Numerical methods for solving open-loop non zero-sum differential Nash games

Dissertation zur Erlangung
des naturwissenschaftlichen Doktorgrades
der Julius-Maximilians-Universität Würzburg

vorgelegt von

Francesca Calà Campana

Würzburg, July 2021

To my beloved parents:
Maria and Salvatore.

Zusammenfassung

Diese Dissertation handelt von eine theoretischen und numerischen Untersuchung von Methoden zur Lösung von Open-Loop-Nicht-Nullsummen-Differential-Nash-Spielen. Diese Probleme treten in vielen Anwendungen auf, z.B., Biologie, Wirtschaft, Physik, in denen die Konkurrenz zwischen verschiedenen Wirkstoffen bzw. Agenten auftritt. In diesem Fall steht das Ziel jedes Agenten im Gegensatz zu dem der anderen und ein Wettbewerbsspiel kann als gekoppeltes Optimierungsproblem interpretiert werden. Im Allgemeinen gibt es keine optimale Lösung für ein solches Spiel. Tatsächlich kann eine optimale Strategie für einen Spieler für den anderen unbefriedigend sein. Aus diesem Grund wird ein Gleichgewicht eines Spiels als Lösung gesucht, und unter den in der Literatur vorgeschlagenen Lösungskonzepten steht das Nash-Gleichgewicht (NE) im Mittelpunkt dieser Arbeit.

Die Bausteine der resultierenden differenziellen Nash-Spiele sind ein dynamisches System mit unterschiedliche Kontrollfunktionen, die verschiedenen Spielern zugeordnet sind, die nicht kooperative Ziele verfolgen. Der Schwerpunkt dieser Arbeit liegt insbesondere auf Differentialmodellen mit linearen oder bilinearen Strukturen.

In diesem Rahmen werden im ersten Kapitel einige bekannte Ergebnisse präsentiert, insbesondere für die nicht kooperativen linear-quadratischen Differential-Nash-Spiele. Anschließend wird ein bilineares Nash-Spiel formuliert und analysiert. Das Hauptresultat des Kapitels ist Theorem 1.4.2 über die Existenz von Nash Gleichgewichten für die nicht kooperativen bilinearen Differentialspiele. Dieses Ergebnis wird für einen ausreichend kleinen Zeithorizont T erhalten, und eine Schätzung der Endzeit T wird in Lemma 1.4.8 erhalten, die auf spezifischen Eigenschaften der regulierten Nikaido-Isoda-Funktion basiert.

Das bilineare Nash-Spiel wird numerisch in Kapitel 2 gelöst, indem ein halbglattes Newton-Schema mit einer klassischen Relaxationsmethode kombiniert wird. Die erste Wahl wird durch das Vorhandensein von Einschränkungen bei den Kontrollmaßnahmen motiviert, die das Problem nicht differenzierbar machen. Die resultierende Methode wird in Satz 2.1 als lokal konvergent bewiesen. Es wird auch eine Schätzung des Relaxationsparameters erhalten, die den Relaxationsfaktor mit dem Zeitpunkt des Enstehens eines Nash-Gleichgewichts und den anderen Parametern des Spiels in Beziehung setzt.

Für das bilineare Nash-Spiel wird auch ein Nash-Verhandlungsproblem eingeführt und diskutiert, um eine Verbesserung der Ziele aller Spieler in Bezug auf das Nash-Gleichgewicht festzustellen. Insbesondere wird in Theorem 2.2.1 eine Charakterisierung einer Verhandlungslösung gegeben und ein auf diesem Ergebnis basierendes numerisches Schema vorgestellt, um diese Lösung an der Pareto-Front zu finden. Es werden Ergebnisse numerischer Experimente vorgestellt, die auf einem Quantenmodell von zwei Spinpartikeln und einem Populationsdynamik Modell mit zwei konkurrierenden Spezies basieren. Somit werden die vorgeschlagenen Algorithmen erfolgreich validieren.

In Kapitel 3 wird eine Funktional Formulierung des klassischen Nash-Spiels mit mörderischem Chauffeur und Flüchtendem vorgestellt und ein neuer numerischer Rahmen für seine Lösung in einer zeitoptimalen Formulierung diskutiert. Diese Methodik kombiniert ein Hamilton-basiertes Schema mit einer proximalen Strafe, um den Zeithorizont zu bestimmen, in dem das Spiel stattfindet, mit einem Lagrange-Ansatz für die optimale Kontrolle, um das Nash-Spiel zu einer festgelegten Endzeit zu lösen. Das resultierende numerische Optimierungsschema hat eine Bilevel-Struktur, die darauf abzielt, die Berechnung der Endzeit von der Lösung des Verfolgung-Ausweich-Spiel zu entkoppeln. Es werden

mehrere numerische Experimente durchgeführt, um die Fähigkeit des vorgeschlagenen Algorithmus zur Lösung des HC-Spiels zu zeigen. Wenn man sich auf den Fall konzentriert, in dem eine Kollision auftreten kann, wird die Zeit für dieses Ereignis bestimmt.

Der letzte Teil dieser Arbeit befasst sich mit der Analyse eines neuartigen sequentiellen quadratischen Hamilton-Schemas (SQH) zur Lösung von Open-Loop-Differential-Nash-Spielen. Diese Methode wurde im Rahmen des Pontryagin Maximumprinzip (PMP) formuliert und stellt eine effiziente und robuste Erweiterung der Strategie für sukzessive Approximationsstrategie im Bereich der Nash-Spiele dar.

Bei der SQH-Methode werden die Hamilton-Pontryagin-Funktionen durch einen quadratischen Strafterm ergänzt und die Nikaido-Isoda-Funktion wird als Auswahlkriterium verwendet. Basierend auf dieser Tatsache besteht die Schlüsselidee dieses SQH-Schemas darin, dass die PMP-Charakterisierung von Nash-Spielen für jede festgelegte Zeit zu einem endlichdimensionalen Nash-Spiel führt. Somit wird eine Klasse von Problemen identifiziert, für die dieses endlichdimensionale Spiel eine eindeutige Lösung zulässt, und für diese Klasse von Spielen werden theoretische Ergebnisse präsentiert, die die Wohldefiniertkeit des vorgeschlagenen Schemas beweisen. Insbesondere Proposition 4.2.1 zeigt, dass das Auswahlkriterium für die Nikaido-Isoda-Funktion erfüllt ist.

Ein Vergleich der benötisten Rechenzeiten des SQH-Schemas und der zuvor diskutierten halb glattes Newton-Relaxationsmethode wird gezeigt.

Es werden Anwendungen für linear-quadratische Nash-Spiele und Varianten mit Kontrollbeschränkungen, gewichteten L^1-Kosten der Aktionen der Spieler und Verfolgungszielen vorgestellt, die die theoretischen Aussagen bestätigen.

Abstract

This thesis is devoted to a theoretical and numerical investigation of methods to solve open-loop non zero-sum differential Nash games. These problems arise in many applications, e.g., biology, economics, physics, where competition between different agents appears. In this case, the goal of each agent is in contrast with those of the others, and a competition game can be interpreted as a coupled optimization problem for which, in general, an optimal solution does not exist. In fact, an optimal strategy for one player may be unsatisfactory for the others. For this reason, a solution of a game is sought as an equilibrium and among the solutions concepts proposed in the literature, that of Nash equilibrium (NE) is the focus of this thesis.

The building blocks of the resulting differential Nash games are a dynamical model with different control functions associated with different players that pursue non-cooperative objectives. In particular, the aim of this thesis is on differential models having linear or bilinear state-strategy structures.

In this framework, in the first chapter, some well-known results are recalled, especially for non-cooperative linear-quadratic differential Nash games. Then, a bilinear Nash game is formulated and analysed. The main achievement in this chapter is Theorem 1.4.2 concerning existence of Nash equilibria for non-cooperative differential bilinear games. This result is obtained assuming a sufficiently small time horizon T, and an estimate of T is provided in Lemma 1.4.8 using specific properties of the regularized Nikaido-Isoda function.

In Chapter 2, in order to solve a bilinear Nash game, a semi-smooth Newton (SSN) scheme combined with a relaxation method is investigated, where the choice of a SSN scheme is motivated by the presence of constraints on the players' actions that make the problem non-smooth. The resulting method is proved to be locally convergent in Theorem 2.1, and an estimate on the relaxation parameter is also obtained that relates the relaxation factor to the time horizon of a Nash equilibrium and to the other parameters of the game.

For the bilinear Nash game, a Nash bargaining problem is also introduced and discussed, aiming at determining an improvement of all players' objectives with respect to the Nash equilibrium. A characterization of a bargaining solution is given in Theorem 2.2.1 and a numerical scheme based on this result is presented that allows to compute this solution on the Pareto frontier. Results of numerical experiments based on a quantum model of two spin-particles and on a population dynamics model with two competing species are presented that successfully validate the proposed algorithms.

In Chapter 3 a functional formulation of the classical homicidal chauffeur (HC) Nash game is introduced and a new numerical framework for its solution in a time-optimal formulation is discussed. This methodology combines a Hamiltonian based scheme, with proximal penalty to determine the time horizon where the game takes place, with a Lagrangian optimal control approach and relaxation to solve the Nash game at a fixed end-time. The resulting numerical optimization scheme has a bilevel structure, which aims at decoupling the computation of the end-time from the solution of the pursuit-evader game. Several numerical experiments are performed to show the ability of the proposed algorithm to solve the HC game. Focusing on the case where a collision may occur, the time for this event is determined.

The last part of this thesis deals with the analysis of a novel sequential quadratic Hamiltonian (SQH) scheme for solving open-loop differential Nash games. This method is formulated in the framework of Pontryagin's maximum principle and represents an efficient and robust extension of the successive approximations strategy in the realm of Nash games. In the SQH method, the Hamilton-Pontryagin functions are augmented by a quadratic penalty term and the Nikaido-Isoda function is used as a selection criterion. Based on this fact, the key idea of this SQH scheme is that the PMP characterization of Nash games leads to a finite-dimensional Nash game for any fixed time. A class of problems for which this finite-dimensional game admits a unique solution is identified and for this class of games theoretical results are presented that prove the well-posedness of the proposed scheme. In particular, Proposition 4.2.1 is proved to show that the selection criterion on the Nikaido-Isoda function is fulfilled. A comparison of the computational performances of the SQH scheme and the SSN-relaxation method previously discussed is shown. Applications to linear-quadratic Nash games and variants with control constraints, weighted L^1 costs of the players' actions and tracking objectives are presented that corroborate the theoretical statements.

Acknowledgements

This PhD has been an incredible journey that deeply changed my attitude to life. This would have not been possible without all the people that helped me during the last years. I wish to thank all of them.

Before any other person, I would thank who made all this possible: my supervisor Prof. Alfio Borzì. I am very grateful for the great opportunity he gave me to study in Würzburg, giving me trust and always encouraging, supporting and helping me. With his invaluable guide, he also taught me to manage the most stressful situations, to be stronger and to never give up. Thank you!

A huge thank is to Dr. Gabriele Ciaramella for being a fixed-point during the hardest time of this PhD. I will never forget the three days spent in Konstanz and all the skype-calls made in the last years. He always had the right words at the right time and strongly contributed to develop my self-esteem. Many thanks, Gabriele!

I would thank Prof. Abderrahmane Habbal for the detailed reading of this manuscript and all the interesting conversations about game and optimization theory. With his constructive criticism, he helped me to improve the quality of my work.

I wish also thank Prof. Matthias Gerdts and Alberto De Marchi for the nice time spent in Munich and Würzburg discussing about mathematics and bilevel optimization.

I want also to thank all my colleagues and the MoWiRe group, in particular, Ms. Petra Markert-Autsch for her invaluable assistance during all these years. Without her constant support, this PhD would have been harder.

My life in Würzburg would have not been the same without the nice people I met that helped me to feel less alone and well-liked. A special thank is to my dear friend and colleague Francesco and to Manon, Lara and Luka. I hope to meet all of you again!

I wish to express all my gratitude to my family for all the love and support given to me. In particular, to my father for always let me think that everything is possible, and to my mother for being an extraordinary example of a strong and courageous woman. An honorable mention is for my beloved sister for convincing me to apply for this PhD, almost four years ago. Thank you for always encouraging me to do all my best and for helping me to get up every time I fall. Grazie!

Last but not least, I would thank Fabrizio for all the love, support and happy moments spent together in these years. Thank you for being always by my side and for never let me feel alone. Grazie!

26.07.2021

Francesca

Contents

Introduction

Since their appearance [69, 70], Nash games have attracted the attention of many scientists as they provide a convenient mathematical framework to investigate problems of competition and cooperation. A competition game can be interpreted as a coupled optimization problem for which the Nash equilibrium (NE) defines a solution concept. In general, new solution concepts have been defined that lead to the concept of equilibrium solutions which allow to model various situations, depending on the information available to the players. For example, one can consider games with asymmetry of information, where one player represents a leader which selects its strategy in advance and the others play their strategies accordingly. These problems lead to a solution in the sense of a Stackelberger equilibrium [87]. On the other hand, there are a lot of symmetric situations where the players have no reasons to cooperate and do not share any information about their strategies. In this framework, a solution is sougth as a NE.

Non-cooperative differential Nash games were introduced in [52], where differential (dynamical) models govern the state of the system that is subject to the action of different controls representing the strategies of the players in the game, and to each player is associated an objective (cost) functional. With this setting, a NE is obtained when no player can improve its objective by unilateral change of its strategy. However, in many applications, players have willingness to cooperate to get an improvement in their costs. To model these situations, one can consider the concept of Pareto optimality. In this framework, a cooperation game can be interpreted as a parameterized optimal control problem [41]. Since, in general, a Pareto solution is not unique, it is reasonable to investigate whether there is a preferable choice with respect to NE. In this sense, there is the so-called bargaining theory. The most commonly bargaining solutions are the Nash bargaining solution [68], the Kalai-Smorodinsky solution [56] and the egalitarian solution [55]. In this thesis, we deal with the Nash bargaining problem.

Since the study of differential games initiated by Isaacs, many authors have focused on games having the so-called zero-sum property, that is, there is only one objective cost that one player tries to maximize and the other tries to minimize. However, later in [89], A.W. Starr and Y.C. Ho introduced noncooperative non zero-sum differential games.

Differential games have received much attention in the field of economics and marketing [37, 54], and are well investigated in the case of linear differential models with linear player's action mechanism and quadratic objectives; see, e.g., [21, 40, 43, 94]. For this class of games, P. Varaiya [94] proved that open-loop Nash equilibria exist for a sufficiently small duration of the game. Few years later, R.C. Scalzo [84] extended the work of Varaiya to linear-quadratic differential games with players' strategies constrained in compact and convex subsets of $\mathbb{R}^n$, proving that a NE exists for any arbitrary time.

On the other hand, much less is known in the case of nonlinear models, especially in the case of nonlinear functions of state and strategy and, in particular, in the case of a bilinear structure, where a function of the state variable multiplies the players' actions. For a past review of works on differential Nash games, we refer to [92], and for more recent and detailed discussion and references see [40, 44, 54]. We also remark that nonlinear differential NE games have been investigated in the framework of Young measures; see, e.g., [5, 92]. We remark that Nash games are related to multi-objective optimization with nonlinear control structure, and therefore we believe that a study of NE game problems would be beneficial also for further development and application in this and related fields. In particular, our work could be extended to differential games with partial differential equations that are of interest in many applications. In fact, they appear in multi-objective shape optimization problems [78, 10], inverse problems [47, 48] and multi-agent problems [81].

A focus of this thesis is dynamical models with bilinear strategy-state structure, also called linear-affine systems. These models play a central role in many applications [72]. In particular, they are omnipresent in the field of quantum control problems [13] and in biology. However, the bilinear structure poses additional theoretical and numerical difficulties that hinder the further application of the NE framework to many envisioned problems. This is in particular true for new problems involving quantum systems [45] and biological systems [22]. For this reason, we discuss two related dynamical models that can both be put in the following general structure

$$y' = f^0(t, y) + F(t, y)u, \quad y(0) = y_0, \tag{1}$$

where $y(t) \in \mathbb{R}^n$ represents the n-dimensional state of the system at the time t, y_0 is the initial condition at time $t = 0$ and $u = (u_1, \ldots, u_N)$ represents the vector of strategy functions. The function f^0 governs the free dynamics of the system, and the function F denotes the interaction coupling between the players' actions and the the state of the system. In the bilinear case, F is linear in y.

We present a theoretical and numerical investigation of methods to solve Nash games governed by (1), where the action components u_i, $i = 1, \ldots, N$, represent N players, choosing their strategy in an admissible set U^i_{ad}, and to each player we associate a different cost functional such that a non-cooperative problem is defined. In particular, we consider the following reduced objectives

$$\tilde{J}_i(u_1, \ldots, u_N) = \int_0^T \ell_i(t, y(t), u_1(t), \ldots, u_N(t))dt + g_i(y(T)), \quad i = 1, \ldots, N, \tag{2}$$

where ℓ_i, g_i, $i = 1, \ldots, N$, are continuous and twice continuously differentiable with respect to the variables y, u_i. In (2), y denotes the solution of the differential model with the given u_i and a specified initial condition.

In this framework, the strategy $u^* = (u_1^*, \ldots, u_N^*) \in U_{ad}$ is a NE for the game (1)-(2) if

$$\tilde{J}_i(u_1^*, \ldots, u_N^*) \leq \tilde{J}_i(u_1^*, \ldots, u_i, \ldots, u_N^*), \quad u_i \in U^i_{ad}, \ i = 1, \ldots, N. \tag{3}$$

Thus, our first goal is to prove existence of a NE for this class of games. With this motivation, we consider the theoretical framework introduced by P. Varaiya in [94] for linear dynamics and convex objectives, which we extend to our nonlinear case. For this purpose, we consider the linearised problem related to (1) and provide some properties of the state function. Then, we consider the regularized Nikaido-Isoda function and prove some fundamental properties which allow us to prove that a NE exists if a sufficiently short time horizon, $T \leq T_0$, is considered, and we provide an estimate of T_0.

With this knowledge, we turn our attention to the numerical realisation of the NE solution. In the literature, an iterative method for solving NE problems is the so-called relaxation scheme proposed by J.B. Krawczyk in [58]. Thus, to solve our bilinear game, we implement and analyse a relaxation scheme that we combine with a semi-smooth Newton (SSN) method [28, 29, 30], where the latter choice is motivated by the presence of constraints on the players' actions that make our game problem non-smooth. We prove that this numerical scheme is locally convergent, further providing an estimate on the relaxation parameter depending on the time T_0 of existence of the NE sought and on the parameters of the game.

However, it is well known that a Nash equilibrium does not provide an efficient solution with respect to the objectives that the players could achieve by agreeing to cooperate negotiating the values of the objectives that they can jointly generate. For this purpose, J.F. Nash proposed in [68] a bargaining strategy of jointly improving efficiency while keeping close to the strategy of a NE point. The idea is to find a point that is Pareto optimal and symmetric (the labelling of the players should not matter) that maximizes the surplus of objective cost for both players. In this way, we obtain strategies that represent an improvement toward the task of approaching the desired targets while keeping their costs as small as possible.

In order to implement this goal, we consider the work of H. Ehtamo [38] that we extend to our model. Hence, we use the Nash characterization of a bargaining solution to construct a method to explore the Pareto frontier and converge to the solution sought. For this task, we determine a Pareto point based on its characterization as a solution of a bilinear optimal control problem with a cost functional resulting from a composition of the players' objectives. Further, we consider the framework illustrated by J. Engwerda in [38], see also [40], to reformulate the characterization of a Nash bargaining (NB) solution and use this characterization to construct an iterative scheme that converges to this solution on the Pareto frontier.

In the framework of relaxation schemes for solving differential Nash games, another topic of this thesis is to develop a numerical scheme for solving a non-zero sum Homicidal Chauffeur game in a time-optimal formulation. The Homicidal Chauffeur (HC) game is a classic pursuit-evader problem in the field of dynamical differential games that was introduced by R. P. Isaacs in [51] and further elaborated in his seminal book [52]. The statement of the problem is that of a car with a limited radius of turn and constant velocity that pursues a pedestrian, whose velocity is bounded by a given value, that tries to prevent collision. We shall refer to the car as the pursuer and to the pedestrian as the evader. Both are players of the HC game.

This is a continuous pursuit-evasion game that can be considered the archetypal of problems of this class and has motivated much research work with early fundamental contribution as in [65]. We refer to [73] for a review of results and a survey of the literature on this topic. We remark that these works focus on a geometrical setting of the HC game and construct solutions based on optimal trajectories and singular lines that disperse, join or refract. Also the numerical algorithms are devoted to computation of the level sets of the value function of the game. In particular, Isaacs investigated the HC game using a particular method for solving partial differential equations based on backward computation of characteristics.

On the other hand, in forthcoming works, different functional settings of pursuit-evasion differential games were considered where the actions of the players is modelled by time-dependent functions, and the aim of each player and the cost of its action are formulated in terms of a cost functional that defines the player's objective. In this framework, zero-sum versions of the HC problem were proposed in [8, 49]; see [9] for a review. Furthermore, and also based on the solution concept of Nash equilibrium (NE), nonzero-sum pursuit-evasion games were introduced in [89]; see also [62] for a recent contribution and further references in this field. However, we notice that in many of the latter references the pursuit-evasion game is considered in a fixed time horizon and only few works address the case of a free end-time; see, e.g., [20]. Furthermore, we remark that, in many research works on the HC game, the focus is on theoretical results as the existence of NE, whereas less effort has been put in the construction of algorithms that accommodate the functional framework and provide a solution.

It is the purpose of Chapter 3 to develop a relaxation scheme with free end-time subproblems and proximal penalty for solving a non-zero sum HC game in a time-optimal formulation. The proposed scheme has a bilevel structure [34] where the outer procedure determines the optimal time horizon in a Hamiltonian framework including proximal penalty, and the inner procedure solves a HC game in a fixed time interval by using an optimal control strategy. Then a relaxation step is performed to get a common end-time for the players and a new players' action couple. At convergence these strategies together with the end-time will give the equilibrium solution of the HC game. Further, by choosing appropriate values for the weights of the cost functionals, we focus on the case where collision may occur and, correspondingly, determine the time for this event.

The third part of this thesis is devoted to the analysis of a sequential quadratic Hamiltonian (SQH) scheme for solving differential Nash games. As we have seen at the beginning of our discussion, differential Nash games where pioneered by R. P. Isaacs [52]. However, contemporary to Isaacs' book, there are the works [74, 75] where differential games are discussed in the framework of the Pontryagin's maximum principle.

With this motivation, we formulate an extension of the sequential quadratic Hamiltonian scheme, proposed in [16, 17, 18, 19] for solving nonsmooth optimal control problems governed by differential models, to solve NE game problems. The SQH scheme belongs to the class of iterative methods known as successive approximations (SA) schemes that are based on the characterisation of optimality in control problems by the Pontryagin's maximum principle (PMP); see [11, 76] and [36] for a recent detailed discussion. The initial development of SA schemes was inspired by the work of L. I. Rozonoér [82], and originally proposed in different variants by H.J. Kelley, R.E. Kopp and H.G. Moyer [57] and by I.A. Krylov and F.L. Chernous'ko [59, 60]; see [27] for an early review.

The working principle of most SA schemes is the point-wise minimisation of the Hamilton-Pontryagin function introduced in the PMP theory. However, in their original formulation, the SA schemes appeared efficient but not robust with respect to the numerical and optimisation parameters. Twenty years later, a great improvement in robustness was achieved by Y. Sakawa and Y. Shindo [83, 86] by introducing a quadratic penalty of the control updates that resulted in an augmented Hamiltonian. This approach has been elaborated further with the SQH scheme where a sequential point-wise optimisation of an augmented Hamiltonian function is considered that defines a suitable update step for the control variable while the state function is updated after the completion of this step. Since the SQH iterative procedure has proved efficient and robust in solving (non-smooth) optimal control problems, we investigate whether this framework can be successfully applied to differential games.

Notice that, in [3, 60] we find early attempts and comments towards the development of a SA scheme for differential games. Unfortunately, less attention was paid to this research direction and the further development of these schemes for differential games was left out.

In Chapter 4, we contribute to this development by investigating a SQH scheme for solving open-loop non zero-sum two-players differential Nash games. In this method, the Hamilton-Pontryagin functions are augmented by a quadratic penalty term and the Nikaido-Isoda function is used as a selection criterion. In particular, we consider linear-quadratic (LQ) Nash games that appear in the field of, e.g., economics and marketing [37, 54], and are very well investigated from the theoretical point of view; see, e.g., [21, 40, 43, 94]. For this class of games, the point-wise PMP characterisation of a Nash equilibrium leads, for any time instant, to a finite-dimensional Nash game. In this framework, we prove the well-posedness of the proposed scheme. Moreover, since the solution of unconstrained LQ Nash games can be readily obtained by solving coupled Riccati equations [12, 40], they provide a convenient benchmark for our method. However, we also consider extension of LQ Nash games to problems with tracking objectives, box constraints on the players' actions, and actions' costs that include L^1 terms.

This thesis is organised as follows. The next chapter is devoted to an introduction to differential Nash games and the characterization of their solutions in the PMP framework. We start our discussion with some results in the field of optimal control theory that are used to analyse differential games. Further, we illustrate the NE solution concept and state the set of necessary conditions for optimality. Moreover, we discuss cooperative differential games, illustrating the Pareto efficient solutions and providing necessary and sufficient conditions for Pareto optimality. We describe the bargaining problem with focus on the Nash bargaining solution. Then, we provide existence results of Nash equilibria. In particular, well-known results for linear-quadratic differential Nash games are recalled. In Section 1.4.2, bilinear Nash games are formulated and analysed. For this purpose, we introduce the related linearised model and present results addressing the Fréchet differentiability of our model and other functional properties of the components of the dynamical Nash game. The regularized Nikaido-Isoda function is introduced and analysed in Lemma 1.4.8. In Theorem 1.4.2, we prove existence of NE for a bilinear game. A section of conclusion completes this chapter.

In Chapter 2, we illustrate the numerical framework for solving a NE differential bilinear game, which requires to introduce first-order optimality conditions that must be satisfied by the NE solution and discuss a semi-smooth Newton scheme that is applied to this system in such a way to obtain partial updates for the game strategies sought. Thereafter, we show how these updates are combined in a relaxation scheme in order to construct an iterative procedure that converges to the NE point. The convergence properties of this relaxation scheme are discussed in Theorem 2.1. In Section 2.2, the analysis of the Nash bargaining problem is presented, where we prove a characterization of a solution to this problem, and correspondingly define a solution procedure for its calculation. In Section 2.3, we present results of numerical experiments based on a quantum model of two spin-particles and in Section 2.4 we apply our numerical methodologies to a Lotka-Volterra model of population dynamics with two competing species. In all these experiments the two algorithms are able to determine the NE and NB solutions sought. A section of summary concludes this chapter.

In Chapter 3, we illustrate the dynamical system modeling the motion of the evader and pursuer and including the player's actions mechanisms. Further, we introduce the functional objectives of these two players and define the corresponding NE problem. Also in this section, we draw a connection with optimal control problems and discuss the (partial) characterization of the NE solution in terms of optimality systems.

In Section 3.4, we present our numerical framework that combines an optimal control strategy and relaxation with a method for determining the time horizon of the game. This scheme represents an extension of methods proposed in previous works [34, 64, 46]. To facilitate the presentation of our algorithm, we first discuss some subproblems. Section

3.5 is devoted to the numerical validation of the proposed NE game formulation and of the solution procedure. With our method, we are able to find different solutions of the HC problem starting from different initialisations. On the other hand, small changes of the optimization weights result in similar solutions. A section of summary concludes this chapter.

In Chapter 4, we formulate another class of differential Nash games and discuss their characterisation in the PMP framework. In particular, we notice that the point-wise PMP characterisation of a Nash equilibrium corresponds to the requirement that, at each time instant, the conditions of equilibrium of a finite-dimensional Nash game with two Hamilton-Pontryagin functions must be satisfied. In Section 4.2, we present our SQH method for solving differential Nash games and discuss its well-posedness. Specifically, we show that the adaptive choice of the weight of a Sakawa-Shindo-type penalisation can be successfully performed in a finite number of steps such that the proposed update criteria based on the Nikaido-Isoda function are satisfied.

At the end of the chapter, we present numerical experiments that successfully validate our computational framework. In the first experiment, we consider a differential LQ Nash game and show that the SQH scheme provides a solution that is identical to that obtained by solving an appropriate Riccati system. In the second experiment, the same problem with the addition of the requirement that the players' actions must belong to given bounded, closed and convex sets is solved by the SQH scheme. In the third experiment, we extend the setting of the second experiment by adding weighted L^1 costs of the players' actions and verify that these costs promote their sparsity. In the fourth experiment, we consider the case where each cost functional corresponds to a tracking problem where the players aim at following different paths. Also in this case, constraints on the players' actions and L^1 costs are considered. We remark that all NE solutions are successfully computed with the same setting of values of the parameters entering in the SQH algorithm, that is, independently of the problem and of the chosen weights in the cost functionals.

In Section 4.6 the computational performances of the relaxation-Newton method and of the SQH scheme are compared by applying these two methodologies to the bilinear quantum game and the competitive Lotka-Volterra problem. It is shown that, the SQH method performs better than the relaxation-Newton method for the bilinear quantum game, whereas the relaxation-Newton scheme appears more efficient in determining a NE for the Lotka-Volterra model. This chapter ends with a section of conclusion.

An Appendix is included to collect auxiliary results on ordinary differential equations, on a modified Crank-Nicolson scheme and on the geometric reformulation of the HC game. Also in the Appendix, the MATLAB codes that implement our schemes are described.

The results presented in this thesis are partially based on the following publications:

- [24] F. Calà Campana, G. Ciaramella, A. Borzì, Nash equilibria and bargaining solutions of differential bilinear games, *Dynamic Games and Applications*, 11:1-28, 2021. https://doi.org/10.1007/s13235-020-00351-2

- [25] F. Calà Campana, A. De Marchi, A. Borzì, M. Gerdts, On the numerical solution of a free end-time Homicidal Chauffeur game. *FGS'2019, ESAIM: Proceedings and Surveys*, 2021.

- [23] F. Calà Campana, A. Borzì, On the SQH method for solving differential Nash games, *Journal of Dynamical and Control Systems*, 2021. https://doi.org/10.1007/s10883-021-09546-1

Chapter 1

Differential Nash games

In this chapter, some results in the field of optimal control problems governed by differential equations are recalled and applied to differential games. The concept of Nash equilibrium (NE) for noncooperative differential games is recalled and a characterization of the NE in the framework of the Pontryagin's maximum principle is discussed. The Nikaido-Isoda function is defined. Cooperative games are also introduced and Pareto and bargaining solutions are defined. Results on existence of NE for the classes of problems discussed in this thesis are given.

1.1 Some results in optimal control theory

In this section, we summarize well known results in the field of optimal control problems concerning necessary optimality conditions and the Pontryagin's maximum principle (PMP) that we shall use throughout this thesis.

For this purpose, the following optimal control problem is considered

$$\begin{aligned} &\min_{y,u} J(y,u) := \int_0^T \ell(t,y(t),u(t))dt + g(y(T)) \\ &\text{s.t. } y' = f(t,y,u), \quad y(0) = y_0 \\ &u \in U_{ad}, \end{aligned} \tag{1.1}$$

where $t \in [0,T], y(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, with $m \leq n$ and $y' := \frac{dy}{dt}$ denotes the time derivative of y. The functions ℓ, g are supposed to be continuous and twice continuously differentiable with respect to the variables y, u and bounded from below by zero. Further, we assume that f is continuous in (t,y,u), Lipschitz-continuous in y and that for any fixed y and $u \in U_{ad}$, the function $f(t,y,u(t))$ is measurable in t. The set U_{ad} is a closed, convex and bounded subset of $L^2(0,T;\mathbb{R}^m)$.

Moreover, for any $c > 0$, we assume the existence of a positive function $m_c(t) \in L^1(0,T)$ such that

$$|f(t,y,u(t))| \leq m_c(t), \qquad (t,y) \in R,$$

where the set R is defined as follows

$$R := \{(t,y) : \ t \in [0,T], \ \ |y - y_0| \leq c\}.$$

With these assumptions we can apply the Carathéodory theorem, see, e.g., [12], to state the existence and uniqueness of a solution $y : [0,T] \to \mathbb{R}^n$ to the Cauchy problem in (1.1) which is absolutely continuous on $[0,T]$ and satisifes the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s), u(s))ds,$$

for any $t \in [0,T]$ and $u \in U_{ad}$. In this thesis, the set U_{ad} represents the admissible set of controls defined as follows

$$U_{ad} := \{u \in L^2(0,T;\mathbb{R}^m) : \, u(t) \in K_{ad}, \, t \in [0,T]\}, \tag{1.2}$$

where K_{ad} is a compact and convex subset of $\mathbb{R}^m$.

Next, we discuss the existence of a solution to the problem (1.1). If the functional J is weakly lower semi-continuous, then the existence of a minimizer can be proved with variational techniques; see, e.g., [12]. In the following, we shall consider continuous and convex cost functionals and linear or bilinear control mechanism. The optimal control problem (1.1) is well defined and admits a solution. For these models, the control-to-state map $u \mapsto y = y(u)$ is well-posed and continuous in u. Therefore, we can introduce the reduced cost functional $\tilde{J}(u) := J(y(u), u)$ and the corresponding reduced problem

$$\begin{aligned} &\min_u \tilde{J}(u) \\ &\text{s.t. } \; u \in U_{ad}. \end{aligned} \tag{1.3}$$

If u^* is a solution to (1.3), then the pair $(y(u^*), u^*)$ solves (1.1).

To continue our discussion, we illustrate the characterization of a solution to (1.1) by using the PMP. For this purpose, we introduce the Hamilton-Pontryagin (HP) function as follows

$$\mathcal{H}(t, y, u, p) := p \cdot f(y, y, u) - \ell(t, y, u), \tag{1.4}$$

where $p(t)$ represents the so-called adjoint variable, which is the unique solution to the following adjoint problem

$$-p'(t) = \left(\partial_y f(t, y(t), u(t))\right)^\top p(t) - \partial_y \ell(t, y(t), u(t)), \tag{1.5}$$

$$p(T) = -\partial_y g(y(T)), \tag{1.6}$$

where $\partial_y \phi(y)$ represents the Jacobian of ϕ with respect to the vector of variables y, and $\top$ means transpose.

Let u^* be an optimal solution to (1.3) and $y^* = y(u^*)$ be the corresponding state. Further, let $p^* = p(u^*)$ be the absolutely continuous function satisfying (1.5)-(1.6) with the given y^*, u^*. Then the PMP states that (y^*, u^*, p^*) must necessarily satisfy the following optimality system

$$\begin{aligned} &(y^*)'(t) = \partial_p \mathcal{H}(t, y^*(t), u^*(t), p^*(t)), \quad y^*(0) = y_0 \\ &(p^*)'(t) = -\partial_y \mathcal{H}(t, y^*(t), u^*(t), p^*(t)), \quad p^*(T) = -\partial_y g(y^*(T)) \\ &\mathcal{H}(t, y^*(t), u^*(t), p^*(t)) \geq \mathcal{H}(t, y^*(t), v, p^*(t)), \;\; v \in K_{ad}, \text{ for almost all } t \in [0,T], \end{aligned} \tag{1.7}$$

where, as above, we denote with $\partial_p \phi(p)$ the Jacobian of ϕ with respect to the vector variable p. Notice that the latter condition means that at optimality the Hamilton-Pontryagin function has an extremum in u^*, $t \in [0,T]$. For a proof of the PMP see [12, 36] and reference therein.

Moreover, in the case that $\mathcal{H}$ is differentiable in u, the last condition is replaced with

$$\frac{\partial \mathcal{H}}{\partial u}(t, y^*(t), u^*(t), p^*(t))(v - u^*(t)) \leq 0, \tag{1.8}$$

for all $v \in K_{ad}$ and almost all $t \in [0, T]$.

In terms of the HP function, the PMP condition states the existence of a multiplier adjoint variable $p : [0, T] \to \mathbb{R}^n$ such that for almost all $t \in [0, T]$, the following holds

$$\max_{v \in K_{ad}} \mathcal{H}(t, y^*(t), v, p^*(t)) = \mathcal{H}(t, y^*(t), u^*(t), p^*(t)). \tag{1.9}$$

1.2 Nash games and PMP characterization

In this section, we illustrate the extension of the results seen above in the field of optimal control problems to differential games. We start with a formulation of differential Nash games and the characterization of their NE solution in the PMP framework. In particular, we consider a dynamic game which takes place over the time interval $[0, T]$ and we discuss the case of two players, which can be readily extended to the case of N players, and assume the following dynamics

$$y'(t) = f(t, y(t), u_1(t), u_2(t)), \qquad y(0) = y_0, \tag{1.10}$$

where $t \in [0, T]$, $y(t) \in \mathbb{R}^n$, $u_1(t) \in \mathbb{R}^m$ and $u_2(t) \in \mathbb{R}^m$, $m \leq n$. As in the previous section, we assume that f is such that for any choice of the initial condition $y_0 \in \mathbb{R}^n$, and any u_1, $u_2 \in L^2(0, T; \mathbb{R}^m)$, the Cauchy problem (1.10) admits a unique solution in the sense of Carathéodory; see, e.g., [12]. Further, we assume that the map $(u_1, u_2) \mapsto y = y(u_1, u_2)$, where $y(u_1, u_2)$ represents the unique solution to (1.10) with fixed initial conditions, is continuous in (u_1, u_2).

We refer to u_1 and u_2 as the game strategies of the players P_1 and P_2, respectively. In our game problem, the goal of P_1 is to minimise the following cost functional

$$J_1(y, u_1, u_2) := \int_0^T \ell_1(t, y(t), u_1(t), u_2(t))\, dt + g_1(y(T)), \tag{1.11}$$

whereas P_2 aims at minimising its own objective given by

$$J_2(y, u_1, u_2) := \int_0^T \ell_2(t, y(t), u_1(t), u_2(t))\, dt + g_2(y(T)). \tag{1.12}$$

We require that the strategies belong to the following admissible sets

$$U_{ad}^{(i)} = \{u \in L^2(0, T; \mathbb{R}^m) : \, u(t) \in K_{ad}^{(i)}, \, t \in [0, T]\}, \quad i = 1, 2, \tag{1.13}$$

Let $U_{ad} = U_{ad}^{(1)} \times U_{ad}^{(2)}$. We assume that each player has complete information on the dynamics of the system represented by the function f, the initial state y_0, the sets $U_{ad}^{(1)}, U_{ad}^{(2)}$, the two cost functionals J_1, J_2, and the current time $t \in [0, T]$.

Further, we shall consider the cases of unconstrained and constrained strategies. We assume $u_1 \in K_{ad}^{(1)}$ and $u_2 \in K_{ad}^{(2)}$, where $K_{ad}^{(i)}$ are some compact and convex subsets of $\mathbb{R}^m$ that we specify in the following chapters.

We consider open-loop strategies, i.e., apart from the initial condition y_0, player P_i, $i = 1, 2$, cannot have any information on the state of the system and on the strategy selected by the other player. For this reason, the strategies implemented by the players are functions of time only, $u_i(t)$, $i = 1, 2$, $t \in [0, T]$. Another type of strategies that we do not developed in this thesis are the feedback ones, where the players can observe also the current state of the system and hence the players' actions are functions of the variables t and y, $u_i(t, y)$, $i = 1, 2$.

By using the map $(u_1, u_2) \mapsto y = y(u_1, u_2)$, we can introduce the reduced objectives $\tilde{J}_1(u_1, u_2) := J_1(y(u_1, u_2), u_1, u_2)$ and $\tilde{J}_2(u_1, u_2) := J_2(y(u_1, u_2), u_1, u_2)$. In this framework, a Nash equilibrium is defined as follows.

Definition 1. *The functions* $(u_1^*, u_2^*) \in U_{ad}$ *are said to form a Nash equilibrium (NE) for the game* $(\tilde{J}_1, \tilde{J}_2; U_{ad}^{(1)}, U_{ad}^{(2)})$*, if it holds*

$$\begin{aligned} \tilde{J}_1(u_1^*, u_2^*) &\leq \tilde{J}_1(u_1, u_2^*), \qquad u_1 \in U_{ad}^{(1)}, \\ \tilde{J}_2(u_1^*, u_2^*) &\leq \tilde{J}_2(u_1^*, u_2), \qquad u_2 \in U_{ad}^{(2)}. \end{aligned} \tag{1.14}$$

Thus, no player can get an improvement in its own objective by unilaterally deviating its strategy from the equilibrium solution.

Notice that, in general, existence of a NE requires local convexity of the maps $u_1 \mapsto \tilde{J}_1(u_1, u_2^*)$ and $u_2 \mapsto \tilde{J}_2(u_1^*, u_2)$ and can be proved subject to appropriate conditions on the structure of the differential game, including the choice of T, as we shall discuss later.

For the purpose of this section, we assume existence of a Nash equilibrium $(u_1^*, u_2^*) \in U_{ad}$. Thus, if a NE exists, the couple (u_1^*, u_2^*) solves simultaneously the following two optimisation problems

$$u_1^* = \arg\min_{u_1 \in U_{ad}^{(1)}} \tilde{J}_1(u_1, u_2^*), \qquad u_2^* = \arg\min_{u_2 \in U_{ad}^{(2)}} \tilde{J}_2(u_1^*, u_2). \tag{1.15}$$

Notice that the optimal solution u_1^* of the first problem enters as a parameter in the second problem and viceversa.

The fact that the equilibrium is obtained by solving two optimisation problems, implies that the NE point $u^* = (u_1^*, u_2^*)$ must satisfy the necessary optimality conditions given by the Pontryagin's maximum principle applied to both optimisation problems. For this purpose, as in the previous section, we introduce the following HP functions

$$\mathcal{H}_i(t, y, u_1, u_2, p_1, p_2) = p_i \cdot f(t, y, u_1, u_2) - \ell_i(t, y, u_1, u_2), \qquad i = 1, 2. \tag{1.16}$$

In terms of these functions, the PMP condition for the NE point $u^* = (u_1^*, u_2^*)$ states the existence of multiplier (adjoint) functions p_1, $p_2 : [0, T] \to \mathbb{R}^n$ such that the following holds

$$\begin{aligned} \max_{v_1 \in K_{ad}^{(1)}} \mathcal{H}_1(t, y^*(t), v_1, u_2^*(t), p_1^*(t), p_2^*(t)) &= \mathcal{H}_1(t, y^*(t), u_1^*(t), u_2^*(t), p_1^*(t), p_2^*(t)), \\ \max_{v_2 \in K_{ad}^{(2)}} \mathcal{H}_2(t, y^*(t), u_1^*(t), v_2, p_1^*(t), p_2^*(t)) &= \mathcal{H}_2(t, y^*(t), u_1^*(t), u_2^*(t), p_1^*(t), p_2^*(t)), \end{aligned} \tag{1.17}$$

for almost all $t \in [0, T]$. In (4.7), we have $y^* = y(u_1^*, u_2^*)$, and the adjoint variables p_1^*, p_2^* are the solutions to the following differential problems

$$-p_i'(t) = \left(\partial_y f(t, y(t), u_1(t), u_2(t))\right)^\top p_i(t) - \partial_y \ell_i(t, y(t), u_1(t), u_2(t)), \tag{1.18}$$

$$p_i(T) = -\partial_y g_i(y(T)), \tag{1.19}$$

where $i = 1, 2$. Similarly to (1.10), we require that (1.18) - (1.19) can be uniquely solved.

Hence, we have the following optimality system

$$\begin{aligned} & y'(t) = f(t, y(t), u_1(t), u_2(t)) \\ & y(0) = y_0 \\ & -p_1(t) = \left(\partial_y f(t, y(t), u_1(t), u_2(t))\right)^\top p_1(t) - \partial_y \ell_1(t, y(t), u_1(t), u_2(t)) \\ & p_1(T) = -\partial_y g_1(y(T)) \\ & -p_2(t) = \left(\partial_y f(t, y(t), u_1(t), u_2(t))\right)^\top p_2(t) - \partial_y \ell_2(t, y(t), u_1(t), u_2(t)) \\ & p_2(T) = -\partial_y g_2(y(T)) \\ & \mathcal{H}_1(t, y(t), u_1(t), u_2(t), p_1(t), p_2(t)) \geq \mathcal{H}_1(t, y(t), v_1, u_2(t), p_1(t), p_2(t)), \qquad v_1 \in K_{ad}^{(1)} \\ & \mathcal{H}_2(t, y(t), u_1(t), u_2(t), p_1(t), p_2(t)) \geq \mathcal{H}_1(t, y(t), u_1(t), v_2, p_1(t), p_2(t)), \qquad v_2 \in K_{ad}^{(2)}. \end{aligned} \tag{1.20}$$

The optimality system (1.20) gives only a necessary condition for optimality.

Moreover, in the case that $\mathcal{H}_i$ are differentiable in u_i, the optimality conditions become

$$(\partial_{u_1} f(\cdot, y, u_1, u_2)p_1 - \partial_{u_1}\ell_1(\cdot, y, u_1, u_2), v_1 - u_1) \leq 0 \qquad v_1 \in K_{ad}^{(1)} \tag{1.21}$$
$$(\partial_{u_2} f(\cdot, y, u_1, u_2)p_2 - \partial_{u_2}\ell_2(\cdot, y, u_1, u_2), v_2 - u_2) \leq 0 \qquad v_2 \in K_{ad}^{(2)}. \tag{1.22}$$

We remark that, if the system is autonomous, the functions $\mathcal{H}_i(t, y, u_1, u_2, p_1, p_2)$ are constant along the optimal solution.

Notice that, at each t fixed, the problem (1.17) corresponds to a finite-dimensional static Nash game. In fact, the players choose simultaneously their strategies $(v_1, v_2) \in K_{ad}^{(1)} \times K_{ad}^{(2)}$ which completely determine the objective costs. Based on the PMP formulation, one can state a procedure for finding an open-loop NE. For this purpose, we need to assume that the obtained family of optimisation problems (1.17) is uniquely solved. Hence, we consider the following theorem [21]

Theorem 1.2.1. *Assume the following structure*

$$f(t, y, u_1, u_2) = f^0(t, y) + F_1(t, y)\, u_1 + F_2(t, y)\, u_2,$$

and

$$\ell_i(t, y, u_1, u_2) = \ell_i^0(t, y) + \ell_i^1(t, u_1) + \ell_i^2(t, u_2), \qquad i = 1, 2.$$

Further, suppose that $K_{ad}^{(1)}$ and $K_{ad}^{(2)}$ are compact and convex sets in $\mathbb{R}^m$, the function f^0, ℓ_i^0 and the matrix functions F_1 and F_2 are continuous in t and y, and the functions $u_1 \to \ell_1^1(t, u_1)$ and $u_2 \to \ell_2^2(t, u_2)$ are strictly convex for any choice of $t \in [0, T]$.

Then, for any $t \in [0, T]$ and any y, p_1, $p_2 \in \mathbb{R}^n$, there exists a unique pair $(\tilde{u}_1, \tilde{u}_2) \in K_{ad}^{(1)} \times K_{ad}^{(2)}$ such that

$$\tilde{u}_1 = \arg\max_{v_1 \in K_{ad}^{(1)}} \Big(p_1 \cdot f(t, y, v_1, \tilde{u}_2) - \ell_1(t, y, v_1, \tilde{u}_2)\Big), \tag{1.23}$$
$$\tilde{u}_2 = \arg\max_{v_2 \in K_{ad}^{(2)}} \Big(p_2 \cdot f(t, y, \tilde{u}_1, v_2) - \ell_2(t, y, \tilde{u}_1, v_2)\Big). \tag{1.24}$$

Proof. For any (t, y, p_1, p_2), consider the HP functions

$$\begin{aligned} \tilde{\mathcal{H}}_1(u_1, u_2) &:= p_1 \cdot f(t, y, u_1, u_2) - \ell_1(t, y, u_1, u_2) \\ \tilde{\mathcal{H}}_2(u_1, u_2) &:= p_2 \cdot f(t, y, u_1, u_2) - \ell_2(t, y, u_1, u_2). \end{aligned} \tag{1.25}$$

For any fixed $u_2 \in K_{ad}^{(2)}$, $\tilde{\mathcal{H}}_1$ is continuous and strictly concave in u_1 on the compact set $K_{ad}^{(1)}$. Hence, it attains a maximum on $K_{ad}^{(1)}$. Similarly, $\tilde{\mathcal{H}}_2$ attains a maximum on the compact set $K_{ad}^{(2)}$.

Since the dynamics f and the running costs ℓ_1, ℓ_2 have the players' actions u_1, u_2 decoupled, for any (t, y, p_1, p_2), the controls strategies $\tilde{u}_1$, $\tilde{u}_2$ are given by

$$\begin{aligned} \tilde{u}_1 &= \arg\max_{\omega_1 \in K_{ad}^{(1)}} \Big[p_1 \cdot F_1(t, y)\omega_1 - \ell_1^1(t, \omega_1)\Big] \\ \tilde{u}_2 &= \arg\max_{\omega_2 \in K_{ad}^{(2)}} \Big[p_2 \cdot F_2(t, y)\omega_2 - \ell_2^2(t, \omega_2)\Big]. \end{aligned} \tag{1.26}$$

From (1.26) we can conclude that $\tilde{u}_1$ is the maximum of a strictly concave function, for any u_2 played by P_2. Hence, $\tilde{u}_1$ is unique. The same holds for $\tilde{u}_2$. Thus the claim is proved. $\square$

With this setting, applying the Pontryagin's maximum principle, we obtain the set of necessary conditions (1.20).

Next, we recall some properties of the free end-time problems that we shall use in Chapter 3 for the numerical solution of a free end-time Homicidal Chauffeur game.

The so-called time-optimal problems include in the cost functionals the length of the time-interval where the problem is defined, which in our case is T, $t \in [0,T]$. One of the purpose is to minimise the lenght of T. Hence, consider the dynamics (1.10) and let

$$J_i(y,u_1,u_2,T) = \int_0^T [\rho_i + \ell_i(t,y(t),u_1(t),u_2(t))]\,dt + g_i(T,y(T)), \qquad i=1,2. \tag{1.27}$$

Assume u_i, $i=1,2$, belong to the admissible sets $U_{ad}^{(i)}$ introduced above and let $\rho_i > 0$ be given parameters, $i=1,2$.

Again, by using the map $(u_1,u_2) \mapsto y = y(u_1,u_2)$ we can introduce the reduced objectives $\tilde{J}_1(u_1,u_2) := J_1(y(u_1,u_2),u_1,u_2)$ and $\tilde{J}_2(u_1,u_2) := J_2(y(u_1,u_2),u_1,u_2)$. In this framework, we define the Nash equilibrium for a free end-time game problem as follows

Definition 2. *The functions $(u_1^*,u_2^*) \in U_{ad}$ are said to form a Nash equilibrium strategy for the game $G = (\tilde{J}_1,\tilde{J}_2,U_{ad}^{(1)},U_{ad}^{(2)})$ for a $T = T^*$ if it holds*

$$\tilde{J}_1(u_1^*,u_2^*,T^*) \le \tilde{J}_1(u_1,u_2^*,T^*) \tag{1.28}$$

$$\tilde{J}_2(u_1^*,u_2^*,T^*) \le \tilde{J}_2(u_1^*,u_2,T^*) \tag{1.29}$$

for all $(u_1,u_2) \in U_{ad}$.

Thus, since the NE solves simultaneously the two optimisation problems

$$u_1^* = \arg\min_{u_1} \tilde{J}_1(u_1,u_2^*,T^*) \tag{1.30}$$

$$u_2^* = \arg\min_{u_2} \tilde{J}_1(u_1^*,u_2,T^*), \tag{1.31}$$

we can use the PMP to state the set of necessary optimality conditions, as seen above.

In particular, notice that for a free end-time problem at the optimality it holds

$$\partial g_i(T,y^*(T)) - \mathcal{H}_i(T,y^*(T),u_1^*(T),u_2^*(T),p_1^*(T),p_2^*(T)) = 0, \tag{1.32}$$

and in the case of autonomous systems we have

$$\mathcal{H}_i(y^*(t),u_1^*(t),u_2^*(t),p_1^*(t),p_2^*(t)) = \mathcal{H}_i(y^*(T),u_1^*(T),u_2^*(T),p_1^*(T),p_2^*(T)) = 0, \tag{1.33}$$

along the optimal solution.

We shall use this property in Chapter 3 to develop a bilevel algorithm for a free end-time Homicidal Chauffeur game.

To conclude this section, we discuss the so-called Nikaido-Isoda function, introduced in [71]. We use this function in the proof of the theorem of existence of Nash equilibria for bilinear Nash games, Theorem 1.4.2, and in Chapter 4 in the implementation of the SQH algorithm, as a selection criterion.

The Nikaido-Isoda function is defined as follows.

Definition 3. *The Nikaido-Isoda function $\psi : U_{ad} \times U_{ad} \to \mathbb{R}$ is defined as*

$$\psi(u,v) := \tilde{J}_1(u_1,u_2) - \tilde{J}_1(v_1,u_2) + \tilde{J}_2(u_1,u_2) - \tilde{J}_2(u_1,v_2), \tag{1.34}$$

where $u = (u_1,u_2) \in U_{ad}$ and $v = (v_1,v_2) \in U_{ad}$.

From the definition, it holds $\psi(u,u) = 0$. Moreover, at the Nash equilibrium $u^* = (u_1^*, u_2^*)$ we have

$$\psi(u^*, v) \le 0, \tag{1.35}$$

for any $v \in U_{ad}$ and $\psi(u^*, u^*) = 0$.

In fact, the Nikaido-Isoda function represents the sum of the changes in the cost functionals when a player decides to change its strategy from u_i to v_i, while the other player continues to play u_j. Therefore, the function is non-positive for any admissible v and at the equilibrium it can be zero at the most.

1.3 Cooperative differential games

It is well known that the Nash equilibrium solution concept is inefficient is the sense that, in many cases, the players can get an improvement in their costs by choosing to cooperate. By cooperation, in general, the objective values that a player may obtain is not uniquely determined. In the following, we focus on the case where a set of players' actions u is sought such that the resulting individual objectives cannot be improved upon by all players simultaneously. That is, we consider the so-called Pareto efficient solution; see, e.g., [40, 41].

Definition 4. *A pair $(\hat{u}_1, \hat{u}_2) \in U_{ad}^{(1)} \times U_{ad}^{(2)}$ is said to be Pareto optimal if*

$$\tilde{J}_i(\hat{u}_1, \hat{u}_2) \le \tilde{J}_i(u_1, u_2), \quad j = 1, 2,$$

for all $(u_1, u_2) \in U_{ad}^{(1)} \times U_{ad}^{(2)}$ and the inequality is strict for at least one i.

The corresponding point $(\tilde{J}_1(\hat{u}), \tilde{J}_2(\hat{u})) \in \mathbb{R}^2$ is called Pareto solution. The set of all Pareto solutions is said Pareto frontier.

In the literature, a way to find Pareto solutions is to solve a parameterized optimal control problem as follows

$$\min_{u \in U_{ad}} \sum_{i=1}^{2} \mu_i \, \tilde{J}_i(u), \tag{1.36}$$

where $u = (u_1, u_2)$, $\mu_i \in [0,1]$ with $\sum_{i=1}^{2} \mu_i = 1$.

In fact, we recall the following result; see [41] for a proof.

Lemma 1.3.1. *Let $\mu_i \in (0,1)$, with $\mu_1 + \mu_2 = 1$ and assume $\hat{u} \in U_{ad}$ is such that*

$$\hat{u} \in \arg\min_{u \in U_{ad}} \sum_{i=1}^{2} \mu_i \tilde{J}_i(u). \tag{1.37}$$

Then $\hat{u}$ is Pareto efficient.

Notice that in this way not all Pareto solutions can be found; see e.g. [41] where, in the spirit of the PMP, necessary and sufficient conditions for a control to be Pareto efficient in a cooperative dynamic differential game are stated.

In particular, we recall the following theorem; see [41].

Theorem 1.3.2. *Assume $(\tilde{J}_1(\hat{u}), \tilde{J}_2(\hat{u}))$ is a Pareto solution, $\hat{u} = (\hat{u}_1, \hat{u}_2)$ and let $\hat{y} = y(\hat{u})$. Then there exist $\mu_1, \mu_2 \in (0,1)$, $\mu_1 + \mu_2 = 1$ and a continuous and piecewise-continuously differentiable function $p : [0,T] \to \mathbb{R}^n$ such that, defining the Hamilton-Pontryagin function*

$$\mathcal{H}(t, y, u_1, u_2, p) = p \cdot f(t, y, u_1, u_2) - \sum_{i=1}^{2} \mu_i \ell_i(t, y, u_1, u_2), \tag{1.38}$$

p satisfies the following system

$$
\begin{aligned}
&-p'(t) = \left[\partial_y f(t, y(t), \hat{u}_1(t), \hat{u}_2(t))^\top p(t) - \sum_{i=1}^{2} \mu_1 \partial_y \ell_i(t, y(t), \hat{u}_1(t), \hat{u}_2(t))\right],\\
&p(T) = -\sum_{i=1}^{2} \mu_i \partial_y g_i(y(T)),\\
&\hat{y}'(t) = f(t, \hat{y}(t), \hat{u}_1(t), \hat{u}_2(t)),\\
&\hat{y}(0) = y_0,\\
&\mathcal{H}(t, \hat{y}(t), \hat{u}_1(t), \hat{u}_2(t), p(t)) \geq \mathcal{H}(t, \hat{y}(t), u_1(t), u_2(t), p(t)) \qquad \textit{almost all } t \in [0, T].
\end{aligned}
\tag{1.39}
$$

The necessary conditions obtained in (1.39) characterize the solution to (1.36). Thus, we can obtain the set of necessary conditions with the PMP applied to

$$
\mathcal{H}(t, y, u_1, u_2, p) = p \cdot f(t, y, u_1, u_2) - \sum_{i=1}^{2} \mu_i \ell_i(t, y, u_1, u_2).
$$

Next, we recall a theorem that gives both necessary and sufficient conditions for a control strategy to be Pareto optimal; see [41] for a proof.

Theorem 1.3.3. *The function $\hat{u} \in U_{ad}$ is Pareto efficient if and only if, for all i, $\hat{u}$ minimizes $\hat{J}_i(u)$ on the constrained set*

$$
U_i := \left\{ u \mid \tilde{J}_j(u) \leq \tilde{J}_j(\hat{u}), j = 1, 2, j \neq i \right\}, \quad i = 1, 2. \tag{1.40}
$$

Notice that, for a fixed player, the constrained set U_i depends on the loss of the other player. Therefore all Pareto solutions can be obtained by solving two constrained optimal control problems.

Next, we provide another theorem stating sufficient conditions for Pareto optimality under convexity assumptions.

Theorem 1.3.4. *If the admissible sets $U_{ad}^{(1)}, U_{ad}^{(2)}$ and the reduced objectives $\widetilde{J}_1(u), \widetilde{J}_2(u)$ are convex, then for any Pareto efficient solution $\hat{u} = (\hat{u}_1, \hat{u}_2)$ there exist $\mu_1, \mu_2 \in (0, 1)$, $\mu_1 + \mu_2 = 1$, such that*

$$
\hat{u} \in \arg\min_{u \in U_{ad}} \sum_{i=1}^{2} \mu_i \tilde{J}_i(u).
$$

Nevertheless, in the next chapter, we assume that $\hat{u}$ can be computed with (1.36) and corresponding to a specific choice of $\mu = (\mu_1, \mu_2)$. Since the treatment of Pareto solutions is more involved, we refer to [40, 41] for further details.

To continue our discussion, we remark that, in general, the Pareto solution is not unique. Hence, it is reasonable to ask whether there is a more advantageous solution and how to determine it. For this purpose, we focus on the so-called bargaining theory.

Consider two players that, after knowing their non-cooperative objectives, decide to cooperate in order to jointly generate and share an improvement in their costs. We refer to the non-cooperative outcome as the threatpoint.

In Figure 1.1 a bargaining problem is depicted. With the threatpoint (d_1, d_2), a possible set of outcomes, called feasible set S, is determined. The set P represents the Pareto-optimal outcomes. When the players unanimously agree on a point in S, then the corresponding outcome is called a solution to the bargaining problem. In the literature

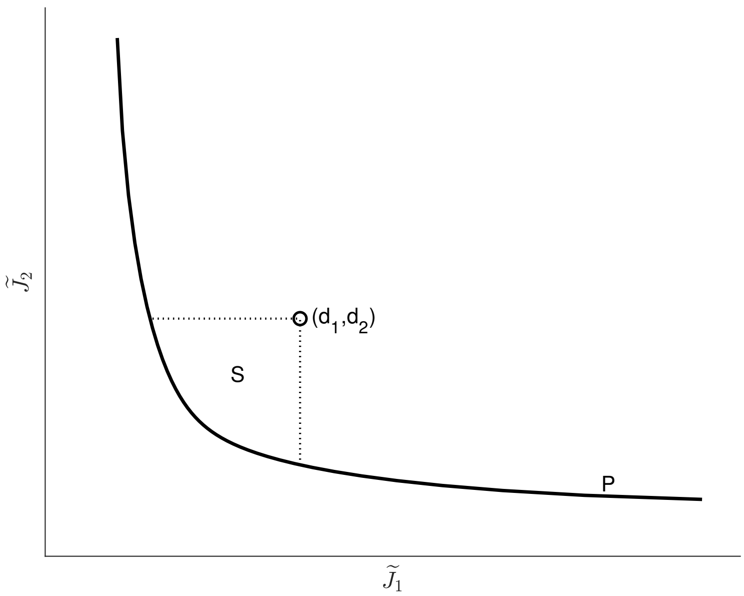

Figure 1.1: The bargaining problem.

there are many ways to determine a point on S; see, e.g., [91]. Following the axiomatic approach, the three most commonly known bargaining solutions are the Nash bargainig (NB) solution, the Kalai-Smorodinsky solution and the egalitarian solution.

The NB solution $u^{NB} = (u_1^{NB}, u_2^{NB})$ requires that the strategies of the players are Pareto optimal, independent of the utility scales used to represent the players' preferences, symmetric (the labelling of the players does not matter) and independent of irrelevant alternatives. Moreover, the NB solution satisfies the following criterion

$$u^{NB} = \arg\max_{u \in S} \prod_{i=1}^{2} \left(\tilde{J}_i(u^{NE}) - \tilde{J}_i(u) \right), \tag{1.41}$$

where $u = (u_1, u_2)$ and S is the set of all feasible values of the functionals, which can be achieved with an admissible set of controls. It is defined as follows

$$S := \{(\tilde{J}_1(u), \tilde{J}_2(u)) \in \mathbb{R}^2 \,|\, u \in U_{ad}\}. \tag{1.42}$$

In this formulation, we choose u^{NE} as the disagreement outcome (the threatpoint), and $\tilde{J}_i(u^{NE})$, $i = 1, 2$, are the values of the objectives of the game if no bargaining takes place (status quo).

Based on (1.41), the players act in order to maximize the Nash product of the excesses (or defects) with respect to the solution corresponding to the disagreement. Moreover, the NB solution concept requires Pareto optimality, in the sense that the NB solution is sought in the Pareto frontier, as discussed by J. Nash in [68].

In Figure 1.2 the Nash bargaining problem is depicted. Geometrically, the NB solution is the point on the feasible set S that maximizes the area of the rectangle (NB, A, d, B), where $d = (d_1, d_2)$ is the threatpoint.

We consider the following Nash bargaining (NB) problem:

$$\max_u \prod_{i=1}^{2} (d_i - \tilde{J}_i(u))$$
$$s.t.\ u \in U_{ad},\ d_i > \tilde{J}_i(u)\ \forall i$$

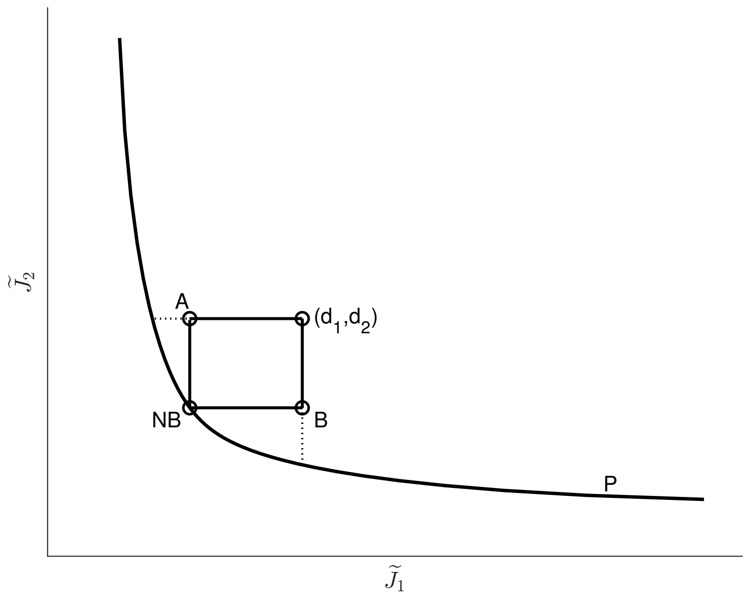

Figure 1.2: The Nash bargaining solution

where $d_i = \widetilde{J}_i(u^{NE})$ is a disagreement outcome for Player i. We shall assume that there exists at least one point $s \in S$ such that $d_i > s_i$, $\forall i$, and suppose that the NB problem has a solution $u^* \in U_{ad}$.

However, we remark that the axiom on independence of irrelevant alternatives has been criticized by some authors and alternative bargaining solutions have been defined. In the Kalai-Smorodinsky (KS) solution the axiom of irrelevant alternatives is replaced by the axiom of monotonicity [56], which states that an expansion of the feasible set in a direction favorable to a particular player always benefits this player. The KS solution determines the gains from the threatpoint proportional to the most optimistic expectations of the competitors, defined as the lowest costs that the players can obtain in S, subject to the constraint that no player gets a cost higher than its threatpoint's coordinate.

On the other hand, the so-called egalitarian solution [55] requires a strong monotonicity axiom but not the scale invariance property. In fact, the players should benefit from any expansion of the feasible set S, this regardless of whether the expansion may be biased in favor of one of them. As result, in the egalitarian solution, the gains are equally divided between the players, see Figure 1.3.

Notice that, in the discussion above the Pareto frontier is required to be convex. We refer, e.g., to [33, 50] for extensions to non-convex problems.

1.4 Existence of Nash equilibria

In this section, we discuss existence of Nash equilibria for the classes of problems considered in this thesis. In particular, we consider linear differential models with linear strategy mechanism and quadratic objectives that are well investigated from a theoretical perspective and we provide an overview of the main results. Then, we formulate a differential game with bilinear state-strategy mechanism and quadratic objectives and we prove the existence of a Nash equilibrium for this type of problems.

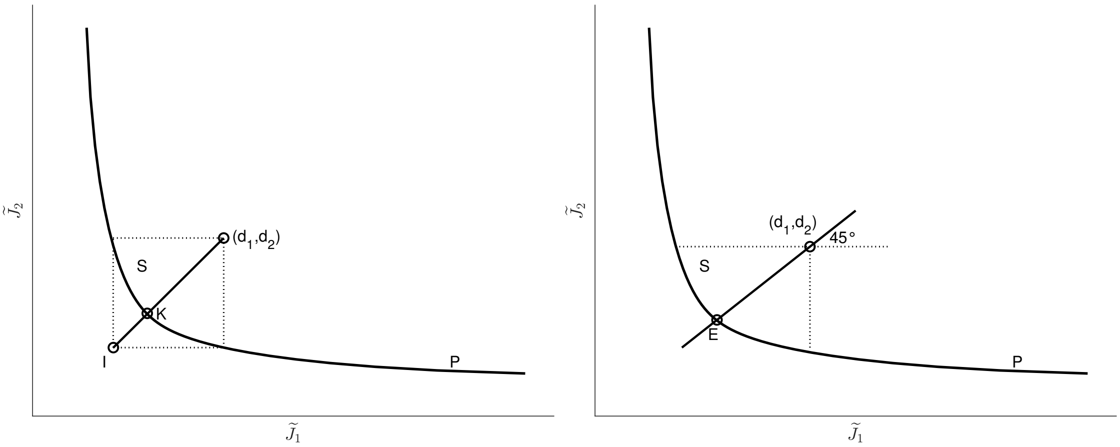

Figure 1.3: The Kalai-Smorodinsky solution K (left) and the egalitarian solution E (right).

1.4.1 Linear-quadratic differential Nash games

We start our discussion recalling well-known results on existence of NE for non-cooperative linear-quadratic differential games. We refer to [40, 39, 6, 94, 84] for a more detailed discussion.

Consider the following linear dynamics

$$y'(t) = A(t)\,y(t) + B_1(t)u_1(t) + B_2(t)u_2(t), \quad y(0) = y_0, \tag{1.43}$$

where $y(t), y_0 \in \mathbb{R}^n$, $u_1(t), u_2(t) \in \mathbb{R}^m$, $m \le n$, $t \in [0, T]$. The matrices $A(t) \in \mathbb{R}^{n\times n}$, $B_i(t) \in \mathbb{R}^{n\times m}$ are continuous.

Moreover, consider the following quadratic objective costs

$$\begin{aligned} J_1(y, u_1, u_2) &= \frac{1}{2}\int_0^T \Big[y(s)^\top M_1(s)y(s) + u_1(s)^\top N_{11}(s)u_1(s) + u_2(s)^\top N_{12}(s)u_2(s)\Big]\,ds \\ &\quad + \frac{1}{2}y(T)^\top D_1\,y(T) \\ J_2(y, u_1, u_2) &= \frac{1}{2}\int_0^T \Big[y(s)^\top M_2(s)y(s) + u_1(s)^\top N_{21}(s)u_1(s) + u_2(s)^\top N_{22}(s)u_2(s)\Big]\,ds \\ &\quad + \frac{1}{2}y(T)^\top D_2\,y(T). \end{aligned} \tag{1.44}$$

The matrices $M_i(s) \in \mathbb{R}^{n\times n}$, $N_{ij}(s) \in \mathbb{R}^{m\times m}$, $D_i \in \mathbb{R}^{n\times n}$, $i, j = 1, 2$, are assumed to be continuous and symmetric with M_i, D_i positive-semidefinite and N_{ii} positive-definite.

As in the previous sections the players' actions must satisfy the following condition

$$u = (u_1, u_2) \in U_{ad} \tag{1.45}$$

where $U_{ad} = U_{ad}^{(1)} \times U_{ad}^{(2)}$, with U_{ad}^i, $i = 1, 2$, given in (1.13).

In the case of integral constraints on the strategies, existence of NE is proved in [94] providing a bound on the duration of the game, whereas existence of an open-loop Nash equilibrium for a LQ differential game with strategies having values in $K_{ad}^{(i)}$, compact and convex subsets of $\mathbb{R}^m$, is proved in [85], without restriction on the duration of the game. In particular in [85] the following theorem is proved.

Theorem 1.4.1. *The linear-quadratic differential game* (1.43)-(1.45) *admits an open-loop Nash equilibrium point.*

Notice that the LQ differential game (1.43)-(1.45) has the structure discussed in Theorem 1.2.1, which gives the existence of a NE for any $t \in [0, T]$ of the corresponding finite-dimensional Nash game (1.23)-(1.24).

If there are no constraints on the players' strategies, it is possible to derive a coupled Riccati problem related to the LQ differential Nash game whose solution is the NE sought; see, e.g, [7].

To illustrate this fact, suppose the existence of two symmetric matrix functions Q_1, Q_2 such that the adjoints variables $p_i, i = 1, 2$, can be written as follows

$$p_1(t) = Q_1(t)y(t), \qquad p_2(t) = Q_2(t)y(t). \tag{1.46}$$

With some manipulations; see e.g. [12] and [7], one arrives at the following coupled Riccati problem

$$\begin{aligned}
&Q_1' + Q_1 A + A^\top Q_1 + Q_1 B_1 N_{11}^{-1} B_1^\top Q_1 + Q_1 B_2 N_{22}^{-1} B_2^\top Q_2 = M_1 \\
&Q_1(T) = -D_1 &&(1.47)\\
&Q_2' + Q_2 A + A^\top Q_2 + Q_2 B_1 N_{11}^{-1} B_1^\top Q_1 + Q_2 B_2 N_{22}^{-1} B_2^\top Q_2 = M_2 \\
&Q_2(T) = -D_2. &&(1.48)
\end{aligned}$$

By solving this system, we get the following open-loop Nash strategies

$$u_1(t) = N_{11}^{-1}(t)B_1(t)^\top Q_1(t)y(t), \qquad u_2(t) = N_{22}^{-1}(t)B_2(t)^\top Q_2(t)y(t), \tag{1.49}$$

where $y(t)$ is the solution to (1.43) at the NE, i.e.,

$$\begin{aligned}
&y'(t) = \left[A(t) + B_1(t)N_{11}^{-1}(t)B_1(t)^\top Q_1(t) + B_2(t)N_{22}^{-1}(t)B_2(t)^\top Q_2(t)\right] y(t) \\
&y(0) = y_0.
\end{aligned}$$

Further, notice that, according to the assumptions made on the cost functionals, J_1, J_2 are strictly convex in u_1 and u_2, for all the admissible u_2 and u_1, and for any initial state of the system y_0. Therefore, the conditions following from the PMP are necessary and sufficient to characterize a NE; see [6, Section 6.5] and the optimality system can be used to generate candidate NE solutions. In particular, for LQ differential games, a unique candidate can be obtained explicitely as above and due to the assumptions on J_i, (1.49) is the open-loop Nash equilibrium sought.

In fact, we have the following theorem; see [39] for a proof.

Theorem 1.4.2. *The following statements are equivalent*

i) For all $t \in [0, T]$ there exists a unique open-loop Nash equlibrium for the linear-quadratic differential game defined on the interval $[0, T]$.

ii) The set of Riccati differential equations (1.47)-(1.48) *has a solution on* $[0, T]$.

Thus, the existence of NE is related to the existence of a solution of the Riccati system (1.47)-(1.48). We refer to [1, 42] for more details on this topic.

1.4.2 Bilinear differential Nash games

In this section, we formulate our bilinear Nash game and prove existence of a Nash equilibrium. Consider the general case of N players and the following bilinear dynamics

$$y'(t) = f^0(y) + \sum_{j=1}^{N} u_j F_j(y), \qquad y(0) = y_0, \tag{1.50}$$

where $y(t) \in \mathbb{R}^n$ represents the n-dimensional state of the system at time t, y_0 is the initial configuration of the system and $u = (u_1, \ldots, u_N)$ represents the vector strategy function. In this section we omit the dependence on t of the functions $f^0, F_j, j = 1, \ldots, N$. Further, $y \in X$ is referred to as the state of the system in the set

$$X := \left\{ x \in H^1(0,T;\mathbb{R}^n) \;:\; x(0) = y_0 \right\},$$

where H^1 is the usual Sobolev space of the subset of L^2 functions such that their first-order weak derivatives have finite L^2 norm; see, e.g., [2]. The functions $f^0, F_j : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be Lipschitz and twice Fréchet differentiable. Whenever necessary, we denote with $F(y) \in \mathbb{R}^{n \times N}$ the matrix with columns the linear functions $F_j(y)$, $j = 1, \ldots, N$. The functions $u_j : [0,T] \to \mathbb{R}$, $j = 1, \ldots, N$, represent the strategies of the players and belong to the following admissible sets

$$\mathcal{U}_j(M) := \left\{ v \in L^2(0,T) \;:\; v(t) \in K_M \text{ a.e. in } (0,T) \right\}, \tag{1.51}$$

where $K_M = [-\sqrt{M}, \sqrt{M}]$ with $M \in (0, \infty]$ given positive constant.We denote $\mathcal{U}(M) = \mathcal{U}_1(M) \times \cdots \times \mathcal{U}_N(M)$.

Notice that, for the purpose of this section, it is more convenient to denote the admissible set with $\mathcal{U}(M)$, instead of U_{ad}, to underline its dependence on M. Moreover, we label the generic player with j. This notation is also used in Chapter 2.

Now, assuming that each player has chosen its strategy, this choice determines a unique solution $y \in X$ of problem (1.50). We call $y(t)$, $t \in [0,T]$, the trajectory corresponding to $u = (u_1, \ldots, u_N)$. For the jth player, each choice of strategies has a cost, defined as follows

$$J_j(y, u_1, \ldots, u_N) = \frac{1}{2}\|y(T) - y_T^{(j)}\|_2^2 + \frac{\nu}{2}\|u_j\|_{L^2}^2, \tag{1.52}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Moreover, in the following we indicate with $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$.

Considering the control-to-state map $u \mapsto y(u)$ we can define the reduced costs $\widetilde{J}_j(u_1, \ldots, u_N) := J_j(y(u_1, \ldots, u_N), u_1, \ldots, u_N)$, that are the objectives of the game.

In the following, we show that the differential bilinear game formulated above admits a Nash equilibrium (NE). For this purpose, we closely follow the approach in [94].

We start our discussion showing some properties of the solution to (1.50). Clearly, this solution depends on the initial condition, and since we focus on open-loop NE games, some of the constants obtained in the following estimates depend on the initial state of the system denoted with y_0. We have the following properties to state the boundedness and Lipschitz-continuity of the state function.

Proposition 1.4.3. *For any $u \in \mathcal{U}(M)$ the solution y of* (1.50) *is bounded in* $[0,T]$, *i.e. there exists a constant $\mathcal{K}$ such that $\|y(t)\|_2 \leq \mathcal{K}$, $t \in [0,T]$.*

Proof. It is well known that $y : [0, T] \to \mathbb{R}^n$ satisfies the following integral equation

$$y(t) = y_0 + \int_0^t \Big[f^0(y(s)) + F(y(s))u(s) \Big] \, ds. \tag{1.53}$$

For any $t \in [0, T]$, one can compute

$$\begin{aligned}
\|y(t)\|_2 &\leq \|y_0\|_2 + \int_0^t \|f^0(y(s))\|_2 + \|F(y(s))\|_2 \|u(s)\|_2 ds \\
&= \|y_0\|_2 + \int_0^t \|f^0(y(s)) - f^0(0) + f^0(0)\|_2 ds \\
&+ \int_0^t \|F(y(s)) - F(0) + F(0)\|_2 \|u(s)\|_2 ds,
\end{aligned}$$

which allows us to estimate

$$\begin{aligned}
\|y(t)\|_2 &\leq \|y_0\|_2 + \int_0^t \|f^0(0)\|_2 + \|F(0)\|_2 \|u(s)\|_2 ds \\
&+ \int_0^t L_{f^0} \|y(s)\|_2 + L_F \|y(s)\|_2 \|u(s)\|_2 ds \\
&\leq \|y_0\|_2 + T\|f^0(0)\|_2 + \|F(0)\|_2 \frac{T}{2} + \frac{1}{2} \|F(0)\|_2 \|u\|_{L^2}^2 \\
&+ \int_0^t \Big(L_{f^0} + \frac{L_F}{2} + \frac{L_F}{2} \|u(s)\|_2^2 \Big) \|y(s)\|_2 ds,
\end{aligned} \tag{1.54}$$

where the Lipschitz continuity of f^0 and F is used, and L_{f^0} and L_F denote the corresponding Lipschitz constants. From estimate (1.54), we obtain

$$\|y(t)\|_2 \leq \alpha_T + \int_0^t \beta(s) \|y(s)\|_2 ds \tag{1.55}$$

where we defined the following quantities

$$\begin{aligned}
\alpha_T &:= \|y_0\|_2 + T\|f^0(0)\|_2 + \|F(0)\|_2 \frac{T}{2} + \frac{1}{2} \|F(0)\|_2 \|u\|_{L^2}^2, \\
\beta(s) &:= \Big(L_{f^0} + \frac{L_F}{2} + \frac{L_F}{2} \|u(s)\|_2^2 \Big).
\end{aligned}$$

Next, applying the Grönwall's lemma, we get

$$\|y(t)\|_2 \leq \alpha_T \exp\Big(\int_0^t \beta(s) ds \Big) \leq \alpha_T \exp\Big(\int_0^T \beta(s) ds \Big).$$

Therefore Proposition 1.4.3 is proved with $\mathcal{K}(T) := \alpha_T \exp\Big(\int_0^T \beta(s) ds \Big)$ which is a monotonically increasing function of T. $\square$

Proposition 1.4.4. *The solution y of* (1.50) *is Lipschitz-continuous in u, for any $u \in \mathcal{U}(M)$, $M \in (0, \infty)$.*

Proof. For any $u_1, u_2 \in \mathcal{U}(M)$, consider the corresponding state equations given by

$$\begin{aligned}
y_1' &= f^0(y_1) + F(y_1)u_1, \quad y_1(0) = y_0, \\
y_2' &= f^0(y_2) + F(y_2)u_2, \quad y_2(0) = y_0.
\end{aligned}$$

Let $z := y_1 - y_2$, we get

$$z' = \left(f^0(y_1) - f^0(y_2)\right) + F(y_1)u_1 - F(y_2)u_2. \tag{1.56}$$

Hence, the function z satifies the following integral equation

$$\begin{aligned} z(t) = \int_0^t \Big[& f^0(y_1(s)) - f^0(y_2(s)) + F(y_1(s))(u_1(s) - u_2(s)) \\ & + [F(y_1(s)) - F(y_2(s))]u_2(s)\Big] ds. \end{aligned} \tag{1.57}$$

Now, considering the 2-norm, we have the following estimates

$$\begin{aligned} \|z(t)\|_2 \le & \int_0^t \Big[\|f^0(y_1(s)) - f^0(y_2(s))\|_2 + \|F(y_1(s))\|_2\|u_1(s) - u_2(s)\|_2 \\ & + \|F(y_1(s)) - F(y_2(s))\|_2\|u_2(s)\|_2\Big] ds \\ \le & \int_0^t \Big[L_{f^0}\|y_1(s) - y_2(s)\|_2 + \|F(y_1(s))\|_2\|u_1(s) - u_2(s)\|_2 \\ & + L_F\|y_1(s) - y_2(s)\|_2\|u_2(s)\|_2\Big] ds \\ \le & K(T)\int_0^t \|u_1(s) - u_2(s)\|_2 ds + (L_{f^0} + ML_F)\int_0^t \|z(s)\|_2 ds, \end{aligned}$$

where, due to the continuity of F and the properties of y, the function $K(T)$ is monotonically increasing in T.

Finally, the Cauchy-Schwarz inequality and the Grönwall's lemma imply

$$\|y_1(t) - y_2(t)\|_2 \le (L_{f^0} + ML_F)K(T)T^{3/2}\|u_1 - u_2\|_{L^2}. \tag{1.58}$$

Therefore y is a Lipschitz function in u and Proposition 1.4.4 is proved. □

Next, consider the linearised problem related to (1.50) for a general reference pair (y, u). This problem can be written as follows

$$\begin{aligned} \delta y' &= \left[\partial_y f^0(y) + \sum_{j=1}^N \partial_y F_j(y)u_j\right]\delta y + \sum_{j=1}^N F_j(y)\delta u_j \quad \text{in } (0, T], \\ \delta y(0) &= 0, \end{aligned} \tag{1.59}$$

where $\partial_y f^0$ and $\partial_y F_j$ denote the Jacobian matrices of f^0 and F_j, $j = 1, \ldots, N$, respectively.

For the linearised problem (1.59), the following properties hold.

Proposition 1.4.5. *Consider $\delta u \in L^2(0, T; \mathbb{R}^N)$ and let $\delta y = \delta y(\delta u)$ be the corresponding unique solution to* (1.59)*. Then δy satisfies the following inequality*

$$\|\delta y(t)\|_2 \le \tilde{C}(T)T^{3/2}N\|\delta u\|_{L^2}, \tag{1.60}$$

in $(0, T)$.

Proof. Consider the linearised problem (1.59) and the corresponding solution δy. Writing δy in integral form and taking the 2-norm, it holds

$$\begin{aligned} \|\delta y(t)\|_2 \le & \int_0^t \Big[\|\partial_y f^0(y(s)) + \sum_{j=1}^N \partial_y F_j(y(s))u_j\|_2\|\delta y(s)\|_2 \\ & + \sum_{j=1}^N \|F_j(y(s))\|_2|\delta u_j(s)|\Big] ds \\ \le & \int_0^t \Big[K_1(T)\|\delta y(s)\|_2 + K_2(T)\|u(s)\|_2\|\delta y(s)\|_2 + C_1\|\delta u(s)\|_2\Big] ds. \end{aligned}$$

As in the previous propositions, applying the Cauchy-Schwarz inequality and the Grönwall's lemma, we get

$$\|\delta y(t)\|_2 \leq \tilde{C}(T) T^{3/2} N \, \|\delta u\|_{L^2},$$

in $(0,T)$, with $\tilde{C}(T)$ monotonically increasing with T. This concludes the proof. $\square$

Next, consider the operator $c(\cdot,\cdot) : H^1(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^N) \to L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ defined as follows

$$c(y,u) := \begin{pmatrix} \frac{d}{dt}y - f^0(y) - \sum_{j=1}^N u_j F_j(y) \\ y(0) - y_0 \end{pmatrix}.$$

In this way (1.50) and (1.59) can be written respectively as

$$c(y,u) = 0, \quad Dc(y,u)(\delta y, \delta u) = 0,$$

where $(\delta y, \delta u) \mapsto Dc(y,u)(\delta y, \delta u)$ is defined as

$$Dc(y,u)(\delta y, \delta u) := \begin{pmatrix} \frac{d}{dt}\delta y - \left[\partial_y f^0(y) + \sum_{j=1}^N \partial_y F_j(y) u_j\right] \delta y - \sum_{j=1}^N F_j(y)\delta u_j \\ \delta y(0) \end{pmatrix}.$$

The operator c is Fréchet differentiable and its Fréchet derivative $D_y c(y,u)$ is invertible. Hence, the equation

$$Dc(y,u)(\delta y, \delta u) = D_y c(y,u)(\delta y) + D_u c(y,u)(\delta u) = 0,$$

and the implicit function theorem (see [31]) imply that $D_u y(u)(\delta u) = \delta y$ is the solution to the linearised problem (1.59).

Moreover, the assumptions on f^0, F, with the implicit function theorem imply that the control-to-state map $u \mapsto y(u)$, where $y(u) \in H^1(0,T;\mathbb{R}^n)$ is the unique solution to (1.50) corresponding to u, is twice Fréchet differentiable. Therefore, denoting by $\delta y(u,\delta u) \in H^1(0,T;\mathbb{R}^n)$ the unique solution to (1.59) corresponding to δu and u, the following expansion holds

$$y(u+\delta u) = y(u) + \delta y(u,\delta u) + \theta(u,\delta u) + R(u,\delta u), \tag{1.61}$$

where $\theta(u,\delta u) := D^2_{uu} y(u)(\delta u, \delta u)$ and $R(u,\delta u) = o(\|\delta u\|^2_{L^2})$.

For our discussion we need to estimate $\theta(u,\delta u)$. For this purpose, the following property is proved.

Proposition 1.4.6. *For any $h \in L^2(0,T;\mathbb{R}^N)$, the function $\theta(u,h) \in H^1(0,T;\mathbb{R}^n)$ solves the following problem*

$$\begin{aligned} \theta' &= \left[\partial_y f^0(y) + \sum_{j=1}^N \partial_y F_j(y) u_j\right] \theta \\ &\quad + \partial^2_{yy} f^0(y)(\delta y, \delta y) + \sum_{j=1}^N u_j \partial^2_{yy} F_j(y)(\delta y, \delta y) \\ &\quad + 2\sum_{j=1}^N h_j \partial_y F_j(y) \delta y \quad \textit{in } (0,T] \\ \theta(0) &= 0, \end{aligned} \tag{1.62}$$

and

$$\|\theta(t)\|_2 \leq \mathcal{C}(T) T^3 \|h\|^2_{L^2}, \quad \textit{a.e. in } (0,T). \tag{1.63}$$

Proof. Consider the operator $c(\cdot,\cdot)$ and compute its second derivative. We get

$$Dc(y(u),u)(h) = D_y c(y(u),u)(D_u y(u)(h)) + D_u c(y(u),u)(h)$$

and hence

$$\begin{aligned} D^2c(y(u),u)(h,h) &= D_{yy}c(y(u),u)(D_u y(u)(h), D_u y(u)(h)) + D_{yu}c(y(u),u)(D_u y(u)(h),h) \\ &+ D_y c(y(u),u)(D^2_{uu}y(u)(h,h)) + D_{uy}c(y(u),u)(h, D_u y(u)(h)) \\ &+ D_{uu}c(y(u),u)(h,h). \end{aligned}$$

If $y(u)$ is the solution of (1.50) corresponding to the strategy u, then $c(y(u),u) = 0$. Therefore $Dc(y(u),u)(h) = 0$ and $D^2c(y(u),u)(h,h) = 0$. Specifically, computing term-by-term, we get

$$D_{uu}c(y(u),u)(h,h) = 0, \tag{1.64}$$

$$D_{yy}c(y(u),u)(\delta y,\delta y) = -\partial^2_{yy} f^0(y)(\delta y,\delta y) - \sum_{j=1}^N u_j \partial^2_{yy} F_j(y)(\delta y,\delta y), \tag{1.65}$$

$$D_{yu}c(y(u),u)(\delta y, h) = -\sum_{j=1}^N h_j \partial_y F_j(y)(\delta y), \tag{1.66}$$

$$D_y c(y(u),u)(\theta) = \theta' - \partial_y f^0(y)(\theta) - \sum_{j=1}^N u_j \partial_y F_j(y)(\theta), \tag{1.67}$$

$$D_{uy}c(y(u),u)(h,\delta y) = -\sum_{j=1}^N h_j \partial_y F_j(y)(\delta y). \tag{1.68}$$

By replacing (1.64)-(1.68) into $D^2c(y(u),u)(h,h) = 0$, we get that θ solves (1.62).

The estimate can be obtained similarly to the previous propositions. In fact, integrating (1.62) over $(0,t)$, we get

$$\begin{aligned} \theta(t) = &\int_0^t \left[\partial_y f^0(y(s)) + \sum_{j=1}^N \partial_y F_j(y(s)) u_j(s) \right] \theta(s) ds \\ &+ \int_0^t \partial^2_{yy} f^0(y(s))(\delta y(s), \delta y(s)) + \sum_{j=1}^N u_j(s) \partial^2_{yy} F_j(y(s))(\delta y(s) \delta y(s)) ds \\ &+ 2 \int_0^t \sum_{j=1}^N h_j(s) \partial_y F_j(y(s)) \delta y(s) ds. \end{aligned}$$

Hence, taking the 2-norm, it holds

$$\begin{aligned} \|\theta(t)\|_2 \le &\int_0^t \left[\|\partial_y f^0(y(s))\|_2 + \sum_{j=1}^N |u_j(s)| \, \|\partial_y F_j(y(s))\|_2 \right] \|\theta(s)\|_2 ds \\ &+ \int_0^t \|\partial^2_{yy} f^0(y(s))(\delta y(s), \delta y(s))\|_2 + \sum_{j=1}^N |u_j(s)| \|\partial^2_{yy} F_j(y(s))(\delta y(s), \delta y(s))\|_2 ds \\ &+ 2 \int_0^t \sum_{j=1}^N |h_j(s)| \, \|\partial_y F_j(y(s))\|_2 \|\delta y(s)\|_2 ds. \end{aligned}$$

Next, define $\Big(M_i(y)\Big)_{kl} := \partial_{y_l}(\partial_{y_k} f_i^0(y))$, $i, j, k = 1, \ldots, n$ and $\Big(N_{ij}(y)\Big)_{kl} := \partial_{y_l}(\partial_{y_k} F_{ij}(y))$, $i, k, l = 1, \ldots, n$ $j = 1, \ldots, N$. It follows

$$
\begin{aligned}
\|\theta(s)\|_2 \leq (L_{f^0} + ML_F) &\int_0^t \|\theta(s)\|_2 ds \\
&+ \int_0^t \max_i \|M_i(y)\|_2 \|\delta y(s)\|_2^2 ds + M \int_0^t \sum_{j=1}^N \max_i \|N_{ij}(y)\|_2 \|\delta y(s)\|_2^2 ds \\
&+ 2L_F \int_0^t \|h(s)\|_2 \|\delta y(s)\|_2 ds.
\end{aligned}
$$

Using Proposition 1.4.5,

$$
\begin{aligned}
\|\theta(s)\|_2 \leq (L_{f^0} + ML_F) &\int_0^t \|\theta(s)\|_2 ds \\
&+ \int_0^t \Big[\max_i \|M_i(y)\|_2 + M \sum_{j=1}^N \max_i \|N_{ij}(y)\|_2\Big] C_{\delta y}^2 T^3 \|h\|_{L^2}^2 ds \\
&+ 2L_F \int_0^t \|h(s)\|_2 C_{\delta y} T^{3/2} \|h\|_{L^2} ds.
\end{aligned}
$$

Applying the Grönwall's lemma, we get

$$
\|\theta(t)\|_2 \leq \mathcal{C}(T) T^3 \|h\|_{L^2}^2, \tag{1.69}
$$

with $\mathcal{C}(T)$ monotonically increasing with T. □

Now that the main properties of the functions y and δy are stated, to continue our discussion, we need to introduce some functions used in the proof of the main result.

For this purpose, define $\bar{\psi} : \mathcal{U}(M) \times \mathcal{U}(M) \to \mathbb{R}$ as

$$
\bar{\psi}((u_1, \ldots, u_N), (v_1, \ldots, v_N)) := \sum_{i=1}^N \widetilde{J}_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_N).
$$

The following lemma is fulfilled [94].

Lemma 1.4.7. *The function $u^* = (u_1^*, \ldots, u_N^*)$ is an equilibrium strategy for the game G if and only if*

$$
\bar{\psi}(u^*, u^*) \leq \bar{\psi}(u^*, v), \tag{1.70}
$$

for all $v \in \mathcal{U}(M)$.

Proof. Suppose $u^* = (u_1^*, \ldots, u_N^*)$ is an equilibrium strategy and let $v = (v_1, \ldots, v_N) \in \mathcal{U}(M)$. Then

$$
\widetilde{J}_i(u_1^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_N^*) \leq \widetilde{J}_i(u_1^*, \ldots, u_{i-1}^*, v_i, u_{i+1}^*, \ldots, u_N^*), \tag{1.71}
$$

for $i = 1, \ldots, N$. Adding these inequalities for $i = 1, \ldots, N$, we obtain (1.70). Conversely, suppose (1.70) holds. Replacing $v = (u_1^*, \ldots, u_{i-1}^*, v_i, u_{i+1}^*, \ldots, u_N^*)$ in (1.70) we obtain (1.71). □

Next, let $v = (v_1, \ldots, v_N) \in \mathcal{U}(M)$ be fixed and consider the regularized Nikaido-Isoda function $\sigma : \mathcal{U}(M) \to \mathbb{R}$ defined by

$$
\sigma(u_1, \ldots, u_N) := \sum_{i=1}^N \left\{ \frac{\nu}{2} \|u_i\|_{L^2}^2 + \frac{1}{2} \|y(T) - y_T^{(i)}\|_2^2 - \frac{1}{2} \|y^i(T) - y_T^{(i)}\|_2^2 \right\},
$$

where $y_T^{(i)}$ are the desired players' targets and $y(t), y^i(t)$, for $t \in [0,T]$, are the trajectories of (1.50) corresponding to the strategies $u := (u_1, \dots, u_N)$ and $u^i := (u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_N)$, respectively.

At this point, we can state the following result which focuses to our NE game with bilinear structure.

Lemma 1.4.8. *There exists $T_0 > 0$ such that for all $T \in [0, T_0)$ it holds that*

i) σ is weakly lower semicontinuous,

ii) σ is convex,

iii) $\sigma(u) \to \infty$ as $\|u\|_{L^2} \to \infty$.

Proof. Since the functions that constitute σ are continuous, then also $\sigma : L^2(0,T;\mathbb{R}) \times C([0,T];\mathbb{R}) \to \mathbb{R}$ is a continuous function. Next, we prove that there exist $T_0 > 0$ and $\epsilon_0(T_0) > 0$ such that

$$D^2\sigma(u)(w,w) \geq \epsilon_0(T_0)\|w\|_{L^2}^2, \tag{1.72}$$

for any $0 < T < T_0$.

For the proof, let $u \in \mathcal{U}(M), w \in L^2(0,T)$ be fixed and compute the derivatives of σ. We have

$$D\sigma(u)(w) = \sum_{i=1}^N \Big[\nu\langle u_i, w_i\rangle + \langle y(u)(T) - y_T^{(i)}, D_u y(u)(w)(T)\rangle - \langle y^i(u)(T) - y_T^{(i)}, D_u y^i(u)(w)(T)\rangle\Big]$$

and

$$\begin{aligned} D^2\sigma(u)(w,w) = \sum_{i=1}^N \Big[&\nu\langle w_i, w_i\rangle + \langle y(u)(T) - y_T^{(i)}, D^2_{uu} y(u)(w,w)(T)\rangle \\ &- \langle y^i(u)(T) - y_T^{(i)}, D^2_{uu} y^i(u)(w,w)(T)\rangle \\ &+ \langle D_u y(u,w)(T), D_u y(u,w)(T)\rangle - \langle D_u y^i(u,w)(T), D_u y^i(u,w)(T)\rangle\Big]. \end{aligned}$$

Then, we get

$$\begin{aligned} D^2\sigma(u)(w,w) = \sum_{i=1}^N \Big[&\int_0^T \nu|w_i(t)|^2 dt + \langle y(u)(T) - y_T^{(i)}, \theta(w,w)(T)\rangle \\ &- \langle y^i(u)(T) - y_T^{(i)}, \theta^i(w,w)(T)\rangle \\ &+ \|\delta y(u,w)(T)\|_2^2 - \|\delta y^i(u^i,w)(T)\|_2^2\Big], \end{aligned}$$

where $\delta y(u,w)$ and $\delta y^i(u^i,w)$ are the solutions of (1.59) with $(u_1, \dots, u_N)$ and $(u_1, \dots, u_{i-1}, w_i, u_{i+1}, \dots, u_N)$.

Therefore, we obtain the following estimate

$$\begin{aligned} D^2\sigma(u)(w,w) &\geq \sum_{i=1}^N \Big[\nu\|w_i\|_{L^2}^2 - k_1(T)T^3\|w\|_{L^2}^2 - k_2(T)T^3\|w\|_{L^2}^2\Big] \\ &= \Big[\nu - \big(k_1(T) + k_2(T)\big)NT^3\Big]\|w\|_{L^2}^2, \end{aligned}$$

where we used Proposition 1.4.5 and Proposition 1.4.6 and we defined $k_1(T) := 2(\mathcal{K}(T) + \|y_T^{(i)}\|_2)\mathcal{C}(T)$, $k_2(T) := (\tilde{C}(T)N)^2$.

The coercivity of σ is then guaranteed if require

$$\nu > \Big(k_1(T) + k_2(T)\Big) N T^3. \tag{1.73}$$

Therefore it is possible to choose ϵ_0 and T_0 such that (1.72) holds and σ is convex [96, Corollary 42.8 page 248]. Moreover, since σ is continuous, it is weakly lower semicontinuous. $\square$

Corollary 1.4.1. *The function $\bar{\psi}(u, v)$, for any $u \in \mathcal{U}(M)$, is convex in $v \in \mathcal{U}(M)$.*

Proof. To prove that $\bar{\psi}(u, v)$ is convex in v for any $u \in \mathcal{U}(M)$, it is sufficient to show that there exist a time $T_0 > 0$ and $\epsilon_0(T_0) > 0$ such that

$$D^2\bar{\psi}(u, v)(w, w) \geq \epsilon_0(T_0)\|w\|^2_{L^2}, \tag{1.74}$$

for any $T \in (0, T_0)$.

For this purpose, consider the function σ defined as follows

$$\sigma(v_1, \dots, v_N) := \sum_{i=1}^{N} \left[\frac{\nu}{2}\|v_i\|^2_{L^2} + \frac{1}{2}\|y^i(T) - y_T^{(i)}\|_2^2\right]. \tag{1.75}$$

Repeating the same calculation seen in the proof of Lemma 1.4.8, one gets the convexity of $\bar{\psi}(u, v)$ in v if the following condition is fulfilled

$$\nu > k_2(T) N T^3, \tag{1.76}$$

with k_2 defined in the proof of Lemma 1.4.8. Thus the proof is completed. $\square$

Next, let $v \in \mathcal{U}(M)$ be fixed and consider the Nikaido-Isoda function $\psi : \mathcal{U}(M) \times \mathcal{U}(M) \to \mathbb{R}$, introduced in the previous sections. According with the definition of $\bar{\psi}$ seen above, it can be written as

$$\psi(u, v) := \bar{\psi}(u, u) - \bar{\psi}(u, v).$$

Define $U_v := \{u : \psi(u, v) > 0\}$. The following property of the Nikaido-Isoda function is proved below.

Lemma 1.4.9. *There exists a $T_0 > 0$ such that if $T < T_0$ then U_v is weakly open and $\psi(u, v) \to \infty$ as $\|u\|_{L^2} \to \infty$.*

Proof. We have

$$\begin{aligned}\psi(u_1, \dots, u_N, v_1, \dots, v_N) = &\sum_{i=1}^{N} \left(\frac{1}{2}\|y(T) - y_T^{(i)}\|_2^2 - \frac{1}{2}\|y^i(T) - y_T^{(i)}\|_2^2\right) \\ &+ \sum_{i=1}^{N} \frac{\nu}{2}\|u_i\|^2_{L^2} - \sum_{i=1}^{N} \frac{\nu}{2}\|v_i\|^2_{L^2}.\end{aligned}$$

In this expression, the first sum is continuous and the third sum is constant. By Lemma 1.4.8 there is a $T_0 > 0$ such that if $T < T_0$ then the first sum plus the second sum is weakly lower semicontinuous and grows indefinitely as $\|u\|_{L^2}$ grows indefinitely. Therefore ψ is weakly lower semicontinuous, so that the complement set of U_v, $U_v^c = \{u : \psi(u, v) \leq 0\}$ is weakly closed and its complement U_v is weakly open. $\square$

Next, we prove existence of a NE for our differential game.

Theorem 1.4.10. *Let $\widetilde{J}_1, \ldots, \widetilde{J}_N$ be the cost functionals and $M \in [0, \infty]$. There is a $T_0 > 0$ such that for any $T < T_0$ there exists at least an equilibrium strategy for the game $G = (\widetilde{J}_1, \ldots, \widetilde{J}_N; M)$.*

Proof. Suppose the theorem is false. Then, using Lemma 1.4.7, for each $u \in \mathcal{U}(M)$, there is a $v \in \mathcal{U}(M)$ such that $\psi(u, v) := \bar{\psi}(u, u) - \bar{\psi}(u, v) > 0$, i.e. $u \in U_v$, where U_v is defined above. Therefore $\mathcal{U}(M)$ has the following weakly open cover

$$\mathcal{U}(M) \subset \cup_{v \in \mathcal{U}(M)} U_v. \tag{1.77}$$

Next, we show that there is a finite subset of vector functions $\{v^1, \ldots, v^p\}$ of $\mathcal{U}(M)$ such that

$$\mathcal{U}(M) \subset \cup_{i=1}^p U_{v^i}. \tag{1.78}$$

First suppose $M < \infty$. Then $\mathcal{U}(M)$ is a convex, bounded and closed subset of $L^2(0, T; \mathbb{R}^N)$ so that it is weakly compact. Then (1.77) must have a finite subcover (1.78).

Now, suppose $M = \infty$. Let $v^1 \in \mathcal{U}(M)$. Then by Lemma 1.4.9, $\psi(u, v^1) \to \infty$ as $\|u\|_{L^2} \to \infty$. Hence, there is $M_1 < \infty$ such that $\psi(u, v^1) > 0$ whenever $\|u\|_{L^2} > M_1$. That is, $\{u : \|u\|_{L^2}^2 > M_1\} \subset U_{v^1}$. Now, since $\mathcal{U}(M_1)$ is weakly compact, there exist $v^2, \ldots, v^p \in \mathcal{U}(M)$ such that $\mathcal{U}(M_1) \subset \cup_{i=2}^p U_{v^i}$. Thus, we have

$$\mathcal{U}(\infty) = \mathcal{U}(M_1) \cup \Big\{u : \|u\|_{L^2}^2 > M_1\Big\} \subset \cup_{i=1}^p U_{v^i},$$

so that once again (1.78) holds. Note that (1.78) implies that for each $u \in \mathcal{U}(M)$ there is a $j \in \{1, \ldots, p\}$ such that $\psi(u, v^j) > 0$.
Now, let V be a convex hull of $\{v^1, \ldots, v^p\}$, i.e.,

$$V = \left\{ \sum_{i=1}^p \lambda_i v^i : \lambda_i \geq 0, \sum_{i=1}^p \lambda_i = 1 \right\}.$$

Notice that $V \subset \mathcal{U}(M)$. Define the functions $\gamma_j : V \to \mathbb{R}$ by

$$\gamma_j(v) := \max(\psi(v, v^j), 0).$$

Since V is finite-dimensional, the weak and strong topology coincide. Notice that ψ is strongly continuous on $\mathcal{U}(M)$, hence it is continuous on V, and so γ_j is continuous on V. Finally, since we have seen that for all $v \in V$ there is at least one $j \in \{1, \ldots, p\}$ such that $\psi(v, v^j) > 0$, then the function

$$\gamma := \sum_{j=1}^p \gamma_j$$

satisfies the condition

$$\gamma(v) > 0, \quad v \in V.$$

Now, define the function $\eta : V \to V$ by

$$\eta(v) = \sum_{j=1}^p \frac{\gamma_j(v)}{\gamma(v)} v^j.$$

Then η is continuous and we can apply the Brouwer fixed-point theorem to get the existence of a point $v^* \in V$ such that $\eta(v^*) = v^*$. Suppose $\gamma_j(v^*) > 0$, $j = 1, \ldots, \ell$, $\ell < p$, and $\gamma_j(v^*) = 0$, $j > \ell$. Then

$$\gamma(v^*) = \sum_{j=1}^{\ell} \gamma_j(v^*),$$

and the fixed-point condition becomes

$$v^* = \sum_{j=1}^{\ell} \frac{\gamma_j(v^*)}{\gamma(v^*)} v^j.$$

Since $\gamma_j(v^*) > 0$ is equivalent to $\bar{\psi}(v^*, v^*) > \bar{\psi}(v^*, v^j)$, we have

$$\begin{aligned} \bar{\psi}(v^*, v^*) &= \sum_{j=1}^{p} \frac{\gamma_j(v^*)}{\gamma(v^*)} \bar{\psi}(v^*, v^*) = \sum_{j=1}^{\ell} \frac{\gamma_j(v^*)}{\gamma(v^*)} \bar{\psi}(v^*, v^*) \\ &> \sum_{j=1}^{\ell} \frac{\gamma_j(v^*)}{\gamma(v^*)} \bar{\psi}(v^*, v^j). \end{aligned} \tag{1.79}$$

Finally, from the convexity of $\bar{\psi}(v^*, v)$ in v proved in Corollary 1.4.1 we get that if

$$\nu \geq \max\left\{k_1(T)NT^3 + k_2(T)NT^3, k_2(T)NT^3\right\} = k_1(T)NT^3 + k_2(T)NT^3,$$

then

$$\bar{\psi}(v^*, v^*) = \bar{\psi}\left(v^*, \sum_{j=1}^{\ell} \frac{\gamma_j(v^*)}{\gamma(v^*)} v^j\right) \leq \sum_{j=1}^{\ell} \frac{\gamma_j(v^*)}{\gamma(v^*)} \bar{\psi}(v^*, v^j),$$

which is in contrast with (1.79). Therefore the theorem is proved. □

As already remarked at the beginning of this section, our results concern open-loop NE games. Thus, in particular, the value of the time horizon T_0 specified in Theorem 1.4.2 may depend on the choice of the initial condition.

1.5 Summary

In this chapter, differential Nash games were discussed. For this purpose, some results of optimal control theory were recalled and a PMP characterization of Nash games was given. Cooperative games were also introduced and Pareto and bargaining solutions discussed.

Well-known results of existence of a Nash equilibrium for linear-quadratic Nash games were recalled. A Nash game with bilinear state-strategy mechanism was formulated and analysed. Existence of a Nash equilibrium for a sufficiently small time horizon was proved.

Chapter 2

A relaxation method for solving differential Nash games

This chapter deals with the analysis of a relaxation scheme combined with a semi-smooth Newton method. In particular, the locally convergence of the proposed scheme is proved and an estimate on the relaxation parameter is provided. Moreover, a characterisation of the solution to the Nash bargaining problem on the Pareto frontier is given and, based on this result, an algorithm for the computation of a bargaining solution on the Pareto frontier is introduced. Results of numerical experiments based on a quantum model of two spin-particles and a Lotka-Volterra model of population dynamics conclude the chapter.

2.1 A two player's bilinear game

In this section, we consider two players, $N = 2$, and we illustrate a numerical procedure for solving the bilinear differential Nash games, introduced in the previous chapter, whose dynamics is given by

$$y'(t) = f^0(y(t)) + F_1(y(t))u_1(t) + F_2(y(t))u_2(t)\ ,\ t \in (0, T] \tag{2.1}$$

where $y \in X$ is the state of the system in the set

$$X := \left\{x \in H^1(0, T; \mathbb{R}^n)\ :\ x(0) = y_0\right\} ,$$

with H^1 the usual Sobolev space. The functions $f^0, F_j : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be as in Section 1.4.2. The two players, P_1, P_2, are identified with their strategies $u_j : [0, T] \to \mathbb{R}$, $j = 1, 2$, belonging to the following admissible sets

$$\mathcal{U}_j(M) := \left\{v \in L^2(0, T)\ :\ v(t) \in K_M \text{ a.e. in } (0, T)\right\} , \tag{2.2}$$

where $K_M = [-\sqrt{M}, \sqrt{M}]$ and $M \in (0, \infty]$ is a given positive constant.

As we have seen, each player aims at minimising its own objective cost given by

$$J_j(y, u_1, u_2) = \frac{1}{2}\|y(T) - y_T^{(j)}\|_2^2 + \frac{\nu}{2}\|u_j\|_{L^2}^2, \quad j = 1, 2. \tag{2.3}$$

Thus, if we define the map $(u_1, u_2) \mapsto y = y(u_1, u_2)$, we can introduce the reduced objectives $\widetilde{J}_1(u_1, u_2) = J_1(y(u_1, u_2), u_1, u_2)$ and $\widetilde{J}_2(u_1, u_2) = J_2(y(u_1, u_2), u_1, u_2)$. In this framework, a Nash equilibrium is sought as a solution for the game

$G = (\widetilde{J}_1, \widetilde{J}_2, \mathcal{U}_1(M), \mathcal{U}_2(M))$.

We have proved that for this game there exists a NE for a sufficiently small time horizon T. In this section, we want to illustrate a procedure for solving the bilinear differential NE game numerically. For this purpose, assume that $u^* = (u_1^*, u_2^*)$ is a Nash equilibrium for the given time T. Then the following holds [21].

1. The strategy u_1^* is optimal for Player 1, in the sense that it solves the following optimal control problem

$$\begin{aligned} &\min_{u_1} J_1(y, u_1, u_2^*) \\ &s.t. \quad y' = f^0(y) + u_1 F_1(y) + u_2^* F_2(y), \qquad y(0) = y_0, \end{aligned} \tag{2.4}$$

2. The strategy u_2^* is optimal for Player 2, that is, it is a solution of the following optimal control problem

$$\begin{aligned} &\min_{u_2} J_2(y, u_1^*, u_2) \\ &s.t. \quad y' = f^0(y) + u_1^* F_1(y) + u_2 F_2(y), \qquad y(0) = y_0. \end{aligned} \tag{2.5}$$

Therefore, as we have seen in Chapter 1, $u^* = (u_1^*, u_2^*)$ solves simultaneously two optimal control problems whose solutions are characterized by the following optimality system

$$\begin{aligned} -p_1' &= \left[\left(\partial_y f^0(y)\right)^\top + \sum_{n=1}^{2} u_n \left(\partial_y F_n(y)\right)^\top \right] p_1, \quad p_1(T) = -(y(T) - y_T^{(1)}), \\ -p_2' &= \left[\left(\partial_y f^0(y)\right)^\top + \sum_{n=1}^{2} u_n \left(\partial_y F_n(y)\right)^\top \right] p_2, \quad p_2(T) = -(y(T) - y_T^{(2)}), \\ y' &= f^0(y) + \sum_{n=1}^{2} u_n F_n(y), \qquad y(0) = y_0, \\ u_n(t) &= \mathcal{P}_{\mathcal{U}_n(M)}\left(\frac{1}{\nu}\langle F_n(y(t)), p_n(t)\rangle\right), \text{ a.e. in } [0,T], \quad n = 1, 2, \end{aligned} \tag{2.6}$$

where $y_T^{(n)}$, $n = 1, 2$, are the final targets to the players and $\mathcal{P}_{\mathcal{U}_n(M)}$ denotes the L^2 projector operator on $\mathcal{U}_n(M)$; see [30] for all details.

As we have seen in the previous chapter, introducing the reduced cost functionals $\widetilde{J}_1(u_1, u_2) := J_1(y(u_1, u_2), u_1, u_2)$ and $\widetilde{J}_2(u_1, u_2) := J_2(y(u_1, u_2), u_1, u_2)$, the solution to (2.4) can be written as $u_1^* = \arg\min_{u_1} \widetilde{J}_1(u_1, u_2^*)$, and the solution to (2.5) can be written as $u_2^* = \arg\min_{u_2} \widetilde{J}_2(u_1^*, u_2)$. In this framework, a classical iterative method for solving a NE problem is the relaxation scheme discussed in [58] and implemented in Algorithm 1.

In the relaxation scheme, τ is a relaxation factor that we specify in our numerical experiments. In general, there is no a priori choice of τ available. However, in our numerical experiments we always observe convergence of this scheme by a moderate choice of the relaxation factor. Nevertheless, at the end of this section, a proof of convergence of Algorithm 1 is given.

In Algorithm 1 the two minimisation problems are solved separately and the strategy of the other player enter as a parameter. Then, a relaxation step is performed to get a new update of the strategy functions u_1^{k+1}, u_2^{k+1} until an appropriate stopping criterion is met.

Algorithm 1 Relaxation scheme

1: Initialize (u_1^0, u_2^0); set $\tau \in (0,1)$, tol_u and $k = 0$.
2: **repeat**
3: Compute $\bar{u}_1 = \arg\min_{u_1} \widetilde{J}_1(u_1, u_2^k)$
4: Compute $\bar{u}_2 = \arg\min_{u_2} \widetilde{J}_2(u_1^k, u_2)$
5: Set $(u_1^{k+1}, u_2^{k+1}) := \tau\,(u_1^k, u_2^k) + (1-\tau)\,(\bar{u}_1, \bar{u}_2)$
6: $k := k + 1$
7: **until** $\|u^{k+1} - u^k\| < tol_u$

The main advantage of the above algorithm is that we can compute $\bar{u}_1$ and $\bar{u}_2$ separately (in parallel) using an efficient optimization scheme. Specifically, given u_2^k, we compute $\bar{u}_1$ by solving the optimality system given by

$$\begin{aligned} -p_1' &= \Big[\Big(\partial_y f^0(y)\Big)^\top + u_1\,(\partial_y F_1(y))^\top + u_2^k\,(\partial_y F_2(y))^\top \Big]\, p_1, \quad p_1(T) = -(y(T) - y_T^{(1)}), \\ y' &= f^0(y) + u_1 F_1(y) + u_2^k F_2(y), \qquad y(0) = y_0, \\ u_1(t) &= \mathcal{P}_{\mathcal{U}_1(M)}\Big(\frac{1}{\nu}\langle F_1(y(t)), p_1(t)\rangle\Big), \text{ a.e. in } [0,T]. \end{aligned} \tag{2.7}$$

In a similar way, given u_1^k, we can compute $\bar{u}_2$ by solving the following optimality system

$$\begin{aligned} -p_2' &= \Big[\Big(\partial_y f^0(y)\Big)^\top + u_1^k\,(\partial_y F_1(y))^\top + u_2\,(\partial_y F_2(y))^\top \Big]\, p_2, \quad p_2(T) = -(y(T) - y_T^{(2)}), \\ y' &= f^0(y) + u_1^k F_1(y) + u_2 F_2(y), \qquad y(0) = y_0, \\ u_2(t) &= \mathcal{P}_{\mathcal{U}_2(M)}\Big(\frac{1}{\nu}\langle F_2(y(t)), p_2(t)\rangle\Big), \text{ a.e. in } [0,T]. \end{aligned} \tag{2.8}$$

Next, we give a short description of the semi-smooth Newton scheme that we implement for solving (separately) the optimality systems (2.7) and (2.8); for more details, see [29, 30]. Denote with $\eta := (y, u_1, p_1)$ and define the map $\mathcal{F}(\eta) := \Big(\mathcal{F}_1(\eta), \mathcal{F}_2(\eta), \mathcal{F}_3(\eta)\Big)^\top$, which represent the residual of the adjoint, state, and optimality condition equations. Therefore, the solution to the optimality systems corresponds to the root of $\mathcal{F}(\eta) = 0$, which can be determined by a Newton procedure. However, for this purpose, we need the Jacobian of $\mathcal{F}$, which is not differentiable in a classical sense with respect to u_n, because of the projection function.

On the other hand, by sub-differential calculus, it is possible to construct a generalized Jacobian, such that the following Newton equation is obtained [30]

$$\Big(\nabla_\eta \mathcal{F}(\eta_k)\Big)\Big(\eta_{k+1} - \eta_k\Big) + \mathcal{F}(\eta_k) = 0.$$

This equation can be solved by a Krylov method that requires to implement the action of the Jacobian on an input vector, and thus allows to avoid the assembling of $\nabla_\eta \mathcal{F}$, which leads to the possibility to define a Krylov and semi-smooth Newton matrix-free procedure. This is the method that we use in our calculations in the Steps 3 and 4 of Algorithm 1, and for computing the Pareto points discussed in the following section.

Notice that we have discussed our solution methodology at a functional level. However, its numerical realisation requires to approximate the optimality system by appropriate numerical schemes. In particular, we approximate our model using the so-called modified Crank-Nicolson (MCN) scheme. For this purpose, the time domain $[0,T]$ is subdivided in uniform intervals of size h and N_t points, such that $t^j = (j-1)h$ and $0 = t^1 < \cdots < t^{N_t} = T$. This scheme is norm-preserving and second order accurate. For a complete and detailed analysis of this discretization scheme see, e.g., [13].

Next, we prove that Algorithm 1, where Steps 3 and 4 are solved with the semi-smooth Newton method described above, is locally convergent. For this purpose, we need the following preliminary result which states a similar result as in Propositions 1.4.3-1.4.4 for the backward equations p_i, $i = 1, 2$.

Lemma 2.1.1. *The maps $u \mapsto p_i(u)$, $i = 1, 2$, are bounded in $[0, T]$ and Lipschitz continuous in $u \in \mathcal{U}(M), M \in (0, \infty)$, that is,*

$$\|p_i(t)\|_2 \leq \mathcal{K}_{p_i}(T), \tag{2.9}$$

$$\|p_i(u^{(1)})(t) - p_i(u^{(2)})(t)\|_2 \leq 2\mathcal{K}_{p_i}(T) L_F \Big(L_{f^0} + M L_F\Big) T^{3/2} \|u^{(1)} - u^{(2)}\|_{L^2}, \tag{2.10}$$

where $\mathcal{K}_{p_i}, i = 1, 2$, are monotonically increasing with T.

Proof. Similarly to the proof of Propositions 1.4.3 and 1.4.4, consider p_i, the unique solution to the corresponding adjoint equation, written in integral form. We have

$$p_i(t) = p_i(T) + \int_t^T \left[(\partial_y f^0(y(s)))^\top + \sum_{n=1}^{2} u_n(s) (\partial_y F_n(y(s))^\top \right] p_i(s) ds. \tag{2.11}$$

Taking the 2-norm, we get

$$\|p_i(t)\|_2 \leq \|y(T) - y_T^{(i)}\|_2 + \int_t^T \left[\|(\partial_y f^0(y(s)))^\top\|_2 + \sum_{n=1}^{2} |u_n(s)| \|(\partial_y F_n(y(s)))^\top\|_2 \right] \|p_i(s)\|_2 ds.$$

Now, with the same argument seen in the proof of Proposition 1.4.3, from the Lipschitz continuity of f^0 and F_n in y, it follows the boundedness of their derivatives. Hence, we arrive at the following estimate

$$\|p_i(t)\|_2 \leq \|y(T) - y_T^{(i)}\|_2 + \int_t^T [L_{f^0} + L_F \|u(s)\|_2] \, \|p_i(s)\| ds. \tag{2.12}$$

With the Grönwall's lemma we get the needed estimate on the p_i.

Next, to prove the second part of the lemma, we proceed as is the proof the Lemma 1.4.4. Therefore, for any $u^{(1)}, u^{(2)} \in \mathcal{U}(M)$, considering the correspondig adjoints equations and taking their difference we arrive at the following equation

$$-(p_i(u^{(1)}) - p_i(u^{(2)}))' = (\partial_y f^0(y))^\top (p_i(u^{(1)}) - p_i(u^{(2)})) + \left[\sum_{n=1}^{2} (\partial_y F_n(y))^\top (u_j^{(1)} - u_j^{(2)}) \right] p_i(u^{(1)})$$
$$+ \left[\sum_{n=1}^{2} (\partial_y F_n(y))^\top u_j^{(2)} \right] (p_i(u^{(1)}) - p_i(u^{(2)})).$$

Integrating over (t, T) and taking the 2-norm, with similar estimates done as above, and applying the Grönwall's lemma, we get the following

$$\|p_i(u^{(1)})(t) - p_i(u^{(2)})(t)\|_2 \leq 2\mathcal{K}_{p_i}(T) L_F \Big(L_{f^0} + M L_F\Big) T^{3/2} \|u^{(1)} - u^{(2)}\|_{L^2}.$$

□

Now, we can prove the main result of this chapter concerning the convergence of Algorithm 1.

Theorem 2.1.2. *Let $\mathcal{B}(u^*)$ be the largest closed ball of $\mathcal{U}(M)$, $M \in (0,\infty)$, centered in u^*, a NE for the game, where the optimality systems* (2.7)-(2.8) *are assumed to be uniquely solvable.*

Then, the relaxation scheme given in Algorithm 1 is convergent in $\mathcal{B}(u^*)$.

Proof. In the following we omit the dependence on t of the functions y and p_i, $i = 1, 2$.

Consider $\mathcal{B}(u^*)$, the largest closed ball of $\mathcal{U}(M)$ centered in u^*, a NE solution for the game, in which (2.7)-(2.8) can be uniquely solved.

We want to show that the map $A : \mathcal{B}(u^*) \to \mathcal{B}(u^*)$, defined as

$$A(v) := \tau v + (1-\tau)\bar{u}(v) \tag{2.13}$$

is a contraction, where $\bar{u}(v)$ is the solution of (2.7)-(2.8), i.e.,

$$\bar{u}(v) = \begin{pmatrix} \mathcal{P}_{\mathcal{U}_1(M)}\Big(\frac{1}{\nu}\langle F_1(y(\bar{v}_1, v_2)), p_1(\bar{v}_1, v_2)\rangle\Big) \\ \mathcal{P}_{\mathcal{U}_2(M)}\Big(\frac{1}{\nu}\langle F_2(y(v_1, \bar{v}_2)), p_2(v_1, \bar{v}_2)\rangle\Big) \end{pmatrix}, \tag{2.14}$$

with $v := (v_1, v_2)$ and $\bar{v}_1 := \bar{u}_1(v), \bar{v}_2 := \bar{u}_2(v)$.

Now, let $v \in \mathcal{U}(M)$. For the assumption on $\mathcal{B}(u^*)$, the map A is well defined.

Next, we prove that A is a contraction in $\mathcal{B}(u^*)$.

Let $v := (v_1, v_2), w := (w_1, w_2) \in \mathcal{U}(M)$ and compute $\|A(v) - A(w)\|_{L^2}$. Since

$$\|A(v) - A(w)\|_{L^2} \leq \tau\|v - w\|_{L^2} + (1-\tau)\|\bar{u}(v) - \bar{u}(w)\|_{L^2},$$

as first, we show the Lipschitz continuity of the map $v \mapsto \bar{u}(v)$, to get

$$\|A(v) - A(w)\|_{L^2} \leq \Big(\tau + (1-\tau)L_{\bar{u}}\Big)\|v - w\|_{L^2}.$$

For this purpose and recalling (2.14), we need to estimate

$$|\langle F_1(y(\bar{v}_1, v_2)), p_1(\bar{v}_1, v_2)\rangle - \langle F_1(y(\bar{w}_1, w_2)), p_1(\bar{w}_1, w_2)\rangle| \tag{2.15}$$

and

$$|\langle F_2(y(v_1, \bar{v}_2)), p_2(v_1, \bar{v}_2)\rangle - \langle F_2(y(w_1, \bar{w}_2)), p_2(w_1, \bar{w}_2)\rangle|. \tag{2.16}$$

We estimate (2.15) and the same holds for (2.16). The aim is to use the boundedness and Lipschitz continuity of the functions y and p_1 proved in Proposition 1.4.3, Proposition 1.4.4, Lemma 2.1.1, and of F_1. Adding and subtracting the following quantities from (2.15),

$$\begin{gathered} \langle F_1(y(\bar{v}_1, w_2)), p_1(\bar{v}_1, v_2)\rangle, \\ \langle F_1(y(\bar{w}_1, w_2)), p_1(\bar{v}_1, v_2)\rangle \end{gathered}$$

we get,

$$\begin{aligned} &|\langle F_1(y(\bar{v}_1, v_2)), p_1(\bar{v}_1, v_2)\rangle - \langle F_1(y(\bar{w}_1, w_2)), p_1(\bar{w}_1, w_2)\rangle| \\ &\leq |\langle F_1(y(\bar{v}_1, v_2)) - F_1(y(\bar{v}_1, w_2)), p_1(\bar{v}_1, v_2)\rangle| + \\ &|\langle F_1(y(\bar{w}_1, w_2)), p_1(\bar{v}_1, v_2) - p_1(\bar{w}_1, w_2)\rangle| + \\ &|\langle F_1(y(\bar{v}_1, w_2)) - F_1(y(\bar{w}_1, w_2)), p_1(\bar{v}_1, v_2)\rangle|. \end{aligned}$$

Since y, p_1 are Lipschitz continuous in u and bounded, and F_1 is Lipschitz in y and bounded, it follows

$$\begin{aligned}
&|\langle F_1(y(\bar{v}_1, v_2)), p_1(\bar{v}_1, v_2)\rangle - \langle F_1(y(\bar{w}_1, w_2)), p_1(\bar{w}_1, w_2)\rangle| \\
&\leq C\Big[L_{F,y,p}\, T^{3/2}\Big(\|v_2 - w_2\|_{L^2} + \|\bar{v}_1 - \bar{w}_1\|_{L^2} + \|v_2 - w_2\|_{L^2} + \|\bar{v}_1 - \bar{w}_1\|_{L^2}\Big)\Big] \\
&\leq 2C\, L_{F,y,p}\, T^{3/2}\Big[\|v - w\|_{L^2} + \|\bar{v} - \bar{w}\|_{L^2}\Big],
\end{aligned}$$

where $C := C(T)$ is the maximum between the functions bounds of p_1, F_1 and $L_{F,y,p} := L_{F,y,p}(T)$ is the maximum between the Lipschitz constants of F_1, y, p_1. Note that the bound of F_1 depends on $\mathcal{K}(T)$ defined in Proposition 1.4.3. Moreover C and $L_{F,y,p}$ are monotonically increasing functions of T.

Repeating the same calculations with (2.16), it holds

$$\|\bar{u}(v) - \bar{u}(w)\|_{L^2} \leq \frac{2}{\nu}\hat{C}T^{3/2}\Big[\|v - w\|_{L^2} + \|\bar{u}(v) - \bar{u}(w)\|_{L^2}\Big],$$

and hence

$$\left|1 - \frac{2}{\nu}\hat{C}T^{3/2}\right| \|\bar{u}(v) - \bar{u}(w)\|_{L^2} \leq \frac{2}{\nu}\hat{C}T^{3/2}\|v - w\|_{L^2},$$

where $\hat{C} := 2CL_{F,y,p}$, i.e. it is a monotonically increasing funtion of T. Hence, the map $v \mapsto \bar{u}(v)$ is Lipschitz continuous with constant

$$L_{\bar{u}} := \frac{\frac{2}{\nu}\hat{C}T^{3/2}}{\left|1 - \frac{2}{\nu}\hat{C}T^{3/2}\right|}. \tag{2.17}$$

Therefore A is a contraction if

$$\tau + (1-\tau)\frac{\frac{2}{\nu}\hat{C}T^{3/2}}{\left|1 - \frac{2}{\nu}\hat{C}T^{3/2}\right|} < 1. \tag{2.18}$$

Since $\mathcal{B}(u^*) \subset L^2(0,T)$ is a complete space, choosing the parameters τ, ν and T such that (2.18) holds, the map A admits a unique fixed point. Hence, Algorithm 1 converges. □

Notice that the estimate (2.18) relates the choice of the relaxation factor τ to the parameters of the model and to the time T of existence of a Nash equilibrium.

2.2 Pareto and bargaining solutions

In this section, we discuss a numerical methodology to determine a Nash bargaining solution of a differential Nash game. For this purpose, we first discuss the Pareto efficient solutions. In Chapter 1, we stated necessary and sufficient conditions for a couple of controls u_1, u_2 to be Pareto optimal. In particular, we have seen that a way to find Pareto solutions is to solve a parameterized optimal control problem. To this end, the following optimal control problem is considered

$$\begin{aligned}
&\min_{u_1,u_2}\ J(y, u_1, u_2) := \mu_1 J_1(y, u_1) + \mu_2 J_2(y, u_2) \\
&s.t. \quad y' = f^0(y) + u_1 F_1(y) + u_2 F_2(y), \qquad y(0) = y_0,
\end{aligned} \tag{2.19}$$

where $\mu_1, \mu_1 \in (0,1)$, with $\mu_1 + \mu_2 = 1$.
A solution to (2.19) for the bilinear problem is characterized by the following optimality system

$$
\begin{aligned}
-p' &= \left[\left(\partial_y f^0(y)\right)^\top + \sum_{n=1}^{2} \left(\partial_y F_n(y)\right)^\top u_n \right] p, \quad p(T) = -\mu_1(y(T) - y_T^{(1)}) - \mu_2(y(T) - y_T^{(2)}) \\
y' &= f^0(y) + \sum_{n=1}^{2} u_n F_n(y), \qquad y(0) = y_0 \\
u_n(t) &= \mathcal{P}_{\mathcal{U}_n(M)}\left(\frac{1}{\nu} \langle F_n(y(t)), p_n(t) \rangle \right), \text{ a.e. in } [0,T], \quad n = 1, 2.
\end{aligned}
\tag{2.20}
$$

Introducing the control-to-state map $(u_1, u_2) \mapsto y(u_1, u_2)$ as in the previous section and the reduced cost functional $\widetilde{J}(u_1, u_2) := J(y(u_1, u_2), u_1, u_2)$, the solution to (2.19) can be written as

$$
u^* = \underset{(u_1,u_2)\in\mathcal{U}(M)}{\arg\min} \; \widetilde{J}(u_1, u_2).
$$

This optimal control problem is solved with the semismooth Newton method discussed in Section 2.1.

Next, we discuss a numerical methodology to determine bargaining solutions. In particular, we assume that the players agree to follow a Nash's bargaining scheme. However, we remark that there exist also other bargaining solutions such the Kalai-Smorodinsky and the egalitarian solutions, introduced in Chapter 1.

Consider the set S of all feasible values of the functionals, which can be achieved with an admissible set of controls,

$$
S = \{(\widetilde{J}_1(u), \widetilde{J}_2(u)) \in \mathbb{R}^2 \,|\, u \in \mathcal{U}(M)\}. \tag{2.21}
$$

The purpose of the players is to share an improvement in their costs by choosing to cooperate. A solution to the bargaining problem is obtained when both players agree on a point in S.

The NB solution $u^{NB} = (u_1^{NB}, u_2^{NB})$ requires that the strategies of the players are Pareto optimal and satisfy the following criterion

$$
u^{NB} = \arg\max_{u\in S} \prod_{i=1}^{2} \left(\tilde{J}_i(u^{NE}) - \tilde{J}_i(u) \right), \tag{2.22}
$$

where $u = (u_1, u_2)$.

Geometrically, the NB solution is the point on the feasible set S that maximizes the area of the rectangle (NB, A, d, B), where $d = (d_1, d_2)$, see Figure 2.1.

Next, we focus on the following Nash bargaining (NB) problem:

$$
\begin{aligned}
&\max_u \prod_{i=1}^{2} (d_i - \widetilde{J}_i(u)) \\
&s.t. \; u \in \mathcal{U}(M), \; d_i > \widetilde{J}_i(u) \; \forall i
\end{aligned}
$$

where $d_i = \widetilde{J}_i(u^{NE})$ is a disagreement outcome for Player i. As already explained in the previous chapter, we assume that there exists at least one point $s \in S$ such that $d_i > s_i$, $i = 1, 2$, and suppose that the NB problem has a solution $u^* \in \mathcal{U}(M)$.

The solution to the NB problem is sought on the Pareto frontier; see, e.g., [68].

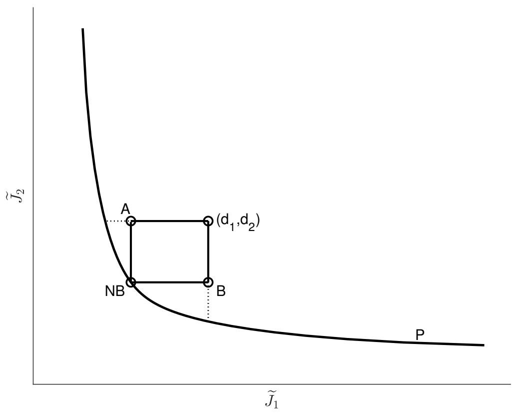

Figure 2.1: The Nash bargaining solution

For this purpose, we present a theorem that provides a characterisation of the solution to the NB problem on the Pareto frontier. This result is essential to formulate an algorithm for the computation of the NB solution; see also [38].

Specifically, we show that if u^* minimizes a weighted sum of the reduced cost functionals $\widetilde{J}_1, \widetilde{J}_2$, with weights chosen with a fixed-point procedure, then the strategy u^* provides a Nash bargaining solution for the game.

Theorem 2.2.1. *Let $u^* \in \mathcal{U}(M)$ be such that*

$$\mu_1 \widetilde{J}_1(u^*) + \mu_2 \widetilde{J}_2(u^*) \leq \mu_1 \widetilde{J}_1(u) + \mu_2 \widetilde{J}_2(u), \tag{2.23}$$

for all $u \in \mathcal{U}(M)$, where

$$\mu_1 = \frac{d_2 - \widetilde{J}_2(u^*)}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))} \tag{2.24}$$

and

$$\mu_2 = \frac{d_1 - \widetilde{J}_1(u^*)}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))}. \tag{2.25}$$

Then

$$\prod_{i=1}^{2} \Big(d_i - \widetilde{J}_i(u)\Big) \leq \prod_{i=1}^{2} \Big(d_i - \widetilde{J}_i(u^*)\Big). \tag{2.26}$$

Proof. Consider the following equation

$$-\mu_1 d_1 - \mu_2 d_2 = -\mu_1 d_1 - \mu_2 d_2. \tag{2.27}$$

Adding term by term with (2.23), it follows

$$\mu_1(d_1 - \widetilde{J}_1(u^*)) + \mu_2(d_2 - \widetilde{J}_2(u^*)) \geq \mu_1(d_1 - \widetilde{J}_1(u)) + \mu_2(d_2 - \widetilde{J}_2(u)).$$

Replacing the values of μ_1, μ_2, we get

$$\frac{(d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u^*))}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))} + \frac{(d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u^*))}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))} \geq$$
$$\frac{(d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u))}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))} + \frac{(d_2 - \widetilde{J}_2(u))(d_1 - \widetilde{J}_1(u^*))}{(d_1 - \widetilde{J}_1(u^*)) + (d_2 - \widetilde{J}_2(u^*))}.$$

Multiplying both sides for $(d_2 - \widetilde{J}_2(u^*)) + (d_1 - \widetilde{J}_1(u^*)) > 0$, it follows

$$2(d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u^*)) \geq (d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u)) + (d_2 - \widetilde{J}_2(u))(d_1 - \widetilde{J}_1(u^*)).$$

Then, dividing by $(d_2 - \widetilde{J}_2(u^*))(d_1 - \widetilde{J}_1(u^*)) > 0$, the following holds

$$\frac{d_1 - \widetilde{J}_1(u)}{d_1 - \widetilde{J}_1(u^*)} + \frac{d_2 - \widetilde{J}_2(u)}{d_2 - \widetilde{J}_2(u^*)} \leq 2$$

and hence

$$2\sqrt{\frac{d_1 - \widetilde{J}_1(u)}{d_1 - \widetilde{J}_1(u^*)}\frac{d_2 - \widetilde{J}_2(u)}{d_2 - \widetilde{J}_2(u^*)}} \leq \frac{d_1 - \widetilde{J}_1(u)}{d_1 - \widetilde{J}_1(u^*)} + \frac{d_2 - \widetilde{J}_2(u)}{d_2 - \widetilde{J}_2(u^*)} \leq 2.$$

Therefore we obtain

$$\prod_{i=1}^{2}\Big(d_i - \widetilde{J}_i(u)\Big) \leq \prod_{i=1}^{2}\Big(d_i - \widetilde{J}_i(u^*)\Big).$$

□

Notice that in (2.24) and (2.25), $\mu_1, \mu_2 \in (0,1)$ and $\mu_1 + \mu_2 = 1$. Theorem 2.2.1 states that it is possible to find a point on the Pareto frontier that maximizes the product $\prod_{i=1}^{2}\Big(d_i - \widetilde{J}_i(u)\Big)$.

Moreover, if u^* satisfies Theorem 2.2.1, then the corresponding optimal outcome $(\widetilde{J}_1(u^*), \widetilde{J}_2(u^*))$ is a Pareto solution and hence the bargaining strategy function $u^* = (u_1^*, u_2^*)$ must satisfy the necessary first-order optimality conditions given in Theorem 1.3.2.

Next, based on the result of Theorem 2.2.1, we can introduce a computational method to obtain a solution for the bargaining problem; see [38, 40] for other variants.

Algorithm 2 Computation of Nash bargaining solution

1: Initialize $\mu_1^{(0)}$, then $\mu_2^{(0)} = 1 - \mu_1^{(0)}$; set $\alpha \in (0,1)$ and $k = 0$.
2: Compute the disagreement outcome d_i for player i.
3: **repeat**
4: For fixed $\mu_1^{(k)}$, solve the optimality system and compute the corresponding objective costs.
5: Update the weight according to the following formula

$$\mu_1^{(k+1)} = (1-\alpha)\mu_1^{(k)} + \alpha\frac{d_2 - \widetilde{J}_2(u(\mu^{(k)}))}{d_2 - \widetilde{J}_2(u(\mu^{(k)})) + d_1 - \widetilde{J}_1(u(\mu^{(k)}))}. \tag{2.28}$$

6: $k := k + 1$
7: **until** (2.24) holds
8: Set $\mu^* = \mu^{(k)}$.

The value of the initial guess of μ_1, namely, μ_1^0, is chosen arbitrarly by the user and it depends on the goal of the problem. We remark that a converge proof of this algorithm is lacking. However, numerical experiments indicate that the algorithm performs well and converges to the NB solution sought.

Notice that, in step 4 of Algorithm 2 we need to solve the optimization problem (2.19) discussed above.

2.3 A bilinear quantum game

Consider a model of two uncoupled spin-1/2 particles, whose state configuration represents the density matrix operator and the corresponding dynamics is governed by the Liouville – von Neumann master equation. This is a basic model of importance in nuclear magnetic resonance (NMR) spectroscopy. We remark that in the case of quantum dynamics constrained on the energy ground state, only transitions between magnetic/spin states are possible. In this case, as discussed in detail in [13], the Pauli-Schrödinger equation represented in a basis of spherical-harmonics becomes

$$i\, a' = \left[B_z \tilde{H}_0 + B_x \tilde{H}_x + i B_y \tilde{H}_y \right] a \, , \tag{2.29}$$

where the complex-valued vector $a(t)$ represents the time-dependent coefficients of the spectral discretization. In the Hamiltonian defined by the operator in the square brackets, $\tilde{H}_0$ and $\tilde{H}_x$ are Hermitian matrices, and $\tilde{H}_y$ is a skew-symmetric matrix. In an experimental setting, we can have that the z-component of the magnetic field, B_z, is fixed, and we would like to manipulate the spin orientation of the particle by acting with the transversal magnetic fields B_x and B_y, which we identify with u_1 and u_2, respectively. In this case the controlled dynamics of each spin-1/2 particle is described by the Hamiltonian

$$H = \hat{\nu}\, I_z + u_1\, I_x + u_2\, I_y,$$

where $\hat{\nu}$ is the Larmor-frequency, u is the control, and I_x, I_y and I_z are the Pauli matrices. To analyse this model, it results very convenient to choose a frame rotating with the Larmor frequency and use the so-called real-matrix representation [13]. We obtain the following model

$$y' = 2\pi \left[\widetilde{A} + u_1 \widetilde{B}_1 + u_2 \widetilde{B}_2 \right] y, \tag{2.30}$$

where

$$\widetilde{A} = c \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad \widetilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & B_i \end{pmatrix}.$$

We choose $c = 483$ and

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $y_0 = (0, 0, 1, 0, 0, 1)^\top$ be the initial state of the system and consider the following targets for the two players

$$y_T^{(1)} = (1, 0, 0, 1, 0, 0)^\top \;\text{ and }\; y_T^{(2)} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^\top.$$

Notice that these vectors must be normalized. Our problem is to find $u_1 \in \mathcal{U}_1(M)$, with $M = 60$, such that the system aims at performing a transition from the initial state y_0, where both spins are pointing in the z-direction, to a target state $y_T^{(1)}$ where both spins are pointing in the x-direction, and to find $u_2 \in \mathcal{U}_2(M)$ that has the aim to drive the system to $y_T^{(2)}$, where an inversion of orientation is desired. Hence, subject to (2.30), the following players' objectives are considered

$$J_1(y, u_1, u_2) = \frac{1}{2} \| y(T) - y_T^{(1)} \|_2^2 + \frac{\nu}{2} \| u_1 \|_{L^2}^2, \quad J_2(y, u_1, u_2) = \frac{1}{2} \| y(T) - y_T^{(2)} \|_2^2 + \frac{\nu}{2} \| u_2 \|_{L^2}^2.$$

As in [4], we choose $T = 0.008$, and based on our estimate (1.73), we take $\nu = 2 \cdot 10^{-1}$ in order to guarantee the existence of a NE for the considered quantum game.

Let $u^0 := (u_1^0, u_2^0)$. We chose $u^0(t) = (0.1, 0.1)$, $t \in [0, T]$. To solve this NE problem, we use the relaxation scheme of Algorithm 1 with $\tau = 0.5$ and $tol_u = 10^{-3}$. In this method, the semismooth Newton scheme [28, 30] is employed to solve the optimality systems corresponding to the two players. In this implementation, the differential equations are approximated by the MCN scheme with $N_t = 1000$.

The tolerances for the convergence of the semismooth Newton and for the Krylov linear solver are 10^{-7} and 10^{-8}, respectively. These tolerances are meat always before the maximum number of allowed iterations, which is set equal to 100, is reached. With this procedure, we obtain the NE strategies depicted in Figure 2.2, which give NE point that is shown in Figure 2.3, as a '*'-point. At NE solution, $\|y(T) - y_T^{(1)}\|_2 = 0.4967$ and $\|y(T) - y_T^{(2)}\|_2 = 0.5996$.

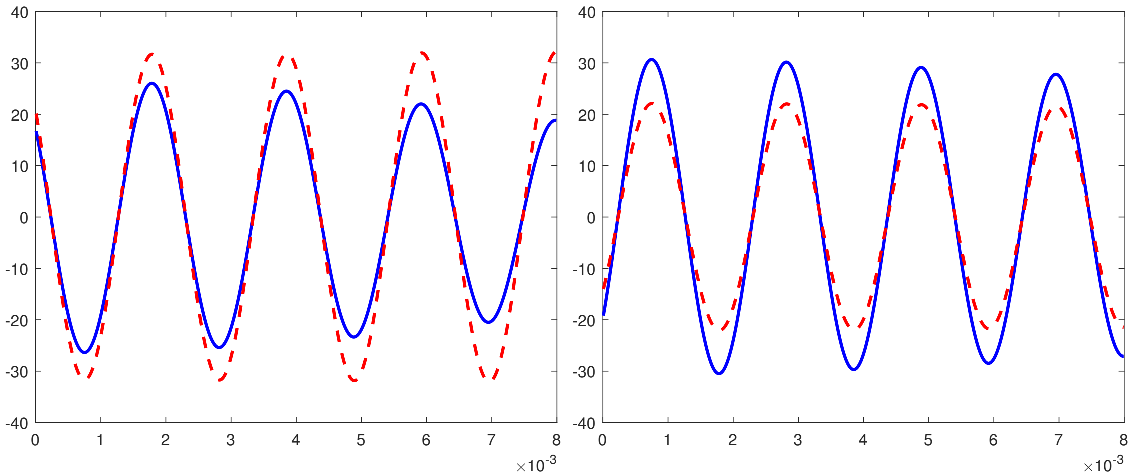

Figure 2.2: NE strategies for the first experiment (solid line) and NB strategies (dashed line); u_1 on the left-hand side, and u_2 on the right-hand side. As expected, at Nash equilibrium, u_2 assumes higher values than u_1 because it has to make a greater effort to reach its own target.

With the NE point, we can consider the problem of Nash bargaining that assumes cooperation in order to get an improvement of the players' objectives. We solve the bargaining problem using Algorithm 2 with $\mu_1^{(0)} = 0.55$, $\alpha = 0.2$. We obtain $\mu^* = 0.47$, and the corresponding solution is depicted also in Figure 2.3 with a $\circ$-point. Furthermore, in this figure we present our computation of the Pareto frontier to which this point belongs.

Now, in the second experiment with our quantum model, we replace the final target of the second player with

$$y_T^{(2)} = (0, 1, 0, 0, 1, 0)^\top,$$

and let $M = 15$. All other parameters are as in the previous experiment. In this case, the targets for the two players present similar difficulty. By taking a lower value of M, we see that the constraints on the NE strategies become active as shown in Figure 2.4. With this solution, $\|y(T) - y_T^{(1)}\|_2 = 0.4155$ and $\|y(T) - y_T^{(2)}\|_2 = 0.4155$.

Furthermore, we consider the Nash bargaining problem. In this case, Algorithm 2 provides the solution $\mu^* = 0.5$, which is plotted in Figure 2.5.

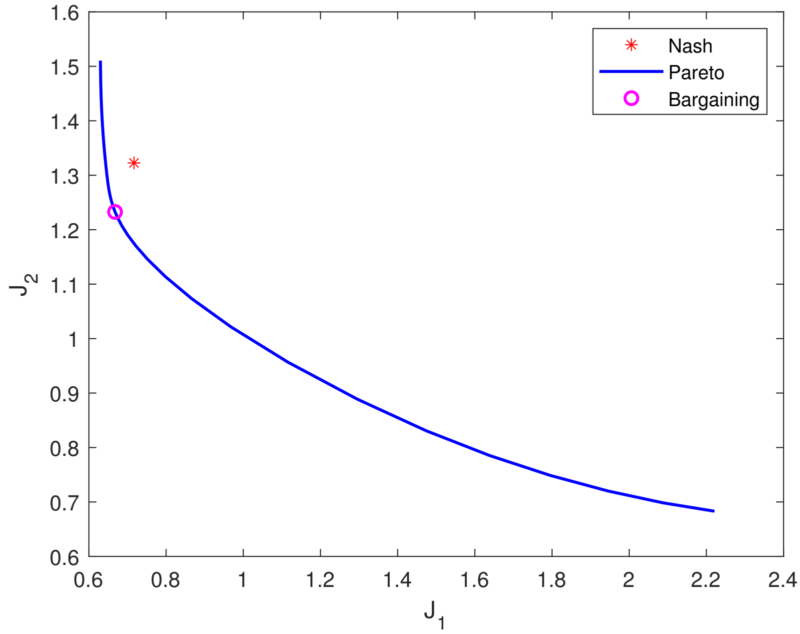

Figure 2.3: The NE point, the NB point and the Pareto frontier of the first experiment.

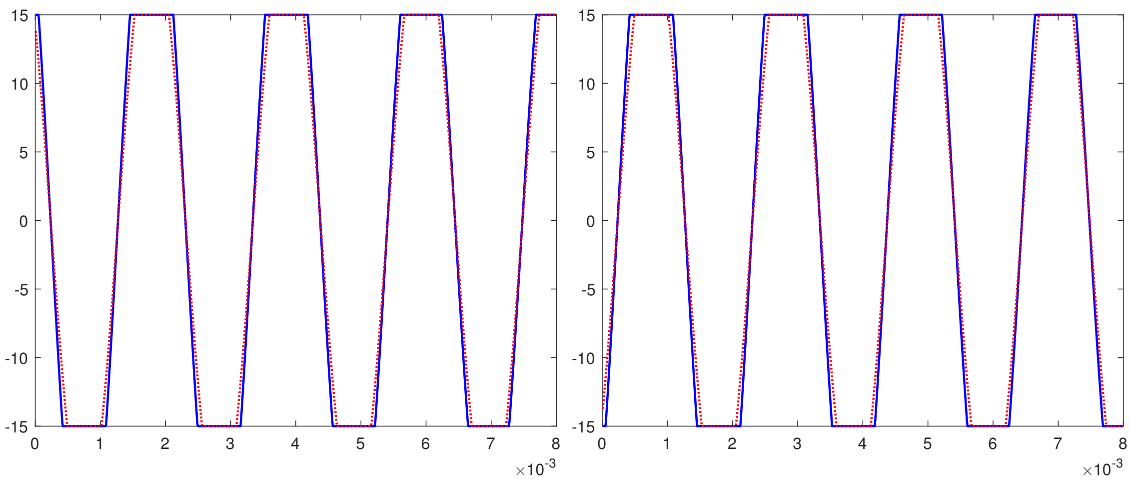

Figure 2.4: NE strategies (solid line), NB strategies (dashed line): u_1 on the left and u_2 on the right: as expected the two functions have the same range of values. Moreover, due to the symmetry of the problem, the two solutions are similar.

A comparison of the results of the two experiments is conveniently done based on the Figures 2.3 and 2.5. It is clear that, in the second experiment, the two targets are placed on the same hemisphere and therefore the game is more balanced than the game of the first experiment. This fact can be seen also in the positioning of the NE points and NB points with respect to the Pareto frontier.

Next, in order to provide more insight in the performance of our algorithms, we report some results concerning the relaxation procedure and the semi-smooth Newton scheme. Concerning the relaxation scheme in Algorithm 1, we give in Table 2.1 and Table 2.2 the convergence history of this scheme towards a NE point for the first and second experiment, respectively. In both tables, we see that initially both functionals are minimized, and thereafter we see that the values of the norms of the two gradients decrease until both

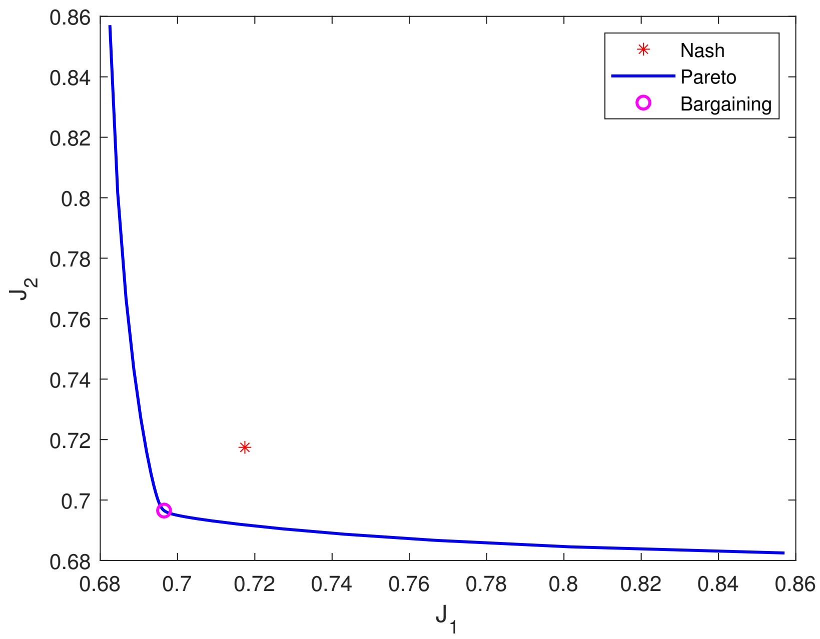

Figure 2.5: The NE point, the NB point and the Pareto frontier of the second experiment.

reach zero to high accuracy. Algorithm 1 stops when the updates of the players' actions become smaller than a given tolerance tol_u. These tables show the history of the game up to reaching the Nash equilibrium.

To conclude this experiment, we show that our semi-smooth Newton method provides a quadratic convergent behaviour to the solution of the given optimality system. For this purpose, we consider the case of solving (2.19) corresponding to the setting of the two experiments above and choosing $\mu = \mu^*$. With this setting, we obtain the convergence behaviours shown in Table 2.3.

2.4 A competitive Lotka-Volterra model

In this section we present an application with the so-called competitive Lotka-Volterra equations; see [66] for more details. This model describes the case of two species competing for the same limited food resources. For example, one can consider a competition for the territory which is directly related to food resources. In particular, each species has logistic growth in the absence of the other. Then, we focus on the following system

$$\begin{aligned} y_1' &= b_1 y_1 - a_{11} y_1^2 - a_{21} y_1 y_2 \\ y_2' &= b_2 y_1 - a_{12} y_1 y_2 - a_{22} y_2^2 \end{aligned} \tag{2.31}$$

where b_i (> 0), $i = 1, 2$, are the birth rates of the two species and a_{ij} (> 0), $i, j = 1, 2$, are the competition efficiencies.

Suppose a_{ii} and a_{ji} can be controlled by player i, i.e. $a_{ij} = a_{ij}^* + u_i$, with $a_{ij}^* > 0$. Then, the system (2.31) can be written as (1) in the following way

$$y' = \begin{pmatrix} b_1 y_1 - a_{11}^* y_1^2 - a_{21}^* y_1 y_2 \\ b_2 y_2 - a_{12}^* y_1 y_2 - a_{22}^* y_2^2 \end{pmatrix} + \begin{pmatrix} -y_1^2 \\ -y_1 y_2 \end{pmatrix} u_1 + \begin{pmatrix} -y_1 y_2 \\ -y_2^2 \end{pmatrix} u_2. \tag{2.32}$$

Table 2.1: Relaxation scheme for the first experiment with $\nu = 2 \cdot 10^{-1}$ and $\tau = 0.5$

iter	$\|\nabla \widetilde{J}_1\|_{L^2}$	$\|\nabla \widetilde{J}_2\|_{L^2}$	$\widetilde{J}_1(u_1)$	$\widetilde{J}_2(u_2)$
0	$3.9315 \cdot 10^{-1}$	$2.7790 \cdot 10^{-1}$	1.000164	1.707004
1	$2.0549 \cdot 10^{-1}$	$1.5969 \cdot 10^{-1}$	0.763244	1.455984
2	$9.6836 \cdot 10^{-2}$	$7.8763 \cdot 10^{-2}$	0.716240	1.366769
3	$4.4333 \cdot 10^{-2}$	$3.6226 \cdot 10^{-2}$	0.711989	1.337804
4	$2.0480 \cdot 10^{-2}$	$1.6241 \cdot 10^{-2}$	0.713750	1.328051
5	$9.7224 \cdot 10^{-3}$	$7.2830 \cdot 10^{-3}$	0.715280	1.324587
6	$4.7624 \cdot 10^{-3}$	$3.3284 \cdot 10^{-3}$	0.716117	1.323325
7	$2.3945 \cdot 10^{-3}$	$1.5725 \cdot 10^{-3}$	0.716527	1.322872
8	$1.2244 \cdot 10^{-3}$	$7.7321 \cdot 10^{-4}$	0.716719	1.322718
9	$6.3168 \cdot 10^{-4}$	$3.9452 \cdot 10^{-4}$	0.716808	1.322672
10	$3.2689 \cdot 10^{-4}$	$2.0683 \cdot 10^{-4}$	0.716848	1.322662
⋮	⋮	⋮	⋮	⋮
15	$1.2014 \cdot 10^{-5}$	$9.1261 \cdot 10^{-6}$	0.716882	1.322672
⋮	⋮	⋮	⋮	⋮
20	$4.7112 \cdot 10^{-7}$	$3.8636 \cdot 10^{-7}$	0.716882	1.322674
⋮	⋮	⋮	⋮	⋮
24	$7.4457 \cdot 10^{-8}$	$5.8082 \cdot 10^{-8}$	0.716882	1.322674

Table 2.2: Relaxation scheme for the second experiment with $\nu = 2 \cdot 10^{-1}$ and $\tau = 0.5$

iter	$\|\nabla \widetilde{J}_1\|_{L^2}$	$\|\nabla \widetilde{J}_2\|_{L^2}$	$\widetilde{J}_1(u_1)$	$\widetilde{J}_2(u_2)$
0	$3.9315 \cdot 10^{-1}$	$3.9299 \cdot 10^{-1}$	1.001643	0.999851
1	$2.6703 \cdot 10^{-1}$	$2.6694 \cdot 10^{-1}$	0.813612	0.813495
2	$1.9647 \cdot 10^{-1}$	$1.9642 \cdot 10^{-1}$	0.752994	0.752950
3	$1.1935 \cdot 10^{-1}$	$1.1925 \cdot 10^{-1}$	0.731974	0.731956
4	$6.3067 \cdot 10^{-2}$	$6.3013 \cdot 10^{-2}$	0.723889	0.723882
5	$3.2329 \cdot 10^{-2}$	$3.2302 \cdot 10^{-2}$	0.720458	0.720456
6	$1.6347 \cdot 10^{-2}$	$1.6339 \cdot 10^{-2}$	0.718893	0.718894
7	$8.2180 \cdot 10^{-3}$	$8.2117 \cdot 10^{-3}$	0.718148	0.718150
8	$4.1205 \cdot 10^{-3}$	$4.1172 \cdot 10^{-3}$	0.717785	0.717787
9	$2.0642 \cdot 10^{-3}$	$2.0625 \cdot 10^{-3}$	0.717605	0.717608
10	$1.0330 \cdot 10^{-3}$	$1.0321 \cdot 10^{-3}$	0.717516	0.717519
⋮	⋮	⋮	⋮	⋮
15	$3.2417 \cdot 10^{-5}$	$3.2392 \cdot 10^{-5}$	0.717429	0.717432
⋮	⋮	⋮	⋮	⋮
20	$1.0191 \cdot 10^{-6}$	$1.0183 \cdot 10^{-6}$	0.717427	0.717430
⋮	⋮	⋮	⋮	⋮
24	$6.4214 \cdot 10^{-8}$	$6.4164 \cdot 10^{-8}$	0.717426	0.717430

Table 2.3: Convergence of the semi-smooth Newton method for the solution of (2.19) for the two experiments with $\mu = \mu^*$.

	first experiment	second experiment
iter	$\|\nabla \widetilde{J}\|_{L^2}$	$\|\nabla \widetilde{J}\|_{L^2}$
0	$5.3748 \cdot 10^{-2}$	$4.3944 \cdot 10^{-2}$
1	$3.4197 \cdot 10^{-3}$	$8.5453 \cdot 10^{-2}$
2	$1.0608 \cdot 10^{-5}$	$2.1218 \cdot 10^{-3}$
3	$1.0596 \cdot 10^{-10}$	$1.2197 \cdot 10^{-9}$

Notice that in (2.32) there are no restriction on the sign of a_{ij}, i.e. also phenomena of mutualism or symbiosis are admitted.

In the dynamical system (2.32), the function $f^0(y)$ has the steady states $(0,0)^\top$, $(0, \frac{b_2}{a^*_{22}})^\top$, $(\frac{b1}{a^*_{11}}, 0)^\top$ and

$$\left(\frac{b_1 a^*_{22} - b_2 a^*_{21}}{-a^*_{12}a^*_{21} + a^*_{22}a^*_{11}}, \frac{b_2 a^*_{11} - b_1 a^*_{12}}{-a^*_{12}a^*_{21} + a^*_{22}a^*_{11}} \right)^\top,$$

where the latter represents co-existence.

Now, in our NE setting with the players' objectives given by

$$J_1(y,u_1,u_2) = \frac{1}{2}\|y(T) - y_T^{(1)}\|_2^2 + \frac{\nu}{2}\|u_1\|^2_{L^2}, \quad J_2(y,u_1,u_2) = \frac{1}{2}\|y(T) - y_T^{(2)}\|_2^2 + \frac{\nu}{2}\|u_2\|^2_{L^2},$$

we choose

$$y_T^{(1)} = (\frac{b_1}{a^*_{11}}, 0)^\top \text{ and } y_T^{(2)} = (0, \frac{b_2}{a^*_{22}})^\top,$$

that is, each species aims at the extinction of the other one.

In our numerical simulations, the parameters are chosen as in [67], i.e. $b_1 = 1, b_2 = 1, a^*_{11} = 2, a^*_{22} = 2, a^*_{12} = 1, a^*_{21} = 1$. Therefore $y_T^{(1)} = (\frac{1}{2}, 0)^\top$ and $y_T^{(2)} = (0, \frac{1}{2})^\top$. Furthermore, let $y_0 = (1.5, 1)^\top$, $\nu = 1$, $T = 0.25$ and $u_1^0(t) = 0, u_2^0(t) = 0, t \in [0,T]$.

To solve this NE problem, we use Algorithm 1 with $\tau = 0.5$ and $tol_u = 10^{-4}$. All the parameters in the semismooth Newton method and in the Krylov linear solver are as in the previous experiments. The differential equations are approximated by the MCN scheme with $N_t = 1000$.

We obtain the NE strategies depicted in Figure 2.6, which give the NE point shown in Figure 2.7, as a '*'-point. At the Nash equilibrium solution, we get $\|y(T) - y_T^{(1)}\|_2 = 0.6457$ and $\|y(T) - y_T^{(2)}\|_2 = 0.7981$ with $y(T) = (0.7947, 0.5746)^\top$ and $\widetilde{J}_1(u^{NE}) = 0.2291$, $\widetilde{J}_2(u^{NE}) = 0.3324$.

Concerning the convergence behaviour of Algorithm 1, we obtain results very similar to those shown in the previous experiments. Therefore they are omitted.

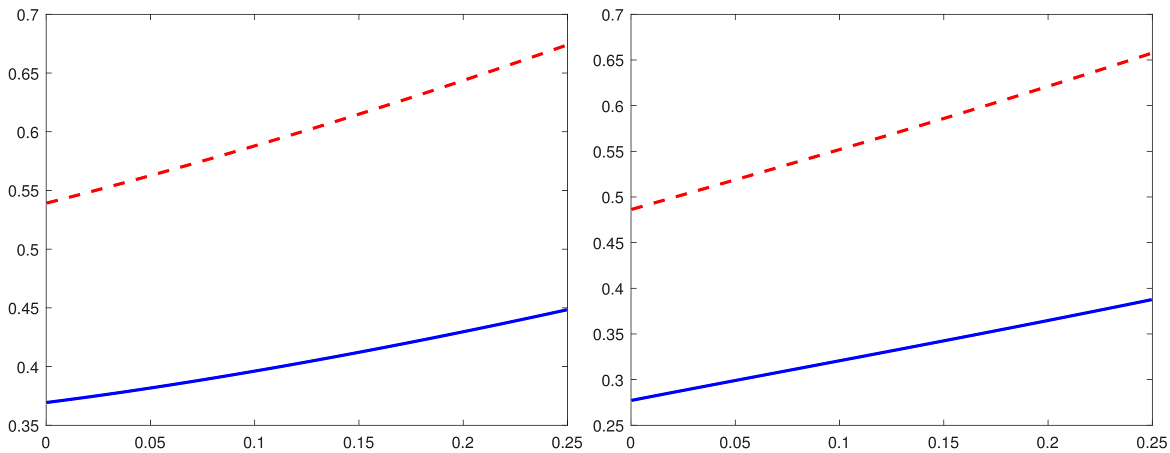

Figure 2.6: Strategies corresponding to the NE point (solid line) and at the NB point (dotted line)

With the NE point, we compute the bargaining solution using Algorithm 2 with $\mu_1^{(0)} = 0.75$ and $\alpha = 0.01$. The corresponding point, obtained for $\mu^* = 0.57$, is drawn as a $\circ$-point in Figure 2.7.

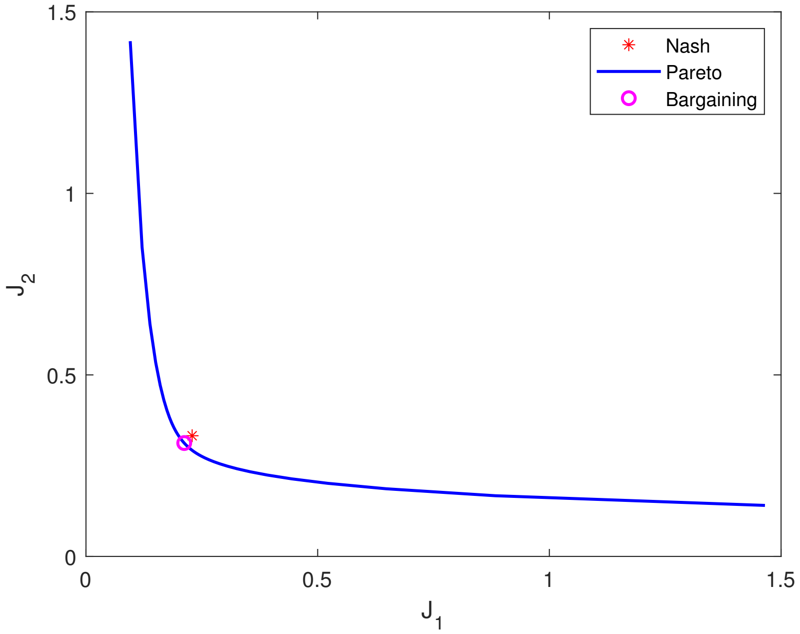

Figure 2.7: The NE point, the NB point and the Pareto frontier of the Lotka-Volterra model.

At the bargaining solution, we get $\|y(T) - y_T^{(1)}\|_2 = 0.5925$, $\|y(T) - y_T^{(2)}\|_2 = 0.7478$ with $y(T) = (0.7468, 0.5386)^\top$ and $\widetilde{J}_1(u^{NB}) = 0.2120$, $\widetilde{J}_2(u^{NB}) = 0.3123$.

2.5 Summary

In this chapter a numerical investigation of Nash equilibria (NE) and Nash bargaining (NB) problems governed by bilinear differential models was presented. In particular, a NE was computed with a semismooth Newton scheme combined with a relaxation method. A converge theorem for the proposed algorithm was demonstrated. A related Nash bargaining problem was discussed and a computational procedure for its determination was presented. Results of numerical experiments were presented that demonstrated the effectiveness of the present NE and NB computational framework.

Chapter 3

Numerical solution of a free end-time Homicidal Chauffeur game

In this chapter, a numerical scheme for solving a non-zero sum Homicidal Chauffeur (HC) game in a time-optimal formulation is introduced.

The HC game is a classic problem in the field of differential (dynamical) games that was introduced by R. P. Isaacs.The statement of the problem is that of a car with a limited radius of turn and constant velocity that pursues a pedestrian, whose velocity is bounded by a given value, that tries to prevent collision. In the following, the car represents the pursuer and the pedestrian the evader. Both are players of the HC game.

3.1 A Homicidal-Chauffeur game

Consider a planar system of a pursuer and an evader whose positions, with respect to an inertial Cartesian reference frame, are subject to the following dynamics [73]

$$\begin{aligned} x_p' &= \sin\theta, & x_e' &= v_1, \\ y_p' &= \cos\theta, & y_e' &= v_2, \\ \theta' &= u, |u| \le \nu_u, & |v_i| &\le \bar{\nu}_v,\ i = 1, 2, \end{aligned} \tag{3.1}$$

with given initial conditions. In this system, (x_p, y_p) represents the position of the pursuer (P) that has a unit-vector velocity while the orientation of this vector with respect to the y-axis (clockwise) is given by θ. On the other hand, the position of the evader (E) is given by (x_e, y_e) and its velocity is denoted with (v_1, v_2).

In the original work of Isaacs in [52], it appears convenient to reformulate this model in a reference frame moving with the pursuer. In this setting, the origin of the reference system corresponds to the position of P, and the direction of the (new) y-axis coincides with the velocity vector of the pursuer. By coordinate transformation (see the Appendix), we obtain the following bilinear dynamical system for the position of the evader in the reference frame of the pursuer

$$\begin{aligned} x' &= -u\,y + v_x, & x(0) &= x_0, \\ y' &= u\,x - 1 + v_y, & y(0) &= y_0, \end{aligned} \tag{3.2}$$

where (x_0, y_0) represents the initial position of the evader in the moving reference frame at time $t = 0$.

Notice that $u = u(t)$ denotes the scalar strategy function of P, and $v = v(t)$ represents the vector strategy function of the evader E with components (v_x, v_y). As in (3.1), these functions are required to lay in given admissible sets: $u \in U_{ad}$, and $v \in V_{ad}$. Specifically, we have

$$U_{ad} = \{u \in L^2(0,T) : u(t) \in K_u \text{ a.e. in } (0,T)\},$$

and

$$V_{ad} = \{v \in L^2(0,T;\mathbb{R}^2) : v_s(t) \in K_s \text{ a.e. in } (0,T),\ s \in \{x,y\}\}.$$

The sets $K_u := [-\nu_u, \nu_u]$, $K_s := [-\nu_v, \nu_v]$, $s = x, y$ where the numbers ν_u and ν_v are given, and $T > 0$ is the final time that will be determined by the algorithm. Notice that the components of v refer to the new reference frame and are different than (v_1, v_2) in (3.1) and the value of ν_v possibly differs from $\bar{\nu}_v$ in (3.1) because the absolute value is considered, and not the 2-norm.

The HC model (3.2) can be put in a compact form as follows. Denote with $z := (x, y)^\top$, $z_0 := (x_0, y_0)^\top$, $B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b := (0, -1)^\top$ then we can write (3.2) in the following form

$$z' = u\,B\,z + v + b, \qquad z(0) = z_0. \tag{3.3}$$

The objective of player P is to come as close as possible to player E, while minimizing the L^2 cost of its action. The player E wants to prevent its capture, while also minimizing the cost of its action. These objectives are modelled by the following functionals

$$\begin{aligned} J_p(u, v, z, T) &:= \frac{\nu_p}{2}\|u\|^2_{L^2} + \frac{r_p}{2}\|z(T)\|_2^2 + \omega_p T, \\ J_e(u, v, z, T) &:= \frac{\nu_e}{2}\|v\|^2_{L^2} - \frac{r_e}{2}\|z(T)\|_2^2 - \omega_e T, \end{aligned} \tag{3.4}$$

where $\nu_p, \nu_e > 0$, r_p, r_e represent the relative strength of the interaction with $r_p, r_e \geq 0$ and $\omega_p > 0, \omega_e \geq 0$.

One can prove that, for given $u^* \in U_{ad}$ and $v^* \in V_{ad}$, the control-to-state maps $u \mapsto z(u, v^*)$ and $v \mapsto z(u^*, v)$ are well defined; where $z(u, v)$ is the unique solution to (3.3), with the given initial condition $z(0) = z_0$, and $u \in U_{ad}$, and $v \in V_{ad}$. With these maps, we can introduce the reduced cost functionals

$$\widetilde{J}_p(u, v, T) := J_p(u, v, z(u, v), T), \qquad \widetilde{J}_e(u, v, T) := J_e(u, v, z(u, v), T). \tag{3.5}$$

Notice that both functionals depend on both strategies and the solution to the differential game is sought as a Nash equilibrium, defined in Chapter 1.

Moreover, (1.28) can be written as $u^* = \arg\min_u \widetilde{J}_p(u, v^*, T^*)$, and (1.29) can be written as $v^* = \arg\min_v \widetilde{J}_e(u^*, v, T^*)$. This definition allows to interpret our Nash game as two coupled optimal control problems. If (u^*, v^*, T^*) is the Nash equilibrium, then u^* is optimal for player P, in the sense that it solves the following optimal control problem

$$\begin{aligned} \min \quad & J_p(u, v^*, z, T^*) \\ \text{s.t.} \quad & z' = uBz + v^* + b, \quad z(0) = z_0. \end{aligned} \tag{3.6}$$

On the other hand, the function v^* is optimal for player E, that is, it is a solution of the following optimal control problem

$$\begin{aligned} \min \quad & J_e(u^*, v, z, T^*) \\ \text{s.t.} \quad & z' = u^*Bz + v + b, \quad z(0) = z_0. \end{aligned} \tag{3.7}$$

Therefore (u^*, v^*) solves simultaneously two optimal control problems whose solutions are characterized by the following optimality system

$$\begin{aligned} z' &= u\,B\,z + v + b, & z(0) &= z_0, \\ -p_p' &= u\,B\,p_p, & p_p(T^*) &= -r_p\,z(T^*), \\ -p_e' &= u\,B\,p_e, & p_e(T^*) &= r_e\,z(T^*), \\ u &= \mathcal{P}_{U_{ad}}\left(\nu_p^{-1}\langle Bz, p_p\rangle\right), & v &= \mathcal{P}_{V_{ad}}\left(\nu_e^{-1}p_e\right), \end{aligned} \tag{3.8}$$

where $\mathcal{P}_{U_{ad}}$ and $\mathcal{P}_{V_{ad}}$ represent the L^2 projections onto U_{ad} and V_{ad}, respectively, and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$. We refer to the functions p_p and p_e as the adjoint variables, and the corresponding differential equations are the adjoint equations.

Further, corresponding to (3.6) and (3.7), we define the following Hamilton-Pontryagin (HP) functions

$$\begin{aligned} H_p(z,u,v,p_p) &:= p_p\cdot(uBz + v + b) - \frac{\nu_p}{2}u^2 - \omega_p, \\ H_e(z,u,v,p_e) &:= p_e\cdot(uBz + v + b) - \frac{\nu_e}{2}|v|^2 + \omega_e. \end{aligned}$$

3.2 Fixed end-time HC optimal control problem

At the core of our iterative numerical scheme, where a tentative end-time T is available and iterates u^k and v^k have been computed, we consider the solutions to the two optimal control problems (3.6) and (3.7). Specifically, given v^k, we aim at computing $\bar{u}$ by solving the optimality system given by

$$\begin{aligned} z' &= uBz + v^k + b, & z(0) &= z_0, \\ -p_p' &= uBp_p, & p_p(T) &= -r_p z(T), \\ u &= \mathcal{P}_{U_{ad}}\left(\nu_p^{-1}\langle Bz, p_p\rangle\right). \end{aligned} \tag{3.9}$$

In a similar way, given u^k, we can compute $\bar{v}$ by solving the following optimality system

$$\begin{aligned} z' &= u^kBz + v + b, & z(0) &= z_0, \\ -p_e' &= u^kBp_e, & p_e(T) &= r_e z(T), \\ v &= \mathcal{P}_{V_{ad}}\left(\nu_e^{-1}p_e\right). \end{aligned} \tag{3.10}$$

In practice, to compute $\bar{u}$ and $\bar{v}$, we use these optimality systems to determine reduced gradients, $\nabla_u\tilde{J}_p$ and $\nabla_v\tilde{J}_e$, that are required to implement the well-known nonlinear conjugate gradient (NCG) method. We employ the Polak-Ribiére variant of NCG to determine $\bar{u}$ and $\bar{v}$; see [14] for more details.

This approach requires the numerical solution of the HC model and its adjoint. For this purpose, we use the so-called modified Crank-Nicolson (MCN) scheme [13], which requires a time grid that results by subdividing the interval $[0,T]$ in uniform intervals of size h and N_t points, such that $t_j = (j-1)h$ and $0 = t_1 < \cdots < t_{N_t} = T$. In this setting, the MCN approximation to our governing HC model is given by

$$\frac{z_{j+1}-z_j}{h} = \frac{u_{j+1}+u_j}{2}B\frac{z_{j+1}+z_j}{2} + \frac{v_{j+1}+v_j}{2} + b, \tag{3.11}$$

where $j = 1, \ldots, N_t - 1$, $h = T/(N_t - 1)$, and z_j, v_j, etc., represent the numerical approximation to $z(t_j)$, $v(t_j)$, etc.. The initial point $z_1 = z(0)$ is given. A similar scheme is used to solve the adjoint equations. We recall that, as in [13], one can prove that this scheme is stable and second-order accurate.

For ease of presentation, we summarize the numerical procedure discussed above with the following algorithms.

Algorithm 3 Solution of fixed end-time optimal control problem for the pursuer

1: **input** u^k, v^k, T;
2: Compute $\bar{u} = \arg\min_u \widetilde{J}_p(u, v^k, T)$

Algorithm 4 Solution of fixed end-time optimal control problem for the evader

1: **input** u^k, v^k, T;
2: Compute $\bar{v} = \arg\min_v \widetilde{J}_e(u^k, v, T)$

3.3 Free end-time optimal control problems

The free end-time version of the optimal control problems (3.6) and (3.7) are solved through a bilevel approach, which aims at decoupling the determination of the end-time from the solution of the optimal control problems. This allows avoiding time transformation techniques, commonly used in automotive and robotics applications [15, 63, 93, 95], which usually require solving a problem very sensitive to the initial guess, see [64].

On the other hand, this bilevel iterative solution procedure leads to a two-nested-loops algorithm. The outer loop is devoted to computing the end-time, while the inner loop solves the optimal control problems for a given, tentative time horizon. The key result supporting this approach, in particular how the two levels interact, is drawn in [34, Thm. 10] for linear-quadratic problems and extended in [64] to nonlinear problems. In fact, the interface between the two levels is based on the relation between the Hamilton-Pontryagin function and the cost functional. The following theorem holds; see [64] for a proof.

Theorem 3.3.1. *Consider the free-final time optimal control problems* (3.6) *and* (3.7)*. Assume the corresponding fixed end-time optimal control problems can be uniquely solved for a given time T and denote with $u[T], v[T], z[T], p_p[T], p_e[T]$ the solution for the given time T. If the reduced costs $\widetilde{J}_p, \widetilde{J}_e$ are continuously differentiable in T, then the following holds*

$$\frac{d\widetilde{J}_p}{dT}(u[T], v, T) = H_p(z[T], u[T], v, p_p[T]) \tag{3.12}$$

$$\frac{d\widetilde{J}_e}{dT}(u, v[T], T) = H_e(z[T], u, v[T], p_e[T]). \tag{3.13}$$

Notice that the above theorem is motivated by the fact that the Hamilton-Pontryagin function of a free end-time optimal control problem at the final time is zero along an optimal solution. Moreover, to get local optimality, as necessary condition, one expects that the derivative of the reduced cost is null. However, by using this theorem, we only need the values of the two Hamilton-Pontryagin functions to share with the outer-level to get a new estimate of the final time T.

We remark that the bilevel optimization problems are in general not equivalent to the original problems (3.6), (3.7), unless the solution to the fixed end-time optimal control problems is unique, then the two formulations are equivalent; see [32, 35].

Next, in order to illustrate the outer loop, we assume that at the k-th iterate the estimates u^k, v^k and T^k are available. Further, let us denote with $u[T] \in U_{ad}$ and $v[T] \in V_{ad}$ the optimal controls associated with final time T as follows

$$\begin{aligned} u[T] &= \arg\min_u \widetilde{J}_p(u, v^k, T), \\ v[T] &= \arg\min_v \widetilde{J}_e(u^k, v, T). \end{aligned} \tag{3.14}$$

Clearly, this construction requires to extend or restrict the functions u^k and v^k defined on $[0, T^k]$ to the interval $[0, T]$. With this preparation, we can consider the following optimization problems, which formally involve only one decision variable: the final time.

$$\begin{aligned} T_u &= \arg\min_T \widetilde{J}_p(u[T], v^k, T) + \frac{\mu_k}{2} \left(T - T^k\right)^2, \\ T_v &= \arg\min_T \widetilde{J}_e(u^k, v[T], T) + \frac{\mu_k}{2} \left(T - T^k\right)^2. \end{aligned} \tag{3.15}$$

For the purpose of defining a robust iteration procedure, a (quadratic) proximal regularization term is introduced, with parameters $\mu_k > 0$. Corresponding to (3.6) and (3.7), consider the HP functions given by

$$\begin{aligned} H_p(z, u, v^k, p_p) &= p_p \cdot \left(uBz + v^k + b\right) - \frac{\nu_p}{2} u^2 - \omega_p, \\ H_e(z, u^k, v, p_e) &= p_e \cdot \left(u^k Bz + v + b\right) - \frac{\nu_e}{2} |v|^2 + \omega_e. \end{aligned} \tag{3.16}$$

Then, considering (3.15) and (3.16), we have the following augmented HP functions

$$\widetilde{H}_p(u[T], v^k, T) = H_p(z, u, v^k, p_p) + \mu_k \left(T - T^k\right), \tag{3.17}$$
$$\widetilde{H}_e(u^k, v[T], T) = H_e(z, u^k, v, p_e) + \mu_k \left(T - T^k\right),$$

where the term $\mu_k(T - T^k)$ is associated to the quadratic proximal penalty in (3.15). In order to implement a gradient-based scheme for determining T, we consider the sensitivity of the cost functional with respect to this variable. This can be evaluated using the HP function of the underlying problem along a fixed-time solution [34, Thm. 10], namely

$$\begin{aligned} \frac{\mathrm{d}\widetilde{J}_p}{\mathrm{d}T}(u[T], v^k, T) &= H_p(z[T], u[T], v^k, p_p[T]), \\ \frac{\mathrm{d}\widetilde{J}_e}{\mathrm{d}T}(u^k, v[T], T) &= H_e(z[T], u^k, v[T], p_e[T]). \end{aligned} \tag{3.18}$$

By exploiting the fact that the (non-augmented) system is autonomous and thus the Hamiltonian is constant along a solution, these sensitivities can be evaluated at any $t \in [0, T]$ at the outer loop of our solution procedure. For more details see [64, Algorithm 3].

Summarizing, the method for solving the following free end-time optimal control problem

$$(u, T_u) = \arg\min_{u,T} \widetilde{J}_p(u, \bar{v}, T),$$

is given in Algorithm 5 below. A similar procedure applies for the optimal control problem corresponding to the evader. If required in Step 8 of Algorithm 5, an extrapolation step is performed to extend the control function to the given time interval.

Algorithm 5 Free end-time optimal control problem and proximal penalty

input u^0,$\bar{v}$, T^0; parameters $\epsilon > 0$, $\{\mu_k\} \to \infty$
Set $k = 0$, $T^{-1} = T^0$
repeat
  Compute $\bar{u} = \arg\min_u \widetilde{J}_p(u, \bar{v}, T^k)$
  Compute z^k, p_p^k
  Evaluate $\tilde{H}_p^k = H_p(z^k, \bar{u}, \bar{v}, p_p^k)\Big|_{T^k} + \mu_k(T^k - T^{k-1})$
  Set $T^{k+1} = T^k - s\,\tilde{H}_p^k$ ($s > 0$ suff. small step size)
  If needed, extend functions $\bar{u}, \bar{v}$ to $[0, T^{k+1}]$
  Set $k := k + 1$
until $|H_p\Big|_{T^k}| < \epsilon$

3.4 Free end-time Nash games

In this section, we assemble the numerical optimization schemes discussed above to define our algorithm for solving the HC Nash game. We can say that it is a relaxation method with free end-time sub-problems and proximal penalty. A numerical issue that arises when considering the different free end-time optimal control problems for P and E is that the corresponding optimal final times may be different during iterates. However, the relaxation step requires to combine different approximations of the strategy functions that are given on different intervals. Thus, before the relaxation step is performed, an extrapolation/restriction procedure is applied to define all functions involved on the same time horizon. Our free end-time HC Nash game solver is given by Algorithm 6.

Algorithm 6 Relaxation scheme with free end-time sub-problems and proximal penalty

1: **input** u^0, v^0, T^0; parameters $\tau \in (0,1)$, $\alpha \in (0,1)$, $\epsilon > 0$, $\{\mu_k\} \to \infty$
2: Set $k = 0$
3: **repeat**
4:   Compute $(\bar{u}, T_u) = \arg\min_{u,T}\ \widetilde{J}_p(u, v^k, T) + \frac{\mu_k}{2}\,|T - T^k|^2$
5:   Compute $(\bar{v}, T_v) = \arg\min_{v,T}\ \widetilde{J}_e(u^k, v, T) + \frac{\mu_k}{2}\,|T - T^k|^2$
6:   Compute $\bar{T} := (T_u + T_v)/2$
7:   Set $T^{k+1} := \alpha T^k + (1 - \alpha)\bar{T}$
8:   If needed, extend functions u^k, v^k, $\bar{u}$, and $\bar{v}$ to $[0, T^{k+1}]$
9:   Set $(u^{k+1}, v^{k+1}) := \tau(u^k, v^k) + (1 - \tau)(\bar{u}, \bar{v})$
10:   Set $k := k + 1$
11: **until** $\max(|H_e|, |H_p|) < \epsilon$

In this algorithm, the parameters τ and α are relaxation factors that we specify in our numerical experiments. The main advantage of Algorithm 6 is that we can compute $(\bar{u}, T_u)$ and $(\bar{v}, T_v)$ separately (in parallel) using an efficient optimization scheme. In this respect, Algorithm 6 is of Jacobi, and not of Gauß-Seidel type.

Notice that Algorithm 6 stops when the two Hamilton functions are smaller than a given tolerance ϵ, i.e., the solution is optimal.

A different approach, not further investigated in this thesis, may consider replacing the free end-time optimal control problems at Steps 4–5 of Algorithm 6 with fixed end-time problems, with final time T^k. Successively, lacking of estimates of T_u and T_v, Step 6 could find a $\bar{T}$ based on $\bar{u}$, $\bar{v}$, and T^k. This approach would avoid solving free-time optimization problems, which may be a valuable property. On the other hand, this still needs extrapolation or truncation to match the time domains. Also, further decoupling the optimization problem may lead to a slower, less robust convergence.

To conclude this section, we remark that we prefer avoiding time scaling techniques in our game framework and rather use a bilevel approach. However, time scaling provides useful tools for studying free end time optimal control problems. The core idea is to transform the original problem, with time variable $t \in [0, T]$, $T > 0$, into a fixed-time one by treating the final time T as a parameter (or a constant state). By introducing the mapping $s \mapsto t(s) := Ts$, the new time variable is $s \in [0, 1]$. The system dynamics have to be transformed as well, scaled by T. Then, the problem is of finding the optimal control over the scaled time domain $[0, 1]$ and the optimal final time T. Given a final time $T > 0$, the (original) control $u \in U_{ad}$ can be recovered as $u(t) = u_s(t/T)$ for any $t \in [0, T]$. See [61, 46] for more details.

3.5 Numerical experiments

In this section, we present results of numerical experiments to validate our HC Nash game formulation and the ability of our numerical framework to solve the resulting problems.

In our numerical experiments, we choose the initial state of the system $z_0 = (3, 2)^\top$ and initialize $T^0 = 10$, $\nu_u = 1$ and $\nu_v = 0.3$. Moreover, let $u^{00}(t) = 0$ and $v^{00}(t) = (0.1, 0.1)^\top$, $t \in [0, T^0]$. With these parameters, we solve the HC game for fixed end-time T^0, in order to get a better initial guess for the players' actions, namely u^0 and v^0, in our algorithm for a faster convergence. The extrapolation step is achieved by using the MATLAB function `interp1` with the `nearest` method [90]. Similar results are obtained with the `linear` method.

Next, assume that player P has an advantage on player E. This situation results from an appropriate choice of the parameters of the game. For example, let $\nu_e = 10^{-6}$, $\nu_p = 10^{-6}$, $r_e = 10^{-4}$, $r_p = 1$ and $\omega_e = 0$, $\omega_p = 10^{-6}$. Notice that this choice of parameters aims at having the pursuer catch the evader. Otherwise, the game will stop only by prescribing a maximun number of iterations, at which P and E will have different positions.

To solve this NE problem, we use Algorithm 6 with $\tau = 0.5$, $\alpha = 0.5$ and $\epsilon = 10^{-5}$. The regularization parameters are $\mu_k = \mu = 10^{-10}$. In this implementation, the differential HC model and its adjoint are approximated by the MCN scheme with a grid of $N_t = 500$ points. Notice that further experiments have been performed to verify that the results reported in this section are not mesh dependent.

With this setting, we obtain the P and E strategies depicted in Figure 3.1, and the time when the pursuer catches the evader results to be $T^* = 8.88$. This result is obtained after 15 iterations in Algorithm 6.

Starting the algorithm with a different end-time initialization, e.g., $T^0 = 9$, we obtain $T^* = 8.81$ and the strategy functions are very similar to those obtained above, hence omitted for brevity. We want comment on the fact that, even using the tolerance $\epsilon = 10^{-5}$ in the stopping condition, we get relatively different values for T^* that means the end-time changes much more than the two Hamilton functions. Hence we do not need a very strict tolerance. However, it is well-known that Nash games admit many solutions, and these may result by different initializations. In fact, choosing $T^0 = 7$, we obtain $T^* = 10.14$ and the corresponding strategies and trajectories are shown in Figure 3.2.

To conclude this section, we show that, if we change the regularization terms $\nu_e = 10^{-6}$,

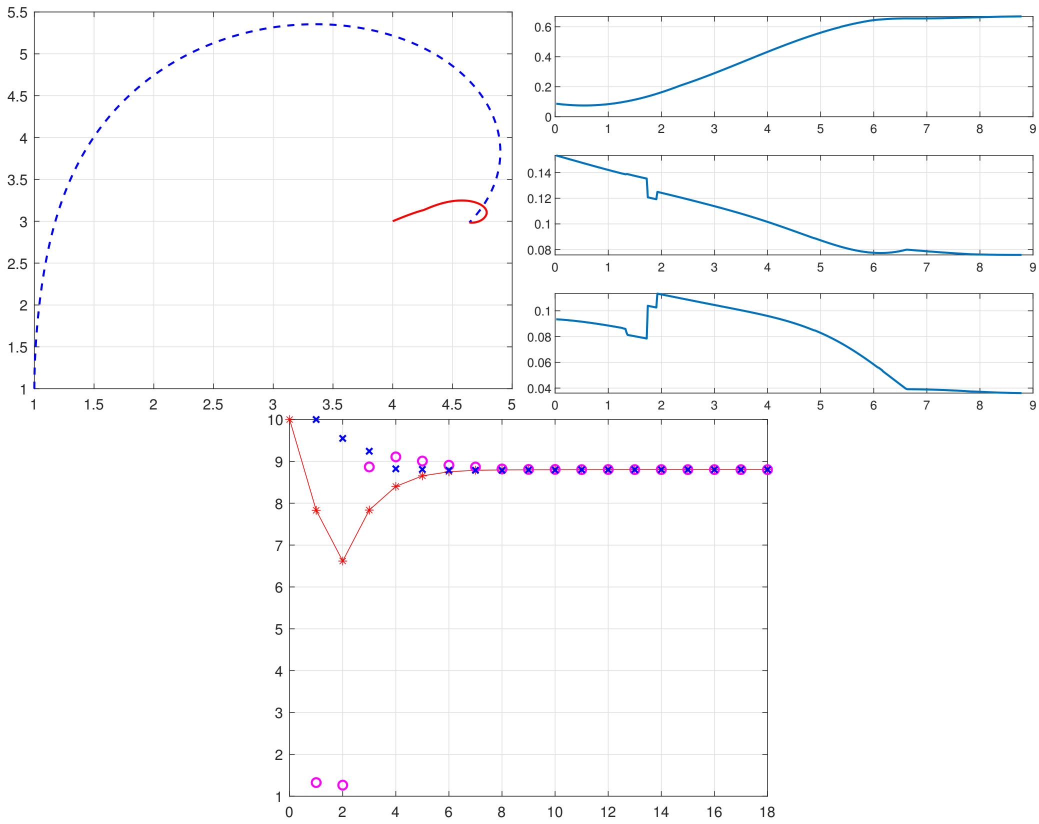

Figure 3.1: Test 1: (above-left) players' trajectories: pursuer in dashed line and evader in solid line; (above-right) strategies corresponding to the two players u above and v_j below; and (below) final time during iterations: 'o' for the evader and 'x' for the pursuer.

$\nu_p = 10^{-4}$, and start with $T^0 = 9$ (all the others parameters are as mentioned above), we obtain (again) the optimal final time $T^* = 8.79$ and similar trajectories, as depicted in Figure 3.3.

The results of these experiments can be interpreted as follows. At the beginning of the optimization procedure, the evader tries to take the time to the lower bound of the time interval to avoid the chance to be captured. At the same time, the pursuer tries to get enough time to do it. After few iterations it holds the opposite, i.e., the evader aims to get more time to escape, while the pursuer tries to catch it earlier.

Moreover, as explained in [51], if the pursuer turns straight in the direction of the evader, then he can frustrate the pursuer by entering in the circle of maximal curvature. Hence, the pursuer should minimize their distance by moving around the evader until the capture.

3.6 Summary

In this chapter, a numerical framework for solving a pursuit-evader homicidal chauffeur Nash game was presented. The Nash game was formulated in a functional setting involving cost functionals for the two players and the classical homicidal chauffeur differential models. In this model, the strategies of the pursuer and evader are represented by control functions that are subject to constraints. The numerical solution procedure was obtained combining a Hamiltonian based scheme with proximal penalty to determine the time horizon where the game takes place with a Lagrangian optimal control approach and

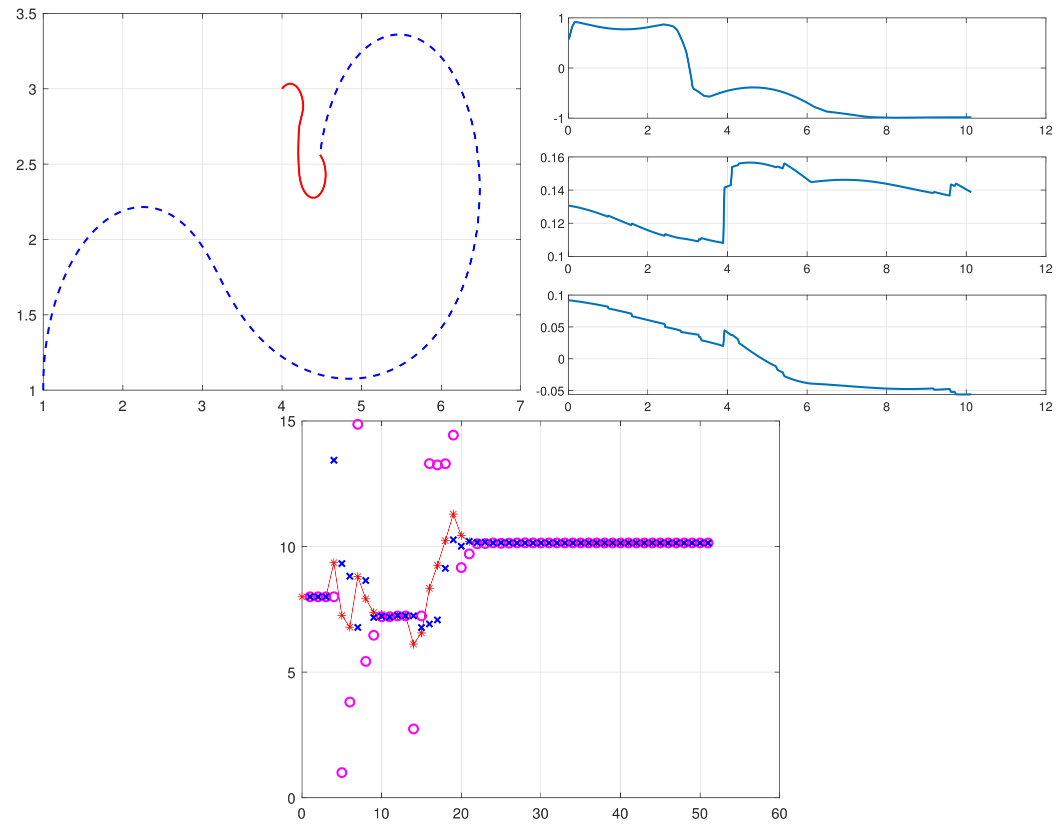

Figure 3.2: Test 2: (top-left) players' trajectories: pursuer in dashed line and evader in solid line; (top-right) strategies corresponding to the two players: u above and v_j below; and (bottom) final time during iterations: 'o' for the evader and 'x' for the pursuer.

relaxation to solve the Nash game at a fixed end-time.

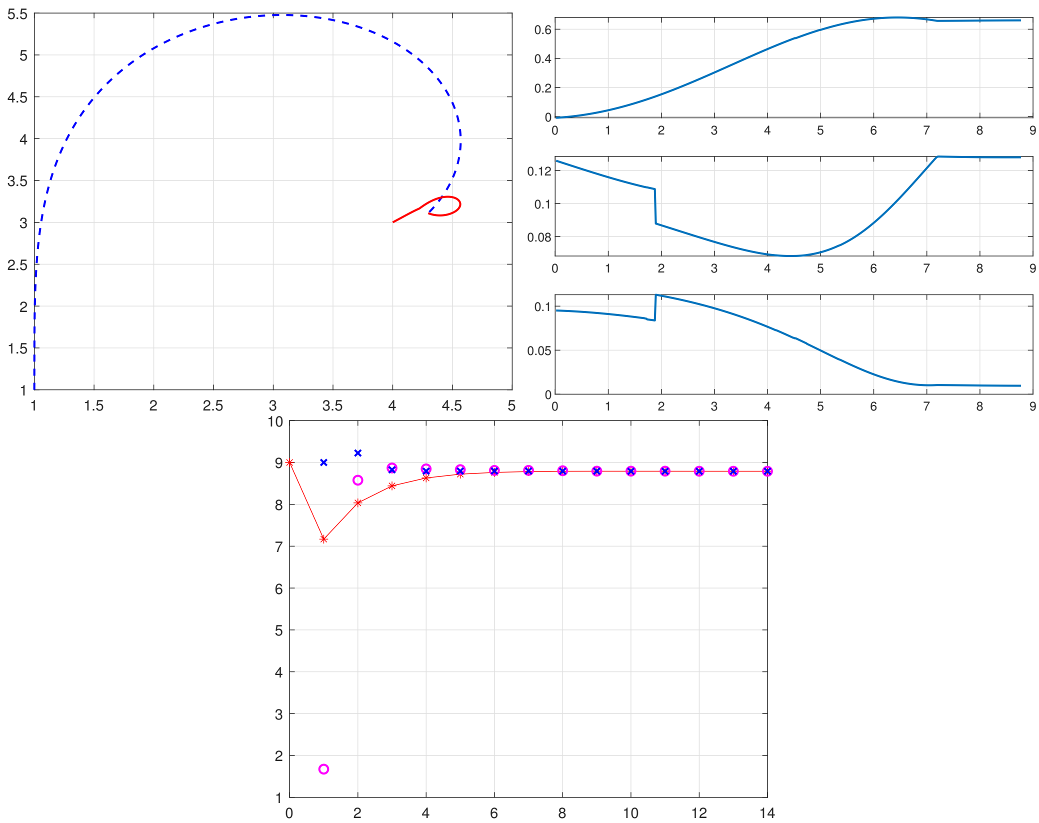

Figure 3.3: Test 3: (top-left) players' trajectories: pursuer in dashed line and evader in solid line; (top-right) strategies corresponding to the two players: u above and v_j below; and (bottom) final time during iterations: 'o' for the evader and 'x' for the pursuer.

Chapter 4

A sequential quadratic Hamiltonian scheme for solving differential Nash games

It is the purpose of this chapter to investigate a sequential quadratic Hamiltonian (SQH) scheme for solving open-loop non zero-sum two-players differential Nash games. In particular, linear-quadratic (LQ) Nash games are considered since they are well investigated from the theoretical point of view. The proposed method is formulated in the framework of Pontyagin's maximum principle. Theoretical results are presented that prove the well-posedness of the proposed scheme. Several numerical tests are performed at the end of the chapter, including extension of LQ Nash games to problems with tracking objectives, box constraints on the players' actions, and actions' costs that include L^1 terms to corroborate its computational performance.

Further, a comparison on the computational performances of the SQH and relaxation-Newton methods is presented.

4.1 PMP characterization of Nash games

In the following, we discuss the case of two players, represented by their strategies u_1 and u_2, which can be readily extended to the case of N players, and assume the dynamics (1.10), introduced in Chapter 1, that for convenience of the reader we recall. Consider

$$y'(t) = f(t, y(t), u_1(t), u_2(t)), \qquad y(0) = y_0, \tag{4.1}$$

where $t \in [0, T]$, $y(t) \in \mathbb{R}^n$, and $u_1(t) \in \mathbb{R}^m$ and $u_2(t) \in \mathbb{R}^m$, $m \leq n$. We remark that f is chosen as in Chapter 1 such that for any choice of the initial condition $y_0 \in \mathbb{R}^n$, and any u_1, $u_2 \in L^2(0, T; \mathbb{R}^m)$, the Cauchy problem (4.1) admits a unique solution in the sense of Carathéodory. Further, we assume that the map $(u_1, u_2) \mapsto y = y(u_1, u_2)$, where $y(u_1, u_2)$ represents the unique solution to (4.1) with fixed initial conditions is continuous in (u_1, u_2).

In this framework, we introduce the following cost functionals

$$J_1(y, u_1, u_2) := \int_0^T \ell_1(t, y(t), u_1(t), u_2(t))\, dt + g_1(y(T)), \tag{4.2}$$

and

$$J_2(y, u_1, u_2) := \int_0^T \ell_2(t, y(t), u_1(t), u_2(t))\, dt + g_2(y(T)). \tag{4.3}$$

We consider the cases of unconstrained and constrained strategies. In the former case, we assume u_1, $u_2 \in L^2(0,T;\mathbb{R}^m)$, whereas in the latter case we assume that u_1 and u_2 belong, respectively, to the following admissible sets

$$U_{ad}^{(i)} = \{u \in L^2(0,T;\mathbb{R}^m) : \, u(t) \in K_{ad}^{(i)}, \, t \in [0,T]\}, \quad i = 1,2, \tag{4.4}$$

where $K_{ad}^{(i)}$ are compact and convex subsets of $\mathbb{R}^m$. We denote with $U_{ad} = U_{ad}^{(1)} \times U_{ad}^{(2)}$ and $U = L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^m)$.

Once again, by using the map $(u_1,u_2) \mapsto y = y(u_1,u_2)$, we can introduce the reduced objectives $\tilde{J}_1(u_1,u_2) := J_1(y(u_1,u_2),u_1,u_2)$ and $\tilde{J}_2(u_1,u_2) := J_2(y(u_1,u_2),u_1,u_2)$. In this framework, a Nash equilibrium is sought as solution for the game $G = (\tilde{J}_1, \tilde{J}_2, U_{ad}^{(1)}, U_{ad}^{(2)})$.

As seen in Chapter 1, existence of a NE point can be proved subject to appropriate conditions on the structure of the differential game, including the choice of T or, in the case of unconstrained strategies, the existence of NE can be proved and the NE point computed explicitely, with the help of a related Riccati problem.

Assuming the game satisfies the requirements seen in Section 1.4.1, to continue our discussion, we consider as Nash equilibrium the point $(u_1^*, u_2^*) \in U_{ad}$.

Therefore, if $u^* = (u_1^*, u_2^*)$ is a NE for the game, then it solves simultaneously the following two optimisation problems

$$u_1^* = \underset{u_1 \in U_{ad}^{(1)}}{\arg\min}\, \tilde{J}_1(u_1, u_2^*), \qquad u_2^* = \underset{u_2 \in U_{ad}^{(2)}}{\arg\min}\, \tilde{J}_2(u_1^*, u_2). \tag{4.5}$$

This fact implies that the NE point $u^* = (u_1^*, u_2^*)$ must satisfy the necessary optimality conditions given by the Pontryagin's maximum principle applied to both optimisation problems. For this purpose, we introduce the following Hamilton-Pontryagin (HP) functions

$$\mathcal{H}_i(t,y,u_1,u_2,p_1,p_2) = p_i \cdot f(t,y,u_1,u_2) - \ell_i(t,y,u_1,u_2), \qquad i = 1,2. \tag{4.6}$$

In terms of these functions, the PMP condition for the NE point $u^* = (u_1^*, u_2^*)$ states the existence of multiplier (adjoint) functions p_1, $p_2 : [0,T] \to \mathbb{R}^n$ such that the following holds

$$\begin{aligned} \max_{w_1 \in K_{ad}^{(1)}} \mathcal{H}_1(t, y^*(t), w_1, u_2^*(t), p_1^*(t), p_2^*(t)) &= \mathcal{H}_1(t, y^*(t), u_1^*(t), u_2^*(t), p_1^*(t), p_2^*(t)), \\ \max_{w_2 \in K_{ad}^{(2)}} \mathcal{H}_2(t, y^*(t), u_1^*(t), w_2, p_1^*(t), p_2^*(t)) &= \mathcal{H}_2(t, y^*(t), u_1^*(t), u_2^*(t), p_1^*(t), p_2^*(t)), \end{aligned} \tag{4.7}$$

for almost all $t \in [0,T]$. In (4.7), we have $y^* = y(u_1^*, u_2^*)$, and the adjoint variables p_1^*, p_2^* are the solutions to the following differential problems

$$-p_i'(t) = (\partial_y f(t, y(t), u_1(t), u_2(t)))^\top \, p_i(t) - \partial_y \ell_i(t, y(t), u_1(t), u_2(t)), \tag{4.8}$$

$$p_i(T) = -\partial_y g_i(y(T)), \tag{4.9}$$

with $i = 1,2$.

Similarly to (4.1), we require that (4.8) - (4.9) can be uniquely solved. Notice that, at each t fixed, the formulation (4.7) corresponds to a finite-dimensional Nash game.

4.2 The SQH scheme for solving Nash games

In the spirit of the successive approximations (SA) scheme proposed by Krylov and Chernous'ko [59], a SA methodology for solving our Nash game $(\tilde{J}_1, \tilde{J}_2; U_{ad}^{(1)}, U_{ad}^{(2)})$ consists of an iterative process, starting with an initial guess $(u_1^0, u_2^0) \in U_{ad}$, and followed by the solution of our governing model (4.1) and of the adjoint problems (4.8) - (4.9) for $i = 1, 2$. Thereafter, a new approximation to the strategies u_1 and u_2 is obtained by solving, at each t fixed, the Nash game (4.7) and assigning the values of $(u_1(t), u_2(t))$ equal the solution of this game.

We remark that this update step is well posed if this solution exists for $t \in [0, T]$ and the resulting functions u_1 and u_2 are measurable. Clearly, this issue requires to identify classes of problems for which we can guarantee existence and uniqueness (or the possibility of selection) of a NE point. In this respect, a large class can be identified based on Theorem 1.2.1, stated in Chapter 1.

With the setting of this theorem, the map $(t, y, p_1, p_2) \mapsto (u_1^*, u_2^*)$ is continuous [21]. Moreover, based on results given in [80], one can prove that the functions $(u_1(t), u_2(t))$ resulting from the SA update, starting from measurable $(u_1^0(t), u_2^0(t))$, are measurable. Therefore the proposed SA update is well-posed and it can be repeated in order to construct a sequence of functions $\left((u_1^k, u_2^k)\right)_{k=0}^{\infty}$.

However, as already pointed out in [59] in the case of optimal control problems, it is difficult to find conditions that guarantee convergence of SA iterations to the solution sought. Furthermore, results of numerical experiments show a lack of robustness of the SA scheme with respect to the choice of the initial guess and of the numerical and optimisation parameters.

For this reason, further research effort was put in the development of the SA strategy, and an advancement was achieved by Sakawa and Shindo considering a quadratic penalty on the Hamiltonian [83, 86]. We remark that these authors related their penalisation strategy to that proposed by B. Järmark in [53], which is similar to the proximal scheme of R. T. Rockafellar discussed in [79].

For our purpose, we follow the same path of [83] and extend it to the case of Nash games as follows. Consider the following augmented HP functions

$$\mathcal{K}_\epsilon^{(i)}(t, y, u_1, u_2, v_1, v_2, p_1, p_2) := \mathcal{H}_i(t, y, u_1, u_2, p_1, p_2) - \epsilon \, \|u - v\|_2^2, \quad i = 1, 2, \tag{4.10}$$

where, in the iteration process, $u = (u_1, u_2)$ is subject to the update step, and $v = (v_1, v_2)$ corresponds to the previous strategy approximation.The parameter $\epsilon > 0$ represents the augmentation weight that is chosen adaptively along the iteration as discussed below.

Now, similar to the SA update illustrated above, suppose that the kth function approximation (u_1^k, u_2^k) and the corresponding y^k and p_1^k, p_2^k have been computed. For any fixed $t \in [0, T]$ and $\epsilon > 0$, consider the following finite-dimensional Nash game

$$\begin{aligned} \mathcal{K}_\epsilon^{(1)}(t, y^k, \tilde{u}_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) &= \max_{u_1 \in K_{ad}^{(1)}} \mathcal{K}_\epsilon^{(1)}(t, y^k, u_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k), \\ \mathcal{K}_\epsilon^{(2)}(t, y^k, \tilde{u}_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) &= \max_{u_2 \in K_{ad}^{(2)}} \mathcal{K}_\epsilon^{(2)}(t, y^k, \tilde{u}_1, u_2, u_1^k, u_2^k, p_1^k, p_2^k), \end{aligned} \tag{4.11}$$

where $y^k = y^k(t)$, $p_1^k = p_1^k(t)$, $p_2^k = p_2^k(t)$, and $(u_1^k, u_2^k) = (u_1^k(t), u_2^k(t))$.

It is clear that, assuming the structure specified in Theorem 1.2.1, the Nash game (4.11) admits a unique NE point, $(\tilde{u}_1, \tilde{u}_2) \in K_{ad}^{(1)} \times K_{ad}^{(2)}$, and the sequence constructed recursively by the procedure:

$$(u_1^k(t), u_2^k(t)) \to (u_1^{k+1}(t), u_2^{k+1}(t)) = (\tilde{u}_1, \tilde{u}_2)$$

is well defined.

Notice that, in this procedure, the solution to (4.11) depends on the value of ϵ. Therefore the issue arises whether, corresponding to the step $k \to k+1$, we can choose the value of this parameter such that the strategy function $u^{k+1} = (u_1^{k+1}, u_2^{k+1})$ represents an improvement on $u^k = (u_1^k, u_2^k)$, in the sense that some convergence criteria towards the solution to our differential Nash problem are fulfilled.

For this purpose, we define a criterion that is based on the Nikaido-Isoda function $\psi : U_{ad} \times U_{ad} \to \mathbb{R}$. We require that

$$\psi(u^{k+1}, u^k) \leq -\xi \, \|u^{k+1} - u^k\|^2_{L^2(0,T;\mathbb{R}^m)},$$

for some chosen $\xi > 0$. This is a consistency criterion in the sense that ψ must be non positive, and if $(u^{k+1}, u^k) \to (u^*, u^*)$, then we must have $\lim_{k\to\infty} \psi(u^{k+1}, u^k) = 0$. Thus, we require that the absolute value $|\psi(u^{k+1}, u^k)|$ monotonically decreases in the SQH iteration process.

In our SQH scheme, if the strategy update meets the two requirements above, then the update is taken and the value of ϵ is diminished by a factor $\zeta \in (0,1)$. If not, the update is discarded and the value of ϵ is increased by a factor $\sigma > 1$, and the procedure is repeated. Below, we show that a value of ϵ can be found such that the update is successful and the SQH iteration proceeds until an appropriate stopping criterion is met.

Our SQH scheme for differential Nash games is implemented as follows.

Algorithm 7 SQH scheme for differential Nash games

1: Choose $\epsilon > 0, \mathcal{K} > 0, \sigma > 1, \zeta \in (0,1), \xi \in (0,\infty)$, and initial guess: $\Psi^0 > 0$, (u_1^0, u_2^0); compute y^0 and p_1^0, p_2^0, set $k = 0$.

2: Solve (4.11):

$$\tilde{u}_1 = \arg\max_{u_1 \in K_{ad}^{(1)}} \mathcal{K}_\epsilon^{(1)}(t, y^k, u_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k)$$

and

$$\tilde{u}_2 = \arg\max_{u_2 \in K_{ad}^{(2)}} \mathcal{K}_\epsilon^{(2)}(t, y^k, \tilde{u}_1, u_2, u_1^k, u_2^k, p_1^k, p_2^k)$$

for all $t \in [0,T]$.

3: Calculate $\tilde{y}$ corresponding to $\tilde{u} := (\tilde{u}_1, \tilde{u}_2)$ and compute $\tau := \|\tilde{u} - u^k\|^2_{L^2}$.

4: Calculate the Nikaido-Isoda function $\psi(\tilde{u}, u^k)$.

5: If $\psi(\tilde{u}, u^k) \leq -\xi\,\tau$ and $|\psi(\tilde{u}, u^k)| \leq \Psi^k$: choose $\epsilon = \zeta\,\epsilon$, $y^{k+1} = \tilde{y}$, $u^{k+1} = \tilde{u}$ and $\Psi^{k+1} = |\psi(u^{k+1}, u^k)|$; compute p_1^{k+1} and p_2^{k+1} corresponding to y^{k+1} and u^{k+1}.
Else: choose $\epsilon = \sigma\,\epsilon$. Set $k = k+1$.

6: If $\tau < \mathcal{K}$: STOP and return u^k. Else go to 2.

In the following proposition, we prove that the Steps 1 - 6 of the SQH scheme are well posed. For the proof, we consider the assumptions of Theorem 1.2.1 with further simplifying hypothesis, which can be relaxed at the cost of more involved calculations.

Our purpose is to show that it is possible to find an ϵ in Algorithm 7 such that $\tilde{u}$ generated in Step 2 satisfies the criterion required in Step 5 for a successful update. We have

Proposition 4.2.1. *Let the assumptions of Theorem 1.2.1 hold, and suppose that f^0, g_i, ℓ_i, $i = 1,2$ are twice continuous differentiable and strictly convex in y and u for all*

$t \in [0,T]$. *Moreover, let* F_1, F_2 *dependent only on* t *and suppose that* f^0, ℓ_i *and* g_i, $i=1,2$, *represent quadratic forms in* u *and* y *such that their Hessians are constant.*

Let $(\tilde{y}, \tilde{u}_1, \tilde{u}_2)$, (y^k, u_1^k, u_2^k) *be generated by Algorithm 7, Steps 2-3, and denote* $\delta u = \tilde{u} - u^k$. *Then, there exists a* $\theta > 0$ *independent of* ϵ *such that, for* $\epsilon > 0$ *currently chosen in Step 2, the following inequality holds*

$$\psi(\tilde{u}, u^k) \leq -(\epsilon - \theta) \left\| \delta u \right\|^2_{L^2(0,T)}. \tag{4.12}$$

In particular, if $\epsilon > \theta$ *then* $\psi(\tilde{u}, u^k) \leq 0$.

Proof. Recall the definition of the Nikaido-Isoda function:

$$\psi(\tilde{u}, u^k) := \tilde{J}_1(\tilde{u}_1, \tilde{u}_2) - \tilde{J}_1(u_1^k, \tilde{u}_2) + \tilde{J}_2(\tilde{u}_1, \tilde{u}_2) - \tilde{J}_2(\tilde{u}_1, u_2^k).$$

We focus on the first two terms involving J_1; however, the same calculation applies to the last two terms with J_2.

Notice that, with the our assumptions, the function $y(u_1, u_2)$ results differentiable. Denote $y^k = y(u_1^k, u_2^k)$, $\tilde{y} = y(\tilde{u}_1, \tilde{u}_2)$ and $\tilde{y}_1^k = y(u_1^k, \tilde{u}_2)$, $p_1^k = p_1(u_1^k, u_2^k)$, and define $\delta y_1 := \tilde{y} - \tilde{y}_1^k$, and $\delta \tilde{y}_1 := \tilde{y}_1^k - y^k$.

Consider the augmented Hamiltonian $\mathcal{K}_\epsilon^{(1)}$ of player P_1. Similar computations can be done with $\mathcal{K}_\epsilon^{(2)}$. It holds

$$\mathcal{K}_\epsilon^{(1)}(t, y^k, \tilde{u}_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) \geq \mathcal{K}_\epsilon^{(1)}(t, y^k, w, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k), \tag{4.13}$$

for any $w \in K_{ad}^{(1)}$. Hence, choosing $w = u_1^k$, we have

$$\begin{aligned}
\mathcal{K}_\epsilon^{(1)}(t, y^k, \tilde{u}_1, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) &\geq \mathcal{K}_\epsilon^{(1)}(t, y^k, u_1^k, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) \\
&= \mathcal{H}_1(t, y^k, u_1^k, \tilde{u}_2, p_1^k, p_2^k) - \epsilon \left\| \tilde{u}_2 - u_2^k \right\|_2^2.
\end{aligned}$$

Now, compute

$$\begin{aligned}
& J_1(\tilde{y}, \tilde{u}_1, \tilde{u}_2) - J_1(\tilde{y}_1^k, u_1^k, \tilde{u}_2) = \\
& = \int_0^T \Big(\ell_1(t, \tilde{y}, \tilde{u}_1, \tilde{u}_2) - \ell_1(t, \tilde{y}_1^k, u_1^k, \tilde{u}_2) \Big)\, dt + g_1(\tilde{y}(T)) - g_1(\tilde{y}_1^k(T)) \\
& + \int_0^T \Big(p_1^k \cdot f(t, \tilde{y}, \tilde{u}_1, \tilde{u}_2) - p_1^k \cdot f(t, \tilde{y}, \tilde{u}_1, \tilde{u}_2) \Big)\, dt \\
& + \int_0^T \Big(p_1^k \cdot f(t, \tilde{y}_1^k, u_1^k, \tilde{u}_2) - p_1^k \cdot f(t, \tilde{y}_1^k, u_1^k, \tilde{u}_2) \Big)\, dt \\
& = \int_0^T \Big(- \mathcal{H}_1(t, \tilde{y}, \tilde{u}_1, \tilde{u}_2, p_1^k, p_2^k) + p_1^k \cdot f(t, \tilde{y}, \tilde{u}_1, \tilde{u}_2) \Big)\, dt \\
& + \int_0^T \Big(\mathcal{H}_1(t, \tilde{y}_1^k, u_1^k, \tilde{u}_2, p_1^k, p_2^k) - p_1^k \cdot f(t, \tilde{y}_1^k, u_1^k, \tilde{u}_2) \Big)\, dt \\
& + g_1(\tilde{y}(T)) - g_1(\tilde{y}_1^k(T)).
\end{aligned}$$

Notice that $\tilde{y} = \tilde{y}_1^k + \delta y_1$. By applying the mean value theorem to second order to the maps $y \mapsto \mathcal{H}_1(\cdot, y, \cdot, \cdot, \cdot, \cdot)$ and $y \mapsto f(\cdot, y, \cdot, \cdot)$, and $u_1 \mapsto f(\cdot, \cdot, u_1, \cdot)$, we have

$$\begin{aligned}
& \mathcal{H}_1(t, \tilde{y}_1^k + \delta y_1, \tilde{u}_1, \tilde{u}_2, p_1^k, p_2^k) = \mathcal{H}_1(t, \tilde{y}_1^k, \tilde{u}_1, \tilde{u}_2, p_1^k, p_2^k) \\
& + \Big((p_1^k)^\top \partial_y f^0(t, \tilde{y}_1^k) - \partial_y \ell_1^0(t, \tilde{y}_1^k) \Big)\, \delta y_1 + \frac{1}{2} \delta y_1^\top \partial^2_{yy} \mathcal{H}_1 \delta y_1,
\end{aligned}$$

and

$$f(t,\tilde y_1^k+\delta y_1,\tilde u_1,\tilde u_2)=f(t,\tilde y_1^k,\tilde u_1,\tilde u_2)+\partial_y f^0(t,\tilde y_1^k)\,\delta y_1+\frac{1}{2}\delta y_1^\top\partial^2_{yy}f^0\,\delta y_1$$

and

$$f(t,\tilde y_1^k,u_1^k+\delta u_1,\tilde u_2)=f(t,\tilde y_1^k,u_1^k,\tilde u_2)+F_1(t)\,\delta u_1.$$

With these estimates, we continue the calculation above as follows. Notice that we add and subtract the term $\epsilon\,\|\delta u\|_2^2$.

$$\begin{aligned}
&J_1(\tilde y,\tilde u_1,\tilde u_2)-J_1(\tilde y_1^k,u_1^k,\tilde u_2)=\\
&\int_0^T\Big(-\mathcal{H}_1(t,\tilde y_1^k,\tilde u_1,\tilde u_2,p_1^k,p_2^k)+\mathcal{H}_1(t,\tilde y_1^k,u_1^k,\tilde u_2,p_1^k,p_2^k)\\
&+\partial_y\ell_1^0(t,\tilde y_1^k)\,\delta y_1-\frac{1}{2}\delta y_1^\top\partial^2_{yy}\mathcal{H}_1\delta y_1+(p_1^k)^\top F_1(t)\delta u_1+\frac{1}{2}(p_1^k)^\top\delta y_1^\top\partial^2_{yy}f^0\,\delta y_1\Big)\,dt\\
&+g_1(\tilde y(T))-g_1(\tilde y_1^k(T))+\epsilon\,\|\delta u\|_2^2-\epsilon\,\|\delta u\|_2^2\\
&=\int_0^T\Big(-\mathcal{K}_\epsilon^{(1)}(t,\tilde y_1^k,\tilde u_1,\tilde u_2,u_1^k,u_2^k,p_1^k,p_2^k)-\epsilon\|\delta u\|_2^2+\mathcal{H}_1(t,\tilde y_1^k,u_1^k,\tilde u_2,p_1^k,p_2^k)\\
&+\partial_y\ell_1^0(t,\tilde y_1^k)\,\delta y_1-\frac{1}{2}\delta y_1^\top\partial^2_{yy}\mathcal{H}_1\delta y_1+(p_1^k)^\top F_1(t)\delta u_1+\frac{1}{2}(p_1^k)^\top\delta y_1^\top\partial^2_{yy}f^0\,\delta y_1\Big)\,dt\\
&+\partial_y g_1(\tilde y_1^k(T))\delta y_1(T)+\frac{1}{2}\delta y_1(T)^\top\partial^2_{yy}g_1\,\delta y_1(T).
\end{aligned}$$

Now, we consider the following integration by parts

$$\begin{aligned}
&\int_0^T\Big((p_1^k)^\top F_1(t)\delta u_1+\frac{1}{2}(p_1^k)^\top\delta y_1^\top\partial^2_{yy}f^0\,\delta y_1\Big)\,dt=\\
&-\partial_y g_1(\tilde y_1^k(T))\,\delta y_1(T)-\int_0^T\Big[(p_1^k)'+(\partial_y f^0(t,\tilde y_1^k))^\top p_1^k\Big]\,\delta y_1\,dt.
\end{aligned}$$

Using this result in the previous calculation, we have

$$\begin{aligned}
&\int_0^T\Big(-\mathcal{K}_\epsilon^{(1)}(t,\tilde y_1^k,\tilde u_1,\tilde u_2,u_1^k,u_2^k,p_1^k,p_2^k)-\epsilon\|\delta u\|_2^2+\mathcal{H}_1(t,\tilde y_1^k,u_1^k,\tilde u_2,p_1^k,p_2^k)\\
&-\frac{1}{2}\delta y_1^\top\partial^2_{yy}\mathcal{H}_1\delta y_1\Big)\,dt+\frac{1}{2}\delta y_1(T)^\top\partial^2_{yy}g_1\,\delta y_1(T)\\
&+\int_0^T\Big[-(p_1^k)'-(\partial_y f^0(t,\tilde y_1^k))^\top p_1^k+\partial_y\ell_1^0(t,\tilde y_1^k)\Big]\,\delta y_1\,dt.
\end{aligned}\tag{4.14}$$

Applying again the mean value theorem to second order to the maps $y\mapsto\mathcal{H}_1(\cdot,y,\cdot,\cdot,\cdot,\cdot)$ and $y\mapsto\mathcal{K}_\epsilon^{(1)}(\cdot,y,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$, we get

$$\begin{aligned}
&\mathcal{H}_1(t,y^k+\delta\tilde y_1,u_1^k,\tilde u_2,p_1^k,p_2^k)=\mathcal{H}_1(t,y^k,u_1^k,\tilde u_2,p_1^k,p_2^k)\\
&+\Big((p_1^k)^\top\partial_y f^0(t,y^k)-\partial_y\ell_1^0(t,y^k)\Big)\,\delta\tilde y_1+\frac{1}{2}\delta\tilde y_1^\top\partial^2_{yy}\mathcal{H}_1\delta\tilde y_1,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{K}_\epsilon^{(1)}(t,y^k+\delta\tilde y_1,\tilde u_1,\tilde u_2,u_1^k,u_2^k,p_1^k,p_2^k)=\mathcal{K}_\epsilon^{(1)}(t,y^k,\tilde u_1,\tilde u_2,u_1^k,u_2^k,p_1^k,p_2^k)\\
&+\Big((p_1^k)^\top\partial_y f^0(t,y^k)-\partial_y\ell_1^0(t,y^k)\Big)\,\delta\tilde y_1+\frac{1}{2}\delta\tilde y_1^\top\partial^2_{yy}\mathcal{H}_1\delta\tilde y_1,
\end{aligned}$$

With these results and replacing the expression of $-(p_1^k)'$, we continue the calculation as follows

$$
\begin{aligned}
&J_1(\tilde{y},\tilde{u}_1,\tilde{u}_2) - J_1(\tilde{y}_1^k, u_1^k, \tilde{u}_2) = \\
&\int_0^T \Big(-\mathcal{K}_\epsilon^{(1)}(t,y^k,\tilde{u}_1,\tilde{u}_2,u_1^k,u_2^k,p_1^k,p_2^k) + \mathcal{H}_1(t,y^k,u_1^k,\tilde{u}_2,p_1^k,p_2^k) - \epsilon\|\delta u\|_2^2 \\
&- \frac{1}{2}\delta y_1^\top \partial_{yy}^2 \mathcal{H}_1 \delta y_1 \Big)\, dt + \frac{1}{2}\delta y_1(T)^\top \partial_{yy}^2 g_1\, \delta y_1(T) \\
&+ \int_0^T \Big((p_1^k)^\top \left[\partial_y f^0(t,y^k) - \partial_y f^0(t,\tilde{y}_1^k)\right] + \partial_y \ell_1^0(t,\tilde{y}_1^k) - \partial_y \ell_1^0(t,y^k) \Big)\delta y_1\, dt \\
&\leq \int_0^T \Big(-\epsilon\|\delta u_1\|_2^2 - \frac{1}{2}\delta y_1^\top \partial_{yy}^2 \mathcal{H}_1 \delta y_1 \Big)\, dt + \frac{1}{2}\delta y_1(T)^\top \partial_{yy}^2 g_1\, \delta y_1(T) \\
&+ \int_0^T \Big(\delta y_1^\top \partial_{yy}^2 \ell_1^0 \delta\tilde{y}_1 - (p_1^k)^\top \delta y_1^\top \partial_{yy}^2 f^0 \delta \tilde{y}_1 \Big)\, dt,
\end{aligned}
\tag{4.15}
$$

where in the last integral we applied the mean value theorem to the maps $y \mapsto \partial_y \ell_1^0(\cdot, y)$ and $y \mapsto \partial_y f^0(\cdot, y)$.

We have shown that

$$
\begin{aligned}
J_1(\tilde{y},\tilde{u}_1,\tilde{u}_2) - J_1(\tilde{y}_1^k, u_1^k, \tilde{u}_2) \leq \int_0^T \Big(&-\epsilon\,\|\delta u_1\|_2^2 - \frac{1}{2}\delta y_1^\top \partial_{yy}^2 \mathcal{H}_1\, \delta y_1 \\
&+\delta y_1^\top \partial_{yy}^2 \ell_1^0\, \delta\tilde{y}_1 - (p_1^k)^\top \delta y_1^\top \partial_{yy}^2 f^0\, \delta\tilde{y}_1 \Big)\, dt \\
&+ \frac{1}{2}\, \delta y_1(T)^\top \partial_{yy}^2 g_1\, \delta y_1(T).
\end{aligned}
$$

With a similar computation, we also obtain

$$
\begin{aligned}
J_2(\tilde{y},\tilde{u}_1,\tilde{u}_2) - J_2(\tilde{y}_2^k, \tilde{u}_1, u_2^k) \leq \int_0^T \Big(&-\epsilon\,\|\delta u_2\|_2^2 - \frac{1}{2}\delta y_2^\top \partial_{yy}^2 \mathcal{H}_2\, \delta y_2 \\
&+\delta y_2^\top \partial_{yy}^2 \ell_2^0\, \delta\tilde{y}_2 - (p_2^k)^\top \delta y_2^\top \partial_{yy}^2 f^0\, \delta\tilde{y}_2 \Big)\, dt \\
&+ \frac{1}{2}\, \delta y_2(T)^\top \partial_{yy}^2 g_2\, \delta y_2(T),
\end{aligned}
$$

where $\tilde{y}_2^k = y(\tilde{u}_1, u_2^k)$, $\delta y_2 := \tilde{y} - \tilde{y}_2^k$ and $\delta\tilde{y}_2 := \tilde{y}_2^k - y^k$. Thus, we arrive at the following inequality

$$
\begin{aligned}
\psi(\tilde{u}, u^k) \leq \int_0^T \Big(&-\epsilon\left[\|\delta u_1\|_2^2 + \|\delta u_2\|_2^2\right] - \frac{1}{2}\delta y_1^\top \partial_{yy}^2 \mathcal{H}_1\, \delta y_1 - \frac{1}{2}\delta y_2^\top \partial_{yy}^2 \mathcal{H}_2\, \delta y_2 \\
&+\delta y_1^\top \partial_{yy}^2 \ell_1^0\, \delta\tilde{y}_1 - (p_1^k)^\top \delta y_1^\top \partial_{yy}^2 f^0\, \delta\tilde{y}_1 + \delta y_2^\top \partial_{yy}^2 \ell_2^0\, \delta\tilde{y}_2 - (p_2^k)^\top \delta y_2^\top \partial_{yy}^2 f^0\, \delta\tilde{y}_2 \Big)\, dt \\
&+ \frac{1}{2}\delta y_1(T)^\top \partial_{yy}^2 g_1\, \delta y_1(T) + \frac{1}{2}\delta y_2(T)^\top \partial_{yy}^2 g_2\, \delta y_2(T).
\end{aligned}
\tag{4.16}
$$

Next, we notice that, as we have seen in Chapter 1, the solutions to the state and adjoint problems are uniformly bounded in $[0, T]$ for any choice of $u \in U_{ad}$, and the following estimates hold

$$
\|\delta y_1(t)\|_2 \leq \mathcal{C}_{11}\, \|\delta u_1\|_{L^2(0,T)}, \quad \|\delta y_2(t)\|_2 \leq \mathcal{C}_{22}\, \|\delta u_2\|_{L^2(0,T)}, \qquad t \in (0,T). \tag{4.17}
$$

Similarly,

$$
\|\delta \tilde{y}_1(t)\|_2 \leq \mathcal{C}_{12}\, \|\delta u_2\|_{L^2(0,T)}, \quad \|\delta \tilde{y}_2(t)\|_2 \leq \mathcal{C}_{21}\, \|\delta u_1\|_{L^2(0,T)}, \qquad t \in (0,T); \tag{4.18}
$$

By using these estimates in (4.16), we obtain

$$\psi(\tilde{u}, u^k) \leq -(\epsilon - \theta)\, \|\delta u\|^2_{L^2(0,T)},$$

where θ depends on the functions computed at the kth iteration but not on ϵ. Thus the claim is proved.

□

We remark that in Step 2 of the SQH algorithm, the NE solution $\tilde{u}$ obtained in Step 2 depends on ϵ so that $\|\tilde{u} - u^k\|^2_{L^2(0,T)}$ decreases as $O(1/\epsilon^2)$. In order to illustrate this fact, consider the following optimisation problem

$$\max f_\epsilon(u) := b\,u - \frac{\nu}{2}\,u^2 - \epsilon\,(u - v)^2,$$

where ν, $\epsilon > 0$. Clearly, the function f_ϵ is concave and its maximum is attained at $\tilde{u} = (b + 2\,\epsilon\,v)/(\nu + 2\,\epsilon)$. Further, we have

$$\|\tilde{u} - v\|_2 = \frac{\|b - \nu\,v\|_2}{(\nu + 2\,\epsilon)}.$$

Now, subject to the assumptions of Proposition 4.2.1 and using the estimates in its proof, we can state that there exists a constant $C > 0$ such that

$$|\psi(\tilde{u}, u^k)| \leq C\,\|\tilde{u} - u^k\|^2_{L^2(0,T)},$$

where C increases linearly with ϵ. On the other hand, since the HP functions are concave, we have that $\|\tilde{u} - u^k\|^2_{L^2(0,T)}$ decreases as $O(1/\epsilon^2)$. Therefore, given the value Ψ^k in Step 5 of the SQH algorithm, it is always possible to choose ϵ sufficiently large such that $|\psi(\tilde{u}, u^k)| \leq \Psi^k$.

In Algorithm 7, we have that $\psi(u^{k+1}, u^k) \to 0$ as $k \to \infty$. Thus, since $\psi(u^{k+1}, u^k) \leq -\xi\,\|u^{k+1} - u^k\|^2_{L^2}$, it follows that $\lim_k \|u^{k+1} - u^k\|^2_{L^2} = 0$ and hence the convergence criterion in Step 6 can be satisfied.

Next, we show that, if u_1^k and u_2^k satisfy the PMP conditions, then Algorithm 7 stops and returns these functions.

Proposition 4.2.2. *Subject to the assumptions of Proposition 4.2.1, let* (u_1^k, u_2^k) *be generated by Algorithm 7. If this pair satisfies the PMP conditions, then Algorithm 7 stops returning* (u_1^k, u_2^k).

Proof. Suppose that u_1^k, u_2^k satisfy the PMP conditions as follows

$$\mathcal{H}_1(t, y^k, u_1^k, u_2^k, p_1^k, p_2^k) = \max_{w_1 \in K_{ad}^{(1)}} \mathcal{H}_1(t, y^k, w_1, u_2^k, p_1^k, p_2^k)$$
$$\mathcal{H}_2(t, y^k, u_1^k, u_2^k, p_1^k, p_2^k) = \max_{w_2 \in K_{ad}^{(2)}} \mathcal{H}_2(t, y^k, u_1^k, w_2, p_1^k, p_2^k),$$

where we omit the argument t for easy of notation.

Now, consider player P_1, it holds

$$\begin{aligned}
&\mathcal{K}_\epsilon^{(1)}(t, y^k, u_1^k, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) = \mathcal{H}_1(t, y^k, u_1^k, u_2^k, p_1^k, p_2^k)\\
&\geq \mathcal{H}_1(t, y^k, w_1, u_2^k, p_1^k, p_2^k) \geq \mathcal{H}_1(t, y^k, w_1, u_2^k, p_1^k, p_2^k) - \epsilon\|(w_1, u_2^k) - (u_1^k, u_2^k)\|_2^2\\
&= \mathcal{K}_\epsilon^{(1)}(t, y^k, w_1, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k), \quad \text{for any } w_1 \in K_{ad}^{(1)}.
\end{aligned}$$

Analogously, $\mathcal{K}_\epsilon^{(2)}(t, y^k, u_1^k, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) \geq \mathcal{K}_\epsilon^{(2)}(t, y^k, u_1^k, w_2, u_1^k, u_2^k, p_1^k, p_2^k)$, for any $w_2 \in K_{ad}^{(2)}$. Hence, u_1^k, u_2^k are among those selected by the algorithm. Next, we show that no other pair can be found. For this purpose, we suppose that there exist $\tilde{t}$ and $\tilde{u}_1, \tilde{u}_2$ such that

$$\begin{aligned}\mathcal{K}_\epsilon^{(1)}(\tilde{t}, y^k, \tilde{u}_1, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) &\geq \mathcal{K}_\epsilon^{(1)}(\tilde{t}, y^k, u_1^k, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) \\ &= \mathcal{H}_1(\tilde{t}, y^k, u_1^k, u_2^k, p_1^k, p_2^k),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\mathcal{K}_\epsilon^{(2)}(\tilde{t}, y^k, u_1^k, \tilde{u}_2, u_1^k, u_2^k, p_1^k, p_2^k) &\geq \mathcal{K}_\epsilon^{(2)}(\tilde{t}, y^k, u_1^k, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) \\ &= \mathcal{H}_2(\tilde{t}, y^k, u_1^k, u_2^k, p_1^k, p_2^k).\end{aligned} \tag{4.20}$$

Since u_1^k satisfies the PMP condition, it holds

$$\mathcal{H}_1(\tilde{t}, y^k, u_1^k, u_2^k, p_1^k, p_2^k) \geq \mathcal{H}_1(\tilde{t}, y^k, w_1, u_2^k, p_1^k, p_2^k) \quad \text{for any} \quad w_1 \in K_{ad}^{(1)}. \tag{4.21}$$

If we take $w_1 = \tilde{u}_1$, we get

$$\begin{aligned}&\mathcal{H}_1(\tilde{t}, y^k, u_1^k, u_2^k, p_1^k, p_2^k) - \epsilon\|\tilde{u}_1 - u_1^k\|_2^2 \\ &\geq \mathcal{H}_1(\tilde{t}, y^k, \tilde{u}_1, u_2^k, p_1^k, p_2^k) - \epsilon\|\tilde{u}_1 - u_1^k\|_2^2 \\ &= \mathcal{K}_\epsilon^{(1)}(\tilde{t}, y^k, \tilde{u}_1, u_2^k, u_1^k, u_2^k, p_1^k, p_2^k) \\ &\geq \mathcal{H}_1(\tilde{t}, y^k, u_1^k, u_2^k, p_1^k, p_2^k).\end{aligned}$$

Hence, it follows $-\epsilon\|\tilde{u}_1 - u_1^k\|_2^2 \geq 0$, which means $\tilde{u}_1 = u_1^k$. In the same way we obtain $\tilde{u}_2 = u_2^k$, which concludes the proof. □

4.3 Application to a LQ-game

In this section, we present results of our experiments to test the ability of the proposed SQH scheme to determine a NE for the given game. The first experiment exploits the possibility to compute open-loop NE solutions to linear-quadratic Nash games by solving a coupled system of Riccati equations [40]. Thus, we use this solution for comparison to the solution of the same Nash game obtained with the SQH method. We remark that in all the experiments performed in this chapter, the structure of the corresponding problems is such that in Step 2 of the SQH algorithm the update $\tilde{u}(t)$ at any fixed t can be determined analytically.

Consider a linear-quadratic Nash game formulated as follows

$$y'(t) = A\, y(t) + B_1 u_1(t) + B_2 u_2(t), \quad y(0) = y_0, \tag{4.22}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore $y(t) \in \mathbb{R}^2$ and $u_i(t) \in \mathbb{R}^2$, $t \in [0, T]$.

The cost functionals are as follows

$$J_i(y, u_1, u_2) = \frac{1}{2}\int_0^T \Big(y(s)^\top L_i(s) y(s) + u_i(s)^\top N_i(s) u_i(s) \Big)\, ds + \frac{1}{2} y(T)^\top D_i y(T), \tag{4.23}$$

$i = 1, 2$, where the matrices $L_i(s)$, D_i, $N_i(s)$ are given by $L_1 = \alpha_1 I_2$, $L_2 = \alpha_2 I_2$, $N_1 = \nu_1 I_2$, $N_2 = \nu_2 I_2$, and $D_1 = \gamma_1 I_2$, and $D_2 = \gamma_2 I_2$, where I_2 is the identity matrix in $\mathbb{R}^2$. In the following experiment, we choose $\alpha_1 = 1$, $\alpha_2 = 10$, $\nu_1 = 0.1$, $\nu_2 = 0.1$, $\gamma_1 = 0$ and $\gamma_2 = 0$.

We consider the time interval $[0, T]$, with $T = 0.2$, subdivided into $N = 2000$ subintervals and, on this grid, the state and adjoint equations are solved numerically by a midpoint scheme [12].

The initial guess u^0 for the SQH iteration are zero functions, and we choose $\epsilon = 10$, $\zeta = 0.95$, $\sigma = 1.05$, $\xi = 10^{-8}$, $\Psi^0 = 10$, and $\mathcal{K} = 10^{-14}$. With this setting, we obtain the Nash strategies (u_1, u_2) depicted in Figure 4.1 (left), which are compared with the solution obtained by solving the Riccati system as shown in Figure 4.1 (right). We can see that the two sets of solutions overlap.

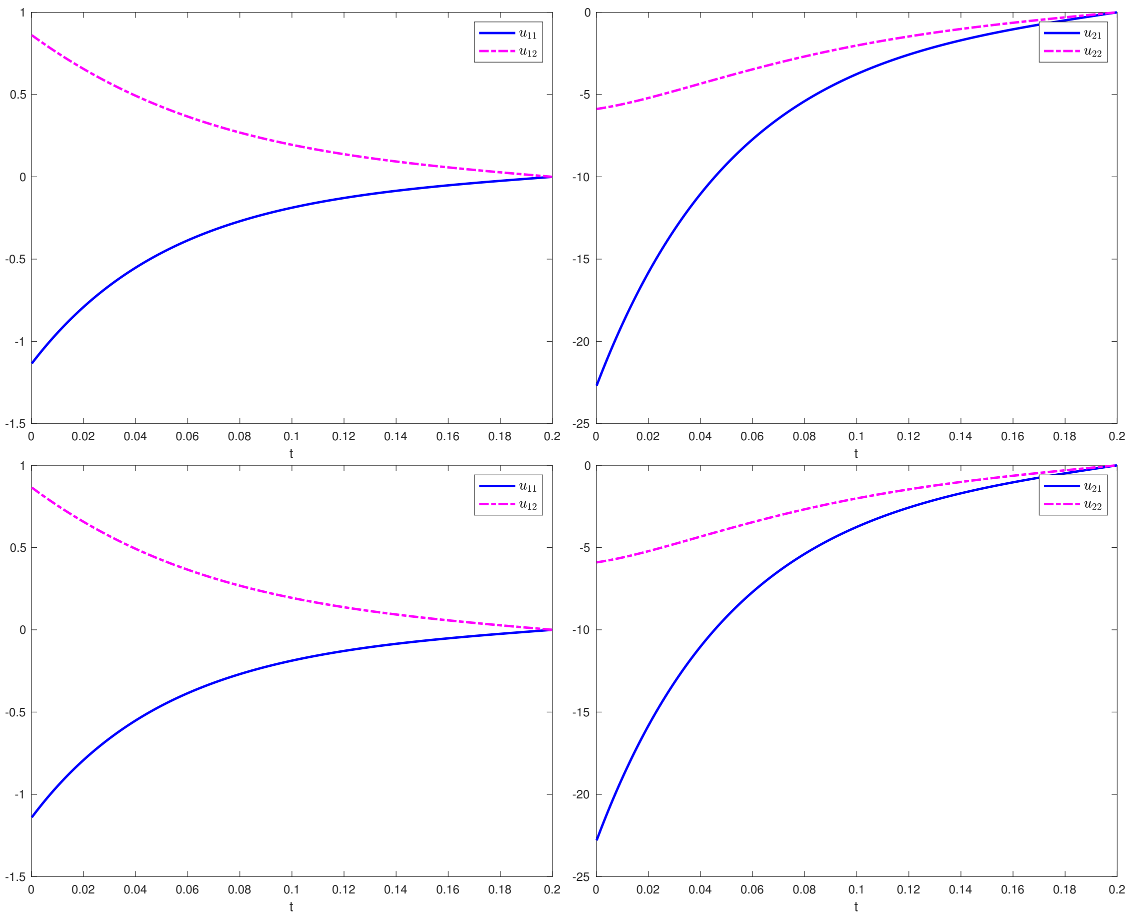

Figure 4.1: Strategies u_1 (left) and u_2 for the LQ Nash game obtained with the SQH scheme (top) and by solving the Riccati system (bottom).

Next, we consider the same setting but require that the players' strategies are constrained by choosing $K_{ad}^{(i)} = [-2, 2] \times [-2, 2]$, $i = 1, 2$. With this setting, we obtain the strategies depicted in Figure 4.2.

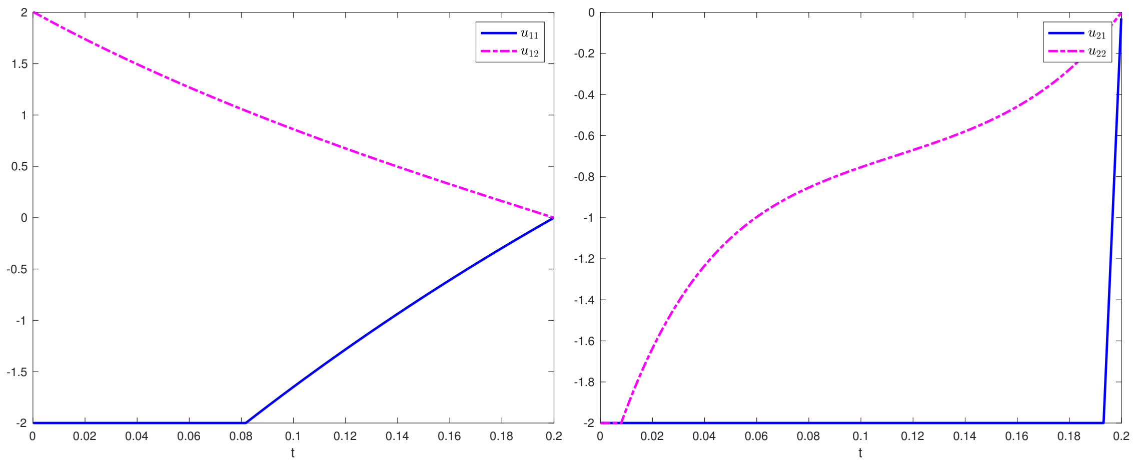

Figure 4.2: Strategies u_1 (left) and u_2 for the LQ Nash game with constraints on u as obtained by the SQH scheme.

4.4 Application to weighted L^1 costs on the players' strategies

In this section, as third experiment, we consider a setting similar to the previuos experiment with constrained strategies, but we add to the cost functionals a weighted L^1 cost of the strategies. We have (written in a more compact form)

$$J_i(y, u_1, u_2) = \frac{1}{2} \int_0^T \left(\alpha_i \, |y(s)|_2^2 + \nu_i \, |u_i(s)|^2 + 2\beta_i \, |u_i(s)| \right) ds, \tag{4.24}$$

where $i = 1, 2$; the terms with D_i, $i = 1, 2$, are omitted. We choose $\nu_1 = 0.1$, $\nu_2 = 0.01$, and $\beta_1 = 0.01$, $\beta_2 = 0.1$; the other parameters are set as in the first experiment. Further, we require that the players' strategies are constrained by choosing $K_{ad}^{(i)} = [-10, 10] \times [-10, 10]$, $i = 1, 2$. The strategies obtained with this setting are depicted in Figure 4.3. Notice that the addition of L^1 costs of the players' actions promotes their sparsity.

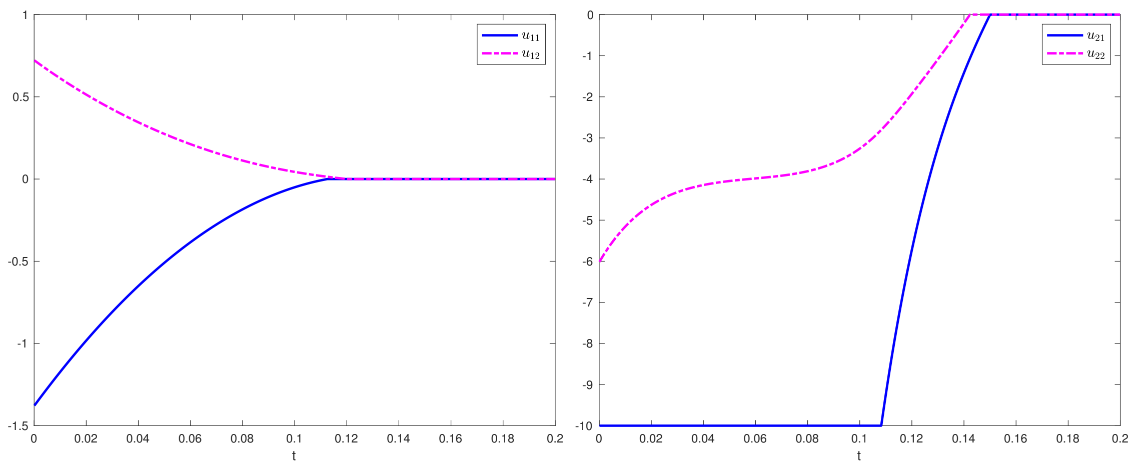

Figure 4.3: Strategies u_1 (left) and u_2 for the Nash game with L^2 and L^1 costs and constraints on u as obtained by the SQH scheme.

4.5 Application to a tracking problem

In this section, we consider a tracking problem where the cost functionals have the following structure

$$J_i(y, u_1, u_2) = \frac{1}{2}\int_0^T \Big(\alpha_i \left|y(s) - \bar{y}^i(s)\right|^2 + \nu_i \left|u_i(s)\right|^2 + 2\beta_i \left|u_i(s)\right|\Big)\, ds \\ + \frac{\gamma_i}{2}\left\|y(T) - \bar{y}^i(T)\right\|_2^2, \tag{4.25}$$

where $\bar{y}^i$ denotes the trajectory desired by the Player P_i, $i = 1, 2$. Specifically, we take

$$\bar{y}^1(t) = \begin{pmatrix}1\\1\end{pmatrix} \sin(2\pi t), \qquad \bar{y}^2(t) = \begin{pmatrix}1\\1\end{pmatrix} \cos(2\pi t).$$

Notice that these trajectories are orthogonal to each other, that is, the two players have very different purposes. For the initial state, we take $y_0 = (1/2, 1/2)$.

In this fourth experiment, the values of the game parameters are given by $\alpha_1 = 1$, $\alpha_2 = 10$, $\nu_1 = 10^{-8}$, $\nu_2 = 10^{-6}$, $\beta_1 = 10^{-8}$, $\beta_2 = 10^{-6}$, and $\gamma_1 = 1$ and $\gamma_2 = 1$. Further, we require that the players' strategies are constrained by choosing $K_{ad}^{(i)} = [-4, 4] \times [-4, 4]$, $i = 1, 2$. In this experiment, we take $T = 1$ and $N = 10^4$ subdivision of $[0, T]$ for the numerical approximation. The parameters of the SQH scheme remain unchanged. The results of this experiment are depicted in Figure 4.4.

For this concluding experiment, we report that the convergence criterion is achieved after 3593 SQH iterations, whereas the number of successful updates is 1686. Correspondingly, we see that ψ is always negative and its absolute value monotonically decreases, with $\psi = -1.70 \times 10^{-10}$ at convergence. On the other hand, we can see that the value of ϵ is changed along the iteration, while the values of the cost functionals reach the Nash equilibrium. The CPU time for this experiment is 1151.4 seconds on a laptop computer.

4.6 Comparison of the relaxation-Newton and SQH methods

In this section, we compare the SQH scheme with the relaxation method combined with a semismooth Newton scheme, introduced in Chapter 2, to address the question of how the SQH scheme performs with respect to the relaxation-Newton method.

For this purpose, the numerical experiments introduced in Chapter 2 are performed with the SQH scheme. As expected, both methods successfully determine the sought NE. We show that, for a not too small time step, the SQH method outperforms the relaxation-Newton method in all the numerical experiments. Hovewer, also when the relaxation-Newton scheme performs better than the SQH method (see the application to the Lotka-Volterra equations), we remark that the second method is easier to implement.

4.6.1 Application to a bilinear quantum game

In this section, we consider the bilinear quantum Nash game, introduced in Section 2.3 and we determine the NE for the game with the SQH scheme. We show that both methods are able to determine the NE solution and that for this problem, the SQH method outperforms the relaxation-Newton scheme.

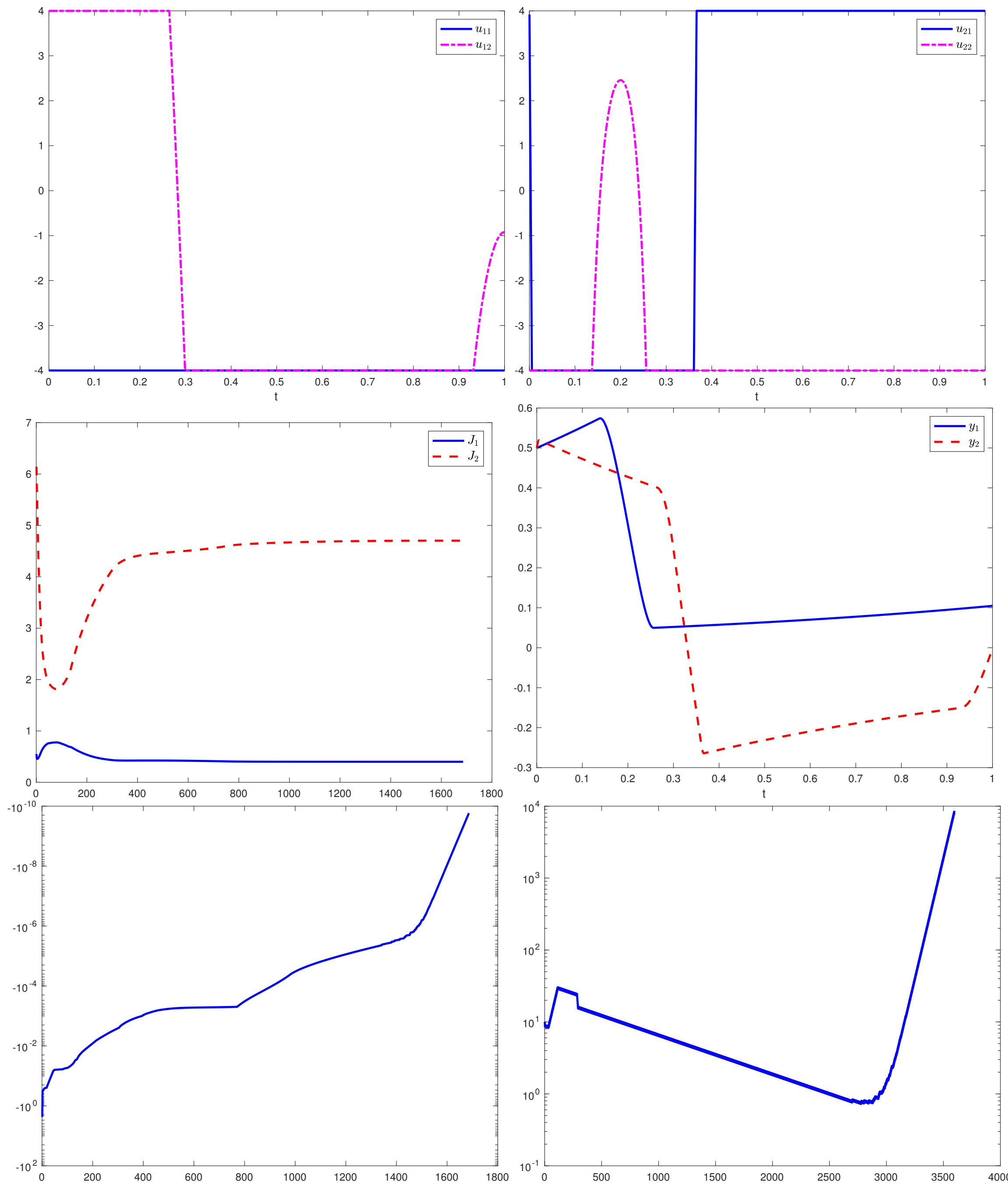

Figure 4.4: Strategies u_1 (top, left) and u_2 (top, right) for the Nash game with tracking costs as obtained by the SQH scheme. In the middle figures the values of J_1 and J_2 along the SQH iterations (left) and the evolution of y corresponding to u_1 and u_2 (right). In the bottom figures the values of ψ (left) and of ϵ along the SQH iterations.

For this purpose, consider the model of two uncoupled spin$-1/2$ particles, introduced in Chapter 2 where at the initial state, the two spins are both pointing in the $z-$direction.

We recall that the first player P_1 aims at driving the system from the initial state to a target state $y_T^{(1)}$ where the spins are pointing in the $x-$direction and the player P_2 aims at driving the system to $y_T^{(2)}$ requiring an inversion of the orientation of the particles.

To implement the SQH scheme for the quantum bilinear game, we need the two augmented Hamilton-Pontryagin functions. Thus, with the same notation of Chapter 2, consider the following HP functions

$$\mathcal{H}_j(t,y,u_1,u_2,p_1,p_2) := p_j \cdot \left[f^0(y) + \sum_{j=1}^{2} u_j F_j(y) \right] - \frac{\nu}{2} u_j^2, \tag{4.26}$$

where p_j, $j = 1,2$ represent the adjoint variables.

In the PMP framework, we get that the NE point (u_1^*, u_2^*) must satisfy the following system

$$u_1^* = \arg\max_{v \in [-M,M]} \mathcal{H}_1(t, y^*(t), v, u_2^*(t), p_1^*(t), p_2^*(t)) \quad \text{a.e. in } [0,T]; \tag{4.27}$$

$$u_2^* = \arg\max_{v \in [-M,M]} \mathcal{H}_2(t, y^*(t), u_1^*(t), v, p_1^*(t), p_2^*(t)) \quad \text{a.e. in } [0,T]. \tag{4.28}$$

Next, consider the corresponding augmented HP functions

$$\mathcal{K}_\epsilon^{(i)}(t,y,u_1,u_2,v_1,v_2,p_1,p_2) := \mathcal{H}_i(t,y,u_1,u_2,p_1,p_2) - \epsilon \left\| u - v \right\|_2^2, \quad i = 1,2, \tag{4.29}$$

where, in the iteration process, $u = (u_1, u_2)$ is subject to the update step, and $v = (v_1, v_2)$ corresponds to the previous strategy approximation.

Notice that, due to the differentiability of the HP functions, in Step 2 of Algorithm 7 it appears that the only points where $\mathcal{K}_\epsilon^{(j)}$ can attain a maximum are given by

$$\tilde{u}_1 = \max\left(\min\left(\frac{2\epsilon v_1 + p_1^\top B_1 y}{\nu + 2\epsilon}, M \right), -M \right), \tag{4.30}$$

$$\tilde{u}_2 = \max\left(\min\left(\frac{2\epsilon v_2 + p_2^\top B_2 y}{\nu + 2\epsilon}, M \right), -M \right). \tag{4.31}$$

Moreover, the following necessary second-order equilibrium conditions are fulfilled, see [77]

$$\partial^2_{u_1 u_1} \mathcal{K}_\epsilon^{(1)}(u_1^*, u_2^*) \leq 0 \tag{4.32}$$

$$\partial^2_{u_2 u_2} \mathcal{K}_\epsilon^{(2)}(u_1^*, u_2^*) \leq 0. \tag{4.33}$$

In our numerical experiments, let $M = 60$, that means $K_{ad}^{(i)} = [-60, 60]$, $i = 1,2$, and choose $u^0(t) = (0.1, 0.1), t \in [0,T]$ with $T = 0.008$. The time interval is subdivided into $N = 1000$ grid points. On these points, the state and adjoint equations are approximated by a MCN scheme. The problem parameters are as discussed in the first test of Section 2.3.

Next, we define the SQH parameters. Let $\epsilon = 0.0005, \zeta = 0.8, \sigma = 2, \xi = 10^{-8}, \Psi^0 = 10$, and $\mathcal{K} = 10^{-15}$.

With this setting we obtain the Nash strategies showed in Figure 4.5.

As one can see, the obtained strategies are the same of those computed with the relaxation scheme. Correspondingly, the Nash equilibrium is the point $(\tilde{J}_1(u^{NE}), \tilde{J}_2(u^{NE})) = (0.7170, 1.3227)$. This solution corresponds to a value of $\psi = -6.6613 \times 10^{-16}$. The convergence criterion is fulfilled after 10 successful updates in 1.6316 seconds, see also Table 4.1.

Next, we perform the second experiment with the quantum dynamics, where the two players have the same difficulty in reaching their final targets. Specifically, $y_T^{(2)} =$

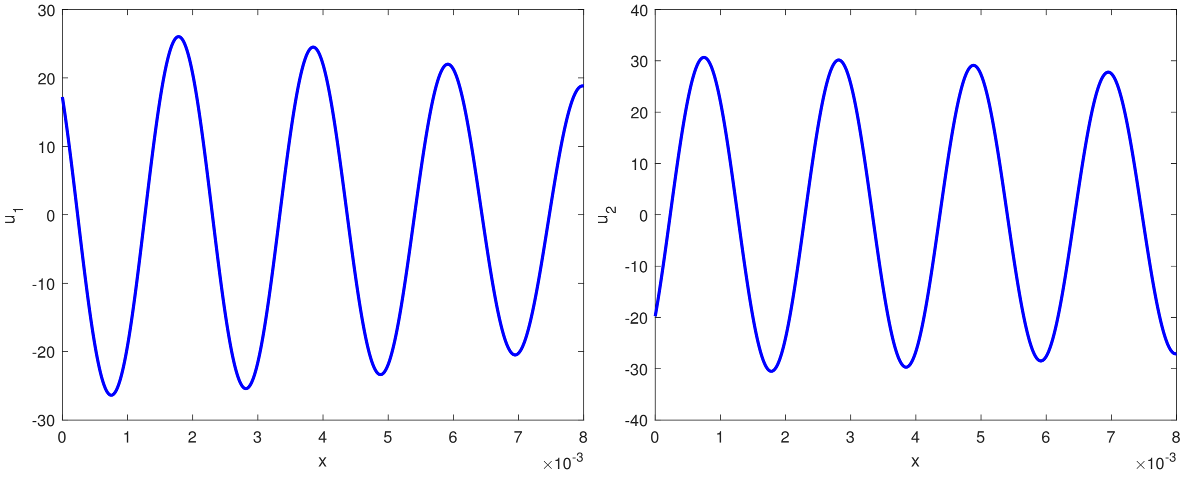

Figure 4.5: Strategies u_1 (left) and u_2 for the asymetric bilinear quantum game as obtained by the SQH scheme.

Table 4.1: Comparison of the computation time of the solution to the asymmetric bilinear quantum differential game obtained with the two methods.

Δt	method	CPU time/s	iteration
$8 \cdot 10^{-5}$	Relaxation-Newton	29.5166	22
	SQH	0.4479	30
$8 \cdot 10^{-6}$	Relaxation-Newton	36.2392	24
	SQH	1.6316	10
$8 \cdot 10^{-7}$	Relaxation-Newton	477.3287	22
	SQH	34.2804	27

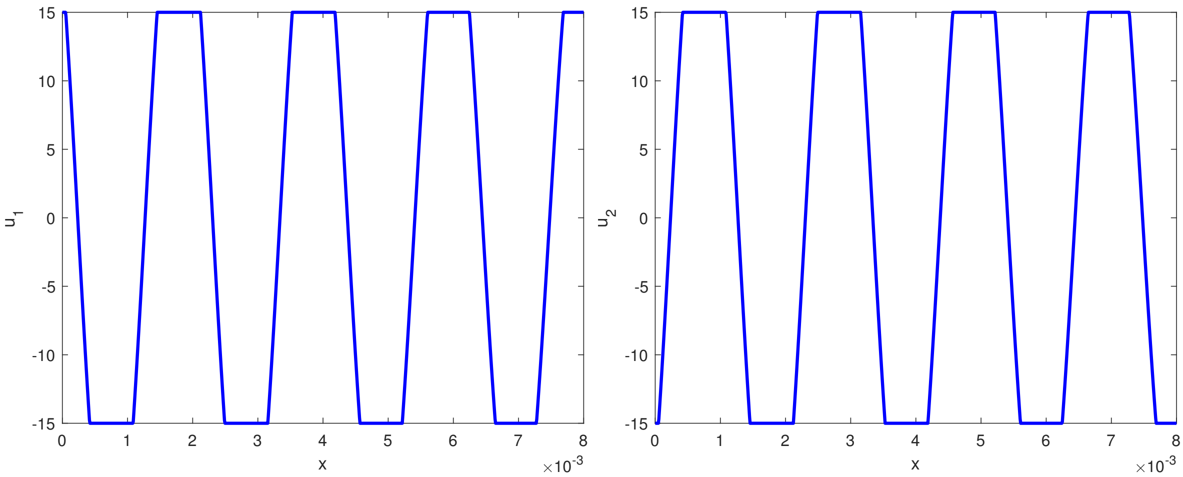

Figure 4.6: Strategies u_1 (left) and u_2 for the symetric bilinear quantum game as obtained by the SQH scheme.

$(0, 1, 0, 0, 1, 0)^\top$. Moreover, let $K_{ad}^{(i)} = [-15, 15]$, $i = 1, 2$ to see that the constraints on the players' actions become active, as in Section 2.3.

We set the parameters of the SQH algorithm as above. The corresponding Nash strategies are given in Figure 4.6.

Notice that the corresponding NE is $(\tilde{J}_1(u^{NE}), J_2(u^{NE})) = (0.7174, 0.7174)$ as obtained by using the relaxation scheme. Moreover, at equilibrium we get $\psi = -1.6653 \times 10^{-15}$,

Table 4.2: Comparison of the computation time of the solution to the symmetric bilinear quantum differential game obtained with the two methods.

Δt	method	CPU time/s	iteration
$8 \cdot 10^{-5}$	Relaxation-Newton	47.0250	21
	SQH	0.5286	4
$8 \cdot 10^{-6}$	Relaxation-Newton	35.4941	24
	SQH	1.0903	6
$8 \cdot 10^{-7}$	Relaxation-Newton	605.8600	21
	SQH	7.9212	8

value obtained after 6 successful updates. The elapsed time of the whole procedure is 1.0903 seconds, see Table 4.2.

The results in Tables 4.1 and 4.2 show that, the SQH performs better than the relaxation-Newton method.

4.6.2 Application to the competitive Lotka-Volterra problem

In this section, we use the SQH scheme to solve the competitive Lotka-Volterra equations introduced in Section 2.4. Specifically, we consider two species, that we label with y_1 and y_2 , competing for the same resources. The dynamics is given by (2.32). The parameters of the game are chosen as seen in Section 2.4.

To implement the SQH scheme, we consider the two HP functions whose general structure is given in the previous section in (4.26) and we proceed as above. In fact, by using the differentiability of the augmented Hamilton-Pontryagin functions with respect to the players' actions variables, Step 2 of Algorithm 7 reduces to the following computation

$$\tilde{u}_1 = \max\left(\min\left(\frac{2\epsilon v_1 + p_1^\top F_1(y)}{\nu + 2\epsilon}, M\right), -M\right) \tag{4.34}$$

$$\tilde{u}_2 = \max\left(\min\left(\frac{2\epsilon v_2 + p_2^\top F_2(y)}{\nu + 2\epsilon}, M\right), -M\right). \tag{4.35}$$

With this setting, we take the following SQH parameters, $\epsilon = 5$, $\zeta = 0.8$, $\sigma = 2$, $\xi = 10^{-8}$, $\Psi^0 = 10$ and $\mathcal{K} = 10^{-15}$.

Moreover, as in section 2.4, let $K_{ad}^{(i)} = [-4, 4]$, and $u_i^0(t) = 0$, $t \in [0, T]$, $i = 1, 2$, $T = 0.25$, $y_0 = (1.5, 1)^\top$ and as final targets $y_T^{(1)} = (\frac{1}{2}, 0)^\top$ and $y_T^{(2)} = (0, \frac{1}{2})^\top$, that is each species aims at the extinction of the other one.

In this framework, we obtain the equilibrium strategies depicted in Figure 4.7.

As one can see, the NE strategies are the same of those listed in Figure 2.6. Moreover, with the SQH scheme we obtain the same Nash equilibrium which is $(\tilde{J}_1(u^{NE}), \tilde{J}_2(u^{NE})) = (0.2291, 0.3324)$ which corresponds to the Nikaido-Isoda function $\psi = -9.4368 \times 10^{-16}$, achieved in 15 iterations, see also Table 4.3.

From Table 4.3 we can deduce that, for the Lotka-Volterra problem, the relaxation-Newton method outplays the SQH scheme in CPU time and in the number of iterations.

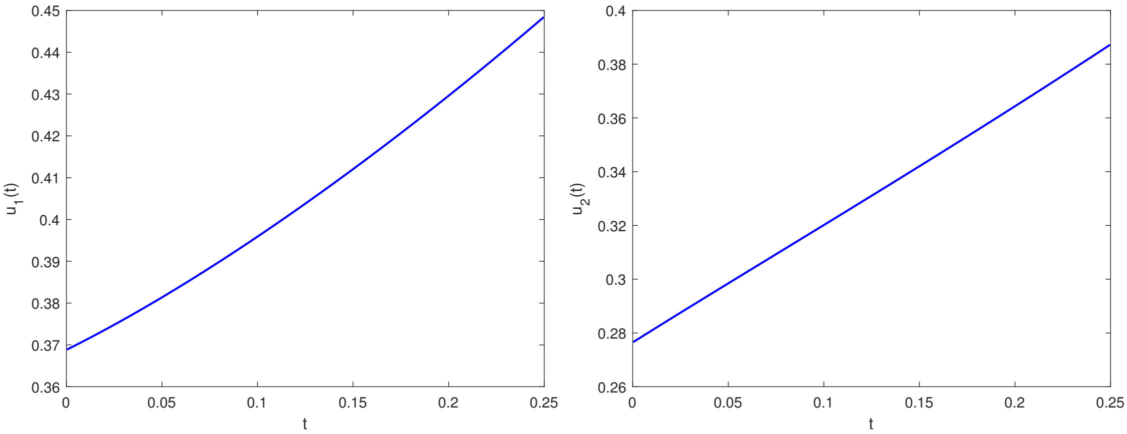

Figure 4.7: Strategies u_1 (left) and u_2 for the competitive Lotka-Volterra problem as obtained by the SQH scheme.

Table 4.3: Comparison of the computation time of the solution to the Lotka-Volterra differential game obtained with the two methods.

Δt	method	CPU time/s	iteration
$2.5 \cdot 10^{-3}$	Relaxation-Newton	8.4925	8
	SQH	1.7324	14
$2.5 \cdot 10^{-4}$	Relaxation-Newton	33.3315	14
	SQH	138.7536	15
$2.5 \cdot 10^{-5}$	Relaxation-Newton	250.8320	15
	SQH	9306.8543	19

4.7 Summary

In this chapter, a sequential quadratic Hamiltonian (SQH) scheme for solving open-loop differential Nash games was discussed. Theoretical and numerical results were presented that successfully demonstrated the well-posedness and computational performance of the SQH method applied to different Nash games governed by ordinary differential equations.

A comparison with the relaxation-Newton method was performed to show also the ability of the SQH scheme to solve Nash games having non linear dynamics.

However, the applicability of the proposed method seems not restricted to these models and appears to be a promising technique for solving infinite-dimensional differential Nash game problems that have been considered recently in different fields of applied mathematics as, for example, in [26, 78, 81].

Appendix

In this Appendix supplementary results used in this thesis are given.

4.8 The Carathéodory theorem

The following theorem of existence and uniqueness of a solution of a Cauchy problem of the form (1.10) is stated; see e.g, [12], [88].

Theorem 4.8.1. *For any fixed $u_1 \in U_{ad}^{(1)}, u_2 \in U_{ad}^{(2)}$, let $f : [0,T] \times \mathbb{R}^n \times K_{ad}^{(1)} \times K_{ad}^{(2)} \to \mathbb{R}^n$, $t \mapsto \tilde{f}(t,y) := f(t,y,u_1(t),u_2(t))$ be measurable for any fixed $y \in R^n$ and continuous in y for each fixed t. If there exist a Lebesgue-integrable function $m_1 : [0,T] \to \mathbb{R}$ such that*

$$\|\tilde{f}(t,y_1) - \tilde{f}(t,y_2)\| \leq m_1(t)\|y_1 - y_2\|$$

holds for a norm defined in $\mathbb{R}^n$, for each $t \in [0,T]$ and any $y_1, y_2 \in \mathbb{R}^n$, and a non-negative Lebesgue-integrable function $m_2 : [0,T] \to \mathbb{R}^+$ such that

$$\|\tilde{f}(t,y)\| \leq m_2(t)$$

for almost all t, then there exists a unique absolutely-continuous function $y : [0,T] \to \mathbb{R}^n$ such that

$$\begin{aligned} y(t) &= y_0 + \int_0^T \tilde{f}(s,y(s))ds \\ &= y_0 + \int_0^T f(s,y(s),u_1(s),u_2(s))ds. \end{aligned}$$

4.9 The Grönwall's lemma

In this section, we recall the Grönwall's lemma used in many proofs of Chapter 1; see, e.g., [12] and reference therein. For this purpose, denote with $C_{pw}([t_0,\infty];\mathbb{R}^+)$ the set of the continuous functions $u : [t_0,\infty) \to \mathbb{R}^+$ on (t_k,t_{k+1}), with discontinuity of the first kind at the point t_k, $k \in \mathbb{N}$. We are assuming $0 \leq t_0 < t_1 < \dots$ with $\lim_{k\to\infty} t_k = \infty$.

Lemma 4.9.1. *Assume that, for $t \geq t_0$, the following inequality holds*

$$u(t) \leq a(t) + \int_{t_0}^t g(t,s)u(s)ds + \sum_{t_0<t_k<t} \beta_k(t)u(t_k),$$

where $\beta_k(t)$, $k \in \mathbb{N}$ are non-decreasing functions for $t \geq t_0$, $a \in C_{pw}([t_0,\infty];\mathbb{R}^+)$ is a non-decreasing function, $u \in C_{pw}([t_0,\infty];\mathbb{R}^+)$ and $g(t,s)$ is a continuous non-negative *function for t, $t \geq t_0$, and non-decreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$ the following inequality is fulfilled*

$$u(t) \leq a(t) \prod_{t_0<t_k<t} (1+\beta_k(t)) \exp\left(\int_{t_0}^{t} g(t,s)ds\right).$$

4.10 The implicit function theorem

This section is devoted to the implicit function theorem used to study some properties of the control-to-state map in Chapter 1; see [12] for a proof.

Theorem 4.10.1. *Let X, Y and Z be three Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ and the function $c : X \times Y \to Z$ maps an open subset of $X \times Y$ into Z.*

Let $(x_0, y_0) \in X \times Y$ with $c(x_0, y_0) = 0$ be given. Assume the map c is m times continuously Fréchet differentiable in a neighbourhood of (x_0, y_0) and that the Fréchet derivative $D_y c(x_0, y_0) \in \mathcal{L}(Y, Z)$ is continuously invertible, where $\mathcal{L}(Y, Z)$ denotes the set of all bounded linear operator from Y to Z.

Then, there exist a function $f : X \to Y$ and $\delta, \epsilon > 0$ such that for all $(x, y) \in B_\delta(x_0) \times B_\epsilon(y_0)$ the two statements

$$y = f(x) \quad \text{and} \quad c(x, y) = 0$$

are equivalent.

Furthermore, the map $f : X \to Y$ is m times Freéchet differentiable in $B_\delta(x_0)$ and its derivative is given by

$$D_y f(x) = -(D_y c(x, f(x)))^{-1} D_x c(x, f(x)).$$

4.11 The Modified Crank-Nicolson scheme

In this section, we derive the so-called modified Crank-Nicolson (MCN) scheme for the discretization of a system with a bilinear control mechanism, defined as follows

$$y' = \left[A + \sum_{i=1}^{N} u_i B_i\right] y, \quad y(0) = y_0, \tag{4.36}$$

where $y(t) \in \mathbb{R}^n$, $A, B_i \in \mathbb{R}^{n\times n}$, N is the number of controls.

Consider the time interval $[0, T]$ subdivided into N_t points with uniform mesh size δt, such that $0 = t^1 < \cdots < t^{N_t} = T$.

Let y^j be the discrete approximation to $y(t^j)$, $j = 1, \ldots, N_t$. Then the MCN scheme reads as follows

$$y^{j+1} - y^j = \frac{\delta t}{4} \left[2A + \sum_{i=1}^{N} \left(u_i(t^{j+1}) + u_i(t^j)\right) B_i\right] \left(y^{j+1} + y^j\right), \tag{4.37}$$

$j = 1, \ldots, N_t - 1$, $\delta t = \frac{T}{N_t - 1}$ and the initial point $y^1 = y(0)$ is given.

We refer to [13] and reference therein for a more detailed analysis.

4.12 Reformulation of the Homicidal Chauffeur game

In this section, we derive (3.2) from (3.1). For this purpose, we consider the following equations of the pursuer's coordinates in the evader's reference system with an angle of θ measured clockwise from the y-axis.

$$\begin{aligned} x &= (x_e - x_p)\cos\theta - (y_e - y_p)\sin\theta, \\ y &= (y_e - y_p)\cos\theta + (x_e - x_p)\sin\theta. \end{aligned} \tag{4.38}$$

By differentiation of (4.38) we get

$$\begin{aligned} x' &= (x_e' - x_p')\cos\theta - (x_e - x_p)\sin\theta\,\theta' - (y_e' - y_p')\sin\theta - (y_e - y_p)\cos\theta\,\theta', \\ y' &= (y_e' - y_p')\cos\theta - (y_e - y_p)\sin\theta\,\theta' + (x_e' - x_p')\sin\theta + (x_e - x_p)\cos\theta\,\theta'. \end{aligned} \tag{4.39}$$

Next, by using the dynamics (3.1) we can replace the differences $x_e' - x_p'$, $y_e' - y_p'$ and θ' with the corresponding values. It holds

$$\begin{aligned} x' &= (v_1 - \sin\theta)\cos\theta - (x_e - x_p)\sin\theta\,u - (v_2 - \cos\theta)\sin\theta - (y_e - y_p)\cos\theta\,u, \\ y' &= (v_2 - \cos\theta)\cos\theta - (y_e - y_p)\sin\theta\,u + (v_1 - \sin\theta)\sin\theta + (x_e - x_p)\cos\theta\,u. \end{aligned} \tag{4.40}$$

By using the following geometric equations

$$\begin{aligned} x_e - x_p &= y\sin\theta + x\cos\theta, \\ y_e - y_p &= y\cos\theta - x\sin\theta, \end{aligned}$$

we obtain the system

$$\begin{aligned} x' &= v_1\cos\theta - (y\sin\theta + x\cos\theta)\sin\theta\,u - v_2\sin\theta - (y\cos\theta - x\sin\theta)\cos\theta\,u, \\ y' &= v_2\cos\theta - (y\cos\theta - x\sin\theta)\sin\theta\,u + v_1\sin\theta + (y\sin\theta + x\cos\theta)\cos\theta\,u - 1. \end{aligned} \tag{4.41}$$

Therefore, the following holds

$$\begin{aligned} x' &= v_1\cos\theta - v_2\sin\theta - yu, \\ y' &= v_2\cos\theta + v_1\sin\theta + xu - 1. \end{aligned} \tag{4.42}$$

Denoting with v_x and v_y the components of the velocity of the evader in this new reference system, equations (3.2) are obtained. To start the calculations the main file must be run.

4.13 Description of the MATLAB files

In this section, we describe the MATLAB files provided with this thesis, that we used in the numerical experiments performed in the previous chapters.

The file BilinearQuantumGame.zip contains the codes related to Section 2.3. In particular,

- The folder Nash contains the codes to get the Nash equilibrium for the bilinear quantum game. The main file is nash.m which implements the relaxation scheme combined with a semi-smooth Newton method. The problem parameters are set in Test2GL.m . The code is set to perform the second experiment of the application to a bilinear quantum game which leads to Table 2.2.

- The folder Bargaining contains the codes to determine the bargaining solution on the Pareto frontier. The main file is bargaining.m and the parameters are set in Test2GL.m. The code is set to perform the second experiment of the application to a bilinear quantum game. The file plot_NBP.m gives the NE strategies compared with the NB (Figure 2.4) and Figure 2.5.
- The folder Pareto contains the codes to construct the Pareto frontier. The main function in pareto.m which is set to get the second experiment of the quantum bilinear game. The parameters are given in Test2GL.m.
- The folder Plots collects the data of the performed numerical experiments. With the function plot_NBP.m, the figures related to the quantum games are obtained.

The file BilinearLotkaVolterraGame.zip contains the codes related to Section 2.4. In particular,

- The folder Nash contains the codes to get the Nash equilibrium for the Lotka-Volterra game. The main file is nash.m which implements the relaxation scheme combined with a semi-smooth Newton method. The problem parameters are set in Test2GL.m . The code is set to perform the experiment described in Section 2.4 which leads to the NE depicted in Figure 2.6.
- The folder Bargaining contains the codes to determine the bargaining solution on the Pareto frontier. The main file is bargaining.m and the parameters are set in Test2GL.m. The file plot_NBP.m gives the NE strategies compared with the NB (Figure 2.6) and Figure 2.7..
- The folder Pareto contains the codes to construct the Pareto frontier. The main function in pareto.m. The parameters are set in Test2GL.m.

The file HCgame.zip contains the codes related to Chapter 3. In particular, the file main.m contains the main code which is set to get Figure 3.1, obtained with the function game_plot.m. The parameters are given in parameters.m .

The file Code2DimLQ.zip contains the codes related to the first experiment of Section 4.3. The main function is SQH.m and it is set to get Figure 4.1, which is obtained by using the function game_plot.m.

The file Code2DimLQuconstr.zip contains the codes related to the second experiment of Section 4.3. The main function is SQH.m and it is set to get Figure 4.2, which is obtained by using the function game_plot.m.

The file Code2DimLQuconstrL1.zip contains the codes related to Section 4.4. The main function is SQH.m and it is set to get Figure 4.3, which is obtained by using the function game_plot.m.

The file Code2DimLQcomplete.zip contains the codes related to Section 4.5. The main function is SQH.m and it is set to get Figure 4.4, which is obtained by using the function game_plot.m.

The file SQHQuantum.zip contains the codes related to Section 4.6.1. The main function is game.m and it is set to get Figure 4.6.

The file SQHLV.zip containts the codes related to Section 4.6.2. The main function is game.m and it is set to get Figure 4.7.

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