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ORIGINAL ARTICLE

A local input-to-state stability result w.r.t. attractors of nonlinear reaction–diffusion equations

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Abstract
We establish the local input-to-state stability of a large class of disturbed nonlinear reaction–diffusion equations w.r.t. the global attractor of the respective undisturbed system.

Keywords Local input-to-state stability · Global attractor · Nonlinear reaction–diffusion equations

1 Introduction

In this paper, we are concerned with disturbed nonlinear reaction–diffusion equations of the form

$$\begin{aligned} \partial_t y(t,\zeta) &= \Delta y(t,\zeta) + g(y(t,\zeta)) + h(\zeta)u(t) \quad (\zeta \in \Omega) \\ y(t,\zeta) &= 0 \quad (\zeta \in \partial\Omega) \end{aligned} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$, where $g \in C^1(\mathbb{R},\mathbb{R})$ and $h \in L^2(\Omega,\mathbb{R})$ and the disturbance u belongs to $\mathcal{U} := L^\infty([0,\infty),\mathbb{R})$. It is well known

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[24] that the corresponding undisturbed equation

$$\begin{aligned} \partial_t y(t,\zeta) &= \Delta y(t,\zeta) + g(y(t,\zeta)) \qquad (\zeta \in \Omega) \\ y(t,\zeta) &= 0 \qquad (\zeta \in \partial\Omega) \end{aligned} \tag{1.2}$$

has a unique global attractor $\Theta \subset X := L^2(\Omega, \mathbb{R})$ under suitable growth and upper-boundedness conditions on the nonlinearity g and its derivative g', respectively. As usual, a global attractor for (1.2) is defined to be a compact subset of X that is invariant and uniformly attractive for (1.2). Also, it can be shown [13] that the global attractor Θ of (1.2) is a stable set for (1.2).

What we show in this paper is that the disturbed reaction–diffusion equations (1.1) are locally input-to-state stable w.r.t. the global attractor Θ of the undisturbed equation (1.2). So, we show that there exist comparison functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and radii $r_{0x}, r_{0u} > 0$ such that for every initial value $y_0 \in X$ with $\|y_0\|_\Theta \le r_{0x}$ and every disturbance $u \in \mathcal{U}$ with $\|u\|_\infty \le r_{0u}$ the global weak solution

$$[0,\infty) \ni t \mapsto y(t,\cdot) = y(t, y_0, u) \in X$$

of the boundary value problem (1.1) with initial condition $y(0,\cdot) = y_0 \in X$ satisfies the following estimate:

$$\|y(t, y_0, u)\|_\Theta \le \beta(\|y_0\|_\Theta, t) + \gamma(\|u\|_\infty) \quad (t \in [0,\infty)). \tag{1.3}$$

See [18] for the analogous definition in the special case $\Theta = \{0\}$. In the above relations, we use the standard notation

$$\|x\|_\Theta := \operatorname{dist}(x,\Theta) := \inf_{\theta\in\Theta} \|x - \theta\| \quad (x \in X) \tag{1.4}$$

and the standard definitions for the comparison function classes $\mathcal{KL}$ and $\mathcal{K}$, which are recalled in (1.5). In words, the local input-to-state stability estimate (1.3) means that

(i) the invariant set Θ for (1.2) is locally stable and attractive for the undisturbed system (1.2) and
(ii) these local stability and attractivity properties are affected only slightly in the presence of disturbances of small magnitude $\|u\|_\infty$.

In order to achieve the estimate (1.3), we will construct a suitable local input-to-state Lyapunov function V.

In the finite-dimensional case, input-to-state stability properties w.r.t. attractors have been studied by many authors, see for example [9,28]. As far as we know, however, our result is the first (local) input-to-state stability result w.r.t. attractors Θ of an infinite-dimensional system given by concrete partial differential equations. All previous concrete pde results we are aware of—like those from [4,11,12,15–17], [22,25,29,32,33], for instance—establish input-to-state stability only w.r.t. an equilibrium point θ, which without loss of generality is assumed to be $\theta = 0$. In particular, all these previous results require their nonlinearity g to be such that $g(\theta) = g(0) = 0$

and such that the undisturbed system has the singleton $\Theta := \{\theta\} = \{0\}$ as an attractor. With our result, by contrast, we can treat much more general nonlinearities: we can treat nonlinearities g with $g(0) \neq 0$ and, more importantly, nonlinearities g for which the undisturbed system (1.2) has only a non-singleton attractor $\Theta \supsetneq \{0\}$. A simple example of such a nonlinearity is given by $g(r) := -r^3 + br$, $b > 0$ which leads to the Chafee–Infante equation. It possesses a global attractor Θ, which is the set consisting of all equilibrium points and all trajectories connecting these points, see [24] (Section 11.5). Also, the lower bound of the fractional dimension (capacity) of Θ depends on the parameter b [30] (Section VII.5). We point out that there is no way to conclude anything about an input-to-state stability-like property of Θ by studying stability properties of the single equilibrium points only. Also note that some of these equilibrium points can be unstable, but an input-to-state stability-like property may still be true with respect to Θ. We refer to [5,7,8,14] for other interesting results about non-trivial global attractors of nonlinear, impulsive, or even multi-valued semigroups.

In the entire paper, we will use the following conventions and notations. As above, $X := L^2(\Omega, \mathbb{R})$ and $\mathcal{U} := L^\infty(\mathbb{R}_0^+, \mathbb{R})$ with $\mathbb{R}_0^+ := [0, \infty)$ and with the standard norm of X being denoted simply by $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$. As usual,

$$B_r(x_0) = B_r^X(x_0), \quad \overline{B}_r(x_0) = \overline{B}_r^X(x_0) \quad \text{and} \quad B_r(u_0) = B_r^{\mathcal{U}}(u_0), \quad \overline{B}_r(u_0) = \overline{B}_r^{\mathcal{U}}(u_0)$$

denote the open and closed balls in X or $\mathcal{U}$ of radius r around $x_0 \in X$ or $u_0 \in \mathcal{U}$, respectively. We will often use the notation (1.4) and

$$B_r(\Theta) := \{x \in X : \|x\|_\Theta < r\} \quad \text{and} \quad \overline{B}_r(\Theta) := \{x \in X : \|x\|_\Theta \leq r\},$$

as well as the notation $\operatorname{dist}(M, \Theta) := \sup_{x \in M} \|x\|_\Theta$ for subsets $M, \Theta \subset X$. Also, $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ will denote the following standard classes of comparison functions:

$$\begin{aligned}
&\mathcal{K} := \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) : \gamma \text{ strictly increasing with } \gamma(0) = 0\} \\
&\mathcal{K}_\infty := \{\gamma \in \mathcal{K} : \gamma \text{ unbounded}\} \\
&\mathcal{KL} := \{\beta \in C(\mathbb{R}_0^+ \times \mathbb{R}_0^+, \mathbb{R}_0^+) : \beta(\cdot, t) \in \mathcal{K} \text{ for } t \geq 0 \\
&\qquad\qquad\qquad \text{and } \beta(s, \cdot) \in \mathcal{L} \text{ for } s > 0\}, \qquad (1.5)
\end{aligned}$$

where $\mathcal{L} := \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) : \gamma \text{ strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0\}$. And finally, upper right-hand Dini derivatives will be denoted by

$$\overline{\partial}_t^+ v(t) := \overline{\lim_{\tau \to 0+}} \frac{v(t + \tau) - v(t)}{\tau}.$$

2 Some preliminaries

In this section, we provide the necessary preliminaries for our local input-to-state stability result. We begin by recalling the definition of weak solutions of initial boundary value problems of the form

$$\begin{aligned} &\partial_t y(t,\zeta) = \Delta y(t,\zeta) + g(y(t,\zeta)) + h(\zeta)u(t) \quad ((t,\zeta) \in [s,\infty) \times \Omega) \\ &y(t,\cdot)|_{\partial\Omega} = 0 \quad \text{and} \qquad y(s,\cdot) = y_s \qquad (t \in [s,\infty)). \end{aligned} \tag{2.1}$$

In fact, we will have to consider initial boundary value problems with more general inhomogeneities of the form

$$\begin{aligned} &\partial_t y(t,\zeta) = \Delta y(t,\zeta) + \overline{g}(y(t,\zeta)) + \overline{h}(t,\zeta) \qquad ((t,\zeta) \in [s,\infty) \times \Omega) \\ &y(t,\cdot)|_{\partial\Omega} = 0 \quad \text{and} \quad y(s,\cdot) = y_s \qquad (t \in [s,\infty)), \end{aligned} \tag{2.2}$$

where $\overline{g}, \overline{h}$ satisfy the following conditions.

Condition 2.1 (i) *Ω is a bounded domain in $\mathbb{R}^d$ for some $d \in \mathbb{N}$ with smooth boundary $\partial\Omega$ and, moreover, $p \in [2,\infty)$, $q \in (1,2]$ are dual exponents: $1/p + 1/q = 1$*
(ii) *$\overline{g} \in C^1(\mathbb{R},\mathbb{R})$ and there exist constants $\alpha_1, \alpha_2, \kappa, \lambda \in (0,\infty)$ such that*

$$-\kappa - \alpha_1|r|^p \le \overline{g}(r)r \le \kappa - \alpha_2|r|^p \quad \textit{and} \quad \overline{g}'(r) \le \lambda \quad (r \in \mathbb{R}) \tag{2.3}$$

and, moreover, $\overline{h} \in L^q_{\mathrm{loc}}(\mathbb{R}^+_0, L^q(\Omega))$.

A bit more explicitly, the first two inequalities in (2.3) mean that $\overline{g}|_{(0,\infty)}$ lies between $r \mapsto -\kappa/|r| - \alpha_1|r|^{p-1}$ and $r \mapsto \kappa/|r| - \alpha_2|r|^{p-1}$ and that $\overline{g}|_{(-\infty,0)}$ lies between $r \mapsto -\kappa/|r| + \alpha_2|r|^{p-1}$ and $r \mapsto \kappa/|r| + \alpha_1|r|^{p-1}$. A simple class of functions $\overline{g}$ satisfying the three inequalities from (2.3) is given by the polynomials of odd degree with negative leading coefficient:

$$\overline{g}(r) = \sum_{i=0}^{2m-1} c_i r^i \quad (r \in \mathbb{R})$$

with $c_{2m-1} < 0$, where $m \in \mathbb{N}$. (Choose $p := 2m$ in order to see that Condition 2.1 is satisfied here). In particular, the nonlinearity of the Chafee–Infante equation given by $\overline{g}(r) := -r^3 + br,\ b > 0$ falls into that class (Section 11.5 of [24]).

Suppose that Condition 2.1 is satisfied and let $s \in \mathbb{R}^+_0$ and $y_s \in X$. A function $y \in C([s,\infty), X)$ is called a *global weak solution of* (2.2) iff $y(s) = y_s$ and for every $T \in (s,\infty)$ one has

$$y|_{[s,T]} \in L^2([s,T], H^1_0(\Omega)) \cap L^p([s,T], L^p(\Omega)) \tag{2.4}$$

and there exists a (then unique) $z \in L^2([s,T], H_0^1(\Omega)^*) + L^q([s,T], L^q(\Omega))$ such that

$$\begin{aligned}\int_s^T \big(z(t), \varphi(t)\big)\,\mathrm{d}t = &-\int_s^T \int_\Omega \nabla y(t)(\zeta)\cdot\nabla\varphi(t)(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}t \\ &+\int_s^T \int_\Omega \overline{g}\big(y(t)(\zeta)\big)\,\varphi(t)(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}t \\ &+\int_s^T \int_\Omega \overline{h}(t)(\zeta)\,\varphi(t)(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}t \end{aligned} \tag{2.5}$$

for every $\varphi \in L^2([s,T], H_0^1(\Omega)) \cap L^p([s,T], L^p(\Omega))$. See [31] or [13] and, for more background information, [2] or [3]. In this equation, $(\cdot,\cdot\cdot)$ stands for the dual pairing of $H_0^1(\Omega)^* + L^q(\Omega)$ and $H_0^1(\Omega)\cap L^p(\Omega)$, that is,

$$(z,\varphi) = (z_1,\varphi)_{H_0^1(\Omega)^*, H_0^1(\Omega)} + (z_2,\varphi)_{L^q(\Omega), L^p(\Omega)} \tag{2.6}$$

for every $z = z_1 + z_2 \in H_0^1(\Omega)^* + L^q(\Omega)$ and $\varphi \in H_0^1(\Omega)\cap L^p(\Omega)$, where $(\cdot,\cdot\cdot)_{H_0^1(\Omega)^*, H_0^1(\Omega)}$ and $(\cdot,\cdot\cdot)_{L^q(\Omega), L^p(\Omega)}$ denote the respective dual pairings. See [1] (Theorem 2.7.1) and [6] (Theorem IV.1.1 and Corollary III.2.13), for instance, to get that $H_0^1(\Omega)^* + L^q(\Omega)$, $H_0^1(\Omega)\cap L^p(\Omega)$ and

$$L^2([s,T], H_0^1(\Omega)^*) + L^q([s,T], L^q(\Omega)), \quad L^2([s,T], H_0^1(\Omega)) \cap L^p([s,T], L^p(\Omega))$$

are dual to each other. We point out that if y is a global weak solution to (2.2), then for every $T \in (s,\infty)$ there is only one $z \in L^2([s,T], H_0^1(\Omega)^*) + L^q([s,T], L^q(\Omega))$ satisfying (2.5). And this z is called the *weak or generalized derivative* of $y|_{[s,T]}$. It is denoted by $\partial_t y|_{[s,T]}$ or simply by $\partial_t y$ in the following.

Lemma 2.2 *Suppose that Condition 2.1 is satisfied and let $s \in \mathbb{R}_0^+$ and $y_s \in X$. Then, the initial boundary value problem (2.2) has a unique global weak solution y and, moreover, $t \mapsto \|y(t)\|^2$ is absolutely continuous (hence differentiable almost everywhere) with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|^2 = 2\big(\partial_t y(t), y(t)\big) \tag{2.7}$$

for almost every $t \in [s,\infty)$, where $(\cdot,\cdot\cdot)$ is the dual pairing from (2.6).

Proof It is clear from the first two inequalities in (2.3) that

$$|\overline{g}(r)r| \le \kappa + \alpha_1 |r|^p \quad (r \in \mathbb{R}). \tag{2.8}$$

Since $\sup_{|r|\le 1} |\overline{g}(r)| < \infty$ by the continuity of $\overline{g}$, it follows from (2.8) that for some constant $C_1 \in (0,\infty)$

$$|\overline{g}(r)| \le C_1(1 + |r|^{p-1}) \quad (r \in \mathbb{R}), \tag{2.9}$$

and therefore, condition (2) from [31] is satisfied. Also, in view of the second and third inequalities in (2.3), condition (3) and condition (4) from [31] with $M = 0$ are satisfied. Consequently, the assertions of the lemma follow from the remarks made in Section 2 (up to Remark 1) of [31]. □

With this lemma at hand, it is easy to see that the initial boundary value problem (2.1) generates a semiprocess family $(S_u)_{u\in\mathcal{U}}$ on X (Lemma 2.4). A *semiprocess family on* X is a family of maps $S_u : \Delta \times X \to X$ for every $u \in \mathcal{U}$ such that

$$S_u(s, s, x) = x \quad \text{and} \quad S_u\big(t, s, S_u(s, r, x)\big) = S(t, r, x) \tag{2.10}$$

$$S_u(t + \tau, s + \tau, x) = S_{u(\cdot+\tau)}(t, s, x) \tag{2.11}$$

for all $(t, s), (s, r) \in \Delta$, $\tau \in \mathbb{R}_0^+$, $x \in X$ and $u \in \mathcal{U}$, where we used the abbreviation $\Delta := \{(s, t) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : t \geq s\}$. See [3], for instance, for more information on semiprocess families.

Condition 2.3 (i) *Ω is a bounded domain in $\mathbb{R}^d$ for some $d \in \mathbb{N}$ with smooth boundary $\partial\Omega$ and, moreover, $p \in [2, \infty)$*

(ii) *$g \in C^1(\mathbb{R}, \mathbb{R})$ and there exist constants $\alpha_1, \alpha_2, \kappa, \lambda \in (0, \infty)$ such that*

$$-\kappa - \alpha_1|r|^p \leq g(r)r \leq \kappa - \alpha_2|r|^p \quad \textit{and} \quad g'(r) \leq \lambda \quad (r \in \mathbb{R}) \tag{2.12}$$

and, moreover, $h \in X\backslash\{0\}$.

Lemma 2.4 *Suppose that Condition 2.3 is satisfied. Then, for every $s \in \mathbb{R}_0^+$ and every $(y_s, u) \in X \times \mathcal{U}$ the initial boundary value problem (2.1) has a unique global weak solution $y(\cdot, s, y_s, u)$. Additionally, $(S_u)_{u\in\mathcal{U}}$ defined by*

$$S_u(t, s, y_s) := y(t, s, y_s, u) \tag{2.13}$$

is a semiprocess family on X.

Proof In order to see the unique global weak solvability, simply apply Lemma 2.2 with $\overline{g} := g$ and with $\overline{h} \in L^2_{\mathrm{loc}}(\mathbb{R}_0^+, L^2(\Omega)) \subset L^q_{\mathrm{loc}}(\mathbb{R}_0^+, L^q(\Omega))$ defined by $\overline{h}(t)(\zeta) := h(\zeta)u(t)$. In order to see the semiprocess property, use the definition of weak solutions and the uniqueness statement from Lemma 2.2. □

In the following, $(S_u)_{u\in\mathcal{U}}$ will always denote the semiprocess family from the previous lemma. Also, we will often refer to $(S_u)_{u\in\mathcal{U}}$ and S_0 as the disturbed and the undisturbed system, respectively. In proving our local input-to-state stability result, the following estimates will play an important role.

Lemma 2.5 *Suppose that Condition 2.3 is satisfied. Then,*

$$\|S_0(t, 0, y_{01}) - S_0(t, 0, y_{02})\| \leq \mathrm{e}^{\lambda t}\, \|y_{01} - y_{02}\| \quad (t \in \mathbb{R}_0^+) \tag{2.14}$$

$$\|S_u(t, 0, y_0) - S_0(t, 0, y_0)\| \leq 2\mathrm{e}^{2\lambda}\, \|h\|\, \|u\|_\infty\, t \quad (t \in [0, 1]) \tag{2.15}$$

for all $y_0, y_{01}, y_{02} \in X$ and all $u \in \mathcal{U}$.

Proof As a first step, we show that for every $y_{01}, y_{02} \in X$ and $u \in \mathcal{U}$ the function

$$y_{12}^u := y_1^u - y_2^0 \quad \text{with} \quad y_1^u := S_u(\cdot, 0, y_{01}) \quad \text{and} \quad y_2^0 := S_0(\cdot, 0, y_{02}) \tag{2.16}$$

is a global weak solution of the initial boundary value problem

$$\begin{aligned} &\partial_t y(t,\zeta) = \Delta y(t,\zeta) + \overline{g}(y(t,\zeta)) + \overline{h}(t,\zeta) \quad ((t,\zeta) \in [0,\infty) \times \Omega) \\ &y(t,\cdot)|_{\partial\Omega} = 0 \quad \text{and} \quad y(0,\cdot) = y_{01} - y_{02} \quad (t \in [0,\infty)), \end{aligned} \tag{2.17}$$

where $\overline{g} := g$ and $\overline{h}(t)(\zeta) := g(y_1^u(t)(\zeta)) - g(y_2^0(t)(\zeta)) - g(y_{12}^u(t)(\zeta)) + h(\zeta)u(t)$. So, let $y_{01}, y_{02} \in X$ and $u \in \mathcal{U}$ and adopt the abbreviations from (2.16). It is not difficult—using (2.9) and $q(p-1) = p$—to see from Condition 2.3 that with $\overline{g}, \overline{h}$ as defined above, Condition 2.1 is satisfied. Since y_1^u, y_2^0 are global weak solutions, we have $y_{12}^u \in C(\mathbb{R}_0^+, X)$ and for every $T \in (0,\infty)$ we have

$$y_{12}^u|_{[0,T]} \in L^2([0,T], H_0^1(\Omega)) \cap L^p([0,T], L^p(\Omega))$$

and $\partial_t y_1^u|_{[0,T]} - \partial_t y_2^0|_{[0,T]} \in L^2([0,T], H_0^1(\Omega)^*) + L^q([0,T], L^q(\Omega))$ as well as

$$\begin{aligned} \int_0^T \big(\partial_t y_1^u(t) - \partial_t y_2^0(t), \varphi(t)\big)\, \mathrm{d}t &= -\int_0^T \int_\Omega \nabla y_{12}^u(t)(\zeta) \cdot \nabla\varphi(t)(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \overline{g}\big(y_{12}^u(t)(\zeta)\big)\, \varphi(t)(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \overline{h}(t)(\zeta)\, \varphi(t)(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}t \end{aligned} \tag{2.18}$$

for every $\varphi \in L^2([0,T], H_0^1(\Omega)) \cap L^p([0,T], L^p(\Omega))$. And therefore, y_{12}^u is a weak solution of (2.17), as desired.

As a second step, we show that for every $y_{01}, y_{02} \in X$ and $u \in \mathcal{U}$ the function y_{12}^u from (2.16) satisfies the estimate

$$\begin{aligned} \sup_{T\in[0,t]} \left\| y_{12}^u(T) \right\|^2 \le \mathrm{e}^{2\lambda t} \Big(&\left\| y_{01} - y_{02} \right\|^2 \\ &+ 2\,\|h\|\,\|u\|_\infty \cdot t \cdot \sup_{T\in[0,t]} \left\| y_{12}^u(T) \right\| \Big) \end{aligned} \tag{2.19}$$

for every $t \in \mathbb{R}_0^+$. Indeed, by the first step and Lemma 2.2, the function $t \mapsto \left\| y_{12}^u(t) \right\|^2$ is absolutely continuous with

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\left\| y_{12}^u(t) \right\|^2}{2} = \big(\partial_t y_{12}^u(t), y_{12}^u(t)\big) = \big(\partial_t y_1^u(t) - \partial_t y_2^0(t), y_{12}^u(t)\big)$$

for almost every $t \in \mathbb{R}_0^+$. And therefore, by virtue of (2.18) with $\varphi := y_{12}^u$, we get

$$
\begin{aligned}
\frac{\left\|y_{12}^u(T)\right\|^2}{2} - \frac{\left\|y_{12}^u(0)\right\|^2}{2} &= \int_0^T \big(\partial_t y_1^u(t) - \partial_t y_2^0(t), y_{12}^u(t)\big)\, \mathrm{d}t \\
&\le \int_0^T \int_\Omega \big(g(y_1^u(t)(\zeta)) - g(y_2^0(t)(\zeta))\big) y_{12}^u(t)(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega h(\zeta) u(t) y_{12}^u(t)(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}t \\
&\le \lambda \int_0^T \left\|y_{12}^u(t)\right\|^2 \mathrm{d}t + \|h\| \, \|u\|_\infty \int_0^T \left\|y_{12}^u(t)\right\| \mathrm{d}t
\end{aligned} \tag{2.20}
$$

for every $T \in (0, \infty)$. In the last inequality, we used that $(g(r) - g(s))(r - s) \le \lambda |r - s|^2$ for all $r, s \in \mathbb{R}$ due to (2.12). So, for every $t_0 \in (0, \infty)$, we obtain

$$
\left\|y_{12}^u(T)\right\|^2 \le \|y_{01} - y_{02}\|^2 + 2\,\|h\| \, \|u\|_\infty \cdot t_0 \cdot \sup_{t \in [0, t_0]} \left\|y_{12}^u(t)\right\| + 2\lambda \int_0^T \left\|y_{12}^u(t)\right\|^2 \mathrm{d}t
$$

for every $T \in [0, t_0]$. And from this, in turn, the claimed estimate (2.19) immediately follows by Grönwall's lemma.

As a third step, it is now easy to conclude the desired estimates (2.14) and (2.15) from the second step. Indeed, (2.14) immediately follows from (2.19) with the special choice $u := 0 \in \mathcal{U}$ and (2.15) follows from (2.19) with the special choice $y_{01} = y_{02} := y_0 \in X$. □

We remark for later reference that our semiprocess family $(S_u)_{u \in \mathcal{U}}$, like any other semiprocess family [27], satisfies the following so-called cocycle property:

$$
S_u(t + \tau, 0, x) = S_{u(\cdot + \tau)}\big(t, 0, S_u(\tau, 0, x)\big) \tag{2.21}
$$

for all $t, \tau \in \mathbb{R}_0^+$, $x \in X$ and $u \in \mathcal{U}$. (Just combine (2.10) and (2.11) to see this). In particular, S_0 satisfies the following (nonlinear) semigroup property [23]:

$$
S_0(t + \tau, 0, x) = S_0\big(t, 0, S_0(\tau, 0, x)\big) \quad (t, \tau \in \mathbb{R}_0^+ \text{ and } x \in X). \tag{2.22}
$$

We conclude this section with some remarks on the asymptotic behavior of this semigroup S_0 in terms of attractors [24,30]. A *global attractor* of S_0 is a compact subset Θ of X such that

(i) Θ is invariant under S_0, that is, $S_0(t, 0, \Theta) = \Theta$ for every $t \in \mathbb{R}_0^+$
(ii) Θ is uniformly attractive for S_0, that is, for every bounded subset $B \subset X$ one has

$$
\operatorname{dist}\big(S_0(t, 0, B), \Theta\big) = \sup_{x \in B} \|S_0(t, 0, x)\|_\Theta \longrightarrow 0 \quad (t \to \infty). \tag{2.23}
$$

It directly follows from this definition that a global attractor of S_0 is minimal among all closed uniformly attractive sets of S_0 and maximal among all bounded invariant sets of S_0. And from this, in turn, it immediately follows that if S_0 has any global attractor then it is already unique.

Lemma 2.6 *Suppose that Condition 2.3 is satisfied. Then, the undisturbed system S_0 has a unique global attractor Θ and, moreover, Θ is uniformly globally asymptotically stable for S_0, that is, there exists a comparison function $\beta_0 \in \mathcal{KL}$ such that*

$$\|S_0(t,0,x)\|_\Theta \leq \beta_0(\|x\|_\Theta\,, t) \quad (t \in \mathbb{R}_0^+ and x \in X). \tag{2.24}$$

Proof It is well known that S_0 has a global attractor Θ (by Theorem 11.4 of [24], for instance) and that global attractors when existent are already unique (by the remarks preceding the lemma). So, we have only to show that Θ is uniformly globally asymptotically stable for S_0. And in order to do so, we will proceed in three steps, applying results from [20] to the system $S_0 = (S_u)_{u\in\mathcal{U}_0}$ with trivial disturbance space $\mathcal{U}_0 := \{0\}$. (In this context, it should be noticed that by (2.22) and the continuity of weak solutions, $(S_u)_{u\in\mathcal{U}_0}$ is a forward-complete system in the sense of [20,21,26]).

As a first step, we show that Θ is uniformly globally stable for $(S_u)_{u\in\mathcal{U}_0} = S_0$, that is, there exists a comparison function $\sigma_0 \in \mathcal{K}$ such that

$$\|S_0(t,0,x)\|_\Theta \leq \sigma_0(\|x\|_\Theta) \quad (t \in \mathbb{R}_0^+) \tag{2.25}$$

for every $x \in X$ (Definition 2.8 of [20]). Indeed, it immediately follows from the invariance of Θ under S_0 and from the estimate (2.14) that for every $\varepsilon > 0$ and every $T \in (0,\infty)$ there exists a $\delta \in (0,1]$ such that

$$\|S_0(t,0,x)\|_\Theta \leq \inf_{\theta\in\Theta} \|S_0(t,0,x) - S_0(t,0,\theta)\| < \varepsilon \quad (t \in [0,T] \text{ and } x \in B_\delta(\Theta)).$$

And from this and the uniform attractivity (2.23) of Θ for S_0 (with $B := B_1(\Theta)$), in turn, it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|S_0(t,0,x)\|_\Theta < \varepsilon \quad (t \in \mathbb{R}_0^+) \tag{2.26}$$

for every $x \in B_\delta(\Theta)$. Also, it is well known that

$$\|S_0(t,0,x)\|^2 \leq \mathrm{e}^{-2\omega t}\,\|x\|^2 + \frac{\lambda|\Omega|}{\omega} \quad (t \in \mathbb{R}_0^+) \tag{2.27}$$

for all $x \in X$, where $\omega \in (0,\infty)$ is the smallest eigenvalue of $-\Delta$, the negative Dirichlet Laplacian on Ω. (See the very last equation on p. 286 of [24], for instance). Since $\|S_0(t,0,x)\|_\Theta \leq \|S_0(t,0,x)\| + \|\Theta\|$ and $\|x\| \leq \|x\|_\Theta + \|\Theta\|$ with $\|\Theta\| := \sup_{\theta\in\Theta}\|\theta\|$, it follows from (2.27) that there exists a comparison function $\sigma \in \mathcal{K}$ and a constant $c \in (0,\infty)$ such that

$$\|S_0(t,0,x)\|_\Theta \leq \sigma(\|x\|_\Theta) + c \quad (t \in \mathbb{R}_0^+) \tag{2.28}$$

for every $x \in X$. In the terminology of [20], the relations (2.26) and (2.28) mean that Θ is uniformly locally stable and Lagrange-stable for $(S_u)_{u\in\mathcal{U}_0}$, respectively. And therefore, Θ is uniformly globally stable for $(S_u)_{u\in\mathcal{U}_0} = S_0$ by virtue of Remark 2.9 of [20], as desired.

As a second step, we show that Θ is uniformly globally attractive for $(S_u)_{u\in\mathcal{U}_0} = S_0$, that is, for every $\varepsilon > 0$ and $r > 0$ there exists a time $\tau(\varepsilon, r) \in \mathbb{R}_0^+$ such that

$$\|S_0(t, 0, x)\|_\Theta < \varepsilon \quad (t \geq \tau(\varepsilon, r)) \tag{2.29}$$

for every $x \in \overline{B}_r(\Theta)$ (Definition 2.8 of [20]). Indeed, this immediately follows from the uniform attractivity (2.23) of Θ for S_0 with $B := \overline{B}_r(\Theta)$.

As a third step, we can now conclude the desired uniform global asymptotic stability of Θ for $(S_u)_{u\in\mathcal{U}_0} = S_0$ from Theorem 4.2 of [20] and the first two steps. □

3 A local input-to-state stability result

In this section, we establish our local input-to-state stability result for the disturbed reaction–diffusion system (1.1). We begin by showing that the undisturbed system (1.2) has a local Lyapunov function and, for that purpose, we will argue in a similar way as [10] (Theorem 4.2.1).

Lemma 3.1 *Suppose that Condition 2.3 is satisfied and let Θ be the global attractor of the undisturbed system S_0. Then, for every $r_0 > 0$ there exists a Lipschitz continuous function $V : \overline{B}_{r_0}(\Theta) \to \mathbb{R}_0^+$ with Lipschitz constant 1 and comparison functions $\underline{\psi}, \overline{\psi}, \alpha \in \mathcal{K}_\infty$ such that*

$$\underline{\psi}(\|x\|_\Theta) \leq V(x) \leq \overline{\psi}(\|x\|_\Theta) \quad (x \in \overline{B}_{r_0}(\Theta)) \tag{3.1}$$

$$\dot{V}_0(x) \leq -\alpha(\|x\|_\Theta) \qquad (x \in B_{r_0}(\Theta)), \tag{3.2}$$

where $\dot{V}_0(x) := \overline{\lim}_{t\to 0+} \frac{1}{t}\big(V(S_0(t, 0, x)) - V(x)\big)$.

Proof Choose an arbitrary $r_0 \in (0, \infty)$ and fix it for the rest of the proof. Also, choose $\beta_0 \in \mathcal{KL}$ as in Lemma 2.6 and, for every $\varepsilon > 0$, let $T(\varepsilon) = T_{r_0}(\varepsilon)$ be a time such that

$$\beta_0(r_0, t) \leq \varepsilon \quad (t \in [T(\varepsilon), \infty)). \tag{3.3}$$

Set now, for every given $\varepsilon > 0$,

$$V^\varepsilon(x) := \mathrm{e}^{-(\lambda+c_0)T(\varepsilon)} \sup_{t\in[0,\infty)} \Big(\mathrm{e}^{c_0 t}\, \eta_\varepsilon\big(\, \|S_0(t, 0, x)\|_\Theta\, \big)\Big) \quad (x \in \overline{B}_{r_0}(\Theta)), \tag{3.4}$$

where $c_0 \in (0, \infty)$ is an arbitrary constant (which is fixed throughout the proof) and $\eta_\varepsilon(r) := \max\{0, r - \varepsilon\}$ for every $r \in \mathbb{R}_0^+$. In view of (2.24) and (3.3), the supremum in (3.4) for $x \in \overline{B}_{r_0}(\Theta)$ actually extends only over a compact interval, namely

$$V^\varepsilon(x) = \mathrm{e}^{-(\lambda+c_0)T(\varepsilon)} \sup_{t\in[0,T(\varepsilon)]} \Big(\mathrm{e}^{c_0 t}\, \eta_\varepsilon\big(\,\|S_0(t,0,x)\|_\Theta\,\big)\Big) \quad (x \in \overline{B}_{r_0}(\Theta)). \tag{3.5}$$

In particular, $V^\varepsilon : \overline{B}_{r_0}(\Theta) \to \mathbb{R}_0^+$ is a well-defined map (with finite values) and

$$\begin{aligned} V^\varepsilon(x) &\le \mathrm{e}^{-\lambda T(\varepsilon)} \sup_{t\in[0,T(\varepsilon)]} \Big(\eta_\varepsilon\big(\,\|S_0(t,0,x)\|_\Theta\,\big)\Big) \\ &\le \beta_0(\|x\|_\Theta\,, 0) \quad (x \in \overline{B}_{r_0}(\Theta)) \end{aligned} \tag{3.6}$$

because $\eta_\varepsilon(r) \le r$ for all $r \in \mathbb{R}_0^+$. Since, moreover, $|\eta_\varepsilon(r) - \eta_\varepsilon(s)| \le |r-s|$ for all $r, s \in \mathbb{R}_0^+$, we see from (3.5) and (2.14) that

$$\begin{aligned} |V^\varepsilon(x) - V^\varepsilon(y)| &\le \mathrm{e}^{-(\lambda+c_0)T(\varepsilon)} \sup_{t\in[0,T(\varepsilon)]} \Big|\mathrm{e}^{c_0 t}\, \eta_\varepsilon\big(\,\|S_0(t,0,x)\|_\Theta\,\big) \\ &\quad - \mathrm{e}^{c_0 t}\, \eta_\varepsilon\big(\,\|S_0(t,0,y)\|_\Theta\,\big)\Big| \\ &\le \mathrm{e}^{-\lambda T(\varepsilon)} \sup_{t\in[0,T(\varepsilon)]} \Big|\,\|S_0(t,0,x)\|_\Theta - \|S_0(t,0,y)\|_\Theta\,\Big| \\ &\le \mathrm{e}^{-\lambda T(\varepsilon)} \sup_{t\in[0,T(\varepsilon)]} \|S_0(t,0,x) - S_0(t,0,y)\| \\ &\le \|x-y\| \quad (x, y \in \overline{B}_{r_0}(\Theta)). \end{aligned} \tag{3.7}$$

(In the first inequality above, we used the elementary fact that $|\sup_{t\in I} a_t - \sup_{t\in I} b_t| \le \sup_{t\in I} |a_t - b_t|$ for arbitrary bounded functions $t \mapsto a_t, b_t$ on an arbitrary set I, and in the third inequality above, we used the elementary fact that $|\,\|\xi\|_\Theta - \|\eta\|_\Theta\,| \le \|\xi - \eta\|$ for arbitrary $\xi, \eta \in X$). Additionally, for every $x \in B_{r_0}(\Theta)$, we have $S_0(\tau, 0, x) \in B_{r_0}(\Theta)$ for τ small enough and thus, by (3.4) and the semigroup property (2.22),

$$V^\varepsilon(S_0(\tau,0,x)) = \mathrm{e}^{-(\lambda+c_0)T(\varepsilon)} \sup_{t\in[0,\infty)} \Big(\mathrm{e}^{c_0 t}\, \eta_\varepsilon\big(\,\|S_0(t+\tau,0,x)\|_\Theta\,\big)\Big) \le \mathrm{e}^{-c_0\tau} V^\varepsilon(x)$$

for every $x \in B_{r_0}(\Theta)$ and all sufficiently small times τ. Consequently,

$$\dot{V}_0^\varepsilon(x) = \overline{\lim_{\tau\to 0+}} \frac{1}{\tau}\big(V^\varepsilon(S_0(\tau,0,x)) - V^\varepsilon(x)\big) \le -c_0 V^\varepsilon(x) \quad (x \in B_{r_0}(\Theta)). \tag{3.8}$$

With the help of the auxiliary functions V^ε, we can now construct a function $V : \overline{B}_{r_0}(\Theta) \to \mathbb{R}_0^+$ with the desired properties. Indeed, let

$$V(x) := \sum_{k=1}^{\infty} 2^{-k} V^{1/k}(x) \quad (x \in \overline{B}_{r_0}(\Theta)). \tag{3.9}$$

We then conclude from (3.6), (3.7), (3.8) that

$$V(x) \leq \beta_0(\|x\|_\Theta, 0) \quad (x \in \overline{B}_{r_0}(\Theta)), \tag{3.10}$$

$$|V(x) - V(y)| \leq \sum_{k=1}^{\infty} 2^{-k} |V^{1/k}(x) - V^{1/k}(y)| \leq \|x - y\| \quad (x, y \in \overline{B}_{r_0}(\Theta)), \tag{3.11}$$

$$\dot{V}_0(x) \leq \sum_{k=1}^{\infty} 2^{-k} \dot{V}_0^{1/k}(x) \leq -c_0 V(x) \quad (x \in B_{r_0}(\Theta)). \tag{3.12}$$

Since $\sup_{t \in [0,\infty)} (\mathrm{e}^{c_0 t} \eta_{1/k}(\|S_0(t, 0, x)\|_\Theta)) \geq \eta_{1/k}(\|x\|_\Theta)$ for all $x \in X$, we also conclude from (3.4) and (3.9) that

$$V(x) \geq \sum_{k=1}^{\infty} 2^{-k} \mathrm{e}^{-(\lambda + c_0) T(1/k)} \eta_{1/k}(\|x\|_\Theta) \quad (x \in \overline{B}_{r_0}(\Theta). \tag{3.13}$$

In view of these estimates, we now define the comparison functions $\overline{\psi}$, $\underline{\psi}$ and α in the following way:

$$\overline{\psi}(r) := \beta_0(r, 0) + r \quad \text{and} \quad \underline{\psi}(r) := \sum_{k=1}^{\infty} 2^{-k} \mathrm{e}^{-(\lambda + c_0) T(1/k)} \eta_{1/k}(r)$$

and $\alpha(r) := c_0 \underline{\psi}(r)$ for $r \in \mathbb{R}_0^+$. It is easy to verify that $\overline{\psi}$, $\underline{\psi}$ and hence α belong to $\mathcal{K}_\infty$. And, moreover, by virtue of (3.10), (3.11), (3.12), (3.13), the desired estimates (3.1) and (3.2) follow. □

It should be noticed that the functions $V, \underline{\psi}, \alpha$ constructed in the proof above all depend on the chosen radius $r_0 \in (0, \infty)$ because these functions are defined in terms of the times $T(\varepsilon) = T_{r_0}(\varepsilon)$ from (3.3). With the next lemma, we show that the local Lyapunov function V for the undisturbed system is also a local input-to-state Lyapunov function for the disturbed system w.r.t. Θ. (See [4] for the definition of local input-to-state Lyapunov functions w.r.t. an equilibrium point).

Lemma 3.2 *Suppose that Condition 2.3 is satisfied and let Θ be the global attractor of the undisturbed system S_0. Also, let $r_0 > 0$ and let $V : \overline{B}_{r_0}(\Theta) \to \mathbb{R}_0^+$ be chosen as in the previous lemma. Then, there exist comparison functions $\alpha, \sigma \in \mathcal{K}$ such that for every $u \in \mathcal{U}$*

$$\dot{V}_u(x) := \overline{\lim_{t \to 0+}} \frac{1}{t} \big(V(S_u(t, 0, x)) - V(x) \big) \leq -\alpha(\|x\|_\Theta) + \sigma(\|u\|_\infty) \quad (x \in B_{r_0}(\Theta)).$$

Proof Choose $\alpha = \alpha_{r_0} \in \mathcal{K}_\infty$ as in Lemma 3.1 and define $\sigma \in \mathcal{K}_\infty$ by $\sigma(r) := 2\mathrm{e}^{2\lambda} \|h\| r$ for all $r \in \mathbb{R}_0^+$. We then see from Lemma 3.1 and from (2.15) that for every $x \in B_{r_0}(\Theta)$ and every $u \in \mathcal{U}$

$$\dot{V}_u(x) \leq \overline{\lim_{t\to 0+}} \frac{1}{t}\big(V(S_0(t,0,x)) - V(x)\big) + \overline{\lim_{t\to 0+}} \frac{1}{t}\big(V(S_u(t,0,x)) - V(S_0(t,0,x))\big)$$
$$\leq -\alpha(\|x\|_\Theta) + \overline{\lim_{t\to 0+}} \frac{1}{t} \|S_u(t,0,x) - S_0(t,0,x)\|$$
$$\leq -\alpha(\|x\|_\Theta) + \sigma(\|u\|_\infty), \tag{3.14}$$

as desired. □

With these lemmas at hand, we can now establish the local input-to-state stability of the disturbed reaction–diffusion system (1.1) w.r.t. the global attractor of the undisturbed system (1.2). It is an open question—left to future research—whether this result can actually be extended to a semi-global input-to-state stability result. See the remarks after the proof for a discussion of the obstacles to such an extension.

Theorem 3.3 *Suppose that Condition 2.3 is satisfied and let Θ be the global attractor of the undisturbed system S_0. Then, the disturbed system $(S_u)_{u\in\mathcal{U}}$ is locally input-to-state stable w.r.t. Θ, that is, there exist comparison functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and radii $r_{0x}, r_{0u} > 0$ such that*

$$\|S_u(t,0,x_0)\|_\Theta \leq \beta(\|x_0\|_\Theta, t) + \gamma(\|u\|_\infty) \quad (t \in \mathbb{R}_0^+) \tag{3.15}$$

for all $(x_0, u) \in X \times \mathcal{U}$ with $\|x_0\|_\Theta \leq r_{0x}$ and $\|u\|_\infty \leq r_{0u}$.

Proof Choose an arbitrary $r_0 \in (0,\infty)$ and fix it for the entire proof. Also, take $V = V_{r_0}$ and $\underline{\psi} = \underline{\psi}_{r_0}$, $\overline{\psi}$ as in Lemma 3.1. It then immediately follows from Lemma 3.2 that there exist comparison functions $\alpha = \alpha_{r_0} \in \mathcal{K}$ and $\chi = \chi_{r_0} \in \mathcal{K}$ such that for all $(x_0, u) \in X \times \mathcal{U}$ with $r_0 \geq \|x_0\|_\Theta \geq \chi(\|u\|_\infty)$ one has

$$\dot{V}_u(x_0) \leq -\alpha(\|x_0\|_\Theta). \tag{3.16}$$

(Simply choose $\chi(r) := \alpha_0^{-1}(2\sigma_0(r))$ and $\alpha(r) := \alpha_0(r)/2$, where $\alpha_0, \sigma_0 \in \mathcal{K}_\infty$ are as in Lemma 3.2). According to the comparison lemma from [19] (Corollary 1), we can then choose a comparison function $\overline{\beta} = \overline{\beta}_{\alpha\circ\overline{\psi}^{-1}}$ in such a way that for every $T \in (0,\infty]$ and every function $v \in C([0,T), \mathbb{R}_0^+)$ with

$$\overline{\partial}_t^+ v(t) \leq -(\alpha \circ \overline{\psi}^{-1})(v(t)) \quad (t \in [0,T))$$

one has $v(t) \leq \overline{\beta}(v(0), t)$ for all $t \in [0,T)$. We now define

$$\beta(r,t) := \underline{\psi}^{-1}\big(\overline{\beta}(\overline{\psi}(r), t)\big) \quad \text{and} \quad \gamma(r) := \underline{\psi}^{-1}\big(\overline{\psi}(\chi(r))\big) \tag{3.17}$$

for $r, t \in \mathbb{R}_0^+$ and choose $r_{0x}, r_{0u} \in (0,\infty)$ so small that

$$r_{0x} < r_0 \quad \text{and} \quad \beta(r_{0x}, 0) < r_0 \quad \text{and} \quad \gamma(r_{0u}) < r_0. \tag{3.18}$$

Also, we will write

$$M_u := \left\{x \in \overline{B}_{r_0}(\Theta) : V(x) \le \overline{\psi}(\chi(\|u\|_\infty))\right\} \tag{3.19}$$

for $u \in \mathcal{U}$. Clearly, $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and M_u is closed for every $u \in \mathcal{U}$. Additionally, for every $u \in \overline{B}_{r_{0u}}(0)$ we have by (3.18) that

$$M_u \subset \left\{x \in \overline{B}_{r_0}(\Theta) : \|x\|_\Theta \le \gamma(\|u\|_\infty)\right\} \subset B_{r_0}(\Theta). \tag{3.20}$$

After these preliminary considerations, we now prove that

$$\|S_u(t, 0, x_0)\|_\Theta \le \beta(\|x_0\|_\Theta, t) + \gamma(\|u\|_\infty) \quad (t \in \mathbb{R}_0^+) \tag{3.21}$$

for all $(x_0, u) \in \overline{B}_{r_{0x}}(\Theta) \times \overline{B}_{r_{0u}}(0)$ and thus obtain the desired local input-to-state stability. So, let $(x_0, u) \in \overline{B}_{r_{0x}}(\Theta) \times \overline{B}_{r_{0u}}(0)$ be fixed for the rest of the proof. We will distinguish two cases in the following, namely the case where $x_0 \in M_u$ treated in part (i) of the proof and the case where $x_0 \notin M_u$ treated in part (ii) of the proof.

(i) Suppose we are in the case $x_0 \in M_u$. In order to establish (3.21) in that case, we will show—in two steps—that for every $t_0 \in [0, \infty)$ one has

$$S_u(t, t_0, M_u) \in M_u \quad (t \in [t_0, \infty)). \tag{3.22}$$

So, let $t_0 \in [0, \infty)$ and $x_{t_0} \in M_u$ and

$$T := \sup\left\{T' \in (t_0, \infty) : \|x(t)\|_\Theta < r_0 \text{ for all } t \in [t_0, T')\right\}, \tag{3.23}$$

where we use the abbreviation $x(t) := S_u(t, t_0, x_{t_0})$. Since $x(t_0) = x_{t_0} \in M_u$ and thus $\|x(t_0)\|_\Theta < r_0$ by (3.20), we observe that $T \in (t_0, \infty]$ and that

$$\|x(t)\|_\Theta < r_0 \quad (t \in [t_0, T)). \tag{3.24}$$

As a first step, we show that $x(t) \in M_u$ at least for all $[t_0, T)$. Assuming the contrary, we find a $t \in [t_0, T)$ and an $\varepsilon > 0$ such that $V(x(t)) > \overline{\psi}(\chi(\|u\|_\infty)) + \varepsilon$. Since $x(t_0) = x_{t_0} \in M_u$ and thus $V(x(t_0)) \le \overline{\psi}(\chi(\|u\|_\infty)) + \varepsilon$, we observe that

$$t_1 := \inf\left\{t \in [t_0, T) : V(x(t)) > \overline{\psi}(\chi(\|u\|_\infty)) + \varepsilon\right\} \tag{3.25}$$

belongs to the interval (t_0, T) and, moreover, $V(x(t_1)) = \overline{\psi}(\chi(\|u\|_\infty)) + \varepsilon$. So,

$$\overline{\psi}(\|x(t_1)\|_\Theta) \ge V(x(t_1)) > \overline{\psi}(\chi(\|u\|_\infty)) \ge \overline{\psi}\big(\chi(\|u(\cdot + t_1)\|_\infty)\big),$$

and therefore, we get by virtue of (3.16) that

$$\begin{aligned} \overline{\lim_{t\to 0+}} \frac{1}{t}\Big(V(x(t_1+t)) - V(x(t_1))\Big) &= \overline{\lim_{t\to 0+}} \frac{1}{t}\Big(V\big(S_{u(\cdot+t_1)}(t, 0, x(t_1))\big) - V(x(t_1))\Big) \\ &= \dot{V}_{u(\cdot+t_1)}(x(t_1)) \le -\alpha(\|x(t_1)\|_\Theta) \\ &< 0. \end{aligned} \tag{3.26}$$

Consequently, there exists a $\delta > 0$ such that $V(x(t_1 + t)) \leq V(x(t_1)) = \overline{\psi}(\chi(\|u\|_\infty)) + \varepsilon$ for all $t \in [0, \delta)$. Contradiction to the definition of t_1!
As a second step, we show that $T = \infty$. Indeed, assuming $T < \infty$, we would get by the first step and continuity that even $x(T) \in M_u$ and thus $\|x(T)\|_\Theta < r_0$ by (3.20). And from this, in turn, it would follow again by continuity that $\|x(t)\|_\Theta < r_0$ for all $t \in [T, T + \delta)$ with some $\delta > 0$. In conjunction with (3.24), this would yield a contradiction to the definition (3.23) of T!
Combining now the first and the second step, we finally obtain the desired invariance (3.22), which clearly implies (3.21) in the case $x_0 \in M_u$.

(ii) Suppose we are in the case $x_0 \notin M_u$. In order to establish (3.21) in that case, we will show—in three steps—that for some $t_0 \in (0, \infty]$ one has

$$\|S_u(t, 0, x_0)\|_\Theta \leq \beta(\|x_0\|_\Theta, t) \quad (t \in [0, t_0)) \tag{3.27}$$

$$\|S_u(t, 0, x_0)\|_\Theta \leq \gamma(\|u\|_\infty) \quad (t \in (t_0, \infty)). \tag{3.28}$$

Indeed, let $t_0 := \inf\{t \in \mathbb{R}_0^+ : x(t) \in M_u\}$ and

$$T := \sup\left\{T' \in (0, t_0) : \|x(t)\|_\Theta < r_0 \text{ for all } t \in [0, T')\right\}, \tag{3.29}$$

where we use the abbreviation $x(t) := S_u(t, 0, x_0)$. (In view of the standard convention $\inf \emptyset := \infty$, we have $t_0 = \infty$ in case $x(t) \notin M_u$ for all $t \in \mathbb{R}_0^+$). Since $x(0) = x_0 \in (X \backslash M_u) \cap \overline{B}_{r_{0x}}(\Theta)$ and thus $\|x(0)\|_\Theta < r_0$ by (3.18), we observe that $t_0 \in (0, \infty]$ and $T \in (0, t_0]$ and that

$$x(t) \notin M_u \quad (t \in [0, t_0)) \quad \text{and} \quad \|x(t)\|_\Theta < r_0 \quad (t \in [0, T)). \tag{3.30}$$

As a first step, we show that $\|x(t)\|_\Theta \leq \beta(\|x_0\|_\Theta, t)$ at least for all $t \in [0, T)$. Indeed, in view of (3.30.a) and (3.30.b) we have

$$\overline{\psi}(\|x(t)\|_\Theta) \geq V(x(t)) > \overline{\psi}(\chi(\|u\|_\infty)) \geq \overline{\psi}\big(\chi(\|u(\cdot + t)\|_\infty)\big) \quad (t \in [0, T))$$

and therefore we get by virtue of (3.16) that

$$\begin{aligned} \overline{\partial}_t^+ V(x(t)) &= \overline{\lim_{\tau \to 0+}} \frac{1}{\tau}\Big(V(x(t+\tau)) - V(x(t))\Big) \\ &= \overline{\lim_{\tau \to 0+}} \frac{1}{\tau}\Big(V\big(S_{u(\cdot+t)}(\tau, 0, x(t))\big) - V(x(t))\Big) = \dot{V}_{u(\cdot+t)}(x(t)) \\ &\leq -\alpha(\|x(t)\|_\Theta) \leq -\big(\alpha \circ \overline{\psi}^{-1}\big)\big(V(x(t))\big) \quad (t \in [0, T)). \end{aligned} \tag{3.31}$$

Consequently, by our choice of $\overline{\beta}$ we see that

$$V(x(t)) \leq \overline{\beta}(V(x(0)), t) \quad (t \in [0, T)).$$

In view of (3.30.b) and our definition (3.17) of β, the assertion of the first step is then clear.

As a second step, we show that $T = t_0$. Indeed, assuming $T < t_0$, we would get by the first step and continuity that even $\|x(T)\|_\Theta \leq \beta(\|x_0\|_\Theta, T) \leq \beta(r_{0x}, 0)$ and thus $\|x(T)\|_\Theta < r_0$ by (3.18). And from this, in turn, it would follow that $\|x(t)\|_\Theta < r_0$ for all $t \in [T, T+\delta)$ with some $\delta > 0$. In conjunction with (3.30.b), this would yield a contradiction to the definition (3.29) of T!

As a third step, we show that $\|x(t)\|_\Theta \leq \gamma(\|u\|_\infty)$ for all $t \in [t_0, \infty)$. We can assume $t_0 < \infty$ because in the case $t_0 = \infty$ the assertion is empty. So, by the definition of t_0 it then follows that $x(t_0) \in M_u$ and therefore by virtue of (3.22)

$$x(t) = S_u(t, 0, x_0) = S_u(t, t_0, x(t_0)) \in M_u \quad (t \in [t_0, \infty)).$$

In view of (3.20), the assertion of the third step is then clear.

Combining now the first, second and third step, we finally obtain the desired estimates (3.27) and (3.28), which clearly imply (3.21) in the case $x_0 \notin M_u$. □

An inspection of the above proof shows that we actually proved a bit more than local input-to-state stability, namely we have: for every $r_0 > 0$ there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and $r_{0x}, r_{0u} > 0$ such that the estimate (3.21) holds true for all $\|x\|_\Theta \leq r_{0x}$ and $\|u\|_\infty \leq r_{0u}$. So, if by choosing r_0 large enough we could also ensure that r_{0x} and r_{0u} with (3.18) can be chosen arbitrarily large, we would even have semi-global input-to-state stability. Yet, this is not so clear because the functions $\beta = \beta_{r_0}$ and $\gamma = \gamma_{r_0}$ from (3.18) which determine our choice of r_{0x} and r_{0u} depend on r_0 themselves (basically because $V = V_{r_0}$ and $\underline{\psi} = \underline{\psi}_{r_0}$ depend on r_0 as was pointed out after Lemma 3.1). We, therefore, leave the question of semi-global input-to-state stability to future research.

We would finally like to remark that a similar parabolic system in one space dimension was considered in [4] (Example 4.1). In that example, the nonlinear function was chosen such that the origin is the only attracting point of the unperturbed system, and the input-to-state stability was established by means of a rather standard Lyapunov function that cannot be used here for two reasons: first, we have a wider state space and second, we need a Lyapunov function that vanishes not only in the origin but on the entire attractor, which is in our case larger than just one point. This requires a more sophisticated construction and rather different estimations.

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