline{\Omega}.$$

First-order optimality conditions can be derived with Theorem 2.49, see also Remark 2.50.

Theorem 3.4 (First-order necessary optimality conditions). *Let Assumption 3.3 be satisfied. Then $\bar{u}$ is a solution of (P) if and only if there exists an adjoint state $\bar{p} \in W^{1,s}(\Omega)$, $s \in (1, d/(d-1))$ and a Lagrange multiplier $\bar{\mu} \in \mathcal{M}(\overline{\Omega})$ with $\bar{\mu} = \bar{\mu}|_\Omega + \bar{\mu}|_\Gamma$, such that the following optimality system is satisfied:*

$$\begin{aligned} A\bar{y} &= \bar{u} \quad \text{in } \Omega, \\ \partial_{\nu_A}\bar{y} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.2a}$$

$$\begin{aligned} A^*\bar{p} &= \bar{y} - y_d + \bar{\mu}_\Omega \quad &\text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p} &= \bar{\mu}_\Gamma \quad &\text{on } \Gamma, \end{aligned} \tag{3.2b}$$

$$(\bar{p} + \alpha\bar{u}, u - \bar{u}) \geq 0 \quad \forall u \in U_{\text{ad}}, \tag{3.2c}$$

$$\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{\mu} \geq 0, \quad \bar{y} \leq \psi. \tag{3.2d}$$

3.2 The Augmented Lagrangian Method

Since the Lagrange multiplier corresponding to the pointwise state constraint is only a measure, its numerical treatment causes difficulties. To overcome these, we replace the inequality constraint $y \leq \psi$ by an augmented penalization term as presented in Section 2.6. The complete algorithm will be presented in Section 3.2.3 below.

3.2.1 The Augmented Lagrangian Optimal Control Problem

Following the argumentation from Section 2.7.3, we consider the augmented Lagrangian subproblem

$$\begin{aligned} \underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad & f_{AL}(u_\rho, \mu, \rho) := f(u_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(Su_\rho - \psi))_+ \right\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_\rho \in U_{\text{ad}}. \end{aligned} \tag{P_{AL}}$$

We know from Theorem 2.52 that (P_{AL}) admits a unique global solution.

Theorem 3.5 (Existence of solutions of the augmented Lagrangian subproblem). *For every $\rho > 0$ and every $\mu \in L^2(\Omega)$ the augmented Lagrangian control problem (P_{AL}) admits a unique global solution $\bar{u}_\rho \in U_{ad}$ with associated optimal state $\bar{y}_\rho = S\bar{u}_\rho$.*

Since the problem (P_{AL}) does not have to satisfy state constraints, the first-order optimality system is fulfilled without any further regularity assumptions, see Theorem 2.53.

Theorem 3.6 (First-order necessary optimality conditions). *The point $\bar{u}_\rho$ is the unique solution of (P_{AL}) if and only if there exists an associated state $\bar{y}_\rho \in H^1(\Omega) \cap C(\overline{\Omega})$ and a unique adjoint state $\bar{p}_\rho \in H^1(\Omega)$, satisfying the following system.*

$$\begin{aligned} A\bar{y}_\rho &= \bar{u}_\rho \quad \text{in } \Omega, \\ \partial_{\nu_A}\bar{y}_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.3a}$$

$$\begin{aligned} A^*\bar{p}_\rho &= \bar{y}_\rho - y_d + \bar{\mu}_\rho \quad \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p}_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.3b}$$

$$(\bar{p}_\rho + \alpha\bar{u}_\rho, u - \bar{u}_\rho) \geq 0 \quad \forall u \in U_{\mathrm{ad}}, \tag{3.3c}$$

$$\bar{\mu}_\rho := \big(\mu + \rho(\bar{y}_\rho - \psi)\big)_+ . \tag{3.3d}$$

Due to the choice of $\bar{\mu}_\rho$ in (3.3d), the optimality system (3.3) of the augmented problem is very similar to optimality system (3.2) of the original problem (P). In fact, if $(\bar{y}_\rho, \bar{u}_\rho, \bar{p}_\rho, \bar{\mu}_\rho)$ solves (3.3), $\bar{y}_\rho$ is feasible, and $(\bar{\mu}_\rho, \bar{y}_\rho - \psi) = 0$ holds, then the point $(\bar{y}_\rho, \bar{u}_\rho, \bar{p}_\rho, \bar{\mu}_\rho)$ is a KKT point of the original problem. Another observation is that it is enough to control the $L^1(\Omega)$-norm of $\bar{\mu}_\rho$ in order to derive bounds on the solution $(\bar{y}_\rho, \bar{u}_\rho, \bar{p}_\rho)$ of (3.3). Here, we have the following theorem.

Theorem 3.7. *Let $\rho > 0$ and $\mu \in L^2(\Omega)$ be given. Let $s \in (1, d/(d-1))$. Then there exists a constant $c > 0$ independent of ρ and μ such that for all solutions $(\bar{y}_\rho, \bar{u}_\rho, \bar{p}_\rho, \bar{\mu}_\rho)$ of* (3.3) *it holds*

$$\|\bar{y}_\rho\|_{H^1(\Omega)} + \|\bar{y}_\rho\|_{C(\overline{\Omega})} + \|\bar{u}_\rho\|_{L^2(\Omega)} + \|\bar{p}_\rho\|_{W^{1,s}(\Omega)} \leq c(\|\bar{\mu}_\rho\|_{L^1(\Omega)} + 1).$$

Proof. Let us test the state equation (3.3a) with $\bar{p}_\rho$ and the adjoint equation (3.3b) with $\bar{y}_\rho$. This yields

$$(\bar{p}_\rho, \bar{u}_\rho) = (\bar{y}_\rho - y_d, \bar{y}_\rho) + (\bar{\mu}_\rho, \bar{y}_\rho).$$

Using the optimal control $\bar{u}$ of the original problem as test function in (3.3c), we obtain

$$(\bar{y}_\rho - y_d, \bar{y}_\rho) + (\bar{\mu}_\rho, \bar{y}_\rho) \leq (\alpha\bar{u}_\rho, \bar{u} - \bar{u}_\rho) + (\bar{p}_\rho, \bar{u}).$$

This inequality can be rewritten equivalently as

$$\|\bar{y}_\rho\|^2_{L^2(\Omega)} + \alpha\|\bar{u}_\rho\|^2_{L^2(\Omega)} \leq (\alpha\bar{u}_\rho, \bar{u}) + (\bar{p}_\rho, \bar{u}) + (y_d, \bar{y}_\rho) - (\bar{\mu}_\rho, \bar{y}_\rho).$$

Applying Young's inequality and the estimate $-(\bar{\mu}_\rho, \bar{y}_\rho) \leq \left\|\bar{\mu}_\rho\right\|_{L^1(\Omega)} \left\|\bar{y}_\rho\right\|_{C(\overline{\Omega})}$, we have

$$\begin{aligned} \frac{1}{2}\|\bar{y}_\rho\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u}_\rho\|^2_{L^2(\Omega)} \leq{}& \frac{\alpha}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \|\bar{p}_\rho\|_{L^2(\Omega)}\|\bar{u}\|_{L^2(\Omega)} \\ &+ \frac{1}{2}\|y_d\|^2_{L^2(\Omega)} + \|\bar{\mu}_\rho\|_{L^1(\Omega)}\|\bar{y}_\rho\|_{C(\overline{\Omega})}. \end{aligned}$$

Let us fix $\bar{s} \in (1, d/(d-1))$ such that $W^{1,\bar{s}}(\Omega)$ is continuously embedded in $L^2(\Omega)$. Then, we obtain from the Theorems 2.22 and 2.25 that there exist constants $c_1, c_2 > 0$, which are independent of ρ and μ, such that

$$\begin{aligned} \frac{1}{2}\|\bar{y}_\rho\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u}_\rho\|^2_{L^2(\Omega)} \leq{}& \frac{\alpha}{2}\|\bar{u}\|^2_{L^2(\Omega)} + \frac{1}{2}\|y_d\|^2_{L^2(\Omega)} + c_1 \left\|\bar{\mu}_\rho\right\|_{L^1(\Omega)} \left\|\bar{u}_\rho\right\|_{L^2(\Omega)} \\ &+ c_2 \left(\left\|\bar{y}_\rho\right\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + \left\|\bar{\mu}_\rho\right\|_{L^1(\Omega)}\right) \|\bar{u}\|_{L^2(\Omega)} . \end{aligned}$$

With Young's inequality we arrive at

$$\|\bar{y}_\rho\|^2_{L^2(\Omega)} + \alpha\|\bar{u}_\rho\|^2_{L^2(\Omega)} \leq c\left(\|\bar{\mu}_\rho\|^2_{L^1(\Omega)} + \|\bar{u}\|^2_{L^2(\Omega)} + \|y_d\|^2_{L^2(\Omega)}\right).$$

This implies the bound on the L^2-norms of $\bar{u}_\rho$ and $\bar{y}_\rho$. Using again the regularity results from Theorems 2.22 and 2.25, the claim is proven. □

3.2.2 The Multiplier Update Rule

In the following, let $(P_{AL})_k$ denote the augmented Lagrangian subproblem (P_{AL}) for given penalty parameter $\rho := \rho_k$ and multiplier $\mu := \mu_k$. We will denote its solution by $\bar{u}_k$ with corresponding state $\bar{y}_k$, adjoint state $\bar{p}_k$ and updated multiplier $\bar{\mu}_k$, which is given by (3.3d). For convenience, let us restate the optimality system of $(P_{AL})_k$, which is solved by $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\mu}_k)$:

$$\begin{aligned} A\bar{y}_k &= \bar{u}_k \quad \text{in } \Omega, \\ \partial_{\nu_A}\bar{y}_k &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.4a}$$

$$\begin{aligned} A^*\bar{p}_k &= \bar{y}_k - y_d + \bar{\mu}_k \quad \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p}_k &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.4b}$$

$$(\bar{p}_k + \alpha\bar{u}_k, u - \bar{u}_k) \geq 0 \quad \forall u \in U_{\text{ad}}, \tag{3.4c}$$

$$\bar{\mu}_k := (\mu_k + \rho_k(\bar{y}_k - \psi))_+ \,. \tag{3.4d}$$

Let us start this section with a basic estimate, which will be useful in the sequel.

Lemma 3.8. *Let* $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ *be a solution of* (3.2)*, and let* $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\mu}_k)$ *solve* (3.4)*. Then it holds*

$$\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} \leq (\bar{\mu}_k, \psi - \bar{y}_k) + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}.$$

Proof. Using (3.2b),(3.2c) and (3.4b),(3.4c), we obtain

$$\begin{aligned} \|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} &= (A^*(\bar{p} - \bar{p}_k), \bar{y} - \bar{y}_k) - (\bar{\mu} - \bar{\mu}_k, \bar{y} - \bar{y}_k) \\ &= ((\bar{p} - \bar{p}_k), \bar{u} - \bar{u}_k) - (\bar{\mu} - \bar{\mu}_k, \bar{y} - \bar{y}_k) \\ &\leq -\alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} - (\bar{\mu} - \bar{\mu}_k, \bar{y} - \bar{y}_k), \end{aligned}$$

which implies

$$\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} \leq (\bar{\mu}_k - \bar{\mu}, \bar{y} - \bar{y}_k). \tag{3.5}$$

The term on the right-hand side of equation (3.5) can be split into two parts:

$$(\bar{\mu}_k, \bar{y} - \bar{y}_k) = (\bar{\mu}_k, \bar{y} - \psi) + (\bar{\mu}_k, \psi - \bar{y}_k) \leq (\bar{\mu}_k, \psi - \bar{y}_k) \tag{3.6}$$

and

$$\begin{aligned} -\langle \bar{\mu}, \bar{y} - \bar{y}_k \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} &= -\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} - \langle \bar{\mu}, \psi - \bar{y}_k \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \\ &= \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}. \end{aligned} \tag{3.7}$$

Here, we used the complementarity relation (3.2d) as well as $\bar{y} \leq \psi$ and $\bar{\mu}_k \geq 0$. Putting the inequalities (3.5), (3.6), and (3.7) together, we get

$$\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} \leq (\bar{\mu}_k, \psi - \bar{y}_k) + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})},$$

which is the claim. □

Our multiplier update decision is motivated by the following result, which estimates the difference of solutions of the augmented Lagrangian subproblem to the solution of the original problem. The upper bound of the error contains the violation of the state constraint and the mismatch in the complementarity condition.

Lemma 3.9. *Let $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ and $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\mu}_k)$ be given as in Lemma 3.8. Then it holds*

$$\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} \leq \|\bar{\mu}\|_{\mathcal{M}(\Omega)} \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + (\bar{\mu}_k, \psi - \bar{y}_k).$$

Proof. The claim follows directly from Lemma 3.8 using the estimate

$$\langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \leq \|\bar{\mu}\|_{\mathcal{M}(\Omega)} \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})}. \qquad \square$$

This result shows that the iterates $(\bar{y}_k, \bar{u}_k)$ will converge to the solution of the original problem if the quantity

$$\|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)|$$

tends to zero for $k \to \infty$. We will say that a step of Algorithm 1 is successful if this quantity decreases sufficiently fast. Specifically, we will say that step k was successful if the condition

$$\|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)| \leq \tau \left(\|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)| \right)$$

is satisfied with $\tau \in (0, 1)$. Here, we denoted by step n, $n < k$, the previous successful step with corresponding iterates y_n^+, u_n^+, p_n^+ and μ_n^+. Moreover, the quantity above can be used as termination criterion, where the iteration is stopped if this quantity is small enough.

3.2.3 The Augmented Lagrangian Algorithm in Detail

Let us now state the concrete algorithm with the update rule as described in the previous section.

Algorithm 3.1 Augmented Lagrangian Algorithm for (P)

Let $(\bar{y}_0, \bar{u}_0, \bar{p}_0) \in (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega) \times W^{1,s}(\Omega)$, $\rho_1 > 0$ and $\mu_1 \in L^2(\Omega)$ be given with $\mu_1 \geq 0$. Choose $\theta > 1$, $\tau \in (0, 1)$, $\epsilon \geq 0$, $R_0^+ \gg 1$. Set $k = 1$ and $n = 1$.

1: Compute a solution $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ of $(P_{AL})_k$.

2: Set $\bar{\mu}_k := (\mu_k + \rho_k(\bar{y}_k - \psi))_+$.

3: Compute $R_k := \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)|$.

4: If $R_k \leq \tau R_{n-1}^+$, then the step k is successful. Set

$$\mu_{k+1} := \bar{\mu}_k, \quad \rho_{k+1} := \rho_k$$

and define

$$(y_n^+, u_n^+, p_n^+) := (\bar{y}_k, \bar{u}_k, \bar{p}_k), \quad \mu_n^+ := \mu_{k+1}, \quad R_n^+ := R_k.$$

Set $n := n + 1$.

5: Otherwise, the step k is not successful. Set $\mu_{k+1} := \mu_k$ and increase the penalty parameter $\rho_{k+1} := \theta \rho_k$.

6: If $R_{n-1}^+ \leq \epsilon$, then stop. Otherwise set $k := k + 1$ and go to step 1.

Although the algorithm is well-defined, we still have to prove that infinitely many steps are successful. Otherwise the algorithm is caught in an infinite loop between steps 1, 2, 3, and 5, see the discussion at the end of Section 2.7.3. In the sequel, we analyze Algorithm 3.1 with tolerance $\epsilon = 0$. If for some n it holds $R_n^+ = 0$, then by Lemma 3.9 the current iterate is a solution of the original problem. Otherwise, the method will iterate infinitely. In order to prove that infinitely many steps are successful, we will investigate the solutions of the augmented Lagrangian KKT system (3.3) with fixed multiplier approximation μ and penalization parameter ρ tending to infinity. In this situation, the method reduces to a penalty method with additional shift parameter μ. Such a scheme was already investigated in [55]. However, in this publication a much stronger regularity condition was imposed, which can only be fulfilled if the state constraints are considered in $H^2(\Omega)$.

Lemma 3.10. *Let $(\rho_k)_k$ be a sequence of positive numbers with $\rho_k \to \infty$. Let $\mu \in L^2(\Omega)$ with $\mu \geq 0$ be given. Let $\bar{u}$ be a solution of (P) with corresponding state $\bar{y}$ and $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ be solutions of the optimality system* (3.4)*. Then for $k \to \infty$, it holds*

$$(\bar{y}_k, \bar{u}_k) \to (\bar{y}, \bar{u}) \text{ in } (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega).$$

Proof. The general idea of the proof follows [55]. Using an observation from the proof of [65, Theorem 3.1], we find

$$\begin{aligned}(\bar{\mu}_k, \bar{y} - \bar{y}_k) &= (\bar{\mu}_k, -\frac{\mu}{\rho_k} - \bar{y}_k + \psi - \psi + \bar{y} + \frac{\mu}{\rho_k}) \\ &= -\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{\rho_k}(\bar{\mu}_k, \mu) + (\bar{\mu}_k, \bar{y} - \psi) \\ &\leq -\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{2\rho_k}\left(\|\bar{\mu}_k\|^2_{L^2(\Omega)} + \|\mu\|^2_{L^2(\Omega)}\right) \\ &= -\frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.8}$$

From inequality (3.5) in the proof of Lemma 3.8, we get

$$\begin{aligned}\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} &\leq (\bar{\mu}_k, \bar{y} - \bar{y}_k) - \langle \bar{\mu}, \bar{y} - \bar{y}_k \rangle \\ &\leq (\bar{\mu}_k, \bar{y} - \bar{y}_k) + \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \|\bar{y} - \bar{y}_k\|_{C(\overline{\Omega})} \\ &\leq (\bar{\mu}_k, \bar{y} - \bar{y}_k) + \frac{\alpha}{2} \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} + \frac{c^2}{2\alpha} \|\bar{\mu}\|^2_{\mathcal{M}(\overline{\Omega})},\end{aligned} \tag{3.9}$$

where we used Young's inequality and the regularity result from Theorem 2.22. With inequality (3.8) this leads to

$$\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} + \frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} \leq \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)} + \frac{c^2}{2\alpha} \|\bar{\mu}\|^2_{\mathcal{M}(\overline{\Omega})}. \tag{3.10}$$

Hence, the sequence $(\bar{u}_k)_k$ is bounded in $L^2(\Omega)$, which implies the boundedness of $(\bar{y}_k)_k$ in $H^1(\Omega) \cap C(\overline{\Omega})$. This allows to extract weakly convergent subsequences $\bar{u}_{k'} \rightharpoonup u^*$ in $L^2(\Omega)$ and $\bar{y}_{k'} \rightharpoonup y^*$ in $H^1(\Omega)$. Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the sequence $(\bar{y}_{k'})_{k'}$ converges strongly in $L^2(\Omega)$. With the compactness of S we obtain strong convergence $\bar{y}_{k'}$ to y^* in $C(\overline{\Omega})$. In order to prove $y^* \leq \psi$, we use the identity

$$\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} = \rho_k \left\| \max\left(0, \frac{\mu}{\rho_k} + \bar{y}_k - \psi\right) \right\|^2_{L^2(\Omega)}, \tag{3.11}$$

which is bounded because of (3.10). As the term $\max\left(0, \frac{\mu}{\rho_{k'}} + \bar{y}_{k'} - \psi\right)$ converges to the limit $\max(0, y^* - \psi)$ in $L^2(\Omega)$ for $k' \to \infty$, we obtain $y^* \leq \psi$ by passing to the limit in (3.11). This shows that y^* is feasible. To argue that $y^* = \bar{y}$ and $u^* = \bar{u}$, we use again inequality (3.5) and

(3.8) to conclude

$$\begin{aligned}\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} &+ \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} \\ &\leq (\bar{\mu}_k, \bar{y} - \bar{y}_k) - \langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \\ &\leq \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)} + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}.\end{aligned}$$

Passing to the limit $k' \to \infty$ yields

$$0 \leq \lim_{k' \to \infty} \|\bar{y} - \bar{y}_{k'}\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_{k'}\|^2_{L^2(\Omega)} \leq \langle \bar{\mu}, y^* - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \leq 0,$$

and consequently $\bar{u}_{k'} \to \bar{u}$ in $L^2(\Omega)$. The compactness of S immediately yields the strong convergence $\bar{y}_{k'} \to \bar{y}$ in $H^1(\Omega) \cap C(\overline{\Omega})$. As the limit is independent of the taken subsequence, we obtain convergence of the whole sequences $(u_k)_k$ and $(y_k)_k$ to $\bar{u}$ and $\bar{y}$, respectively. □

Lemma 3.11. *Under the same assumptions as in Lemma 3.10, it holds*

$$\lim_{k \to \infty} (\bar{\mu}_k, \psi - \bar{y}_k) = 0.$$

Proof. First, we estimate

$$\begin{aligned}(\bar{\mu}_k, \psi - \bar{y}_k) &= \frac{1}{\rho_k}(\bar{\mu}_k, -\mu + \rho_k(\psi - \bar{y}_k) + \mu) = -\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{\rho_k}(\bar{\mu}_k, \mu) \\ &\leq -\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} + \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)},\end{aligned}$$

which proves

$$\limsup_{k \to \infty} (\bar{\mu}_k, \psi - \bar{y}_k) \leq 0. \tag{3.12}$$

From Lemma 3.8 we get

$$(\bar{\mu}_k, \psi - \bar{y}_k) \geq \|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - \bar{u}_k\|^2_{L^2(\Omega)} + \langle \bar{\mu}, \psi - \bar{y}_k \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})},$$

which leads with Lemma 3.10 to

$$\liminf_{k \to \infty} (\bar{\mu}_k, \psi - \bar{y}_k) \geq 0. \tag{3.13}$$

The inequalities (3.12) and (3.13) yield the claim. □

Using these two results, we can show that an infinite number of successful steps are taken.

Lemma 3.12. *Algorithm 3.1 makes infinitely many successful steps.*

Proof. We assume the algorithm to do a finite number of successful steps only. Then there is an index m such that all steps k with $k > m$ are not successful. According to Algorithm 3.1 it holds $\mu_k = \mu_m$ for all $k > m$, $R_k > \tau R_m > 0$ and $\rho_k \to \infty$. However, by Lemma 3.10 and Lemma 3.11 we get

$$\lim_{k \to \infty} R_k = \lim_{k \to \infty} \|(\bar{y}_k - \psi)_+\|_{C(\bar{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)| = 0,$$

yielding a contradiction. □

3.3 Convergence of the Algorithm

Let us recall that the sequence $(y_n^+, u_n^+, p_n^+)_n$ denotes the solution of the n-th successful iteration of Algorithm 3.1 with μ_n^+ being the corresponding approximation of the Lagrange multiplier. We want to show convergence of the algorithm next. The most important part is proving $L^1(\Omega)$-boundedness of the Lagrange multipliers μ_n^+, which is accomplished in Lemma 3.14 below.

Lemma 3.13. *Let y_n^+, μ_n^+ be given as defined in Algorithm 2. Then it holds*

$$|(\mu_n^+, \psi - y_n^+)| \leq \tau^{n-1} \left(\|(y_1^+ - \psi)_+\|_{C(\overline{\Omega})} + \|\mu_1^+\|_{L^2(\Omega)} \|(\psi - y_1^+)_+\|_{L^2(\Omega)} \right).$$

Proof. By definition of a successful step in Algorithm 3.1, we get the result directly by induction and the Cauchy-Schwarz inequality. ☐

Let us now show the $L^1(\Omega)$-boundedness of the sequence of Lagrange multipliers $(\mu_n^+)_n$.

Lemma 3.14 (Boundedness of the iterates). *Let Assumption 3.3 be fulfilled. Then Algorithm 3.1 generates an infinite sequence of bounded iterates, i.e., there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ it holds*

$$\|y_n^+\|_{H^1(\Omega)} + \|y_n^+\|_{C(\overline{\Omega})} + \|u_n^+\|_{L^2(\Omega)} + \|p_n^+\|_{W^{1,s}(\Omega)} + \|\mu_n^+\|_{L^1(\Omega)} \leq C.$$

Proof. Let $(\hat{y}, \hat{u})$ be the Slater point given by Assumption 3.3, i.e., there exists $\sigma > 0$, such that $\hat{y} + \sigma \leq \psi$. Then we can estimate

$$\begin{aligned} \sigma||\mu_n^+||_{L^1(\Omega)} &= \int_\Omega \sigma\mu_n^+ \,\mathrm{d}x \leq \int_\Omega \mu_n^+(\psi - \hat{y}) \,\mathrm{d}x \\ &= \int_\Omega \mu_n^+(\psi - y_n^+ + y_n^+ - \hat{y}) \,\mathrm{d}x \\ &= \underbrace{\int_\Omega \mu_n^+(\psi - y_n^+) \,\mathrm{d}x}_{\text{(I)}} + \underbrace{\int_\Omega \mu_n^+(y_n^+ - \hat{y}) \,\mathrm{d}x}_{\text{(II)}}. \end{aligned}$$

The first part (I) can be estimated with Lemma 3.13 yielding

$$\begin{aligned} \int_\Omega \mu_n^+(\psi - y_n^+) \,\mathrm{d}x &\leq |(\mu_n^+, \psi - y_n^+)| \\ &\leq \tau^{n-1} \left(\|(y_1^+ - \psi)_+\|_{C(\bar{\Omega})} + \|\mu_1^+\|_{L^2(\Omega)} \|(\psi - y_1^+)_+\|_{L^2(\Omega)} \right) \\ &= \tau^{n-1} C. \end{aligned} \tag{3.14}$$

Applying the Cauchy–Schwarz and Young's inequality we observe that

$$\begin{aligned} \alpha(u_n^+, \hat{u} - u_n^+) &= \alpha(u_n^+ - \hat{u}, \hat{u} - u_n^+) + \alpha(\hat{u}, \hat{u} - u_n^+) \\ &\leq -\alpha \|\hat{u} - u_n^+\|_{L^2(\Omega)}^2 + \alpha \|\hat{u}\|_{L^2(\Omega)} \|\hat{u} - u_n^+\|_{L^2(\Omega)} \\ &\leq -\alpha \|\hat{u} - u_n^+\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\hat{u}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\hat{u} - u_n^+\|_{L^2(\Omega)}^2 \\ &\leq -\frac{\alpha}{2} \|\hat{u} - u_n^+\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\hat{u}\|_{L^2(\Omega)}^2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(y_n^+ - y_d, \hat{y} - y_n^+) &= (y_n^+ - \hat{y}, \hat{y} - y_n^+) + (\hat{y} - y_d, \hat{y} - y_n^+)\\ &\leq -\frac{1}{2}\left\|\hat{y} - y_n^+\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\hat{y} - y_d\right\|^2_{L^2(\Omega)}.\end{aligned}$$

The second part (II) can now be estimated as follows

$$\begin{aligned}\int_\Omega \mu_n^+ (y_n^+ - \hat{y})\,\mathrm{d}x &= \langle A^* p_n^+ - (y_n^+ - y_d), y_n^+ - \hat{y}\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}\\ &= (p_n^+, u_n^+ - \hat{u}) - (y_n^+ - y_d, y_n^+ - \hat{y})\\ &\leq \alpha(u_n^+, \hat{u} - u_n^+) + (y_n^+ - y_d, \hat{y} - y_n^+)\\ &\leq -\frac{\alpha}{2}\left\|\hat{u} - u_n^+\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|\hat{y} - y_n^+\right\|^2_{L^2(\Omega)}\\ &\quad + \frac{\alpha}{2}\left\|\hat{u}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\hat{y} - y_d\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$\begin{aligned}&\left\|\mu_n^+\right\|_{L^1(\Omega)} + \frac{\alpha}{2\sigma}\left\|\hat{u} - u_n^+\right\|^2_{L^2(\Omega)} + \frac{1}{2\sigma}\left\|\hat{y} - y_n^+\right\|^2_{L^2(\Omega)}\\ &\qquad \leq \frac{\tau^{n-1}}{\sigma}C + \frac{\alpha}{2\sigma}\left\|\hat{u}\right\|^2_{L^2(\Omega)} + \frac{1}{2\sigma}\left\|\hat{y} - y_d\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $\tau \in (0,1)$ by assumption, the right-hand side is bounded. Consequently, we get boundedness of $(u_n^+)_n$ in $L^2(\Omega)$ and boundedness of $(\mu_n^+)_n$ in $L^1(\Omega)$. By the regularity result Theorem 2.22, the sequence $(y_n^+)_n$ is uniformly bounded in $H^1(\Omega) \cap C(\overline{\Omega})$. Boundedness of $(p_n^+)_n$ follows directly from Theorem 3.7. □

Let us note that the proof of the previous Lemma 3.14 yields boundedness of $(u_n^+)_n$ without using boundedness of the admissible set U_{ad}.

Theorem 3.15 (Convergence of solutions of the augmented Lagrangian algorithm). *Let $\bar{u}$ be the solution of (P) with corresponding state $\bar{y}$. As $n \to \infty$ we have for the sequence $(y_n^+, u_n^+)_n$ generated by Algorithm 3.1*

$$(y_n^+, u_n^+) \to (\bar{y}, \bar{u}) \quad \text{in } (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega).$$

Proof. Since the algorithm yields an infinite number of successful steps (Lemma 3.12) we get

$$\lim_{n\to\infty} R_n^+ = \lim_{n\to\infty} \left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)| = 0. \tag{3.16}$$

From Lemma 3.8 we get the following inequality

$$\begin{aligned}\left\|\bar{y} - y_n^+\right\|^2_{L^2(\Omega)} + \alpha\left\|\bar{u} - u_n^+\right\|^2_{L^2(\Omega)} &\leq \langle \bar{\mu}, y_n^+ - \psi\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|\\ &\leq \left\|\bar{\mu}\right\|_{\mathcal{M}(\overline{\Omega})}\left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|.\end{aligned}$$

With (3.16) from above, we conclude

$$0 \leq \lim_{n\to\infty} \left\|\bar{y} - y_n^+\right\|^2_{L^2(\Omega)} + \alpha\left\|\bar{u} - u_n^+\right\|^2_{L^2(\Omega)} \leq 0,$$

yielding $y_n^+ \to \bar{y}$ in $L^2(\Omega)$ and $u_n^+ \to \bar{u}$ in $L^2(\Omega)$. In addition, we get strong convergence of $y_n^+ \to \bar{y}$ in $H^1(\Omega) \cap C(\overline{\Omega})$ since S is compact. □

The next step in the convergence analysis is to show the convergence of the sequences of the dual quantities $(\mu_n^+)_n$ and $(p_n^+)_n$ to multipliers and adjoint states of the original problem (P). Since these sequences are bounded in $L^1(\Omega)$ and $W^{1,s}(\Omega)$, $s \in (1, d/(d-1))$, we can extract weak-* and weakly convergent subsequences in $\mathcal{M}(\overline{\Omega})$ and $W^{1,s}(\Omega)$, respectively. These weak subsequential limits are indeed Lagrange multipliers for the original problem.

Theorem 3.16 (Subsequential convergence of dual quantities). *Let subsequences $(p_{n_j}^+, \mu_{n_j}^+)_{n_j}$ of $(p_n^+, \mu_n^+)_n$ be given such that $\mu_{n_j}^+ \rightharpoonup^* \bar{\mu}$ in $\mathcal{M}(\overline{\Omega})$ and $p_{n_j}^+ \rightharpoonup \bar{p}$ in $W^{1,s}(\Omega)$, $s \in (1, d/(d-1))$. Then $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ satisfies the optimality system* (3.2) *of the original problem* (P).

Proof. The proof that the limits satisfy the adjoint equation (3.2b) can be found in [61, Lemma 2.6]. It remains to prove that the weak-* limit of $\mu_{n_j}^+$ is indeed a Lagrange multiplier. First, we prove the positivity property $\langle \bar{\mu}, \varphi \rangle \geq 0 \quad \forall \varphi \in C(\overline{\Omega})$ with $\varphi \geq 0$. By construction of the update of the Lagrange multiplier we obtain that $\mu_n^+ \geq 0$ pointwise, which implies

$$\int_\Omega \mu_n^+ \varphi \, \mathrm{d}x \geq 0 \quad \forall \varphi \in C(\overline{\Omega}) \text{ with } \varphi \geq 0,$$

which in turn yields

$$0 \leq \int_\Omega \mu_{n_j}^+ \varphi \, \mathrm{d}x \to \langle \bar{\mu}, \varphi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \quad \forall \varphi \in C(\overline{\Omega}) \text{ with } \varphi \geq 0.$$

Next, we show that the complementary slackness condition $\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0$ is fulfilled. From Theorem 3.15 we get $y_{n_j} \to \bar{y}$ in $C(\overline{\Omega})$. Hence, we obtain

$$0 = \lim_{j \to \infty} \left| (\mu_{n_j}^+, \psi - y_{n_j}^+) \right| = \left| \langle \bar{\mu}, \psi - \bar{y} \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \right|,$$

and, thus, the validity of the complementary condition. The inequality $(\bar{p} + \alpha \bar{u}, u - \bar{u}) \geq 0$ for $u \in U_{\text{ad}}$ follows with $u_{n_j}^+ \to \bar{u}$ in $L^2(\Omega)$ and $p_{n_j}^+ \rightharpoonup \bar{p}$ in $L^2(\Omega)$ from (3.4c). This shows that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ satisfies (3.2). □

Let us put this convergence result into perspective. Similar results, i.e., strong convergence of primal quantities and weak/weak-* convergence of dual quantities, are available for many other methods. For instance, such results were established for Lavrentiev-regularization [61], penalization-based approaches combined with path-following methods [56], and interior point methods [82, 112].

Since Lagrange multipliers are not uniquely determined in general, we cannot expect weak convergence of the whole sequences $(\mu_n^+)_n$ and $(p_n^+)_n$. If we assume uniqueness of multipliers then this is possible indeed.

Corollary 3.17. *Let $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ satisfy* (3.2). *Let us assume that $(\bar{p}, \bar{\mu})$ are uniquely determined Lagrange multipliers. Then it holds with $s \in (1, d/(d-1))$*

$$\begin{aligned} p_n^+ &\rightharpoonup \bar{p} \quad \text{in } W^{1,s}(\Omega), \\ \mu_n^+ &\rightharpoonup^* \bar{\mu} \quad \text{in } \mathcal{M}(\overline{\Omega}). \end{aligned}$$

Let us now prove that bounded penalty parameters imply existence of Lagrange multipliers in $L^2(\Omega)$.

Theorem 3.18 (Boundedness of penalty parameters implies multipliers in $L^2(\Omega)$). *Let the assumptions of the previous Theorem 3.16 be satisfied. Assume that $(\rho_n)_n$ is a bounded sequence. Then $(\mu_n^+)_n$ is bounded in $L^2(\Omega)$, and the multiplier $\bar{\mu}$ given by Theorem 3.16 belongs to $L^2(\Omega)$.*

Proof. Inequality (3.5) in the proof of Lemma 3.8 and the complementarity condition (3.2d) imply

$$\left\|\bar{y} - y_n^+\right\|_{L^2(\Omega)}^2 + \alpha \left\|\bar{u} - u_n^+\right\|_{L^2(\Omega)}^2 \leq \langle \bar{\mu}, y_n^+ - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} + (\mu_n^+, \bar{y} - y_n^+).$$

Then inequality (3.8) from the proof of Lemma 3.9 yields

$$\begin{aligned} \left\|\bar{y} - y_n^+\right\|_{L^2(\Omega)}^2 + \alpha \left\|\bar{u} - u_n^+\right\|_{L^2(\Omega)}^2 &\leq \langle \bar{\mu}, y_n^+ - \psi \rangle + (\mu_n^+, \bar{y} - y_n^+) \\ &\leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} + \frac{1}{2\rho_n} \left\|\mu_{n-1}^+\right\|_{L^2(\Omega)}^2 - \frac{1}{2\rho_n} \left\|\mu_n^+\right\|_{L^2(\Omega)}^2 . \end{aligned}$$

Rearranging the terms, we obtain

$$\frac{1}{2\rho_n} \left\|\mu_n^+\right\|_{L^2(\Omega)}^2 - \frac{1}{2\rho_n} \left\|\mu_{n-1}^+\right\|_{L^2(\Omega)}^2 \leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} .$$

Since $\rho_n \geq \rho_{n-1}$, we get

$$\frac{1}{2\rho_n} \left\|\mu_n^+\right\|_{L^2(\Omega)}^2 - \frac{1}{2\rho_{n-1}} \left\|\mu_{n-1}^+\right\|_{L^2(\Omega)}^2 \leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} .$$

Summing up we obtain

$$\begin{aligned} \frac{1}{2\rho_n} \left\|\mu_n^+\right\|_{L^2(\Omega)}^2 &\leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \sum_{j=1}^{n} \left\|(y_j^+ - \psi)_+\right\|_{C(\overline{\Omega})} \leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \sum_{j=1}^{n} R_j^+ \\ &\leq \|\bar{\mu}\|_{\mathcal{M}(\overline{\Omega})} \frac{\tau}{1-\tau} R_0^+ . \end{aligned}$$

This implies the boundedness of $(\mu_n^+)_n$ in $L^2(\Omega)$. Arguing as in the proof of Theorem 3.16, we can prove that weak accumulation points of $(\mu_n^+)_n$ in $L^2(\Omega)$ are multipliers to the state constraint. □

Let us emphasize that this result constitutes a remarkable difference to augmented Lagrangian methods in the finite dimensional setting, where the penalty parameter does not need to go to infinity. It is an open problem to modify the augmented Lagrangian scheme to obtain a method with this property.

3.4 Numerical Examples

In this section, we report on numerical results for the solution of an elliptic pointwise state constrained optimal control problem in two dimensions. All optimal control problems have been solved using Algorithm 3.1 implemented with FEniCS [86] using the DOLFIN [87] Python interface. We solved the discretized subproblems $(P_{AL})_k$ by applying an active-set method, which can also be interpreted as a semi-smooth Newton method, see [64] for solving state constrained optimal control problems and [116, Section 2.12.4] for an application to control constraints. The exact solution of the subproblem is obtained if there is no change in the active set [64, Proposition 2.1]. For all examples, the augmented Lagrangian algorithm was stopped as soon as

$$R_n^+ = \left\|(y_n^+ - \psi)_+\right\|_{C(\bar{\Omega})} + |(\mu_n^+, \psi - y_n^+)| \leq 10^{-6}$$

was satisfied, i.e., the violation of feasibility and complementarity is sufficiently small. Since the discretized version of the subproblems (3.3) are solved almost exactly, this yields a discrete KKT point of the original problem, which is, due to convexity, a global solution. In the following, $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_h)$ denotes the calculated solution after the stopping criterion is reached.

3.4.1 Example 1 – Control and State Constrained Problem

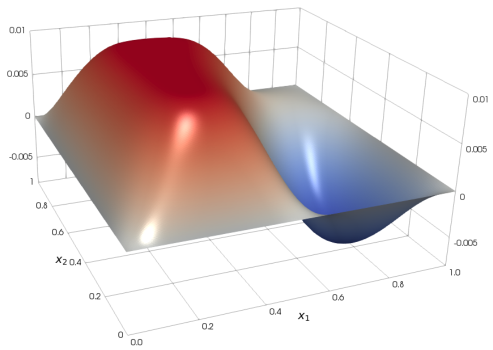
We consider an optimal control problem with with $\Omega = (0,1)^2$ given by

$$\begin{aligned}
\text{minimize} \quad & J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \\
\text{subject to} \quad & -\Delta y = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma, \\
& y \leq \psi \quad \text{in } \overline{\Omega}, \\
& u_a(x) \leq u(x) \leq u_b(x) \quad \text{a.e. in } \Omega.
\end{aligned}$$

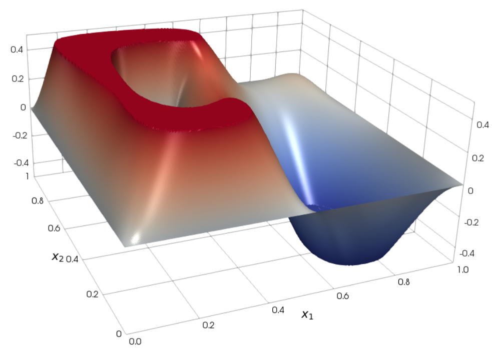
This setting differs slightly from the setting in which our convergence theory has been developed. Here, the given PDE has to satisfy homogeneous Dirichlet boundary conditions. Since the state equation of this example admits $H^2(\Omega)$-regular solutions, regularity results similar to Theorem 2.22 and Theorem 2.25 are satisfied, see [26, Theorem 4]. Moreover, KKT conditions analogous to (3.2) can be established, see [26, Theorem 2]. The convergence analysis of the augmented Lagrangian method is not affected at all and can be transferred line by line to this problem setting. For this test case we adapt the numerical example from [55]. With $x := (x_1, x_2) \in \Omega$ we set

$$\begin{aligned}
y_d(x) := 10(\sin(2\pi x_1) + x_2), \quad \psi(x) := 0.01, \quad \alpha := 0.15, \\
u_a(x) := -0.5, \quad u_b(x) := 0.5.
\end{aligned}$$

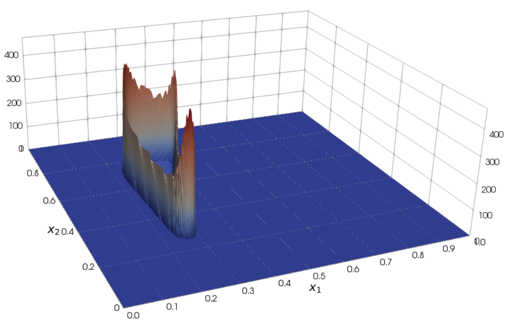
The algorithm was initialized with $\rho_1 := 100$ and $(\bar{y}_o, \bar{u}_0, \bar{p}_0, \mu_1)$ equal to zero. We choose $\tau := 0.1$ as the parameter in the decision concerning successful steps. If a step has not been successful, the penalization parameter is increased by the factor $\theta := 5$. Figure 3.1 shows the numerical solution of Example 1. All figures depict results gained for a triangular mesh with 10^5 degrees of freedom (dofs).



Optimal state $\bar{y}_h$



Optimal control $\bar{u}_h$



Optimal multiplier $\bar{\mu}_h$

Figure 3.1: (Example 1) Computed results for approximately 10^5 degrees of freedom.

3.4.2 Example 2 – State Constrained Problem with Exact Solution

In [107] an example has been presented such that the state constraint $y \geq \psi$ has to be satisfied. We modify the given example such that the constraints suit our setting. Thus, we consider the domain $\Omega = (-1,2)^2$ and the optimal control problem

$$\begin{aligned} &\text{minimize } J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \\ &\text{subject to } \quad -\Delta y = u + f \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma, \\ &\qquad\qquad y \leq \psi \quad \text{in } \overline{\Omega}. \end{aligned}$$

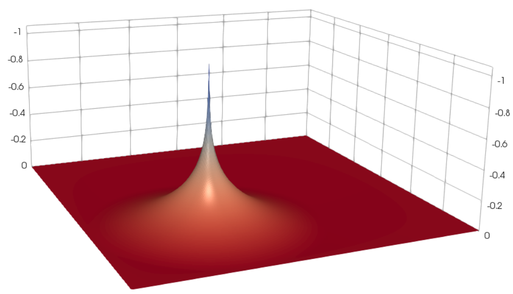
Since the function f belongs to $L^2(\Omega)$, the state equation admits solutions in $H^2(\Omega)$, and the remarks for Example 1 are valid here as well. Hence, our convergence theory readily transfers to this problem setting. To shorten our notation we set $r := r(x_1, x_2) := (x_1^2 + x_2^2)^{1/2}$. For the functions

$$\begin{aligned} y_d(r) &:= \bar{y}(r) - \frac{1}{2\pi}\chi_{r\leq 1}(4 - 9r), \\ \psi(r) &:= -\frac{1}{2\pi\alpha}\left(\frac{1}{4} - \frac{r}{2}\right), \\ f(r) &:= -\frac{1}{8\pi}\chi_{r\leq 1}(4 - 9r + 4r^2 - 4r^3), \end{aligned}$$

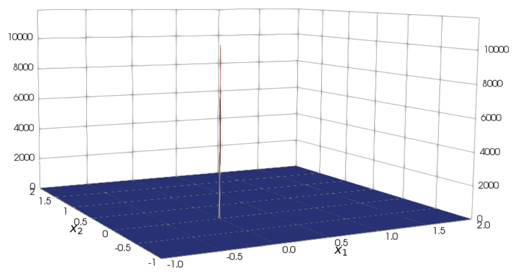
the exact solution of the optimal control problem is given by

$$\begin{aligned} \bar{y}(r) &:= -\frac{1}{2\pi\alpha}\chi_{r\leq 1}\left(\frac{r^2}{4}(\log r - 2) + \frac{r^3}{4} + \frac{1}{4}\right), \\ \bar{u}(r) &:= \frac{1}{2\pi\alpha}\chi_{r\leq 1}(\log r + r^2 - r^3), \\ \bar{p}(r) &:= -\alpha\bar{u}(r), \\ \bar{\mu}(r) &:= \delta_0(r). \end{aligned}$$

For this example we choose $\alpha := 1.0$. We started the algorithm with $\rho_1 := 0.5$ and $(\bar{y}_0, \bar{u}_0, \bar{p}_0, \mu_1)$ equal to zero and set $\tau := 0.11$, as well as $\theta := 10$. Figure 3.2 and 3.3 depict our numerical results for Example 2 for a triangular mesh with approximately 10^5 degrees of freedom. The computed Lagrange multiplier behaves like expected, approximating $\delta_0(r)$.



Optimal control $\bar{u}_h$.



Optimal multiplier $\bar{\mu}_h$.

Figure 3.2: (Example 2) Computed optimal control $\bar{u}_h$ (left) and optimal multiplier $\bar{\mu}_h$ (right).

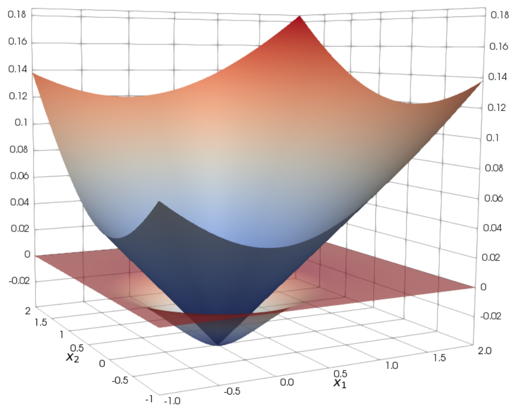


Figure 3.3: (Example 2) State constraint ψ and optimal state $\bar{y}_h$ (transparent).

Since the exact solution of the problem is known, the errors of the control $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$ and the error of the state $\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}$ can be evaluated. Figure 3.4 depicts the errors depending on the numbers of degrees of freedom, showing once again convergence of our algorithm.

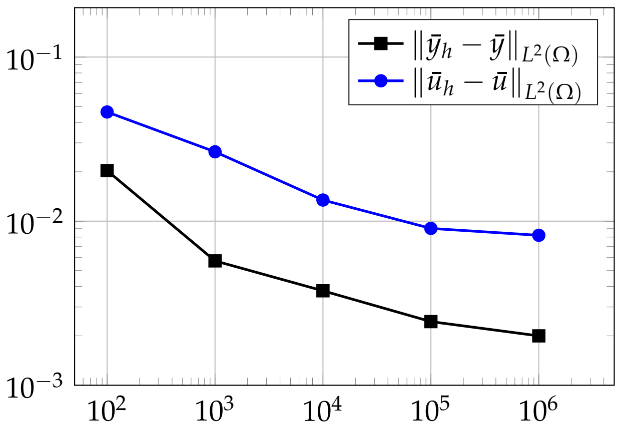


Figure 3.4: (Example 2) Errors $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$ and $\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}$ vs. degrees of freedom.

3.4.3 Iteration Numbers and Penalization Parameter

Finally, let us report about the number of iterations and the final penalization parameter for different refinements of the mesh in both examples. Table 3.1 shows the number of outer iterations, i.e., the iteration numbers of the augmented Lagrangian algorithm until the stopping criterion is reached for our two examined examples. Further, the accumulated inner iteration numbers, that are needed to solve the subproblems using a primal-dual active set method is given. It also shows the penalization parameter ρ_{max} of the final iteration and the $L^1(\Omega)$-norm of the approximated Lagrange multiplier.
Figure 3.5 illustrates the $L^1(\Omega)$-norm of the approximated Lagrange multiplier μ_k during the iterations. Clearly, this sequence is bounded in $L^1(\Omega)$. In addition, the values of the penalization parameters ρ_k are depicted in logarithmic scale. As can be seen, this sequence is not bounded. If it would be bounded, then the sequence $(\mu_k)_k$ would be bounded in $L^2(\Omega)$ due to Theorem 3.18.

Degrees of freedom		10^2	10^3	10^4	10^5
Example 1	outer it	11	13	15	15
	inner it	29	44	55	65
	ρ_{max}	$1.6 \cdot 10^6$	$3.9 \cdot 10^7$	$9.8 \cdot 10^8$	$9.8 \cdot 10^8$
	$\lVert\mu_h\rVert_{L^1(\Omega)}$	2.927	2.905	2.900	2.901
Example 2	outer it	10	10	12	14
	inner it	16	23	35	43
	ρ_{max}	$5 \cdot 10^4$	$5 \cdot 10^5$	$5 \cdot 10^6$	$5 \cdot 10^8$
	$\lVert\mu_h\rVert_{L^1(\Omega)}$	0.911	0.9996	1.060	1.070

Table 3.1: Iteration history for both examples and different discretizations.

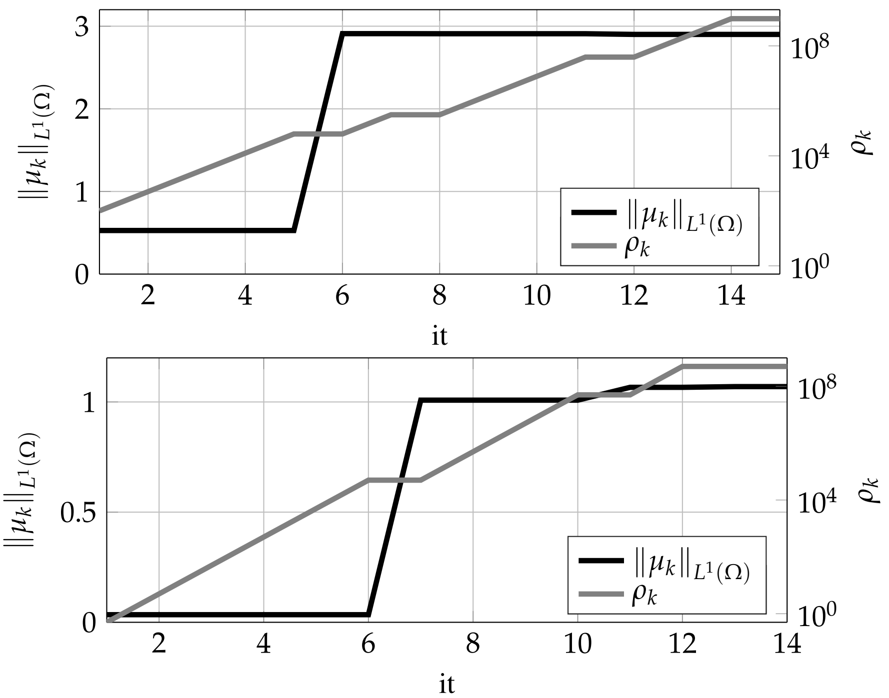


Figure 3.5: (Example 1 & 2) $L^1(\Omega)$-norm of discrete multipliers μ_k, penalty parameters ρ_k vs. iteration number. Top: Example 1, Bottom: Example 2.

CHAPTER 4

ILL-POSED STATE CONSTRAINED OPTIMAL CONTROL PROBLEMS WITH SPARSE CONTROLS

In this chapter, we extend the results from the previous chapter by replacing the regularizing Tikhonov term in the objective functional by an $L^1(\Omega)$-norm term, which causes sparsity of the control.

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded Lipschitz domain with boundary Γ. We consider the following optimal control problem:

$$\underset{u \in L^2(\Omega)}{\text{minimize}} \; J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \beta \|u\|_{L^1(\Omega)} \tag{4.1}$$

subject to

$$\begin{aligned} Ay &= u && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma, \\ y &\leq \psi && \text{in } \overline{\Omega}, \\ u_a \leq u &\leq u_b && \text{a.e. in } \Omega. \end{aligned}$$

Here, $\beta \geq 0$ is a positive parameter, A is a second order elliptic operator, $\psi \in C(\overline{\Omega})$, $y_d \in L^2(\Omega)$ and u_a, u_b are functions in $L^\infty(\Omega)$. For abbreviation, we set $j(u) := \|u\|_{L^1(\Omega)}$. The main difficulties in this problem are the pointwise state constraints $y(x) \leq \psi(x)$ and the convex but non-differentiable term $\|u\|_{L^1(\Omega)}$. Note, that there is no additional $L^2(\Omega)$-regularization term present in (4.1), which makes the problem ill-posed and numerically challenging.

Our aim is to modify and extend the method presented in Chapter 3 to obtain a numerical scheme to solve (4.1). The main idea is the following: To deal with the ill-posedness of (4.1), we add a Tikhonov regularization term, which results in

$$\text{minimize } J^\alpha(y,u) := J(y,u) + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}. \tag{4.2}$$

Here, $\alpha > 0$ is positive regularization parameter. Moreover, to overcome the problems, that arise due to the pointwise state constraints, we apply an augmented Lagrangian method. By penalizing the state constraint, in every iteration the following optimal control problem has to be solved:

$$\text{minimize } J^\alpha_\rho(y,u) := J(y,u) + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\left\|(\mu + \rho(y - \psi))_+\right\|^2 \tag{4.3}$$

subject to an elliptic partial differential equation and bilateral control constraints. Here, again $\alpha > 0$ denotes the regularization parameter of the Tikhonov term, while ρ is the penalization parameter of the augmented state constraints. Both variables are coupled in our method. During the algorithm we decrease the regularization parameter $\alpha \to 0$ while increasing the penalization parameter $\rho \to \infty$. The coupling is described in detail in Section 4.5. Since the decrease of α is a classical Tikhonov regularization approach, we aim to achieve strong convergence against the solution of (4.1). Now, let $\bar{u}$ denote the solution of (4.1), u^α the solution of (4.2) and u^α_ρ the solution of (4.3). Similar to [89] we split the error into the *Tikhonov error* and the *Lagrange error* in order to show convergence of the algorithm

$$\|\bar{u} - u^\alpha_\rho\|_{L^2(\Omega)} \le \underbrace{\|\bar{u} - u^\alpha\|_{L^2(\Omega)}}_{\text{Tikhonov error}} + \underbrace{\|u^\alpha - u^\alpha_\rho\|_{L^2(\Omega)}}_{\text{Lagrange error}}.$$

This chapter is organized as follows: We start by collecting results for the original problem in Section 4.1 and for the regularized problem in Section 4.2. The augmented Lagrangian subproblem, as well as the augmented Lagrangian method are introduced in Section 4.3. Section 4.4 aims at showing convergence of the algorithm. A detailed description of the numerical solution of the augmented Lagrangian subproblem can be found in Section 4.5. The results of this chapter have been published in [75].

4.1 The Original Problem

Throughout this chapter let A satisfy Assumption 2.19. Then, the following result holds true.

Theorem 4.1 ([29, Theorem 2.1]). *For every $u \in L^2(\Omega)$ there exists a unique weak solution $y \in H^1_0(\Omega) \cap C(\overline{\Omega})$ of the state equation and it holds*

$$\|y\|_{H^1_0(\Omega)} + \|y\|_{C(\overline{\Omega})} \le c\,\|u\|_{L^2(\Omega)}$$

with a constant $c > 0$ independent of u. Moreover, the control-to-state mapping $S\colon L^2(\Omega) \to H^1_0(\Omega) \cap C(\overline{\Omega})$, $u \mapsto y$ is a linear, continuous and compact operator.

We introduce the reduced formulation of problem (4.1).

$$\begin{aligned} \underset{u\in L^2(\Omega)}{\text{minimize}} \quad & f(u) := \frac{1}{2}\,\|Su - y_d\|^2_{L^2(\Omega)} + \beta\,\|u\|_{L^1(\Omega)} \\ \text{subject to} \quad & u_a(x) \le u(x) \le u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \le \psi(x) \qquad\qquad\quad \text{in } \overline{\Omega}, \end{aligned} \tag{P}$$

In the following, we will use the *admissible set* and the *feasible set* with respect to the state and control constraints denoted by

$$\begin{aligned} U_{\text{ad}} &:= \{u \in L^\infty(\Omega) \mid u_a(x) \le u(x) \le u_b(x) \text{ a.e. in } \Omega\}, \\ F_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u \in U_{\text{ad}},\ Su \le \psi \text{ in } \overline{\Omega}\}. \end{aligned}$$

Theorem 4.2. *Assume that the set F_{ad} is non-empty. Then, there exists a unique solution $\bar{u}$ of (P).*

Proof. Since S is injective, we obtain that the reduced cost functional f is strictly convex and continuous. As the set F_{ad} is weakly compact, see Theorem 2.47, existence of solutions follows directly from Corollary 2.33. □

To guarantee the existence of Lagrange multipliers, we will throughout this chapter assume that the Slater condition from Assumption 3.3 is satisfied. This allows us to derive first-order optimality conditions, see also Lemma 2.37.

Theorem 4.3 (First-order optimality conditions [34, Theorem 2.5]). *Let $\bar{u}$ be a solution of the problem (P). Furthermore, let the Slater condition from Assumption 3.3 be fulfilled. Then, there exists an adjoint state $\bar{p} \in W_0^{1,s}(\Omega)$, $s \in [1, d/(d-1))$, a Lagrange multiplier $\bar{\mu} \in \mathcal{M}(\overline{\Omega})$ and a subgradient $\bar{\lambda} \in \partial j(\bar{u})$ such that the following optimality system*

$$\begin{aligned} A\bar{y} &= \bar{u} \quad \text{in } \Omega, \\ \bar{y} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} A^*\bar{p} &= \bar{y} - y_d + \bar{\mu} \quad \text{in } \Omega, \\ \bar{p} &= 0 \qquad\qquad\quad \text{on } \Gamma, \end{aligned} \tag{4.4b}$$

$$(\bar{p} + \beta\bar{\lambda}, u - \bar{u}) \geq 0 \quad \forall u \in U_{\text{ad}}, \tag{4.4c}$$

$$\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{\mu} \geq 0, \quad \bar{y} \leq \psi, \tag{4.4d}$$

is fulfilled.

The next theorem shows the relation between the adjoint state and the control. One can see, that if β is large, the control will be zero on large parts of Ω. Hence, $\bar{u}$ is sparse.

Lemma 4.4 ([28, Theorem 3.1]). *Let $(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda}, \bar{\mu})$ satisfy the optimality system* (4.4a)-(4.4d). *Then the following relations hold for $\theta > 0$:*

$$\bar{u}(x) \begin{cases} = u_a(x) & \text{if } \bar{p}(x) > \beta \\ = u_b(x) & \text{if } \bar{p}(x) < -\beta \\ = 0 & \text{if } |\bar{p}(x)| < \beta \\ \in [u_a(x), u_b(x)] & \text{if } |\bar{p}(x)| = \beta \end{cases},$$

$$\bar{\lambda}(x) = P_{[-1,+1]}\left(-\frac{1}{\beta}\bar{p}(x)\right),$$

$$\bar{u}(x) = P_{[u_a(x),u_b(x)]}\left(\bar{u}(x) - \theta(\bar{p}(x) + \beta\bar{\lambda}(x))\right).$$

From the second formula it follows that $\bar{\lambda}$ is unique if the multiplier $\bar{\mu}$ and adjoint state $\bar{p}$ are unique.

4.2 The Regularized Problem

Solving problem (P) directly is challenging for mainly two reasons. First, the multiplier corresponding to the state constraints is only a measure. The second challenge is its ill-posedness. Small perturbations of the given data y_d may lead to large errors in the associated optimal controls. To deal with this issue we will use the well known Tikhonov regularization technique with some positive regularization parameter $\alpha > 0$. The regularized problem is in its reduced formulation given by

$$\begin{aligned} \underset{u \in L^2(\Omega)}{\text{minimize}} \quad & f^\alpha(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \beta\|u\|_{L^1(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_a(x) \leq u(x) \leq u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \leq \psi(x) \qquad\qquad\quad \text{in } \overline{\Omega}, \end{aligned} \tag{P^α}$$

Since f^α is strongly convex, it is clear, that (P^α) admits a unique solution u^α with associated state y^α. One can expect that u^α converges to the solution of (P) as $\alpha \to 0$. Similar results can be found in the literature, e.g. [118].

Lemma 4.5. *Let u^α be the unique solution of (P^α) with $\alpha > 0$ with associated state y^α. Furthermore let $\bar{u}$ be the unique solution of (P) and $\bar{y}$ its associated optimal state. Then, we have for $\alpha \to 0$*

$$\|u^\alpha - \bar{u}\|_{L^2(\Omega)} \to 0,$$
$$\frac{1}{\alpha}\|y^\alpha - \bar{y}\|^2_{L^2(\Omega)} \to 0.$$

Proof. We first show that $\|u^\alpha\|_{L^2(\Omega)} \le \|\bar{u}\|_{L^2(\Omega)}$ for all $\alpha > 0$. We start with

$$f(u^\alpha) + \frac{\alpha}{2}\|u^\alpha\|^2_{L^2(\Omega)} = f_\alpha(u^\alpha) \le f_\alpha(\bar{u}) = f(\bar{u}) + \frac{\alpha}{2}\|\bar{u}\|^2_{L^2(\Omega)} \le f(u^\alpha) + \frac{\alpha}{2}\|\bar{u}\|^2_{L^2(\Omega)},$$

where we exploited the optimality of u^α for (P^α) and the optimality of $\bar{u}$ for (P). This yields $\|u^\alpha\|_{L^2(\Omega)} \le \|\bar{u}\|_{L^2(\Omega)}$. Now we use that the set U_{ad} is weakly compact and extract a subsequence $u^{\alpha_i} \rightharpoonup u^* \in U_{\text{ad}}$. Since the operator S is compact, see Theorem 4.1, we obtain strong convergence of the state on the subsequence $y^{\alpha_i} \to y^* = Su^*$ in $H^1_0(\Omega) \cap C(\overline{\Omega})$. Now let $u \in U_{\text{ad}}$ be arbitrary, then

$$f(u^*) = \lim_{i\to\infty} f(u^{\alpha_i}) = \lim_{i\to\infty} f_{\alpha_i}(u^{\alpha_i}) \le \lim_{i\to\infty} f_{\alpha_i}(u) = f(u).$$

Hence u^* is a minimizer of f. The solution $\bar{u}$ of (P) is unique, thus we obtain $\bar{u} = u^*$. As the norm is weakly lower semicontinuous we get

$$\limsup_{i\to\infty} \|u^{\alpha_i}\| \le \|u^*\|_{L^2(\Omega)} \le \liminf_{i\to\infty} \|u^{\alpha_i}\|_{L^2(\Omega)} \le \limsup_{i\to\infty} \|u^{\alpha_i}\|_{L^2(\Omega)},$$

which shows $\|u^{\alpha_i}\|_{L^2(\Omega)} \to \|u^*\|_{L^2(\Omega)}$. As a well known fact, weak and norm convergence yield strong convergence and, hence, we obtain $u^{\alpha_i} \to u^*$ in $L^2(\Omega)$. As the sequence u^{α_i} was arbitrarily chosen we obtain convergence of the whole sequence $u^\alpha \to \bar{u}$. We now want to show improved convergence results for the states. Since the function

$$J_y : L^2(\Omega) \to \mathbb{R}, \quad y \mapsto \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)}$$

is a strongly convex function in y we know that the following inequality holds for all $t \in [0,1]$ and $y_1, y_2 \in L^2(\Omega)$ with some $m > 0$

$$J_y(ty_1 + (1-t)y_2) \le tJ_y(y_1) + (1-t)J_y(y_2) - m \cdot t(1-t)\|y_1 - y_2\|^2_{L^2(\Omega)}.$$

Now let $u \in F_{\text{ad}}$ and define $t := \frac{1}{2}$, $y_1 := \bar{y} = S\bar{u}$ and $y_2 := y = Su$. Furthermore, note that with $u, \bar{u} \in F_{\text{ad}}$ the convex combination is also feasible. To be precise, we obtain with the optimality of $\bar{u}$ that

$$J_y(\bar{y}) \le J_y\left(\frac{1}{2}\bar{y} + \frac{1}{2}y\right) \le \frac{1}{2}J_y(\bar{y}) + \frac{1}{2}J_y(y) - \frac{m}{4}\|\bar{y} - y\|^2_{L^2(\Omega)}.$$

Rearranging this inequality above yields the following growth condition

$$f(\bar{u}) + c\|\bar{y} - y\|^2_{L^2(\Omega)} \le f(u) \quad \forall u \in F_{\text{ad}}.$$

This growth condition can now be used to establish improved convergence results for the states y^α. Recall that $f_\alpha(u^\alpha) \le f_\alpha(\bar{u})$ and estimate

$$\begin{aligned} f(\bar{u}) + c\|y^\alpha - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u^\alpha\|^2_{L^2(\Omega)} &\le f(u^\alpha) + \frac{\alpha}{2}\|u^\alpha\|^2_{L^2(\Omega)} = f_\alpha(u^\alpha) \\ &\le f_\alpha(\bar{u}) = f(\bar{u}) + \frac{\alpha}{2}\|\bar{u}\|^2_{L^2(\Omega)}. \end{aligned}$$

This implies

$$\|y^\alpha - \bar{y}\|^2_{L^2(\Omega)} \le c \cdot \alpha \left(\|\bar{u}\|^2_{L^2(\Omega)} - \|u^\alpha\|^2_{L^2(\Omega)} \right).$$

Using the already established strong convergence $u^\alpha \to \bar{u}$, we get

$$\lim_{\alpha \to 0} \frac{1}{\alpha}\|y^\alpha - \bar{y}\|^2_{L^2(\Omega)} = 0,$$

which finishes the proof. □

With the Slater condition from Assumption 3.3, first-order necessary optimality conditions can be established for the regularized problem.

Theorem 4.6 (First-order necessary optimality conditions [34, Theorem 2.5]). *Let u^α be the solution of the problem (P^α) with corresponding state y^α. Furthermore, let Assumption 3.3 be satisfied. Then, there exists an adjoint state $p^\alpha \in W_0^{1,s}(\Omega)$, $s \in [1, d/(d-1))$, a Lagrange multiplier $\mu^\alpha \in \mathcal{M}(\overline{\Omega})$, and a subdifferential $\lambda^\alpha \in \partial j(u^\alpha)$ such that the following optimality system holds:*

$$\begin{aligned} Ay^\alpha &= u^\alpha \quad \text{in } \Omega, \\ y^\alpha &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.5a}$$

$$\begin{aligned} A^* p^\alpha &= y^\alpha - y_d + \mu^\alpha \quad \text{in } \Omega, \\ p^\alpha &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.5b}$$

$$(p^\alpha + \alpha u^\alpha + \beta\lambda^\alpha, u - u^\alpha) \ge 0 \quad \forall u \in U_{\text{ad}}, \tag{4.5c}$$

$$\langle \mu^\alpha, y^\alpha - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \mu^\alpha \ge 0, \quad y^\alpha \le \psi. \tag{4.5d}$$

In the following, we collect some results similar to Lemma 4.4.

Lemma 4.7. *Let $(y^\alpha, u^\alpha, p^\alpha, \lambda^\alpha, \mu^\alpha)$ satisfy the optimality system* (4.5a)-(4.5d). *Then the following relations hold:*

$$u^\alpha(x) = \begin{cases} u_a(x) & \text{if } \beta - \alpha u_a(x) < p^\alpha(x) \\ \frac{1}{\alpha}(\beta - p^\alpha(x)) & \text{if } \beta \le p^\alpha(x) \le \beta - \alpha u_a(x) \\ 0 & \text{if } |p^\alpha(x)| < \beta \\ \frac{1}{\alpha}(-\beta - p^\alpha(x)) & \text{if } -\alpha u_b(x) - \beta \le p^\alpha(x) \le -\beta \\ u_b(x) & \text{if } p^\alpha(x) < -\alpha u_b(x) - \beta, \end{cases}$$

$$\lambda^\alpha(x) = P_{[-1,1]}\left(-\frac{1}{\beta}(p^\alpha(x) + \alpha u^\alpha(x)) \right),$$

$$u^\alpha(x) = P_{[u_a(x), u_b(x)]}\left(-\frac{1}{\alpha}(p^\alpha(x) + \beta\lambda^\alpha(x)) \right).$$

Proof. These results can be proven by using a pointwise interpretation of the optimality condition (4.5c). □

In the subsequent analysis we will need that the multipliers for the problem (P^α) are uniformly bounded in $\mathcal{M}(\overline{\Omega})$ for all $\alpha \geq 0$. We will make use of the Slater condition to prove this. Note that for $\alpha = 0$ the problem (P^α) reduces to problem (P).

Lemma 4.8. *Let* $0 \leq \alpha \leq \tilde{c}$ *and define the set*

$$M^\alpha := \{\mu^\alpha \in \mathcal{M}(\overline{\Omega}) : \ (y^\alpha, u^\alpha, p^\alpha, \lambda^\alpha, \mu^\alpha) \text{ satisfy } (4.5a) - (4.5d)\}$$

of all multipliers associated with problem (P^α). *Then the multipliers are uniformly bounded, i.e., there exists a constant* $C > 0$ *independent from* α *such that*

$$\|\mu^\alpha\|_{\mathcal{M}(\overline{\Omega})} \leq C, \quad \forall \alpha \geq 0 \; \forall \mu^\alpha \in M^\alpha.$$

Proof. We follow the book of Tröltzsch [116] and consider the solution mapping $S\colon L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$. Then the dual operator is a mapping $S^*\colon \mathcal{M}(\overline{\Omega}) \to L^2(\Omega)$. Let $\alpha \geq 0$ be given and u^α be the solution of (P^α) with corresponding state y^α and associated multiplier μ^α. We now use the Slater condition and compute for any $f \in C(\overline{\Omega})$ with $\|f\|_\infty = 1$:

$$\begin{aligned}
\sigma\left|\int_\Omega f d\mu^\alpha\right| &\leq \sigma \int_\Omega |f| d\mu^\alpha \leq \int_\Omega \sigma d\mu^\alpha \leq \int_\Omega (\psi - \hat{y}) d\mu^\alpha \\
&= \underbrace{\langle \mu^\alpha, \psi - y^\alpha \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}}_{=0 \text{ by } (4.5d)} + \langle \mu^\alpha, y^\alpha - \hat{y} \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \\
&= \langle \mu^\alpha, S(u^\alpha - \hat{u}) \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \\
&= \int_\Omega (S^*\mu^\alpha)(u^\alpha - \hat{u})\, \mathrm{d}x.
\end{aligned}$$

Now, with Theorem 2.25 recall that the adjoint equation (4.5b) can be rewritten as

$$S^*\mu^\alpha = S^*(y_d - Su^\alpha) + p^\alpha.$$

Furthermore, by assumption $u^\alpha \in U_{\mathrm{ad}}$ and Theorem 4.1, we obtain that u^α and y^α are uniformly bounded in $L^2(\Omega)$. This now yields

$$\begin{aligned}
\sigma\|\mu^\alpha\|_{\mathcal{M}(\overline{\Omega})} &= \sigma \sup_{f \in C(\overline{\Omega}),\, \|f\|_\infty = 1} \left|\int_\Omega f\, \mathrm{d}\mu^\alpha\right| \\
&= \int_\Omega (S^*(y_d - Su^\alpha))(u^\alpha - \hat{u})\, \mathrm{d}x + \int_\Omega p^\alpha (u^\alpha - \hat{u})\, \mathrm{d}x.
\end{aligned}$$

We now apply the optimality condition (4.5c) and obtain with the boundedness of λ^α, see Lemma 4.7, and the boundedness of α that the following holds

$$\begin{aligned}
\sigma\|\mu^\alpha\|_{\mathcal{M}(\overline{\Omega})} &\leq c\|u^\alpha - \hat{u}\|_{L^2(\Omega)} \|y_d - y^\alpha\|_{L^2(\Omega)} + \int_\Omega (\alpha u^\alpha + \beta\lambda^\alpha)(\hat{u} - u^\alpha)\, \mathrm{d}x \\
&\leq c\|u^\alpha - \hat{u}\|_{L^2(\Omega)} \left(\|y_d - y^\alpha\|_{L^2(\Omega)} + \|\alpha u^\alpha + \beta\lambda^\alpha\|_{L^2(\Omega)}\right) \\
&\leq C.
\end{aligned}$$

Dividing the above inequality by $\sigma > 0$ finishes the proof. □

4.3 The Augmented Lagrangian Method

In the following we want to solve the regularized problem (P^α) for $\alpha \searrow 0$. For fixed α we follow the idea presented in Chapter 3 and replace the inequality constraint $y \leq \psi$ by an augmented penalization term. In that way we get rid of the measure in the corresponding optimality system and work with a more regular approximation instead.

4.3.1 The Augmented Lagrangian Optimal Control Problem

In each step of the augmented Lagrangian method the following subproblem has to be solved:

$$\begin{aligned} &\underset{u \in L^2(\Omega)}{\text{minimize}} \quad f^\alpha_\rho(u,\mu,\rho) := f(u) + \frac{1}{2\rho} \left\| (\mu + \rho(Su - \psi))_+ \right\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u \in U_{\text{ad}}. \end{aligned} \tag{P^α_{AL}}$$

with $\alpha, \rho > 0$, subject to the control constraints $u \in U_{\text{ad}}$. A solution of (P^α_{AL}) will be denoted by u^α_ρ with associated state y^α_ρ and adjoint state p^α_ρ. We know from Corollary 2.33 that the subproblem is uniquely solvable.

Theorem 4.9 (Existence of solutions of the augmented Lagrangian subproblem). *For every $\rho > 0$, $\mu \in L^2(\Omega)$ with $\mu \geq 0$ the augmented Lagrangian subproblem (P^α_{AL}) admits a unique solution $u^\alpha_\rho \in U_{\text{ad}}$.*

First-order optimality conditions can be established in a straight forward manner, see Lemma 2.37.

Theorem 4.10 (First-order necessary optimality conditions). *Let u^α_ρ be the solution of (P^α_{AL}) with associated state y^α_ρ. Then there exists a unique adjoint state $p^\alpha_\rho \in H^1_0(\Omega)$ associated with the optimal control u^α_ρ and a subdifferential $\lambda^\alpha_\rho \in \partial j(u^\alpha_\rho)$, satisfying the following system*

$$\begin{aligned} Ay^\alpha_\rho &= u^\alpha_\rho \quad \text{in } \Omega, \\ y^\alpha_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.6a}$$

$$\begin{aligned} A^* p^\alpha_\rho &= y^\alpha_\rho - y_d + \mu^\alpha_\rho \quad \text{in } \Omega, \\ p^\alpha_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.6b}$$

$$(p^\alpha_\rho + \alpha u^\alpha_\rho + \beta \lambda^\alpha_\rho, u - u^\alpha_\rho) \geq 0 \quad \forall u \in U_{\text{ad}}, \tag{4.6c}$$

$$\mu^\alpha_\rho := \Big(\mu + \rho(y^\alpha_\rho - \psi)\Big)_+ . \tag{4.6d}$$

4.3.2 The Augmented Lagrangian Algorithm

In the following, let $(P^\alpha_{AL})_k$ denote the augmented Lagrangian subproblem (P^α_{AL}) for given penalty parameter $\rho := \rho_k$, multiplier $\mu := \mu_k$ and regularization parameter $\alpha := \alpha_k$. We will denote its solution by $\bar{u}_k$ with corresponding state $\bar{y}_k$, adjoint state $\bar{p}_k$ and updated multiplier $\bar{\mu}_k$, which is given by (4.6d). We continue with a technical estimate, which will be useful in the subsequent analysis.

Lemma 4.11. *Let $\alpha_k > 0$ be given and let $(y^{\alpha_k}, u^{\alpha_k}, p^{\alpha_k}, \lambda^{\alpha_k}, \mu^{\alpha_k})$ be the solution of* (4.5) *and let $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\lambda}_k, \bar{\mu}_k)$ solve* (4.6). *Then it holds*

$$\begin{aligned} \|y^{\alpha_k} - \bar{y}_k\|^2_{L^2(\Omega)} + \alpha_k \|u^{\alpha_k} - \bar{u}_k\|^2_{L^2(\Omega)} &\leq (\bar{\mu}_k - \mu^{\alpha_k}, y^{\alpha_k} - \bar{y}_k) && (4.7) \\ &\leq (\bar{\mu}_k, \psi - \bar{y}_k) + \langle \mu^{\alpha_k}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\bar{\Omega}), C(\bar{\Omega})}. && (4.8) \end{aligned}$$

Proof. We have $\lambda^{\alpha_k} \in \partial j(u^{\alpha_k})$ and $\bar{\lambda}_k \in \partial j(\bar{u}_k)$. Exploiting that the subdifferential is a monotone operator, i.e., $(\lambda^{\alpha_k} - \bar{\lambda}_k, \bar{u}_k - u^{\alpha_k}) \leq 0$, the proof can be done like the proof of Lemma 3.8. □

The following result motivates the update rule and is a direct consequence of Lemma 4.11 .

Lemma 4.12. *Let $(y^{\alpha_k}, u^{\alpha_k}, p^{\alpha_k}, \lambda^{\alpha_k}, \mu^{\alpha_k})$ and $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\lambda}_k, \bar{\mu}_k)$ be given as in Lemma 4.11. Then it holds*

$$\frac{1}{\alpha_k} \|y^{\alpha_k} - \bar{y}_k\|^2_{L^2(\Omega)} + \|u^{\alpha_k} - \bar{u}_k\|^2_{L^2(\Omega)} \leq \frac{C}{\alpha_k} \big(\|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)| \big), \tag{4.9}$$

where C is the constant from Lemma 4.8.

Proof. We estimate the second term from the right-hand side of (4.8) from Lemma 4.11 via

$$\langle \mu^{\alpha_k}, \bar{y}_k - \psi \rangle \leq \|\mu^{\alpha_k}\|_{\mathcal{M}(\overline{\Omega})} \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} .$$

The result now follows using the uniform boundedness of μ^{α_k}, see Lemma 4.8. □

This result shows that the iterates $\bar{y}_k, \bar{u}_k$ will converge to the solution of the regularized problem (P^α) for fixed α_k, if the quantity

$$\frac{1}{\alpha_k} \big(\|(\bar{y}_k - \psi)_+\|_{C(\bar{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)| \big)$$

tends to zero for $k \to \infty$. Motivated by this finding, we adapt the update rule from Step 4 of Algorithm 3.1 and end up with the following algorithm:

Algorithm 4.1 Augmented Lagrangian Algorithm for (P)

Let $(\bar{y}_0, \bar{u}_0, \bar{p}_0) \in (H^1_0(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega) \times W^{1,s}_0(\Omega)$, $\alpha_1 > 0$, $\rho_1 > 0$ and $\mu_1 \in L^2(\Omega)$ be given with $\mu_1 \geq 0$. Choose $\theta > 1, 0 < \omega < 1$, $\tau \in (0,1)$. Set $k := 1$ and $n := 1$.

1: Compute a solution $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\lambda}_k)$ of $(P^\alpha_{AL})_k$.

2: Set $\bar{\mu}_k := (\mu_k + \rho_k(\bar{y}_k - \psi))_+$.

3: Compute $R_k := \frac{1}{\alpha_k} \big(\|(\bar{y}_k - \psi)_+\|_{C(\bar{\Omega})} + |(\bar{\mu}_k, \psi - \bar{y}_k)| \big)$.

4: If $R_k \leq \tau R^+_{n-1}$ then the step k is successful, set

$$\begin{cases} \alpha_{k+1} := \omega \alpha_k \\ \mu_{k+1} := \bar{\mu}_k \\ \rho_{k+1} := \rho_k \end{cases}$$

and define $(y^+_n, u^+_n, p^+_n, \lambda^+_n) := (\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\lambda}_k)$, as well as $\mu^+_n := \mu_{k+1}$, $R^+_n := R_k$ and $\alpha^+_n := \alpha_k$. Set $n := n + 1$.

5: Otherwise, the step k is not successful, set $\mu_{k+1} := \mu_k$ and $\alpha_{k+1} := \alpha_k$, and increase the penalty parameter $\rho_{k+1} := \theta \rho_k$.

6: If a stopping criterion is satisfied stop, otherwise set $k := k + 1$ and go to step 1.

Note that the regularization parameter α_k is only decreased when the algorithm produces a successful step. We will take advantage of this in the subsequent analysis. Moreover, in Chapter 3 the quantity R_k was also used as a stopping criterion. However, this is not possible here, as we proceed to let α go to zero. Instead, we will check the first-order optimality conditions for problem (P) as a stopping criterion. This will be described in detail in Section 4.5.

4.4 Convergence Results

The main aim of this section is to prove that the proposed algorithm produces infinitely many successful steps. In order to prove this, we consider the augmented Lagrangian KKT system of the minimization problem (P^{α}_{AL}). We fix the multiplier approximation μ, the regularization parameter α and let the penalization parameter ρ tend to infinity.

Lemma 4.13. *Let $\mu \in L^2(\Omega)$ with $\mu \geq 0$ and $\alpha > 0$ be given. Let u^{α}_{ρ} be a solution of (P^{α}_{AL}), $y^{\alpha}_{\rho} = Su^{\alpha}_{\rho}$ with $\rho > 0$ and u^{α} be the solution of (P^{α}) with $y^{\alpha} = Su^{\alpha}$. Then it holds*

a) $u^{\alpha}_{\rho} \to u^{\alpha}$ *in* $L^2(\Omega)$ *and* $y^{\alpha}_{\rho} \to y^{\alpha}$ *in* $H^1_0(\Omega) \cap C(\overline{\Omega})$ *for* $\rho \to \infty$.

b) $\lim_{\rho\to\infty}(\mu^{\alpha}_{\rho}, \psi - y^{\alpha}_{\rho}) = 0$.

Proof. The first assertion follows from the estimate (4.7) with the same proof strategy as in Lemma 3.10. The second one can be proven as in Lemma 3.11. □

With the help of this result, we can show that our algorithm produces infinitely many successful steps. This will be crucial in the convergence analysis in the next section.

Lemma 4.14. *The augmented Lagrangian algorithm makes infinitely many successful steps.*

Proof. Since α_k remains constant during all not successful steps, this can be proven with the help of Lemma 4.13 as in the proof of Lemma 3.12. □

In this section we want to show convergence of Algorithm 4.1. Let us recall that the sequence $(y^+_n, u^+_n, p^+_n)_n$ denotes the solution of the n-th successful iteration of Algorithm 4.1 with μ^+_n being the corresponding approximation of the Lagrange multiplier. We start with proving $L^1(\Omega)$-boundedness of the Lagrange multipliers μ^+_n, which is accomplished in Lemma 4.16 below. To prove this result we need an auxiliary estimate first.

Lemma 4.15. *Let y^+_n, μ^+_n be given as defined in Algorithm 4.1. Then it holds*

$$\frac{1}{\alpha^+_n}|(\mu^+_n, \psi - y^+_n)| \leq \frac{\tau^{n-1}}{\alpha^+_1}\left(\left\|(y^+_1 - \psi)_+\right\|_{C(\overline{\Omega})} + \left\|\mu^+_1\right\|_{L^2(\Omega)}\left\|(\psi - y^+_1)_+\right\|_{L^2(\Omega)}\right). \tag{4.10}$$

Proof. This follows directly from the definition of a successful step. □

Let us now show the $L^1(\Omega)$-boundedness of the sequence of Lagrange multipliers $(\mu^+_n)_n$.

Lemma 4.16 (Boundedness of the Lagrange multiplier). *Let Assumption 3.3 be fulfilled. Then Algorithm 4.1 generates an infinite sequence of bounded iterates, i.e., there is a constant $C > 0$ such that for all n it holds $\|\mu^+_n\|_{L^1(\Omega)} \leq C$.*

Proof. Since u^+_n and y^+_n are uniformly bounded, the subgradient $\lambda^+_n \in L^{\infty}(\Omega)$ is uniformly bounded by construction and the sequence $(\alpha^+_n)_n$ is monotonically decreasing, the proof can be done like the proof of Lemma 3.14, see also the proof of Lemma 4.8. □

Theorem 4.17 (Convergence of solutions). *As $n \to \infty$ we have for the sequence $(y^+_n, u^+_n)_n$ generated by Algorithm 4.1*

$$(y^+_n, u^+_n) \to (\bar{y}, \bar{u}) \quad \text{in } (H^1_0(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega),$$

where $\bar{u}$ denotes the unique solution of problem (P).

Proof. Since the algorithm yields an infinite number of successful steps (Lemma 4.14) we arrive at

$$\lim_{n\to\infty} R_n^+ = \lim_{n\to\infty} \frac{1}{\alpha_n^+}\left(\|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|\right) = 0. \tag{4.11}$$

with $\alpha_n^+ \to 0$. Let $(y^{\alpha_n^+}, u^{\alpha_n^+}, p^{\alpha_n^+}, \lambda^{\alpha_n^+}, \mu^{\alpha_n^+})$ be a solution of (4.5) for $\alpha := \alpha_n^+$. Then we obtain from Lemma 4.11 the following inequality

$$\begin{aligned}
\frac{1}{\alpha_n^+}\left\|y^{\alpha_n^+} - y_n^+\right\|^2_{L^2(\Omega)} + \left\|u^{\alpha_n^+} - u_n^+\right\|^2_{L^2(\Omega)} &\le \frac{1}{\alpha_n^+}\left(\langle \mu^{\alpha_n^+}, y_n^+ - \psi\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|\right)\\
&\le \frac{1}{\alpha_n^+}\left(\left\|\mu^{\alpha_n^+}\right\|_{\mathcal{M}(\overline{\Omega})}\|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|\right)\\
&\le \frac{c}{\alpha_n^+}\left(\|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} + |(\mu_n^+, \psi - y_n^+)|\right).
\end{aligned}$$

Note, that in the last step we used Lemma 4.8. With (4.11) from above, we conclude

$$\lim_{n\to\infty} \frac{1}{\alpha_n^+}\left\|y^{\alpha_n^+} - y_n^+\right\|^2_{L^2(\Omega)} + \left\|u^{\alpha_n^+} - u_n^+\right\|^2_{L^2(\Omega)} = 0. \tag{4.12}$$

We now split the error as described in the introduction

$$\|u_n^+ - \bar{u}\|_{L^2(\Omega)} \le \underbrace{\|u_n^+ - u^{\alpha_n^+}\|_{L^2(\Omega)}}_{\text{Lagrange error}} + \underbrace{\|u^{\alpha_n^+} - \bar{u}\|_{L^2(\Omega)}}_{\text{Tikhonov error}}. \tag{4.13}$$

Using (4.12) we obtain that the first term on the right hand side of (4.13) converges to zero. Now, we use the fact that our algorithm creates infinitely many successful steps, which gives $\alpha_n^+ \to 0$ as $n \to \infty$. We therefore conclude that $u^{\alpha_n^+} \to \bar{u}$, see Lemma 4.5. Hence, also the second term on the right hand side of (4.13) converges to zero. So in total, we obtain $u_n^+ \to \bar{u}$ in $L^2(\Omega)$. Convergence of $y_n^+ \to \bar{y}$ follows from Theorem 4.1, which finishes the proof. □

Corollary 4.18. *For the sequence $(y_n^+)_n$ generated by Algorithm 4.1 we obtain*

$$\frac{1}{\alpha_n^+}\|y_n^+ - \bar{y}\|^2_{L^2(\Omega)} \to 0,$$

which is similar to the results obtained for a Tikhonov regularization without state constraints, see [118] and Lemma 4.5.

Proof. We split the error to obtain with some $c > 0$ independent from n:

$$\frac{1}{\alpha_n^+}\|y_n^+ - \bar{y}\|^2_{L^2(\Omega)} \le \frac{c}{\alpha_n^+}\|y_n^+ - y^{\alpha_n^+}\|^2_{L^2(\Omega)} + \frac{c}{\alpha_n^+}\|y^{\alpha_n^+} - \bar{y}\|^2_{L^2(\Omega)}.$$

The result is now an immediate consequence of (4.12) and Lemma 4.5. □

Theorem 4.19. *Let $s \in (1, d/(d-1))$ such that the embedding $W_0^{1,s}(\Omega) \to L^2(\Omega)$ is compact. Moreover, let subsequences $(y_{n_j}^+, u_{n_j}^+, p_{n_j}^+, \lambda_{n_j}^+, \mu_{n_j}^+)_{n_j}$ of $(y_n^+, u_n^+, p_n^+, \lambda_n^+, \mu_n^+)_n$ be given, such that $p_{n_j}^+ \rightharpoonup \bar{p}$ in $W_0^{1,s}$ and $\mu_{n_j}^+ \overset{*}{\rightharpoonup} \bar{\mu}$ in $\mathcal{M}(\overline{\Omega})$. Then, $(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda}, \bar{\mu})$, where $\bar{\lambda} = P_{[-1,1]}(-\beta^{-1}\bar{p}(x))$, satisfy the optimality system* (4.4).

Proof. The proof mainly follows Lemma 3.16. The only thing, which remains to show is the validity of the variational inequality (4.4c). By the compact embedding $W_0^{1,s}(\Omega) \to L^2(\Omega)$ we obtain strong convergence of $p_{n_j}^+ \to \bar{p}$ in $L^2(\Omega)$. Further, the representation from Lemma 4.7 yields

$$\lambda_{n_j}^+ = P_{[-1,1]}\left(-\frac{1}{\beta}(p_{n_j}^+ + \alpha_n^+ u_{n_j}^+)(x)\right).$$

The facts, that $u_{n_j}^+$ is bounded and $\alpha_{n_j}^+ \searrow 0$, yield $\lambda_{n_j}^+ \to \bar{\lambda}$ in $L^2(\Omega)$. The strong convergence of the control $u_{n_j}^+ \to \bar{u}$ in $L^2(\Omega)$ now allows us to conclude

$$(p_{n_j}^+ + \alpha_{n_j}^+ u_{n_j}^+ + \beta\lambda_{n_j}^+, u - u_{n_j}^+) \to (\bar{p} + \beta\bar{\lambda}, u - \bar{u}) \geq 0 \quad \forall u \in U_{\text{ad}}. \qquad \square$$

Let us now assume that the adjoint state $\bar{p}$ and the multiplier corresponding to the state constraint $\bar{\mu}$ are unique, then the following result is a immediate consequence of the precedent theorem.

Theorem 4.20. *Let $(\bar{u}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{\mu})$ satisfy the KKT-system* (4.4). *Let us assume that $(\bar{p}, \bar{\mu})$ are uniquely given. Pick $s \in (1, d/(d-1))$ such that the embedding $W_0^{1,s}(\Omega) \to L^2(\Omega)$ is compact. Then $\bar{\lambda}$ is also unique and it holds*

$$\begin{aligned} p_n^+ &\rightharpoonup \bar{p} && \text{in } W_0^{1,s}(\Omega), \\ \mu_n^+ &\overset{*}{\rightharpoonup} \bar{\mu} && \text{in } \mathcal{M}(\overline{\Omega}), \\ \lambda_n^+ &\to \bar{\lambda} && \text{in } L^2(\Omega). \end{aligned}$$

4.5 The Numerical Method in Detail

In this section we want to introduce an active-set method for the solution of the subproblems arising in the augmented Lagrangian method stated in Algorithm 4.1. The subproblem is given as

$$\min_{u \in U_{\text{ad}}} \quad \frac{1}{2}||y - y_d||_{L^2(\Omega)}^2 + \beta\|u\|_{L^1(\Omega)} + \frac{\alpha}{2}||u||_{L^2(\Omega)}^2 + \frac{1}{2\rho}\,\|(\mu + \rho(y - \psi))_+\|_{L^2(\Omega)}^2. \quad (P_{AL}^\alpha)$$

We follow [113] and introduce multipliers for the bilateral inequality constraints for the control. Hence, the optimal solution $(\bar{y}, \bar{u}, \bar{p}) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ of (P_{AL}^α) is characterized by the existence of $\lambda, \lambda^a, \lambda^b \in L^2(\Omega)$ such that

$$\begin{cases} A\bar{y} = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases} \tag{4.14a}$$

$$\begin{cases} A^*\bar{p} = \bar{y} - y_d + \bar{\mu} & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \Gamma, \end{cases} \tag{4.14b}$$

$$\bar{p} + \alpha\bar{u} + \lambda + \lambda^b - \lambda^a = 0, \tag{4.14c}$$

$$\lambda^a \geq 0, \quad \bar{u} - u_a \geq 0, \quad \lambda^a(\bar{u} - u_a) = 0, \tag{4.14d}$$

$$\lambda^b \geq 0, \quad u_b - \bar{u} \geq 0, \quad \lambda^b(u_b - \bar{u}) = 0, \tag{4.14e}$$

$$\begin{cases} \lambda = \beta & \text{on } \{x \in \Omega : \ \bar{u} > 0\}, \\ |\lambda| \leq \beta & \text{on } \{x \in \Omega : \ \bar{u} = 0\}, \\ \lambda = -\beta & \text{on } \{x \in \Omega : \ \bar{u} < 0\}, \end{cases} \tag{4.14f}$$

$$\bar{\mu} := (\mu + \rho(\bar{y} - \psi))_+ . \tag{4.14g}$$

Here (4.14a) is the state equation, (4.14b) characterizes the adjoint state, (4.14d)-(4.14e) define the multipliers for the control constraints and (4.14f) reflects the fact that $\lambda \in \partial\beta\|\bar{u}\|_{L^1(\Omega)}$. The arising subproblems (P^{α}_{AL}) are solved by combining two methods. The first method is the active-set method presented by Stadler [113], where optimal control problems of type (P^{α}_{AL}) were solved, but without augmented state constraints. The second is the method established by Ito and Kunisch [64], who presented an active-set method for optimal control problems with state constraints but without an $L^1(\Omega)$-cost term. Like in [113] we set

$$\xi := \lambda - \lambda^a + \lambda^b,$$

where λ denotes the subdifferential of $\beta\,\|\bar{u}\|_{L^1(\Omega)}$, λ^a the corresponding multiplier to the lower control constraints $u_a - u \le 0$ and λ^b the multiplier corresponding to the upper control constraint $u - u_b \le 0$. Then (4.14a)-(4.14g) can be rewritten [113, Lemma 2.2] as

$$S^*(S\bar{u} - y_d + (\mu + \rho(\bar{y} - \psi))_+) + \alpha\bar{u} + \xi = 0, \tag{4.15a}$$

$$\begin{aligned} &\bar{u} - \max(0, \bar{u} + c(\xi - \beta) - \min(0, \bar{u} + c(\xi + \beta) \\ &\quad + \max(0, \bar{u} - u_b + c(\xi - \beta)) + \min(0, (\bar{u} - u_a) + c(\xi + \beta)) = 0, \end{aligned} \tag{4.15b}$$

with $c > 0$, where the multipliers can be derived via the formula

$$\begin{aligned} \lambda &= \min(\beta, \max(-\beta, \xi)), \\ \lambda^a &= -\min(0, \xi + \beta), \\ \lambda^b &= \max(0, \xi - \beta). \end{aligned} \tag{4.16}$$

Solving (4.15a) for ξ and inserting the solution in (4.15b), the arising equation can be solved with a semi-smooth Newton method, or equivalently an active-set method, see [113, Sec. 4.3]. In the following y_k, u_k, p_k and λ_k are iterates generated by the active-set method, which is described in Algorithm 4.2 below. We define the following sets, see also Lemma 4.7:

$$\begin{aligned} \mathcal{Y}^k_+ &= \{x \in \Omega : \; (\mu + \rho(y_k - \psi)) > 0\}, \\ \mathcal{Y}^k_- &= \Omega \setminus \mathcal{Y}^k_+, \\ \mathcal{A}^k_a &= \{x \in \Omega : \; p_k \ge \beta - \alpha u_a\}, \\ \mathcal{A}^k_0 &= \{x \in \Omega : \; |p_k| < \beta\}, \\ \mathcal{A}^k_b &= \{x \in \Omega : \; p_k \le -\alpha u_b - \beta\}, \\ \mathcal{I}^k_- &= \{x \in \Omega : \; \beta \le p_k < \beta - \alpha u_a\}, \\ \mathcal{I}^k_+ &= \{x \in \Omega : \; -\alpha u_b - \beta < p_k \le -\beta\}. \end{aligned}$$

The sets $\mathcal{A}^k_a$, $\mathcal{A}^k_0$ and $\mathcal{A}^k_b$ are called active sets, as on $\mathcal{A}^k_a$ we obtain $u_k = u_a$, on $\mathcal{A}^k_b$ we get $u_k = u_b$ and on $\mathcal{A}^k_0$ we have $u_k = 0$. Obviously, the five sets $\mathcal{A}^k_a$, $\mathcal{A}^k_0$, $\mathcal{A}^k_b$, $\mathcal{I}^k_-$ and $\mathcal{I}^k_+$ are disjoint and their union is Ω. The sets $\mathcal{Y}^k_-$ and $\mathcal{Y}^k_+$ are motivated by (4.14g). Note that (4.17b) in Algorithm 4.2 can be equivalently written as

$$u_{k+1} = \begin{cases} u_a & \text{on } \mathcal{A}^k_a, \\ 0 & \text{on } \mathcal{A}^k_0, \\ u_b & \text{on } \mathcal{A}^k_b, \end{cases} \qquad \xi_{k+1} = \begin{cases} -\beta & \text{on } \mathcal{I}^k_-, \\ \beta & \text{on } \mathcal{I}^k_+, \end{cases}$$

but it is more accessible in this form. The computation of the $L^1(\Omega)$-subgradient follows from the reconstruction formula (4.16). Further, the termination criterion yields a solution of the augmented Lagrangian subproblem (P^{α}_{AL}).

Algorithm 4.2 Active-set method for solving (P_{AL}^{α})

Choose initial data u_0, p_0 and parameters α, ρ, compute the sets $\mathcal{Y}_-^0$, $\mathcal{Y}_+^0$, $\mathcal{A}_a^0$, $\mathcal{A}_0^0$, $\mathcal{A}_b^0$, $\mathcal{I}_-^0, \mathcal{I}_+^0$.

1: Solve for $(y_{k+1}, u_{k+1}, p_{k+1}, \xi_{k+1})$ satisfying

$$\begin{aligned} Ay_{k+1} - u_{k+1} &= 0, \\ -A^* p_{k+1} + y_{k+1} - y_d + \mu_{k+1} &= 0, \\ p_{k+1} + \alpha u_{k+1} + \xi_{k+1} &= 0, \end{aligned} \tag{4.17a}$$

$$\begin{aligned} (1 - \chi_{\mathcal{A}_a^k} - \chi_{\mathcal{A}_b^k} - \chi_{\mathcal{A}_0^k})\xi_{k+1} &+ (1 - \chi_{\mathcal{I}_-^k} - \chi_{\mathcal{I}_+^k})u_{k+1} \\ &= \chi_{\mathcal{A}_a^k} u_a + \chi_{\mathcal{A}_b^k} u_b - \chi_{\mathcal{I}_-^k} \beta + \chi_{\mathcal{I}_+^k} \beta, \end{aligned} \tag{4.17b}$$

$$\mu_{k+1} = \begin{cases} 0 & \text{on } \mathcal{Y}_-^k, \\ \mu + \rho(y_{k+1} - \psi) & \text{on } \mathcal{Y}_+^k. \end{cases} \tag{4.17c}$$

2: Compute the sets $\mathcal{Y}_-^{k+1}, \mathcal{Y}_+^{k+1}, \mathcal{A}_a^{k+1}, \mathcal{A}_0^{k+1}, \mathcal{A}_b^{k+1}, \mathcal{I}_-^{k+1}, \mathcal{I}_+^{k+1}$.

3: If the following equalities hold: $\mathcal{A}_a^{k+1} = \mathcal{A}_a^k$, $\mathcal{A}_0^{k+1} = \mathcal{A}_0^k$, $\mathcal{A}_b^{k+1} = \mathcal{A}_b^k$, $\mathcal{I}_-^{k+1} = \mathcal{I}_-^k$, $\mathcal{I}_+^{k+1} = \mathcal{I}_+^k$, $\mathcal{Y}_-^{k+1} = \mathcal{Y}_-^k$ and $\mathcal{Y}_+^{k+1} = \mathcal{Y}_+^k$ then go step 4. Otherwise set $k := k + 1$ and go to step 2.

4: Compute the subdifferential $\lambda_{k+1} := \min(\beta, \max(-\beta, \xi_{k+1}))$ and stop the algorithm.

Lemma 4.21. *If the following equalities hold*

$$\begin{array}{llllllll} \mathcal{A}_a^{k+1} = \mathcal{A}_a^k, & \mathcal{A}_0^{k+1} = \mathcal{A}_0^k, & \mathcal{A}_b^{k+1} = \mathcal{A}_b^k, & \mathcal{I}_-^{k+1} = \mathcal{I}_-^k, \\ \mathcal{I}_+^{k+1} = \mathcal{I}_+^k, & \mathcal{Y}_-^{k+1} = \mathcal{Y}_-^k, & \mathcal{Y}_+^{k+1} = \mathcal{Y}_+^k, \end{array}$$

then $(u_{k+1}, y_{k+1}, p_{k+1}, \mu_{k+1}, \lambda_{k+1})$ *is a solution to* (4.6) *with* α, μ *and* β *fixed.*

Proof. For a detailed proof we refer to [100, Lemma 7.1.1]. □

However, high values of the penalty parameter ρ paired with small values of the Tikhonov parameter α may evoke bad stability during solution of the subproblem. As a termination criterion, we check the optimality conditions of the current iterate given by $(u_n^+, y_n^+, p_n^+, \lambda_n^+, \mu_n^+)$, i.e., we stop the algorithm if the following inequality is satisfied:

$$\left\| u_n^+ - P_{[u_a,u_b]}(u_n^+ - (p_n^+ + \beta\lambda_n^+)) \right\|_{L^2(\Omega)} + \|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} + |(\mu_n^+, y_n^+ - \psi)| \le \varepsilon.$$

4.6 Numerical Examples

In the following we want to present several numerical examples for Algorithm 4.1. The implementation was done with FEniCS [86] using the DOLFIN [87] Python interface. We apply our method for problems of the following form:

$$\text{minimize } J(y,u) := \frac{1}{2}||y - y_d||_{L^2(\Omega)}^2 + \beta\|u\|_{L^1(\Omega)}$$

subject to

$$\begin{aligned} Ay &= u + f && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma, \\ y &\leq \psi && \text{in } \overline{\Omega}, \\ u_a \leq u &\leq u_b && \text{a.e. in } \Omega. \end{aligned}$$

The additional variable $f \in L^2(\Omega)$ allows us to construct test problems with known solutions.

4.6.1 Example 1: Bang-Bang-Off Example in Two Space Dimension

We set $u_a := -1$, $u_b := 1$. Let Ω be the circle around 0 with radius 2. We now define the following functions. For clarity and to shorten our notation we set $r := r(x,y) := \sqrt{x^2+y^2}$.

$$\bar{y}(x,y) := \begin{cases} 1 & \text{if } r < 1 \\ 32 - 120 \cdot r + 180 \cdot r^2 - 130 \cdot r^3 + 45 \cdot r^4 - 6 \cdot r^5 & \text{if } r \geq 1 \end{cases}$$

$$\bar{p}(x,y) := \sin(x) \cdot \sin(y) \cdot \left(1 - \frac{5}{4} r^3 + \frac{15}{16} r^4 - \frac{3}{16} r^5\right)$$

$$\bar{u}(x,y) := -\text{sign}(\bar{p}(x,y))$$

$$\bar{\mu}(x,y) := \begin{cases} \text{Exp}\left(-\frac{1}{1-r^2}\right) & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}$$

$$\psi(x,y) := 1.$$

Some calculation show that $\bar{\mu}, \bar{p} \in C^2(\bar{\Omega})$ and $\bar{\mu} \in C(\overline{\Omega})$. Furthermore, $\bar{y} = \bar{p} = 0$ on Γ. We now set

$$\begin{aligned} f(x,y) &:= -\Delta \bar{y}(x,y) - \bar{u}(x,y), \\ y_d(x,y) &:= \Delta \bar{p}(x,y) + \bar{y}(x,y) + \bar{\mu}(x,y). \end{aligned}$$

One now can check that for $\beta = 0$ the functions $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ satisfy the KKT conditions defined in Theorem 4.3 leading to a bang-bang solution. For $\beta \neq 0$ we expect the optimal solution to exhibit a bang-bang-off structure. Here no exact solution is known. We computed the solution of this problem for different values of β on a regular triangular grid with approximately $1.3 \cdot 10^5$ degrees of freedom. The parameter used for this computation are $\tau := 0.8$, $\omega := 0.9$, $\theta := 5$ and $\varepsilon := 5 \cdot 10^{-5}$. We started with $\alpha := 0.1$, $\rho := 1$ and $(\bar{y}_o, \bar{u}_o, \bar{p}_0, \bar{\lambda}_0, \mu_1)$ equal to zero. Additional information for the calculations can be found in Table 4.1 while the computed controls can be seen in Figure 4.1. As expected we observe that the solution becomes more sparse as β becomes large. Taking a look at the final values of the regularization parameter α and penalization parameter ρ we see, that they are of the same order of magnitude for all β.

β	final α	final ρ	outer iterations	accumulated inner iterations
0.05	$9.7 \cdot 10^{-4}$	$6.1 \cdot 10^{9}$	58	165
0.1	$5.7 \cdot 10^{-4}$	$3.1 \cdot 10^{10}$	64	176
0.2	$2.5 \cdot 10^{-4}$	$3.1 \cdot 10^{10}$	65	171
0.5	$4.6 \cdot 10^{-4}$	$3.1 \cdot 10^{10}$	66	167

Table 4.1: (Example 1) Additional information for the computation for different β.

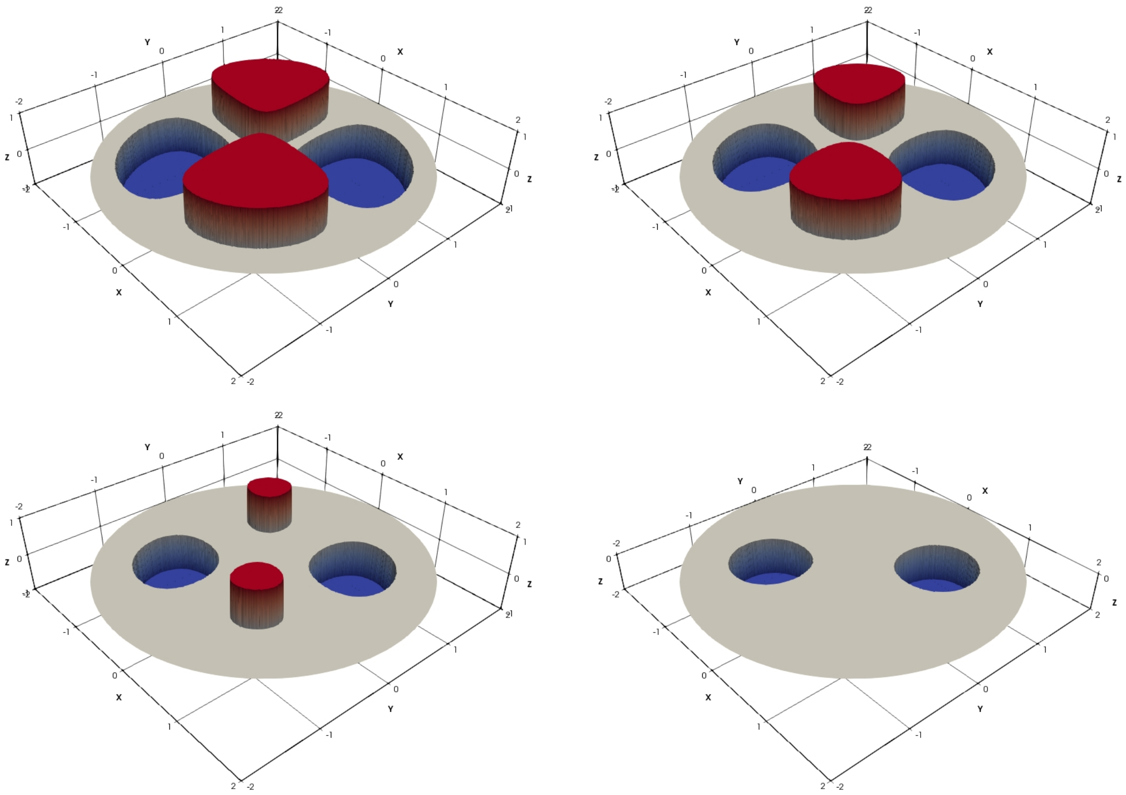


Figure 4.1: (Example 1) Computed discrete control $\bar{u}_h$ for different values of β. From left to right and from top to bottom: $\beta = 0.05$, $\beta = 0.1$, $\beta = 0.2$, $\beta = 0.5$.

4.6.2 Example 2

For the next example we set $\Omega := (0,1)^2$, $u_a := -1$, $u_b := 1$ and $\beta := 10^{-3}$. Furthermore $\tau := 0.8$, $\omega := 0.7$ and $\theta := 5$. Now define

$$\psi(x,y) := 0.01, \qquad y_d(x,y) := \frac{1}{2\pi}\sin(\pi x)\sin(\pi y)$$

Note that here no exact solution is available. If the state constraints and the $L^1(\Omega)$-term are neglected the exact solution is given by

$$\bar{y}(x,y) := y_d(x,y), \qquad \bar{u}(x,y) := \Delta y_d(x,y).$$

This example is taken from [99] and is an example of an optimal control problem where the desired state is reachable and the source condition $\bar{u} = S^*w$ with an element $w \in L^2(\Omega)$ is satisfied if the state constraints are not present. We computed the solution on a regular triangular grid with $6.6 \cdot 10^4$ degrees of freedom and $\varepsilon := 10^{-5}$. As starting values we set $\alpha := 0.01$ and $\rho := 10$. The stopping criterion has been satisfied after 38 outer iterations and (accumulated) 151 inner iterations with the final values $\alpha = 2.3 \cdot 10^{-5}$ and $\rho = 3.9 \cdot 10^9$. The computed results can be seen in Figure 4.2 and Figure 4.3. Clearly, the control $\bar{u}_h$ exhibits a bang-bang-off structure and the state $\bar{y}_h$ satisfies the state constraint.

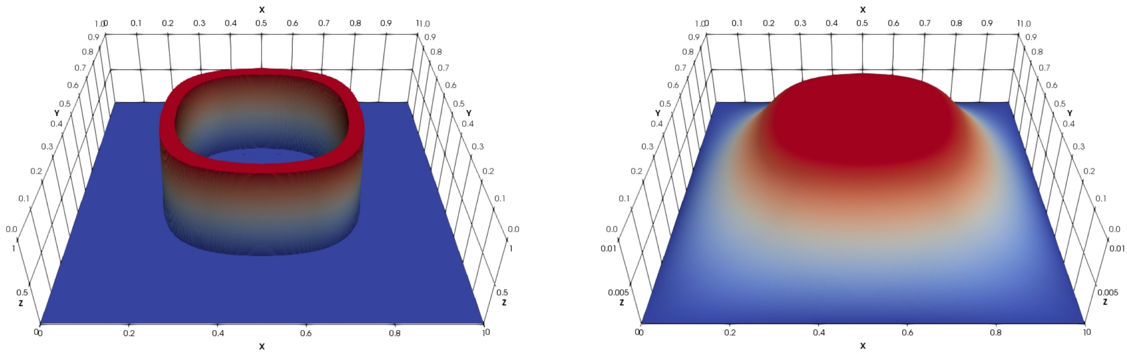


Figure 4.2: (Example 2) Computed results. From left to right: Control $\bar{u}_h$, state $\bar{y}_h$.

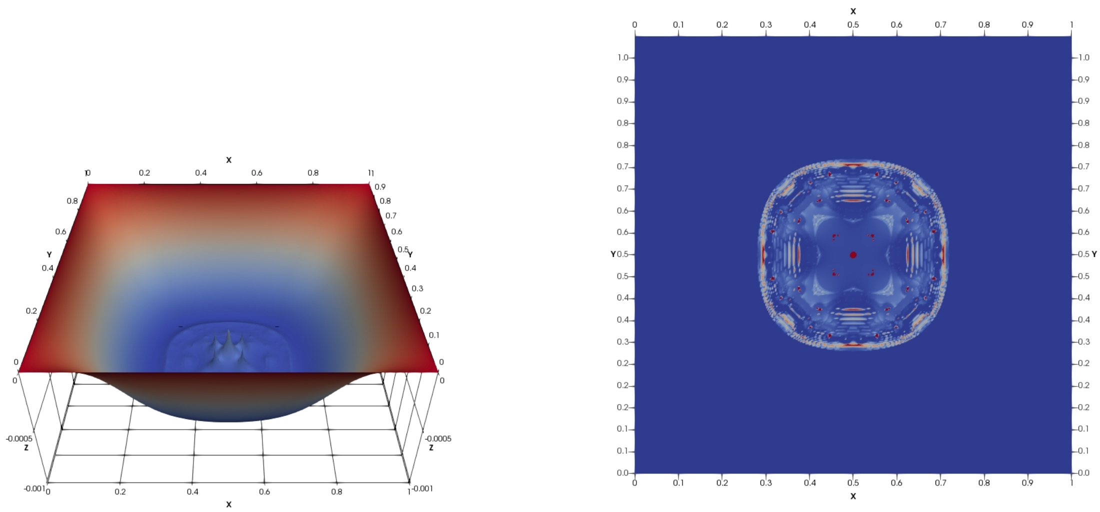


Figure 4.3: (Example 2) Computed results. From left to right: Adjoint state $\bar{p}_h$ and multiplier $\bar{\mu}_h$. The range of $\bar{\mu}_h$ is given by $\bar{\mu}_h(x) \in [0, 40]$.

CHAPTER 5

SEMILINEAR OPTIMAL CONTROL PROBLEMS

In this chapter the solution of an optimal control problem subject to a semilinear elliptic state equation and pointwise control and state constraints will be studied. The control problem is non-convex due to the nonlinearity of the state equation. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open, bounded domain with boundary Γ. The problem under consideration is given by

$$\text{minimize } J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \tag{5.1}$$

subject to

$$\begin{aligned} Ay + d(y) &= u && \text{in } \Omega, \\ \partial_{\nu_A} y &= 0 && \text{on } \Gamma, \\ y &\leq \psi && \text{in } \overline{\Omega}, \\ u_a \leq u &\leq u_b. && \text{a.e. in } \Omega \end{aligned} \tag{5.2}$$

Here, A denotes a second-order elliptic operator while $d(y)$ is a nonlinear term in y. The setting of the optimal control problem will be made precise in Section 5.1.

In order to solve (5.1), we extend the augmented Lagrangian method from Chapter 3 and provide convergence results for the overall iterative solution method. This task is challenging since feasibility of limit points can not be guaranteed for augmented Lagrangian methods, which are only stationary points of the augmented subproblem. In an analogue way to Chapter 3 we perform the classical multiplier update only if a sufficient decrease in the maximal constraint violation and the complementarity condition is achieved and consider these steps as successful. This approach will enable us to conclude feasibility of a weak limit point of iterates if and only if an infinite number of steps is successful.

The outline of this chapter is as follows: In Section 5.1 we start collecting results about the unregularized optimal control problem. Next, in Section 5.2 we present the augmented Lagrangian method. Section 5.2.3 is dedicated to show that every weak limit point of the sequence generated by our algorithm is a KKT point of the original problem. Further, in Section 5.3 we construct an auxiliary problem that claims solutions near a local solution of the original problem. Exploiting appropriate properties of this auxiliary problem, we prove that for ρ sufficiently large solutions of the auxiliary problem are local solutions of the augmented Lagrangian subproblem. In Section 5.4 we consider second-order sufficient conditions. To illustrate our theoretical findings we present numerical examples in Section 5.5.

The results of this chapter are submitted for publication [74].

5.1 The Optimal Control Problem

Let Y denote the space $Y := H^1(\Omega) \cap C(\overline{\Omega})$, and set $U := L^2(\Omega)$. We want to solve the following state constrained optimal control problem: Minimize

$$J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}$$

over all $(y,u) \in Y \times U_{\mathrm{ad}}$ subject to the semilinear elliptic equation

$$\begin{aligned} (Ay)(x) + d(x,y) &= u(x) \quad \text{in } \Omega, \\ (\partial_{\nu_A} y)(x) &= 0 \quad \text{on } \Gamma, \end{aligned}$$

and subject to the pointwise state constraints

$$\begin{aligned} y(x) &\le \psi(x) \quad \text{in } \overline{\Omega}, \\ u_a(x) \le u(x) &\le u_b(x) \quad \text{in } \Omega. \end{aligned}$$

In the sequel, we will work with the following set of standing assumptions.

Assumption 5.1 (Standing assumptions). a) Let $\Omega \subset \mathbb{R}^d$, $d = \{2,3\}$ be a bounded domain with $C^{1,1}$-boundary Γ or a bounded, convex domain with polygonal boundary Γ.

b) The given data satisfy $y_d \in L^2(\Omega)$, $\psi \in C(\overline{\Omega})$.

c) Let the differential operator A satisfy Assumption 2.19.

d) The function $d(x,y)\colon \Omega \times \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$ and twice continuously differentiable with respect to y for almost all $x \in \Omega$. Moreover, for $y = 0$ the function d and its derivatives with respect to y up to order two are bounded, i.e. there exists $C > 0$ such that

$$\|d(\cdot,0)\|_\infty + \left\|\frac{\partial d}{\partial y}(\cdot,0)\right\|_\infty + \left\|\frac{\partial^2 d}{\partial y^2}(\cdot,0)\right\|_\infty \le C$$

is satisfied. Further

$$d_y(x,y) \ge 0 \quad \text{for almost all } x \in \Omega.$$

The derivatives of d with respect to y are uniformly Lipschitz up to order two on bounded sets, i.e, there exists a constant M and a constant $L(M)$, that is dependent of M such that

$$\left\|\frac{\partial^2 d}{\partial y^2}(\cdot,y_1) - \frac{\partial^2 d}{\partial y^2}(\cdot,y_2)\right\|_\infty \le L(M)|y_1 - y_2|$$

for almost every $x \in \Omega$ and all $y_1, y_2 \in [-M, M]$. Finally, there is a subset $E_\Omega \subset \Omega$ of positive measure with $d_y(x,y) > 0$ in $E_\Omega \times \mathbb{R}$.

5.1.1 Analysis of the Optimal Control Problem

5.1.1.1 The State Equation

A function $y \in H^1(\Omega)$ is called a weak solution of the state equation (5.2) if for all $v \in H^1(\Omega)$ there holds

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x)\partial_{x_i} y(x)\partial_{x_j} v(x) + a_0(x)y(x)\,\mathrm{d}x + \int_\Omega d(x,y)v(x)\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

Theorem 5.2 (Existence of solution of the state equation). *Let Assumption 5.1 be satisfied. Then for every $u \in L^2(\Omega)$, the elliptic partial differential equation*

$$\begin{aligned} Ay + d(y) &= u \quad \text{in } \Omega, \\ \partial_{\nu_A} y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{5.3}$$

admits a unique weak solution $y \in H^1(\Omega) \cap C(\overline{\Omega})$, and it holds

$$\|y\|_{H^1(\Omega)} + \|y\|_{C(\overline{\Omega})} \leq c\, \|u\|_{L^2(\Omega)}$$

with $c > 0$ independent of u. If in addition $(u_n)_n$ is such that $u_n \rightharpoonup u \in L^2(\Omega)$ then the corresponding solutions $(y_n)_n$ of (5.3) *converge strongly in $H^1(\Omega) \cap C(\overline{\Omega})$ to the solution y of* (5.3) *to data u.*

Proof. The proof stating existence of a solution, its uniqueness, and the estimates of the norm can be found in [27, Thm. 3.1]. The compact inclusion $L^2(\Omega) \subset H^{-1}(\Omega)$ and the fact that $u \in L^2(\Omega)$ provides solutions in $H^2(\Omega)$ [79, Thm. 5], which can be embedded compactly in $C(\overline{\Omega})$ [1, Thm. 5.4] imply the additional statement. □

We introduce the control-to-state operator

$$S\colon L^2(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega}), \quad u \mapsto y.$$

It is well known [116, Thm. 4.16] that S is locally Lipschitz continuous from $L^2(\Omega)$ to the function space $H^1(\Omega) \cap C(\overline{\Omega})$, i.e., there exists a constant L such that

$$\|y_1 - y_2\|_{H^1(\Omega)} + \|y_1 - y_2\|_{C(\overline{\Omega})} \leq L\, \|u_1 - u_2\|_{L^2(\Omega)} \tag{5.4}$$

is satisfied for all $u_i \in L^2(\Omega)$, $i = 1, 2$, with corresponding states $y_i = S(u_i)$. We define the *admissible* and *feasible* sets

$$\begin{aligned} U_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u_a(x) \leq u(x) \leq u_b(x) \text{ a.e. in } \Omega\}, \\ F_{\text{ad}} &:= \{u \in L^2(\Omega) \mid u \in U_{\text{ad}},\ Su(x) \leq \psi(x) \text{ in } \overline{\Omega}\}. \end{aligned}$$

The reduced formulation of problem (5.1) is given by

$$\begin{aligned} \underset{u \in L^2(\Omega)}{\text{minimize}} \quad & f(u) := \frac{1}{2}\, \|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\, \|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u_a(x) \leq u(x) \leq u_b(x) \qquad \text{a.e. in } \Omega, \\ & Su(x) \leq \psi(x) \qquad\qquad\quad \text{in } \overline{\Omega}, \end{aligned} \tag{P}$$

For further use we want to recall a result concerning differentiability of the nonlinear control-to-state mapping S.

Theorem 5.3 (Differentiability of the solution mapping). *Let Assumption 5.1 be satisfied. Then the mapping $S\colon L^2(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega})$, that is defined by $S(u) = y$, is twice continuously Fréchet differentiable. Furthermore for all $u, h \in L^2(\Omega)$, $y_h = S'(u)h$ is defined as solution of*

$$\begin{aligned} Ay_h + d_y(y) y_h &= h \quad \text{in } \Omega, \\ \partial_{\nu_A} y_h &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Moreover, for every $h_1, h_2 \in L^2(\Omega)$, $y_{h_1,h_2} = S''(u)[h_1, h_2]$ is the solution of

$$\begin{aligned} Ay_{h_1,h_2} + d_y(y) y_{h_1,h_2} &= -d_{yy}(y) y_{h_1} y_{h_2} \quad \text{in } \Omega, \\ \partial_{\nu_A} y_{h_1,h_2} &= 0 \qquad\qquad\qquad\quad \text{on } \Gamma, \end{aligned}$$

where $y_{h_i} = S'(u)h_i, i = 1, 2.$

Proof. The proof for the first derivative of $S\colon L^r(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega}), r > N/2$ can be found in [116, Thm. 4.17]. We refer to [116, Thm. 4.24] for the proof of second-order differentiability of $S\colon L^\infty(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega})$ which is also valid for $S\colon L^2(\Omega) \to H^1(\Omega) \cap C(\overline{\Omega})$. □

5.1.1.2 Existence of Solutions and Optimality Conditions

Under the standing assumptions we can show existence of solutions of the reduced control problem (P). By standard arguments we get the following theorem.

Theorem 5.4 (Existence of solution of the optimal control problem). *Let Assumption 5.1 be satisfied. Assume that the feasible set F_{ad} is nonempty. Then there exists at least one global solution $\bar{u}$ of (P).*

Proof. This follows directly from Theorem 2.32, see also [62, Thm. 1.45]. □

Due to non-convexity, global solutions of problem (P) are not unique in general. Also, in addition there might be local solutions. The existence of Lagrange multipliers for state constrained optimal control problems is not guaranteed without some regularity assumption. In order to formulate first-order necessary optimality conditions we will work with the following linearized Slater condition.

Assumption 5.5 (Linearized Slater condition). A feasible point $\bar{u}$ satisfies the linearized Slater condition, if there exists $\hat{u} \in U_{\mathrm{ad}}$ and $\sigma > 0$ such that it holds

$$S(\bar{u})(x) + S'(\bar{u})(\hat{u} - \bar{u})(x) \leq \psi(x) - \sigma \quad \forall x \in \overline{\Omega}.$$

Based on the linearized Slater condition first-order necessary optimality conditions for problem (P) can be established.

Theorem 5.6 (First-order necessary optimality conditions). *Let $\bar{u}$ be a local solution of problem (P) that satisfies Assumption 5.5. Let $\bar{y} = S(\bar{u})$ denote the corresponding state. Then there exists an adjoint state $\bar{p} \in W^{1,s}(\Omega)$, $s \in (1, d/(d-1))$ and a Lagrange multiplier $\bar{\mu} \in \mathcal{M}(\overline{\Omega})$ with $\bar{\mu} = \bar{\mu}\big|_\Omega + \bar{\mu}\big|_\Gamma$ such that the following optimality system*

$$\begin{aligned} A\bar{y} + d(\bar{y}) &= \bar{u} \quad \text{in } \Omega, \\ \partial_{\nu_A}\bar{y} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} A^*\bar{p} + d_y(\bar{y})\bar{p} &= \bar{y} - y_d + \bar{\mu}_\Omega \quad \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p} &= \bar{\mu}_\Gamma \qquad\qquad\quad \text{on } \Gamma, \end{aligned} \tag{5.5b}$$

$$(\bar{p} + \alpha\bar{u}, u - \bar{u}) \geq 0 \quad \forall u \in U_{\mathrm{ad}}, \tag{5.5c}$$

$$\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{\mu} \geq 0, \quad \bar{y} \leq \psi \tag{5.5d}$$

is fulfilled. Here, the inequality $\bar{\mu} \geq 0$ means $\langle \bar{\mu}, \varphi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \geq 0$ for all $\varphi \in C(\overline{\Omega})$ with $\varphi \geq 0$.

Proof. Since the linearized Slater condition implies the Zowe-Kurcyusz condition (Lemma 2.44), this follows directly with Theorem 2.42. □

Let us emphasize that due to the presence of control as well as state constraints, the adjoint state $\bar{p}$ and the Lagrange multiplier $\bar{\mu}$ need not to be unique.

5.2 The Augmented Lagrangian Method

Like in Chapter 3 we eliminate the explicit state constraint $S(u) \le \psi$ from the set of constraints by adding an augmented Lagrangian term to the cost functional. Let $\rho > 0$ denote a penalization parameter and μ a fixed function in $L^2(\Omega)$. Then in every step k of the augmented Lagrangian method one has to solve subproblems of the type

$$\begin{aligned} &\underset{u_\rho \in L^2(\Omega)}{\text{minimize}} \quad f_{AL}(u_\rho, \mu, \rho) := f(u_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(Su_\rho - \psi))_+ \right\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u_\rho \in U_{\text{ad}}. \end{aligned} \tag{P_{AL}}$$

5.2.1 Analysis of the Augmented Lagrangian Subproblem

In the following, existence of an optimal control and existence of a corresponding adjoint state will be proven. Existence of solutions follows directly by standard theory, see Theorem 2.32.

Theorem 5.7 (Existence of solutions of the augmented Lagrangian subproblem). *For every $\rho > 0$, $\mu \in L^2(\Omega)$ with $\mu \ge 0$ the augmented Lagrangian subproblem (P_{AL}) admits at least one global solution $\bar{u}_\rho \in U_{\text{ad}}$.*

Since the problem (P_{AL}) has no state constraints, the first-order optimality system is fulfilled without any further regularity assumptions.

Theorem 5.8 (First-order necessary optimality conditions). *For given $\rho > 0$ and $0 \le \mu \in L^2(\Omega)$ let $\bar{u}_\rho$ be a solution of (P_{AL}) with corresponding state $\bar{y}_\rho$. Then for every given $\bar{u}_\rho$ there exists a unique adjoint state $\bar{p}_\rho \in H^1(\Omega)$ satisfying the following system*

$$\begin{aligned} A\bar{y}_\rho + d(\bar{y}_\rho) &= \bar{u}_\rho \quad \text{in } \Omega, \\ \partial_{\nu_A}\bar{y}_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{5.6a}$$

$$\begin{aligned} A^*\bar{p}_\rho + d_y(\bar{y}_\rho)\bar{p}_\rho &= \bar{y}_\rho - y_d + \bar{\mu}_\rho \quad \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{y}_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{5.6b}$$

$$(\bar{p}_\rho + \alpha\bar{u}_\rho, u - \bar{u}_\rho) \ge 0, \qquad \forall u \in U_{\text{ad}} \tag{5.6c}$$

$$\bar{\mu}_\rho = \big(\mu + \rho(\bar{y}_\rho - \psi)\big)_+ . \tag{5.6d}$$

Proof. For the existence of an adjoint state $\bar{p}_\rho \in H^1(\Omega)$ that satisfies the KKT system we refer to [62, Cor. 1.3, p.73]. By construction we get a unique $\bar{\mu}_\rho$ for each given $\bar{u}_\rho$. Due to Theorem 2.25 the adjoint equation admits a unique solution. Thus, the adjoint state $\bar{p}_\rho$ is unique for every control $\bar{u}_\rho$. □

Finally, in Algorithm 5.1 we present the augmented Lagrangian algorithm, which is based on the algorithm that has been developed in Chapter 3. The definition of a successful step is a variant of the strategy used in [36, 37].
In the following we will call the step k *successful* if the quantity

$$R_k := \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + (\bar{\mu}_k, \psi - \bar{y}_k)_+$$

shows sufficient decrease (see step 4 of the algorithm). Otherwise we will call the step *not successful*. The first part of R_k measures the maximal constraint violation while the second term quantifies the fulfilment of the complementarity condition in the second part. Since $(\bar{\mu}_k(x), \psi(x) - \bar{y}_k(x))$ is nonnegative for every feasible $\bar{y}_k$ it is enough to check on the smallness of $(\bar{\mu}_k, \psi - \bar{y}_k)_+$ in the

Algorithm 5.1 Augmented Lagrangian Algorithm

Let $(\bar{y}_0, \bar{u}_0, \bar{p}_0) \in (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega) \times W^{1,s}(\Omega)$, $\rho_1 > 0$ and $\mu_1 \in L^2(\Omega)$ be given with $\mu_1 \geq 0$. Choose $\theta > 1$, $\tau \in (0,1)$, $\epsilon \geq 0$, $R_0^+ \gg 1$. Set $k := 1$ and $n := 1$.

1: Solve the optimality system (5.6) for $\mu := \mu_k$, and obtain $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$.
2: Set $\bar{\mu}_k := (\mu_k + \rho_k(\bar{y}_k - \psi))_+$.
3: Compute $R_k := \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + (\bar{\mu}_k, \psi - \bar{y}_k)_+$.
4: If $R_k \leq \tau R_{n-1}^+$ then the step k is successful. Set

$$\mu_{k+1} := \bar{\mu}_k, \quad \rho_{k+1} := \rho_k$$

and define

$$(y_n^+, u_n^+, p_n^+) := (\bar{y}_k, \bar{u}_k, \bar{p}_k), \; \mu_n^+ := \mu_{k+1}, \; R_n^+ := R_k.$$

Set $n := n + 1$.
5: Otherwise the step k is not successful, set $\mu_{k+1} := \mu_k$, increase penalty parameter $\rho_{k+1} := \theta\rho_k$.
6: If $R_{n-1}^+ \leq \epsilon$ then stop, otherwise set $k := k + 1$ and go to step 1.

second term for quantifying whether or not the complementarity condition is satisfied.

From now on let $(P_{AL})_k$ denote the augmented Lagrangian subproblem (P_{AL}) for given penalty parameter $\rho := \rho_k$ and multiplier $\mu := \mu_k$. We will denote its solution by $\bar{u}_k$ with corresponding state $\bar{y}_k$, adjoint state $\bar{p}_k$ and updated multiplier $\bar{\mu}_k$.

5.2.2 Successful Steps and Feasibility of Limit Points

The question of convergence of the algorithm is linked to the question of feasibility of limit points of the iterates $(\bar{u}_k)_k$. Here, it turns out that the feasibility of limit points is tightly linked with the occurrence of infinitely many successful steps.
Let us emphasize that for non-convex optimization problems the feasibility of limit points of augmented Lagrangian methods is *not* guaranteed. Typically, the feasibility of limit points is an additional assumption in convergence results [36, 37, 73]. Or the feasibility is the consequence of a constraint qualification assumed to hold in the limit point [20, 71].

Theorem 5.9. *Let $(\bar{u}_k)_k$ denote the sequence that is generated by Algorithm 5.1. Then $(\bar{u}_k)_k$ has a feasible weak limit point if and only if infinitely many steps in the execution of Algorithm 5.1 were successful.*

Proof. First, suppose that infinitely many steps were successful. Let $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ denote the sequence of successful iterates generated by Algorithm 5.1. By the boundedness of $(u_n^+)_n \in U_{\text{ad}}$ we get existence of a subsequence $u_{n'}^+ \rightharpoonup u^*$ in $L^2(\Omega)$ and $y_{n'}^+ \to y^*$ in $H^1(\Omega) \cap C(\overline{\Omega})$ by Theorem 5.2. Due to the definition of successful steps, we have that $\|(y_n^+ - \psi)_+\|_{C(\overline{\Omega})} \leq R_n^+ \to 0$ and u^* is a feasible control of (P).
Suppose now that only finitely many steps were successful. Let m be the largest index of a successful step. Hence, all steps k with $k > m$ are not successful. According to Algorithm 5.1 it holds $\mu_k = \mu_m$ for all $k > m$. We will prove $\limsup_{k\to\infty} (\bar{\mu}_k, \psi - \bar{y}_k)_+ \leq 0$ first. Let

$$\Omega_k := \{x \in \Omega \colon (\bar{\mu}_k(x), \psi(x) - \bar{y}_k(x)) \geq 0\} .$$

Then the desired estimate follows easily by pointwise evaluation of the contributing quantities in

$$
\begin{aligned}
(\bar{\mu}_k, \psi - \bar{y}_k)_+ &= (\bar{\mu}_k, -\frac{\mu_m}{\rho_k} + \psi - \bar{y}_k + \frac{\mu_m}{\rho_k})_+ \le -\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega_k)} + \frac{1}{\rho_k}(\bar{\mu}_k, \mu_m)_{L^2(\Omega_k)} \\
&\le -\frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega_k)} + \frac{1}{2\rho_k} \|\mu_m\|^2_{L^2(\Omega_k)} \le \frac{1}{2\rho_k} \|\mu_m\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we applied Young's inequality. The algorithm only makes $l \ge 0$ successful steps, which implies $R_k > \tau R_l^+$ for all $k > m$. This proves with $\mu_k = \mu_l^+$

$$
\begin{aligned}
\liminf_{k\to\infty} \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} &= \liminf_{k\to\infty} \left(R_k - (\bar{\mu}_k, \psi - \bar{y}_k)_+\right) \\
&\ge \tau R_l^+ - \limsup_{k\to\infty} \, (\bar{\mu}_k, \psi - \bar{y}_k)_+ \ge \tau R_l^+ > 0.
\end{aligned}
$$

Let u^* be a weak limit of the subsequence $(u_{k'})_{k'}$ with associated state y^*. Then, arguing as in the first part of the proof, we have

$$
\|(y^* - \psi)_+\|_{C(\overline{\Omega})} = \lim_{k'\to\infty} \|(\bar{y}_{k'} - \psi)_+\|_{C(\overline{\Omega})} \ge \tau R_l^+ > 0,
$$

and u^* is not feasible. □

The proof of the previous theorem shows that if the algorithm performs infinitely many successful steps then every limit point of $(u_n^+)_n$ is feasible for the original problem. In case that only finitely many steps are successful, we have the following additional result.

Theorem 5.10. *Let us assume that Algorithm 5.1 does a finite number of successful steps only. Let $(\bar{u}_k)_k$ denote the sequence that is generated by the algorithm and let u^* be a weak limit point of $(\bar{u}_k)_k$. Then u^* is infeasible for (P) and it is a stationary point of the minimization problem*

$$
\min_{u\in U_{\text{ad}}} \|(S(u) - \psi)_+\|^2_{L^2(\Omega)}. \tag{5.7}
$$

Proof. The infeasibility of u^* is a consequence of Theorem 5.9. Let m be the index of the last successful step. Dividing the first-order optimality condition of the augmented Lagrangian subproblem by ρ_k

$$
\left(S'(\bar{u}_k)^* \left(\frac{S(\bar{u}_k) - y_d}{\rho_k} + \left(\frac{\mu_m}{\rho_k} + S(\bar{u}_k) - \psi\right)_+\right) + \alpha\frac{\bar{u}_k}{\rho_k}, v - \bar{u}_k\right) \ge 0 \quad \forall v \in U_{\text{ad}}
$$

and taking the limit $k \to \infty$ yields

$$
(S'(u^*)^*(S(u^*) - \psi)_+), v - u^*) \ge 0 \quad \forall v \in U_{\text{ad}},
$$

which is exactly the optimality condition for (5.7). □

In [20, 71] such a stationarity property together with a suitable constraint qualification was used to prove feasibility of limit points. Another way to obtain feasibility of u^* is to assume the boundedness of the sequence of multipliers $(\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)})_k$. Assumptions of this kind are common for augmented Lagrangian methods. The multiplier update of [37, Algorithm 2] is constructed such that a related boundedness result holds. In safeguarded augmented Lagrangian methods, see, e.g., [18, 73], a bounded sequence of safeguarded multipliers is used to define the multiplier update. In our situation, this would amount to choosing a bounded sequence $(w_k)_k$ in $L^2(\Omega)$ and computing stationary points of $\min_{u\in U_{\text{ad}}} f_{AL}(u, w_k, \rho_k)$ instead of $\min_{u\in U_{\text{ad}}} f_{AL}(u, \mu_k, \rho_k)$, which results in the safe-guarded multiplier update $\mu_{k+1} := (w_k + \rho_k(\bar{y}_k - \psi))_+$.

Theorem 5.11. *Assume that in step 1 of Algorithm 5.1, the solutions* $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ *of* (5.6) *are chosen such that*

$$\frac{1}{\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} = \frac{1}{\rho_k} \|(\mu_k + \rho_k(\bar{y}_k - \psi))_+\|^2_{L^2(\Omega)}$$

is uniformly bounded. Then every weak limit point u^* *of* $(\bar{u}_k)_k$ *is feasible.*

Proof. Suppose first, that $(\rho_k)_k$ is bounded. Then the algorithm performs only finitely many unsuccessful steps. Consequently, the tails of the sequence of iterates $(\bar{u}_k)_k$ and of the sequence of successful iterates $(u_n^+)_n$ coincide. By Theorem 5.9, all weak limit points of $(u_n^+)_n$ and thus of $(\bar{u}_k)_k$ are feasible.
Now, consider the case $\rho_k \to +\infty$. Due to the assumption, there is $M > 0$ such that

$$\frac{1}{\rho_k} \|(\mu_k + \rho_k(\bar{y}_k - \psi))_+\|^2_{L^2(\Omega)} = \rho_k \left\| \left(\frac{\mu_k}{\rho_k} + \bar{y}_k - \psi \right)_+ \right\|^2_{L^2(\Omega)} \leq M,$$

which yields with $\mu_k \geq 0$ the estimate

$$\frac{M}{\rho_k} \geq \left\| \left(\frac{\mu_k}{\rho_k} + \bar{y}_k - \psi \right)_+ \right\|^2_{L^2(\Omega)} \geq \|(\bar{y}_k - \psi)_+\|^2_{L^2(\Omega)}.$$

This proves $\lim_{k\to\infty} \|(\bar{y}_k - \psi)_+\|^2_{L^2(\Omega)} = 0$. By the compactness result of Theorem 5.2, the claim follows. □

Under the assumptions of the previous theorem, Algorithm 5.1 makes infinitely many successful steps by Theorem 5.9. In the case that Algorithm 5.1 chooses $\bar{u}_k$ to be global minimizers of the augmented Lagrangian subproblem the boundedness assumption of Theorem 5.11 is satisfied.

Theorem 5.12. *Let the feasible set* F_{ad} *be non-empty. Assume that in step 1 of Algorithm 5.1,* $\bar{u}_k$ *is chosen to be a global minimizer of the augmented Lagrangian subproblem. Then the augmented Lagrangian algorithm makes infinitely many successful steps.*

Proof. Let $\bar{u}$ be a global solution of the original problem. Assume that algorithm performs only finitely many successful steps. Let $k > m$, where m is the largest index of a successful step. This implies $\mu_k = \mu_m$. Then we obtain

$$\begin{aligned}
\frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} &\leq f(\bar{u}_k) + \frac{1}{2\rho_k} \|\bar{\mu}_k\|^2_{L^2(\Omega)} \\
&\leq f(\bar{u}) + \frac{1}{2\rho_k} \|(\mu_k + \rho_k(S(\bar{u}) - \psi))_+\|^2_{L^2(\Omega)} \\
&= f(\bar{u}) + \frac{1}{2\rho_k} \|(\mu_m + \rho_k(S(\bar{u}) - \psi))_+\|^2_{L^2(\Omega)} \leq f(\bar{u}) + \frac{1}{2\rho_k} \|\mu_m\|^2_{L^2(\Omega)}.
\end{aligned}$$

Hence, all assumptions of Theorem 5.11 are satisfied, and all weak limit points of $(\bar{u}_k)_k$ are feasible. As this sequence is bounded, there exists such weak limit points. This contradicts Theorem 5.9, and the algorithm performs infinitely many successful steps. □

Note that this strategy is only viable if the original problem and thus the augmented Lagrangian subproblems are convex. Then computing stationary points is equivalent to computing global minima. In practice, solutions of the augmented Lagrangian subproblems are obtained by iterative methods. Naturally, these methods use the previous iterate as starting point. Thus, it is a realistic scenario that the iterates stay in a neighbourhood of a local solution of the original problem. One main result of this chapter is to prove that such a situation can occur. To this end, let $\bar{u}$ be a strict local solution of (P). For some radius $r > 0$, let us consider the auxiliary problem

$$\begin{aligned} \underset{u_\rho^r \in L^2(\Omega)}{\text{minimize}} \quad & f_{AL}^r(u_\rho^r, \mu, \rho) := f(u_\rho^r) + \frac{1}{2\rho} \left\| \left(\mu + \rho(S(u_\rho^r) - \psi) \right)_+ \right\|_{L^2(\Omega)}^2 \\ \text{subject to} \quad & u_\rho^r \in U_{\text{ad}}, \qquad \left\| u_\rho^r - \bar{u} \right\|_{L^2(\Omega)} \le r. \end{aligned} \tag{5.8}$$

The radius r is chosen sufficiently small such that a quadratic growth condition is satisfied. This auxiliary problem will be analyzed in detail in Section 5.3. We will show that global solutions of the auxiliary problem are local solutions of the augmented Lagrangian subproblem, provided the penalty parameter ρ is sufficiently large. In addition, we will prove that if the iterates of Algorithm 5.1 are chosen as such a solution then the algorithm performs infinitely many successful steps. We refer to Theorem 5.27 and Theorem 5.28.

Let us close this section with an example demonstrating that augmented Lagrangian methods will not deliver feasible limit points in general. The example is taken from [71]: Consider the minimization problem in $\mathbb{R}$ given by

$$\min x \quad \text{subject to} \quad 1 - x^3 \le 0.$$

Clearly, $x^* = 1$ is the global solution. Note that the inequality constraint is defined by a non-convex function, while the feasible set is the interval $[1, +\infty)$. For penalty parameter $\rho > 0$ and multiplier estimate $\mu \ge 0$, the augmented Lagrangian is defined by

$$L(x, \mu, \rho) := x + \frac{1}{2\rho} \left((\mu + \rho(1 - x^3))_+ \right)^2.$$

As argued in [71], the augmented Lagrangian function admits for all possible values of ρ and μ a local minimum $x_{\rho,\mu} < 0$. If the method chooses these minima as iterates, then limit points are clearly not feasible. This applies equally well to the classical quadratic penalty method, which corresponds to the choice $\mu = 0$.

5.2.3 Convergence towards KKT Points

In the previous section we have investigated several cases and conditions under which Algorithm 5.1 generates infinitely many successful steps. In the following, we will always assume that this is the case, i.e., the method produces an infinite sequence of successful iterates $(y_n^+, u_n^+, p_n^+)_n$. By Theorem 5.9, we know that $(u_n^+)_n$ has a feasible weak limit point. However, we do not know yet, if u^* is a stationary point, i.e., if $(p_n^+, \mu_n^+)_n$ converges in some sense to (p^*, μ^*) such that (y^*, u^*, p^*, μ^*) satisfies the optimality system (5.5). To achieve this aim, we have to suppose additional properties of the weak limit point u^*. In the rest of this section, we will prove convergence of the dual quantities $(p_n^+, \mu_n^+)_n$ under the assumption that the weak limit point u^* satisfies the linearized Slater condition Assumption 5.5. We start with several auxiliary results.

Lemma 5.13. *Let $(u_k)_k, (h_k)_k$ denote sequences in $L^2(\Omega)$ that converge weakly to the limits u^*, h^*, respectively. Then for $k \to \infty$ we have*

$$\left\| S'(u_k)h_k - S'(u^*)h^* \right\|_{C(\overline{\Omega})} \to 0.$$

Proof. From Theorem 5.2 we know that $y_k := S(u_k)$ is the unique weak solution of the state equation

$$\begin{aligned} Ay_k + d(y_k) &= u_k \quad && \text{in } \Omega, \\ \partial_{\nu_A} y_k &= 0 && \text{on } \Gamma. \end{aligned}$$

Further, for $u_k \rightharpoonup u^*$ in $L^2(\Omega)$ we get $y_k \to y^*$ in $H^1(\Omega) \cap C(\overline{\Omega})$. Let now z_k denote the linearized state $z_k := S'(u_k)h_k$. Then by Theorem 5.3 we know that z_k is the unique solution of

$$\begin{aligned} Az_k + d_y(y_k)z_k &= h_k && \text{in } \Omega, \\ \partial_{\nu_A} z_k &= 0 && \text{on } \Gamma. \end{aligned}$$

Further, let $z^* := S'(u^*)h^*$ solve the equation

$$\begin{aligned} Az^* + d_y(y^*)z^* &= h^* && \text{in } \Omega, \\ \partial_{\nu_A} z^* &= 0 && \text{on } \Gamma. \end{aligned}$$

We subtract both PDEs and set $e_k := S'(u_k)h_k - S'(u^*)h^*$

$$\begin{aligned} Ae_k + d_y(y_k)z_k - d_y(y^*)z^* &= h_k - h^* && \text{in } \Omega, \\ \partial_{\nu_A} e_k &= 0 && \text{on } \Gamma. \end{aligned}$$

Inserting the identity $d_y(y_k)z_k - d_y(y^*)z^* = \left(d_y(y_k) - d_y(y^*)\right) z_k + d_y(y^*)(z_k - z^*)$ we obtain

$$\begin{aligned} Ae_k + d_y(y^*)e_k &= (h_k - h^*) - (d_y(y_k) - d_y(y^*))z_k && \text{in } \Omega, \\ \partial_{\nu_A} e_k &= 0 && \text{on } \Gamma. \end{aligned}$$

From Assumption 5.1 we know that d_y is locally Lipschitz continuous, i.e.,

$$\left\|d_y(y_1) - d_y(y_2)\right\|_{L^\infty(\Omega)} \le L \left\|y_1 - y_2\right\|_{L^\infty(\Omega)}.$$

Concluding, for $y_k \to y^*$ in $L^\infty(\Omega)$ we have $d_y(y_k) \to d_y(y^*)$ in $L^\infty(\Omega)$. Due to $h_k \rightharpoonup h^*$ in $L^2(\Omega)$ and the boundedness of z_k in $L^2(\Omega)$ we gain $e_k \to 0$ in $H^1(\Omega) \cap C(\overline{\Omega})$. Hence,

$$\left\|S'(u_k)h_k - S'(u^*)h^*\right\|_{C(\overline{\Omega})} \to 0$$

and the proof is done. □

Let us recall that $(y_n^+, u_n^+, p_n^+, \mu_n^+)$ denotes the solution of the n-th successful iteration of Algorithm 5.1. We want to investigate the convergence properties of the algorithm for a weak limit point u^* of $(u_n^+)_n$. A point $u^* \in U_{\text{ad}}$ satisfies the linearized Slater condition if there exists a $\hat{u} \in U_{\text{ad}}$ and $\sigma > 0$ such that

$$S(u^*)(x) + S'(u^*)(\hat{u} - u^*)(x) \le \psi(x) - \sigma \quad \forall x \in \overline{\Omega}. \tag{5.9}$$

Lemma 5.14. *Let u^* denote a weak limit point of $(u_n^+)_n$ that satisfies the linearized Slater condition* (5.9). *Then there exists an $N_0 \in \mathbb{N}$ such that for all $n' > N_0$ the control $u_{n'}^+$ satisfies*

$$S(u_{n'}^+) + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) \le \psi - \frac{\sigma}{2}. \tag{5.10}$$

Proof. By Theorem 5.3 we have strong convergence $S(u_{n'}^+) \to S(u^*)$ in $H^1(\Omega) \cap C(\overline{\Omega})$. By Theorem 5.13 we obtain $S'(u_{n'}^+)(\hat{u} - u_{n'}^+) \to S'(u^*)(\hat{u} - u^*)$ in $C(\overline{\Omega})$. Using the identity

$$\begin{aligned} S(u_{n'}^+) + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) = {} & S(u^*) + S'(u^*)(\hat{u} - u^*) \\ & + S(u_{n'}^+) - S(u^*) \\ & + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) - S'(u^*)(\hat{u} - u^*) \end{aligned}$$

and exploiting the specified convergence results, we conclude existence of an $N_0 \in \mathbb{N}$ such that

$$S(u_{n'}^+) + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) \le \psi - \frac{\sigma}{2}, \quad \forall n' > N_0.$$

□

We recall an estimate for the second term of the update rule, see Lemma 3.13 that is necessary to state L^1-boundedness of the Lagrange multiplier. This estimate does not require any additional assumption, it just results from the structure of the update rule.

Lemma 5.15. *Let y_n^+, μ_n^+ be given as defined in Algorithm 5.1. Then for all $n > 1$ it holds*

$$(\mu_n^+, \psi - y_n^+)_+ \leq \tau^{n-1} \left(\left\| (y_1^+ - \psi)_+ \right\|_{C(\overline{\Omega})} + \left\| \mu_1^+ \right\|_{L^2(\Omega)} \left\| (\psi - y_1^+)_+ \right\|_{L^2(\Omega)} \right).$$

Lemma 5.16 (Boundedness of the Lagrange multiplier). *Let $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ denote the sequence that is generated by Algorithm 5.1. Let $(u_{n'}^+)_{n'}$ denote a subsequence of $(u_n^+)_n$ that converges weakly to u^*. If u^* satisfies the linearized Slater condition from* (5.9)*, then the corresponding sequence of multipliers $(\mu_{n'}^+)_{n'}$ is bounded in $L^1(\Omega)$, i.e., there is a constant $C > 0$ independent of n' such that for all n' it holds*

$$\left\| \mu_{n'}^+ \right\|_{L^1(\Omega)} \leq C.$$

Proof. Writing (5.6c) in variational form we see

$$(p_{n'}^+ + \alpha u_{n'}^+, u - u_{n'}^+) \geq 0 \quad \forall u \in U_{\text{ad}}.$$

Using the identity

$$p_{n'}^+ = S'(u_{n'}^+)^* (y_{n'}^+ - y_d + \mu_{n'}^+)$$

we obtain

$$(S'(u_{n'}^+)^* (y_{n'}^+ - y_d + \mu_{n'}^+) + \alpha u_{n'}^+, u - u_{n'}^+) \geq 0 \quad \forall u \in U_{\text{ad}}.$$

Rearranging terms yields

$$(\mu_{n'}^+, S'(u_{n'}^+)(u_{n'}^+ - u)) \leq (y_{n'}^+ - y_d, S'(u_{n'}^+)(u - u_{n'}^+)) + (\alpha u_{n'}^+, u - u_{n'}^+).$$

Testing the left hand side of the previous inequality with the test function $u := \hat{u} \in U_{\text{ad}}$ we get

$$\begin{aligned}(\mu_{n'}^+, S'(u_{n'}^+)(u_{n'}^+ - \hat{u})) &= (\mu_{n'}^+, S'(u_{n'}^+)(u_{n'}^+ - \hat{u})) + (\mu_{n'}^+, S(u_{n'}^+) - \psi) - (\mu_{n'}^+, S(u_{n'}^+) - \psi) \\ &= -(\mu_{n'}^+, S(u_{n'}^+) + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) - \psi) + (\mu_{n'}^+, S(u_{n'}^+) - \psi).\end{aligned}$$

By Lemma 5.14 we know that there exists an N_0 such that for all $n' > N_0$ the control $u_{n'}^+$ satisfies (5.10). Hence for all $n' > N_0$ we obtain

$$\frac{\sigma}{2} \left\| \mu_{n'}^+ \right\|_{L^1(\Omega)} \leq -(\mu_{n'}^+, S(u_{n'}^+) + S'(u_{n'}^+)(\hat{u} - u_{n'}^+) - \psi).$$

Thus, we estimate

$$\begin{aligned}\frac{\sigma}{2} \left\| \mu_{n'}^+ \right\|_{L^1(\Omega)} \leq &(\mu_{n'}^+, \psi - S(u_{n'}^+)) + (y_{n'}^+ - y_d, S'(u_{n'}^+)(\hat{u} - u_{n'}^+)) + (\alpha u_{n'}^+, \hat{u} - u_{n'}^+) \\ \leq &(\mu_{n'}^+, \psi - y_{n'}^+)_+ + \left\| y_{n'}^+ - y_d \right\|_{L^2(\Omega)} \left\| S'(u_{n'}^+)(\hat{u} - u_{n'}^+) \right\|_{L^2(\Omega)} \\ &+ \alpha \left\| u_{n'}^+ \right\|_{L^2(\Omega)} + \left\| \hat{u} - u_{n'}^+ \right\|_{L^2(\Omega)}.\end{aligned}$$

From Theorem 5.3 we know that $y_h := S'(u_{n'}^+)(\hat{u} - u_{n'}^+)$ is the weak solution of a uniquely solvable partial differential equation with right-hand side $\hat{u} - u_{n'}^+$. Hence, it is norm bounded by $c \left\| \hat{u} - u_{n'}^+ \right\|_{L^2(\Omega)}$ with $c > 0$ independent of n. Further, exploiting Theorem 5.2 and Lemma 5.15, the boundedness of the terms on the right-hand side now follows directly from the boundedness of the admissible set U_{ad}. This yields the assertion. □

Let us conclude this section with the following result on convergence.

Theorem 5.17 (Convergence towards KKT points). *Let $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ denote the sequence that is generated by Algorithm 5.1. Let u^* denote a weak limit point of $(u_n^+)_n$. If u^* satisfies the linearized Slater condition from* (5.9)*, then there exist subsequences $(y_{n'}^+, u_{n'}^+, p_{n'}^+, \mu_{n'}^+)_{n'}$ of $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ such that*

$$u_{n'}^+ \to u^* \quad \text{in } L^2(\Omega), \qquad y_{n'}^+ \to y^* \quad \text{in } H^1(\Omega) \cap C(\overline{\Omega}),$$
$$p_{n'}^+ \rightharpoonup p^* \quad \text{in } W^{1,s}(\Omega),\ s \in [1, d/(d-1)) \qquad \mu_{n'}^+ \rightharpoonup^* \mu^* \quad \text{in } \mathcal{M}(\overline{\Omega})$$

and (y^*, u^*, p^*, μ^*) is a KKT point of the original problem (P).

Proof. Since $(u_n^+)_n$ is bounded in $L^2(\Omega)$ we can extract a weak convergent subsequence $u_{n'}^+ \rightharpoonup u^*$ in $L^2(\Omega)$, thus $y_{n'}^+ \to y^*$ in $H^1(\Omega) \cap C(\overline{\Omega})$ due to Theorem 5.2. Hence, (5.5a) ist satisfied. Since $u_{n'}^+$ satisfies a linearized Slater condition by Lemma 5.14 for n' sufficiently large, Lemma 5.16 yields $L^1(\Omega)$-boundedness of $(\mu_{n'}^+)_{n'}$. By the Eberlein-Šmulyan Theorem 2.5 we can extract a weak-* convergent subsequence in $\mathcal{M}(\overline{\Omega})$ denoted w.l.o.g. by $\mu_{n'} \rightharpoonup^* \mu^*$. Convergence of $p_{n'}^+ \rightharpoonup p^*$ in $W^{1,s}(\Omega), s \in [1, d/(d-1))$ can now be shown as in [80, Lem. 11]. Thus, the adjoint equation (5.5b) is satisfied. The space $W^{1,s}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Hence $p_{n'}^+ \to p^*$ in $L^2(\Omega)$ and we get

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} (p_{n'}^+ + \alpha u_{n'}^+, u - u_{n'}^+) \le (p^*, u - u^*) - \liminf_{k\to\infty} (\alpha u_{n'}^+, u_{n'}^+ - u) \\ &\le (p^*, u - u^*) - (\alpha u^*, u^* - u) = (p^* + \alpha u^*, u - u^*), \end{aligned}$$

where we exploited the weak lower semicontinuity of $(\alpha u_{n'}^+, u - u_{n'}^+)$, $u \in L^2(\Omega)$. Hence, (5.5c) is satisfied. Due to the structure of the update rule we have

$$\lim_{n'\to\infty} R_{n'}^+ = \lim_{n'\to\infty} \left\| (y_{n'}^+ - \psi)_+ \right\|_{C(\overline{\Omega})} + (\mu_{n'}^+, \psi - y_{n'}^+)_+ = 0.$$

This implies $y^* \le \psi$ and $\langle \mu^*, \psi - y^* \rangle_+ = 0$. In addition, $y^* \le \psi$ and $\mu^* \ge 0$ implies $\langle \mu^*, \psi - y^* \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \ge 0$, and we get $\langle \mu^*, \psi - y^* \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0$. Thus (5.5d) is satisfied. We have proven that (y^*, u^*, p^*, μ^*) is a KKT point of (P), i.e., (y^*, u^*, p^*, μ^*) solves (5.5). It remains to show strong convergence of $u_{n'}^+ \to u^*$ in $L^2(\Omega)$. Testing (5.5c) with $u_{n'}^+$ and (5.6c) with u^* and adding both inequalities we arrive at

$$(p^* - p_{n'}^+ + \alpha(u^* - u_{n'}^+), u_{n'}^+ - u^*) \ge 0.$$

Hence,

$$\alpha \left\| u_{n'}^+ - u^* \right\|_{L^2(\Omega)}^2 \le (p^* - p_{n'}^+, u_{n'}^+ - u^*).$$

Since we already know that $p_{n'}^+ \to p^*$ in $L^2(\Omega)$ and $u_{n'}^+ \rightharpoonup u^*$ in $L^2(\Omega)$ this directly yields $u_{n'}^+ \to u^*$ in $L^2(\Omega)$. □

Remark 5.18. The proof of Theorem 5.17 above requires $R_n^+ \to 0$ only. This opens up the possibility to modify the decision about successful steps in algorithm 5.1. We report about such a modification in Section 5.5.

5.3 Convergence towards Local Solutions

So far, we have been able to show that a weak limit point that has been generated by Algorithm 5.1 is a stationary point of the original problem (P) if it satisfies the linearized Slater condition. If a weak limit point satisfies a second-order condition, we gain convergence to a local solution. However, the convergence result from Theorem 5.17 yields convergence of a subsequence of $(u_n^+)_n$ only. Accordingly, during all other steps the algorithm might choose solutions of the KKT system (5.6) that are far away from a desired local minimum $\bar{u}$. Here the following questions arise:

1. For every fixed μ does there exist a KKT point of the arising subproblem that satisfies $\bar{u}_k \in B_r(\bar{u})$?
   and
2. Is an infinite number of steps successful if the algorithm chooses these KKT points in step 1?

Indeed these questions can be answered positively. We will show in this section that for every fixed μ there exists a KKT point of the augmented Lagrangian subproblem such that for ρ sufficiently large $\bar{u}_k \in B_r(\bar{u})$. One should keep in mind, that also in this case there is no warranty that forces the algorithm to choose exactly these solutions. However, if the previous iterates are used in numerical computations as a starting point for the computation of the next iterate, the remaining iterates are likely located in $B_r(\bar{u})$. In order to reach this result we need the following assumption which is rather standard.

Assumption 5.19 (Quadratic growth condition (QGC)). Let $\bar{u} \in U_{\text{ad}}$ be a control satisfying the first-order necessary optimality conditions (5.5). We assume that there exist $\beta > 0$ and $r_{\bar{u}} > 0$ such that the quadratic growth condition

$$f(u) \geq f(\bar{u}) + \beta \|u - \bar{u}\|^2_{L^2(\Omega)} \tag{5.11}$$

is satisfied for all feasible $u \in U_{\text{ad}}$, $S(u) \leq \psi$ with $\|u - \bar{u}\|_{L^2(\Omega)} \leq r_{\bar{u}}$. Hence, $\bar{u}$ is a local solution in the sense of $L^2(\Omega)$ for problem (P).

Let us mention that the quadratic growth condition can be implied by some well known second-order sufficient condition (SSC). We refer the reader to Section 5.4 for more details.

Our idea now is the following: In order to show that in every iteration of the algorithm there exists $\bar{u}_k \in B_r(\bar{u})$ we want to estimate the error norm $\|\bar{u}_k - \bar{u}\|^2_{L^2(\Omega)}$. Here we want to exploit the quadratic growth condition from Assumption 5.19. However, this condition requires a control $u \in U_{\text{ad}}$ that is feasible for the original problem (P), which has explicit state constraints. Since the solutions of the augmented Lagrangian subproblems cannot be expected to be feasible for the original problem in general, we consider an auxiliary problem. Due to the special construction of this problem one can construct an auxiliary control that is feasible for the original problem (P). This idea has been presented for instance in [33] for a finite-element approximation as well as in [80] for regularizing a semilinear elliptic optimal control problem with state constraints by applying a virtual control approach.

5.3.1 The Auxiliary Problem

Let $\bar{u}$ be a local solution of (P) that satisfies the first-order necessary optimality conditions (5.5) of Theorem 5.6 and the quadratic growth condition from Assumption 5.19. Following the idea from [33, 80] we consider the following auxiliary problem

$$\begin{aligned} &\underset{u_\rho^r \in L^2(\Omega)}{\text{minimize}} \quad f_{AL}^r(u_\rho^r, \mu, \rho) := f(u_\rho^r) + \frac{1}{2\rho} \left\| \Big(\mu + \rho(S(u_\rho^r) - \psi)\Big)_+ \right\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u_\rho^r \in U_{\text{ad}}, \qquad \left\| u_\rho^r - \bar{u} \right\|_{L^2(\Omega)} \leq r. \end{aligned} \tag{5.12}$$

We choose $r < r_{\bar{u}}$ such that the quadratic growth condition from Assumption 5.19 is satisfied. In the following we define the set of admissible controls of (5.12) by

$$U_{\text{ad}}^r := \{u \in U_{\text{ad}} \mid \|u - \bar{u}\|_{L^2(\Omega)} \leq r\}.$$

Since the set U^r_{ad} is closed, convex and bounded, the auxiliary problem admits at least one (global) solution. Moreover, replacing U_{ad} with U^r_{ad}, first-order necessary optimality conditions can be derived as for the augmented Lagrangian subproblem, see Theorem 5.6.

5.3.2 Construction of a Feasible Control

In this section we want to construct a control $u^{r,\delta} \in U^r_{\text{ad}}$ that is feasible for the original problem (P), i.e., $u^{r,\delta} \in U_{\text{ad}}$ and $S(u^{r,\delta}) \leq \psi$. Based on a Slater point assumption controls of this type have already been constructed in [91] for obtaining error estimates of finite element approximation of linear elliptic state constrained optimal control problems. In [80] these techniques were combined with the idea of the auxiliary problem presented for nonlinear optimal control problems in [33].

We follow the strategy from [80]. This work applied the virtual control approach in order to solve (P). In this approach the state constraints are relaxed by considering mixed control-state-constraints instead of pure state constraints. To obtain optimality conditions for the corresponding auxiliary problem the authors showed that the linearized Slater condition of the original problem can be carried over to feasible controls of the auxiliary problem. This transferred linearized Slater condition is also the main ingredient for the construction of feasible controls of the original problem. In our case, the state constraints have been removed from the set of explicit constraints by augmentation. Thus it is not necessary to establish a linearized Slater condition for the auxiliary problem in order to establish optimality conditions. However the Slater-type inequality that is deduced in the following lemma is still needed for our analysis, see Lemma 5.21.

Lemma 5.20. *Let $\bar{u}$ satisfy Assumption 5.5 with $\sigma > 0$ and associated linearized Slater point $\hat{u}$. For $r > 0$ let us define*

$$\hat{u}^r := \bar{u} + t(\hat{u} - \bar{u}), \qquad t := \frac{r}{\max(r, \|\hat{u} - \bar{u}\|_{L^2(\Omega)})}, \qquad \sigma_r := t\sigma.$$

Then it holds $\|\hat{u}^r - \bar{u}\|_{L^2(\Omega)} \leq r$ and there exists an $\bar{r} > 0$ such that for all $r \in (0, \bar{r})$ and all $\bar{u}^r_\rho \in U^r_{\text{ad}}$ the following inequality is satisfied

$$S(\bar{u}^r_\rho) + S'(\bar{u}^r_\rho)(\hat{u}^r - \bar{u}^r_\rho) \leq \psi - \frac{\sigma_r}{2}. \tag{5.13}$$

Proof. By definition of $\hat{u}^r$ and t it holds $\|\hat{u}^r - \bar{u}\|_{L^2(\Omega)} \leq r$. Inserting the definition of $\hat{u}^r$ we obtain

$$\begin{aligned} S(\bar{u}) + S'(\bar{u})(\hat{u}^r - \bar{u}) &= S(\bar{u}) + tS'(\bar{u})(\hat{u} - \bar{u}) \\ &= (1-t)S(\bar{u}) + t\left(S(\bar{u}) + S'(\bar{u})(\hat{u} - \bar{u})\right) \\ &\leq \psi - t\sigma =: \psi - \sigma_r. \end{aligned}$$

Note, that we exploited the feasibility of $S(\bar{u})$ and the linearized Slater condition in the last step. Hence, $\hat{u}^r$ is a linearized Slater point of the original problem (P) in the neighborhood of $\bar{u}$. We have $\|\hat{u}^r - \bar{u}\| \leq r$, $\left\|\bar{u} - \bar{u}^r_\rho\right\| \leq r$ and hence $\left\|\hat{u}^r - \bar{u}^r_\rho\right\| \leq 2r$. Since S and S' are Lipschitz we obtain (if r sufficiently small) $\left\|S(\bar{u}^r_\rho) - S(\bar{u})\right\|_{C(\overline{\Omega})} \leq \sigma_r/6$, $\left\|S'(\bar{u})(\bar{u} - \bar{u}^r_\rho)\right\|_{C(\overline{\Omega})} \leq \sigma_r/6$ and

$\left\|(S'(\bar{u}^r_\rho) - S'(\bar{u}))(\hat{u}^r - \bar{u}^r_\rho)\right\|_{C(\overline{\Omega})} \le \sigma_r/6$. Hence,

$$\begin{aligned} S(\bar{u}^r_\rho) + S'(\bar{u}^r_\rho)(\hat{u}^r - \bar{u}^r_\rho) &= S(\bar{u}) + S'(\bar{u})(\hat{u}^r - \bar{u}) \\ &\quad + S(\bar{u}^r_\rho) - S(\bar{u}) \\ &\quad + (S'(\bar{u}^r_\rho) - S'(\bar{u}))(\hat{u}^r - \bar{u}^r_\rho) + S'(\bar{u})(\bar{u} - \bar{u}^r_\rho) \\ &\le \psi - \frac{\sigma_r}{2}. \end{aligned}$$

Thus, $\hat{u}^r$ satisfies (5.13) and the proof is done. □

In the following lemma we will construct feasible controls for (P) to be used in the sequel for our convergence analysis. The construction of an admissible control $u^{r,\delta} \in U^r_{\mathrm{ad}}$ that is also feasible for (P) is based on the fact that $\bar{u}^r_\rho$ satisfies Lemma 5.20.

We define the maximal violation of $\bar{u}^r_\rho$ with respect to the state constraints $\bar{y}^r_\rho \le \psi$ by

$$d[\bar{u}^r_\rho, (P)] := \left\|(\bar{y}^r_\rho - \psi)_+\right\|_{C(\overline{\Omega})}, \tag{5.14}$$

where $\bar{y}^r_\rho = S(\bar{u}^r_\rho)$.

Lemma 5.21. *Let all assumptions from Lemma 5.20 be satisfied and define $\delta_\rho \in (0,1)$ via*

$$\delta_\rho := \frac{d[\bar{u}^r_\rho, (P)]}{d[\bar{u}^r_\rho, (P)] + \frac{\sigma_r}{4}}.$$

Then there exists $\bar{r} > 0$ such that for all $r \in (0, \bar{r})$ and $\bar{u}^r_\rho \in U^r_{\mathrm{ad}}$ the auxiliary control

$$u^{r,\delta} := \bar{u}^r_\rho + \delta(\hat{u}^r - \bar{u}^r_\rho)$$

is feasible for the original problem (P), i.e., $S(u^{r,\delta}) \le \psi$ for all $\delta \in [\delta_\rho, 1]$.

Proof. Applying (5.13) the proof follows the argumentation from [80, Lem. 7]. □

The error between the auxiliary control $u^{r,\delta}$ and the global solution $\bar{u}^r_\rho$ of (5.12) is bounded by the maximal constraint violation.

Lemma 5.22. *The constructed feasible control $u^{r,\delta}$ from Lemma 5.21 satisfies for $\delta := \delta_\rho$ the estimate*

$$\left\|\bar{u}^r_\rho - u^{r,\delta}\right\|_{L^2(\Omega)} \le c\, d[\bar{u}^r_\rho, (P)].$$

Proof. We estimate δ_ρ from Lemma 5.21 by

$$\delta_\rho = \frac{d[\bar{u}^r_\rho, (P)]}{d[\bar{u}^r_\rho, (P)] + \frac{\sigma_r}{4}} \le 4\frac{d[\bar{u}^r_\rho, (P)]}{\sigma_r}.$$

Together with $\left\|\hat{u}^r - \bar{u}^r_\rho\right\|_{L^2(\Omega)} \le 2r$ and the definition of σ_r from Lemma 5.20 we arrive at

$$\begin{aligned} \left\|\bar{u}^r_\rho - u^{r,\delta}\right\|_{L^2(\Omega)} &= \left\|\delta_\rho(\hat{u}^r - \bar{u}^r_\rho)\right\|_{L^2(\Omega)} \le 8r\frac{d[\bar{u}^r_\rho, (P)]}{\sigma_r} \\ &\le 8\frac{\max\{r, \|\hat{u}^r - \bar{u}\|_{L^2(\Omega)}\}}{\sigma} d[\bar{u}^r_\rho, (P)] \le c\, d[\bar{u}^r_\rho, (P)] \end{aligned}$$

and the proof is done. □

Finally we are able to apply the quadratic growth condition from Assumption 5.19.

Lemma 5.23. *Let $\bar{u}$ be a local solution of (P) that satisfies the quadratic growth condition Assumption 5.19 and the linearized Slater condition Assumption 5.5. Let $\mu \in L^2(\Omega)$ be fixed. Then there exists $\bar{r} \in (0, r_{\bar{u}})$ such that for all $r \in (0, \bar{r})$, the global solution $\bar{u}^r_\rho$ of the auxiliary problem* (5.12) *satisfies*

$$\beta \left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} \le c \left\|(\bar{y}^r_\rho - \psi)_+\right\|_{C(\bar{\Omega})} + \frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}, \tag{5.15}$$

with a constant $c > 0$ that is independent of ρ and μ.

Proof. As has been shown in Lemma 5.21 there exists $\bar{r} \in (0, r_{\bar{u}})$ such that for all $r \in (0, \bar{r})$ the control $u^{r,\delta}$ is feasible for (P). We insert the special choice $u = u^{r,\delta}$ with $\delta := \delta_\rho$ in the quadratic growth condition (5.11) and obtain

$$\begin{aligned} f(u^{r,\delta}) &\ge f(\bar{u}) + \beta \left\|u^{r,\delta} - \bar{u}\right\|^2_{L^2(\Omega)} \\ &= f(\bar{u}) + \beta \left\|u^{r,\delta} - \bar{u}^r_\rho + \bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} \\ &\ge f(\bar{u}) + \beta \left(\left\|u^{r,\delta} - \bar{u}^r_\rho\right\|^2_{L^2(\Omega)} - 2|(u^{r,\delta} - \bar{u}^r_\rho, \bar{u}^r_\rho - \bar{u})| + \left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} \right) \\ &\ge f(\bar{u}) + \beta \left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} - c\left\|\bar{u}^r_\rho - u^{r,\delta}\right\|_{L^2(\Omega)}, \end{aligned} \tag{5.16}$$

where we exploited that $\left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} \le r^2$ and $\left\|\bar{u}^r_\rho - u^{r,\delta}\right\|_{L^2(\Omega)}$ is bounded by the maximal constraint violation (Lemma 5.22). Rearranging the terms of (5.16) and applying Lemma 5.22 we get

$$\begin{aligned} \beta \left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} &\le f(u^{r,\delta}) - f(\bar{u}) + c\left\|u^{r,\delta} - \bar{u}^r_\rho\right\|_{L^2(\Omega)} \\ &\le f(u^{r,\delta}) - f(\bar{u}^r_\rho) + f(\bar{u}^r_\rho) - f(\bar{u}) + cd[\bar{u}^r_\rho, (P)]. \end{aligned}$$

We recall the definition of the reduced cost functional of the auxiliary problem (5.12)

$$f^r_{AL}(\bar{u}^r_\rho, \mu, \rho) := f(\bar{u}^r_\rho) + \frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)}, \quad \bar{\mu}^r_\rho = (\mu + \rho(S(\bar{u}^r_\rho) - \psi))_+.$$

Exploiting the Lipschitz continuity of the solution operator S for the estimate

$$|f(u^{r,\delta}) - f(\bar{u}^r_\rho)| \le c\left\|u^{r,\delta} - \bar{u}^r_\rho\right\|_{L^2(\Omega)},$$

with a constant c which is only dependent on $\bar{u}$ and exploiting the optimality of $\bar{u}^r_\rho$ for (5.12) as well as applying the definition of the reduced cost functional and the feasibility of $\bar{u}$ for the auxiliary problem, we get

$$\begin{aligned} \beta \left\|\bar{u}^r_\rho - \bar{u}\right\|^2_{L^2(\Omega)} \le& f(\bar{u}^r_\rho) - f(\bar{u}) + cd[\bar{u}^r_\rho, (P)] \\ \le& f^r_{AL}(\bar{u}^r_\rho, \mu, \rho) - f^r_{AL}(\bar{u}, \mu, \rho) - \frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2\rho}\|(\mu + \rho(S(\bar{u}) - \psi))_+\|^2_{L^2(\Omega)} + cd[\bar{u}^r_\rho, (P)] \\ \le& -\frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} + \frac{1}{2\rho}\|(\mu + \rho(S(\bar{u}) - \psi))_+\|^2_{L^2(\Omega)} + cd[\bar{u}^r_\rho, (P)]. \end{aligned}$$

Noting that it holds

$$\frac{1}{2\rho}\left\|(\mu+\rho(S(\bar{u})-\psi))_+\right\|^2_{L^2(\Omega)} \leq \frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}$$

we get with (5.14)

$$\begin{aligned}\beta\left\|\bar{u}^r_\rho-\bar{u}\right\|^2_{L^2(\Omega)}+\frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} &\leq cd[\bar{u}^r_\rho,(P)]+\frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}\\ &= c\left\|(\bar{y}^r_\rho-\psi)_+\right\|_{C(\overline{\Omega})}+\frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}\end{aligned}$$

which yields the claim. □

5.3.3 An Estimate of the Maximal Constraint Violation

In this section we will derive an estimate on the maximal constraint violation. We recall an estimate from [81, Lem. 4].

Lemma 5.24. *Let $f \in C^{0,1}(\overline{\Omega})$ be given with $\|f\|_{C^{0,1}(\Omega)} \leq L$. Then there exists a constant $c_L > 0$, which is only dependent on L, such that the following estimate is satisfied*

$$\|f\|_{C(\overline{\Omega})} \leq c_L\|f\|^{\frac{2}{2+N}}_{L^2(\Omega)}.$$

Theorem 5.25. *Let $\mu \in L^2(\Omega)$ be fixed and $\bar{r}$ be given as in Lemma 5.23. Further, let $\bar{u}^r_\rho$ be an optimal control of the auxiliary problem (5.12) with $r \in (0,\bar{r})$. Then the maximal violation $d[\bar{u}^r_\rho,(P)]$ of $\bar{u}^r_\rho$ with respect to (P) can be estimated by*

$$d[\bar{u}^r_\rho,(P)] \leq c\left(\frac{1}{\rho}\right)^{1/(2+N)}\left(1+\frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}\right)^{1/(2+N)},$$

where $c > 0$ is independent of r, ρ, μ.

Proof. Since $\bar{u}^r_\rho \in L^\infty(\Omega)$ we get with a regularity result [79, Thm. 5] that $\bar{y}^r_\rho \in W^{2,q}(\Omega)$ for all $1 < q < \infty$. Due to the embedding $W^{2,q}(\Omega) \hookrightarrow C^{0,1}(\Omega)$ for $q > N$ we can apply Lemma 5.24 and get the following estimate

$$\begin{aligned}d[\bar{u}^r_\rho,(P)] &= \left\|(S(\bar{u}^r_\rho)-\psi)_+\right\|_{C(\overline{\Omega})} \leq c_L\left\|(\bar{y}^r_\rho-\psi)_+\right\|^{2/(2+N)}_{L^2(\Omega)}\\ &\leq c_L\left\|\frac{1}{\rho}\left(\mu+\rho(\bar{y}^r_\rho-\psi)\right)_+\right\|^{2/(2+N)}_{L^2(\Omega)} = c_L\left(\frac{1}{\rho}\left\|\bar{\mu}^r_\rho\right\|_{L^2(\Omega)}\right)^{2/(2+N)}.\end{aligned} \tag{5.17}$$

From Lemma 5.23 we obtain

$$\frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} \leq c\left\|(\bar{y}^r_\rho-\psi)_+\right\|_{C(\overline{\Omega})}+\frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}.$$

Since $\left\|y^r_{\rho'}\right\|_{C(\overline{\Omega})}$ is uniformly bounded by Theorem 5.2 and $u^r_{\rho'} \in U^r_{\mathrm{ad}}$, there is $c > 0$ independent of r, ρ, μ such that

$$\frac{1}{2\rho}\left\|\bar{\mu}^r_\rho\right\|^2_{L^2(\Omega)} \leq c+\frac{1}{2\rho}\|\mu\|^2_{L^2(\Omega)}.$$

Straight forward calculations now yield

$$\left(\frac{1}{\rho}\left\|\bar{\mu}_\rho^r\right\|_{L^2(\Omega)}\right)^{2/(2+N)} = \left(\frac{1}{\rho}\right)^{1/(2+N)}\left[\frac{1}{\rho}\left\|\bar{\mu}_\rho^r\right\|^2_{L^2(\Omega)}\right]^{1/(2+N)}$$
$$\leq c\left(\frac{1}{\rho}\right)^{1/(2+N)}\left(1+\frac{1}{2\rho}\left\|\mu\right\|^2_{L^2(\Omega)}\right)^{1/(2+N)},$$

which is the desired estimate. □

5.3.4 Main Results

We can now formulate our main results of this section. Let us start with a result that shows that a local solution $\bar{u}$ of the original problem (P) can be approximated by a sequence of successful iterates $(u_n^+)_n$, which are KKT points of the augmented Lagrangian subproblem in an arbitrary small neighborhood of a local solution $\bar{u}$ of (P). Since the successful iterates basically are found by fixing μ and letting ρ_k tend to infinity, this investigation basically reduces to the investigation of a quadratic penalty method with a fixed shift.

Throughout this section we assume that $\bar{u}$ is a local solution of (P) satisfying the QGC from Assumption 5.19 and the linearized Slater condition from Assumption 5.5.

Theorem 5.26. *Let $\mu \in L^2(\Omega)$ be fix, $\bar{r}$ be given as in Lemma 5.23, and let $\bar{u}_\rho^r$ denote a global solution of the auxiliary problem* (5.12).

Then we have:

a) For every $r \in (0, \bar{r})$ there is a $\bar{\rho}$, which is dependent on μ, such that for all $\rho > \bar{\rho}$ it holds $\left\|\bar{u}_\rho^r - \bar{u}\right\|_{L^2(\Omega)} < r$.

b) For every $r \in (0, \bar{r})$ the points $\bar{u}_\rho^r$ are local solutions of the augmented Lagrangian subproblem $(P_{AL})_k$, provided that $\rho > \bar{\rho}$.

Proof. a) The first statement follows directly from Lemma 5.23, the estimate of the maximal constraint violation from Theorem 5.25 and the Lipschitz continuity of the solution operator (5.4).
b) Let $u \in U_{\text{ad}}$ be chosen arbitrarily such that $\left\|u - \bar{u}_\rho^r\right\|_{L^2(\Omega)} \leq \frac{r}{2}$. Applying statement a) we obtain

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq \left\|u - \bar{u}_\rho^r\right\|_{L^2(\Omega)} + \left\|\bar{u}_\rho^r - \bar{u}\right\|_{L^2(\Omega)} \leq \frac{r}{2} + \frac{r}{2} = r$$

for ρ sufficiently large. Thus, $u \in U_{\text{ad}}^r$. Since $\bar{u}_\rho^r$ is the global solution of the auxiliary problem we obtain $f_{AL}(u) \geq f_{AL}(\bar{u}_\rho^r)$ for all $u \in U_{\text{ad}}$ with $\left\|u - \bar{u}_\rho^r\right\|_{L^2(\Omega)} \leq \frac{r}{2}$. □

In Theorem 5.26 we have accomplished to prove that it is at least possible to approximate a local solution of the original problem (P) by a sequence of stationary points of the augmented Lagrangian subproblem. Moreover, Theorem 5.26 is the basis of the further analysis of the behavior of Algorithm 5.1 if in step 1 $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ is chosen as a global solution of the auxiliary problem (5.12).

Theorem 5.27. *Assume that in step 1 of Algorithm 5.1 $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ is chosen as a global solution of the auxiliary problem* (5.12) *if this global solution solves the optimality system of the augmented Lagrangian subproblem* (5.6). *Then Algorithm 5.1 makes infinitely many successful steps.*

Proof. Theorem 5.26 justifies that global solutions of the auxiliary problem (5.12) are local solutions and hence KKT points of the augmented Lagrangian subproblem. The remaining part of the proof follows the proof strategy of Theorem 5.12. □

Moreover, if the penalty parameter remains bounded, the resulting multiplier $\bar{\mu}$ is a function in $L^2(\Omega)$ and $(\frac{1}{\rho_k}\|\bar{\mu}_k\|^2_{L^2(\Omega)})_k$ is uniformly bounded.

Theorem 5.28. *Assume that the sequence of penalty parameters $(\rho_k)_k$ is bounded. Suppose further that $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ is chosen as a global solution of the auxiliary problem* (5.12) *for all k large enough. Then the sequences $(\|\bar{\mu}_k\|_{L^2(\Omega)})_k$ and $(\frac{1}{\rho_k}\|\bar{\mu}_k\|^2_{L^2(\Omega)})_k$ are bounded. The multiplier $\bar{\mu}$ given by Theorem 5.17 belongs to $L^2(\Omega)$.*

Proof. By assumption, the algorithm makes only finitely many unsuccessful steps, and $\rho_k = \bar{\rho}$ holds for all k large enough. In addition, for all k large enough the iterates are global solutions of the auxiliary problem (5.12). Rearranging the terms from Lemma 5.23, we obtain for large k

$$\frac{1}{2\bar{\rho}}\|\bar{\mu}_k\|^2_{L^2(\Omega)} - \frac{1}{2\bar{\rho}}\|\bar{\mu}_{k-1}\|^2_{L^2(\Omega)} \le c\,\|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} \le c\,R_k.$$

By definition of successful steps, we have $\sum_k R_k < +\infty$. Hence, summing the above inequality yields the boundedness of $(\bar{\mu}_k)_k$ in $L^2(\Omega)$. Since $\rho_k \ge \rho_0$, the sequence $(\frac{1}{\rho_k}\|\bar{\mu}_k\|^2_{L^2(\Omega)})_k$ is bounded as well. □

One has to keep in mind that the quadratic growth condition is only a local condition. Hence, the result of Theorem 5.26 is actually the best we can expect. In particular, the subproblems $(P_{AL})_k$ may have solutions arbitrarily far from $\bar{u}$ and we cannot exclude the possibility that these solutions are chosen in the subproblem solution process from Algorithm 5.1. However, one can prevent this kind of scenario by using the previous iterate $\bar{u}_k$ as a starting point for the computation of $\bar{u}_{k+1}$. In this way it is reasonable to expect that as soon as one of the iterates $\bar{u}_k$ lies in $B_r(\bar{u})$ (with r as above) and the penalty parameter is sufficiently large, the remaining iterates will stay in $B_r(\bar{u})$ and converge to $\bar{u}$. In practice, the occurring subproblems will be solved with a semi-smooth Newton method, see Section 5.5, which is only locally superlinear convergent. In order to obtain convergence of the overall method, it is necessary to assume that the initial value of the augmented Lagrangian method is close enough the solution of the penalized subproblem. As soon as as the algorithm has once computed a KKT point of this subproblem, which is sufficiently close to a local solution $\bar{u}$, it is reasonable to expect the whole method to converge.

5.4 Second-Order Sufficient Conditions

We take up the quadratic growth condition from Assumption 5.19. This condition is implied by a second-order sufficient condition, see [29]. We define the Lagrangian function

$$\min_{u\in U_{\text{ad}}} \mathcal{L}(u,\mu) = f(u) + \int_\Omega (S(u) - \psi)\ \mathrm{d}\mu$$

where $y = S(u)$ and assume that for all $(\bar{y}, \bar{p}, \bar{\mu})$ satisfying the first-order necessary optimality conditions (5.5) to $\bar{u}$ it holds

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})[h,h] \ge 0, \qquad \forall h \in C_{\bar{u}}\backslash\{0\}, \tag{5.18}$$

where $C_{\bar{u}}$ denotes the cone of critical directions as defined in [29]. Since the solution operator S (Theorem 5.3) and the cost functional $J : L^2(\Omega) \to \mathbb{R}$ are of class C^2 (see [29, 31]), inequality (5.18) together with the first-order necessary conditions implies the quadratic growth condition from Assumption 5.19, see [29, Thm. 4.1, Remark 4.2] and [116]. Note, that the multiplier $\bar{\mu}$ does not need to be unique. That is why (5.18) is imposed for every multiplier.

Let us return to the convergence analysis of Algorithm 5.1. If in addition to the assumptions of Theorem 5.17, u^* satisfies the QGC from Assumption 5.19, then u^* obviously is a local solution. Second-order sufficient conditions not only allow us to prove convergence to a local solution but also to show local uniqueness of stationary points of the augmented Lagrangian subproblem. This is an important issue for numerical methods. In [79] the authors proved that the Moreau-Yosida regularization without additional shift parameter is equivalent to the virtual control problem for a specific choice of therein appearing parameters. This equivalence can be transferred to the augmented Lagrangian subproblem (P_{AL}).

Remark 5.29. Let $\bar{u} \in U_{\text{ad}}$ be a control that satisfies the first-order necessary optimality conditions (5.5) and let $\bar{\mu}$ be the unique Lagrange multiplier w.r.t. the state constraints. We assume that there exists a constant $\delta > 0$ such that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})[h,h] \geq \delta \, \|h\|^2_{L^2(\Omega)}, \qquad \forall h \in L^2(\Omega). \tag{5.19}$$

One can prove that the SSC (5.19) can be carried over to the augmented Lagrangian subproblems. Let $\mu \in L^2(\Omega)$ and $\rho > 0$ be fixed. Let $\bar{u}_\rho \in U_{\text{ad}}$ be a control that satisfies $\bar{u}_\rho \in B_r(\bar{u})$ and the first-order necessary optimality conditions (5.6). Let the SSC (5.19) be satisfied. Then there exists a constant $\delta' > 0$, which is independent of μ such that for all $h \in L^2(\Omega)$ the following condition

$$f''(\bar{u}_\rho)h^2 + ((\mu + \rho(S(\bar{u}_\rho) - \psi)_+, S''(\bar{u}_\rho)h^2) \geq \delta' \, \|h\|^2_{L^2(\Omega)}$$

or equivalently

$$\int_\Omega (y_h^2 - \bar{p}_\rho d_{yy}(x, \bar{y}_\rho) y_h^2 + \alpha h^2) \, \mathrm{d}x \geq \delta' \, \|h\|^2_{L^2(\Omega)}$$

is fulfilled for all $(h, y_h) \in L^2(\Omega) \times H^1(\Omega)$ provided that ρ is sufficiently large. Here, $y_h = S'(\bar{u}_\rho)h$ and $\bar{p}_\rho$ is the solution of the adjoint equation of the augmented Lagrangian subproblem. Thus, there exists a constant $\beta > 0$ and $\gamma > 0$ such that the quadratic growth condition

$$f_{AL}(u, \mu, \rho) \geq f_{AL}(\bar{u}_\rho, \mu, \rho) + \beta \, \|u - \bar{u}_\rho\|^2_{L^2(\Omega)}$$

holds for all $u \in U_{\text{ad}}$ with $\|u - \bar{u}_\rho\|_{L^2(\Omega)} \leq \gamma$ and $\bar{u}_\rho$ is a local solution with corresponding state $\bar{y}_\rho$ of the augmented Lagrangian subproblem. Here, Theorem 13 from [80] yields the carried over version of the second-order condition for a virtual control problem. In [79, Prop. 3] it is proved that this condition implies a quadratic growth condition for the virtual control problem. Further, following the arguments as in [79, Thm. 5] this results can be adapted to the augmented Lagrangian subproblem.

5.5 Numerical Examples

In this section we report on numerical results for the solution of a semilinear elliptic pointwise state constrained optimal control problem in two dimensions. All optimal control problems have been solved using the above stated augmented Lagrangian algorithm implemented with FEniCS [86] using the DOLFIN [87] Python interface.

In every outer iteration of the augmented Lagrangian algorithm the KKT system (5.6) has to be solved for given μ and ρ. This is done by applying a semi-smooth Newton method. We define the sets

$$\begin{gathered} \mathcal{A}_k^a := \left\{ x \in \Omega \colon -\frac{1}{\alpha}\bar{p}_k \leq u_a \right\}, \qquad \mathcal{A}_k^b := \left\{ x \in \Omega \colon -\frac{1}{\alpha}\bar{p}_k \geq u_b \right\}, \\ \mathcal{Y}_k := \{ x \in \Omega \colon (\mu + \rho(\bar{y}_k - \psi))(x) > 0 \}. \end{gathered} \tag{5.20}$$

Then system (5.6) can be stated as

$$\begin{aligned} A\bar{y}_k + d(\bar{y}_k) &= \bar{u}_k \\ A^*\bar{p}_k + d_y(\bar{y}_k)\bar{p}_k &= \bar{y}_k - y_d + \chi_{\mathcal{Y}_k}(\mu + \rho(\bar{y}_k - \psi)) \\ \bar{u}_k + (1 - \chi_{\mathcal{A}_k^a} - \chi_{\mathcal{A}_k^b})\frac{1}{\alpha}\bar{p}_k &= \chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b. \end{aligned} \tag{5.21}$$

The semi-smooth Newton method for solving (5.6) is given in Algorithm 5.2.

Algorithm 5.2 Semi-smooth Newton method for the augmented Lagrangian subproblem

1: Set $k = 0, \rho > 0, \alpha > 0$, set $\mu \in L^2(\Omega), y_d \in L^2(\Omega), \psi \in C(\overline{\Omega})$.
Choose (y_0, u_0, p_0) in $(H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega) \times H^1(\Omega)$
2: **repeat**
3: Set $\mathcal{A}_k^a$, $\mathcal{A}_k^b$and $\mathcal{Y}_k$ as defined in (5.20)
4: Solve for $\delta_y, \delta_u, \delta_p$ by solving

$$G(y_k, u_k, p_k)(\delta_y, \delta_u, \delta_p) = -F(y_k, u_k, p_k)$$

where

$$G(y_k, u_k, p_k) := \begin{pmatrix} A + d_y(y_k) & -\mathrm{Id} & 0 \\ -(\mathrm{Id} + \chi_{\mathcal{Y}_k}\rho \cdot \mathrm{Id}) + d_{yy}(y_k)p_k & 0 & A^* + d_y(y_k) \\ 0 & \mathrm{Id} & \frac{1}{\alpha}(1 - \chi_{\mathcal{A}_k^a} - \chi_{\mathcal{A}_k^b}) \end{pmatrix}$$

and

$$F(y_k, u_k, p_k) := \begin{pmatrix} Ay_k + d(y_k) - u_k \\ A^*p_k + d_y(y_k)p_k - y_k + y_d - \chi_{\mathcal{Y}_k}(\mu + \rho(y_k - \psi)) \\ u_k + (1 - \chi_{\mathcal{A}_k^a} - \chi_{\mathcal{A}_k^b})\frac{1}{\alpha}p_k - \chi_{\mathcal{A}_k^a}u_a - \chi_{\mathcal{A}_k^b}u_b \end{pmatrix}$$

5: Set $y_{k+1} =: y_k + \delta_y, u_{k+1} := u_k + \delta_u$ and $p_{k+1} := p_k + \delta_p$,
6: Set $k := k + 1$.
7: **until** a suitable stopping criterion is satisfied.

Since the linear parts of the system can be solved exactly, we choose the error that arises during the linearization of the discretized system (5.21) as a stopping criterion. We terminate the semi-smooth Newton method as soon as

$$\max(r_1, r_2, r_3) \leq 10^{-6},$$

where

$$r_1 := \left\| d(y_k) - \left(d_y(y_{k-1})(y_k - y_{k-1}) + d(y_{k-1})\right)\right\|_{L^2(\Omega)},$$

$$r_2 := \left\| d_y(y_k) - (d_y(y_{k-1})p_k + d_{yy}(y_{k-1})p_{k-1}(y_k - y_{k-1})) + (\chi_{\mathcal{Y}_k} - \chi_{\mathcal{Y}_{k-1}})(\mu + \rho(y_k - \psi)\right\|_{L^2(\Omega)},$$

$$r_3 := \left\| u_k - P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}p_k\right)\right\|_{L^2(\Omega)}$$

is satisfied. In the following, $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_h)$ denote the calculated solutions after the stopping criterion is reached. We consider optimal control problems like

$$\begin{aligned} &\text{minimize} \quad J(y,u) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \\ &\text{subject to} \quad y = Su, \qquad y \leq \psi, \qquad u \in U_{\mathrm{ad}}, \end{aligned}$$

where $\Omega = (0,1) \times (0,1)$. As not mentioned otherwise, we initialize $(\bar{y}_0, \bar{u}_0, \bar{p}_0, \mu_1)$ equal to zero and the penalty parameter with $\rho_0 := 1.0$. The parameter in the decision concerning successful steps τ is chosen dependent on the example. If a step has not been successful, the penalization parameter is increased by the factor $\theta := 10$. We stopped the algorithm as soon as

$$R_n^+ := \left\|(y_n^+ - \psi)_+\right\|_{C(\overline{\Omega})} + (\mu_n^+, \psi - y_n^+)_+ \leq 10^{-4} \tag{5.22}$$

was satisfied. Since the stopping criterion from Algorithm 5.2 yields $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ that satisfies (5.5a)-(5.5c) with the desired accuracy this is a suitable stopping criterion.

We compare our method to the plain penalty method. In order to do so, we penalize the state constraint via the standard Moreau-Yosida regularization $(\rho/2)\left\|(y-\psi)_+\right\|^2$ and increase the penalty parameter in every iteration of the arising algorithm via the factor θ, which is the same as for the augmented Lagrangian method. The algorithm is stopped as soon as (5.22) is satisfied. In this situation, all iterates are successful iterates corresponding to the notation $y_n^+, u_n^+, p_n^+, \mu_n^+$ and the approximation of the multiplier μ_n^+ is computed via $\mu_n^+ := \rho(y_n^+ - \psi)_+$. We will refer to this method as the MY method.

Moreover, we will examine the behavior of the algorithm, in particular the behavior of the penalty parameter ρ, dependent on the different choices of τ. The natural choice of $\tau < 1$ as a constant, postulates a linear decrease of the quantity R_n^+. We will refer to this choice of τ as the method AL I. Additionally, we want to investigate the case that the choice of τ is modified such that no linear decrease is required any more. In this way, due to construction of the algorithm, one would expect more successful steps, hence, more updates of the multiplier and less increase of the penalty parameter. In the following we set

$$\tau_k := \begin{cases} \tau_0 \in (0,1), & \text{if } k = 0, \\ \tau_{k-1}, & \text{if the step } k \text{ been successful,} \\ \frac{c_n}{c_n + \frac{1-\tau_0}{\tau_0}}, & \text{if the step } k \text{ has not been successful.} \end{cases}$$

Thus, τ remains unchanged, if the step has been successful, otherwise, we increase its value according to the third case, where c_n is the number of successful steps until the k-th iteration. Clearly, this sequence is monotonically increasing with limit 1. Note, that this choice of τ entails a slight change in Lemma 5.15, where the factor τ^n has to be replaced via $\prod_{j=1}^n \tau_j$. However, since $\tau_j \in (0,1)$ the remaining convergence analysis, see in particular Lemma 5.16, is not influenced. We will indicate this choice of τ as the AL II method.

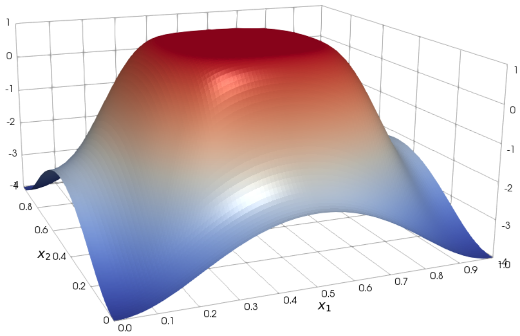
Let us briefly comment on the influence of the tuning parameter τ on the number of successful updates. For a constant choice of τ, one would naturally expect a higher number of successful steps and a smaller value of the final penalty parameter ρ for a large value of τ. We checked all of our numerical examples for different values of τ. As expected, a larger value of τ leads to more successful updates. However, enlarging τ had no influence on the final penalty parameter. Thus, for the subsequent comparison of the different numerical methods in our examples we rely on the choice of τ that yields the best results concerning low iteration numbers and final value of ρ.

Example 1

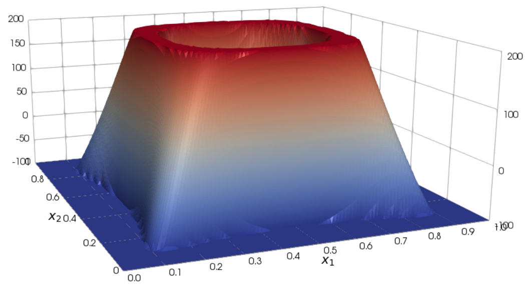
Let $\Omega := (0,1) \times (0,1)$. Let us first consider an optimal control problem that is governed by the following partial differential equation

$$\begin{aligned} -\Delta y + y + \exp(y) &= u \quad \text{in } \Omega, \\ \partial_\nu y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

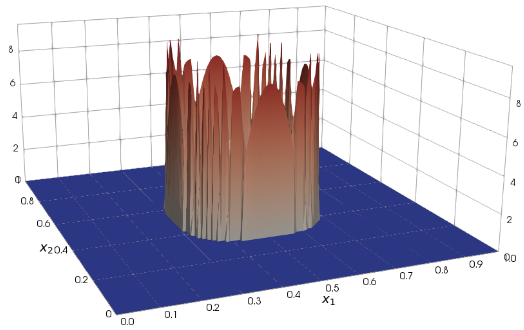
Clearly $d(y) := \exp(y)$ satisfies the required assumptions from Assumption 5.1. We set $y_d(x) := 8\sin(\pi x_1)\sin(\pi x_2) - 4$, $\psi(x) := 1.0$ and $U_{\mathrm{ad}} := \{u \in L^\infty(\Omega) \colon -100 \le u(x) \le 200\}$. We choose $\alpha := 10^{-5}$. Figure 5.1 illustrates the computed results for the augmented Lagrangian method with constant $\tau := 0.4$ for a degree of freedom of 10^4. Table 5.1 shows iterations numbers for the Moreau-Yosida method compared with the augmented Lagrangian method for two different choices of τ. For the constant choice of τ in AL I the augmented Lagrangian method converges nearly as fast as the Moreau-Yosida regularization method, however the penalty parameter is smaller. The value of the final penalty parameter can even be decreased more for AL II. Figure 5.2 depicts the behavior of the penalty parameter for AL I and AL II for a degree of freedom of 10^5. While the penalty parameter tends towards infinity pretty fast for the constant choice of τ in AL I, it can be more controlled for AL II. However, the large percentage of successful steps results in high iteration numbers compared to the other two methods.



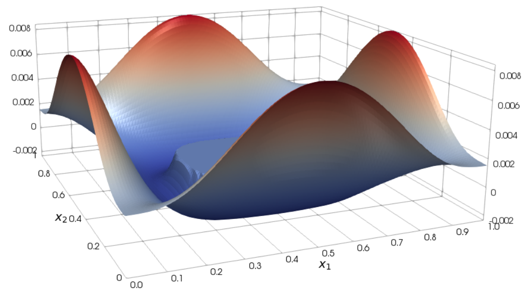
Optimal state $\bar{y}_h$



Optimal control $\bar{u}_h$



Optimal multiplier $\bar{\mu}_h$



Optimal adjoint state $\bar{p}_h$.

Figure 5.1: (Example 1) Computed results for approximately 10^4 degrees of freedom.

	MY			AL I			AL II		
dof	Outer	Inner	Final ρ	Outer	Inner	Final ρ	Outer	Inner	Final ρ
10^2	6	12	10^6	7	14	10^1	7	14	10^1
10^3	6	17	10^6	11	23	10^3	23	36	10^2
10^4	6	23	10^6	11	28	10^4	31	53	10^3
10^5	6	25	10^6	12	35	10^5	45	73	10^3

Table 5.1: (Example 1) Iteration numbers and final value of the penalty parameter ρ with the parameters $\theta = 10$, $\tau := 0.4$ for AL I and $\tau_0 := 0.5$ for AL II.

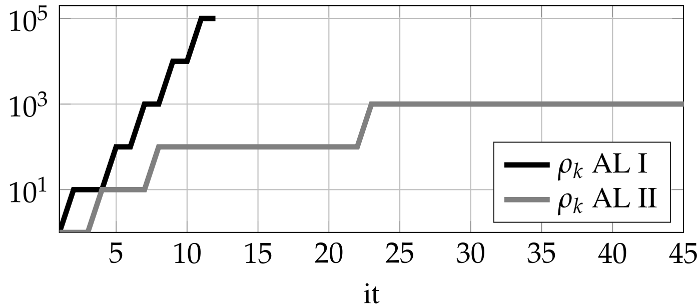


Figure 5.2: (Example 1) $L^1(\Omega)$-norm of discrete multipliers μ_k, penalty parameters ρ_k vs. iteration number for a degree of freedom of 10^5.

Example 2

Next, we consider the partial differential equation

$$\begin{aligned} -\Delta y + y^3 &= u + f \quad &&\text{in } \Omega, \\ \partial_\nu y &= 0 &&\text{on } \Gamma \end{aligned}$$

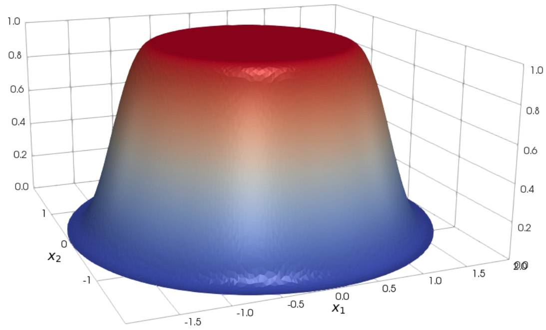
and construct $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ that satisfy the KKT system (5.5). Let $\Omega := B_2(0)$. We consider box constraints and set $u_a := -5$, $u_b := 5$. For clarity and to shorten our notation we set $r := r(x_1, x_2) := \sqrt{x_1^2 + x_2^2}$ and define the following functions

$$\begin{aligned} \bar{y}(x_1,x_2) &:= \begin{cases} 1 & \text{if } r < 1 \\ 32 - 120 \cdot r + 180 \cdot r^2 - 130 \cdot r^3 + 45 \cdot r^4 - 6 \cdot r^5 & \text{if } r \geq 1 \end{cases}, \\ \bar{p}(x_1,x_2) &:= 2\cos\left(\frac{3}{4}\pi x_1\right)\cos\left(\frac{3}{4}\pi x_2\right) \cdot \left(1 - \frac{5}{4}r^3 + \frac{15}{16}r^4 - \frac{3}{16}r^5\right), \\ \bar{u}(x_1,x_2) &:= P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}\bar{p}(x_1,x_2)\right), \\ \bar{\mu}(x_1,x_2) &:= \begin{cases} \exp\left(-\frac{1}{1-r^2}\right) & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \end{cases}, \\ \psi(x_1,x_2) &:= 1. \end{aligned}$$

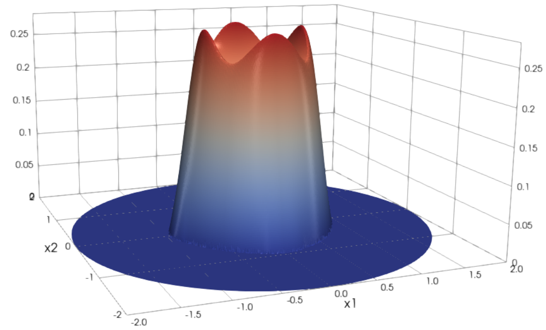
Some calculation show that $\bar{y}, \bar{p} \in C^2(\overline{\Omega})$ and $\bar{\mu} \in C(\overline{\Omega})$. Furthermore $\partial_\nu \bar{y} = \partial_\nu \bar{p} = 0$ on Γ. We now set

$$\begin{aligned} f(x_1,x_2) &:= -\Delta \bar{y}(x_1,x_2) + \bar{y}^3(x_1,x_2) - \bar{u}(x_1,x_2), \\ y_d(x_1,x_2) &:= \Delta \bar{p}(x_1,x_2) - 3\bar{y}^2(x_1,x_2)\bar{p}(x_1,x_2) + \bar{y}(x_1,x_2) + \bar{\mu}(x_1,x_2). \end{aligned}$$

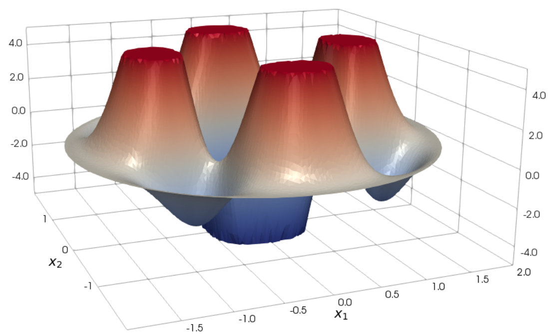
We start the algorithm with $\alpha := 0.1$, $\rho_0 := 1$, and $\tau := 0.5$. Figure 5.3 depicts the computed result for constant $\tau := 0.1$ and a degree of freedom of 10^5. The iteration numbers given in Table 5.2 indicate once more that the augmented Lagrangian method is a suitable method to solve state constrained optimal control problems with a resulting low value of the final penalty parameter ρ compared to the quadratic penalty method. Moreover, in this example the iteration numbers scale well with increasing dimension. This might be due to the case that the multiplier enjoys a higher regularity. In fact $\bar{\mu}$ is an $L^2(\Omega)$-function. Furthermore, Figure 5.4 supports Theorem 5.28 by emphasizing the likely boundedness of the penalty parameter for AL II.



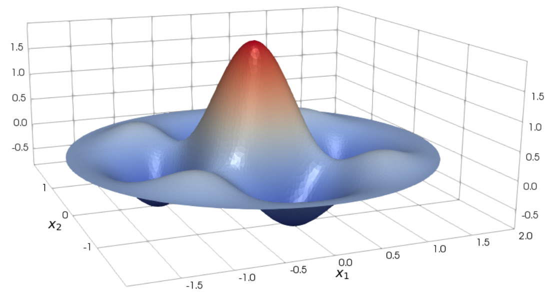
Optimal state $\bar{y}_h$



Optimal multiplier $\bar{\mu}_h$



Optimal control $\bar{u}_h$



Optimal adjoint state $\bar{p}_h$

Figure 5.3: (Example 2) Computed results for approximately 10^5 degrees of freedom.

	MY			AL I			AL II		
dof	Outer	Inner	Final ρ	Outer	Inner	Final ρ	Outer	Inner	Final ρ
10^2	6	16	10^6	7	18	10^3	10	22	10^3
10^3	6	23	10^6	9	29	10^5	12	29	10^3
10^4	5	20	10^5	6	19	10^3	10	24	10^2
10^5	5	19	10^5	6	20	10^3	10	24	10^2

Table 5.2: (Example 2) Iteration numbers and final value of the penalty parameter ρ with the parameters $\theta = 10.0$, $\tau := 0.1$ for AL I and $\tau_0 := 0.5$ for AL II.

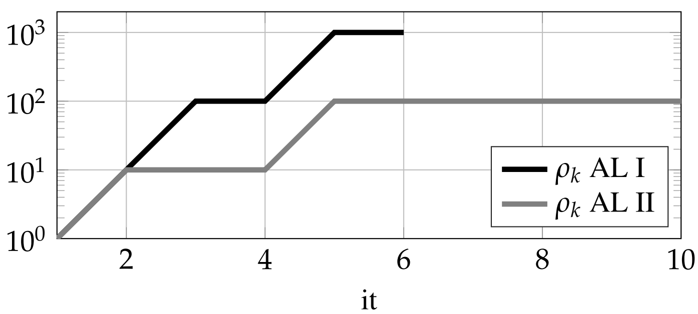


Figure 5.4: (Example 2) penalty parameters ρ_k vs. iteration number for different choices of θ for a degree of freedom of 10^5.

Example 3

We adapt an example from Chapter 3, which can also be found in [107] for state constraints given by $y \geq \psi$. In this case $\Omega := (-1,2) \times (-1,2)$. This example does not include constraints on the control. The optimal control problem is governed by the semilinear partial differential equation

$$\begin{aligned} -\Delta y + y^5 &= u + f \quad \text{in } \Omega, \\ \partial_\nu y &= 0 \quad \text{on } \Gamma \end{aligned}$$

which satisfies Assumption 5.1. We set $r := r(x_1, x_2) := \sqrt{x_1^2 + x_2^2}$. The state constraint is given by $\psi(r) := -\frac{1}{2\pi\alpha}\left(\frac{1}{4} - \frac{r}{2}\right)$. Further, we have

$$\bar{y}(r) := -\frac{1}{2\pi\alpha}\chi_{r\leq 1}\left(\frac{r^2}{4}(\log r - 2) + \frac{r^3}{4} + \frac{1}{4}\right), \qquad \bar{u}(r) := \frac{1}{2\pi\alpha}\chi_{r\leq 1}(\log r + r^2 - r^3),$$

$$\bar{p}(r) := -\alpha\bar{u}(r), \qquad \bar{\mu}(r) := \delta_0(r).$$

It can be checked easily that $\bar{y}$ and $\bar{p}$ satisfy the Neumann boundary. We consider the auxiliary functions

$$\tilde{y}_d(r) := \bar{y}(r) - \frac{1}{2\pi}\chi_{r\leq 1}(4 - 9r), \quad \tilde{f}(r) := -\frac{1}{8\pi}\chi_{r\leq 1}(4 - 9r + 4r^2 - 4r^3)$$

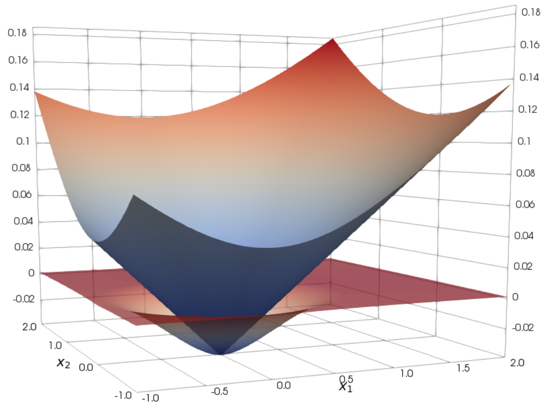
and set

$$y_d(r) := \tilde{y}_d(r) - 5\bar{y}^4\bar{p}, \quad f(r) := \tilde{f}(r) - \bar{y}^5.$$

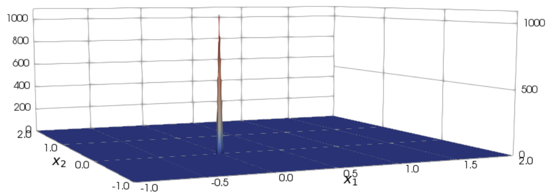
We start the algorithm with $\alpha := 1.0$, $\rho_0 := 0.5$ and $\tau := 0.3$. The computed results can be seen in Figure 5.5 for the choice of constant $\tau := 0.3$ and a degree of freedom of 10^4.
Concerning the performance of the algorithm, all methods behave very similarly, see Table 5.3. While the Moreau-Yosida method holds an advantage concerning iteration numbers, the augmented Lagrangian method requires a smaller value of the penalty parameter at the expense of higher iteration numbers. In this example, the multiplier $\bar{\mu}$ is only a function in $\mathcal{M}(\overline{\Omega})$, i.e., compared to Example 1 and Example 2 it is the most challenging example. This becomes apparent in the larger values of the final penalty parameter ρ as well as the higher iteration numbers that are needed to solve the problem numerically. Moreover, it is surprising that Figure 5.6 indicates the boundedness of the penalty parameter, which we would not expect in general from Theorem 5.28.

	MY			AL I			AL II		
dof	Outer	Inner	Final ρ	Outer	Inner	Final ρ	Outer	Inner	Final ρ
10^2	6	12	10^6	9	17	10^4	12	20	10^3
10^3	7	22	10^7	10	26	10^5	21	37	10^4
10^4	8	32	10^8	12	37	10^7	37	62	10^5
10^5	9	38	10^9	14	45	10^8	84	116	10^6

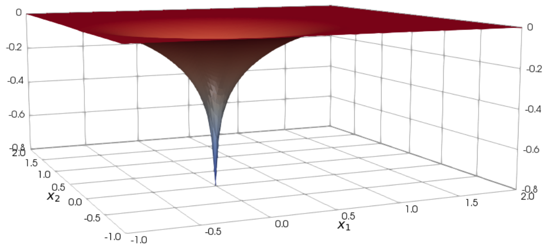
Table 5.3: (Example 3) Iteration numbers and final value of the penalty parameter ρ with the parameters $\theta = 10.0$, $\tau := 0.3$ for AL I and $\tau_0 := 0.4$ for AL II.



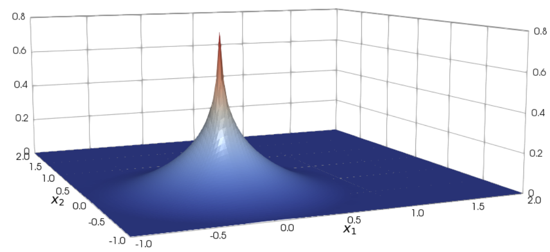
Opt. state $\bar{y}_h$ (transparent) with constraint ψ



Optimal multiplier $\bar{\mu}_h$



Optimal control $\bar{u}_h$



Optimal adjoint state $\bar{p}_h$

Figure 5.5: (Example 3) Computed results for approximately 10^4 degrees of freedom.

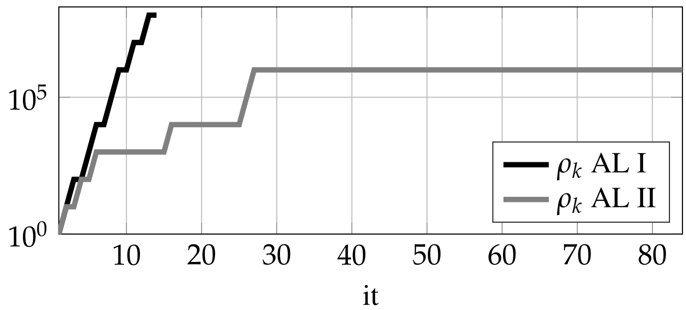


Figure 5.6: (Example 3) $L^1(\Omega)$-norm of discrete multipliers μ_k, penalty parameters ρ_k vs. iteration number for a degree of freedom of 10^5.

Part II

Generalized Nash Equilibrium Problems

CHAPTER 6

JOINTLY CONVEX MULTI-PLAYER OPTIMAL CONTROL PROBLEMS

This chapter is dedicated to the investigation of an N-player generalized Nash equilibrium problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ denote a bounded Lipschitz domain with boundary Γ. Let $Y := H_0^1(\Omega) \cap C(\overline{\Omega})$ and $U := U^1 \times \cdots \times U^N := L^2(\Omega)^N$. Each player aims at solving the following optimal control problem

$$\underset{y^\nu \in Y, u^\nu \in U^\nu}{\text{minimize}} \; J_\nu(y^\nu, u^\nu) := \frac{1}{2} \|y^\nu - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u^\nu\|_{L^2(\Omega)}^2 \tag{6.1}$$

subject to

$$\begin{aligned} Ay &= Bu && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma, \\ u^\nu &\in U_{\text{ad}}^\nu && \text{a.e. in } \Omega, \\ y &\leq \psi && \text{in } \overline{\Omega}, \end{aligned}$$

where A denotes a second-order elliptic operator and $B \in L(L^2(\Omega)^N, L^2(\Omega))$ a linear and continuous mapping. Each player's control u^ν affects the state y via the right hand side of a given linear elliptic partial differential equation. The state y is forced to satisfy the constraint $y \leq \psi$, where $\psi \in C(\overline{\Omega})$. The controls $u^\nu \in L^2(\Omega)$ have to be located in a closed and convex but not necessarily bounded set $U_{\text{ad}}^\nu \in L^2(\Omega)$. In particular, we can choose $U_{\text{ad}}^\nu = L^2(\Omega)$.
The outline of this chapter is as follows. We will collect results about the jointly convex GNEP (6.1) in Section 6.1. The augmented Lagrangian subproblem is introduced in Section 6.2. We continue by presenting our solution method, the augmented Lagrangian method, in Section 6.3. This section is also dedicated to the corresponding convergence analysis. Finally, a numerical example is given in Section 6.4.

6.1 The Multi-Player Control Problem

Throughout this chapter we will work with the following standing assumptions.

Assumption 6.1. For all ν the given data satisfy $y_d^\nu \in L^2(\Omega)$, $\alpha_\nu > 0$, $u_a^\nu, u_b^\nu \in L^2(\Omega)$ with $u_a^\nu \leq u_b^\nu$, $\psi \in C(\overline{\Omega})$. The differential operator A satisfies Assumption 2.19. The linear and continuous operator $B\colon L^2(\Omega)^N \to L^2(\Omega)$ is given by $B := \sum_{\nu=1}^N B_\nu u^\nu$, where $B_\nu \in L(L^2(\Omega), L^2(\Omega))$.

At the first glance, (6.1) does not reveal the structure of a generalized Nash equilibrium problem. In order to fit the corresponding setting for this type of problems we have to consider the reduced formulation of each player's optimal control problem. Let $G\colon L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega})$ denote the solution mapping of $Ay = f$ with $f \in L^2(\Omega)$ and define

$$S_\nu\colon L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad S_\nu u^\nu := GB_\nu u^\nu,$$
$$S\colon L^2(\Omega)^N \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad Su := GBu = \sum_{\nu=1}^N GB_\nu u^\nu = \sum_{\nu=1}^N S_\nu u^\nu.$$

Due to Assumption 6.1, we can deduce by [29, Theorem 2.1] that the operators S_ν and S are linear, continuous and compact. The reduced formulation of (6.1) is now given by

$$\begin{aligned} \underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad & f^\nu(u) := \frac{1}{2}\|Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & u^\nu \in U_{\text{ad}}^\nu \\ & Su(x) \le \psi(x) \text{ in } \overline{\Omega}. \end{aligned} \tag{P^{JC}}$$

This formulation exposes the structure of a jointly convex generalized Nash equilibrium as introduced in Section 2.8.2. This class of problems admits two different solution concepts. First, problem (P^{JC}) can be treated as generalized Nash equilibrium as presented in Section 2.8.1. We will investigate this approach in Chapter 7. Moreover, (P^{JC}) admits the more restrictive solution concept of *normalized Nash equilibria*. In particular, normalized Nash equilibria of (P^{JC}) are unique, which is a great advantage for numerical computations.

For further use, we define the *admissible* and *feasible sets*

$$U_{\text{ad}} := U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N,$$
$$F_{\text{ad}} := \{u \in L^2(\Omega)^N \mid u \in U_{\text{ad}},\ Su(x) \le \psi(x) \text{ in } \overline{\Omega}\}.$$

A point $u \in L^2(\Omega)^N$ is called *feasible* if $u \in F_{\text{ad}}$. Let us recall the definition of a normalized Nash equilibrium.

Definition 6.2 (Normalized equilibrium). Let $\bar{u}$ be a feasible point. Then $\bar{u}$ is called a *normalized solution* or *variational equilibrium* of (P^{JC}) if

$$\sum_{\nu=1}^N f^\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \le \sum_{\nu=1}^N f^\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v \in F_{\text{ad}}. \tag{6.2}$$

From Section 2.8.2 we know that a normalized Nash equilibrium $\bar{u}$ can equivalently be characterized as a point $\bar{u} \in F_{\text{ad}}$ that solves the following variational inequality:

$$(F(\bar{u}), v - \bar{u}) \ge 0 \quad \forall v \in F_{\text{ad}}, \text{ where } F(u) := \left(D_{u^1} f^1(u), \ldots, D_{u^N} f^N(u)\right). \tag{6.3}$$

Theorem 6.3 (Existence of solution). *The generalized Nash equilibrium problem* (P^{JC}) *admits a unique normalized solution.*

Proof. Like in [69] it can be shown that the operator F is strongly monotone in F_{ad} and we obtain the existence of a unique solution of (P^{JC}) from Theorem 2.63. □

For bounded sets U_{ad}^ν, it is also possible to prove existence of solutions by applying the Kakutani-Fan-Glicksberg-Theorem (Theorem 2.59), see [60, Theorem 3.4]. However, this proof does not imply uniqueness of normalized solutions. Similarly to Chapter 3, we require the existence of a Slater point in order to obtain first-order optimality conditions.

Assumption 6.4 (Slater condition). We assume that there exists a $\sigma > 0$ and $\hat{u} \in U_{\text{ad}}$ such that

$$S(\hat{u})(x) \leq \psi(x) - \sigma \quad \text{in } \overline{\Omega}.$$

The Slater condition from Assumption 6.4 implies the following first-order necessary optimality conditions, see [60, Theorem 3.7], which are also sufficient.

Lemma 6.5 (First-order necessary optimality conditions). *Let Assumption 6.4 be satisfied. Let $\bar{u} \in L^2(\Omega)^N$ denote a normalized solution of (P^{JC}) with corresponding state $\bar{y} \in H_0^1(\Omega) \cap C(\overline{\Omega})$. Then there exist adjoint states $\bar{p}^\nu \in W_0^{1,s}(\Omega), s \in [1, d/(d-1))$ and a multiplier $\bar{\mu} \in \mathcal{M}(\overline{\Omega})$ such that for all ν the following system is satisfied.*

$$\begin{aligned} A\bar{y} &= B\bar{u} \quad \text{in } \Omega, \\ \bar{y} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{6.4a}$$

$$\begin{aligned} A^*\bar{p}^\nu &= \bar{y} - y_d^\nu + \bar{\mu} \quad \text{in } \Omega, \\ \bar{p}^\nu &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{6.4b}$$

$$(B_\nu^*\bar{p}^\nu + \alpha_\nu \bar{u}^\nu, v^\nu - \bar{u}^\nu) \geq 0 \quad \forall v^\nu \in U_{\text{ad}}^\nu, \tag{6.4c}$$

$$\langle \bar{\mu}, \bar{y} - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{y} \leq \psi, \quad \bar{\mu} \geq 0. \tag{6.4d}$$

6.2 The Augmented Nash Equilibrium Problem

In a similar way to the single-player optimal control problem from Chapter 3, we augment the pointwise constraints on the state variable y and obtain the following class of parameter dependent Nash equilibrium problems

$$\begin{aligned} &\underset{u_\rho^\nu \in L^2(\Omega)}{\text{minimize}} \quad f_{AL}^\nu(u_\rho, \mu, \rho) := f^\nu(u_\rho) + \frac{1}{2\rho} \left\| (\mu + \rho(Su_\rho - \psi))_+ \right\|_{L^2(\Omega)}^2 \\ &\text{subject to} \quad u_\rho^\nu \in U_{\text{ad}}^\nu. \end{aligned} \tag{P_{AL}^{JC}}$$

We refer to (P_{AL}^{JC}) as the *augmented NEP*. Note that the constraints do no more depend on the other players' strategies. The problem under consideration now turned into a more simple Nash equilibrium problem. We will refer to this kind of problem as the *augmented NEP*. Moreover, u_ρ is called *admissible* for the augmented NEP (P_{AL}^{JC}) if $u_\rho \in U_{\text{ad}}$.

Definition 6.6 (Nash equilibrium). Let $\bar{u}_\rho \in L^2(\Omega)^N$ be admissible for (P_{AL}^{JC}). Then $\bar{u}_\rho$ is a *Nash equilibrium* (NE) or a *solution* of the augmented NEP if and only if for every ν it holds

$$f_{AL}^\nu(\bar{u}_\rho^\nu, \bar{u}_\rho^{-\nu}, \mu, \rho) \leq f_{AL}^\nu(v^\nu, \bar{u}_\rho^{-\nu}, \mu, \rho) \quad \forall v^\nu \in U_{\text{ad}}^\nu.$$

For bounded sets U_{ad}^ν, existence, but not uniqueness, of solutions can be shown applying the Kakutani-Fan-Glicksberg-Theorem, see [59, Theorem 2.3]. For unbounded sets we investigate existence of solutions of the corresponding variational inequality.

Lemma 6.7 (Existence of solution). *The augmented NEP (P_{AL}^{JC}) admits a unique Nash equilibrium.*

Proof. Concatenating each players optimality conditions, we see that solutions of the NEP (P^{JC}_{AL}) can be characterized via the solution u of the variational inequality

$$(F_{AL}(u), v-u) \geq 0 \quad \forall v \in U_{\text{ad}}, \text{ where } F_{AL}(u) := (D_{u^1} f^1_{AL}(u,\mu,\rho), \ \dots \ D_{u^N} f^N_{AL}(u,\mu,\rho)).$$

The operator can be split in $F_{AL} = F + M$ where F is given as in (6.3) and M is defined by

$$M(u) := (S_1^*(\mu + \rho(Su-\psi))_+, \dots, S_N^*(\mu + \rho(Su-\psi))_+).$$

Since M is the gradient of the convex penalty function $u \mapsto \frac{1}{2\rho}\|(\mu + \rho(Su-\psi))_+\|^2_{L^2(\Omega)}$, we know that M is monotone in u. Since F is strongly monotone and M is monotone, F_{AL} is strongly monotone. Thus, the claim follows from Theorem 2.66. ☐

Moreover, first-order necessary optimality conditions of (P^{JC}_{AL}) can be established without any further regularity assumptions, see [59, Proposition 2.8]. Due to convexity these conditions are also sufficient.

Theorem 6.8 (First-order necessary optimality conditions). *Let $\bar{u}_\rho \in L^2(\Omega)^N$ denote a solution of the augmented NEP (P^{JC}_{AL}) and $\bar{y}_\rho \in H^1_0(\Omega) \cap C(\overline{\Omega})$ the corresponding state. Then there exist unique adjoint states $\bar{p}^\nu_\rho \in H^1_0(\Omega)$ such that for all ν the following system is satisfied:*

$$\begin{aligned} A\bar{y}_\rho &= B\bar{u}^\nu_\rho \quad \text{in } \Omega, \\ \bar{y}_\rho &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{6.5a}$$

$$\begin{aligned} A^*\bar{p}^\nu_\rho &= \bar{y}_\rho - y^\nu_d + \bar{\mu}_\rho \quad \text{in } \Omega, \\ \bar{p}^\nu_\rho &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{6.5b}$$

$$(B^*_\nu \bar{p}^\nu_\rho + \alpha_\nu \bar{u}^\nu_\rho, v^\nu - \bar{u}^\nu_\rho) \geq 0 \quad \forall v^\nu \in U^\nu_{\text{ad}}, \tag{6.5c}$$

$$\bar{\mu}_\rho = \left(\mu + \rho(\bar{y}_\rho - \psi)\right)_+. \tag{6.5d}$$

6.3 Convergence Analysis

In the following let $(P^{JC}_{AL})_k$ denote the augmented NEP for given $\rho := \rho_k$ and $\mu := \mu_k$. Its solution will be denoted by $\bar{u}_k$ with corresponding state $\bar{y}_k$ and adjoint states $\bar{p}_k$. The augmented Lagrangian algorithm is given in Algorithm 6.1.

Algorithm 6.1 Augmented Lagrangian Algorithm for (P^{JC})

Let $(\bar{y}_0, \bar{u}_0, \bar{p}_0) \in (H^1(\Omega) \cap C(\overline{\Omega})) \times L^2(\Omega)^N \times W^{1,s}(\Omega)^N$, $\rho_1 > 0$ and $0 \leq \mu_1 \in L^2(\Omega)$ be given. Choose $\theta > 1$, $\tau \in (0,1)$, $\epsilon \geq 0$, $R^+_0 \gg 1$. Set $k := 1$ and $n := 1$.

1: Solve the KKT system (6.5) corresponding to $(P^{JC}_{AL})_k$ and obtain $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$.
2: Set $\bar{\mu}_k := (\mu_k + \rho_k(\bar{y}_k - \psi))_+$.
3: Compute $R_k := \|(\bar{y}_k - \psi)_+\|_{C(\overline{\Omega})} + (\bar{\mu}_k, \psi - \bar{y}_k)_+$.
4: If $R_k \leq \tau R^+_{n-1}$ then the step k is successful. Set

$$\mu_{k+1} := \bar{\mu}_k, \quad \rho_{k+1} := \rho_k$$

and define for all ν:

$$(y^+_n, u^{\nu,+}_n, p^{\nu,+}_n) := (\bar{y}_k, \bar{u}^\nu_k, \bar{p}^\nu_k), \quad \mu^+_n := \bar{\mu}_k, \quad R^+_n := R_k.$$

Set $n := n+1$.
5: Otherwise the step k is not successful. Set $\mu_{k+1} := \mu_k$, $\rho_{k+1} := \theta\rho_k$.
6: If $R^+_{n-1} \leq \epsilon$ then stop, otherwise set $k := k+1$ and go to step 1.

The convergence analysis for jointly convex GNEPs can in wide parts be done almost identically like for a standard optimal control problem, see Chapter 3, which is basically a single-player problem. Several players can be easily incorporated by summing up the related inequalities. Following Lemma 3.8, it is in this way straight forward to obtain the following essential estimate.

Lemma 6.9. *Let* $(\bar{y}_k, \bar{u}_k, \bar{p}_k, \bar{\mu}_k)$ *solve* (6.5) *and* $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ *solve* (6.4). *Then we have the following estimate*

$$\begin{aligned}\|\bar{y}-\bar{y}_k\|^2_{L^2(\Omega)} + \sum_{\nu=1}^{N} \alpha_\nu \|\bar{u}^\nu - \bar{u}^\nu_k\|^2_{L^2(\Omega)} &\leq (\bar{\mu}_k - \bar{\mu}, \bar{y} - \bar{y}_k) \\ &\leq (\bar{\mu}_k, \psi - \bar{y}_k) + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}. \end{aligned} \tag{6.6}$$

Proof. The proof can be done in a similar way to Lemma 3.8. Testing (6.4c) with $\bar{u}^\nu_k$ and (6.5c) with $\bar{u}^\nu$, adding both variational inequalites and applying the definition of the adjoint states $\bar{p}^\nu, \bar{p}^\nu_k$,

$$\bar{p}^\nu := G^*(S\bar{u} - y^\nu_d + \bar{\mu}), \qquad \bar{p}^\nu_k := G^*(S\bar{u}_k - y^\nu_d + \bar{\mu}_k),$$

we obtain for each ν

$$\begin{aligned} 0 &\leq (B^*_\nu(\bar{p}^\nu - \bar{p}^\nu_k) + \alpha_\nu(\bar{u}^\nu - \bar{u}^\nu_k), \bar{u}^\nu_k - \bar{u}^\nu) \\ &= (S^*_\nu(S\bar{u} - S\bar{u}_k + \bar{\mu} - \bar{\mu}_k), \bar{u}^\nu_k - \bar{u}^\nu) - \alpha_\nu \|\bar{u}^\nu - \bar{u}^\nu_k\|^2_{L^2(\Omega)} \\ &= (S\bar{u} - S\bar{u}_k + \bar{\mu} - \bar{\mu}_k, S_\nu(\bar{u}^\nu_k - \bar{u}^\nu)) - \alpha_\nu \|\bar{u}^\nu - \bar{u}^\nu_k\|^2_{L^2(\Omega)}. \end{aligned}$$

Adding this inequality for all ν implies

$$\begin{aligned} \sum_{\nu=1}^{N} \alpha_\nu \|\bar{u}^\nu - \bar{u}^\nu_k\|^2_{L^2(\Omega)} &\leq \sum_{\nu=1}^{N} (S\bar{u} - S\bar{u}_k + \bar{\mu} - \bar{\mu}_k, S_\nu(\bar{u}^\nu_k - \bar{u}^\nu)) \\ &= (S(\bar{u} - \bar{u}_k) + \bar{\mu} - \bar{\mu}_k, S(\bar{u}_k - \bar{u})) \\ &= -\|\bar{y} - \bar{y}_k\|^2_{L^2(\Omega)} + (\bar{\mu}_k - \bar{\mu}, \bar{y} - \bar{y}_k). \end{aligned} \tag{6.7}$$

The second term of (6.7) can be simplified and estimated by

$$(\bar{\mu}_k - \bar{\mu}, \bar{y} - \bar{y}_k) \leq (\bar{\mu}_k, \psi - \bar{y}_k) + \langle \bar{\mu}, \bar{y}_k - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}. \qquad \square$$

Adapting Lemma 3.10, the proof of Theorem 5.9 and Lemma 3.12 to the jointly convex setting, Lemma 6.9 is the essential estimate to arrive at the following result.

Theorem 6.10. *Algorithm 6.1 makes infinitely many successful steps.*

Proof. Suppose the Lagrange multiplier estimate μ is fixed. Then, following Lemma 3.10, it is easy to see that the generated sequence of states $(\bar{y}_k)_k$ converges strongly to $\bar{y}$ in $H^1_0(\Omega) \cap C(\overline{\Omega})$. Thus, in particular, the limit is feasible. Moreover, following the proof of Theorem 5.9 we obtain

$$(\bar{\mu}_k, \psi - \bar{y}_k)_+ \leq \frac{1}{2\rho_k} \|\mu\|^2_{L^2(\Omega)}.$$

Combining these two results, the same proof strategy as in Lemma 3.12 yields that the algorithm makes infinitely many successful step. $\square$

Recall that $(y^+_n, u^+_n, p^+_n, \mu^+_n)$ denotes the solution of the n-th successful iteration. We can now prove $L^1(\Omega)$-boundedness of the multiplier. Let us emphasize that this result yields boundedness of $(u^+_n)_n$ even if U^ν_{ad} is unbounded.

Lemma 6.11. *Let $\hat{u} \in U_{\text{ad}}$ denote the Slater point from Assumption 6.4 with corresponding $\sigma > 0$. Let $\hat{y} := S(\hat{u})$. Then the following estimate is satisfied with a constant $C > 0$ independent of n*

$$\|\mu_n^+\|_{L^1(\Omega)} + \frac{1}{\sigma}\|\hat{y} - y_n^+\|_{L^2(\Omega)}^2 + \sum_{\nu=1}^N \frac{\alpha_\nu}{2\sigma}\|\hat{u}^\nu - u_n^{\nu,+}\|_{L^2(\Omega)}^2 \le C.$$

Proof. Like in the proof of Lemma 3.14 we arrive with the Slater condition from Assumption 6.4 at

$$\sigma\|\mu_n^+\|_{L^1(\Omega)} \le (\mu_n^+, \psi - y_n^+)_+ + (\mu_n^+, y_n^+ - \hat{y}). \tag{6.8}$$

Since the solution operators S_ν are the same for each player we obtain for all ν the estimate

$$\|S_\nu(u_n^{\nu,+} - \hat{u}^\nu)\|_{L^2(\Omega)} \le c\,\|S_\nu(u_n^{\nu,+} - \hat{u}^\nu)\|_{H_0^1(\Omega)} \le c\,\|u_n^{\nu,+} - \hat{u}^\nu\|_{L^2(\Omega)},$$

where $c > 0$ is a constant independent of $u_n^{\nu,+}$ and $\hat{u}^\nu$. Estimating the second term of (6.8) via

$$\begin{aligned}
(\mu_n^+, y_n^+ - \hat{y}) =& (\mu_n^+, \sum_{\nu=1}^N S_\nu(u_n^{\nu,+} - \hat{u}^\nu)) = \sum_{\nu=1}^N \left(B_\nu^* p_n^{\nu,+} - S_\nu^*(Su_n^+ - y_d^\nu), u_n^{\nu,+} - \hat{u}^\nu\right) \\
\le& \sum_{\nu=1}^N \alpha_\nu (u_n^{\nu,+}, \hat{u}^\nu - u_n^{\nu,+}) + \sum_{\nu=1}^N (y_d^\nu - Su_n^+, S_\nu(u_n^{\nu,+} - \hat{u}^\nu)) \\
\le& -\|\hat{y} - y_n^+\|_{L^2(\Omega)}^2 + \sum_{\nu=1}^N \frac{c^2}{\alpha_\nu}\|y_d^\nu - \hat{y}\|_{L^2(\Omega)}^2 \\
&+ \sum_{\nu=1}^N \left(\alpha_\nu (u_n^{\nu,+}, \hat{u}^\nu - u_n^{\nu,+}) + \frac{\alpha_\nu}{4}\|u_n^{\nu,+} - \hat{u}^\nu\|_{L^2(\Omega)}^2\right) \\
\le& -\|\hat{y} - y_n^+\|_{L^2(\Omega)}^2 + \sum_{\nu=1}^N -\frac{\alpha_\nu}{2}\|\hat{u}^\nu - u_n^{\nu,+}\|_{L^2(\Omega)}^2 \\
&+ \sum_{\nu=1}^N \left(\alpha_\nu \|\hat{u}^\nu\|_{L^2(\Omega)}^2 + \frac{c^2}{\alpha_\nu}\|y_d^\nu - \hat{y}\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

we obtain the desired result directly with a similar result like Lemma 3.13. □

Theorem 6.12 (Convergence of solutions). *Let $\bar{u}$ denote the unique normalized solution of (P^{JC}) with associated state $\bar{y}$. The sequence $(y_n^+, u_n^+)_n$ that is generated by Algorithm 6.1 satisfies the following convergence properties:*

$$u_n^+ \to \bar{u} \quad \text{in } L^2(\Omega)^N, \qquad y_n^+ \to \bar{y} \quad \text{in } H_0^1(\Omega) \cap C(\overline{\Omega}).$$

Proof. This is due to inequality (6.6) and the update rule from Algorithm 6.1. □

By Lemma 6.11, we obtain $L^1(\Omega)$-boundedness of the sequence of multipliers $(\mu_n^+)_n$. As we know from [26, Theorem 4] that

$$\|p_n^{\nu,+}\|_{W_0^{1,s}(\Omega)} \le c\left(\|y_n^{\nu,+}\|_{L^2(\Omega)} + \|y_d^\nu\|_{L^2(\Omega)} + \|\mu_n^{\nu,+}\|_{\mathcal{M}(\overline{\Omega})}\right),$$

is satisfied, Lemma 6.11 yields a suitable upper bound and the sequence $(p_n^{\nu,+})_n$ is bounded in $W_0^{1,s}(\Omega), s \in [1, d/(d-1))$. Thus, we can extract weak-* and weakly convergent subsequences of $(\mu_n^+)_n$ in $\mathcal{M}(\overline{\Omega})$, $(p_n^{\nu,+})_n$ in $W_0^{1,s}(\Omega)$, respectively. The corresponding weak limits are indeed multipliers for the original jointly convex GNEP. The proof follows Lemma 3.16 and is omitted here.

Theorem 6.13 (Convergence of dual variables). *Let subsequences* $(p_{n'}^{\nu,+}, \mu_{n'}^{+})_{n'}$ *of* $(p_n^{\nu,+}, \mu_n^{+})$ *be given such that* $\mu_{n'}^{+} \rightharpoonup^* \bar{\mu}$ *in* $\mathcal{M}(\overline{\Omega})$ *and* $p_{n'}^{\nu,+} \rightharpoonup \bar{p}$ *in* $W_0^{1,s}(\Omega)$*,* $s \in [1, d/(d-1))$*. Then* $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ *satisfy the optimality system* (6.4) *of the original problem* (P^{JC}).

6.4 Numerical Example

Let us present a numerical example. All implementations have been done in Fenics [86] using the DOLFIN Python interface [87].

6.4.1 Solution of the Subproblem

Let us briefly comment on the solution of the arising augmented Lagrangian subproblems. These problems are solved by applying a semi-smooth Newton or active set method, respectively. In a similar way to single player optimal control problems, we define for each player the sets

$$\mathcal{A}_k^{\nu,a} := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^* p^\nu(u_k) \le u_a^\nu \right\}, \quad \mathcal{A}_k^{\nu,b} := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^* p^\nu(u_k) \ge u_b^\nu \right\}, \tag{6.9}$$

$$\mathcal{I}_k^\nu := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^* p^\nu(u_k) \in (u_a^\nu, u_b^\nu) \right\}, \mathcal{Y}_k := \{ x \in \Omega \colon (\mu + \rho(y_k - \psi))(x) > 0 \}.$$

The inner loop of Algorithm 6.1 is given in Algorithm 6.2. If the regularization parameters α_ν coincide for all players, one can show that the active set method is equivalent to a semi-smooth Newton method, which converges locally superlinear. Moreover, the stopping criterion yields $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$ such that the residual of (6.5) is equal zero, see Chapter 8. Consequently, $R_n^+ \le \varepsilon$ is a suitable choice for a stopping criterion for the outer loop of the algorithm.

Algorithm 6.2 Algorithm for solving the subproblem of $(P_{AL}^{JC,k})$

1: Set $k := 0$, choose $(y_0, u_0, p_0) \in Y \times L^2(\Omega)^N \times L^2(\Omega)^N$.
2: **repeat**
3: Set $\mathcal{A}_k^{\nu,a}, \mathcal{A}_k^{\nu,b}, \mathcal{I}_k^\nu$ and $\mathcal{Y}_k$ as defined in (6.9).
4: Solve for $(y_{k+1}, u_{k+1}, p_{k+1}) \in Y^N \times L^2(\Omega)^N \times L^2(\Omega)^N$ by solving

$$Ay_{k+1} = Bu_{k+1} \quad \text{in } \Omega,$$

$$A^* p_{k+1}^\nu = y_{k+1} - y_d^\nu + \chi_{\mathcal{Y}_k}(\mu + \rho(y_{k+1} - \psi)) \quad \text{in } \Omega,$$

$$u_{k+1}^\nu + \chi_{\mathcal{I}_k^\nu}\left(\frac{1}{\alpha} B_\nu^* p_{k+1}^\nu\right) = \chi_{\mathcal{A}_k^{\nu,a} k} u_a^\nu + \chi_{\mathcal{A}_k^{\nu,b}} u_b^\nu.$$

5: Set $k := k + 1$
6: **until** $\mathcal{A}_{k-1}^{\nu,b} = \mathcal{A}_{k-2}^{\nu,b}, \mathcal{A}_{k-1}^{\nu,a} = \mathcal{A}_{k-2}^{\nu,a}, \mathcal{I}_{k-1}^\nu = \mathcal{I}_{k-2}^\nu$ and $\mathcal{Y}_{k-1} = \mathcal{Y}_{k-2}$.

6.4.2 Example 1 - Four Player Problem with Known Exact Solution

We adapt a single-player example which has been presented in [93] to the jointly convex setting. Let $\Omega := (0,1)^2$ and $N := 4$. Each player's optimization problem is given by

$$\begin{aligned}
\underset{y\in Y, u^\nu \in L^2(\Omega)}{\text{minimize}} \quad & J^\nu(y,u^\nu) := \frac{1}{2}||y - y_d^\nu||^2_{L^2(\Omega)} + \frac{\alpha_\nu}{2}\,\|u^\nu\|^2_{L^2(\Omega)} \\
\text{subject to} \quad & -\Delta y + y = \sum_{\nu=1}^{N} u^\nu \quad \text{in } \Omega \quad \text{and} \quad \partial y = 0 \quad \text{on } \Gamma, \\
& y(x) \geq \psi(x) \qquad \text{in } \overline{\Omega}.
\end{aligned}$$

This setting differs slightly from the setting given in (P^{JC}). First, we have to deal with an elliptic partial differential equation with homogeneous Neumann boundary conditions. Hence, we obtain $Y := H^1(\Omega) \cap C(\overline{\Omega})$. Second, the state constraints are given in the form $y(x) \geq \psi(x)$. Last, no control constraints are given. However, all results from this chapter can be transferred to this kind of problem. Due to Theorem 6.3 the normalized solution of the jointly convex GNEP is unique. Hence, we are able to construct a numerical example with a known exact and unique solution. We set $\bar{y} := c$ and $\bar{p} = (\alpha_1 b_1, \alpha_2 b_2, \alpha_3 b_3, \alpha_4 b_4)$, where c, b_1, b_2, b_3, b_4 are constant functions. Since this example does not include control constraints, we obtain directly $\bar{u}^\nu = -\alpha_\nu^{-1}\bar{p}^\nu$. Then, we know that c and b_ν have to satisfy

$$c = -\Delta\bar{y} + \bar{y} = \sum_{\nu=1}^{N} \bar{u}^\nu = -\sum_{\nu=1}^{N} b_\nu. \tag{6.11}$$

Defining

$$\psi(x_1,x_2) := \min(c, -20(x_1-0.5)^2 - 20(x_2-0.5)^2 + 1 + c)$$

we know that the Lagrange multiplier has to satisfy

$$\bar{\mu}(x_1,x_2) = \max(-20(x_1-0.5)^2 - 20(x_2-0.5)^2 + 1, 0).$$

The desired states can now be constructed via the adjoint equation

$$y_d^\nu := \Delta\bar{p}^\nu - \bar{p}^\nu + \bar{y} - \bar{\mu} = -\alpha_\nu b_\nu + c - \bar{\mu}.$$

For our numerical test we have chosen $c := 2$, $b := (-0.2, -0.4, -0.6, -0.8)$ and $\alpha_\nu := 1.0$ for all ν. Hence, (6.11) is satisfied and the exact solution is given by

$$\bar{y} = 2, \quad \bar{u} := (0.2, 0.4, 0.6, 0.8), \quad \bar{p} := (-0.2, -0.4, -0.6, -0.8).$$

The algorithm has been initialized with (y_0, u_0, p_0, μ_1) equal to zero. Further, we choose the parameters $\rho_0 := 1.0$, $\theta := 10$, and $\tau := 0.1$. We set $\tilde{\varepsilon} := 10^{-6}$ and stop the algorithm as soon as

$$R_n^+ := \left\|(\psi - y_n^+)_+\right\|_{C(\overline{\Omega})} + (\mu_n^+, y_n^+ - \psi)_+ \leq 10^{-6}$$

is satisfied. Some iteration numbers are given in Table 6.1. The iteration numbers indicate that the augmented Lagrangian algorithm behaves nicely for this type of problem. Independent of the different mesh sizes, we obtain nearly constant iteration numbers and a consistent value of the maximal value of the penalization parameter ρ. Figure 6.1 shows the computed results for $n = 256$.

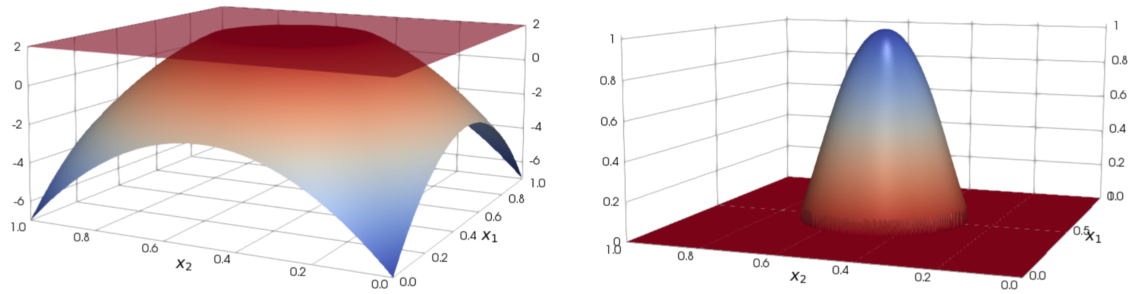


Figure 6.1: (Example 1) Left: State constraint ψ and computed state $\bar{y}_h$ (transparent), Right: Computed multiplier $\bar{\mu}_h$.

n	8	16	32	64	128	256
outer it	11	11	11	11	11	11
inner it	13	14	17	17	19	20
ρ_{max}	10^5	10^5	10^5	10^5	10^5	10^5

Table 6.1: (Example 1) Iteration numbers.

CHAPTER 7

GENERALIZED MULTI-PLAYER OPTIMAL CONTROL PROBLEMS

This chapter aims at extending the results of the previous chapter to the larger class of generalized Nash equilibrium problems. In particular, existence of solutions is not trivial for this type of problems. Under a Slater-type constraint qualification, which will be shown to imply weak Mosco-continuity of the feasible set, we prove an existence result. Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ be a bounded Lipschitz domain with boundary Γ. We consider an N-player game. Let $Y := H_0^1(\Omega) \cap C(\overline{\Omega})$ and $U := U^1 \times \cdots \times U^N := L^2(\Omega)^N$. Throughout this chapter each player aims at solving the optimal control problem

$$\underset{y^\nu \in Y, u^\nu \in U^\nu}{\text{minimize}} \; J_\nu(y^\nu, u^\nu) := \frac{1}{2}\,\|y^\nu - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\,\|u^\nu\|_{L^2(\Omega)}^2 \tag{7.1}$$

subject to

$$\begin{aligned} A_\nu y^\nu &= B_\nu u && \text{in } \Omega, \\ y^\nu &= 0 && \text{on } \partial\Omega, \\ u^\nu &\in U_{\text{ad}}^\nu && \text{a.e. in } \Omega, \\ y^\nu &\le \psi_\nu && \text{in } \overline{\Omega}, \end{aligned} \tag{7.2}$$

where A_ν denotes a second-order elliptic operator and $B_\nu \in L(L^2(\Omega)^N, L^2(\Omega))$ a linear and continuous mapping. The precise setting is given in Section 6.1 below. Let us emphasize that each player's control affects the other players' states y^ν via the right hand side of a given linear elliptic partial differential, i.e., $y^\nu = y^\nu(u^\nu, u^{-\nu})$. Note that this dependence also influences each player's inequality constraints $y^\nu \le \psi_\nu$.

The outline of this chapter is as follows. In Section 7.1 we introduce the GNEP to be solved and specify the problem setting. We state an existence result and establish optimality conditions under a Slater type condition. We connect this condition to the notion of weak Mosco-continuity in Section 7.2. After collecting results about the augmented NEP in Section 7.3, we introduce the augmented Lagrangian algorithm in Section 7.4. In the same section we state our main convergence result (Theorem 7.24). To finish, we illustrate our theoretical findings by some numerical experiments in Section 7.5.

7.1 The Generalized Nash Equilibrium Problem

Throughout this chapter, we will work with the following set of standing assumptions.

Assumption 7.1. a) For all ν the given data satisfy $y_d^\nu \in L^2(\Omega)$, $\alpha_\nu > 0$, $u_a^\nu, u_b^\nu \in L^2(\Omega)$ with $u_a^\nu \le u_b^\nu$, $\psi_\nu \in C(\overline{\Omega})$.

b) The differential operator A_ν satisfies Assumption 2.19.
c) The linear and continuous operator $B_\nu \colon L^2(\Omega)^N \to L^2(\Omega)$ is given by $B_\nu u := B_\nu^1 u^\nu + B_\nu^2 u^{-\nu}$, where

$$B_\nu^1 u^\nu \in L(L^2(\Omega), L^2(\Omega))$$
$$B_\nu^2 u^{-\nu} := \sum_{j=1, j\neq\nu}^{N} B_{\nu,j}^2 u^j \quad \text{with } B_{\nu,j}^2 \in L(L^2(\Omega), L^2(\Omega)).$$

Due to the standing assumptions on A_ν, it is well known that the PDE $A_\nu y = f$ admits a unique weak solution $y \in H_0^1(\Omega) \cap C(\overline{\Omega})$ for every $f \in L^2(\Omega)$. We use the linear and continuous solution operator

$$G_\nu \colon L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad f \mapsto y$$

to define

$$S_\nu^1 \colon L^2(\Omega) \to H_0^1(\Omega) \cap C(\overline{\Omega}), \qquad S_\nu^1 u^\nu := G_\nu B_\nu^1 u^\nu,$$
$$S_\nu^2 \colon L^2(\Omega)^{N-1} \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad S_\nu^2 u^{-\nu} := G_\nu B_\nu^2 u^{-\nu}$$

and set

$$\begin{aligned} S_\nu \colon L^2(\Omega)^N &\to H_0^1(\Omega) \cap C(\overline{\Omega}), \\ S_\nu u := S_\nu^1 u^\nu &+ S_\nu^2 u^{-\nu} = G_\nu B_\nu u. \end{aligned} \tag{7.3}$$

Lemma 7.2. *Let Assumption 7.1 be satisfied. Then the state equation* (7.2) *admits a unique weak solution. The control-to-state mapping S_ν from* (7.3) *is linear, continuous, and compact. Thus, there exists a constant $c > 0$ independent of u such that*

$$\|y^\nu\|_{H_0^1(\Omega)} + \|y^\nu\|_{C(\overline{\Omega})} \le c \left(\|u^\nu\|_{L^2(\Omega)} + \left\|u^{-\nu}\right\|_{L^2(\Omega)^{N-1}} \right). \tag{7.4}$$

Proof. It is well known that the solution operator G_ν is linear and continuous [26]. Since B_ν^1, B_ν^2 are linear and continuous as well, we arrive at (7.4). Since G_ν is completely continuous [29, Theorem 2.1] and continuity of B_ν^1, B_ν^2 implies their weak continuity, we conclude complete continuity of S_ν. □

Inserting the mapping S_ν in (7.1) we obtain the reduced formulation of the optimal control problem

$$\begin{aligned} \underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \quad & f^\nu(u) := \frac{1}{2} \|S_\nu u - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u^\nu\|_{L^2(\Omega)}^2 \\ \text{subject to} \quad & u_a^\nu(x) \le u^\nu(x) \le u_b^\nu(x) \qquad \text{a.e. in } \Omega, \\ & S_\nu(u^\nu, u^{-\nu})(x) \le \psi_\nu(x) \qquad \text{in } \overline{\Omega}. \end{aligned} \tag{P}$$

We define the *admissible* and the *feasible sets*

$$\begin{aligned} U_{\text{ad}} &:= U_{\text{ad}}^1 \times \cdots \times U_{\text{ad}}^N, \\ F_{\text{ad}}(u) &:= F_{\text{ad}}^1(u^{-1}) \times \cdots \times F_{\text{ad}}^N(u^{-N}), \end{aligned} \tag{7.5}$$

where

$$U_{\text{ad}}^\nu := \{u^\nu \in L^2(\Omega) \mid u_a^\nu(x) \le u^\nu(x) \le u_b^\nu(x) \text{ a.e. in } \Omega\},$$
$$F_{\text{ad}}^\nu(u^{-\nu}) := \{u^\nu \in L^2(\Omega) \mid u^\nu \in U_{\text{ad}}^\nu,\ S_\nu(u^\nu, u^{-\nu})(x) \le \psi_\nu(x) \text{ in } \overline{\Omega}\}.$$

A point $\bar{u} \in L^2(\Omega)^N$ is called *feasible*, if $\bar{u} \in F_{\text{ad}}(\bar{u})$. Let us recall the definition of a solution of a generalized Nash equilibrium problem.

Definition 7.3 (Generalized Nash equilibrium). Let $\bar{u} \in L^2(\Omega)^N$ be feasible. We say that $\bar{u}$ is a *generalized Nash equilibrium (GNE)* or a *solution* of the GNEP (P) if and only if for all ν it holds

$$f_\nu(\bar{u}^\nu, \bar{u}^{-\nu}) \leq f_\nu(v^\nu, \bar{u}^{-\nu}) \quad \forall v^\nu \in F_{\text{ad}}^\nu(\bar{u}^{-\nu}).$$

In order to derive first-order optimality conditions a constraint qualification is needed. Throughout this chapter we will assume that a solution $\bar{u}$ of the GNEP satisfies the following Slater condition.

Definition 7.4 (Slater condition). We say that a point $u^* \in U_{\text{ad}}$ satisfies the Slater condition if for every ν there exists a point $\hat{u}^\nu \in U_{\text{ad}}^\nu$ and a $\sigma_\nu > 0$ such

$$S_\nu(\hat{u}^\nu, u^{*-\nu})(x) \leq \psi_\nu(x) - \sigma_\nu \quad \text{in } \overline{\Omega}. \tag{7.6}$$

Assumption 7.5. We assume that a solution $\bar{u} \in L^2(\Omega)^N$ of the GNEP (P) satisfies the Slater condition from Definition 7.4.

A similar constraint qualification has already been presented in [60] for providing optimality conditions of the therein treated GNEP. Incorporating the varying solution operators S_ν, the proof of [60, Theorem 2.5] can easily be adapted to our setting.

Lemma 7.6 (First-order necessary optimality conditions). *Let $\bar{u} \in L^2(\Omega)^N$ be a solution of the GNEP (P) that satisfies Assumption 7.5 and $\bar{y} \in (H_0^1(\Omega) \cap C(\overline{\Omega}))^N$ the corresponding state. Then there exist adjoint states $\bar{p}^\nu \in W_0^{1,s}(\Omega), s \in [1, d/(d-1))$ and multipliers $\bar{\mu}^\nu \in \mathcal{M}(\overline{\Omega})$ such that for all ν the system*

$$\begin{aligned} A_\nu \bar{y}^\nu &= B\bar{u} && \text{in } \Omega, \\ \bar{y}^\nu &= 0 && \text{on } \Gamma, \end{aligned} \tag{7.7a}$$

$$\begin{aligned} A_\nu^* \bar{p}^\nu &= \bar{y}^\nu - y_d^\nu + \bar{\mu}^\nu && \text{in } \Omega, \\ \bar{p}^\nu &= 0 && \text{on } \Gamma, \end{aligned} \tag{7.7b}$$

$$(B_\nu^{1*} \bar{p}^\nu + \alpha_\nu \bar{u}^\nu, v^\nu - \bar{u}^\nu) \geq 0 \quad \forall v^\nu \in U_{\text{ad}}^\nu, \tag{7.7c}$$

$$\langle \bar{\mu}^\nu, \bar{y}^\nu - \psi_\nu \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad \bar{y}^\nu \leq \psi_\nu, \quad \bar{\mu}^\nu \geq 0 \tag{7.7d}$$

is satisfied. Here, $\bar{\mu}^\nu \geq 0$ means that $\bar{\mu}^\nu$ lies in the dual of the nonnegative cone of $C(\overline{\Omega})$.

Due to the convexity of the reduced cost functional and of the sets $F_{\text{ad}}^\nu(\bar{u}^{-\nu})$ the first-order necessary optimality conditions of each player's problem are also sufficient. Hence, every tuple $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ that satisfies the optimality system (7.7) is a generalized Nash equilibrium. The Slater condition from Definition 7.4 requires a Slater point $\hat{u}^\nu$ dependent of $u^{*-\nu}$ for each player's optimization problem. However, for proving existence of a generalized Nash equilibrium problem it is not enough to claim this property in one single point u^*. In fact, one needs the following stronger *uniform Slater condition*.

Assumption 7.7 (Uniform Slater condition). We assume that for every ν there exists a $\sigma_\nu > 0$ such that for all $u^{-\nu} \in U_{\text{ad}}^{-\nu}$ there exists a point $\hat{u}^\nu \in U_{\text{ad}}^\nu$ satisfying

$$S_\nu(\hat{u}^\nu, u^{-\nu})(x) \leq \psi_\nu(x) - \sigma_\nu \quad \text{in } \overline{\Omega}.$$

An assumption of this type is not completely new. In [59] it has been used for proving existence of a normalized equilibrium of a jointly convex GNEP and showing convergence of the therein applied Moreau-Yosida penalty method. Note that this is a much stronger assumption than the Slater condition from Assumption 7.5. Obviously, the uniform Slater condition implies the Slater condition.

For each tuple $u^{-\nu} \in U_{\text{ad}}^{-\nu}$ let $M_\nu(u^{-\nu})$ denote the optimal solution set of the ν-th player's optimization problem. Thus, a GNE $\bar{u}$ has to satisfy $\bar{u}^\nu \in M_\nu(\bar{u}^{-\nu})$ for all ν. Based on [98], which deals with finite dimensional generalized Nash equilibrium problems, we postulate for every player ν the following constraint qualification in order to establish an existence result.

Definition 7.8 (Weak Convergence Constraint Qualification). Let $(u_k)_k$ in U_{ad} denote an arbitrary weakly convergent sequence such that $u_k^\nu \in M_\nu(u_k^{-\nu})$ for every k. We say that $(u_k)_k$ satisfies the *weak convergence constraint qualification (WCCQ)* if there exist sequences $(\mu_k^\nu)_k$ in $\mathcal{M}(\overline{\Omega})$ with weak-* convergent subsequences $(\mu_{k'}^\nu)_{k'}$ such that $(y_{k'}^\nu, u_{k'}^\nu, p_{k'}^\nu, \mu_{k'}^\nu)$ satisfies the ν-th player's optimality system (7.7), where $y_{k'}^\nu = S_\nu(u_{k'}^\nu, u_{k'}^{-\nu})$ and $p_{k'}^\nu = G_\nu^*(y_{k'}^\nu - y_d + \mu_{k'}^\nu)$.

The WCCQ is naturally satisfied by the uniform Slater condition.

Lemma 7.9. *The uniform Slater condition from Assumption 7.7 implies the* WCCQ *from Definition 7.8.*

Proof. We choose an arbitrary sequence $u_k \rightharpoonup u^* \in U_{\text{ad}}$ such that for every k the control u_k^ν solves the optimization problem (P) for given $u_k^{-\nu}$, i.e., $u_k^\nu \in M_\nu(u_k^{-\nu})$. Due to Assumption 7.7, we obtain existence of a sequence of adjoint states $(p_k^\nu)_k$ and multipliers $(\mu_k^\nu)_k$ such that $(y_k^\nu, u_k^\nu, p_k^\nu, \mu_k^\nu)$ satisfies the KKT system (7.7) for every k. It remains to show that a subsequence of $(\mu_k^\nu)_k$ converges weak-* in $\mathcal{M}(\overline{\Omega})$. We use the uniform Slater condition from Assumption 7.7 to show $L^1(\Omega)$-boundedness of the corresponding sequence of the multipliers. Since $u_k \rightharpoonup u^*$ in $L^2(\Omega)^N$, we can conclude boundedness of $(u_k)_k, (y_k)_k$ in $L^2(\Omega)^N$. Now, recall that the definition of the adjoint state yields the identity

$$S_\nu^{1^*}\mu_k^\nu = B_\nu^{1^*}G_\nu^*\mu_k^\nu = B_\nu^{1^*}G_\nu^*(y_d^\nu - S_\nu u_k) + B_\nu^{1^*}p_k^\nu = S_\nu^{1^*}(y_d^\nu - S_\nu u_k) + B_\nu^{1^*}p_k^\nu.$$

Further, the uniform Slater condition yields existence of a constant $\sigma_\nu > 0$ such that for every $u_k^{-\nu}$ of the sequence $(u_k)_k$ there exists a corresponding Slater point $\hat{u}_k^\nu \in U_{\text{ad}}^\nu$ satisfying

$$S_\nu(\hat{u}_k^\nu, u_k^{-\nu})(x) \leq \psi_\nu(x) - \sigma_\nu \quad \text{in } \overline{\Omega}.$$

Hence, we obtain

$$\begin{aligned}
\sigma_\nu \|\mu_k^\nu\|_{L^1(\Omega)} &= \int_\Omega \sigma_\nu \mu_k^\nu \,\mathrm{d}x \leq \int_\Omega \mu_k^\nu(\psi - S_\nu(\hat{u}_k^\nu, u_k^{-\nu}))\,\mathrm{d}x \\
&= \underbrace{\langle \mu_k^\nu, \psi - y_k^\nu\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}}_{=0} + \langle \mu_k^\nu, S_\nu(u_k^\nu, u_k^{-\nu}) - S_\nu(\hat{u}_k^\nu, u_k^{-\nu})\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \\
&= (S_\nu^{1^*}\mu_k^\nu, u_k^\nu - \hat{u}_k^\nu) = (S_\nu^{1^*}(y_d^\nu - S_\nu u_k) + B_\nu^{1^*}p_k^\nu, u_k^\nu - \hat{u}_k^\nu) \\
&\leq (S_\nu^{1^*}(y_d^\nu - y_k^\nu), u_k^\nu - \hat{u}_k^\nu) + \alpha_\nu(u_k^\nu, \hat{u}_k^\nu - u_k^\nu) \\
&\leq \|u_k^\nu - \hat{u}_k^\nu\|_{L^2(\Omega)}(c\|y_d^\nu - y_k^\nu\|_{L^2(\Omega)} + \alpha_\nu\|u_k^\nu\|_{L^2(\Omega)}).
\end{aligned}$$

Dividing the above inequality by $\sigma_\nu > 0$ and exploiting boundedness of u_k, y_k as well as $\hat{u}_k^\nu \in U_{\text{ad}}^\nu$, we obtain boundedness of $(\mu_k^\nu)_k$ in $L^1(\Omega)$. Hence, we can extract a weak-* convergent subsequence $\mu_{k'}^\nu \rightharpoonup^* \mu^{\nu*}$ in $\mathcal{M}(\overline{\Omega})$. □

Existence of solutions of the reduced GNEP (P) can be shown by applying the Kakutani-Fan-Glicksberg Theorem (Theorem 2.59). Defining

$$\Phi : U_{\text{ad}} \rightrightarrows U_{\text{ad}}, \quad \Phi(u) = \prod_{\nu=1}^N M_\nu(u^{-\nu}) \tag{7.8}$$

and following the argumentation from Section 2.8.1 we know that it is enough to check if graph(Φ) is weakly sequentially closed.

Theorem 7.10 (Existence of solutions of the GNEP). *Let the uniform Slater condition from Assumption 7.7 be satisfied. Then the GNEP* (P) *admits a solution.*

Proof. Let Φ be given as in (7.8). Due to the uniform Slater condition from Assumption 7.7 the set $M_\nu(u^{-\nu})$ is nonempty for all $u^{-\nu} \in U_{\text{ad}}^{-\nu}$. Due to strong convexity for given $u^{-\nu}$ the solution sets $M_\nu(u^{-\nu})$ are single-valued. Hence, we know that $\Phi(u)$ is nonempty and convex. Let $u_k \rightharpoonup u^*$ in U_{ad}, $w_k \rightharpoonup w^*$ in U_{ad} with $w_k^\nu \in M_\nu(u_k^{-\nu})$ for all ν. Hence, $w_k \in \Phi(u_k)$. We have to show that these conditions imply $w^* \in \Phi(u^*)$.

Let us fix an arbitrary ν. For each k we define $L^2(\Omega)^N \ni z_k = \left(z_k^{\nu'}\right)_{\nu'=1}^N$ via

$$z_k^{\nu'} := \begin{cases} w_k^\nu & \text{if } \nu' = \nu, \\ u_k^{\nu'} & \text{else.} \end{cases}$$

Since $u_k^{\nu'} \rightharpoonup u^{*\nu'}$ and $w_k^\nu \rightharpoonup w^{*\nu}$ in U_{ad}^ν, z_k is weakly convergent, i.e., $z_k \rightharpoonup z^*$ in U_{ad} with

$$z^{*\nu'} := \begin{cases} w^{*\nu} & \text{if } \nu' = \nu, \\ u^{*\nu'} & \text{else.} \end{cases}$$

By Lemma 7.9 the uniform Slater condition from Assumption 7.7 implies the WCCQ from Definition 7.8. We consider the weakly convergent sequence $(z_k)_k$. Due to construction of this sequence we have $z_k := (w_k^\nu, u_k^{-\nu})$ with $w_k^\nu \in M_\nu(u_k^{-\nu})$. Hence, $z_k^\nu \in M_\nu(z_k^{-\nu})$. Thus, by the WCCQ we find sequences $(\mu_k^\nu)_k$ in $\mathcal{M}(\overline{\Omega})$ with weak-* convergent subsequences $(\mu_{k'}^\nu)_{k'}$ such that $(S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}), w_{k'}^\nu, G_\nu^*(S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}) - y_d^\nu + \mu_{k'}^\nu), \mu_{k'}^\nu)$ satisfies the ν-th player's optimality system (7.7) for every k'

$$\begin{gathered} (S_\nu^{1*}(S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}) - y_d^\nu + \mu_{k'}^\nu) + \alpha_\nu w_{k'}^\nu, v - w_{k'}^\nu) \geq 0 \quad \forall v \in U_{\text{ad}}^\nu, \\ \langle \mu_{k'}^\nu, S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}) - \psi_\nu \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}) \leq \psi_\nu, \quad \mu_{k'}^\nu \geq 0. \end{gathered}$$

Due to Lemma 7.2 we have $S_\nu(w_{k'}^\nu, u_{k'}^{-\nu}) \to S_\nu(w^{*\nu}, u^{*-\nu})$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Further, we know $S_\nu^1(w_{k'}^\nu) \to S_\nu^1(w^{*\nu})$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Exploiting the weak-* convergence $\mu_{k'}^\nu \overset{*}{\rightharpoonup} \mu^{*\nu}$ of $(\mu_k^\nu)_k$ in $\mathcal{M}(\overline{\Omega})$ and the weak lower semicontinuity of $\alpha_\nu(w_{k'}^\nu, w_{k'}^\nu - v)$, we can conclude that the limit $k' \to \infty$ satisfies

$$\begin{gathered} (S_\nu^{1*}(S_\nu(w^{*\nu}, u^{*-\nu}) - y_d^\nu + \mu^{*\nu}) + \alpha_\nu w^{*\nu}, v - w^{*\nu}) \geq 0 \quad \forall v \in U_{\text{ad}}^\nu, \\ \langle \mu^{*\nu}, S_\nu(w^{*\nu}, u^{*-\nu}) - \psi_\nu \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0, \quad S_\nu(w^{*\nu}, u^{*-\nu}) \leq \psi_\nu, \quad \mu^{*\nu} \geq 0. \end{gathered}$$

Hence, $w^{*\nu} \in M_\nu(u^{*-\nu})$. Thus, $w^* \in \Phi(u^*)$ and we obtain existence of a solution of the GNEP. □

7.2 Connection to Weak Mosco-Continuity

Besides the WCCQ condition the uniform Slater condition implies the weak Mosco-continuity of the feasible set F_{ad}. Introduced by Mosco [96], the weak Mosco-continuity is until now the most common assumption for showing existence of solutions or convergence of algorithms concerning QVIs, see for instance [72, 83, 85]. In this section, we will illustrate the connection of the uniform Slater condition from Assumption 7.7 and the weak Mosco-continuity of the feasible set F_{ad}. This shows even more that the uniform Slater condition is a reasonable assumption.

Definition 7.11 (Mosco-convergence and weak Mosco-continuity). Let Q and Q_k, $k \in \mathbb{N}$ be subsets of a Banach space X. We say that $(Q_k)_k$ *Mosco-converges* to Q, and write $Q_k \overset{M}{\to} Q$, if

a) for every $v \in Q$, there is a sequence $v_k \in Q_k$ such that $v_k \to v$ in X, and
b) whenever $v_k \in Q_k$ for all k and v is a weak limit point of v_k, then $v \in Q$.

Let further $\Upsilon\colon X \rightrightarrows X$ denote a set-valued mapping and let $u \in X$. We say that Υ is *weakly Mosco-continuous* in u if $u_k \rightharpoonup u$ implies $\Upsilon(u_k) \overset{M}{\to} \Upsilon(u)$. If this holds for every $u \in X$, we say that Υ is weakly Mosco-continuous.

Besides the WCCQ condition, the uniform Slater condition also implies weak Mosco-continuity of the set-valued mapping F_{ad} in U_{ad}, which strengthens the motivation to apply this condition. It is straight forward to adapt the proof of [62, Thm. 1.55] in order to show that the uniform Slater condition implies Robinson's regularity condition.

Lemma 7.12. *Let U, Y denote Banach spaces. Let $C \subset U$ be a closed, convex set, $K \subset Y$ be a closed convex cone. If $G\colon U^2 \to Y$ is differentiable and convex w.r.t. K in the first argument and there exists $\hat{u} \in C$ such that*

$$G(\hat{u}, u) \in \operatorname{int}(K) \quad \forall u \in C$$

is satisfied, then Robinson's regularity condition *holds at all points $v \in C$ such that $G(v,u) \in K$ with respect to the mapping $G(\cdot, u)$, i.e.,*

$$0 \in \operatorname{int}\left(G(v,u) + D_v G(v,u)(C - v) - K\right) \quad \forall v, u \in C \textit{ with } G(v,u) \in K. \tag{7.9}$$

Based on Robinson's condition it is possible to state weak Mosco-continuity of the feasible set in the admissible set. The following result is based on [72, Cor. 3.5], where weak Mosco-continuity in a feasible point has been investigated under the assumption that

$$0 \in \operatorname{int}\left(G(\bar{u}, \bar{u}) + D_v G(\bar{u}, \bar{u})(C - \bar{u}) - K\right) \tag{7.10}$$

is satisfied, i.e., Robinson's condition is satisfied in a feasible point $\bar{u}$. The proof exploits that (7.10) implies that (7.9) is satisfied [114, Prop. 3.21]. Thus, we arrive directly at the following theorem.

Theorem 7.13. *Let U, Y denote Banach spaces. Let $C \subset U$ denote a closed, convex set and $K \subset Y$ a closed, convex cone. Let $G\colon U^2 \to Y$ denote an operator that is linear, completely continuous in $X \times U$, and convex w.r.t. K in the first variable. Further, let there exist $\hat{u} \in C$ such that for all $u \in C$ the condition $G(\hat{u}, u) \in \operatorname{int}(K)$ is satisfied. Then the set valued mapping $\Upsilon : U \rightrightarrows U$*

$$\Upsilon(u) = \{v \in C \mid G(v,u) \in K\}$$

is weakly Mosco-continuous in C.

We can now apply Theorem 7.13 to the set-valued mapping F_{ad} from (7.5).

Corollary 7.14. *Let the uniform Slater condition from Assumption 7.7 be satisfied. Then the set-valued mapping $F_{\text{ad}}\colon L^2(\Omega)^N \rightrightarrows L^2(\Omega)^N$ from* (7.5) *is weakly Mosco-continous in all $u \in U_{\text{ad}}$.*

Proof. We set $U := L^2(\Omega)^N, Y := (H_0^1(\Omega) \cap C(\overline{\Omega}))^N$ and define the operator G via

$$G\colon U^2 \to Y, \quad G(v,u) = \begin{pmatrix} S_1(v^1, u^{-1}) - \psi_1 \\ S_2(v^2, u^{-2}) - \psi_2 \\ \vdots \\ S_N(v^N, u^{-N}) - \psi_N \end{pmatrix}.$$

Using this representation we can rewrite F_{ad} as $F_{\text{ad}}(u) = \{v \in U_{\text{ad}} \mid G(v,u) \in K\}$, where K denotes the cone of non-positive continuous functions which is closed and convex. Thus, convexity with respect to K is the standard definition of convexity. Further, the set $U_{\text{ad}} \subset L^2(\Omega)^N$ is a closed convex set. From Lemma 7.2 we know that operator G is linear, hence convex, and compact. Last, the uniform Slater condition requires $\hat{u} \in U_{\text{ad}}$ such that for all $u \in U_{\text{ad}}$ the condition $G(\hat{u},u) \in \text{int}(K)$ is satisfied. Thus, all assumptions of Lemma 7.13 are satisfied and we immediately obtain the weak Mosco-continuity of F_{ad} in all $u \in U_{\text{ad}}$. □

7.3 The Augmented NEP

We define a class of parameter dependent Nash equilibrium problems (NEPs) by augmenting the pointwise constraint on the state variable y^ν. For every player ν this leads to a component problem given by

$$\begin{aligned} &\underset{u^\nu_\rho \in L^2(\Omega)}{\text{minimize}} \quad f^\nu_{AL}(u_\rho, \mu^\nu, \rho_\nu) := f^\nu(u_\rho) + \frac{1}{2\rho_\nu} \left\|(\mu^\nu + \rho_\nu(S_\nu u_\rho - \psi_\nu))_+\right\|^2_{L^2(\Omega)} \qquad (P_{AL}) \\ &\text{subject to} \quad u^\nu_\rho \in U^\nu_{\text{ad}}, \end{aligned}$$

which is a Nash equilibrium problem. We will refer to this kind of problem as the *augmented NEP* and call $\bar{u}$ *admissible* if $\bar{u}_\rho \in U_{\text{ad}}$. Solutions of the augmented NEP (P_{AL}) are characterized as for the jointly convex case, see Definition 6.6. The existence of solutions follows from the Kakutani-Fan-Glicksberg Theorem.

Theorem 7.15 (Existence of solutions of the augmented NEP). *The augmented NEP (P_{AL}) admits a Nash equilibrium for all $\rho := (\rho_1, \ldots, \rho_N)$ and for all $\mu \in L^2(\Omega)^N$.*

Proof. Taking into account the different solution operators S_ν the proof follows exactly [59, Theorem 2.3]. □

Since U^ν_{ad} is convex, one can easily derive first-order necessary optimality conditions for the ν-th problem for any fixed $\bar{u}^{-\nu}_\rho \in L^2(\Omega)^{N-1}$. Note, that these optimality conditions do not require the fulfillment of any constraint qualification.

Lemma 7.16. *Let $\bar{u}_\rho \in L^2(\Omega)^N$ denote a solution of the augmented NEP (P_{AL}) and $\bar{y}_\rho \in (H^1_0(\Omega) \cap C(\overline{\Omega}))^N$ the corresponding state. Then there exist unique adjoint states $\bar{p}^\nu_\rho \in H^1_0(\Omega)$ such that for all ν the following system is satisfied*

$$\begin{aligned} A_\nu \bar{y}^\nu_\rho &= B\bar{u} \quad \text{in } \Omega, \\ \bar{y}^\nu_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{7.11a}$$

$$\begin{aligned} A^*_\nu \bar{p}^\nu_\rho &= \bar{y}^\nu_\rho - y^\nu_d + \bar{\mu}^\nu_\rho \quad \text{in } \Omega, \\ \bar{p}^\nu_\rho &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{7.11b}$$

$$(B^{1*}_\nu \bar{p}^\nu_\rho + \alpha_\nu \bar{u}^\nu_\rho, v^\nu - \bar{u}^\nu_\rho) \geq 0 \quad \forall v^\nu \in U^\nu_{\text{ad}}, \tag{7.11c}$$

$$\bar{\mu}^\nu_\rho = \Big(\mu^\nu + \rho_\nu(\bar{y}^\nu_\rho - \psi_\nu)\Big)_+. \tag{7.11d}$$

Due to the convexity of each player's problem these optimality conditions are also sufficient.

7.4 Convergence Analysis

From now on let $(P_{AL})_k$ denote the augmented NEP (P_{AL}) for given penalty parameters $\rho_\nu := \rho_{\nu,k}$ and $\mu^\nu := \mu^\nu_k$. Its solution are given by $\bar{u}^\nu_k$ with corresponding states $\bar{y}_k$ and adjoint states $\bar{p}^\nu_k$.

7.4.1 The Augmented Lagrangian Method

We introduce a modified version of the augmented Lagrangian method that has been presented in Chapter 5. The algorithm differs from the classical augmented Lagrangian method, by updating the Lagrange multipliers only if the quantity

$$\sum_{\nu=1}^N R_k^\nu, \quad R_k^\nu := \left\|(\bar{y}_k^\nu - \psi_\nu)_+\right\|_{C(\overline{\Omega})} + (\bar{\mu}_k^\nu, \psi_\nu - \bar{y}_k^\nu)_+$$

shows sufficient decrease. This term measure the maximal constraint violation and the fulfilment of the complementarity condition. The augmented Lagrangian algorithm is given in Algorithm 7.1.

Algorithm 7.1 Augmented Lagrangian Algorithm for (P)

Let $(\bar{y}_0, \bar{u}_0, \bar{p}_0) \in (H_0^1(\Omega) \cap C(\overline{\Omega}))^N \times L^2(\Omega)^N \times W_0^{1,s}(\Omega)^N$, $\rho_{\nu,1} > 0$ and $0 \le \mu_1^\nu \in L^2(\Omega)$ be given. Choose $\theta > 1$, $\tau \in (0,1)$, $\varepsilon \ge 0$, $R_0^{\nu,+} \gg 1$. Set $k := 1$ and $n := 1$.

1: Solve the KKT system (7.11) corresponding to $(P_{AL})_k$ and obtain $(\bar{y}_k, \bar{u}_k, \bar{p}_k)$.
2: For all ν set $\bar{\mu}_k^\nu := \big(\mu_k^\nu + \rho_{\nu,k}(\bar{y}_k^\nu - \psi_\nu)\big)_+$.
3: For all ν compute $R_k^\nu := \left\|(\bar{y}_k^\nu - \psi_\nu)_+\right\|_{C(\bar{\Omega})} + (\bar{\mu}_k^\nu, \psi_\nu - \bar{y}_k^\nu)_+$.
4: If

$$\sum_{\nu=1}^N R_k^\nu \le \tau \sum_{\nu=1}^N R_{n-1}^{\nu,+},$$

then the step k is successful. For all ν set $\mu_{k+1}^\nu := \bar{\mu}_k^\nu$, $\rho_{\nu,k+1} := \rho_{\nu,k}$ and define for all ν

$$(y_n^{\nu,+}, u_n^{\nu,+}, p_n^{\nu,+}) := (\bar{y}_k^\nu, \bar{u}_k^\nu, \bar{p}_k^\nu), \quad \mu_n^{\nu,+} := \mu_{k+1}^\nu, \quad R_n^{\nu,+} := R_k^\nu.$$

Set $n := n+1$.
5: Otherwise the step k is not successful. Set for all ν

$$\mu_{k+1}^\nu := \mu_k^\nu, \quad \rho_{\nu,k+1} := \theta\rho_{\nu,k}.$$

6: If $\max_\nu R_{n-1}^{\nu,+} \le \varepsilon$ then stop, otherwise set $k := k+1$ and go to step 1.

7.4.2 Successful Steps and Feasibility of Limit Points

We have already faced in Chapter 5 that the question of convergence of the algorithm is tightly linked to the question of feasibility of limit points of the iterates $(\bar{u}_k)_k$ and the occurrence of infinitely many successful steps.

Theorem 7.17. *Let $(\bar{u}_k)_k$ denote the sequence that is generated by Algorithm 7.1. Then $(\bar{u}_k)_k$ has a feasible weak limit point if and only if infinitely many steps in the execution of Algorithm 7.1 were successful.*

Proof. The proof is inspired from Theorem 5.9. First, suppose that infinitely many steps were successful. Let $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ denote the sequence of successful iterates generated by Algorithm 7.1. By the boundedness of $(u_n^+)_n \in U_{\text{ad}}$ we get existence of a subsequence $u_{n'}^{\nu,+} \rightharpoonup u^{*\nu}$ in U_{ad}. The compactness of the solution operator (Lemma 7.2) yields $y_{n'}^{\nu,+} \to y^{*\nu}$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$. Due to the definition of successful steps, we have that $\sum_{\nu=1}^N \left\|(y_n^{\nu,+} - \psi_\nu)_+\right\|_{C(\overline{\Omega})} \le \sum_{\nu=1}^N R_n^{\nu,+} \to 0$ and u^* is a feasible strategy of (P).
Suppose now that only finitely many steps were successful. Let m be the largest index of a successful step. Hence, all steps k with $k > m$ are not successful. According to Algorithm 7.1 it

holds $\mu_k^\nu = \mu_m^\nu$ for all ν and all $k > m$. Like in the proof of Theorem 5.9 we obtain for all ν

$$\limsup_{k\to\infty} (\bar{\mu}_k^\nu, \psi_\nu - \bar{y}_k^\nu)_+ \leq 0.$$

The algorithm only makes $l \geq 0$ successful steps, which implies $\sum_{\nu=1}^N R_k^\nu > \tau \sum_{\nu=1}^N R_l^{\nu,+}$ for all $k > m$. This proves

$$\begin{aligned}\sum_{\nu=1}^N \liminf_{k\to\infty} \|(\bar{y}_k^\nu - \psi_\nu)_+\|_{C(\overline{\Omega})} &= \sum_{\nu=1}^N \liminf_{k\to\infty} \left(R_k^\nu - (\bar{\mu}_k^\nu, \psi_\nu - \bar{y}_k^\nu)_+\right) \\ &\geq \tau \sum_{\nu=1}^N R_l^{\nu,+} - \sum_{\nu=1}^N \limsup_{k\to\infty} (\bar{\mu}_k^\nu, \psi_\nu - \bar{y}_k^\nu)_+ \geq \sum_{\nu=1}^N \tau R_l^{\nu,+} > 0.\end{aligned}$$

Let u^* be the weak limit of the subsequence $(\bar{u}_{k'})_{k'}$ with associated states $y^{*\nu}$. Then, arguing as in the first part of the proof, we have

$$\sum_{\nu=1}^N \left\|(y^{*\nu} - \psi_\nu)_+\right\|_{C(\overline{\Omega})} = \sum_{\nu=1}^N \lim_{k'\to\infty} \|(\bar{y}_{k'}^\nu - \psi_\nu)_+\|_{C(\overline{\Omega})} \geq \tau \sum_{\nu=1}^N R_l^{\nu,+} > 0,$$

and u^* is not feasible. □

The proof of the previous theorem shows that if the algorithm performs infinitely many successful steps then every limit point of $(u_n^+)_n$ is a feasible strategy for the original problem. In case that only finitely many steps are successful, we have the following result.

Theorem 7.18. *Let us assume that Algorithm 7.1 does a finite number of successful steps only. Let $(\bar{u}_k)_k$ denote the sequence that is generated by the algorithm and let u^* be a weak limit point of $(\bar{u}_k)_k$. Then u^* is infeasible for (P) and $u^{*\nu}$ is a solution to the minimization problem*

$$\min_{u^\nu \in U_{\mathrm{ad}}^\nu} \left\|(S_\nu(u^\nu, u^{*-\nu}) - \psi_\nu)_+\right\|_{L^2(\Omega)}^2. \tag{7.12}$$

Proof. The infeasibility of u^* is a consequence of Theorem 7.17. Let m be the index of the last successful step. Dividing each player's first-order optimality condition of the augmented Lagrangian NEP by $\rho_{\nu,k}$ and concatenating all inequalities yields

$$\sum_{\nu=1}^N \left(S_\nu^1(\bar{u}_k^\nu)^* \left(\frac{S_\nu(\bar{u}_k^\nu, \bar{u}_k^{-\nu}) - y_d^\nu}{\rho_{\nu,k}} + \left(\frac{\mu_m}{\rho_{\nu,k}} + S_\nu(\bar{u}_k^\nu, \bar{u}_k^{-\nu}) - \psi_\nu\right)_+ \right) + \alpha_\nu \frac{\bar{u}_k^\nu}{\rho_{\nu,k}}, v^\nu - \bar{u}_k^\nu \right) \geq 0$$

for all $v^\nu \in U_{\mathrm{ad}}^\nu$. Taking the limit $k \to \infty$ we obtain

$$\sum_{\nu=1}^N (S_\nu^1(u^{*\nu})^*(S(u^*) - \psi)_+), v^\nu - u^{*\nu}) \geq 0 \quad \forall v^\nu \in U_{\mathrm{ad}}^\nu.$$

Exploiting Lemma 2.16 we obtain

$$\begin{aligned}&\sum_{\nu=1}^N \left\|(S_\nu(v^\nu, u^{*-\nu}) - \psi_\nu)_+\right\|_{L^2(\Omega)}^2 - \left\|(S_\nu(u^{*\nu}, u^{*-\nu}) - \psi_\nu)_+\right\|_{L^2(\Omega)}^2 \\ &\geq \sum_{\nu=1}^N (S_\nu^1(u^{*\nu})^*(S(u^*) - \psi)_+), v^\nu - u^{*\nu}) \geq 0 \quad \forall v^\nu \in U_{\mathrm{ad}}^\nu.\end{aligned}$$

Inserting the points $v := (v^\nu, u^{*-\nu})$, where $v^\nu \in U_{\mathrm{ad}}^\nu$ into the inequality above yields

$$\left\|(S_\nu(u^{*\nu}, u^{*-\nu}) - \psi_\nu)_+\right\|_{L^2(\Omega)}^2 \leq \left\|(S_\nu(v^\nu, u^{*-\nu}) - \psi_\nu)_+\right\|_{L^2(\Omega)}^2 \quad \forall v^\nu \in U_{\mathrm{ad}}^\nu.$$

Thus, $u^{*\nu}$ is the solution of the minimization problem (7.12). □

The remaining part of this section is devoted to the verification that the uniform Slater condition from Assumption 7.7 guarantees an infinite number of successful steps of Algorithm 7.1. We need the following auxiliary result that ensures that the Slater condition from Assumption 7.5 transfers to the elements of a weakly convergent sequence $(u_k)_k$ provided that k is sufficiently large.

Lemma 7.19. *Let $u_k \rightharpoonup u^*$ in $L^2(\Omega)^N$ and let the weak limit u^* satisfy the Slater condition* (7.6). *Then there exists $N_0 \in \mathbb{N}$ such that for all $k > N_0$, u_k satisfies the Slater condition, i.e.,*

$$S_\nu(\hat{u}^\nu, u_k^{-\nu})(x) \leq \psi_\nu(x) - \frac{\sigma_\nu}{2} \quad \text{in } \overline{\Omega},$$

where $\hat{u}^\nu$ denotes the Slater point from Assumption 7.5 with corresponding σ_ν.

Proof. By Assumption we know that $u_k^{-\nu} \rightharpoonup u^{*-\nu}$ in $L^2(\Omega)^{N-1}$. Hence, with Lemma 7.2 we have $S_\nu^2(u_k^{-\nu}) \to S_\nu^2(u^{*-\nu})$ in $C(\overline{\Omega})$ and the Slater condition implies

$$\begin{aligned} S_\nu(\hat{u}^\nu, u_k^{-\nu}) &= S_\nu^1(\hat{u}^\nu) + S_\nu^2(u^{*-\nu}) + S_\nu^2(u_k^{-\nu}) - S_\nu^2(u^{*-\nu}) \\ &\leq S_\nu(\hat{u}^\nu, u^{*-\nu}) + \frac{\sigma_\nu}{2} \\ &\leq \psi_\nu - \frac{\sigma_\nu}{2}, \end{aligned}$$

provided k is large enough. □

Lemma 7.20. *Let the uniform Slater condition from Assumption 7.7 be satisfied. Let $0 \leq \mu^\nu \in L^2(\Omega)$ be given as a fixed function and let $\rho_{\nu,k} \to \infty$. Let $(\bar{u}_k)_k$ denote the corresponding sequence of solutions of the KKT system* (7.11) *with a weak convergent subsequence $u_{k'} \rightharpoonup u^*$. Then*

$$\frac{1}{\rho_{\nu,k'}} \|\bar{\mu}_{k'}^\nu\|_{L^2(\Omega)}^2 = \rho_{\nu,k'} \left\| \left(\frac{\mu^\nu}{\rho_{\nu,k'}} + S_\nu(\bar{u}_{k'}) - \psi_\nu\right)_+ \right\|_{L^2(\Omega)}^2$$

is for all ν uniformly bounded.

Proof. The uniform Slater condition obviously yields that the weak limit u^* satisfies (7.6). By the definition of a solution of the augmented NEP, we obtain

$$\begin{aligned} 0 \leq \frac{1}{2\rho_{\nu,k'}} \|\bar{\mu}_{k'}^\nu\|^2 &\leq f_{AL}^\nu(\bar{u}_{k'}, \mu^\nu, \rho_\nu) \leq f_{AL}^\nu(\hat{u}^\nu, \bar{u}_{k'}^{-\nu}, \mu^\nu, \rho_\nu) \\ &= f^\nu(\hat{u}^\nu, \bar{u}_{k'}^{-\nu}) + \frac{1}{2\rho_{\nu,k'}} \left\|(\mu^\nu + \rho_{\nu,k'}(S_\nu(\hat{u}^\nu, \bar{u}_{k'}^{-\nu}) - \psi_\nu))_+\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{7.13}$$

From Lemma 7.19 we obtain for all k' sufficiently large $S_\nu(\hat{u}^\nu, \bar{u}_{k'}^{-\nu}) - \psi_\nu \leq -\frac{\sigma_\nu}{2}$. Hence, for all k' sufficiently large we can estimate

$$\frac{1}{2\rho_{\nu,k'}} \left\|(\mu^\nu + \rho_{\nu,k'}(S_\nu(\hat{u}^\nu, \bar{u}_{k'}^{-\nu}) - \psi_\nu))_+\right\|_{L^2(\Omega)}^2 \leq \frac{1}{2\rho_{\nu,k'}} \|\mu^\nu\|_{L^2(\Omega)}^2.$$

Taking the limit in (7.13) we arrive at

$$0 \leq \lim_{k' \to \infty} \frac{1}{2\rho_{\nu,k'}} \|\bar{\mu}_{k'}^\nu\|^2 \leq f^\nu(\hat{u}^\nu, u^{*-\nu}).$$

Since $u^\nu, u^{*-\nu}$ in U_{ad}, $U_{\mathrm{ad}}^{-\nu}$, respectively, the claim follows. □

We are now ready to prove that the uniform Slater condition implies that Algorithm 7.1 does an infinite number of successful steps.

Theorem 7.21. *Assume that Assumption 7.7 is satisfied. Then the sequence $(\bar{u}_k)_k$ has a feasible weak limit point and Algorithm 7.1 makes infinitely many successful steps.*

Proof. First, let $(\rho_k)_k$ be bounded. This results in a finite number of not successful steps, which lets the tails of sequences $(\bar{u}_k)_k$ and $(u_n^+)_n$ coincide. By Theorem 7.17 we obtain that all weak limit points of $(u_n^+)_n$ and thus of $(\bar{u}_k)_k$ are feasible. Suppose now that $\rho_k \to \infty$. According to the construction of Algorithm 7.1 it is in this scenario not possible to determine if an infinite or finite number of steps are successful. However, if an infinite number of steps is successful, we can conclude that the sequence $(u_n^+)_n$ and thus $(\bar{u}_k)_k$ admits a feasible weak limit point by Theorem 7.17. Let us now assume that only a finite number of steps is successful. Then, we find an index m that denotes the last successful step and we know that for all $k > m$ it holds $\mu_k^\nu = \mu_m^\nu$. Further, we can extract a subsequence $\bar{u}_{k'} \rightharpoonup u^*$. By Lemma 7.20, for all $k' > m$ the identity

$$\frac{1}{\rho_{\nu,k'}} \|\bar{\mu}_{k'}^\nu\|_{L^2(\Omega)}^2 = \rho_{\nu,k'} \left\| \left(\frac{\mu_m^\nu}{\rho_{\nu,k'}} + \bar{y}_{k'}^\nu - \psi_\nu \right)_+ \right\|_{L^2(\Omega)}^2$$

is bounded. Dividing by $\rho_{\nu,k'}$ and taking the limit $k' \to \infty$ we can argue that u^* is feasible. However, by Theorem 7.17 this yields a contradiction. Thus an infinite number of successful steps is done and we can argue as before that $(\bar{u}_k)_k$ admits a feasible weak limit point. □

7.4.3 Convergence to a Generalized Nash Equilibrium

From now on, we will always assume that Assumption 7.7 is satisfied. Thus, the algorithm makes infinitely many successful steps and every weak limit point of $(u_n^+)_n$ is feasible. We recall an estimate for the second term of the update rule, see Lemma 3.13, which simply results from the structure of the update rule.

Lemma 7.22. *Let $y_n^{\nu,+}, \mu_n^{\nu,+}$, $\nu = 1, ..., N$ be given as defined in Algorithm 7.1. Then it holds*

$$(\mu_n^{\nu,+}, \psi_\nu - y_n^{\nu,+})_+ \leq \tau^{n-1} \left(\|(y_1^{\nu,+} - \psi_\nu)_+\|_{C(\bar{\Omega})} + \|\mu_1^{\nu,+}\|_{L^2(\Omega)} \|(\psi_\nu - y_1^{\nu,+})_+\|_{L^2(\Omega)} \right).$$

With the help of Lemma 7.22, $L^1(\Omega)$-boundedness of the multipliers can now be proven.

Lemma 7.23 (Boundedness of multipliers). *Let $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ denote the sequence that is generated by Algorithm 7.1. Let $(u_{n'}^+)_{n'}$ denote a subsequence of $(u_n^+)_n$ that converges weakly to u^*. If u^* satisfies the Slater condition* (7.6)*, then there exists a constant $C > 0$, which is independent of n, such that the corresponding sequences of multipliers $(\mu_{n'}^{\nu,+})_{n'}$ satisfy*

$$\|\mu_{n'}^{\nu,+}\|_{L^1(\Omega)} \leq C.$$

Proof. Consider the subsequence $u_{n'}^+ \rightharpoonup u^*$ in $L^2(\Omega)^N$ with corresponding multiplier $\mu_{n'}^+$. Due to the Slater condition (7.6) and Lemma 7.19, we obtain for all n' sufficiently large

$$\begin{aligned}
\frac{\sigma_\nu}{2} \|\mu_{n'}^{\nu,+}\|_{L^1(\Omega)} &= \int_\Omega \frac{\sigma_\nu}{2} \mu_{n'}^{\nu,+} \, \mathrm{d}x \leq \int_\Omega \mu_{n'}^{\nu,+} (\psi_\nu - S_\nu(\hat{u}^\nu, u_{n'}^{-\nu,+})) \, \mathrm{d}x \\
&= \int_\Omega \mu_{n'}^{\nu,+} (\psi_\nu - y_{n'}^{\nu,+}) \, \mathrm{d}x + \int_\Omega \mu_{n'}^{\nu,+} (y_{n'}^{\nu,+} - S_\nu(\hat{u}^\nu, u_{n'}^{-\nu,+})) \, \mathrm{d}x \\
&\leq (\mu_{n'}^{\nu,+}, \psi_\nu - y_{n'}^{\nu,+})_+ + (\mu_{n'}^{\nu,+}, y_{n'}^{\nu,+} - S_\nu(\hat{u}^\nu, u_{n'}^{-\nu,+})).
\end{aligned}$$

The first term can be estimated via Lemma 7.22. Exploiting the linearity of S_ν and substituting the definition of the adjoint state

$$B_\nu^{1^*} p_{n'}^{\nu,+} = S_\nu^{1^*} (y_{n'}^{\nu,+} - y_d^\nu + \mu_{n'}^{\nu,+}),$$

the second term simplifies to

$$\begin{aligned}\left(\mu_{n'}^{\nu,+}, y_{n'}^{\nu,+} - S_\nu(\hat{u}^\nu, u_{n'}^{-\nu,+})\right) &= \left(\mu_{n'}^{\nu,+}, S_\nu\left[(u_{n'}^{\nu,+}, u_n^{-\nu,+}) - (\hat{u}^\nu, u_{n'}^{-\nu,+})\right]\right)\\ &= \left(\mu_{n'}^{\nu,+}, S_\nu^1(u_{n'}^{\nu,+} - \hat{u}^\nu)\right) = \left(S_\nu^{1^*}\mu_{n'}^{\nu,+}, u_{n'}^{\nu,+} - \hat{u}^\nu\right)\\ &= \left(B_\nu^{1^*} p_n^{\nu,+} - S_\nu^{1^*}(y_n^{\nu,+} - y_d^\nu), u_{n'}^{\nu,+} - \hat{u}^\nu\right)\\ &\le \left(\alpha u_{n'}^{\nu,+}, \hat{u}^\nu - u_{n'}^{\nu,+}\right) - \left(y_n^+ - y_d^\nu, S_\nu^1(u_{n'}^{\nu,+} - \hat{u}^\nu)\right).\end{aligned}$$

All together, we obtain

$$\|\mu_{n'}^{\nu,+}\|_{L^1(\Omega)} \le \frac{2}{\sigma_\nu}\left((\mu_{n'}^{\nu,+}, \psi_\nu - y_{n'}^{\nu,+})_+ + \left(\alpha u_{n'}^{\nu,+}, \hat{u}^\nu - u_{n'}^{\nu,+}\right) - \left(y_n^+ - y_d^\nu, S_\nu^1(u_{n'}^{\nu,+} - \hat{u}^\nu)\right)\right),$$

which is bounded due to Lemma 7.22, $\hat{u}^\nu, u_{n'}^{\nu,+} \in U_{\text{ad}}^\nu$ and Lemma 7.2. □

This immediately leads us to our final convergence result.

Theorem 7.24 (Convergence of the algorithm). *Let $\bar{u}$ denote a generalized Nash equilibrium of the GNEP (P) with corresponding state $\bar{y}$, adjoint state $\bar{p}$ and multiplier $\bar{\mu}$. Let*

$$(y_n^+, u_n^+, p_n^+, \mu_n^+)_n \in (H_0^1(\Omega) \cap C(\overline{\Omega}))^N \times L^2(\Omega)^N \times W_0^{1,s}(\Omega)^N \times \mathcal{M}(\overline{\Omega})^N,$$

$s \in [1, d/(d-1))$ denote the sequence that is generated by Algorithm 7.1 under Assumption 7.7. Then every weak limit point of $(u_n^+)_n$ is a solution of the GNEP and there exist subsequences $(y_{n'}^+, u_{n'}^+, p_{n'}^+, \mu_{n'}^+)_{n'}$ of $(y_n^+, u_n^+, p_n^+, \mu_n^+)_n$ such that

$$\begin{aligned}&u_{n'}^+ \to \bar{u} \quad \text{in } L^2(\Omega)^N, && y_{n'}^+ \to \bar{y} \quad \text{in } (H_0^1(\Omega) \cap C(\overline{\Omega}))^N,\\ &p_{n'}^+ \rightharpoonup \bar{p} \quad \text{in } W_0^{1,s}(\Omega)^N, && \mu_{n'}^+ \rightharpoonup^* \bar{\mu} \quad \text{in } \mathcal{M}(\overline{\Omega})^N.\end{aligned}$$

Proof. With the boundedness of U_{ad} we obtain $u_{n'}^+ \rightharpoonup u^*$ in $L^2(\Omega)^N$ and $y_{n'}^+ \to y^*$ in $H_0^1(\Omega) \cap C(\overline{\Omega})^N$. Hence, (7.7a) is satisfied. Since the uniform Slater condition implies that the Slater condition (7.6) is satisfied in u^*, we obtain $L^1(\Omega)$-boundedness of $(\mu_{n'}^{\nu,+})_{n'}$ by Lemma 7.23. Hence, we can extract weak-* convergent subsequences in $\mathcal{M}(\overline{\Omega})$ denoted w.l.o.g. by $\mu_{n'}^{\nu,+} \rightharpoonup^* \mu^{*\nu}$. As we know from [26, Theorem 4] that

$$\|p_n^{\nu,+}\|_{W_0^{1,s}(\Omega)} \le c\left(\|y_n^{\nu,+}\|_{L^2(\Omega)} + \|y_d^\nu\|_{L^2(\Omega)} + \|\mu_n^{\nu,+}\|_{\mathcal{M}(\overline{\Omega})}\right)$$

is satisfied, Lemma 7.23 yields a suitable upper bound and the sequence $(p_{n'}^{\nu,+})_{n'}$ is bounded in $W_0^{1,s}(\Omega), s \in [1, d/(d-1))$ and we obtain w.l.o.g. a weakly convergent subsequence $p_{n'}^+ \rightharpoonup p^*$ in $W_0^{1,s}(\Omega)^N$. By the Rellich-Kondrachov embedding theorem we get strong convergence $p_{n'}^+ \to p^*$ in $L^2(\Omega)^N$. It can be shown as in [61, Lemma 2.6] that $p^{*\nu}$ satisfies the respective adjoint equations (7.7b). Furthermore, for every ν we get with $v \in U_{\text{ad}}^\nu$

$$\begin{aligned}0 \le \liminf_{n\to\infty}\,(B_\nu^{1^*} p_{n'}^{\nu,+} + \alpha_\nu u_{n'}^{\nu,+}, v - u_{n'}^{\nu,+}) &\le (B_\nu^{1^*} p^{*\nu}, v - u^{*\nu}) - \liminf_{k\to\infty}\,(\alpha_\nu u_{n'}^{\nu,+}, u_{n'}^{\nu,+} - v)\\ &\le (B_\nu^{1^*} p^{*\nu} + \alpha_\nu u^{*\nu}, v - u^{*\nu}),\end{aligned}$$

where we exploited the strong convergence $p_{n'}^{\nu,+} \to p^{*\nu}$ in $L^2(\Omega)$ and the weak lower semicontinuity of $(\alpha_\nu u_{n'}^{\nu,+}, v - u_{n'}^{\nu,+})$, $v \in L^2(\Omega)$. Hence, the variational inequality (7.7c) is satisfied. By construction of the update of the Lagrange multiplier it is easy to show that the positivity property

$\langle \mu^{*\nu}, \varphi \rangle \geq 0 \quad \forall \varphi \in C(\overline{\Omega})$ with $\varphi \geq 0$ is satisfied for all ν, see Lemma 3.16. Due to the structure of the update rule we have

$$\lim_{n'\to\infty} \sum_{\nu=1}^{N} R_{n'}^{\nu,+} = \lim_{n'\to\infty} \sum_{\nu=1}^{N} \left(\|(y_{n'}^{\nu,+} - \psi_\nu)_+\|_{C(\overline{\Omega})} + (\mu_{n'}^{\nu,+}, \psi_\nu - y_{n'}^{\nu,+})_+ \right) = 0.$$

Hence, $y^{*\nu} \leq \psi_\nu$ and consequently $\langle \mu^{*\nu}, \psi_\nu - y^{*\nu} \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \geq 0$. Since $(\mu^{*\nu}, \psi_\nu - y^{*\nu})_+ = 0$ we obtain $\langle \mu^{*\nu}, \psi_\nu - y^{*\nu} \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} = 0$. Thus, (7.7d) is satisfied and (y^*, u^*, p^*, μ^*) solves the optimality system (7.7). Due to convexity and differentiability properties of each player's cost functional (y^*, u^*, p^*, μ^*) is also a solution $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu})$ of the GNEP (P). Adding the variational inequalities (7.7c) and (6.5c) and applying the definition of the adjoint states $\bar{p}^\nu, p_{n'}^{\nu,+}$, we obtain for all ν

$$\alpha_\nu \left\| \bar{u}^\nu - u_{n'}^{\nu,+} \right\|_{L^2(\Omega)}^2 \leq (B_\nu^{1*}(\bar{p}^\nu - p_{n'}^{\nu,+}), u_{n'}^{\nu,+} - \bar{u}^\nu).$$

Due to the strong convergence of $p_{n'}^{\nu,+} \to \bar{p}^\nu$ in $L^2(\Omega)$ and the weak convergence $u_{n'}^{\nu,+} \rightharpoonup \bar{u}^\nu$ in $L^2(\Omega)$, we finally obtain strong convergence of $u_{n'}^{+} \to \bar{u}$ in $L^2(\Omega)^N$. ☐

7.5 Numerical Examples

7.5.1 Solution of the Subproblem

Let us briefly comment on the solution of the arising augmented Lagrangian subproblems. These problems are solved by a semi-smooth Newton. We define the sets

$$\mathcal{A}_k^{\nu,a} := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^{1*} p^\nu(u_k) \leq u_a^\nu \right\},\ \mathcal{A}_k^{\nu,b} := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^{1*} p^\nu(u_k) \geq u_b^\nu \right\}, \tag{7.14}$$

$$\mathcal{I}_k^\nu := \left\{ x \in \Omega \colon -\frac{1}{\alpha_\nu} B_\nu^{1*} p^\nu(u_k) \in (u_a^\nu, u_b^\nu) \right\}, \mathcal{Y}_k^\nu := \left\{ x \in \Omega \colon (\mu^\nu + \rho_\nu(y_k^\nu - \psi_\nu))(x) > 0 \right\}.$$

Thus, solving the KKT system (7.11) in step 1 of the outer loop of Algorithm 7.1 results in Algorithm 7.2. Here, Y denotes the state space which corresponds to the underlying PDE, i.e., $Y := H_0^1(\Omega) \cap C(\overline{\Omega})$ for a linear elliptic PDE with homogeneous Dirichlet boundary conditions. Let us justify the applied stopping criterion.

Lemma 7.25. *The stopping criterion from Algorithm 7.2 yields a point $(y_{k-1}, u_{k-1}, p_{k-1})$ that solves the KKT system* (7.11) *for given $\mu^\nu \in L^2(\Omega)$ and penalty parameters ρ_ν up to the precision $\tilde{\varepsilon}$.*

Proof. The first equation of the system (7.15) coincides with the state equation (7.11a) and can be solved exactly. Further, the equation $u_{k-1}^\nu - P_{U_{\mathrm{ad}}^\nu}\left(-\frac{1}{\alpha_\nu} B_\nu^{1*} p_{k-1}^\nu\right) = 0$ is a rewritten form of the variational inequality (7.11c). Moreover, we know that $(y_{k-1}, u_{k-1}, p_{k-1})$ solves the second equation of (7.15) for $\chi_{\mathcal{Y}_{k-2}^\nu}$. Hence, we obtain with $\mu_k^\nu := \mu^\nu + \rho_\nu(y_k^\nu - \psi_\nu)$

$$\begin{aligned} &||A_\nu^* p_{k-1}^\nu - y_{k-1}^\nu + y_d^\nu - \chi_{\mathcal{Y}_{k-1}^\nu} \mu_{k-1}^\nu||_{L^2(\Omega)} \\ &\leq \underbrace{\left\| A_\nu^* p_{k-1}^\nu - y_{k-1}^\nu + y_d^\nu - \chi_{\mathcal{Y}_{k-2}^\nu} \mu_{k-1}^\nu \right\|_{L^2(\Omega)}}_{=0} + \left\| \chi_{\mathcal{Y}_{k-2}^\nu} \mu_{k-1}^\nu - \chi_{\mathcal{Y}_{k-1}^\nu} \mu_{k-1}^\nu \right\|_{L^2(\Omega)} \leq \tilde{\varepsilon}. \square \end{aligned}$$

Thanks to Lemma 7.25 it is reasonable to choose $\max_\nu R_n^{\nu,+} \leq \varepsilon, \varepsilon \geq \tilde{\varepsilon}$, as a stopping criterion for the outer loop of our algorithm. The implementation has been done in Fenics [86] using the DOLFIN Python interface [87].

Algorithm 7.2 Algorithm for solving the subproblem of (P_{AL}^k)

1: Set $k := 0$, choose $(y_0, u_0, p_0) \in Y^N \times L^2(\Omega)^N \times L^2(\Omega)^N$.
2: **repeat**
3: Set $\mathcal{A}_k^{\nu,a}, \mathcal{A}_k^{\nu,b}, \mathcal{I}_k^\nu$ and $\mathcal{Y}_k^\nu$ as defined in (7.14).
4: Solve for $(y_{k+1}, u_{k+1}, p_{k+1}) \in Y^N \times L^2(\Omega)^N \times L^2(\Omega)^N$ by solving

$$\begin{aligned} A_\nu y_{k+1}^\nu &= B u_{k+1} \quad \text{in } \Omega, \\ A_\nu^* p_{k+1}^\nu &= y_{k+1}^\nu - y_d^\nu + \chi_{\mathcal{Y}_k^\nu}(\mu^\nu + \rho_\nu(y_{k+1}^\nu - \psi_\nu)) \quad \text{in } \Omega, \\ u_{k+1}^\nu + \chi_{\mathcal{I}_k^\nu}\left(\frac{1}{\alpha_\nu} B_\nu^{1*} p_{k+1}^\nu\right) &= \chi_{\mathcal{A}_k^{\nu,a}} u_a^\nu + \chi_{\mathcal{A}_k^{\nu,b}} u_b^\nu. \end{aligned} \tag{7.15}$$

5: Set $k := k+1$
6: **until** $\max_\nu \left\| u_{k-1}^\nu - P_{U_{\text{ad}}^\nu}\left(-\frac{1}{\alpha_\nu} B_\nu^{1*} p_{k-1}^\nu\right)\right\|_{L^2(\Omega)} \le \tilde{\varepsilon}$

and $\max_\nu \left\| \chi_{\mathcal{Y}_{k-2}^\nu} - \chi_{\mathcal{Y}_{k-1}^\nu}(\mu^\nu + \rho_\nu(y_{k-1}^\nu - \psi_\nu)) \right\|_{L^2(\Omega)} \le \tilde{\varepsilon}.$

7.5.2 Example 1 - Four Player Problem

We start with a generalized Nash equilibrium problem with four players. Let $\Omega := (0,1)^2$ with the subsets $\Omega_\nu \subset \Omega$

$$\begin{aligned} \Omega_1 &:= (0, 0.5) \times (0, 0.5), & \Omega_2 &:= (0.5, 1) \times (0, 0.5), \\ \Omega_3 &:= (0, 0.5) \times (0.5, 1), & \Omega_4 &:= (0.5, 1) \times (0.5, 1). \end{aligned}$$

The state equation is given by

$$-\Delta y^\nu = u^\nu + \sum_{j=1, j\neq\nu}^N \chi_{\Omega_\nu} u^j \quad \text{in } \Omega, \qquad y^\nu = 0 \quad \text{on } \Gamma.$$

Thus, $B_\nu^1 := \text{Id}$ for all ν and $B_{\nu,j}^2 := \chi_{\Omega_\nu}$. We define $z^1 := (0.25, 0.75, 0.25, 0.75)$ and $z^2 := (0.25, 0.25, 0.75, 0.75)$. Further, we set

$$\begin{gathered} \xi_\nu(x_1, x_2) := 500 \max(0, 4(0.25 - \max(|x_1 - z_\nu^1|, |x_2 - z_\nu^2|))), \\ y_d^1 := \xi_1 - \xi_4, \qquad y_d^2 := \xi_2 - \xi_3, \qquad y_d^3 := \xi_3 - \xi_2, \qquad y_d^4 := \xi_4 - \xi_1. \end{gathered}$$

The state constraints ψ_ν are given by

$$\psi_\nu := -\max(0, -(5(x_1 - z_\nu^1))^2 - 5(x_2 - z_\nu^2))^2 + 0.4 + 0.1\nu).$$

We set the control constraints equal for all players, namely $u_a^\nu := -40$ and $u_b^\nu := 40$. For all ν the Tikhonov parameters are given by $\alpha_\nu := 1.0$. The algorithm has been initialized with (y_0, u_0, p_0, μ_1) equal to zero. Further, we chose the following parameters $\rho_0^\nu := 1.0$, $\tau := 0.1$, and $\gamma := 10$. We choose $\tilde{\varepsilon} := 10^{-6}$ and stop the algorithm as soon as the following stopping criterion is satisfied

$$\max_\nu R_n^{\nu,+} := \max_\nu \left(\left\| (y_n^{\nu,+} - \psi_\nu)_+ \right\|_{C(\overline{\Omega})} + (\mu_n^{\nu,+}, \psi_\nu - y_n^{\nu,+})_+ \right) \le 10^{-6}.$$

We divide each axis of the the unit square in n intervals. The Figures 7.1-7.3 depict the computed results for $n = 256$ which corresponds to a mesh size $h \approx 5.5 \cdot 10^{-3}$ and to an approximate number of $6.6 \cdot 10^4$ degrees of freedom. Table 7.1 denotes some iteration numbers for the outer and inner loop as well as the maximal value of the penalty parameter ρ^ν for given n. Note, that the number of inner iterations has been accumulated over all outer iterations.

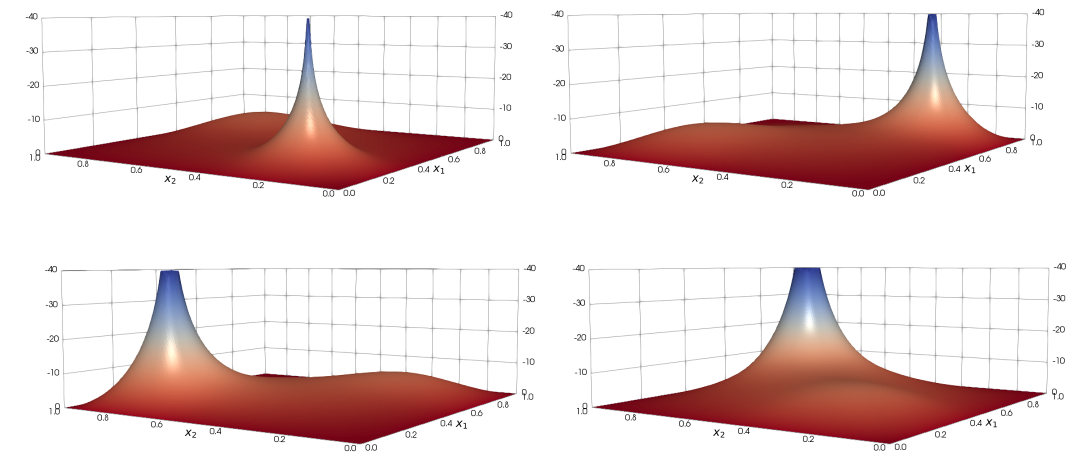


Figure 7.1: (Example 1) Computed optimal controls $\bar{u}_h^1, \bar{u}_h^2, \bar{u}_h^3, \bar{u}_h^4$.

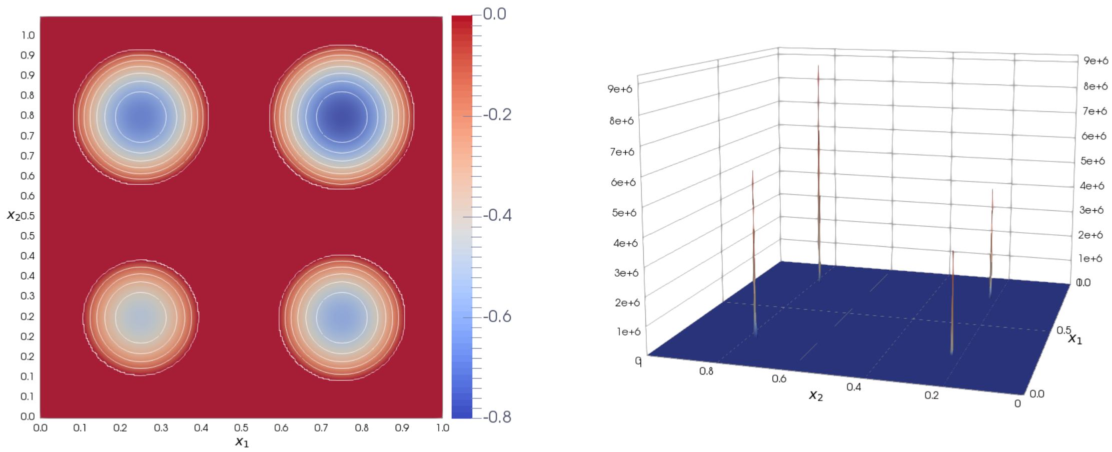


Figure 7.2: (Example 1) Sum of state constraints ψ_ν and sum of computed multipliers $\bar{\mu}_h^\nu$.

n	8	16	32	64	128	256
outer it	12	13	13	17	17	18
inner it	17	24	29	43	46	59
ρ_{max}	10^6	10^7	10^8	10^{10}	10^{11}	10^{12}

Table 7.1: (Example 1) Iteration numbers.

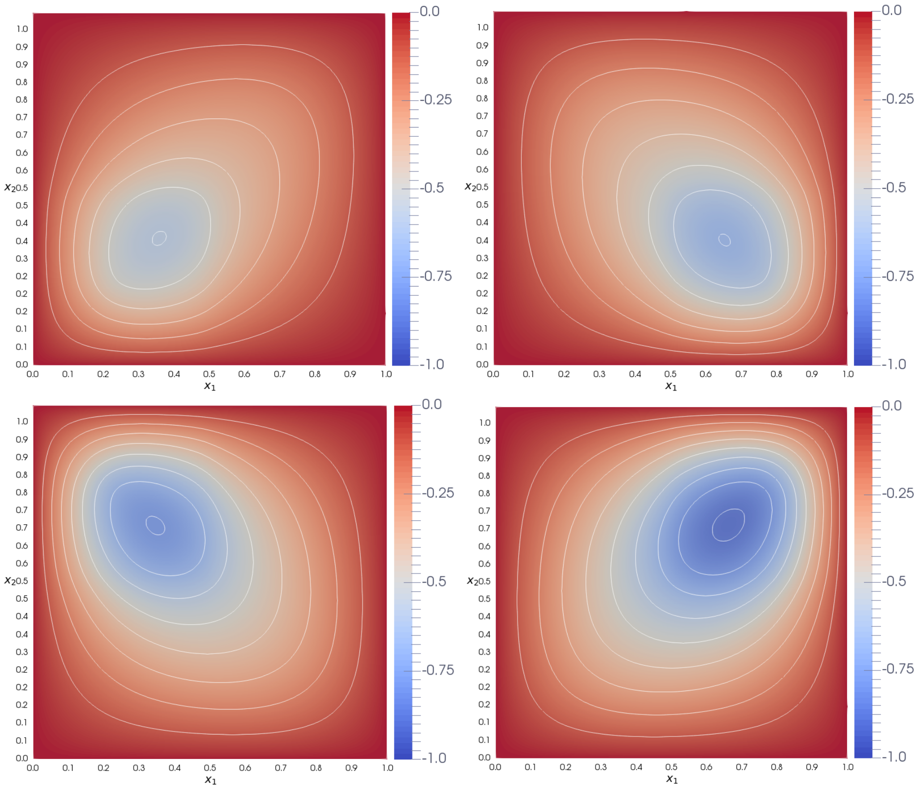


Figure 7.3: (Example 1) Computed states $\bar{y}_h^1, \bar{y}_h^2, \bar{y}_h^3, \bar{y}_h^4$.

CHAPTER 8

NON-REDUCIBLE MULTI-PLAYER OPTIMAL CONTROL PROBLEMS

This chapter considers a special class of Nash equilibrium problems that can not be reduced to a single-player control problem. Let $\Omega \subseteq \mathbb{R}^d, d \in \{1,2,3\}$ denote an open bounded domain. We investigate the generalized Nash equilibrium, where each player wants to solve the following optimal control problem

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \ \frac{1}{2} \|C_\nu Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u^\nu\|^2_{L^2(\Omega)} \\ &\text{subject to} \ \ u^\nu \in U^\nu_{\text{ad}}, \quad g(u) \in K, \end{aligned} \tag{8.1}$$

where the operator $S\colon L^2(\Omega) \to Y$ denotes the solution operator of an underlying linear elliptic partial differential equation with a suitable Banach space Y. Further, $C_\nu \in L(Y, L^2(\Omega))$ and $g \in L(L^2(\Omega)^N, Y)$ are given linear and continuous mappings, $y_d^\nu \in L^2(\Omega)$ and α a non-negative regularization parameter. Moreover, the set $U^\nu_{\text{ad}} \subset L^2(\Omega)$ is bounded, closed, convex and $K \subseteq Y$ is a closed, convex cone. The joint constraint $g(u) \in K$ coincides for each player, which makes the problem a so-called jointly convex GNEP.

The investigation of multi-player control problems in the function space setting often considers the case $C_\nu := C \in L(Y, L^2(\Omega))$. In this setting existence and uniqueness of solution is well-known and can be derived by fixed point theorems [49], the theory of strongly monotone variational inequalities [78, Chap. III], or, by reducing the problem to a single convex PDE constrained optimization problem, by standard arguments from optimization theory [60, Prop. 3.10]. However, for varying $C_\nu \in L(Y, L^2(\Omega))$, problem (8.1) cannot be reduced to a single control problem and it can not expected in general that the resulting first-order optimality system is a (strongly) monotone VI. Thus, uniqueness of solutions is not clear. The main aim of this chapter is to find a sufficient condition on the regularization parameter α, that proves the uniqueness of solutions.

Assuming K to be the cone of non-positive continuous functions, solving the GNEP (8.1) via an augmented Lagrangian method requires in each iteration the solution of the following augmented Nash equilibrium problem

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \ \frac{1}{2} \|C_\nu Su - y_d^\nu\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u^\nu\|^2_{L^2(\Omega)} + \frac{1}{2\rho} \|(\mu + \rho g(u))_+\|^2_{L^2(\Omega)} \\ &\text{subject to} \ \ u^\nu \in U^\nu_{\text{ad}}. \end{aligned} \tag{8.2}$$

Nash equilibrium problems of this type can be solved by applying a semi-smooth Newton method. Applying the same condition on α as needed for the uniqueness of solutions, we derive superlinear convergence for the associated Newton method and the equivalent active-set method.

The outline of this chapter is as follows: In Section 8.1, we give a precise formulation of the problem setting and collect some results about reducible problems. Further, we introduce the reader to non-reducible NEPs. Here, our main results state existence and uniqueness of solutions, see Theorem 8.4 and Theorem 8.7. In Section 8.2, we introduce the semi-smooth Newton method and contribute Lemma 8.9 that proves semi-smoothness of $u \mapsto \max(a, u)$ from $L^q(\Omega)$ to $L^p(\Omega)$ even if $a \in L^r(\Omega)$, with $1 \leq p \leq r < q \leq \infty$. In Section 8.3, we apply the semi-smooth Newton method to the augmented NEP (8.2), state a convergence result and give a detailed description of the implementation applying a finite element discretization. The equivalence of the semi-smooth Newton method and the active-set method is treated in Section 8.4. To illustrate our theoretical findings and to compare the two presented methods we study numerical examples in detail. All results of this chapter can be found in the publication [76].

8.1 Uniqueness of Variational Equlibria

In this section, we will introduce the reader to the non-reducible GNEP. Moreover, we will derive a sufficient condition that allows us to prove existence and uniqueness of solutions.

8.1.1 Problem Setting

Throughout this chapter let $\Omega \subset \mathbb{R}^n, n \in \{1, 2, 3\}$ denote an open bounded domain with boundary $\partial\Omega$. We assume that either (i) Ω is a convex polyhedron or (ii) $\partial\Omega$ is a $C^{1,1}$-boundary. The operator A is given as in Assumption 2.19. We consider the the linear partial differential equation

$$\begin{aligned} Ay &= \sum_{\nu=1}^{N} u^\nu && \text{in } \Omega, \\ y &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{8.3}$$

where $u^\nu \in L^2(\Omega)$. By well known regularity results [50, Thm. 2.2.2.5, Thm. 3.2.1.2], we know that for each $u \in L^2(\Omega)^N$ the weak solution y of (8.3) satisfies $y \in H^2(\Omega) \cap H_0^1(\Omega)$. The Sobolev embedding theorem [1, Thm. 5.4] yields the continuous embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. We introduce the solution operator

$$S\colon u \mapsto y = A^{-1} \sum_{\nu=1}^{N} u^\nu, \quad S\colon H^{-1}(\Omega)^N \to H_0^1(\Omega)$$

and set $Y := H_0^1(\Omega)$. Since for $n = 1$ we have the embedding $H_0^1(\Omega) \hookrightarrow C(\overline{\Omega})$, for $n = 2$ it holds $H^1(\Omega) \hookrightarrow L^q(\Omega)$ with $1 \leq q < \infty$ and for $n = 3$ we obtain $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we can consider S as an operator that maps into $L^q(\Omega)$, $q > 2$. Due to the linearity of A^{-1} we have

$$Su = \sum_{\nu=1}^{N} A^{-1}u^\nu := \sum_{\nu=1}^{N} S_\nu u^\nu, \quad S_\nu\colon H^{-1}(\Omega) \to H_0^1(\Omega) \hookrightarrow L^q(\Omega),\ S_\nu u^\nu := A^{-1}u^\nu.$$

8.1.2 The Reducible Case

We start with the generalized Nash equilibrium problem: Here, each player aims at solving the optimal control problem

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}}\ f^\nu(u) := \frac{1}{2}\|CSu - y_d^\nu\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2}\|u^\nu\|_{L^2(\Omega)}^2 \\ &\text{subject to}\ \ u^\nu \in U_{\text{ad}}^\nu, \quad g(u^\nu, u^{-\nu}) \in K, \end{aligned} \tag{8.4}$$

We define the feasible set $F_{\text{ad}} := \{u \in U_{\text{ad}},\ g(u) \in K\}$, where $U_{\text{ad}} := U^1_{\text{ad}} \times \dots U^N_{\text{ad}}$. Clearly, the set F_{ad} is closed, bounded and convex, hence weakly compact. Problems of this type can be reduced to a single convex minimization problem.

Lemma 8.1. *The GNEP* (8.4) *admits a unique variational equilibrium, which is the unique solution of the single PDE constrained convex optimization problem*

$$\begin{aligned} &\underset{u\in L^2(\Omega)^N}{\text{minimize}}\ \hat{f}(u) := \frac{1}{2}\|CSu\|^2_{L^2(\Omega)} + \sum_{\nu=1}^{N}\left(-(u^\nu, S^*_\nu C^* y^\nu_d) + \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)}\right) \\ &\text{subject to}\ \ u \in F_{\text{ad}}. \end{aligned} \tag{8.5}$$

Proof. This can be proven analogously to [60, Prop. 3.10]. Since F_{ad} is convex, we know that $\bar{u}$ is a minimizer of (8.5) if and only if $(\hat{f}'(\bar{u}), v - \bar{u}) \geq 0 \quad \forall v \in F_{\text{ad}}$. Each component of $\hat{f}'(\bar{u})$ is given by

$$\left(\hat{f}'(\bar{u})\right)_\nu = S^*_\nu C^*(CSu - y^\nu_d) + \alpha_\nu u^\nu$$

and coincides with the corresponding component of $F(\bar{u}) = (D_{u^1} f^1(\bar{u}), ..., D_{u^N} f^N(\bar{u}))$. Thus, $\bar{u}$ is a minimizer of (8.5) if and only if $\bar{u}$ is a variational equilibrium of (8.4). □

The corresponding augmented NEP can be treated in the same way. We adapt the definition of Monderer [95, p. 128] to the infinite dimensional case.

Definition 8.2 (Exact potential game). We say that the GNEP (8.4) is an *exact potential game* if there exists a *potential function* $P\colon L^2(\Omega)^N \to \mathbb{R}$, such that for every ν it holds

$$f^\nu(u) - f^\nu(v, u^{-\nu}) = P(u) - P(v, u^{-\nu}) \quad \forall u, (v, u^{-\nu}) \in F_{\text{ad}}.$$

Let us now check if the GNEP (8.4) is an exact potential game. For given $(u^\nu, u^{-\nu}), (v, u^{-\nu}) \in F_{\text{ad}}$ it holds that

$$\begin{aligned} f^\nu(u) - f^\nu(v, u^{-\nu}) =& \frac{1}{2}\|CSu\|^2_{L^2(\Omega)} - (u^\nu, S^*_\nu C^* y^\nu_d) + \frac{\alpha_\nu}{2}\|u^\nu\|^2_{L^2(\Omega)} \\ &- \frac{1}{2}\left\|CS(v, u^{-\nu})\right\|^2_{L^2(\Omega)} + (v, S^*_\nu C^* y^\nu_d) - \frac{\alpha_\nu}{2}\|v\|^2_{L^2(\Omega)} \\ =& \hat{f}(u) - \hat{f}(v, u^{-\nu}). \end{aligned}$$

Thus, $\hat{f}$ satisfies the definition of an exact potential.

8.1.3 The Non-Reducible Case

Let us now investigate problems, where for each player the following infinite dimensional PDE constrained optimization problem is considered:

$$\begin{aligned} &\underset{u^\nu\in L^2(\Omega)}{\text{minimize}}\ f^\nu(u) := \frac{1}{2}\|C_\nu Su - y^\nu_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u^\nu\|^2_{L^2(\Omega)} \\ &\text{subject to}\ \ u^\nu \in U^\nu_{\text{ad}}, \quad g(u) \in K. \end{aligned} \tag{P_ν}$$

In this case, due to the distinct operators $C_\nu \in L(Y, L^2(\Omega))$, the reduction to a single control problem is not possible and we will refer to this type of problem as a *non-reducible GNEP*. Moreover, it is in general not clear if the resulting first-order optimality system is a (strongly) monotone VI. Problems of this type are included in the rather general setting from [69], see also [60]. The authors proved existence of a variational equilibrium by using a fixed point-argument ([69, Thm.

2.3] and [60, Thm. 3.4]). However, they do not deal with the uniqueness these solutions. Our aim is to study under which conditions (P_ν) admits a unique variational equilibrium. In Theorem 8.4 we will prove existence and uniqueness of solution, provided that the regularization parameter α is sufficiently large. For further use we define $U := L^2(\Omega)^N$ and the operator

$$F\colon U \to U^*, \quad F^\nu(u) := D_{u^\nu} f^\nu(u) = S_\nu^* C_\nu^* (C_\nu S u - y_d^\nu) + \alpha u^\nu, \tag{8.6}$$

where D_{u^ν} denotes the partial Gâteaux derivative with respect u^ν. Due to the convexity of the objective functional, variational equilibria of the GNEP (P_ν) can be characterized via controls $\bar{u} \in U$ that solve the variational inequality

$$\begin{aligned} & (F(\bar{u}), v - \bar{u})_U \geq 0 \quad \forall v \in F_{\text{ad}} \\ \Leftrightarrow & \sum_{\nu=1}^{N} \left(S_\nu^* C_\nu^* (C_\nu S \bar{u} - y_d^\nu) + \alpha \bar{u}^\nu, v^\nu - \bar{u}^\nu \right) \geq 0 \quad \forall v \in F_{\text{ad}}. \end{aligned} \tag{8.7}$$

We will exploit this relation to prove uniqueness of solutions of problem (P_ν).

8.1.3.1 Existence and Uniqueness of Solutions

If the variational inequality (8.7) is uniquely solvable, then the GNEP (P_ν) admits a unique solution. It is well known that this is the case if F is strongly monotone [78, Thm III.1.4]. The next theorem states that it is enough to choose the regularization parameter α large enough, depending on the operator C_ν. We need the following assumption.

Assumption 8.3. Assume that the regularization parameter α satisfies the inequality

$$\alpha > \frac{1}{4} \sum_{\nu=1}^{N} \|(S_\nu - C_\nu^* C_\nu S_\nu)\|^2_{L^2(\Omega) \to L^2(\Omega)}. \tag{8.8}$$

Let us now start to exploit Assumption 8.3.

Theorem 8.4 (Uniqueness of solutions). *Let Assumption 8.3 be satisfied. Then there exists a unique variational equilibrium of the non-reducible GNEP (P_ν).*

Proof. It is enough to show that the operator F as defined in (8.6) is strongly monotone. A calculation reveals for arbitrary $u, v \in U$

$$\begin{aligned} & (F(u) - F(v), u - v)_U \\ & \quad = \sum_{\nu=1}^{N} (S_\nu^* C_\nu^* (C_\nu S u - y_d^\nu) + \alpha u^\nu - S_\nu^* C_\nu^* (C_\nu S v - y_d^\nu) - \alpha v^\nu, u^\nu - v^\nu)_{L^2(\Omega)} \\ & \quad = \sum_{\nu=1}^{N} (Su - Sv, C_\nu^* C_\nu S_\nu (u^\nu - v^\nu))_{L^2(\Omega)} + \alpha \|u - v\|_U^2. \end{aligned} \tag{8.9}$$

We now use the decomposition

$$\sum_{\nu=1}^{N} C_\nu^* C_\nu S_\nu = \sum_{\nu=1}^{N} S_\nu - \sum_{\nu=1}^{N} (S_\nu - C_\nu^* C_\nu S_\nu)$$

and Young's inequality to obtain the following estimate

$$
\begin{aligned}
&\Big(F(u) - F(v), u - v\Big)_U \\
&= \|Su - Sv\|^2_{L^2(\Omega)} - \left(Su - Sv, \sum_{\nu=1}^{N} (S_\nu - C_\nu^* C_\nu S_\nu)(u^\nu - v^\nu)\right)_{L^2(\Omega)} + \alpha \|u - v\|^2_U \\
&\geq -\frac{1}{4} \left\| \sum_{\nu=1}^{N} (S_\nu - C_\nu^* C_\nu S_\nu)(u^\nu - v^\nu) \right\|^2_{L^2(\Omega)} + \alpha \|u - v\|^2_U \\
&\geq -\frac{1}{4} \left(\sum_{\nu=1}^{N} \|S_\nu - C_\nu^* C_\nu S_\nu\|^2_{L^2(\Omega)\to L^2(\Omega)} \right) \left(\sum_{\nu=1}^{N} \|u^\nu - v^\nu\|^2_{L^2(\Omega)} \right) + \alpha \|u - v\|^2_U \\
&= \left(\alpha - \frac{1}{4} \sum_{\nu=1}^{N} \|S_\nu - C_\nu^* C_\nu S_\nu\|^2_{L^2(\Omega)\to L^2(\Omega)} \right) \|u - v\|^2_U.
\end{aligned}
$$

Due to our assumption on α we now conclude that the operator F is strongly monotone. □

Let us commit ourselves to the special structure of (8.9). Rewriting this equation yields

$$
\begin{aligned}
(F(u) - F(v), u - v)_U &= \sum_{\nu=1}^{N} (Su - Sv, C_\nu^* C_\nu S_\nu (u^\nu - v^\nu))_{L^2(\Omega)} + \alpha \|u - v\|^2_U \\
&= \sum_{j=1}^{N} \sum_{i=1}^{N} \Big(S_j(u^j - v^j), C_i^* C_i S_i (u^i - v^i) \Big)_{L^2(\Omega)} + \alpha \|u - v\|^2_U \\
&= \int_\Omega (u - v)^T (D + R)(u - v) \, \mathrm{dx},
\end{aligned}
$$

where

$$
D = \begin{pmatrix} R_{11} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & R_{NN} \end{pmatrix}, \quad R = \begin{pmatrix} \alpha & R_{12} & R_{13} & \dots & R_{1N} \\ R_{21} & \alpha & R_{23} & \dots & R_{2N} \\ R_{31} & R_{32} & \alpha & \ddots & \vdots \\ \vdots & & \ddots & \ddots & R_{N-1,N} \\ R_{N1} & R_{N2} & \dots & R_{N,N-1} & \alpha \end{pmatrix}
$$

and $R_{ij} := S_i^* C_j^* C_j S_j$. Adapting our setting to Rosen's notion of strict diagonal convexity, it would be sufficient to check if $D + R$ is positive definite. Clearly, D is positive definite. However, since R is not symmetric, it is hard to determine if R is positive definite as well.

The condition on the regularization parameter α is needed to guarantee the existence of a unique solution of (P_ν). If α is chosen too small the resulting operator F might not be strongly monotone. Let us briefly investigate the special case that all C_ν coincide. In this situation, the GNEP (P_ν) turns into a reducible GNEP. Thus, existence and uniqueness of solutions follow from Lemma 8.1. Moreover, it is easy to check that the corresponding VI is strongly monotone.

Corollary 8.5. *Let $C_\nu := C \in L(Y, L^2(\Omega))$ for all ν. Then, F is strongly monotone and the GNEP (P_ν) admits a unique variational equilibrium for all $\alpha > 0$.*

Proof. By an easy calculation

$$
\begin{aligned}
(F(u) - F(v), u - v)_U &= \left(C(Su - Sv), C \sum_{\nu=1}^{N} S_\nu (u^\nu - v^\nu) \right)_{L^2(\Omega)} + \alpha \|u - v\|^2_U \\
&= \|CS(u - v)\|^2_{L^2(\Omega)} + \alpha \|u - v\|^2_U > 0,
\end{aligned}
$$

we see that F is strongly monotone for $\alpha > 0$. Hence, the claim follows with [78, Thm. III.1.4]. □

Let us now consider subsets $\Omega_\nu \subset \Omega$. We define the characteristic function

$$\chi_\nu \colon \Omega \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{else} \end{cases}$$

and the set

$$Z := \bigcup_{\nu=1}^{N} \Omega_\nu$$

with associated characteristic function χ_Z. Let us now assume that all C_ν are given as characteristic functions χ_ν of subsets $\Omega_\nu \subset \Omega$. Note, that this setting would include GNEPs with an objective functional of the type

$$f^\nu(u) := \frac{1}{2}\,\|Su - y_d^\nu\|^2_{L^2(\Omega_\nu)} + \frac{\alpha_\nu}{2}\,\|u^\nu\|^2_{L^2(\Omega)}\,.$$

Here, we have the following result:

Lemma 8.6. *Let $C_\nu := \chi_\nu$ for all ν. Let*

$$\alpha > \frac{1}{4}\sum_{\nu=1}^{N} \|\chi_Z(S_\nu - \chi_\nu S_\nu)\|^2_{L^2(\Omega)\to L^2(\Omega)}. \tag{8.10}$$

be satisfied. Then the GNEP (P_ν) admits a unique variational equilibrium.

Proof. The proof basically follows the one from Theorem 8.4. Let $u, v \in U$ be arbitrary.

$$\begin{aligned}
(F(u) - F(v), u - v)_U &= \sum_{\nu=1}^{N} (Su - Sv, \chi_\nu S_\nu(u^\nu - v^\nu))_{L^2(Z)} + \alpha\|u - v\|^2_U \\
&= \|Su - Sv\|^2_{L^2(Z)} - \left(Su - Sv, \sum_{\nu=1}^{N} (S_\nu - \chi_\nu S_\nu)\,(u^\nu - v^\nu)\right)_{L^2(Z)} + \alpha\|u - v\|^2_U \\
&\geq -\frac{1}{4}\left\|\sum_{\nu=1}^{N} \chi_Z\,(S_\nu - \chi_\nu S_\nu)\,(u^\nu - v^\nu)\right\|^2_{L^2(\Omega)} + \alpha\|u - v\|^2_U \\
&= \left(\alpha - \frac{1}{4}\sum_{\nu=1}^{N} \|\chi_Z(S_\nu - \chi_\nu S_\nu)\|^2_{L^2(\Omega)\to L^2(\Omega)}\right)\|u - v\|^2_U.
\end{aligned}$$

Thus, our assumption on α implies that the operator F is strongly monotone. □

8.1.3.2 Estimate of the Parameter α

Let us analyze the right hand side of (8.10). Let w denote an arbitrary function in $L^2(\Omega)$. Due to the definition of the operator norm we obtain

$$\begin{aligned}
\|\chi_Z(S_\nu - \chi_\nu S_\nu)\|_{L^2(\Omega)\to L^2(\Omega)} &= \sup_{\|w\|_{L^2(\Omega)}=1} \|\chi_Z(S_\nu - \chi_\nu S_\nu)w\|_{L^2(\Omega)} \\
&= \sup_{\|w\|_{L^2(\Omega)}=1} \left(\int_Z (1 - \chi_\nu)^2 (S_\nu w)^2\,\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned} \tag{8.11}$$

Due to the continuous embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, the solution operator S_ν is continuous from $L^2(\Omega)$ to $C(\overline{\Omega})$ and we know that

$$c_{L^\infty} := \max_{\nu=1,\dots,N} \sup_{\|w\|_{L^2(\Omega)}=1} \|S_\nu w\|_{L^\infty(\Omega)} < \infty$$

exists. Hence, we obtain that

$$\begin{aligned}\|\chi_Z(S_\nu - \chi_\nu S_\nu)\|_{L^2(\Omega)\to L^2(\Omega)} &\leq \sup_{\|w\|_{L^2(\Omega)}=1} \|S_\nu w\|_{L^\infty(\Omega)} \left(\int_Z (1-\chi_\nu)^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \\ &\leq c_{L^\infty} \left(\int_Z (1-\chi_\nu)^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = c_{L^\infty} \sqrt{\operatorname{meas}(Z\backslash\Omega_\nu)}.\end{aligned}$$

Thus,

$$\frac{1}{4}\sum_{\nu=1}^{N} \|\chi_Z(S_\nu - \chi_\nu S_\nu)\|^2_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{c_{L^\infty}^2}{4} \sum_{\nu=1}^{N} \operatorname{meas}(Z\backslash\Omega_\nu). \tag{8.12}$$

Hence, we can interpret the right-hand side of (8.10) as the maximum difference of the set Z and the sets Ω_ν. Again, if $\Omega_\nu := \Omega$ for all ν this value is obviously zero and we are in the setting of a reducible GNEP. However, if the right-hand side of (8.10) is too large, the existence of minimizers can not be guaranteed by our theory for all $\alpha > 0$. For the special case $A := -\Delta$, the constant c_{L^∞}, which is dependent on the domain Ω , can be computed [106] and is given by $c_{L^\infty} \approx 1.3596(\operatorname{meas}(\Omega))^{1/6}$. Thus, we can conclude that

$$\alpha > 0.4621(\operatorname{meas}(\Omega))^{1/3} \sum_{\nu=1}^{N} \operatorname{meas}(Z\backslash\Omega_\nu)$$

satisfies (8.10) and, hence, Assumption 8.3 in the case that $C_\nu := \chi_\nu$. Moreover, for the special case $A := -\Delta$ we know by the Lax Milgram Theorem, that

$$\|S_\nu w\|_{H_0^1(\Omega)} \leq (1 + c_P)\, \|w\|_{L^2(\Omega)}\,,$$

where c_P denotes the Poincaré constant. The Poincaré-Friedrich inequality yields the basic estimate

$$\|S_\nu w\|_{L^2(\Omega)} \leq \sqrt{c_P}\, \|S_\nu w\|_{H_0^1(\Omega)} \leq \sqrt{c_P}(1 + c_P)\, \|w\|_{L^2(\Omega)}\,.$$

Thus,

$$\begin{aligned}\|\chi_Z(S_\nu - \chi_\nu S_\nu)\|_{L^2(\Omega)\to L^2(\Omega)} &= \sup_{\|w\|_{L^2(\Omega)}=1} \|\chi_Z(S_\nu - \chi_\nu S_\nu)w\|_{L^2(\Omega)} \\ &= \sup_{\|w\|_{L^2(\Omega)}=1} \left(\int_Z (1-\chi_\nu)^2 (S_\nu w)^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \sup_{\|w\|_{L^2(\Omega)}=1} \|S_\nu w\|_{L^2(\Omega)} \\ &\leq \sqrt{c_P}(1 + c_P).\end{aligned}$$

Hence, for $A := -\Delta$ we arrive at

$$\frac{1}{4}\sum_{\nu=1}^{N} \|\chi_Z(S_\nu - \chi_\nu S_\nu)\|^2_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{N}{4} c_P (1 + c_P)^2. \tag{8.13}$$

Since the Poincaré constant c_P is only dependent on the domain Ω, we can use (8.13) to check if Assumption 8.3 is satisfied. For instance, for a rectangular domain $\Omega = (a,b) \times (c,d)$ the Poincaré constant is given by

$$c_P = \left(\frac{2}{(b-a)^2} + \frac{2}{(d-c)^2} \right)^{-1}.$$

Note that (8.12) depends on the difference between Z and Ω_ν. Consequently, this quantity yields a better approximation for α if the subsets Ω_ν are large compared to Z. The second estimate does not depend on that difference. Hence, this estimate is probably more appropriate if Ω_ν is relatively small. Moreover, (8.12) depends crucially on $\mathrm{meas}(\Omega)$. Thus, this estimate may in practice only be suitable for special domains like $\Omega = (0,1)^2$.

8.1.3.3 The Augmented NEP

$$\begin{aligned} &\underset{u^\nu \in L^2(\Omega)}{\text{minimize}} \; f^\nu_{AL}(u,\mu,\rho) := f^\nu(u) + \frac{1}{2\rho} \|(\mu + \rho g(u))_+\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u^\nu \in U^\nu_{\text{ad}}, \end{aligned} \qquad (P^\nu_{AL})$$

We refer to problems of this type as augmented NEP. These NEPs are arising during the process of solving (P_ν), if K is the cone of non-positive continuous functions, by applying a penalty or augmented Lagrangian method. For the sake of simplicity we assume that $g(u) := Su - \psi$, where $\psi \in C(\overline{\Omega})$. Defining

$$F_{AL}(u) := \Big(D_{u^1} f^1_{AL}(u,\mu,\rho), \ldots, D_{u^N} f^N_{AL}(u,\mu,\rho) \Big)$$

it is again the convexity of the objective functional that allows us to characterize the solution of the augmented NEP via controls $\bar{u} \in U$ that solve the variational inequality

$$\begin{aligned} &(F_{AL}(\bar{u}), v - \bar{u})_U \geq 0 \quad \forall v \in U_{\text{ad}} \\ \Leftrightarrow &\sum_{\nu=1}^{N} (S^*_\nu(C^*_\nu(C_\nu S\bar{u} - y^\nu_d) + (\mu + \rho(Su - \psi))_+) + \alpha \bar{u}^\nu, v^\nu - \bar{u}^\nu) \geq 0 \quad \forall v^\nu \in U^\nu_{\text{ad}}. \end{aligned} \qquad (8.14)$$

From Theorem 8.4 we know that the mapping

$$u \mapsto \Big(D_{u^1} f^1(u^1), \ldots, D_{u^N} f^N(u^N) \Big)$$

is strongly monotone if Assumption 8.3 is satisfied. Furthermore, we know that the function

$$u \mapsto \frac{1}{2\rho} \|(\mu + \rho(Su - \psi))_+\|^2_{L^2(\Omega)}$$

is convex and its derivative is monotone. Hence, F_{AL} is strongly monotone and Theorem 8.4 can easily be adapted to that case.

Theorem 8.7. *Let Assumption 8.3 be satisfied. Then there exists an unique solution of problem (P^ν_{AL}). Further, if $C_\nu := C \in L(Y, L^2(\Omega))$ for all ν, then the NEP (P^ν_{AL}) is uniquely solvable for all $\alpha > 0$.*

Note, that the strong monotonicity of F does not only imply uniqueness of variational equilibria. Moreover, it is directly related to the convergence analysis of solving the GNEP (P_ν) with a Lagrange multiplier method, where the subproblems are given by the augmented NEPs (P^ν_{AL}).

In particular, the strong monotonicity of F implies strong convergence of the primal iterates and weak-* convergence of the corresponding multiplier on a subsequence, provided a suitable constraint qualification is satisfied [69, Section 5.1].

We will deepen our studies of problem (P^{ν}_{AL}) in Section 8.3. Here, we will among others derive the corresponding Newton iteration that allows us to solve the problem numerically with superlinear convergence.

8.2 Semi-Smoothness of the Projection Operator

This section aims at proving semi-smoothness of $u \mapsto \max(a,u)$ from $L^q(\Omega)$ to $L^p(\Omega)$ even if $a \in L^r(\Omega)$, with $1 \le p \le r < q \le \infty$. This property is crucial for obtaining superlinear convergence of the semi-smooth Newton method applied to (P^{ν}_{AL}).

For our later application we will need semi-smoothness of the mapping

$$u \mapsto (\mu + \rho(Su - \psi))_+,$$

see Section 8.3.1. Since μ is only an $L^2(\Omega)$ function we cannot expect from the known result [117, Thm. 4.4] that the mapping

$$\max(0, \mu + \rho(Su - \psi)) = \mu - \rho\psi + \max(-\mu + \rho\psi, Su)$$

is semi-smooth from $L^q(\Omega)$ to $L^2(\Omega)$. In [66, Ex. 8.12] Ito and Kunisch investigated the semi-smoothness of superposition operators

$$F\colon L^q(\Omega) \to L^p(\Omega), \quad F(u)(x) = f(u(x)) \text{ for a.e. } x \in \Omega,$$

where $1 \le p < q \le \infty$ and $f\colon \mathbb{R} \to \mathbb{R}$ is semi-smooth and globally Lipschitz continuous. However, due to the dependence of a and b on the x-variable the mapping $u \mapsto \max(a, \min(u,b))$ cannot be built via superposition. Nevertheless, since the regularity of the functions a and b isn't needed in the proof one can apply similar arguments.

Theorem 8.8. *Let $a,b \in L^r(\Omega)$ with $a \le b$ and $1 \le p \le r < q \le \infty$. The mapping $m\colon L^q(\Omega) \to L^p(\Omega)$, $u \mapsto \max(a, \min(u,b))$ is semi-smooth with Newton derivative*

$$L^s(\Omega) \ni h(u)(x) = \begin{cases} 0 & \text{if} \quad u(x) \ge b(x), \\ 1 & \text{if} \quad u(x) \in (a(x), b(x)), \\ 0 & \text{if} \quad u(x) \le a(x), \end{cases} \tag{8.15}$$

where s is chosen such that $\frac{1}{p} = \frac{1}{s} + \frac{1}{q}$ holds.

Proof. A similar proof can be found in the PhD-Thesis [114]. Let $u \in L^q(\Omega)$ be arbitrary and $(s_k)_k \subset L^q(\Omega)$ be a (strong) nullsequence. Furthermore, define $u_k := u + s_k$ and $d_k := h(u_k)$. We have to check the condition

$$\|m(u_k) - m(u) - D_N m(u_k) s_k\|_{L^p(\Omega)} = o(\|s_k\|_{L^q(\Omega)}).$$

First we extract a subsequence $(s_k)_{k\in I}$ with an index set I such that $s_k(x) \to_I 0$ for almost all $x \in \Omega$. To shorten the notation we furthermore define $v := m(u)$ and $v_k := m(u_k)$. It is known [62, Ex. 2.5] that the mapping $\tilde{m} : \mathbb{R} \to \mathbb{R},\ x \mapsto \max(a, \min(x,b))$ with $a,b \in \mathbb{R}$ is semi-smooth. Hence, we obtain

$$s_k(x)^{-1}(v_k(x) - v(x) - d_k(x)s_k(x)) \to_I 0$$

for almost all $x \in \Omega$. The quotient on the left side is understood to be zero whenever $s_k(x) = 0$. Now we use that the projection m is nonexpansive and obtain

$$\begin{aligned}|v_k(x) - v(x) - d_k(x)s_k(x)| &\le |v_k(x) - v(x)| + |d_k(x)s_k(x)| \\ &\le |u(x) + s_k(x) - u(x)| + |s_k(x)| \\ &\le 2|s_k(x)|.\end{aligned}$$

By applying Lebesgue's dominated convergence theorem we obtain

$$s_k^{-1}(v_k - v - d_k s_k) \to_I 0$$

in $L^r(\Omega)$ for all $r \in [1, \infty)$. Hence, by applying Hölder's inequality we get with $\frac{1}{p} = \frac{1}{s} + \frac{1}{q}$

$$\frac{\|v_k - v - d_k s_k\|_{L^p(\Omega)}}{\|s_k\|_{L^q(\Omega)}} \le \|s_k^{-1}(v_k - v - d_k s_k)\|_{L^s(\Omega)} \to_I 0. \tag{8.16}$$

Since this argumentation can be repeated for any subsequence of $(s_k)_k$ the limit in (8.16) holds in fact for the whole sequence. □

In the same manner we obtain the following result.

Lemma 8.9. *Let $a \in L^r(\Omega)$ and $1 \le p \le r < q \le \infty$. The mapping $m : L^q(\Omega) \to L^p(\Omega)$, $u \mapsto \max(a, u)$ is semi-smooth with Newton derivative*

$$L^s(\Omega) \ni h(u)(x) = \begin{cases} 1 & \text{if } \; u(x) > a(x), \\ 0 & \text{if } \; u(x) \le a(x), \end{cases} \tag{8.17}$$

where s is chosen such that $\frac{1}{p} = \frac{1}{s} + \frac{1}{q}$ holds.

Note that the norm gap $p < q$ is indispensable for Newton differentiability of the projection operator, see for instance [66, Ex. 8.14]. Hence, the functions defined in (8.15) and (8.17) can in general not serve as a Newton derivative for $m\colon L^2(\Omega) \to L^2(\Omega)$, see [54, Prop. 4.1]. To bridge this norm gap, one needs additional structure. For problems that involve partial differential equations this structure is often given by smoothing properties of the corresponding solution operators. To finish, let us briefly comment on the semi-smoothness of the projection operator $P_{U_{\text{ad}}} : L^q(\Omega)^N \to L^2(\Omega)^N$. The mapping

$$\Pi_\nu \colon L^q(\Omega)^N \to L^q(\Omega),\ u \mapsto u^\nu.$$

is linear and continuously Fréchet differentiable, hence semi-smooth. Applying the chain rule [62, Thm. 2.10 c)], we now obtain that

$$P_{U_{\text{ad}}}^\nu(u) = \min(\max(a^\nu, \Pi_\nu(u)), b^\nu)$$

is a composition of semi-smooth functions, hence semi-smooth from $L^q(\Omega)^N \to L^2(\Omega)$, see Lemma 8.8. Using [62, Thm. 2.10 a)], we obtain that $P_{U_{\text{ad}}}$ is semi-smooth from $L^q(\Omega)^N \to L^2(\Omega)^N$.

8.3 Newton Iteration for the Non-Reducible NEP

We now want to study the semi-smooth Newton method applied to problem (P_{AL}^ν). This investigation basically reduces to the application of the semi-smooth Newton method to strongly monotone VIs. To simplify our notation let us introduce $d \in U$ with components $d^\nu \in L^2(\Omega)$ and $M \in L(U, U)$. We define the product $d \cdot M = dM \in L(U, U)$ in a component-wise manner

$$(d \cdot M(u))^\nu := d^\nu M^\nu(u) \in L^2(\Omega). \tag{8.18}$$

Hence, $d \cdot M : U \to U$. In a similar way we define $d \cdot u \in U$ for some $u \in U$.

8.3.1 Newton Iteration and Convergence Result

It is well known, that (8.14) can be equivalently formulated using the projection operator P onto the set U_{ad}. A solution $\bar{u} \in U_{\text{ad}}$ of (P_{AL}^{ν}) can be characterized by the equation

$$G(u) := u - P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}p(u)\right) = 0, \tag{8.19}$$

where $p : U \to L^q(\Omega)$, $q > 2$ and the ν-th component is given as the adjoint state

$$p(u)^{\nu} = S_{\nu}^{*}\left(C_{\nu}^{*}(C_{\nu}Su - y_d^{\nu}) + (\mu + \rho(Su - \psi))_{+}\right).$$

We aim at solving (8.19) with the semi-smooth Newton method, which is given in the following algorithm

Algorithm 8.1 Semi-smooth Newton method

Choose $x_0 \in X$
For $k = 0, 1, 2, ...$ repeat:
1: Choose a Newton derivative $D_N G(x_k)$.
2: Compute δ_k by solving $D_N G(x_k)\delta_k = -G(x_k)$.
3: Set $x_{k+1} := x_k + \delta_k$.

Due to the chain rule [62, Thm. 2.10 c)], Example [62, Ex. 2.5] and Lemma 8.9 it is clear, that a suitable Newton derivative of G from (8.19) at u_k in direction $h \in U$ is given by

$$(D_N G(u_k)h)^{\nu} = h_{\nu} + \frac{1}{\alpha}\chi_{\mathcal{I}_k^{\nu}}\left(S_{\nu}^{*}(C_{\nu}^{*}C_{\nu}S + \chi_{\mathcal{Y}_k}\rho S)h\right),$$

where the components of $\chi_{\mathcal{I}}(u_k)$ are given as

$$(\chi_{\mathcal{I}}(u_k))^{\nu}(x) := \begin{cases} 0 & \text{if} \quad -\frac{1}{\alpha}p(u_k)^{\nu}(x) \geq b^{\nu}(x), \\ 1 & \text{if} \quad -\frac{1}{\alpha}p(u_k)^{\nu}(x) \in (a^{\nu}(x), b^{\nu}(x)), \\ 0 & \text{if} \quad -\frac{1}{\alpha}p(u_k)^{\nu}(x) \leq a^{\nu}(x), \end{cases} \tag{8.20}$$

for almost all $x \in \Omega$. In the further, we will use the notation: $\chi_{\mathcal{I}_k} := \chi_{\mathcal{I}}(u_k)$ and $\chi_{\mathcal{I}_k}^{\nu} = \chi_{\mathcal{I}}(u_k)^{\nu}$. We denote by $u_{k+1} := u_k + \delta_k$ the next iterate of the semi-smooth Newton method from Algorithm 8.1 and define $M := S_{\nu}^{*}(C_{\nu}^{*}C_{\nu}S + \chi_{\mathcal{Y}_k}\rho S)$. A Newton step is given by

$$\begin{aligned} D_N G(u_k)\delta_k = -G(u_k) \;&\Leftrightarrow\; \left(\text{Id} + \frac{1}{\alpha}\chi_{\mathcal{I}_k^{\nu}}M\right)\delta_k = -u_k + P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}p(u_k)\right) \\ &\Leftrightarrow\; \left(\text{Id} + \frac{1}{\alpha}\chi_{\mathcal{I}_k^{\nu}}M\right)u_{k+1} = P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}p(u_k)\right) + \frac{1}{\alpha}\chi_{\mathcal{I}_k^{\nu}}Mu_k. \end{aligned}$$

Using this representation we see that

$$u_{k+1}^{\nu}(x) = \begin{cases} u_a^{\nu}(x) & \text{if} \quad x \in \mathcal{A}_k^{\nu,a}, \\ -\frac{1}{\alpha}\left(S_{\nu}^{*}(C_{\nu}^{*}(C_{\nu}Su_{k+1} - y_d^{\nu}) + \chi_{\mathcal{Y}_k}(\mu + \rho(Su_{k+1} - \psi)))\right)(x) & \text{if} \quad x \in \mathcal{I}_k^{\nu}, \\ u_b^{\nu}(x) & \text{if} \quad x \in \mathcal{A}_k^{\nu,b}. \end{cases} \tag{8.21}$$

where the sets $\mathcal{A}_k^{\nu,a}, \mathcal{A}_k^{\nu,b}, \mathcal{I}_k^{\nu}$ and $\mathcal{Y}_k$ are defined by

$$\begin{aligned} \mathcal{A}_k^{\nu,a} &:= \left\{x \in \Omega\colon -\frac{1}{\alpha}p(u_k)^{\nu} \leq u_a^{\nu}\right\}, & \mathcal{A}_k^{\nu,b} &:= \left\{x \in \Omega\colon -\frac{1}{\alpha}p(u_k)^{\nu} \geq u_b^{\nu}\right\}, \\ \mathcal{I}_k^{\nu} &:= \left\{x \in \Omega\colon -\frac{1}{\alpha}p(u_k)^{\nu} \in (u_a^{\nu}, u_b^{\nu})\right\}, & \mathcal{Y}_k &:= \{x \in \Omega\colon (\mu + \rho(Su_k - \psi)) > 0\}. \end{aligned} \tag{8.22}$$

Thus, on the set $\mathcal{I}_k^\nu$ we obtain

$$\chi_{\mathcal{I}_k^\nu}\left(u_{k+1}^\nu + \frac{1}{\alpha}\left(S_\nu^*(C_\nu^*(C_\nu S u_{k+1} - y_d^\nu) + \chi_{\mathcal{Y}_k}(\mu + \rho(S u_{k+1} - \psi)))\right)\right) = 0. \tag{8.23}$$

Let us introduce the function $u_{k+1}^{\mathcal{I}} \in U$ with components $u_{k+1}^{\nu,\mathcal{I}} := \chi_{\mathcal{I}_k^\nu} u_{k+1}^\nu$ for $\nu = 1, ..., N$. Hence, we can write $u_{k+1}^{\mathcal{I}} = \chi_{\mathcal{I}_k} u_{k+1}$. In a similar way we define $\chi_{A_k^a}$ and $\chi_{A_k^b}$. Using this definitions we can now write (8.23) as a linear equation for the ν-th component of $u_{k+1}^{\mathcal{I}}$ and we obtain

$$\begin{aligned} u_{k+1}^{\nu,\mathcal{I}} &+ \frac{1}{\alpha}\chi_{\mathcal{I}_k^\nu}\left(S_\nu^*(C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S) u_{k+1}^{\mathcal{I}}\right) \\ &= -\frac{1}{\alpha}\chi_{\mathcal{I}_k^\nu}\left(S_\nu^*((C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S)(\chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b) - C_\nu^* y_d^\nu + \chi_{\mathcal{Y}_k}(\mu - \rho\psi))\right). \end{aligned}$$

The Newton step can now be written in the following compact form.

Lemma 8.10. *The solution u_{k+1} of one step of the semi-smooth Newton method is given by*

$$u_{k+1} = u_{k+1}^{\mathcal{I}} + \chi_{A_k^a} u_a + \chi_{A_k^b} u_b,$$

where $u_{k+1}^{\mathcal{I}}$ is given as the solution of the linear system

$$\left(\mathrm{Id} + \chi_{\mathcal{I}_k} T^k\right) u_{k+1}^{\mathcal{I}} = \chi_{\mathcal{I}_k} g_k, \tag{8.24}$$

with the operator $T^k : U \to U$ and function $g_k \in U$ given by

$$\begin{aligned} (T^k h)^\nu &:= \frac{1}{\alpha} S_\nu^*(C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S) h, \\ (g_k)^\nu &:= -\frac{1}{\alpha}\chi_{\mathcal{I}_k^\nu}\left(S_\nu^*((C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S)(\chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b) - C_\nu^* y_d^\nu + \chi_{\mathcal{Y}_k}(\mu - \rho\psi))\right). \end{aligned}$$

Here $\mathrm{Id} : U \to U$ *denotes the identity mapping.*

The complete semi-smooth Newton method is given in the following algorithm.

Algorithm 8.2 Semi-smooth Newton method for problem (P_{AL}^ν)

1: Set $k = 0$, choose u_0 in $L^2(\Omega)^N$
2: **repeat**
3: Set $\mathcal{A}_k^{\nu,a}, \mathcal{A}_k^{\nu,b}, \mathcal{I}_k^\nu$ and $\mathcal{Y}_k$ as defined in (8.22)
4: Solve for $u_{k+1}^{\mathcal{I}} \in L^2(\Omega)^N$ by solving (8.24)
5: Set $u_{k+1} := u_{k+1}^{\mathcal{I}} + \chi_{A_k^a} u_a + \chi_{A_k^b} u_b$
6: Set $k := k + 1$
7: **until** $\mathcal{A}_k^{\nu,a} = \mathcal{A}_{k-1}^{\nu,a}, \mathcal{A}_k^{\nu,b} = \mathcal{A}_{k-1}^{\nu,b}, \mathcal{I}_k^\nu = \mathcal{I}_{k-1}^\nu$ and $\mathcal{Y}_k = \mathcal{Y}_{k-1}$.

Theorem 8.11 (Convergence of the semi-smooth Newton method). *Let Assumption 8.3 hold and let $\bar{u}$ denote the normalized solution of (P_{AL}^ν). Then the semi-smooth Newton method from Algorithm 8.2 has the following properties*

a) Let $\|u_0 - \bar{u}\|_{L^2(\Omega)^N}$ be sufficiently small. Then the iterates u_k converge for $k \to \infty$ superlinearly to $\bar{u}$ which is the normalized solution of (P_{AL}^ν).

b) Let u_k be generated by Algorithm 8.2 such that the stopping criterion from step 7 is satisfied. Then u_k is a solution of (8.19).

Proof. a) Since G is semi-smooth from $L^2(\Omega)^N$ to $L^2(\Omega)^N$, it suffices to show that $(D_N G)^{-1}$ is uniformly bounded. Applying standard arguments for semi-smooth Newton methods the characterization (8.21) can be reformulated in

$$\left(\mathrm{Id} + \frac{1}{\alpha}\chi_{\mathcal{I}_k} M \chi_{\mathcal{I}_k}\right) u_{k+1} = \chi_{\mathcal{I}_k}\left(-\frac{1}{\alpha}p(u_k)\right) + \frac{1}{\alpha}\chi_{\mathcal{I}_k} M u_k - \frac{1}{\alpha}\chi_{\mathcal{I}_k} M\left(\chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b\right) + \chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b,$$

where $(Mw)^\nu = S_\nu^*(C_\nu^* C_\nu S + \rho \chi_{\mathcal{Y}_k} S) w$. Defining the bilinear form

$$a(w,v) = \left(\left(\mathrm{Id} + \frac{1}{\alpha}\chi_{\mathcal{I}_k} M \chi_{\mathcal{I}_k}\right) w, v\right)_U,$$

we obtain

$$a(w,v) = (w,v)_U + \frac{1}{\alpha}(\chi_{\mathcal{I}_k} M \chi_{\mathcal{I}_k} w, v)_U \le c\,\|w\|_U\,\|v\|_U\,.$$

Using the decomposition from the proof of Theorem 8.4 we arrive at

$$\begin{aligned}
a(w,w) &= \left(\left(\mathrm{Id} + \frac{1}{\alpha}\chi_{\mathcal{I}_k} M \chi_{\mathcal{I}_k}\right) w, w\right)_U \\
&= \sum_{\nu=1}^N \left(w^\nu + \frac{1}{\alpha}\chi_{\mathcal{I}_k} S_\nu^*(C_\nu^* C_\nu S + \rho\chi_{\mathcal{Y}_k} S)\chi_{\mathcal{I}_k} w, w^\nu\right)_{L^2(\Omega)} \\
&= \|w\|_U^2 + \frac{1}{\alpha}\sum_{\nu=1}^N (S\chi_{\mathcal{I}_k} w, C_\nu^* C_\nu S_\nu \chi_{\mathcal{I}_k} w^\nu)_{L^2(\Omega)} + \frac{\rho}{\alpha}\sum_{\nu=1}^N (\chi_{\mathcal{Y}_k} S\chi_{\mathcal{I}_k} w, S_\nu \chi_{\mathcal{I}_k} w^\nu)_{L^2(\Omega)} \\
&= \|w\|_U^2 + \frac{1}{\alpha}\left(S\chi_{\mathcal{I}_k} w, \sum_{\nu=1}^N C_\nu^* C_\nu S_\nu \chi_{\mathcal{I}_k} w^\nu\right)_{L^2(\Omega)} + \frac{\rho}{\alpha}\left\|\chi_{\mathcal{Y}_k} S\chi_{\mathcal{I}_k} w\right\|_U^2 \\
&\ge \|w\|_U^2 + \frac{1}{\alpha}\left\|S\chi_{\mathcal{I}_k} w\right\|_{L^2(\Omega)}^2 - \frac{1}{\alpha}\left(S\chi_{\mathcal{I}_k} w, \sum_{\nu=1}^N (S_\nu - C_\nu^* C_\nu S_\nu)\chi_{\mathcal{I}_k} w^\nu\right)_{L^2(\Omega)} \\
&\ge \|w\|_U^2 - \frac{1}{4\alpha}\left\|\sum_{\nu=1}^N (S_\nu - C_\nu^* C_\nu S_\nu)\,\chi_{\mathcal{I}_k} w^\nu\right\|_{L^2(\Omega)}^2 \\
&\ge \|w\|_U^2 - \frac{1}{4\alpha}\left(\sum_{\nu=1}^N \|S_\nu - C_\nu^* C_\nu S_\nu\|_{L^2(\Omega)\to L^2(\Omega)}^2\right)\left(\sum_{\nu=1}^N \left\|\chi_{\mathcal{I}_k} w^\nu\right\|_{L^2(\Omega)}^2\right) \\
&\ge \left(1 - \frac{1}{4\alpha}\sum_{\nu=1}^N \|S_\nu - C_\nu^* C_\nu S_\nu\|_{L^2(\Omega)\to L^2(\Omega)}^2\right)\|w\|_U^2\,.
\end{aligned}$$

Choosing α as in Assumption 8.3 implies that $a(w,v)$ is coercive and satisfies the conditions of the Lax-Milgram Theorem, which yields boundedness of $\left\|D_N G(u_k)^{-1}\right\|_{U\to U}$.

b) We know that the solution of (8.24) is unique for fixed sets $\mathcal{A}_k^{\nu,a}, \mathcal{A}_k^{\nu,b}, \mathcal{I}_k^\nu$ and $\mathcal{Y}_k$. We set $\mathcal{A}_k^{\nu,a} := \mathcal{A}_{k+1}^{\nu,a}, \mathcal{A}_k^{\nu,b} := \mathcal{A}_{k+1}^{\nu,b}, \mathcal{I}_k^\nu := \mathcal{I}_{k+1}^\nu$ and $\mathcal{Y}_k := \mathcal{Y}_{k+1}$ in (8.24) and get

$$u_{k+1}^{\nu,\mathcal{I}} + \frac{\chi_{\mathcal{I}_{k+1}^\nu}}{\alpha}\left(S_\nu^*(C_\nu^* C_\nu S + \chi_{\mathcal{Y}_{k+1}}\rho S) u_{k+1}^{\mathcal{I}}\right) = - \frac{\chi_{\mathcal{I}_{k+1}^\nu}}{\alpha}\left(S_\nu^*(C_\nu^* C_\nu S + \chi_{\mathcal{Y}_{k+1}}\rho S)(\chi_{\mathcal{A}_{k+1}^a} u_a + \chi_{\mathcal{A}_{k+1}^b} u_b) - S_\nu^*(C_\nu^* y_d^\nu + \chi_{\mathcal{Y}_{k+1}}(-\mu + \rho\psi))\right).$$

Which is equivalent to

$$u_{k+1}^{\nu,\mathcal{I}} + \frac{\chi_{\mathcal{I}_{k+1}^{\nu}}}{\alpha}\left(S_{\nu}^{*}(C_{\nu}^{*}(C_{\nu}Su_{k+1} - y_{d}^{\nu}) + \chi_{\mathcal{Y}_{k+1}}(\mu + \rho(Su_{k+1} - \psi))\right) = 0$$
$$\Leftrightarrow u_{k+1}^{\nu,\mathcal{I}} + \frac{\chi_{\mathcal{I}_{k+1}^{\nu}}}{\alpha} p^{\nu}(u_{k+1}) = 0.$$

Together with $u_{k+1} = u_a$ on $\mathcal{A}_{k+1}^{a}$ and $u_{k+1} = u_b$ on $\mathcal{A}_{k+1}^{b}$ we get

$$u_{k+1}^{\nu} - P_{[u_a^{\nu}, u_b^{\nu}]}\left(-\frac{1}{\alpha}p^{\nu}(u_{k+1})\right) = 0.$$

Hence, u_{k+1} is a solution of (8.19).

□

Again, we can drop the assumption on α if the operators C_{ν} coincide for all ν.

Corollary 8.12. *Let $C_{\nu} := C \in L(Y, L^2(\Omega))$ for all ν and let $\alpha > 0$. Then the Newton method associated to the NEP (P_{AL}^{ν}) converges superlinear.*

It is obvious, that the convergence result from Theorem 8.11 is still valid for NEPs, where each player's cost functional is given by

$$f^{\nu}(u) := \frac{1}{2}\|C_{\nu}Su - y_{d}^{\nu}\|_{L^2(\Omega)}^{2} + \frac{\alpha}{2}\|u^{\nu}\|_{L^2(\Omega)}^{2}.$$

In this situation $M := S_{\nu}^{*}C_{\nu}^{*}C_{\nu}S$ and the only change in the proof is that the additional augmentation term has to be neglected.

8.3.2 Implementation

Let us now focus on the details of an implementation using finite elements. To illustrate the implementation we focus on problem (P_{AL}^{ν}), where S denotes the solution operator of (8.3) with $A := -\Delta$. Using standard methods the corresponding optimality system is given by

$$-\Delta\bar{y} = \sum_{\nu=1}^{N}\bar{u}^{\nu} \quad \text{in } \Omega, \tag{8.25a}$$

$$-\Delta\bar{p}^{\nu} = C_{\nu}^{*}(C_{\nu}\bar{y} - y_{d}^{\nu}) + (\mu + \rho(Su - \psi))_{+} \quad \text{in } \Omega, \tag{8.25b}$$

$$(\bar{p}^{\nu} + \alpha\bar{u}^{\nu}, v^{\nu} - \bar{u}^{\nu}) \geq 0 \quad \forall v^{\nu} \in U_{\text{ad}}^{\nu}, \tag{8.25c}$$

where the state and adjoint equation satisfy suitable boundary conditions. We are interested in a finite element discretization. Let $\mathcal{T}_h$ be a regular mesh which consists of closed cells T. For $T \in \mathcal{T}_h$ we define $h_T := \operatorname{diam}(T)$. Furthermore, we set $h := \max_{T\in\mathcal{T}_h} h_T$. We assume that there exists a constant $R > 0$ such that $\frac{h_T}{R_T} \leq R$ for all $T \in \mathcal{T}$. Here, we define R_T to be the diameter of the largest ball contained in T. In the further, we use a regular triangulation of the domain Ω with mesh size h. For this mesh $\mathcal{T}$ we define an associated finite dimensional space $\mathbb{V}_h := \operatorname{span}\{\phi_1, ..., \phi_m\}$ with basis functions ϕ_j, such that the restriction of a function $v \in \mathbb{V}_h$ to a cell $T \in \mathcal{T}$ is a linear polynomial. Let us now consider a discretized version of (8.25). We define the bilinear form

$$a(w, v) := \int_{\Omega} \nabla w \nabla v \,\mathrm{d}x.$$

Then, the discretized version of (8.25) is given by the solution (y_h, u_h, p_h) of the system

$$\begin{aligned}
a(y_h, v_h) &= \left(\sum_{\nu=1}^{N} u_h^\nu, v_h\right) && \forall v_h \in \mathbb{V}_h, \\
a(p_h^\nu, v_h) &= (C_\nu^*(C_\nu y_h - y_d^\nu) + (\mu + \rho(y_h - \psi))_+, v_h) && \forall v_h \in \mathbb{V}_h, \\
u_h^\nu &= P_{[u_a^\nu, u_b^\nu]}\left(-\frac{1}{\alpha} p_h^\nu\right).
\end{aligned} \tag{8.26}$$

Since for a given u_h there exists a unique $y_h(u_h)$ and unique adjoint states $p_h^\nu(u_h)$, system (8.26) can be reduced to the single equation

$$u_h^\nu = P_{[u_a^\nu, u_b^\nu]}\left(-\frac{1}{\alpha} p_h^\nu(u_h)\right) \qquad \forall v_h \in \mathbb{V}_h.$$

Again we define the active and inactive sets for the discrete function $u_{k,h}$:

$$\begin{aligned}
\mathcal{A}_k^{\nu,a} &:= \left\{x \in \Omega\colon -\frac{1}{\alpha} p_h^\nu(u_{k,h}) \le u_a^\nu\right\}, \qquad \mathcal{A}_k^{\nu,b} := \left\{x \in \Omega\colon -\frac{1}{\alpha} p_h^\nu(u_{k,h}) \ge u_b^\nu\right\}, \\
\mathcal{I}_k^\nu &:= \left\{x \in \Omega\colon -\frac{1}{\alpha} p_h^\nu(u_{k,h}) \in (u_a^\nu, u_b^\nu)\right\}, \ \mathcal{Y}_k := \{x \in \Omega\colon (\mu + \rho(Su_{k,h} - \psi)) > 0\}.
\end{aligned}$$

We now define the functions $u_{k+1,h}^{\mathcal{I}} := \chi_{\mathcal{I}_k} u_{k+1,h}$, where $\chi_{\mathcal{I}^k} := \chi_{\mathcal{I}}(u_k)$. Following the lines of the proof of Section 8.3.1 we can establish a linear equation for the components of $u_{k+1,h}^{\mathcal{I}}$:

$$\begin{aligned}
&u_{k+1,h}^{\nu,\mathcal{I}} + \frac{1}{\alpha}\chi_{\mathcal{I}_k^\nu}\left(S_\nu^*(C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S) u_{k+1,h}^{\mathcal{I}}\right) \\
&\qquad = -\frac{1}{\alpha}\chi_{\mathcal{I}_k^\nu}\left(S_\nu^*((C_\nu^* C_\nu S + \chi_{\mathcal{Y}_k}\rho S)(\chi_{\mathcal{A}_k^a} u_a + \chi_{\mathcal{A}_k^b} u_b) - C_\nu^* y_d^\nu + \chi_{\mathcal{Y}_k}(\mu - \rho\psi))\right).
\end{aligned}$$

We want to solve this system by testing it with a function $v_h \in \mathbb{V}_h$. Note that we have $u_{k+1,h}^{\nu,\mathcal{I}} \notin \mathbb{V}_h$ in general, but it can be calculated as a projection $u_{k+1,h}^{\nu,\mathcal{I}} = \chi_{\mathcal{I}_k^\nu} \tilde{u}_{k+1,h}^\nu$ of a function $\tilde{u}_{k+1,h}^\nu \in \mathbb{V}_h$, see (8.26). In the following denote $\underline{u_h} \in \mathbb{R}^m$ the coefficient vector of a function $u_h \in \mathbb{V}_h$, where m denotes the dimension of the space $\mathbb{V}_h$. Furthermore, we assume that $u_a^\nu, u_b^\nu \in \mathbb{V}_h$. We can reformulate the Newton step as a linear system in the coefficient vectors of $\tilde{u}_{k+1,h}^\nu$.

Lemma 8.13. *The coefficient vectors* $\underline{\tilde{u}_{k+1,h}^\nu}$ *for* $1 \le \nu \le N$ *satisfy the linear system*

$$\begin{pmatrix}
E_{1,1} & E_{1,2} & \dots & \dots & E_{1,N} \\
E_{2,1} & E_{2,2} & E_{2,3} & \dots & E_{2,N} \\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & E_{N-1,N} \\
E_{N,1} & \dots & \dots & E_{N,N-1} & E_{N,N}
\end{pmatrix}
\begin{pmatrix}
\underline{\tilde{u}_{k+1,h}^1} \\ \underline{\tilde{u}_{k+1,h}^2} \\ \vdots \\ \underline{\tilde{u}_{k+1,h}^N}
\end{pmatrix}
=
\begin{pmatrix}
R_1 \\ R_2 \\ \vdots \\ R_{N-1} \\ R_N
\end{pmatrix}, \tag{8.27}$$

where $E_{i,j} \in \mathbb{R}^{m \times m}$*, with*

$$E_{i,j} := \left\{\begin{array}{ll}
M_{\mathcal{I}_k^i} + \frac{1}{\alpha} M_{\mathcal{I}_k^i} K^{-1} M_{C_i^*} M_{C_i} K^{-1} M_{\mathcal{I}_k^j} + \frac{\rho}{\alpha} M_{\mathcal{I}_k^i} K^{-1} M_{\mathcal{Y}_k} K^{-1} M_{\mathcal{I}_k^j} & \textit{if } i = j \\
\frac{1}{\alpha} M_{\mathcal{I}_k^i} K^{-1} M_{C_i^*} M_{C_i} K^{-1} M_{\mathcal{I}_k^j} + \frac{\rho}{\alpha} M_{\mathcal{I}_k^i} K^{-1} M_{\mathcal{Y}_k} K^{-1} M_{\mathcal{I}_k^j} & \textit{else}
\end{array}\right\},$$

as well as

$$R_i := -\frac{1}{\alpha} M_{\mathcal{I}_k^i} K^{-1} \Big[M_{\mathcal{Y}^k}(\underline{\mu} - \rho \underline{\psi}) - M_{C_i^*} \underline{y_d^i} + (M_{C_i^*} M_{C_i} + \rho M_{\mathcal{Y}^k}) K^{-1} \sum_{i=1}^{N} \Big(M_{\mathcal{A}_k^{i,a}} \underline{u_a^i} + M_{\mathcal{A}_k^{i,b}} \underline{u_b^i} \Big) \Big] \in \mathbb{R}^m,$$

and matrices $K, C_\nu, M_{\mathcal{I}_k^\nu}, M_{\mathcal{A}_k^{\nu,a}}, M_{\mathcal{A}_k^{\nu,b}}$ *and* $M_{\mathcal{Y}_k}$ *of the size* $\mathbb{R}^{m \times m}$ *with*

$$K_{ij} := \left[\int_\Omega \nabla \phi_i \cdot \nabla \phi_j \right]_{ij}, \quad (M_{C_\nu})_{ij} := \left[\int_\Omega C(\phi_i) \phi_j \right]_{ij}, \quad \left(M_{\mathcal{I}_k^\nu} \right)_{ij} := \left[\int_{\mathcal{I}_k^\nu} \phi_i \phi_j \right]_{ij},$$

$$\left(M_{\mathcal{A}_k^{\nu,a}} \right)_{ij} := \left[\int_{\mathcal{A}_k^{\nu,a}} \phi_i \phi_j \right]_{ij}, \quad \left(M_{\mathcal{A}_k^{\nu,b}} \right)_{ij} := \left[\int_{\mathcal{A}_k^{\nu,b}} \phi_i \phi_j \right]_{ij}, \quad (M_{\mathcal{Y}_k})_{ij} := \left[\int_{\mathcal{Y}_k} \phi_i \phi_j \right]_{ij},$$

where ϕ_i, ϕ_j *denote the finite element basis functions of* $\mathbb{V}_h$.

We can reconstruct the state and the adjoint states using the coefficient vectors $\underline{\tilde{u}_{k+1,h}^\nu}$.

Corollary 8.14. *The coefficient vector of the state* $y_{k+1,h}$ *satisfies*

$$\underline{y_{k+1,h}} = K^{-1} \sum_{\nu=1}^{N} \Big(M_{\mathcal{I}_k^\nu} \underline{\tilde{u}_{k+1,h}^\nu} + M_{\mathcal{A}_k^{\nu,a}} \underline{u_a^\nu} + M_{\mathcal{A}_k^{\nu,b}} \underline{u_b^\nu} \Big)$$

and the coefficient vector of the adjoint state $p_{k+1,h}^\nu$ *can be computed by*

$$\underline{p_{k+1,h}^\nu} = K^{-1} \Big(M_{C_\nu^*} (\underline{M_{C_\nu} y_{k+1,h}} - \underline{y_d^\nu}) + M_{\mathcal{Y}_k} (\underline{\mu} + \rho(\underline{y_{k+1,h}} - \underline{\psi})) \Big).$$

The control $u_{k+1,h}^\nu$ can be computed by

$$u_{k+1,h}^\nu = \chi_{\mathcal{I}_k^\nu} \tilde{u}_{k+1,h}^\nu + \chi_{\mathcal{A}_k^{\nu,a}} u_a^\nu + \chi_{\mathcal{A}_k^{\nu,b}} u_b^\nu.$$

We only need the adjoint states to update our active sets, hence kinks and discontinuities in the control will not be accumulated during the algorithm. This is an advantage over the discrete version of the active-set method. However, the expressions arising in the Newton method are more complicated than the expressions in the active-set method.

8.4 Active-Set Method

In this section we want to introduce an active-set method which is equivalent to the semi-smooth Newton method. For additional information regarding active-set methods, we want to refer to [16, 54, 64, 65, 113] and the references therein.

Let us establish the relation between the semi-smooth Newton method and the active-set method. We consider the problem's first-order optimality conditions (8.25). Reformulating (8.25c) by applying the projection formula one has to solve systems of this type in the active-set method which is defined below.

Algorithm 8.3 Active-set method for problem (P^ν_{AL})

1: Set $k = 0$, choose $(y_0, u_0, p_0) \in Y \times L^2(\Omega)^N \times L^2(\Omega)^N$
2: **repeat**
3: Set $\mathcal{A}^{\nu,a}_k, \mathcal{A}^{\nu,b}_k, \mathcal{I}^\nu_k$ and $\mathcal{Y}_k$ as defined in (8.22)
4: Solve for $(y_{k+1}, u_{k+1}, p_{k+1}) \in Y \times L^2(\Omega)^N \times L^2(\Omega)^N$ by solving

$$-\Delta y_{k+1} = \sum_{\nu=1}^{N} u^\nu_{k+1} \quad \text{in } \Omega, \tag{8.28a}$$

$$-\Delta p^\nu_{k+1} = C^*_\nu \left(C_\nu y_{k+1} - y^\nu_d\right) + \chi_{\mathcal{Y}_k}(\mu + \rho(y_{k+1} - \psi)) \quad \text{in } \Omega, \tag{8.28b}$$

$$u^\nu_{k+1} + \chi_{\mathcal{I}^\nu_k}\left(\frac{1}{\alpha} p^\nu_{k+1}\right) = \chi_{\mathcal{A}^{\nu,a}_k k} u^\nu_a + \chi_{\mathcal{A}^{\nu,b}_k} u^\nu_b \tag{8.28c}$$

5: Set $k = k + 1$
6: **until** $\mathcal{A}^{\nu,b}_k = \mathcal{A}^{\nu,b}_{k-1}, \mathcal{A}^{\nu,a}_k = \mathcal{A}^{\nu,a}_{k-1}, \mathcal{I}^\nu_k = \mathcal{I}^\nu_{k-1}$ and $\mathcal{Y}_k = \mathcal{Y}_{k-1}$.

First realized in [54], it is easy to see, that the semi-smooth Newton method from Algorithm 8.2 and the active set method from Algorithm 8.3 are equivalent. Let us also present a numerical implementation of the active-set method.

Lemma 8.15. *One step of the active-set method from Algorithm 8.3 can be computed by solving the system*

$$\begin{pmatrix} K & E_1 & 0 \\ E_2 & 0 & E_3 \\ 0 & E_4 & E_5 \end{pmatrix} \begin{pmatrix} \underline{y} \\ \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ l_1 \\ l_2 \end{pmatrix} \tag{8.29}$$

where $E_1 := \begin{pmatrix} -M & \cdots & -M \end{pmatrix} \in \mathbb{R}^{m \times Nm}$ *and*

$$E_2 := \begin{pmatrix} -M_{C_1^*} M_{C_1} - \rho M_{\mathcal{Y}_k} \\ \vdots \\ -M_{C_N^*} M_{C_N} - \rho M_{\mathcal{Y}_k} \end{pmatrix} \in \mathbb{R}^{Nm \times m}, \qquad E_3 := \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix} \in \mathbb{R}^{Nm \times Nm},$$

$$E_4 := \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix} \in \mathbb{R}^{Nm \times Nm}, \quad E_5 := \begin{pmatrix} \alpha^{-1} M_{\mathcal{I}^1_k} & & \\ & \ddots & \\ & & \alpha^{-1} M_{\mathcal{I}^N_k} \end{pmatrix} \in \mathbb{R}^{Nm \times Nm},$$

as well as

$$\underline{u} := \begin{pmatrix} \underline{u^1_{k+1,h}} \\ \vdots \\ \underline{u^N_{k+1,h}} \end{pmatrix} \in \mathbb{R}^{Nm}, \qquad \underline{y} := \underline{y_{k+1,h}} \in \mathbb{R}^m, \qquad \underline{p} := \begin{pmatrix} \underline{p^1_{k+1,h}} \\ \vdots \\ \underline{p^N_{k+1,h}} \end{pmatrix} \in \mathbb{R}^{Nm},$$

and right hand side

$$l_1 := \begin{pmatrix} -M_{C_1^*} \underline{y^1_d} + M_{\mathcal{Y}_k}(\underline{\mu} - \rho \underline{\psi}) \\ \vdots \\ -M_{C_N^*} \underline{y^N_d} + M_{\mathcal{Y}_k}(\underline{\mu} - \rho \underline{\psi}) \end{pmatrix}, \quad l_2 := \begin{pmatrix} M_{\mathcal{A}^{1,a}_k} \underline{u^1_a} + M_{\mathcal{A}^{1,b}_k} \underline{u^1_b} \\ \vdots \\ M_{\mathcal{A}^{N,a}_k} \underline{u^N_a} + M_{\mathcal{A}^{N,b}_k} \underline{u^N_b} \end{pmatrix}, \quad \mathbf{0} \in \mathbb{R}^m$$

with the notation used in Lemma 8.13 and $M \in \mathbb{R}^{m \times m}$, $(M)_{ij} := \left[\int_\Omega \phi_i \phi_j\right]_{ij}$.

Let us now compare the discrete Newton step (8.27) and the discrete active-set method (8.29). The entries on the diagonal of the matrix on the left hand side of (8.27) $E_{\nu,\nu}$ are symmetric. However, for $N > 1$ the resulting system is not symmetric. Note that the matrix in (8.27) should not be computed explicitly due to the appearance of K^{-1}. Still it is possible to compute its matrix-vector multiplication. This makes it impossible to apply a direct solver or a preconditioner which is based on decomposition, i.e., LU-factorisation. However, it can be solved by iterative methods, i.e., GMRES or BiCGSTAB. The resulting system for the active-set method (8.29) is not even for $N = 1$ symmetric, but it can be solved by a direct solver with a preconditioner, i.e. incomplete LU-factorisation.

8.5 Numerical Examples

The matrices are computed using the DOLFIN [87,88] Python interface, which is part of the open-source computing platform FEniCS [4,86]. The arising linear systems are solved with NumPy and SciPy. We used an 8× Intel® Core™i7-2600 CPU @ 3.40 Ghz and 7,7 GiB RAM.

8.5.1 Example 1 - Four Player Game

We consider a four player game like (P^{ν}_{AL}) on the domain $\Omega = (0,1)^2$ with observation domains

$$\Omega_1 := \left(0,\frac{1}{2}\right) \times \left(0,\frac{1}{2}\right), \quad \Omega_2 := \left(\frac{1}{2},1\right) \times \left(0,\frac{1}{2}\right),$$
$$\Omega_3 := \left(\frac{1}{2},1\right) \times \left(\frac{1}{2},1\right), \quad \Omega_4 := \left(0,\frac{1}{2}\right) \times \left(\frac{1}{2},1\right).$$

In this example $C_\nu := \chi_\nu$ and S is the solution mapping of the state equation $-\Delta y = \sum_{\nu=1}^{N} u^\nu$ with homogeneous Dirichlet boundary conditions. The desired states are given by constant functions

$$y_d^1 := 0, \;\; y_d^2 := 1, \; y_d^3 := 2, \; y_d^4 := 3$$

and we choose $\psi(x_1,x_2) := -2x_1 + 2x_2 + 2$, where $(x_1,x_2) \in \Omega$. For the approximation of the multiplier we set y_0, u_0, p_0 and μ equal zero as well as $\alpha := 10^{-5}$, and $\rho := 100$. Let us introduce the quantity

$$\kappa(u_k) := \log\left(\frac{\|u_{k+1} - u_k\|_U}{\|u_k - u_{k-1}\|_U}\right)\left(\log\left(\frac{\|u_k - u_{k-1}\|_U}{\|u_{k-1} - u_{k-2}\|_U}\right)\right)^{-1},$$

which is an approximation for the numerical order of convergence. If the sequence $(u_k)_k \subset U$ converges superlinear we expect $\kappa(u_k) \in (1,2)$ for k large enough. Note that we do not have an exact solution available to compute the order of convergence, but in practice $\kappa(u_k)$ will give a good approximation. We use a regular triangulation with different mesh sizes h. We applied both, the semi-smooth Newton method and the active-set method to this type of problem. The system that arises if the active-set method is applied has been solved directly by using the `spsolve` method from the `scipy.sparse.linalg` library. The Newton equation instead has to be solved by an iterative method. Here we make use of the `gmres` method from the same library and use a tolerance of 10^{-12}. Applying our estimates on α we obtain from (8.12) that $\alpha > 1.3863$ satisfies Assumption 8.3. Moreover, we have $c_P = 1/4$ and we obtain from (8.13) that is enough to choose $\alpha > 25/64 \approx 0.4$. However, the algorithm still works nice for $\alpha = 10^{-5}$. This could happen because 0.4 is just an approximation from above and Assumption 8.3 is just a sufficient condition. Hence, there might be much smaller α such that the problem is still uniquely solvable and the Newton method converges superlinear. Since both methods are equivalent it is not surprising that the approximated order of convergence κ and the change of the active sets coincide for both methods. Table 8.1 shows the computed results dependent on h for the active-set and the semi-smooth Newton method, respectively. Clearly, the computed orders of convergence support the superlinear convergence.

We are using linear finite elements for the controls, adjoints and state variable. Let us quickly comment on our stopping criterion from step 7 of Algorithm 8.2 or step 6 of Algorithm 8.3. Both algorithms stop when the active and inactive sets coincide. Due to the use of linear finite elements we compare the values on the nodes to check this condition. Let us illustrate this on the example of the set $\mathcal{A}_k^{\nu,a}$, which is defined by the inequality $\alpha^{-1} p_h^{\nu}(u_{k,h}) \leq u_a^{\nu}$. We now count all the nodes which lie in the symmetric difference of $\mathcal{A}_{k+1}^{\nu,a}$ and $\mathcal{A}_k^{\nu,a}$. If this returns zero, we conclude that $\mathcal{A}_{k+1}^{\nu,a} \approx \mathcal{A}_k^{\nu,a}$ holds good enough. We count these nodes for all the active and inactive sets in each iteration and sum them up. This calculation can be found in the row labeled "nodes".

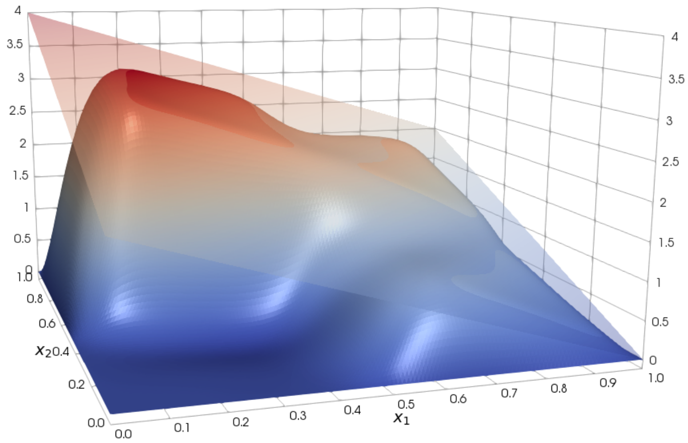


Figure 8.1: (Example 1) Computed state and state constraint (transparent).

	$h \approx 0.02$, dof $\approx 4.2 \cdot 10^3$					$h \approx 0.01$, dof $\approx 1.6 \cdot 10^4$				
	CPU_{AS}= 90.0s CPU_{N} = 807.6s					CPU_{AS} = 803.4s CPU_{N} =4149.0s				
k	$\kappa(u_k)$	nodes	opt AS	opt N	gmres	$\kappa(u_k)$	nodes	opt AS	opt N	gmres
1		1254	3.5e-14	6.0e-09	38		5042	3.5e-14	9.1e-09	36
2		307	5.1e-14	1.5e-06	93		1125	5.2e-14	1.4e-06	93
3		263	5.0e-14	1.3e-07	86		918	5.3e-14	1.2e-06	86
4	0.2037	177	5.3e-14	1.0e-06	79	0.1884	667	5.3e-14	1.1e-06	79
5	1.2962	124	5.2e-14	9.4e-07	72	1.3861	473	5.4e-14	7.5e-07	73
6	1.4221	75	5.1e-14	9.4e-07	66	1.2810	266	5.5e-14	7.2e-07	67
7	1.3647	27	5.4e-14	6.2e-07	62	1.4401	111	5.5e-14	6.1e-07	62
8	1.3660	7	5.3e-14	8.2e-07	57	1.5216	29	5.5e-14	8.2e-07	57
9	1.6705	2	5.5e-14	7.0e-06	52	1.6333	6	5.5e-14	8.2e-07	51
10	1.4939	0	5.2e-14	6.5e-07	44	1.6965	0	5.5e-14	7.8e-07	40

Table 8.1: (Example 1) Computed order of convergence $\kappa(u_k)$, change of nodes of the respective active sets, optimality of the problem, i.e., $\left\| u_k - P_{U_{\text{ad}}}(-\frac{1}{\alpha} p_k) \right\|_{L^2(\Omega)}$ for the Newton method (opt N) and the active-set method (opt AS) and number of GMRES iterations for solving the Newton system.

8.5.2 Example 2 - Four Player Game with Known Exact Solution

Next, we aim at solving (P_{AL}^{ν}) where S denotes the solution operator of

$$-\Delta y = \sum_{\nu=1}^{N} u^{\nu} + f \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega,$$

where f denotes a function in $L^2(\Omega)$. This setting differs slightly from the one presented above. However, it is easy to see that this does not have any impact on our convergence analysis. We investigate a four player game on the domain $\Omega = (-1,1)^2$. We set $C_\nu := \chi_\nu$, where

$$\begin{aligned} \Omega_1 &:= (-1,0) \times (-1,0)\,, & \Omega_2 &:= (0,1) \times (-1,0)\,, \\ \Omega_3 &:= (-1,0) \times (0,1)\,, & \Omega_4 &:= (0,1) \times (0,1)\,. \end{aligned}$$

With $(x_1, x_2) \in \Omega$, we set the optimal state

$$\bar{y}(x_1,x_2) := \begin{cases} 0 & \text{if } |x| < 0.2, \\ -192(|x|-0.2)^5 + 240(|x|-0.2)^4 - 80(|x|-0.2)^3 & \text{if } 0.2 < |x| < 0.7, \\ 1 & \text{if } |x| > 0.7. \end{cases}$$

With $\xi^1 := (0.5, -0.5, 0.5, -0.5)$ and $\xi^2 := (0.5, 0.5, -0.5, -0.5)$ we set

$$r_\nu := r_\nu(x_1,x_2) := \sqrt{(x_1 + \xi_\nu^1)^2 + (x_2 + \xi_\nu^2)^2}$$

and define for $\nu = 1, ..., N$ the optimal adjoint states via

$$\bar{p}^\nu := (-1)(-r_\nu^2 + 0.25)(16 r_\nu^4 - 8 r_\nu^2 + 1).$$

Choosing a regularization parameter α, we construct the optimal control via $\bar{u}^\nu := -(1/\alpha)\bar{p}^\nu$. Due to the construction of the adjoint states we obtain $\bar{u}^\nu = 0$ in $\Omega \backslash \Omega_\nu$ We set $f := -\Delta \bar{y} - \sum_{\nu=1}^{N} \bar{u}^\nu$ such that $\bar{y}$ and $\bar{u}^\nu$ satisfy the state equation. It remains to construct y_d^ν. Due to the adjoint equation we obtain

$$y_d^\nu := \begin{cases} \bar{y} + \Delta \bar{p}^\nu + (\mu + \rho(\bar{y} - \psi))_+ & \text{in } \Omega_\nu, \\ 0 & \text{else.} \end{cases}$$

For our numerical experiments we use $\rho := 100.0$, $\mu := 0$ and $\psi := 1.0$. In order to solve this problem we apply the active-set method using the initial values $(y_0, u_0, p_0) := (1,0,0)$. Due to the knowledge of the exact solution the rate R and order of convergence κ can be estimated via

$$\lim_{k\to\infty} \frac{\|u_{k+1} - \bar{u}\|_U}{\|u_k - \bar{u}\|_U} = R, \qquad \kappa^{ex}(u_k) = \left(\log \frac{\|u_{k+1} - \bar{u}\|_U}{\|u_k - \bar{u}\|_U}\right) \left(\log \frac{\|u_k - \bar{u}\|_U}{\|u_{k-1} - \bar{u}\|_U}\right)^{-1}.$$

We solved the problems for $h \approx 0.02$ which corresponds to approximately $4.2 \cdot 10^3$ degrees of freedom and used a tolerance of 10^{-8} for the `gmres` method. For determining the rate of convergence we compute in each iteration $R(u_k) := \frac{\|u_{k+1} - \bar{u}\|_U}{\|u_k - \bar{u}\|_U}$ and denote the corresponding value of the active-set method by $R_{AS}(u_k)$ and the one of the semi-smooth Newton method by $R_N(u_k)$. Let us check on the convergence properties corresponding to different regularization parameters α. Due to the discussion from Section 8.1.3.2 we know that $\alpha > 8.8025$ satisfies Assumption (8.3). However, also in this example (8.13) yields the better estimate. In this situation it is enough to choose $\alpha > 4$. The convergence rates from Table 8.2 confirm our theoretical finding. For $\alpha < 4$ we see that the the semi-smooth Newton method still behaves nicely for $\alpha = 1.0$. However, we do not obtain superlinear convergence for $\alpha = 0.1$ and $\alpha = 0.01$. This indicates that $\alpha = 1.0$ may still satisfy Assumption 8.3, while the other values of α may be too small. Finally, Figure 8.2 depicts the sum of the computed controls and the computed state for $\alpha = 5$.

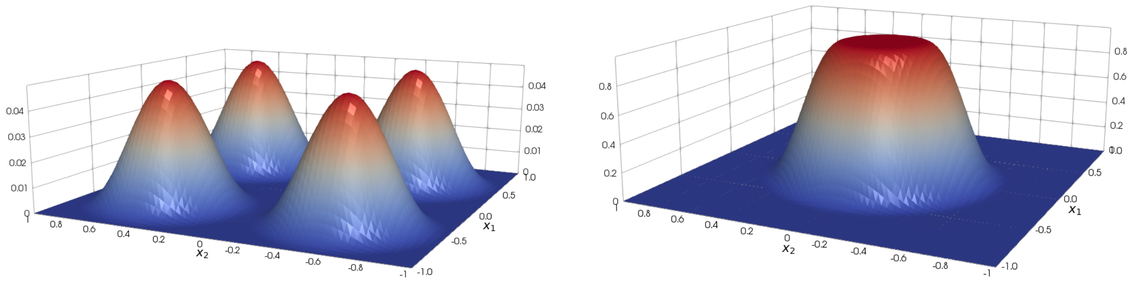


Figure 8.2: (Example 2) Computed sum of controls (left), computed state (right).

	$\alpha = 10.0$			$\alpha = 5.0$		
k	$R_{AS}(u_k)$	$\kappa^{ex}_{AS}(u_k)$	nodes AS	$R_{AS}(u_k)$	$\kappa^{ex}_{AS}(u_k)$	nodes N
1			641			717
2	0.0941		348	0.1256		368
3	0.0523		130	0.0930		156
4	0.1026	0.7718	163	0.0935	0.9980	193
5	0.1298	0.8968	0	0.0850	1.0401	0

	$\alpha = 1.0$			$\alpha = 0.1$		
k	$R_{AS}(u_k)$	$\kappa^{ex}_{AS}(u_k)$	nodes AS	$R_{AS}(u_k)$	$\kappa^{ex}_{AS}(u_k)$	nodes N
1			845			1380
2	0.1466		408	0.5747		844
3	0.1441		176	0.2075		305
4	0.1289	1.05748	80	0.2165	0.9731	140
5	0.2469	0.68273	116	0.1454	1.2599	80
6	0.5354	0.4466	65	0.2464	0.7266	48
7	0.5285	1.0210	0	0.8596	0.1080	89
8				0.9832	0.1123	48
9				0.9728	1.6223	4
10				1.0	0. 0	0

	$\alpha = 0.01$		
k	$R_{AS}(u_k)$	$\kappa_{AS}^{ex}(u_k)$	nodes AS
1			2044
2	0.5663		1008
3	0.7962		544
4	0.7469	1.28073	378
5	0.2210	5.17141	230
6	0.3985	0.60945	88
7	0.2734	1.40977	92
8	0.4582	0.60164	44
9	0.8973	0.13883	62
10	0.998	0.0185	42
11	1.002	-1.099	24
12	1.012	4.8085	4
13	1.0	0	0

Table 8.2: (Example 2) Computed rates $R_{AS}(u_k)$, order of convergence $\kappa_{AS}^{ex}(u_k)$ and change of nodes of the respective active sets for different values of α.

CHAPTER 9

CONCLUSION AND OUTLOOK

In this thesis we investigated an augmented Lagrangian method in order to solve state constrained optimal control problems governed by linear and semilinear partial differential equations. Let us summarize the main results of this thesis and discuss some possible topics for future research.

Optimal Control Problems

In the first part of this thesis an augmented Lagrangian algorithm has been applied to optimal control problems, where the objective function is given by

$$f(u) = \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + j(u).$$

Additional constraints consist of box constraints for the control and pointwise inequality constraints for the state.

While augmented Lagrangian methods are well-known in optimization, only a limited number of publications is devoted to the application of such methods to state constrained optimal control problems. We first focused on the convex case, i.e., S is a linear and continuous operator. To deal with problems of this type naturally requires that the Slater condition is satisfied.

In Chapter 3 we considered the case $j(u) := (\alpha/2)\|u\|^2_{L^2(\Omega)}$ and provided a full convergence analysis of the corresponding augmented Lagrangian algorithm, which relied on the Slater condition only, see Theorem 3.15. While augmented Lagrangian methods in the finite-dimensional setting do not require that the penalty parameter tends to infinity, we proved that the penalty parameters are in the infinite dimensional setting bounded only if there is a multiplier to the state constraint in $L^2(\Omega)$, which is not the case in general, see Theorem 3.18.

In Chapter 4, we chose $j(u) := \|u\|_{L^1(\Omega)}$, which made the problem under consideration ill-posed. We combined a Tikhonov regularization approach with the augmented Lagrangian method an coupled the corresponding regularization and penalization parameters in order to derive a convergence result, see Theorem 4.17 and Theorem 4.20. Let us emphasize that also for this class of problems the Slater condition has been the only assumption needed for carrying out our convergence analysis.

While convex optimal control problems have been studied extensively in the last years, the situation changes considerably for non-convex state constrained optimal control problems, i.e., S is a nonlinear operator. Since the convergence analysis of solution algorithms of non-convex optimization problems suffers significantly from non-uniqueness of local and global solutions, only few contributions can be found in the literature. In Chapter 5 we presented a detailed elaboration of the augmented Lagrangian method applied to state constrained optimal control problems governed by semilinear partial differential equations. It turned out that the crucial point here was to prove feasibility of weak limit points of the sequence that is generated by the algorithm, which can not be expected in general for augmented Lagrangian methods. However, we were able to argue that this property is obtained in different scenarios: Either the algorithm chooses the global solution of the augmented Lagrangian subproblem, or some kind of auxiliary problem that allows solutions that are located in an arbitrary small neighbourhood of a local solution of the original problem, or some kind of boundedness property of the Lagrange multiplier sequence is satisfied (Theorem 5.11). Moreover, the choice of global solutions of the auxiliary problem as iterates of the algorithm allowed us to prove that the sequence of multipliers is bounded, supposed the penalty parameter is bounded in $L^2(\Omega)$, see Theorem 5.28.

Multi-Player Control Problems

The second part of this thesis extended the results from the first part to multi-player optimal control problems.
As a start Chapter 6 investigated the augmented Lagrangian method applied to jointly convex multi-player optimal control problems. We benefited from the advantageous structure of this type of problem by elaborating a comparatively simple convergence analysis (Theorem 6.12 and Theorem 6.13), which included feasibility of weak limit points without any additional assumption besides the Slater condition.

The generalized Nash equilibrium problem that has been investigated in Chapter 7 can be reformulated as a QVI. Until now only few existence results can be found for QVIs. We contributed an existence result (Theorem 7.10) under a Slater-type constraint qualification, which also implied Mosco-continuity (Theorem 7.13). This constraint qualification has also been the main ingredient for the corresponding convergence analysis. To be more precise, the uniform Slater condition implies the same kind of boundedness condition on the Lagrange multiplier as needed in Chapter 5, see Lemma 7.20.

Chapter 8 has been devoted to the investigation of the uniqueness of normalized solutions of jointly convex generalized Nash equilibrium problems, which can not be expected in general. We contributed a new condition on the regularization parameter α, which ensured uniqueness of this type of equilibria, see Theorem 8.4. Moreover, we proved that the same condition implies superlinear convergence of the semi-smooth Newton applied to the augmented Lagrangian subproblem (Theorem 8.11), which is crucial for the numerical solution of GNEPs.

Outlook

Finally, let us report on some ideas concerning possible extensions and open questions of the augmented Lagrangian method that has been introduced in this thesis. First, in order to bridge the gap between finite and infinite dimensions, it would be favourable to modify the presented augmented Lagrangian scheme in such a way that we obtain a method where the penalization parameter does not need to go to infinity. This would in particular be advantageous for numerical computations, since numerical experiments often suffer from too large penalization parameters. Moreover, it would be interesting to investigate the non-differentiable optimal control problem with sparse controls from Chapter 4 for a nonlinear solution operator S. Combining the strategies that have been elaborated for semilinear state constrained optimal control problems in Chapter 5 with a Tikhonov regularization as in the spirit of Chapter 4, we would expect that the corresponding convergence analysis can readily be transferred to ill-posed state constrained optimal control problems. The extension to the nonlinear case would also be a challenging task for multi-player control problems. Last, it would be appealing to establish convergence rates for the augmented Lagrangian method presented in this thesis.

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