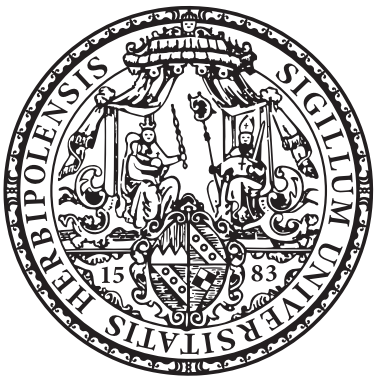
On the rôle of entanglement in quantum field theory

Julius-Maximilians-Universität Würzburg
Fakultät für Physik und Astronomie



Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades der Julius-Maximilians-Universität Würzburg

vorgelegt von

Pascal Fries
aus Würzburg

Würzburg, 2020

Eingereicht am ..... bei der Fakultät für Physik und Astronomie.

1. Gutachter: Prof. Dr. Haye Hinrichsen .....

2. Gutachter: Prof. Dr. Thorsten Ohl .....

3. Gutachter: .....

der Dissertation.

Vorsitzende(r): .....

1. Prüfer: Prof. Dr. Haye Hinrichsen .....

2. Prüfer: Prof. Dr. Thorsten Ohl .....

3. Prüfer: .....

im Promotionskolloquium.

Tag des Promotionskolloquiums: .....

Doktorurkunde ausgehändigt am: .....

to Larissa

Half of the people can be part right all of the time
Some of the people can be all right part of the time
But all of the people can't be all right all of the time
I think Abraham Lincoln said that
"I'll let you be in my dreams if I can be in yours"
I said that

Bob Dylan

Abstract

In this thesis, I study entanglement in quantum field theory, using methods from operator algebra theory. More precisely, the thesis covers original research on the entanglement properties of the free fermionic field.

After giving a pedagogical introduction to algebraic methods in quantum field theory, as well as the modular theory of Tomita-Takesaki and its relation to entanglement, I present a coherent framework that allows to solve Tomita-Takesaki theory for free fermionic fields in any number of dimensions. Subsequently, I use the derived machinery on the free massless fermion in two dimensions, where the formulae can be evaluated analytically. In particular, this entails the derivation of the resolvent of restrictions of the propagator, by means of solving singular integral equations. In this way, I derive the modular flow, modular Hamiltonian, modular correlation function, Rényi entanglement entropy, von-Neumann entanglement entropy, relative entanglement entropy, and mutual information for multi-component regions. All of this is done for the vacuum and thermal states, both on the infinite line and the circle with (anti-)periodic boundary conditions. Some of these results confirm previous results from the literature, such as the modular Hamiltonian and entanglement entropy in the vacuum state. The non-universal solutions for modular flow, modular correlation function, and Rényi entropy, however are new, in particular at finite temperature on the circle. Additionally, I show how boundaries of spacetime affect entanglement, as well as how one can define relative (entanglement) entropy and mutual information in theories with superselection rules.

The findings regarding modular flow in multi-component regions can be summarised as follows: In the non-degenerate vacuum state, modular flow is multi-local, in the sense that it mixes the field operators along multiple trajectories, with one trajectory per component. This was already known from previous literature but is presented here in a more explicit form. In particular, I present the exact solution for the dynamics of the mixing process. What was not previously known at all, is that the modular flow of the thermal state on the circle is infinitely multi-local even for a connected region, in the sense that it mixes the field along an infinite, discretely distributed set, of trajectories. In the limit of high temperatures, all trajectories but the local one are pushed towards the boundary of the region, where their amplitude is damped exponen-

tially, leaving only the local result. At low temperatures, on the other hand, these trajectories distribute densely in the region to either—for anti-periodic boundary conditions—cancel, or—for periodic boundary conditions—recover the non-local contribution due to the degenerate vacuum state.

Proceeding to spacetimes with boundaries, I show explicitly how the presence of a boundary implies entanglement between the two components of the Dirac spinor. By computing the mutual information between the components inside a connected region, I show quantitatively that this entanglement decreases as an inverse square law at large distances from the boundary. In addition, full conformal symmetry (which is explicitly broken due to the presence of a boundary) is recovered from the exact solution for modular flow, far away from the boundary. As far as I know, all of these results are new, although related results were published by another group during the final stage of this thesis.

Finally, regarding relative entanglement entropy in theories with superselection sectors, I introduce charge and flux resolved relative entropies, which are novel measures for the distinguishability of states, incorporating a charge operator, central to the algebra of observables. While charge resolved relative entropy has the interpretation of being a "distinguishability per charge sector", I argue that it is physically meaningless without placing a cutoff, due to infinite short-distance entanglement. Flux resolved relative entropy, on the other hand, overcomes this problem by inserting an Aharonov-Bohm flux and thus passing to a variant of the grand canonical ensemble. It takes a well defined value, even without putting a cutoff, and I compute its value between various states of the free massless fermion on the line, the charge operator being the total fermion number.

Zusammenfassung

In dieser Dissertation untersuche ich quantenmechanische Verschränkung mittels Methoden aus Theorie der Operatoralgebren. Genauer gesagt stelle ich eigene Forschung über die Verschränkungseigenschaften des freien Fermions vor.

Die Arbeit beginnt mit einer pädagogischen Einführung in algebraische Quantenfeldtheorie und stellt die modulare Theorie nach Tomita und Takesaki, sowie ihre Verbindung zu Verschränkung vor. Darauffolgend stelle ich einen vollständigen Satz an Werkzeugen vor, mit dem Tomita-Takesaki-Theorie für freie fermionische Felder in beliebiger Anzahl von Dimenionen gelöst werden kann. Diese Werkzeuge wende ich dann auf das freie, masselose Dirac-Fermion in zwei Dimensionen an, wo die hergeleiteten Formeln exakt gelöst werden können. Dies beinhaltet insbesondere die Herleitung der Resolvente von Einschränkungen des Propagators mittels der analytischen Lösung singulärer Integralgleichungen. Daraus ergeben sich schließlich der modulare Fluss, der modulare Hamiltonian, der modulare Korrelator, Rényi Verschränkungsentropien, von-Neumann Verschränkungsentropien, relative Verschränkungsentropie und Transinformation für nicht-zusammenhängende Verschränkungsgebiete. Dies alles wird im Vakuum und bei endlicher Temperatur ausgearbeitet, für ein Fermion sowohl auf der Geraden, als auch auf dem Kreis mit (anti-)periodischen Randbedingungen. Einige der Ergebnisse, besipielsweise der modulare Hamiltonian und von-Neumann Verschränkungsentropie, bestätigen Resultate aus bereits existierender Literatur. Die nicht-universellen Lösungen für den modularen Fluss, den modularen Korrelator und die Rényi Verschränkungsentropie dagegen sind neu, insbesondere für den Fall des thermischen Zustandes auf dem Kreis. Zusätzlich demonstriere ich den Einfluss von Rändern der Raumzeit auf Verschränkung und zeige, wie man relative Entropie und Transinformation in Theorien mit Superselektionsregeln definieren kann.

Die Ergebnisse bezüglich modularen Flusses in nicht-zusammenhängenden Gebieten lassen sich wie folgt zusammenfassen: Im nicht-entarteten Vakuum ist der modulare Fluss multi-lokal, was bedeutet, dass er Feldoperatoren entlang mehrerer Trajektorien – eine pro Zusammenhangskomponente der Region – untereinander vermischt. Dies war bereits vorher bekannt, allerdings folgt es sich hier in expliziter Form aus exakten Lösungen. Ein vollkommen neues Ergbnis ist, dass der modulare Fluss des thermischen Zustandes auf

dem Kreis sogar für zusammenhängende Regionen multi-lokal ist: Er mischt Feldoperatoren entlang unendlich vieler, diskret verteilter Trajektorien in der Verschränkungsregion. Im Hochtemperaturgrenzwert befinden sich alle diese Trajektorien, bis auf die lokale, nahe am Rand der Region, wo ihre Amplitude exponentiell gedämpft wird – es bleibt nur die lokale Lösung. Bei tiefen Temperaturen dagegen sind die Trajektorien dicht in der Region verteilt, sodass sie entweder (bei antiperiodischen Randbedingungen) sich durch destruktive Interferenz gegenseitig aufheben oder (bei periodischen Randbedingungen) durch konstruktive Interferenz einen nicht-lokalen Term erzeugen, der auf das entartete Vakuum zurückgeführt werden kann.

Im Falle von Raumzeiten mit Rand zeige ich explizit, wie der Rand Verschränkung zwischen beiden Komponenten des Dirac-Spinors impliziert. Mit zunehmdendem Abstand vom Rand nimmt diese Verschränkung invers quadratisch ab, wie ich quantitativ durch Berechnung der Transinformation zwischen den Komponenten in einem zusammenhängenden Gebiet zeige. Zusätzlich lässt sich die volle konforme Symmetrie der Theorie (die durch den Rand explizit gebrochen wird) aus der exakten Lösung für den modularen Fluss wiederherstellen, indem man den Grenzwert eines weit entfernten Randes betrachtet. Meines Wissens sind alle diese Resultate neu, allerdings wurden während der Fertigstellung dieser Dissertation verwandte Ergebnisse von einer anderen Arbeitsgruppe veröffentlicht.

Die letzten Resultate in dieser Arbeit beziehen sich auf die Untersuchung relativer Entropie in Systemen mit Superselektionsregeln. Hier führe ich neue informationstheoretische Maße für die Unterscheidbarkeit von Zuständen ein: Die ladungs- und flussbezogenen relativen Entropien. Beide werden mittels eines Ladungsoperators aus dem Zentrum der Observablenalgebra definiert. Während die ladungsbezogene relative Entropie sich physikalisch als „Unterscheidbarkeit pro Ladungssektor“ interpretieren lässt, argumentiere ich, dass sie nur innerhalb eines Regularisierungsschemas physikalisch bedeutsam ist, da die universell unendliche Verschränkung auf kurzen Längenskalen sonst zu Widersrpüchen führt. Flussbezogene relative Entropie dagegen hat dieses Problem nicht: Durch das Hinzufügen eines Aharonov-Bohm-Flusses betrachtet man hier eine lokale Variante des großkanonischen Ensembles, wodurch sie sich auch ohne Regularisierung definieren und berechnen lässt. Ich berechne ihren Wert zwischen verschiedenen Zuständen des freien masselosen Fermions auf der Geraden. Die erhaltene Ladung ist hierbei die Gesamtzahl der Fermionen im System.

Contents

1 Introduction

Since its foundations in the early twentieth century, quantum physics has perplexed the minds of probably everyone who came across it. Appearing at scales much smaller than we are accustomed to deal with naturally, its probabilistic laws dictate that some properties of a system can not be measured simultaneously with arbitrary precision[1] and that you can not perform any measurement at all without disturbing the state of said system.[2] While such peculiarities are, of course, in conflict with our human intuition, quantum physics is not witchcraft nor wizardry – already in 1925, Heisenberg discovered[3] that it can be formulated axiomatically as a theory where the observables are *non-commutative*.

This insight paved the way for our modern understanding of quantum mechanics, although some implications of the axioms were not yet clear. In particular, as was famously argued by Einstein, Podolsky, and Rosen (EPR),[4] the non-commutative formulation allows for states showing a novel kind of correlation: In their proposed thought experiment, they constructed a paradoxical situation in which the information about the state of a system is not "stored" locally, in the sense that you can not determine the state of a subsystem without disturbing its environment. To cite them directly, they were "thus led to conclude that the description of reality as given by a wave function is not complete". Here, by "complete", the authors mean locally complete in the above sense. Soon after, these EPR-type correlations were baptised *entanglement* by Schrödinger[5] but it was still unknown, whether they were a *real physical phenomenon* or just an absurd artefact of a theory that needs to be fixed, as suggested by EPR. It took a long time until, almost thirty years later, Bell[6] proved mathematically that it was possible to detect entanglement experimentally and discriminate it from classical correlations and experimental evidence was given a few years later by Freedman and Clauser.[7] Since then, entanglement has become an integral part of our understanding of quantum physics, in particular in low-dimensional systems that are relevant in quantum information processing.[8]

All of this was initially done for theories involving a *fixed, finite number of particles*, which are easily tractable in full detail in Heisenberg's or Schrödinger's[9] formalism. The development of a quantum version of the theory of fields, a *quantum field theory* (QFT), happened somewhat independently

and was limited, for a long time, to exactly solvable models that we now call free theories.[10–14] Furthermore, due to the infinite number of degrees of freedom in a QFT, studying entanglement by the usual methods is difficult in this framework. Nevertheless, as was shown by Reeh and Schlieder in 1961,[15] entanglement is a universal feature in QFT and is the source of particle fluctuations in the vacuum. As a result, entanglement is at the heart of phenomena such as black hole evaporation through Hawking radiation[16] and the Unruh effect.[17]

The omnipresence of entanglement in quantum physics is also a key ingredient in our current understanding of quantum gravity: As first proposed by Maldacena in what is now the most cited paper in high energy physics of all time,[18] a weakly coupled theory of quantum gravity (supergravity) in an Anti-de-Sitter background is equivalent to a strongly coupled conformal field theory ($\mathcal{N} = 4$ supersymmetric SU(N)-Yang-Mills theory at large N) on the flat conformal boundary. This results in a *holographic dictionary* that relates between quantities in the "bulk" and "boundary" theory. In particular, (massive) minimal surfaces in the bulk are dual to (Rényi) entanglement entropies in the boundary.[19–21] Even more so: The ever increasing number of discovered relationships between entanglement and geometry[22–35] indicates that the study of entanglement in QFT, even in flat spacetime, might pave the way to a quantum theory of gravity.[36,37]

The question is thus: What do we know about entanglement in QFT? Unfortunately, the short answer is: Not much. Apart from a few universal results,[38,39] the entanglement properties of a QFT depend crucially on the details (Hilbert space, observables, symmetries, ...) of the theory under consideration. In addition, the above mentioned quantum Yang-Mills theories are still poorly understood non-perturbatively, in the sense that we do not even know how to rigorously define, let alone study them.[40] Instead, I think it makes sense to step back a bit and consider entanglement in theories that are better behaved, in order to obtain results that might serve as a guidance on the way to more involved challenges.

Therefore, in this thesis, I study in detail the entanglement properties of one of the oldest QFTs – the free Dirac fermion. As you might have noticed already, this thesis is written in the style of a very long research paper, albeit slightly more personal. This is because it is intended to serve two purposes: First, it contains details to and extensions of the results in the research papers[41–43] that I was working on in my time as a PhD student, as well as preliminary results of works in progress.[44,45] Secondly, it contains a pedagogical introduction to the methods used in the study of entanglement in QFT, a topic that is generally difficult to access due to the sparsity of "physicist-friendly" literature.[46]

Finally, I would like to mention that, during my time as a PhD student, I

also worked on applications of signal processing methods in QFT,[47] continuing previous work from my time as a master's student of our group.[48] This work is not included in this thesis, in order to stick with one, coherent topic.

1.1 Structure and main results of this thesis

In order to keep the presentation self-contained, I give a short pedagogical summary of the algebraic formulations of QFT and entanglement in section 1.2. Of course, this is not intended as a thorough introduction to these fields of research – nevertheless this section may give a somewhat unusual point of view on things and I hope it helps in building an intuition for the *meaning* of entanglement.

Chapter 2 presents the main body of this thesis. It starts in with an introduction to the modular theory of Tomita and Takesaki section 2.1.2, providing a physics-oriented approach to the field rather than the usual abstract mathematical treatment. It is this framework that I will use in the study of entanglement in QFT. To get a better feeling for it, in section 2.2, I rederive previously known universal results that can be related to geometric symmetries of the theory. Again, I try to keep the explanations as physical as possible. Section 2.3 then specialises to free fermions and introduced the mathematical machinery that is needed in order to solve Tomita-Takesaki theory in the free fermion QFT. Most of the formulae there were previously known – the main objective of this section is to put them into a coherent framework and derive the new eqs. (2.47) and (2.49), which we introduced in our most recent paper.[43] The formulae derived in this section are summarised in tables 2.1 and 2.2 for later reference.

Building on this machinery, section 2.4 presents the derivation and extensions of non-universal results in free fermion entanglement: Sections 2.4.1, 2.4.2 and 2.4.5 summarise and extend the results from our paper on the modular Hamiltonian on the circle at finite temperature.[41] This includes the solution of a singular integral equation, which is of fundamental importance to all results in this thesis. The solutions of this equation for different boundary conditions are summarised in table 2.3 for later reference. Section 2.4.6 summarises our work on entanglement entropy[42] and includes a new extension of the results to Rényi entanglement entropy. Sections 2.4.3 and 2.4.4 gives detailed derivations of the results in our paper on the implementation of modular flow and the Kubo-Martin-Schwinger condition.[43] In particular, section 2.4.4 contains a new, detailed discussion of the analyticity structure of the modular correlation function. Section 2.4.7 extends the previous results to free fermions on a spacetime with boundaries – the results there are completely new and not yet

published.[45] Finally, in section 2.5, I introduce (and evaluate) a novel notion of relative entanglement entropy that is adapted to the study of entanglement in the presence of conserved charges. These results are also not published yet and present a growing work in progress.[44]

The main results of this thesis are as follows: As I will show in eqs. (2.137) and (2.149), modular flow of a free fermion is not a *local*, geometric symmetry but rather *multi-local*, in the sense that it mixes the fields between discretely distributed trajectories. In the non-degenerate vacuum state and the thermal state on the line, these trajectories are distributed as "one per component" of the entangling region (compare eq. (2.137)). At finite temperature on the circle, however, we find an infinity of coupled trajectories within each component (compare eq. (2.149)), which "condense" at low temperatures to a completely non-local flow if the vacuum is degenerate. Furthermore, this picture extends naturally to spacetimes with a boundary by virtue of the boundary condition (2.100): Here, modular flow of a single interval couples between the two chiralities along a local trajectory. While this trajectory is not given by a conformal transformation, it approaches one when the boundary is very far away, so that it yields a way to recover full conformal symmetry. The amount of entanglement between the two chiralities is quantified by eq. (2.205). Finally, as shown in eqs. (2.233) and (2.234), we will find that adding an *Aharonov-Bohm flux*—a local flux corresponding to a conserved charge—to the entangling region, will always *increase* the amount of correlations between the different components of the region (or the different chiralities, in the presence of a boundary), indicating that vacuum correlations are unevenly distributed with respect to the different superselection sectors.

I conclude in chapter 3, with a discussion and physical interpretation of the main results.

1.2 Initial questions

1.2.1 What is theoretical physics?

Before we get going, I would like discuss what I think theoretical physics is all about. The line of thought presented in this subsection is strongly inspired by the great works of von Neumann[2] and Dirac.[49] There are several reasons why I think that this motivation belongs here: Firstly, I feel like whenever we want to talk about a subject in scientific terms, we should properly define what this subject is about. Secondly, during both my undergraduate and graduate studies, the question about *what physics actually is* was always answered differently by different people, depending on individual own experiences in their

respective field. Finally, I believe that this diversity of opinions is key to discovering the beauty of physics – depending on what physics means to you, different aspects of it will be more exciting to you than to someone else. My hope for this thesis is to convey to you my perspective of physics, so that you will hopefully agree that entanglement is one of the most exciting things imaginable.

So, what is theoretical physics? In my opinion, theoretical physics comprises of the mathematical analysis of physical models, each of which usually contains at least three crucial elements:

1. **Observables** – Every physical model should clearly state a collection of observables, i.e., things that *can be measured*. In classical mechanics,[50] these can be arbitrary functions on phase space, such as energy, velocity, angular momentum etc. In quantum physics,[2,49,51] the observables are usually (though not necessarily[52–55]) given as self adjoint operators on a Hilbert space $\mathcal{H}$, examples are the position and momentum operators in quantum mechanics or current operators in quantum field theory. In statistical mechanics, they are random variables on a sample space, e.g., the magnetisation of an Ising magnet.[56]

   Note that, in all these examples, the collections of observables always have the structure of an (associative) algebra. Whether this algebra is commutative or not marks the difference between classical and quantum physics. Note that the variables which we use in our description of a physical system need not be observables themselves – a phenomenon which is called gauge invariance. For example, the Riemannian metric $g_{\mu\nu}$ in general relativity is not an observable, because it is not invariant under the gauge group of diffeomorphisms. On a final note, there are physical models, where the collection of observables do not form an algebra: Prominent examples are large N gauge theories and effective field theories, where we choose to consider only observables on a specific energy scale. Taking arbitrary powers of such observables, we might exit said scale, hence we are not dealing with an algebra. This was most brilliantly explained by Papadodimas and Raju,[57,58] where it was proposed as a starting point for the resolution of the black hole information paradox.[16,59,60]

2. **States** – Starting from a collection of observables, we need to know how to generate *measurement results* a.k.a. *expectation values* by specifying which state the system is in. In classical mechanics, a state can be given by a point $(q, p) \in T^*Q$ in phase space. An observable $f \in C^\infty(T^*Q)$ can then be evaluated to yield the measured value $f(q, p)$. More generally, we

can consider an arbitrary probability measure P on phase space, whose corresponding expectation value is

$$\langle f \rangle_P = \int \mathrm{d}P(q,p)\, f(p,q).$$

Note that the choice of a point in phase space can be recovered from this by taking P to be a Dirac (point) measure. These states are special in the sense that they can not be written as a convex combination[1] of different states, hence we call them *pure*.[61] In contrast, all states that can be written as a convex combination are called *mixed*.

In quantum physics, pure states are usually given by vectors $\Omega \in \mathcal{H}$ in the Hilbert space of unit norm $\langle \Omega | \Omega \rangle = 1$ and an expectation value of the observable A is computed by evaluating the inner product $\langle \Omega | A\Omega \rangle$. Since this expression is *quadratic* in the state, rather than linear as in the classical case, it has the meaning of an overlap $\langle \Omega | \tilde{\Omega} \rangle$ between different states Ω and $\tilde{\Omega}$. This overlap, or rather its absolute value, can also be interpreted as a *transition amplitude*[51] and is a unique feature of quantum theories. Mixed states, on the other hand, are positive trace class operators $\rho > 0$ with unit trace $\mathrm{Tr}[\rho] = 1$, commonly referred to as *density matrices*. Expectation values are generated by taking the trace

$$\langle A \rangle_\rho = \mathrm{Tr}[\rho A]$$

and we can consider pure states as a special case of this by assigning to them the rank one projection density matrix $|\Omega\rangle\langle\Omega|$.

Of course, depending on the particular implementation of the observables, states of a model always have to be chosen accordingly. For example, in gauge theories, observables have several equivalent representations as phase space functions or operators, each for a different choice of gauge. To compensate for this redundancy, we have to require that expectation values agree on these equivalence classes, so that they only depend on the choice of observables. In the quantum case, this leads to the discrimination between the kinematical Hilbert space, which supports the operators, and a physical Hilbert space, on which all gauge equivalent operators agree.[62]

3. **Symmetries** – So far, I listed *static* features of physical systems without any relation to one another. The symmetries of a system now give a way

[1] We say that P is a convex combination of Q and R, if we can write $P = \lambda Q + (1-\lambda)R$ with $0 < \lambda < 1$.

to transform different observables into one another. For example, in classical mechanics, translations $q \to q + \alpha$ are trivially implemented on the observables by transforming $f(q, p) \to f(q + \alpha, p)$.

However, not all possible transformations deserve to be called a symmetry: The general practice is to single out a specific set of transformations which are *defined to be* the fundamental symmetries. One then requires all other symmetries to be compatible with them, meaning that they have to commute with one another.[51] In almost all cases, these fundamental symmetries consist of *time translations*, possibly augmented to the full Galilean (non-relativistic physics) or Poincaré (relativistic physics) group. The implementation of these symmetries, i.e., their action on observables, is part of the definition of the model and given by the *equations of motion*. Since additional symmetries are required to commute with the fundamental symmetries, equivalently, you can define them as those transformations that leave invariant the equations of motion.[2,49]

Since observables and states pair up expectation values, we can require invariance of this pairing under symmetries and dually think of symmetries as transformations of states. In quantum mechanics, for example, instead of working in the Heisenberg picture, which describes the time evolution of operators, one can equivalently work in the Schrödinger picture, where instead states evolve in time. Additionally, one then requires the aforementioned transition amplitudes $\langle\Omega|\tilde{\Omega}\rangle$ to be invariant under symmetries, which implies that symmetries are necessarily implemented as (anti-)unitary operators on the Hilbert space $\mathcal{H}$.[63] Furthermore, the action of symmetries on states allows to search for *invariant* states and minimum energy states that *spontaneously break* a symmetry.[64]

Finally, I would like to add that there are physical theories, where the symmetries do not form a group. For example, in statistical mechanics[56] or its non-commutative variant, open quantum systems,[65] time evolution is not reversible, hence, only forms a semi-group. As it turns out, this means that transition amplitudes are not invariant either, allowing *non-unitarity*, *decoherence*, and *thermalisation*.[66,67]

Of course, these three constituents are not enough to cover every physical theory, nor does every physical model allow for a strict separation of them. However, I found that thinking about physics in terms of the three pillars *observables*, *states*, and *symmetries* helps clearly recognising similarities and differences between different theories.

1.2.2 What is quantum field theory?

Let us now be a little bit more specific and wander into the realm of QFT. To do this, I will use the approach by Haag and Kastler,[64,68] since it fits very nicely in the above discussion. This approach builds on the historical vision of Faraday: The idea of fields as *localised* observables. In fact, the founders of this framework baptised it *local quantum physics* to highlight the key rôle of locality.

The axioms of this approach are usually given in a rather formal fashion, while I would like to elaborate a bit on their physical meaning. To keep the discussion brief, I will restrict to QFT in d+1 dimensional Minkowski spacetime and only mention that variants exist for QFT in curved spacetime,[69] spacetime with boundaries,[70] and additional symmetries.[71] The key axioms are:

1. **Isotony** – This axiom is concerned with the localisation of observables. To each open subregion $\mathcal{R}$ of spacetime, we associate a C^*-algebra $\mathcal{A}(\mathcal{R})$ of observables that are *localised* within $\mathcal{R}$. Additionally, we require that for any two regions

$$\mathcal{R} \subset \tilde{\mathcal{R}}, \quad \text{there exists an inclusion} \quad \mathcal{A}(\mathcal{R}) \to \mathcal{A}(\tilde{\mathcal{R}}), \tag{1.1}$$

   implying that whenever an obervable is localised in $\mathcal{R}$, it is also localised in the bigger $\tilde{\mathcal{R}}$. In mathematical terms, this is equivalent to saying that the assignment $\mathcal{R} \mapsto \mathcal{A}(\mathcal{R})$ is required to be a *net*, indexed by spacetime regions. For the sake of brevity, I will often omit the argument of $\mathcal{A}$ if the region in question is clear from context or not relevant.

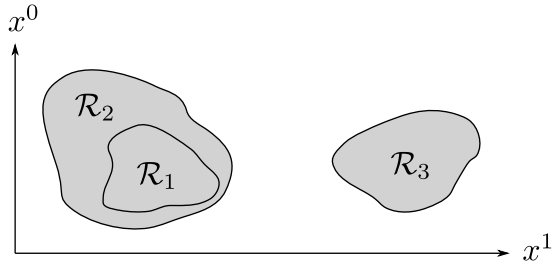


Figure 1.1: Different regions in spacetime and their relevance in AQFT: Isotony states that $\mathcal{A}(\mathcal{R}_1) \subset \mathcal{A}(\mathcal{R}_2)$ since $\mathcal{R}_1 \subset \mathcal{R}_2$. Einstein causality means that $[\mathcal{A}(\mathcal{R}_2), \mathcal{A}(\mathcal{R}_3)]$ since $\mathcal{R}_2$ and $\mathcal{R}_3$ have spacelike separation.

   Note that we do not require $\mathcal{A}$ to be an algebra of operators on a Hilbert space, nor do we require the existence of a Hilbert space at all. The

only technical requirement that we put on $\mathcal{A}$ is that it is a C^*-algebra. Among other things, this entails that every observable $A \in \mathcal{A}$ has finite norm $\|A\| < \infty$, i.e., is *bounded*, as every physical observable should be.

2. **Einstein causality** – This axiom makes sure that no information can travel faster than light. It is implemented at the level of observables and states that the algebras of spacelike separated regions (see fig. 1.1) commute. To write as a formula, please note that throughout this thesis, I will use the *mostly negative* convention for the Minkowski metric

$$(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, ..., -1). \tag{1.2}$$

Einstein causality then means that

$$[\mathcal{A}(\mathcal{R}), \mathcal{A}(\tilde{\mathcal{R}})] = 0 \quad \text{if} \quad \|x - \tilde{x}\|^2 < 0 \quad \text{for all} \quad x \in \mathcal{R}, \tilde{x} \in \tilde{\mathcal{R}}. \tag{1.3}$$

Physically, this means that measurements at spacelike separated points can not interfere with one another.

3. **Covariance** – This is where symmetry comes into play. We require that the Poincaré group $\mathfrak{P} = \mathrm{O}(1, d) \ltimes \mathbb{R}^{1,d}$, consisting of relativistic rotations and translations, acts on the algebras in a consistent way. Namely, we require

$$g \triangleright \mathcal{A}(\mathcal{R}) = \mathcal{A}(g\mathcal{R}) \quad \text{for all} \quad g \in \mathfrak{P}. \tag{1.4}$$

Furthermore, this group action allows to forbid non-local observables by requiring *additivity* of the local algebras: For any region $\mathcal{R}$, the global algebra $\mathcal{A}(\mathbb{R}^{1,d})$ should be coverable by the translations of $\mathcal{A}(\mathcal{R})$, i.e.,

$$\mathcal{A}(\mathbb{R}^{1,d}) = \bigvee_{v \in \mathbb{R}^{1,d}} \Big[v \triangleright \mathcal{A}(\mathcal{R}) \Big], \tag{1.5}$$

where $\bigvee$ denotes the union of C^*-algebras.

4. **The vacuum** – Finally, we talk about states. In this algebraic language, a state ω is a positive, normalised functional on the algebras, i.e.,

$$\omega : \mathcal{A} \to \mathbb{C}, \quad \omega(A^*A) \geq 0, \quad \text{and} \quad \omega(1) = 1. \tag{1.6}$$

Introducing states at this stage is also the main reason for choosing $\mathcal{A}$ to be a C^*-algebra: For any such state on a C^*-algebra, one can use the celebrated Gelfand-Naimark-Segal (GNS) construction[61, 72, 73] to obtain a Hilbert space $\mathcal{H}_\omega$ carrying a representation π_ω of $\mathcal{A}$. As a result, there

is a designated vector $\Omega_\omega \in \mathcal{H}_\omega$ with

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle \quad \text{for all} \quad A \in \mathcal{A}, \tag{1.7}$$

which is also cyclic, in the sense that

$$\pi_\omega(\mathcal{A})\Omega_\omega \quad \text{is dense in} \quad \mathcal{H}_\omega. \tag{1.8}$$

Out of all the possible states that one might choose, a *vacuum state* ω_0 has to satisfy further properties, which in a physically sound theory, should single out a *unique* vacuum state:

Firstly, it should be invariant with respect to the action of the Poincaré group $\mathfrak{P}$, in the sense that

$$\omega_0(g \triangleright A) = \omega_0(A) \quad \text{for all} \quad g \in \mathfrak{P} \quad \text{and} \quad A \in \mathcal{A}. \tag{1.9}$$

This implies that the corresponding GNS vector $\Omega_0 \in \mathcal{H}_0$ is Poincaré invariant, allowing to define a unitary representation of $\mathfrak{P}$ via

$$U(g)\pi_0(A)\Omega_0 := \pi_0(g \triangleright A)\Omega_0. \tag{1.10}$$

Secondly, and this will turn out to be an extremely important in doing actual computations, the spectrum of the generator

$$P_\mu := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s} U(1, s\hat{e}_\mu)\Big|_{s=0}. \tag{1.11}$$

of translations should be contained in the forward light cone. To see why this is a property of the state ω_0, note that $U(g)$, as defined in eq. (1.10), depends on the GNS representation, which in turn is state dependent. We can formalise this so called *spectrum condition*[64] as

$$\langle \psi | P_0 \psi \rangle > 0 \quad \text{and} \quad \langle \psi | P_\mu P^\mu \psi \rangle > 0 \quad \text{for all} \quad \psi \in \mathcal{H}. \tag{1.12}$$

Thirdly, a vacuum state should be faithful, i.e.,

$$\omega_0(A) = 0 \quad \text{if and only if} \quad A = 0. \tag{1.13}$$

As a result, Ω_0 is separating[72]

$$\pi_0(A)\Omega_0 = 0 \quad \text{if and only if} \quad A = 0, \tag{1.14}$$

which has the physical interpretation that Ω_0 is highly entangled, as we will discuss later in section 1.2.3.

Lastly, a vacuum should be pure in the sense discussed in section 1.2.1. Equivalently,[72] one could require the corresponding GNS representation to be irreducible. This requirement fails in the case of spontaneous symmetry breaking: In spontaneously broken symmetry, you are dealing with a invariant state that is a mixture of non-invariant *false vacua*. We will also have to consider mixed states later on when talking about states of finite temperature.

Together, the above axioms give a solid foundation to modern QFT. In contrast to, e.g., path integral formulations,[74,75] the focus on algebraic structure allows for a mathematically rigorous treatment of the subject and has been fruitful both for physics and mathematics: On the physical side, great insights into the structure of QFT can be deduced directly from the axioms, such as the spin-statistics theorem[64] and CPT invariance[76] (see also section 2.2), which we will link to entanglement of the vacuum in section 2.2. On the mathematical side, the programme of algebraic QFT (AQFT) has been a driving force in the study and classification of von-Neumann algebras,[77] as well as non-commutative integration theory.[61]

1.2.3 What is entanglement?

The last big question that I want to discuss in this introduction is what entanglement actually is. As explained in section 1.2.1, quantum physics is a probabilistic theory, in the sense that a pairing between observable and state (even a pure one) yields an *expectation value* for the corresponding measurement, instead of a sharp prediction. This probabilistic nature allows to ask whether there might be correlations between different components of the same system, even in a pure state: These *quantum correlations* are called entanglement. They are in contrast to *classical correlations*, which are due to the probabilistic nature of the "mixedness" of a state, be it classical or quantum.

For mixed quantum states, it is often hard to discriminate classical from quantum correlations – there even are several non-equivalent definitions of what these terms precisely mean.[8,78,79] While this discussion opens up an incredibly rich fauna of quantum information theoretical concepts, such as entanglement witnesses,[78] entanglement distillability,[80] quantum discord[79] and generalised measurements,[8] I do not want to go down this rabbit hole and instead focus on entanglement for pure states, where only one kind of correlation is present.

To give a proper definition of entanglement for pure states, consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Intuitively, $\mathcal{H}_i$ are the local Hilbert spaces of different parts of a system, e.g., the individual polarisation vector spaces of two photons. A state $\psi \in \mathcal{H}$ is called *separable* if it can be written as a product

$$\psi = \psi_1 \otimes \psi_2 \quad \text{for some} \quad \psi_i \in \mathcal{H}_i, \tag{1.15}$$

while every state which does not factorise in this form is called entangled.

In a more general context, we can define separability in terms of algebras of observables:[81] Assume a global algebra of observables $\mathcal{A}$ and two commuting subalgebras $\mathcal{A}_i \subset \mathcal{A}$, $[\mathcal{A}_1, \mathcal{A}_2] = 0$. The intuition is that $\mathcal{A}_i$ are local operations that can be performed on the individual portions of the system. In this framework, a state $\omega \in \mathcal{A}^*$ is a functional on the global algebra $\mathcal{A}$ and a pure state is called separable *with respect to the given subalgebras* if all expectation values factorise as

$$\omega(A_1 A_2) = \omega(A_1)\omega(A_2) \quad \text{for all} \quad A_i \in \mathcal{A}_i, \tag{1.16}$$

while entanglement is again the absence of such a factorisation. This definition is equivalent to the one in eq. (1.15) if the subalgebras $\mathcal{A}_1 = B(\mathcal{H}_1) \otimes 1$ and $\mathcal{A}_2 = 1 \otimes B(\mathcal{H}_2)$ consist of all bounded operators on the individual Hilbert spaces. This is usually the case in quantum information theory, where all Hilbert spaces under consideration are finite dimensional. In QFT however, the vast number of degrees of freedom usually makes it impossible to "measure everything". Additionally, symmetries and superselection rules might exclude large classes of operators from actually being observable. In this case, a state might be entangled with respect to definition (1.15), while these quantum correlations are not measurable by any physical process. We will come back to this in section 2.5. For now, just note that eq. (1.16) provides a strictly more general definition of entanglement than what the usual Hilbert space approach is able to provide.

Anyway, since entanglement is a kind of correlation, it seems desirable to have a measure for "how much" entanglement is present in a state. In the Hilbert space approach to separability (1.15), one such measure is the so called *entanglement entropy*,[78] which is obtained by tracing out one system and computing the von-Neumann entropy of the resulting reduced density matrix via

$$S_{\text{EE}} = -\operatorname{Tr}\left[\rho_1 \log \rho_1\right] \quad \text{with} \quad \rho_1 = \operatorname{Tr}_{\mathcal{H}_2}\left[|\psi\rangle\langle\psi|\right]. \tag{1.17}$$

This measure is symmetric in the sense that it does not make a difference which subsystem we choose to trace out. Furthermore, it is non-negative and vanishes precisely for separable states, as follows from Jensen's inequality.[82]

Note that eq. (1.17) can also be evaluated for mixed quantum states, albeit picking up classical correlations as well and thus losing its symmetry. I will discuss entanglement entropy in section 2.4, where I present its computation for different states in a QFT of free fermions.

Another measure for entanglement of a pure state is the *Schmidt rank* r_{S}, which is the minimum number of extra terms that have to be added to the right hand side of eq. (1.15), i.e., the minimal number r_{S}, such that

$$\psi = \sum_{k=1}^{r_{\mathrm{S}}} \sqrt{p_k}\, \psi_1^{(k)} \otimes \psi_2^{(k)} \quad \text{for some} \quad \psi_i^{(k)} \in \mathcal{H}_i, 0 < p_k < 1. \tag{1.18}$$

Obviously, the Schmidt rank is bounded from above by the dimensions of $\mathcal{H}_i$ and is larger than one if and only if a state is entangled. Notably, you can always make sure that the $\psi_i^{(k)}$ in eq. (1.18) are orthonormal to one another, meaning that ψ can be written as a "Bell-like" superposition. Such a decomposition, also called *Schmidt decompostion*, is unitarily unique,[8] as can be shown using the singular value decompostion. This uniqueness implies that the *Schmidt numbers* p_k are identical to the eigenvalues of the reduced density matrices ρ_i, hence the Schmidt rank measures the number of non-zero eigenvalues of the reduced density matrix, while entanglement entropy gives the corresponding entropy. Both can be obtained from the entanglement Rény entropy[83]

$$S_\alpha = \frac{1}{1-\alpha} \log \operatorname{Tr} \rho_1^\alpha = \frac{1}{1-\alpha} \log \sum_{k=1}^{r_{\mathrm{S}}} p_k^\alpha, \quad \Re(\alpha) > 0, \alpha \neq 1 \tag{1.19}$$

via the limits

$$S_{\mathrm{EE}} = \lim_{\alpha \to 1} S_\alpha \quad \text{and} \quad \log r_{\mathrm{S}} = \lim_{\alpha \searrow 0} S_\alpha. \tag{1.20}$$

For the sake of completeness, let me mention that there have also been efforts to generalise the Schmidt decomposition to mixed states,[84] however, uniqueness is lost and classical correlations are picked up as in the case of entanglement entropy.

To get a better picture of the meaning of S_{EE} and r_{S}, I would like to present a short discussion for finite dimensional systems, roughly based on the excellent presentations by Witten[46] and Papadodimas and Raju.[58] Consider first the Bell state

$$\psi = \frac{1}{\sqrt{2}} \left[\psi^{(0)} \otimes \psi^{(0)} + \psi^{(1)} \otimes \psi^{(1)} \right] \in \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{1.21}$$

and an arbitrary operator

$$A : \mathbb{C}^2 \to \mathbb{C}^2 \quad \text{with matrix elements} \quad A_{ij} = \langle \psi^{(i)} | A \psi^{(j)} \rangle.$$

Acting with A on the first qubit is equivalent to acting with $A^\intercal$ on the second qubit, as

$$(A \otimes 1)\psi = \frac{1}{\sqrt{2}} \sum_{ij} A_{ij} \psi^{(i)} \otimes \psi^{(j)} = \frac{1}{\sqrt{2}} \sum_{ij} (A^\intercal)_{ji} \psi^{(i)} \otimes \psi^{(j)} = (1 \otimes A^\intercal)\psi.$$

This is the essence of entanglement: *In an entangled state, an action on one subsystem is equivalent to an action on the other subsystem.* Note that this does not mean that an action on one subsystem has some kind of instant reaction in the other system – the actions are just equivalent. All non-locality is contained in the state, while measurements still happen locally within a single subsystem, resolving Einstein's issue with entanglement being a "spooky action at a distance".[85]

To generalise slightly, consider now a state ψ with general Schmidt decompostion as in eq. (1.18) and an arbitrary operator A on the space spanned by the $\psi_1^{(k)}$. The same reasoning as above leads to

$$\begin{aligned}(A \otimes 1)\psi &= \sum_{kl} \sqrt{p_k} A_{lk} \psi_1^{(l)} \otimes \psi_2^{(k)} \\ &= \sum_{kl} \sqrt{p_k} \sqrt{\frac{p_l}{p_k}} A_{kl} \psi_1^{(k)} \otimes \psi_2^{(l)} = (1 \otimes \tilde{A})\psi,\end{aligned}$$

where $\tilde{A}$ is an operator in the space spanned by the $\psi_2^{(k)}$ with matrix elements

$$\tilde{A}_{lk} = \sqrt{\frac{p_l}{p_k}} A_{kl}. \tag{1.22}$$

This relation has recently been proposed as the starting point for the unitarisation of black hole evaporation.[57] It also admits a generalisation to the operator algebraic setting and is the basis for entanglement in AQFT, as will be discussed in section 2.1.2. Furthermore, the above discussion gives a nice interpretation to S_{EE} and r_{S}: We already established that entanglement is the equivalence of actions on different subsystems. Now, from relation (1.22), you can directly see that r_{S} is the size of the operator algebras where this equivalence holds, while S_{EE} tells us how similar the equivalent operators are, since large entanglement entropy means that the prefactor $(p_l/p_k)^{1/2}$ is approximately unity.

2 Entanglement in quantum field theory

Let us now proceed to the main concern of this thesis: the rôle of entanglement in QFT. To this end, I would first like to introduce some universal results in QFT that might seem non-intuitive at first glance, but make perfect sense when interpreted in terms of entanglement. After that, beginning in section 2.3, I will specialise to free fermions, where much stronger variants of these results can be derived. Finally, in section 2.5, I will discuss which adaptions can (and should) be made if the theory under consideration has conserved charges. Again, I will make these adaptions explicit for free fermions with a conserved number of fermions.

2.1 Entanglement and modular flow

2.1.1 The failure of the reduced density matrix

In section 1.2.3, I already discussed the interpretation of entanglement as an equivalence of local operations in different subsystems. Interestingly, entanglement in this form can be shown to be a direct consequence of the Haag-Kastler axioms from section 1.2.2, hence, is present in any vacuum state of a QFT.

This fact is known as the Reeh-Schlieder theorem[15,64] and is formulated as follows: Assume you are dealing with a QFT in the sense of section 1.2.2 and a vacuum state ω_0 on the global algebra of observables $\mathcal{A}(\mathbb{R}^{1,d})$. In particular, this means that the corresponding GNS vector Ω_0 is cyclic and separating for $\mathcal{A}(\mathbb{R}^{1,d})$ in the sense of eqs. (1.8) and (1.14). Then, surprisingly, Ω_0 is also cyclic and separating for the local algebra $\mathcal{A}(\mathcal{R})$ of *any* open subregion $\mathcal{R} \subset \mathbb{R}^{1,d}$. Here, $\mathcal{A}(\mathcal{R})$ is considered to be a subalgebra of $\mathcal{A}(\mathbb{R}^{1,d})$ by virtue of the isotony axiom.

In particular, cyclicity means that you can generate the entire Hilbert space $\mathcal{H}_0$ (or, more precisely, a dense subset) by acting on the vacuum only with operators localised in $\mathcal{R}$.

The Reeh-Schlieder theorem is usually proven with a rather technical argument, the key points of which I would like to highlight here: Since $\mathcal{A}(\mathcal{R}) \subset \mathcal{A}$

is a subalgebra of the global algebra $\mathcal{A}$, the separating property of Ω_0 is immediate. The cyclic property is proven by contradiction. If $\mathcal{A}(\mathcal{R})\Omega_0$ is not dense in $\mathcal{H}_0$, then there has to be a vector $\psi \neq 0 \in \mathcal{H}_0$ with

$$\langle \psi | A\Omega_0 \rangle = 0 \quad \text{for all} \quad A \in \mathcal{A}(\mathcal{R}).$$

Please note that I kept the GNS representation $\pi_0(A)$ implicit here and just write A, as I will continue to do for the rest of this thesis. The idea of proof is that, because of covariance (1.4) and the spectrum condition (1.12), you can move the A in this relation to any different region in $\mathbb{R}^{1,d}$ of your choice. To see how this works, pick an even smaller subregion $\tilde{\mathcal{R}} \subset \mathcal{R}$ such that you can move it around in time by $\pm\epsilon$ without leaving $\mathcal{R}$ (see fig. 2.1. Formally this means that

$$t\hat{e}_0 \triangleright \tilde{\mathcal{R}} \subset \mathcal{R} \quad \text{for} \quad t \in [-\epsilon, \epsilon],$$

where $t\hat{e}_0 \triangleright$ denotes time translation by t.

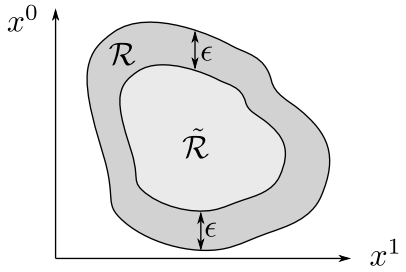


Figure 2.1: The two regions $\mathcal{R}$ (grey) and $\tilde{\mathcal{R}}$ (light grey) in the proof of the Reeh-Schlieder theorem. The smaller region $\tilde{\mathcal{R}}$ can be translated in time by $\pm\epsilon$ without leaving the larger region $\mathcal{R}$.

Now, since the Poincaré group has a unitary implementation U on $\mathcal{H}_0$, you can pick any $A \in \mathcal{A}(\tilde{\mathcal{R}})$ to find

$$\langle \psi | U(t\hat{e}_0) A U(t\hat{e}_0)^{-1} \Omega_0 \rangle = \langle \psi | (t\hat{e}_0 \triangleright A)\Omega_0 \rangle = 0 \quad \text{for} \quad t \in [-\epsilon, \epsilon].$$

Introducing the Hamiltonian H, i.e., the generator of time translations, and using Poincaré invariance of the vacuum, this can be rewritten as

$$\langle \psi | \mathrm{e}^{\mathrm{i}tH} A\Omega_0 \rangle = 0 \quad \text{for} \quad t \in [-\epsilon, \epsilon].$$

This correlation function has a remarkable property: Since $H > 0$ by the

spectrum condition, $\mathrm{e}^{\mathrm{i}tH}$ is a bounded operator for $\Im(t) > 0$, hence

$$\langle\psi|\mathrm{e}^{\mathrm{i}tH}A\Omega_0\rangle \quad \text{is holomorphic for} \quad \Im(t) > 0.$$

Therefore, $\langle\psi|\mathrm{e}^{\mathrm{i}tH}A\Omega_0\rangle$ is a function of t that is holomorphic in the upper half plane and vanishes along a finite interval of the real axis, hence, vanishes along the entire real axis because of the *edge of the wedge theorem*.[64,86] As a result,

$$\langle\psi|(t\hat{e}_0 \triangleright A)\Omega_0\rangle = 0 \quad \text{for any} \quad t \in \mathbb{R}, A \in \mathcal{A}(\tilde{\mathcal{R}}).$$

This argument can of course be repeated with any other timelike vector instead of $\hat{e}_0$ and since any point in $\mathbb{R}^{1,d}$ can be reached by a combination of two timelike translations (one into the future, one into the past), you find

$$\langle\psi|(v \triangleright A)\Omega_0\rangle = 0 \quad \text{for any} \quad v \in \mathbb{R}^{1,d}, A \in \mathcal{A}(\tilde{\mathcal{R}}),$$

implying that $\langle\psi|A\Omega_0\rangle$ vanishes for any $A \in \mathcal{A}(\mathbb{R}^{1,d})$ because of the additivity axiom (1.5). This is, of course, in contradiction to the assumption that Ω_0 is cyclic for the global algebra and completes the proof.

The reason why I included this proof is twofold: Firstly, I would like to mention that the proof does not require the state to be pure – all we need is Poincaré invariance and the spectrum condition, along with the usual axioms for the net of observable algebras. This means that the Reeh-Schlieder theorem also holds for mixed states, most prominently thermal states, which we will come back to later in this chapter. Secondly, the proof showcases the most ubiquitous technique in all of QFT: Using the spectrum condition (positivity of energy) to obtain holomorphic correlation functions. This technique will be of great use in section 2.1.2.

So what does the Reeh-Schlieder theorem tell us about entanglement in QFT? As discussed at the end of section 1.2.3, the fact that the vacuum state vector Ω_0 is cyclic for any open subregion $\mathcal{R} \subset \mathbb{R}^{1,d}$, implies an equivalence all local algebras, no matter how small they are. In the interpretation of entangled states as non-local entities, this means that all degrees of freedom in a QFT are delocalised! Let us try to model such behaviour in a simpler system. To this end, begin with the archetypical example of delocalised degrees of freedom: the Bell state $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ as given in eq. (1.21). This state is separating for the set of operators on the first subsystem, all of which are of the form $A \otimes 1$ for some matrix $A \in \mathbb{C}^{2\times 2}$. You can now build a Hilbert space

$$\mathcal{H}_\psi := (\mathbb{C}^{2\times 2} \otimes 1)\psi = \mathbb{C}^2 \otimes \mathbb{C}^2,$$

such that ψ is cyclic and separating for the action of $\mathcal{A}_1 := \mathbb{C}^{2\times 2} \otimes 1$ on the

first subsystem. As explained in section 1.2.3, this action is equivalent to that of $\mathcal{A}_2 := 1 \otimes \mathbb{C}^{2\times 2}$ on the second subsystem via eq. (1.22) with $p_0 = p_1 = 1/2$ and, indeed, ψ is also cyclic and separating for the action of $\mathcal{A}_2$. It is also noteworthy that these algebras are linked by a specific property: $\mathcal{A}_1$ consists of all bounded operators on $\mathcal{H}_\psi$ that commute with everything in $\mathcal{A}_2$ – we say that

$$\mathcal{A}_1 = \mathcal{A}_2' := \Big\{ A \in B(\mathcal{H}_\psi) \Big| [A, \mathcal{A}_2] = 0 \Big\} \tag{2.1}$$

is the *commutant* of $\mathcal{A}_2$. Of course, since we are working in finite dimensions, $\mathcal{A}_2$ is also the commutant of $\mathcal{A}_1$.

Now, in a QFT, we are not dealing with finite dimensional operator algebras. Instead, we have to switch gears and consider infinite dimensions. To get an idea of important implications without unnecessary complications, take a look at the following model of a so called von-Neumann factor of type II, which can be found, e.g., in Witten's pedagogical article:[46] Consider the tensor product

$$\psi^{(\infty)} := \bigotimes_{k\in\mathbb{N}} \psi \in \left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)^{\otimes_{\text{alg}}\infty} \tag{2.2}$$

of an infinite number of Bell states. The subscript in $\otimes_{\text{alg}}$ denotes that we are working in the algebraic tensor product, i.e., that the vector space on the right hand side has no Hilbert space structure. This is because we have not yet defined an inner product or sense of convergence, with respect to which the Hilbert space would have to be complete. Instead, we first define the infinite variant of the algebras $\mathcal{A}_i$ by allowing for an individual copy of $\mathcal{A}_i$ to act on each copy of ψ in $\psi^{(\infty)}$. In order to have a well defined action on $\psi^{(\infty)}$, we allow only for algebra elements that have a finite number of non-unity tensor factors, i.e., we define

$$\mathcal{A}_1^{(\infty)} := \Big\{ \bigotimes_{k\in\mathbb{N}} A_k \Big| A_k \in \mathcal{A}_1, \quad A_k = 1 \text{ up to finite exceptions} \Big\} \tag{2.3}$$

and similarly for $\mathcal{A}_2$. As a result, the vector space

$$\mathcal{H}_\psi^{(\infty)} := \mathcal{A}_1^{(\infty)}\psi^{(\infty)} = \mathcal{A}_2^{(\infty)}\psi^{(\infty)} \tag{2.4}$$

is well defined and can be endowed with the obvious inner product, given by tensor contraction, as sketched in fig. 2.2.

This inner product is finite by construction, since all but a finite number of the contractions are of the form $\langle\psi|\psi\rangle$. We can use it to complete $\mathcal{H}_\psi^{(\infty)}$ to a Hilbert space. Similarly, we can complete $\mathcal{A}_i^{(\infty)}$ to von-Neumann algebras: A *von-Neumann algebra* is an algebra of bounded operators which is complete in

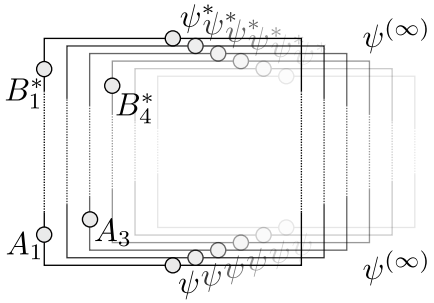


Figure 2.2: Sketch of the Hilbert space structure of the type II von-Neumann factor (2.3) as an infinite tensor network. The inner product of two vectors $A\psi^{(\infty)}$ and $B\psi^{(\infty)}$ is given by tensor contraction (dotted vertical lines).

the weak (or, equivalently, strong) operator topology. Equivalently,[61,72] it is an operator algebra that is its own bi-commutant, i.e., to pass from $\mathcal{A}_i^{(\infty)}$ to the smallest von-Neumann algebra containing $\mathcal{A}_i^{(\infty)}$, we just have to take the commutant twice

$$\mathcal{A}_i^{(\infty)} \to (\mathcal{A}_i^{(\infty)})'' := \Big((\mathcal{A}_i^{(\infty)})'\Big)'.$$

Finally, $\psi^{(\infty)}$ is cyclic and separating for $\mathcal{A}_i^{(\infty)}$ and both algebras are commutants, as they were in the finite dimensional case.

The interesting thing is now that, while we still have an equivalence of local algebras, tracing out the second subsystem is technically impossible, as there is no such thing as a local Hilbert space of the first subsystem – the definition of the inner product and completion were only performed at the level of the global Hilbert space. Out of pure curiosity, we could put a *cutoff* to the number of degrees of freedom, by restricting the range of the tensor products in eqs. (2.2) and (2.3) to $k = 1, \ldots, N$. We would then obtain a diagonal reduced density matrix with eigenvalues $p_l = 2^{-N}$, yielding an entanglement entropy of $S_{\text{EE}} = N \log 2$, which is clearly divergent. Indeed, one can show[46] that for *any* state in $\mathcal{H}_\psi^{(\infty)}$, the entanglement entropy diverges as $N \log 2$ to leading order in N. This implies that entanglement entropy and reduced density matrices can only ever be understood in the presence of a cutoff, while the interpretation of entanglement as an equivalence of the local algebras of observables works in all settings: For any cyclic vector in $\mathcal{H}_\psi^{(\infty)}$ the prefactors $(p_l/p_k)^{1/2}$ in eq. (1.22) turn out to be finite in the limit $N \to \infty$.

2.1.2 Tomita-Takesaki theory

In the previous subsection, we have seen that the equivalence (1.22) of local observable algebras holds even in limits where a reduced density matrix is not definable as an operator on a Hilbert space. You could therefore expect that the equivalence can be established without introducing a cutoff in the first place. This is done in the mathematical framework of *Tomita-Takesaki modular theory*,[61] which lies at the heart of structure theory of von-Neumann algebras,[61,77] is a cornerstone of AQFT,[87,88] and is a key element in our current understanding of the interplay between QFT and gravity,[27,58,89–95] as well as the celebrated AdS/CFT correspondence.[19,22–26,28–35,57,96–99] To discuss its most important features, let me first present some definitions, leading to the infinite dimensional variant of eq. (1.22). After that we will quickly proceed to its important rôle in QFT and relations to thermal states.

To get started, consider a von-Neumann algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ and a designated vector $\Omega \in \mathcal{H}$. As discussed before in sections 1.2.3 and 2.1.1, for an equivalence between $\mathcal{A}$ and its commutant $\mathcal{A}'$, it is important that Ω is cyclic separating for $\mathcal{A}$ – we say that $\mathcal{A}$ is given in *normal form* with respect to Ω. We already saw that Ω is cyclic separating for $\mathcal{A}'$ as well and, indeed, a vector is cyclic for a von-Neumann algebra if and only if it is separating for its commutant. To see why this is the case, let Ω be cyclic for $\mathcal{A}$ and consider $A' \in \mathcal{A}'$ with $A'\Omega = 0$. Then

$$0 = AA'\Omega = A'A\Omega \quad \text{for all} \quad A \in \mathcal{A},$$

i.e., A' vanishes on the dense subspace $\mathcal{A}\Omega \subset \mathcal{H}$ and Ω is separating for $\mathcal{A}'$. On the other hand, if Ω is not cyclic for $\mathcal{A}$, consider the orthogonal projection Π onto $\mathcal{A}\Omega$ and its commutator

$$[A, \Pi] = (A\Pi - \Pi A)\Pi = (1 - \Pi)A\Pi.$$

with an arbitrary operator $A \in cA$. By definition of Π, this vanishes on both $\mathcal{A}'\Omega$ and its orthogonal complement. This implies that $1 - \Pi \neq 0$ is an element of $\mathcal{A}'$ with $(1 - \Pi)\Omega = 0$, hence Ω is not separating for $\mathcal{A}'$, which completes the proof.

Let now Ω be cyclic separating for $\mathcal{A}$. We define the *Tomita operator* S by

$$SA\Omega := A^*\Omega \quad \text{for every} \quad A \in \mathcal{A}. \tag{2.5}$$

This definition is non-contradictory because Ω is separating and it defines S on a dense subspace of $\mathcal{H}$ because Ω is cyclic. A technical argument shows that S is actually a closable operator[61,72] and we will denote its closure by the

same symbol. Note that the definition (2.5) depends both on $\mathcal{A}$ and Ω. As argued in the previous paragraph, Ω cyclic separating for $\mathcal{A}'$, too – in fact, we the Tomita operators S and S' of $\mathcal{A}$ and $\mathcal{A}'$ are adjoint one another:

$$S' = S^*, \quad \text{i.e.,} \quad \langle\psi|S'\phi\rangle = \langle\phi|S\psi\rangle \quad \text{for all} \quad \psi, \phi \in \mathcal{H}. \tag{2.6}$$

This follows directly from the definition of S on the dense subspaces $\psi \in \mathcal{A}\Omega$ and $\phi \in \mathcal{A}'\Omega$, since

$$\langle A\Omega|S'A'\Omega\rangle = \langle\Omega|A^*(A')^*\Omega\rangle = \langle\Omega|(A')^*A^*\Omega\rangle = \langle A'\Omega|SA\Omega\rangle \quad \text{for all}$$

for all $A \in \mathcal{A}, A' \in \mathcal{A}'$. Since S is a closed anti-linear operator, it has a unique polar decomposition

$$S = J\Delta^{1/2}, \tag{2.7}$$

where J is anti-unitary in the sense that

$$J^*J = JJ^* = 1, \quad \text{i.e.,} \quad \langle J\psi|J\phi\rangle = \langle\phi|\psi\rangle = \langle J^*\psi|J^*\phi\rangle \quad \text{for all} \quad \psi, \phi \in \mathcal{H}$$

and $\Delta > 0$ is positive self-adjoint. We call J and Δ the *modular conjugation* and *modular operator*, respectively. Since $\Delta = S^*S = S'S$ and both S and S' preserve Ω, we obtain the important property

$$\Delta\Omega = \Omega = J\Omega. \tag{2.8}$$

Furthermore, since $S^2 = 1$, you see that

$$\Delta^{-1/2} = J\Delta^{1/2}J \tag{2.9}$$

and multiplying this equation with its adjoint yields

$$\Delta^{-1} = J^*\Delta^{1/2}J^*J\Delta^{1/2}J = J^*\Delta J.$$

You can take arbitrary powers of this equation to find

$$\overline{f}(\Delta^{-1}) = J^*f(\Delta)J \tag{2.10}$$

for any complex function f that can be expressed as a convergent power series. In particular, choose $f(\Delta) = \Delta^{-1/2}$ and multiply by eq. (2.9) a to find

$$1 = \Delta^{-1/2}(\Delta^{-1})^{(-1/2)} = J\Delta^{1/2}JJ^*\Delta^{-1/2}J = J^2, \tag{2.11}$$

hence, J is self-adjoint anti-unitary. Furthermore, you can choose $f(\Delta) = \Delta^{it}$

to obtain

$$\Delta^{\mathrm{i}t} J = J \Delta^{\mathrm{i}t}. \tag{2.12}$$

In addition to leaving Ω invariant, the modular conjugation and modular operator encode the relationship between $\mathcal{A}$ and $\mathcal{A}'$. This is the content of *Tomita's theorem*,[61,77,100] which says that

$$J\mathcal{A}J = \mathcal{A}' \quad \text{and} \quad \Delta^{\mathrm{i}t}\mathcal{A}\Delta^{-\mathrm{i}t} = \mathcal{A} \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.13}$$

To see why this is true, note that[72]

$$SASBC\Omega = SAC^*B^*\Omega = BCA^*\Omega = BCSAS\Omega \quad \text{for all} \quad A, B, C \in \mathcal{A}.$$

Since the set of all $C\Omega$ is dense in $\mathcal{H}$, this means that $[SAS, B] = 0$, hence

$$S\mathcal{A}S \subset \mathcal{A}' \quad \text{and, similarly,} \quad S^*\mathcal{A}'S^* \subset \mathcal{A}. \tag{2.14}$$

Note that the above equations have to be taken with a grain of salt, since S is unbounded and operator products SAS might be ill defined or yield unbounded operators. Nevertheless I would like to continue with this line of thought to give some intuition and refer to, e.g. Takesaki's books[61] for a more rigorous treatment. Since $\Delta = S^*S$ and $\Delta^{-1} = J\Delta J = SS^*$, eq. (2.14) implies

$$\Delta\mathcal{A}\Delta^{-1} = S^*S\mathcal{A}SS^* \subset S^*\mathcal{A}'S^* \subset \mathcal{A}.$$

Again, this extends to arbitrary powers series of Δ, hence,

$$\Delta^{\mathrm{i}t}\mathcal{A}\Delta^{-\mathrm{i}t} \subset \mathcal{A} \quad \text{for all} \quad t \in \mathbb{R}.$$

Replacing $t \to -t$ in this expression reverses the direction of inclusion, such that the left and right hand side are actually equal. Finally, the inclusion $J\mathcal{A}J \subset \mathcal{A}'$ follows as a direct result of $\Delta^{1/2}\mathcal{A}\Delta^{-1/2} \subset \mathcal{A}$ and eq. (2.14), implying the first part of eq. (2.13) and completing the sketch of proof.

So the question is, what *are* J and Δ and why are they interesting? To get a feeling for them, let us consider the finite dimensional case: Starting from a state ψ with Schmidt decomposition (1.18) and a local algebra of operators $\mathcal{A} = B(\mathcal{H}) \otimes 1$, we have

$$(A \otimes 1)\psi = \sum_{ij} \sqrt{p_j} A_{ij} \psi_1^{(i)} \otimes \psi_2^{(j)} \quad \text{and} \quad (A^* \otimes 1)\psi = \sum_{ij} \sqrt{p_i} \overline{A_{ij}} \psi_1^{(j)} \otimes \psi_2^{(i)},$$

implying that S is antilinear with

$$S(\psi_1^{(i)} \otimes \psi_2^{(j)}) = \sqrt{\frac{p_i}{p_j}} \psi_1^{(j)} \otimes \psi_2^{(i)}.$$

Note how much this resembles eq. (1.22)! As for the polar decomposition (2.7), note that the operators defined by

$$\Delta(\psi_1^{(i)} \otimes \psi_2^{(j)}) := \frac{p_i}{p_j} \psi_1^{(i)} \otimes \psi_2^{(j)} \quad \text{and} \quad J(\psi_1^{(i)} \otimes \psi_2^{(j)}) := \psi_1^{(j)} \otimes \psi_2^{(i)} \tag{2.15}$$

do the trick. Without referring to the particular bases $\psi_k^{(i)}$, this means that

$$\Delta = \rho_1 \otimes \rho_2^{-1} \tag{2.16}$$

can be expressed in terms of the reduced density matrices

$$\rho_1 = \mathrm{Tr}_2[\psi \otimes \psi^*], \quad \rho_2 = \mathrm{Tr}_1[\psi \otimes \psi^*],$$

while

$$J(A \otimes 1)J = 1 \otimes \overline{A} \tag{2.17}$$

exchanges the algebra with its commutant. Finally, note that the equivalence (1.22) of local algebras can be expressed directly in terms of the modular data: For any $A \in \mathcal{A}$, we have

$$A\Omega = SA^*S\Omega = J\Delta^{1/2}A^*\Delta^{-1/2}J\Omega =: \tilde{A}\Omega \quad \text{with} \quad \tilde{A} \in \mathcal{A}'. \tag{2.18}$$

As before, this expression is not entirely kosher in the infinite dimensional setting, as S might be (and often is) an unbounded operator. However, eq. (2.18) at least formally describes an equivalence of the local algebras and indeed one recovers eq. (1.22) when inserting eqs. (2.16) and (2.17).

2.1.3 Thermal states and the KMS condition

Considering that the modular operator associated to an algebra $\mathcal{A}$ preserves the vacuum via eq. (2.8), as well as $\mathcal{A}$ itself in the sense of Tomita's theorem (2.13), it constitutes a *symmetry* in the sense of section 1.2.1. This is remarkable, since the modular data is a purely algebraic entity and, as discussed in section 2.1.2, originates in the entanglement of a state with respect to an algebra of observables. In the spirit of reducing properties of physical models to first principles, it thus makes sense to ask if there are cases where

the modular flow

$$\sigma_t(A) := \Delta^{\mathrm{i}t} A \Delta^{-\mathrm{i}t}, \quad A \in \mathcal{A} \tag{2.19}$$

recovers a known symmetry of the model.

To this end, consider a quantum system in a thermal state ω_β. As discussed in section 1.2.1, we can take the *global* algebra of observables to be the set of bounded operators on a Hilbert space $\mathcal{H}$. Expectation values are generated by tracing against the thermal density matrix ρ_β via

$$\omega_\beta(A) := \mathrm{Tr}[\rho_\beta A], \quad \rho_\beta = \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}[\mathrm{e}^{-\beta H}]}, \quad A \in B(\mathcal{H}), \tag{2.20}$$

where the Hamiltonian $H = H^* \in B(\mathcal{H})$ is the generator of time translations. Of course ρ_β fails to be cyclic separating for $B(\mathcal{H})$ as it is not even a vector in $\mathcal{H}$ but rather an operator. To resolve this issue, you can use a construction known as *purification*[8] in the framework of quantum information theory. The idea is that any operator $\mathcal{H} \to \mathcal{H}$ can equivalently be considered as an element of $\mathcal{H} \otimes \mathcal{H}^*$. Applying this reasoning to the operator $\sqrt{\rho_\beta}$, you obtain a vector $\psi_\beta \in \mathcal{H} \otimes \mathcal{H}^*$ that satisfies

$$\langle \psi_\beta | (A \otimes 1) \psi_\beta \rangle = \mathrm{Tr}[\rho_\beta A] = \omega_\beta(A) = \langle \psi_\beta | (1 \otimes A^\intercal) \psi_\beta \rangle, \quad A \in B(\mathcal{H}), \tag{2.21}$$

as can be seen pictographically in fig. 2.3.

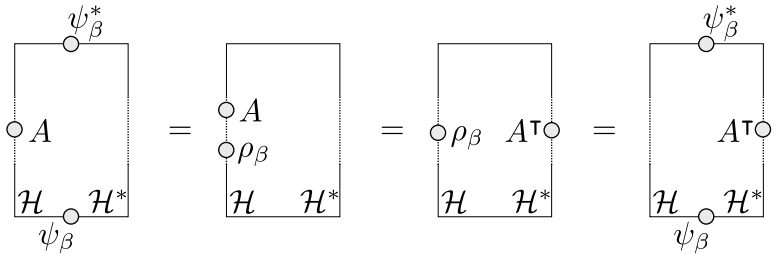


Figure 2.3: Tensor network diagram of the purification (2.21). The vector $\psi_\beta \in \mathcal{H} \otimes \mathcal{H}^*$ yields the same expectation values as the density matrix ρ_β on $\mathcal{H}$. Furthermore, the actions of the operators $Atimes1$ and $1 \otimes A^\intercal$ are equivalent on ψ_β. Contraction is again denoted by dotted lines, while the dual pairing $\mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C}$ is given by a "bent" line.

By construction, the Schmidt numbers of ψ_β coincide with the eigenvalues of ρ_β, which are non-zero because of eq. (2.20). This means that ψ_β is cyclic separating for the observable algebra $\mathcal{A} = B(\mathcal{H}) \otimes 1$ and its commutant $\mathcal{A}' =$

$1 \otimes B(\mathcal{H})$. Note that we could equivalently have arrived at this situation by means of the GNS construction, as discussed in Haag's book.[64]

The discussion that lead to eq. (2.16) remains valid in this case and we find

$$\Delta = \rho_\beta \otimes \rho_\beta^{-1} = \mathrm{e}^{-\beta H} \otimes \mathrm{e}^{\beta H}$$

for the modular operator associated to $\mathcal{A}$ and ψ_β. As a result, the modular flow

$$\sigma_t(A \otimes 1) = \Delta^{\mathrm{i}t}(A \otimes 1)\Delta^{-\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}t\beta H} A \mathrm{e}^{\mathrm{i}t\beta H} \otimes 1, \quad A \in B(\mathcal{H}) \tag{2.22}$$

coincides with a rescaled time evolution!

As will be discussed in section 2.2, even more physical symmetries can be recovered from modular flow. At this point, however, I would like to highlight an extremely important aspect of modular flows and eq. (2.22) in particular.

Assume again that we are in a setting where the modular operator is given by eq. (2.16). For any operator in $B \otimes 1 \in \mathcal{A}$, we then have

$$\sigma_t(B \otimes 1) = \Delta^{\mathrm{i}t}(B \otimes 1)\Delta^{-\mathrm{i}t} = \rho_V^{\mathrm{i}t} B \rho_V^{-\mathrm{i}t} \otimes 1 \in \mathcal{A}.$$

This means that the correlation function

$$\langle \psi | (A \otimes 1)\sigma_t(B \otimes 1)\psi \rangle = \mathrm{Tr}[A\rho_V^{\mathrm{i}t} B \rho_V^{1-\mathrm{i}t}]$$

is analytic on the strip $-1 < \Im(t) < 0$. This is because, on this strip, both $\mathrm{i}t$ and $1 - \mathrm{i}t$ have positive real part and as a result $\|\rho_V^{\mathrm{i}t}\|, \|\rho_V^{1-\mathrm{i}t}\| < 1$. Similarly, the correlation function

$$\langle \psi | \sigma_t(B \otimes 1)(A \otimes 1)\psi \rangle = \mathrm{Tr}[\rho_V^{1+\mathrm{i}t} B \rho_V^{-\mathrm{i}t} A]$$

is analytic on the strip $0 < \Im(t) < 1$ and the two are related by

$$\langle \psi | \sigma_{t+\mathrm{i}}(B \otimes 1)(A \otimes 1)\psi \rangle = \langle \psi | (A \otimes 1)\sigma_t(B \otimes 1)\psi \rangle. \tag{2.23}$$

Remarkably, this relation continues to hold for general modular flows,[46,72] even in settings where a reduced density matrix does not exist. Explicitly, for operators $A, B \in \mathcal{A}$ in a von-Neumann algebra $\mathcal{A}$ and a cyclic separating vector Ω, we have that $\langle \Omega | A\sigma_t(B)\Omega \rangle$ is analytic on $-1 < \Im(t) < 0$, while $\langle \Omega | \sigma_t(B)A\Omega \rangle$ is analytic on $0 < \Im(t) < 1$ and both are related by

$$\langle \Omega | \sigma_{t+\mathrm{i}}(B)A\Omega \rangle = \langle \Omega | A\sigma_t(B)\Omega \rangle \tag{2.24}$$

In the context of QFT, this relation is known as the Kubo-Martin-Schwinger

(KMS) condition.[64,101] Its fundamental importance was first discovered in the context of thermal states by Haag, Hugenholtz, and Winnink.[102] The KMS condition provides a neat characterisation of the modular flow, as stated by Takesaki's theorem:[61,100] Let $U(t)$ be *any* strongly continuous one-parameter group on $\mathcal{H}$ preserving Ω. If $\tilde{\sigma}_t(B) := U(t)AU(t)^*$ preserves $\mathcal{A}$ and has analytic continuations satisfying eq. (2.24), then $U(t) = \Delta^{\mathrm{i}t}$.

Defining the *modular correlation function* by

$$G_t(A,B) := \begin{cases} \langle\Omega|A\sigma_t(B)\Omega\rangle & \text{for} \quad -1 < \Im(t) < 0 \\ \langle\Omega|\sigma_t(B)A\Omega\rangle & \text{for} \quad 0 < \Im(t) < 1, \end{cases} \tag{2.25}$$

you can use the KMS condition (2.24) to periodically extend $G_t(A,B)$ to arbitrary non-integer imaginary parts. The resulting function might be discontinuous along the real axis (hence at all integer imaginary parts of t), its behaviour being linked to the expectation value of the commutator

$$\Big[G_{t-\mathrm{i}0}(A,B) - G_{t+\mathrm{i}0}(A,B)\Big] = \langle\Omega|[A,\sigma_t(B)]\Omega\rangle \tag{2.26}$$

for real t. A variant of this link will be of great use later on in sections 2.3 and 2.4.

2.2 Universal results for modular flow

2.2.1 The argument by Bisognano and Wichmann

In section 2.1.3, I already explained that time evolution of a physical system is the same as modular flow of the thermal state. The key element of this connection is that the thermal state encodes the structure of the Hamiltonian in thermal correlations, which then show up as entanglement in the purification. While this may seem trivial at first glance—going from $\mathrm{e}^{-\beta H}$ to $\mathrm{e}^{-\mathrm{i}\beta t H}$—the construction of symmetries from entanglement is extremely powerful in the context of QFT. An important example of how it works in non-thermal cases is the celebrated Unruh effect.[17,59] I will discuss it in the context of Tomita-Takesaki theory, where it was discovered by Bisognano and Wichmann.[38,76]

While the result can be derived for an arbitrary Wightman QFT in any number of dimensions, I will stick to a simple example for the sake of clarity: Consider the vacuum state of a complex scalar field Φ in $1+1$ dimensions. We will derive the modular data of the algebra associated to the right Rindler wedge

$$\mathcal{R}_{\text{Rindler}} := \{(x^0,x^1) \in \mathbb{R}^{1,1} | x^1 > |x^0|\} = \{(x^+,x^-) \in \mathbb{R}^{1,1} | x^- < 0 < x^+\},$$

where I switched to lightray coordinates $x^{\pm} = x^0 \pm x^1$, as shown in fig. 2.4.

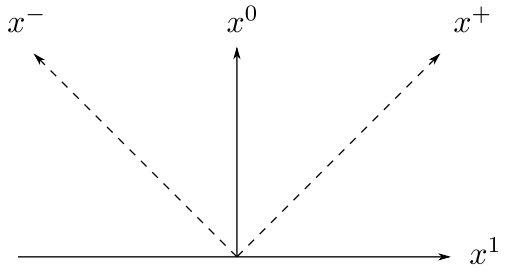


Figure 2.4: Axes (dashed) of the lightray coordinate system $x^{\pm} = x^0 \pm x^1$.

Since modular flow has to preserve $\mathcal{A}(\mathcal{R}_{\text{Rindler}})$, it makes sense to ask whether there is a geometric symmetry preserving $\mathcal{R}_{\text{Rindler}}$ and, indeed, there is one: The Lorentz boost Λ_θ with rapidity θ transforms the lightray coordinates as

$$\begin{bmatrix} x^+ \\ x^- \end{bmatrix} \overset{\Lambda_\theta}{\longmapsto} \begin{bmatrix} \mathrm{e}^{\theta} & \\ & \mathrm{e}^{-\theta} \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix},$$

hence, preserves the signs of $x^{\pm}$. As in any QFT, elements $(g, \alpha) \in \mathfrak{P}$ of the Poincaré group $\mathfrak{P} = \mathrm{O}(1,1) \ltimes \mathbb{R}^{1,1}$ have a unitary representation $U(g,\alpha)$ on the vacuum Hilbert space $\mathcal{H}_0$ leaving the cyclic vector Ω_0 invariant, i.e.,

$$U(g,\alpha)\Phi(x)U(g,\alpha)^* = \Phi(gx+\alpha)\Omega_0 \quad \text{and} \quad U(g,\alpha)\Omega_0 = \Omega_0.$$

The aim is to relate $U(\Lambda_\theta, 0)$ to the modular flow $\Delta^{\mathrm{i}\tau}$ and we will achieve it by working out some analyticity properties similar to those in section 2.1.3. First, note that within the Poincaré group $\mathfrak{P}$, we have the commutation relation

$$(\Lambda_\theta, 0)(0, x) = (0, \Lambda_\theta x)(\Lambda_\theta, 0),$$

hence,

$$\begin{aligned} U(\Lambda_\theta,0)\Phi(x)\Omega_0 &= U(\Lambda_\theta,0)U(1,x)\Phi(0)\Omega_0 \\ &= U(0,\Lambda_\theta x)U(\Lambda_\theta,0)\Phi(0)\Omega_0 \\ &= U(0,\Lambda_\theta x)\Phi(0)\Omega_0 \\ &= \mathrm{e}^{\mathrm{i}\mathrm{e}^{\theta}x^+P_+}\mathrm{e}^{\mathrm{i}\mathrm{e}^{-\theta}x^-P_-}\Phi(0)\Omega_0, \end{aligned} \tag{2.27}$$

where $P_\pm$ are the generators of translations in the coordinate directions $x^{\pm}$.

Because of the spectrum condition, the spectrum of P_μ has to lie in the forward light-cone, implying $P_\pm > 0$. Now, since $x^- < 0 < x^+$ in $\mathcal{R}_{\text{Rindler}}$, we can actually perform an analytic continuation of eq. (2.27) to $\Im(\mathrm{e}^{-\theta}) < 0 < \Im(\mathrm{e}^{\theta}) > 0$, which translates to $0 < \Im(\theta) < \pi$. In particular, expressing

$$U(\Lambda_\theta, 0) = \mathrm{e}^{\mathrm{i}\theta K}$$

in terms of the boost generator K, we find

$$\begin{aligned}\mathrm{e}^{-\pi K}\Phi(x)\Omega_0 &= \lim_{\theta\to\mathrm{i}\pi} U(\Lambda_\theta, 0)\Phi(x)\Omega_0 \\ &= \mathrm{e}^{-\mathrm{i}x^+P_+}\mathrm{e}^{\mathrm{i}x^-P_-}\Phi(0)\Omega_0 \\ &= \Phi(-x)\Omega_0.\end{aligned}$$

Using the same reasoning, you can extend this result to arbitrary products of the field operators, e.g., you can check that

$$\begin{aligned}U(\Lambda_\theta, 0)\Phi(x_1)\Phi(x_2)\Omega_0 &= U(\Lambda_\theta, 0)U(1, x_1)\Phi(0)U(1, x_1)^*U(1, x_2)\Phi(0)\Omega_0 \\ &= U(0, \Lambda_\theta x_1)U(\Lambda_\theta, 0)\Phi(0)U(1, x_1)^*U(1, x_2)\Phi(0)\Omega_0 \\ &= U(0, \Lambda_\theta x_1)\Phi(0)U(\Lambda_\theta, 0)U(1, x_2 - x_1)\Phi(0)\Omega_0 \\ &= \mathrm{e}^{\mathrm{i}\mathrm{e}^\theta x_1^\mu P_\mu}\Phi(0)\mathrm{e}^{\mathrm{i}\mathrm{e}^\theta(x_2^\mu - x_1^\mu)P_\mu}\Phi(0)\Omega_0,\end{aligned}$$

which again possesses an analytic continuation to $0 < \Im(\theta) < \pi$ with

$$\mathrm{e}^{-\pi K}\Phi(x_1)\Phi(x_2)\Omega_0 = \Phi(-x_1)\Phi(-x_2)\Omega_0 = \Phi(-x_2)\Phi(-x_1)\Omega_0,$$

as long as $x_1, x_2 - x_1 \in \mathcal{R}_{\text{Rindler}}$. The reordering on the right hand side is legal due to Einstein causality, because x_i have spacelike separation. By cyclicity of Ω_0, this shows that $\mathrm{e}^{-\pi K}$ is well defined on a dense subset of $\mathcal{H}_0$ and allows to define an operator Θ on a dense subset of $\mathcal{H}_0$ via

$$\Theta A\Omega_0 := \mathrm{e}^{-\pi K}A^*\Omega_0 \quad \text{for every} \quad A = \Phi(x_1)\cdots\Phi(x_n) \tag{2.28}$$

with $x_1, x_{i+1} - x_i \in \mathcal{R}_{\text{Rindler}}$.

We can now show that this charge, parity, and time (CPT) reversing operation Θ is anti-unitary. To do this, first note that for each A of the above form we have

$$\Theta A\Omega_0 = \Theta\Phi(-x_1)^*\cdots\Phi(-x_n)^*\Omega_0 =: A'\Omega_0,$$

with A' in the algebra of the left Rindler wedge. Also note that, for any such A', the vector valued function $\mathrm{e}^{\mathrm{i}\theta K}A'\Omega_0$ admits an analytic continuation to $-\pi < \Im(\theta) < 0$, by the same argument as before, such that $\mathrm{e}^{\pi\theta K}$ is well

defined on the dense set of states of the form $A'\Omega_0$. As a result, we can compute

$$\begin{aligned}\langle B\Omega_0|\Theta^*\Theta A\Omega_0\rangle &= \langle \Theta A\Omega_0|\Theta B\Omega_0\rangle \\ &= \langle A'\Omega_0|\mathrm{e}^{-\pi K}B^*\Omega_0\rangle \\ &= \lim_{\theta\to \mathrm{i}\pi}\langle A'\Omega_0|\mathrm{e}^{\mathrm{i}\theta K}B^*\Omega_0\rangle \\ &= \lim_{\theta\to \mathrm{i}\pi}\langle \mathrm{e}^{-\mathrm{i}\theta K}A'\Omega_0|B^*\Omega_0\rangle \\ &= \langle \mathrm{e}^{\pi K}\mathrm{e}^{-\pi K}A^*\Omega_0|B^*\Omega_0\rangle \\ &= \langle B\Omega_0|A\Omega_0\rangle,\end{aligned}$$

hence Θ is anti-unitary. This fact is known in QFT as the CPT theorem[38,46,64] and, as shown here, it can be traced back to the entanglement of the vacuum state.

Finally, anti-unitarity of Θ implies that we can rewrite eq. (2.28) as

$$SA\Omega_0 = A^*\Omega_0 = \Theta^*\mathrm{e}^{-\pi K}A\Omega_0$$

which gives a polar decomposition of the Tomita operator $S = J\Delta^{1/2}$ of $\mathcal{A}(\mathcal{R}_{\text{Rindler}})$. Since such a decomposition is unique, we must have

$$\Delta^{\mathrm{i}t} = \mathrm{e}^{-2\pi \mathrm{i}tK} = U(\Lambda_{-2\pi t}, 0) \quad \text{and} \quad J = \Theta^* = \Theta. \tag{2.29}$$

Since the above argument does not depend on any details of the model, eqs. (2.28) and (2.29) are valid for arbitrary Wightman QFTs. In fact, the only place where we used that Φ is a scalar field, was back in eq. (2.27). If Φ had values in some *finite dimensional* representation of $\mathfrak{P}$, you would expect some finite dimensional matrices in this equation, which would however not affect the analyticity argument at all. Note the emphasis on finite dimensionality: For fields in infinite dimensional $\mathfrak{P}$-representations such as large spin limits, there is no guarantee that eqs. (2.28) and (2.29) are still valid, hence, there might be no CPT theorem.[64]

2.2.2 Stronger results in conformal field theory

The idea of Bisognano and Wichmann hinges on using geometric symmetries to construct the modular operator. This means that we can expect even stronger results if we have more symmetry and so it is no surprise that, in a QFT with conformal symmetry (conformal field theory, CFT), Hislop and Longo[39] found a general result for the modular flow of a double cone. To illustrate this, let us again restrict to two dimensions. To match standard CFT notation, I will

denote the lightray coordinates by

$$z = x^+ \quad \text{and} \quad \bar{z} = x^-. \tag{2.30}$$

As we saw before, $z, \bar{z}$ transform independently, so we can first restrict to z and deal with $\bar{z}$ later. We consider the algebra associated to the interval $z \in [a, b]$. To derive the modular flow, we will proceed as follows: First, we map $[a, b]$ to $[0, \infty]$ via symmetry transformations. Since $[0, \infty]$ is the z component of the right Rindler wedge, we can employ the Bisognano-Wichmann result (2.29). Finally, we use symmetry transformations to map back to the interval $[a, b]$.

To this end let us first recall that the symmetry group of a CFT on a lightray is given by the projective special linear group

$$\mathrm{PSL}(2, \mathbb{R}) := \{g \in \mathbb{R}^{2\times 2} | \det g = 1\} \big/ \{1, -1\}, \tag{2.31}$$

acting on the coordinates via Möbius transformations, which amounts to the group action

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \triangleright z = \frac{\alpha z + \beta}{\gamma z + \delta}. \tag{2.32}$$

In generalisation of the covariance axiom from section 1.2.2, this group has a strongly continuous unitary representation on the Hilbert space of the CFT, such that

$$U(g)\mathcal{O}(z)U^\dagger(g) = \Big[(g \triangleright z)'\Big]^h \mathcal{O}(g \triangleright z) = (\gamma z + \delta)^{-2h} \mathcal{O}\Big(\frac{\alpha z + \beta}{\gamma z + \delta}\Big) \tag{2.33}$$

for any primary field $\mathcal{O}$ of scaling dimension h. It is instructive to check that eqs. (2.32) and (2.33) indeed give a group action and group representation.

Having discussed the general structure, we can now map $[a, b]$ to $[0, \infty]$ in three simple steps

$$[a, b] \xrightarrow{\text{translation}} [a - b, 0] \xrightarrow{\text{inversion}} [1/(b - a), \infty] \xrightarrow{\text{translation}} [0, \infty]. \tag{2.34}$$

The fist step, a translation by $-b$, is given by the action of the $\mathrm{PSL}(2, \mathbb{R})$ element

$$\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, \quad \text{since} \quad \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} \triangleright z = \frac{1z - b}{0z + 1} = z - b.$$

Proceeding similarly for the other steps and multiplying the corresponding

group elements, we find that

$$I := \begin{bmatrix} 1 & -1/(b-a) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/(b-a) & a/(b-a) \\ 1 & -b \end{bmatrix}.$$

indeed maps the interval $[a, b]$ to the half line $[0, \infty]$.

Now, from eq. (2.29), we already know that modular flow on $[0, \infty]$ implements the symmetry transformation $z \mapsto \mathrm{e}^{-2\pi t} z$, which coincides with the action of the $\mathrm{PSL}(2, \mathbb{R})$ element

$$\begin{bmatrix} \mathrm{e}^{-\pi t} & 0 \\ 0 & \mathrm{e}^{\pi t} \end{bmatrix}, \quad \text{because} \quad \begin{bmatrix} \mathrm{e}^{-\pi t} & 0 \\ 0 & \mathrm{e}^{\pi t} \end{bmatrix} \triangleright z = \frac{\mathrm{e}^{-\pi t} z + 0}{0 z + \mathrm{e}^{\pi t}} = \mathrm{e}^{-2\pi t} z.$$

Putting it all together, we find that modular flow on $[a, b]$ has to implement the $\mathrm{PSL}(2, \mathbb{R})$ element

$$I^{-1} \begin{bmatrix} \mathrm{e}^{-\pi t} & 0 \\ 0 & \mathrm{e}^{\pi t} \end{bmatrix} I = \frac{\mathrm{e}^{\pi t}}{b-a} \begin{bmatrix} b\mathrm{e}^{-2\pi t} - a & ab(1 - \mathrm{e}^{-2\pi t}) \\ \mathrm{e}^{-2\pi t} - 1 & b - a\mathrm{e}^{-2\pi t} \end{bmatrix}.$$

Finally, you can plug this into eqs. (2.32) and (2.33) to find the Hislop-Longo result[22, 39, 64, 103]

$$\Delta^{\mathrm{i}t} \mathcal{O}(z) \Delta^{-\mathrm{i}t} = \Big[\frac{\mathrm{e}^{\pi t}(b - z) + \mathrm{e}^{-\pi t}(z - a)}{b - a} \Big]^{-2h} \times \mathcal{O}\Big(\frac{\mathrm{e}^{2\pi t} a(b - z) + b(z - a)}{\mathrm{e}^{2\pi t}(b - z) + (z - a)} \Big). \tag{2.35}$$

Again, to obtain the modular conjugation J, first consider the analytic continuation

$$\begin{aligned} \Delta^{1/2} \mathcal{O}(z) \Omega_0 &= \lim_{t \to -\mathrm{i}/2} \Delta^{\mathrm{i}t} \mathcal{O}(z) \Omega_0 \\ &= \Big[- \Big(\frac{2z - (a+b)}{b-a} \Big)^2 \Big]^{-h} \mathcal{O}\Big(\frac{z(a+b) - 2ab}{2z - (a+b)} \Big) \Omega_0. \end{aligned}$$

Combining this with

$$\Delta^{1/2}\mathcal{O}(z)\Omega_0 = JJ\Delta^{1/2}\mathcal{O}(z)\Omega_0 = J\mathcal{O}^*(z)J\Omega_0,$$

you then find the CFT generalisation of the CPT theorem

$$J\mathcal{O}(z)J = \left[-\left(\frac{2z-(a+b)}{b-a}\right)^2\right]^{-h}\mathcal{O}^*\left(\frac{z(a+b)-2ab}{2z-(a+b)}\right). \tag{2.36}$$

It is instructive to show from this that J is indeed an involution.

We can now recover the second variable $\bar{z}$ by replacing all quantities by their barred siblings, i.e., the total modular flow is given by

$$\begin{aligned}\Delta^{\mathrm{i}t}\mathcal{O}(z,\bar{z})\Delta^{-\mathrm{i}t} = &\left[\frac{\mathrm{e}^{\pi t}(b-z)+\mathrm{e}^{-\pi t}(z-a)}{b-a}\right]^{-2h}\\ &\times\left[\frac{\mathrm{e}^{\pi t}(\bar{b}-\bar{z})+\mathrm{e}^{-\pi t}(\bar{z}-\bar{a})}{\bar{b}-\bar{a}}\right]^{-2\bar{h}}\\ &\times\mathcal{O}\left(\frac{\mathrm{e}^{2\pi t}a(b-z)+b(z-a)}{\mathrm{e}^{2\pi t}(b-z)+(z-a)},\frac{\mathrm{e}^{2\pi t}\bar{a}(\bar{b}-\bar{z})+\bar{b}(\bar{z}-\bar{a})}{\mathrm{e}^{2\pi t}(\bar{b}-\bar{z})+(\bar{z}-\bar{a})}\right).\end{aligned}$$

It is, however, important to keep in mind that $[a,b]$ is the projection of a double cone to the z axis, while $[\bar{a},\bar{b}]$ is its projection to the $\bar{z}$ axis, as pictured in fig. 2.5. In particular, we always require that $a<b$ and $\bar{a}<\bar{b}$, such that the ∞ in eq. (2.34) is positive.

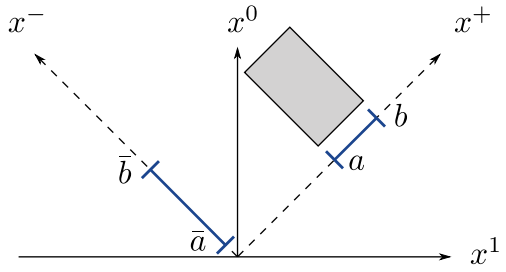


Figure 2.5: A double cone (grey) and its projections to the lightray coordinate axes (solid dark blue). Our convention for the order of boundary points is $a<b$ and $\bar{a}<\bar{b}$.

As an example, consider the unit double cone, whose left and right corners are located at $x^1_{\mathrm{L/R}} = \mp 1$, $x^0_{\mathrm{L/R}} = 0$. The top and bottom corners are then

given by $x^1_{\mathrm{T/B}} = 0$, $x^0_{\mathrm{T/B}} = \pm 1$, so that

$$a = x^+_{\mathrm{L}} = -1, \quad b = x^+_{\mathrm{T}} = 1, \quad \bar{a} = x^-_{\mathrm{R}} = -1, \quad \bar{b} = x^-_{\mathrm{T}} = 1.$$

Modular flow then transports an operator initially localised at $(z, \bar{z})$ along the trajectory

$$\big(z(t), \bar{z}(t)\big) = \Big(\frac{z\cosh \pi t - \sinh \pi t}{\cosh \pi t - z \sinh \pi t}, \frac{\bar{z}\cosh \pi t - \sinh \pi t}{\cosh \pi t - \bar{z}\sinh \pi t}\Big).$$

A plot of this trajectory in coordinates x^0, x^1 for various initial locations is given in fig. 2.6. You can see that modular flow approaches x_{T} in the limit $t \to -\infty$ and x_{B} in the limit $t \to \infty$. It never leaves the unit double cone, in accordance with Tomita's theorem (2.13).

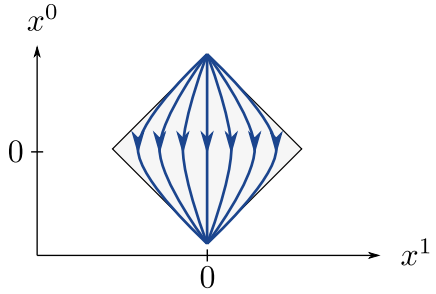


Figure 2.6: Trajectories (solid dark blue) of the modular flow (2.35) in standard coordinates for the unit double cone (light grey).

2.3 A framework for free fermion entanglement

In the previous section, we discussed vacuum modular flows of wedge and double cone geometries, which were universally given by eqs. (2.29) and (2.35) for any QFT and CFT, respectively. The aim of this section is to show how to extend these results to thermal states and different geometries in theories of free fermions. The reason why this is possible is because the thermal and vacuum state of a free theory are gaussian, which means that it is characterised by a functional kernel. As I will show in the following, computations of entanglement entropy, modular flow, the modular correlation function, and many others can be expressed in terms of this kernel only. This fact was initially discovered by Araki[104] but it was only made accessible to the general physics audience 30 years later by Peschel.[105,106] The first analytic results were presented by Casini and Huerta[107] and subsequently generalised by a small group

of people, including myself.[41,42,108–114] By now, after finishing the presentation of the method,[43] I feel like the case of free massless fermions in two dimensions (without boundaries) can be considered completely solved.

To explain the general procedure, consider a free fermion ψ in $d+1$ dimensions. For the sake of simplicity, we shall assume that the d spacial directions are compact. Assume now a generic quadratic Hamiltonian operator

$$H := \int \mathrm{d}^d x \, \mathrm{d}^d y \, \psi^*(x) h(x,y) \psi(y), \tag{2.37}$$

where $h(x,y) = \overline{h(y,x)}$ is some kernel which is hermitian to ensure self-adjointness of H. Since h is hermitian, there is some orthonormal basis of eigenfunctions f_k and eigenenergies ω_k such that

$$\int \mathrm{d}^d x \, \overline{f_k(x)} f_{k'}(x) = \delta_{k-k'}, \quad \sum_k f_k(x) \overline{f_k(y)} = \delta(x-y), \quad \text{and}$$
$$\int \mathrm{d}^d y \, h(x,y) f_k(y) = \omega_k f_k(x).$$

Here, the discreteness of the sum in the second equation is a result of compactness. Note that the last equality is equivalent to the eigendecomposition

$$h(x,y) = \sum_k \omega_k f_k(x) \overline{f_k(y)}. \tag{2.38}$$

The above implies that

$$\psi(x) = \int \mathrm{d}^d y \, \delta(x-y) \psi(y) = \sum_k \int \mathrm{d}^d y \, f_k(x) \overline{f_k(y)} \psi(y) = \sum_k f_k(x) c_k$$

with

$$c_k := \int \mathrm{d}^d y \, \overline{f_k(y)} \psi(y),$$

such that

$$H = \sum_{k,k'} c_k^* c_{k'} \int \mathrm{d}^d x \, \mathrm{d}^d y \, \overline{f_k(x)} h(x,y) f_{k'}(y) = \sum_k \omega_k c_k^* c_k. \tag{2.39}$$

Since the terms in this series commute among each other, this allows to exponentiate H to obtain

$$\mathrm{e}^{-H} = \prod_k \mathrm{e}^{-\omega_k c_k^* c_k} = \prod_k \left[1 + (\mathrm{e}^{-\omega_k} - 1) c_k^* c_k\right] = \prod_k \left[c_k c_k^* + \mathrm{e}^{-\omega_k} c_k^* c_k\right].$$

Here, the last equality is a result of the canonical anti-commutation relations

$$\{c_k, c_{k'}^*\} := c_k c_{k'}^* + c_{k'}^* c_k = \delta_{k-k'}. \tag{2.40}$$

Again, since the terms $c_k^* c_k$ commute among another, we can assume that they act on individual Hilbert spaces $\mathcal{H}_k$, i.e., the total Hilbert space $\mathcal{H}$ has the form $\mathcal{H} = \bigotimes_k \mathcal{H}_k$. In particular, to achieve an irreducible representation of eq. (2.40), each $\mathcal{H}_k$ can be taken to be two dimensional, carrying eigenvalues 0 and 1 of $c_k^* c_k$. As a result, we can define a trace on $\mathcal{H}$ and obtain the *thermal state*

$$\frac{\mathrm{e}^{-H}}{\mathrm{Tr}[\mathrm{e}^{-H}]} = \prod_k \left[\frac{1}{1+\mathrm{e}^{-\omega_k}} c_k c_k^* + \frac{\mathrm{e}^{-\omega_k}}{1+\mathrm{e}^{-\omega_k}} c_k^* c_k \right]. \tag{2.41}$$

Finally, you can compute the propagator G in this state via

$$\begin{aligned} G(x,y) &= \frac{\mathrm{Tr}[\mathrm{e}^{-H}\psi(x)\psi^*(y)]}{\mathrm{Tr}[\mathrm{e}^{-H}]} \\ &= \sum_{k,k'} f_k(x)\overline{f_{k'}(y)} \frac{\mathrm{Tr}[\mathrm{e}^{-H} c_k c_{k'}^*]}{\mathrm{Tr}[\mathrm{e}^{-H}]} \\ &= \sum_{k,k'} f_k(x)\overline{f_{k'}(y)} \delta_{k-k'} \frac{1}{1+\mathrm{e}^{-\omega_k}} \\ &= \sum_k \frac{1}{1+\mathrm{e}^{-\omega_k}} f_k(x)\overline{f_k(y)}. \end{aligned}$$

Compare this with eq. (2.38) to see that G and h have the same eigenfunctions and their eigenvalues are related by

$$\omega_k \leftrightarrow \frac{1}{1+\mathrm{e}^{-\omega_k}},$$

which means that the same relation also must hold on the level of operators, i.e.,

$$G = \frac{1}{1+\mathrm{e}^{-h}}, \tag{2.42}$$

where the functional dependence has to be defined in terms of convergent power series or some functional calculus, such as the one introduced in the next subsection.

Now, going back to entanglement, we essentially reverse the above argument. Consider a fermionic theory in d dimensions and a gaussian state, given by the density matrix ρ. By Wick's theorem,[74] the gaussian state ρ is entirely characterised by its propagator on a Cauchy slice. Now pick a (compact)

subregion V of this Cauchy slice and consider the algebra of observables

$$\mathcal{A}(V) := \{\int_V \mathrm{d}^d x\, f(x)\psi(x) : f \text{ test functions on } V\}'' \tag{2.43}$$

associated to it. As in section 2.1.1, we have to take a bi-commutant—denoted by the double prime—in order to obtain a von-Neumann algebra. The assignment $V \mapsto \mathcal{A}(V)$ satisfies all the axioms of a Haag-Kastler net in the sense of section 1.2.2, we only have to generalise the axiom of Einstein causality to mean *graded* commutativity of spacially separated regions. While V is only defined as a subset of the chosen Cauchy slice, we can luckily identify $\mathcal{A}(V)$ with the algebra localised on the minimal union $\mathcal{R}$ of double cones that cover V, as sketched in fig. 2.7. This is possible because a field localised anywhere inside $\mathcal{R}$ may be evolved in time until it is localised entirely inside V.

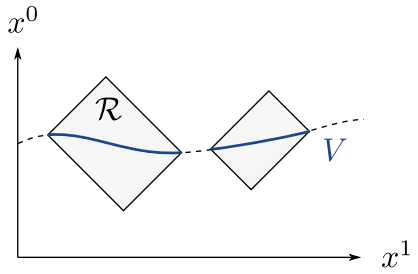


Figure 2.7: A Cauchy slice (dashed black) with subregion V (solid dark blue). The region V can be covered by a minimal union $\mathcal{R}$ of double cones (light grey), whose corresponding local algebra we denote by $\mathcal{A}(V)$.

We are now interested in the modular data of $\mathcal{A}(V)$ with respect to ρ. To obtain it, we will assume that a reduced density matrix ρ_V exists, instead of working with analyticity arguments as in section 2.2.1. If you are worried that this may not be a justified assumption, I would like to refer you to the rigorous paper by Hollands,[112] where some of the following computations were indeed derived from the basic axioms of AQFT.

So, if a reduced density matrix ρ_V exists, what can we say about it? Certainly, it should be expressible as an element of the local algebra $\mathcal{A}(V)$ and it should yield the same expectation values for algebra elements as the original state ρ, i.e.,

$$\mathrm{Tr}[\rho_V A] = \mathrm{Tr}[\rho A] \quad \text{for all} \quad A \in \mathcal{A}(V). \tag{2.44}$$

This implies that ρ_V is gaussian, since ρ is gaussian. We can thus write

$$\rho_V = \frac{\mathrm{e}^{-K}}{\mathrm{Tr}[\mathrm{e}^{-K}]} \quad \text{with} \quad K = \int_V \mathrm{d}^d x\, \mathrm{d}^d y\, \psi^*(x) k(x,y) \psi(y) \tag{2.45}$$

for some hermitian kernel k, where the integration is performed only over V, as $\rho_V \in \mathcal{A}(V)$. In addition, inserting $A = \psi(x)\psi^*(y)$ for $x, y \in V$ in eq. (2.44), you see that

$$\mathrm{Tr}[\rho_V \psi(x)\psi^*(y)] = \mathrm{Tr}[\rho \psi(x)\psi^*(y)] = G|_V(x,y),$$

where the right hand side is the restriction of the propagator G to V. However, as we saw earlier in eq. (2.42), this already implies that[105]

$$k = -\log(G|_V^{-1} - 1) = -\log\Big[\frac{1 - G|_V}{G|_V}\Big], \tag{2.46}$$

which is a formula for the reduced density matrix in terms of the restricted propagator. This formula is at the heart of modular theory for free fermions.[104] In fact, because of eq. (2.16), we could say that the reduced density matrix ρ_V and the modular operator Δ coincide *inside* V. In this sense, K is actually the (negative) logarithm of the modular operator and we will call it the *modular Hamiltonian* from hereon.

We can now do additional computations in the same way: The modular flow (2.19) is given by

$$\begin{aligned}
\sigma_t(\psi^*(x)) &= \Delta^{\mathrm{i}t} \psi^*(x) \Delta^{-\mathrm{i}t} \\
&= \rho_V^{\mathrm{i}t} \psi^*(x) \rho_V^{-\mathrm{i}t} \\
&= \mathrm{e}^{-\mathrm{i}tK} \psi^*(x) \mathrm{e}^{\mathrm{i}tK} \\
&= \mathrm{e}^{-\mathrm{i}t[K,-]} \psi^*(x) \\
&= \psi^*(x) + \frac{(-\mathrm{i}t)^1}{1!}[K, \psi^*(x)] + \frac{(-\mathrm{i}t)^2}{2!}[K,[K,\psi^*(x)]] + \dots
\end{aligned}$$

and since

$$\begin{aligned}
[K, \psi^*(x)] &= \int_V \mathrm{d}^d y\, \mathrm{d}^d z\, k(y,z)[\psi^*(y)\psi(z), \psi^*(x)] \\
&= \int_V \mathrm{d}^d y\, \mathrm{d}^d z\, k(y,z)\Big(-\{\psi^*(x), \psi^*(y)\}\psi(z) + \psi^*(y)\{\psi^*(x), \psi(z)\}\Big) \\
&= \int_V \mathrm{d}^d y\, \psi^*(y) k(y,x),
\end{aligned}$$

this yields

$$\sigma_t(\psi^*(x)) = \int_V \mathrm{d}^d y\, \psi^*(y)\Sigma_t(y,x) \quad \text{with} \quad \Sigma_t = \mathrm{e}^{-\mathrm{i}tk} = \Big[\frac{1-G|_V}{G|_V}\Big]^{\mathrm{i}t}. \tag{2.47}$$

As a direct consequence, the kernel of the modular Hamiltonian can be recovered from Σ_t via

$$k = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t\Big|_{t=0}. \tag{2.48}$$

Furthermore, for the modular correlation function (2.25) of two fields, we find

$$\begin{aligned} G_t(x,y) &:= \mathrm{Tr}\left[\rho_V \psi(x)\sigma_t\Big(\psi^*(y)\Big)\right] \\ &= \int_V \mathrm{d}^d z\ \mathrm{Tr}[\rho_V \psi(x)\psi^*(z)]\Sigma_t(z,y) \\ &= \int_V \mathrm{d}^d z\, G(x,z)\Sigma_t(z,y) \\ &= \Big(G|_V\Big[\frac{1-G|_V}{G|_V}\Big]^{\mathrm{i}t}\Big)(x,y). \end{aligned} \tag{2.49}$$

As far as I know, this formula as well as eq. (2.47), were first introduced by our group at Würzburg.[43]

An observation that was also presented in our paper[43] is the following: Sometimes it might be easier to compute the modular correlation function eq. (2.49) than the modular flow of the fields eq. (2.47). In such a case, or also just as a sanity check of results, you can use the following graded variant of eq. (2.26): For fermionic fields, the analytic continuation of the modular correlation function is actually defined via[112]

$$G_t(x,y) := \begin{cases} \langle\Omega|\psi(x)\sigma_t(\psi^*(y))\Omega\rangle & \text{for} \quad -1 < \Im(t) < 0 \\ -\langle\Omega|\sigma_t(\psi^*(y))\psi(x)\Omega\rangle & \text{for} \quad 0 < \Im(t) < 1. \end{cases} \tag{2.50}$$

Note the peculiar negative sign in the second case, which yields, in analogy to eq. (2.26),

$$\begin{aligned} &G_{t-\mathrm{i}0}(x,y) - G_{t+\mathrm{i}0}(x,y) \\ &\qquad = \mathrm{Tr}\left[\rho_V\{\psi(x), \sigma_t(\psi(y))\}\right] \\ &\qquad = \int_V \mathrm{d}^d z\, \Sigma_t(z,y)\, \mathrm{Tr}\left[\rho_V\{\psi(x), \psi^*(z)\}\right] \\ &\qquad = \Sigma_t(x,y), \end{aligned} \tag{2.51}$$

which gives a direct way for retrieving Σ_t from the modular correlation func-

tion.

Entanglement entropy (1.17) and entanglement Rényi entropy (1.19) can be computed similarly to the above: From eq. (2.41), we know that the reduced density matrix takes the form

$$\rho_V = \prod_k \Big[g_k c_k c_k^* + (1-g_k) c_k^* c_k \Big],$$

where g_k are the eigenvalues of $G|_V$. With regards to entanglement Rényi entropy, this means that

$$\operatorname{Tr}[\rho_V^\alpha] = \prod_k \operatorname{Tr} \Big[g_k^\alpha c_k c_k^* + (1-g_k^\alpha) c_k^* c_k \Big] = \prod_k \Big[g_k^\alpha + (1-g_k)^\alpha \Big],$$

hence

$$\log \operatorname{Tr}[\rho_V^\alpha] = \sum_k \log \Big[g_k^\alpha + (1-g_k)^\alpha \Big] = \operatorname{Tr} \log \Big[G|_V^\alpha + (1-G|_V)^\alpha \Big].$$

It is important to keep in mind that the trace on the right hand side is a trace of a distributional kernel, while the trace on the left hand side is a trace on the Hilbert space of the theory. Plugging this into the formula (1.19) for entanglement Rényi entropy, you obtain

$$S_\alpha = \frac{1}{1-\alpha} \operatorname{Tr} \log \Big[G|_V^\alpha + (1-G|_V)^\alpha \Big]. \tag{2.52}$$

As a result of this and eq. (1.20), you also find that entanglement entropy is given by[106, 107]

$$S_{\mathrm{EE}} = -\operatorname{Tr} \log \Big[G|_V \log G|_V + (1-G|_V) \log(1-G|_V) \Big]. \tag{2.53}$$

This concludes the derivation of the most important formulae for free fermion entanglement. For later convenience, they are summarised in table 2.1.

Table 2.1: Functional formulae for deriving entanglement data of free fermions from the restricted propagator $G = G|_V$ for later reference, along with their locations in the main text to find their derivation.

Formula	**Location**
$\sigma_t(\psi^*(x)) = \int_V \mathrm{d}^d y\, \psi^*(y)\Sigma_t(y,x), \quad \Sigma_t = [(1-G)/G)]^{\mathrm{i}t}$	(2.47)
$G_t(x,y) = \Big(G[(1-G)/G]^{\mathrm{i}t}\Big)(x,y)$	(2.49)
$k = -\log[(1-G)/G)]$	(2.46)
$S_\alpha = \frac{1}{1-\alpha} \operatorname{Tr} \log\Big[G^\alpha + (1-G)^\alpha\Big]$	(2.52)
$S_{\mathrm{EE}} = -\operatorname{Tr} \log\Big[G \log G + (1-G)\log(1-G)\Big]$	(2.53)

2.3.1 Resolvent method

So far, we have seen in eqs. (2.46), (2.49), (2.52) and (2.53), that the entanglement structure of free fermions can be recovered in functional dependence from the restricted propagator $G|_V$. The next question is now, how these equations can be evaluated: How do you actually compute arbitrary functions $f(G|_V)$ of the propagator $G|_V$ and how are these even defined? To this end, one has to introduce some notion of functional calculus.[115]

The easiest way to do this is by the functional generalisation of Cauchy's integral formula. To this end, note first that eq. (2.42) implies that the spectrum of G and $G|_V$ are contained in the open unit interval $(0, 1)$. For any complex function f that is analytic on this interval, we therefore have[115]

$$f(G|_V) = \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \searrow 0} \oint_{[\epsilon, 1-\epsilon]^{\circlearrowleft}} \mathrm{d}\lambda \, \frac{f(\lambda)}{\lambda - G|_V}, \tag{2.54}$$

where $[\epsilon, 1-\epsilon]^{\circlearrowleft}$ denotes a contour that wraps tightly around the interval $[\epsilon, 1-\epsilon]$ in a counter-clockwise manner. In the limit $\epsilon \searrow 0$, the above contour encircles all points in the open unit interval $(0, 1)$. The points $\lambda = 0, 1$ are never contained, which makes sense since eq. (2.42) tells us that they correspond to infinite eigenenergies of the kernel of the modular Hamiltonian. Typically in QFT, you want to avoid such points, which is exactly what the ϵ in eq. (2.54) is for. This reduces the computation of $f(G|_V)$ to finding the *resolvent* $1/(\lambda - G|_V)$ and performing a complex integral.

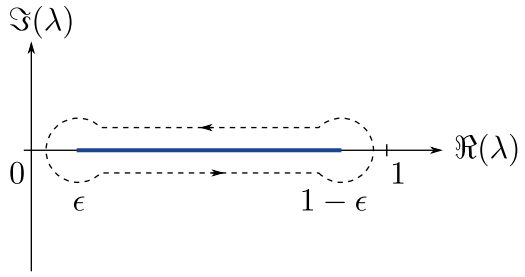


Figure 2.8: Contour of integration (dashed black) in eq. (2.54), wrapping around the interval $[\epsilon, 1-\epsilon]$ (solid dark blue). As pictured, it can be decomposed into circular and horizontal contributions, the latter of which lead to the spectral calculus (2.55).

As sketched in fig. 2.8, the above contour can be decomposed into four pieces: two portions running *along* the interval just above and below the real axis, as well as two circles around the endpoints $\epsilon, 1-\epsilon$. Let us assume now that

$G|_V$ has continuous spectrum (this will be the case for all propagators that we consider). This implies that $1/(\lambda - G|_V)$ has no poles of order ≥ 1 inside the open interval $(0,1)$, hence, you immediately see that the contribution from the circular pieces of the contour vanishes.[41,43] In this case, eq. (2.54) reduces to

$$f(G|_V) = \frac{1}{2\pi\mathrm{i}} \int_0^1 \mathrm{d}\lambda\, f(\lambda) \Big[\frac{1}{\lambda - \mathrm{i}0 - G|_V} - \frac{1}{\lambda + \mathrm{i}0 - G|_V} \Big], \tag{2.55}$$

where the $\pm\mathrm{i}0$-terms denote that the integral is performed as the (weak) limit

$$\lim_{\epsilon \searrow 0} \int_0^1 \mathrm{d}\lambda\, f(\lambda) \Big[\frac{1}{\lambda - \mathrm{i}\epsilon - G|_V} - \frac{1}{\lambda + \mathrm{i}\epsilon - G|_V} \Big].$$

Considering now the spectral calculus[115]

$$f(G|_V) = \int_{[0,1]} \mathrm{d}E_\lambda\, f(\lambda) = \int_0^1 \mathrm{d}\lambda\, \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} f(\lambda) \tag{2.56}$$

of $G|_V$ with the corresponding *spectral measure* E_λ originating in the spectral decomposition

$$G|_V = \int_{[0,1]} \mathrm{d}E_\lambda\, \lambda,$$

you see that eq. (2.55) allows to express the Radón-Nikodym derivative[116] of E_λ by

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{1}{2\pi\mathrm{i}} \Big[\frac{1}{\lambda - \mathrm{i}0 - G|_V} - \frac{1}{\lambda + \mathrm{i}0 - G|_V} \Big]. \tag{2.57}$$

In particular, if you are interested in computing the trace of a function of $G|_V$, as in eqs. (2.52) and (2.53), you can write

$$\mathrm{Tr}[f(G|_V)] = \int_0^1 \mathrm{d}\lambda\ \mathrm{Tr} \left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} \right] f(\lambda) \tag{2.58}$$

and the trace only has to be performed on the (derivative of the) spectral measure. This is very useful since this trace is done by a separate integration over V via[107]

$$\mathrm{Tr} \left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} \right] = \int_V \mathrm{d}x\, \lim_{y \to x} \frac{\mathrm{d}E_\lambda(x,y)}{\mathrm{d}\lambda}. \tag{2.59}$$

As stated before, the problem of evaluating arbitrary functions of the restricted propagator is now reduced to finding the resolvent and a (generally non-trivial) complex integration. So how to find the resolvent $1/(\lambda - G|_V)$? As it turns out, this problem can again be reduced to a well understood one in the following way: By its very definition, the resolvent has to satisfy the

functional equation

$$(\lambda - G|_V)\frac{1}{\lambda - G|_V} = 1.$$

You can now insert the ansatz

$$\frac{1}{\lambda - G|_V} = \frac{1}{\lambda} + \frac{F_\lambda}{\lambda^2} \tag{2.60}$$

with some kernel F_λ to find the functional equation[41,109]

$$-G|_V + F_\lambda - \frac{1}{\lambda} G|_V F_\lambda = 0.$$

Since multiplication in this equation is actually given by integration of the free variables, this translates to the integral equation[41,107]

$$-G(x,y) + F_\lambda(x,y) - \frac{1}{\lambda}\int_V \mathrm{d}^d z\, G(x,z)F_\lambda(z,y) = 0 \quad \text{for} \quad x,y \in V. \tag{2.61}$$

Remarkably, all of this subsection is valid for fairly general fermionic theories – the only requirement being that the state under consideration is gaussian. In particular, eq. (2.61) yields the correct resolvent in arbitrary dimensions, as well as for massive theories. However, and this is the bad news, eq. (2.61) has been solved analytically only for massless theories in two dimensions, because there the specific form of the propagator G allows to use complex analysis methods,[117] as will be discussed in section 2.4.1.

2.3.2 Deriving the propagator

As already stated in the previous subsection, from now on, we will restrict to massless fermions in two dimensions in order to be able to solve the integral equation (2.61). I would like to begin this discussion at its very start: The quantisation of the Dirac field. To this end, consider the Dirac action

$$S[\psi,\bar\psi] := \int \mathrm{d}^2x\, \mathcal{L} \quad \text{with} \quad \mathcal{L} = \mathrm{i}\bar\psi\gamma^\mu\partial_\mu\psi, \tag{2.62}$$

where the Dirac adjoint is given by $\bar\psi = \psi^*\gamma^0$ and the Dirac matrices γ^μ have to satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}.$$

As before, I am using the convention of *mostly negative* signature $(\eta_{\mu\nu}) = \mathrm{diag}(1,-1)$, even though this term does not make much sense in two dimen-

sions. A possible real-valued choice for the γ^μ is

$$\gamma^0 = \sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \text{and} \quad \gamma^1 = \mathrm{i}\sigma_y = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{2.63}$$

To derive the Hamiltonian corresponding to eq. (2.62), first note that

$$\frac{\partial \mathcal{L}}{\partial \partial_0 \psi} = \mathrm{i}\bar{\psi}\gamma^0 = \mathrm{i}\psi^*,$$

hence

$$\mathcal{H} = \mathrm{i}\psi^*\partial_0\psi - \mathcal{L} = -\mathrm{i}\psi^*\gamma^0\gamma^1\partial_1\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}^* \begin{bmatrix} \mathrm{i}\partial_1 & \\ & -\mathrm{i}\partial_1 \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}.$$

This means that

$$H = H_+ + H_- \quad \text{with} \quad H_\pm = \pm \int \mathrm{d}x\, \mathrm{i}\psi_\pm^*(x)\partial_x\psi_\pm(x), \tag{2.64}$$

so that the equations of motion are

$$\partial_t \psi_\pm = \frac{\partial \mathcal{H}_\pm}{\partial(\mathrm{i}\psi_\pm^*)} = \pm\partial_x\psi_\pm$$

or, equivalently,

$$\partial_\mp \psi_\pm = \frac{1}{2}(\partial_0 \mp \partial_1)\psi_\pm = 0 \tag{2.65}$$

in terms of the lightray coordinates $x^\pm = x^0 \pm x^1$. As a result, $\psi_\pm$ is a function of $x^\pm$ only and we can rewrite

$$H_\pm = \pm \int \mathrm{d}x^\pm\, \mathrm{i}\psi_\pm^*(\partial_+ - \partial_-)\psi_\pm = \int \mathrm{d}x^\pm\, \mathrm{i}\psi_\pm^*\partial_\pm\psi_\pm :, \tag{2.66}$$

such that $H_\pm$ are formally identical. In this setting $H_\pm$ generate translations along the lightray coordinates $x^\pm$. Furthermore, since $H_\pm$ commute, they can be quantised individually and the quantum theory consists of two identical copies of a single one dimensional theory, the two so-called *chiralities*. From now on, I will drop the $\pm$ and deal with just one of the two chiralities. The full result can be restored later on in the same way as discussed at the end of section 2.2.2.

We are now ready to derive the propagator for different geometries. Let us begin with the propagator on the real line $x^{(\pm)} \in \mathbb{R}$ at inverse temperature $\beta > 0$. It can be derived from eq. (2.42) as

$$G = \frac{1}{1 + \mathrm{e}^{-\mathrm{i}\beta\partial}} \tag{2.67}$$

but of course, the right hand side looks rather cryptic. To evaluate it, we use the following Fourier technique: Since

$$\delta(x-y) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k(x-y)},$$

we have

$$\begin{aligned} G(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}z\, G(x,z)\mathrm{e}^{-\mathrm{i}k(z-y)} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}z\, \frac{1}{1+\mathrm{e}^{-\mathrm{i}\beta\partial}}(x,z)\mathrm{e}^{-\mathrm{i}k(z-y)} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, \frac{1}{1+\mathrm{e}^{-\mathrm{i}\beta(-\mathrm{i}k)}}\mathrm{e}^{-\mathrm{i}k(x-y)} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, \frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{1+\mathrm{e}^{-\beta k}}. \end{aligned} \tag{2.68}$$

Note that

$$G(x,y) = G(x-y) \tag{2.69}$$

is translationally invariant, as was to be expected since G is a function of ∂ via eq. (2.67). Note also that the integral in eq. (2.68) converges absolutely for $-\beta < \Im(x-y) < 0$ and thus yields an analytic function on this strip, as already implied from the KMS condition (2.24).

Now, before evaluating eq. (2.68) for general β, first consider the low temperature limit $\beta \to \infty$. In this limit the integrand vanishes for negative k due to the $\mathrm{e}^{-\beta k}$ in the denominator. For positive k, the denominator becomes unity and we find the vacuum propagator on the line

$$G(z) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kz} = \frac{1}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}kz}}{-\mathrm{i}z}\Big|_{k=0}^{k\to\infty} = \frac{1}{2\pi\mathrm{i}z}. \tag{2.70}$$

For the case of finite temperature in eq. (2.68), first change variables to $s = \mathrm{e}^{-\beta k}$ with $\mathrm{d}s/s = -\beta \mathrm{d}k$ to find

$$G(z) = \frac{1}{2\pi\beta}\int_0^\infty \frac{\mathrm{d}s}{s}\frac{z^{\mathrm{i}z/\beta}}{1+s}.$$

This can be related to the contour integral

$$I = \frac{1}{2\pi\beta} \oint_\Gamma \frac{\mathrm{d}s}{s} \frac{(-s)^{\mathrm{i}z/\beta}}{1+s}$$

along the Hankel contour Γ as shown in fig. 2.9.

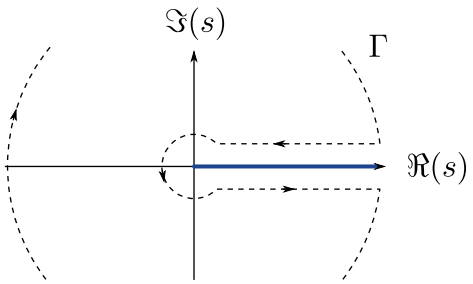


Figure 2.9: Hankel contour Γ (dashed black), encircling all of the complex plane except for the origin and the positive real axis $\mathbb{R}_+$ (solid dark blue). This contour is particularly useful for evaluating integrals whose integrand possesses a multiplicative branch cut along $\mathbb{R}_+$, as explained in the main text.

You can decompose Γ into four pieces: One just above and one just below the real axis, as well as circular contributions around the origin and at infinity. The contribution at the origin $s \to 0$ vanishes, since the integrand behaves as $\sim s^{-1-\Im(z)/\beta}$ and we always have $\Im(z) < 0$ by assumption. Similarly, the contribution at infinity $|s| \to \infty$ vanishes as well, because the integrand behaves as $\sim |s|^{-2-\Im(z)/\beta}$ and we have $-\beta < \Im(z)$ by assumption. You thus find

$$\begin{aligned} I &= \frac{1}{2\pi\beta} \int_0^\infty \frac{\mathrm{d}s}{s(1+s)} \Big[(-s+\mathrm{i}0)^{\mathrm{i}z/\beta} - (-s-\mathrm{i}0)^{\mathrm{i}z/\beta} \Big] \\ &= \frac{1}{2\pi\beta} \int_0^\infty \frac{\mathrm{d}s}{s(1+s)} s^{\mathrm{i}z/\beta} \Big[\mathrm{e}^{-\pi z/\beta} - \mathrm{e}^{\pi z/\beta} \Big] \\ &= -2 \sinh \frac{\pi z}{\beta} \times G(z), \end{aligned}$$

On the other hand, I can be evaluated via the residue theorem for the interior of the Hankel contour. As a result,

$$I = -\frac{1}{2\pi\beta} 2\pi\mathrm{i} \lim_{s\to-1} \frac{(-s)^{\mathrm{i}z/\beta}}{s} = -\frac{1}{\mathrm{i}\beta}$$

and, hence,[118]

$$G(z) = \frac{1}{2\mathrm{i}\beta \sinh \pi z/\beta}. \tag{2.71}$$

Note that eq. (2.71) looks like eq. (2.70) for $|z| \ll \beta$, in the sense that

$$\frac{1}{2\mathrm{i}\beta \sinh \pi z/\beta} = \frac{1}{2\pi \mathrm{i} z} + \mathcal{O}(1) \quad \text{as} \quad |z|/\beta \to 0.$$

This is a general feature of CFTs:[118] For very small $|z|$, we approach the ultraviolet (UV) fixed point of the theory, where any scale such as β is infinite. In fact, we could have used this reasoning to arrive at eq. (2.71) in an entirely different way: At finite temperature, $G(z)$ has to be analytic on the strip $-\beta < \Im(z) < 0$ and satisfy

$$G(z) = -G(z - \mathrm{i}\beta) \tag{2.72}$$

due to the KMS condition (2.24). Near the real axis, we can employ eq. (2.51), where translation in t equals translation in $z = x - y$ because modular flow of the thermal state coincides with $\mathrm{e}^{-\mathrm{i}t\beta H}$. Since $\Sigma_0(x, y) = 0$ except at $x - y = 0$, we find that $G(z)$ is continuous, hence, analytic by the Schwarz reflection principle,[116] except at $z = 0$. Finally, as seen above, $G(z) \sim 1/2\pi \mathrm{i} z$ near $z = 0$. This puts enough restrictions on G to uniquely fix it due to the Riemann-Roch theorem,[119] which is basically a statement about the uniqueness of solutions of the Cauchy-Riemann differential equations. Since $1/2\mathrm{i}\beta \sinh(\pi z/\beta)$ satisfies these restrictions, it has to coincide with $G(z)$ due to uniqueness.

Let us now proceed to the propagator on the circle $x^{(\pm)} \in [0, 1]$ at inverse temperature β. Since we are dealing with fermions, only even powers of the field operators have to be periodic on the circle. As a result, the fermion field could be periodic (P) or antiperiodic (A), i.e,

$$\psi(x + 1) = \pm\psi(x). \tag{2.73}$$

Historically these two spin structures are known as the Ramond (P) and Neveu-Schwarz (A) sectors of the free fermion CFT.[118] Note also that we restrict to a circle circumference of 1. This, however, does not limit generality, as we are dealing with a CFT and a different circumference can be recovered by a scale transformation. For the propagator G, eq. (2.73) implies (anti-)periodic behaviour in both arguments. This means that we can obtain it from eq. (2.67) by another Fourier method: Note that on the unit interval $[0, 1]$, the (anti-

)periodic Dirac distributions are given by[82]

$$\delta_{\mathrm{P}}(x-y)=\sum_{k\in\mathbb{Z}}\mathrm{e}^{-2\pi\mathrm{i}k(x-y)}\quad\text{and}\quad\delta_{\mathrm{A}}(x-y)=\sum_{k\in\mathbb{Z}+1/2}\mathrm{e}^{-2\pi\mathrm{i}k(x-y)}.$$

This implies that

$$\begin{aligned}G(x,y)&=\sum_k\int_0^1\mathrm{d}z\,G(x,z)\mathrm{e}^{-2\pi\mathrm{i}k(z-y)}\\&=\sum_k\int_0^1\mathrm{d}z\,\frac{1}{1+\mathrm{e}^{-\mathrm{i}\beta\partial}}(x,z)\mathrm{e}^{-2\pi\mathrm{i}k(z-y)}\\&=\sum_k\frac{\mathrm{e}^{-2\pi\mathrm{i}k(x-y)}}{1+\mathrm{e}^{-2\pi\beta k}},\end{aligned}\tag{2.74}$$

where the sum runs over $\mathbb{Z}$ ($\mathbb{Z}+1/2$) in the periodic (anti-periodic) case and you see that translational invariance (2.69) sill holds. Also, as you can easily check, the above series are absolutely convergent on the strip $-\beta<\Im(x-y)<0$, in accordance with the KMS condition (2.24).

Again, let us first have a look at the limit $\beta\to\infty$ of low temperature. In this case we obtain

$$G_{\mathrm{P}}(z)=\frac{1}{2}+\sum_{k=1}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}kz}\quad\text{and}\quad G_{\mathrm{A}}(z)=\sum_{k=1/2}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}kz},\tag{2.75}$$

where the first term in G_{P} arises for $k=0$. These series converge absolutely for $\Im(z)<0$ and evaluate to[118]

$$G_{\mathrm{P}}(z)=\frac{1}{2\mathrm{i}}\cot\pi z\quad\text{and}\tag{2.76}$$

$$G_{\mathrm{A}}(z)=\frac{1}{2\mathrm{i}}\csc\pi z,\tag{2.77}$$

respectively. There are two ways to see why this is the case: First, similarly to the case of eq. (2.71), $G_{\mathrm{P/A}}$ has the correct UV behaviour $\sim 1/2\pi\mathrm{i}z$, the required analyticity, and the correct periodicity. It is thus the unique solution of eq. (2.75) because of the Riemann-Roch theorem. Second and more directly, one can compute the Fourier series of $G_{\mathrm{P/A}}$ and verify that it coincides with eq. (2.75). Let us do this exercise explicitly for G_{A}. Because of anti-periodicity, there has to be a Fourier decomposition of the type

$$\frac{1}{2\mathrm{i}}\csc\pi z=\sum_{k\in\mathbb{Z}+1/2}g_k\mathrm{e}^{-2\pi\mathrm{i}kz},$$

with Fourier coefficients g_k. Note that we require this representation to be valid for z in the lower half plane. I will make this explicit by replacing $z \to z - \mathrm{i}\epsilon$ for some small $\epsilon > 0$ that will be sent to 0 in the end of the computation. The Fourier coefficients are then given by

$$g_k = \int_0^1 \mathrm{d}z\, \mathrm{e}^{2\pi \mathrm{i} k z} \frac{1}{2\mathrm{i}} \csc \pi(z - \mathrm{i}\epsilon) = \int_0^1 \mathrm{d}z\, \frac{\mathrm{e}^{(2k-1)\pi \mathrm{i} k z}}{1 - \mathrm{e}^{-2\pi \mathrm{i} z} + \epsilon},$$

where the second step is valid since ϵ is very small. Change variables to $w = \mathrm{e}^{-2\pi \mathrm{i} z}$ with $\mathrm{d}w/w = -2\pi \mathrm{i} \mathrm{d}z$ to find

$$g_k = \frac{1}{2\pi \mathrm{i}} \oint_{|w|=1} \frac{\mathrm{d}w}{w^{(2k+1)/2}} \frac{1}{1 + \epsilon - w}.$$

Note that $(2k+1)/2 \in \mathbb{Z}$, hence the integrand is analytic and the integral can be evaluated via residues. Since the pole at $w = 1 + \epsilon$ is outside of the contour $|w| = 1$, you only have to deal with the pole at $w = 0$. As a result, you find that $g_k = 0$ for negative k, since then there is no pole at all. For positive k, Cauchy's integral formula yields

$$g_k = \frac{1}{[(2k-1)/2]!} \frac{\mathrm{d}^{(2k-1)/2}}{\mathrm{d}w^{(2k-1)/2}} \frac{1}{1-w}\bigg|_{w=0},$$

where I already sent $\epsilon \to 0$, since it is not necessary anymore. To compute the higher derivatives, you can use the fact that

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k$$

is a geometric series, hence

$$g_k = \begin{cases} 1 & \text{for } k > 0 \\ 0 & \text{for } k < 0, \end{cases}$$

which validates the equality of eqs. (2.75) and (2.77) for the anti-periodic case. The periodic case is entirely analogous and I will omit it here.

Finally, we can proceed to the evaluation of eq. (2.74) for finite inverse temperature β. Again, I will state the solution first and verify it afterwards. Before doing so, however, note that eqs. (2.72) and (2.73) imply that G will be (anti-)periodic both in imaginary and real direction. This makes it necessary

to introduce the Jacobi theta function[120]

$$\vartheta_3(z|\tau) := \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi k^2\tau}\mathrm{e}^{2\pi\mathrm{i}kz}. \tag{2.78}$$

This series converges absolutely and is analytic for $\Im(\tau) > 0$ and $z \in \mathbb{C}$. We will usually set $\tau = \mathrm{i}\beta$ and think of eq. (2.78) as defining an entire function in z. Please note that ϑ_3 is periodic with respect to $z \to z + 1$ by construction. Furthermore, you can show that

$$\vartheta_3(z+\tau|\tau) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi(k^2+2k)\tau}\mathrm{e}^{2\pi\mathrm{i}kz} = \mathrm{e}^{-\mathrm{i}\pi\tau}\mathrm{e}^{-2\pi\mathrm{i}z}\vartheta_3(z|\tau)$$

by completing the square. Introducing in addition the auxiliary theta functions[120]

$$\vartheta_4(z|\tau) := \vartheta_3(z+1/2|\tau) \tag{2.79}$$

$$\vartheta_2(z|\tau) := \mathrm{e}^{\mathrm{i}\pi\tau/4}\mathrm{e}^{\mathrm{i}\pi z}\vartheta_3(z+\tau/2|\tau) \tag{2.80}$$

$$\vartheta_1(z|\tau) := -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\tau/4}\mathrm{e}^{\mathrm{i}\pi z}\vartheta_3(z+1/2+\tau/2|\tau), \tag{2.81}$$

you have

$$\vartheta_\nu(z+1|\tau) = \pm\vartheta_\nu(z|\tau) \quad \text{and} \quad \vartheta_\nu(z+\tau|\tau) = (\pm)'\mathrm{e}^{-\mathrm{i}\pi\tau}\mathrm{e}^{-2\pi\mathrm{i}z}\vartheta_\nu(z|\tau), \tag{2.82}$$

where the signs depend on $\nu = 1, 2, 3, 4$ and should be obvious from eqs. (2.79) to (2.81). In particular, you can easily verify that

$$G_{\mathrm{P/A}}(z) \propto \frac{\vartheta_{2/3}(z|\mathrm{i}\beta)}{\vartheta_1(z|\mathrm{i}\beta)} \tag{2.83}$$

satisfies

$$G_{\mathrm{P/A}}(z+1) = \pm G_{\mathrm{P/A}}(z) \quad \text{and} \quad G_{\mathrm{P/A}}(z+\mathrm{i}\beta) = -G_{\mathrm{P/A}}(z), \tag{2.84}$$

hence, has the correct (anti-)periodicities to solve eq. (2.74). Furthermore, the analyticity of ϑ_ν implies that $G_{\mathrm{P/A}}$ is also analytic, except at points where the denominator ϑ_1 vanishes.

We will now show that this only happens for $z \in \mathbb{Z}$, yielding a simple pole at $z = 0$ and integer translates by (anti-)periodicity. To see why this is the case, make use of the *Jacobi triple product*[120,121]

$$\vartheta_3(z|\tau) = \prod_{k=1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}k\tau})(1+\mathrm{e}^{(2k-1)\pi\mathrm{i}\tau}\mathrm{e}^{2\pi\mathrm{i}z})(1+\mathrm{e}^{(2k-1)\pi\mathrm{i}\tau}\mathrm{e}^{-2\pi\mathrm{i}z}), \tag{2.85}$$

which is a corollary of Euler's pentagonal number theorem.[122] As a result of eq. (2.85), we have

$$\vartheta_1(z|\tau) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\tau/4}\mathrm{e}^{\mathrm{i}\pi z}\prod_{k=1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}k\tau})(1-\mathrm{e}^{2\pi\mathrm{i}k\tau}\mathrm{e}^{2\pi\mathrm{i}z})(1-\mathrm{e}^{2\pi\mathrm{i}(k-1)\tau}\mathrm{e}^{-2\pi\mathrm{i}z}).$$

In the fundamental domain of the periodicities (2.84), this product only vanishes at $z = 0$, as you can see from the last factor for $k = 1$. Furthermore, you can compute

$$\left.\frac{\partial\vartheta_1(z|\tau)}{\partial z}\right|_{z=0} = 2\pi\left[\mathrm{e}^{\mathrm{i}\pi\tau/12}\prod_{k=1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}k\tau})\right]^3 = 2\pi\eta^3(\tau),$$

where I introduced the *Dedekind eta function*[121,122]

$$\eta(\tau) := \mathrm{e}^{\mathrm{i}\pi\tau/12}\prod_{k=1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}k\tau}). \tag{2.86}$$

As a result, we can fix the factor of proportionality in eq. (2.83) can be fixed to[118]

$$G_{\mathrm{P/A}}(z) = \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)}\frac{\vartheta_{2/3}(z|\mathrm{i}\beta)}{\vartheta_{2/3}(0|\mathrm{i}\beta)} \tag{2.87}$$

in order to normalise the UV pole to $1/2\pi\mathrm{i}z$. Again, eq. (2.87) is uniquely fixed by the Riemann-Roch theorem, but you can also verify explicitly that its Fourier series is given by eq. (2.74). I will only sketch this computation for G_{A} here. The details were worked out in collaboration with Christine Bott and can be found in her Bachelor's thesis.[123] First off, note that eqs. (2.78), (2.81) and (2.87) imply that G_{A} is actually a function of $w = \mathrm{e}^{\mathrm{i}\pi z}$. Furthermore, because of the pole $1/2\pi\mathrm{i}z$ at $z = 0$, you can see that

$$G_{\mathrm{A}}(z) = \frac{1}{w - w^{-1}} + g_{\mathrm{A}}(w),$$

where g_{A} is an odd meromorphic function of w and has no poles on the annulus $\mathrm{e}^{-\pi\beta} < |w| < \mathrm{e}^{\pi\beta}$. This is equivalent to $z = 0$ being the only zero of ϑ_1 in the fundamental domain. Since we already know the Fourier series of

$$\frac{1}{w - w^{-1}} = \frac{1}{2\mathrm{i}}\csc\pi z,$$

we only have to deal with g_{A}. As an odd meromorphic function of w, it

certainly has a convergent Laurent expansion

$$g_{\mathrm{A}}(w) = \sum_{k\in\mathbb{Z}} g_{2k+1} w^{2k+1}$$

with coefficients

$$g_{2k+1} = \frac{1}{2\pi\mathrm{i}} \oint_{|w|=1} \frac{\mathrm{d}w}{w} \frac{g_{\mathrm{A}}(w)}{w^{2k+1}} = \frac{1}{2\pi\mathrm{i}} \oint_{|w|=\mathrm{e}^{-\pi\beta/2}} \frac{\mathrm{d}w}{w} \frac{g_{\mathrm{A}}(w)}{w^{2k+1}}, \tag{2.88}$$

where I moved the contour of integration to $|w| = \mathrm{e}^{-\pi\beta/2}$. This is possible because the integrand is analytic on the annulus $\mathrm{e}^{-\pi\beta} < |w| < \mathrm{e}^{\pi\beta}$. Now, change variables $w \to \mathrm{e}^{-\pi\beta} w$ to find

$$g_{2k+1} = \frac{1}{2\pi\mathrm{i}} \mathrm{e}^{(2k+1)\pi\beta} \oint_{|w|=\mathrm{e}^{\pi\beta/2}} \frac{\mathrm{d}w}{w} \frac{g_{\mathrm{A}}(\mathrm{e}^{-\pi\beta} w)}{w^{2k+1}}.$$

Next, note that eq. (2.84) implies

$$\begin{aligned} g_{\mathrm{A}}(\mathrm{e}^{-\pi\beta} w) &= G_{\mathrm{A}}(z + \mathrm{i}\beta) - \frac{1}{\mathrm{e}^{-\pi\beta} w - \mathrm{e}^{\pi\beta} w^{-1}} \\ &= -G_{\mathrm{A}}(z) - \frac{1}{\mathrm{e}^{-\pi\beta} w - \mathrm{e}^{\pi\beta} w^{-1}} \\ &= -g_{\mathrm{A}}(w) - \frac{1}{w - w^{-1}} - \frac{1}{\mathrm{e}^{-\pi\beta} w - \mathrm{e}^{\pi\beta} w^{-1}}, \end{aligned}$$

hence

$$\begin{aligned} g_{2k+1} = &- \frac{1}{2\pi\mathrm{i}} \mathrm{e}^{(2k+1)\pi\beta} \oint_{|w|=\mathrm{e}^{\pi\beta/2}} \frac{\mathrm{d}w}{w} \frac{1}{w^{2k+1}} \\ &\times \left[g_{\mathrm{A}}(w) + \frac{1}{w - w^{-1}} + \frac{1}{\mathrm{e}^{-\pi\beta} w - \mathrm{e}^{\pi\beta} w^{-1}} \right] \\ = &- \mathrm{e}^{(2k+1)\pi\beta} g_{2k+1} \\ &- \frac{1}{2\pi\mathrm{i}} \mathrm{e}^{(2k+1)\pi\beta} \oint_{|w|=\mathrm{e}^{\pi\beta/2}} \frac{\mathrm{d}w}{w^{2(k+1)}} \left[\frac{1}{w - w^{-1}} + \frac{1}{\mathrm{e}^{-\pi\beta} w - \mathrm{e}^{\pi\beta} w^{-1}} \right]. \end{aligned}$$

Rearranging this a little, you find

$$g_{2k+1} = \frac{1}{1 + \mathrm{e}^{-(2k+1)\pi\beta}} \frac{1}{2\pi\mathrm{i}} \oint_{|w|=\mathrm{e}^{\pi\beta/2}} \frac{\mathrm{d}w}{w^{2k+1}} \left[\frac{1}{1 - w^2} + \frac{\mathrm{e}^{-\pi\beta}}{1 - \mathrm{e}^{-2\pi\beta} w^2} \right],$$

which can be evaluated by residues. As a result[123]

$$g_{2k+1} = \frac{1}{1+\mathrm{e}^{-(2k+1)\pi\beta}} \times \begin{cases} -1 & \text{for} \quad k < 0 \\ \mathrm{e}^{-(2k+1)\pi\beta} & \text{for} \quad k \geq 0, \end{cases}$$

hence

$$g_{\mathrm{A}}(w) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-(2k+1)\pi\beta}}{1+\mathrm{e}^{-(2k+1)\pi\beta}} \Big[w^{2k+1} - w^{-(2k+1)} \Big],$$

so that

$$G_{\mathrm{A}}(z) = \frac{1}{2\mathrm{i}} \csc \pi z + \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-(2k+1)\pi\beta}}{1+\mathrm{e}^{-(2k+1)\pi\beta}} \Big[\mathrm{e}^{(2k+1)\mathrm{i}\pi z} - \mathrm{e}^{-(2k+1)\mathrm{i}\pi z} \Big]. \tag{2.89}$$

Finally, replacing the cosecant in this equation by its Fourier series (2.75) verifies the equality between eq. (2.74) and eq. (2.87). Again, the periodic case is entirely analogous and yields

$$G_{\mathrm{P}}(z) = \frac{1}{2\mathrm{i}} \cot \pi z + \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-2\pi k\beta}}{1+\mathrm{e}^{-2\pi k\beta}} \Big[\mathrm{e}^{2\pi\mathrm{i}kz} - \mathrm{e}^{-2\pi\mathrm{i}kz} \Big], \tag{2.90}$$

which is also identical to eq. (2.74) because of eq. (2.75).

You can find a summary of all the propagators that we just derived in table 2.2.

Table 2.2: Propagators $G(z) = G(z, 0)$ on the line and circle, in the vacuum and thermal states, along with their locations in the main text to find their derivation.

Geometry/State	**Formula**	**Location**
line/vacuum	$G(z) = 1/2\pi \mathrm{i} z$	(2.70)
line/thermal	$G(z) = 1/[2\mathrm{i}\beta \sinh(\pi z/\beta)]$	(2.71)
circle/vacuum	$G_{\mathrm{P}}(z) = 1/2\mathrm{i} \times \cot \pi z$	(2.76)
	$G_{\mathrm{A}}(z) = 1/2\mathrm{i} \times \csc \pi z$	(2.77)
circle/thermal	$G_{\mathrm{P/A}}(z) = \dfrac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z\|\mathrm{i}\beta)} \dfrac{\vartheta_{2/3}(z\|\mathrm{i}\beta)}{\vartheta_{2/3}(0\|\mathrm{i}\beta)}$	(2.87)

2.3.3 Adding a chemical potential

Later on in section 2.4, we will need the thermal propagator on the circle with a chemical potential μ. This means that, instead of taking the density matrix (2.20) of the *canonical* ensemble, we instead work in the *grand canonical* ensemble

$$\rho = \frac{\mathrm{e}^{-\beta H-\mu N}}{\mathrm{Tr}[\mathrm{e}^{-\beta H+\mu N}]},$$

where

$$N := \int \mathrm{d}x\, \psi^*(x)\psi(x) \tag{2.91}$$

is the fermion-number operator. Since N is quadratic, all computations leading to eq. (2.42) go through unchanged, hence, the corresponding propagator is given by

$$G = \frac{1}{1+\mathrm{e}^{-\mathrm{i}\beta\partial+\mu}}.$$

Evaluating this on the circle leads to the series representation (compare eq. (2.74))

$$G(x-y) = \sum_k \frac{\mathrm{e}^{-2\pi\mathrm{i}k(x-y)}}{1+\mathrm{e}^{-2\pi\beta k+\mu}}, \tag{2.92}$$

where the summation is over $k \in \mathbb{Z}$ for periodic and $k \in \mathbb{Z}+1/2$ for anti-periodic boundary conditions. This propagator satisfies a variant of the KMS condition (2.24): Since $[H,N]=0$, we have

$$\rho = \frac{\mathrm{e}^{-\beta H}\mathrm{e}^{\mu N}}{\mathrm{Tr}[\mathrm{e}^{-\beta H+\mu N}]},$$

hence,

$$G(z) = \frac{\mathrm{Tr}[\mathrm{e}^{\mu N}\psi(0)\psi^*(-z)\mathrm{e}^{-\beta H}]}{\mathrm{Tr}[\mathrm{e}^{-\beta H+\mu N}]} = \frac{\mathrm{Tr}[\mathrm{e}^{\mu N}\psi(0)\mathrm{e}^{-\mathrm{i}zH}\psi^*(0)\mathrm{e}^{(\mathrm{i}z-\beta)H}]}{\mathrm{Tr}[\mathrm{e}^{-\beta H+\mu N}]},$$

which is analytic on the strip $-\beta < \Im(z) < 0$, as already discussed in section 2.1.3. The behaviour at the lower boundary of this strip is given by

$$\begin{aligned}\lim_{s\nearrow\beta} G(x-\mathrm{i}s) &= \frac{\mathrm{Tr}[\mathrm{e}^{\mu N}\psi(0)\mathrm{e}^{(-\mathrm{i}x-\beta)H}\psi^*(0)\mathrm{e}^{\mathrm{i}xH}]}{\mathrm{Tr}[\mathrm{e}^{-\beta H+\mu N}]}\\ &= \frac{\mathrm{Tr}[\mathrm{e}^{\mu N}\mathrm{e}^{(-\mathrm{i}x-\beta)H}\psi^*(0)\mathrm{e}^{\mathrm{i}xH}\mathrm{e}^{\mu N}\psi(0)\mathrm{e}^{-\mu N}]}{\mathrm{Tr}[\mathrm{e}^{-\beta H-\mu N}]}\\ &= \mathrm{Tr}[\rho\psi^*(-x)\mathrm{e}^{\mu N}\psi(0)\mathrm{e}^{-\mu N}], \quad x\in\mathbb{R}.\end{aligned}$$

Now, since

$$\mathrm{e}^{\mu N}\psi(0)\mathrm{e}^{-\mu N} = \psi(0) + [N, \psi(0)] + \ldots = \psi(0) - \mu\psi(x) + \ldots = \mathrm{e}^{-\mu}\psi(0),$$

you see that

$$\lim_{s\nearrow\beta} G(x - \mathrm{i}s) = \mathrm{e}^{-\mu}\big(\delta(x) - G(x)\big), \quad x \in \mathbb{R},$$

where the Dirac delta appears because of the anti-commutation relations (2.40) and is (anti-)periodic on the circle. As a result, G has an analytic continuation satisfying

$$G(z+1) = \pm G(z), \quad G(z + \mathrm{i}\beta) = -\mathrm{e}^{\mu}G(z), \quad z \in \mathbb{C} \setminus (\mathbb{Z} + \mathrm{i}\beta\mathbb{Z}). \tag{2.93}$$

In the same spirit as before, we can use this quasi-periodicity to pin down the analytic form of

$$G_{\mathrm{P/A}}(z) = \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_{2/3}(z + \mathrm{i}\mu/2\pi|\mathrm{i}\beta)}{\vartheta_{2/3}(\mathrm{i}\mu/2\pi|\mathrm{i}\beta)}. \tag{2.94}$$

As you can easily verify from eq. (2.82), this satisfies the modified KMS condition (2.93) and has the correct pole $\sim 1/2\pi\mathrm{i}z$ as $z \to 0$. You could also verify that its Fourier series is given by eq. (2.92) in the same way as before, but I will omit this computation here.

2.3.4 The case of boundaries

To finish this section, I would like to discuss how to treat spacetimes with boundaries. The arguments presented here present the starting point of a work in progress with Ignacio Reyes,[45] devoted to the study of entanglement in boundary CFT[124–126] with free fermion methods. The first important question that has to be answered is how boundaries can be implemented in a QFT at all. This can be answered easily in the case of a free scalar field ϕ, where the equations of motion are *second order* and one can simply require Dirichlet or Neumann boundary conditions ($\phi(x) = 0$ or $\phi'(x) = 0$ along the boundary, respectively). For the free fermion, however, the equations of motion (2.65) are *first order* and would trivialise under such conditions. Instead, we go along a more abstract route and implement the boundary at the level of symmetries[126] to ultimately derive compatible boundary conditions.

To this end, we start with the Dirac action (2.62) in Minkowski space *without a boundary*. This action is invariant with respect to the Poincaré group $\mathfrak{P}$ up to boundary terms. In particular, this means that translations in x^μ-direction

leave the Lagrangian $\mathcal{L}$ invariant up to a total divergence

$$\delta^\mu \mathcal{L} = \partial_\nu \Lambda^{\mu\nu}.$$

As a result, the associated Noether current

$$T^{\mu\nu} = \Lambda^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi)} \delta^\mu \psi - (\delta^\mu \bar{\psi}) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\psi})}, \tag{2.95}$$

also known as the *energy-momentum tensor*, is conserved, in the sense that

$$\partial_\nu T^{\mu\nu} = 0 \tag{2.96}$$

along solutions of the equations of motion. This means that we can define the generator of translations (i.e., momentum) by integrating

$$P_\mu := \int_C \mathrm{d}n^\nu \, T_{\mu\nu} \tag{2.97}$$

along any spacelike surface C that extends to infinity with positively timelike normal vector n^μ. Due to Gauss' law, eq. (2.96) guarantees that this definition will not depend on the specific choice of C, as sketched in fig. 2.10.

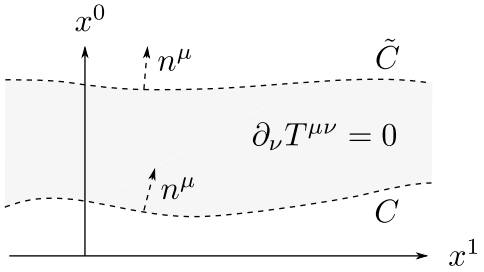


Figure 2.10: Argument for conservation of momentum (2.97): P_μ is given as the integral of $T_{\mu\nu}$ over a Cauchy slice C or $\tilde{C}$ (dashed black). Due to Gauss' law, it is independent of the choice of C vs. $\tilde{C}$.

More explicitly, for the action (2.62), a translation in x^μ-direction changes the fields as

$$\delta^\mu \psi = -\partial^\mu \psi, \quad \text{and} \quad \delta^\mu \bar{\psi} = -\partial^\mu \bar{\psi},$$

hence, $\delta^\mu \mathcal{L} = -\partial^\mu \mathcal{L}$, such that

$$\Lambda^{\mu\nu} = -\eta^{\mu\nu} \mathcal{L}.$$

The energy-momentum tensor takes the form

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \mathrm{i}\bar{\psi}\gamma^\nu\partial^\mu\psi = \mathrm{i}\bar{\psi}[\gamma^\nu\partial^\mu - \eta^{\mu\nu}\gamma^\kappa\partial_\kappa]\psi,$$

or, in components,

$$T^{00} = \mathrm{i}\psi^*\sigma_z\partial_1\psi, \quad T^{01} = -\mathrm{i}\psi^*\sigma_z\partial_0\psi, \quad T^{10} = -\mathrm{i}\psi^*\partial_1\psi, \quad T^{11} = \mathrm{i}\psi^*\partial_0\psi,$$

where I inserted the Dirac matrices (2.63). In lightray coordinates $x^\pm = x^0 \pm x^1$, this is

$$\begin{aligned} T^{++} &= 4\mathrm{i}\psi_-^*\partial_-\psi_-, \quad T^{--} = 4\mathrm{i}\psi_+^*\partial_+\psi_+. \\ T^{+-} &= -4\mathrm{i}\psi_-^*\partial_+\psi_-, \quad T^{-+} = -4\mathrm{i}\psi_+^*\partial_-\psi_+ \end{aligned}$$

where the off-diagonal terms in the second line vanish due to the equations of motion. As a result, the energy-momentum tensor is symmetric with[118]

$$T_{\pm\pm} = \mathrm{i}\psi_\pm^*\partial_\pm\psi_\pm \quad \text{and} \quad T_{\pm\mp} = 0, \tag{2.98}$$

which is obviously conserved.

We are now in the position to state what happens when we introduce a boundary: At the very least, the Poincaré symmetry $\mathfrak{P}$ has to be broken down to the subgroup of $\mathfrak{P}$ preserving the boundary. In particular, this includes translations parallel to the boundary, hence momenta $P_\parallel$ parallel to the boundary still need to be conserved. These take the form

$$P_\parallel := \int_C \mathrm{d}n^\mu\, T_{\parallel\mu}$$

for some spacelike surface C, which either extends to infinity or ends at the boundary. If this definition is to be independent of the specific choice of C, you can use eq. (2.96) and Gauss' law to see that the parallel-perpendicular component $T_{\parallel\perp}$ of the energy-momentum tensor has to vanish along the boundary, as sketched in fig. 2.11. The requirement

$$T_{\parallel\perp} = 0 \quad \text{along the boundary}, \tag{2.99}$$

also known as the *conformal boundary condition*,[126,127] is what we will use to derive sensible conditions of the fields.

The conformal boundary condition (2.99) can be interpreted in two different ways: On one hand, you could say that the theory with boundary effectively only consists of one chirality and the other one is just an alias for the first one. This interpretation, suggested by Longo and Rehren,[125,128] takes eq. (2.99) to

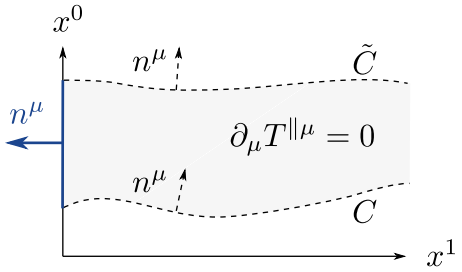


Figure 2.11: Argument leading to the conformal boundary condition (2.99): The generator of translations along the boundary (solid blue) is given as the integral of $T_{\parallel\mu}$ over a Cauchy slice C or $\tilde{C}$ (dashed black). Due to Gauss' law, it is independent of the choice of C vs. $\tilde{C}$ if $T^{\parallel\mu} = 0$ along the boundary.

be a statement of operators and makes it easy to deal with timelike boundaries.

On the other hand, you could say that the chiralities are still independent as operators, but eq. (2.99) only holds on a *subspace* of the theory without boundary – the space of so-called *boundary states*.[126] This interpretation is well suited for the treatment of spacelike boundaries[127,129,130] and allows for one key observation: The actions of the energy-momentum tensors $T_{\pm\pm}(z)$ are related with one another on boundary states, hence boundary states show entanglement between the two chiralities! In section 2.4.7, we will verify this statement quantitatively.

While the full discussion is still in preparation,[45] I will explicitly work out the example of a free fermion on the right half plane $x^1 > 0$, i.e., $x^+ > x^-$, in this thesis. The conformal boundary condition (2.99) then states that

$$0 = T_{01} = T_{++}(x^+) - T_{--}(x^-) \quad \text{along} \quad x^+ = x^-,$$

hence, $T_{++}(z) = T_{--}(z)$ for all $z \in \mathbb{R}$. Working in the interpretation that this is a statement of operators and recalling eq. (2.98), the obvious solutions to this condition are

$$\psi_+(z) = \eta\psi_-(z), \quad \eta = \pm 1 \quad \text{for any} \quad z \in \mathbb{R} \tag{2.100}$$

with a nice physical interpretation: Left moving modes ψ_- are reflected off the boundary to become outgoing modes ψ_+, maybe undergoing a sign change.

Next, to uphold the spectrum condition (1.12), we have to require that the

Hamiltonian

$$\mathcal{H} = \int_0^\infty \mathrm{d}x^1\, T_{00}(x^0 = 0, x^1) = \int_0^\infty \mathrm{d}z\, [T_{++}(z) + T_{--}(-z)]$$

is a positive operator. Luckily, since $T_{--}(z) = T_{++}(z)$ for any $z \in \mathbb{R}$, this Hamiltonian can be rewritten as

$$\mathcal{H} = \int_{\mathbb{R}} \mathrm{d}z\, T_{++}(z) = \int_{\mathbb{R}} \mathrm{d}z\, T_{--}(z),$$

which is exactly the Hamiltonian of a single chirality, hence, positive on the Fock space of a single chirality.

To compute the propagator G, first note that the chiralities no longer being independent implies that G is a matrix of the form

$$\begin{aligned} G(x,y) &= \langle \Omega_0 | \begin{bmatrix} \psi_+(x^+) \\ \psi_-(x^-) \end{bmatrix} \begin{bmatrix} \psi_+(y^+) \\ \psi_-(y^-) \end{bmatrix}^* \Omega_0 \rangle \\ &= \begin{bmatrix} \langle \Omega_0 | \psi_+(x^+) \psi_+^*(y^+) \Omega_0 \rangle & \langle \Omega_0 | \psi_+(x^+) \psi_-^*(y^-) \Omega_0 \rangle \\ \langle \Omega_0 | \psi_-(x^-) \psi_+^*(y^+) \Omega_0 \rangle & \langle \Omega_0 | \psi_-(x^-) \psi_-^*(y^-) \Omega_0 \rangle \end{bmatrix} \end{aligned}$$

for $x^1, y^1 > 0$. As always, we work on the time slice $x^0 = y^0 = 0$, such that $x^\pm = \pm x^1$ and $y^\pm = \pm y^1$. Because of eq. (2.65), the equal-time propagator then takes the form

$$G(x^1, y^1) = \begin{bmatrix} \langle \Omega_0 | \psi_+(x^1) \psi_+^*(y^1) \Omega_0 \rangle & \langle \Omega_0 | \psi_+(x^1) \psi_-^*(-y^1) \Omega_0 \rangle \\ \langle \Omega_0 | \psi_-(-x^1) \psi_+^*(y^1) \Omega_0 \rangle & \langle \Omega_0 | \psi_-(-x^1) \psi_-^*(-y^1) \Omega_0 \rangle \end{bmatrix}$$

Dropping the ^{1}-superscript and imposing eq. (2.100), we find

$$G(x,y) = \begin{bmatrix} \langle \Omega_0 | \psi_+(x) \psi_+^*(y) \Omega_0 \rangle & \eta \langle \Omega_0 | \psi_+(x) \psi_+^*(-y) \Omega_0 \rangle \\ \eta \langle \Omega_0 | \psi_+(-x) \psi_+^*(y) \Omega_0 \rangle & \langle \Omega_0 | \psi_+(-x) \psi_+^*(-y) \Omega_0 \rangle \end{bmatrix}$$

or, more concisely,

$$G(x,y) = \begin{bmatrix} G_{\text{w/o bdy}}(x,y) & \eta G_{\text{w/o bdy}}(x,-y) \\ \eta G_{\text{w/o bdy}}(-x,y) & G_{\text{w/o bdy}}(-x,-y) \end{bmatrix}$$

with $G_{\text{w/o bdy}}$ being the propagator of the corresponding state in the single-chirality-theory without boundary. As you can see, the diagonal entries of G coincide with those from the theory without a boundary but there are non-vanishing off-diagonal entries implying correlations between the two chiralities – a clear sign of entanglement! In particular, you could choose the vacuum propagator on the line (2.70) to find the vacuum propagator on the half line

$$G(x,y) = \frac{1}{2\pi \mathrm{i}} \begin{bmatrix} 1/(x-y) & \eta 1/(x+y) \\ \eta 1/(-x-y) & 1/(-x+y) \end{bmatrix}, \tag{2.101}$$

which shows diminishing correlations between the chiralities with increasing distance from the boundary.

Finally, I would like to say word about the algebraic properties of boundary states: You might be worried that the requirement (2.100) could break the separating property of local algebras, since $\psi_+(z) - \eta\psi_-(z)$ annihilates boundary states. Fortunately, this is not the case,[125] as shown in fig. 2.12. To see this, consider an open double cone $\mathcal{O}$ that has a finite distance $d > 0$ from the boundary $x^1 = 0$. This means that every point point $x \in \mathcal{O}$ satisfies $x^+ \geq d$ and $x^- \leq -d$, so that

$$x^+ - y^- \geq 2d$$

for any two points $x, y \in \mathcal{O}$. Since $\psi_\pm$ depends on $x^\pm$ only, terms like $\psi_+(z) - \eta\psi_-(z')$ can only be contained in the local algebra $\mathcal{A}(\mathcal{O})$ if $z - z' \geq 2d$. Operators of this form, however, only annihilate boundary states for $z = z'$, as argued above. This can be traced back to the fact that the conformal boundary condition (2.99) only holds exactly at the boundary, such that no smearing of $T_{\parallel\perp}$ with finite support ever annihilates boundary states.

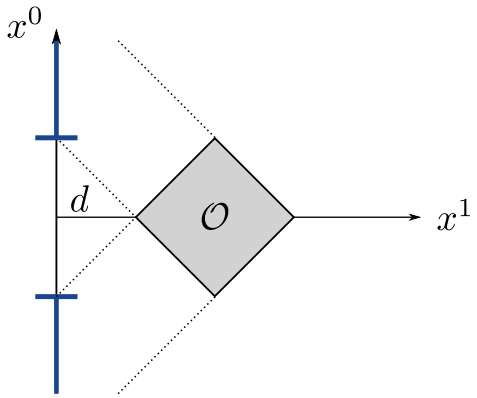


Figure 2.12: Double cone $\mathcal{O}$ (shaded) at a finite distance d from the boundary at $x^1 = 0$. Its projections onto the boundary (solid blue) always have a finite distance $2d$ from one another, so that $\mathcal{O}$ can not contain operators that annihilate boundary states.

2.4 Results

Now that we have covered all prerequisites that are needed in order to understand what we are doing, we can dive into the world of entanglement of the free fermion CFT (2.62): In section 2.3, I already presented how to derive modular data of a gaussian state from the corresponding propagator G. These formulae, summarised in table 2.1, require the computation of functional dependencies of the (restricted) propagators $G|_V$. While this is generally a hard task, it can be overcome by the introduction of a spectral calculus, as done in eqs. (2.56) to (2.59). This basically reduces all computations to the determination of the resolvent $[\lambda - G|_V]^{-1}$ and subsequently performing some complex integrals in order to evaluate table 2.1 for all propagators in table 2.2.

In its complete form, this procedure was first worked out by our group,[41–43] although many crucial bits and pieces already existed in the literature beforehand[107–111] and other groups contributed greatly to the completion of this task with similar[113,114] and completely different[112] methods.

2.4.1 Solving the singular integral equation

Let us now turn to the integral equation (2.61) and how to solve it. For the time being, we will only consider the propagators from table 2.2, which belong to a single chirality of the two dimensional free fermion CFT (2.62). Since the equations of motion imply that the fields $\psi_\pm$ only depend on one lightray

coordinate $x^{\pm} = x^0 \pm x^1$ each, we are effectively dealing with a one-dimensional problem (per chirality). Because of this and translational invariance (2.69), the integral equation takes the form

$$-G(x-y) + F_\lambda(x,y) - \frac{1}{\lambda}\int_V \mathrm{d}z\, G(x-z)F_\lambda(z,y) = 0 \quad \text{for} \quad x, y \in V, \tag{2.102}$$

where V denotes the projection of the considered spacetime region onto the corresponding lightray coordinate axis, as sketched in figure fig. 2.13. The region of spacetime under consideration shall be a (finite) union of spacelike separated double cones, so that

$$V = \bigcup_n [a_n, b_n], \quad a_1 < b_1 < a_2 < \dots \tag{2.103}$$

is a union of disjoint intervals on the line.

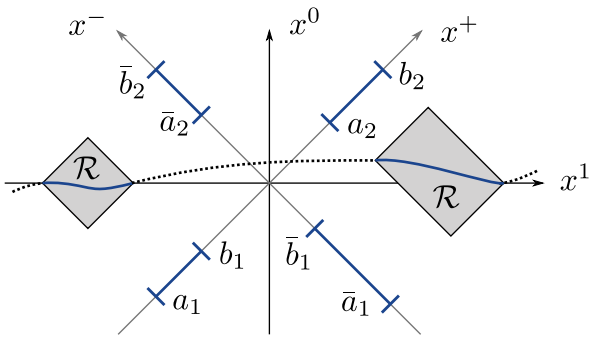


Figure 2.13: A region $\mathcal{R}$ comprised two double cones (shaded) on a Cauchy slice (dotted black), along with its projections (solid dark blue) onto the lightray coordinate axes. This figure shows the conventions on the order of boundary points a_n, b_n of the projection V.

At this point, I would like to say one word of caution about G and F_λ: As stated before, G and F_λ are *distributional kernels*, i.e., expressions such as $G(z)$ have no meaning *pointwise*. They only make sense when integrated against some suitable test function, similar to how the Dirac delta $\delta(z)$ can not be treated like a function. This comes about simply from the definition of G as an expectation value of field operators $\psi(x)$, which themselves are operator valued distributions.[64] Furthermore, since $G(z)$ has a singularity for $z \to 0$, the corresponding integrals have to be regulated in some sense and, as a distribution, this regulation is part of the definition of G. In our case, the

integrals may be regulated by replacing

$$G(z) \to G(z - \mathrm{i}\epsilon)$$

for some small $\epsilon > 0$ inside the integral and performing the limit $\lim_{\epsilon \searrow 0}$ after the integration. This scheme is valid because we know that $G(z)$ has to have an analytical continuation to the lower half plane because of the KMS condition (2.24) and, indeed, we already imposed it in the proof of eqs. (2.76) and (2.77). More mathematically, the act of performing the limit $\epsilon \searrow 0$ only after integration can be described as the distributional kernel G being a *weak limit*

$$G(x - y) := \operatorname*{w\text{-}lim}_{\epsilon \searrow 0} G(x - y - \mathrm{i}\epsilon) \quad \text{for} \quad x - y \in \mathbb{R}. \tag{2.104}$$

As we will see soon, the solution F_λ of eq. (2.102) needs to be regulated in the same way by setting

$$F_\lambda(x, y) := \operatorname*{w\text{-}lim}_{\epsilon \searrow 0} F_\lambda(x, y + \mathrm{i}\epsilon) \quad \text{for} \quad x, y \in V. \tag{2.105}$$

Similarly to eq. (2.104), the functions F_λ on both sides have different meanings and can be distinguished by their arguments: The distributional kernel F_λ on the left hand side can only take real valued arguments $x, y \in V \subset \mathbb{R}$, supposedly having singularities at specific values of x, y. The complex function F_λ on the right hand side, however, is not necessarily defined for $x, y \in V$ but rather in a complex neighbourhood of V, in order to get around the aforementioned singularities. In this sense, we are looking for a solution of

$$\begin{aligned} &G(x - y) - F_\lambda(x, y) \\ &\qquad + \operatorname*{w\text{-}lim}_{\epsilon \searrow 0} \operatorname*{w\text{-}lim}_{\epsilon' \searrow 0} \frac{1}{\lambda} \int_V \mathrm{d}z \, G(x - z - \mathrm{i}\epsilon) F_\lambda(z, y + \mathrm{i}\epsilon') = 0. \end{aligned} \tag{2.106}$$

So how do we solve eq. (2.102) or, rather, eq. (2.106)? Historically, this was first achieved for the case of the vacuum state on the line (2.70), where the propagator is just $1/2\pi\mathrm{i}z$. This case is actually a standard problem from the theory of singular integral equations[117] and was rediscovered in this physical context by Casini and Huerta.[107] After that, further progress[108,109] was basically limited to rank-one perturbations of the propagator (2.70) and its conformally compactified cousin (2.77). In the following, I would like to present a method,[41] based on previous ideas of Arias, Casini, and Huerta,[111] which generalises to a much more general class of propagators, namely to which behave *asymptotically* like $\sim 1/2\pi\mathrm{i}z$ as $z \to 0$. The key idea is to rephrase eq. (2.106) as a statement about contour integrals, since the simple pole of G allows to

evaluate such integrals elegantly. More precisely, I will first list some analytic properties that F_λ will have to satisfy such that eq. (2.106) will be solved by a simple application of the residue theorem. Subsequently, I will present a systematic way for constructing an F_λ that satisfies all these properties.

As a first step, let us require that F_λ is such that

$$\oint_\gamma \mathrm{d}z\, G(x - z - \mathrm{i}\epsilon) F_\lambda(z, y + \mathrm{i}\epsilon') = 0 \quad \text{for} \quad x, y \in V, \tag{2.107}$$

for some contour γ encircling V within a finite distance, as sketched in fig. 2.14.

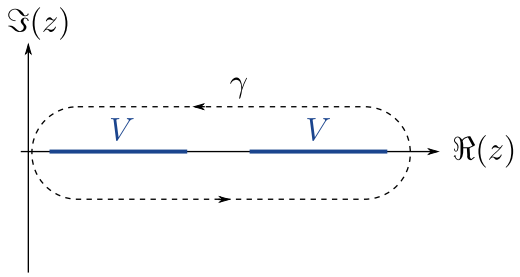


Figure 2.14: Sketch for the general method of solution of the singular integral equation (2.106): The contour γ (dashed black) encircles V (solid blue) with a finite distance.

The other assumptions will be chosen such that we can deform the left hand side of eq. (2.107) to the left hand side of eq. (2.106): First, we would like $F_\lambda(z, z')$ to be analytic in z (except at $z \in V$ and $z = z'$), such that we can deform the contour γ freely. This allows to contract γ to be snug along V, crossing the first order pole of G at $x - z - \mathrm{i}\epsilon$ as we do so. By the residue theorem, this creates a contribution of

$$\mathrm{Res}_{z=x-\mathrm{i}\epsilon}\, G(x - z - \mathrm{i}\epsilon) F_\lambda(z, y + \mathrm{i}\epsilon') = -F_\lambda(x - \mathrm{i}\epsilon, y + \mathrm{i}\epsilon'),$$

which already looks much like the F_λ outside of the integral in eq. (2.106). To recover it from the weak limit, you need to require continuity in the first argument from the lower half plane, i.e,

$$\lim_{\epsilon \searrow 0} F_\lambda(x - \mathrm{i}\epsilon, z) = F_\lambda(x, z), \quad x \in V.$$

You can recover the G in the same way if $F_\lambda(z, z')$ has a simple pole $\sim 1/2\pi\mathrm{i}(z-$

z') at $z \to z'$. In order to keep track of further non-analyticities, let us write

$$F_\lambda(z, z') = H_\lambda(z - z') \frac{S_\lambda(z)}{S_\lambda(z')} \tag{2.108}$$

with $H_\lambda(z - z')$ analytic everywhere except for a simple pole

$$H_\lambda(z - z') \sim \frac{1}{2\pi \mathrm{i}(z - z')} \quad \text{at} \quad z \to z' \tag{2.109}$$

and S_λ analytic up to a possible discontinuity along V. The above continuity requirement translates to

$$\lim_{\epsilon \searrow 0} S_\lambda(x - \mathrm{i}\epsilon) = S_\lambda(x) \quad \text{along} \quad V. \tag{2.110}$$

As a result,

$$\begin{aligned} &\mathrm{Res}_{z=y+\mathrm{i}\epsilon'} G(x - z - \mathrm{i}\epsilon) F_\lambda(z, y + \mathrm{i}\epsilon') \\ &\quad = \mathrm{Res}_{z=y+\mathrm{i}\epsilon'} G(x - z - \mathrm{i}\epsilon) H_\lambda(z - y - \mathrm{i}\epsilon') \frac{S_\lambda(z)}{S_\lambda(y + \mathrm{i}\epsilon')} \\ &\quad = G\Big(x - y - \mathrm{i}(\epsilon + \epsilon')\Big), \end{aligned}$$

which indeed recovers the G outside of the integral in eq. (2.106) as the weak limit is performed. Note that in all of the above, the sign of ϵ' is crucial, as a term like $\epsilon - \epsilon'$ would lead to a dependence on the order of limits. This is the reason for the sign choices in eq. (2.105).

To sum up everything so far, we have deformed eq. (2.107) to

$$\begin{aligned} &G(x - y) - F_\lambda(x, y) \\ &\qquad + \operatorname*{w\text{-}lim}_{\epsilon \searrow 0} \operatorname*{w\text{-}lim}_{\epsilon' \searrow 0} \oint_{V^{\circlearrowleft}} \mathrm{d}z \, G(x - z - \mathrm{i}\epsilon) H_\lambda(z - y - \mathrm{i}\epsilon') \frac{S_\lambda(z)}{S_\lambda(y + \mathrm{i}\epsilon')} = 0, \end{aligned}$$

where $V^{\circlearrowleft}$ denotes a contour that wraps tightly around V. In particular, the poles of the integrand lie outside the contour of integration. This already looks strikingly similar to the integral equation (2.106), but we still have to turn the contour integral *around* V into an ordinary integral *along* V. To this end, note that each component $[a_n, b_n]$ of $V^{\circlearrowleft}$ can be deformed to a "bone"-like, consisting of four pieces, as sketched in fig. 2.15: One contribution going to the right just below $[a_n, b_n]$, one going to the left just above $[a_n, b_n]$, and two circular contributions around the endpoints a_n, b_n.

Let us assume that $S_\lambda(z)$ does not diverge too quickly near these endpoints,

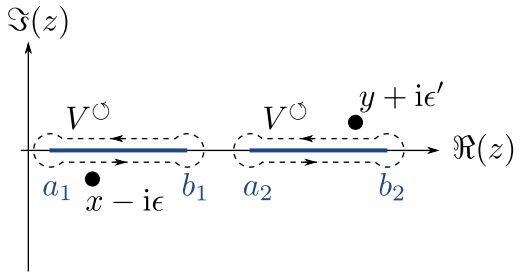


Figure 2.15: Another sketch for the solution of the singular integral equation (2.106): The contour $V^{\circlearrowleft}$ (dashed black) snugly wraps around V (solid blue), excluding the poles (black dots) of the integrand at $x - \mathrm{i}\epsilon$ and $y + \mathrm{i}\epsilon'$.

such that the circular contributions vanish. Mathematically, this is guaranteed if

$$\lim_{z \to a_n} (z - a_n) S_\lambda(z) = 0 = \lim_{z \to b_n} (z - b_n) S_\lambda(z). \tag{2.111}$$

We then have

$$\begin{aligned} &\oint_{V^{\circlearrowleft}} \mathrm{d}z\, G(x - z - \mathrm{i}\epsilon) H_\lambda(z - y - \mathrm{i}\epsilon') \frac{S_\lambda(z)}{S_\lambda(y + \mathrm{i}\epsilon')} \\ &= \int_V \mathrm{d}z\, G(x - z - \mathrm{i}\epsilon) H_\lambda(z - y - \mathrm{i}\epsilon') \lim_{\epsilon'' \searrow 0} \frac{S_\lambda(z - \mathrm{i}\epsilon'') - S_\lambda(z + \mathrm{i}\epsilon'')}{S_\lambda(y + \mathrm{i}\epsilon')}, \end{aligned}$$

which is equal to the

$$\frac{1}{\lambda} \int_V \mathrm{d}z\, G(x - z - \mathrm{i}\epsilon) F_\lambda(z, y + \mathrm{i}\epsilon')$$

from eq. (2.106) if

$$\frac{1}{\lambda} S_\lambda(z) = \lim_{\epsilon'' \searrow 0} \left[S_\lambda(z - \mathrm{i}\epsilon'') - S_\lambda(z + \mathrm{i}\epsilon'') \right] \quad \text{along} \quad V.$$

Combined with the continuity condition (2.110), this is equivalent to

$$\lim_{\epsilon'' \searrow 0} \frac{S_\lambda(x + \mathrm{i}\epsilon'')}{S_\lambda(x - \mathrm{i}\epsilon'')} = -\frac{1 - \lambda}{\lambda} \quad \text{along} \quad V, \tag{2.112}$$

i.e., S_λ has to have a multiplicative branch cut along V. Combined with

eq. (2.109), this also implies that F_λ takes the form

$$F_\lambda(x,y) = -\frac{\lambda}{1-\lambda}\,\underset{\epsilon\searrow 0}{\text{w-lim}}\, H_\lambda(x-y-\mathrm{i}\epsilon)\times\frac{S_\lambda(x)}{S_\lambda(y)},$$

which concludes our derivation of F_λ. Finally, note that we can generally solve the multiplicative branch cut (2.112) by setting

$$S_\lambda(z) := \Big[-\frac{1-\lambda}{\lambda}\Big]^{\mathrm{i}Z(z)} \tag{2.113}$$

for some function Z which is continuous from the lower half plane

$$\lim_{\epsilon\searrow 0} Z(x-\mathrm{i}\epsilon) = Z(x) \quad \text{along} \quad V \tag{2.114}$$

and has an additive branch cut

$$\lim_{\epsilon\searrow 0}\Big[Z(x-\mathrm{i}\epsilon) - Z(x+\mathrm{i}\epsilon)\Big] = \mathrm{i} \quad \text{along} \quad V. \tag{2.115}$$

We have thus established that

$$F_\lambda(x,y) = -\frac{\lambda}{1-\lambda}\,\underset{\epsilon\searrow 0}{\text{w-lim}}\, H_\lambda(x-y-\mathrm{i}\epsilon)\times\Big[-\frac{1-\lambda}{\lambda}\Big]^{\mathrm{i}[Z(x)-Z(y)]} \tag{2.116}$$

solves the integral equation (2.102) if the requirements from eqs. (2.107), (2.109), (2.111), (2.114) and (2.115) are fulfilled.

2.4.2 Solutions for different propagators

We can now turn to the explicit construction of an F_λ that satisfies all these requirements for each of the propagators in table 2.2. We begin with the vacuum on line, given by eq. (2.70), i.e,

$$G(z) = \frac{1}{2\pi\mathrm{i}z}.$$

Let us try to take the γ in eq. (2.107) to be a circle of infinite radius. Then eq. (2.107), as well as eq. (2.109), can be fulfilled by setting

$$H_\lambda(z) := G(z) = \frac{1}{2\pi\mathrm{i}z}, \tag{2.117}$$

as long as $S(z)$ satisfies the growth condition

$$\lim_{|z|\to\infty} z^{-1}S(z) = 0, \tag{2.118}$$

since then

$$\lim_{R\to\infty} \Big| \oint_{|z|=R} \mathrm{d}z\, G(x-z)F_\lambda(z,y) \Big| \leq \lim_{R\to\infty} \frac{2\pi R}{(2\pi R)^2} \sup_{|z|=R} |S(z)| = 0.$$

The branch cut (2.115) can be realised by a natural logarithm, via

$$Z(z) := \frac{1}{2\pi} \log \prod_n \frac{z-a_n}{z-b_n} \quad \text{for} \quad z \in \mathbb{C} \setminus V.$$

Here, we take the branch cut of the natural logarithm to be along the negative axis. To verify eq. (2.115), first note that the argument of the logarithm in $Z(x)$ is negative for all $x \in V$. Furthermore, it is also strictly decreasing

$$0 > \frac{\mathrm{d}}{\mathrm{d}x} \prod_n \frac{x-a_n}{x-b_n} \quad \text{along} \quad V$$

and meromorphic, so that we can employ the Cauchy-Riemann differential equations to find

$$0 > \lim_{\epsilon\searrow 0} \frac{\partial}{\partial x} \Re\Big[\prod_n \frac{x+\mathrm{i}\epsilon-a_n}{x+\mathrm{i}\epsilon-b_n} \Big] = \lim_{\epsilon\searrow 0} \frac{\partial}{\partial \epsilon} \Im\Big[\prod_n \frac{x+\mathrm{i}\epsilon-a_n}{x+\mathrm{i}\epsilon-b_n} \Big] \quad \text{along} \quad V,$$

so that

$$\Im\Big[\prod_n \frac{x\pm\mathrm{i}\epsilon-a_n}{x\pm\mathrm{i}\epsilon-b_n} \Big] \lesseqgtr 0 \quad \text{along} \quad V$$

for sufficiently small $\epsilon > 0$. As a result,

$$\lim_{\epsilon\searrow 0} Z(x\pm\mathrm{i}\epsilon) = \frac{1}{2\pi} \log \prod_n \frac{x-a_n}{b_n-x} \mp \frac{\mathrm{i}}{2} \quad \text{along} \quad V,$$

so that eq. (2.115) is indeed satisfied and we can take eq. (2.114) as the definition of

$$Z(x) = \frac{1}{2\pi} \log \Big[-\prod_n \frac{x-a_n}{x-b_n} \Big] + \text{const.} \quad \text{along} \quad V, \tag{2.119}$$

where the specific value $\mathrm{i}/2$ of the additive constant is irrelevant since it cancels in eq. (2.116). This concludes the derivation of H_λ and Z. As a final step, we still have to verify the growth conditions (2.111) and (2.118). This was

already done by Arias, Casini, and Huerta,[111] but I would like to include the argument here for the sake of completeness. First, note that

$$|S(z)| = \mathrm{e}^{\mathrm{i}[\xi Z(z) - \overline{\xi Z(z)}]/2} = \mathrm{e}^{-\Im[\xi Z(z)]}$$

with $\xi := \log(1 - \lambda^{-1})$. Recall now from eq. (2.42) that the spectrum of $G|_V$ is contained in $[0, 1]$, so that the resolvent (hence, F_λ) exists for $\lambda \in \mathbb{C} \setminus [0, 1]$. This means that ξ covers the open strip $-\pi < \Im(\xi) < \pi$. As a result

$$|S(z)| = \mathrm{e}^{-\Re(\xi)\Im[Z(z)]}\mathrm{e}^{-\Im(\xi)\Re[Z(z)]},$$

where the first factor

$$\mathrm{e}^{-\Re(\xi)\Im[Z(z)]} < e^{|\Re(\xi)|/2}$$

is bounded for fixed ξ. The second factor satisfies

$$\mathrm{e}^{-\Im(\xi)\Re[Z(z)]} \leq \exp\left[\pi \times \frac{1}{2\pi} \log \prod_n \left|\frac{z - a_n}{z - b_n}\right|\right] = \sqrt{\prod_n \left|\frac{z - a_n}{z - b_n}\right|},$$

so that

$$\limsup_{|z|\to\infty} |S(z)| \leq e^{|\Re(\xi)|/2} \quad \text{and} \quad \lim_{z\to a_n} |S(z)| = 0 = \lim_{z\to b_n} |z - b_n||S(z)|$$

in accordance with eqs. (2.111) and (2.118).

Now that we are done with the vacuum on the line, let us proceed to the thermal state on the line. The corresponding propagator

$$G(z) = \frac{1}{2\mathrm{i}\beta \sinh \pi z/\beta}$$

was already derived in eq. (2.71). As stated before, G satisfies the KMS condition (2.72), which will help greatly in solving the requirement (2.107): Let us assume that F_λ also satisfies the KMS condition (2.72) in its first argument, so that

$$G(x - z - \mathrm{i}\beta)F_\lambda(z + \mathrm{i}\beta, y) = G(x - z)F_\lambda(z, y),$$

i.e., the integrand in eq. (2.107) is *periodic* in imaginary direction. It then makes sense to choose a rectangular

$$\gamma : (-R - \mathrm{i}\beta/2) \to (R - \mathrm{i}\beta/2) \to (R + \mathrm{i}\beta/2) \to (-R + \mathrm{i}\beta/2),$$

with $R \to \infty$, as sketched in fig. 2.16.

This has the advantage that the upper and lower contributions to the in-

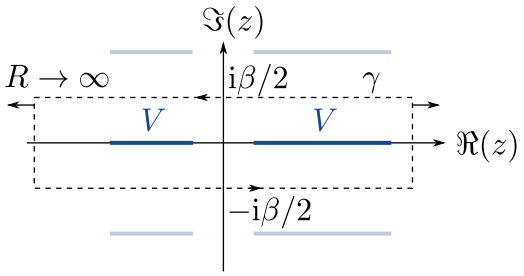


Figure 2.16: Solving eq. (2.106) for the thermal sate on the line: The contour γ (dashed black) is chosen to run along the lines $\Im(z) = \pm\mathrm{i}\beta/2$ in order to make use of the imaginary periodicity of the propagator. The left and right vertical contributions are sent to infinity. Periodic copies of the region V (solid dark blue) are shown in transparent blue.

tegral (2.107) cancel because of symmetry. To ensure that the left and right contributions also vanish, we will again need some growth condition on F_λ. A good candidate for H_λ is again

$$H_\lambda(z) = G(z) = \frac{1}{2\mathrm{i}\beta \sinh \pi z/\beta}, \tag{2.120}$$

because it has the correct pole and the resulting F_λ will satisfy eq. (2.72), as long as

$$S_\lambda(z + \mathrm{i}\beta) = S_\lambda(z)$$

is periodic in imaginary direction. For eq. (2.107) to hold, S_λ should satisfy the growth condition

$$\lim_{R\to\pm\infty} \mathrm{e}^{-2\pi R/\beta} S_\lambda(R + \mathrm{i}s) = 0 \quad \text{for} \quad |s| \leq \frac{\beta}{2},$$

which is very weak indeed! Similar to the vacuum case, we can choose

$$Z(z) := \frac{1}{2\pi} \log \prod_n \frac{\sinh \pi(z - a_n)/\beta}{\sinh \pi(z - b_n)/\beta}$$

for $z \in \mathbb{C} \setminus (V + \mathrm{i}\beta\mathbb{Z})$. All arguments from the vacuum go through unchanged and I will not repeat them here – this was the reason for being so general in the first place. This function Z is indeed periodic in imaginary direction, has the correct branch cut (2.115) and satisfies all necessary growth conditions.

From eq. (2.114), you again find

$$Z(x) = \frac{1}{2\pi} \log \Big[- \prod_n \frac{\sinh \pi(x - a_n)/\beta}{\sinh \pi(x - b_n)/\beta} \Big] + \text{const.} \quad \text{along} \quad V, \tag{2.121}$$

which already concludes the derivation for the thermal state on the line.

Next, let us turn to the vacuum state on the circle. Depending on the boundary conditions, the propagators are given by eqs. (2.76) and (2.77). Let us start with the anti-periodic case

$$G(z) = \frac{1}{2\mathrm{i}} \csc \pi z.$$

Here, we can repeat the argument from the thermal state on the line, albeit with all periodicities in real direction. The rectangular contour is now

$$\gamma : (-1/2 - \mathrm{i}R) \to (1/2 - \mathrm{i}R) \to (1/2 + \mathrm{i}R) \to (-1/2 + \mathrm{i}R)$$

and we can choose

$$H_\lambda(z) := G(z) = \frac{1}{2\mathrm{i}} \csc \pi z \tag{2.122}$$

as well as

$$Z(z) := \frac{1}{2\pi} \log \prod_n \frac{\sin \pi(z - a_n)}{\sin \pi(z - b_n)}$$

for $z \in \mathbb{C} \setminus (V + \mathbb{Z})$, which yields

$$Z(z) = \frac{1}{2\pi} \log \Big[- \prod_n \frac{\sin \pi(x - a_n)}{\sin \pi(x - b_n)} \Big] \quad \text{along} \quad V. \tag{2.123}$$

The growth condition for S_λ is

$$\lim_{R \to \pm\infty} \mathrm{e}^{-2\pi R} S_\lambda(s + \mathrm{i}R) = 0 \quad \text{for} \quad |s| \leq \frac{1}{2},$$

which is obviously satisfied.

The first propagator where things get tricky is the periodic one

$$G(z) = \frac{1}{2\mathrm{i}} \cot \pi z.$$

It is tempting to again just try $F_\lambda \stackrel{?}{=} G$, together with $Z(z)$ as above, but it will not work this time: The periodic propagator does not vanish for large $\Im(z)$ because of the constant zero-mode contribution in eq. (2.75). This implies that

the growth condition on S_λ would have to be

$$\lim_{R\to\pm\infty} S_\lambda(s+\mathrm{i}R) = 0 \quad \text{for} \quad |s| \leq \frac{1}{2},$$

but that would imply

$$\oint_\gamma \mathrm{d}z\, S_\lambda(z) = 0,$$

which is in conflict with the requirements (2.111) and (2.112). The actual solution to this problem was first found by Klich, Vaman, and Wong[109] using rank-one perturbation theory. Here, I will show a more direct approach within the scope of the methods discussed so far. The idea is to modify H_λ slightly, so that eq. (2.107) is satisfied for our above choice of Z. Since the whole trouble originates from a constant term in G we will make the ansatz

$$H_\lambda(z) = \frac{1}{2\mathrm{i}} \cot \pi z + \frac{C_\lambda}{2},$$

where C_λ is a constant that we wish to determine. Again, we choose γ as above. Since the imaginary-direction contributions cancel by symmetry, eq. (2.107) reads

$$\lim_{R\to\infty} \Big[\int_{-1/2-\mathrm{i}R}^{1/2-\mathrm{i}R} \mathrm{d}z\, G(x-z)G(z-y) \Big[-\frac{1-\lambda}{\lambda} \Big]^{\mathrm{i}Z(z)} - \int_{-1/2+\mathrm{i}R}^{1/2+\mathrm{i}R} \mathrm{d}z\, \mathrm{d}z\, G(x-z)G(z-y) \Big[-\frac{1-\lambda}{\lambda} \Big]^{\mathrm{i}Z(z)} \Big] = 0.$$

We can now just compute this integral in the limit $R \to 0$, in order to fix C_λ. To do this, change variables to $u = \mathrm{e}^{2\pi\mathrm{i}x}$, $v = \mathrm{e}^{2\pi\mathrm{i}y}$, and $w = \mathrm{e}^{2\pi\mathrm{i}z}$, such that

$$\lim_{\rho\to\infty} \Big(\oint_{|w|=\rho} \frac{\mathrm{d}w}{w} \frac{u+w}{u-w} \Big[\frac{w+v}{w-v} + C_\lambda \Big] \Big[-\frac{1-\lambda}{\lambda} \Big]^{\mathrm{i}\tilde{Z}(w)} - \oint_{|w|=\rho^{-1}} \frac{\mathrm{d}w}{w} \frac{u+w}{u-w} \Big[\frac{w+v}{w-v} + C_\lambda \Big] \Big[-\frac{1-\lambda}{\lambda} \Big]^{\mathrm{i}\tilde{Z}(w)} \Big) = 0,$$

with

$$\tilde{Z}(w) = \frac{1}{2\pi} \log \prod_n \mathrm{e}^{\mathrm{i}\pi(b_n-a_n)} \frac{\mathrm{e}^{2\pi\mathrm{i}a_n} - w}{\mathrm{e}^{2\pi\mathrm{i}b_n} - w}.$$

You can now easily verify that

$$\lim_{|w|\to\infty} \tilde{Z}(w) = \frac{\mathrm{i}}{2}|V| \quad \text{and} \quad \lim_{|w|\to 0} \tilde{Z}(w) = -\frac{\mathrm{i}}{2}|V|,$$

where

$$|V| := \sum_n (b_n - a_n) \tag{2.124}$$

denotes the total size of the entangling region. Our requirement thus reduces to

$$\lim_{\rho\to\infty} \Big(\oint_{|w|=\rho} \frac{\mathrm{d}w}{w} \frac{u+w}{u-w} \Big[\frac{w+v}{w-v} + C_\lambda \Big] - \Big[-\frac{1-\lambda}{\lambda} \Big]^{|V|} \oint_{|w|=\rho^{-1}} \frac{\mathrm{d}w}{w} \frac{u+w}{u-w} \Big[\frac{w+v}{w-v} + C_\lambda \Big] \Big) = 0,$$

The rest is a simple application of the residue theorem. You find

$$-1 - C_\lambda + \Big[-\frac{1-\lambda}{\lambda} \Big]^{|V|} (1 - C_\lambda) = 0$$

and therefore

$$C_\lambda = \tanh \frac{\xi_\lambda |V|}{2} \quad \text{with} \quad \xi_\lambda = \log \Big[-\frac{1-\lambda}{\lambda} \Big].$$

The correct H_λ thus takes the form

$$H_\lambda(z) = \frac{1}{2\mathrm{i}} \cot \pi z + \frac{1}{2} \tanh \frac{\xi_\lambda |V|}{2}, \quad \xi_\lambda = \log \Big[-\frac{1-\lambda}{\lambda} \Big]. \tag{2.125}$$

Note that this is the first time that H_λ actually depends on λ. This λ-dependence will be important soon, when we compute entanglement properties. There, it will be the source of non-trivial and non-local behaviour.

We will now proceed to the thermal state on the circle, which was the most recent to be solved.[41,113] The propagator from eq. (2.87) is

$$G(z) = \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_\nu(z|\mathrm{i}\beta)}{\vartheta_\nu(0|\mathrm{i}\beta)},$$

where $\nu = 2$ for periodic and $\nu = 3$ for anti-periodic boundary conditions. As we seen above, it makes sense to choose γ in a way that is compatible with the (anti-)periodicities. As explained in eq. (2.84), we have

$$G(z+1) = \pm G(z) \quad \text{and} \quad G(z + \mathrm{i}\beta) = -G(z)$$

as a result of the KMS condition (2.72), hence we should γ to enclose one

fundamental region, i.e.,

$$\gamma : (-1/2 - \mathrm{i}\beta/2) \to (1/2 - \mathrm{i}\beta/2) \to (1/2 + \mathrm{i}\beta/2) \to (-1/2 + \mathrm{i}\beta/2),$$

as sketched in fig. 2.17.

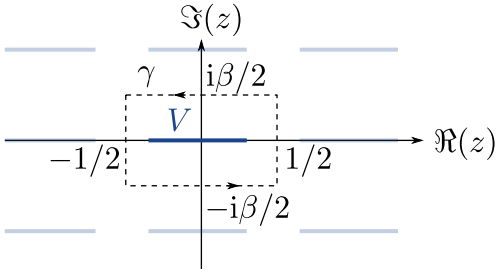


Figure 2.17: Solving eq. (2.106) for the thermal sate on the circle: The contour γ (dashed black) is chosen to run along one fundamental region of the two periodicities $z \sim z+\mathrm{i}\beta \sim z+1$. Periodic copies of the region V (solid dark blue) are shown in transparent blue.

The fact, that there are now two independent periods, is of great use to us: If we manage to find an F_λ which satisfies the same (anti-)periodicities as G in its first argument, then the integral in eq. (2.107) vanishes trivially, as opposite contributions cancel! Finding such a function, however, is a daunting task. It does not make sense to try $H_\lambda \stackrel{?}{=} G$, since that would imply that S_λ needs to be doubly periodic

$$S_\lambda(z+1) = S_\lambda(z) = S_\lambda(z+\mathrm{i}\beta)$$

and thus

$$\oint_\gamma \mathrm{d}z\, S_\lambda(z) = 0,$$

which is in conflict with the requirements (2.111) and (2.112), similarly to the problem that we encountered for the periodic vacuum. This means that we again have to choose Z and then find an appropriate modification for H_λ. An educated guess for Z is

$$Z(z) := \frac{1}{2\pi} \log \prod_n \frac{\vartheta_1(z - a_n|\mathrm{i}\beta)}{\vartheta_1(z - b_n|\mathrm{i}\beta)} \tag{2.126}$$

for all $z \in \mathbb{C} \setminus (V + \mathbb{Z} + \mathrm{i}\beta\mathbb{Z})$. Again, since

$$\prod_n \frac{\vartheta_1(x - a_n|\mathrm{i}\beta)}{\vartheta_1(x - b_n|\mathrm{i}\beta)}$$

is negative and strictly decreasing along V (you can show this directly from the series representation (2.78)), Z has the correct branch cut (2.115). Also, as shown in the derivation of eq. (2.87), we have

$$\vartheta_1(z|\mathrm{i}\beta) \sim 2\pi\eta^3(\mathrm{i}\beta)z \quad \text{as} \quad z \to 0,$$

which implies that the previous argument for the asymptotic behaviour of $Z(z)$ near a_n, b_n goes through unchanged. As a result, eq. (2.111) is also satisfied, hence, we only have to find the correct H_λ. To this end, note first that, while it is impossible for Z to be doubly periodic, it is *almost* doubly periodic in the sense that $Z(z+1) = Z(z)$ and

$$\begin{aligned} Z(z + \mathrm{i}\beta) &= \frac{1}{2\pi} \log \prod_n \frac{\vartheta_1(z + \mathrm{i}\beta - a_n|\mathrm{i}\beta)}{\vartheta_1(z + \mathrm{i}\beta - b_n|\mathrm{i}\beta)} \\ &= \frac{1}{2\pi} \log \prod_n \frac{\mathrm{e}^{\pi\beta}\mathrm{e}^{-2\pi\mathrm{i}(z-a_n)}\vartheta_1(z - a_n|\mathrm{i}\beta)}{\mathrm{e}^{\pi\beta}\mathrm{e}^{-2\pi\mathrm{i}(z-b_n)}\vartheta_1(z - b_n|\mathrm{i}\beta)} \\ &= Z(z) - \mathrm{i}|V|. \end{aligned} \tag{2.127}$$

This means that

$$S_\lambda(z+1) = S_\lambda(z) \quad \text{and} \quad S_\lambda(z + \mathrm{i}\beta) = \Big[-\frac{1-\lambda}{\lambda} \Big]^{|V|} S_\lambda(z),$$

hence we require that

$$H_\lambda(z+1) = \pm H_\lambda(z) \quad \text{and} \quad H_\lambda(z + \mathrm{i}\beta) = -\Big[-\frac{1-\lambda}{\lambda} \Big]^{-|V|} H_\lambda(z)$$

so that F_λ has the correct (anti-)periodicities for eq. (2.107) to be fulfilled. This, however, coincides exactly with the conditions (2.93) for the propagator of a thermal state on the circle with chemical potential

$$\mu = -|V| \log \Big[-\frac{1-\lambda}{\lambda} \Big],$$

so that we can simply choose

$$H_\lambda(z) = \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_\nu(z - \mathrm{i}\xi_\lambda|V|/2\pi|\mathrm{i}\beta)}{\vartheta_\nu(-\mathrm{i}\xi_\lambda|V|/2\pi|\mathrm{i}\beta)}, \quad \xi_\lambda = \log\Big[-\frac{1-\lambda}{\lambda}\Big]. \tag{2.128}$$

Together with eq. (2.126), this satisfies all requirements that we need and we can again use eq. (2.114) to find

$$Z(x) = \frac{1}{2\pi}\log\Big[-\prod_n \frac{\vartheta_1(z-a_n|\mathrm{i}\beta)}{\vartheta_1(z-b_n|\mathrm{i}\beta)}\Big] \quad \text{along} \quad V, \tag{2.129}$$

which concludes the derivation of F_λ in this last case.

We could now go through all of the above again in order to solve eq. (2.102) again for a thermal state on the circle with added chemical potential. However, this will not introduce anything new – since we only slightly change the periodicities from eq. (2.84) to eq. (2.93), the result is given by keeping Z as in eq. (2.129) and modifying[42]

$$H_\lambda(z) \to \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_\nu(z + \mathrm{i}(\mu - \xi_\lambda|V|)/2\pi|\mathrm{i}\beta)}{\vartheta_\nu(\mathrm{i}(\mu-\xi_\lambda|V|)/2\pi|\mathrm{i}\beta)}, \quad \xi_\lambda = \log\Big[-\frac{1-\lambda}{\lambda}\Big].$$

As already stated, this does not produce any qualitative change in any of the computations that follow, so I will not use this solution for the rest of this thesis. It is only included for the sake of completeness.

Finally, I would like to show how the vacuum result can be recovered from the above: The computation that lead us to the series expansions eqs. (2.89) and (2.90) for the thermal state on the circle without chemical potential can of course be repeated to find

$$\begin{aligned}\frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_3(z+\mathrm{i}\mu/2\pi|\mathrm{i}\beta)}{\vartheta_3(\mathrm{i}\mu/2\pi|\mathrm{i}\beta)} &= \frac{1}{2\mathrm{i}}\csc\pi z \\ &+ \sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta}\Big[\frac{\mathrm{e}^{(2k+1)\mathrm{i}\pi z}}{\mathrm{e}^{\mu}+\mathrm{e}^{-(2k+1)\pi\beta}} - \frac{\mathrm{e}^{-(2k+1)\mathrm{i}\pi z}}{\mathrm{e}^{-\mu}+\mathrm{e}^{-(2k+1)\pi\beta}}\Big]\end{aligned} \tag{2.130}$$

and

$$\begin{aligned}\frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z|\mathrm{i}\beta)} \frac{\vartheta_2(z+\mathrm{i}\mu/2\pi|\mathrm{i}\beta)}{\vartheta_2(\mathrm{i}\mu/2\pi|\mathrm{i}\beta)} &= \frac{1}{2\mathrm{i}}\cot\pi z - \frac{1}{2}\tanh\frac{\mu}{2} \\ &+ \sum_{k=1}^{\infty} \mathrm{e}^{-2\pi k\beta}\Big[\frac{\mathrm{e}^{2\pi\mathrm{i}kz}}{\mathrm{e}^{\mu}+\mathrm{e}^{-2\pi k\beta}} - \frac{\mathrm{e}^{-2\pi\mathrm{i}kz}}{\mathrm{e}^{-\mu}+\mathrm{e}^{-2\pi k\beta}}\Big]\end{aligned} \tag{2.131}$$

Now, in the limit of low temperatures $\beta \to \infty$, you see that $\mathrm{e}^{-2\pi\beta} \to 0$, so that

(2.128) yields

$$H_\lambda(z) \to \begin{cases} \frac{1}{2\mathrm{i}} \csc \pi z & \text{(anti-periodic)} \\ \frac{1}{2\mathrm{i}} \cot \pi z + \frac{1}{2} \tanh \frac{\xi_\lambda |V|}{2} & \text{(periodic)} \end{cases}$$

in accordance with eqs. (2.122) and (2.125). Note that, while we did not need any special treatment for the derivation of eq. (2.128) for periodic boundary conditions, we are able to easily recover the zero-mode contribution in the vacuum, which was hard to find directly.

Again, you can find summary of all solutions to eq. (2.102) in table 2.3.

Table 2.3: Solutions to the integral equation (2.102) for the propagators in table 2.2. Every solution F_λ is a regulated distributional kernel (2.105), which can be written in terms of two functions H_λ and Z, as shown in eq. (2.116). Presented are the functions H_λ and Z, along with their locations in the main text to find their derivation.

Geometry/State	**Formulae**	**Location**
line/vacuum	$H_\lambda(z) = 1/2\pi\mathrm{i}z$	(2.117)
	$Z(x) = 1/2\pi \times \log\Big[-\prod_n \frac{x-a_n}{x-b_n}\Big]$	(2.119)
line/thermal	$H(z) = 1/[2\mathrm{i}\beta \sinh(\pi z/\beta)]$	(2.120)
	$Z(x) = 1/2\pi \times \log\Big[-\prod_n \frac{\sinh \pi(x-a_n)/\beta}{\sinh \pi(x-b_n)\beta}\Big]$	(2.121)
circle/vacuum	$H^{(\mathrm{P})}_\lambda(z) = 1/2\mathrm{i} \times \cot \pi z$	(2.125)
	$H^{(\mathrm{A})}_\lambda(z) = 1/2\mathrm{i} \times \csc \pi z + 1/2 \times \tanh(\xi_\lambda \lvert V\rvert/2)$	(2.122)
	$Z(x) = 1/2\pi \times \log\Big[-\prod_n \frac{\sin \pi(x-a_n)}{\sin \pi(x-b_n)}\Big]$	(2.123)
	$\lvert V\rvert = \sum_n (b_n - a_n)$	(2.124)
	$\xi_\lambda = \log(1-\lambda^{-1})$	(2.125)
circle/thermal	$H^{(\mathrm{P/A})}_\lambda(z) = \frac{\eta^3(\mathrm{i}\beta)}{\mathrm{i}\vartheta_1(z\lvert\mathrm{i}\beta)} \frac{\vartheta_{2/3}(z - \mathrm{i}\xi_\lambda\lvert V\rvert/2\pi\lvert\mathrm{i}\beta)}{\vartheta_{2/3}(-\mathrm{i}\xi_\lambda\lvert V\rvert/2\pi\lvert\mathrm{i}\beta)}$	(2.128)
	$Z(x) = 1/2\pi \times \log\Big[-\prod_n \frac{\vartheta_1(x-a_n\lvert\mathrm{i}\beta)}{\vartheta_1(x-b_n\lvert\mathrm{i}\beta)}\Big]$	(2.129)

2.4.3 Modular flow of operators

Now that we solved the singular integral equation (2.102), we can use the solutions to compute various quantities associated with entanglement. Let us begin with the modular flow, defined in eq. (2.19). In the present case of free fermions, we already established in eq. (2.47) that

$$\sigma_t(\psi^*(x)) = \int_V \mathrm{d}y\, \psi^*(y)\Sigma_t(y,x), \quad \text{with} \quad \Sigma_t = \Big[\frac{1-G|_V}{G|_V}\Big]^{\mathrm{i}t}.$$

To evaluate this, we shall use the spectral calculus from eqs. (2.56) and (2.57), i.e., we have

$$\Sigma_t = \int_0^1 \mathrm{d}\lambda\, \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}\Big[\frac{1-\lambda}{\lambda}\Big]^{\mathrm{i}t}$$

with

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{1}{2\pi\mathrm{i}}\Big[\frac{1}{\lambda - \mathrm{i}0 - G|_V} - \frac{1}{\lambda + \mathrm{i}0 - G|_V}\Big].$$

According to eq. (2.60), the resolvent takes the form

$$\frac{1}{\lambda - G|_V} = \frac{1}{\lambda} + \frac{F_\lambda}{\lambda^2}$$

so that the spectral measure is determined by the branch cut of F_λ, i.e.,

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{1}{2\pi\mathrm{i}}\frac{F_{\lambda-\mathrm{i}0} - F_{\lambda+\mathrm{i}0}}{\lambda^2}.$$

Inserting F_λ from eq. (2.116) you find

$$\frac{F_\lambda}{\lambda^2} = -\frac{1}{(1-\lambda)\lambda}H_\lambda\Big[-\frac{1-\lambda}{\lambda}\Big]^{\mathrm{i}\Delta Z}$$

where I suppressed the weak limit for the sake of brevity and used the shorthand notation

$$\Delta Z(x,y) := Z(x) - Z(y). \tag{2.132}$$

It is now clear that the branch cut of F_λ is better expresses in terms of the variable $\zeta = (1-\lambda)/\lambda$, where we have

$$\frac{F_{\lambda\pm\mathrm{i}0}}{\lambda^2} = \frac{\mathrm{d}\zeta}{\mathrm{d}\lambda}\frac{1}{\zeta}H_{1/(1+\zeta\mp\mathrm{i}0)}[-\zeta \pm \mathrm{i}0]^{\mathrm{i}\Delta Z},$$

so that

$$\Sigma_t = \int_0^\infty \mathrm{d}\zeta \, \frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} \zeta^{\mathrm{i}t} \tag{2.133}$$

with

$$\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = -\frac{1}{2\pi\mathrm{i}} \Big[\mathrm{e}^{\pi\Delta Z} H_{1/(1+\zeta+\mathrm{i}0)} - \mathrm{e}^{-\pi\Delta Z} H_{1/(1+\zeta-\mathrm{i}0)} \Big] \zeta^{\mathrm{i}\Delta Z - 1}, \tag{2.134}$$

where the negative sign at the beginning comes a change in orientation due to the change of variables. We are now in the position to evaluate eqs. (2.133) and (2.134) for the various solutions in table 2.3.

As a warm-up, let us again begin with the vacuum state on the line

$$G(z) = \frac{1}{2\pi\mathrm{i}z},$$

where the solution is

$$H_\lambda(z) = G(z) \quad \text{and} \quad Z(x) = \frac{1}{2\pi} \log \Big[-\prod_n \frac{x - a_n}{x - b_n} \Big].$$

Since H_λ does not depend on λ, eq. (2.134) simplifies to

$$\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = \frac{\mathrm{i}}{\pi} \sinh(\pi\Delta Z) G \zeta^{\mathrm{i}\Delta Z - 1}, \tag{2.135}$$

hence

$$\begin{aligned} \Sigma_t &= \frac{\mathrm{i}}{\pi} \sinh(\pi\Delta Z) G \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta} \zeta^{\mathrm{i}(t+\Delta Z)} \\ &= \frac{\mathrm{i}}{\pi} \sinh(\pi\Delta Z) G \int_{\mathbb{R}} \mathrm{d}s \, \mathrm{e}^{\mathrm{i}s(t+\Delta Z)} \\ &= -2\mathrm{i} \sinh(\pi t) G \times \delta(t + \Delta Z). \end{aligned}$$

We have thus established that the kernel associated to the action of modular flow for the vacuum state on the line is[43]

$$\Sigma_t(x, y) = -2\mathrm{i} \sinh(\pi t) G(x - y) \delta(t + Z(x) - Z(y)), \tag{2.136}$$

the support of which is located on solutions of the equation

$$t + Z(x) - Z(y) = 0. \tag{2.137}$$

This equation is of fundamental importance for the free fermion CFT,[112,131] as

it shows that modular flow is *multi-local*. To see what this means, recall from our previous derivations that the function $Z(x)$ is strictly increasing in each interval $[a_n, b_n]$ with $Z(a_n) = -\infty$ and $Z(b_n) = \infty$, as illustrated in fig. 2.18. This implies that, for fixed y and t, there is one solution of eq. (2.137) *per interval*!

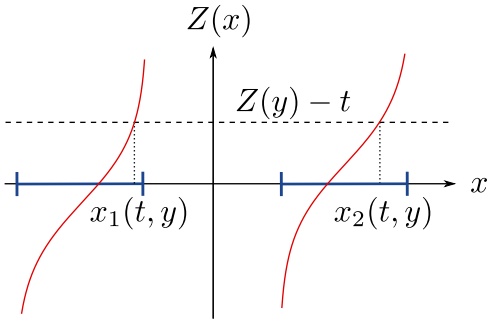


Figure 2.18: Sketch of the structure of solutions to eq. (2.137): For any given value of $Z(y) - t$ (dotted black), eq. (2.137) has two solutions $x_n(t, y)$, one per interval (solid dark blue). The graph of the function $Z(x)$ is given in solid red.

Defining $x_n(t, y)$ as the solution in the n-th interval, i.e.,

$$t + Z(x_n(t, y)) - Z(y) = 0, \quad x_n(t, y) \in [a_n, b_n],$$

you can see from eq. (2.136) that

$$\sigma_t(\psi^*(y)) = -2\mathrm{i}\sinh(\pi t) \sum_n \frac{G(x_n(t, y) - y)}{Z'(x_n)} \psi^*(x_n(t, y)) \tag{2.138}$$

is a superposition of one localised field operator per interval for any $t \neq 0$. In particular, consider the case of just a single interval $[a, b]$, such that

$$Z(x) = \frac{1}{2\pi} \log \frac{x - a}{b - x}, \quad Z'(x) = \frac{1}{2\pi} \frac{b - a}{(x - a)(b - x)}.$$

We can then solve eq. (2.137) explicitly: It takes the form

$$\frac{(x - a)(b - y)}{(b - x)(y - a)} = \mathrm{e}^{-2\pi t},$$

yielding

$$x(t, y) = \frac{\mathrm{e}^{2\pi t} a(b - y) + b(y - a)}{\mathrm{e}^{2\pi t}(b - y) + (y - a)}$$

and (after some algebra)

$$-2\mathrm{i}\sinh(\pi t)\frac{G(x(t,y)-y)}{Z'(x)} = \frac{b-a}{\mathrm{e}^{\pi t}(b-y)+\mathrm{e}^{-\pi t}(y-a)},$$

so that we recover the Hislop-Longo result (2.35) with $h = 1/2$.

For two intervals, eq. (2.137) is quadratic and still analytically solvable. The general structure of the solutions was already presented by Casini and Huerta[107] and subsequently formally analysed by Rehren and Tedesco.[131] Here, I will not cover them quantitatively, since for larger numbers of intervals, eq. (2.137) becomes analytically intractable anyway. Nevertheless, the key observation is always the same: There is one solution per interval, so that $\sigma_t(\psi^*(y))$ is multi-local with one localised field operator $\psi^*(x_n(t,y))$ per interval. Each of these localised operators comes with prefactor of

$$-2\mathrm{i}\sinh(\pi t)\frac{G(x_n(t,y)-y)}{Z'(x_n(t,y))},$$

which, due to the appearance of Z', looks like some multi-local version of the action of the conformal group from eq. (2.33). Recalls also that eq. (2.138) is just the result for the one-dimensional problem associated to one chirality. The actual two-dimensional result can be retrieved by changing back from lightray coordinates $x^\pm$ to the original x^0, x^1, as discussed at the end of section 2.2.2. In this, however, we will have to be careful: The location of the Dirac fermion field

$$\psi(x^0,x^1) = \begin{bmatrix}\psi_+(x^+)\\ \\ \psi_-(x^-)\end{bmatrix}$$

can only be restored given a pair of chiral fermions $\psi_\pm$. The question is then, how do we group a multi-local expression, such as

$$\begin{bmatrix}c_1\psi_+(x_1^+)+c_2\psi_+(x_2^+)\\ \\ \tilde{c}_1\psi_-(x_1^-)+\tilde{c}_2\psi_-(x_2^-)\end{bmatrix}, \quad x_n^\pm \in [a_n^\pm, b_n^\pm]$$

into pairs of chiral fermions? For *general* multi-local expressions, there is no

unique answer to this problem and, e.g.,

$$\begin{bmatrix} c_1\psi_+(x_1^+) \\ \\ \tilde{c}_2\psi_-(x_2^-) \end{bmatrix} + \begin{bmatrix} c_c\psi_+(x_c^+) \\ \\ \tilde{c}_1\psi_-(x_1^-) \end{bmatrix}$$

would qualify as a valid split. However, the resulting Dirac fermion fields would be located outside of the double cones corresponding to the intervals $[a_n^\pm, b_n^\pm]$, so Tomita's theorem would not apply. Indeed, there is only one unique way of pairing the fields correctly and that is to take

$$\psi(x_n^0, x_n^1) = \begin{bmatrix} c_n\psi_+(x_n^+) \\ \\ \tilde{c}_n\psi_-(x_n^-) \end{bmatrix}, \quad x_n^\pm \in [a_n^\pm, b_n^\pm]. \tag{2.139}$$

We can now repeat the analysis for the thermal state on the line, given by solutions (2.120) and (2.121). Again, H_λ does not depend on λ, so the previous derivation goes through without a single change and eqs. (2.136) and (2.138) are still valid, albeit with G and Z given by eqs. (2.71) and (2.121). Modular flow is again multi-local with support at the solutions of eq. (2.137). The only effect of finite temperature is that these solutions are shifted away from their vacuum positions. For low temperatures ($\beta \to \infty$), you see that

$$\frac{\sinh \pi(x-a)/\beta}{\sinh \pi(b-x)/\beta} \sim \frac{x-a}{b-x},$$

which recovers the vacuum results. For high temperatures ($\beta \to 0$) on the other hand, assume that $x \in [a_m, b_m]$ for some m. We then have

$$Z(x) \sim \frac{1}{\beta}\Big[x - \frac{a_m + b_m}{2} - \sum_{n>m} \frac{b_n - a_n}{2} + \sum_{n<m} \frac{b_n - a_n}{2}\Big]$$

and if you insert this in eq. (2.137), you quickly see that it yields $x = y - \beta t$ if x, y are in the same interval, i.e., we verified that modular flow coincides with time evolution in this limit. The multi-local solutions (those where x, y are in different intervals) also show linear transport of the form $x = y\beta t + \text{const.}$ but their amplitude in eq. (2.138) scales asymptotically $\propto 1/\beta$. The case of finite β is an interpolation between these cases: As β gets smaller, for fixed t, the multi-local solutions are pushed towards the boundary and their amplitude gets smaller. In the context of AdS/CFT, this can be given the interpretation

of a red-shift of entanglement near boundaries/horizons.[41,43]

Next on the list is the vacuum state on the circle, given by eqs. (2.122), (2.123) and (2.125). Again, H_λ is independent of λ for anti-periodic boundary conditions, so that the previous analysis applies. Obviously, the key equation (2.137) is now periodic, so that its solutions shift as the intervals are moved around the circle. The interesting case is that of periodic boundary conditions, where there is an additional contribution to the spectral measure (2.134), originating in the zero-mode contribution

$$\delta H_{1/(1+\zeta)} = \frac{1}{2}\tanh\frac{|V|\log(-\zeta)}{2} = \frac{1}{2}\frac{(-\zeta)^{|V|}-1}{(-\zeta)^{|V|}+1}.$$

The extra term of the spectral measure thus takes the form

$$\begin{aligned}\delta\Big[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\Big] &= -\frac{1}{4\pi\mathrm{i}}\Big[\mathrm{e}^{\pi\Delta Z}\frac{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}-1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1} - \mathrm{e}^{-\pi\Delta Z}\frac{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}-1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\Big]\zeta^{\mathrm{i}\Delta Z-1}\\ &= -\frac{1}{2\pi\mathrm{i}}\sinh(\pi\Delta Z)\zeta^{\mathrm{i}\Delta Z-1} \qquad (2.140)\\ &\quad + \frac{1}{2\pi\mathrm{i}}\Big[\frac{\mathrm{e}^{\pi\Delta Z}}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1} - \frac{\mathrm{e}^{-\pi\Delta Z}}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\Big]\zeta^{\mathrm{i}\Delta Z-1}. \qquad (2.141)\end{aligned}$$

While the first of these terms can be evaluated easily in eq. (2.133), the second one looks scary. Fortunately, we can use the relation[112]

$$\frac{1}{1+y} = \frac{\mathrm{i}}{2}\int_{\mathbb{R}}\mathrm{d}s\,\frac{y^{\mathrm{i}s}}{\sinh\pi s+\mathrm{i}0}, \quad y\in\mathbb{C}\setminus\mathbb{R}_-, \qquad (2.142)$$

which can be derived from complex integral along the Hankel contour. Inserting $y = \mathrm{e}^{\pm\mathrm{i}\pi|V|}\zeta^{|V|}$, you obtain

$$\begin{aligned}\delta\Big[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\Big] &= -\frac{1}{2\pi\mathrm{i}}\sinh(\pi\Delta Z)\zeta^{\mathrm{i}\Delta Z-1}\\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}s\,\frac{\sinh\pi(\Delta Z+s|V|)}{\sinh\pi s+\mathrm{i}0}\zeta^{\mathrm{i}(s|V|+\Delta Z)-1}, \qquad (2.143)\end{aligned}$$

such that eq. (2.133) yields

$$\begin{aligned}\delta\Sigma_t &= -\frac{1}{2\pi\mathrm{i}}\sinh(\pi\Delta Z)\int_0^\infty \mathrm{d}\zeta\,\zeta^{\mathrm{i}(t+\Delta Z)-1} \\ &\qquad + \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}s\,\frac{\sinh\pi(\Delta Z+s|V|)}{\sinh\pi s+\mathrm{i}0}\int_0^\infty \mathrm{d}\zeta\,\zeta^{\mathrm{i}(t+s|V|+\Delta Z)-1} \\ &= \mathrm{i}\sinh(\pi\Delta Z)\delta(t+\Delta Z) \\ &\qquad + \int_{\mathbb{R}}\mathrm{d}s\,\frac{\sinh\pi(\Delta Z+s|V|)}{\sinh\pi s+\mathrm{i}0}\delta(t+s|V|+\Delta Z) \\ &= -\mathrm{i}\sinh(\pi t)\delta(t+\Delta Z)+\frac{1}{|V|}\frac{\sinh\pi t}{\sinh\pi(t+\Delta Z)/|V|-\mathrm{i}0},\end{aligned}$$

where the $-\mathrm{i}0$ in the denominator is still present to regularise σ_t as a distribution. To obtain a more compact result, we shall use a different regularisation, namely the Cauchy principal value[116] $\mathcal{P}$, which is related to our previous regularisation by the Kramers-Kronig relation[116]

$$\frac{1}{z\pm\mathrm{i}0} = \operatorname*{w-lim}_{\epsilon\searrow 0}\frac{z\mp\mathrm{i}\epsilon}{z^2+\epsilon^2} = \mathcal{P}\frac{1}{z}\mp\mathrm{i}\pi\delta(z). \tag{2.144}$$

Combining this with the above, you see that the Dirac delta distributions cancel, hence,[43]

$$\delta\Sigma_t(x,y) = \frac{1}{|V|}\,\mathcal{P}\,\frac{\sinh\pi t}{\sinh\pi[t+Z(x)-Z(y)]/|V|} \tag{2.145}$$

and correspondingly

$$\delta\sigma_t(\psi^*(y)) = \frac{1}{|V|}\,\mathcal{P}\int_V \mathrm{d}x\,\frac{\sinh\pi t}{\sinh\pi[t+Z(x)-Z(y)]/|V|}\psi^*(x). \tag{2.146}$$

As you see, the zero-mode contribution to the modular flow is *completely non-local*, i.e., $\delta\sigma_t(\psi^*(y))$ is smeared our everywhere in the entangling region V, presenting a complete breakdown of the Hislop-Longo result (2.35). However, similar to the derivation of the resolvent, we can recover the behaviour from a slightly better behaved situation, namely the one of finite temperature on the circle.

To treat the case of finite temperature, we insert the solutions (2.128) and (2.129) in eq. (2.134). Here, the branch cut of H_λ is completely non-trivial. In order to handle it, we use the series representation

$$H_{1/(1+\zeta)}(x-y) = \sum_k \frac{\mathrm{e}^{-2\pi\mathrm{i}k(x-y)}}{1+\mathrm{e}^{-2\pi\beta k+\mu}},\quad \mu = -|V|\log(-\zeta)$$

from eq. (2.92), where the summation is over $k \in \mathbb{Z}$ for periodic and $k \in \mathbb{Z}+1/2$ for anti-periodic boundary conditions. This allows to write

$$\begin{aligned} H_{1/(1+\zeta\pm \mathrm{i}0)}(x-y) &= \sum_k \frac{\mathrm{e}^{-2\pi \mathrm{i}k(x-y)}}{1+\mathrm{e}^{-2\pi\beta k}\mathrm{e}^{\pm \mathrm{i}\pi|V|}\zeta^{-|V|}} \\ &= \frac{\mathrm{i}}{2}\sum_k \mathrm{e}^{-2\pi \mathrm{i}k(x-y)} \int_{\mathbb{R}} \mathrm{d}s\, \frac{\mathrm{e}^{-2\pi \mathrm{i}\beta k s}\mathrm{e}^{\mp \pi s|V|}}{\sinh \pi s + \mathrm{i}0} \zeta^{-\mathrm{i}s|V|}, \end{aligned}$$

where I used eq. (2.142) with $y = \mathrm{e}^{-2\pi\beta k}\mathrm{e}^{\pm \mathrm{i}\pi|V|}\zeta^{-|V|}$. Plugging this into eq. (2.134), you find

$$\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = -\frac{1}{2\pi}\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi \mathrm{i}k(x-y+\beta s)} \frac{\sinh \pi(\Delta Z - s|V|)}{\sinh \pi s + \mathrm{i}0} \zeta^{\mathrm{i}(\Delta Z - s|V|)-1}, \tag{2.147}$$

so that eq. (2.133) evaluates to

$$\begin{aligned} \Sigma_t(x,y) &= -\frac{1}{2\pi}\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi \mathrm{i}k(x-y+\beta s)} \\ &\qquad \times \frac{\sinh \pi(\Delta Z(x,y) - s|V|)}{\sinh \pi s + \mathrm{i}0} \int_0^\infty \mathrm{d}\zeta\, \zeta^{\mathrm{i}(t+\Delta Z(x,y) - s|V|)-1} \\ &= -\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi \mathrm{i}k(x-y+\beta s)} \\ &\qquad \times \frac{\sinh \pi[\Delta Z(x,y) - s|V|]}{\sinh \pi s + \mathrm{i}0} \delta(t + \Delta Z(x,y) - s|V|) \\ &= \frac{1}{|V|}\frac{\sinh \pi t}{\sinh \pi[t+\Delta Z(x,y)]/|V|} \sum_k \mathrm{e}^{-2\pi \mathrm{i}k(x-y+\beta[t+\Delta Z(x,y)]/|V|)}. \end{aligned}$$

The remaining series yields an (anti-)periodic Dirac delta, depending on whether the summation is performed over integers or half-integers. As a result, you obtain[43]

$$\begin{aligned} \Sigma_t(x,y) = \frac{1}{|V|}&\frac{\sinh \pi t}{\sinh \pi[t+Z(x)-Z(y)]/|V|} \\ &\times \sum_{k\in\mathbb{Z}} (\pm 1)^k \delta(x-y+\beta[t+Z(x)-Z(y)]/|V| + k), \end{aligned} \tag{2.148}$$

which has support on solutions of

$$x - y + \frac{\beta}{|V|}[t + Z(x) - Z(y)] + k = 0, \quad k \in \mathbb{Z}. \tag{2.149}$$

In contrast to the solutions that we had so far, for fixed y, t, this equation

has one solution per interval for each value of $k \in \mathbb{Z}$, hence, modular flow is "infinitely" multi-local even for a single interval, as sketched in fig. 2.19.

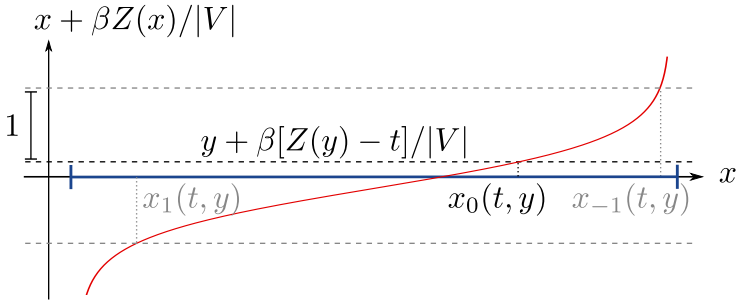


Figure 2.19: Sketch of the structure of solutions to eq. (2.149): For any given value of $y+\beta[Z(y)-t]/|V|-k$ with $k \in \mathbb{Z}$—here, $k = 0$ (dotted black) and $k = \pm 1$ (dotted grey)—, eq. (2.149) has a unique solution $x_k(t, y)$ in the interval (solid dark blue), culminating in infinite multi-locality, even for a single interval.

Denoting the solution for fixed k in the n-th interval by $x_{nk}(t, y)$, the above can be rewritten as

$$\Sigma_t(x, y) = \sum_{nk} (\pm 1)^k \frac{\sinh \pi t}{\sinh \pi[t + Z(x_{nk}(t, y)) - Z(y)]/|V|} \frac{\delta(x - x_{nk}(t, y))}{|V| + \beta Z'(x_{nk}(t, y))},$$

so that

$$\sigma_t(\psi^*(y)) = \sum_{nk} (\pm 1)^k \frac{\sinh \pi t}{\sinh \pi[t + Z(x_{nk}(t, y)) - Z(y)]/|V|} \times \frac{\psi^*(x_{nk}(t, y))}{|V| + \beta Z'(x_{nk}(t, y))}. \quad (2.150)$$

To finish this derivation, let us discuss the behaviour of eq. (2.148) as we vary the temperature: For large temperatures ($\beta \to 0$), similarly to our observations for the thermal state on the line, eq. (2.149) has a localised solution $x = y - \beta t/|V|$, as well as infinitely many solutions near the boundaries of the entangling region, whose amplitudes in eq. (2.148) are at least exponentially suppressed because of the sinh in the denominator. This behaviour indicates that the thermal state of small inverse temperature $\beta \approx 0$ "looks like" a state of inverse temperature $\beta/|V|$ if you only have access to a region of size $|V|$.[132] This is because heat capacity can be defined on a system of finite size, such as a circle, and is of course extensive. The limit of low temperatures ($\beta \to \infty$) is

interesting as well since, in this limit, eq. (2.149) becomes

$$t + Z(x) - Z(y) + k\frac{|V|}{\beta} = 0, \quad k \in \mathbb{Z},$$

whose solutions are spread densely in all intervals. Indeed, in the periodic variant of eq. (2.148), we could treat $\omega := k/\beta$ as a continuous variable so that the summation over k actually becomes a Riemann integral

$$\sigma_t(\psi^*(y)) \sim \sum_n \int_{\mathbb{R}} \mathrm{d}\omega \, \frac{\sinh \pi t}{\sinh \pi[t + Z(x_{n\omega}(t,y)) - Z(y)]/|V|} \frac{\psi^*(x_{n\omega}(t,y))}{Z'(x_{n\omega}(t,y))}$$

with $x_{n\omega}(t,y)$ the solutions to

$$t + Z(x) - Z(y) + \omega|V| = 0.$$

This integral can be brought to a familiar form by changing variables to $x_n = x_{n\omega}(t,y)$. Since ω takes on every value on the real axis, x_n covers all of $[a_n, b_n]$. Furthermore, we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}x_n} = -\frac{Z'(x_n)}{|V|}, \quad \text{hence,} \quad \frac{\mathrm{d}\omega}{Z'(x_n)} = -\frac{\mathrm{d}x_n}{|V|},$$

so that you finally obtain

$$\sigma_t(\psi^*(y)) \sim \frac{1}{|V|} \int_V \mathrm{d}x \, \frac{\sinh \pi t}{\sinh \pi[t + Z(x) - Z(y)]/|V|} \psi^*(x),$$

which is exactly the zero-mode contribution from eq. (2.146). Finally, the multi-local contributions can be recovered from the solutions with $k = 0$. This, however, requires a much more subtle analysis of the limit $\beta \to \infty$ and I will omit it here. Again, finite temperature interpolates between the above extreme cases. As the temperature increases from absolute zero, the non-local zero-mode contribution breaks apart into a discrete infinity of multi-local contributions, which are then pushed towards the boundary of V, where their amplitude is damped. For anti-periodic boundary conditions, there is no-zero mode contribution – nevertheless the infinity of multi-local contributions at finite temperature remains.

2.4.4 Modular correlation function

Now that we computed the action of modular flow on free fermionic fields $\sigma_t(\psi^*(y))$, let us study it in more detail and derive the modular correlation

function

$$G_t(x,y) = \mathrm{Tr}\left[\rho_V \psi(x) \sigma_t\big(\psi^*(y)\big)\right].$$

We could try to do this by substituting the results from eqs. (2.138), (2.146) and (2.150) and directly compute, e.g.,

$$\begin{aligned} G_t(x,y) &= -2\mathrm{i}\sinh(\pi t) \sum_n \frac{G(x_n(t,y)-y)}{Z'(x_n)} \mathrm{Tr}\left[\rho_V \psi(x) \psi^*(x_n(t,y))\right] \\ &= -2\mathrm{i}\sinh(\pi t) \sum_n \frac{G(x_n(t,y)-y)}{Z'(x_n)} G(x - x_n(t,y)) \end{aligned}$$

for the vacuum state on the line with $x_n(t,y)$ the solution of eq. (2.137) in the interval $[a_n, b_n]$. While this might actually work in the cases of up to four intervals, where eq. (2.137) can be solved analytically, the general case of an arbitrary number of intervals seems elusive. In particular, the corresponding problem (2.149) on the circle at finite temperature is a transcendental equation with no hope for a closed solution. Fortunately, we can just as well compute the modular correlation function from eq. (2.49) via the derived spectral calculus and, as it turns out, obtain a closed form result for an arbitrary number of intervals.

Let us again start with the solution on the real line, where the spectral measure of the restricted propagator is given by

$$\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = \frac{\mathrm{i}}{\pi}\sinh(\pi\Delta Z) G \zeta^{\mathrm{i}\Delta Z - 1}, \quad \lambda = \frac{1}{1+\zeta} \in \mathrm{spec}(G|_V) = (0,1),$$

as derived in eq. (2.135). The function Z and propagator G are again given by eq. (2.119) and eq. (2.70), respectively. From eqs. (2.49) and (2.56), you find

$$\begin{aligned} G_t &= \int_0^\infty \mathrm{d}\zeta \, \frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} \frac{\zeta^{\mathrm{i}t}}{1+\zeta} \\ &= \frac{\mathrm{i}}{\pi}\sinh(\pi\Delta Z) G \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)} \zeta^{\mathrm{i}(t+\Delta Z)}. \end{aligned}$$

This integral can be readily evaluated by the same trick that we used before in the derivation of eq. (2.71): Consider the integral

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)} (-\zeta)^{\mathrm{i}(t+\Delta Z)}$$

along the Hankel contour Γ from fig. 2.9. The circular contributions to this integral vanish as long as the modular parameter t lies inside the strip $-1 <$

$\Im(t) < 0$, that we already know from the KMS condition (2.24). Therefore,

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)}(-\zeta)^{\mathrm{i}(t+\Delta Z)} = \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)}\Big[(-\zeta+\mathrm{i}0)^{\mathrm{i}(t+\Delta Z)} - (-\zeta-\mathrm{i}0)^{\mathrm{i}(t+\Delta Z)}\Big]$$
$$= -2\sinh\pi(t+\Delta Z) \times \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta(1+\zeta}\zeta^{\mathrm{i}(t+\Delta Z)},$$

so that

$$G_t = \frac{\sinh\pi\Delta Z}{\sinh\pi(t+\Delta Z)}G \times \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)}(-\zeta)^{\mathrm{i}(t+\Delta Z)}.$$

To evaluate the contour integral, note that the only pole inside the Hankel contour lies at $\zeta = -1$ and is encircled clockwise, hence, as independently derived by Hollands,[112]

$$G_t(x,y) = \frac{\sinh\pi[Z(x)-Z(y)]}{\sinh\pi[t+Z(x)-Z(y)]-\mathrm{i}0}G(x-y), \tag{2.151}$$

where the regulator $-\mathrm{i}0$ is present in the denominator because the previous derivation only holds on the strip $-1 < \Im(t) < 0$. To see how useful this equation is, let us verify eq. (2.51) by explicitly computing the branch cut behaviour

$$G_{t-\mathrm{i}0}(x,y) - G_{t+\mathrm{i}0}(x,y)$$

because sinh is strictly increasing along the real axis and satisfies the Cauchy-Riemann differential equations, we have

$$\sinh\pi[t\pm\mathrm{i}0+Z(x)-Z(y)] = \sinh\pi[t+Z(x)-Z(y)]\pm\mathrm{i}0,$$

so that you can use the Kramers-Kronig relation (2.144) to derive

$$\frac{1}{\sinh\pi(x-\mathrm{i}0)} - \frac{1}{\sinh\pi(x+\mathrm{i}0)} = 2\mathrm{i}\delta(x).$$

This implies

$$G_{t-\mathrm{i}0}(x,y) - G_{t+\mathrm{i}0}(x,y) = -2\mathrm{i}\sinh(\pi t)G(x-y)\delta(t+Z(x)-Z(y)),$$

indeed reproducing eq. (2.136).

By the same reasoning as before, the result (2.151) continues to hold for the thermal state on the line, with Z and G given by eq. (2.121) and eq. (2.71), respectively. Again, the poles of G_t are precisely at the solutions of eq. (2.137) and we can use eq. (2.51) to provide an independent derivation of Σ_t. Furthermore, since the state under consideration is thermal, the same arguments that

led to eq. (2.51) can be used to derive a *second* KMS condition, with respect to translations in time / along the real axis: It takes the form

$$\begin{aligned}\mathrm{Tr}[\rho\{\psi(x)\sigma_t(\psi^*(y))\}] \\ &= \mathrm{Tr}[\rho\psi(x)\sigma_t(\psi^*(y))] + \mathrm{Tr}[\rho\sigma_t(\psi^*(y))\psi(x)] \\ &= \mathrm{Tr}[\rho\psi(x-\mathrm{i}0)\sigma_t(\psi^*(y))] + \mathrm{Tr}[\rho\psi(x-\mathrm{i}\beta+\mathrm{i}0)\sigma_t(\psi^*(y))] \\ &= G_t(x-\mathrm{i}0,y) + G_t(x-\mathrm{i}\beta+\mathrm{i}0,y),\end{aligned}$$

where I used eq. (2.24) in the second equality. While the left hand side again equals $\Sigma_t(x,y)$, we can plug the result (2.151) into the right hand side to obtain

$$\begin{aligned}\Sigma_t(x,y) = \frac{\sinh\pi[Z(x)-Z(y)]}{\sinh\pi[t+Z(x)-Z(y)]-\mathrm{i}0} G(x-y-\mathrm{i}0) \\ - \frac{\sinh\pi[Z(x)-Z(y)]}{\sinh\pi[t+Z(x)-Z(y)]+\mathrm{i}0} G(x-y+\mathrm{i}0).\end{aligned}$$

Using the Kramers-Kronig relation eq. (2.144) again, this yields (for $t \neq 0$)

$$\Sigma_t(x,y) = -2\mathrm{i}\sinh(\pi t) G(x-y)\delta(t+Z(x)-Z(y)),$$

providing yet another independent derivation of eq. (2.136) on the line at finite temperature.

The result (2.151) also holds for the vacuum state on the circle with anti-periodic boundary conditions, with Z from eq. (2.123) and G from eq. (2.77). For periodic boundary conditions, the zero-mode causes a non-trivial correction to the spectral measure of the form

$$\begin{aligned}\delta\Big[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\Big] = -\frac{1}{2\pi\mathrm{i}}\sinh(\pi\Delta Z)\zeta^{\mathrm{i}\Delta Z-1} \\ + \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}s\,\frac{\sinh\pi(\Delta Z+s|V|)}{\sinh\pi s+\mathrm{i}0}\zeta^{\mathrm{i}(s|V|+\Delta Z)-1},\end{aligned}$$

as derived in eq. (2.143). The corresponding correction to the modular correlation function is

$$\begin{aligned}\delta G_t = -\frac{1}{2\pi\mathrm{i}}\sinh(\pi\Delta Z)\int_0^\infty \frac{\mathrm{d}\zeta}{z(1+\zeta)}\zeta^{\mathrm{i}(t+\Delta Z)} \\ + \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}s\,\frac{\sinh\pi(\Delta Z+s|V|)}{\sinh\pi s+\mathrm{i}0}\int_0^\infty\frac{\mathrm{d}\zeta}{z(1+\zeta)}\zeta^{\mathrm{i}(t+s|V|+\Delta Z)}\end{aligned}$$

and can be partly evaluated to

$$\delta G_t = \frac{1}{2}\frac{\sinh \pi\Delta Z}{\sinh \pi(t+\Delta Z)} + \frac{1}{2\mathrm{i}}\int_{\mathbb{R}} \frac{\mathrm{d}s}{\sinh \pi s + \mathrm{i}0}\frac{\sinh \pi(\Delta Z + s|V|)}{\sinh \pi(t+s|V|+\Delta Z)-\mathrm{i}0} \tag{2.152}$$

by the same methods as before. Note that I included a regulating $-\mathrm{i}0$ in the denominator of the last fraction, coming from the fact that δG_t is analytic on the strip $-1 < \Im(t) < 0$. Sadly, the integral over s can not be done analytically for a general choice of V, but only for rational entangling region size $|V|$. To see how this works, first change variables to $w = \mathrm{e}^{\pi s}$, $\mathrm{d}w/w = \pi \mathrm{d}s$ so that the remaining integral becomes

$$\frac{\mathrm{e}^{-\pi t}}{\mathrm{i}\pi}\int_0^\infty \frac{\mathrm{d}w}{w^2-1+\mathrm{i}0}\frac{w^{2|V|}-\mathrm{e}^{-2\pi\Delta Z}}{w^{2|V|}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}.$$

Assume now a $|V| = p/q$ with $\gcd(p,q) = 1$. You can then change variables once more to $v = w^{1/q}$, $\mathrm{d}w = qv^{q-1}$ to obtain

$$\frac{q\mathrm{e}^{-\pi t}}{\mathrm{i}\pi}\int_0^\infty \frac{\mathrm{d}v\, v^{q-1}}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}. \tag{2.153}$$

For odd q, the integrand is even, so that the integral may be rewritten as

$$\frac{q\mathrm{e}^{-\pi t}}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{\mathrm{d}v\, v^{q-1}}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}$$

and since the integrand scales like $\sim v^{-q-1}$ for large $|v|$, you may even close the range along a semi-circle of infinite radius. Choosing to close the contour in the positive half plane, you can compute the integral via residues. Let us start with the poles at $v = \mathrm{e}^{\mathrm{i}\pi k/q}$ with $k = 0, \dots q-1$, whose contributions to the integral are

$$\begin{aligned}
&q\mathrm{e}^{-\pi t}\,\mathrm{Res}_{v=\mathrm{e}^{\mathrm{i}\pi k/q}} \frac{1}{2\pi\mathrm{i}}\frac{v^{q-1}}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\\
=&q\mathrm{e}^{-\pi t}\lim_{v\to\mathrm{e}^{\mathrm{i}\pi k/q}}\frac{v^{q-1}(v-\mathrm{e}^{\mathrm{i}\pi k/q})}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\\
=&q\mathrm{e}^{-\pi t}(-1)^k\mathrm{e}^{-\mathrm{i}\pi k/q}\frac{\mathrm{e}^{2\pi\mathrm{i}k|V|}-\mathrm{e}^{-2\pi\Delta Z}}{\mathrm{e}^{2\pi\mathrm{i}k|V|}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\left[\frac{\mathrm{d}(v^{2q}-1)}{\mathrm{d}v}\Big|_{v=\mathrm{e}^{\mathrm{i}\pi k/q}}\right]^{-1}\\
=&\frac{1}{2}(-1)^k\frac{\sinh\pi(\mathrm{i}k|V|+\Delta Z)}{\pi(t+\mathrm{i}k|V|+\Delta Z)-\mathrm{i}0}.
\end{aligned}$$

Similarly, the poles at $v = \mathrm{e}^{-\pi(t+\Delta Z)/p}\mathrm{e}^{\mathrm{i}\pi k/p}$ with $k = 0, \dots p-1$ yield the contributions

$$\begin{aligned}
&q\mathrm{e}^{-\pi t}\,\mathrm{Res}_{v=\mathrm{e}^{-\pi(t+\Delta Z-\mathrm{i}k)/p}}\frac{1}{2\pi\mathrm{i}}\frac{v^{q-1}}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\\
=&q\mathrm{e}^{-\pi t}\lim_{v\to\mathrm{e}^{-\pi(t+\Delta Z-\mathrm{i}k)/p}}\frac{v^{q-1}(v^{2p}-\mathrm{e}^{-2\pi\Delta Z})}{v^{2q}-1+\mathrm{i}0}\frac{v-\mathrm{e}^{\mathrm{i}\pi k/q}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\\
=&\frac{1}{2|V|}\mathrm{e}^{-\pi t}\frac{\mathrm{e}^{-\pi(t+\Delta Z-\mathrm{i}k)(q-1)/p}(\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{e}^{-2\pi\Delta Z})}{\mathrm{e}^{-2\pi(t+\Delta Z-\mathrm{i}k)/|V|}-1+\mathrm{i}0}\mathrm{e}^{\pi(t+\Delta Z-\mathrm{i}k)(2p-1)/p}\\
=&\frac{1}{2|V|}\frac{\sinh\pi t}{\sinh\pi(t+\Delta Z-\mathrm{i}k)/|V|-\mathrm{i}0}.
\end{aligned}$$

Finally, there is a pole at $v = -1+\mathrm{i}0$, which yields a contribution of

$$\begin{aligned}
&q\mathrm{e}^{-\pi t}\,\mathrm{Res}_{v=-1+\mathrm{i}0}\frac{1}{2\pi\mathrm{i}}\frac{v^{q-1}}{v^{2q}-1+\mathrm{i}0}\frac{v^{2p}-\mathrm{e}^{-2\pi\Delta Z}}{v^{2p}-\mathrm{e}^{-2\pi(t+\Delta Z)}-\mathrm{i}0}\\
=&-\frac{1}{2}\frac{\sinh\pi\Delta Z}{\sinh\pi(t+\Delta Z)-\mathrm{i}0},
\end{aligned}$$

which exactly cancels the first term in δG_t that we already computed. As a result, we obtain

$$\begin{aligned}
\delta G_t(x,y) = &\frac{1}{2}\sum_{k=0}^{q-1}(-1)^k\frac{\sinh\pi[\mathrm{i}k|V|+Z(x)-z(y)]}{\pi[t+\mathrm{i}k|V|+Z(x)-Z(y)]-\mathrm{i}0}\\
&+\frac{1}{2|V|}\sum_{k=0}^{p-1}\frac{\sinh\pi t}{\sinh\pi[t+Z(x)-Z(y)-\mathrm{i}k]/|V|-\mathrm{i}0}
\end{aligned}\tag{2.154}$$

for rational $|V| = p/q$, $\gcd(p,q) = 1$ with odd q.

For even denominator q, the evaluation of eq. (2.153) can be done by partial fractions. This is even more tedious than the previous analysis and I will leave it as an exercise. The result for $|V| = 1/2$ is[43]

$$\delta G_t = \frac{\sinh\pi t}{\sinh 2\pi[t+Z(x)-Z(y)]-\mathrm{i}0}\Big(1-2\mathrm{i}[t+Z(x)-Z(y)]\Big),\tag{2.155}$$

which is the first time that the function Z appears outside of an exponential. Note that this implies that δG_t now has a branch cut along V since Z has one there. This makes it particularly interesting to check the validity of eq. (2.51), i.e., we would like to compute

$$\delta G_{t-\mathrm{i}0}(x,y) - \delta G_{t+\mathrm{i}0}(x,y)$$

and check if it matches the non-local contribution eq. (2.145). To do this, we first have to find the correct analytic continuation of G_t to the upper half plane, since eq. (2.155) is only valid on the strip $-1 < \Im(t) < 0$. By the very definition (2.50) of G_t and the KMS condition (2.24), we know

$$\delta G_{t+\mathrm{i}0}(x,y) = -\delta G_{t-\mathrm{i}+\mathrm{i}0}(x,y),$$

so that

$$\begin{aligned}\delta G_{t-\mathrm{i}0}(x,y) - \delta G_{t+\mathrm{i}0}(x,y) &= \delta G_{t-\mathrm{i}0}(x,y) + \delta G_{t-\mathrm{i}+\mathrm{i}0}(x,y) \\ &= 2\,\mathcal{P}\,\frac{\sinh \pi t}{\sinh 2\pi[t + Z(x) - Z(y)]},\end{aligned}$$

where I used the Kramers-Kronig relation (2.144) on the pole of the sinh. This exactly reproduces the term eq. (2.145). While this was of course to be expected, I would like to stress that, to the best of my knowledge, our work[43] was the first time that the branch cut behaviour of a modular correlation function was ever analysed analytically in a non-trivial case.

Finally, let us consider the thermal state on the circle. Here the spectral measure is given by

$$\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = -\frac{1}{2\pi\zeta}\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi \mathrm{i} k(x-y+\beta s)} \frac{\sinh \pi(\Delta Z(x,y) - s|V|)}{\sinh \pi s + \mathrm{i}0} \zeta^{\mathrm{i}(\Delta Z(x,y) - s|V|)}$$

as already derived in eq. (2.147). The same steps that led to eq. (2.151) then yield

$$\begin{aligned} G_t &= \int_0^\infty \mathrm{d}\zeta\, \frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} \frac{\zeta^{\mathrm{i}t}}{1+\zeta} \\ &= -\frac{1}{2\pi}\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-2\pi \mathrm{i} k(x-y+\beta s)} \frac{\sinh \pi(\Delta Z - s|V|)}{\sinh \pi s + \mathrm{i}0} \int_0^\infty \mathrm{d}\zeta\, \frac{\zeta^{\mathrm{i}(t+\Delta Z - s|V|)}}{\zeta(1+\zeta)} \\ &= \frac{\mathrm{i}}{2}\sum_k \int_{\mathbb{R}} \mathrm{d}s\, \frac{\mathrm{e}^{-2\pi \mathrm{i} k(x-y+\beta s)}}{\sinh \pi(t + \Delta Z - s|V|) - \mathrm{i}0} \frac{\sinh \pi(\Delta Z - s|V|)}{\sinh \pi s + \mathrm{i}0} \\ &= \frac{\mathrm{i}}{2}\sum_k \int_{\mathbb{R}} (\pm 1)^k \mathrm{d}s\, \frac{\delta(x - y + \beta s + k)}{\sinh \pi(t + \Delta Z - s|V|) - \mathrm{i}0} \frac{\sinh \pi(\Delta Z - s|V|)}{\sinh \pi s + \mathrm{i}0}. \end{aligned}$$

Performing the summation over k yields an (anti-)periodic Dirac delta, so that the integral over s can be evaluated as well. When the dust settles, you are

left with[43]

$$G_t(x,y) = \frac{1}{2\mathrm{i}\beta}\sum_{k\in\mathbb{Z}} \frac{(\pm 1)^k}{\sinh \pi(x-y+k)/\beta - \mathrm{i}0} \times \frac{\sinh \pi[Z(x)-Z(y)+(x-y+k)|V|/\beta]}{\sinh \pi[t+Z(x)-Z(y)+(x-y+k)|V|/\beta] - \mathrm{i}0}. \quad (2.156)$$

Note that this again satisfies two KMS conditions, with respect to x and t and you can recover eq. (2.148) from either of them. For the KMS condition with respect to x, this is achieved by the extra factor of $|V|/\beta$ in the second fraction, which cancels the quasi-periodicity (2.127) of Z.

Again, it is instructive so see what happens for low temperatures ($\beta \to 0$) and periodic boundary conditions: In this limit, the summation over k turns into an integral over $s = k/\beta$, so that

$$G_t \sim \frac{1}{2\mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}s}{\sinh \pi s - \mathrm{i}0} \frac{\sinh \pi(\Delta Z + s|V|)}{\sinh \pi(t+\Delta Z + s|V|) - \mathrm{i}0},$$

which is exactly the non-local term in eq. (2.152). Again, the multi-local terms require a more careful analysis of the limit and I will not discuss them here.

2.4.5 Modular Hamiltonian

Next, let us compute the kernel of the modular Hamiltonian

$$k = -\log\left[\frac{1-G|_V}{G|_V}\right]$$

from eq. (2.46). Of course, we could do this in a similar fashion as before, by evaluating via the derived spectral calculus. However, since we already derived the kernel Σ_t of the action of modular flow, it is much easier to just compute

$$k = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t\Big|_{t=0},$$

as shown in eq. (2.48).

For the result (2.136), which is valid on the line in the vacuum and thermal state, as well as the circle with anti-periodic boundary conditions in the

vacuum state, this yields

$$\begin{aligned} k &= \frac{\mathrm{d}}{\mathrm{d}t} 2\sinh(\pi t) G(x-y)\delta(t+Z(x)-Z(y))\Big|_{t=0} \\ &= 2\pi G(x-y)\delta(Z(x)-Z(y)) \\ &= 2\pi \sum_n \frac{1}{Z'(x_n(y))} G(x_n(y)-y)\delta(x-x_n(y)), \end{aligned}$$

where $x_n(y) \in [a_n, b_n]$ is the solution of the equation

$$Z(x) - Z(y) = 0 \tag{2.157}$$

in the n-th interval, as you could have already expected from eq. (2.137). The solutions with x, y in different intervals of this equation yield multi-local contributions to k, while the solution $x = y$ needs further discussion. Since $G(z) \sim 1/2\pi\mathrm{i}(x-y)$, this solution has the structure

$$\frac{\delta(x-y)}{x-y},$$

which is ill defined as it stands. We can however use the following trick[107] to regularise it: From the distributional properties of δ, we see that $\delta(x-y)/(x-y)$ has to vanish for $x \neq y$, while it also needs to satisfy

$$\int \mathrm{d}y \, \frac{\delta(x-y)}{x-y}(x-y)f(y) = f(x)$$

for any suitably regular test function $f(x)$. Now, since

$$-\int \mathrm{d}y \, \delta'(x-y)(x-y)f(y) = \int \mathrm{d}y \, \delta(x-y)[f(y) - (x-y)f'(y)] = f(x),$$

via integration by parts, we see that the sought after regularisation needs to be of the form

$$\frac{\delta(x-y)}{x-y} = -\delta'(x-y) + r(x)\delta(x-y),$$

fore some $r(x)$ that can be fixed by requiring hermiticity of k. In the above case, this amounts to

$$\frac{1}{\mathrm{i}Z'(x)} \frac{\delta(x-y)}{x-y} = \frac{1}{\mathrm{i}Z'(x)} \big[-\delta'(x-y) + r(x)\delta(x-y) \big]$$

being hermitian. Since

$$\int \mathrm{d}x\,\mathrm{d}y\, f(x)\frac{1}{Z'(x)}\Big[-\delta'(x-y)+r(x)\delta(x-y)\Big]g(y)$$
$$=\int \mathrm{d}x\, f(x)\Big[-\frac{Z''(x)}{[Z'(x)]^2}+\frac{r(x)}{Z'(x)}\Big]g(x)+\int \mathrm{d}x\,\frac{1}{Z'(x)}f'(x)g(x),$$

for arbitrary test functions f, g, we thus have to require

$$\int \mathrm{d}x\, f(x)\Big[-\frac{Z''(x)}{[Z'(x)]^2}+\frac{r(x)}{Z'(x)}\Big]g(x)+\int \mathrm{d}x\,\frac{1}{Z'(x)}f'(x)g(x)$$
$$=-\int \mathrm{d}x\, g(x)\Big[-\frac{Z''(x)}{[Z'(x)]^2}+\frac{r(x)}{Z'(x)}\Big]f(x)-\int \mathrm{d}x\,\frac{1}{Z'(x)}g'(x)f(x).$$

You can now integrate the second integral on the right hand side by parts to find

$$r(x)=\frac{1}{2}\frac{Z''(x)}{Z'(x)}.$$

Putting everything together, you obtain[107]

$$k(x,y)=\frac{1}{\mathrm{i}Z'(x)}\Big[-\delta'(x-y)+\frac{1}{2}\frac{Z''(x)}{Z'(x)}\delta(x-y)\Big]$$
$$+2\pi\sum_{x_n(y)\neq y}\frac{1}{Z'(x_n(y))}G(x_n(y)-y)\delta(x-x_n(y)). \qquad (2.158)$$

Note that, in the regularisation of $G(x-y)\delta(x-y)$, we only used the *asymptotic* behaviour of $G(z)$ as $z\to\infty$. This means that eq. (2.158) is also valid at finite temperature and so on. We already discussed the properties of the multi-local couplings before, so I would only like to talk about the local piece

$$\frac{1}{\mathrm{i}Z'(x)}\Big[-\delta'(x-y)+\frac{1}{2}\frac{Z''(x)}{Z'(x)}\delta(x-y)\Big]$$

here: Because of eq. (2.45), the corresponding piece of the modular Hamiltonian is given by

$$K_{\mathrm{loc}}=\mathrm{i}\int_V \mathrm{d}x\,\mathrm{d}y\,\frac{1}{Z'(x)}\psi^*(x)\delta'(x-y)\psi(y)+\ldots,$$

where the ellipses correspond to the r-dependent term, i.e., an anti-self-adjoint operator that is added to ensure self-adjointness. Performing another integra-

tion by parts, this yields

$$K_{\text{loc}} = \int_V \mathrm{d}x \, \frac{1}{Z'(x)} T_{00}(x) + \ldots, \tag{2.159}$$

where T_{00} is the energy-momentum tensor from eq. (2.98). This shows that K_{loc} is related to a symmetry transformation. Furthermore, writing the reduced density matrix as $\rho = \mathrm{e}^{-K}$, you see that $Z'(x)$ takes the rôle of a "local temperature", which is why its inverse is also called the inverse *entanglement temperature*[132]

$$\beta(x) = \frac{1}{Z'(x)}. \tag{2.160}$$

This gives a physical interpretation to the fact that Z diverges at the boundary of the entangling region: The divergence of entanglement temperature implies that the state locally "looks" like a very hot thermal state, originating in strong fluctuations that are caused by entanglement with the exterior of the region.

Let us now proceed to see the effect of the zero-mode of the periodic circle in the vacuum state. From the extra contribution (2.145), we obtain

$$\begin{aligned} \delta k(x,y) &= \frac{\mathrm{i}}{|V|} \mathcal{P} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\sinh \pi t}{\sinh \pi [t + Z(x) - Z(y)]/|V|} \Big|_{t=0} \\ &= \frac{\mathrm{i}\pi}{|V|} \mathcal{P} \frac{1}{\sinh \pi [Z(x) - Z(y)]/|V|}, \end{aligned} \tag{2.161}$$

as independently derived by Klich, Vaman, and Wong.[109] This non-local term is present even if $V = [a, b]$ is just a single interval. It yields a long ranged coupling in the modular Hamiltonian that behaves like

$$\delta k(x,y) \sim \mathcal{P} \frac{\mathrm{i}}{Z(x) - Z(y)}$$

close to solutions of eq. (2.157).

in the thermal state on the circle, we can use eq. (2.148) to obtain

$$\begin{aligned} k(x,y) &= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{|V|}\frac{\sinh \pi t}{\sinh \pi[t+Z(x)-Z(y)]/|V|} \\ &\qquad \times \sum_{k\in\mathbb{Z}} (\pm 1)^k \delta(x-y+\beta[t+Z(x)-Z(y)]/|V|+k)\Big|_{t=0} \\ &= \frac{\mathrm{i}\pi}{|V|}\frac{1}{\sinh \pi[Z(x)-Z(y)]/|V|} \\ &\qquad \times \sum_{k\in\mathbb{Z}} (\pm 1)^k \delta(x-y+\beta[Z(x)-Z(y)]/|V|+k). \end{aligned}$$

Again, denoting by $x_{nk}(y)$ the solution of

$$x-y+\beta[Z(x)-Z(y)]/|V|+k=0, \quad k\in\mathbb{Z}$$

in the n-th interval, the above can be rewritten as

$$k(x,y) = -\mathrm{i}\pi \sum_{nk} \frac{(\pm 1)^k}{|V|+\beta Z'(x_{nk})}\frac{\delta(x-x_{nk}(y))}{\sinh \pi(x_{nk}(y)-y+k)/\beta}.$$

Similarly to the vacuum case, there is a local contribution coming from the solution $x_{n0}=y$, taking the form

$$\begin{aligned} &-\frac{\mathrm{i}\pi}{|V|+\beta Z'(x)}\frac{\delta(x-y)}{\sinh \pi(x-y)/\beta} \\ &\qquad = -\frac{\mathrm{i}\beta}{|V|+\beta Z'(x)}\Big[-\delta'(x-y)+\frac{1}{2}\frac{\beta Z''(x)}{|V|+\beta Z'(x)}\delta(x-y)\Big]. \end{aligned}$$

The kernel of the modular Hamiltonian is therefore[41,113]

$$\begin{aligned} k(x,y) = &-\frac{\mathrm{i}\beta}{|V|+\beta Z'(x)}\Big[-\delta'(x-y)+\frac{1}{2}\frac{\beta Z''(x)}{|V|+\beta Z'(x)}\delta(x-y)\Big] \\ &+\frac{\pi}{\mathrm{i}\beta}\sum_{x_{nk}\neq y}\frac{(\pm 1)^k}{Z'(x_{nk})}\frac{\delta(x-x_{nk}(y))}{\sinh \pi(x_{nk}(y)-y+k)/\beta}. \end{aligned} \tag{2.162}$$

Again, the local term is proportional to the energy-momentum tensor, with inverse entanglement temperature

$$\beta(x) = \frac{\beta}{|V|+\beta Z'(x)}.$$

Let us once more discuss the high and low temperature limits of this result:

For low temperatures ($\beta \to \infty$), we have

$$\beta(x) \sim \frac{1}{Z'(x)},$$

which recovers the vacuum result (2.160). For high temperatures ($\beta \approx 0$), the inverse entanglement temperature asymptotically behaves like

$$\beta(x) \sim \frac{\beta}{|V|},$$

which is constant with respect to x and confirms our observation from the discussion of eq. (2.150), that the reduced density matrix of a thermal state at high temperature "looks like" a thermal state at inverse temperature $\beta/|V|$.

2.4.6 Entanglement entropy

Now that we computed the modular data of free fermions, i.e., the *dynamical* structure of entanglement, let us proceed to the study of the *static* quantities characterising entanglement, i.e, the entanglement Rényi entropies (1.19) and their limits (1.20). To do this, we will evaluate the relation

$$S_\alpha = \frac{1}{1-\alpha} \operatorname{Tr} \log \left[G|_V^\alpha + (1 - G|_V)^\alpha \right]$$

form eq. (2.52) via the formula

$$\operatorname{Tr}[f(G|_V)] = \int_0^1 \mathrm{d}\lambda \, \operatorname{Tr} \left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} \right] f(\lambda)$$

that we derived in eq. (2.58), during our study of the spectral calculus associated to the restricted propagator. Combining the above equations, you find

$$S_\alpha = \frac{1}{1-\alpha} \int_0^1 \mathrm{d}\lambda \, \operatorname{Tr} \left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} \right] \left[\log \left(1 + \left[\frac{1-\lambda}{\lambda} \right]^\alpha \right) + \alpha \log \lambda \right], \tag{2.163}$$

where the spectral measure E_λ was already derived before.

Beginning again with the vacuum state on the real line, we know from eq. (2.135) that the spectral measure takes a particularly nice form when expressed in terms of the variable $\zeta = (1-\lambda)/\lambda$. To evaluate the above integral,

it makes sense to first compute the trace

$$\begin{aligned}\operatorname{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] &= \frac{\mathrm{i}}{\pi\zeta}\int_V \mathrm{d}x \lim_{y\to x} \sinh \pi[Z(x)-Z(y)]G(x-y)\zeta^{\mathrm{i}[Z(x)-Z(y)]} \\ &= \frac{1}{2\pi\zeta}\int_V \mathrm{d}x \lim_{y\to x}\frac{Z(x)-Z(y)}{x-y} \\ &= \frac{1}{2\pi\zeta}\int_V \mathrm{d}x\, Z'(x).\end{aligned}$$

Obviously, since Z diverges at the boundary of V, this trace is divergent. On the other hand, this was to expected from the very beginning, since we are trying to compute an entanglement measure in QFT, where we know that every region shares an infinite amount of entanglement with its complement due to the Reeh-Schlieder theorem. This goes back all the way to the discussion in section 2.1.1 where we found that the Hilbert space in QFT does not simply factorise

$$\mathcal{H} \neq \mathcal{H}_V \otimes \mathcal{H}_{V'}$$

into factor spaces associated to V and its complement V', since the algebra of observables $\mathcal{A}(V)$ is a so-called von-Neumann factor of type III.[46,72] In particular, the non-existence of such a factorisation means that the concept of a reduced density matrix to V is nonsensical.

Fortunately, there is a way around this issue if we make a mild assumption on the algebraic structure of the Haag-Kastler net: Pick a slightly larger region $\tilde{V}$ that encloses V with a finite distance, as pictured in fig. 2.20. Our aim is to use $\tilde{V}$ as a "cushion" to shield V from correlations with the exterior $\tilde{V}'$.

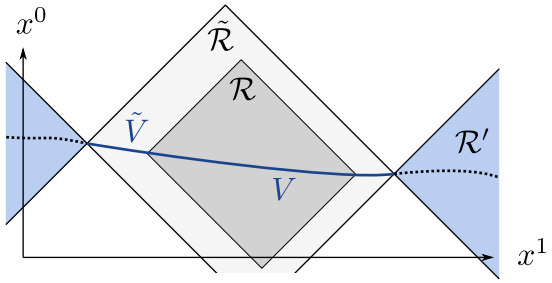


Figure 2.20: Sketch of the spacetime regions involved in a split inclusion (2.164): The region $\tilde{\mathcal{R}}$ (shaded grey) acts as a "cushion" for the considered region $\mathcal{R}$ (shaded light grey), shielding it from short-range correlations with $\mathcal{R}'$ (shaded light blue). Pulling everything back to a Cauchy slice (dotted black), the same holds for the slices $\tilde{V}$ and V (solid dark blue).

To this end, we invoke the assumption that the inclusion $\mathcal{A}(V) \subset \mathcal{A}(V')$ is *split*,[64,133,134] meaning that we assume the existence of an intermediate von-Neumann factor $\mathcal{F}$ of type I

$$\mathcal{A}(V) \subset \mathcal{F} \subset \mathcal{A}(\tilde{V}). \tag{2.164}$$

A von-Neumann factor of type I is a von-Neumann algebra naturally isomorphic to the set of bounded operators. This means that there is a unitary map $\mathcal{H} \xrightarrow{U} \mathcal{K} \otimes \mathcal{K}'$ for some $\mathcal{K}, \mathcal{K}'$ with

$$U\mathcal{F}U^* = B(\mathcal{K}) \otimes 1_{\mathcal{K}'}. \tag{2.165}$$

While the assumption of split inclusions might seem rather technical, it can actually be deduced from very physical assumptions about the thermodynamical behaviour of the algebras, including the existence of bounds on the number of localised degrees of freedom.[135–137] To see how the split property helps in defining a reduced density matrix, note that eq. (2.164) implies[69]

$$U\mathcal{A}(V)U^* \subset B(\mathcal{K}) \otimes 1_{\mathcal{K}'}, \quad U\mathcal{A}(\tilde{V})'U^* \subset 1_{\mathcal{K}} \otimes B(\mathcal{K}'). \tag{2.166}$$

Furthermore, for every density matrix ρ on $\mathcal{H}$, we find a density matrix

$$\rho^{(U)} := U\rho U^* \tag{2.167}$$

on $\mathcal{K} \otimes \mathcal{K}'$ and the partial traces

$$\rho^{(U)}_{\mathcal{A}(V)} := \mathrm{Tr}_{\mathcal{K}'}[U\rho U^*], \quad \rho^{(U)}_{\mathcal{A}(\tilde{V})'} := \mathrm{Tr}_{\mathcal{K}}[U\rho U^*] \tag{2.168}$$

are well defined. Unfortunately, the intermediate type I factor $\mathcal{F}$ is far from unique, since we could chose any other cushioning region between V, $\tilde{V}$ and the split property would still hold. This means that, if we wish to define entanglement (Rényi) entropies via $\rho^{(U)}_{\mathcal{A}(V)}$, we will have to either minimise the result over all possible choices for $\mathcal{F}$ or find a canonical such choice. While both of these possibilities have been studied by Longo and Xu for free theories[138] and were found to yield finite results, I do not want to dive too deep into the details. Rather, I would like to emphasise the central message of this approach: A reduced density matrix can be defined by providing a cushioning region around V, placing a cutoff on short range entanglement.

This implies that, in order to obtain a workable result for the trace of the spectral measure, we can use the function $\tilde{Z}$ of a region $\tilde{V}$ which is slightly

than V, while still integrating only over V. As a result,

$$\mathrm{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] = \frac{1}{2\pi\zeta}\sum_n[\tilde{Z}(b_n) - \tilde{Z}(a_n)] =: \frac{1}{2\pi^2\zeta}\log\ell_V, \tag{2.169}$$

where $\tilde{Z}$ takes the form

$$\tilde{Z}(x) = \frac{1}{2\pi}\log\Big[-\prod_n \frac{x - a_n + \epsilon}{x - b_n - \epsilon}\Big]$$

and I defined the shorthand notation ℓ_V. Note that, for a single interval,

$$\ell_{[a,b]} = \frac{b-a}{\epsilon},$$

which is the reason for the choice of prefactors in eq. (2.169). More generally, ℓ_V is a dimensionless quantity that summarises the geometry of V. As a result of the above, we have

$$\begin{aligned} S_\alpha &= \frac{1}{1-\alpha}\int_0^\infty \mathrm{d}\zeta\ \mathrm{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right][\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)] \\ &= \log\ell_V \times \frac{1}{1-\alpha}\frac{1}{2\pi^2}\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta}\Big[\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)\Big] \end{aligned}$$

which simplifies to

$$S_\alpha = \log\ell_V \times \frac{1}{1-\alpha}\frac{1}{\pi^2}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta}\Big[\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)\Big],$$

since the integrand is invariant under exchanging $\zeta \leftrightarrow 1/\zeta$. To evaluate this integral, note that $|\zeta^\alpha| < 1$ on the range of integration, as long as $\Re(\alpha) > 0$. We can thus insert the power series expansion

$$\log(1+\zeta^\alpha) = \sum_{k=1}^\infty (-1)^{k-1}\frac{\zeta^{\alpha k}}{k} \tag{2.170}$$

to find

$$\begin{aligned} S_\alpha &= \log\ell_V \times \frac{1}{1-\alpha}\frac{1}{\pi^2}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}\int_0^1 \mathrm{d}\zeta[\zeta^{(\alpha k-1}} - \alpha\zeta^{k-1}] \\ &= \log\ell_V \times \frac{1+\alpha}{\alpha}\frac{1}{\pi^2}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^2}. \end{aligned}$$

Finally, you can use the relation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12}$$

to obtain the entanglement Rényi entropy

$$S_\alpha = \frac{1}{12}\frac{1+\alpha}{\alpha}\log \ell_V. \tag{2.171}$$

The limit $\alpha \to 1$ yields the entanglement (von-Neumann) entropy

$$S = \frac{1}{6}\log \ell_V, \tag{2.172}$$

as independently derived by Casini and Huerta,[107] recovering the celebrated single-interval result

$$S = \frac{c}{3}\log\frac{b-a}{\epsilon}$$

by Calabrese and Cardy[139] for central charge $c = 1/2$. On the other hand, the limit $\alpha \to 0$ of eq. (2.171) is divergent, which was to be expected in the first place since the Schmidt rank only counts the number of entangled degrees of freedom (of which there are infinitely many in a QFT), no matter how small the amount of entanglement may be.

Before proceeding to the computation of the Rényi entropies for other states and geometries, let us derive a few more quantities from the result eq. (2.171). To be more specific, we will derive the notion of *relative entropy* in this framework: Recall from section 2.1.2, that we defined the Tomita operator via

$$SA\Omega := A^*\Omega, \quad A \in \mathcal{A}$$

for some choice of state von-Neumann algebra $\mathcal{A}$ and cyclic separating vector Ω. Now, note that if we have another cyclic separating vector $\tilde{\Omega}$, we can define the *relative Tomita operator* $S_{\Omega\|\tilde{\Omega}}$ via[46,140]

$$S_{\Omega\|\tilde{\Omega}} A\Omega := A^*\tilde{\Omega}, \quad A \in \mathcal{A}. \tag{2.173}$$

Again, $S_{\Omega\|\tilde{\Omega}}$ is closable and has a unique polar decomposition

$$S_{\Omega\|\tilde{\Omega}} = J_{\Omega\|\tilde{\Omega}}\Delta^{1/2}_{\Omega\|\tilde{\Omega}} \tag{2.174}$$

with anti-unitary $J_{\Omega\|\tilde{\Omega}}$ and positive $\Delta^{1/2}_{\Omega\|\tilde{\Omega}}$. We can use this to define the *Connes*

Radón-Nikodym cocycle[141,142]

$$\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t} := \Delta^{\mathrm{i}t}_{\Omega\|\tilde{\Omega}}\Delta_{\Omega}^{-\mathrm{i}t} \tag{2.175}$$

where Δ_ω is the modular operator associated to $\mathcal{A}$ and the vector Ω. As shown by Connes,[141] this one-parameter family of unitaries is always an element of $\mathcal{A}$ (in contrast to the modular flow $\Delta^{\mathrm{i}t}_\Omega$ itself) and intertwines the modular flows $\sigma_t, \tilde{\sigma}_t$ associated to $\Omega, \tilde{\Omega}$, in the sense that

$$\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t}\sigma_t(A) = \tilde{\sigma}_t(A)\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t}, \quad A \in \mathcal{A}. \tag{2.176}$$

Furthermore, the Connes cocycle satisfies a chain rule

$$\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t}\left[\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}\hat{\Omega}}\right]_{\mathrm{i}t} = \left[\frac{\mathrm{d}\Omega}{\mathrm{d}\hat{\Omega}}\right]_{\mathrm{i}t},$$

which is the reason for the above choice of notation. Finally, note that definition (2.175) implies the cocycle condition

$$\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}(t+s)} = \left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t}\sigma_t\Big(\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}s}\Big).$$

Now that we have defined $[\mathrm{d}\Omega/\mathrm{d}\tilde{\Omega}]_{\mathrm{i}t} \in \mathcal{A}$, we can use it to define the relative entropy of $\tilde{\Omega}$ with respect to Ω by

$$S(\Omega\|\tilde{\Omega}) := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle\Omega|\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{\mathrm{i}t}\Omega\rangle\Big|_{t=0}. \tag{2.177}$$

While this is very abstract, we can get a better glimpse of what is going on in the finite dimensional case: There, the modular operator takes the form

$$\Delta_\Omega = \rho_1 \otimes \rho_2^{-1}$$

in terms of the reduced density matrices

$$\rho_1 = \mathrm{Tr}_2[\Omega \otimes \Omega^*], \quad \rho_2 = \mathrm{Tr}_1[\Omega \otimes \Omega^*],$$

as already demonstrated eq. (2.16). In an entirely analogous way, you can show that the relative modular operator is given by[46]

$$\Delta_{\Omega\|\tilde{\Omega}} = \tilde{\rho}_1 \otimes \rho_2^{-1},$$

with $\tilde{\rho}_1$ the reduced density matrix of the vector $\tilde{\Omega}$. As a result, the Connes cocycle is

$$\left[\frac{\mathrm{d}\Omega}{\mathrm{d}\tilde{\Omega}}\right]_{it} = \tilde{\rho}_1^{it}\rho_1^{-it} \otimes 1 \in \mathcal{A}$$

and the definition (2.177) of relative entropy evaluates to the more standard formula[8]

$$S(\Omega\|\tilde{\Omega}) = \mathrm{Tr}[\rho_1 \log \rho_1] - \mathrm{Tr}[\rho_1 \log \tilde{\rho}_1]. \tag{2.178}$$

The first term in this expression is precisely entanglement entropy (1.17) and we already derived its value in QFT in eq. (2.172), using split inclusions to give a somewhat rigorous meaning to the result. The second term, however, seems hopeless to we will resort to a standard trick again, by rewriting[23,25]

$$\begin{aligned} S(\Omega\|\tilde{\Omega}) &= \mathrm{Tr}[\rho_1 \log \rho_1] - \mathrm{Tr}[\tilde{\rho}_1 \log \tilde{\rho}_1] \\ &\quad + \mathrm{Tr}[\tilde{\rho}_1 \log \tilde{\rho}_1] - \mathrm{Tr}[\rho_1 \log \tilde{\rho}_1] \\ &= \Delta\langle \tilde{K}\rangle - \Delta S, \end{aligned} \tag{2.179}$$

where

$$\Delta S := S(\rho_1) - S(\tilde{\rho}_1) = \mathrm{Tr}[\rho_1 \log \rho_1] - \mathrm{Tr}[\tilde{\rho}_1 \log \tilde{\rho}_1]$$

is the difference in entanglement entropies and

$$\Delta\langle \tilde{K}\rangle = \mathrm{Tr}[\rho_1(-\log \tilde{\rho}_1)] - \mathrm{Tr}[\tilde{\rho}_1(-\log \tilde{\rho}_1)]$$

is the difference in expectation values of the modular Hamiltonian

$$\tilde{K} = -\log \tilde{\rho}_1 + \text{const.}$$

Finally, we can use eq. (2.179) to compute the relative entropy $S(\beta\|\tilde{\beta})$ between thermal states of different temperatures $\beta, \tilde{\beta}$ on the real line. To simplify notation, I will restrict to a single interval $V = [-a, a]$. From eq. (2.172), we directly obtain

$$\Delta S = \frac{1}{6}\log\frac{\ell_V}{\tilde{\ell}_V} = \frac{1}{6}\log\frac{\beta \sinh 2\pi a/\beta}{\tilde{\beta}\sinh 2\pi a/\tilde{\beta}}. \tag{2.180}$$

because

$$\ell_V = \frac{1}{2}\log\frac{\sinh \pi(x+a+\epsilon)/\beta}{\sinh \pi(a+\epsilon-x)/\beta}\Big|_{x=-a}^{x=a} = \log\frac{\beta}{\pi\epsilon}\sinh\frac{2\pi a}{\beta}.$$

Note that ΔS is finite as long as $\beta/\tilde{\beta}$ is finite. The arguably more difficult computation is that of $\Delta\langle\tilde{K}\rangle$. First, recall from eq. (2.159), that $\tilde{K}$ is given

by

$$\tilde{K} = \int_{-a}^{a} \mathrm{d}x\, \tilde{\beta}(x) \tilde{T}_{00}(x)$$

with the regularised energy momentum tensor

$$T_{00}(x) = \mathrm{i}[\psi^*(x)\psi'(x) - \tilde{r}(x)\psi^*(x)\psi(x)]$$

and inverse entanglement temperature

$$\tilde{\beta}(x) = \frac{1}{\tilde{Z}'(x)} = \tilde{\beta}\Big[\coth\frac{2\pi a}{\tilde{\beta}} - \frac{\cosh 2\pi x/\tilde{\beta}}{\sinh 2\pi a/\tilde{\beta}}\Big].$$

As stated before, the term involving $\tilde{r}(x)$ is regulation dependent and present to make $\tilde{K}$ self-adjoint. Since T_{00} involves products of fields at the same point, we have to regularise, e.g., by setting

$$T_{00}(x) = \lim_{y\to x} \mathrm{i}\Big[\frac{\mathrm{d}}{\mathrm{d}y} - \tilde{r}(x)\Big] : \psi^*(x)\psi(y) :,$$

where the normal ordering $: \quad :$ means that we subtract the expectation value $\mathrm{EV}(x,y)$ of this expression in some fixed state. We can now compute

$$\begin{aligned}
\mathrm{Tr}[\rho T_{00}(x)] &= \lim_{y\to x} \mathrm{i}\Big[\frac{\mathrm{d}}{\mathrm{d}y} - \tilde{r}(x)\Big]\, \mathrm{Tr}[\rho : \psi^*(x)\psi(y) :] \\
&= -\lim_{y\to x} \mathrm{i}\Big[\frac{\mathrm{d}}{\mathrm{d}y} - \tilde{r}(x)\Big]\Big(\mathrm{Tr}[\rho\psi(y)\psi^*(x)] - \mathrm{EV}(x,y)\Big) \\
&= -\lim_{y\to x} \mathrm{i}\Big[\frac{\mathrm{d}}{\mathrm{d}y} - \tilde{r}(x)\Big][G(y,x) - \mathrm{EV}(x,y)],
\end{aligned}$$

where

$$G(y,x) = G(y-x) = \frac{1}{2\mathrm{i}\beta}\frac{1}{\sinh\pi(y-x)\beta}$$

is the thermal propagator from eq. (2.71). Since we will perform a limit $y \to x$, we are only interested in the leading terms in the UV-expansion

$$G(z) = \frac{1}{2\pi\mathrm{i}z} + \frac{\mathrm{i}\pi}{12\beta^2} z + \mathcal{O}(z^3).$$

As a result, we obtain

$$\mathrm{Tr}[\rho\tilde{T}_{00}(x)] = \frac{\pi}{12\beta^2} + \ldots,$$

where the ellipses denote terms that do not depend on β. In particular, this is independent of x, so that

$$\begin{aligned}\Delta\langle\tilde{K}\rangle &= \Big(\operatorname{Tr}[\rho\tilde{T}_{00}(x)] - \operatorname{Tr}[\tilde{\rho}\tilde{T}_{00}(x)]\Big) \times \int_{-a}^{a} \mathrm{d}x\, \tilde{\beta}(x) \\ &= \frac{1}{12}\Big[\Big(\frac{\tilde{\beta}}{\beta}\Big)^2 - 1\Big]\Big[\frac{2\pi a}{\tilde{\beta}} \coth \frac{2\pi a}{\tilde{\beta}} - 1\Big],\end{aligned} \tag{2.181}$$

hence, relative entropy is given by[42]

$$S(T\|\tilde{T}) = \frac{1}{12}\Big[\Big(\frac{T}{\tilde{T}}\Big)^2 - 1\Big]\Big[\tilde{T}\coth\tilde{T} - 1\Big] - \frac{1}{6}\log\frac{\tilde{T}\sinh T}{T\sinh\tilde{T}} \tag{2.182}$$

in terms of the dimensionless temperatures

$$T := \frac{2\pi a}{\beta} \quad \text{and} \quad \tilde{T} := \frac{2\pi a}{\tilde{\beta}}. \tag{2.183}$$

Note that $S(T\|\tilde{T})$ vanishes if and only if $T = \tilde{T}$ and you can easily verify that its gradient vanishes there as well. Finally, you can compute the Hessian

$$\frac{\partial^2 S(T\|\tilde{T})}{\partial T^2}\Big|_{T=\tilde{T}} = \frac{1}{6}\Big[\frac{(T\coth T - 1)(T\coth T + 2)}{T^2} - 1\Big], \tag{2.184}$$

also known as *Fisher information*.[8] As you can see, eq. (2.184) is always positive, hence $T = \tilde{T}$ is a global minimum of $S(\beta\|\tilde{\beta})$, which is one of the key characteristics of relative entropy.[46,140] You can find plots of the relative entropy (2.182) and Fisher information (2.184) in figs. 2.21 and 2.22.

Another related quantity that we can compute is that of *mutual information*.[8] To define it in QFT, recall that the split property (2.164) implies the existence of reduced density matrices $\rho^{(U)}_{\mathcal{A}(V)}$ and $\rho^{(U)}_{\mathcal{A}(\tilde{V})'}$ on some auxiliary Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, as defined in eq. (2.168). This however means that we can define the density matrix

$$\tilde{\rho} := U^* \rho^{(U)}_{\mathcal{A}(V)} \otimes \rho^{(U)}_{\mathcal{A}(\tilde{V})'} U, \tag{2.185}$$

on $\mathcal{H}$, which is a product state in the sense of eq. (1.16) with respect to the commuting algebras $\mathcal{A}(V)$ and $\mathcal{A}(\tilde{V})'$! To verify this, let $A \in \mathcal{A}(V)$ and $A' \in \mathcal{A}(\tilde{V})'$. Then

$$\operatorname{Tr}[\tilde{\rho}AA'] = \operatorname{Tr}[\rho^{(U)}_{\mathcal{A}(V)} \otimes \rho^{(U)}_{\mathcal{A}(\tilde{V})'} UAU^*UA'U^*]$$

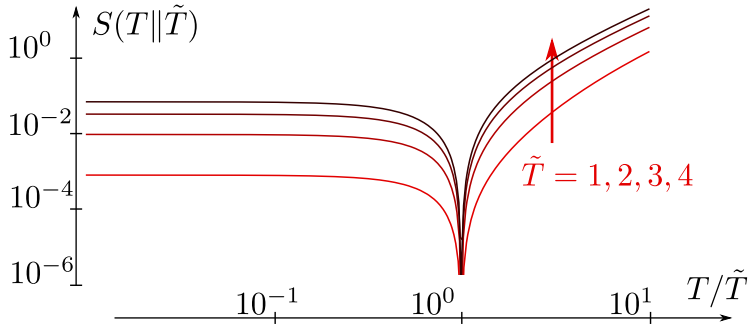


Figure 2.21: Relative entropy (solid red) between thermal states on the line of different reduced temperatures T, $\tilde{T}$, as given by eq. (2.182). As expected, it is always positive and vanishes if and only if $T = \tilde{T}$. Distinguishability is generally larger at higher temperatures and the relative entropy between thermal states and the vacuum ($T = 0$) is always finite.

and since $UAU^* \in B(\mathcal{K}) \otimes 1$ and $UA'U^* \in 1 \otimes B(\mathcal{K}')$ we find

$$\mathrm{Tr}[\tilde{\rho} AA'] = \mathrm{Tr}[\rho^{(U)} UAU^*] \, \mathrm{Tr}[\rho^{(U)} UA'U^*] = \mathrm{Tr}[\tilde{\rho} A] \, \mathrm{Tr}[\tilde{\rho} A'].$$

The density matrix $\tilde{\rho}$ is thus indistinguishable from ρ by algebra elements within either $\mathcal{A}(V)$ or $\mathcal{A}(\tilde{V})'$, while all correlations between the two are removed. We can now choose any region $\hat{V} \subset \tilde{V}'$ in the complement of $\tilde{V}$, so that $\mathcal{A}(\hat{V}) \subset \mathcal{A}(\tilde{V})'$ because of the isotony axiom (1.1), and compute the relative entropy between the states ρ and $\tilde{\rho}$ with respect to the algebra $\mathcal{A}(V \cup \hat{V})$. Since $\tilde{\rho}$ is a product state, the result will quantify the amount of correlations between the two regions present in ρ. To compute $S(\rho\|\tilde{\rho})$, we will use the finite dimensional result eq. (2.178) again. Now, since $\tilde{\rho}$ is actually a product state and entropy is additive on product states, we find

$$S(\rho\|\tilde{\rho}) = S(\rho_V) + S(\rho_{\hat{V}}) - S(\rho_{V\cup\hat{V}}) =: I(V : \hat{V}), \tag{2.186}$$

which is the more standard definition of mutual information.[8] At last, eq. (2.172) allows to obtain

$$I(V : \hat{V}) = \frac{1}{6} \log \frac{\ell_V \ell_{\hat{V}}}{\ell_{V\cup\hat{V}}} \tag{2.187}$$

and it is again instructive to evaluate this for $V, \hat{V}$ given by single intervals of size l with distance d from one another. In the vacuum state on the line, you

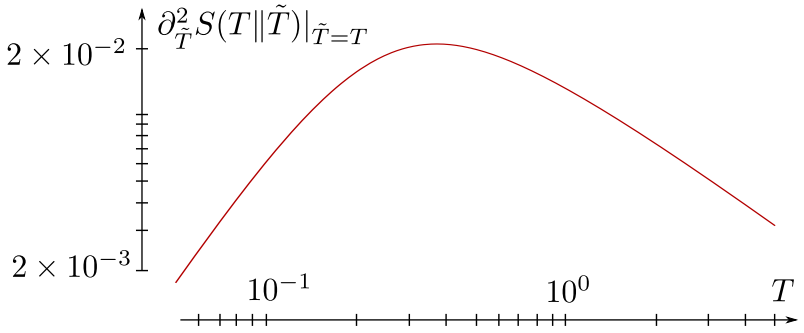


Figure 2.22: Fischer information (solid red) for thermal states on the line, again expressed in terms of the reduced temperature $T = 2\pi a/\beta$, as in eq. (2.184).

then find

$$I(V:\hat{V}) = \frac{1}{6}\log\frac{(1+l/d)^2}{1+2l/d} \sim \frac{1}{6} \times \begin{cases} \log l/d & \text{for } d \ll l \\ (l/d)^2 & \text{for } d \gg l, \end{cases} \tag{2.188}$$

hence, correlations decay quadratically at large distances and diverge logarithmically at short distances. Note that the former could already be expected from twist-field computations in CFT,[143] which suggest an asymptotic decay of

$$I(V:\hat{V}) \sim (l/d)^{4c} \quad \text{for} \quad d \gg l,$$

where c is the conformal charge.

To finish this subsection, let us discuss the computation of entanglement Rényi entropy on the circle with a zero-mode contribution and at finite temperature. Beginning with the former, recall that we already derived the additional term to the spectral measure in eq. (2.141). Its trace can be trivially computed as

$$\begin{aligned} \delta \operatorname{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] &= \int_V \mathrm{d}x \lim_{y\to x} \delta \operatorname{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}(x,y)\right] \\ &= \frac{|V|}{2\pi \mathrm{i}\zeta}\left[\frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1} - \frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\right], \end{aligned}$$

so that eq. (2.163) with $\lambda = 1/(1+\zeta)$ yields

$$\delta S_\alpha = \frac{1}{1-\alpha}\frac{|V|}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta}\left[\frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1} - \frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\right] \times \Big[\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)\Big].$$

Again, you can change variables $\zeta \to 1/\zeta$ for $\zeta \in [1,\infty]$ in order to find

$$\delta S_\alpha = \frac{1}{1-\alpha}\frac{|V|}{\mathrm{i}\pi}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta}\left[\frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1} - \frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\right] \times \Big[\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)\Big], \qquad (2.189)$$

which can again only be evaluated analytically for rational entangling region size $|V|$. To do this, you can insert the Taylor series (2.170) of the logarithm and then use partial fractions to evaluate the integral. I will not include this computation here, since it provides no new insight. Nevertheless, eq. (2.189) shows that δS_α is finite for every α and that it only depends on the size $|V|$ of the entangling region. Somewhat surprisingly you can obtain a closed form solution for $\alpha = 2, 3, \ldots$ in the following way: Note that you can rewrite eq. (2.189) as

$$\delta S_\alpha = -\frac{1}{1-\alpha}\frac{|V|}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta}\frac{1}{1+(-\zeta)^{|V|}}\Big[\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)\Big],$$

where Γ is again the Hankel contour from fig. 2.9. This is possible because the circular contributions of Γ again vanish for $\Re(\alpha), |V| > 0$. You can now contract the outer circular contribution of Γ towards the inner one, leaving behind trails along the branch cuts of $\log(1+\zeta^\alpha)$ and $\log(1+z)$. The corresponding branch points (of the first term) are located at solutions of $1+\zeta^\alpha = 0$, i.e,

$$\zeta_m = -\mathrm{e}^{2\pi\mathrm{i}m/\alpha}, \quad m = -\frac{\alpha-1}{2}, -\frac{\alpha-1}{2}+1, \ldots, \frac{\alpha-1}{2}$$

and we can choose the cuts to extend linearly to infinity. The difference between the values of $\log(1+\zeta^\alpha)$ directly left and right of the cuts is given by $-2\pi\mathrm{i}$, and a similar argument can be made for the branch cut of $\log(1+\zeta)$ (whose branch point is ζ_0). Parameterising the cuts by $\zeta = \zeta_m \mathrm{e}^{s/|V|}$, $\mathrm{d}\zeta/\zeta = \mathrm{d}s/|V|$,

you thus find

$$\begin{aligned}\delta S_\alpha &= \frac{1}{1-\alpha}\int_0^\infty \mathrm{d}s \sum_{m=-(\alpha-1)/2}^{(\alpha-1)/2}\Big[\frac{1}{1+\mathrm{e}^{2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^s}-\frac{1}{1+\mathrm{e}^s}\Big]\\ &= \frac{1}{1-\alpha}\log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2}\frac{1+\mathrm{e}^{-s}}{1+\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-s}}\Big]\Big|_{s=0}^{s=\infty}\\ &= \frac{1}{1-\alpha}\log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2}\frac{1+\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}}{2}\Big]\\ &= \frac{1}{1-\alpha}\log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2}\cos\frac{\pi|V|m}{\alpha}\Big],\end{aligned} \tag{2.190}$$

which is the low temperature limit of a previously known result due to Herzog and Nishioka,[144] who derived it via twist-field methods.

Returning to eq. (2.189), the zero-mode contribution to entanglement von-Neumann entropy can be derived as the limit $\alpha \to 1$, with the result[109]

$$\begin{aligned}\delta S = \frac{|V|}{\mathrm{i}\pi}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)}\Big[\frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|}+1}-\frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|}+1}\Big]\\ \times\big[(1+\zeta)\log(1+\zeta)-\zeta\log\zeta\big],\end{aligned} \tag{2.191}$$

yet another finite integral that we can only evaluate analytically for rational $|V|$. Again, the limit $\alpha \to 0$ makes no sense due to discrete nature of the Schmidt rank.

Finally, for the thermal state on the circle, the spectral measure is given in eq. (2.147). Taking its trace however is difficult, since the summation over k does not converge for $x \to y$. To resolve this issue perform the summation first to obtain an (anti-)periodic Dirac delta and find

$$\begin{aligned}\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta} = \frac{1}{2\pi\beta\zeta}\sum_k(\pm1)^k\frac{\sinh\pi[Z(x)-Z(y)+(x-y+k)|V|/\beta]}{\sinh\pi(x-y+k)/\beta}\\ \times\zeta^{\mathrm{i}[Z(x)-Z(y)+(x-y+k)|V|/\beta]}.\end{aligned}$$

Now, the limit $y \to x$ is easy for all terms with $k \neq 0$, since the denominator may not diverge there. For $k = 0$, you have

$$\lim_{y\to x}\frac{\sinh\pi[Z(x)-Z(y)+(x-y)|V|/\beta]}{\sinh\pi(x-y)/\beta} = \beta Z'(x)+|V|,$$

so that the trace of the spectral measure is given by

$$\mathrm{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] = \frac{1}{2\pi^2\zeta}\log\ell_V + \frac{|V|}{2\pi\beta\zeta}\left[|V| + \sum_{k\neq 0}(\pm 1)^k \frac{\sinh \pi k|V|/\beta}{\sinh \pi k/\beta}\zeta^{\mathrm{i}k|V|/\beta}\right]$$

with ℓ_V as defined in eq. (2.169), albeit with Z from eq. (2.129). Plugging this into eq. (2.163) with $\lambda = 1/(1+\zeta)$, you find that

$$S_\alpha = S_\alpha^{(0)} + S_\alpha^{(1)}, \tag{2.192}$$

where $S_\alpha^{(0)}$ is given by eq. (2.171) and

$$\begin{aligned} S_\alpha^{(1)} = {} & \frac{1}{12}\frac{1+\alpha}{\alpha}\frac{\pi|V|^2}{\beta} \\ & + \frac{1}{1-\alpha}\frac{|V|}{\pi\beta}\sum_{k=1}^{\infty}(\pm 1)^k \frac{\sinh \pi k|V|/\beta}{\sinh \pi k/\beta} \\ & \times \int_0^1 \frac{\mathrm{d}\zeta}{\zeta}[\zeta^{\mathrm{i}k|V|/\beta} + \zeta^{-\mathrm{i}k|V|/\beta}][\log(1+\zeta^\alpha) - \alpha\log(1+\zeta)] \end{aligned} \tag{2.193}$$

is again finite, yet can no be evaluated analytically in general. Note that the series converges exponentially fast, as long as $\Re(\alpha) > 0$, since the coefficient scales as $\sim \mathrm{e}^{-\pi k(1-|V|)/\beta}$ for large k, so numerical evaluation can be done efficiently, in particular for $\Re(\alpha) > 1$, where the integrand is bounded on $[0,1]$. Furthermore we can take the limit $\alpha \to 1$ of the new piece (2.193) to find the extra contribution to entanglement von-Neumann entropy as

$$\begin{aligned} S^{(1)} = {} & \frac{1}{6}\frac{\pi|V|^2}{\beta} \\ & + \frac{|V|}{\pi\beta}\sum_{k=1}^{\infty}(\pm 1)^k \frac{\sinh \pi k|V|/\beta}{\sinh \pi k/\beta} \\ & \times \int_0^1 \frac{\mathrm{d}\zeta}{\zeta(1+\zeta)}[\zeta^{\mathrm{i}k|V|/\beta} + \zeta^{-\mathrm{i}k|V|/\beta}][(1+\zeta)\log(1+\zeta) - \zeta\log\zeta]. \end{aligned} \tag{2.194}$$

We could now repeat the discussion of relative entropy and mutual information for the results (2.191) and (2.194). However, this discussion would not lead to much new insight, as all integrals can only be evaluated numerically. Rather, I would like to note that eq. (2.194) presents a high temperature expansion of entanglement entropy, since a smaller β leads to faster convergence of the series. We could also have arrived at a low temperature expansion by inserting the low temperature expansions of the propagator (2.130) and (2.131) in the definition of the spectral measure, as we did in our original paper.[42] The

resulting expansion for entanglement (Rényi) entropy would then also yield a term-by-term finite result, which too can be evaluated analytically for rational $|V|$ only. To show its general form, let me sketch what happens in the anti-periodic case in detail: From eqs. (2.128) and (2.130), we have

$$H_\lambda(z) = \frac{1}{2\mathrm{i}} \csc \pi z + \sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta} \times \left[\frac{\mathrm{e}^{(2k+1)\mathrm{i}\pi z}}{(-\zeta)^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} - \frac{\mathrm{e}^{-(2k+1)\mathrm{i}\pi z}}{(-\zeta)^{|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right],$$

again with $\lambda = 1/(1+\zeta)$. Combined with eq. (2.129), this can be inserted in eq. (2.134) to yield the low temperature expansion

$$\begin{aligned} \mathrm{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] = \frac{1}{2\pi^2\zeta} \log \ell_V - \frac{|V|}{2\pi\mathrm{i}\zeta} \sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta} \\ \times \left[\frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} - \frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right. \\ \left. - \frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} + \frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right]. \end{aligned}$$

Insert this into eq. (2.163) to obtain again the decomposition eq. (2.192), where the correction term $S_\alpha^{(1)}$ is given by[42]

$$\begin{aligned} S_\alpha^{(1)} = -\frac{1}{1-\alpha} \frac{|V|}{2\pi\mathrm{i}} \sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta} \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta} [\log(1+\zeta^\alpha) - \alpha \log(1+\zeta)] \\ \times \left[\frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} - \frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right. \\ \left. - \frac{1}{\mathrm{e}^{-\mathrm{i}\pi|V|}\zeta^{|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} + \frac{1}{\mathrm{e}^{\mathrm{i}\pi|V|}\zeta^{|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right]. \end{aligned}$$

Similar to the zero-mode correction (2.189), this can be evaluated analytically for $\alpha = 2, 3, \ldots$ by rewriting it as a contour integral

$$\begin{aligned} S_\alpha^{(1)} = \frac{1}{1-\alpha} \frac{|V|}{2\pi\mathrm{i}} \sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta} \oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta} [\log(1+\zeta^\alpha) - \alpha \log(1+\zeta)] \\ \times \left[\frac{1}{(-\zeta)^{-|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} - \frac{1}{(-\zeta)^{|V|} + \mathrm{e}^{-(2k+1)\pi\beta}} \right] \end{aligned}$$

along the Hankel contour from fig. 2.9. Contracting the Hankel contour in

complete analogy to the evaluation of eq. (2.189) then yields

$$S_\alpha^{(1)} = -\frac{1}{1-\alpha}\sum_{k=0}^{\infty} \mathrm{e}^{-(2k+1)\pi\beta} \int_0^\infty \mathrm{d}s \sum_{m=-(\alpha-1)/2}^{(\alpha-1)/2} \Big[\frac{1}{\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-s}+\mathrm{e}^{-(2k+1)\pi\beta}} - \frac{1}{\mathrm{e}^{2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{s}+\mathrm{e}^{-(2k+1)\pi\beta}} - \frac{1}{\mathrm{e}^{-s}+\mathrm{e}^{-(2k+1)\pi\beta}} + \frac{1}{\mathrm{e}^{s}+\mathrm{e}^{-(2k+1)\pi\beta}}\Big].$$

Again, integration can be done elementarily with the result

$$\begin{aligned} S_\alpha^{(1)} &= \frac{1}{1-\alpha}\sum_{k=0}^{\infty} \log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2} \frac{1+\mathrm{e}^{-(2k+1)\pi\beta}\mathrm{e}^{s}}{\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}+\mathrm{e}^{-(2k+1)\pi\beta}\mathrm{e}^{s}} \times \frac{1+\mathrm{e}^{-(2k+1)\pi\beta}\mathrm{e}^{-s}}{\mathrm{e}^{2\pi\mathrm{i}|V|m/\alpha}+\mathrm{e}^{-(2k+1)\pi\beta}\mathrm{e}^{-s}}\Big]\Big|_{s=0}^{s=\infty} \\ &= \frac{1}{1-\alpha}\log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2} \mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha} \times \prod_{k=0}^{\infty} \frac{(1+\mathrm{e}^{2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-(2k+1)\pi\beta})(1+\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-(2k+1)\pi\beta})}{(1+\mathrm{e}^{-(2k+1)\pi\beta})^2}\Big]. \end{aligned}$$

Finally, note that

$$\log \prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2} \mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha} = -\frac{2\pi\mathrm{i}|V|}{\alpha} \sum_{m=-(\alpha-1)/2}^{(\alpha-1)/2} m$$

vanishes due to symmetry, while the remaining infinite product is exactly

$$\prod_{k=0}^{\infty} \frac{(1+\mathrm{e}^{2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-(2k+1)\pi\beta})(1+\mathrm{e}^{-2\pi\mathrm{i}|V|m/\alpha}\mathrm{e}^{-(2k+1)\pi\beta})}{(1+\mathrm{e}^{-(2k+1)\pi\beta})^2} = \frac{\vartheta_3(|V|m/\alpha|\mathrm{i}\beta)}{\vartheta_3(0|\mathrm{i}\beta)}$$

by means of the Jacobi triple product (2.85). As a result, you find

$$S_\alpha^{(1)} = \frac{1}{1-\alpha}\log\Big[\prod_{m=-(\alpha-1)/2}^{(\alpha-1)/2} \frac{\vartheta_3(|V|m/\alpha|\mathrm{i}\beta)}{\vartheta_3(0|\mathrm{i}\beta)}\Big], \tag{2.195}$$

which recovers exactly the result of Herzog and Nishioka,[144] albeit derived from a formula that is valid for *any* α with positive real part. The corresponding computation for periodic boundary conditions is identical, with ϑ_3 replaced by ϑ_2 in the final result.

2.4.7 The case of boundaries

This subsection introduces a work in progress together with Ignacio Reyes,[45] dealing with the computation of entanglement in the free fermion *boundary* CFT. In everything that we have done so far, the two chiralities of the free fermion CFT have been completely independent, so that we could limit discussions to one chirality at a time. This is no longer possible in the presence of a boundary, since we found in section 2.3.4 that the conformal boundary condition (2.99) implies an equivalence between the chiralities through eq. (2.100). As a result, e.g., the vacuum propagator on the line takes the form

$$G(x,y) = \frac{1}{2\pi \mathrm{i}} \begin{bmatrix} 1/(x-y) & \pm 1/(x+y) \\ \pm 1/(-x-y) & 1/(-x+y) \end{bmatrix}$$

with $\eta \in \{0,1\}$ as shown in eq. (2.101). Since this example illustrates all major points about entanglement in the presence of a boundary, I will stick with it for the rest of this subsection.

Please note that x, y here are actually spacial coordinates x^1, y^1 of two points on the standard time slice $x^0 = y^0 = 0$. This is in strong contrast to all previous computations, where we were dealing with one-dimensional problems on a lightray. Nevertheless, the formulae in table 2.1 and the discussion from section 2.3.1 remain valid, as they do not make any assumptions about the number of dimensions. In particular, we can still define the resolvent via eq. (2.60) in terms of a solution F_λ of the integral equation (2.61). This time, however, since G is a matrix, so are F_λ and the resolvent $1/(\lambda - G|_V)$. Denoting the components of G by

$$G(x,y) = \begin{bmatrix} G^0{}_0(x,y) & G^0{}_1(x,y) \\ G^1{}_0(x,y) & G^1{}_1(x,y) \end{bmatrix} \tag{2.196}$$

and similarly for F_λ, eq. (2.61) then takes the form

$$-G^a{}_b(x,y) + (F_\lambda)^a{}_b(x,y) - \frac{1}{\lambda} \sum_{c=0}^{1} \int_V \mathrm{d}z \, G^a{}_c(x,z) (F_\lambda)^c{}_b(z,y) = 0. \tag{2.197}$$

for $x, y \in V \subset \mathbb{R}_+$ and $a, b \in \{0,1\}$. Solving this equation might seem hard indeed but luckily there is a trick that significantly simplifies the problem: As

we derived in section 2.3.4, the components of the propagator have the form

$$G^a{}_b(x,y) = \eta^{a+b}\hat{G}((-1)^a x,(-1)^b y),$$

where

$$\hat{G}(x,y) = \hat{G}(x-y) = \frac{1}{2\pi\mathrm{i}(x-y)}$$

is the vacuum propagator of a single chirality on the lightray. This suggests making the ansatz

$$(F_\lambda)^a{}_b(x,y) = \eta^{a+b}\hat{F}_\lambda((-1)^a x,(-1)^b y) \tag{2.198}$$

for some yet to determine scalar kernel $\hat{F}_\lambda$ that is defined on $V \cup -V$. Inserting this into eq. (2.197), you obtain

$$-\hat{G}(x,y) + \hat{F}_\lambda(x,y) - \frac{1}{\lambda}\int_{\hat{V}} \mathrm{d}z\, \hat{G}(x,z)\hat{F}_\lambda(z,y) = 0 \quad \text{with} \quad \hat{V} = V \cup -V,$$

which is exactly eq. (2.102), i.e., the corresponding integral equation on the lightray, albeit with *mirror-symmetric* entangling region $\hat{V}$. Since we already solved this equation for arbitrary regions, we also solved eq. (2.197).

To explain the implications of this, I will discuss the example of a single interval $V = [a,b]$, $0 < a < b$ at a finite distance from the boundary. We can directly look up the solution for $\hat{F}_\lambda$ from table 2.3 – it is given by eq. (2.116) with $\hat{H}_\lambda(z) = 1/2\pi\mathrm{i}z$ and

$$\hat{Z}(x) = \frac{1}{2\pi}\log\frac{(x-a)(b+x)}{(b-x)(x+a)},$$

where the second factors in the fraction appear because of the mirrored interval $-V = [-b,-a]$. The resolvent is

$$\left[\frac{1}{\lambda - G|_V}\right]^a{}_b(x,y) = \frac{\delta^a_b\delta(x-y)}{\lambda} + \frac{\eta^{a+b}\hat{F}_\lambda((-1)^a x,(-1)^b y)}{\lambda^2}$$

and could, in principle, redo all computations from before.

However, there is an even easier route: Note that the above assignment

$$A^a{}_b(x,y) := \eta^{a+b}\hat{A}((-1)^a x,(-1)^b y) \tag{2.199}$$

defines an isomorphism between the algebra of 2×2-matrix-valued kernels on V on the one side and the algebra of scalar kernels on $\hat{V}$ on the other side. This amounts to the fact it is well behaved with respect to products, i.e., we

have

$$\begin{aligned}(AB)^a{}_b(x,y) &= \sum_c \int_{\mathbb{R}_+} \mathrm{d}z\, A^a{}_c(x,z) B^c{}_b(z,y) \\ &= \sum_c \int_{\mathbb{R}_+} \mathrm{d}z\, \eta^{a+c} \hat{A}^a{}_c((-1)^a x,(-1)^c z) \eta^{c+b} \hat{B}^c{}_b((-1)^c z,(-1)^b y) \\ &= \eta^{a+b} \int_{\hat{V}} \mathrm{d}z\, \hat{A}((-1)^a x, z) \hat{B}(z,(-1)^b y) \\ &= \eta^{a+b} (\hat{A}\hat{B})((-1)^a x,(-1)^b y).\end{aligned}$$

This implies that it also extends to arbitrary functions of these kernels (by expanding them as power series), which is the reason why eq. (2.197) works. It also implies that we can apply the assignment (2.199) to every result on the lightray that we derived so far, to immediately obtain the corresponding result on the standard time slice of right half plane.

As an example, consider the modular flow of operators with respect to the algebra of the interval $V = [a, b]$. The corresponding result on the lightray is

$$\sigma_t(\psi^*(y)) = \int_{\hat{V}} \mathrm{d}x\, \psi^*(x) \hat{\Sigma}_t(x,y)$$

with

$$\hat{\Sigma}_t(x,y) = -\frac{\sinh(\pi t)}{\pi(x-y)} \delta(t + \hat{Z}(x) - \hat{Z}(y)),$$

as shown in eq. (2.136). This means that the result with a boundary has to be

$$\sigma_t\left(\begin{bmatrix} \psi_+(y) \\ \psi_-(-y) \end{bmatrix}^* \right) = \int_V \mathrm{d}x \begin{bmatrix} \psi_+(x) \\ \psi_-(-x) \end{bmatrix}^* \Sigma_t(x,y) \tag{2.200}$$

with

$$(\Sigma_t)^a{}_b(x,y) = \eta^{a+b} \hat{\Sigma}_t((-1)^a x,(-1)^b y).$$

As $\hat{Z}(-x) = -\hat{Z}(x)$, this implies

$$\Sigma_t(x,y) = -\frac{1}{\pi} \sinh(\pi t) \begin{bmatrix} \frac{\delta(t+\hat{Z}(x)-\hat{Z}(y))}{x-y} & \eta \frac{\delta(t+\hat{Z}(x)+\hat{Z}(y))}{x+y} \\ \eta \frac{\delta(t-\hat{Z}(x)-\hat{Z}(y))}{-x-y} & \frac{\delta(t-\hat{Z}(x)+\hat{Z}(y))}{-x+y} \end{bmatrix}, \tag{2.201}$$

hence,

$$\sigma_t(\psi_+^*(y)) = -\frac{1}{\pi}\sinh(\pi t)\int_{V\cup -V} \mathrm{d}x\, \psi_+^*(x)\frac{\delta(t+\hat{Z}(x)-\hat{Z}(y))}{x-y},$$

where I used (2.100). For a single interval $V = [a, b]$ this will yield a bi-local result, with one contribution for $x \in V$ and one contribution for $x \in -V$. So far, this latter contribution is not in conflict with our restriction to the right half plane, as $\psi_+^*(x) = \psi_+^*(x^+)$ is actually defined on a light ray, so even a negative argument might belong to a point in our domain of interest. However, as shown in eq. (2.139), this contribution will have to be paired with a chiral fermion of opposite chirality in order to yield a localised Dirac fermion. This Dirac fermion will then be localised inside the double cone spanned by $-V$ in the left half plane, so we have to discard it by hand. To do so, we can discard the $-V$-part of the above integral, yielding

$$\sigma_t(\psi_+^*(y)) = -\frac{\sinh(\pi t)}{\pi}\frac{1}{\hat{Z}'(x(t,y))}\frac{1}{x(t,y)-y}\psi_+^*(x(t,y)), \tag{2.202}$$

where $x(t, y)$ is the unique positive root of the quadratic equation

$$\frac{(x-a)(b+x)}{(b-x)(x+a)} = \mathrm{e}^{-2\pi t}\frac{(y-a)(b+y)}{(b-y)(y+a)}. \tag{2.203}$$

Note that this recovers the Hislop-Longo result (2.35) in the limit of large $(b+a)/(b-a)$, i.e., when V is small or far away from the boundary. Note also that eq. (2.203) explicitly breaks translational invariance due to the presence of the mirrored interval $-V$, while it is still invariant under scale transformations. Similarly, you can obtain expressions for the modular Hamiltonian and correlator in the boundary system from their mirror symmetric counterparts on the lightray, which will be invariant under scale transformations but not translations.

Proceeding to the discussion of entropy, recall from eq. (2.52) that you can compute S_α from the trace of the spectral measure. Since this is again a kernel that can be expressed in terms of the propagator, we can apply eq. (2.199) to

obtain

$$\begin{aligned}\mathrm{Tr}\left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}\right] &= \int_V \mathrm{d}x \lim_{y\to x}\sum_a \left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}\right]^a{}_a(x,y)\\ &= \int_{V\cup -V}\mathrm{d}x \lim_{y\to x}\left[\frac{\mathrm{d}\hat{E}_\lambda}{\mathrm{d}\lambda}\right](x,y)\\ &= \mathrm{Tr}\left[\frac{\mathrm{d}\hat{E}_\lambda}{\mathrm{d}\lambda}\right],\end{aligned}$$

hence,

$$S_\alpha = \hat{S}_\alpha \quad \text{and} \quad S = \hat{S}. \tag{2.204}$$

This allows for a key observation in boundary CFT: Assume again that we are dealing with the double cone generated by a single interval $V = [a, b]$ on the x^1-axis. Its projections onto the lightray axes $x^\pm$ are given by $\pm V$. Therefore, the corresponding algebra of observables is generated by $\psi_\pm(x)$ with support in $\pm V$. Because of eq. (2.100), the fields $\psi_\pm(x)$ are equivalent, hence, we can equivalently with the algebra generated by $\psi_+(x)$ with support in $\hat{V} = V \cup -V$. In the vacuum state, we will generally find entanglement between V and $-V$, which causes the off-diagonal terms in the propagator (2.101). On the other hand, we already discussed that the split property of QFT allows to construct a product state (2.185), which is indistinguishable from the vacuum state within $V_\pm$ on their own, but has all correlations removed – this product state $\tilde{\rho}$ generates, by construction, the same expectation values as the vacuum state of the theory *without boundary*! In this sense, the theory without boundary is included in the theory with boundary by means of the split property.[128] In particular, this allows to compare the vacua of the theories with and without boundary. The relative entropy between them is precisely the mutual information between V and $-V$ in the vacuum state of the boundary CFT and coincides with that of the vacuum state of a single chirality on the lightray because of eq. (2.204).

As a result, the relative entropy between the vacua is given by eq. (2.188) with $l = b - a$ and $d = 2a$, i.e., denoting the vacuum of the boundary theory with ρ and its split product state by $\tilde{\rho}$,

$$S(\rho\|\tilde{\rho}) = S(\tilde{\rho}) - S(\rho) = \frac{1}{6}\log\frac{1}{4}\left[2 + \frac{a}{b} + \frac{b}{a}\right], \tag{2.205}$$

which is always positive, as relative entropy should be. The fact that entanglement entropy is smaller in the theory with boundary has a nice physical interpretation: Since both chiralities are entangled with each other inside V, they can not be entangled with the complement V' as much as they could if

they were independent. This is known as *monogamy of entanglement*.[8] In fact, since the different chiralities $\psi_\pm$ are in $\pm V$, eq. (2.205) also gives the mutual information between both chiralities, quantifying how much entanglement there actually is between them.

2.5 Entanglement and superselection

In this section, I would like to present another work in progress together with Ignacio Reyes,[44] that aims at a symmetry-aware generalisation of relative entropy and its operational interpretation in terms of distinguishability of states.

Before we study entropy, let me first clarify what I mean by symmetry-aware. Often in QFT, we find ourselves in a situation where our *global* algebra $\mathcal{A}$ of observables has a central element Q, you could call it *charge*. Since Q is central, it commutes with all algebra elements, i.e., it is conserved algebraically. In the QFT literature, the existence of such central elements is denoted as a *superselection rule*.[64,145,146] This is in contrast to a *selection rule*, which would state that Q is conserved dynamically, i.e., commutes with the unitary implementers of other symmetries as discussed in section 1.2.1. Equivalently, you could say that the GNS representation of $\mathcal{A}$ is reducible – its Hilbert space decomposes as a direct sum

$$\mathcal{H} = \bigoplus_q \mathcal{H}_q, \tag{2.206}$$

where each *superselection sector* $\mathcal{H}_q$ is invariant under $\mathcal{A}$. As an example of such a decomposition recall the definition of the Fermi net from eq. (2.43) on a lightray. It contains smeared field operators of the form

$$\int \mathrm{d}x\, f(x)\psi(x)$$

as well as sums, products, adjoints, and weak limits of them. The GNS representation of the corresponding global algebra is the usual fermionic Fock space with creation and annihilation operators and, as a result, irreducible. While this is great for doing computations, it does not really reflect physical reality: Physically reasonable operators have to be contained in the sub-net[147]

$$\mathcal{A}_{\text{phys}}(V) := \{\int_V \mathrm{d}x\, \mathrm{d}y\, \psi^*(x) f(x,y)\psi(y) : f \text{ test kernel } V\}'' \tag{2.207}$$

which means that the normal ordered operator

$$Q := \int_{\mathbb{R}} \mathrm{d}x\ :\psi^*(x)\psi(x): \tag{2.208}$$

is central in the global physical algebra $\mathcal{A}_{\text{phys}}$. Now, let $\omega \in \mathcal{A}^*_{\text{phys}}$ be the vacuum state on this algebra. The corresponding GNS representation will take the form (2.206) with a cyclic separating vector

$$\Omega = \sum_q c_q \Omega_q, \quad \Omega_q \in \mathcal{H}_q. \tag{2.209}$$

Here, Ω_q is cyclic separating in cH_q for $\mathcal{A}_{\text{phys}}/Q$ and it is also an eigenvector of Q with

$$Q\Omega_q = q\Omega_q.$$

In our above example, Q measures the number of fermions in the system and we usually only work in the sector $\mathcal{H}_0$ with Ω_0 the Fock vacuum. However, we could of course also consider the space $\mathcal{H}_n$ containing n Fermions, or $\mathcal{H}_{-m}$ containing m positrons. Let us pick any one of these spaces, i.e., we assume that all coefficients but one vanish in eq. (2.209). Then, of course, the associated density matrix ρ of the GNS vector Ω commutes with Q. Furthermore, this also holds for partial traces of ρ if the total Hibert space splits into local subspaces as guaranteed by the split property (2.166): Since Q is locally defined, we have

$$Q = Q_V \otimes 1 + 1 \otimes Q_{V'}, \quad Q_{V^{(\prime)}} := \int_{V^{(\prime)}} \mathrm{d}x \; :\psi^*(x)\psi(x):,$$

so that

$$\begin{aligned}
[Q_V, \rho_V] &= \mathrm{Tr}_{V'}\Big([Q_V \otimes 1, \rho]\Big) \\
&= \mathrm{Tr}_{V'}\Big([Q_V \otimes 1, \rho]\Big) + \mathrm{Tr}_{V'}\Big((1+Q_{V'})\rho - (1+Q_{V'})\rho\Big) \\
&= \mathrm{Tr}_{V'}\Big([Q_V \otimes 1, \rho]\Big) + \mathrm{Tr}_{V'}\Big(1+Q_{V'}, \rho]\Big) \\
&= \mathrm{Tr}_{V'}\Big([Q, \rho]\Big) \\
&= 0,
\end{aligned}$$

where the first equality comes from the definition of the partial trace and the third one is due to the cyclicity of the trace. As a result, reduced density matrices are block diagonal in the presence of superselection rules and it is interesting to see what this means on an information theoretical level. In the following, I will drop all subscripts from reduced density matrices, since we will always be working in a fixed subregion V.

2.5.1 Symmetry resolved relative entropy

In section 2.4.6, we discussed relative entropy and its definition in the context of QFT. As I argued, relative entropy allows to compare different states ρ, $\tilde{\rho}$ – it supposedly is a measure of "how similar" they are. However, there is one caveat to this distinguishability, if relative entropy is defined by

$$S(\rho\|\tilde{\rho}) = \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\tilde{\rho}],$$

as done in eq. (2.178). Here and for the rest of this subsection, I use the finite dimensional version of relative entropy opposed to its definition in terms of the Connes cocyle, in order to keep this discussion as clear as possible. As already stated, this $S(\rho\|\tilde{\rho})$ vanishes if and only if ρ and $\tilde{\rho}$ are indistinguishable by measurements of *all* operators $B(\mathcal{H})$ on the Hilbert space that they are defined on.

From an experimental perspective, the assumption that we could implement *every operator* as a measurement is of course optimistic at best – even if we had a perfect array of measurement devices, we just saw above the superselection rules prohibit the measurement of amplitudes between different charge sectors $\mathcal{H}_q$, $\mathcal{H}_{p\neq q}$, as there are no physical operators that cause transitions between them. This implies that eq. (2.178) overestimates the operational distinguishability of ρ and $\tilde{\rho}$! Indeed, as Q is central, we can only compare states within one sector at a time, so it makes more sense to consider the *charge resolved relative entropy*[44]

$$S_q(\rho\|\tilde{\rho}) := S(\rho_q\|\tilde{\rho}_q). \tag{2.210}$$

Here, ρ_q and $\tilde{\rho}_q$ are the resulting states after performing a projective measurement of Q with result q,

$$\rho_q := \frac{\Pi_q\rho\Pi_q}{\mathrm{P}(q)} \tag{2.211}$$

with

$$\mathrm{P}(q) := \mathrm{Tr}[\Pi_q\rho] \tag{2.212}$$

the probability of obtaining the result q from a projective measurement of Q in the state ρ. The definitions of $\tilde{\rho}_q$ and $\tilde{\mathrm{P}}(q)$ are entirely analogous.

Note that in eq. (2.211), we have

$$\rho_q = \frac{\Pi_q\rho}{\mathrm{P}(q)},$$

since $\Pi_q = \Pi_q^2$ commutes with ρ. Inserting eq. (2.178) into definition (2.210),

we find

$$S_q(\rho\|\tilde{\rho}) = \frac{1}{\mathrm{P}(q)}\Big(\mathrm{Tr}[\Pi_q\rho\log\rho] - \mathrm{Tr}[\Pi_q\rho\log\tilde{\rho}]\Big) - \log\frac{\mathrm{P}(q)}{\tilde{\mathrm{P}}(q)}, \tag{2.213}$$

resulting in the ensemble average

$$\sum_q \mathrm{P}_\rho(q) S_q(\rho\|\tilde{\rho}) = S(\rho\|\tilde{\rho}) - D(\mathrm{P}\,\|\,\tilde{\mathrm{P}}).$$

Here, I used the completeness relation (2.206) and recognised the classical *Kullback-Leibler divergence*[148]

$$D(\mathrm{P}\,\|\,\tilde{\mathrm{P}}) := \sum_q \mathrm{P}(q)\log\frac{\mathrm{P}(q)}{\tilde{\mathrm{P}}(q)},$$

which is always non-negative and vanishes if and only if $\mathrm{P} = \tilde{\mathrm{P}}$. This shows, that eq. (2.178) on average overestimates the distinguishability of ρ and $\tilde{\rho}$ if and only if ρ, $\tilde{\rho}$ yield different statistics for the measurement of Q.

At this point, the important question is of course if eq. (2.210) makes sense in the context of quantum field theory and, if so, how you can compute it. To see how this works, note that the hard part in evaluating eq. (2.213) is the term

$$\mathrm{Tr}[\Pi_q\rho\log\rho] - \mathrm{Tr}[\Pi_q\rho\log\tilde{\rho}]. \tag{2.214}$$

To tackle it, we will use methods that were introduced by Goldstein and Sela[149] in the study of the non-relative version of eq. (2.210), the charge resolved entanglement entropy.[150–152] For the rest of this subsection, let us assume that Q has integer spectrum, so that we can make use of the Fourier identity[149]

$$\Pi_q = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}\phi(Q-q)}. \tag{2.215}$$

This reduces the computation of eq. (2.214) to that of

$$\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho\log\rho] - \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho\log\tilde{\rho}]. \tag{2.216}$$

Next, note that in QFT the two terms in eq. (2.216) can be expected to diverge individually, due to their similarity to entanglement entropy – we can only expect their difference to be finite. In order to obtain finite intermediate steps nonetheless, we use a trick akin to the one in eq. (2.179), by rewriting

eq. (2.216) as

$$\begin{aligned}
&\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho\log\rho] - \frac{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]}{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}]}\,\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}\log\tilde{\rho}] \\
&\qquad\qquad - \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho\log\tilde{\rho}] + \frac{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]}{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}]}\,\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}\log\tilde{\rho}] \\
&= \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]\times\Big[\Big(\mathrm{Tr}[\rho_\phi\log\rho] - \mathrm{Tr}[\tilde{\rho}_\phi\log\tilde{\rho}]\Big) + \\
&\qquad\qquad \Big(\mathrm{Tr}[\rho_\phi\tilde{K}] - \mathrm{Tr}[\tilde{\rho}_\phi\tilde{K}]\Big)\Big] \qquad (2.217)
\end{aligned}$$

with the *grand canonical* density matrices

$$\rho_\phi := \frac{\mathrm{e}^{\mathrm{i}\phi Q}\rho}{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]}, \quad \tilde{\rho}_\phi := \frac{\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}}{\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\tilde{\rho}]}$$

and $\tilde{K} = -\log\tilde{\rho} + \text{const.}$ the modular Hamiltonian of $\tilde{\rho}$.

Putting everything together, we thus find that eq. (2.214) can be expressed as

$$\mathrm{Tr}[\Pi_q\rho\log\rho] - \mathrm{Tr}[\Pi_q\rho\log\tilde{\rho}] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\phi\ \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi(Q-q)}\rho]S_\phi(\rho\|\tilde{\rho}), \qquad (2.218)$$

where I introduced the *flux resolved relative entropy* $S_\phi(\rho\|\tilde{\rho})$ as the (generally finite) quantity[44]

$$S_\phi(\rho\|\tilde{\rho}) := \Big(\mathrm{Tr}[\rho_\phi\tilde{K}] - \mathrm{Tr}[\tilde{\rho}_\phi\tilde{K}]\Big) - \Big(-\mathrm{Tr}[\rho_\phi\log\rho] + \mathrm{Tr}[\tilde{\rho}_\phi\log\tilde{\rho}]\Big). \qquad (2.219)$$

The other missing pieces in eq. (2.213) are the probability distributions P and $\tilde{\mathrm{P}}$. Again, you can use eq. (2.215) to find

$$\mathrm{P}(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\phi\ \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi(Q-q)}\rho] \qquad (2.220)$$

and similarly for $\tilde{\rho}$. To conclude the general discussion, note that in eq. (2.219), the *flux resolved entanglement entropy*[149]

$$S_\phi := -\mathrm{Tr}[\rho_\phi\log\rho]$$

can be evaluated via the replica trick as[139]

$$-\operatorname{Tr}[\rho_\phi \log \rho] = -\frac{\operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho \log \rho]}{\operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]} = -\frac{\mathrm{d}}{\mathrm{d}\alpha} \log \operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho^\alpha]\Big|_{\alpha=1}. \tag{2.221}$$

and similarly for $\tilde{\rho}$.

2.5.2 Results for free fermions

Let us now proceed to the evaluation of eq. (2.219) for different states of free fermions. I will do this explicitly for thermal states of different inverse temperatures β,$\tilde{\beta}$ on the lightray, restricted to the region V, in order to obtain a result that we can compare with eq. (2.182). As stated previously, the conserved charge Q shall be the number of fermions inside V, i.e.,

$$Q = \int_V \mathrm{d}x \; :\psi^*(x)\psi(x): .$$

To get started, note that we can use the methods from section 2.3 to obtain an expression for $\operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]$ in terms of the restricted propagator $G|_V$. Since

$$\rho_V = \prod_k \Big[g_k c_k c_k^* + (1-g_k)c_k^* c_k\Big]$$

with g_k the eigenvalues of $G|_V$, you see that

$$\begin{aligned}\log \operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho^\alpha] &= \sum_k \log\Big[g_k^\alpha + \mathrm{e}^{\mathrm{i}\phi}(1-g_k^\alpha)\Big] \\ &= \operatorname{Tr}\log[G|_V^\alpha + \mathrm{e}^{\mathrm{i}\phi}(1-G|_V)^\alpha],\end{aligned} \tag{2.222}$$

if we ignore the normal ordering constant that has to be subtracted from Q. Applying the replica trick (2.221) then yields the first difference in eq. (2.219). To evaluate eq. (2.222), we can again use the spectral calculus (2.58) with the spectral measure (2.135) to find

$$\begin{aligned}\log \operatorname{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho^\alpha] &= \int_0^1 \mathrm{d}\lambda \; \operatorname{Tr}\left[\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}\right] \log[\lambda^\alpha + \mathrm{e}^{\mathrm{i}\phi}(1-\lambda)^\alpha] \\ &= \int_0^\infty \mathrm{d}\zeta \; \operatorname{Tr}\left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] [\log(1+\mathrm{e}^{\mathrm{i}\phi}\zeta^\alpha) - \alpha\log(1+\zeta)] \\ &= \log \ell_V \times \frac{1}{2\pi^2}\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta}[\log(1+\mathrm{e}^{\mathrm{i}\phi}\zeta^\alpha) - \alpha\log(1+\zeta)].\end{aligned}$$

This integral is very similar to the one we evaluated to obtain entanglement Rényi entropy (2.171). Let us use the same trick of changing variables $\zeta \to 1/\zeta$ on $[1, \infty]$. The result is

$$\log \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho^\alpha] = \log \ell_V \times \frac{1}{2\pi^2}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta}\Big[\mathrm{i}\phi + \log(1+\mathrm{e}^{\mathrm{i}\phi}\zeta^\alpha) + \log(1+\mathrm{e}^{-\mathrm{i}\phi}\zeta^\alpha) - 2\alpha\log(1+\zeta)\Big].$$

Note that the contribution coming from the $\mathrm{i}\phi$ in the integrand is divergent! This should come at no surprise since it yields exactly the expectation value of Q via

$$\mathrm{i}\,\mathrm{Tr}[\rho Q] = \frac{\mathrm{d}}{\mathrm{d}\phi}\log \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi Q}\rho]\Big|_{\phi=0} = \log \ell_V \times \frac{\mathrm{i}}{2\pi^2}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta}.$$

This is exactly the constant which we need to subtract from Q due to its normal ordered definition. Making this explicit by writing

$$:Q: := Q - \mathrm{Tr}[\rho Q],$$

we are thus left with the computation of

$$\log \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi:Q:}\rho^\alpha] = \log \ell_V \times \frac{1}{2\pi^2}\int_0^1 \frac{\mathrm{d}\zeta}{\zeta}\Big[\log(1+\mathrm{e}^{\mathrm{i}\phi}\zeta^\alpha) + \log(1+\mathrm{e}^{-\mathrm{i}\phi}\zeta^\alpha) - 2\alpha\log(1+\zeta)\Big].$$

You can now insert the Taylor expansion (2.170) again, so that the integral yields

$$\frac{1}{2\pi^2\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\mathrm{e}^{\mathrm{i}\phi k}+\mathrm{e}^{-\mathrm{i}\phi k}-2\alpha^2}{k^2} = -\frac{\mathrm{Li}_2(-\mathrm{e}^{\mathrm{i}\phi})+\mathrm{Li}_2(-\mathrm{e}^{-\mathrm{i}\phi})}{2\pi^2\alpha} - \frac{\alpha}{12},$$

where

$$\mathrm{Li}_2(z) := \sum_{k=1}^{\infty}\frac{z^k}{k^2}$$

is the dilogarithm (Spence's function).[153] Finally, you can use the magical relation[154]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z),$$

to find

$$\log \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi:Q:}\rho^\alpha] = \frac{1}{\alpha}\Big[\frac{1-\alpha^2}{12} - \Big(\frac{\phi}{2\pi}\Big)^2\Big]\log \ell_V. \tag{2.223}$$

In particular, we recover the result of Goldstein and Sela[149]

$$-\operatorname{Tr}[\rho_\phi \log \rho] = \left[\left(\frac{1}{6} - \frac{\phi}{2\pi}\right)^2\right] \log \ell_V, \tag{2.224}$$

albeit for an arbitrary number of intervals. For the single interval $V = [-a, a]$, this means that the "entropy" difference in eq. (2.219) is

$$-\operatorname{Tr}[\rho_\phi \log \rho] + \operatorname{Tr}[\tilde{\rho}_\phi \log \tilde{\rho}] = \left[\frac{1}{6} - \left(\frac{\phi}{2\pi}\right)^2\right] \log \frac{\beta \sinh 2\pi a/\beta}{\tilde{\beta} \sinh 2\pi a/\tilde{\beta}}. \tag{2.225}$$

Finally, in order to evaluate eq. (2.219), we need to compute the "modular Hamiltonian" difference

$$\operatorname{Tr}[\rho_\phi \tilde{K}] - \operatorname{Tr}[\tilde{\rho}_\phi \tilde{K}].$$

Again, since K, $\tilde{K}$ are quadratic with kernels k,$\tilde{k}$, this amounts to evaluating an integral of the form

$$\int \mathrm{d}x\, \mathrm{d}y\, \tilde{k}(x,y) \Big(\operatorname{Tr}[\rho_\phi \psi^\dagger(x)\psi(y)] - \operatorname{Tr}[\tilde{\rho}_\phi \psi^\dagger(x)\psi(y)] \Big),$$

where the traces are plain propagators. To evaluate them, note that the modular Hamiltonian of ρ_ϕ is $K - \mathrm{i}\phi Q$, as is obvious from eq. (2.211). This means that it is again quadratic with kernel $k - \mathrm{i}\phi$ and of course the same relations hold for $\tilde{\rho}$, $\tilde{K}$, and $\tilde{k}$. As a result, we can use the formula $G_\phi = 1/(1 + \mathrm{e}^{-k+\mathrm{i}\phi})$ from eq. (2.42) to obtain

$$\begin{aligned} \operatorname{Tr}[\rho_\phi \tilde{K}] - \operatorname{Tr}[\tilde{\rho}_\phi \tilde{K}] = -\int \mathrm{d}x\, \mathrm{d}y\, \tilde{k}(x,y) \\ \times \left[\frac{1}{1 + \mathrm{e}^{-k+\mathrm{i}\phi}}(y,x) - \frac{1}{1 + \mathrm{e}^{-\tilde{k}+\mathrm{i}\phi}}(y,x)\right], \end{aligned} \tag{2.226}$$

where

$$\frac{1}{1 + \mathrm{e}^{-k+\mathrm{i}\phi}} = \int_0^\infty \mathrm{d}\zeta \left[\frac{\mathrm{d}E_\zeta}{\mathrm{d}\zeta}\right] \frac{1}{1 + \mathrm{e}^{\mathrm{i}\phi}\zeta} \tag{2.227}$$

by the spectral calculus eq. (2.56) with $\lambda = 1/1 + \zeta$ and similarly for $1/(1 + \mathrm{e}^{-\tilde{k}+\mathrm{i}\phi})$. Upon insertion of the spectral measure (2.135), the integral (2.227) turns into

$$\frac{1}{1 + \mathrm{e}^{-k+\mathrm{i}\phi}} = \frac{\mathrm{i}}{\pi} \sinh(\pi \Delta Z) G \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta} \frac{\zeta^{\mathrm{i}\Delta Z}}{1 + \mathrm{e}^{\mathrm{i}\phi}\zeta},$$

which is exactly the integral representation of a propagator with chemical potential $\mathrm{i}\phi$ (hence the name "grand canonical" density matrix). To evaluate,

consider again the contour integral

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta}\frac{(-\zeta)^{\mathrm{i}\Delta Z}}{1+\mathrm{e}^{\mathrm{i}\phi}\zeta},$$

along the Hankel contour from fig. 2.9. For $-1 < \Im(\Delta z) < 0$, the contributions at $|\zeta| \to 0, \infty$ vanish, so that

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta}\frac{(-\zeta)^{\mathrm{i}\Delta Z}}{1+\mathrm{e}^{\mathrm{i}\phi}\zeta} = -2\sinh(\pi\Delta Z)\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta}\frac{\zeta^{\mathrm{i}\Delta Z}}{1+\mathrm{e}^{\mathrm{i}\phi}\zeta}$$

and, hence,

$$\frac{1}{1+\mathrm{e}^{-k+\mathrm{i}\phi}} = \frac{1}{2\pi\mathrm{i}}G\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta}\frac{(-\zeta)^{\mathrm{i}\Delta Z}}{1+\mathrm{e}^{\mathrm{i}\phi}\zeta} = -G\mathrm{e}^{-\mathrm{i}\phi}\frac{(-\zeta)^{\mathrm{i}\Delta Z}}{\zeta}\Big|_{\zeta=-\mathrm{e}^{-\mathrm{i}\phi}} = G\mathrm{e}^{\phi\Delta Z}.$$

We can now evaluate eq. (2.226) for a single interval $V = [-a, a]$. To this end, recall from eq. (2.158) that the kernel $\tilde{k}$ of the modular Hamiltonian is

$$\tilde{k}(x,y) = \frac{1}{\mathrm{i}\tilde{Z}'(x)}\Big[-\delta'(x-y) + \frac{1}{2}\frac{\tilde{Z}''(x)}{\tilde{Z}'(x)}\delta(x-y)\Big]$$

with

$$\tilde{Z}(x) = \frac{1}{2\pi}\log\frac{\sinh\pi(a+x)/\tilde{\beta}}{\sinh\pi(a-x)/\tilde{\beta}}.$$

As a result, we have

$$\begin{aligned}\mathrm{Tr}[\rho_\phi\tilde{K}] - \mathrm{Tr}[\tilde{\rho}_\phi\tilde{K}] = \mathrm{i}\int_{-a}^{a}\mathrm{d}x\,\frac{1}{\tilde{Z}'(x)}\lim_{y\to x}\Big[-\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\frac{\tilde{Z}''(x)}{\tilde{Z}'(x)}\Big]\\ \times\Big[\frac{1}{1+\mathrm{e}^{-k+\mathrm{i}\phi}}(y,x) - \frac{1}{1+\mathrm{e}^{-\tilde{k}+\mathrm{i}\phi}}(y,x)\Big],\end{aligned}$$

so that we again only need the leading terms of the UV-expansion

$$\begin{aligned}\frac{1}{1+\mathrm{e}^{-k+\mathrm{i}\phi}}(x,y) = \Big[\frac{1}{2\pi\mathrm{i}(x-y)} + \frac{\mathrm{i}\pi}{12\beta^2}(x-y)\Big]\\ \times\Big[1+\phi Z'(y)(x-y) + \frac{1}{2}\Big([\phi Z'(y)]^2 + \phi Z''(y)\Big)(x-y)^2\Big]\\ + \mathcal{O}\Big((x-y)^2\Big).\end{aligned} \tag{2.228}$$

Inserting this into the above integral, you may drop a lot of terms because

$\tilde{Z}(x)$ is odd. The remaining integrals are

$$\mathrm{Tr}[\rho_\phi \tilde{K}] - \mathrm{Tr}[\tilde{\rho}_\phi \tilde{K}] = \frac{\pi}{12}\Big[\frac{1}{\beta^2} - \frac{1}{\tilde{\beta}^2}\Big]\int_{-a}^{a} \mathrm{d}x\, \frac{1}{\tilde{Z}'(x)} - 2\pi\Big(\frac{\phi}{2\pi}\Big)^2 \int_0^a \mathrm{d}x\, \frac{[Z'(x)]^2 - [\tilde{Z}'(x)]^2}{\tilde{Z}'(x)}.$$

The first of these integrals coincides with the corresponding term $\Delta\langle\tilde{K}\rangle$ from eq. (2.181), i.e.,

$$\frac{\pi}{12}\Big[\frac{1}{\beta^2} - \frac{1}{\tilde{\beta}^2}\Big]\int_{-a}^{a} \mathrm{d}x\, \frac{1}{\tilde{Z}'(x)} = \frac{1}{12}\Big[\Big(\frac{T}{\tilde{T}}\Big)^2 - 1\Big]\Big[\tilde{T}\coth\tilde{T} - 1\Big].$$

with the dimensionless temperatures (2.183). The second integral is new and can be rewritten as

$$-T\sinh T\Big(\frac{\phi}{2\pi}\Big)^2 \int_0^1 \frac{\mathrm{d}X}{\cosh T - \cosh XT} \times \Big(1 - \Big[\frac{\tilde{T}\sinh\tilde{T}}{T\sinh T} \times \frac{\cosh T - \cosh XT}{\cosh\tilde{T} - \cosh X\tilde{T}}\Big]^2\Big),$$

which you can easily check to be finite. Unfortunately, this integral seems to have no analytical solution, so that the final result for flux resolved relative entropy on the interval $V = [-a, a]$ between thermal states of temperatures $\beta, \tilde{\beta}$ on the lightray is[44]

$$S_\phi(T\|\tilde{T}) = S(T\|\tilde{T}) + \Big(\frac{\phi}{2\pi}\Big)^2 S^{(2)}(T\|\tilde{T}), \tag{2.229}$$

where the usual relative entropy $S(\beta\|\tilde{\beta})$ was already computed in eq. (2.182) and the coefficient of the flux-dependent quadratic correction is

$$S^{(2)}(T\|\tilde{T}) = \log\frac{\tilde{T}\sinh T}{T\sinh\tilde{T}} - T\sinh T\int_0^1 \frac{\mathrm{d}X}{\cosh T - \cosh XT} \times \Big(1 - \Big[\frac{\tilde{T}\sinh\tilde{T}}{T\sinh T} \times \frac{\cosh T - \cosh XT}{\cosh\tilde{T} - \cosh X\tilde{T}}\Big]^2\Big). \tag{2.230}$$

Note that $S^{(2)}$ also vanishes precisely for $T = \tilde{T}$, but it is neither strictly positive nor negative. The physical interpretation of this fact is currently not clear and it will be discussed in our upcoming work.[44]

The big question is now, how do we obtain the *charge* resolved relative entropy? Trying to evaluate eq. (2.218), we will get stuck at the term

$$\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi(:Q:-q)}\rho] = \mathrm{e}^{-\mathrm{i}\phi q}(\ell_{[-a,a]})^{-(\phi/2\pi)^2},$$

which we already derived in eq. (2.223). This is problematic, since

$$\ell_{[-a,a]} = \frac{\beta}{\pi\epsilon}\sinh\frac{2\pi a}{\beta}$$

diverges when we send the cushioning region distance ϵ to zero. To see why this is the case, let us compute another quantity that diverges in the same way, namely the fluctuations of the number of fermions in $[-a, a]$, given by

$$\mathrm{Var}(Q) = -\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\log\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi:Q:}\rho]\Big|_{\phi=0} = \frac{1}{2\pi^2}\log\ell_{[-a,a]}.$$

This allows to rewrite

$$\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\phi(:Q:-q)}\rho] = \mathrm{e}^{-\mathrm{i}\phi q-\phi^2\,\mathrm{Var}(Q)/2}$$

and $\mathrm{Var}(Q)$ of course diverges for small cushioning due to infinite correlations near the boundary. This also means that the probability distributions from eqs. (2.212) and (2.220) are meaningless in QFT: the probability of finding *exactly* q fermions in the interval is zero, due to extremely high fluctuations. Of course, in any *physical* measurement, we can only probe our system with finite accuracy ϵ, which we can then take as our cushioning size. This means, that charge resolved relative entropy (2.210) will always depend on the spacial resolution of our measurements, while flux resolved relative entropy (2.219) can be defined without referencing any scale.

2.5.3 Symmetry resolved mutual information

Similarly to the what we did above, we can define a symmetry-aware version of mutual information (2.186) as well. To this end, let me introduce Alice and Bob, who are lab neighbours. One day they are in the lucky but rare situation that their department has some extra funding for joint projects, hence, they decide that it is time to collaborate. The aim of their project is to pin down the amount of entanglement of a quantum field in some quantum wire that stretches between their labs. As they both enjoyed an excellent education in QFT, they know that entanglement entropy is universally divergent, hence will yield larger and larger results at finer and finer scales of measurement.

In order to avoid this problem, they each mark down a portion of the wire in their respective labs and measure only the mutual information between the two. From now on, let us call these portions A and B. Unfortunately, this exact measurement is impossible – since the total particle number in the wire is conserved, they can only perform measurements within a single superselection sector, so they derive the following protocol:

Starting from a single state ρ of the wire, Alice and Bob perform a projective *joint* measurement of the particle number in AB. They repeat this procedure, until they achieve some previously chosen target value q. In the sense of eq. (2.211), the state after measuring the total fermion number $Q_{AB} = q$ is given by

$$\rho_q = \frac{\Pi_q \rho}{\mathrm{P}_\rho(q)}.$$

In order to obtain another state $\tilde{\rho}_q$ to which they might compare ρ_q, they repeat this procedure without joint measurements. This means that they measure the particle numbers in A and B *individually*, so that Alice only accesses the partial trace ρ_A and Bob only accesses ρ_B. After measuring, they meet in the coffee room to check if the total number is q again. If this is not the case, they discard their quantum states and try again. If it is, the resulting state $\tilde{\rho}_q$ takes the form

$$\tilde{\rho}_q = \frac{1}{\mathrm{P}_{\rho_A \otimes \rho_B}(q)} \sum_{q'} \Pi_{q'} \rho_A \otimes \Pi_{q-q'} \rho_B = \frac{\Pi_q(\rho_A \otimes \rho_B)}{\mathrm{P}_{\rho_A \otimes \rho_B}(q)} = (\rho_A \otimes \rho_B)_q.$$

As soon as this setup is done, they compare the states ρ_q and $\tilde{\rho}_q$, again with *possibly joint* measurements, to obtain the charge resolved relative entropy (2.210) in superselection sector q between them. Due to the similarity to eq. (2.186), we call the result[44]

$$I_q(A:B) := S_q(\rho_{AB} \| \rho_A \otimes \rho_B). \tag{2.231}$$

the *charge resolved mutual information* in sector q.

Of course, by virtue of eq. (2.218), you can express this in terms of the flux resolved mutual information

$$I_\phi(A:B) := S_\phi(\rho_{AB} \| \rho_A \otimes \rho_B). \tag{2.232}$$

Using the same methods as before, you can easily evaluate this for the vacuum state of a free fermion on the lightray. In order to obtain an analogous result to eq. (2.187), we choose the regions A, B as single intervals. As a first step,

note that

$$(\rho_A \otimes \rho_B)_\phi = (\rho_A)_\phi \otimes (\rho_B)_\phi,$$

so that eq. (2.219) yields

$$\begin{aligned} I_\phi(A:B) = {} & \operatorname{Tr}[(\rho_{AB})_\phi (K_A \otimes 1 + 1 \otimes K_B)] \\ & - \operatorname{Tr}[(\rho_A)_\phi K_A] - \operatorname{Tr}[(\rho_B)_\phi K_B] + \left[\frac{1}{6} - \left(\frac{\phi}{2\pi}\right)^2\right] \log \frac{\ell_A \ell_B}{\ell_{AB}}. \end{aligned}$$

Note that, in contrast to ordinary mutual information, the "modular Hamiltonian" difference does not cancel, since

$$[(\rho_{AB})_\phi]_n \neq (\rho_n)_\phi, \quad i = A, B$$

due to fluctuations of $Q_{A/B}$. Instead, we can use the above machinery (K_A and K_B are local) to find

$$\begin{aligned} & \operatorname{Tr}[(\rho_{AB})_\phi (K_A \otimes 1 + 1 \otimes K_B)] - \operatorname{Tr}[(\rho_A)_\phi K_A] - \operatorname{Tr}[(\rho_B)_\phi K_B] \\ & \quad = \sum_{n=A,B} \mathrm{i} \int_n \mathrm{d}x \, \frac{1}{Z_n'(x)} \lim_{y \to x} \Big[-\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2} \frac{Z_n''(x)}{Z_n'(x)} \Big] \\ & \qquad \times \left[\frac{1}{1 + \mathrm{e}^{-k_{AB} + \mathrm{i}\phi}}(y,x) - \frac{1}{1 + \mathrm{e}^{-k_n + \mathrm{i}\phi}}(y,x) \right] \end{aligned}$$

with

$$Z_n'(x) = \frac{1}{2\pi} \frac{b_n - a_n}{(x - a_n)(b_n - x)}, \quad Z_{AB}'(x) = Z_A'(x) + Z_B'(x).$$

Inserting the UV-expansion (2.228) (with $\beta \to \infty$), the above integral simplifies to

$$\begin{aligned} & \frac{\phi}{4\pi} \sum_{n=A,B} \int_n \mathrm{d}x \, \frac{1}{Z_n'(x)} \Big[\frac{Z_n''(x)}{Z_n'(x)} [Z_{AB}'(x) - Z_n'(x)] \\ & \qquad - [Z_{AB}''(x) - Z_n''(x)] - \phi \Big([Z_{AB}'(x)]^2 - [Z_n'(x)]^2 \Big) \Big] \end{aligned}$$

where the first two brackets cancel because

$$\begin{aligned} & \frac{Z_n''(x)}{[Z_n'(x)]^2} [Z_{AB}'(x) - Z_n'(x)] - \frac{1}{Z_n'(x)} [Z_{AB}''(x) - Z_n''(x)] \\ & \qquad = -\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{Z_n'(x)} [Z_{AB}'(x) - Z_n'(x)] \end{aligned}$$

is the total derivative of a function that vanishes at the boundaries. As a result

$$\begin{aligned}\mathrm{Tr}[(\rho_{AB})_\phi(K_A\otimes 1+1\otimes K_B)]-\mathrm{Tr}[(\rho_A)_\phi K_A]-\mathrm{Tr}[(\rho_B)_\phi K_B]\\ =-\pi\Big(\frac{\phi}{2\pi}\Big)^2\sum_{n=A,B}\int_n \mathrm{d}x\,\frac{[Z'_{AB}(x)]^2-[Z'_n(x)]^2}{Z'_n(x)},\end{aligned}$$

which can actually be evaluated analytically. Similar to the computation in eq. (2.188), let us take A, B with length l each and distance d from one another. For convenience, assume that they are placed as $A=[0,l]$ and $B=[d{+}l, d{+}2l]$. Then the above evaluates to

$$\Big(\frac{\phi}{2\pi}\Big)^2\Big(2-\Big[1-4\frac{d}{l}-2\Big(\frac{d}{l}\Big)^2\Big]\log\frac{d(d+2l)}{(d+l)^2}\Big),$$

so that[44]

$$I_\phi(A:B)=I(A:B)+\Big(\frac{\phi}{2\pi}\Big)^2 I^{(2)}(A:B) \tag{2.233}$$

with the regular mutual information $I(A:B)$ from eq. (2.188) and a flux dependent quadratic correction whose coefficient is given by

$$\begin{aligned}I^{(2)}(A:B)&=2\Big(1-\frac{d}{l}\Big[2+\frac{d}{l}\Big]\log\frac{(1+l/d)^2}{1+2l/d}\Big)\\ &\sim\begin{cases}2-4(d/l)\log l/d & \text{for } d\ll l\\ (l/d)^2 & \text{for } d\gg l.\end{cases}\end{aligned} \tag{2.234}$$

A plot of eq. (2.234) is given in fig. 2.23. Note that this is finite, even in the limit $d/l\to 0$, where $I(A:B)$ diverges logarithmically. Furthermore, eq. (2.234) is strictly positive, indicating that adding an Aharonov-Bohm flux $\mathrm{e}^{\mathrm{i}\phi Q_{AB}}$ will always *increase* the amount of correlations between A and B. We can also apply this line of reasoning to the boundary CFT solution (2.205) to find that the two chiralities get more entangled when adding a flux $\mathrm{e}^{\mathrm{i}\phi(Q_+ + Q_-)}$. In particular, eq. (2.234) tells us that these additional correlations strictly decrease with increasing distance from the boundary.

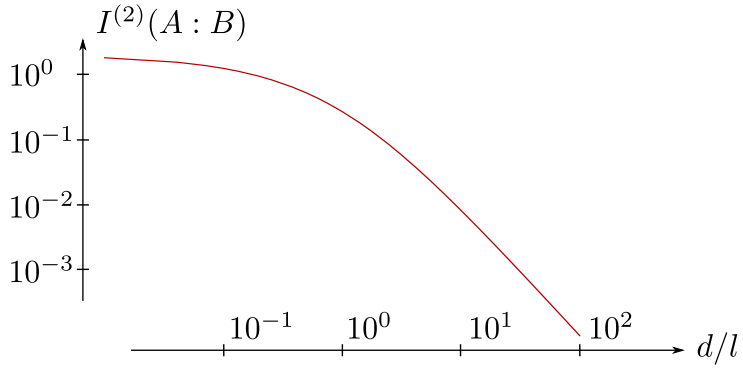


Figure 2.23: Coefficient of the flux dependent correction (solid red) to the flux resolved mutual information between two intervals of size l and distance d, as given by eq. (2.234). Note that it is finite as $d/l \to 0$, in contrast to the flux independent piece (2.187).

3 Conclusion

In this thesis, I studied in details the entanglement properties of free fermions. As shown, specialising to a well understood theory allows to obtain very strong results regarding the associated modular data – much stronger in fact than the Bisognano-Wichmann theorem (2.29) in general QFT or even the Hislop-Longo theorem (2.35) in CFT, since we can deal with almost arbitrary regions, thermal states, as well as states containing zero-mode excitations. After the admittedly computationally heavy main body of this thesis, I would like to provide a summary and physical interpretation of the derived results and give an outlook on possible future extensions of the work in the following two final sections.

3.1 Summary and interpretation

Let me begin by summarising the results from sections 2.4 and 2.5: Using the mathematical machinery derived in section 2.3, I presented how to obtain the dynamical description of entanglement, i.e., the modular Hamiltonian, modular flow, and modular correlation function. Additionally, we also derived the static description of entanglement in terms of (Rényi) entanglement entropy and associated quantities such as relative entropy and mutual information. All of this was done in $1+1$ dimensions for free massless fermions, either in the vacuum state or at finite temperature. Since both chiralities decouple—this can be seen either as a consequence of conformal symmetry,[118] or directly from the Hamiltonian, as derived in eqs. (2.64) and (2.65)—, the discussion is effectively reduced to a two one dimensional problems, one on the left and one on the right pointing lightray. Each of the considered states was studied on both the full and compactified lightray, corresponding in usual coordinates to a free massless Dirac fermion on the line or the circle, respectively. In the latter case, there are two possible spin structures (2.73), corresponding to periodic and anti-periodic boundary conditions in spacial direction. An addition, in section 2.4.7, I presented the discussion of the vacuum state on the half plane with a boundary at $x^1 = 0$. In this setting, the chiralities can no longer be independent, due to the conformal boundary condition (2.99) with solutions (2.100) which is needed in order for conservation laws to still

be valid, as was discussed in section 2.3.4. Finally, in section 2.5, I introduced charge (2.210) and flux resolved relative entropy (2.219) for general quantum systems and QFTs – an entanglement measure that is adapted for the study of systems with conserved charges (and associated superselection rules). I then computed flux resolved relative entropy of a free fermion on the line, between thermal states of different temperatures.

Many of the results in section 2.4 were previously known in some form.[107–110,144] The first achievement of this thesis is putting them in a coherent framework. The second achievement of this thesis is generalising the existent results: Chronologically, the first new result was the modular Hamiltonian (2.162) at finite temperature on the circle, as derived by our group[41] and others[112,113] nearly simultaneously. The other dynamical aspects of entanglement in this setting (modular flow (2.150) and modular correlation function (2.156)) were also discovered by our group.[43] Furthermore, the formula (2.194) for entanglement entropy in this setting was also achieved in a head-to-head race between our[42] and another group.[114] The non-trivial analytic continuations (2.189), (2.192), and (2.193) of the Rényi entropies on the circle are also novel and first published in this thesis, as is the extension to spacetimes with a boundary in section 2.4.7. Finally, most of the results in section 2.5 are completely new, albeit inspired by previous works on charge[150] and flux resolved (non-relative) entanglement entropy.[149,151,152]

Now, let us proceed to stating the core insights of this thesis and their physical interpretation: Starting with the modular correlation function (2.151), valid in the vacuum and thermal states on the line, as well as in the non-degenerate vacuum state on the circle, we found that it has discretely distributed poles at the solutions of the eq. (2.137),

$$t + Z(x) - Z(y) = 0,$$

where the precise form of the function Z depends on the boundary conditions but it is always strictly increasing in each component of the entangling region (in lightray coordinates). This means, that for fixed t, y there is one pole at the solution $x = x_n(t, y)$ in the n-th component, hinting at a multi-local modular flow along the trajectories $t \to x_n(t, y)$, which are rooted at $x_n(0, y) = y$ for some n. Indeed, owing to the KMS condition (2.24), it is possible to recover the modular flow of the field operators from the modular correlation function via eq. (2.51) and we found in eq. (2.138) that

$$\sigma_t(\psi^*(y)) = -2\mathrm{i}\sinh(\pi t)\sum_n \frac{G(x_n(t,y)-y)}{Z'(x_n)}\psi^*(x_n(t,y))$$

is a multi-local operator for any $t \neq 0$, in the sense that it is a superposition of one localised field operator per interval. In parallel to how the universal results (2.29) and (2.35) recover the fundamental geometric symmetries of QFT and CFT, respectively, the above implies that the massless Fermi net (2.43) in two dimensions has an additional symmetry which turns our to be multi-local. Indeed, this symmetry can be verified explicitly for the net, as was done by Rehren and Tedesco.[131] Furthermore, the above result implies that the degenerate vacuum state on both the line and circle are invariant with respect to this symmetry. To be explicit, this means that the algebra of one fermion in n intervals is isomorphic to the algebra of n fermions in a single interval, where the isomorphism is multi-local (maps local fields to multi-local fields), vacuum preserving, and intertwines the corresponding modular flows. Since the modular flow of a single interval is always geometric due to eq. (2.35), this is the reason for the "almost-geometric" form of eq. (2.138). As shown, a variant of this statement also has to hold for the thermal state on the line.

On the other hand, on the circle at finite temperature, we find an entirely different picture: Here, modular flow couples the field operators at solutions of eq. (2.149),

$$x - y + \frac{\beta}{|V|}[t + Z(x) - Z(y)] + k = 0, \quad k \in \mathbb{Z},$$

where β is the inverse temperature and $|V|$ is the total size of the entangling region. The qualitative difference between this equation and eq. (2.137) is that there is one solution per interval and value of $k \in \mathbb{Z}$, as sketched in fig. 2.19. In particular, this implies that there is an *infinite* discrete set of trajectories per interval, all of which are coupled by modular flow, as sketched in fig. 3.1. At very high temperatures, all of these trajectories but the local one are pushed towards the boundary of the entangling region, where their amplitude in eq. (2.150) diminishes exponentially. The resulting flow is local and linear, indicating that the reduced density matrix of a thermal state at high temperature "looks like" a thermal state at inverse temperature $\beta/|V|$. This phenomenon is, in fact, universal[132] and has a nice physical interpretation: At high temperatures, thermal fluctuate dominate, hence the effects of entanglement are suppressed. The reduction of effective temperature $\beta \to \beta/|V|$ happens because the subregion V has a smaller heat capacity than the full system. The limit of low temperatures is particularly interesting. As $\beta \to \infty$, the solutions of eq. (2.149) "condense" in the entangling region. While they cancel for anti-periodic boundary conditions due to the alternating sign in eq. (2.150), they recover a Riemann sum for periodic boundary conditions, resulting in a continuously non-local contribution (2.146) to the flow, which can be inter-

preted as arising from the degenerate ground state.[109] The question as to why this infinite multi-locality is there in the first place, is also very interesting: From a mathematical point of view, one could argue that this is a result of the analytic structure of the corresponding resolvent – as I argued, the problem is highly constrained and, in fact, a local solution that has two periodicities (boundary conditions and KMS condition) can not exist by simple arguments from complex analysis. From a physical point of view, the multi-locality hints at yet another hidden symmetry of the Fermi net, which, as far as I know, is yet to be discovered.

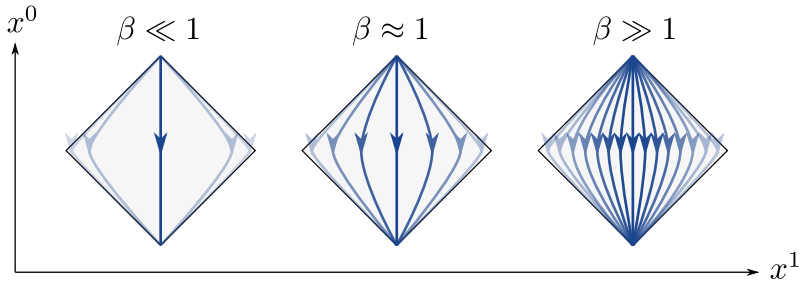


Figure 3.1: Trajectories (solid blue) of the modular flow (2.150) in the thermal state with periodic boundary conditions inside a single double cone, for a field initially localised in the centre of the double cone. The amplitude of the multi-local contributions is visualised by the opacity of the trajectories. At high temperatures (left) all trajectories but the local one are pushed towards the boundary, while at low temperatures (right), they "condense" to a continuous non-locality, smeared everywhere inside the double cone.

Proceeding to the novel solutions (2.189), (2.192), and(2.193) for the Rényi entanglement entropies for a degenerate ground state and on the circle at finite temperature, respectively, we see that these formulae present analytic continuations, valid for general $\Re(\alpha) > 0$, of previous results from CFT[144] that were only known for integer values ≥ 2. For these integer values, the Rényi entanglement entropies were originally derived via twist operator methods, as is customary in two dimensional CFT,[143] in order to then apply the replica trick[139] and obtain the von-Neumann entanglement entropy as the limit $\alpha \to 1$, for which the analytic continuation is essential. As far as I know, this was one of only a few times where the analytic continuation of the Rényi entropies was not obvious from their integer values. Nevertheless, as shown in section 2.4.6, it is possible to retrieve closed form expressions for $\alpha = 2, 3, \ldots$ from eqs. (2.189) and (2.193), matching the previously known results.

The next important result concerns free fermion entanglement in spacetimes with a boundary in section 2.4.7. Here, I demonstrated in eq. (2.202) that the modular flow associated to a single double cone is still local, yet it does not coincide with a conformal transformation. This is because the presence of a boundary breaks full conformal invariance. The fact that the flow is still local can be traced back to the aforementioned addition symmetry that is particular to the free fermion CFT: This multi-local symmetry is actually local in the boundary theory because both chiralities are identified. As a result, you can recover full conformal covariance from the modular flow of regions far away from the boundary, as shown in fig. 3.2. Futhermore, in eq. (2.205), I presented the mutual information between the two chiralities inside a connected region. The fact that this scales as an inverse square law at large distance from the boundary shows quantitatively that entanglement between the two chiralities is largest close to the boundary. The fact that

$$I(+:-) = S(\text{no boundary}) - S(\text{boundary}) > 0$$

can also be reinterpreted as a monogamy statement about entanglement: Since both chiralities are entangled with each other inside the region, they can not be entangled with the complement as much as they could if they were independent.

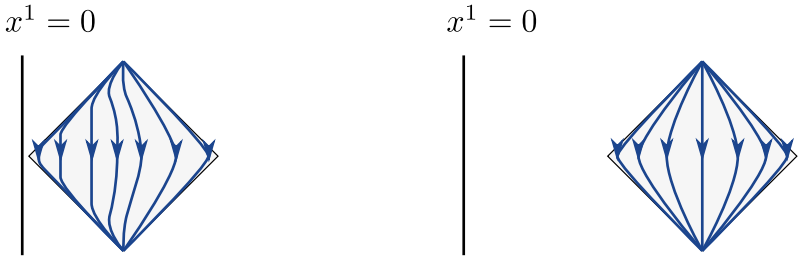


Figure 3.2: Trajectories (solid dark blue) of the modular flow (2.202) of free fermions in the vacuum state with a boundary at $x^1 = 0$. For regions far away from the boundary (right) modular flow recovers the Hislop-Longo result (2.35) and, with it, conformal covariance. For regions very close to the boundary (left) the flow is still local, albeit distorted.

The final result that I would like to discuss concerns section 2.5, where I introduced and studied charge and flux resolved relative entropy in a general context. The aim there was to derive a version of relative entropy that measures the *physical* distinguishability of different states in a setting where there are superselection rules, i.e., where the total Hilbert space of the theory decom-

poses in such a way that we may not measure transition amplitudes between different sectors. Physically, this means that we are dealing with a theory with a *charge* operator, which is preserved under all physical operations, i.e, *central*. Thus, in order to obtain a reasonable result for the distinguishability between two states, it makes sense to first project them onto a sector where this charge takes on a fixed value, as done in eq. (2.210). To evaluate the resulting expression, I then introduced flux resolved relative entropy in eq. (2.219) as an auxiliary quantity, which basically measures the distinguishability between "states" that contain an additional Aharonov-Bohm flux (here, I wrote "states" because the resulting corresponding are not positive definite). These "states" behave somewhat like a local *grand canonical ensemble*, in the sense that they allow to fix the value of the conserved charge. Subsequently, I worked out exact results for free fermions in eqs. (2.229), (2.230), (2.233) and (2.234), which proved one important point: The flux resolved quantities are actually more than auxiliary quantities – in fact, they can be defined and computed in the absence of a cutoff, which is impossible for the corresponding charge resolved quantities. This is because, without a cutoff, the charge fluctuations diverge because of infinite entanglement close to the boundary of the entangling region. As a result, the projection of the state to a sector of constant charge is nonsensical, while we can still fix the *expectation value* of the charge operator by considering the grand canonical ensemble. In addition, I showed in eq. (2.234) that the inclusion of an Aharonov-Bohm flux in the non-degenerate fermionic vacuum always *increases* correlations between disconnected subregions, which is a structural result about the details of vacuum correlations, as it is a trace of their uneven distribution over the superselection sectors.

3.2 Outlook

I would like to conclude with an outlook on current work in progress and possible future works. Currently, I am working on two projects with Ignacio Reyes, parts of which were already included in this thesis.

In the first of these projects,[44] we aim to extend the results from section 2.5 about relative entropy in theories with superselection rules. In particular, we are trying to extend the results (2.229) and (2.230) for relative entropy between two thermal states to arbitrary conformal field theories with additional symmetries. Since the flux resolved entropies were already derived in general,[151] the computation boils down to computing the expectation value of the modular Hamiltonian in the thermal state with an inserted flux. Luckily, this should be possible within the framework of two dimensional CFT, as there the modular Hamiltonian is universally given by the Unruh type expres-

sion (2.159), due to the work by Hislop and Longo.[39] Furthermore, the fact that flux resolved relative entropy is well defined in the absence of a cutoff, suggests that it might be possible to rigorously define a flux resolved variant of the Connes cocycle (2.175), which would allow for the study of entanglement in the presence of superselection sectors with mathematically sound methods.

The second work in progress[45] aims to extend the results of section 2.4.7 on entanglement in the presence of a boundary. In particular, we would like to understand the influence of *accelerating* boundaries on modular flow and mutual information, as these can be viewed as a natural toy model for Hawking radiation.[155] To this end, we first need to solve the conformal boundary condition (2.99) for a moving boundary in order to obtain the correct boundary conditions of the fields. After that, we will have to solve the integral equation (2.61) for the resulting "vacuum" propagator, although the term vacuum is a little bit difficult here because virtual particles are constantly created due to the accelerating boundary. Additionally, in the final stages of preparing this thesis, another group published a paper[156] discussing modular flow with static boundaries. I will be extremely interesting to study the connections between their work and the results in section 2.4.7.

Continuing to possible future directions, the first possibility would be to extend the results of section 2.4 to excited states in the free fermion CFT, since the machinery in section 2.3 applies to them as well and the corresponding resolvents are known.[108] From what you have seen in section 2.4, it is to be expected that you find additional non-local terms there. Similarly, progress could also be made in higher dimensions or for massive theories, if an analytic solution of the corresponding integral equation (2.61) can be found. Success in this direction is possible, as was already proven by Casini and Huerta,[107] who obtained perturbative results in the small mass expansion.

Finally, the in my opinion most interesting extension of the present results would be to consider free theories of higher spin fields, with possible applications in holography. This might actually be possible because the free higher spin Haag-Kastler nets embed into the Fermi net.[157,158] Thus, to obtain the correct modular data of these theories, it is sufficient to make this embedding explicit due to Takesaki's theorem.[61]

Acknowledgements

Throughout my time as a PhD student and during the preparation of this thesis, I was always supported by a lot of people, for which I am immensely grateful. First of all, I gratefully acknowledge the DFG project HI 744/9-1, for contributing to the funding of my PhD position at the Julius-Maximilians-Universität Würzburg.

I am also very happy to have Haye Hinrichsen as my supervisor: His lectures on statistical mechanics sparked my interest in information theory and, starting with my bachelor's thesis, he has been guiding me through the world of quantum information and quantum field theory. I am very thankful for his patience and guidance – whenever I had a question or needed any help, I found answers in his office. Including my time as a master's student, I have been under his supervision at the chair of theoretical physics III for over five years now and I could not imagine a better place.

I would also like to thank many others for inspiring discussions: Johanna Erdmenger, Souvik Banerjee, and Ro Jefferson taught me about the rôle of entanglement in the AdS/CFT duality. René Meyer explained many features of CFT to me and his incredibly broad knowledge of high energy physics always guarantees a different point of view on things. The lectures by Thorsten Ohl about the algebraic structure of quantum physics where my first contact with AQFT, his lectures on classical mechanics sparked my interest in theoretical physics. I am thankful that I could always (and very often did) come to his office for discussions on algebra and physics. Christian Northe guided me to the best literature on boundary CFT and modular functions and discussions with him led to some of the explanations in section 2.3.4. I had many helpful discussions with Marius Gerbershagen about modular theory and its physical interpretations, which undoubtedly inspired my discussion of the split property in section 2.4.6. Discussions with Stefan Hollands and Rainer Verch helped me understand the significance of the KMS condition during my visit at the Universität Leipzig. Talking to Martin Ammon at the Friedrich-Schiller-Universität Jena about generalisations of holography was inspiring. I am particularly thankful to him for explaining the structure of the academic world to me and giving me the chance to present the information theoretical interpretation of entanglement to high energy physicists. My talks in Jena were the origin of the presentation in section 1.2.3. In addition, a great amount

of the explanations in this thesis originate in my answers to great questions of my students, in particular Christian Simon and Christine Bott – I am very grateful to them.

Finally, the most inspiring discussions I had with my friend Ignacio Reyes: It was his incredible physical intuition that started our work on free fermion entanglement in the first place. I am indebted to him for countless hours of joyful collaboration which have not ended yet and, as I hope, will not end for another while. I am also grateful to him for inviting me to the Max-Planck-Institut für Gravitationsphysik Potsdam (Albert-Einstein-Institut), where I could present my work and meet Michal Heller and Ro Jefferson.

In addition, I am happy to have had the colleagues I had at the chair: The warm atmosphere that we have can not be taken for granted and I always enjoyed our common lunch breaks and meetings in the coffee room. The same of course is true for Björn Trauzettel and the rest of the chair of theoretical physics IV, who have always been supportive and great colleagues. Additionally, I am thankful to Nelia Meyer, without whom I (and probably many others at our chair) would have been hopelessly lost in paperwork.

I would also like to thank my parents for their great support, encouragement, and interest in my work. Finally, I would not have been able to finish my thesis without the support, patience, and love of my wonderful wife Larissa. I am forever grateful to her.

Bibliography

[1] Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift fur Physik*, **43**(3-4):172–198 (1927). DOI:10.1007/BF01397280.

[2] Von Neumann, J. *Mathematische Grundlagen der Quantenmechanik*. Springer-Verlag, Berlin, Germany, 2 edition (1996).

[3] Heisenberg, W. Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen. *Zeitschrift fur Physik*, **33**(1):879–893 (1925). DOI:10.1007/BF01328377.

[4] Einstein, A., Podolsky, B. and Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, **47**:777–780 (1935). DOI:10.1103/PhysRev.47.777.

[5] Schrödinger, E. Discussion of probability relations between separated systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, **31**(4):555–563 (1935). DOI:10.1017/S0305004100013554.

[6] Bell, J. On the Einstein-Podolsky-Rosen paradox. *Physics Physique Fizika*, **1**:195–200 (1964). DOI:10.1103/PhysicsPhysiqueFizika.1.195.

[7] Freedman, S. J. and Clauser, J. F. Experimental Test of Local Hidden-Variable Theories. *Phys. Rev. Lett.*, **28**(14):938–941 (1972). DOI:10.1103/PhysRevLett.28.938.

[8] Nielsen, M. and Chuang, I. *Quantum Computation and Quantum Information*. Cambridge University Press, United Kingdom, 2 edition (2003).

[9] Schrödinger, E. Quantisierung als Eigenwertproblem. *Annalen Phys.*, **384**(4):361–376 (1926). DOI:10.1002/andp.19263840404.

[10] Born, M., Heisenberg, W. and Jordan, P. Zur Quantenmechanik. II. *Zeitschrift fur Physik*, **35**(8-9):557–615 (1926). DOI:10.1007/BF01379806.

[11] Klein, O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift fur Physik*, **37**(12):895–906 (1926). DOI:10.1007/BF01397481.

[12] Gordon, W. Der Comptoneffekt nach der Schrödingerschen Theorie. *Zeitschrift fur Physik*, **40**(1-2):117–133 (1926). DOI:10.1007/BF01390840.

[13] Dirac, P. The Quantum Theory of the Emission and Absorption of Radiation. *Proceedings of the Royal Society of London Series A*, **114**(767):243–265 (1927). DOI:10.1098/rspa.1927.0039.

[14] Dirac, P. A Theory of Electrons and Protons. *Proc. Roy. Soc. Lond. A*, **126**(801):360–365 (1930). DOI:10.1098/rspa.1930.0013.

[15] Reeh, H. and Schlieder, S. Bemerkungen zur unitäräquivalenz von lorentzinvarianten Feldern. *Nuovo Cim.*, **22**(5):1051–1068 (1961). DOI:10.1007/BF02787889.

[16] Hawking, S. Particle Creation by Black Holes. *Commun. Math. Phys.*, **43**:199–220 (1975). DOI:10.1007/BF02345020. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[17] Unruh, W. G. Notes on black hole evaporation. *Phys. Rev. D*, **14**:870 (1976). DOI:10.1103/PhysRevD.14.870.

[18] Maldacena, J. M. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, **38**:1113–1133 (1999). DOI:10.1023/A:1026654312961, 10.4310/ATMP.1998.v2.n2.a1. [Adv. Theor. Math. Phys.2,231(1998)].

[19] Ryu, S. and Takayanagi, T. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, **96**:181602 (2006). DOI:10.1103/PhysRevLett.96.181602.

[20] Ryu, S. and Takayanagi, T. Aspects of Holographic Entanglement Entropy. *JHEP*, **08**:045 (2006). DOI:10.1088/1126-6708/2006/08/045.

[21] Dong, X. The Gravity Dual of Renyi Entropy. *Nature Commun.*, **7**:12472 (2016). DOI:10.1038/ncomms12472.

[22] Casini, H., Huerta, M. and Myers, R. C. Towards a derivation of holographic entanglement entropy. *JHEP*, **05**:036 (2011). DOI:10.1007/JHEP05(2011)036.

[23] Blanco, D. D., Casini, H., Hung, L.-Y. and Myers, R. C. Relative Entropy and Holography. *JHEP*, **08**:060 (2013). DOI:10.1007/JHEP08(2013)060.

[24] Jafferis, D. L. and Suh, S. J. The Gravity Duals of Modular Hamiltonians. *JHEP*, **09**:068 (2016). DOI:10.1007/JHEP09(2016)068.

[25] Jafferis, D. L., Lewkowycz, A., Maldacena, J. and Suh, S. J. Relative entropy equals bulk relative entropy. *JHEP*, **06**:004 (2016). DOI:10.1007/JHEP06(2016)004.

[26] Lashkari, N., McDermott, M. B. and Van Raamsdonk, M. Gravitational dynamics from entanglement 'thermodynamics'. *JHEP*, **04**:195 (2014). DOI:10.1007/JHEP04(2014)195.

[27] Koeller, J., Leichenauer, S., Levine, A. and Shahbazi-Moghaddam, A. Local modular hamiltonians from the quantum null energy condition. *Phys. Rev. D*, **97**(6):065011 (2018). DOI:10.1103/PhysRevD.97.065011.

[28] Czech, B., Lamprou, L., Mccandlish, S. and Sully, J. Modular Berry Connection for Entangled Subregions in AdS/CFT. *Phys. Rev. Lett.*, **120**(9):091601 (2018). DOI:10.1103/PhysRevLett.120.091601.

[29] Belin, A., Iqbal, N. and Lokhande, S. F. Bulk entanglement entropy in perturbative excited states. *SciPost Phys.*, **5**(3):024 (2018). DOI:10.21468/SciPostPhys.5.3.024.

[30] Abt, R. and Erdmenger, J. Properties of Modular Hamiltonians on Entanglement Plateaux. *JHEP*, **11**:2 (2018). DOI:10.1007/JHEP11(2018)002.

[31] Faulkner, T., Li, M. and Wang, H. A modular toolkit for bulk reconstruction. *JHEP*, **04**:119 (2019). DOI:10.1007/JHEP04(2019)119.

[32] Jefferson, R. Comments on black hole interiors and modular inclusions. *SciPost Phys.*, **6**:042 (2019). DOI:10.21468/SciPostPhys.6.4.042.

[33] Czech, B., De Boer, J., Ge, D. and Lamprou, L. A modular sewing kit for entanglement wedges. *JHEP*, **11**:094 (2019). DOI:10.1007/JHEP11(2019)094.

[34] De Boer, J. and Lamprou, L. Holographic Order from Modular Chaos. *JHEP*, **06**:024 (2020). DOI:10.1007/JHEP06(2020)024.

[35] Arias, R., Botta-Cantcheff, M., Martinez, P. J. and Zarate, J. F. Modular Hamiltonian for (holographic) excited states. *arXiv preprint* (2020).

[36] Van Raamsdonk, M. Building up spacetime with quantum entanglement. *Gen. Rel. Grav.*, **42**:2323–2329 (2010). DOI:10.1142/S0218271810018529.

[37] Maldacena, J. and Susskind, L. Cool horizons for entangled black holes. *Fortsch. Phys.*, **61**:781–811 (2013). DOI:10.1002/prop.201300020.

[38] Bisognano, J. J. and Wichmann, E. H. On the Duality Condition for Quantum Fields. *J. Math. Phys.*, **17**:303–321 (1976). DOI:10.1063/1.522898.

[39] Hislop, P. D. and Longo, R. Modular structure of the local algebras associated with the free massless scalar field theory. *Comm. Math. Phys.*, **84**(1):71–85 (1982).

[40] 't Hooft, G. *50 Years of Yang-Mills theory*. World Scientific, Singapore (2005). URL: `https://cds.cern.ch/record/780270`.

[41] Fries, P. and Reyes, I. A. Entanglement Spectrum of Chiral Fermions on the Torus. *Phys. Rev. Lett.*, **123**(21):211603 (2019). DOI:10.1103/PhysRevLett.123.211603.

[42] Fries, P. and Reyes, I. A. Entanglement and relative entropy of a chiral fermion on the torus. *Phys. Rev. D*, **100**(10):105015 (2019). DOI:10.1103/PhysRevD.100.105015.

[43] Erdmenger, J., Fries, P., Reyes, I. A. and Simon, C. P. Resolving modular flow: a toolkit for free fermions. *accepted for publication in JHEP* (2020).

[44] Fries, P. and Reyes, I. A. Relative entropy and symmetry in quantum field theory. *in preparation* (2020).

[45] Fries, P. and Reyes, I. A. Modular flow in with a boundary. *in preparation* (2020).

[46] Witten, E. APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory. *Rev. Mod. Phys.*, **90**(4):045003 (2018). DOI:10.1103/RevModPhys.90.045003.

[47] Fries, P., Reyes, I., Erdmenger, J. and Hinrichsen, H. Renormalization of lattice field theories with infinite-range wavelets. *J. Stat. Mech.*, **1906**(6):064001 (2019). DOI:10.1088/1742-5468/ab14d8.

[48] Epple, H., Fries, P. and Hinrichsen, H. Implementing the sine transform of fermionic modes as a tensor network. *Phys. Rev. A*, **96**:032308 (2017). DOI:10.1103/PhysRevA.96.032308.

[49] Dirac, P. A. M. *The Principles of Quantum Mechanics*. Oxford University Press, United Kingdom, 4 edition (1988).

[50] Arnol'd, V. I. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, 2 edition (1989).

[51] Shankar, R. *Principles of Quantum Mechanics*. Springer-Verlag, United States, 2 edition (1994).

[52] Mackey, G. W. *The Mathematical Foundations of Quantum Mechanics*. W.A. Benjamin, New York (1963).

[53] Ludwig, G. *Foundations of Quantum Mechanics*. Springer-Verlag, Berlin Heidelberg, 1 edition (1985).

[54] Kraus, K. States, effects, and operations. In A. Böhm, J. D. Dollard and W. W. H., editors, *Fundamental Notions of Quantum Theory*. Springer-Verlag, Berlin, Germany (1983).

[55] Janotta, P. and Hinrichsen, H. Generalized probability theories: what determines the structure of quantum theory? *J. Phys. A*, **47**:323001 (2014). DOI:10.1088/1751-8113/47/32/323001.

[56] Greiner, W., Neise, L. and Stöcker, H. *Thermodynamics and Statistical Mechanics*. Springer-Verlag, New York, 1 edition (1995).

[57] Papadodimas, K. and Raju, S. An Infalling Observer in AdS/CFT. *JHEP*, **10**:212 (2013). DOI:10.1007/JHEP10(2013)212.

[58] Papadodimas, K. and Raju, S. State-Dependent Bulk-Boundary Maps and Black Hole Complementarity. *Phys. Rev. D*, **89**(8):086010 (2014). DOI:10.1103/PhysRevD.89.086010.

[59] Hawking, S. W. Euclidean Quantum Gravity. *NATO Sci. Ser. B*, **44**:145 (1979).

[60] Verlinde, E. P. Black hole evaporation and complementarity. In *11th International Conference on Mathematical Physics (ICMP-11) (Satellite colloquia: New Problems in the General Theory of Fields and Particles, Paris, France, 25-28 Jul 1994)*, pages 668–673 (1994).

[61] Takesaki, M. *Theory of Operator Algebras I-III*. Springer, Berlin, Germany (2003).

[62] Henneaux, M. and Teitelboim, C. *Quantization of gauge systems*. Princeton University Press, Princeton, United States (1992). ISBN 978-0-691-03769-1.

[63] Wigner, E. P. *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*. Vieweg+Teubner, Germany, 1 edition (1931).

[64] Haag, R. *Local quantum physics: Fields, particles, algebras*. Texts and monographs in physics. Springer, Berlin, Germany (1992).

[65] Davies, E. *Quantum Theory of Open Systems*. Academic Press, Cambridge, United States (1976).

[66] Schlosshauer, M. A. *Decoherence and the Quantum-To-Classical Transition*. Springer-Verlag, Berlin, Germany, 1 edition (2007).

[67] Gemmer, J., Michel, M. and Gúnter, M. *Quantum Thermodynamics*. Springer-Verlag, Berlin, Germany, 2 edition (2007).

[68] Haag, R. and Kastler, D. An Algebraic approach to quantum field theory. *J. Math. Phys.*, **5**:848–861 (1964). DOI:10.1063/1.1704187.

[69] Fewster, C. J. and Verch, R. Algebraic quantum field theory in curved spacetimes. In R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason, editors, *Advances in Algebraic Quantum Field Theory*, pages 125–189. Springer-International, Switzerland (2015). DOI:10.1007/978-3-319-21353-8_4.

[70] Benini, M., Dappiaggi, C. and Schenkel, A. Algebraic quantum field theory on spacetimes with timelike boundary. *Annales Henri Poincare*, **19**(8):2401–2433 (2018). DOI:10.1007/s00023-018-0687-1.

[71] Brunetti, R., Dappiaggi, C., Fredenhagen, K. and Yngvason, J., editors. *Advances in algebraic quantum field theory*. Mathematical Physics Studies. Springer (2015). ISBN 978-3-319-21352-1, 978-3-319-21353-8. DOI:10.1007/978-3-319-21353-8.

[72] Brattelli, O. and Robinson, D. W. *Operator Algebras and Quantum Statistical Mechanics I-II*. Springer-Verlag, Berlin, Germany, 2 edition (1997).

[73] Balachandran, A., Govindarajan, T., de Queiroz, A. R. and Reyes-Lega, A. Algebraic Approach to Entanglement and Entropy. *Phys. Rev. A*, **88**(2):022301 (2013). DOI:10.1103/PhysRevA.88.022301.

[74] Peskin, M. E. and Schroeder, D. V. *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA (1995). ISBN 978-0-201-50397-5.

[75] Weinberg, S. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press (2005). ISBN 978-0-521-67053-1, 978-0-511-25204-4.

[76] Bisognano, J. and Wichmann, E. On the Duality Condition for a Hermitian Scalar Field. *J. Math. Phys.*, **16**:985–1007 (1975). DOI:10.1063/1.522605.

[77] Takesaki, M. *Tomita's Theory of Modular Hilbert Algebras and its Applications*. Lecture Notes in Mathematics. Springer, Berlin, Germany (1970).

[78] Horodecki, M., Horodecki, P. and Horodecki, R. On the necessary and sufficient conditions for separability of mixed quantum states. *Phys. Lett. A*, **223**:1 (1996). DOI:10.1016/S0375-9601(96)00706-2.

[79] Ollivier, H. and Zurek, W. H. Introducing Quantum Discord. *Phys. Rev. Lett.*, **88**(1):017901 (2001). DOI:10.1103/PhysRevLett.88.017901.

[80] Bennett, C. H., Bernstein, H. J., Popescu, S. and Schumacher, B. Concentrating partial entanglement by local operations. *Phys. Rev. A*, **53**:2046–2052 (1996). DOI:10.1103/PhysRevA.53.2046.

[81] Verch, R. and Werner, R. F. Distillability and positivity of partial transposes in general quantum field systems. *Rev. Math. Phys.*, **17**:545–576 (2005). DOI:10.1142/S0129055X05002364.

[82] Rudin, W. *Principles of Mathematical Analysis*. International Series in Pure & Applied Mathematics. McGraw Hill, New York, 3 edition (1976).

[83] Rényi, A. *Probability Theory*. Dover Publications, United States (2007).

[84] Rudolph, O. Some properties of the computable cross-norm criterion for separability. *Phys. Rev. A*, **67**:032312 (2003). DOI:10.1103/PhysRevA.67.032312.

[85] Einstein, A., Born, M. and Born, H. *The Born-Einstein letters: correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955, with commentaries by Max Born.* Macmillan, London (1971).

[86] Vladimirov, V. S. *Methods of the Theory of Functions of Many Complex Variables.* M.I.T. Press, Cambridge, United States (1966).

[87] Borchers, H. On revolutionizing quantum field theory with Tomita's modular theory. *J. Math. Phys.*, **41**:3604–3673 (2000). DOI:10.1063/1.533323.

[88] Casini, H., Teste, E. and Torroba, G. Modular Hamiltonians on the null plane and the Markov property of the vacuum state. *J. Phys. A: Math. Gen.*, **50**(36):364001 (2017). DOI:10.1088/1751-8121/aa7eaa.

[89] Casini, H. Relative entropy and the Bekenstein bound. *Class. Quant. Grav.*, **25**:205021 (2008). DOI:10.1088/0264-9381/25/20/205021.

[90] Blanco, D. D. and Casini, H. Localization of Negative Energy and the Bekenstein Bound. *Phys. Rev. Lett.*, **111**(22):221601 (2013). DOI:10.1103/PhysRevLett.111.221601.

[91] Faulkner, T., Leigh, R. G., Parrikar, O. and Wang, H. Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition. *JHEP*, **09**:038 (2016). DOI:10.1007/JHEP09(2016)038.

[92] Balakrishnan, S., Faulkner, T., Khandker, Z. U. and Wang, H. A General Proof of the Quantum Null Energy Condition. *JHEP*, **09**:020 (2019). DOI:10.1007/JHEP09(2019)020.

[93] Blanco, D., Casini, H., Leston, M. and Rosso, F. Modular energy inequalities from relative entropy. *JHEP*, **01**:154 (2018). DOI:10.1007/JHEP01(2018)154.

[94] Ceyhan, F. and Faulkner, T. Recovering the QNEC from the ANEC. *Commun. Math. Phys.*, **377**(2):999–1045 (2020). DOI:10.1007/s00220-020-03751-y.

[95] Lashkari, N. Constraining Quantum Fields using Modular Theory. *JHEP*, **01**:059 (2019). DOI:10.1007/JHEP01(2019)059.

[96] Azeyanagi, T., Nishioka, T. and Takayanagi, T. Near Extremal Black Hole Entropy as Entanglement Entropy via AdS(2)/CFT(1). *Phys. Rev. D*, **77**:064005 (2008). DOI:10.1103/PhysRevD.77.064005.

[97] Faulkner, T., Guica, M., Hartman, T., Myers, R. C. and Van Raamsdonk, M. Gravitation from Entanglement in Holographic CFTs. *JHEP*, **03**:051 (2014). DOI:10.1007/JHEP03(2014)051.

[98] Sárosi, G. and Ugajin, T. Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields. *JHEP*, **01**:012 (2018). DOI:10.1007/JHEP01(2018)012.

[99] Blanco, D., Leston, M. and Pérez-Nadal, G. Gravity from entanglement for boundary subregions. *JHEP*, **18**:130 (2018). DOI:10.1007/JHEP06(2018)130.

[100] Rieffel, M. A. and van Daele, A. A bounded operator approach to tomita-takesaki theory. *Pacific J. Math.*, **69**(1):187–221 (1977).

[101] Kubo, R. Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems. *J. Phys. Soc. Jap.*, **12**:570–586 (1957). DOI:10.1143/JPSJ.12.570.

[102] Haag, R., Hugenholtz, N. M. and Winnink, M. On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.*, **5**(3):215–236 (1967).

[103] Cardy, J. and Tonni, E. Entanglement hamiltonians in two-dimensional conformal field theory. *J. Stat. Mech.*, **1612**(12):123103 (2016). DOI:10.1088/1742-5468/2016/12/123103.

[104] Araki, H. On quasifree states of car and bogoliubov automorphisms. *Publ. Res. Inst. Math. Sci. Kyoto*, **6**:385–442 (1971). DOI:10.2977/prims/1195193913.

[105] Peschel, I. Calculation of reduced density matrices from correlation functions. *J. Phys. A: Math. Gen.*, **36**(14):205–208 (2003). DOI:10.1088/0305-4470/36/14/101.

[106] Peschel, I. and Eisler, V. Reduced density matrices and entanglement entropy in free lattice models. *J. Phys. A: Math. Gen.*, **42**(50):504003 (2009). DOI:10.1088/1751-8113/42/50/504003.

[107] Casini, H. and Huerta, M. Reduced density matrix and internal dynamics for multicomponent regions. *Class. Quant. Grav.*, **26**:185005 (2009). DOI:10.1088/0264-9381/26/18/185005.

[108] Wong, G., Klich, I., Pando Zayas, L. A. and Vaman, D. Entanglement Temperature and Entanglement Entropy of Excited States. *JHEP*, **12**:020 (2013). DOI:10.1007/JHEP12(2013)020.

[109] Klich, I., Vaman, D. and Wong, G. Entanglement Hamiltonians for chiral fermions with zero modes. *Phys. Rev. Lett.*, **119**(12):120401 (2017). DOI:10.1103/PhysRevLett.119.120401.

[110] Klich, I., Vaman, D. and Wong, G. Entanglement Hamiltonians and entropy in 1+1D chiral fermion systems. *Phys. Rev.*, **B98**:035134 (2018). DOI:10.1103/PhysRevB.98.035134.

[111] Arias, R. E., Casini, H., Huerta, M. and Pontello, D. Entropy and modular Hamiltonian for a free chiral scalar in two intervals. *Phys. Rev. D*, **98**(12):125008 (2018). DOI:10.1103/PhysRevD.98.125008.

[112] iHollands, S. On the modular operator of mutli-component regions in chiral CFT. *arXiv preprint* (2019).

[113] Blanco, D. and Pérez-Nadal, G. Modular Hamiltonian of a chiral fermion on the torus. *Phys. Rev. D*, **100**(2):025003 (2019). DOI:10.1103/PhysRevD.100.025003.

[114] Blanco, D., Garbarz, A. and Pérez-Nadal, G. Entanglement of a chiral fermion on the torus. *JHEP*, **09**:076 (2019). DOI:10.1007/JHEP09(2019)076.

[115] Conway, J. B. *A Course in Functional Analysis*. Springer-Verlag, New York, United States, 1 edition (1985).

[116] Rudin, W. *Real and Complex Analysis*. International Series in Pure & Applied Mathematics. McGraw Hill, New York, 3 edition (1987).

[117] Muskhelishvili, N. I. *Singular Integral Equations*. Springer Netherlands, 1 edition (1958).

[118] Di Francesco, P., Mathieu, P. and Senechal, D. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York (1997). ISBN 9780387947853, 9781461274759. DOI:10.1007/978-1-4612-2256-9. URL: `http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997`.

[119] Hartshorne, R. *Algebraic Geometry*. Springer-Verlag, New York, 1 edition (1977).

[120] Whittaker, E. T. and Watson, G. N. *A Course of Modern Analysis*. Cambridge University Press, 4 edition (2009).

[121] Ginsparg, P. H. Applied Conformal Field Theory. In *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*, pages 1–168 (1988).

[122] Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers*. Oxford University Press, 6 edition (2008).

[123] Bott, C. *Freie Fermionen bei hohen Temparaturen*. Bachelor's thesis, Julius-Maximilians-Universität Würzburg (2020).

[124] Cardy, J. L. Conformal Invariance and Surface Critical Behavior. *Nucl. Phys. B*, **240**:514–532 (1984). DOI:10.1016/0550-3213(84)90241-4.

[125] Longo, R. and Rehren, K.-H. Local fields in boundary conformal QFT. *Rev. Math. Phys.*, **16**:909 (2004). DOI:10.1142/S0129055X04002163.

[126] Cardy, J. L. Boundary conformal field theory. *arXiv preprint* (2004).

[127] Blumenhagen, R. and Plauschinn, E. *Introduction to conformal field theory: with applications to String theory*, volume 779. Springer Verlag, Heidelberg, Germany (2009). DOI:10.1007/978-3-642-00450-6.

[128] Longo, R. and Rehren, K.-H. How to remove the boundary in CFT: An operator algebraic procedure. *Commun. Math. Phys.*, **285**:1165–1182 (2009). DOI:10.1007/s00220-008-0459-8.

[129] Polchinski, J. and Cai, Y. Consistency of Open Superstring Theories. *Nucl. Phys. B*, **296**:91–128 (1988). DOI:10.1016/0550-3213(88)90382-3.

[130] Ishibashi, N. The Boundary and Crosscap States in Conformal Field Theories. *Mod. Phys. Lett. A*, **4**:251 (1989). DOI:10.1142/S0217732389000320.

[131] Rehren, K.-H. and Tedesco, G. Multilocal fermionization. *Lett. Math. Phys.*, **103**:19–36 (2013). DOI:10.1007/s11005-012-0582-5.

[132] Bhattacharya, J., Nozaki, M., Takayanagi, T. and Ugajin, T. Thermodynamical Property of Entanglement Entropy for Excited States. *Phys. Rev. Lett.*, **110**(9):091602 (2013). DOI:10.1103/PhysRevLett.110.091602.

[133] Doplicher, S. and Longo, R. Standard and split inclusions of von Neumann algebras. *Invent. Math.*, **75**:493–536 (1984). DOI:10.1007/BF01388641.

[134] Buchholz, D., Fredenhagen, K. and D'Antoni, C. The Universal Structure of Local Algebras. *Commun. Math. Phys.*, **111**:123 (1987). DOI:10.1007/BF01239019.

[135] Buchholz, D. and Wichmann, E. H. Causal Independence and the Energy Level Density of States in Local Quantum Field Theory. *Commun. Math. Phys.*, **106**:321 (1986). DOI:10.1007/BF01454978.

[136] Buchholz, D., D'Antoni, C. and Longo, R. Nuclear Maps and Modular Structures. 2. Applications to Quantum Field Theory. *Commun. Math. Phys.*, **129**:115 (1990). DOI:10.1007/BF02096782.

[137] Fewster, C. J. The split property for quantum field theories in flat and curved spacetimes. *arXiv preprint* (2016).

[138] Longo, R. and Xu, F. Von Neumann Entropy in QFT. *arXiv preprint* (2019). DOI:10.s00220-020-03702-7.

[139] Calabrese, P. and Cardy, J. L. Entanglement entropy and quantum field theory. *J. Stat. Mech.*, **0406**:P06002 (2004). DOI:10.1088/1742-5468/2004/06/P06002.

[140] Araki, H. Inequalities in von neumann algebras. *Les rencontres physiciens-mathématiciens de Strasbourg -RCP25*, **22** (1975). Talk:1.

[141] Connes, A. Une classification des facteurs de type iii. *Annales scientifiques de l'École Normale Supérieure*, **4e série, 6**(2):133–252 (1973). DOI:10.24033/asens.1247.

[142] Araki, H. One-parameter family of radon-nikodym theorems for states of a von neumann algebra. *Publications of The Research Institute for Mathematical Sciences*, **10**:1–10 (1974).

[143] Calabrese, P. and Cardy, J. Entanglement entropy and conformal field theory. *J. Phys. A*, **42**:504005 (2009). DOI:10.1088/1751-8113/42/50/504005.

[144] Herzog, C. P. and Nishioka, T. Entanglement Entropy of a Massive Fermion on a Torus. *JHEP*, **03**:077 (2013). DOI:10.1007/JHEP03(2013)077.

[145] Doplicher, S., Haag, R. and Roberts, J. E. Fields, observables and gauge transformations I. *Commun. Math. Phys.*, **13**:1–23 (1969). DOI:10.1007/BF01645267.

[146] Doplicher, S., Haag, R. and Roberts, J. E. Fields, observables and gauge transformations II. *Commun. Math. Phys.*, **15**:173–200 (1969). DOI:10.1007/BF01645674.

[147] Casini, H., Huerta, M., Magán, J. M. and Pontello, D. Entanglement entropy and superselection sectors. Part I. Global symmetries. *JHEP*, **02**:014 (2020). DOI:10.1007/JHEP02(2020)014.

[148] Kullback, S. and Leibler, R. A. On information and sufficiency. *Ann. Math. Statist.*, **22**(1):79–86 (1951). DOI:10.1214/aoms/1177729694.

[149] Goldstein, M. and Sela, E. Symmetry-resolved entanglement in many-body systems. *Phys. Rev. Lett.*, **120**(20):200602 (2018). DOI:10.1103/PhysRevLett.120.200602.

[150] Laflorencie, N. and Rachel, S. Spin-resolved entanglement spectroscopy of critical spin chains and luttinger liquids. *Journal of Statistical Mechanics: Theory and Experiment*, **2014**(11):P11013 (2014). DOI:10.1088/1742-5468/2014/11/p11013.

[151] Bonsignori, R., Ruggiero, P. and Calabrese, P. Symmetry resolved entanglement in free fermionic systems. *J. Phys. A*, **52**(47):475302 (2019). DOI:10.1088/1751-8121/ab4b77.

[152] Fraenkel, S. and Goldstein, M. Symmetry resolved entanglement: Exact results in 1D and beyond. *J. Stat. Mech.*, **2003**(3):033106 (2020). DOI:10.1088/1742-5468/ab7753.

[153] Abramowitz, M. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables,*. Dover Publications, Inc., USA (1974). ISBN 0486612724.

[154] Zagier, D. The dilogarithm function. In P. Cartier, P. Moussa, B. Julia and P. Vanhove, editors, *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, pages 3–65. Springer Berlin Heidelberg, Berlin, Heidelberg (2007). ISBN 978-3-540-30308-4. DOI:10.1007/978-3-540-30308-4_1. URL: `https://doi.org/10.1007/978-3-540-30308-4_1`.

[155] Akal, I., Kusuki, Y., Shiba, N., Takayanagi, T. and Wei, Z. Entanglement entropy in holographic moving mirror and Page curve. *ArXiv preprint* (2020).

[156] Mintchev, M. and Tonni, E. Modular Hamiltonians for the massless Dirac field in the presence of a boundary. *ArXiv preprint* (2020).

[157] Tanimoto, Y. Construction of wedge-local nets of observables through Longo-Witten endomorphisms. *Commun. Math. Phys.*, **314**:443–469 (2012). DOI:10.1007/s00220-012-1462-7.

[158] Bischoff, M. and Tanimoto, Y. Construction of wedge-local nets of observables through Longo-Witten endomorphisms. II. *Commun. Math. Phys.*, **317**:667–695 (2013). DOI:10.1007/s00220-012-1593-x.