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# Existence of weak solutions of diffuse interface models for magnetic fluids

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In this article we collect some recent results on the global existence of weak solutions for diffuse interface models involving incompressible magnetic fluids. We consider both the cases of matched and unmatched specific densities. For the model involving fluids with identical densities we consider the free energy density to be a double well potential whereas for the unmatched density case it is crucial to work with a singular free energy density.



## 1 Introduction to the model and existence results

In this article we present a summary of some recent results on the global existence of weak solutions of diffuse interface models involving two incompressible fluids with different magnetic properties. The results presented here are based on our recent research detailed in [2,3]. To the best of our knowledge there are only a few articles so far dealing with diffuse interface magnetic fluids. The models we present here (see also [2, 3]) differ from the ones introduced in [5, 6] for modeling diffuse interface for magnetic fluids.

Concerning the fluids undergoing partial mixing, we consider two cases: $(a)$ the fluids have identical densities and $(b)$ the fluids have unmatched specific densities.

We will use a unified set up to introduce the models corresponding to the two cases mentioned above. The fluids undergoing partial mixing are confined in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with the boundary $\partial\Omega$ of class $C^2$. Let $Q_T = \Omega \times (0, T)$ be the space time cylinder for a fixed final time $T > 0$. The lateral boundary of $Q_T$ is denoted by $\Sigma_T = \partial\Omega \times (0, T)$.

We use an order parameter (which might be considered as the difference of the volume fractions of the fluids involved) $\phi : Q_T \to \mathbb{R}$ to describe the partial mixing of the two fluids involved. We denote by $v : Q_T \to \mathbb{R}^d$ the mean fluid velocity, by $\rho = \rho(\phi) : Q_T \to \mathbb{R}$ the mean mass density, $p : Q_T \to \mathbb{R}$ the pressure, $M : Q_T \to \mathbb{R}^3$ the magnetization and $\mu : Q_T \to \mathbb{R}$ the chemical potential. The diffuse interface model we consider reads as follows

$$\left\{\begin{aligned} &\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) - \mathrm{div}(2\nu(\phi)\mathrm{D}(v)) + \mathrm{div}(v \otimes J) + \nabla p \\ &\qquad\qquad = \mu\nabla\phi + \frac{\xi(\phi)}{\alpha^2}((|M|^2 - 1)M)\nabla M - \mathrm{div}(\xi(\phi)\nabla M)\nabla M && \text{in } Q_T, \\ &\mathrm{div} v = 0 && \text{in } Q_T, \\ &\partial_t M + (v \cdot \nabla)M = \mathrm{div}(\xi(\phi)\nabla M) - \frac{\xi(\phi)}{\alpha^2}(|M|^2 - 1)M && \text{in } Q_T, \\ &\partial_t\phi + (v \cdot \nabla)\phi = \Delta\mu && \text{in } Q_T, \\ &\mu = -\eta\Delta\phi + \Psi'(\phi) + \xi'(\phi)\frac{|\nabla M|^2}{2} + \frac{\xi'(\phi)}{4\alpha^2}(|M|^2 - 1)^2 && \text{in } Q_T, \\ &v = 0,\ \partial_n M = 0,\ \partial_n\phi = \partial_n\mu = 0 && \text{on } \Sigma_T, \\ &(v, M, \phi)(\cdot, 0) = (v_0, M_0, \phi_0) && \text{in } \Omega. \end{aligned}\right. \tag{1}$$

where $J$ is a relative flux related to the diffusion of the components and is defined as follows

$$J = -\frac{\widetilde{\rho}_2 - \widetilde{\rho}_1}{2}\nabla\mu. \tag{2}$$

In (2), $\widetilde{\rho}_i$ $(i = 1, 2)$ denotes the specific density of the $i-$th fluid and the mean mass density $\rho$ is related to the phase field $\phi$ via

$$\rho(\phi) = \frac{1}{2}(\widetilde{\rho}_1 + \widetilde{\rho}_2) + \frac{1}{2}(\widetilde{\rho}_2 - \widetilde{\rho}_1)\phi \text{ in } \overline{Q}_T. \tag{3}$$

The factor $\alpha > 0$ penalizes the saturation condition of the length of the magnetization vector $|M|$ from 1 and $\eta > 0$ measures the thickness of the region where the two fluids mix. In our case the viscosity coefficient of the mixture $\nu(\phi)$ is concentration dependent and $\mathrm{D}(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^\top\right)$ is the symmetric velocity gradient. The function $\xi(\phi)$ denotes the exchange parameter reflecting the tendency of the magnetization to align in one direction. The free energy density of the fluid mixture

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is denoted by $\Psi(\phi)$. We consider two different kinds of potentials $\Psi(\phi)$ and we will specify them later. The derivation of the model from the total energy of the system can be found in [2] (the matched density case) and [3] (the case of unmatched density). We specify that the energetic variational approach used in our modeling is motivated from [4] and the density dependence is inspired from [1].
From now on we assume the following concerning the boundedness and non-degeneracy of $\xi(\cdot)$, $\xi'(\cdot)$ and $\nu(\phi)$ :

**Assumption 1.1** The function $\xi \in C^1(\mathbb{R})$ satisfies

$$\begin{aligned} &0 < c_1 \leq \xi \leq c_2 \text{ on } \mathbb{R}, \text{ for some } c_1, c_2 > 0,\\ &\xi' \leq c_3 \text{ on } \mathbb{R}, \text{ for some } c_3 > 0. \end{aligned} \tag{4}$$

The viscosity coefficient $\nu \in C^1(\mathbb{R})$ satisfies

$$0 < \nu_1 \leq \nu \leq \nu_2 \text{ on } \mathbb{R}, \text{ for some } \nu_1, \nu_2 > 0. \tag{5}$$

First we present our result on the global existence of weak solutions for the model (1) in case of matched densities.

**Theorem 1.2** *[2, Theorem 2.2] Let $T > 0$, $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^2$, the fluids involved have matched specific densities i.e. $\widetilde{\rho}_1 = \widetilde{\rho}_2 = 1$ (which implies $J = 0$ and $\rho = 1$ via* (2) *and* (3) *respectively), the free energy density $\Psi(\phi)$ is of double well shape and is given by $\Psi(\phi) = \frac{1}{4\eta}(|\phi|^2 - 1)^2$ for $\eta > 0$, the assumptions in* (4) *and* (5) *hold and the initial data*

$$(v_0, M_0, \phi_0) \in L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega) \tag{6}$$

*be given. Then there exits a weak solution to* (1)–(2)–(3) *in the functional framework*

$$\begin{aligned} v &\in C_w\big([0,T]; L^2_{\mathrm{div}}(\Omega)\big) \cap L^2\big(0,T; W^{1,2}_{0,\mathrm{div}}(\Omega)\big),\\ M &\in C_w\big([0,T]; W^{1,2}(\Omega)\big) \cap C^0\big([0,T]; L^2(\Omega)\big) \cap W^{1,2}\big(0,T; L^{\frac{3}{2}}(\Omega)\big),\\ \phi &\in C_w\big([0,T]; W^{1,2}(\Omega)\big) \cap C^0\big([0,T]; L^2(\Omega)\big), \quad \mu \in L^2\big(0,T; W^{1,2}(\Omega)\big), \end{aligned} \tag{7}$$

*where*

$$L^2_{\mathrm{div}}(\Omega) = \overline{\{v \in C^\infty_c(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega}^{\|\cdot\|_{L^2}} \quad \text{and} \quad W^{1,2}_{0,\mathrm{div}}(\Omega) = \overline{\{v \in C^\infty_c(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega}^{\|\cdot\|_{W^{1,2}}} \tag{8}$$

*and for some Banach space $X$, $C_w([0,T]; X)$ denotes a subspace of $L^\infty(0,T; X)$ containing functions $f$ for which the mapping $t \mapsto \langle \phi, f(t) \rangle$ is continuous on $[0,T]$ for each $\phi \in X'$.*

Our next result concerns the global existence of weak solutions for the model (1) for the case of unmatched densities.

**Theorem 1.3** *[3, Theorem 1.1] Let $T > 0$, $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^2$ and let the initial datum $(v_0, M_0, \phi_0)$ satisfy* (6) *and $|\phi_0| \leq 1$. Further suppose that the free energy density $\Psi \in C([-1,1]) \cap C^2((-1,1))$ solves*

$$\lim_{s \to -1} \Psi'(s) = -\infty, \quad \lim_{s \to 1} \Psi'(s) = \infty \text{ and } \Psi''(s) \geq -\kappa \text{ for some } \kappa \in \mathbb{R} \tag{9}$$

*and the mean density is given by* (3). *Then under the assumptions* (4) *and* (5) *there exists a quadruple $(v, M, \phi, \mu)$ which solves* (1)–(2)–(3) *in the functional framework* (7) *and $|\phi| \leq 1$ on $Q_T$. Moreover there exists a $p > 2$ such that the triplet $(M, \phi, \Psi'(\phi))$ enjoys the following improved regularity*

$$M \in L^2\big(0,T; W^{1,p}(\Omega)\big), \quad \phi \in L^2\big(0,T; W^{2,\frac{2p}{p+2}}(\Omega)\big) \quad \text{and} \quad \Psi'(\phi) \in L^2\big(0,T; L^{\frac{2p}{p+2}}(\Omega)\big). \tag{10}$$

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