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Eigenvalues of zero-divisor graphs of finite commutative rings

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Abstract

We investigate eigenvalues of the zero-divisor graph $\Gamma(R)$ of finite commutative rings R and study the interplay between these eigenvalues, the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$. The graph $\Gamma(R)$ is defined as the graph with vertex set consisting of all nonzero zero-divisors of R and adjacent vertices x, y whenever $xy = 0$. We provide formulas for the nullity of $\Gamma(R)$, i.e., the multiplicity of the eigenvalue 0 of $\Gamma(R)$. Moreover, we precisely determine the spectra of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ for a prime number p. We introduce a graph product $\times_\Gamma$ with the property that $\Gamma(R) \cong \Gamma(R_1) \times_\Gamma \cdots \times_\Gamma \Gamma(R_r)$ whenever $R \cong R_1 \times \cdots \times R_r$. With this product, we find relations between the number of vertices of the zero-divisor graph $\Gamma(R)$, the compressed zero-divisor graph, the structure of the ring R and the eigenvalues of $\Gamma(R)$.

Keywords EJMA-D-19-00287 · Zero-divisor graphs · Graph eigenvalues · Graph nullity · Graph products · Local rings

1 Introduction

Let R be a finite commutative ring with $1 \neq 0$ and let $Z(R)$ denote its set of zero-divisors. As introduced by Anderson and Livingston [3] in 1999, the *zero-divisor graph* $\Gamma(R)$ is defined as the graph with vertex set $Z^*(R) = Z(R)\backslash\{0\}$ where two vertices x, y are adjacent if and only if $xy = 0$. The aim of considering these graphs is to study the interplay between graph theoretic properties of $\Gamma(R)$ and the ring properties of R. In order to simplify the representation of $\Gamma(R)$, it is often useful to consider the so-called *compressed zero-divisor graph* $\Gamma_E(R)$. This graph was first introduced by Mulay [10] and further studied in [2,12,14,17].

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Definition 1.1 (*Compressed zero-divisor graph*) For an element $r \in R$ let $[r]_R = \{s \in R \mid \mathrm{ann}_R(r) = \mathrm{ann}_R(s)\}$ and $R_E = \{[r]_R \mid r \in R\}$. Then, the *compressed zero-divisor graph* $\Gamma_E(R)$ is defined as the graph $\Gamma(R_E)$.

Note that $[0]_R = \{0\}$, $[1]_R = R \backslash Z(R)$ and $[r]_R \subseteq Z(R)\backslash\{0\}$ for every $r \in R\backslash([0]_r \cup [1]_R)$. The notations are adopted from Spiroff and Wickham [14].

The *spectrum* of a graph G is defined as the multiset of eigenvalues, i.e., the roots of the characteristic polynomial of the adjacency matrix $A(G)$. The aim of studying eigenvalues of graphs is to find relations between those values and structural properties of the graph. The author refers to [5,7] for good introductions to graph theory and spectral graph theory, respectively.

The *nullity* $\eta(G)$ of a graph G is defined as the multiplicity of the eigenvalue 0 of G. It is easy to see that

$$\eta(G) = \dim A(G) - \operatorname{rank} A(G),$$

where $\dim A(G)$ denotes the dimension of the domain of the linear transformation associated to the matrix $A(G)$, i.e., the number of columns of $A(G)$. Background and further results on the nullity of graphs are summarized in [8].

Within spectral graph theory, most graphs are considered to be *simple*, i.e., to be undirected finite graphs without loops or multiple edges. By definition, $\Gamma(R)$ has no multiple edges, and we can easily see that $\Gamma(R)$ is undirected if and only if R is commutative. Moreover, as already proven by Anderson and Livingston [3, Theorem 2.2], the graph $\Gamma(R)$ is finite if and only if R is finite or an integral domain. In the latter case, though, R has no zero-divisors at all and is just the empty graph. Hence, all our rings are assumed to be finite and commutative. However, in contrast to the original definition of Anderson and Livingston [3], we do not want to eliminate potential loops of our zero-divisor graphs since these loops provide important information about the structure of the ring R.

Definition 1.2 (*Zero-divisor graph*) Let R be a finite commutative ring with $1 \neq 0$ and let $Z(R)$ denote its set of zero-divisors. Then, the *zero-divisor graph* $\Gamma(R)$ is defined as the graph with vertex set $Z^*(R) = Z(R)\backslash\{0\}$ where two (not necessarily distinct) vertices x, y are adjacent if and only if $xy = 0$.

In order to determine the eigenvalues of a graph, it often can be useful to consider graph products. For example, in [1], the spectra of unitary Cayley graphs of finite rings could easily be determined by observing that these graphs are isomorphic to direct products of unitary Cayley graphs of finite local rings.

Definition 1.3 (*Direct product*) The *direct product* $G_1 \times G_2$ of graphs G_1 and G_2 is defined as the graph with vertex set $V(G_1) \times V(G_2)$ where two vertices $(v_1, v_2), (v_1', v_2') \in V(G_1) \times V(G_2)$ are adjacent in $G_1 \times G_2$ if and only if v_1 is adjacent to v_1' in G_1 and v_2 is adjacent to v_2' in G_2.

It is well-known that the adjacency matrix of the direct product $G_1 \times G_2$ equals the Kronecker product $A(G_1) \otimes A(G_2)$. Therefore, if λ_i and μ_i are the eigenvalues of G_1 and G_2, respectively; then, the eigenvalues of $G_1 \times G_2$ are exactly the products $\lambda_i \mu_j$.

Moreover, we want to introduce the following two graph products:

Definition 1.4 (*Complete product and point identification*)

(i) The *complete product* $G_1 \nabla G_2$ consists of the vertex set $V(G_1) \cup V(G_2)$, where v_1 and v_2 are adjacent in $G_1 \nabla G_2$ if and only if either $v_1 \in V_1$ and $v_2 \in V_2$ or v_1 is adjacent to v_2 in G_1 or G_2, respectively.

(ii) For $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ the *point identification* (or *coalescence*) $G_1 \bullet G_2$ arises from setting $v_1 = v_2$. If $v \in V(G_1)$ and $v \in V(G_2)$, we write $G_1 \overset{v}{\bullet} G_2$ in order to make clear that the graphs were coalesced at v.

In this paper, we study the interplay between graph-theoretic properties of the zero-divisor graph $\Gamma(R)$, the spectrum of $\Gamma(R)$ and the ring properties of R. By now, surprisingly little is known about the eigenvalues and adjacency matrices of zero-divisor graphs. First research in this direction was done by Sharma et. al. [13] in 2011. They made some observations on the adjacency matrices and eigenvalues of the graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$. Further results were found by Young [18] in 2015 and independently by Surendranath Reddy et. al. [15] in 2017. Both studied the graphs $\Gamma(\mathbb{Z}_n)$ and precisely determined the eigenvalues of $\Gamma(\mathbb{Z}_p)$, $\Gamma(\mathbb{Z}_{p^2})$, $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^2q})$ for p and q being prime numbers. Other recent papers on that topic are [11,16]. Note that in most of these papers the corresponding zero-divisor graphs were also considered with loops.

Our main approach is the following: since R is a finite ring, it can be written as $R \cong R_1 \times \cdots \times R_r$, where each R_i is a finite local ring. A proof for this and further results within the theory of finite commutative rings can be found in [4]. In Sect. 2, we introduce a graph product x_Γ with the property that

$$\Gamma(R) \cong \Gamma(R_1) \times_\Gamma \cdots \times_\Gamma \Gamma(R_r)$$

whenever $R \cong R_1 \times \cdots \times R_r$. With this graph product, in Sect. 3, we find a relation between the number of vertices of $\Gamma_E(R)$ and the property of R being a local ring. Moreover, we derive formulas for the number of vertices of the zero-divisor graph $\Gamma(R)$ and the compressed zero-divisor graph $\Gamma_E(R)$ in terms of the local rings R_i. From these formulas, we can deduce a lower bound for the nullity of $\Gamma(R)$. In Sect. 4, we restrict our considerations to rings which are isomorphic to direct products of rings of integers modulo n, i.e., $R \cong \mathbb{Z}_{{p_1}^{t_1}} \times \cdots \times \mathbb{Z}_{{p_r}^{t_r}}$ for (not necessarily distinct) prime numbers p_i and positive integers r, t_i. For these rings, we find a criterion which may detect a local ring by considering its zero-divisor graph and the respective nullity. Moreover, we find the exact nullity of $\Gamma(R)$ and present an easy approach to determine also the nonzero eigenvalues of $\Gamma(R)$. For example, we precisely determine the spectra of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ in terms of a prime number p. We also provide the characteristic polynomials of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q)$ for primes $q \neq p$. This generalizes the results of Sharma et. al. [13], Young [18] and Surendranath Reddy et. al. [15].

Throughout this paper, we denote edges as sets of two vertices. For a graph G, we write $A(G)$ for the adjacency matrix of G , $V(G)$ for the set of vertices of G and

$\chi_G(x) = \det(xI - A(G))$ for the characteristic polynomial of G. If λ is an eigenvalue of G of multiplicity x, then we denote this by $\lambda^{[x]}$. The number of elements in a set S is denoted by $\#S$, and φ denotes Euler's totient function. For the set of units of a ring R, we write $U(R)$.

2 Products of zero-divisor graphs

Let $R \cong R_1 \times \cdots \times R_r$ be a ring, where each R_i is a finite local ring. Note that in this case $\#R_i = p_i^{t_i}$ for some prime numbers p_i and $t_i \in \mathbb{N}$. Our aim is to define a graph product $\times_\Gamma$ such that

$$\Gamma(R) \cong \Gamma(R_1) \times_\Gamma \cdots \times_\Gamma \Gamma(R_r)$$

whenever

$$R \cong R_1 \times \cdots \times R_r.$$

Since two vertices $(v_1, \ldots, v_r), (v_1', \ldots, v_r') \in R_1 \times \cdots \times R_r$ are adjacent in $\Gamma(R_1 \times \cdots \times R_r)$ if and only if $v_i, v_i' \in Z^*(R_i)$ or either $v_i = 0$ or $v_i' = 0$, our idea is the following: we first add the vertex $0 \in R_i$ and the units of R_i to the vertices of each zero-divisor graph $\Gamma(R_i)$, as well as edges from 0 to every other vertex. Then, we take the direct product of these somehow *extended zero-divisor graphs*, each of which we will denote by $\mathrm{E}\,\Gamma(R_i)$, which yields the extended zero-divisor graph $\mathrm{E}\,\Gamma(R)$. Finally, by removing the vertex $0 \in R$ with all its edges, as well as all units of R, we end up with the zero-divisor graph $\Gamma(R)$.

To formalize this, we define the *unit graph* $\mathrm{U}(R_i)$ of R_i as the graph with vertex set $U(R_i)$ and empty edge set. Moreover, let $\mathrm{Z}(R_i)$ and $\mathrm{Z_L}(R_i)$ be the *zero graphs* with vertex set $\{0\}$ (where $0 \in R_i$) and empty edge set or edge set $\{\{0, 0\}\}$, respectively (i.e., both graphs consist of one vertex only, and, in contrast to $\mathrm{Z}(R)$, the graph $\mathrm{Z_L}(R)$ also has a loop at that vertex; we need this distinction for our result in Sect. 5).

Definition 2.1 (*Extended zero-divisor graph*) Let R be a finite commutative ring with $1 \neq 0$. Then, the *extended zero-divisor graph* $\mathrm{E}\,\Gamma(R)$ is defined as the graph with vertex set R where two (not necessarily distinct) vertices $x, y \in R$ are adjacent if and only if $xy = 0$.

In view of those definitions, the extended zero-divisor graph $\mathrm{E}\,\Gamma(R_i)$ is given by

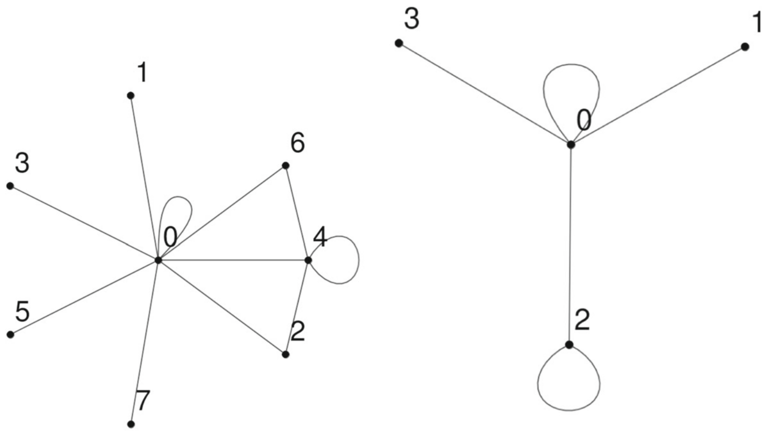
$$\mathrm{E}\,\Gamma(R_i) = \big(\Gamma(R_i) \nabla \mathrm{Z}(R_i)\big) \overset{\{0\}}{\bullet} \big(\mathrm{U}(R_i) \nabla \mathrm{Z_L}(R_i)\big),$$

and we have that

$$\Big(\Gamma(R) \cup \big(\mathrm{U}(R_1) \times \cdots \times \mathrm{U}(R_r)\big)\Big) \nabla \mathrm{Z}(R) \cong \mathrm{E}\,\Gamma(R_1) \times \cdots \times \mathrm{E}\,\Gamma(R_r).$$

(a) $\Gamma(\mathbb{Z}_8)$ **(b)** $\Gamma(\mathbb{Z}_4)$

Fig. 1 Zero-divisor graphs $\Gamma(\mathbb{Z}_8)$ and $\Gamma(\mathbb{Z}_4)$



(a) $\mathrm{E}\Gamma(\mathbb{Z}_8)$ **(b)** $\mathrm{E}\Gamma(\mathbb{Z}_4)$

Fig. 2 Extended zero-divisor graphs $\mathrm{E}\,\Gamma(\mathbb{Z}_8)$ and $\mathrm{E}\,\Gamma(\mathbb{Z}_4)$

Hence, we define the associative product $\times_\Gamma$ by

$$\Gamma(R_1) \times_\Gamma \Gamma(R_2) := \Big(\mathrm{E}\,\Gamma(R_1) \times \mathrm{E}\,\Gamma(R_2)\Big) \backslash \Big(V\big(\mathrm{Z}(R_1 \times R_2)\big) \cup V\big(\mathrm{U}(R_1 \times R_2)\big)\Big),$$

where $G\backslash\{v\}$ denotes the graph G without the vertex $v \in V(G)$ and all its adjacent edges. Note that $\mathrm{Z}(R_1 \times R_2) \cong \mathrm{Z}(R_1) \times \mathrm{Z}(R_2)$ and $\mathrm{U}(R_1 \times R_2) \cong \mathrm{U}(R_1) \times \mathrm{U}(R_2)$. The product $\times_\Gamma$ is illustrated in the following example:

Example 2.2 Let $R = \mathbb{Z}_8 \times \mathbb{Z}_4$. Figure 1 shows the zero-divisor graphs $\Gamma(\mathbb{Z}_8)$ and $\Gamma(\mathbb{Z}_4)$ and Fig. 2 the extended zero-divisor graphs $\mathrm{E}\,\Gamma(\mathbb{Z}_8)$ and $\mathrm{E}\,\Gamma(\mathbb{Z}_4)$. In Fig. 3, we see the direct product $\mathrm{E}\,\Gamma(\mathbb{Z}_8) \times \mathrm{E}\,\Gamma(\mathbb{Z}_4) \cong \mathrm{E}\,\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_4)$ and Fig. 4 finally illustrates the graph product $\Gamma(\mathbb{Z}_8) \times_\Gamma \Gamma(\mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_8 \times \mathbb{Z}_4)$ arising from removing the vertices $(0, 0)$ and $V(\mathrm{U}(\mathbb{Z}_8 \times \mathbb{Z}_4))$ from the graph $\mathrm{E}\,\Gamma(\mathbb{Z}_8) \times \mathrm{E}\,\Gamma(\mathbb{Z}_4)$.

The same also holds for the compressed zero-divisor graph, i.e., we have that $\Gamma_E(R) \cong \Gamma_E(R_1) \times_\Gamma \cdots \times_\Gamma \Gamma_E(R_r)$ whenever $R_E \cong R_{1E} \times \cdots \times R_{rE}$. In Sect. 5, we deduce a relation between the characteristic polynomial of $\Gamma(R)$ and the one of the extended zero-divisor graph $\mathrm{E}\,\Gamma(R)$.

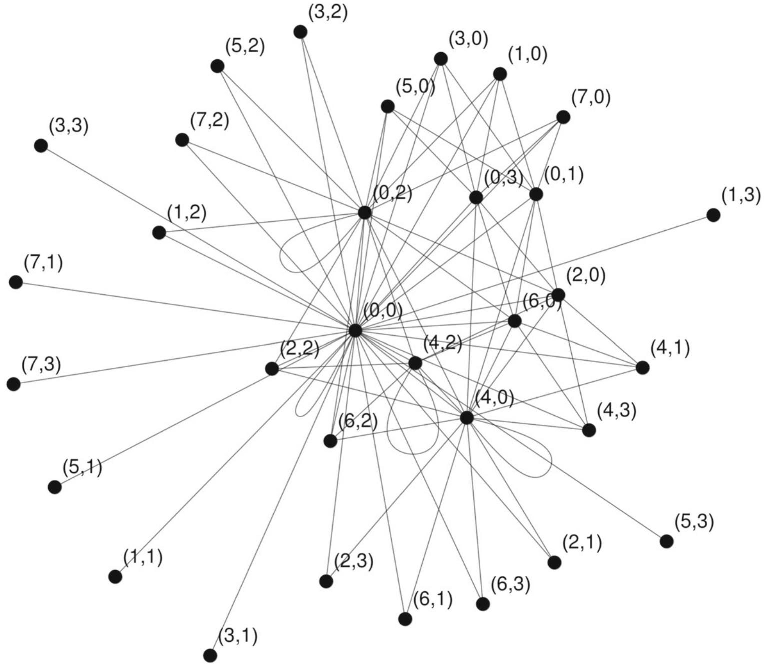


Fig. 3 Direct product $\mathrm{E}\,\Gamma(\mathbb{Z}_8) \times \mathrm{E}\,\Gamma(\mathbb{Z}_4) \cong \mathrm{E}\,\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_4)$

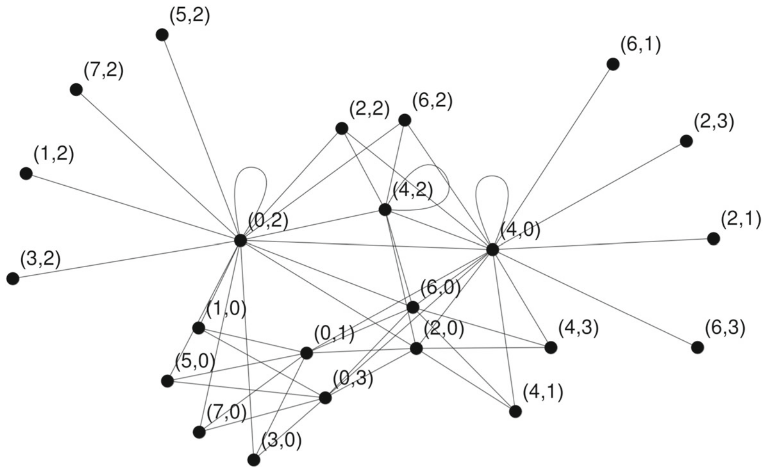


Fig. 4 Zero-divisor graph $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_8) \times_{\Gamma} \Gamma(\mathbb{Z}_4)$

3 Nullity of zero-divisor graphs of finite commutative rings

The following lemma follows directly from the construction of the product $\times_\Gamma$:

Lemma 3.1 *Let $R \cong R_1 \times \cdots \times R_r$ with local rings R_i. Then, the number of nonzero zero-divisors of R, i.e., the number of vertices of the zero-divisor graph $\Gamma(R)$ equals*

$$\begin{aligned}\#V(\Gamma(R)) &= \prod_{i=1}^{r} \#R_i - \prod_{i=1}^{r} \#V(\mathrm{U}(R_i)) - 1\\ &= \prod_{i=1}^{r} \#R_i - \prod_{i=1}^{r} \#U(R_i) - 1.\end{aligned}$$

Proof We have that $\#V(\mathrm{E}\,\Gamma(R_i)) = \#Z^*(R_i) + \#U(R_i) + 1 = \#R_i$ since $R_i = Z^*(R_i) \cup U(R_i) \cup \{0\}$. Taking into account that $\mathrm{E}\,\Gamma(R) \cong \mathrm{E}\,\Gamma(R_1) \times \cdots \times \mathrm{E}\,\Gamma(R_r)$, we therefore get that $\#V(\mathrm{E}\,\Gamma(R)) = \prod_{i=1}^{r} \#R_i$. Finally, since $\Gamma(R)$ arises from $\mathrm{E}\,\Gamma(R)$ by removing the vertex $0 \in R$ and all units of R (where each unit of R is a direct product of units of the R_i's), the statement follows. □

Moreover, we get a similar result for the number of vertices of the compressed zero-divisor graph:

Lemma 3.2 *Let $R \cong R_1 \times \cdots \times R_r$ with local rings R_i. Then, the number of vertices of the compressed zero-divisor graph $\Gamma_E(R)$ equals*

$$\begin{aligned}\#V(\Gamma_E(R)) &= \prod_{i=1}^{r} \big(\#V(\Gamma_E(R_i)) + 2\big) - 2\\ &= \prod_{i=1}^{r} \#R_{i\,E} - 2.\end{aligned}$$

Proof Since the R_i's are finite rings, each element of R_i is either a zero-divisor or a unit. Thus, the elements of $R_{i\,E}$ are exactly the vertices of $\Gamma_E(R_i)$ together with $[0]_{R_i}$ and $[1]_{R_i}$ (since the elements of $[1]_{R_i}$ are exactly the units of R_i). The statement follows from the construction of $\times_\Gamma$. □

Of course, those results are not very surprising since every nonzero-divisor of a finite commutative ring is a unit, i.e., $\#Z^*(R) = \#R - \#U(R) - 1$. However, from the latter lemma we observe that if $\#V(\Gamma_E(R)) + 2$ is a prime number, then R must be a local ring. This provides a notable relation between combinatorial objects (the zero-divisor graphs) and algebraic structures (the respective rings).

Example 3.3 Let $R = \mathbb{Z}_3[[X, Y]]/(XY, X^3, Y^3, X^2 - Y^2)$. The corresponding compressed zero-divisor graph $\Gamma_E(R)$ has five vertices, see Fig. 5. Since $5 + 2 = 7$ is a prime number, R has to be a local ring.

Moreover, we can derive a lower bound for the nullity of zero-divisor graphs:

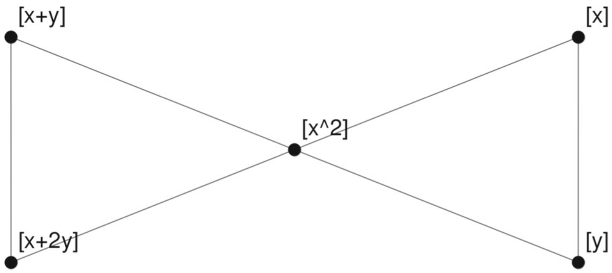


Fig. 5 Compressed Zero-divisor graph $\Gamma_E(R)$ for $R = \mathbb{Z}_3[[X, Y]]/(XY, X^3, Y^3, X^2 - Y^2)$

Theorem 3.4 *Let $R \cong R_1 \times \cdots \times R_r$ with local rings R_i. Then, the nullity of the zero-divisor graph $\Gamma(R)$ is at least*

$$\eta(\Gamma(R)) \geq \prod_{i=1}^{r} \#R_i - \prod_{i=1}^{r} \#U(R_i) - \prod_{i=1}^{r} \big(\#V(\Gamma_E(R_i)) + 2\big) + 1.$$

Proof Each element of $[r]_R \in V(\Gamma_E(R))$ contributes exactly the same row to the adjacency matrix $A(\Gamma(R))$. Thus, rank $A(\Gamma(R)) \leq \#Z^*(R_E) = \#V(\Gamma_E(R))$. Since $\eta(\Gamma(R)) = \dim A(\Gamma(R)) - \operatorname{rank}(A(\Gamma(R)))$ and $\dim A(\Gamma(R)) = \#Z^*(R) = \#V(\Gamma(R))$, we have that $\eta(\Gamma(R)) \geq \#V(\Gamma(R)) - \#V(\Gamma_E(R))$. The statement follows with Theorems 3.1 and 3.2 . □

4 Spectra of zero-divisor graphs of direct products of rings of integers modulo *n*

As already observed by Young [18], the adjacency matrix of the compressed zero-divisor graph is a so-called *equitable partition* of the adjacency matrix of $\Gamma(R)$. A formal definition for this is given in [5]. We define the *weighted adjacency matrix* $\mathcal{A}(\Gamma_E(R))$ of the compressed zero-divisor graph as the matrix with (i, j)-th entry

$$\mathcal{A}(\Gamma_E(R))_{i,j} = \begin{cases} 0, & \text{if } A(\Gamma_E(R))_{i,j} = 0, \\ \#[j]_R, & \text{else.} \end{cases}$$

From [5, Lemma 2.3.1], it follows that every eigenvalue of $\mathcal{A}(\Gamma_E(R))$ is also an eigenvalue of $A(\Gamma(R))$. In general, it is not clear whether these eigenvalues are exactly the nonzero eigenvalues of $\Gamma(R)$, i.e., whether $\mathcal{A}(\Gamma_E(R))$ always has full rank. But assuming that R is a product of rings of integers modulo n, we can prove the following:

Theorem 4.1 *Let $R \cong \mathbb{Z}_{p_1{}^{t_1}} \times \cdots \times \mathbb{Z}_{p_r{}^{t_r}}$ for prime numbers p_j and $r, t_j \in \mathbb{N}$. Then,*

$$\operatorname{rank} A(\Gamma(R)) = \operatorname{rank} \mathcal{A}(\Gamma_E(R)) = \#V(\Gamma_E(R)).$$

Proof We can easily see that rank $A(\Gamma(R)) = \operatorname{rank} \mathcal{A}(\Gamma_E(R))$ since for every $r \in R$, each element of $[r]_R$ contributes exactly the same row to the adjacency matrix $A(\Gamma(R))$. Thus, it suffices to show that rank $\mathcal{A}(\Gamma_E(R)) = \#V(\Gamma_E(R))$. The matrix $\mathcal{A}(\Gamma_E(R_i))$ for $R_i = \mathbb{Z}_{{p_i}^{t_i}}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & p_i-1 \\ 0 & 0 & 0 & \cdots & 0 & p_i(p_i-1) & p_i-1 \\ 0 & 0 & 0 & \cdots & p_i^2(p_i-1) & p_i(p_i-1) & p_i-1 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & p_i^{t_i-4}(p_i-1) & \cdots & p_i^2(p_i-1) & p_i(p_i-1) & p_i-1 \\ 0 & p_i^{t_i-3}(p_i-1) & p_i^{t_i-4}(p_i-1) & \cdots & p_i^2(p_i-1) & p_i(p_i-1) & p_i-1 \\ p_i^{t_i-2}(p_i-1) & p_i^{t_i-3}(p_i-1) & p_i^{t_i-4}(p_i-1) & \cdots & p_i^2(p_i-1) & p_i(p_i-1) & p_i-1 \end{pmatrix}$$

since $V(\Gamma_E(R_i)) = \{[p_i]_{R_i}, [p_i^2]_{R_i}, \ldots, [p_i^{t_i-1}]_{R_i}\}$ and

$$\#[p_i^k]_{R_i} = \#\{x \mid \gcd(x, p_i^{t_i}) = p_i^k\} = \varphi({p_i}^{t_i}/p_i^k) = {p_i}^{t_i-k-1}(p_i-1).$$

Obviously, this matrix has full rank $\#V(\Gamma_E(R_i))$. Now, the graph $\mathrm{E}\,\Gamma_E(R_i)$ arises from $\Gamma_E(R_i)$ by adding the vertices $[1]_{R_i}$ and $[0]_{R_i}$ and the edges $\{[1]_{R_i}, [0]_{R_i}\}$,$\{[0]_{R_i}, [r]_{R_i}\}$ for all $[r]_{R_i} \in V(\Gamma_E(R_i))$. With an appropriate enumeration of the vertices of $\mathrm{E}\,\Gamma_E(R_i)$, it follows that the matrix $\mathcal{A}(\mathrm{E}\,\Gamma_E(R_i))$ equals

$$\begin{pmatrix} 0 & 0 \ldots 0 & 1 \\ 0 & & 1 \\ \vdots & \mathcal{A}(\Gamma_E(R_i)) & \vdots \\ 0 & & 1 \\ p_i^{t_i-1}(p-1) \;\; p_i^{t_i-2}(p-1) & \ldots \;\; p_i-1 & 1 \end{pmatrix}.$$

This matrix has full rank, too. Since $\mathrm{E}\,\Gamma_E(R) \cong \mathrm{E}\,\Gamma_E(R_1) \times \cdots \times \mathrm{E}\,\Gamma_E(R_r)$, the matrix $\mathcal{A}(\mathrm{E}\,\Gamma_E(R))$ equals the Kronecker product $\mathcal{A}(\mathrm{E}\,\Gamma_E(R_1)) \otimes \cdots \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(R_r))$ which has the form

$$\begin{pmatrix} 0 & 0 \ldots 0 & 1 \\ 0 & & 1 \\ \vdots & \mathcal{A}(\Gamma_E(R)) & \vdots \\ 0 & & 1 \\ x_1 \;\; x_2 & \ldots \;\; x_{\#V(\Gamma_E(R))+1} & 1 \end{pmatrix}$$

for nonzero entries x_j. By the fact that the rank of the Kronecker product of two matrices equals the product of the ranks of these two matrices, we finally conclude that $\mathcal{A}(\mathrm{E}\,\Gamma_E(R))$, and therefore also $\mathcal{A}(\Gamma_E(R))$ has full rank, i.e., rank $\mathcal{A}(\Gamma_E(R)) = \#V(\Gamma_E(R))$. $\square$

With this result, we can easily prove the following corollary, which illustrates the interplay between rings of integers modulo n, zero-divisor graphs and their eigenvalues:

Corollary 4.2 *Let $R \cong \mathbb{Z}_{p_1{}^{t_1}} \times \cdots \times \mathbb{Z}_{p_r{}^{t_r}}$ for prime numbers p_j and $r, t_j \in \mathbb{N}$ and let $\Gamma(R)$ be its zero-divisor graph. If*

$$\#V(\Gamma(R)) - \eta(\Gamma(R)) + 2$$

is a prime number, then R is a local ring (i.e., $r = 1$).

Proof Since $\eta(\Gamma(R)) = \dim A(\Gamma(R)) - \operatorname{rank}(A(\Gamma(R))) = \#V(\Gamma(R)) - \operatorname{rank}(A(\Gamma(R)))$ and $\operatorname{rank}(A(\Gamma(R))) = \#V(\Gamma_E(R))$ by Theorem 4.1, we get that $\#V(\Gamma_E(R)) = \#V(\Gamma(R)) - \eta(\Gamma(R))$. As already mentioned above, if $\#V(\Gamma_E(R)) + 2$ is a prime number, then R must be a local ring. Thus, the statement follows. $\square$

Moreover, we are able to improve Theorem 3.4:

Theorem 4.3 *Let $R \cong \mathbb{Z}_{p_1{}^{t_1}} \times \cdots \times \mathbb{Z}_{p_r{}^{t_r}}$ for prime numbers p_j and $r, t_j \in \mathbb{N}$. Then, the zero-divisor graph $\Gamma(R)$ has*

$$\prod_{i=1}^{r}(t_i + 1) - 2$$

nonzero eigenvalues, and the nullity of $\Gamma(R)$ equals

$$\eta(\Gamma(R)) = \prod_{i=1}^{r} p_i^{t_i-1}\Big(\prod_{i=1}^{r} p_i - \prod_{i=1}^{r}(p_i - 1)\Big) - \prod_{i=1}^{r}(t_i + 1) + 1.$$

Proof By Theorem 4.1, the number of nonzero eigenvalues of $\Gamma(R)$ equals the number of vertices of the compressed zero-divisor graph $\Gamma_E(R)$. Since $V(\Gamma_E(R_i)) = \{[p_i]_{R_i}, [p_i^2]_{R_i}, \ldots, [p_i^{t_i-1}]_{R_i}\}$, we deduce form Theorem 3.2 that

$$\#V(\Gamma_E(R)) = \prod_{i=1}^{r}(t_i + 1) - 2.$$

Similar as in the proof of Theorem 3.4, we see that $\eta(\Gamma(R)) = \#V(\Gamma(R)) - \#V(\Gamma_E(R))$. The number of units in R_i ($= \#U(R_i) = \#V(\mathrm{U}(R_i))$) is given by $\varphi(p_i^{t_i}) = p_i^{t_i-1}(p_i - 1)$. Thus, by Theorem 3.1, we get that

$$\begin{aligned}\#V(\Gamma(R)) &= \prod_{i=1}^{r} p_i^{t_i} - \prod_{i=1}^{r} p_i^{t_i-1}(p_i - 1) - 1\\ &= \prod_{i=1}^{r} p_i^{t_i-1}\left(\prod_{i=1}^{r} p_i - \prod_{i=1}^{r}(p_i - 1)\right) - 1,\end{aligned}$$

and, therefore, the statement follows. □

Note that the number of nonzero eigenvalues of $\Gamma(\mathbb{Z}_{p_1{}^{t_1}} \times \cdots \times \mathbb{Z}_{p_r{}^{t_r}})$ does not depend on the prime numbers p_i but on the powers t_i only.

Now, we can easily determine the eigenvalues of $\Gamma(R)$ for $R \cong \mathbb{Z}_{p_1{}^{t_1}} \times \cdots \times \mathbb{Z}_{p_r{}^{t_r}}$. Theorem 4.3 gives us the number of eigenvalues equal to zero. The nonzero eigenvalues can be computed as in the proof of Theorem 4.1. We illustrate this in the following examples. Note that the eigenvalues of the graphs $\Gamma(\mathbb{Z}_{p^2})$, $\Gamma(\mathbb{Z}_{p^3})$, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ for prime numbers $p \neq q$ were already determined by Young [18] and Surendranath Reddy et. al. [15], and the ones of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ by Sharma et. al. [13].

Example 4.4 Let p be a prime number and $R \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 4.3 the multiplicity of the eigenvalue 0 of $\Gamma(R)$ equals

$$\eta(\Gamma(R)) = \left(p^3 - (p-1)^3\right) - 2^3 + 1 = 3(p+1)(p-2).$$

The ring $\mathbb{Z}_p$ has no zero-divisors and, therefore, $\Gamma(\mathbb{Z}_p)$ is the empty graph. Thus, the matrix $\mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p))$ is given by

$$\mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) = \begin{pmatrix} 0 & 1 \\ p-1 & 1 \end{pmatrix}.$$

Now, we compute the Kronecker product

$$\mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p))$$

which yields the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & p-1 & 1 \\
0 & 0 & 0 & 0 & 0 & p-1 & 0 & 1 \\
0 & 0 & 0 & 0 & (p-1)^2 & p-1 & p-1 & 1 \\
0 & 0 & 0 & p-1 & 0 & 0 & 0 & 1 \\
0 & 0 & (p-1)^2 & p-1 & 0 & 0 & p-1 & 1 \\
0 & (p-1)^2 & 0 & p-1 & 0 & p-1 & 0 & 1 \\
(p-1)^3 & (p-1)^2 & (p-1)^2 & (p-1) & (p-1)^2 & p-1 & p-1 & 1
\end{pmatrix}.$$

Hence, $\mathcal{A}(\Gamma_E(R))$ equals the submatrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & p-1 \\
0 & 0 & 0 & 0 & p-1 & 0 \\
0 & 0 & 0 & (p-1)^2 & p-1 & p-1 \\
0 & 0 & p-1 & 0 & 0 & 0 \\
0 & (p-1)^2 & p-1 & 0 & 0 & p-1 \\
(p-1)^2 & 0 & p-1 & 0 & p-1 & 0
\end{pmatrix},$$

which has characteristic polynomial

$$\chi_{\Gamma(R)}(x) = \begin{aligned}[t] &-(-1+3p-3p^2+p^3+(1-p)x-x^2)^2 \\ &(-1+3p-3p^2+p^3+2(p-1)x-x^2). \end{aligned}$$

The roots of this polynomial, i.e., the nonzero eigenvalues of $\Gamma(R)$, are

$$\lambda_{1,2} = \frac{1}{2}\big(1-p \pm (p-1)\sqrt{4p-3}\big), \quad \lambda_{3,4} = p-1 \pm \sqrt{p-2p^2+p^3},$$

and, therefore, the spectrum of $\Gamma(R)$ equals

$$\operatorname{spec}(\Gamma(R)) = \left\{\lambda_1^{[2]}, \lambda_2^{[2]}, \lambda_3^{[1]}, \lambda_4^{[1]}, 0^{[3(p+1)(p-2)]}\right\} \quad \text{for } p > 2,$$

and

$$\operatorname{spec}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = \left\{\lambda_1^{[2]}, \lambda_2^{[2]}, \lambda_3^{[1]}, \lambda_4^{[1]}\right\}.$$

Example 4.5 Let p be a prime number and $R \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 4.3, the multiplicity of the eigenvalue 0 of $\Gamma(R)$ equals

$$\eta(\Gamma(R)) = p^4 - (p-1)^4 - 2^4 + 1.$$

Analogously as in Example 4.4, we find the matrix $\mathcal{A}(\Gamma_E(R))$ as a submatrix of the Kronecker product

$$\mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) \otimes \mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)).$$

The characteristic polynomial of this matrix is

$$\begin{aligned} \chi_{\Gamma(R)}(x) = &- (1-2p+p^2-x)^5(1-2p+p^2+x)\times \\ &\times \big(1-4p+6p^2-4p^3+p^4+(1+p-2p^2)x+x^2\big)\times \\ &\times \big(1-4p+6p^2-4p^3+p^4+(1-3p+2p^2)x+x^2\big)^3 \end{aligned}$$

and has roots

$$\lambda_1 = (p-1)^2, \quad \lambda_2 = -p^2+p-1,$$

$$\lambda_{3,4} = \frac{1}{2}\big(-2p^2+3p-1 \pm (p-1)\sqrt{4p-3}\big),$$

$$\lambda_{5,6} = \frac{1}{2}\big(2p^2-p-1 \pm \sqrt{3}\sqrt{4p^3-9p^2+6p-1}\big).$$

Hence, the spectrum of $\Gamma(R)$ is given by

$$\operatorname{spec}(\Gamma(R)) = \left\{\lambda_1^{[5]}, \lambda_2^{[1]}, \lambda_3^{[3]}, \lambda_4^{[3]}, \lambda_5^{[1]}, \lambda_6^{[1]}, 0^{[p^4-(p-1)^4-2^4+1]}\right\} \quad \text{for } p > 2,$$

and

$$\operatorname{spec}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = \left\{\lambda_1^{[5]}, \lambda_2^{[1]}, \lambda_3^{[3]}, \lambda_4^{[3]}, \lambda_5^{[1]}, \lambda_6^{[1]}\right\}.$$

Example 4.6 Unfortunately, if we consider not only products of the ring $\mathbb{Z}_p$ but also of rings of the form $\mathbb{Z}_{p^t}$ for $t > 1$ or of the form $\mathbb{Z}_q$ for a prime $q \neq p$, the eigenvalues of $\Gamma(R)$ get very cumbersome. However, at least we want to include the characteristic polynomials of the graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q)$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. Note that

$$\mathcal{A}(\mathrm{E}\,\Gamma_E(\mathbb{Z}_p)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & p-1 & 1 \\ p(p-1) & p-1 & 1 \end{pmatrix}.$$

With the same method as in the latter examples, we find the polynomials

$$\begin{aligned}\chi_{\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q}(x) &= -(p-1)^6(q-1)^3 + (p-1)^3(q-1)\big(p(3q-2)-q\big)x^2 - \\ &\quad -2(p-1)^2(q-1)x^3 - (p-1)\big(p(3q-2)-q\big)x^4 + x^6, \\ \chi_{\mathbb{Z}_{p^2} \times \mathbb{Z}_p}(x) &= (p-1)^5 p + (p-1)^3 px - 2(p-1)^2 px^2 - (p-1)x^3 + x^4.\end{aligned}$$

Remark 4.7 It is clear that two rings are isomorphic only if their respective zero-divisor graphs are isomorphic. Moreover, two graphs are isomorphic only if they have the same characteristic polynomial. Thus, in order to see that two rings are non-isomorphic, it might help to compare the characteristic polynomials of their corresponding zero-divisor graphs.

5 A relation between $\chi_{\Gamma(R)}$ and $\chi_{\mathrm{E}\,\Gamma(R)}$

The main interest in spectral graph theory of building graph products is that there are relations between the eigenvalues of graphs and the eigenvalues of their product. For example, we already mentioned that if λ_i, μ_i denote the eigenvalues of graphs G_1 and G_2, respectively, then the eigenvalues of the direct product $G_1 \times G_2$ are exactly the values $\lambda_i \mu_j$. Similar results are also known for the point identification and the complete product of simple graphs: let $G - v$ denote the graph arising from removing the vertex v of G together with all its edges, and let $\overline{G}$ be the complement of G (that is, the graph with same vertex set as G, where two distinct vertices are adjacent whenever they are non-adjacent in G), then the following holds:

Lemma 5.1 ([6, p. 159]) *Let G_1 and G_2 be simple graphs with $v \in V(G_1)$ and $w \in V(G_2)$. The point-identification $v = w$ yields*

$$\chi_{G_1 \bullet G_2}(x) = \chi_{G_1}(x)\chi_{G_2-w}(x) + \chi_{G_1-v}(x)\chi_{G_2}(x) \\ -x\chi_{G_1-v}(x)\chi_{G_2-w}(x).$$

Lemma 5.2 ([6, Theorem 2.7]) *Let G_1 and G_2 be simple graphs with $\#V(G_1) = n_1$ and $\#V(G_2) = n_2$. Then, the characteristic polynomial of the complete product of G_1 and G_2 equals*

$$\chi_{G_1 \nabla G_2}(x) = (-1)^{n_2}\chi_{G_1}(x)\chi_{\overline{G_2}}(-x-1) + (-1)^{n_1}\chi_{G_2}(x)\chi_{\overline{G_1}}(-x-1) \\ -(-1)^{n_1+n_2}\chi_{\overline{G_1}}(-x-1)\chi_{\overline{G_2}}(-x-1).$$

We can easily see that the formula in Lemma 5.1 still holds for graphs with loops if the graphs do not have loops on both vertices, v and w. Moreover, Hwang and Park [9] generalized the result of Lemma 5.2:

Lemma 5.3 ([9, Theorem 2.5]) *Let $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$, $a, c \in \mathbb{R}^m$, $b, d \in \mathbb{R}^n$,*

$$M = \begin{pmatrix} A & ad^t \\ bc^t & B \end{pmatrix}$$

and $\tilde{A} = ac^t - A$, $\tilde{B} = bd^t - B$. Then

$$\chi_M(x) = (-1)^m \chi_{\tilde{A}}(-x)\chi_B(x) + (-1)^n \chi_A(x)\chi_{\tilde{B}}(-x) \\ -(-1)^{m+n}\chi_{\tilde{A}}(-x)\chi_{\tilde{B}}(-x).$$

Therefore, in the following, let $\overline{G}$ denote the *generalized complement* of G, i.e., the graph with same vertex set as G, where two not necessarily distinct vertices are adjacent in $\overline{G}$ whenever they are non-adjacent in G. That is, the graph with adjacency matrix $A(\overline{G}) = J - A(G)$, where J denotes the all-1 matrix. Now, we are able to prove the following:

Theorem 5.4 *Let R be a finite commutative ring and let $n = \#U(R)$, i.e., the number of units in R. Then, we have that*

$$\chi_{\mathrm{E}\,\Gamma(R)}(x) = x^{n-1}\big((-1)^{n+1}\chi_{\overline{\Gamma(R)}}(-x)x + \chi_{\Gamma(R)}(x)(x^2-n)\big).$$

Proof We recall that

$$\mathrm{E}\,\Gamma(R) = \big(\Gamma(R) \nabla \mathrm{Z}(R)\big) \overset{\{0\}}{\bullet} \big(\mathrm{U}(R) \nabla \mathrm{Z}_{\mathrm{L}}(R)\big).$$

We first determine the characteristic polynomial of $\mathrm{U}(R) \nabla \mathrm{Z}_{\mathrm{L}}(R)$ by applying Lemma 5.3 for $A = \big(1\big)$ and B being the zero-matrix of dimension $n \times n$. We can easily see that $\chi_A(x) = x - 1$, $\chi_{\tilde{A}}(x) = x$, $\chi_B(x) = x^n$ and $\chi_{\tilde{B}}(x) = x^{n-1}(x - n)$.

Thus, we get

$$\begin{aligned}\chi_{\mathrm{U}(R)\nabla \mathrm{Z_L}(R)}(x) &= \chi_M(x)\\ &= (-1)(-x)x^n + (-1)^n(x-1)(-x)^{n-1}(-x-n)-\\ &\quad -(-1)^{n+1}(-x)(-x)^{n-1}(-x-n)\\ &= x^{n-1}(x^2-x-n).\end{aligned}$$

Analogously, we find the characteristic polynomial of $\Gamma(R)\nabla\,\mathrm{Z}(R)$ for $A = \begin{pmatrix}0\end{pmatrix}$ and $B = A(\Gamma(R))$ to be

$$\chi_{\Gamma(R)\nabla\,\mathrm{Z}(R)}(x) = (-1)^{n+1}\chi_{\overline{\Gamma(R)}}(-x) + \chi_{\Gamma(R)}(x)(x+1).$$

Finally, with Lemma 5.1 we get

$$\begin{aligned}\chi_{\mathrm{E}\,\Gamma(R)}(x) &= \chi_{\mathrm{U}(R)\nabla \mathrm{Z_L}(R)}(x)\chi_{\Gamma(R)} + x^n\chi_{\Gamma(R)\nabla\,\mathrm{Z}(R)}(x) - x\cdot x^n\chi_{\Gamma(R)}\\ &= x^{n-1}(x^2-x-n)\chi_{\Gamma(R)}(x)+\\ &\quad + x^n\big((-1)^{n+1}\chi_{\overline{\Gamma(R)}}(-x) + \chi_{\Gamma(R)}(x)(x+1)\big)-\\ &\quad - x^{n+1}\chi_{\Gamma(R)}(x)\\ &= x^{n-1}\big((-1)^{n+1}\chi_{\overline{\Gamma(R)}}(-x)x + \chi_{\Gamma(R)}(x)(x^2-n)\big).\end{aligned}$$

□

Remark 5.5 If $R \cong R_1 \times \cdots \times R_r$, we can apply Theorem 5.4 to each of the rings R_i, which gives us the characteristic polynomials $\chi_{\mathrm{E}\,\Gamma(R_i)}$. By computing the roots of $\chi_{\mathrm{E}\,\Gamma(R_i)}$, we find the eigenvalues of $\mathrm{E}\,\Gamma(R)$ to be all possible products of these roots, since $\mathrm{E}\,\Gamma(R) \cong \mathrm{E}\,\Gamma(R_1)\times\cdots\times\mathrm{E}\,\Gamma(R_r)$. Unfortunately, it is difficult to extrapolate the eigenvalues of $\Gamma(R)$ from the ones of $\mathrm{E}\,\Gamma(R)$, since the characteristic polynomial $\chi_{\Gamma(R)}$ not only depends on $\chi_{\mathrm{E}\,\Gamma(R)}$ but also on the characteristic polynomial of the generalized complement of $\Gamma(R)$.

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