Received: 21 February 2019 | Revised: 14 September 2020 | Accepted: 21 September 2020

DOI: 10.1002/rnc.5273

RESEARCH ARTICLE

WILEY

Uniform bounded input bounded output stability of fractional-order delay nonlinear systems with input

R. Almeida[1] | S. Hristova[2] | S. Dashkovskiy[3]

[1]Department of Mathematics, Center for Research and Development in Mathematics ad Applications (CIDMA), University of Aveiro, Aveiro, Portugal

[2]University of Plovdiv "Paisii Hilendarski", Plovdiv, Bulgaria

[3]Institute of Mathematics, University of Würzburg, Würzburg, Germany

Correspondence
S. Dashkovskiy, Institute of Mathematics, University of Würzburg, Würzburg, Germany.
Email: sergey.dashkovskiy@uni-wuerzburg.de

Funding information
Center for Re- search and Development in Mathematics and Applications; Deutsche Forschungsgemeinschaft, Grant/Award Number: DA 767/12-1; Fundação para a Ciência e a Tecnologia, Grant/Award Number: UID/MAT/04106/2019

Summary

The bounded input bounded output (BIBO) stability for a nonlinear Caputo fractional system with time-varying bounded delay and nonlinear output is studied. Utilizing the Razumikhin method, Lyapunov functions and appropriate fractional derivatives of Lyapunov functions some new bounded input bounded output stability criteria are derived. Also, explicit and independent on the initial time bounds of the output are provided. Uniform BIBO stability and uniform BIBO stability with input threshold are studied. A numerical simulation is carried out to show the system's dynamic response, and demonstrate the effectiveness of our theoretical results.

KEYWORDS

bounded input bounded output stability, Caputo fractional derivative, Lyapunov functions, Razumikhin method, time-varying delay

1 | INTRODUCTION

In nonlinear systems control, the Lyapunov direct method provides a way to analyze the stability of the system without having to explicitly solve it. Fractional calculus helped modeling systems in a different way. Many of the real systems have fractional behavior, therefore they can be adequately described through fractional models (see viscoelastic polymers,[1] semi-infinite transmission lines with losses,[2] dielectric polarization,[3] etc). In engineering, for example, the digital fractional-order controller was designed to control temperature in Reference 4, the fractional-order PID controller was used in Reference 5 to control the trajectory of the flight path and stabilization of a fractional-order time delay nonlinear systems in Reference 6. For stability and stabilization of fractional-order systems we refer to the series of papers.[7-10]

Recently, Lyapunov stability theory and Razumikhin method were modified and applied to fractional systems with time dependent delays in References 11-13.

The analysis of bounded input bounded output (BIBO) stability of systems is very important for its possible application in many aspects such as single/double-loop modulators, or issues connected with bilinear input/output maps, and so forth. The BIBO stability for 2D discrete delayed systems is studied in Reference 14, for networked control systems with short time-varying delays in Reference 15, for retarded systems in Reference 16, for switched uncertain neutral systems with constant delay is considered in Reference 17, for perturbed interconnected power systems in Reference 18, for feedback control system with time delays,[2] and for Lurie system with time-varying delay in Reference 19. Recently, input-to-state stability was extended to Caputo fractional models in Reference 20, robust stability to uncertain multiorder fractional systems in Reference 21 and BIBO stability to fractional-order controlled nonlinear systems in References 22,23.

In this article we study a nonlinear Caputo fractional system with bounded time-varying delay and nonlinear output. Regarding the significant dependence of the Caputo fractional derivative on the initial time, we define uniform bounded input-bounded output (UBIBO) stability and UBIBO with input threshold. In the case of UBIBO with input threshold, a special number provides a threshold so that, for any input with the norm below this threshold, the UBIBO estimate holds. We apply quadratic Lyapunov functions and their fractional derivatives among the given system to study the stability properties as in Reference 23. But the presence of the delay into the system requires the application of the fractional modification of Razumikhin method and so called Razumikhin condition. Note that the direct Lyapunov method is not applicable for systems with any types of delays. Also, we study very general case of bounded delay which includes the case of constant delays[17] and variable delays.[2,19] Several types of sufficient conditions for UBIBO are obtained. Explicit bounds of the output are given. All bounds depend on the fractional order and on the bound of the input.

2 | STATEMENT OF THE PROBLEM

In what follows, we denote by t_0 the initial time. The physical meaning of the independent variable t is time in differential equations, so we will assume $t_0 \in \mathbb{R}_+ = [0, \infty)$.

In this article we will use *Caputo fractional derivative* of order $q \in (0, 1)$[24,25]

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \ge t_0, $$

where $m \in C^1([t_0, \infty), \mathbb{R})$ and $\Gamma(.)$ is the Gamma function.

The fractional derivatives for scalar functions could be easily generalized to the vector case, by taking fractional derivatives with the same fractional order for all components.

We will use the following norm

$$ \|x\|_n = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}, $$

with $x = (x_1, x_2, \dots, x_n)$. For any m-dimensional function $u \in L_\infty$, we use the supremum norm

$$ \|u\|_\infty = \sup_{t \ge 0} \|u(t)\|_m, $$

and we consider the Frobenius norm

$$ \|A\|_{n\times m} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2} $$

for a matrix $A \in \mathbb{R}^{n\times m}$. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimal and the maximal eigenvalue of matrix A, respectively.

Let $r > 0$ be a given number and let $C_0 = \{u \in C([-r, 0], \mathbb{R}^n)\}$, with a norm

$$ \|u\|_0 = \max_{t \in [-r,0]} \|u(t)\|_n, $$

and $BC_0(\beta) = \{u \in C([-r, 0], \mathbb{R}^n) : \|u\|_0 \le \beta\}$, where β is a positive real number.

Consider the following nonlinear Caputo fractional delay differential equation with input (FrDDEI) and fractional-order $q \in (0, 1)$:

$$ \begin{aligned} {}_{t_0}^{C}D_t^q x(t) &= \mathcal{F}(t, x(t), x_t, u(t)), \quad \text{for } t > t_0, \\ y(t) &= F(t, x(t)) + h(t, x(t))u(t), \quad \text{for } t > t_0, \end{aligned} \tag{1} $$

where

1. $x \in C^1([0,\infty),\mathbb{R}^n)$ and $x_t(s)=x(t+s),\ s\in[-r,0]$;
2. $\mathcal{F}:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, $F:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, and $h:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$;
3. $u:[0,\infty)\to\mathbb{R}^m,\ u\in L_\infty$, is the input and $y:[0,\infty)\to\mathbb{R}^n$ is the output of the system.

Remark 1. Note that if $x\in C([0,\infty),\mathbb{R}^n)$ then for any fixed $t\geq 0$ the function $x_t\in C_0$.

Remark 2. Note that, for a fixed initial time t_0, the functions in the right side parts of (1) are necessarily to be defined only for $t\geq t_0$, but in connection with the following study of the stability properties, we will assume that all of them are defined for $t\geq 0$.

We will study the following special cases of (1):

Case 1: Function $\mathcal{F}$ is given by

$$\mathcal{F}(t,x(t),x_t,u(t))=f(t,x(t))+G(t,x_t)+g(t,x_t)u(t),$$

where $f,G:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $g:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, that is, the system is

$$\begin{aligned}&{}_{t_0}^{C}D_t^q x(t)=f(t,x(t))+G(t,x_t)+g(t,x_t)u(t),\\&y(t)=F(t,x(t))+h(t,x(t))u(t),\quad \text{for } t>t_0;\end{aligned}\tag{2}$$

Case 2: Function $\mathcal{F}$ is given by

$$\mathcal{F}(t,x(t),x_t,u(t))=Ax(t)+H(x(t))+G(t,x_t)+g(t,x_t)u(t),$$

where A is $n\times n$ dimensional matrix, $H:\mathbb{R}^n\to\mathbb{R}^n$, $G:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $g:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, that is,

$$\begin{aligned}&{}_{t_0}^{C}D_t^q x(t)=Ax(t)+H(x(t))+G(t,x_t)+g(t,x_t)u(t),\\&y(t)=F(t,x(t))+h(t,x(t))u(t),\quad \text{for } t>t_0.\end{aligned}\tag{3}$$

For the FrDDEI (2) and (3), we consider the initial condition

$$x(t+t_0)=\phi_0(t),\quad t\in[-r,0],\tag{4}$$

where $\phi_0\in C_0$.

We denote the solution of the initial value problem (IVP) (2) and (4) (respectively, (3) and (4)) by (x,y), with $x(t)=x(t;t_0,\phi_0,u)$ and $y(t)=y(t;t_0,\phi_0,u)$, for $t\geq t_0-r$.

Remark 3. We will assume that, for any initial function $\phi_0\in C_0$, for any initial time $t_0\geq 0$, and for any input $u\in L_\infty$, the corresponding IVP for FrDDEI (2) and (4) (respectively, (3) and (4)) has a solution. For existence and uniqueness results to Caputo fractional differential equations with delay, see, for example, References 26-28.

Next, we will give the definition of bounded input bounded output (BIBO) stability to FrDDEI (2) and FrDDEI (3). Note that Caputo fractional derivatives depend significantly on the initial time, making the behavior of the solutions different than the case of ordinary derivatives. This requires slight changes in the classical definitions of BIBO stability. Motivated by the definition of BIBO stability presented and studied in References 2,14, we introduce the next definition.

Definition 1. The FrDDEI (2) (respectively, FrDDEI (3)) is said to be

- *uniformly bounded input bounded output (UBIBO) stable with input threshold* γ_u, if there exist positive constants α_1,β such that, for any initial time $t_0\geq 0$, any initial function $\phi_0\in BC_0(\alpha_1)$, and any input $u\in L_\infty$ with $\|u\|_\infty\leq\gamma_u$, the corresponding output y satisfies $\|y(t)\|\leq\beta$, for all $t\geq t_0$;

– uniformly bounded input bounded output (UBIBO) stable if, for any initial time t_0, any positive constants α_1, α_2, any initial function $\phi_0 \in BC_0(\alpha_1)$, and any input $u \in L_\infty$ with $\|u\|_\infty < \alpha_2$, the corresponding output y is bounded, that is, there exists a positive constant $\beta = \beta(\alpha_1, \alpha_2) > 0$ such that $\|y(t)\| \leq \beta$, for all $t \geq t_0$.

Remark 4. The bounds α_1, α_2 in the definition for UBIBO stability are arbitrary and independent on the initial time t_0. Also, the constant β depends on the chosen bounds α_1, α_2 and it is increasing w.r.t. both.

Remark 5. One of the main properties of Caputo fractional derivative is its significant dependence on the initial time t_0. The change of the initial time points leads to a change not only in the initial condition but also in the differential equations, therefore in the solution. Uniform BIBO defined in Definition 1 provides bounds for the solutions of equations with different fractional derivatives.

Remark 6. In the definition for UBIBO with input threshold, if the input is larger than the threshold, then the estimate for the output is not guaranteed.

Note that sometimes BIBO stability is called externally input-output stability.

One approach to study BIBO stability properties of nonlinear Caputo fractional differential equations with input is based on using Lyapunov functions. Note that there are two basic approaches in applying Lyapunov method for studying stability properties. One is the Krasovskii method based on the application of Lyapunov functionals, the other is the Razumikhin method based on the application of Lyapunov functions. In this article we will use the second one. In connection with this we need an appropriate definition of its derivative among the studied fractional differential equation.

3 | LYAPUNOV FUNCTIONS AND COMPARISON RESULTS FOR CAPUTO FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

In this section, we will give some known results concerning the application of Lyapunov functions to the following IVP for the Caputo fractional delay differential equation (FrDDE)

$$\begin{aligned} &{}_{t_0}^{C}D^q x(t) = \mathcal{G}(t, x_t), \quad \text{for } t \in (t_0, T], \\ &x(t_0 + s) = \phi(s), \quad s \in [-r, 0], \end{aligned} \tag{5}$$

where $x \in C^1([t_0, \infty), \mathbb{R}^n)$, $\phi \in C_0$, and $\mathcal{G} : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$.

Note that $T \leq \infty$ and, in the case $T = \infty$, the interval $J = [t_0 - r, T]$ is half-open.

Definition 2. We will say that function $V : J \times \Delta \to \mathbb{R}_+$, $\Delta \subset \mathbb{R}^n$, belongs to the class $\Lambda(J, \Delta)$ if V is continuous on $J \times \Delta$ and it is locally Lipschitzian with respect to its second argument.

Let $x(t) \in \Delta$, $t \in [t_0, T)$, be a solution of the FrDDE (5). We will use *Caputo fractional derivative* of Lyapunov function $V \in \Lambda(J, \Delta)$ among FrDDE (5) defined by

$$ {}_{t_0}^{c}D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \frac{d}{ds}(V(s, x(s)))ds, \quad t \in (t_0, T]. \tag{6}$$

In our study we will use the Razumikhin condition for the Lyapunov function $V \in \Lambda(J, \Delta)$ and any $\psi \in C([-\tau, 0], \mathbb{R}^n)$:

$$V(t + \Theta, \psi(\Theta)) \leq V(t, \psi(0)), \quad \Theta \in [-r, 0].$$

Lemma 1 (Comparison result). *Assume that:*

1. *The function $x = x(t; t_0, \phi) \in C^1([t_0, T], \Delta)$ is a solution of the IVP for FrDDE (5).*
2. *The function $V \in \Lambda([t_0 - r, T], \Delta)$, $\Delta \subset \mathbb{R}^n$ and for any point $t \in [t_0, T]$, such that*

$$V(t + s, x(t + s)) < V(t, x(t)), \quad s \in [-r, 0),$$

the fractional derivative ${}^{c}_{t_0}D^q V(t,x(t))$ *exists, and the inequality*

$$ {}^{c}_{t_0}D^q V(t,x(t)) \leq -\alpha V(t,x(t)) + \xi \tag{7} $$

holds, where $\alpha > 0$, $\xi \geq 0$ *are constants.*

Then,

$$ V(t,x(t)) \leq \max_{s\in[-r,0]} V(t_0+s,\phi(s)) + \frac{\xi}{\alpha}, $$

for all $t \in [t_0, T]$.

Proof. Denote

$$ B = \max_{s\in[-r,0]} V(t_0+s,\phi(s)) + \frac{\xi}{\alpha}, $$

and $m(t) = V(t,x(t))$, for $t \in [t_0 - r, T]$. Let $\varepsilon > 0$ be an arbitrary number. We will prove that

$$ m(t) < B + \varepsilon, \quad t \in [t_0, T]. \tag{8} $$

Let us assume that inequality (8) does not hold. Since

$$ m(t_0+s) = V(t_0+s,\phi(s)) \leq \max_{s\in[-r,0]} V(t_0+s,\phi(s)) \leq \max_{s\in[-r,0]} V(t_0+s,\phi(s)) + \frac{\xi}{\alpha} = B < B+\varepsilon, $$

there exists a point $t^* \in (t_0, T)$ such that

$$ m(t) < B+\varepsilon, \ t \in [t_0 - r, t^*), \quad \text{and} \quad m(t^*) = B + \varepsilon. \tag{9} $$

From inequality (9) and the definition of function $m(t)$, it follows that $V(t^*, x(t^*)) > V(t, x(t))$, for all $t \in [t^* - r, t^*)$. According to condition 2, the fractional derivative ${}^{c}_{t_0}D^q V(t^*, x(t^*))$ exists. Therefore, the fractional derivative ${}^{c}_{t_0}D^q m(t^*)$ exists and ${}^{c}_{t_0}D^q m(t^*) = {}^{c}_{t_0}D^q (m(t^*) - B - \varepsilon)$. Then

$$ {}^{c}_{t_0}D^q (m(t^*) - B - \varepsilon) > 0 \quad \text{or} \quad {}^{c}_{t_0}D^q m(t^*) > 0. $$

Therefore, $V(t^*, x(t^*)) > V(t, x(t))$, for all $t \in [t^* - r, t^*)$. According to condition 2, we get

$$ \begin{aligned} {}^{c}_{t_0}D^q V(t^*,x(t^*)) &\leq -\alpha V(t^*,x(t^*)) + \xi = -\alpha(B+\varepsilon) + \xi \\ &= -\alpha \left(\max_{s\in[-r,0]} V(t_0+s,\phi(s) \right) + \frac{\xi}{\alpha} + \varepsilon) + \xi \\ &= -\alpha \max_{s\in[-r,0]} V(t_0+s,\phi(s)) - \alpha\varepsilon < 0. \end{aligned} \tag{10} $$

The obtained contradiction proves the claim of Lemma 1. ■

We will also need the following result:

Lemma 2 *Reference 29, let* $P \in \mathbb{R}^{n\times n}$ *be a constant, symmetric and positive definite matrix and* $x : \mathbb{R}_+ \to \mathbb{R}^n$ *be a function for which Caputo derivative is defined. Then,*

$$ \frac{1}{2}\, {}_{0}^{C}D_t^q (x^T(t)Px(t)) \leq x^T(t)P\, {}_{0}^{C}D_t^q x(t), \quad \forall t \geq 0. $$

4 | BOUNDED INPUT BOUNDED OUTPUT STABILITY

4.1 | UBIBO stability of FrDDEI (2)

We will introduce the following assumptions:

Assumption (A1). There exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and constants $\gamma > 0, M \geq 0$, such that

$$x^T Pf^T(t,x) \leq -\gamma x^T x + M, \quad \text{for all } t \geq 0,\ x \in \mathbb{R}^n.$$

Assumption (A2). The function $G \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and there exists a constant $C > 0$ such that

$$\|G(t,x)\|_n \leq C, \quad \text{for all } t \geq 0,\ x \in \mathbb{R}^n.$$

Assumption (A3). The function $G \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and there exist a constant $L_G > 0$ such that

$$\|G(t,x)\|_n \leq L_G \|x\|_n, \quad \text{for all } t \geq 0,\ x \in \mathbb{R}^n.$$

Assumption (A4). There exist constants $a, b, L \geq 0$, $p_1 \geq 1$, and $0 < p_2 < 1$ such that

$$\|F(t,x)\|_n \leq a\|x\|_n^{p_1} + b\|x\|_n^{p_2} + L, \quad \text{for all } t \geq 0,\ x \in \mathbb{R}^n.$$

We will obtain several sufficient conditions depending on the type of the matrix function which are coefficients before the input vector in FrDDEI (2).

4.1.1 | Bounded coefficients function before the input

For this section, consider the following new assumption:

Assumption (A5). The matrix functions $g, h \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^{n\times m})$ are bounded, that is, there exist constants $K_1, K_2 \geq 0$ such that

$$\|g(t,x)\|_{n\times m} \leq K_1, \quad \|h(t,x)\|_{n\times m} \leq K_2 \quad \text{for all } t \geq 0,\ x \in \mathbb{R}^n.$$

In the case when both functions g, h in (2), as well as the delay term, are bounded, we obtain the following result:

Theorem 1. *Let the assumptions (A1), (A2), (A4), (A5) be satisfied. Then, the FrDDEI (2) is UBIBO stable.*

Proof. Let the positive constants α_1, α_2 be arbitrary fixed and the initial function $\phi_0 \in BC_0(\alpha_1)$, and the input $u \in L_\infty$ with $\|u\|_\infty < \alpha_2$. Let $t_0 > 0$ be an arbitrary point and consider the solution $(x(t), y(t))$, $t \geq t_0$, of the IVP for FrDDEI (2) and (4), with $x(t) = x(t; t_0, \phi_0, u)$, $y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

Consider the quadratic function $V(t,x) = x^T Px$ for $x \in \mathbb{R}^n$. Then $\lambda_{\min}(P)\, ||x||_n^2 \leq V(t,x) \leq \lambda_{\max}(P)\, ||x||_n^2$, $V(t,x) \leq ||P||_{n\times n}\, ||x||_n^2$, $||P||_{n\times n} \geq \lambda_{\max}(P)$ and for $\phi_0 \in BC_0(\mu)$ we have the inequality $\sup_{\Theta\in[-r,0]} V(t_0 + \Theta, \phi_0(\Theta)) \leq \lambda_{\max}(P)||\phi_0||_0$, $||x(t)||_n^2 \geq \frac{V(t,x(t))}{\lambda_{\max}(P)}$ and $||x(t)||_n^2 \leq \frac{V(t,x(t))}{\lambda_{\min}(P)}$ for $t \geq t_0$.

We will apply Lemma 1 to prove the claim. In connection with this we have to check condition 2 of Lemma 1.

Let the point $t \geq t_0$ be such that the Razumikhin condition be satisfied, that is, the inequality $V(t+\Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or $x(t+\Theta)^T Px(t+\Theta) \leq x(t)^T Px(t)$ for $\Theta \in [-r, 0]$. Then from Lemma 2 and inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$ we get for the above chosen point t :

$$\begin{aligned}
{}_{t_0}^{C}D_t^q V(t,x(t)) &\leq 2x^T(t)P\, {}_{t_0}^{C}D_t^q x(t) \\
&= 2x^T(t)Pf(t,x(t)) + 2x^T(t)PG(t,x_y) + 2x^T(t)Pg(t,x_t)u(t) \\
&\leq -2\gamma x^T(t)x(t) + 2M + 2(\sqrt{0.25\gamma}\|x(t)\|_n)\, \frac{C\|P\|_{n\times n}}{\sqrt{0.25\gamma}}
\end{aligned}$$

$$
\begin{aligned}
&+ 2(\sqrt{0.25\gamma}\|x(t)\|_n)\ \frac{\|P\|_{n\times n}\ K_1\ \|u(t)\|_m}{\sqrt{0.25\gamma}} \\
&\le -\gamma\|x(t)\|_n^2 + 2M + \frac{4C^2\|P\|_{n\times n}^2}{\gamma} + \frac{4\|P\|_{n\times n}^2\ K_1^2\ \|u\|_\infty^2}{\gamma} \\
&\le -\gamma\|x(t)\|_n^2 + 2M + \frac{4\|P\|_{n\times n}^2\ (C^2 + K_1^2\alpha_2^2)}{\gamma} \\
&\le -\frac{\gamma}{\lambda_{\max}(P)}V(t,x(t)) + 2M + \frac{4\|P\|_{n\times n}^2\ (C^2 + K_1^2\alpha_2^2)}{\gamma}.
\end{aligned}
\tag{11}
$$

According to the inequality (11) condition 2 of Lemma 1 is satisfied with the constants

$$
\alpha = \frac{\gamma}{\lambda_{\max}(P)}, \quad \xi = 2M + \frac{4\|P\|_{n\times n}^2\ (C^2 + K_1^2\alpha_2^2)}{\gamma}.
$$

According to Lemma 1 we obtain that for all $t \ge t_0$,

$$
\begin{aligned}
V(t,x(t)) &\le \sup_{\Theta\in[-r,0]} V(t_0+\Theta, x(\Theta)) + \frac{2M + \frac{4\|P\|_{n\times n}^2\ (C^2+K_1^2\alpha_2^2)}{\gamma}}{\frac{\gamma}{\lambda_{\max}(P)}} \\
&\le \lambda_{\max}(P)\|\phi_0\|_0 + \frac{\lambda_{\max}(P)\left(2M + \frac{4\|P\|_{n\times n}^2\ (C^2+K_1^2\alpha_2^2)}{\gamma}\right)}{\gamma}.
\end{aligned}
\tag{12}
$$

Therefore,

$$
\|x(t)\|_n \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\alpha_1 + \frac{2M + \frac{4\|P\|_{n\times n}^2\ (C^2+K_1^2\alpha_2^2)}{\gamma}}{\gamma}\right)},
\tag{13}
$$

and from assumptions (A4) and (A5), we obtain an upper bound for the output:

$$
\begin{aligned}
\|y(t)\|_n \le\ & a\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\alpha_1 + \frac{4\|P\|_{n\times n}^2K_1^2}{\gamma^2}\alpha_2^2 + \frac{2M\gamma + 4\|P\|_{n\times n}^2C^2}{\gamma^2}\right)}\right)^{p_1} \\
&+ b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\alpha_1 + \frac{4\|P\|_{n\times n}^2K_1^2}{\gamma^2}\alpha_2^2 + \frac{2M\gamma + 4\|P\|_{n\times n}^2C^2}{\gamma^2}\right)}\right)^{p_2} \\
&+ L + K_2\alpha_2, \quad \text{for } t \ge t_0.
\end{aligned}
\tag{14}
$$

Inequality (14) proves that the FrDDEI (2) is UBIBO stable. ■

Remark 7. Note that all conditions (A1)-(A5) do not depend on the delay. Therefore, the sufficient condition in Theorem 1 is true for any type of a bounded delay.

Example 1. Let $n = 1, m = 2$ and consider the scalar nonlinear FrDDEI

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^q x(t) = -x(t) + M + A_1\cos(x(t-\tau(t)))u_1(t) + A_2\sin(x(t-\tau(t)))u_2(t), \\
&y(t) = ax(t) + B_1\frac{x(t)}{1+|x(t)|}u_1(t) + B_2\frac{x(t)}{1+|x(t)|}u_2(t), \quad \text{for } t > t_0,
\end{aligned}
\tag{15}
$$

where $x \in C^1([0,\infty),\mathbb{R})$, $q \in (0,1)$, and $\tau \in C([0,\infty),[0,r])$ is the delay. Also, M, C, A_i, B_i are constants, for $i = 1, 2$, $u = (u_1, u_2)$ is the control input, and y is the output of the system. In this case, $b = 0, p_1 = 1, L = 0,\ P = 1,\ g(t,x) = (g_1, g_2)$, where

$$g_1(t,x_t) = A_1\cos(x(t-\tau(t))),\quad g_2(t,x_t) = A_2\sin(x(t-\tau(t))),$$

$h(t,x)=(h_1,h_2)$, where

$$h_1(t,x) = B_1\frac{x}{1+|x|},\quad h_2(t,x) = B_2\frac{x}{1+|x|}.$$

Note that

$$\|g(t,x)\|_2 \le \sqrt{A_1^2\cos^2(x)+A_2^2\sin^2(x)} \le K_1$$

and

$$\|h(t,x)\|_2 \le \sqrt{B_1^2+B_2^2} = K_2.$$

Then, according to Theorem 1, system (15) is UBIBO stable, that is, for any initial time t_0, any $\phi_0 \in BC_0(\alpha_1)$ and any input $u\in L_\infty$ with $\|u\|_\infty \le \alpha_2$ the corresponding output y is bounded by

$$|y(t)| \le a\sqrt{\alpha_1+\frac{2M+\frac{K_1^2\alpha_2^2}{\gamma}}{\gamma}} + \sqrt{B_1^2+B_2^2}\,\alpha_2. \tag{16}$$

Note that the fractional equation and the fractional derivative depend significantly on the initial time t_0 but the bound (16) for the output depends only on the bound of the initial function and on the bound of the input. It does not depend on the initial time.

Consider the particular case of the FrDDEI (15):

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{0.5}x(t) = -x(t)+1+\cos\left(x\left(t-\frac{2}{t+1}\right)\right)u_1(t) - 2\sin\left(x\left(t-\frac{2}{t+1}\right)\right)u_2(t),\quad t\ge t_0,\\
&x(t+t_0) = -e^t,\quad t\in[-2,0],\\
&y(t) = 2x(t) - 3\frac{x(t)}{1+|x(t)|}u_1(t) + 4\frac{x(t)}{1+|x(t)|}u_2(t),\quad \text{for } t>t_0,
\end{aligned} \tag{17}$$

that is, $q=0.5, M=1, a=2, A_1=1, A_2=-2, B_1=-3, B_2=4, P=1$, and $\tau\in C([0,\infty),[0,2])$ is given by $\tau(t)=\frac{2}{t+1}$. Then,

$$\sqrt{(\cos(x))^2+(-2\sin(x))^2} \le 2 = K_1 \quad\text{and}\quad K_2 = 5.$$

Case 1. Choose the bounds $\alpha_1 = 1$, $\alpha_2 = 0.6$. Then, according to (16), the output is bounded:

$$|y(t)| \le 2\sqrt{1+(2+4\,(0.6)^2)}+3 \approx 8.32813.$$

First, let $u=(u_1,u_2)$ with $u_1(t)=\frac{t}{1+e^t}$, $u_2(t)=\frac{t}{1+t^2}$, and $\phi(t)=-e^t$. Then,

$$\|u(t)\| = \sqrt{\left(\frac{t}{1+e^t}\right)^2+\left(\frac{t}{1+t^2}\right)^2} \le \alpha_2 = 0.6$$

and

$$|\phi(t)| = |-e^t| \le \alpha_1 = 1.$$

Denote the corresponding output by y_1.

Second, let $u=(u_1,u_2)$ with $u_1(t)=\frac{t}{1+t}$ and $u_2(t)=0.4\cos(t)$. Then,

$$\|u(t)\| \le \alpha_2 = 0.6,$$

and denote the corresponding output by y_2.

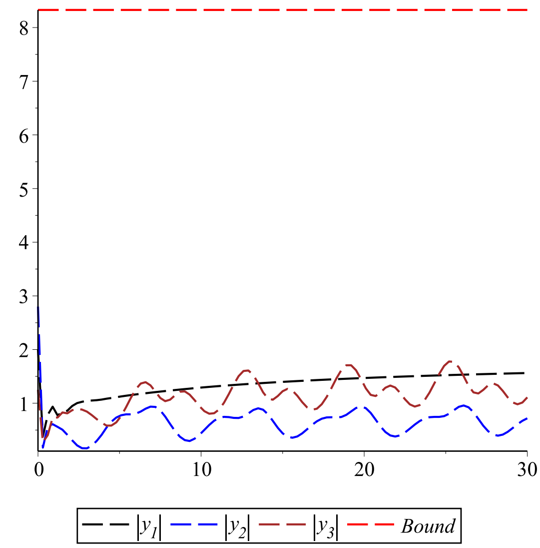

FIGURE 1 The graphs of the outputs $|y_1|, |y_2|, |y_3|$ for $t_0 = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

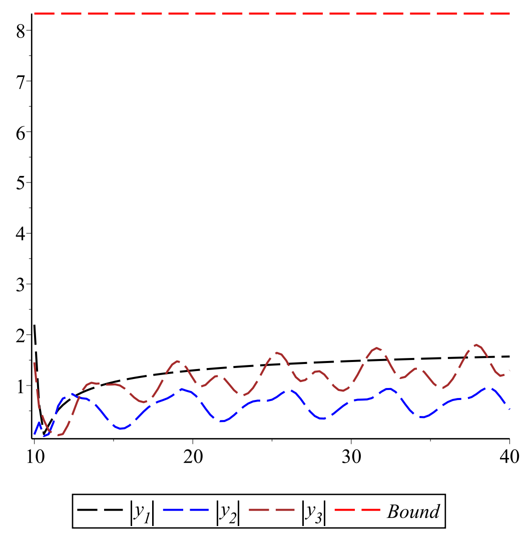

FIGURE 2 The graphs of the outputs $|y_1|, |y_2|, |y_3|$ for $t_0 = 10$ [Colour figure can be viewed at wileyonlinelibrary.com]

Finally, let $u = (u_1, u_2)$ with $u_1(t) = 0.5e^{-t}$ and $u_2(t) = 0.5\sin(t)$. Then,

$$\|u(t)\| \leq \alpha_2 = 0.6,$$

and denote the corresponding output by y_3.

The graphs of the absolute values of the outputs y_1, y_2, y_3, for the initial times $t_0 = 0$ and also $t_0 = 10$, are given on Figures 1 and 2, respectively. They show that, regardless the initial time, if the input is bounded, then the corresponding output is also bounded and this bound does not depend on the initial time.

Case 2. Let the initial function in FrDDEI (17) be changed by $\phi(t) = \cos(4t)$. As in Case 1, we consider the outputs y_1, y_2, y_3 corresponding to the three different bounded inputs. Figure 3 shows that the type of the bounded initial function does not change the boundedness of the output.

Case 3. Consider FrDDEI (17) with unbounded input $u_1(t) = e^t$ and $u_2(t) = t^4$, and two different initial functions, $\phi(t) = \cos(4t)$ and $\phi(t) = e^t$, with corresponding outputs y_1 and y_2, respectively. The graphs of both outputs y_1, y_2 on Figure 4 show that unbounded inputs does not guarantee bounded outputs.

In the case when the delay term has a linear bound, we obtain the following result:

Theorem 2. *Let the assumptions (A1), (A3), (A4), (A5) be satisfied with a Lipschitz constant*

$$L_G < \frac{\gamma}{\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)}.$$

Then, the FrDDEI (2) is UBIBO stable.

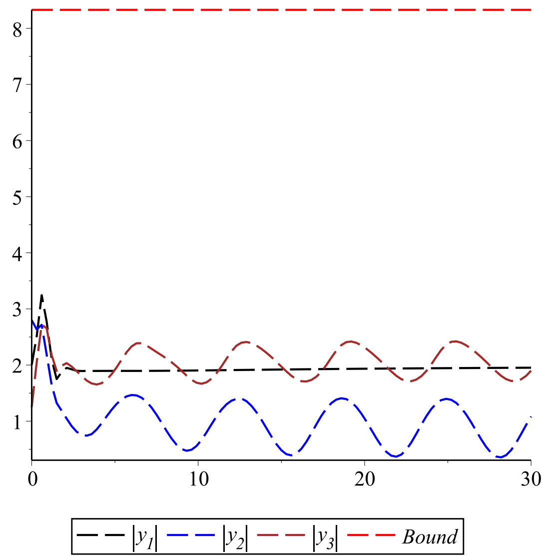

FIGURE 3 The graphs of the outputs $|y_1|, |y_2|, |y_3|$ with $\phi(t) = \cos(4t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

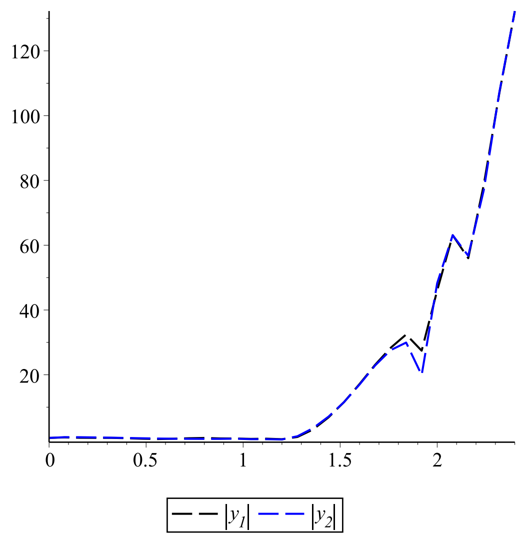

FIGURE 4 The graphs of the outputs $|y_1|, |y_2|$ [Colour figure can be viewed at wileyonlinelibrary.com]

Proof. Let the positive constants α_1, α_2 be arbitrary fixed and the initial function $\phi_0 \in BC_0(\alpha_1)$, and the input $u \in L_\infty$ with $\|u\|_\infty < \alpha_2$. Let $t_0 > 0$ be an arbitrary point and consider the solution $(x(t), y(t)),\ t \geq t_0$, of the IVP for FrDDEI (2) and (4), with $x(t) = x(t; t_0, \phi_0, u), y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

Consider the quadratic function $V(t,x) = x^T P x$. Then, for any $x \in \mathbb{R}^n$ we have

$$\lambda_{\min}(P)\ \|x\|_n^2 \leq V(t,x) \leq \lambda_{\max}(P)\ \|x\|_n^2,$$

$$V(t,x) \leq \|P\|_{n\times n}\ \|x\|_n^2, \quad \|P\|_{n\times n} \geq \lambda_{\max}(P).$$

We will use Lemma 1 to prove these inequalities. In connection with this we have to check condition 2 of Lemma 1.

Let the point $t \geq t_0$ be such that the Razumikhin condition be satisfied, that is, the inequality $V(t+\Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or $x(t+\Theta)^T P x(t+\Theta) \leq x(t)^T P x(t)$ for $\Theta \in [-r, 0]$. Therefore, for that t we have

$$\|x(t+\Theta)\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|x(t)\|^2, \quad \text{for } \Theta \in [-r, 0].$$

Apply Lemma 2 and the inequality $2ab \leq a^2 + b^2,\ a, b \in \mathbb{R}$ and we get for the above chosen t:

$$
\begin{aligned}
{}_{t_0}^{C}D_t^q V(t, x(t)) \leq\ & 2x^T(t)P\, {}_{t_0}^{C}D_t^q x(t) \\
=\ & 2x^T(t)Pf(t, x(t)) + 2x^T(t)PG(t, x_y) + 2x^T(t)Pg(t, x(t), x_t)u(t) \\
\leq\ & -2\gamma x^T(t)x(t) + 2M + 2L_G\|P\|_{n\times n}\|x(t)\|_n\|x_t\|_n \\
& + 2(\sqrt{\gamma}\|x(t)\|_n)\ \frac{\|P\|_{n\times n}\ K_1\ \|u(t)\|_m}{\sqrt{\gamma}} \\
\leq\ & -\gamma\|x(t)\|_n^2 + 2M + L_G\|P\|_{n\times n}\|x(t)\|_n^2 + L_G\|P\|_{n\times n}\|x_t\|_n^2 \\
& + \frac{\|P\|_{n\times n}^2\ K_1^2\ \|u\|_\infty^2}{\gamma} \\
\leq\ & -(\gamma - L_G\|P\|_{n\times n})\|x(t)\|_n^2 + 2M + L_G\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(t)\|^2 \\
& + \frac{\|P\|_{n\times n}^2\ K_1^2\ \|u\|_\infty^2}{\gamma} \\
\leq\ & -\left(\gamma - L_G\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)\right)\|x(t)\|_n^2 + 2M \\
& + \frac{\|P\|_{n\times n}^2\ K_1^2\ \|u\|_\infty^2}{\gamma}.
\end{aligned}
\tag{18}
$$

Similarly to the proof of Theorem 1 by the application of Lemma 1 , we get

$$
\|x(t)\|_n \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\|\phi_0\|_0 + \frac{2M + \frac{\|P\|_{n\times n}^2 K_1^2\|u(t)\|_\infty^2}{\gamma}}{\gamma - L_G\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)}\right)},
\tag{19}
$$

and for the output,

$$
\begin{aligned}
\|y(t)\|_n \leq\ & a\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\|\phi_0\|_0 + \frac{2M + \frac{\|P\|_{n\times n}^2 K_1^2}{\gamma}\|u(t)\|_\infty^2}{\gamma - L_G\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)}\right)}\right)^{p_1} \\
& + b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\|\phi_0\|_0 + \frac{2M + \frac{\|P\|_{n\times n}^2 K_1^2}{\gamma}\|u(t)\|_\infty^2}{\gamma - L_G\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)}\right)}\right)^{p_2} \\
& + L + K_2\|u(t)\|_\infty \quad \text{for } t \geq t_0.
\end{aligned}
\tag{20}
$$

■

4.1.2 | Linear estimate of the coefficient functions before the input

Let us introduce the following assumption:

Assumption (A6). The matrix functions $g, h \in C([0, \infty) \times \mathbb{R}^n, \mathbb{R}^{n\times m})$ and there exist constants $L, l \geq 0$ such that

$$
\|g(t, x)\|_{n\times m} \leq L\|x\|_n \quad \text{and} \quad \|h(t, x)\|_{n\times m} \leq l\|x\|_n,
$$

for all $t \geq 0,\ x \in \mathbb{R}^n$.

In the case when the coefficient functions before the input are linearly bounded and the delay term is bounded, we obtain the following result:

Theorem 3. *Let the assumptions (A1), (A2), (A4), (A6) be satisfied. Then, the FrDDEI (2) is UBIBO stable with input threshold*

$$\gamma_u = \frac{\gamma \lambda_{\min}(P)}{2L\|P\|_{n\times n}\lambda_{\max}(P)}.$$

Proof. Let $t_0 > 0$ be an arbitrary point and the input $u \in L_\infty$ be such that $||u||_\infty < \gamma_u$. Consider the solution $(x(t), y(t))$, $t \geq t_0$, of the IVP for FrDDEI (2), (4) for this input and with $x(t) = x(t; t_0, \phi_0, u)$ and $y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

Consider the quadratic functions $V(t, x) = x^T Px,\ x \in \mathbb{R}^n$.

Again we will apply Lemma 1 to prove the claim. In connection with this we have to check condition 2 of Lemma 1

Let the point $t \geq t_0$ be such that the Razumikhin condition be satisfied, that is, the inequality $V(t+\Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or $x(t+\Theta)^T Px(t+\Theta) \leq x(t)^T Px(t)$ for $\Theta \in [-r, 0]$. By the application of Lemma 2 we obtain for the chosen above point t:

$$\begin{aligned}
{}_{t_0}^{C}D_t^q V(t, x(t)) &\leq -2\gamma x^T(t)x(t) + 2M + 2C\|x(t)\|_n\ \|P\|_{n\times n} \\
&\quad + 2\|x(t)\|_n\ \|P\|_{n\times n}\ \|g(t, x_t)\|_{n\times m}\ \|u(t)\|_m \\
&\leq -2\gamma\|x(t)\|_n^2 + 2M + \sqrt{\gamma}\|x(t)\|_n\ \frac{C\|P\|_{n\times n}}{\sqrt{\gamma}} \\
&\quad + L\|u\|_\infty\ \|P\|_{n\times n}\ 2(\|x(t)\|_n\ \|x_t\|_n) \\
&\leq -\gamma\|x(t)\|_n^2 + 2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma} \\
&\quad + L\gamma_u\|P\|_{n\times n}\ \|x(t)\|_n^2 + L \qquad (21)\\
gg_u\|P\|_{n\times n}\ \|x_t\|_n^2 &\leq -\gamma\|x(t)\|_n^2 + 2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma} + L\gamma_u\|P\|_{n\times n}\|x(t)\|_n^2 \\
&\quad + L\gamma_u\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(t)\|_n^2 \\
&\leq -\left(\gamma - 2L\gamma_u\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)\ \|x(t)\|_n^2 + 2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma} \\
&\leq -\left(\frac{\gamma}{\lambda_{\max}(P)} - 2\frac{L\gamma_u\|P\|_{n\times n}}{\lambda_{\min}(P)}\right) V(t, x(t)) + 2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma}.
\end{aligned}$$

Using inequality (21), together with Lemma 1, considering

$$\alpha = \left(\frac{\gamma}{\lambda_{\max}(P)} - 2\frac{L\gamma_u\|P\|_{n\times n}}{\lambda_{\min}(P)}\right) \quad \text{and} \quad \xi = 2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma},$$

we get

$$\begin{aligned}
V(t, x(t)) &\leq \sup_{\Theta\in[-r,0]} V(t_0+\Theta, x(\Theta)) + \frac{2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma}}{\frac{\gamma}{\lambda_{\max}(P)} - \frac{2L\gamma_u\|P\|_{n\times n}}{\lambda_{\min}(P)}} \\
&= \lambda_{\max}(P)\|\phi_0\|_0 + \frac{2M + \frac{C^2\|P\|_{n\times n}^2}{\gamma}}{\frac{\gamma}{\lambda_{\max}(P)} - \frac{2L\gamma_u\|P\|_{n\times n}}{\lambda_{\min}(P)}}. \qquad (22)
\end{aligned}$$

Therefore,

$$\|x(t)\| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0 + \frac{2M + \frac{C^2\|P\|^2_{n\times n}}{\gamma}}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - 2L\gamma_u\|P\|_{n\times n}}}, \tag{23}$$

and, from assumption (A6), we get for the output

$$\begin{aligned}\|y(t)\|_n \le a&\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0 + \frac{2M + \frac{C^2\|P\|^2_{n\times n}}{\gamma}}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - 2L\|P\|_{n\times n}\gamma_u}}\right)^{p_1}\\ &+ b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0 + \frac{2M + \frac{C^2\|P\|^2_{n\times n}}{\gamma}}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - 2L\|P\|_{n\times n}}\gamma_u}\right)^{p_2} + L\\ &+ l\gamma_u\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0 + \frac{2M + \frac{C^2\|P\|^2_{n\times n}}{\gamma}}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - 2L\|P\|_{n\times n}\gamma_u}}, \quad t \ge t_0.\end{aligned} \tag{24}$$

Therefore, FrDDEI (2) is UBIBO stable with input threshold $\frac{\gamma\lambda_{\min}(P)}{2L\|P\|_{n\times n}\lambda_{\max}(P)}$. ■

Corollary 1. *Let the conditions (A1), (A2), (A4), (A6) be satisfied with $M = 0$. Then, the FrDDEI (2) is UBIBO and the output is bounded by*

$$\|y(t)\|_n \le a\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0}\right)^{p_1} + b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0}\right)^{p_2} + L + l\|u\|_\infty\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\phi_0\|_0}.$$

Example 2. Let $n = 3, m = 2$ and consider the IVP for the nonlinear FrDDEI

$$\begin{aligned}&{}_{t_0}^{C}D_t^q x(t) = f(x(t)) + g(t, x(t-\tau(t))u(t), \quad t > t_0,\\ &x(t+t_0) = \phi(t), \quad t \in [-r, 0],\\ &y(t) = Dx(t) + h(t, x(t))u(t), \quad \text{for } t > t_0,\end{aligned} \tag{25}$$

where $x = (x_1, x_2, x_3)^T, \phi = (\phi_1, \phi_2, \phi_3)^T$, function $f = (f_1, f_2, f_3)^T$ is defined as

$$f_1(x) = -x_1, f_2(x) = -1.5x_2, f_3(x) = -3x_3,$$

$D = diag(d),\ d > 0,\ \tau \in C(\mathbb{R}_+, [0, r])$ is the delay, $u = (u_1, u_2)^T$, matrices $g, h \in \mathbb{R}^{3\times 2}$ are given by

$$g(t,x) = \begin{bmatrix}\eta_{11}(t)x_1 & \eta_{12}(t)x_2\\ \eta_{21}(t)x_2 & \eta_{22}(t)x_3\\ \eta_{31}(t)x_3 & \eta_{32}(t)x_1\end{bmatrix}, \quad h(t,x) = \begin{bmatrix}h_{11}(t)x_1 & h_{12}(t)x_1\\ h_{21}(t)x_2 & h_{22}(t)x_2\\ h_{31}(t)x_3 & h_{32}(t)x_3\end{bmatrix}$$

and the matrices $\Lambda, H \in \mathbb{R}^{3\times 2}$

$$\Lambda(t) = \begin{bmatrix}\eta_{11}(t) & \eta_{12}(t)\\ \eta_{21}(t) & \eta_{22}(t)\\ \eta_{31}(t) & \eta_{32}(t)\end{bmatrix}, \quad H(t) = \begin{bmatrix}h_{11}(t) & h_{12}(t)\\ h_{21}(t) & h_{22}(t)\\ h_{31}(t) & h_{32}(t)\end{bmatrix}$$

are such that $\|\Lambda(t)\|_{3\times 2} \le \delta < \infty$ and $\|H(t)\|_{3\times 2} \le H < \infty$, for all $t \ge 0$.

Then, condition (A6) is satisfied with

$$L=\max_{t\geq 0}\{\eta_{11}^2(t)+\eta_{32}^2(t),\eta_{12}^2(t)+\eta_{21}^2(t),\eta_{22}^2(t)+\eta_{31}^2(t)\}$$

and

$$l=\max_{t\geq 0}\{h_{11}^2(t)+h_{12}^2(t),h_{21}^2(t)+h_{22}^2(t),h_{31}^2(t)+h_{32}^2(t)\}.$$

Also, assumption (A1) is satisfied with $\gamma=3$ and

$$P=\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix},$$

with eigenvalues $\lambda_{\max}=3$ and $\lambda_{\min}=1$, and $||P||_{3\times 3}=\sqrt{14}$.

According to Corollary 1, the scalar nonlinear FrDDEI (25) is UBIBO, that is, for any initial function

$$\sqrt{\phi_2^2(t)+\phi_3^2(t)+\phi_1^2(t)}\leq \alpha_1,\quad t\in[-r,0],$$

($\alpha_1>0$ is an arbitrary constant), any input

$$\sqrt{u_1^2(t)+u_2^2(t)}<\alpha_2<\frac{1}{L\sqrt{14}},\quad t\geq 0,$$

and any $t_0\geq 0$, the norm of the corresponding output $||y(t)||_3$ is bounded by

$$\beta=(c+l\alpha_2)\sqrt{3\alpha_1},$$

for $t\geq t_0$.

Consider the particular case of the system (25)

$$\begin{aligned}
&{}_0^CD_t^q x_1(t)=-x_1(t)+\sin(t)x_1\left(t-\frac{2}{t+1}\right)u_1(t)-\cos(t)x_2\left(t-\frac{2}{t+1}\right)u_2(t),\\
&{}_0^CD_t^q x_2(t)=-1.5x_2(t)+0.2\sin(t)x_2\left(t-\frac{2}{t+1}\right)u_1(t)+\cos(t)x_3\left(t-\frac{2}{t+1}\right)u_2(t),\\
&{}_0^CD_t^q x_3(t)=-3x_3(t)+0.2\sin(t)x_3\left(t-\frac{2}{t+1}\right)u_1(t)-0.2\cos(t)x_1\left(t-\frac{2}{t+1}\right)u_2(t),\\
&x_1(t)=-e^t,\quad x_2(t)=-\sin(t),\quad x_3(t)=-\cos(t),\quad t\in[-2,0],\\
&y(t)=2x(t)+h(t,x(t))u(t),\quad \text{for } t>0,
\end{aligned}\tag{26}$$

with the matrices $\Lambda,h\in\mathbb{R}^{3\times 2}$ given by

$$\Lambda(t)=\begin{bmatrix}\sin(t) & -\cos(t)\\ 0.2\sin(t) & \cos(t)\\ 0.2\sin(t) & -0.2\cos(t)\end{bmatrix},\quad h(t,x)=\begin{bmatrix}0.5e^{-t}x_1 & 0.5\sin(t)x_1\\ 0.5\cos(t)x_2 & 0.5\arctan(t)x_2\\ 0.5\cos(t)x_3 & \sin(t)x_3\end{bmatrix}.$$

Then $a=2$,

$$L=\max\{\sin^2(t)+0.04\cos^2(t),\cos^2(t)+0.04\sin^2(t),\cos^2(t)+0.04\sin^2(t)\}=1,$$

$\alpha_1=0.5$, $\alpha_2=0.26$, and $l=1.5$.

Case 1. Let us consider two different bounded inputs.

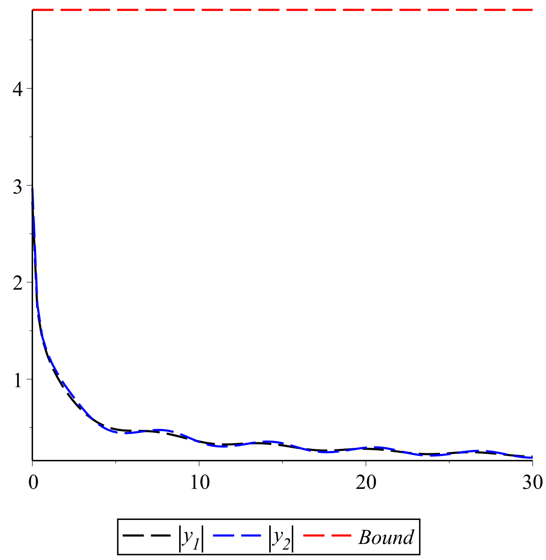

FIGURE 5 The graphs of the output $|y_1|$, $|y_2|$, and the bound β [Colour figure can be viewed at wileyonlinelibrary.com]

Case 1.1. Let the input

$$u_1(t) = \frac{t}{1+3e^t}, \quad u_2(t) = \frac{t}{1+4t}.$$

Then,

$$\|u(t)\| = \sqrt{\left(\frac{t}{1+3e^t}\right)^2 + \left(\frac{t}{1+4t}\right)^2} \leq \alpha_2 = \frac{1}{\sqrt{14}} \approx 0.267261.$$

Denote the output by $y_1 = (y_1^{(1)}, y_2^{(1)}, y_3^{(1)})$.

Case 1.2. Let the input

$$u_1(t) = \frac{1}{5+0.3\sin(t)}, \quad u_2(t) = \frac{1}{7+0.4\cos(t)}.$$

Then,

$$\|u(t)\| \leq \alpha_2 = \frac{1}{\sqrt{14}}.$$

Denote the output by $y_2 = (y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$.

The norm of the outputs $|y_1|$ and $|y_2|$, and the bound $\beta = (2 + \frac{1}{\sqrt{14}})\sqrt{4.5}$, are represented in Figure 5.

Case 2. Let the input $u_1(t) = 3e^{-t}$, $u_2(t) = 4\sin(t)$. Then, the input is not bounded by the threshold $\alpha_2 = \frac{1}{\sqrt{14}}$. In this case, the norm of the output $|y_1|$ is also not bounded by β, as it is shown in Figure 6.

In the case when the coefficient functions before the input and the delay term are linearly bounded, we obtain the following result:

Theorem 4. *Let the assumptions (A1), (A3), (A4), (A6) be satisfied, with Lipschitz constant such that*

$$\frac{L_G}{L} < \frac{\gamma \lambda_{\min}(P)}{L\|P\|_{n\times n}\lambda_{\max}(P)}.$$

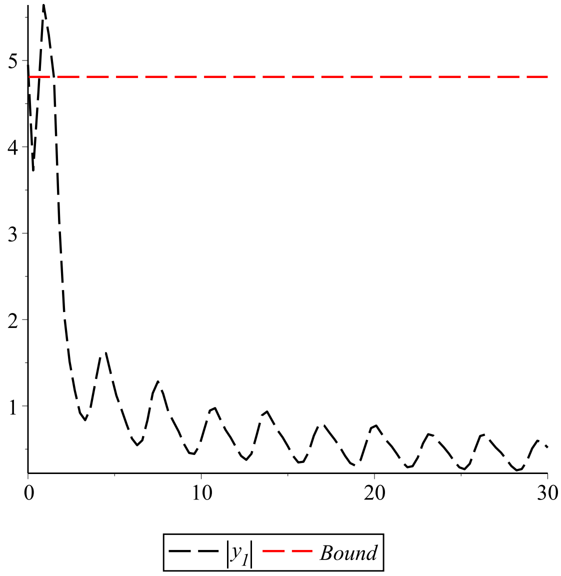

FIGURE 6 The graphs of the output $|y_1|$ and the bound β [Colour figure can be viewed at wileyonlinelibrary.com]

Then, the FrDDEI (2) is UBIBO stable with input threshold

$$\gamma_u = \frac{\gamma\lambda_{\min}(P)}{L\|P\|_{n\times n}\lambda_{\max}(P)} - \frac{L_G}{L}.$$

Proof. Let $t_0 > 0$ be an arbitrary point and the input $u \in L_\infty$ be such that $||u||_\infty < \gamma_u$. Consider the solution $(x(t), y(t))$ of the IVP for FrDDEI (2), (4) for this input.

Let $V(t,x) = x^T Px$. Let the point $t \geq t_0$ be such that the Razumikhin condition be satisfied, that is, the inequality $V(t + \Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or $x(t+\Theta)^T Px(t+\Theta) \leq x(t)^T Px(t)$ for $\Theta \in [-r, 0]$.

Then for this t by the application of Lemma 2 we get:

$$\begin{aligned}
{}_{t_0}^{C}D_t^q V(t,x(t)) &\leq -2\gamma x^T(t)x(t) + 2M + 2L_G\|x(t)\|_n\|x_t\|_n\ \|P\|_{n\times n} \\
&\quad + 2\|x(t)\|_n\ \|P\|_{n\times n}\ \|g(t,x_t)\|_{n\times m}\ \|u(t)\|_m \\
&\leq -2\gamma\|x(t)\|_n^2 + 2M + (L_G + L\|u\|_\infty)\ \|P\|_{n\times n}\ 2(\|x(t)\|_n\ \|x_t\|_n) \\
&\leq -2\gamma\|x(t)\|_n^2 + 2M + (L_G + L\gamma_u)\|P\|_{n\times n}\|x(t)\|_n^2 \\
&\quad + (L_G + L\gamma_u)\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(t)\|_n^2 \\
&\leq -2\gamma\|x(t)\|_n^2 + 2M + (L_G + L\gamma_u)\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\ \|x(t)\|_n^2 \\
&= -2\left(\gamma - (L_G + L\gamma_u)\|P\|_{n\times n}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)\ \|x(t)\|_n^2 + 2M \\
&\leq -2\left(\frac{\gamma}{\lambda_{\max}(P)} - \frac{(L_G + L\gamma_u)\|P\|_{n\times n}}{\lambda_{\min}(P)}\right)V(t,x(t)) + 2M.
\end{aligned} \tag{27}$$

Therefore, condition 2 of Lemma 1 is satisfied and similarly to the proof of Theorem 3, with

$$\alpha = 2\left(\frac{\gamma}{\lambda_{\max}(P)} - \frac{(L_G + L\gamma_u)\|P\|_{n\times n}}{\lambda_{\min}(P)}\right) \quad \text{and} \quad \xi = 2M,$$

we obtain

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha_1 + \frac{M}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - (L_G + L\gamma_u)\|P\|_{n\times n}}}, \tag{28}$$

and, from Assumption (A6), we get for the output

$$\begin{aligned}\|y(t)\|_n \le\ & a\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha_1 + \frac{M}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - (L_G + L\gamma_u)\|P\|_{n\times n}}}\right)^{p_1} \\ & + b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha_1 + \frac{M}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - (L_G + L\gamma_u)\|P\|_{n\times n}}}\right)^{p_2} + L \\ & + l\|u\|_\infty\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\alpha_1 + \frac{M}{\frac{\gamma\lambda_{\min}(P)}{\lambda_{\max}(P)} - (L_G + L\gamma_u)\|P\|_{n\times n}}}. \end{aligned} \tag{29}$$

■

Corollary 2. *Let the conditions (A1), (A3), (A4), (A6) be satisfied, with $M = 0$. Then, the FrDDEI (2) is UBIBO.*

4.2 | UBIBO stability of FrDDEI (3)

We will consider the following assumption:

Assumption (A7). There exist symmetric positive definite matrices $P, B \in \mathbb{R}^{n\times n}$ such that

$$\|H(x)\|_n \le x^T Bx, \quad \text{for } x \in \mathbb{R}^n,$$

and the $(2n)\times(2n)$ dimensional matrix

$$Q = \begin{bmatrix} A^T P + PA + B & P \\ 0 & -I \end{bmatrix}$$

has negative eigenvalues, where I is the $n \times n$ dimensional identity matrix.

Remark 8. The assumption (A7) was used in Reference 23 for a fractional system without delay.

We will obtain several sufficient conditions depending on the type of the matrix function, which is a coefficient before the input vector in FrDDEI (2).

4.2.1 | Bounded coefficients before the input

Introduce the assumption:

Assumption (A8). The function $G \in C([0,\infty)\times\mathbb{R}^n, \mathbb{R}^n)$ and there exist a constant K_3, with

$$0 < K_3 < -\lambda_{\max}(Q)\sqrt{a(1-2a)\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}},$$

where $a \in (0, 0.5)$ is a given real, such that

$$\|G(t,x)\|_n \le K_3\|x\|_n, \quad \text{for } t \ge 0,\ x \in \mathbb{R}^n,$$

where the numbers $\lambda_{\max}(Q), \lambda_{\min}(P), \lambda_{\max}(P)$ are the corresponding eigenvalues of the matrices defined in the assumption (A7).

In the case when both functions g, h in (3) are bounded, we obtain the following result:

Theorem 5. *Let the assumptions (A4), (A5), (A7), and (A8) be satisfied. Then, the system (3) is UBIBO stable.*

Proof. Let the positive constants α_1, α_2 be arbitrary fixed and the initial function $\phi_0 \in BC_0(\alpha_1)$, and the input $u \in L_\infty$ with $\|u\|_\infty < \alpha_2$. Let $t_0 > 0$ be an arbitrary point and consider the solution $(x(t), y(t))$, $t \geq t_0$, of the IVP for FrDDEI (3) and (4), with $x(t) = x(t; t_0, \phi_0, u)$, $y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

Then, we obtain

$$\begin{aligned} x^T(t)P(Ax(t) + H(x(t)) + (Ax(t) + H(x(t))^T Px(t) &\leq x^T(t)P(Ax(t) + H(x(t)) + x^T(t)Bx(t) \\ &\quad + (Ax(t) + H(x(t))^T Px(t) - H^T(x(t))H(x(t)) \\ &= u^T(t)Qv(t) \leq -\eta\|v(t)\|_{2n} \leq -\eta\|x(t)\|_n, \end{aligned} \tag{30}$$

where $\eta = -\lambda_{\max}(Q) > 0$ and $v \in R^{2n}: v = (x_1, x_2, \ldots, x_n, H_1(x), H_2(x), \ldots, H_n(x))$.

Consider the quadratic function $V(t, x) = x^T Px$, $x \in \mathbb{R}^n$, and let the point $t \geq t_0$ be such that the Razumikhin condition be satisfied, that is, the inequality $V(t + \Theta, x(t + \Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or

$$x(t + \Theta)^T Px(t + \Theta) \leq x(t)^T Px(t) \quad \text{for } \Theta \in [-r, 0].$$

Therefore,

$$\|x(t + \Theta)\|_n^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x(t)\|_n^2 \quad \text{for } \Theta \in [-r, 0]$$

and by Lemma 2 for the point t satisfying the above conditions we get

$$\begin{aligned} {}_{t_0}^{C}D_t^q V(t, x(t)) &\leq 2x^T(t)P\, {}_{t_0}^{C}D_t^q x(t) \\ &= 2x^T(t)P(Ax(t) + H(x(t)) + G(t, x_t)) + 2x^T(t)Pg(t, x_t)u(t) \\ &\leq -\eta\|x(t)\|_n^2 + 2K_3\|x(t)\|_n\ \|x_t\|_n + 2K_1\|P\|_{n\times n}\|x(t)\|_n\ \|u(t)\|_m \\ &= -\eta\|x(t)\|_n^2 + 2(\sqrt{a\eta}\|x(t)\|_n)\left(\sqrt{\frac{1}{a\eta}}K_3\|x_t\|_n\right) \\ &\quad + 2(\sqrt{a\eta}\|x(t)\|_n)\ \frac{K_1\|P\|_{n\times n}\ \|u\|_\infty}{\sqrt{a\eta}} \\ &\leq \left(-(1 - 2a)\eta + \frac{K_3^2\lambda_{\max}(P)}{a\eta\lambda_{\min}(P)}\right)\|x(t)\|_n^2 + \frac{K_1^2\|P\|_{n\times n}^2\alpha_2^2}{a\eta} \\ &\leq -\frac{\mu\eta}{\lambda_{\max}(P)}V(t, x(t)) + \frac{K_1^2\|P\|_{n\times n}^2\alpha_2^2}{a\eta}, \end{aligned} \tag{31}$$

where $\mu = (1 - 2a) - \frac{K_3^2\lambda_{\max}(P)}{a\lambda_{\min}^2(Q)\lambda_{\min}(P)} \geq a(1 - 2a) - \frac{K_3^2\lambda_{\max}(P)}{a\lambda_{\min}^2(Q)\lambda_{\min}(P)} > 0$ according to the assumption (A8).

Similarly to the proofs of inequalities (12) and (13) in Theorem 1, we obtain

$$\|x(t)\|_n \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}(\|\phi_0\|_0 + \frac{K_1\|P\|_{n\times n}\alpha_2}{\mu\eta})^2}, \tag{32}$$

and we get for the output

$$\begin{aligned} \|y(t)\|_n \leq a&\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\alpha_1 + \left(\frac{K_1\|P\|_{n\times n}\alpha_2}{\mu\lambda_{\max}(Q)}\right)^2\right)}\right)^{p_1} \\ &+ b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(\alpha_1 + \left(\frac{K_1\|P\|_{n\times n}\alpha_2}{\mu\lambda_{\max}(Q)}\right)^2\right)}\right)^{p_2} + L + K_2\alpha_2 \quad \text{for } t \geq t_0. \end{aligned}$$

∎

Example 3. Let $n=2, m=2$ and consider the IVP for the nonlinear FrDDEI

$$
\begin{aligned}
&{}_{0}^{C}D_t^{0.9}x(t) = Ax(t) + g\left(t, x\left(t - \frac{2}{t+1}\right)\right)u(t), \quad t > t_0,\\
&x(t) = \phi(t), \quad t \in [-2,0],\\
&y(t) = Dx(t) + h(t, x(t))u(t), \quad t > t_0,
\end{aligned} \tag{33}
$$

where $x = (x_1, x_2)^T, \phi = (\phi_1, \phi_2)^T$, $D = diag(d)$ with $d > 0$, $\tau \in C(\mathbb{R}_+, [-r, 0])$, $u = (u_1, u_2)^T$, matrices $g, h \in \mathbb{R}^{2\times 2}$ are given by

$$
g(t,x) = \begin{bmatrix} \cos(x_1) & -\sin(x_2) \\ -\cos(x_1) & \arctan(x_2) \end{bmatrix}, \quad h(t,x) = \begin{bmatrix} \frac{x_1}{1+|x_1|} & 0 \\ 0 & \frac{x_2}{1+|x_2|} \end{bmatrix},
$$

$$
A = \begin{bmatrix} -\frac{27}{22} & -1 \\ \frac{31}{11} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}
$$

with $\lambda_{\min}(P) = 4 - \sqrt{5}$, $\lambda_{\max}(P) = 4 + \sqrt{5}$, and $\|P\|_{2\times 2} = \sqrt{42}$. Then,

$$
Q = \begin{bmatrix} A^TP + PA & P \\ 0 & -I \end{bmatrix} = \begin{bmatrix} -1 & 1 & 5 & 2 \\ 1 & -4 & 2 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
$$

with all negative eigenvalues $\{0.5(-5 \pm \sqrt{13}), -1, -1\}$, and $\eta = -\lambda_{\max}(Q) = 1$. Therefore, the assumptions (A7) and (A8) are satisfied because $K_3 = 0$. Also, $K_1 = \sqrt{3}$ and $K_2 = \sqrt{2}$ in the assumption (A6). According to Theorem 5, the nonlinear FrDDEI (33) is UBIBO stable. Choose $a = 0.005$ and, therefore, $\mu = 1 - 2a = 0.99$. Then, Equation (33) is reduced to

$$
\|y(t)\|_2 \le c\sqrt{\frac{4+\sqrt{5}}{4-\sqrt{5}}\left(\alpha_1 + \left(\frac{\sqrt{3}\sqrt{49}\,\alpha_2}{0.99}\right)^2\right)} + \alpha_2\sqrt{2}.
$$

Let $\alpha_1 = 1$, $\alpha_2 = 0.6$ and $c = 2$.

Case 1. Let the input be

$$
u_1(t) = \frac{t}{1+e^t}, \quad u_2(t) = \frac{t}{1+t^2},
$$

and the initial functions be

$$
\phi_1(t) = -0.1e^{-t}, \quad \phi_2(t) = 0.15t.
$$

Then,

$$
\|u(t)\|_2 = \sqrt{\left(\frac{t}{1+e^t}\right)^2 + \left(\frac{t}{1+t^2}\right)^2} \le \alpha_2 = 0.6
$$

and

$$
\|\phi(t)\|_2 = \sqrt{(-0.1e^{-t})^2 + (0.15t)^2} < \alpha_1 = 1,
$$

for $t \in [-2, 0]$. Denote the corresponding output by $y_1 = (y_1^{(1)}, y_2^{(1)})$.

Case 2. Let the input be

$$
u_1(t) = 0.5\sin(t), \quad u_2(t) = 0.5e^{-t},
$$

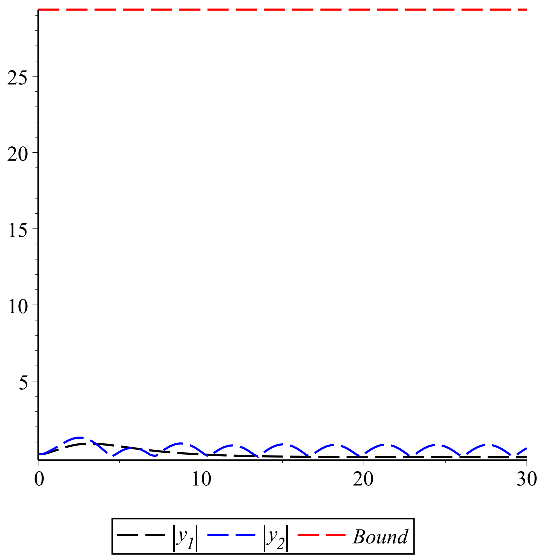

FIGURE 7 The graphs of the output $|y_1|$, $|y_2|$, and the bound β [Colour figure can be viewed at wileyonlinelibrary.com]

and the initial functions be

$$\phi_1(t) = -0.1e^{-t}, \quad \phi_2(t) = 0.15t.$$

Then,

$$\|u(t)\|_2 \le \alpha_2 = 0.6$$

and

$$\|\phi(t)\|_2 < \alpha_1 = 1,$$

for $t \in [-2, 0]$. Denote the corresponding output by $y_2 = (y_1^{(2)}, y_2^{(2)})$.

According to Theorem 5, in both cases, the norm of the outputs are bounded by

$$\beta = 2\sqrt{\frac{4+\sqrt{5}}{4-\sqrt{5}}\left(1+\left(\frac{\sqrt{3}\sqrt{49}\,0.6}{0.99}\right)^2\right)} + 0.6\sqrt{2} \approx 29.4.$$

The graphs of the norm of the outputs $|y_1|$ and $|y_2|$, as well of the bound, are given on Figure 7.

4.2.2 | Linear estimate of the coefficients before the input

Introduce the following assumption:

Assumption (A9). The function $G \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and there exist a constant K_3 with

$$0 \le K_3 < \frac{-\lambda_{\max}(Q)}{\|P\|_{n\times n}(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)})},$$

such that

$$\|G(t,x)\|_n \le K_3\|x\|_n, \quad \text{for } t \ge 0,\ x \in \mathbb{R}^n,$$

where the numbers $\lambda_{\max}(Q), \lambda_{\min}(P), \lambda_{\max}(P)$ are the corresponding eigenvalues of the matrices defined in the assumption (A7).

Theorem 6. *Let the assumptions (A4), (A6), (A7), and (A9) be satisfied. Then, the FrDDEI (3) is UBIBO stable with input threshold* $\gamma_u = \frac{-\lambda_{\max}(Q)}{L\|P\|_{n\times n}(1+\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)})} - \frac{K_3}{L}$.

Proof. Let $t_0 > 0$ be an arbitrary point and the input $u \in L\infty$ be such that $||u||_\infty < \gamma_u$. Consider the solution $(x(t), y(t))$ of the IVP for FrDDEI (3) and (4) for this input and $x(t) = x(t; t_0, \phi_0, u)$ and $y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

For the quadratic function $V(t, x) = x^T Px$ and any point $t \geq t_0$ such that the Razumikhin condition be satisfied, that is, the inequality $V(t+\Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or $x(t+\Theta)^T Px(t+\Theta) \leq x(t)^T Px(t)$, for $\Theta \in [-r, 0]$ applying inequality (30) we get

$$\begin{aligned}
{}_{t_0}^{C}D_t^q V(t, x(t)) &\leq 2x^T(t)P\, {}_{t_0}^{C}D_t^q x(t) \\
&= 2x^T(t)P(Ax(t) + H(x(t)) + G(t, x_t)) + 2x^T(t)Pg(t, x_t)u(t) \\
&\leq -\eta\|x(t)\|_n^2 + 2\|x(t)\|\ \|P\|_{n\times n}\|G(t, x_t)\|_n \\
&\quad + 2\|x(t)\|_n\|P\|_{n\times n}\|g(t, x_t)\|_{n\times n}\|\ \|u(t)\|_m \\
&\leq -\eta\|x(t)\|_n^2 + 2\|x(t)\|\ \|P\|_{n\times n}K_3\|x_t\|_n \\
&\quad + L\gamma_u\|x(t)\|_n^2\|P\|_{n\times n} + L\gamma_u\|x_t\|_n^2\|P\|_{n\times n} \\
&\leq \left(-\eta + (K_3 + L\gamma_u)\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)\right)\|x(t)\|_n^2 \\
&\leq -\frac{\mu}{\lambda_{\max}(P)}V(t, x(t)),
\end{aligned} \tag{34}$$

where

$$\mu = -\lambda_{\min}(Q) - (K_3 + L\gamma_u)\|P\|_{n\times n}\left(1 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right).$$

According to Lemma 1, with

$$\alpha = \frac{\mu}{\lambda_{\max}(P)} \quad \text{and} \quad \xi = 0,$$

we get $V(t, x(t)) \leq \lambda_{\max}(P)||\phi_0||_0$ and

$$\|x(t)\|_n \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\phi_0||_0. \tag{35}$$

Moreover, we get for the output the following upper bound:

$$\|y(t)\|_n \leq a\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\phi_0||_0\right)^{p_1} + b\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\phi_0||_0\right)^{p_2} + L + l\gamma_u\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\phi_0||_0.$$

■

Assumption (A10). The function $G \in C([0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and there exist a constant K_3 with

$$0 \leq K_3 < \frac{-\lambda_{\max}(Q)}{\|P\|_{n\times n}}\sqrt{(1-a)a\frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)}},$$

for some $a \in (0, 0.5)$, such that

$$\|G(t, x)\|_n \leq K_3\|x\|_n, \quad \text{for } t \geq 0,\ x \in \mathbb{R}^n,$$

where the numbers $\lambda_{\max}(Q), \lambda_{\min}(P), \lambda_{\max}(P)$ are the corresponding eigenvalues of the matrices defined in the assumption (A7).

Theorem 7. *Let the assumptions (A4), (A6), (A7), and (A10), be satisfied. Then, the FrDDEI (3) is UBIBO stable with input threshold*

$$\gamma_u = \sqrt{(1-a)a\lambda_{\max}^2(Q)\frac{\lambda_{\min}(P)}{2\lambda_{\max}(P)\|P\|_{n\times n}^2 L^2} - \left(\frac{K_3}{L}\right)^2}.$$

Proof. Let $t_0 > 0$ be an arbitrary point. Choose the input $u \in L_\infty$ such that $\|u\|_\infty < \gamma_u$, and consider the solution (x, y) of the IVP for FrDDEI (3) and (4), with $x(t) = x(t; t_0, \phi_0, u)$ and $y(t) = y(t; t_0, \phi_0, u)$, for $t \geq t_0 - r$.

For the quadratic function $V(t,x) = x^T Px$ and any point $t \geq t_0$ such that the Razumikhin condition be satisfied, that is, the inequality $V(t+\Theta, x(t+\Theta)) \leq V(t, x(t))$ holds for $\Theta \in [-r, 0]$, or

$$x(t+\Theta)^T Px(t+\Theta) \leq x(t)^T Px(t),$$

for $\Theta \in [-r, 0]$ by Lemma 2 and inequalities (30) we get

$$\begin{aligned}
{}_{t_0}^{C}D_t^q V(t, x(t)) &\leq -\eta\|x(t)\|_n^2 + 2\|x(t)\|\ \|P\|_{n\times n}\|G(t, x_t)\|_n \\
&\quad + 2\|x(t)\|_n\|P\|_{n\times n}\|g(t, x_t)\|_{n\times n}\|\ \|u(t)\|_m \\
&\leq -\eta\|x(t)\|_n^2 + 2(\sqrt{0.5a\eta}\|x(t)\|_n)\ \frac{\|P\|_{n\times n}K_3\|x_t\|_n}{\sqrt{0.5a\eta}} \\
&\quad + 2(\sqrt{0.5a\eta}\|x(t)\|_n)\frac{2\|P\|_{n\times n}L\gamma_u\|x_t\|_n}{\sqrt{0.5a\eta}} \\
&\leq -\eta\|x(t)\|_n^2 + 0.5a\eta\|x(t)\|_n^2 + \frac{\|P\|_{n\times n}^2 K_3^2\|x_t\|_n^2}{0.5a\eta} \\
&\quad + 0.5a\eta\|x(t)\|_n^2 + \frac{\|P\|_{n\times n}^2 L^2\gamma_u^2\|x_t\|_n^2}{0.5a\eta} \\
&\leq -(1-a)\eta\|x(t)\|_n^2 + \frac{2\|P\|_{n\times n}^2(K_3^2 + L^2\gamma_u^2)\|x_t\|_n^2}{a\eta} \\
&\leq -\left((1-a)\eta - \frac{2\|P\|_{n\times n}^2(K_3^2 + L^2\gamma_u^2)\frac{\lambda_{\max}}{\lambda_{\min}}}{a\eta}\right)\|x(t)\|_n^2 \\
&\leq -\frac{\mu}{\lambda_{\max}(P)}V(t, x(t)),
\end{aligned} \tag{36}$$

where $\eta = -\lambda_{\max}(Q)$,

$$\mu = (1-a)\eta - \frac{2\|P\|_{n\times n}^2(K_3^2 + L^2\gamma_u^2)\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}{a\eta}.$$

The rest of the proof is similar to the one of Theorem 6. ∎

Example 4. We consider a slight changed Example 3. Consider the IVP for the scalar nonlinear FrDDEI (33), with the matrices $g, h \in \mathbb{R}^{2\times 2}$ given by

$$g(t,x) = \begin{bmatrix} \sin(t)x_1 & -\cos(t)x_2 \\ 0.2\cos(t)x_1 & 0.2\sin(t)x_2 \end{bmatrix}, \quad h(t,x) = \begin{bmatrix} 0.5e^{-t}x_1 & 0.5\sin(t)x_1 \\ 0.5\cos(t)x_2 & 0.5\arctan(t)x_2 \end{bmatrix}.$$

The functions g, h are not bounded, so the result of Theorem 5 cannot be applied. But the assumption (A4) is satisfied, with

$$L = \max_{t\geq 0}\{\sin^2(t) + 0.04\cos^2(t), \cos^2(t) + 0.04\sin^2(t)\} = 1$$

and

$$l = \max_{t\geq 0}\{0.25e^{-2t} + 0.25\sin^2(t), 0.25\cos^2(t) + 0.25\arctan^2(t)\} = 1.5.$$

Also, $K_1 = \sqrt{3}$ and $K_2 = \sqrt{2}$ in the assumption (A4). The assumption (A9) is satisfied because $K_3 = 0$. Then, Equation (36) is reduced to

$$\|y(t)\|_2 \leq (2 + ||u||_\infty \sqrt{2})\sqrt{\frac{4+\sqrt{5}}{4-\sqrt{5}}}||\phi_0||_0.$$

Case 1. Let the input be

$$u_1(t) = \frac{t}{1+e^t}, \quad u_2(t) = \frac{t}{1+t^2},$$

and the initial functions be

$$\phi_1(t) = 0.8e^t, \quad \phi_2(t) = 0.5\sin(t).$$

Then,

$$\|u(t)\|_2 = \sqrt{\left(\frac{t}{1+e^{-t}}\right)^2 + \left(\frac{t}{1+t^2}\right)^2} \leq \alpha_2 = 0.25$$

and

$$\|\phi(t)\|_2 = \sqrt{(0.8e^t)^2 + (0.5\sin(t))^2} < \alpha_1 = 1,$$

for $t \in [-2, 0]$. Denote the corresponding output by $y_1 = (y_1^{(1)}, y_2^{(1)})$.

Case 2. Let the input be

$$u_1(t) = 0.25\sin(t^2), \quad u_2(t) = \frac{t}{1+t^2}$$

and the initial functions be

$$\phi_1(t) = -\cos(t), \quad \phi_2(t) = 0.5\arctan(t).$$

Then,

$$\|u(t)\|_2 \leq \alpha_2 = 0.35 \quad \text{and} \quad \|\phi(t)\|_2 \leq \alpha_1 = 1,$$

for $t \in [-2, 0]$. Denote the corresponding output by $y_2 = (y_1^{(2)}, y_2^{(2)})$.

According to Theorem 6, in both cases, the norm of the outputs is bounded by

$$\beta = (2 + 0.35\sqrt{2})\sqrt{\frac{4+\sqrt{5}}{4-\sqrt{5}}} \approx 4.42526.$$

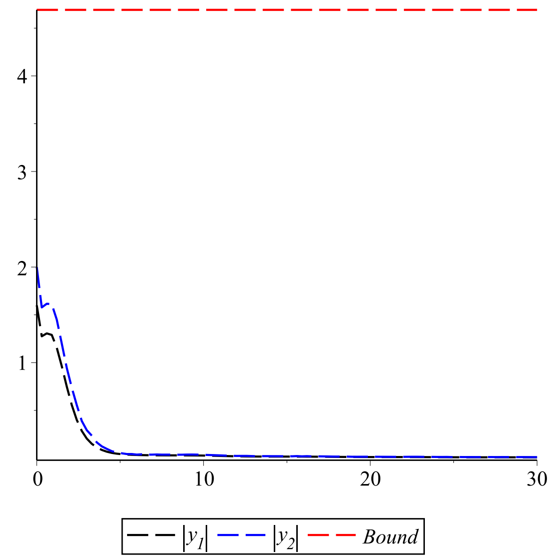

FIGURE 8 The graphs of the output $|y_1|$, $|y_2|$, and the bound β [Colour figure can be viewed at wileyonlinelibrary.com]

The graphs of the norm of the outputs $|y_1|$ and $|y_2|$, and of the bound, are given on Figure 8.

ACKNOWLEDGEMENTS

R. Almeida was supported by Portuguese funds through the CIDMA-Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), within project UIDB/04106/2020, S. Hristova was supported by the Bulgarian National Science Fund under Project KP-06-N32/7. S. Dashkovskiy was partially supported by the German Research Foundation (DFG) via grant number DA 767/12-1. Open access funding enabled and organized by Projekt DEAL.

ORCID

S. Dashkovskiy https://orcid.org/0000-0001-7049-012X

REFERENCES

1. Hilfer R. *Applications of Fractional Calculus in Physics*. Vol 35. Singapore, Asia: World Scientific; 2000:87-130.
2. Awwad E, Gyori I, Hartung F. BIBO stabilization of feedback control systems with time dependent delays. *Appl Math Comput*. 2012;219:3664-3676.
3. Sun H, Abdelwahab A, Onaral B. Linear approximation of transfer function with a pole of fractional power. *IEEE Trans Autom Control*. 1984;29(5):441-444.
4. Petras I, Vinagre BM. Practical application of digital fractional-order controller to temperature control. *Acta Montan Slovaca*. 2002;7:131-137.
5. Aboelela MAS, Ahmed MF, Dorrah HT. Design of aerospace control systems using fractional PID controller. *J Adv Res*. 2012;3:225-232.
6. Echi N, Basdouri I, Benali H. A separation principle for stabilisation of a class of fractional order time delay nonlinear systems. *Bull Austr Math Soc*. 2019;99(1):161-173.
7. Jmal A, Naifar O, Ben Makhlouf A, et al. Robust sensor fault estimation for fractional-order systems with monotone nonlinearities. *Nonlinear Dyn*. 2017;90:2673-2685.
8. Jmal A, Naifar O, Ben Makhlouf A, Derbel N, Hammami MA. On observer design for nonlinear caputo fractional-order systems. *Asian J Control*. 2018;20:1533-1540.
9. Naifar O, Ben Makhlouf A, Hammami MA, Chen L. Global practical Mittag Leffler stabilization by output feedback for a class of nonlinear fractional-order systems. *Asian J Control*. 2018;20:599-607.
10. Naifar O, Nagy AM, Makhlouf AB, Kharrat M, Hammami MA. Finite-time stability of linear fractional-order time-delay systems. *Int J Robust Nonlinear Control*. 2019;29:180-187.
11. Agarwal R, Hristova S, O'Regan D. Lyapunov functions and stability of Caputo fractional differential equations with delays. *Differ Equ Dyn Syst*. 2018;2018. https://doi.org/10.1007/s12591-018-0434-6.
12. Chen B, Chen J. Razumikhin-type stability theorems for functional fractional-order differential systems and applications. *Appl Math Comput*. 2015;254:63-69.
13. Wang Y, Li T. Stability analysis of fractional-order nonlinear systems with delay. *Math Probl.Eng*. 2014;301235 8p.2014.

14. Wang L, Chen X, Gao F. Delay-range-dependent robust BIBO stabilization of 2D discrete delayed systems via LMI approach. *IFAC Proc*. 2014;47:10994-10999.
15. Zhang WA, Yu L. BIBO stability and stabilization of networked control systems with short time-varying delays. *Int J Robust Nonlinear Control*. 2011;21(3):295-308.
16. Gill MI. On bounded input bounded output stability of a class of nonlinear retarded systems. *Int J Robust Nonlinear Control*. 2000;10(15):1337-1344.
17. Xiong L, Zhong S, Ye M. Delay-dependent BIBO stability analysis of switched uncertain neutral systems. *Math Comput Model*. 2011;53:1607-1620.
18. Pham T, Nahavandi S, Trinh H, Hien L. Decentralized bounded input bounded output stabilization of perturbed interconnected time-delay power systems with energy storages. *Int J Electr Power Energy Syst*. 2017;93:51-64.
19. Wang B, Cheng J, Zhong SM. Bounded input bounded output stability for Lurie system with time-varying delay. *Adv Differ Equ*. 2018;2018:57.
20. Li C, Zhang J. Synchronisation of a fractional-order chaotic system using finite-time input-to-state stability. *Int J Syst Sci*. 2016;47(10):2440-2448.
21. Taghavian H, Tavazoei MS. Robust stability analysis of uncertain multiorder fractional systems: Young and Jensen inequalities approach. *Int J Robust Nonlinear Control*. 2018;28(4):1127-1144.
22. Abusaksaka AB, Partington JR. BIBO stability of some classes of delay systems and fractional systems. *Syst Control Lett*. 2014;64:43-46.
23. Liang S, Wu R, Chen L. BIBO stability of fractional-order controlled nonlinear systems. *Int J Syst Sci*. 2017;48(7):1507-1514.
24. Diethelm K. *The Analysis of Fractional Differential Equations*. Berlin, Heidelberg/Germany: Springer-Verlag; 2010.
25. Podlubny I. *Fractional Differential Equations*. San Diego, CA: Academic Press; 1999.
26. Cong ND, Tuan HT. Existence, uniqueness, and exponential boundedness of global solutions to delay fractional differential equations. *Mediterr J Math*. 2017;14:193.
27. Maraaba T, Baleanu D, Jarad F. Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives. *J Math Phys*. 2008;49:083507.
28. Sin CS. Existence and uniqueness of global solutions of fractional functional differential equations with bounded delay; 2017. arXiv:1705.06046v1.
29. Duarte-Mermoud MA, Aguila-Camacho N, Gallegos JA, Castro-Linares R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Commun Nonlinear Sci Numer Simul*. 2015;22(1–3):650-659.

How to cite this article: Almeida R, Hristova S, Dashkovskiy S. Uniform bounded input bounded output stability of fractional-order delay nonlinear systems with input. *Int J Robust Nonlinear Control*. 2021;31:225–249. https://doi.org/10.1002/rnc.5273