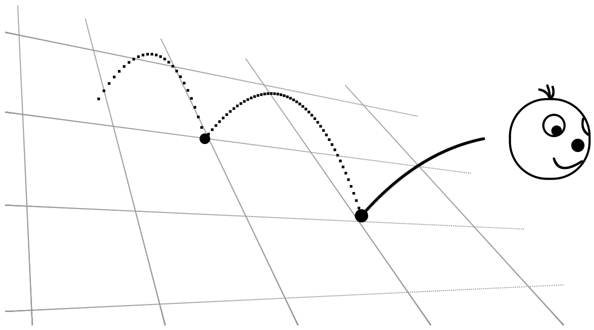
Dissertation
zur Erlangung des naturwissenschaftlichen Doktorgrades
der Julius-Maximilians-Universität Würzburg

On Hydrodynamic Limits and Conservation Laws



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Würzburg, July 2009

Acknowledgments

In the first place I would like to express my sincere gratitude to my supervisor Prof. Christian Klingenberg for his patient support, guidance and encouragement during the years of this work. He provided me with an outstanding scientific surrounding and a sure instinct always leading me in the right direction. This thesis would not have been possible without him.

I am very grateful to Prof. Gui-Qiang Chen for many hours of fruitful discussions and warm hospitality during my annual visits at Northwestern University in Evanston and his visits in Heidelberg. His energy, scientific contribution and ingenious intuition have been essential for my motivation and the development of this thesis.

I thank Prof. Stefano Olla from Université Dauphine in Paris. During the past half year we had many helpful discussions there that contributed to the completion of the second part of my work.

I also owe my gratitude to several people whose constructive comments and remarks were very helpful to me. In particular these are Prof. Fraydoun Rezakhanlou and Prof. József Fritz.

Finally, my special thanks go to my family for a continuous and unconditional support, at any time and during all my ups and downs! I thank my parents Marie-Josée and Norbert, my sister Lynn, my brother Patrick and my friend Stephan for always keeping me in their mind.

Contents

Introduction

" In reality we know nothing, since the truth is at the bottom" (Demokritus, Fr. 117, ∼400 BC), a statement made 2500 years ago in ancient Greece. In this spirit people back then started to construct "microscopic" models by introducing atoms and thereby tried to explain nature. Even though nowadays this philosophy became a science and our understanding in physics is remarkable, there are still many open questions.

I am interested in modeling phenomena from nature both microscopically and macroscopically, motivated by the quest to understand the macroscopic equations of continuum mechanics by deriving them from microscopic statistical mechanics

The microscopic models consist of moving particles. They can be constructed in several ways. One way is to impose on particles a probability law which defines their movement. These interacting particle systems are for example useful to describe traffic flow, percolation, movement of sand piles or flow through porous media.
In other models particles are moved by laws of classical thermodynamics. These models describe for example gas dynamics, heat conduction, harmonic oscillators or elasticity.
In both cases it turns out, that the corresponding macroscopic characterization are partial differential equations where in particular the nonlinear conservation laws arouse my interest. Here one challenge is to discover solutions which are well posed and physically relevant.

One of my motivation for considering macroscopic as well as microscopic approaches is motivated by the difference in information in these two modelings and the relationship between them. My hope is that analytically one can use the microscopic information of a system to be naturally led to a physically meaningful solution to a conservation law. An example here are certain descriptions of flow through porous media where appropriate macroscopic entropy conditions are not clear. One step in this direction has been done in the first part of my thesis (see also [10, 11, 19]).

The description at the microscopic level typically involves stochastic elements. Here we are in the realm of statistical thermodynamics. The description at the macroscopic level typically involves nonlinear partial differential equations. One should mention that various limits can be taken when going from the discrete to the continuum description. Technically the least challenging is the so called moderate limit of Oelschlaeger. Another limit is the parabolic limit when diffusion dominates advection. Most challenging though is the hyperbolic limit where advection dominates diffusion. This typically leads to the nonlinear PDEs of the conservation law type, for which on the macroscopic level the well posedness of solutions in many cases can not be shown. The probably most popular example here are the Euler Equations.

In my work I have tried to marry statistical physics with hyperbolic conservation laws:

- In Chapter 1 of my thesis I am considering flow through porous media, i.e the macroscopic description is a scalar conservation law. Here the new feature is that we allow sudden changes in porosity and thereby the flux may have discontinuities in space. Microscopically this is described through an interacting particle system having only one conserved quantity namely the total mass.

- In Chapter 2 of my thesis I am considering an Hamiltonian system with boundary conditions. Microscopically this is described through a system of coupled oscillators and hence besides the density of particles also momenta and energy play a role. Macroscopically this will lead to a system of conservation laws.

Nonlinear scalar conservation laws with discontinuous fluxes and hydrodynamic limit of interacting particle systems [11].

Macroscopically flow through porous media is given by the following hyperbolic class of scalar conservation laws:

$$\partial_t \rho + \partial_x F(x, \rho(t,x)) = 0 \tag{0.0.1}$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x), \tag{0.0.2}$$

where $F(\cdot, \rho)$ is continuous except on a set of measure zero.

The difficulty of (0.0.1) is the discontinuity of the flux function F in the space variable x arising from sudden changes in porosity. Recall that for fluxes without discontinuities, this partial differential equation is well studied by Kruzkov in [26]: Let $F^\epsilon(x, \rho)$ be the standard mollification of $F(x, \rho)$ in $x \in \mathbb{R}$ defined by (1.2.1) and consider the following Cauchy problem:

$$\begin{cases} \partial_t \rho + \partial_x F^\epsilon(x, \rho) = 0, \\ \rho|_{t=0} = \rho_0(x) \geq 0, \end{cases} \tag{0.0.3}$$

then Kruzkov proved the existence and uniqueness of an L^∞ solution $\rho : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfying the following two properties:

(i) ρ satisfies the entropy inequality

$$\partial_t |\rho(t,x) - c| + \partial_x \big(\text{sign}\,(\rho(t,x) - c)\,(F^\epsilon(x, \rho(t,x)) - F^\epsilon(x,c))\big)$$
$$+\text{sign}(\rho(t,x) - c)\partial_x F^\epsilon(x,c) \leq 0 \tag{0.0.4}$$

for any constant $c \in \mathbb{R}$ in the sense of distributions, that means that for any smooth, positive function $J : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}^+$ we have the following:

$$\int |\rho(t,x) - c| \partial_t J dx dt$$
$$+ \int \text{sign}(\rho(t,x) - c)\,(F^\varepsilon(x,\rho) - F^\varepsilon(x,c))\,\partial_x J dx dt$$
$$+ \int \text{sign}(\rho(t,x) - c)\partial_x F^\varepsilon(x,c) J(t,x) dx dt + \int |\rho(0,x) - c| J(0,x) dx \geq 0$$

(ii) $\rho(t,\cdot)$ converges in $L^1(\mathbb{R})$ to $\rho_0(\cdot)$ as t decreases to 0:

$$\lim_{t\to 0}\int_{\mathbb{R}} |\rho(t,x)-\rho_0(x)|\,du = 0$$

Notice that if the flux function F has a discontinuity in the space the derivative in the third term of (0.0.4) does not make sense and thus the Kruzkov approach does not apply anymore. Therefore discontinuity in space causes new important difficulties in conservation laws.

Several different entropy conditions have been suggested in the literature (see [1, 3, 5, 7, 14, 23, 25, 32] and the references therein). One type of entropy conditions involves a rule how the solution should behave at the jump wave induced by the discontinuity in the flux, that is, the solution is required to satisfy an additional condition on its traces at the discontinuous points of the flux function, for which the existence of traces of the solution is needed. An alternative entropy condition in [3, 5] is an adapted entropy condition that uses steady state solutions to (0.0.1) to replace the constant parameter in the Kruzkov entropy inequality. For this class of entropy solutions uniqueness has been shown, but not existence in many cases.The replacement of the constant c by a steady state solution has the advantage, that the bad third term of (0.0.4) disappears. The entropy inequality has the form

$$\partial_t|\rho(t,x)-m_\alpha(x)| + \partial_x\left(\mathrm{sign}\left(\rho(t,x)-m_{\alpha(x)}\right)\left(F^\varepsilon(x,\rho(t,x))-F^\varepsilon(x,m_{\alpha(x)})\right)\right)\le 0$$

in the distributional sense. Here $m_\alpha:\mathbb{R}\to\mathbb{R}$ denotes a steady state solution to (0.0.1) (for more details of this, see Section 1.1) such that for a constant α

$$F(x,m_\alpha(x))=\alpha.$$

This is quite an attractive notion since it does not require the traces of the entropy solution, which allows the solution only in L^∞. In Chapter 1, we establish the well-posedness in L^∞ for conservation laws with a certain class of flux functions (cf. conditions (H1)–(H2) and (H3) or (H3') in Section 1.1) by providing an existence proof to supplement the uniqueness result in [3].

It happens, that different classes of entropy conditions proposed in the literature may lead to different unique solutions for the same initial data. For example the entropy condition based on the traces of solutions at the jump waves has lead to the existence and uniqueness of the solutions for a wider class of flux functions than those satisfying (H1)–(H2) and (H3) or (H3') in Section 1.1. The Cauchy problem (even the Riemann problem) may lead to different solutions depending on which choice of the conditions on the traces of solutions is made (for an example, see [3]). If one restricts oneself to the flux functions satisfying (H1)–(H2) and (H3') in Section 1.1 (in which $F(x,\cdot)$ in (0.0.1) is monotone) and to the entropy solutions in the class of functions of bounded variation, the two notions of entropy conditions addressed above will lead to the same solution. This is not the case for the flux functions satisfying (H1)–(H2) and (H3) in which $F(x,\cdot)$ may be non-monotone.

On the other hand, in statistical mechanics, some microscopic interacting particle systems with discontinuous speed-parameter $\lambda(x)$, in the hydrodynamic limit, formally lead to scalar hyperbolic conservation laws with discontinuous flux of the form

$$\partial_t\rho + \partial_x\left(\lambda(x)h(\rho)\right) = 0 \tag{0.0.5}$$

and with initial data (0.0.2), where $\lambda(x)$ is continuous except on a set of measure zero and $h(\rho)$ is Lipschitz continuous. Here the discontinuity in space is given by the speed parameter λ, which is continuous except on a set of measure zero. In particular the hydrodynamic limit naturally gives rise to an entropy condition described in [3, 5], for which as already mentioned uniqueness has been shown, but not the existence. Rezakhanlou in [33] first established the hydrodynamic limit of the processus des misanthropes (PdM) with constant speed-parameter. Covert-Rezakhanlou [16] provided a proof of the hydrodynamic limit of a PdM with nonconstant but continuous speed-parameter λ. In both proofs, the most important step is to show an entropy inequality at microscopic level leading to the (macroscopic) Kruzkov entropy inequality, in the limit when the distance between particles tends to zero, and thereby implies the uniqueness of limit points. Bahadoran in [4] proved the hydrodynamic limit of a special case of the processus des misanthropes, the so called simple exclusion process with continuous speed parameter. The methods in [16] and [4] are essentially the same, but they used different characterizations of microscopic entropy inequalities. Seppäläinen in [34] proved the hydrodynamic limit of a K-exclusion process with constant speed parameter, which also is again a special case of the processus des misanthropes using a coupling of a process with an arbitrary initial configuration with a family of processes with simple initial configurations. This technique has the advantage that the hydrodynamic limit can be derived without knowledge of invariant product measures which is crucial in [33, 16] and [4], but it has the disadvantage, that it is not possible to express the flux as a function of ρ.

Equation (0.0.5) is equivalent to the following 2×2 hyperbolic system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x(\lambda h(\rho)) = 0, \\ \partial_t \lambda = 0. \end{cases} \tag{0.0.6}$$

In particular, when $h(\rho)$ is not strictly monotone, system (0.0.6) is nonstrictly hyperbolic, one of the main difficulties in conservation laws (cf. [9, 13]). The natural question is which entropy solution the hydrodynamic limit selects, thereby leading to a suitable, physical relevant notion of entropy solutions of this class of conservation laws. Chapter 1 in this work is a first step in this direction and provides an answer to this question for a family of discontinuous flux functions via an interacting particle system, namely, the attractive zero range process (ZRP). The ZRP leads to a conservation law of the form (0.0.2) with $\lambda(x) > 0$ and $h(\rho)$ being monotone in ρ. Furthermore, its hydrodynamic limit naturally gives rise to an entropy condition of the type described in [3, 5].

Motivated by the hydrodynamic limit of the ZRP we adopt the notion of entropy solutions in the sense of Audusse-Perthame [3] for a class of conservation laws with discontinuous flux functions, including the non-monotone case, and establish the existence of such an entropy solution via the method of compensated compactness in Section 1.2. This completes the well-posedness in L^∞ by combining the uniqueness result established in [3] for this class of conservation laws under their notion of entropy solutions.

In order to establish the hydrodynamic limit of large particle systems and the convergence of other approximate solutions to (0.0.1) rigorously, we establish a compactness framework for (0.0.1)–(0.0.2) in Section 1.1. This mathematical framework is based on the notion and reduction of measure-valued entropy solutions developed in Section 1.1, which is also applied for another proof of the existence of entropy solutions for the non-monotone case in Section 1.2.

In Section 1.3, we establish the hydrodynamic limit for a ZRP with discontinuous speed-parameter $\lambda(x)$ governed by the unique entropy solution of the Cauchy problem (0.0.2)–(0.0.5).

Hydrodynamic limit of Hamiltonian systems with boundary conditions and systems of conservation laws.

The zero range process considered in Chapter 1 is a purely stochastic process, there are no physical assumptions, and hence there is no velocity or energy associated to the particles. In Chapter 2 we consider Hamiltonian systems with boundary conditions.

The microscopic model we consider is a one dimensional chain of N coupled oscillators. The interaction between particles is now defined through a spring with a potential energy V. Thus to each particle denoted by i, there is associated a position x_i and a momentum p_i. The boundary conditions we impose are the following: we attach the first particle to a wall and on the last particle we apply a force $\tau(t)$ depending on time, which is a pressure or a tension. Then the Hamiltonian reads as

$$\mathcal{H}_N^\tau(\mathbf{x},\mathbf{p}) := \frac{1}{2}\sum_{i=1}^N p_i^2 + \sum_{i=1}^N \left(V(x_i - x_{i-1}) - \tau(t)(x_i - x_{i-1})\right).$$

and in contrast to the ZRP, besides total mass we now must handle with two more physical quantities namely momentum and energy. Our goal is to prove that in the hyperbolic limit, that means after a rescaling of time and space in the same way, the conserved quantities in time satisfy a system of partial differential equations which we will specify below. The technics we use for the derivation of the macroscopic equation are based based on [31]. In this paper, they prove the hydrodynamic limit of a Hamiltonian system with weak noise but without boundary conditions. They chose the much more complicated system on the continuum and in 3 dimensions. This system is governed by the full system of Euler equations in 3 dimensions:

$$\begin{cases} \frac{d}{dt}\rho + \sum_{j=1}^3 \frac{\partial}{\partial x}\left(\rho u_j\right) & = & 0 \\ \frac{d}{dt}(\rho u_j) + \sum_{j=1}^3 \frac{\partial}{\partial x}\left(\rho u_j u_j + \delta_{ij}P(\rho,e)\right) & = & 0 \\ \frac{d}{dt}(\rho E) + \sum_{j=1}^3 \frac{\partial}{\partial x}\left(\rho E u_j - u_j P(\rho,e)\right) & = & 0. \end{cases} \tag{0.0.7}$$

This system describes gas dynamics. Here ρ denotes the density, u_j are the 3 components of velocity, E denotes the energy and P is the pressure as a function of the density and the internal energy e. Thus there is conservation of total mass, momentum and energy.
To give an overview on the subject we would like to touch on the main difficulties the authors encountered in deriving the hydrodynamic limit of this system:

(i) The first difficulty is that the deterministic Hamiltonian system has not enough ergodicity: indeed there exists a family of spatially homogenous Gibbs measures for the system, but there is no hope to prove the strong ergodicity hypotheses, which roughly says, that every stationary, translation invariant measure of the infinite stochastic dynamics is a superposition of Gibbs distributions. This problem has been addressed by adding some randomness to the system in terms of a weak conservative noise. This noise exchanges the momenta of nearby particles. It is chosen such that it provides the system with enough ergodicity, but does not change the hydrodynamic limit, that means the three conservation laws are still satisfied in the limit. This technique, on the macroscopic level, can be compared to the vanishing viscosity method, which is a powerful method of proving existence and uniqueness of solutions to hyperbolic systems of conservation laws in many cases (See [13]).

(ii) The second difficulty is rather technical: during the proof it is important, that the kinetic energy is uniformly bounded. Since there are no effective truncation techniques to handle the large velocities, this has been dealt with by replacing the natural kinetic energy, by a function of the velocity, having bounded gradient.

(iii) The main part of the proof relies on the relative entropy method of [38]. The relative entropy measures the distance between the distribution of the actual evolution of the system and the distribution of a local equilibrium with their relative entropy. The method is based on the principle that mean values of functions of the rescaled process should be calculated by means of a product local equilibrium measure (see [24, 35]), also known as the one-block estimate. Unfortunately this method requires the smoothness of solutions. But even for smooth initial data, a weak solutions to (0.0.7) may produce shocks after a certain time. Therefore in [31] they are forced to restrict their proof to the smooth regime of the Euler equations.
A proof of the hydrodynamic limit which goes beyond the shock is a long standing open problem. Also, even if there is a possibility to show the hydrodynamic limit, it is still open whether this limit is unique: on the macroscopic level it is an open problem to prove well posedness of weak solutions when the solution enters a shock, because there is not enough entropy.

In the second part of my thesis, I am using the technics of [31], to derive the hydrodynamic limit for an Hamiltonian system in one dimension with boundary conditions. As already mentioned above the microscopic model we use is a system of N coupled oscillators in one dimension. This means, that we consider atoms sitting on a one dimensional lattice and moving around their equilibrium position. We chose the one dimensional discrete lattice to be of length 1 and having N points, then to each point of the lattice there is associated an atom with mass equal to one.

Since this is a nearest neighbor interaction, we can rewrite the problem in terms of the deformation $r_i := x_i - x_{i-1}$ also known as Lagrangian coordinates. Of course the introduction of boundary conditions implies several problems. For a better understanding of the problem we therefore changed the conservative noise (see 2.4) , in such a way that particles only exchange velocities randomly. With this noise the total energy is not conserved anymore, and a thermal equilibrium is maintained. Thus in the hydrodynamic limit the particle density and momentum satisfy the so called p-system of two conservation laws:

$$\begin{cases} \partial_t \mathfrak{r} - \partial_x \mathfrak{p} = 0 \\ \\ \partial_t \mathfrak{p} - \partial_x P(\mathfrak{r}) = 0 \end{cases}$$

and with boundary conditions

$$\begin{cases} \mathfrak{r}_0(x) = \mathfrak{r}(x,0),\ \mathfrak{p}_0(x) = \mathfrak{p}(x,0) \\ \\ \mathfrak{p}(0,t) = 0,\ P(\mathfrak{r}(1,t)) = \tau(t) \end{cases}$$

For bounded, smooth initial data $\mathfrak{r}_0, p_0 : [0,1] \to \mathbb{R}$ and the force $\tau(t)$ depending on time t. Now the pressure P is a function of the specific volume $\mathfrak{r}$ only. This system has the advantage that on the macroscopic level it is understood much better than (0.0.7), since here existence of solutions is already proved. Notice also that we may chose the force $\tau(t)$ applied on the last particle such that there is no shock produced. This would mean that we can prove the hydrodynamic limit for all times.

To derive the hydrodynamic limit, in Chapter 2 we proceed as follows:

A detailed description of the microscopic model and its underlying equilibrium measures will be given In Sections 2.1,2.3 and 2.4. In Section 2.2 we give a short sketch of the proof for the existence of C^1 solutions to the initial boundary value problem given above.

The proof of the hydrodynamic limit will be done in Section 2.5. Here we have to handle with additional terms when carefully computing the relative entropy due the work done by the system when we apply the force τ. In the proof of the one-block estimate, which, in view of the strong ergodic hypothesis, requires a characterization of the stationary and translation invariant measures as convex combination combination of Gibbs measures, local averages where necessary to the translation invariance of the measures.

Chapter 1

Hyperbolic Conservation Laws with Discontinuous Fluxes and Hydrodynamic Limit for Particle Systems

1.1 Notion and Reduction of measure-valued entropy solutions

In this section, we first develop the notion of measure-valued entropy solutions and establish their reduction to entropy solutions in L^∞ (provided that they exist) of the Cauchy problem

$$\partial_t \rho + \partial_x F(x, \rho(t, x)) = 0 \tag{1.1.1}$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x), \tag{1.1.2}$$

satisfying that

(H1) $F(x, \rho)$ is continuous at all points of $(\mathbb{R} \backslash \mathcal{N}) \times \mathbb{R}$ with $\mathcal{N}$ a closed set of measure zero;

(H2) $\exists$ continuous functions f, g such that, for any $x \in \mathbb{R}$ and large ρ, $f(\rho) \leq |F(x, \rho)| \leq g(\rho)$ with $f(\rho) \geq 0$ and $f(\pm\infty) = \infty$;

(H3) There exists a function $\rho_m(x)$from $\mathbb{R}$ to $\mathbb{R}$ and a constant M_0 such that, for $x \in \mathbb{R} \backslash \mathcal{N}$, $F(x, \rho)$ is a locally Lipschitz, one to one function from $(-\infty, \rho_m]$ and $[\rho_m, \infty)$ to $[M_0, \infty)$ (or $(-\infty, M_0]$) with $F(x, \rho_m(x)) = M_0$ and with common Lipschitz constant L_I for all $x \in \mathbb{R} \backslash \mathcal{N}$ and all $\rho \in I$ that is any bounded interval in $\mathbb{R}$;

or

(H3') For $x \in \mathbb{R}\backslash\mathcal{N}$, $F(x,\cdot)$ is a locally Lipschitz, one to one function from $\mathbb{R}$ to $\mathbb{R}$ with common Lipschitz constant L_I for all $x \in \mathbb{R}\backslash\mathcal{N}$ and all $\rho \in I$ that is any bounded interval in $\mathbb{R}$.

One example of the flux functions satisfying (H1)–(H2) and (H3) or (H3') is

$$F(x,\rho) = \lambda(x)h(\rho), \tag{1.1.3}$$

where $\lambda(x)$ is continuous in $x \in \mathbb{R}$ with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for some constants λ_1 and λ_2, except on a closed set $\mathcal{N}$ of measure zero, and $h(\rho)$ is locally Lipschitz and is either monotone or convex (or concave) with $h(\rho_m) = 0$ for some ρ_m in which case $M_0 = 0$.

It is easy to check that, if the flux function $F(x,\rho)$ satisfies (H1)–(H3), then, for any constant $\alpha \in [M_0,\infty)$ (or $\alpha \in (-\infty, M_0]$), there are two steady-state solutions m_α^+ from $\mathbb{R}$ to $[\rho_m(x),\infty)$ and m_α^- from $\mathbb{R}$ to $(-\infty,\rho_m(x)]$ of (1.1.1) such that

$$F(x, m_\alpha^\pm(x)) = \alpha \qquad \text{for a.e. } x \in \mathbb{R}. \tag{1.1.4}$$

In the case (H1)–(H2) and (H3'), $m_\alpha^+(x) = m_\alpha^-(x)$ which is even simpler.

1.1.1 Notion of measure-valued entropy solutions

First, the notion of entropy solutions in L^∞ introduced in Audusse-Perthame [3] and Baiti-Jenssen [5] can be further formulated into the following.

Definition 1.1.1 (Notion of entropy solutions in L^∞)**.** *We say that an L^∞ function $\rho : \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is an entropy solution of* (1.1.1)–(1.1.2) *provided that, for each $\alpha \in [M_0,\infty)$ (or $\alpha \in (-\infty, M_0]$) and the corresponding two steady-state solutions $m_\alpha^\pm(x)$ of* (1.1.1),

$$\int \Big(|\rho(t,x) - m_\alpha^\pm(x)|\,\partial_t J \,+ sign(\rho(t,x) - m_\alpha^\pm(x))\big(F(x,\rho(t,x)) - \alpha\big)\,\partial_x J\Big)\,dtdx + \int |\rho_0(x) - m_\alpha^\pm(x)| J(0,x)\,dx \geq 0 \tag{1.1.5}$$

for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$.

It is easy to see that any entropy solution is a weak solution of (1.1.1)–(1.1.2) by choosing α such that $m_\alpha^+(x) \geq \|\rho\|_{L^\infty}$ and $m_\alpha^-(x) \leq -\|\rho\|_{L^\infty}$, respectively, for a.e. $x \in \mathbb{R}$.

From the uniqueness argument in Audusse-Perthame [3] (also see [12]), one can deduce that, for any $L > 0$,

$$\lim_{t\to 0} \int_{|x|\leq L} |\rho(t,x) - \rho_0(x)|\,dx = 0. \tag{1.1.6}$$

Following the notion of entropy solutions, we introduce the corresponding notion of measure-valued entropy solutions. We denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

Definition 1.1.2 (Notion of measure-valued entropy solutions)**.** *We say that a measurable map*

$$\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$$

is a measure-valued entropy solution of (1.1.1)–(1.1.2) *provided that* $\langle \pi_{0,x}; k\rangle = \rho_0(x)$ *for a.e.* $x \in \mathbb{R}$ *and, for each* $\alpha \in [M_0, \infty)$ *(or* $\alpha \in (-\infty, M_0]$*) and the corresponding two steady-state solutions* $m_\alpha^\pm(x)$ *of* (1.1.1)*,*

$$\int \Big(\langle \pi_{t,x}; |k - m_\alpha^\pm(x)|\rangle\, \partial_t J + \langle \pi_{t,x}; sign(k - m_\alpha^\pm(x))\,(F(x,k) - \alpha)\rangle\, \partial_x J\Big)\, dxdt$$
$$+ \int |\rho_0(x) - m_\alpha^\pm(x)|\, J(0,x)\, dx \ge 0 \tag{1.1.7}$$

for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$*.*

If a measure-valued entropy solution $\pi_{t,x}(k)$ is a Dirac mass with the associated profile $\rho(t,x)$, i.e. $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$, then $\rho(t,x)$ is an entropy solution of (1.1.1)–(1.1.2), which is unique as shown in [3].

Note that, when the flux function $F(x,\rho)$ is locally Lipschitz in ρ and globally Lipschitz in x, one can use the Kruzkov entropy inequality, instead of (1.1.7), to formulate the following notion of measure-valued solutions:

$$\partial_t \langle \pi_{t,x}; |k - c|\rangle + \partial_x \langle \pi_{t,x}; \mathrm{sign}(k - c)\,(F(x,k) - F(x,c))\rangle + \langle \pi_{t,x}; \mathrm{sign}(k-c)\partial_x F(x,c)\rangle \le 0 \tag{1.1.8}$$

in the sense of distributions and to establish their reduction as in DiPerna [18]. One of the new features in our formulation (1.1.7) in Definition 1.1.2 is that the constant c in (1.1.8) is replaced by the steady-state solutions $m_\alpha^\pm(x)$ such that the additional third term in (1.1.8) vanishes, as in [3, 5], and thereby allows the discontinuity of the flux functions on a closed set of measure zero for measure-valued entropy solutions.

1.1.2 Reduction of measure-valued entropy solutions

In this section we first establish the reduction of measure-valued entropy solutions of (1.1.1)–(1.1.2) and prove that any measure-valued entropy solution $\pi_{t,x}(k)$ in the sense of Definition 1.1.2 is the Dirac solution such that the associated profile $\rho(t,x)$ is an entropy solution in the sense of Definition 1.1.1. That is, our goal is to establish that, when $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$,

$$\pi_{t,x}(k) = \delta_{\rho(t,x)}(k), \tag{1.1.9}$$

where $\rho : \mathbb{R}_+^2 \to \mathbb{R}$ is the unique entropy solution determined by (1.1.5). The reduction proof is achieved by two theorems. We start with the following theorem which yields the L^1-contraction between the measure-valued entropy solution $\pi_{t,x}$ and the unique entropy solution $\rho(t,x)$ of (1.1.1)–(1.1.2).

Theorem 1.1.3 (L^1-contraction)**.** *Assume that there exists a measure-valued entropy solution* $\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$ *of* (1.1.1) *in the sense of Definition* 1.1.2 *with* $\pi_{t,x}$ *having a fixed compact support for a.e.* (t,x)*. Assume that there exists a function* $\rho : \mathbb{R}_+^2 \to \mathbb{R}$ *with initial data* $\rho_0 \in L^\infty(\mathbb{R})$ *and* $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ *for a.e.* $x \in \mathbb{R}$ *satisfying the following inequality:*

$$\int \Big(\langle \pi_{t,x}; |k - \rho(t,x)|\rangle\, \partial_t J + \langle \pi_{t,x}; sign(k - \rho(t,x))(F(x,k) - F(x,\rho(t,x)))\rangle\, \partial_x J\Big)\, dxdt \ge 0 \tag{1.1.10}$$

for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$*. Then the function* $\int \langle \pi_{t,x}; |k - \rho(t,x)| \rangle \, dx$ *is non-increasing in* $t > 0$*, which implies* $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ *when* $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ *for a.e.* $x \in \mathbb{R}$*. Furthermore,* ρ *is the unique entropy solution of* (1.1.1)–(1.1.2) *in the sense of Definition* 1.1.1.

Proof. In expression (1.1.10), we choose the test function as the product test function $J_j(t)H(x)$, with $J_j(t)$ converging to the indicator function $\mathbb{1}_{[t_1,t_2]}(t)$ as $j \to \infty$ for $t_2 > t_1 \geq 0$. Then (1.1.10) is equivalent to

$$\int H(x)\langle \pi_{t_1,x}(k); |k - \rho(t_1,x)|\rangle \, dx - \int H(x)\langle \pi_{t_2,x}(k); |k - \rho(t_2,x)|\rangle \, dx \\ + \int_{t_1}^{t_2} \int H'(x)\langle \pi_{t,x}(k); \operatorname{sign}(k - \rho(t,x)) \big(F(x,k) - F(x,\rho(t,x))\big)\rangle \, dxdt \geq 0. \quad (1.1.11)$$

In (1.1.11), we choose

$$H(x) = e^{-\gamma\sqrt{1+|x|^2}}\chi(\frac{x}{N}), \qquad \gamma, N > 0,$$

for $\chi \in C_0^\infty(-2,2)$ with $\chi(x) = 1$ when $x \in [-1,1]$ and $\chi(x) \geq 0$. Letting $N \to \infty$ first and $\gamma \to 0$ then yields that, for any $t_2 > t_1 \geq 0$,

$$\int \langle \pi_{t_2,x}; |k - \rho(t_2,x)|\rangle \, dx - \int \langle \pi_{t_1,x}; |k - \rho(t_1,x)|\rangle \, dx \leq 0.$$

In particular, when $t_2 = t > 0, t_1 \to 0$, then $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ implies

$$\int \langle \pi_{t,x}; |k - \rho(t,x)|\rangle dx \leq 0$$

so that $\pi_{t,x}(k) = \delta_{\rho(t,x)}(k)$ for any $t > 0$.

Plugging this into inequality (1.1.7), we obtain inequality (1.1.5). Thus, $\rho(t,x)$ is an entropy solution which is unique by [3]. □

It thus remains to prove inequality (1.1.10).

Theorem 1.1.4. *Assume that* $\rho : \mathbb{R}_+^2 \to \mathbb{R}$ *is the unique entropy solution of* (1.1.1)–(1.1.2) *with initial data* $\rho_0 \in L^\infty(\mathbb{R})$*. Assume that there exists a measure-valued entropy solution* $\pi : \mathbb{R}_+^2 \to \mathcal{P}(\mathbb{R})$ *of* (1.1.1) *in the sense of Definition* 1.1.2 *with* $\pi_{t,x}$ *having a fixed compact support for a.e.* (t,x) *and* $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$ *for a.e.* $x \in \mathbb{R}$*. Then inequality* (1.1.10) *holds for any test function* $J : \mathbb{R}_+^2 \to \mathbb{R}_+$.

Proof. The proof is divided into nine steps.

Step 1. We first notice the following:

- Under assumption (H3'), $F(x,\rho)$ is continuous in x a.e.. Then we can define a function $\tilde{\rho}(s,y,x)$ for a.e. $(s,y,x) \in \mathbb{R}_+ \times \mathbb{R}^2$ such that, for fixed (s,y),

$$F(x, \tilde{\rho}(s,y,x)) := F(x, m_{F(y,\rho(s,y))}(x)) = F(y, \rho(s,y)), \quad (1.1.12)$$

where the last equality follows from (1.1.4). Thus, we define

$$\tilde{\rho}(s,y,x) = m_{\beta(s,y)}(x) \qquad \text{with} \quad \beta(s,y) := F(y,\rho(s,y)).$$

In the same way, we can define a function $\tilde{m}(x,c,y)$ for any constant $c \in \mathbb{R}$ and for a.e. $(x,y) \in \mathbb{R}^2$ such that, for fixed x,

$$F(y,\tilde{m}(x,c,y)) := F(y, m_{F(x,c)}(y)) = F(x,c). \tag{1.1.13}$$

Thus, we define

$$\tilde{m}(x,c,y) = m_{\gamma(x,c)}(y) \qquad \text{with} \quad \gamma(x,c) := F(x,c).$$

- For the case (H3), we define $\tilde{\rho}(s,y,x)$ such that the sign of the difference between $\tilde{\rho}(s,y,x)$ and $\rho_m(y)$ is the same as the sign of the difference between the corresponding solution and $\rho_m(y)$, that is,

$$\operatorname{sign}(\rho(s,y) - \rho_m(y)) = \operatorname{sign}(\tilde{\rho}(s,y,x) - \rho_m(y)). \tag{1.1.14}$$

It can be achieved by defining

$$\tilde{\rho}(s,y,x) := m^+_{\beta(s,y)}(x)\operatorname{sign}_+(\rho(s,y) - \rho_m(y)) + m^-_{\beta(s,y)}(x)\operatorname{sign}_-(\rho(s,y) - \rho_m(y)), \tag{1.1.15}$$

since $\rho_m(y)$ is the minimum (or maximum) point of the flux function with $F(y,\rho_m(y)) = M_0$.

Similarly, we define

$$\tilde{m}(x,c,y) := m^+_{\gamma(x,c)}(y)\operatorname{sign}_+(c - \rho_m(x)) + m^-_{\gamma(x,c)}(y)\operatorname{sign}_-(c - \rho_m(x)). \tag{1.1.16}$$

Then we have as in (1.1.12) and (1.1.13),

$$F(x,\tilde{\rho}(s,y,x)) = F(y,\rho(s,y)) = \beta(s,y),$$

and

$$F(y,\tilde{m}(x,c,y)) = F(x,c) = \gamma(x,c).$$

With these notations, we can rewrite inequality (1.1.7) as follows:

$$\partial_t \langle \pi_{t,x}; |k - \tilde{\rho}(s,y,x)| \rangle + \partial_x \big\langle \pi_{t,x}; \operatorname{sign}(k - \tilde{\rho}(s,y,x))\big(F(x,k) - F(y,\rho(s,y))\big)\big\rangle \le 0 \tag{1.1.17}$$

in the sense of distributions, and inequality (1.1.5) can be rewritten as

$$\partial_s |\rho(s,y) - \tilde{m}(x,k,y)| + \partial_y \left(\operatorname{sign}(\rho(s,y) - \tilde{m}(x,k,y))\big(F(y,\rho(s,y)) - F(x,k)\big)\right) \le 0,$$

for any $k \in \mathbb{R}$, which implies

$$\begin{aligned} &\partial_s \langle \pi_{t,x}; |\rho(s,y) - \tilde{m}(x,k,y)| \rangle \\ &\qquad + \partial_y \big\langle \pi_{t,x}; \operatorname{sign}(\rho(s,y) - \tilde{m}(x,k,y))\big(F(y,\rho(s,y)) - F(x,k)\big)\big\rangle \le 0 \end{aligned} \tag{1.1.18}$$

in the sense of distributions.

Step 2. We next perform an integration by parts against a test function of the form

$$J_{\tau,\omega}(t,x,s,y) = J(\frac{t+s}{2}, \frac{x+y}{2})\bar{H}_\tau(t-s)H_\omega(x-y) \geq 0. \tag{1.1.19}$$

Here $J \in C_0^\infty(\mathbb{R}_+^2)$ and the two families of functions $\bar{H}_\tau,\ H_\omega \in C_0^\infty(\mathbb{R})$ are defined as

$$\bar{H}_\tau(z) = \frac{1}{\tau}\bar{H}(\frac{z}{\tau}) \quad \text{and} \quad H_\omega(z) = \frac{1}{\omega}H(\frac{z}{\omega}) \qquad \text{for } \tau, \omega > 0,$$

for a positive, compactly supported function $H \in C_0^\infty(\mathbb{R})$ and a positive function $\bar{H} \in C_0^\infty(\mathbb{R})$ with compact support in $(-1,1)$ such that $\int_{\mathbb{R}} H(z)dz = \int_{\mathbb{R}} \bar{H}(z)dz = 1$.

We first choose the test function in (1.1.17) as defined above for fixed (s,y) and then integrate the resulting inequality with respect to (s,y) to obtain

$$\begin{aligned}
&\int \langle \pi_{t,x}; |k - \tilde{\rho}(s,y,x)|\rangle\, \partial_t J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+ \int \langle \pi_{t,x}; \operatorname{sign}(k - \tilde{\rho}(s,y,x))\big(F(x,k) - \beta(s,y)\big)\rangle\, \partial_x J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+ \int |\rho_0(x) - \tilde{\rho}(s,y,x)|\, J_{\tau,\omega}(0,x,s,y)\, dxdsdy \ \geq\ 0.
\end{aligned} \tag{1.1.20}$$

Furthermore, after integration, it follows from (1.1.18) that

$$\begin{aligned}
&\int \langle \pi_{t,x}; |\rho(s,y) - \tilde{m}(x,k,y)|\rangle\, \partial_s J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+ \int \langle \pi_{t,x}; \operatorname{sign}(\rho(s,y) - \tilde{m}(x,k,y))\big(F(y,\rho(s,y)) - \gamma(x,k)\big)\rangle\, \partial_y J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+ \int \langle \pi_{t,x}; |\rho_0(y) - m(x,k,y)|\rangle\, J_{\tau,\omega}(t,x,0,y)\, dtdxdy \ \geq\ 0.
\end{aligned} \tag{1.1.21}$$

We next add (1.1.20) and (1.1.21) together to obtain the following inequality:

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \geq 0, \tag{1.1.22}$$

where

$$
\begin{aligned}
T_1 &:= \frac{1}{2}\int \langle \pi_{t,x}; |k-\tilde\rho(s,y,x)|\rangle \partial_t J(\frac{t+s}{2},\frac{x+y}{2})\bar H_\tau(t-s)H_\omega(x-y)\,dtdxdsdy \\
&\quad +\frac{1}{2}\int \langle \pi_{t,x}; \operatorname{sign}(k-\tilde\rho(s,y,x))\big(F(x,k)-\beta(s,y)\big)\rangle \\
&\qquad\qquad \times \partial_x J(\frac{t+s}{2},\frac{x+y}{2})\bar H_\tau(t-s)H_\omega(x-y)\,dtdxdsdy, \\
T_2 &:= \frac{1}{2}\int \langle \pi_{t,x}; |\rho(s,y)-\tilde m(x,k,y)|\rangle \partial_s J(\frac{t+s}{2},\frac{x+y}{2})\bar H_\tau(t-s)H_\omega(x-y)\,dtdxdsdy \\
&\quad +\frac{1}{2}\int \langle \pi_{t,x}; \operatorname{sign}(\rho(s,y)-\tilde m(x,k,y))\big(F(y,\rho(s,y))-\gamma(x,k)\big)\rangle \\
&\qquad\qquad \times \partial_y J(\frac{t+s}{2},\frac{x+y}{2})\bar H_\tau(t-s)H_\omega(x-y)\,dtdxdsdy, \\
T_3 &:= \int \langle \pi_{t,x}; |k-\tilde\rho(s,y,x)|-|\rho(s,y)-\tilde m(x,k,y)|\rangle \\
&\qquad\qquad \times J(\frac{t+s}{2},\frac{x+y}{2})\bar H'_\tau(t-s)H_\omega(x-y)\,dtdxdsdy, \\
T_4 &:= \int \langle \pi_{t,x}; (F(x,k)-F(y,\rho(s,y)))\big(\operatorname{sign}(k-\tilde\rho(s,y,x))+\operatorname{sign}(\rho(s,y)-\tilde m(x,k,y))\big)\rangle \\
&\qquad\qquad \times J(\frac{t+s}{2},\frac{x+y}{2})\bar H_\tau(t-s)H'_\omega(x-y)\,dtdxdsdy, \\
T_5 &:= \int \langle \pi_{t,x}; |\rho_0(y)-\tilde m(x,k,y)|\rangle\, J(\frac{t}{2},\frac{x+y}{2})\bar H_\tau(t)H_\omega(x-y)\,dtdxdy, \\
T_6 &:= \int |\rho_0(x)-\tilde\rho(s,y,x)|\, J(\frac{s}{2},\frac{x+y}{2})\bar H_\tau(-s)H_\omega(x-y)\,dxdsdy.
\end{aligned}
$$

Step 3. We first show that $T_4 = 0$. This requires to show that

$$\operatorname{sign}(k-\tilde\rho(s,y,x)) = \operatorname{sign}\big(\tilde m(x,k,y)-\rho(s,y)\big). \tag{1.1.23}$$

With this result, the integrand of T_4 cancels for a.e. $(t,x,s,y)\in\mathbb{R}^2\times\mathbb{R}^2$, which yields that $T_4=0$ for every $\omega,\tau>0$.

To prove (1.1.23), we apply (1.1.15) and (1.1.16). For a.e. $(t,x,s,y)\in\mathbb{R}^2_+\times\mathbb{R}^2_+$, we obtain

$$F(x,k)-F(x,\tilde\rho(s,y,x)) = F(y,\tilde m(x,k,y))-F(y,\rho(s,y)).$$

Under (H3'), the result follows immediately, since F is monotone in the second variable.

Under (H3), we find from (1.1.14) that

$$
\begin{aligned}
0 &= \operatorname{sign}(k-\rho_m(x))-\operatorname{sign}\big(\tilde m(x,k,y)-\rho_m(x)\big) \\
&= \operatorname{sign}(\rho(s,y)-\rho_m(y))-\operatorname{sign}\big(\tilde\rho(s,y,x)-\rho_m(y)\big).
\end{aligned} \tag{1.1.24}
$$

We have two cases:

If $\operatorname{sign}(k-\rho_m(x)) = \operatorname{sign}(\rho(s,y)-\rho_m(y))$, the problem is reduced to the monotone case since $F(x,\cdot)$ is monotone on each interval $[-\infty,\rho_m(x)]$ and $[\rho_m(x),\infty]$;

If $\operatorname{sign}(k-\rho_m(x)) \neq \operatorname{sign}(\rho(s,y)-\rho_m(y))$, the result follows immediately from (1.1.24).

In Steps 4–6, we will show that, in the limit as $\omega \to 0$ first and $\tau \to 0$ second, inequality (1.1.10) follows from $T_1+T_2+T_3+T_5+T_6 \geq 0$.

Step 4. We first show that

$$\tilde{\rho}(s,y,x) \xrightarrow{x\to y} \tilde{\rho}(s,y,y) = \rho(s,y) \qquad \text{for a.e. } (s,y) \in \mathbb{R}_+^2. \tag{1.1.25}$$

and

$$\tilde{m}(x,k,y) \xrightarrow{y\to x} \tilde{m}(x,k,x) = k \qquad \text{for a.e. } x \in \mathbb{R}. \tag{1.1.26}$$

For the case (H3'), since the flux function is continuous outside a negligible set $\mathcal{N}$, then, for $y \in \mathbb{R}\backslash\mathcal{N}$,

$$F(x,\tilde{\rho}(s,y,y)) \xrightarrow{x\to y} F(y,\tilde{\rho}(s,y,y)).$$

On the other hand, we have $F(y,\tilde{\rho}(s,y,y)) = F(x,\tilde{\rho}(s,y,x))$. Therefore, we have

$$F(x,\tilde{\rho}(s,y,x)) - F(x,\tilde{\rho}(s,y,y)) \xrightarrow{x\to y} 0,$$

and (1.1.25) is a consequence of the fact that $F(x,\cdot)$ is a one to one function.

Similarly, for $x \in \mathbb{R}\backslash\mathcal{N}$, we have

$$F(y,k) \xrightarrow{y\to x} F(x,k),$$

while $F(x,k) = F(y,\tilde{m}(x,k,y))$. Therefore, we have

$$F(y,\tilde{m}(x,k,y)) - F(y,k) \xrightarrow{y\to x} 0,$$

and (1.1.26) is a consequence of the fact that $F(y,\cdot)$ is a one to one function.

For the case (H3), it is clear from the definition of $\tilde{\rho}(s,y,x)$ and $\tilde{m}(x,k,y)$ in (1.1.15) and (1.1.16), respectively.

Step 5. We show that, when $\omega \to 0$ first and $\tau \to 0$ second, T_1 converges to

$$\begin{aligned}\frac{1}{2}\int \Big(&\langle \pi_{t,x}; |k-\rho(t,x)|\rangle \partial_t J(t,x) \\ &+ \langle \pi_{t,x}; \operatorname{sign}(k-\rho(t,x))(F(x,k)-F(x,\rho(t,x)))\rangle \partial_x J(t,x)\Big)\, dtdx.\end{aligned} \tag{1.1.27}$$

Observe that

$$\begin{aligned}&\Big|\int \langle \pi_{t,x}; \big|k-\tilde{\rho}(s,y,x)\big|\rangle \partial_t J(\frac{t+s}{2},\frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy \\ &- \int \langle \pi_{t,x}; \big|k-\tilde{\rho}(s,y,y)\big|\rangle \partial_t J(\frac{t+s}{2},\frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy\Big| \\ &\leq \int \Big(\int \big|\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)\big|\, H_\omega(x-y) |\partial_t J(\frac{t+s}{2},\frac{x+y}{2})|\, dx\Big) \bar{H}_\tau(t-s) dtdsdy \\ &\to 0 \qquad \text{when } \omega \to 0,\end{aligned} \tag{1.1.28}$$

by the Dominated Convergence theorem and the fact that

$$\int \big|\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)\big| H_\omega(x-y)|\partial_t J(\frac{t+s}{2},\frac{x+y}{2})|\,dx \to 0$$

when $\omega \to 0$ for a.e. $(s,y) \in \mathbb{R}^2_+$ since $\tilde{\rho}(s,y,x) \stackrel{x\to y}{\longrightarrow} \tilde{\rho}(s,y,y) = \rho(s,y)$ by Step 4. Furthermore,

$$\begin{aligned}\int \langle \pi_{t,x}; \big|k-\rho(s,y)\big|\rangle \Big|\partial_t J(\frac{t+s}{2},\frac{x+y}{2}) - \partial_t J(t,x)\Big| \bar{H}_\tau(t-s)H_\omega(x-y)\,dtdxdsdy \\ = \mathcal{O}(\omega)+\mathcal{O}(\tau) \to 0, \quad (1.1.29)\end{aligned}$$

when $\omega \to 0$ first and $\tau \to 0$ second. Then, to find the limit of the first part of T_1, it suffices to compute the limit of

$$\int \langle \pi_{t,x}; \big|k-\rho(s,y)\big|\rangle\, \partial_t J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dtdxdsdy. \qquad (1.1.30)$$

Thus, it suffices to show that $\rho(s,y)$ can be replaced by $\rho(t,x)$ in (1.1.30), i.e., when $\omega \to 0$ first and $\tau \to 0$ second,

$$\begin{aligned}&\int \big|\rho(t,x)-\rho(s,y)\big|\partial_t J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dtdxdsdy \\ &= \int \big|\rho(t,x)-\rho(t+\tau r, x+\omega z)\big|\,\partial_t J(t,x)\bar{H}(-r)H(-z)\,dtdxdrdz \to 0. \quad (1.1.31)\end{aligned}$$

This is guaranteed by the fact that

$$\lim_{\tau\to 0}\lim_{\omega\to 0}\int \big|\rho(t,x)-\rho(t+\tau r,x+\omega z)\big|\,dtdx = 0,$$

and the Dominated Convergence theorem since all the functions involved are bounded. This implies that, in (1.1.30), we can indeed replace $\rho(s,y)$ by $\rho(t,x)$.

On the other hand, hypothesis (H2) on $F(x,\rho)$ implies

$$\begin{aligned}&\Big|\operatorname{sign}(k-\tilde{\rho}(s,y,x))\big(F(x,k)-\beta(s,y)\big) \\ &\qquad - \operatorname{sign}(k-\tilde{\rho}(s,y,y))\big(F(x,k)-F(x,\tilde{\rho}(s,y,y))\big)\Big| \\ &= \Big|\operatorname{sign}(k-\tilde{\rho}(s,y,x))\big(F(x,k)-F(x,\tilde{\rho}(s,y,x))\big) \\ &\qquad - \operatorname{sign}(k-\rho(s,y))\big(F(x,k)-F(x,\rho(s,y))\big)\Big| \\ &\qquad\qquad \le C|\tilde{\rho}(s,y,x)-\rho(s,y)|.\end{aligned}$$

Integrating the last expression with respect to x against the function $H_\omega(x-y)$ yields its convergence to 0 by the same argument as above when $\omega \to 0$. Since $J \in C_0^\infty(\mathbb{R}^2_+)$, as above, the limit of the second part of T_1 is the same as the limit of

$$\int \langle \pi_{t,x}; \operatorname{sign}(k-\rho(s,y))\big(F(x,k)-F(x,\rho(s,y))\big)\rangle \partial_x J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\,dtdxdsdy,$$

and it suffices to prove that, when $\omega \to 0$ first and $\tau \to 0$ second,

$$\int \Big\langle \pi_{t,x}; \Big| \operatorname{sign}(k-\rho(s,y))\big(F(x,k)-F(x,\rho(s,y))\big) \\ - \operatorname{sign}(k-\rho(t,x))\big(F(x,k)-F(x,\rho(t,x))\big)\Big| \Big\rangle \\ \times \partial_x J(t,x)\bar{H}_\tau(t-s)H_\omega(x-y)\, dtdxdsdy \to 0.$$

Using the Lipschitz property and fact (1.1.31), we achieve the result for the second part of T_1.

Step 6. T_2 converges to (1.1.27) as well. This follows by the same argument as used already in Step 5 and observing that

$$\Big| \int \langle \pi_{t,x}; |\rho(s,y)-\tilde{m}(x,k,y)|\rangle\, \partial_s J(\frac{t+s}{2},\frac{x+y}{2})\bar{H}_\tau(t-s)H_\omega(x-y)\, dtdxdsdy \\ - \int \langle \pi_{t,x}; |\rho(s,y)-k|\rangle\, \partial_s J(\frac{t+s}{2},\frac{x+y}{2})\bar{H}_\tau(t-s)H_\omega(x-y)\, dtdxdsdy \Big| \\ \le \int \langle \pi_{t,x}; |\tilde{m}(x,k,y)-k|\rangle\, |\partial_s J(\frac{t+s}{2},\frac{x+y}{2})|\, \bar{H}_\tau(t-s)H_\omega(x-y)\, dtdxdsdy.$$

Again the right hand side of the last expression converges to zero when $\omega \to 0$. Using the same argument as in Step 5, we achieve the result for T_2.

Step 7. T_3 converges to 0 when $\omega \to 0$. Since

$$\lim_{\omega\to 0} \Big| \int \langle \pi_{t,x}; |k-\tilde{\rho}(s,y,x)| - |\rho(s,y)-\tilde{m}(x,k,y)|\rangle H_\omega(x-y)\, dxdy \Big| \\ \le \lim_{\omega\to 0} \int \langle \pi_{t,x}; |k-\tilde{m}(x,k,y)|\rangle H_\omega(x-y)\, dxdy \\ + \lim_{\omega\to 0} \int \langle \pi_{t,x}; |\rho(s,y)-\tilde{\rho}(s,y,x)|\rangle H_\omega(x-y)\, dxdy = 0,$$

the result follows as in Steps 5 and 6.

Step 8. T_6 converges to zero when $\tau \to 0$ after $\omega \to 0$: Note that

$$\int \Big| |\rho_0(x)-\tilde{\rho}(s,y,x)| - |\rho_0(x)-\tilde{\rho}(s,y,y)| \Big|\, J(0,\frac{x+y}{2})\bar{H}_\tau(-s)H_\omega(x-y)\, dxdsdy \\ \le \int |\tilde{\rho}(s,y,x)-\tilde{\rho}(s,y,y)|\, J(0,\frac{x+y}{2})\bar{H}_\tau(-s)H_\omega(x-y)\, dxdsdy.$$

Again with (1.1.31), the right hand side converges to zero when $\omega \to 0$. We therefore next compute the limit when $\omega \to 0$ first and $\tau \to 0$ second of

$$\int |\rho_0(x)-\rho(s,y)|\, J(\frac{s}{2},\frac{x+y}{2})\bar{H}_\tau(-s)H_\omega(x-y)\, dxdsdy.$$

As before,

$$\lim_{\omega\to 0} \int |\rho(s,x)-\rho(s,y)|\, J(\frac{s}{2},\frac{x+y}{2})H_\tau(-s)H_\omega(x-y)\, dxdsdy = 0.$$

Therefore, the next goal is to compute the limit when $\tau \to 0$ of

$$\int |\rho_0(x) - \rho(s,x)|\, J(\frac{s}{2}, x)\bar{H}_\tau(-s)\, dxdsdy = \int |\rho_0(x) - \rho(\tau r, x)|\, J(\frac{\tau r}{2}, x)\bar{H}(-r)\, dxdr. \tag{1.1.32}$$

Since all the functions are bounded and $\operatorname{supp} \bar{H} \subset (-1,1)$, by the Dominated Convergence theorem, this converges to 0 when $\tau \to 0$, and thereby (1.1.32) converges to 0.

Step 9. T_5 converges to zero by the analogous argument as in Step 8 and using the fact that $\pi_{0,x}(k) = \delta_{\rho_0(x)}(k)$.

With Steps 3–9 and by (1.1.22), we complete the proof. □

1.2 Existence of entropy solutions

In this section, we establish the existence of entropy solutions (1.1.1)–(1.1.2) in the sense of Definition 1.1.1, as required for the reduction of measure-valued entropy solutions. More precisely, for each fixed $\varepsilon > 0$, ρ^ε denotes the unique Kruzkov solution of (1.1.1)–(1.1.2) in the sense (1.2.3), where the flux function depends smoothly on the space variable x; then it is shown that the sequence ρ^ε converges to an entropy solution of (1.1.1)–(1.1.2).

1.2.1 Existence of entropy solutions when F is smooth

Define $F^\varepsilon(x,\rho)$ the standard mollification of $F(x,\rho)$ in $x \in \mathbb{R}$:

$$F^\varepsilon(x,\rho) := (F(\cdot,\rho) * \theta^\varepsilon)(x) \to F(x,\rho) \quad a.e. \qquad \text{as } \varepsilon \to 0, \tag{1.2.1}$$

with $\theta^\varepsilon(x) := \theta(\frac{x}{\varepsilon})$, $\theta(x) \geq 0$, $\operatorname{supp}\theta(x) \subset [-1,1]$, and $\int_{-1}^{1} \theta(x)dx = 1$. For fixed $\varepsilon > 0$, consider the following Cauchy problem:

$$\begin{cases} \partial_t \rho + \partial_x F^\varepsilon(x,\rho) = 0, \\ \rho|_{t=0} = \rho_0(x) \geq 0. \end{cases} \tag{1.2.2}$$

Kruzkov's result in [26] indicates that there exists a unique solution ρ^ε of (1.2.2) satisfying the Kruzkov entropy inequality:

$$\begin{aligned} \partial_t|\rho^\varepsilon(t,x) - c| + \partial_x\big(\operatorname{sign}(\rho^\varepsilon(t,x) - c)(F^\varepsilon(x,\rho^\varepsilon(t,x)) - F^\varepsilon(x,c))\big) \\ + \operatorname{sign}(\rho^\varepsilon(t,x) - c)\partial_x F^\varepsilon(x,c) \leq 0 \end{aligned} \tag{1.2.3}$$

in the sense of distributions. Notice that, since F^ε is now smooth in the first variable, we can define steady state solutions $m_\alpha^{\varepsilon,\pm}(x)$ for each $x \in \mathbb{R}$. In particular, the steady state solutions $m_\alpha^{\varepsilon,\pm}(x)$ also satisfy the Kruzkov entropy inequality (1.2.3):

$$\begin{aligned} \partial_t|m_\alpha^{\varepsilon,\pm}(x) - c| + \partial_x\big(\operatorname{sign}(m_\alpha^{\varepsilon,\pm}(x) - c)(F^\varepsilon(y, m_\alpha^{\varepsilon,\pm}(x)) - F^\varepsilon(x,c))\big) \\ +\operatorname{sign}(m_\alpha^{\varepsilon,\pm}(x) - c)\partial_x F^\varepsilon(x,c) \leq 0 \end{aligned} \tag{1.2.4}$$

in the distributional sense. This can be also seen as follows: Since the level set $\{x \in \mathbb{R} : m_\alpha^{\varepsilon,\pm}(x) = c\}$ is discrete for a.e. c, α, and this level set coincides with the set $\{x \in \mathbb{R} :$

$F^\varepsilon(x,c)=\alpha\}$, it follows from the Sard theorem that the set of critical values of the function $S(m_\alpha^{\varepsilon,\pm}(x)) := \text{sign}(m_\alpha^{\varepsilon,\pm}(x)-c)(F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(x))-F^\varepsilon(x,c))$ has measure zero, which implies (1.2.4).

We now prove that the entropy solution ρ^ε also satisfies (1.1.5).

Proposition 1.2.1. *Let $\rho^\varepsilon(t,x)$ be a solution of the Cauchy problem* (1.2.2) *satisfying the Kruzkov entropy inequality* (1.2.3)*. Then $\rho^\varepsilon(t,x)$ also satisfies the entropy inequality* (1.1.5) *with steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon,\pm}(x)$.*

Proof. We divide the proof into five steps.

Step 1. In (1.2.3), we choose the constant $c = m_\alpha^{\varepsilon,\pm}(y)$ for any $\alpha \in [M_0,\infty)$ (or $\alpha \in (-\infty, M_0]$) for fixed (s,y), and integrate against the test function (1.1.19) first in (t,x) and then in (s,y) to obtain the following inequality:

$$\begin{aligned}
&\int \left|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right| \partial_t J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+\int \text{sign}\left(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\right)\left(F^\varepsilon(x,\rho^\varepsilon(t,x))-F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))\right)\partial_x J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&\qquad -\int \text{sign}\left(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\right)\partial_x F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y)) J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&\qquad\qquad +\int \left|\rho^\varepsilon(0,x)-m_\alpha^{\varepsilon,\pm}(y)\right| J_{\tau,\omega}(t,0,s,y)\, dxdsdy \geq 0. \quad (1.2.5)
\end{aligned}$$

On the other hand, the Kruzkov entropy inequality (1.2.3) is satisfied for any steady state solution $m_\alpha^{\varepsilon,\pm}$, for any $c\in\mathbb{R}$ and $\alpha\in[M_0,\infty)$ (or $\alpha\in(-\infty,M_0]$). For fixed (t,x), the steady state solutions $m_\alpha^{\varepsilon,\pm}(y)$ as functions in y satisfy in (1.2.4) with (s,y) replacing (t,x) and the constant $c=\rho(t,x)$. We integrate against the test function $J_{\tau,\omega}$ first in (s,y) and then in (t,x) to obtain the following inequality:

$$\begin{aligned}
&\int \left|m_\alpha^{\varepsilon,\pm}(y)-\rho^\varepsilon(t,x)\right| \partial_s J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&+\int \text{sign}\left(m_\alpha^{\varepsilon,\pm}(y)-\rho^\varepsilon(t,x)\right)\left(F^\varepsilon(y,m_\alpha^{\varepsilon,\pm}(y))-F^\varepsilon(y,\rho^\varepsilon(t,x))\right)\partial_y J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&\qquad -\int \text{sign}\left(m_\alpha^{\varepsilon,\pm}(y)-\rho^\varepsilon(t,x)\right)\partial_y F^\varepsilon(y,\rho^\varepsilon(t,x))\, J_{\tau,\omega}(t,x,s,y)\, dtdxdsdy \\
&\qquad\qquad +\int \left|m_\alpha^{\varepsilon,\pm}(y)-\rho^\varepsilon(t,x)\right| J_{\tau,\omega}(t,x,0,y)\, dtdxdy \geq 0. \quad (1.2.6)
\end{aligned}$$

Adding (1.2.5) and (1.2.6) together, we then have

$$I_1+I_2+I_3\geq 0,$$

where

$$\begin{aligned}
I_1 := &\frac{1}{2}\int \left|\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\right|(\partial_t J+\partial_s J)\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)\, dtdxdsdy \\
&+\frac{1}{2}\int \text{sign}\left(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\right)\left(F^\varepsilon(x,\rho^\varepsilon(t,x))-F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))\right) \\
&\qquad\times(\partial_x+\partial_y)J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\bar{H}_\tau(t-s)H_\omega(x-y)dtdxdsdy,
\end{aligned}$$

$$I_2 := \int \left|m_\alpha^{\varepsilon,\pm}(y) - \rho^\varepsilon(t,x)\right| J_{\tau,\omega}(t,x,0,y)\,dtdxdy + \int \left|\rho^\varepsilon(0,x) - m_\alpha^{\varepsilon,\pm}(y)\right| J_{\tau,\omega}(0,x,s,y)\,dxdsdy,$$

$$\begin{aligned} I_3 := \int \operatorname{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \Big(&F^\varepsilon(x,\rho^\varepsilon(t,x)) - F^\varepsilon(x, m_\alpha^{\varepsilon,\pm}(y)) \\ &+ F^\varepsilon(y, m_\alpha^{\varepsilon,\pm}(y)) - F^\varepsilon(y,\rho^\varepsilon(t,x))\Big) \\ &\times J(\frac{t+s}{2}, \frac{x+y}{2})\bar{H}_\tau(t-s)H'_\omega(x-y)\,dtdxdsdy \\ - \int \operatorname{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) &\left(\partial_x F^\varepsilon(x, m_\alpha^{\varepsilon,\pm}(y)) - \partial_y F^\varepsilon(y,\rho^\varepsilon(t,x))\right) \\ &\times J(\frac{t+s}{2}, \frac{x+y}{2})\bar{H}_\tau(t-s)H_\omega(x-y)\,dtdxdsdy. \end{aligned}$$

In order to prove Proposition 1.2.1, we need to show that $I_1 + I_2$ converges to the left hand side of the entropy inequality (1.1.5) and $I_3 \to 0$, when $\tau \to 0$ after $\omega \to 0$.

Step 2. We start with the following two useful identities:

$$\lim_{\omega\to 0} \int H_\omega(x-y)\left|m_\alpha^{\varepsilon,\pm}(y) - m_\alpha^{\varepsilon,\pm}(x)\right| dxdy = 0; \tag{1.2.7}$$

and, for any continuous function G of $m_\alpha^{\varepsilon,\pm}$

$$\lim_{\omega\to 0} \int H'_\omega(x-y)(x-y)\left|G\left(m_\alpha^{\varepsilon,\pm}(y)\right) - G\left(m_\alpha^{\varepsilon,\pm}(x)\right)\right| dxdy = 0. \tag{1.2.8}$$

We first show that the steady state solutions are continuous on $\mathbb{R}$ for each $\alpha \in [M_0,\infty)$ (or $\alpha \in (-\infty, M_0]$).

We start with $\alpha \neq M_0$: Since the flux function is continuous in the first variable,

$$F^\varepsilon(y, m_\alpha^{\varepsilon,+}(x)) \overset{y\to x}{\longrightarrow} F^\varepsilon(x, m_\alpha^{\varepsilon,+}(x)).$$

On the other hand, $F^\varepsilon(y, m_\alpha^{\varepsilon,+}(y)) = F^\varepsilon(x, m_\alpha^{\varepsilon,+}(x))$. Therefore, we have

$$F^\varepsilon(y, m_\alpha^{\varepsilon,+}(y)) - F^\varepsilon(y, m_\alpha^{\varepsilon,+}(x)) \overset{y\to x}{\longrightarrow} 0,$$

and, as a consequence of the fact that $F^\varepsilon(y,\cdot)$ is a one to one function on $[\rho_m(y),\infty)$,

$$m_\alpha^{\varepsilon,+}(y) \overset{y\to x}{\longrightarrow} m_\alpha^{\varepsilon,+}(x) \qquad \text{for any } x \in \mathbb{R}.$$

Similarly, we can show for each $\alpha \neq M_0$ that

$$m_\alpha^{\varepsilon,-}(y) \overset{y\to x}{\longrightarrow} m_\alpha^{\varepsilon,-}(x) \qquad \text{for any } x \in \mathbb{R}.$$

If $\alpha = M_0$, then $m_{M_0}^{\varepsilon,\pm}(x) = \rho_m^\varepsilon(x)$,

$$F^\varepsilon(y, \rho_m^\varepsilon(x)) \overset{y\to x}{\longrightarrow} F^\varepsilon(x, \rho_m^\varepsilon(x)) = M_0.$$

On the other hand, we have $F^\varepsilon(y, \rho_m^\varepsilon(y)) = M_0$. Therefore

$$F^\varepsilon(y, \rho_m^\varepsilon(y)) - F^\varepsilon(y, \rho_m^\varepsilon(x)) \overset{y \to x}{\longrightarrow} 0,$$

and, as a consequence of the fact that F^ε is a continuous function in the second variable, we obtain

$$\rho_m^\varepsilon(y) \overset{y \to x}{\longrightarrow} \rho_m^\varepsilon(x) \qquad \text{for any} \quad x \in \mathbb{R}.$$

With this, as in the proof of Theorem 1.1.4, we obtain (1.2.7).

Notice that, for any continuous function G of $m_\alpha^{\varepsilon,\pm}$, we have

$$\begin{aligned}
&\int H'_\omega(x-y)(x-y) \left| G\left(m_\alpha^{\varepsilon,\pm}(y)\right) - G\left(m_\alpha^{\varepsilon,\pm}(x)\right) \right| dxdy \\
&= \int H'_\omega(-\omega z)(-\omega z) \left| G\left(m_\alpha^{\varepsilon,\pm}(x+\omega z)\right) - G\left(m_\alpha^{\varepsilon,\pm}(x)\right) \right| \omega \, dxdz \\
&= \int z H'(-z) \left| G\left(m_\alpha^{\varepsilon,\pm}(x+\omega z)\right) - G\left(m_\alpha^{\varepsilon,\pm}(x)\right) \right| dxdz \\
&\to 0 \qquad \text{when } \omega \to 0,
\end{aligned}$$

since $m_\alpha^{\varepsilon,\pm}$ is continuous and is in L^∞. Thus, (1.2.8) follows.

Step 3. With (1.2.7) and

$$\left|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right| - \left|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(x)\right| \le \left|m_\alpha^{\varepsilon,\pm}(x) - m_\alpha^{\varepsilon,\pm}(y)\right|,$$

as in the proof of Theorem 1.1.4, we obtain that, when $\omega \to 0$ first and $\tau \to 0$ second, I_1 converges to

$$\begin{aligned}
\int \Big(&\left|\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(x)\right| \partial_t J(t,x) \\
&\qquad + \operatorname{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(x)\right) \left(F^\varepsilon(x, \rho^\varepsilon(t,x)) - \alpha\right) \partial_x J(t,x) \Big) \, dtdx.
\end{aligned}$$

In the same way, we can replace $m_\alpha^{\varepsilon,\pm}(y)$ by $m_\alpha^{\varepsilon,\pm}(x)$ and $\rho^\varepsilon(t,x)$ by $\rho^\varepsilon(0,x)$ in I_2, when $\omega \to 0$ first and $\tau \to 0$ second. Then both terms of I_2 converge to

$$\frac{1}{2} \int \left|\rho^\varepsilon(0,x) - m_\alpha^{\varepsilon,\pm}(y)\right| J(0,x) \, dx.$$

Step 4. It remains to show that $\lim_{\tau \to 0} \lim_{\omega \to 0} I_3 = 0$. To avoid confusion, from now on, we denote the derivative of $F^\varepsilon(x,\cdot)$ with respect to the first variable by $F_x^\varepsilon(x,\cdot)$.

Notice that

$$
\begin{aligned}
I_3 = \int \Big(& -\mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F^\varepsilon(y,\rho^\varepsilon(t,x)) - F^\varepsilon(x,\rho^\varepsilon(t,x)) - F_x^\varepsilon(x,\rho^\varepsilon(t,x))(y-x)\right) \\
& + \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F^\varepsilon(y,m_\alpha^{\varepsilon,\pm}(y)) - F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y)) - F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))(y-x)\right)\Big) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H'_\omega(x-y)\, dtdxdsdy \\
+ \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) F_y^\varepsilon(y,\rho^\varepsilon(t,x))\, J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy \\
- \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) F_x^\varepsilon(x,\rho^\varepsilon(t,x))(y-x) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H'_\omega(x-y)\, dtdxdsdy \\
- \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y)) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy \\
+ \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))(y-x) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy
\end{aligned}
$$

$$
\begin{aligned}
= \int \Big(& -\mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F^\varepsilon(y,\rho^\varepsilon(t,x)) - F^\varepsilon(x,\rho^\varepsilon(t,x)) - F_x^\varepsilon(x,\rho^\varepsilon(t,x))(y-x)\right) \\
& + \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F^\varepsilon(y,m_\alpha^{\varepsilon,\pm}(y)) - F^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y)) - F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))(y-x)\right)\Big) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H'_\omega(x-y)\, dtdxdsdy \\
+ \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F_x^\varepsilon(x,\rho^\varepsilon(t,x)) - F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))\right) \left(H_\omega(x-y) + H'_\omega(x-y)(x-y)\right) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s)\, dtdxdsdy \\
+ \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F_y^\varepsilon(y,\rho^\varepsilon(t,x)) - F_x^\varepsilon(x,\rho^\varepsilon(t,x))\right) \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s) H_\omega(x-y)\, dtdxdsdy.
\end{aligned}
$$

The last term of this expression is of order $\mathcal{O}(\omega)$, since F^ε is at least C^1 in the first variable, all the functions in the integrand are bounded, and the support of H is also bounded. Therefore, this term converges to 0 when $\omega \to 0$.

Step 5. It remains to show that the first and the second term in the last expression vanish in the limit. The first term is equal to

$$
\begin{aligned}
\frac{1}{2} \int & \mathrm{sign}\left(\rho^\varepsilon(t,x) - m_\alpha^{\varepsilon,\pm}(y)\right) \left(F_{xx}^\varepsilon(\xi, m_\alpha^{\varepsilon,\pm}(y)) - F_{xx}^\varepsilon(\xi,\rho^\varepsilon(t,x)) + \mathcal{O}(|y-x|)\right) H'_\omega(x-y)(x-y)^2 \\
& \qquad \times J(\frac{t+s}{2}, \frac{x+y}{2}) \bar{H}_\tau(t-s)\, dtdxdsdy.
\end{aligned}
$$

Since $\rho^\varepsilon \in L^\infty$ and F^ε is smooth in the first variable, by (H2), the first term of the last expression is bounded above by

$$C\int \frac{1}{\omega^2}H'(\frac{x-y}{\omega})|y-x|^2 J(\frac{t+s}{2},\frac{x+y}{2})\bar{H}_\tau(t-s)\,dtdxdsdy$$
$$= C\omega\int H'(z)z^2 J(\frac{t+s}{2},\frac{2x+\omega z}{2})\bar{H}_\tau(t-s)\,dtdxdsdz = \mathcal{O}(\omega)\to 0 \qquad \text{when } \omega\to 0.$$

The second term is equal to

$$\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_x^\varepsilon(x,\rho^\varepsilon(t,x))-F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))\big)$$
$$\times\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)\Big(\big(J(\frac{t+s}{2},\frac{x+y}{2})-J(t,x)\big)+J(t,x)\Big)H_\tau(t-s)\,dtdxdsdy$$

$$= \mathcal{O}(\omega)+\mathcal{O}(\tau)+\int J(t,x)\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)$$
$$\times\Big(\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(y)\big)\big(F_x^\varepsilon(x,\rho^\varepsilon(t,x))-F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(y))\big)$$
$$-\operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(x)\big)\big(F_x^\varepsilon(x,\rho^\varepsilon(t,x))-F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(x))\big)\Big)dtdxdy$$
$$+\int \operatorname{sign}\big(\rho^\varepsilon(t,x)-m_\alpha^{\varepsilon,\pm}(x)\big)\Big(F_x^\varepsilon(x,\rho^\varepsilon(t,x))-F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm}(x))\Big)$$
$$\times\Big(\int\big(H_\omega(x-y)+H'_\omega(x-y)(x-y)\big)dy\Big)J(t,x)\,dtdx.$$

Notice that

$$\operatorname{sign}\big(\rho^\varepsilon-m_\alpha^{\varepsilon,\pm}\big)\big(F_x^\varepsilon(x,\rho^\varepsilon)-F_x^\varepsilon(x,m_\alpha^{\varepsilon,\pm})\big)$$

is a continuous function of $m_\alpha^{\varepsilon,\pm}$. Thus, the third term of the last expression goes to zero if $\omega\to 0$ by (1.2.7) and (1.2.8).

In the remaining last term, the integral with respect to y is equal to 0 because

$$H_\omega(x-y)+H'_\omega(x-y)(x-y) = -\partial_y\big((x-y)H_\omega(x-y)\big).$$

This concludes that I_3 vanishes in the limit when $\omega\to 0$ first and $\tau\to 0$ second. ☐

Thus we conclude the existence of an entropy solution $\rho_\varepsilon(t,x)$ in the sense of Definition 1.1.1 for each F_ε with fixed $\varepsilon>0$.

Remark 1.2.2. Notice that the sequence of approximate entropy solutions converges to a measure-valued entropy solution when $\varepsilon\to 0$: First, since $\rho_0\in L^\infty$, we find that, for α big enough,

$$m_\alpha^{\varepsilon,-}(x)\le\rho_0(x)\le m_\alpha^{\varepsilon,+}(x) \qquad \text{for all } x\in\mathbb{R}.$$

From [3], it then follows that

$$m_\alpha^{\varepsilon,-}(x)\le\rho^\varepsilon(t,x)\le m_\alpha^{\varepsilon,+}(x),$$

which implies the uniform boundedness of $\rho^\varepsilon(t,x)$ in ε since $m_\alpha^{\varepsilon,\pm}(x)$ are uniformly bounded in ε. Then there exists a compactly supported family of probability measures $\pi_{t,x}$ on $\mathbb{R}$ (i.e.

Young measures; see Tartar [36]) and a subsequence (still denoted by) $\rho^\varepsilon(t,x)$ such that, for any continuous function $f(\rho)$,

$$f(\rho^\varepsilon(t,x)) \stackrel{*}{\rightharpoonup} \langle \pi_{t,x}, f(k)\rangle \qquad \text{when } \varepsilon \to 0. \tag{1.2.9}$$

On the other hand, by Section 1.2.1, the sequence $\rho^\varepsilon(t,x)$ satisfies the entropy inequality (1.1.5) for $F_\varepsilon(x,\rho)$ and the steady-state solutions $m_\alpha^\pm = m_\alpha^{\varepsilon,\pm}$. In particular, we use (1.2.9) and the definition of the sequence $F_\varepsilon(x,\rho)$ in (1.2.1) to conclude that, when $\varepsilon \to 0$, the compactly supported family of probability measures $\pi_{t,x}$ satisfies that, for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$\int \big(\langle \pi_{t,x}; \big|k - m_\alpha^\pm(x)\big|\rangle\, \partial_t J + \big\langle \pi_{t,x}; \operatorname{sign}\big(k - m_\alpha^\pm\big)\,\big(F(x,k)-\alpha\big)\big\rangle\, \partial_x J\big)\, dxdt$$
$$+ \int \big|\rho_0(x) - m_\alpha^\pm(x)\big| J(0,x)\, dx \geq 0.$$

Thus, $\pi_{t,x}$ is a measure-valued entropy solution of (1.1.1)–(1.1.2) with compact support for a.e. $(t,x) \in \mathbb{R}_+^2$ in the sense of Definition 1.1.2.

1.2.2 Existence of entropy solutions when F is discontinuous in x

We are now ready to state the main theorem of this section.

Theorem 1.2.3. *Let $F(x,\rho)$ be strictly convex or concave in ρ for a.e. $x \in \mathbb{R}$ and satisfy* (H1)–(H3)*, or let $F(x,\rho)$ satisfy* (H1)–(H2) *and* (H3')*. Let $\rho_0(x) \in L^\infty$. Then the sequence of entropy solutions ρ^ε of the Cauchy problem* (1.2.2) *(in the sense of Definition 1.1.1) converges to the unique entropy solution of the Cauchy problem* (1.1.1)–(1.1.2) *in the sense of Definition 1.1.1.*

Proof. We consider the two cases separately.

For the case (H1)–(H2) and (H3'), that is, the flux function F is monotone in ρ, we apply the compactness framework established in Section 1.1 to establish the convergence. For this case, the existence of entropy solutions has been established in [5]. In Remark 1.2.2, we have shown that the limit of the entropy solutions ρ^ε is determined by a measure-valued entropy solution $\pi_{t,x}$. Then, by Theorems 1.1.3–1.1.4, $\pi_{t,x}$ is the Dirac measure concentrated on the unique entropy solution $\rho(t,x)$ of (1.1.1)–(1.1.2) in the sense of Definition 1.1.1, which implies the whole sequence converges.

For the case (H1)–(H3), since we have not established the existence of an entropy solution, we employ the compensated compactness method to establish the convergence of the entropy solutions of the Cauchy problem (1.2.2), which also yields the existence of a unique entropy solution of the Cauchy problem (1.1.1)–(1.1.2).

From Remark 1.2.2, we know that ρ^ε is uniformly bounded in L^∞ which implies that there exists a subsequence ρ^ε converging weakly to a compactly supported family of probability measures $\nu_{t,x}$ on $\mathbb{R}_+$ such that, for any function $f(\rho,t,x)$ that is continuous in ρ for a.e. (t,x),

$$f(\rho^\varepsilon(t,x),t,x) \stackrel{*}{\rightharpoonup} \langle \nu_{t,x}, f(k,t,x)\rangle \qquad \text{when } \varepsilon \to 0. \tag{1.2.10}$$

In particular,

$$\rho^\varepsilon(t,x) \overset{*}{\rightharpoonup} \langle \nu_{t,x}, k\rangle =: \rho(t,x) \in L^\infty. \tag{1.2.11}$$

Our goal is to prove the strong convergence of $\rho^\varepsilon(t,x)$ to $\rho(t,x)$ a.e., equivalently, $\nu_{t,x} = \delta_{\rho(t,x)}$, which implies that $\rho(t,x)$ is an entropy solution of (1.1.1)–(1.1.2), that is, $\rho(t,x)$ satisfies the entropy inequality in Definition 1.1.1.

From Section 1.2.1, we know that the sequence ρ^ε exists and satisfies

$$E^\varepsilon := \partial_t |\rho_\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)| + \partial_x\big(\operatorname{sign}(\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x))\,(F_\varepsilon(x,\rho^\varepsilon(t,x)) - \gamma(s,y))\big) \le 0$$

in the sense of distributions, where

$$\hat\rho^\varepsilon(s,y,x) := m^{+,\varepsilon}_{\gamma(s,y)}(x)\operatorname{sign}_+(\rho(s,y)-\rho_m(y)) + m^{-,\varepsilon}_{\gamma(s,y)}(x)\operatorname{sign}_-(\rho(s,y)-\rho_m(y)).$$

Notice that $\gamma(s,y) := F(y,\rho(s,y))$ is independent of ε. Thus, for fixed (s,y), we have the strong convergence of $m^{\pm,\varepsilon}_{\gamma(s,y)}(x)$ to a steady-state solution $m^{\pm}_{\gamma(s,y)}(x)$ of (1.1.1)–(1.1.2) when $\varepsilon \to 0$. In particular,

$$\|\hat\rho^\varepsilon\|_{L^\infty} \le M, \qquad M \text{ independent of } \varepsilon;$$

and, for a.e. $(s,y,x) \in \mathbb{R}^2_+ \times \mathbb{R}$,

$$\begin{aligned}\hat\rho^\varepsilon(s,y,x) \;\to\; \hat\rho(s,y,x) :=\\ m^{+}_{\gamma(s,y)}(x)\operatorname{sign}_+(\rho(s,y)-\rho_m(y)) + m^{-}_{\gamma(s,y)}(x)\operatorname{sign}_-(\rho(s,y)-\rho_m(y)),\end{aligned}$$

when $\varepsilon \to 0$. By Schwartz's lemma, E^ε is a sequence of measures; by Murat's lemma [30], E^ε is uniformly bounded measure sequence in the measure space, which implies that

$$E^\varepsilon \qquad \text{is compact in } W^{-1,p}_{loc}(\mathbb{R}^2_+) \qquad \text{for any } p \in (1,2). \tag{1.2.12}$$

On the other hand, since the vector-field sequence

$$\big(\big|\rho^\varepsilon(t,x) - m^{\pm,\varepsilon}_{\gamma(s,y)}(x)\big|,\ \operatorname{sign}\big(\rho^\varepsilon(t,x) - m^{\pm,\varepsilon}_{\gamma(s,y)}(x)\big)\,(F_\varepsilon(x,\rho^\varepsilon(t,x)) - \gamma(s,y))\big)$$

is uniformly bounded in ε for any fixed (s,y), it follows that

$$E^\varepsilon \qquad \text{is bounded in } W^{-1,\infty}_{loc}(\mathbb{R}^2_+). \tag{1.2.13}$$

With (1.2.12)–(1.2.13), we obtain by a compactness interpolation theorem in [8, 17] that

$$E^\varepsilon \qquad \text{is compact in } H^{-1}_{loc}(\mathbb{R}^2_+). \tag{1.2.14}$$

On the other hand,

$$\partial_t\rho^\varepsilon + \partial_x F_\varepsilon(x,\rho^\varepsilon) = 0 \qquad \text{which is automatically compact in } H^{-1}_{loc}(\mathbb{R}^2_+). \tag{1.2.15}$$

Moreover, since $\hat\rho^\varepsilon(s,y,x)$ strongly converges a.e., then we find that, when $\varepsilon \to 0$,

$$\begin{aligned}
\eta_1^\varepsilon(\rho^\varepsilon,s,y,x) &:= |\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x)|\\
&\overset{*}{\rightharpoonup} \langle \nu_{t,x}(k); |k - \hat\rho(s,y,x)|\rangle\\
&=: \langle \nu_{t,x}; \eta_1(k,s,y,x)\rangle,\\
q_1^\varepsilon(\rho^\varepsilon,s,y,x) &:= \operatorname{sign}(\rho^\varepsilon(t,x) - \hat\rho^\varepsilon(s,y,x))\,(F_\varepsilon(x,\rho^\varepsilon) - \gamma(s,y))\\
&\overset{*}{\rightharpoonup} \langle \nu_{t,x}(k); \operatorname{sign}(k - \hat\rho(s,y,x))\,(F(x,k) - \gamma(s,y))\rangle\\
&=: \langle \nu_{t,x}; q_1(k,s,y,x)\rangle,\\
\eta_2^\varepsilon(\rho^\varepsilon(t,x)) &:= \rho^\varepsilon(t,x)\\
&\overset{*}{\rightharpoonup} \langle \nu_{t,x}(k); k\rangle = \rho(t,x)\\
&=: \langle \nu_{t,x}; \eta_2(k)\rangle,\\
q_2^\varepsilon(\rho^\varepsilon(t,x),x) &:= F_\varepsilon(x,\rho^\varepsilon)\\
&\overset{*}{\rightharpoonup} \langle \nu_{t,x}(k); F(x,k)\rangle\\
&:= \langle \nu_{t,x}; q_2(k,x)\rangle,
\end{aligned} \tag{1.2.16}$$

and

$$\begin{vmatrix} \eta_1(\rho^\varepsilon(t,x),s,y,x) & q_1(\rho^\varepsilon(t,x),s,y,x) \\ \eta_2(\rho^\varepsilon(t,x)) & q_2(\rho^\varepsilon(t,x),x) \end{vmatrix} \overset{*}{\rightharpoonup} \left\langle \nu_{t,x}; \begin{vmatrix} \eta_1(k,s,y,x) & q_1(k,s,y,x) \\ \eta_2(k) & q_2(k,x) \end{vmatrix} \right\rangle, \tag{1.2.17}$$

where

$$\begin{aligned} (\eta_1(k,s,y,x),\, q_1(k,s,y,x)) &= (|k-\hat{\rho}(s,y,x)|,\ \operatorname{sign}(k-\hat{\rho}(s,y,x))\,(F(x,k)-\gamma(s,y))), \\ (\eta_2(k), q_2(k,x)) &= (k,\ F(x,k)). \end{aligned}$$

Together (1.2.14)–(1.2.15) with (1.2.16)–(1.2.17), we apply the Div-Curl lemma (see Tartar [36] and Murat [29]) to obtain

$$\left\langle \nu_{t,x}; \begin{vmatrix} \eta_1(k,s,y,x) & q_1(k,s,y,x) \\ \eta_2(k) & q_2(k,x) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}; \eta_1(k,s,y,x)\rangle & \langle \nu_{t,x}; q_1(k,s,y,x)\rangle \\ \langle \nu_{t,x}; \eta_2(k)\rangle & \langle \nu_{t,x}; q_2(k,x)\rangle \end{vmatrix}$$

for all $(s,y),(t,x) \in \mathbb{R}\backslash\mathcal{M}$ with $\mathcal{M}$ a set of measure zero in $\mathbb{R}^2_+$. Thus, we have

$$\begin{aligned} &\langle \nu_{t,x}; |k-\hat{\rho}(s,y,x)|F(x,k) - k\ \operatorname{sign}(k-\hat{\rho}(s,y,x))\,(F(x,k)-\gamma(s,y))\rangle \\ &= \langle \nu_{t,x}; |k-\hat{\rho}(s,y,x)|\rangle\,\langle \nu_{t,x}; F(x,k)\rangle - \langle \nu_{t,x}, k\rangle\,\langle \nu_{t,x}; \operatorname{sign}(k-\hat{\rho}(s,y,x))\,(F(x,k)-\gamma(s,y))\rangle\,. \end{aligned}$$

Equivalently, we have

$$\begin{aligned} &\Big\langle \nu_{t,x}; |k-\hat{\rho}(s,y,x)|\Big(F(x,k) - \langle \nu_{t,x}; F(x,k)\rangle\Big)\Big\rangle \\ &-\Big\langle \nu_{t,x}; (k-\rho(t,x))\operatorname{sign}(k-\hat{\rho}(s,y,x))\,\Big(F(x,k)-F(y,\rho(s,y))\Big)\Big\rangle = 0. \end{aligned}$$

Since this is true for all (s,y) and (t,x) except on a set $\mathcal{M}$ of measure zero, we then choose $(s,y)=(t,x)$ for $(t,x) \in \mathbb{R}\backslash\mathcal{M}$ to obtain

$$\begin{aligned} &\Big\langle \nu_{t,x}; |k-\rho(t,x)|\Big(F(x,k) - \langle \nu_{t,x}; F(x,k)\rangle\Big)\Big\rangle \\ &-\Big\langle \nu_{t,x}; (k-\rho(t,x))\operatorname{sign}(k-\rho(t,x))\,\Big(F(x,k)-F(x,\rho(t,x))\Big)\Big\rangle = 0, \end{aligned}$$

that is,

$$\langle \nu_{t,x}; |k-\rho(t,x)|\rangle\,(F(x,\rho(t,x)) - \langle \nu_{t,x}; F(x,k)\rangle) = 0. \tag{1.2.18}$$

There are two possibilities:

When $\langle \nu_{t,x}; |k-\rho(t,x)|\rangle = 0$, then we have $\nu_{t,x}(k) = \delta_{\rho(t,x)}(k)$.

When $\langle \nu_{t,x}; F(x,k)\rangle - F(x,\rho(t,x)) = 0$, we note that

$$\begin{aligned} &\langle \nu_{t,x}; F(x,k)\rangle - F(x,\rho(t,x)) = \langle \nu_{t,x}; F(x,k) - F(x,\rho(t,x))\rangle \\ &= \langle \nu_{t,x}; F_\rho(x,\rho(t,x))(k-\rho(t,x)) + \frac{1}{2}\int_0^1 \theta F_{\rho\rho}(x,\theta\rho(t,x)+(1-\theta)k)\,d\theta\,(k-\rho(t,x))^2\rangle \\ &= F_\rho(x,\rho(t,x))\,\langle \nu_{t,x}; k-\rho(t,x)\rangle + \frac{1}{2}\langle \nu_{t,x}; \int_0^1 \theta F_{\rho\rho}(x,\theta\rho(t,x)+(1-\theta)k)\,d\theta\,(k-\rho(t,x))^2\rangle \\ &= \frac{1}{2}\langle \nu_{t,x}; \int_0^1 \theta F_{\rho\rho}(x,\theta\rho(t,x)+(1-\theta)k)\,d\theta\,(k-\rho(t,x))^2\rangle. \end{aligned}$$

Since $F(x,\rho)$ is strictly convex or concave in ρ, we conclude

$$\nu_{t,x}(k) = \delta_{\rho(t,x)}(k) \qquad \text{for } (t,x) \text{ a.e.} \tag{1.2.19}$$

Therefore, we have

$$\rho^\varepsilon(t,x) \to \rho(t,x) \qquad a.e. \text{ when } \varepsilon \to 0.$$

Since the limit is unique via the uniqueness result in [3], the whole sequence $\rho^\varepsilon(t,x)$ strongly converges to $\rho(t,x)$ a.e. It is easy to check that $\rho(t,x)$ is the unique entropy solution of the Cauchy problem (1.1.1)–(1.1.2) in the sense of Definition 1.1.1. □

Remark 1.2.4. In [7], the existence of entropy solutions (1.1.1)–(1.1.2) in the sense of Definition 1.1.1 is proven for the case $\lambda(x)u^2$. They used the vanishing viscosity method combined with a mollification for $\lambda(x)$.

Remark 1.2.5. The conditions on the flux function $F(x,\rho)$ in Theorem 1.2.3 for the non-monotone case can be relaxed as follows: $F(x,\rho)$ satisfies (H1)–(H3) and is convex or concave with

$$\mathcal{L}^1\{\rho : F_{\rho\rho}(x,\rho) = 0\} = 0 \qquad \text{for a.e. } x \in \mathbb{R},$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

1.3 Hydrodynamic Limit of a Zero Range Processes with Discontinuous Speed-Parameter

In Section 1.1, we have established a compactness framework for approximate solutions via the reduction of measure-valued entropy solutions of (1.1.1)–(1.1.2) in the sense of Definition 1.1.1. In this section we focus on a microscopic particle system for a Zero Range Process (ZRP) with discontinuous speed-parameter $\lambda(x)$. We apply the compactness framework to show the hydrodynamic limit for the particle system, when the distance between particles tend to zero, to the unique entropy solution of the Cauchy problem

$$\partial_t \rho + \partial_x (\lambda(x)h(\rho)) = 0 \tag{1.3.1}$$

and with initial data:

$$\rho|_{t=0} = \rho_0(x) \geq 0, \tag{1.3.2}$$

where $h(\rho)$ is a monotone function of ρ, and $\lambda(x)$ is continuous in $x \in \mathbb{R}$ except on a closed set $\mathcal{N}$ of measure zero, with $0 < \lambda_1 \leq \lambda(x) \leq \lambda_2 < \infty$ for some constants λ_1 and λ_2. Then $m_\alpha^+ = m_\alpha^- := m_\alpha$ for $\alpha \in [0,\infty)$.

Rezakhanlou in [33] first established the hydrodynamic limit of the processus des misanthropes (PdM) with constant speed-parameter. Covert-Rezakhanlou [16] provided a proof of the hydrodynamic limit of a PdM with nonconstant but continuous speed-parameter λ. In both proofs, the most important step is to show an entropy inequality at microscopic level, which then implies the (macroscopic) Kruzkov entropy inequality, when the distance between particles tends to zero, and thereby implies the uniqueness of limit points. In this section, we generalize this to the case when the speed-parameter $\lambda(x)$ has jumps for the attractive Zero Range Process (ZRP). In Section 1.3.1, we analyze some properties of the ZRP. In Section 1.3.2, we prove the one-dimensional microscopic entropy inequality letting $\varepsilon = \varepsilon(N) = N^{-\sigma}, \sigma \in (0,1)$, for a ZRP with discontinuous speed-parameter when $N \to \infty$. Here ε is as in Section 1.2.1 and N is the inverse of the distance between particles. In Section 1.3.4, we show the existence of measure-valued solutions via the microscopic entropy inequality and how inequality (1.1.5) follows.

1.3.1 Some properties of the microscopic interacting particle system

We consider a system of particles with conserved total mass and evolving on a one-dimensional lattice $\mathbb{Z}$ according to a Markovian law. With the Euler scaling factor N, the microscopic particle density is expected to converge to a deterministic limit when $N \to \infty$, which is characterized by a solution of a conservation law. Under the Euler scaling, $\frac{1}{N}$ represents the distance between sites. Obviously we have two space scales: The discrete lattice $\mathbb{Z}$ as embedded in $\mathbb{R}$ with vertices $\frac{u}{N}$ and $u \in \mathbb{Z}$. In this way, the distances between particles tend to zero if N increases to infinity. Sites of the microscopic scale $\mathbb{Z}$ are denoted by the letters u, v and correspond to the points $\frac{u}{N}$, $\frac{v}{N}$ in the macroscopic scale $\mathbb{R}$. Points of the macroscopic space scale $\mathbb{R}$ are denoted by the letters x, y and correspond to the sites $[xN]$, $[yN]$ in the microscopic space scale, where [z] is the integer part of z. We denote by $\eta_t(u)$ the number of particles at time $t > 0$ at site u. Then the vector $\eta_t = (\eta_t(u) : u \in \mathbb{Z})$ is called a configuration at time t with configuration space $E := \mathbb{N}^{\mathbb{Z}}$.

In general, the ZRP can be described as follows: Infinitely many indistinguishable particles are distributed on a 1-dimensional lattice. Any site of the lattice may be occupied by a finite number of particles. Associated to a given site u there is an exponential clock with rate $\lambda_\varepsilon(\frac{u}{N})g(\eta(u))$ depending on the macroscopic spatial coordinates. Each time the clock rings on the site u, one of the particles jumps to the site v chosen with probability $p(u, v)$. The elementary transition probabilities p: $\mathbb{Z} \to [0, 1]$ are supposed to be

(i) translation invariant: $p(x, y) = p(0, y - x) =: p(y - x)$;

(ii) normalized: $\sum_y p(x, y) = 1$, $p(x, x) = 0$;

(iii) assumed to be of finite range: $p(x, y) = 0$ for $|y - x|$ sufficiently large;

(iv) irreducible: $p(0, 1) > 0$.

Without loss of generality, we assume that $\sum_z p(z)z = \gamma = 1$; otherwise, for $\gamma \neq 1$, we replace the function $h(\rho)$ by $h(\rho)/\gamma$ in the following argument. The rate $g : \mathbb{N} \to \mathbb{R}_+$ is a positive, nondecreasing function with $g(0) = 0$, $g(\infty) = \infty$, and

$$\frac{g(k)}{k^2} \to 0 \qquad \text{when } k \to \infty. \tag{1.3.3}$$

Now consider a test particle with initial position X_0. Since it evolves as a continuous time random walk, if X_t denotes its position at time t, there exists for every $\varepsilon > 0$ an $A = A(t, \varepsilon) > 0$ such that $P[|X_t - X_0| > A] \leq \varepsilon$. That means with probability close to 1 the particle moved a distance of order $\mathcal{O}(\frac{1}{N})$, since the distance between particles on the microscopic scale is of order $\frac{1}{N}$. But on the macroscopic scale it did not have time to evolve. Therefore, to have a macroscopic evolution in macroscopic time, we rescale the time by the Euler factor N.

With this description, the Markov process η_t is generated by

$$NL_\varepsilon^N f(\eta) = N \sum_{u,v} \lambda_\varepsilon(\frac{u}{N})g(\eta(u))p(v - u)(f(\eta^{u,v}) - f(\eta)). \tag{1.3.4}$$

Here N comes from the Euler scaling factor speeding the generator, thus η_t denotes a configuration on which the speeded generator NL_ε^N has acted for time t, and $\eta^{u,v}$ represents the configuration η where one particle jumped from u to v:

$$\eta^{u,v}(w) = \begin{cases} \eta(w) & \text{if } w \neq u, v, \\ \eta(u) - 1 & \text{if } w = u, \\ \eta(v) + 1 & \text{if } w = v. \end{cases}$$

For any $\varepsilon = \varepsilon(N) > 0$ and for any constant $\alpha \geq 0$, we define a product measure given by

$$\tilde{\nu}_\alpha^N(\eta) := \prod_u \frac{1}{Z\big(\alpha/\lambda_\varepsilon(\frac{u}{N})\big)} \frac{\alpha^{\eta(u)}}{(\lambda_\varepsilon(\frac{u}{N}))^{\eta(u)} g(\eta(u))!} := \prod_u \tilde{\nu}_\alpha^N(\eta(u)), \tag{1.3.5}$$

where Z is a partition function equal to

$$Z\big(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\big) = \sum_{n=0}^{\infty} \frac{\alpha^n}{\big(\lambda_\varepsilon(\frac{u}{N})\big)^n g(n)!}. \tag{1.3.6}$$

Then the expected value of the occupation variable $\eta(u)$ is equal to

$$E_{\tilde{\nu}_\alpha^N}[\eta(u)] = \frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})} \frac{Z'\big(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\big)}{Z\big(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\big)} := R\big(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\big).$$

Now let h be the inverse function of R to obtain

$$h\big(R\big(\frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})}\big)\big) = \frac{\alpha}{\lambda_\varepsilon(\frac{u}{N})} \quad \Rightarrow \quad \lambda_\varepsilon(\frac{u}{N}) h\left(E_{\tilde{\nu}_\alpha^N}[\eta(u)]\right) = \alpha \quad \Leftrightarrow \quad E_{\tilde{\nu}_\alpha^N}[\eta(u)] = m_\alpha(\frac{u}{N}),$$

where m_α is a steady-state solution to

$$\partial_t \rho + \partial_x \left(\lambda_\varepsilon(x) h(\rho)\right) = 0. \tag{1.3.7}$$

Furthermore, it follows that

$$E_{\tilde{\nu}_\alpha^N}[g(\eta(u))] = h\left(m_\alpha(\frac{u}{N})\right).$$

From now on, we set

$$\mu_{m_\alpha}^N(\eta) = \prod_u \nu_{m_\alpha(\frac{u}{N})}(\eta(u)) := \prod_u \tilde{\nu}^N_{\lambda_\varepsilon(\frac{u}{N}) h(m_\alpha(\frac{u}{n}))}(\eta(u)). \tag{1.3.8}$$

Thus

$$\mu_{m_\alpha}^N(\eta) = \prod_u \frac{1}{Z\left(h\left(m_\alpha(\frac{u}{N})\right)\right)} \frac{\left(h\left(m_\alpha(\frac{u}{N})\right)\right)^{\eta(u)}}{g(\eta(u))!},$$

The important attribute of the ZRP with nonconstant speed-parameter is that the *product* measure $\mu_{m_\alpha}^N(\eta)$ is invariant under the generator NL_ε^N, i.e.,

$$\int L_\varepsilon^N(f(\eta)) d\mu_{m_\alpha}^N(\eta) = 0. \tag{1.3.9}$$

As initial distribution of our system , we choose the local equilibrium *product measure* $\mu_0^N(\eta)$ *associated to a bounded density profile* defined as follows: For a bounded density profile $\rho_0 \geq 0$, the probability that particles at time $t = 0$ are distributed with configuration η is equal to

$$\mu_0^N(\eta) := \prod_u \frac{1}{Z(h(\rho_{u,N}))} \frac{(h(\rho_{u,N}))^{\eta(u)}}{g(\eta(u))!}, \tag{1.3.10}$$

where $\rho_{u,N} \geq 0$ is a sequence satisfying $\lim_{N\to\infty} \int |\rho_{[Nx],N} - \rho_0(x)|dx = 0$ for $[Nx]$ as the integer part of Nx. With this definition, we say that a sequence of probability measures μ^N is associated to a density profile $\rho \geq 0$ if

$$\lim_{N\to\infty} \langle \mu^N(\eta)\,;\, \Big|\frac{1}{N}\sum_u J(\frac{u}{N})\eta(u) - \int J(x)\rho(x)dx\Big|\rangle = 0 \qquad \text{for every test function } J.$$

Furthermore, let μ_t^N denote the distribution of a configuration at time t initially distributed by μ_0^N:

$$\mu_t^N = S_t^N * \mu_0^N, \tag{1.3.11}$$

where $S_t^N = e^{tNL_\varepsilon^N}$ is the semigroup corresponding to the generator NL_ε^N. Since we consider an attractive ZRP, we have the additional condition that for two initial measures $\mu_{\rho_0}^N$ and $\mu_{\omega_0}^N$ on E with profiles ρ_t and ω_t, respectively, the following monotonicity holds:

$$\mu_{\rho_0}^N \leq \mu_{\omega_0}^N \;\;\Rightarrow\;\; \mu_{\rho_t}^N \leq \mu_{\omega_t}^N. \tag{1.3.12}$$

We say that two measures μ_1 and μ_2 on E satisfy $\mu_1 \leq \mu_2$ if there exists a coupling measure $\bar{\mu}$ on $E \times E$ such that for some $A \subset E$, $\bar{\mu}(A \times E) = \mu(A)$, $\bar{\mu}(E \times A) = \mu_2(A)$ and $\bar{\mu}((\eta, \xi);\, \eta \leq \xi) = 1$, where the partial order $\eta \leq \xi$ is given if $\eta(u) \leq \xi(u)$ for all $u \in \mathbb{Z}$.

For a ZRP attractiveness is satisfied if g is a nondecreasing function. Moreover, it is easy to prove that $\mu_{\rho_0} \leq \mu_{\omega_0}$ in the stochastic sense if $\rho_0 \leq \omega_0$. It then follows by attractiveness that, for any constant α such that $m_\alpha(x) \geq \rho_0(x)$, we obtain that the inequality $\mu_0^N \leq \mu_{m_\alpha}^N$ implies

$$S_t^N \mu_0^N \leq S_t^N \mu_{m_\alpha}^N = \mu_{m_\alpha}^N. \tag{1.3.13}$$

Since our initial distribution has a bounded density profile, then the density profile remains bounded at later time t.

The goal in proving the hydrodynamic limit of a ZRP is that, if we start from a configuration η_0 distributed with an initial measure μ_0^N associated to the bounded density profile ρ_0, then the distribution μ_t^N of the configuration η_t at later time t is associated to the density profile $\rho(t, \cdot)$, where ρ is the solution of the Cauchy problem (1.3.1)–(1.3.2) in the sense of Definition 1.1.1. In other words, our main theorem in this section is the following.

Theorem 1.3.1 (Hydrodynamic limit of an attractive ZRP with discontinuous speed-parameter). *Let η_t be an attractive ZRP with* (1.3.3) *initially distributed by the measure μ_0^N associated to a bounded density profile $\rho_0 : \mathbb{R}_+^2 \to \mathbb{R}_+$ as defined in* (1.3.10). *Let $\varepsilon = \varepsilon(N) = N^{-\sigma}, \sigma \in (0, 1)$. Then, at later time t,*

$$\lim_{N\to\infty} \langle \mu_t^N(\eta);\, \Big|\frac{1}{N}\sum_u J(\frac{u}{N})\eta_t(u) - \int J(x)\rho(t,x)dx\Big|\rangle = 0 \tag{1.3.14}$$

for any test function $J : \mathbb{R}_+^2 \to \mathbb{R}$, where $\rho(t, x)$ is the unique solution of the Cauchy problem (1.3.1)–(1.3.2) *in in the sense of Definition 1.1.1.*

To achieve this, we have to establish an entropy inequality at microscopic level. This will be done in Section 1.3.2 by using the scaling relation $\varepsilon = \varepsilon(N) = N^{-\sigma}, \sigma \in (0,1)$. Associated to each configuration η_t, we may define the empirical measure viewed as a random measure on $\mathbb{R}$ by

$$\chi_t^N(x) := \frac{1}{N}\sum_u \eta_t(u)\delta_{\frac{u}{N}}(x). \tag{1.3.15}$$

Then $\langle \chi_t^N(\cdot), J(\cdot)\rangle = \frac{1}{N}\sum_u J(\frac{u}{N})\eta_t(u)$, and we can rewrite (1.3.14) by

$$\lim_{N\to\infty} \langle \mu_t^N(\eta);\ \big|\langle \chi_t^N(\cdot), J(\cdot)\rangle - \int J(x)\rho(t,x)dx\big|\rangle = 0. \tag{1.3.16}$$

1.3.2 The entropy inequality at microscopic level

The following proposition is essential towards the hydrodynamic limit. The proof relies on coupling arguments and here the assumption of attractiveness of the ZRP is crucial.

Proposition 1.3.2 (Entropy inequality at microscopic level for $\varepsilon = N^{-\sigma}$ with $\sigma \in (0,1)$ when $N \to \infty$). *Let m_α^ε be the steady-state solutions of* (1.2.2) *as defined in* (1.1.2) *with $F_\varepsilon(x,\rho) = \lambda_\varepsilon(x)h(\rho)$. Let η_t be the ZRP generated by NL_ε^N defined by* (1.3.4) *and initially distributed by the measure μ_0^N defined by* (1.3.10). *Let $\eta^l(u)$ be the average density of particles in large microscopic boxes of size $2l+1$ and centered at u:*

$$\eta^l(u) := \frac{1}{2l+1}\sum_{|u-v|\le l}\eta(v).$$

Then, for every test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$\lim_{l\to\infty}\lim_{N\to\infty} \mu_t^N\Big\{\int_0^t \frac{1}{N}\sum_u \Big(\partial_s J(s,\frac{u}{N})\big|\eta_s^l(u) - m_\alpha^\varepsilon(\frac{u}{N})\big| + \partial_x J(s,\frac{u}{N})\big|\lambda_\varepsilon(\frac{u}{N})h(\eta_s^l(u)) - \alpha\big|\Big)ds + \frac{1}{N}\sum_u J(0,\frac{u}{N})\big|\eta_0^l(u) - m_\alpha^\varepsilon(\frac{u}{N})\big| \ge -\delta\Big\} = 1. \tag{1.3.17}$$

Inequality (1.3.17) is the entropy inequality (1.1.5) with ρ replaced by the average density of particles in the microscopic boxes of length $2l+1$. To prove the microscopic entropy inequality, we consider the coupled process (η_t,ξ_t) generated by $N\bar{L}_\varepsilon^N$, where $\bar{L}_\varepsilon^N$ is defined by

$$\begin{aligned}\bar{L}_\varepsilon^N f(\eta,\xi) \;=\; & \sum_{u,v} p(v-u)\lambda_\varepsilon(\frac{u}{N})\min\{g(\eta(u)), g(\xi(u))\}\,(f(\eta^{u,v},\xi^{u,v}) - f(\eta,\xi)) \\ & + \sum_{u,v} p(v-u)\lambda_\varepsilon(\frac{u}{N})\{g(\eta(u)) - g(\xi(u))\}_+\,(f(\eta^{u,v},\xi) - f(\eta,\xi)) \\ & + \sum_{u,v} p(v-u)\lambda_\varepsilon(\frac{u}{N})\{g(\xi(u)) - g(\eta(u))\}_+\,(f(\eta,\xi^{u,v}) - f(\eta,\xi)).\end{aligned} \tag{1.3.18}$$

Furthermore, denote the initial distribution of (η_t, ξ_t) by $\bar{\mu}_0^N = \mu_0^N \times \mu_{m_\alpha^\varepsilon}^N$, where μ_0^N is the initial measure with density profile ρ_0 defined by (1.3.10) and $\mu_{m_\alpha^\varepsilon}^N$ denotes the invariant measure as defined in (1.3.8).

Notice that since the ξ-marginal of $\bar{\mu}_0^N$ is the invariant measure $\mu_{m_\alpha^\varepsilon}^N$, at any time t the marginal remains the same. Thus the measure $\mu_{m_\alpha^\varepsilon}^N$ is always stochastically bounded since m_α^ε is bounded. Therefore by the law of large numbers for any limit point μ_m of $\mu_{m_\alpha^\varepsilon}^N$ and for each u fixed we can define μ_m a.s the density

$$\lim_{l\to\infty} \xi^l(u) := m(x).$$

Then by the equivalence of ensembles we obtain for any local function ψ on $\{u-l,\dots,u+l\}$ and for any u that

$$\lim_{l\to\infty} E_{\mu^l_{m_\alpha^\varepsilon}}\left[\left|\psi(\xi^l(u)) - E_{\mu_m}[\psi(\xi(u))]\right|\right] = 0$$

where the probability measure μ_m^l denotes the projection of μ_m^N to configurations on $\{u-l,\dots,u+l\}$. Thus we obtained the following ergodic result:

$$\lim_{l\to\infty} \mu_m\left\{\left|h(\xi^l(u)) - h(m)\right| > 0\right\} = 0 \tag{1.3.19}$$

and consequently

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{N\to\infty} E_{\bar{\mu}_t^N}\left[\int_0^t \frac{1}{N}\sum_u \left|h(\xi_t^l(u)) - h(m_\alpha^\varepsilon(\frac{u}{N}))\right| ds\right]\\
=\ & t\lim_{l\to\infty}\lim_{N\to\infty} E_{\mu^N_{m_\alpha^\varepsilon}}\left[\frac{1}{N}\sum_u \left|h(\xi^l(u)) - h(m_\alpha^\varepsilon(\frac{u}{N}))\right|\right]\\
\le\ & t\lim_{l\to\infty}\sup_{\mu_m} E_{\mu_m}[\left|h(\xi^l(0)) - h(m)\right|] = 0.
\end{aligned}$$

In the same way we have that

$$\lim_{l\to\infty}\lim_{N\to\infty} E_{\bar{\mu}_t^N}\left[\int_0^t \frac{1}{N}\sum_u \left|\xi_t^l(u) - m_\alpha^\varepsilon(\frac{u}{N})\right| ds\right] = 0$$

Then, to prove Proposition 1.3.2, it suffices to prove the following proposition.

Proposition 1.3.3. *Let (η_t, ξ_t) be the coupled process, starting from $\bar{\mu}_0^N$, generated by $N\bar{L}_\varepsilon^N$ as defined by (1.3.18). Let $\bar{\mu}_t^N = \bar{S}_t^N * \bar{\mu}_0^N$, where $\bar{S}_t^N$ is the semigroup corresponding to the generator $N\bar{L}_\varepsilon^N$. Then, for every test function $J : \mathbb{R}_+^2 \to \mathbb{R}_+$ and every $\varepsilon = N^{-\sigma}$ with $\sigma \in (0,1)$,*

$$\begin{aligned}
&\lim_{l\to\infty}\lim_{N\to\infty}\\
&\bar{\mu}_t^N\left\{\int_0^T \frac{1}{N}\sum_u \left\{\partial_s J(s,\frac{u}{N})\left|\eta_s^l(u) - \xi_s^l(u)\right| + \partial_x J(s,\frac{u}{N})\lambda_\varepsilon(\frac{u}{N})\left|h(\eta_s^l(u)) - h(\xi_s^l(u))\right|\right\}ds\right.\\
&\qquad\left. + \frac{1}{N}\sum_u J(0,\frac{u}{N})\left|\eta_0^l(u) - \xi_0^l(u)\right| \ge -\delta\right\} = 1.
\end{aligned}$$

Recall that a microscopic entropy inequality leading to the Kruzkov entropy inequality has been proved in [16] for the process of PdM with nonconstant but continuous speed-parameter λ_ε. Since there does not exist an invariant product measure for a PdM in general such that $E_{\mu^N_{m^\varepsilon_\alpha}}[\xi(u)] = m^\varepsilon_\alpha(\frac{u}{N})$, to replace the process ξ by $m^\varepsilon_\alpha(\frac{u}{N})$, one has to apply the relative entropy method of Yau [38].

In our case of a space-dependent ZRP, the invariant product measure is available so that we can approximate the steady-state solution m^ε_α by a process ξ distributed by the invariant measure $\mu^N_{m^\varepsilon_\alpha}$ for any $\alpha \in (0,\infty)$. Then, Proposition 1.3.2 indeed directly follows from Proposition 1.3.3.

Proof of Proposition 1.3.3 We split the proof in three steps.

Step 1: Lower bound for the martingale. For a test function J with compact support in $\mathbb{R}^2_+$, define by M^J_t the martingale vanishing at time $t=0$:

$$\begin{aligned} M^J_t &= \frac{1}{N}\sum_u J(t,\frac{u}{N})\,|\eta_t(u)-\xi_t(u)| - \frac{1}{N}\sum_u J(0,\frac{u}{N})\,|\eta_0(u)-\xi_0(u)| \\ &\quad - \int_0^t (\partial_s + N\bar{L}^N_\varepsilon)\big(\frac{1}{N}\sum_u J(s,\frac{u}{N})\,|\eta_s(u)-\xi_s(u)|\big)ds. \end{aligned}$$

Since J has compact support, then, for t large enough,

$$M^J_t = -\frac{1}{N}\sum_u J(0,\frac{u}{N})\,|\eta_0(u)-\xi_0(u)| - \int_0^t (\partial_s + N\bar{L}^N_\varepsilon)\Big(\frac{1}{N}\sum_u J(s,\frac{u}{N})\,|\eta_s(u)-\xi_s(u)|\Big)ds.$$

We now calculate

$$\begin{aligned} &\bar{L}^N_\varepsilon\,|\eta(u)-\xi(u)| \\ &\quad= \sum_{v,w} p(w-v)\lambda_\varepsilon(\frac{v}{N})\Big\{ \min\{g(\eta(v)),g(\xi(v))\}\big(|\eta^{v,w}(u)-\xi^{v,w}(u)| - |\eta(u)-\xi(u)|\big) \\ &\qquad\qquad + \{g(\eta(v))-g(\xi(v))\}_+\big(|\eta^{v,w}(u)-\xi(u)| - |\eta(u)-\xi(u)|\big) \\ &\qquad\qquad + \{g(\xi(v))-g(\eta(v))\}_+\big(|\eta(u)-\xi^{v,w}(u)| - |\eta(u)-\xi(u)|\big)\Big\} \\ &= \sum_v \big(1-G_{u,v}(\eta,\xi)\big)\Big(- p(v-u)\lambda_\varepsilon(\frac{u}{N})\,|g(\eta(u))-g(\xi(u))| \\ &\qquad\qquad + p(u-v)\lambda_\varepsilon(\frac{v}{N})\,|g(\eta(v))-g(\xi(v))|\Big) \\ &- \sum_v G_{u,v}(\eta,\xi)\Big(p(v-u)\lambda_\varepsilon(\frac{u}{N})\,|g(\eta(u))-g(\xi(u))| + p(u-v)\lambda_\varepsilon(\frac{v}{N})\,|g(\eta(v))-g(\xi(v))|\Big), \end{aligned} \tag{1.3.20}$$

where $G_{u,v}$ is the indicator function that equals to 1 if η and ξ are not ordered, i.e.,

$$G_{u,v}(\eta,\xi) = \mathbb{1}\{\eta(u)<\xi(u);\eta(v)>\xi(v)\} + \mathbb{1}\{\eta(u)>\xi(u);\eta(v)<\xi(v)\}.$$

Notice that the second sum is nonpositive. Therefore, plugging in the last expression in the martingale M^J_t and then interchange u and v in the last term, we can bound the martingale below by

$$
\begin{aligned}
&-\frac{1}{N}\sum_u J(0,\frac{u}{N})\,|\eta_0(u)-\xi_0(u)| - \int_0^t \frac{1}{N}\sum_u \partial_s J(s,\frac{u}{N})\,|\eta_s(u)-\xi_s(u)|\,ds \\
&+\int_0^t \sum_{u,v}\big(J(s,\frac{u}{N})-J(s,\frac{v}{N})\big)p(v-u)\big(1-G_{u,v}(\eta_s,\xi_s)\big)\,\lambda_\varepsilon(\frac{u}{N})\,|g(\eta_s(u))-g(\xi_s(u))|\,ds.
\end{aligned}
$$

Since the transition probability p is of finite range, i.e. $p(z)=0$ if $|z|>r$ for some r, then

$$
\Big(J(s,\frac{u}{N})-J(s,\frac{v}{N})\Big)\,p(v-u) = -\frac{1}{N}(v-u)p(v-u)\partial_x J(s,\frac{u}{N}) + O(\frac{1}{N^2}).
$$

With $v=u+y$, it then follows that the martingale is bounded below by

$$
\begin{aligned}
-\int_0^t \frac{1}{N}\sum_u \Big\{&\partial_s J(s,\frac{u}{N})\big|\eta_s(u)-\xi_s(u)\big| \\
&+\partial_x J(s,\frac{u}{N})\lambda_\varepsilon(\frac{u}{N})\tau_u\big(\sum_y yp(y)(1-G_{0,y})\big)\big|g(\eta_s(0))-g(\xi_s(0))\big|\Big\}ds \\
&-\frac{1}{N}\sum_u J(0,\frac{u}{N})\,|\eta_0(u)-\xi_0(u)| + O(\frac{1}{N}).
\end{aligned}
$$

where τ_u denotes the shift operator on configurations by u. *Step 2:* We show

$$
\lim_{N\to\infty} E_{\bar{\mu}_t^N}\big[\,\big(M_t^J\big)^2\,\big]=0. \tag{1.3.21}
$$

Recall that

$$
N_t^J := (M_t^J)^2 - \int_0^t \Big(N\bar{L}_\varepsilon^N(A^J(s,\eta,\xi))^2 - 2A^J(s,\eta,\xi)N\bar{L}_\varepsilon^N(A^J(s,\eta,\xi))\Big)\,ds
$$

is a martingale vanishing at time $t=0$, where A^J is defined by

$$
A^J(t,\eta,\xi) = \frac{1}{N}\sum_u J(t,\frac{u}{N})|\eta_t(u)-\xi_t(u)|.
$$

Then, by definition, $E_{\bar{\mu}_s^N}\left[N_s^J\right]=0$ for all $0\le s\le t$. Thus, it suffices to show that the expectation of the integral term of N_t^J converges to zero when $N\to\infty$. In order to prove this, we first find that, by careful calculation,

$$
\begin{aligned}
&N\bar{L}_\varepsilon^N(A^J(s,\eta,\xi))^2 - 2NA^J(s,\eta,\xi)\bar{L}_\varepsilon^N(A^J(s,\eta,\xi)) \\
&=\sum_{v,w}p(w-v)N\lambda_\varepsilon(\frac{v}{N})\Big\{\,|g(\eta_s(v))-g(\xi_s(v))|\,\frac{1}{N^2}\big(1-G_{v,w}(\eta_s,\xi_s)\big)\big(J(s,\frac{w}{N})-J(s,\frac{v}{N})\big)^2 \\
&\qquad + |g(\xi_s(v))-g(\eta_s(v))|\,\frac{1}{N^2}G_{v,w}(\eta_s,\xi_s)\big(J(s,\frac{v}{N})+J(s,\frac{w}{N})\big)^2\Big\}.
\end{aligned}
$$

Since J is a smooth function, the first term of this expression is less $\mathcal{O}(\frac{g(CN)}{N^2})$ for some constant C depending on the total initial mass and therefore converges to zero when $N\to\infty$ by (1.3.3). For the second term, we know that $(J(s,\frac{v}{N})+J(s,\frac{w}{N}))^2\le 4\,\|J\|_\infty^2$, which implies

$$
\begin{aligned}
&N\bar{L}_\varepsilon^N(A^J(s,\eta,\xi))^2 - 2NA^J(s,\eta,\xi)\bar{L}_\varepsilon^N(A^J(s,\eta,\xi)) \\
&=\mathcal{O}(\frac{g(CN)}{N^2}) + \frac{4\,\|J\|_\infty^2}{N}\sum_{v,w}G_{v,w}(\eta_s,\xi_s)p(w-v)\lambda_\varepsilon(\frac{v}{N})\,|g(\xi_s(v))-g(\eta_s(v))|\,.
\end{aligned}
$$

Then, to conclude the proof of (1.3.21), it suffices to show

$$E_{\bar{\mu}_t^N}\Big[\int_0^t \big(\sum_{v,w} G_{v,w}(\eta_s,\xi_s)p(w-v)\lambda_\varepsilon(\frac{v}{N})\,|g(\xi_s(v))-g(\eta_s(v))|\,\big)\,ds\Big] = \mathcal{O}(1). \qquad (1.3.22)$$

For this, we use the martingale M_t^J vanishing at 0 with $J\equiv 1$, that is,

$$M_t := \frac{1}{N}\sum_u |\eta_t(u)-\xi_t(u)| - \frac{1}{N}\sum_u |\eta_0(u)-\xi_0(u)| - \int_0^t \frac{1}{N}\sum_u N\bar{L}_\varepsilon^N|\eta_s(u)-\xi_s(u)|ds.$$

By (1.3.20), the integral term of the martingale is equal to

$$\int_0^t \frac{2}{N}\sum_{u,v} NG_{u,v}(\eta_s,\xi_s)p(v-u)\lambda_\varepsilon(\frac{u}{N})\,|g\left(\eta_s(u)\right)-g\left(\xi_s(u)\right)|\,ds,$$

by interchanging u and v in some terms. Then we find

$$\begin{aligned}
&E_{\bar{\mu}_t^N}\Big[\int_0^t 2\sum_{u,v} G_{u,v}(\eta_s,\xi_s)p(v-u)\lambda_\varepsilon(\frac{u}{N})\,|g\left(\eta_s(u)\right)-g\left(\xi_s(u)\right)|\,ds\Big]\\
&= E_{\bar{\mu}_t^N}\Big[\int_0^t \frac{1}{N}\sum_u |\eta_0(u)-\xi_0(u)|ds\Big] - E_{\bar{\mu}_t^N}\Big[\int_0^t \frac{1}{N}\sum_u |\eta_t(u)-\xi_t(u)|ds\Big]\\
&\le E_{\bar{\mu}_t^N}\Big[\int_0^t \frac{1}{N}\sum_u |\eta_0(u)-\xi_0(u)|ds\Big].
\end{aligned}$$

Since we assumed that both marginals of $\bar{\mu}_t^N$ are bounded, (1.3.22) follows, which leads to (1.3.21).

With the result of Step 1 and (1.3.21) and using the Chebichev inequality, we obtain

$$\begin{aligned}
&\bar{\mu}_t^N\Big\{\frac{1}{N}\sum_u J(0,\frac{u}{N})\,|\eta_0(u)-\xi_0(u)| + \int_0^t \frac{1}{N}\sum_u \big\{\partial_s J(s,\frac{u}{N})\big|\eta_s(u)-\xi_s(u)\big|\\
&\quad +\partial_x J(s,\frac{u}{N})\lambda_\varepsilon(\frac{u}{N})\tau_u\big(\sum_y yp(y)(1-G_{0,y})(\eta,\xi)\big)\big|g(\eta_s(0))-g(\xi_s(0))\big|\big\}ds + O(\frac{1}{N}) < -\delta\Big\}\\
&\le \quad \bar{\mu}_t^N\left\{M_t^J > \delta\right\} \le \bar{\mu}_t^N\left\{\left|M_t^J\right| > \delta\right\} \le \frac{1}{\delta^2}E_{\bar{\mu}_t^N}\left[\left(M_t^J\right)^2\right],
\end{aligned} \qquad (1.3.23)$$

which converges to 0 when $N\to\infty$, for all $\delta>0$.

Step 3. We next use the following summation by parts formula: *For any bounded function a of $\eta(\cdot)$ with $a(0)=0$ and for any smooth test function $J:\mathbb{R}\to\mathbb{R}$, we obtain that, for any $L>0$,*

$$\frac{1}{N}\sum_{|u|\le LN} J(\frac{u}{N})a(\eta(u)) = \frac{1}{N}\frac{1}{(2l+1)}\sum_{|u|\le LN} J(\frac{u}{N})\sum_{|u-v|\le l} a(\eta(v)) + \mathcal{O}(\frac{l\,\|J\|_{Lip}}{N}). \quad (1.3.24)$$

Since we restrict $\varepsilon = N^{-\sigma}, \sigma\in(0,1)$, then $\|\lambda_\varepsilon\|_{Lip}\le C/\varepsilon = CN^\sigma$ and $\mathcal{O}(\frac{l\|\lambda_\varepsilon\|_{Lip}}{N}) = \mathcal{O}(\frac{l}{N^{1-\sigma}})\to 0$ when $N\to\infty$ so that we can use this summation by parts formula (1.3.24) to replace inequality (1.3.23) by

$$\lim_{l\to\infty}\lim_{N\to\infty}\bar{\mu}_t^N\Big\{\frac{1}{N}\sum_u J(0,\frac{u}{N})\frac{1}{2l+1}\sum_{|z-u|\le l}|\eta_0(z)-\xi_0(z)|$$
$$+\int_0^t\frac{1}{N}\sum_u\partial_s J(s,\frac{u}{N})\frac{1}{2l+1}\sum_{|z-u|\le l}|\eta_s(z)-\xi_s(z)|\,ds$$
$$+\int_0^t\frac{1}{N}\sum_u\partial_x J(s,\frac{u}{N})\lambda_\varepsilon(\frac{u}{N})\frac{1}{2l+1}$$
$$\times\sum_{|z-u|\le l}\tau_z\big(\sum_y yp(y)(1-G_{0,y})(\eta_s,\xi_s)\big)\big|g(\eta_s(0))-g(\xi_s(0))\big|\,ds<-\delta\Big\}$$
$$=0. \tag{1.3.25}$$

Notice that, in (1.3.25), since J is a positive function, by the triangle inequality, we can remove the sum inside the absolute value in the first line. In Section 1.3.3 we will prove the following Theorem

Theorem 1.3.4 (One block estimate).

$$\lim_{l\to\infty}\lim_{N\to\infty}E_{\bar{\mu}_t^N}\Big\{\int_0^t\frac{1}{N}\sum_u\Big|\frac{1}{2l+1}\sum_{|u-z|\le l}|\eta_s(z)-\xi_s(z)|-|\eta_s^l(u)-\xi_s^l(u)|\Big|ds\Big\}=0, \tag{1.3.26}$$

and

$$\lim_{l\to\infty}\lim_{N\to\infty}$$
$$E_{\bar{\mu}_t^N}\Big\{\int_0^t\frac{1}{N}\sum_u\tau_u\Big|\frac{1}{2l+1}\sum_{|z|\le l}\tau_z\big(\sum_y yp(y)(1-G_{0,y})(\eta_s,\xi_s)\big)\big|g(\eta_s(0))-g(\xi_s(0))\big|$$
$$-\big|h(\eta_s^l(0))-h(\xi_s^l(0))\big|\Big|ds\Big\}=0. \tag{1.3.27}$$

Combining (1.3.25) with (1.3.26)–(1.3.27), we complete the proof of Proposition 1.3.3.

□

1.3.3 The one-block estimate

We will prove only (1.3.27) of Theorem 1.3.4, since (1.3.26) follows by the same arguments. The properties coming from the attractiveness of the coupled process will play an essential role in the proof. We therefor make some preliminary observations:

First define for any $L\in\mathbb{N}$ the following space time average:

$$\hat{\mu}_T^N:=\frac{1}{TLN}\int_0^T\sum_{|u|\le LN}\tau_{-u}\bar{\mu}_s^N\,ds.$$

Then we can rewrite the left hand side of (1.3.27) as

$$\lim_{l\to\infty}\lim_{N\to\infty}\int\Big|\frac{1}{2l+1}\sum_{|z|\le l}\tau_z\Big(\sum_y yp(y)(1-G_{0,y})(\eta_s,\xi_s)\Big)\big|g(\eta_s(0))-g(\xi_s(0))\big| \\ -\big|h(\eta_s^l(0))-h(\xi_s^l(0))\big|\Big|d\hat{\mu}_t^N. \qquad (1.3.28)$$

Thus instead of shifting the blocks we now shift the measures and we can concentrated on a fixed block.

Recall that $\bar{\mu}_t^N$ is initially distributed by $\mu_0^N\times\mu_{m_\alpha^\varepsilon}^N$ where μ_0^N is equal to the product measure defined in (1.3.10) with density profile ρ_0. As we already noticed in the Section 1.3.2 the ξ-marginal $\mu_{m_\alpha^\varepsilon}^N$ of $\hat{\mu}_t^N$ by the monotonicity assumption (1.3.12) is always stochastically bounded by $\mu_{\|m_\alpha\|_\infty}^N=\nu_{\|m_\alpha\|_\infty}$. On the other hand by Remark 1.2.2 the η-marginal having as initial density profile ρ_0 is also always less than $\nu_{\|m_\alpha\|_\infty}$. Thus the sequence $(\hat{\mu}_t^N)_N$ is tight. Denote by μ any limit point of $(\hat{\mu}_t^N)_N$.

The following two Lemmata will characterize the measure $\hat{\mu}_t^N$ in the limit as invariant and translation invariant measures:

Lemma 1.3.5. *For any local bounded function ψ and any limit point μ of $(\hat{\mu}_t^N)_N$, μ is translation invariant, that means*

$$\int(\tau_j\psi(\eta)-\psi(\eta))d\mu=0.$$

Proof. With the definition of the measure $\hat{\mu}_t^N$ we obtain

$$\begin{aligned}\left|\int(\psi-\tau_j\psi)d\hat{\mu}_t^N\right| &= \left|\frac{1}{tLN}\int_0^t\int\sum_{|u|\le LN}(\psi-\tau_j\psi)\tau_{-u}d\bar{\mu}_s^N ds\right| \\ &= \left|\frac{1}{tLN}\int_0^t\int\sum_{|u|\le LN}(\tau_u\psi-\tau_{j+u}\psi)d\bar{\mu}_s^N ds\right| \\ &= \left|\frac{1}{tLN}\int_0^t\int\left(\sum_{|u|\le LN}\tau_u\psi-\sum_{|u-j|\le LN}\tau_u\psi\right)d\bar{\mu}_s^N ds\right|\end{aligned}$$

Since ψ is a local function the sum in the integral is a sum of the order of j terms and since ψ is also bounded we obtain, the right hand side is of order $\mathcal{O}(\frac{1}{N})$.

The tightness of $\hat{\mu}_t^N$ concludes the proof since

$$\lim_{N\to\infty}\left|\int(\psi-\tau_j\psi)d\hat{\mu}_t^N\right|=\left|\int(\psi-\tau_j\psi)d\mu\right|.$$

□

Lemma 1.3.6. *Denote by $\bar{L}^{(1)}$ the generator of the coupled process (η_t,ξ_t) defined in* (1.3.18) *but with $\lambda_\varepsilon\equiv 1$. Then for any local function ψ and any limit point μ of $(\hat{\mu}_t^N)_N$, μ is invariant with respect $\bar{L}^{(1)}$, that means*

$$\int\bar{L}^{(1)}fd\mu=0.$$

Proof. Since ψ is a local function

$$\lim_{N\to\infty}\int \bar{L}_N^{\varepsilon}\psi d\mu = \int \bar{L}^{(1)}\psi d\mu. \tag{1.3.29}$$

Notice that this is also true in the case where ψ depends on configurations through sites around a discontinuity point $\frac{w}{N}$ of the speed parameter λ since by Lemma 1.3.5 μ is translation invariant and hence we can always chose an appropriate translation on the configurations such that $\bar{L}_N^{\varepsilon}\psi$ does not depend on w. To prove (1.3.29) we look at

$$\int \bar{L}_N^{\varepsilon}\psi d\hat{\mu}_t^N = \frac{1}{LN}\sum_{|u|\le NL}\int\int_0^t \bar{L}_N^{\varepsilon}\psi d(\tau_{-u}\bar{S}_s^N\bar{\mu}_0^N)ds$$

where we recall that $\bar{S}_t^N$ is the semigroup corresponding to the generatoe $\bar{L}_N^{\varepsilon}$. This is equal to

$$\begin{aligned}\frac{1}{LN}\sum_{|u|\le NL}\int\int_0^t \tau_u\bar{S}_s^N\bar{L}_N^{\varepsilon}\psi d\bar{\mu}_0^N ds &= \frac{1}{LN}\sum_{|u|\le NL}\int\int_0^t \tau_u\left(e^{N\bar{L}_N^{\varepsilon}}\bar{L}_N^{\varepsilon}\psi\right)d\bar{\mu}_0^N ds\\ &= \frac{1}{LN^2}\sum_{|u|\le NL}\int\int_0^t \tau_u\frac{d}{ds}\left(e^{N\bar{L}_N^{\varepsilon}}\psi\right)d\bar{\mu}_0^N ds\\ &= \frac{1}{LN^2}\sum_{|u|\le NL}\int \tau_u(S_t^N\psi-\psi)d\bar{\mu}_0^N ds\end{aligned}$$

Since ψ is a local bounded function, this term is of order $\mathcal{O}(\frac{1}{N})$ an thus vanishes in the limit. Combining this with the tightness of the sequence $\hat{\mu}_t^N$ and the arguments at the beginning of the proof, this concludes the Lemma. □

Since $\hat{\mu}_t^N$ is tight , we can rewrite (1.3.28) as

$$\lim_{l\to\infty}\sup_{\mu}\int\Big|\frac{1}{2l+1}\sum_{|z|\le l}\tau_z\big(\sum_y yp(y)(1-G_{0,y})(\eta,\xi)\big)\big|g(\eta(0))-g(\xi(0))\big| - \big|h(\eta^l(0))-h(\xi^l(0))\big|\Big|d\mu. \tag{1.3.30}$$

By the ergodic result (1.3.19) the ξ marginal of μ is always μ_m. Observe also, that for constant $m\ge 0$, the translation invariant measure

$$\nu_m^N(\eta) := \prod_u \frac{1}{Z(m)}\frac{m^{\eta(u)}}{g(\eta(u))!}, \tag{1.3.31}$$

from (1.3.19) is invariant as well with respect to the generator $L^{(1)}$. In particular we identified for each density m a unique invariant and translation invariant measure with respect to $L^{(1)}$ such that

$$E_{\nu_m}[\xi(0)] = m.$$

In the same way as above it can be shown that any limit point β of the measure

$$\tilde{\mu}_T^N := \frac{1}{TLN}\int_0^T\sum_{|u|\le LN}\tau_{-u}\mu_s^N ds.$$

is invariant with respect to translations and invariant with respect to the generator $L^{(1)}$.

Combining what we obtained, we finally can state the two Lemmata which will allow us to prove Theorem 1.3.4:

The first one asserts that if initially configurations are ordered, then the dynamics keeps them ordered at later times:

Lemma 1.3.7. *For every probability measure μ, translation invariant with respect to the shift operator τ_u, $u \in \mathbb{Z}$ and invariant with respect to the generator $\bar{L}^{(1)}$ we have*

$$\mu\{(\eta, \xi) : \eta \leq \xi \text{ or } \eta \geq \xi\} = 1$$

Proof. A proof of this Theorem can be found for example in [28], it relies in the monotonicty properties coming from the attractiveness of the process. □

The second Lemma is a result from [15] (see also [2], [28]) and it states that the limit point β is a convex combination of product stationary measures:

Lemma 1.3.8. *For every probability measure β, translation invariant with respect to the shift operator τ_u, $u \in \mathbb{Z}$ and invariant with respect to the generator $L^{(1)}$ there exists a probability measure $\gamma(\rho)$ on $\mathbb{R}^+$, such that*

$$d\beta(\eta) = \int d\nu_\rho \gamma(d\rho).$$

Now we are ready to prove the one block estimate

Proof of Theorem 1.3.4

Since $\tau_z(1 - G_{0,y})(\eta_s, \xi_s)$ is the indicator equal to one if particles are ordered at time s, by Lemma 1.3.7 they remain ordered for any time. Hence we can split the integral in (1.3.28) in two parts, one for configurations such that $\eta \leq \xi$ and one for $\eta \geq \xi$. Then, as g and h are monotone functions, we can remove the absolute value. We only consider the case with the partial order $\eta \leq \xi$. Then we obtain

$$\int_{\eta \leq \xi} \Big| \frac{1}{2l+1} \sum_{|z| \leq l} (g(\eta(z)) - g(\xi(z))) - h(\eta^l(0)) + h(\xi^l(0)) \Big| d\mu$$
$$\leq \int_{\eta \leq \xi} \Big| \frac{1}{2l+1} \sum_{|z| \leq l} g(\eta(z)) - h(\eta^l(0)) \Big| d\beta(\eta) + \int_{\eta \leq \xi} \Big| \frac{1}{2l+1} \sum_{|z| \leq l} g(\xi(z)) - h(\xi^l(0)) \Big| d\nu_m(\xi).$$

Since ν_m is translation invariant, the second term converges to zero as l approaches to infinity by (1.3.19) and thus it remains to prove

$$\lim_{l \to \infty} \sup_\beta \int_{\eta \leq \xi} \Big| \frac{1}{2l+1} \sum_{|z| \leq l} g(\eta(z)) - h(\eta^l(0)) \Big| d\beta(\eta) = 0$$

where the supremum is taken over all translation invariant and invariant measures with respect to $L^{(1)}$. Then with Lemma 1.3.8 we can rewrite the integral as

$$\lim_{l\to\infty}\sup_{\beta}\int\left(\int_{\eta\le\xi}\Big|\frac{1}{2l+1}\sum_{|z|\le l}g(\eta(z))-h(\eta^l(0))\Big|d\nu_\rho(\eta)\right)\gamma(d\rho)=0$$

By taking the conditional expectation with respect to η^l and since γ is concentrated on $[0,\|m_\alpha\|_\infty]$, we are left to prove that for each $\rho\le\|m_\alpha\|_\infty$ we have

$$\lim_{l\to\infty}\int_{\eta\le\xi}\Big|\frac{1}{2l+1}\sum_{|z|\le l}g(\eta(z)-h(\rho)\Big|d\nu_\rho(\eta)=0$$

But this is jus a law of large number and thus the last expression converges to 0 as l approaches ∞.

□

1.3.4 Existence of measure-valued entropy solutions

In this section, we prove that Theorem 1.3.2 implies the existence of a measure-valued entropy solution associated to the configuration η_t. We recall the empirical measure $\chi_t^N(x)$ associated to a configuration η_t in (1.3.15). We define the Young measures associated to η_t as follows:

$$\pi_t^{N,l}(x,k):=\frac{1}{N}\sum_u\delta_{\frac{u}{N}}(x)\delta_{\eta_t^l(u)}(k),\tag{1.3.32}$$

which implies $\langle\pi_t^{N,l};J\rangle=\frac{1}{N}\sum_u J(\frac{u}{N},\eta_t^l(u))$ for any $J\in C_0(\mathbb{R}\times\mathbb{R}_+)$. If E is the configuration space, then these two measures are finite positive measures on E and, for any $J\in C_0(\mathbb{R})$, they are related by the formula

$$\langle\pi_t^{N,l};kJ(x)\rangle\approx\langle\chi_t^N(\cdot);J(\cdot)\rangle.\tag{1.3.33}$$

Notice that, since there are jumps, the probability measure μ_t^N defined by (1.3.11) must be defined on the Skorohod space $D[(0,\infty),E]$, which is the space of right continuous functions with left limits taking values in E. Then, using the one to one correspondence between the configuration η_t and the empirical measure $\chi_t^N(\cdot)$, the law of χ^N with respect to μ_t^N will give us a probability measure Q^N on the Skorohod space $D[(0,\infty),\mathcal{M}_+(\mathbb{R})]$, for the space $\mathcal{M}_+(\mathbb{R})$ of finite positive measures on $\mathbb{R}$ endowed with the weak topology.

In the same way, we can associate a probability measure $\tilde{Q}^{N,l}$ on the space $D[(0,\infty),\mathcal{M}_+(\mathbb{R}_+^2)]$.

With these definitions, we can state the main theorem of this section as follows.

Theorem 1.3.9 (Law of large numbers for the Young measures). *Let $(\mu^N)_{N\ge1}$ be a sequence of probability measures, as defined by* (1.3.10), *associated to a bounded density profile $\rho_0:\mathbb{R}\to\mathbb{R}_+$. Then the sequence $\tilde{Q}^{N,l}$ converges, when $N\to\infty$ first and $l\to\infty$ second, to the probability measure $\tilde{Q}$ concentrated on the measure-valued entropy solution $\pi_{t,x}$ in the sense of Definition* 2.2.

Proof. In order to be allowed to take the limit points Q and $\tilde{Q}$ of Q^N and $Q^{N,l}$ respectively, we must know that the sequences are tight (weakly relatively compact). If $Q^{N,l}$ is weakly relatively compact, we can take $\tilde{Q}^l$ as a limit point if $N \to \infty$ for each l. Denote by $\tilde{Q}$ a limit point of $\tilde{Q}^{N,l}$ if $N \to \infty$ first and $l \to \infty$ second. Therefore, the proof consists in two main steps: The first is to show that $\tilde{Q}^{N,l}$ is weakly relatively compact and the second is to show the uniqueness of limit points. The key point in the proof is that these can be achieved independent of the choice of mollification λ_ε of the discontinuous speed-parameter λ with our choice of the notion of measure-valued entropy solutions.

These can be achieved by following exactly the standard argument in [16, 33, 24] since it requires only the uniform boundedness of λ_ε in the proof. That is, we can conclude the following: Let μ_t^N be a measure defined by (1.3.11). Then

(i) The sequence Q^N defined above is tight in $D[(0,\infty), \mathcal{M}_+(\mathbb{R})]$ and all its limit points Q are concentrated on weakly continuous paths $\chi(t,\cdot)$;

(ii) Similarly, the sequence $\tilde{Q}^{N,l}$ is tight in $D[(0,\infty), \mathcal{M}_+(\mathbb{R}\times\mathbb{R}_+)]$ and all its limit points $\tilde{Q}$ are concentrated on weakly continuous paths $\pi(t,\cdot,\cdot)$;

(iii) For every $t \geq 0$, $\pi(t,x,k) := \pi_t(x,k)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, $\tilde{Q}$ a.s.. That is, $\tilde{Q}$ a.s.

$$\pi_t(x,k) = \pi_{t,x}(k) \otimes dx; \tag{1.3.34}$$

(iv) For every $t \in [0,T]$, $\pi_{t,x}(k)$ is compactly supported, that is, there exists $k_0 > 0$ such that

$$\pi_{t,x}([0,k_0]^c) = 0 \qquad \forall\,(t,x) \in [0,T]\times\mathbb{R}.$$

(v) $\pi_{t,x}$ is a measure-valued entropy solution in the sense of Definition 2.2 for any $\alpha \in [M_0,\infty)$, i.e.,

$$\partial_t \langle \pi_{t,x}; |k - m_\alpha(x)|\rangle + \partial_x \langle \pi_{t,x}; |h(k)\lambda(x) - \alpha|\rangle \leq 0 \tag{1.3.35}$$

in the sense of distributions on $\mathbb{R}^2_+$ for any $\alpha \in [M_0,\infty)$ or $\alpha \in (-\infty, M_0]$.

The last result follows from the entropy inequality at microscopic level in Theorem 1.3.2. Indeed, in terms of the Young measures, the expression (1.3.17) of Proposition 1.3.2:

$$\lim_{l\to\infty}\lim_{N\to\infty} \mu_t^N\Big\{ \int_0^\infty \frac{1}{N}\sum_u \{\partial_t H(t,\frac{u}{N})|\eta_t^l(u) - m_\alpha(\frac{u}{N})| + \partial_x H(t,\frac{u}{N})\big|\lambda(\frac{u}{N})h(\eta_t^l(u)) - \alpha\big|\}dt \geq -\delta\Big\} = 1$$

can be restated as

$$\lim_{l\to\infty}\lim_{N\to\infty} \tilde{Q}^{N,l}\Big\{ \int_0^T \big(\langle \pi_t(x,k); |k - m_\alpha(x)|\partial_t H(t,x)\rangle + \langle \pi_t(x,k); |\lambda(x)h(k) - \alpha|\,\partial_x H(t,x)\rangle \big)dt \geq -\delta\Big\} = 1,$$

for every smooth function $H : (0,T)\times\mathbb{R} \to \mathbb{R}_+$ with compact support, any $\alpha \in [M_0,\infty)$ or $\alpha \in (-\infty, M_0]$, and any $\delta > 0$. Since $\tilde{Q}$ is a weak limit point concentrated on absolutely continuous measures and since we already proved that $\pi_{t,x}$ is concentrated on a compact set (and therefore the integrand is a bounded function), we obtain from the last expression that

$$\tilde{Q}\Big\{ \int_0^T \int \Big(\langle \pi_{t,x}; |k - m_\alpha(x)| \rangle \, \partial_t H(t,x) + \langle \pi_{t,x}; |\lambda(x)h(k) - \alpha| \rangle \, \partial_x H(t,x) \Big) \, dxdt \geq -\delta \Big\} = 1.$$

Letting $\delta \to 0$, we have that $\tilde{Q}$ a.s. (1.3.35) holds on $(0,T) \times \mathbb{R}$ in the sense of distributions for every $\alpha \in [0,\infty)$. This proves the uniqueness of $\tilde{Q}$ and thereby concludes the proof of Proposition 1.3.9. □

Then Theorem 1.3.1 follows immediately from this result since the measure-valued entropy solution reduces to the Dirac mass concentrated on the unique entropy solution $\rho(t,x)$ of (1.3.1)–(1.3.2) as we noticed in Section 1.2.2.

Chapter 2

Hydrodynamic limit of an Hamiltonian system with Boundary Conditions

2.1 The Model

We will study a system of N coupled oscillators in one dimension. This means, that we consider atoms sitting on a one dimensional lattice and moving around their equilibrium position. We chose the one dimensional discrete lattice to be of length 1 and having N points. The points of the lattice will be denoted by i with $i \in \{0, 1 \dots N\}$. To each point of the lattice there is associated an atom i with mass equal to one which is oscillating around its equilibrium position i. Then the position of the displaced atom i is denoted by x_i, while its momentum is denoted by p_i, with x_i and $p_i \in \mathbb{R}$. Thus each particle has phase space $\mathbb{R} \times \mathbb{R}$ on the microscopic space scale and the configuration space is $(\mathbb{R} \times \mathbb{R})^N$. We assume particle 0 to be attached to a wall, i.e. $x_0 = p_0 \equiv 0$ while on particle N we apply a force $\tau(t)$ depending on time which is a pressure or a tension. These are the boundary conditions we impose.
Denote by $\mathbf{x} := (x_0, \dots, x_N)$ and $\mathbf{p} := (p_1, \dots, p_N)$. The interaction between two particles i and $i-1$ will be described by the potential energy $V(x_i - x_{i-1})$ of an anharmonic spring relying the particles. Such a model is called unpinned since we do not add a self-potential to the energy. Since the particle density is high, the potential must grow to infinity fast enough, such that there is an high interaction between particles if the absolute value of the increment of their position is large. Therefore we assume V to be a positive smooth function which for large r grows faster than r, that means

$$\lim_{|r|\to\infty} \frac{V(r)}{|r|} = \infty. \tag{2.1.1}$$

Then, as $(\mathbf{x}, \mathbf{p})$ evolve following the Hamiltonian equations of motion, the system is described by the following Hamiltonian $\mathcal{H}_N^\tau$

$$\mathcal{H}_N^\tau(\mathbf{x}, \mathbf{p}) : \;=\; \frac{1}{2}\sum_{i=1}^N p_i^2 + \sum_{i=1}^N \left(V(x_i - x_{i-1}) - \tau(t)(x_i - x_{i-1})\right).$$

Notice that additionally to the kinetic energy and the potential energy we have a third term, which takes into account the boundary conditions. Furthermore we impose to our system a fixed temperature T_0. In this way the total volume $\sum_{i=0}^{N-1} r_i$ and velocity $\sum_{i=0}^{N-1} p_i$ obey to a balance law, while the total energy $\sum_{i=0}^{N-1} e_i$ does not.

Since we focus on a nearest neighbor interaction, we may define the distance between particles by

$$r_i = x_i - x_{i-1}.$$

Thus we look at our system using the Lagrangian coordinates. In the new coordinates and since $p_0 \equiv 0$ the Hamiltonian reads,

$$\begin{aligned} \mathcal{H}\tau_N(\mathbf{r},\mathbf{p}) : &= \sum_{i=1}^{N} e_i \\ &= \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + V(r_i) - \tau(t) r_i \right), \end{aligned}$$

where $\mathbf{r} := (r_1, \ldots, r_N)$, $\mathbf{p} := (p_0, \ldots, p_{N-1})$, while we define the energy e_i of particle i by

$$e_i(r_i, p_i) = \frac{p_i^2}{2} + V(r_i) - \tau(t) r_i.$$

Now the dynamics of the system can be computed as

$$\begin{cases} dr_i &= (p_i - p_{i-1})dt, \quad \text{for} \quad i \in \{2, \ldots, N\} \\ dp_i &= (V'(r_{i+1}) - V'(r_i))dt + \textit{noise} \quad \text{for} \quad i \in \{1, \ldots, N-1\} \end{cases}$$
$$\begin{cases} dr_1 &= p_1 dt \\ dp_N &= (\tau(t) - V'(r_N))\, dt + \textit{noise} \end{cases} \quad \textit{(at the boundary)} \tag{2.1.2}$$

The random forces, which we will specify later, will be chosen in such a way, that the balance of volume and momentum are still in place but the balance of total energy is lost because a thermal equilibrium is maintained by the noise. In this way we obtain the dynamics with the imposed temperature T_0. Without loss of generality, we set $T_0 = 1$. Also the noise should help us with the issue of ergodicity as we will see later. Then we will obtain two balanced quantities which are

$$\sum_{i=1}^{N} r_i = x_N - x_0 \quad : \quad \text{length of the chain}$$
$$\sum_{i=0}^{N} p_i \quad : \quad \text{total momentum}$$

We are interested in the macroscopic behavior of the interdistance and momentum of the particles, at time Nt, as $N \to \infty$. For this, we introduce the empirical measures

$$\begin{aligned} \eta^N(dx,t) &:= \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \frac{i}{N}\right) r_i(Nt)\, dx, \\ \xi^N(dx,t) &:= \frac{1}{N} \sum_{i=0}^{N} \delta\left(x - \frac{i}{N}\right) p_i(Nt)\, dx, \end{aligned}$$

representing the spatial distribution of interdistance and momentum respectively as a function of time and where $x \in [0,1]$. Since during a time t, on the microscopic scale, particles move distances which are of order $\frac{1}{N}$, this displacement can not be seen macroscopicaly. Therefore we scale the time t by N to see a macroscopic evolution of the system. This scaling by N is called Euler scaling.

We expect the measures $\eta^N(dx,t)$ and $\xi^N(dx,t)$ to converge as $N \to \infty$ to measures $\mathfrak{r}(x,t)dx$, $\mathfrak{p}(x,t)dx$, being absolutely continuous with respect to the Lebesgue measure and satisfying the following system of conservation laws:

$$\begin{cases} \partial_t \mathfrak{r} - \partial_x \mathfrak{p} = 0 \\ \partial_t \mathfrak{p} - \partial_x P(\mathfrak{r}) = 0 \end{cases} \tag{2.1.3}$$

$$\begin{cases} \mathfrak{r}_0(x) = \mathfrak{r}(x,0),\ \mathfrak{p}_0(x) = \mathfrak{p}(x,0) \\ \mathfrak{p}(0,t) = 0,\ P(\mathfrak{r}(1,t)) = \tau(t) \end{cases} \quad \textit{(boundary conditions)} \tag{2.1.4}$$

For bounded, smooth initial data $\mathfrak{r}_0, \mathfrak{p}_0 : [0,1] \to \mathbb{R}$ and the force $\tau(t)$ depending on time t. We call P the pressure which is a function of the specific volume $\mathfrak{r}$ only. Furthermore we assume that on the edges $(x,t) = (0,0)$ and $(x,t) = (1,0)$ the following compatibility conditions are satisfied

$$\lim_{x\to 0} \mathfrak{p}_0(x) = \lim_{t\to 0} \mathfrak{p}(0,t) = 0 \quad \text{and} \quad \lim_{x\to 1} P(\mathfrak{r}_0(x)) = \tau(t). \tag{2.1.5}$$

Remark 2.1.1. Since our proof is based on the relative entropy method, it is only valid in the smooth regime of the solution to (2.1.3). Since, even for smooth initial data, the solution will develop shocks, we are forced to restrict our derivation to a time $0 < T < t_s$, where t_s is the time when the solution of the p-system enters the first shock. A sketch of the proof for the existence of smooth solutions to the initial-boundary-value problem (2.1.3) will be given in the next section.

In other words, for any test function $J : [0,1] \to \mathbb{R}$ with compact support, consider the empirical densities

$$\eta^N(t,J) \quad := \quad \langle \eta^N(dx,t); J \rangle \ = \ \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) r_i(Nt) \tag{2.1.6}$$

$$\xi^N(t,J) \quad := \quad \langle \xi^N(dx,t); J \rangle \ = \ \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) p_i(Nt). \tag{2.1.7}$$

Then, starting with an initial distribution such that in the hydrodynamic limit as $N \to \infty$ there exist smooth functions $\mathfrak{r}_0$ and $\mathfrak{p}_0$ with

$$\{\eta^N(0,J), \xi^N(0,J)\} \to \left\{ \int J(x)\mathfrak{r}_0(x)dx, \int J(x)\mathfrak{p}_0(x)dx \right\} \tag{2.1.8}$$

in probability, our goal is to show that at time $t \in [0,T]$ we have the same convergence of $\eta^N(t,J)$ and $\xi^N(t,J)$ to corresponding profiles $\mathfrak{r}(x,t)$ and $\mathfrak{p}(x,t)$ respectively, thus

$$\{\eta^N(t,J), \eta^N(t,J)\} \to \left\{ \int J(x)\mathfrak{r}(x,t)dx, \int J(x)\mathfrak{p}(x,t)dx \right\} \tag{2.1.9}$$

in probability as $N \to \infty$, where $\mathfrak{r}$ and $\mathfrak{p}$ satisfy (2.1.3).

Formally this can be seen if we pretend that no random forces are present and then take the derivative of the empirical densities defined in (2.1.6) and (2.1.7):

For the interdistance we obtain:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) r_i(Nt)\right) &= \frac{1}{N}\sum_{i=1}^{N} NJ\left(\frac{i}{N}\right)\frac{d}{d(Nt)} r_i(Nt)\\
&= \frac{1}{N}\sum_{i=1}^{N} NJ\left(\frac{i}{N}\right)\big(p_i(Nt) - p_{i-1}(Nt)\big)\\
&= \frac{1}{N}\sum_{i=1}^{N-1}\frac{J(\frac{i}{N}) - J(\frac{i+1}{N})}{\frac{1}{N}} p_i(Nt) + J(1)p_N(Nt)\\
&= -\frac{1}{N}\sum_{i=1}^{N-1}\left(J'(\frac{i}{N}) + \mathcal{O}(\frac{1}{N})\right) p_i(Nt) + J(1)p_N(Nt).
\end{aligned}$$

Then, since p_N is the momentum of the last particle located at $x = 1$ and using (2.1.9), we obtain

$$\frac{d}{dt}\int J(x)\mathfrak{r}(x,t)dx + \int J'(x)\mathfrak{p}(x,t)dx = J(1)\mathfrak{p}(1,t)$$

which after an integration by parts gives us the first equation of the system (2.1.3)

$$\partial_t \mathfrak{r}(x,t) - \partial_x \mathfrak{p}(x,t) = 0 \quad \text{with} \quad \mathfrak{p}(0,t) \equiv 0.$$

For the momentum we obtain with (2.1.2)

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) p_i(Nt)\right)\\
&\quad= \frac{1}{N}\sum_{i=1}^{N-1} NJ\left(\frac{i}{N}\right)\frac{d}{d(Nt)} p_i(Nt) + J(1)\frac{d}{d(Nt)} p_N(Nt)\\
&\quad= \frac{1}{N}\sum_{i=1}^{N-1} NJ\left(\frac{i}{N}\right)\left(V'(r_{i+1}(Nt)) - V'(r_i(Nt))\right) + J(1)\left(\tau(t) - V'(r_N)\right)\\
&\quad= \frac{1}{N}\sum_{i=2}^{N}\frac{J\left(\frac{i-1}{N}\right) - J\left(\frac{i}{N}\right)}{\frac{1}{N}} V'(r_i(Nt)) + J(1)\tau(t) - J(\frac{1}{N})V'(r_1)\\
&\quad= -\frac{1}{N}\sum_{i=2}^{N-1}\left(J'\left(\frac{i}{N}\right) + \mathcal{O}(\frac{1}{N})\right) V'(r_i(Nt)) + J(1)\tau(t) - J(\frac{1}{N})V'(r_1)
\end{aligned} \tag{2.1.10}$$

Sending N to infinity here is much more difficult than for the interdistance since V' is not a function of the empirical densities, thus the equation is not closed in terms of the empirical densities. Therefore to be lead to the second equation of (2.1.3) we need to replace V' by an expression, that depends on the conserved quantities. To do this we encounter two main difficulties:

(i) We first have to prove that the system is in local equilibrium. This means we focus on boxes which are small on the macroscopic scale but which contain a large number of particles on the microscopic scale. We will show in Sections 2.5.3 and 2.5.4 that this subsystem is in equilibrium, in the sense that there exists an invariant measure ν, such that for a box of size $2k+1$ around a microscopic site i we can replace $V'(r_i)$ by its equilibrium average

$$\int V'(r_i)d\nu := P(\frac{1}{2k+1}\sum_{|i-j|\leq k} r_j).$$

(ii) The second difficulty to close the equation is that even if

$$\frac{1}{2k+1}\sum_{|i-j|\leq k} r_j \quad \overset{\text{weakly}}{\to} \quad \mathfrak{r}(x,t) \quad \text{as} \quad k\to\infty$$

does not imply in general that

$$P(\frac{1}{2k+1}\sum_{|i-j|\leq k} r_j) \quad \overset{\text{weakly}}{\to} \quad P(\mathfrak{r}(x,t)) \quad \text{as} \quad k\to\infty$$

since P is nonlinear. This will be proved in section 2.5.5

Assume that all this can be done rigorously, then (2.1.10) converges in probability to

$$\frac{d}{dt}\int J(x)\mathfrak{p}(x,t)dx + \int J'(x)P(\mathfrak{r}(t,x))dx = J(1)\tau(t) - J(0)P(\mathfrak{r}(0,t)).$$

With an integration by part we obtain

$$\partial_t\mathfrak{p}(x,t) - \partial_x P(\mathfrak{r}(x,t)) = 0 \quad \text{with} \quad P(\mathfrak{r}(1,t)) = \tau(t).$$

2.2 Existence and Uniqueness of C^1 Solutions to the Initial-Boundary-Value Problem (IBVP)

In view of Remark 2.1.1, we will next show the existence of a unique smooth solution to the initial-boundary value problem (2.1.3)–(2.1.5) up to a time $t = t_s > 0$. The main Theorem of this Section is

Theorem 2.2.1 (Well posedness of the IBVP). *There exists a positive time t_s such that, on the domain $\mathcal{D}_{t_s}$ defined by*

$$\mathcal{D}_{t_s} := \{(x,t) \ : \ x\in[0,1],\ 0\leq t\leq t_s\}, \tag{2.2.1}$$

the IBVP (2.1.3)–(2.1.5) admits a unique continuously differentiable solution $\mathfrak{u} := \mathfrak{u}(x,t)$.

In the following we will only give a sketch of the proof, since all the results can be found in Chapters 1 and 2 of [27] for general quasilinear hyprbolic systems in one dimension.

We first rewrite the p-system (2.1.3) as a system of two decoupled partial differential equations in terms of the Riemann invariants:

Observe that we can rewrite the p-system in a non conservation form in the following way:

$$\begin{cases} \partial_t \mathfrak{r} - \partial_x \mathfrak{p} = 0 \\ \partial_t \mathfrak{p} - \partial_x P = 0 \end{cases} \Leftrightarrow \quad \partial_t \mathfrak{u} - D\mathbf{A}(\mathfrak{u})\partial_x \mathfrak{u} = 0, \tag{2.2.2}$$

where

$$\mathfrak{u}(x,t) := \begin{pmatrix} \mathfrak{r}(x,t) \\ \mathfrak{p}(x,t) \end{pmatrix} \quad \text{and} \quad D\mathbf{A}(\mathfrak{u}) := \begin{pmatrix} 0 & 1 \\ c^2(\mathfrak{r}) & 0 \end{pmatrix}$$

is the Jacobian of the flux $\mathbf{A}(\mathfrak{u}) := \begin{pmatrix} \mathfrak{p} \\ P(\mathfrak{r}) \end{pmatrix}$ and $c(\mathfrak{r}) := \sqrt{P'(\mathfrak{r})}$ is called the sound speed.

The characteristics of the system are given by the Eigenvalues of the matrix $D\mathbf{A}(\mathfrak{u})$, which are

$$\lambda_1(\mathfrak{r}) := -c(\mathfrak{r}) \quad \text{and} \quad \lambda_2(\mathfrak{r}) := c(\mathfrak{r}).$$

In order to maintain hyperbolicity of the system we assume that the pressure P is an increasing function of the specific volume $\mathfrak{r}$, then $P'(\cdot) > 0$ and we have only real Eigenvalues. The right Eigenvectors corresponding to $-c$ and c respectively are

$$R_1(\mathfrak{r}) := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -c \end{pmatrix} \quad \text{and} \quad R_2(\mathfrak{r}) := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ c \end{pmatrix}$$

and the left Eigenvectors

$$L_1(\mathfrak{r}) := \frac{1}{\sqrt{2}} \left(-1, -\frac{1}{c} \right) \quad \text{and} \quad L_2(\mathfrak{r}) := \frac{1}{\sqrt{2}} \left(-1, \frac{1}{c} \right)$$

are chosen such that

$$R_\alpha(\mathfrak{r}) \cdot L_\beta(\mathfrak{r}) = \delta_{\alpha\beta} \quad \text{for} \quad \alpha, \beta = 1, 2.$$

Let us denote by $\mathcal{R}$ and $\mathcal{L}$ the matrices

$$\mathcal{R} := (R_1(\mathfrak{r}), R_1(\mathfrak{r})) = \begin{pmatrix} L_1(\mathfrak{r}) \\ L_1(\mathfrak{r}) \end{pmatrix}^{-1} := \mathcal{L}^{-1}.$$

Then (2.2.2) is equivalent to

$$\partial_t \mathfrak{u} + \mathcal{R} \cdot \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \cdot \mathcal{L} \cdot \partial_x \mathfrak{u} = 0 \quad \Leftrightarrow \quad \mathcal{L} \cdot \partial_t \mathfrak{u} + \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \cdot \mathcal{L} \cdot \partial_x \mathfrak{u} = 0.$$

This means that to decouple our system we need to find $\mathbf{\Gamma} : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ with

$$\mathbf{\Gamma}(\mathfrak{u}) := \begin{pmatrix} \Gamma_1(\mathfrak{u}) \\ \Gamma_2(\mathfrak{u}) \end{pmatrix}$$

such that

$$\partial_t \mathbf{\Gamma}(\mathfrak{u}) + \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \cdot \partial_x \mathbf{\Gamma}(\mathfrak{u}) = 0 = \mathcal{L} \cdot \partial_t \mathfrak{u} + \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \cdot \mathcal{L} \cdot \partial_x \mathfrak{u}.$$

Since we may multiply these equations by any contiuous function f depending on $\mathfrak{u}$ and not identically equal to zero, it is enough to find $\mathbf{\Gamma}$ such that

$$\partial_t\mathbf{\Gamma}(\mathfrak{u}) = f(\mathfrak{u})\mathcal{L}\cdot\partial_t\mathfrak{u} \quad \text{and} \quad \partial_x\mathbf{\Gamma}(\mathfrak{u})f(\mathfrak{u})\mathcal{L}\partial_x = \partial_x\mathfrak{u}$$

$$\Leftrightarrow \quad D\mathbf{\Gamma}(\mathfrak{u}) = f(\mathfrak{u})\mathcal{L} \Leftrightarrow D\mathbf{\Gamma}(\mathfrak{u})\cdot\mathcal{L}^{-1} = f(\mathfrak{u})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Leftrightarrow \quad \begin{cases} -\frac{\partial\Gamma_1(\mathfrak{u})}{\partial\mathfrak{r}} - c\frac{\partial\Gamma_1(\mathfrak{u})}{\partial\mathfrak{p}} = \sqrt{2}f(\mathfrak{u}) \quad , \quad -\frac{\partial\Gamma_1(\mathfrak{u})}{\partial\mathfrak{r}} + c\frac{\partial\Gamma_1(\mathfrak{u})}{\partial\mathfrak{p}} = 0 \\ -\frac{\partial\Gamma_2(\mathfrak{u})}{\partial\mathfrak{r}} + c\frac{\partial\Gamma_2(\mathfrak{u})}{\partial\mathfrak{p}} = \sqrt{2}f(\mathfrak{u}) \quad , \quad -\frac{\partial\Gamma_2(\mathfrak{u})}{\partial\mathfrak{r}} - c\frac{\partial\Gamma_2(\mathfrak{u})}{\partial\mathfrak{p}} = 0 \end{cases}.$$

This is satisfied if

$$\begin{cases} \Gamma_1(\mathfrak{u}) = -\int^{\mathfrak{r}} c(\omega)d\omega - \mathfrak{p} \\ \Gamma_2(\mathfrak{u}) = -\int^{\mathfrak{r}} c(\omega)d\omega + \mathfrak{p} \end{cases} \tag{2.2.3}$$

These new variables Γ_1 and Γ_2 are called the $(-c)-$Riemann invariants and the $(+c)-$Riemann invariant respectively. Notice that there is a one-to-one correspondance between (Γ_1, Γ_2) and $(\mathfrak{r}, \mathfrak{p})$. Now we can rewrite the initial-boundary-value problem (2.1.3)–(2.1.5) in terms of the Riemann invariants $\mathbf{\Gamma}(\mathfrak{u}) := \mathbf{\Gamma}(x,t) : [0,1]\times[0,T] \to \mathbb{R}^2$, for some time $T > 0$, as the following system of two decoupled equations:

$$\begin{cases} \partial_t\Gamma_1 - c(\mathbf{\Gamma})\partial_x\Gamma_1 = 0 \\ \partial_t\Gamma_2 + c(\mathbf{\Gamma})\partial_x\Gamma_2 = 0 \end{cases} \tag{2.2.4}$$

$$\begin{cases} \Gamma_{1,0}(x) = \Gamma_1(\mathfrak{r}_0(x), \mathfrak{p}_0(x)); \quad \Gamma_{2,0}(x) = \Gamma_2(\mathfrak{r}_0(x), \mathfrak{p}_0(x)), \\ \Gamma_2(0,t) = B_l(t); \quad \Gamma_1(1,t) = B_r(t) \qquad \textit{(boundary conditions)}, \end{cases} \tag{2.2.5}$$

where $B_l(\cdot)$ and $B_r(\cdot)$ are smooth solutions satisfying the compatibility conditions

$$\lim_{t\to 0} B_l(t) = \lim_{x\to 0}\Gamma_{2,0}(x) \quad \text{and} \quad \lim_{t\to 0} B_r(t) = \lim_{x\to 1}\Gamma_{1,0}(x). \tag{2.2.6}$$

We will see below how to compute the boundary conditions $B_l(\cdot)$ and $B_r(\cdot)$ from the boundary conditions (2.1.4). Furthermore $\mathbf{\Gamma}_0 := (\Gamma_{1,0}, \Gamma_{2,0}) : [0,1] \to \mathbb{R}^2$ is a bounded, smooth function. We assume that for some constant K

$$\|\mathbf{\Gamma}_0\| := \sup_{x\in[0,1]} \max\{|\Gamma_{1,0}(x)|, |\Gamma_{2,0}(x)|\} := \sup_{x\in[0,1]} |\mathbf{\Gamma}_0| \leq K. \tag{2.2.7}$$

In particular, if we assume that there exists a time $t_s > 0$ for which the solution $\mathbf{\Gamma}$ is in C^1, differating what we obtained in (2.2.4), for a curve $x(t) : [0, t_s] \to \mathbb{R}$ we have

$$\frac{d}{dt}\Gamma_1(x(t),t) = \partial_t\Gamma_1 + \partial_x\Gamma_1 x'(t) = 0 \Leftrightarrow x'(t) = -c(\mathbf{\Gamma})$$

and

$$\frac{d}{dt}\Gamma_2(x(t),t) = \partial_t\Gamma_2 + \partial_x\Gamma_2 x'(t) = 0 \Leftrightarrow x'(t) = c(\mathbf{\Gamma}).$$

Hence the $(-c)$-Riemann invariant Γ_1 is constant along the $(-c)$-characteristic and the $(+c)$-Riemann invariant Γ_2 is constant along the $(+c)$-characteristic. Therefore, if we denote by

$$\kappa_{-c}(s;x,t) \quad , \quad \kappa_c(s;x,t)$$

the $(-c)$- and the $(+c)$-characteristic curve respectively passing through the point (x,t), i.e.

$$\frac{d\kappa_{-c}(s;x,t)}{ds} = -c(\mathbf{\Gamma}(\kappa_{-c}(s;x,t),s)) \quad , \quad \frac{d\kappa_c(s;x,t)}{ds} = c(\mathbf{\Gamma}(\kappa_c(s;x,t),s))$$

and

$$\kappa_{-c}(t;x,t) = x \quad , \quad \kappa_c(t;x,t) = x, \tag{2.2.8}$$

it follows that for any $(x,t) \in ([0,1] \times [0,t_s])$

$$\Gamma_1(x,t) = \Gamma_{1,0}(\kappa_{-c}(0;x,t)) \quad \text{and} \quad \Gamma_2(x,t) = \Gamma_{2,0}(\kappa_c(0;x,t)). \tag{2.2.9}$$

With this observation it is easy to compute $B_l(t)$ and $B_r(t)$ from the boundary conditions (2.1.4):

For any $0 \le t < t_s$, by (2.2.3) with $\mathfrak{p}(0,t) \equiv 0$ we have

$$B_l(t) = \Gamma_2(0,t) = -\int^{\mathfrak{r}(0,t)} c(\omega)d\omega.$$

On the other hand since $\Gamma_1(0,t) = \Gamma_{1,0}(\kappa_{-c(0;0,t)})$, we have

$$-\int^{\mathfrak{r}(0,t)} c(\omega)d\omega = -\int^{\mathfrak{r}_0(\kappa_{-c(0;0,t)})} c(\omega)d\omega - \mathfrak{p}_0(\kappa_{-c(0;0,t)}).$$

Hence we obtained

$$B_l(t) = -\int_0^{\mathfrak{r}_0(\kappa_{-c(0;0,t)})} c(\omega)d\omega - \mathfrak{p}_0(\kappa_{-c(0;0,t)}).$$

To compute $B_r(t)$, recall that there is a one to one correspondence between the pressure P on the specific volume $\mathfrak{r}$. since P is assumed to be an increasing function of $\mathfrak{r}$. Then $\mathfrak{r}(x,t) := \mathfrak{r}(P(x,t))$ and we have

$$B_r(t) = \Gamma_1(1,t) = -\int^{\mathfrak{r}(\tau(t))} c(\omega)d\omega - \mathfrak{p}(1,t).$$

On the other hand since $\Gamma_2(1,t) = \Gamma_{2,0}(\kappa_{c(0;1,t)})$, we have

$$-\int^{\mathfrak{r}(\tau(t))} c(\omega)d\omega + \mathfrak{p}(1,t) = -\int^{\mathfrak{r}_0(\kappa_{c(0;1,t)})} c(\omega)d\omega + \mathfrak{p}_0(\kappa_{c(0;1,t)}).$$

Hence we obtained

$$B_r(t) = -2\int_0^{\mathfrak{r}(\tau(t))} c(\omega)d\omega + \int^{\mathfrak{r}_0(\kappa_{c(0;1,t)})} c(\omega)d\omega - \mathfrak{p}_0(\kappa_{c(0;1,t)}).$$

Notice also that with the compatibility conditions (2.1.5) it follows that also (2.2.6) is satisfied.

Recall that for these arguments, we assumed that the solution $\mathbf{\Gamma}$ is smooth on the considered domain. In other words $\mathbf{\Gamma}$ is a smooth solution to to (2.2.4)–(2.2.5) if and only if (2.2.9) is satisfied.

Motivated by the the geometry of last observations, to prove the well posedness of the initial-boundary-value problem (2.2.4)–(2.2.6) up to a time $t_s > 0$, for some $\delta \geq t_s$, we devide the domain $\mathcal{D}_\delta$ defined by (2.2.1) into three parts:

$$\begin{aligned} \mathcal{L}_\delta &:= \{(x,t) \,: 0 \leq \; x \leq x_c(t), \; 0 \leq t \leq \delta\} && (2.2.10)\\ \mathcal{M}_\delta &:= \{(x,t) \;:\; x_c(t) \leq x \leq x_{-c}(t), \; 0 \leq t \leq \delta\} && (2.2.11)\\ \mathcal{R}_\delta &:= \{(x,t) \;:\; x_{-c}(t) \leq x \leq 1, \; 0 \leq t \leq \delta\}, && (2.2.12) \end{aligned}$$

where we denoted by $x_c(t) := \kappa_c(t,0,0)$ the curve of the $(+c)$-characteristic going through the origin $(x,t) = (0,0)$ and by $x_{-c}(t) := \kappa_{-c}(t;1,0)$ is the curve of the $(-c)$-characteristic going through $(x,t) = (1,0)$ that means x_c and x_{-c} satisfy

$$\begin{aligned} x_c(0) = 0 \quad &\text{and} \quad x_c'(t) = c(\mathbf{\Gamma}) && (2.2.13)\\ x_{-c}(0) = 1 \quad &\text{and} \quad x_{-c}'(t) = -c(\mathbf{\Gamma}). && (2.2.14) \end{aligned}$$

In this way, on the domain $\mathcal{M}_\delta$, no characteristics are crossing the artificial boundaries x_c and x_{-c}, and thereby, together with the the observations we made before, the information at any point $(x,t) \in \mathcal{M}_\delta$ is given through the initial data $(\Gamma_{1,0}, \Gamma_{2,0})$ only. This means that on the domain $\mathcal{M}_\delta$ the initial-boundary-value problem (2.2.4)–(2.2.6) is reduced to a pure initial-value problem. On the other hand on the angular domains $\mathcal{L}_\delta$ and $\mathcal{R}_\delta$ the IBVP reduces to a BVP. In view of this, the strategy of the proof is devided in two parts:

- We first prove the existence of a unique C^1 solution to the Cauchy problem (**IVP**) on the domain $\mathcal{M}_{\delta'}$ with $\delta' \geq t_s$.
- We then prove the existence and uniqueness of a C^1 solution on the angular domain $\mathcal{L}_{\delta''}$ with $\delta'' \geq t_s$ to the boundary value problem (**BVP**) with the boundary $x = 0$ and the artificial characteristic boundary $x_c(t)$ and on the angular domain $\mathcal{R}_{\delta'''}$ for some $\delta''' \geq t_s$ we prove existence and uniqueness of a C^1 solution to the boundary value problem with the boundary $x = 1$ and the artificial characteristic boundary $x_{-c}(t)$.

In other words on the domain $\mathcal{M}_{\delta'}$ defined by (2.2.11) which reduces our IBVP to Cauchy problem, we must prove the following

Theorem 2.2.2 (Well posedness of the IVP). *There exists a positive time δ' such that, on the domain $\mathcal{M}_{\delta'}$ defined by* (2.2.11)*, the IBVP* (2.2.4)–(2.2.6) *admits a unique continuously differentiable solution $\mathbf{\Gamma} = \mathbf{\Gamma}(x,t)$.*

Proof. For a detailed proof see chapter 1 in [27]. Essentially it is devided in three steps:

In a first step the quasilinear problem is redueced to the linear Cauchy problem.

To do this, for some $\delta > 0$ we introduce the set of functions

$$\mathcal{G}(\delta, K, K') := \{\tilde{\mathbf{\Gamma}} \,\Big|\, \tilde{\mathbf{\Gamma}} := (\tilde{\Gamma}_1, \tilde{\Gamma}_2) \in C^1(\mathcal{M}(\delta)); \; \|\tilde{\mathbf{\Gamma}}\| \leq K; \; \|\tilde{\mathbf{\Gamma}}\|_1 \leq K'\}$$

where $\|\tilde{\mathbf{\Gamma}}\|_1$ is a C^1-norm defined by

$$\|\tilde{\mathbf{\Gamma}}\|_1 := \|\tilde{\mathbf{\Gamma}}\| + \|\tilde{\mathbf{\Gamma}}_x\| + \|\tilde{\mathbf{\Gamma}}_t\| \tag{2.2.15}$$

with

$$\|\tilde{\mathbf{\Gamma}}_x\| := \sup_{(x,t)\in\mathcal{M}_\delta} |\partial_x \tilde{\mathbf{\Gamma}}| \quad \text{and} \quad \|\tilde{\mathbf{\Gamma}}_t\| := \sup_{(x,t)\in\mathcal{M}_\delta} |\partial_t \tilde{\mathbf{\Gamma}}|$$

and $K' \geq K$ is a constant satisfying

$$\|\mathbf{\Gamma}_0\| + (1 + \sup_{|\mathbf{\Gamma}|\leq K} \{c(\mathbf{\Gamma})\}) \|\mathbf{\Gamma}_{0,x}\| \leq K'. \tag{2.2.16}$$

The absolute value $|\cdot|$ of a vector and the constant K are defined in the same way as in (2.2.7).

For any $\tilde{\mathbf{\Gamma}} \in \mathcal{G}(\delta, K, K')$ we can formulate the following IVP

$$\begin{cases} \partial_t \Gamma_1 - \tilde{c}(x,t)\partial_x \Gamma_1 &= 0 \\ \partial_t \Gamma_2 + \tilde{c}(x,t)\partial_x \Gamma_2 &= 0 \end{cases} \tag{2.2.17}$$

with the initial conditions given in (2.2.5). Here $\tilde{c}(x,t) := c(\tilde{\mathbf{\Gamma}}(x,t))$. Thus now the characteristics $-\tilde{c}$ and $\tilde{c}$ are independent of the solution $\mathbf{\Gamma}$ and consequently this is an IVP for a linear hyperbolic conservation law.

The second step consists in showing that the linear Cauchy problem admits a unique C^1 solution on $\mathcal{M}_{\delta_0}$ for some $\delta_0 \leq \delta$.

Defining the iterative operator $\mathbf{T}$ with

$$\mathbf{T}\tilde{\mathbf{\Gamma}} = \mathbf{\Gamma}$$

and plug this in (2.2.17), it is easy to see that a fixed point of the operator $\mathbf{T}$ is a solution to (2.2.4)–(2.2.5) on M_δ. It thus remains to prove that, for δ small enough, $\mathbf{T}$ possesses a unique fixed point. This is the third step of the proof.

To prove that the operator $\mathbf{T}$ possesses a unique fixed point for δ small enough we will show that

- $\mathbf{T}$ maps the set $\mathcal{G}(\delta_1, K.K')$ to iteself for some positive $\delta_1 \leq \delta_0$.
- For some positive $\delta' \leq \delta_1$, $\mathbf{T}$ is a contraction with respect to the $\|\cdot\|$-norm.

Then by the Banach fixed point Theorem it follows that there exists a unique fixed point on $\mathcal{M}_{\delta'}$ which concludes the proof of the Theorem. □

On the angular domains $\mathcal{L}_{\delta''}$ and $\mathcal{R}_{\delta''}$ which reduces the IBVP a BVP the Theorem to prove is the following:

Theorem 2.2.3 (Well posedness of the BVP). *There exists a positive time δ'' such that, on the domains $\mathcal{L}_{\delta''}$ and $\mathcal{R}_{\delta''}$ defined by* (2.2.10) *and* (2.2.12) *respectively, the IBVP* (2.2.4)–(2.2.6) *admits a unique continuously differentiable solution $\mathbf{\Gamma} = \mathbf{\Gamma}(x,t)$.*

Proof. For a detailed proof of this, see chapter 2 of [27]. Here the strategy is the same: we first reduce the quasilinear BVP to a linear BVP, and then we show that the operator $\mathbf{S}$ defined by $\mathbf{S}\tilde{\mathbf{\Gamma}}$ possesses a unique fixed point. For the proof of this we need that the number of the boundary conditions is equal to the number of the characteristics departing from the boundary. □

2.3 The Gibbs Measures

2.3.1 The Gibbs equilibrium measures

The generator of the dynamics without random forces is given by the Liouville operator L_N^τ

$$\begin{aligned}
L_N^\tau &= \sum_{i=1}^{N} \frac{dr_i}{dt}\frac{\partial}{\partial r_i} + \sum_{i=0}^{N} \frac{dp_i}{dt}\frac{\partial}{\partial p_i} \\
&= \sum_{i=2}^{N}(p_i - p_{i-1})\frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\frac{\partial}{\partial p_i} \\
&\quad + \left(\tau(t) - V'(r_N)\right)\frac{\partial}{\partial p_N} + p_1\frac{\partial}{\partial r_1} \\
&= \sum_{i=1}^{N}(p_i - p_{i-1})\frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\frac{\partial}{\partial p_i} \\
&\quad + \left(\tau(t) - V'(r_N)\right)\frac{\partial}{\partial p_N} \\
&:= L_N^0 + \tau(t)\frac{\partial}{\partial p_N},
\end{aligned} \tag{2.3.1}$$

where we used the fact that $p_0 \equiv 0$ and we define

$$L_N^0 := \sum_{i=1}^{N}(p_i - p_{i-1})\frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\frac{\partial}{\partial p_i} - V'(r_N)\frac{\partial}{\partial p_N} \tag{2.3.2}$$

Then as already computed in (2.1.2), we have for $i = 2, \ldots, N-1$

$$L_N^\tau(r_i) = p_i - p_{i-1}, \quad L_N^\tau(r_1) = p_1 \quad \text{and} \quad L_N^\tau(r_N) = p_N - p_{N-1}$$

and

$$L_N^\tau(p_i) = V'(r_{i+1}) - V'(r_i), \quad L_N^\tau(p_1) = V'(r_2) - V'(r_1), \quad \text{and} \quad L_N^\tau(p_N) = \tau(t) - V'(r_N).$$

From this it follows that

$$L_N^\tau\Big(\sum_{i=1}^{N} r_i\Big) = p_N \qquad \text{and} \qquad L_N^\tau\Big(\sum_{i=1}^{N} p_i\Big) = \tau(t) - V'(r_1).$$

Let us define the two parameter family of Gibbs equilibrium measures on the space $\Omega^N := (\mathbb{R}_+ \times \mathbb{R})^N$. For any $\boldsymbol{\lambda} := (\lambda_1, \lambda_2) \in (\mathbb{R} \times \mathbb{R})$ it is given by:

$$\nu_{\boldsymbol{\lambda}}^N(d\mathbf{r}, d\mathbf{p}) = \prod_{i=1}^{N} \nu_{\boldsymbol{\lambda}}(dr_i, dp_i)$$

with marginals

$$\nu_{\boldsymbol{\lambda}}(dr_i, dp_i) := \frac{1}{Z(\boldsymbol{\lambda})} e^{-(h_i - \lambda_1 r_i - \lambda_2 p_i)} dr_i dp_i. \tag{2.3.3}$$

Here

$$h_i := \frac{1}{2}p_i^2 + V(r_i)$$

and the normalization Z is equal to

$$Z(\boldsymbol{\lambda}) := \int_\Omega e^{\lambda_1 r_i + \lambda_2 p_i - h_i} dr_i dp_i.$$

This is the grand canonical Gibbs measure at temperature $T_0 \equiv 1$, pressure λ_1 and velocity λ_2. This measure is meaningfull whenever Z is finite. Therefore we chose the positive function V such that for each $\lambda_1 \in \mathbb{R}$

$$Z_1(\lambda_1) := \int e^{-V(r)+\lambda_1 r_i} dr < +\infty. \tag{2.3.4}$$

In what follows we will use the following notation:

$$\nu_{\boldsymbol{\lambda}}^N(d\mathbf{r}, d\mathbf{p}) = \prod_{i=1}^N \frac{e^{\lambda_1 r_i + \lambda_2 p_i}}{Z(\boldsymbol{\lambda})} e^{-h_i} dr_i dp_i := g_{\boldsymbol{\lambda}}^N(\mathbf{r}, \mathbf{p}) d\nu_\star^N(\mathbf{r}, \mathbf{p}),$$

hence we defined the density

$$g_{\boldsymbol{\lambda}}^N(\mathbf{r}, \mathbf{p}) := \prod_{i=1}^N \frac{e^{\lambda_1 r_i + \lambda_2 p_i}}{Z(\boldsymbol{\lambda})}$$

of the probability measure $d\nu_{\boldsymbol{\lambda}}^N$ with respect to the fixed reference measure

$$d\nu_\star^N(\mathbf{r}, \mathbf{p}) := \prod_{i=1}^N e^{-h_i} dr_i dp_i. \tag{2.3.5}$$

Notice that we can rewrite $\nu_{\boldsymbol{\lambda}}^N$ as a product measure with marginals

$$\nu_{\boldsymbol{\lambda}}(dr_i, dp_i) = \mu_{\lambda_1}(dr_i)\pi_{\lambda_2}(dp_i)$$

with

$$\mu_{\lambda_1}(dr_i) := \frac{1}{Z_1(\lambda_1)} e^{\lambda_1 r_i - V(r_i)} dr_i \quad \text{where} \quad Z_1(\lambda_1) := \int_{\mathbb{R}} e^{\lambda_1 r_i - V(r_i)} dr_i$$
$$\pi_{\lambda_2}(dp_i) := \frac{1}{\sqrt{2\pi}} e^{-(p_i - \lambda_2)^2} dp_i.$$

Thus the distribution of momenta is Gaussian with mean λ_2.

The measure $\nu_{\boldsymbol{\lambda}}^N$ is invariant with respect to the generator $L^\tau(t)_N$, if $\boldsymbol{\lambda} = (\tau, 0)$, thus

$$\int L_N^\tau f \nu_{(\tau,0)}^N (d\mathbf{r}, d\mathbf{p}) = 0$$

for any nice local function f. This is a consequence of the boundary conditions we impose: With our definition (2.3.1) of the Liouville operator and applying an integration by parts, we obtain

$$\begin{aligned} \int L_N^\tau f \nu_{\boldsymbol{\lambda}}^N (d\mathbf{r}, d\mathbf{p}) &= -\int f \sum_{i=1}^N (p_i - p_{i-1})(\lambda_1 - V'(r_i)) d\nu_{\boldsymbol{\lambda}}^N \\ &\quad - \int f \sum_{i=1}^{N-1} (\lambda_2 - p_i) \left(V'(r_{i+1}) - V'(r_i)\right) d\nu_{\boldsymbol{\lambda}}^N \\ &\quad - \int f (\lambda_2 - p_N) \left(\tau - V'(r_N)\right) d\nu_{\boldsymbol{\lambda}}^N \\ &= -\int f p_N(\tau - \lambda_1) - \lambda_2(V'(r_1) - \tau) d\nu_{\boldsymbol{\lambda}}^N = 0. \end{aligned}$$

if

$$\lambda_2 = 0 \quad \text{and} \quad \lambda_1 = \tau.$$

Notice furthermore, that there is a one-to-one correspondence between the average interdistance $\bar{r}$ and the parameter λ_1 and between the average momentum $\bar{p}$ and the parameter λ_2. This can be seen by the following:

$$\bar{r} = \int r_i \nu_{\boldsymbol{\lambda}}(dr_i, dp_i) = \int r_i \mu_{\lambda_1}(dr_i), \qquad \bar{p} = \int p_i \nu_{\boldsymbol{\lambda}}(dr_i, dp_i) = \int p_i \pi_{\lambda_2}(dp_i) = \lambda_2.$$

while the pressure P is now given by

$$\begin{aligned} \int V'(r_i)\nu_{\boldsymbol{\lambda}}(dr_i, dp_i) &= \int V'(r_i)\mu_{\lambda_1}(dr_i) \\ &= -\frac{1}{Z_1(\lambda_1)} \int (\lambda_1 - V'(r_i)) e^{\lambda_1 r_i - V(r_i)} dr_i + \lambda_1 = \lambda_1. \end{aligned}$$

From this we learn that the pressure is a function of the interdistance $\bar{r}$ alone:

$$P = \lambda_1 = \lambda_1(\bar{r}). \tag{2.3.6}$$

2.3.2 The local Gibbs measures

As mentioned in Section 2.1, we need to show that the system is in local equilibrium that means (2.1.8) and (2.1.9) have to be satisfied. In view of this, we define a family of product measures with slowly varying parameter $\boldsymbol{\lambda}(\cdot, t) := (\lambda_1(\cdot, t), \lambda_2(\cdot, t))$ in space for $x \in [0, 1]$:

$$\nu^N_{\boldsymbol{\lambda}(\cdot,t)} := \prod_{i=1}^{N} \nu_{\boldsymbol{\lambda}(\frac{i}{N},t)}, \tag{2.3.7}$$

where the marginals are given by

$$\nu_{\boldsymbol{\lambda}(\frac{i}{N},t)} dr_i dp_i = \frac{1}{Z(\boldsymbol{\lambda}(\frac{i}{N}, t))} e^{\lambda_1(\frac{i}{N},t) r_i + \lambda_2(\frac{i}{N},t) p_i - h_i} dr_i dp_i. \tag{2.3.8}$$

Since we know by Section 2.3 that there is a one-to-one correspondance between the average interdistance and the parameter λ_1, as well as between the average velocity and the parameter λ_2, for a given profile $\mathfrak{u}(x, t) := (\mathfrak{r}(x, t), \mathfrak{p}(x, t))$, $x \in [0, 1]$, $t \in \mathbb{R}_+$ we can find a unique corresponding Gibbs measure denoted by $\tilde{\nu}^N_{\mathfrak{u}(\cdot,t)}$ with parameters $\boldsymbol{\lambda}(\cdot, t)$ such that

$$E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}[r_i] = \mathfrak{r}(x, t) = E_{\tilde{\nu}^N_{\mathfrak{u}(\cdot,t)}}[r_i] \quad \text{and} \quad E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}[p_i] = \mathfrak{p}(x, t) = E_{\tilde{\nu}^N_{\mathfrak{u}(\cdot,t)}}[p_i] \tag{2.3.9}$$

where E_ν denotes the expected value with respect to a measure ν. Hereafter we will use the notation $\nu^N_{\mathfrak{u}(\cdot,t)}$ instead of $\tilde{\nu}^N_{\mathfrak{u}(\cdot,t)}$, since there is no possibility of confusion with the $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$. As for the equilibrium measure we introduce the density $g^N_{\mathfrak{u}(\cdot,t)}$ with respect to the reference measure $\nu^N_\star$ defined in (2.3.5). The density reads as

$$g^N_{\mathfrak{u}(\cdot,t)} = \prod_{i=1}^{N} \frac{1}{Z(\boldsymbol{\lambda}(\frac{i}{N}, t))} e^{\lambda_1((\frac{i}{N},t)) r_i + \lambda_2((\frac{i}{N},t)) p_i}.$$

With (2.3.9), for any initial data $\mathfrak{r}_0, \mathfrak{p}_0 : [0,1] \to \mathbb{R}$ in $L^\infty(\mathbb{R})$ of (2.1.3) such that

$$\lim_{N\to\infty} \int \left| \mathfrak{r}_0\left(\frac{i}{N}\right) - \mathfrak{r}_0(x) \right| dx = 0 \quad \text{and} \quad \lim_{N\to\infty} \int \left| \mathfrak{p}_0\left(\frac{i}{N}\right) - \mathfrak{p}_0(x) \right| dx = 0,$$

(2.1.8) is immediately satisfied for the measure $\nu^N_{\mathfrak{u}_0(\cdot)}$ by the law of large numbers, thus

$$\lim_{N\to\infty} \nu^N_{\mathfrak{u}_0(\cdot)} \left\{ \left| \frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) r_i - \int_0^1 J(x)\mathfrak{r}_0(x)dx \right| > \delta \right\} = 0$$

and

$$\lim_{N\to\infty} \nu^N_{\mathfrak{u}_0(\cdot)} \left\{ \left| \frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) p_i - \int_0^1 J(x)\mathfrak{p}_0(x)dx \right| > \delta \right\} = 0$$

for any $\delta > 0$.

In particular we can extend this law of large numbers to any local function ψ on Ω^N, any continuous function $J : [0,1] \to \mathbb{R}$ and any given smooth profile $\mathfrak{u}$

$$\lim_{N\to\infty} \nu^N_{\mathfrak{u}(\cdot,t)} \left\{ \left| \frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) \tau_i \psi(\mathbf{r},\mathbf{p}) - \int_0^1 J(x)\tilde{\psi}(\mathfrak{r}(x,t),\mathfrak{p}(x,t))dx \right| > \delta \right\} = 0$$

where

$$\tilde{\psi}(\mathfrak{u}) = \int \psi(\mathbf{r},\mathbf{p}) d\nu^N_{\boldsymbol{\lambda}}$$

is the expected value with respect to the Gibbs measure and τ_x denotes the spatial shift on the configurations.

2.4 The Conservative Noise

We next introduce the random forces. The perturbation of the system will be such that we still have balance of the total length and momentum of the chain, but conservation of total energy will be lost. Furthermore the random forces provide our system with enough ergodicity. A definition of ergodicity will be given in the Section 2.5.4.

The new dynamics is determined by the generator

$$A_N := L_N^\tau + \gamma S_N = L_N^0 + \frac{\partial}{\partial p_N} + \gamma S_N \tag{2.4.1}$$

on the space of smooth functions on Ω^N, with some parameter controling the strength of the noise. Here L_N^τ is the Liouville operator as defined in 2.3 and S_N is a symmetric operator, namely the generator of the stochastic dynamics, which acts only on the momenta and generates a diffusion on the surface $\mathbb{S}_k := \{(p_i, p_{i-1}) \in \mathbb{R}^2 | p_i + p_{i-1} = k\}$ of constant momentum. This perturbation, in terms of partial differential equations can be compared to the vanishing viscosity method on the macroscopic level. We will chose the noise such that momenta are randomly exchanged.

For any nice local function f we define the vector field

$$\Upsilon_{i,i-1} = \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i-1}}, i = 2, \dots, N. \tag{2.4.2}$$

It is easy to see that for all $i = 2, \dots N$

$$\Upsilon_{i,i-1}(p_{i-1} + p_i) = 0,$$

thus it is tangent to $\mathbb{S}_k$. On the other hand we do not have conservation of total energy since

$$\Upsilon_{i,i-1}(\sum_{j=0}^{N} p_j^2) = p_i - p_{i-1} \neq 0$$

in general.

To construct the diffusion generating operator S_N, we first calculate the adjoint operator of $\Upsilon_{i,i-1}$ with respect to the Gibbs measure $\nu^N_{\boldsymbol{\lambda}}$. As the noise only acts on momenta, we may replace $\nu^N_{\boldsymbol{\lambda}}$ by the Gaussian $\pi^N_{\lambda_2}$. We need

$$\int f(\mathbf{p})\Upsilon_{i,i-1}g(\mathbf{p})\pi_{\lambda_2}(d\mathbf{p}) = \int g(\mathbf{p})\Upsilon^\star_{i,i-1}f(\mathbf{p})\pi^N_{\lambda_2}(d\mathbf{p}).$$

By an integration by parts we obtain

$$\begin{aligned}
\int f(\mathbf{p})\Upsilon_{i,i-1}g(\mathbf{p})\pi^N_{\lambda_2}(d\mathbf{p}) &= \sqrt{\frac{1}{2\pi}}\int f(\mathbf{p})\left(\frac{\partial g(\mathbf{p})}{\partial p_i} - \frac{\partial g(\mathbf{p})}{\partial p_{i-1}}\right)\prod_{k=0}^{N} e^{-\frac{1}{2}(p_k-\lambda_2)^2}dp_k \\
&= \int g(\mathbf{p})\left((p_i - p_{i-1})\, f(\mathbf{p}) - \left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i-1}}\right) f(\mathbf{p})\right)\pi^N_{\lambda_2}(d\mathbf{p}) \\
&\Rightarrow \Upsilon^\star_{i,i-1} = (p_i - p_{i-1}) - \left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i-1}}\right).
\end{aligned}$$

Defining the symmetric operator such that

$$\begin{aligned}
&\int g(\mathbf{r},\mathbf{p})S_N f(\mathbf{r},\mathbf{p})\nu^N_{\boldsymbol{\lambda}}(\mathbf{r},\mathbf{p})^N = \\
&\qquad -\frac{1}{2}\sum_{i=2}^{N}\int\left(\frac{\partial f(\mathbf{r},\mathbf{p})}{\partial p_i} - \frac{\partial f(\mathbf{r},\mathbf{p})}{\partial p_{i-1}}\right)\left(\frac{\partial g(\mathbf{r},\mathbf{p})}{\partial p_i} - \frac{\partial g(\mathbf{r},\mathbf{p})}{\partial p_{i-1}}\right)\nu^N_{\boldsymbol{\lambda}}(\mathbf{r},\mathbf{p}),
\end{aligned}$$

S_N is given through

$$S_N := -\frac{1}{2}\sum_{i=2}^{N}\Upsilon^\star_{i,i-1}\Upsilon_{i,i-1} = \frac{1}{2}\sum_{i=2}^{N}\left\{\left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i-1}}\right)^2 - (p_i - p_{i-1})\left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i-1}}\right)\right\}. \tag{2.4.3}$$

Then

$$\begin{aligned}
S_N(p_i) &= -\frac{1}{2}(p_i - p_{i-1}) + \frac{1}{2}(p_{i+1} - p_i) = \frac{1}{2}\Delta p_i, \quad \forall i \in \{2,\dots,N-1\} \\
S_N(p_1) &= \frac{1}{2}(p_2 - p_1) \\
S_N(p_N) &= -\frac{1}{2}(p_N - p_{N-1})
\end{aligned} \tag{2.4.4}$$

where for any function $\sigma : \mathbb{N} \to \mathbb{R}$

$$\Delta\sigma(i) := \sigma(i+1) + \sigma(i-1) - 2\sigma(i),$$

Δ denotes the discrete Laplacian.

Since S_N is symmetric with respect to the the Gibbs measures $\nu_{\boldsymbol{\lambda}}^N$, $\nu_{\tau,0}^N$ is also invariant with respect to S_N since for a function $f \equiv 1$

$$\int f S_N g d\nu_{\boldsymbol{\lambda}}^N = \int 1 S_N g d\nu_{\boldsymbol{\lambda}}^N = \int g S_N 1 d\nu_{(\tau,0)} = 0.$$

Furthermore S_N indeed conserves the total momentum, since by (2.4.4) we have

$$S_N \left(\sum_{i=1}^{N} p_i \right) = 0.$$

Consequently we obtained

$$A_N(\sum_{i=1}^{N} r_i) = p_N \qquad \text{and} \qquad A_N(\sum_{i=0}^{N} p_i) = \tau(t) - V'(r_1)$$

and the dynamics of our model, for $i \in \{2, \dots, N-1\}$ is now given by

$$\begin{cases} \frac{dr_i}{dt} &= p_i - p_{i-1} \\ \frac{dp_i}{dt} &= V'(r_{i+1}) - V'(r_i) + \frac{1}{2}\gamma\Delta p_i + \sqrt{\frac{\gamma}{2}} \sum_{j=1}^{N} \Upsilon_{j,j-1}(p_i) dW_j \\ &= V'(r_{i+1}) - V'(r_i) + \frac{1}{2}\gamma\Delta p_i + \sqrt{\frac{\gamma}{2}}(dW_i - dW_{i+1}) \end{cases}$$

$$\begin{cases} \frac{dr_1}{dt} &= p_1 \\ \frac{dr_N}{dt} &= p_N - p_{N-1} \\ \frac{dp_1}{dt} &= (V'(r_2) - V'(r_1)) + \frac{1}{2}\gamma(p_2 - p_1) - \sqrt{\frac{\gamma}{2}} dW_2 \\ \frac{dp_N}{dt} &= \tau(t) - V'(r_N) - \gamma\frac{1}{2}N(p_N - p_{N-1}) + \sqrt{\frac{\gamma}{2}} dW_N \end{cases} \quad \textit{(at the boundary)} \tag{2.4.5}$$

where $\{W_i\}_{i\in\{1,\cdot,N\}}$ are independent Wiener processes.

2.5 The Hydrodynamic Limit

2.5.1 Main Theorem and sketch of the proof

On the phase space Ω^N we now have two time dependent families of probability measures. One of them is the local Gibbs measure $\nu_{\mathrm{u}(\cdot,t)}^N$ constructed from the p-system (2.1.3). Its density $g_{\mathrm{u}(\cdot,t)}^N(\mathbf{r}, \mathbf{p})$ is such that

$$\log g_{\mathrm{u}(\cdot,t)}^N = \sum_{i=1}^{N} \left(\lambda_1(\frac{i}{N}, t) r_i + \lambda_2(\frac{i}{N}, t) p_i - \log Z \left(\boldsymbol{\lambda}(\frac{i}{N}, t) \right) \right), \tag{2.5.1}$$

where with (A.1) from Appendix A

$$\log Z(\boldsymbol{\lambda}) = \Theta(\boldsymbol{\lambda}) = \log \int_{\Omega} e^{\lambda_1 r + \lambda_2 p - h} dr dp. \tag{2.5.2}$$

On the other hand we have the actual distribution $f_t^N(\mathbf{r}, \mathbf{p})$ of the noisy dynamics with initial distribution $g^N_{\mathfrak{u}_0(\cdot)}(\mathbf{r}, \mathbf{p})$ which is the solution of the Kolmogorov equation:

$$\begin{cases} \frac{\partial f_t^N}{\partial t}(\mathbf{r}, \mathbf{p}) & = N A_N^* f_t^N(\mathbf{r}, \mathbf{p}) \\ f_0^N(\mathbf{r}, \mathbf{p}) & = g^N_{\mathfrak{u}_0(\cdot)}(\mathbf{r}, \mathbf{p}). \end{cases} \tag{2.5.3}$$

In other words, we denote by $f_t^N(\mathbf{r}, \mathbf{p})$ the distribution on which the speeded generator NA_N. We denoted by $A_N^* = L_N^{0,\star} + \tau(t)(\frac{\partial}{\partial p_n})^\star + S_N$ the adjoint operator of A_N, where $L_N^{0,\star}$ and $(\frac{\partial}{\partial p_N})^\star$ can be computed as

$$(\frac{\partial}{\partial p_N})^\star = p_N - \frac{\partial}{\partial p_N} \tag{2.5.4}$$

and

$$\begin{aligned} L_N^{0,\star} &= -\sum_{i=1}^{N} (p_i - p_{i-1}) \left(\frac{\partial}{\partial r_i} - V'(r_i) \right) - \sum_{i=1}^{N-1} (V'(r_{i+1}) - V'(r_i)) \left(\frac{\partial}{\partial p_i} - p_i \right) \\ &\quad + V'(r_N) \left(\frac{\partial}{\partial p_N} - p_N \right) \\ &= -L_N^0. \end{aligned}$$

Next assume that we have (2.3.9) where $\mathfrak{u}$ satisfies (2.1.3)–(2.1.5). Then since there is a one-to-one correspondance between the parameter $\boldsymbol{\lambda}$ and the solution $\mathfrak{u}$, together with (2.3.6) the flux $\mathbf{A}(\mathfrak{u})$ and its Jacobian $D\mathbf{A}(\mathfrak{u})$ can be rewritten as:

$$\mathbf{A}(\mathfrak{u}) := (\lambda_2(\mathfrak{p}), \lambda_1(\mathfrak{r})) = (\mathfrak{p}, P(\mathfrak{r})) \Rightarrow D\mathbf{A}(\mathfrak{u}) = \begin{pmatrix} 0 & 1 \\ \lambda_1'(\mathfrak{r}) & 0 \end{pmatrix}. \tag{2.5.5}$$

Thus we can rewrite the p-system as:

$$\frac{\partial \mathfrak{u}}{\partial t} = D\mathbf{A}(\mathfrak{u}) \frac{\partial \mathfrak{u}}{\partial x} \tag{2.5.6}$$

On the other hand we obtain the following system of partial differential equations:

$$\frac{\partial \lambda_1}{\partial t} = \lambda_1'(\mathfrak{r}) \frac{\partial \mathfrak{r}}{\partial t} = \lambda_1' \frac{\partial \mathfrak{p}}{\partial x} \quad \text{and} \quad \frac{\partial \lambda_2}{\partial t} = \frac{\partial \mathfrak{p}}{\partial t} = \frac{\partial P(\mathfrak{r})}{\partial x}.$$

$$\Rightarrow \frac{\partial \boldsymbol{\lambda}}{\partial t} = (D\mathbf{A})^T(\mathfrak{u}) \frac{\partial \boldsymbol{\lambda}}{\partial x} \tag{2.5.7}$$

Now we are ready to state our main Theorem:

Theorem 2.5.1 (Main Theorem). *For some $0 < T < t_s$, let $\boldsymbol{\lambda}, \mathfrak{u} : [0,1] \times [0,T] \to \mathbb{R}^2$ be two C^1- functions satisfying the dual relation* (A.6) *from Appendix A.*
Let $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$ be the local Gibbs measure with marginals given by (2.3.8).

Furthermore for any time $t \in [0,T]$, denote by ν_t^N the probability measure on the path space $C([0,1],\Omega^N]$ of our process with generator A_N speeded up by N and starting from $\nu^N_{\boldsymbol{\lambda}(\cdot,0)}$.

If $\mathfrak{u} := (u_1,u_2) = (\mathfrak{r},\mathfrak{p})$ is a C^1- solution to the system of conservation laws (2.1.3)–(2.1.5), then for any $t \in [0,T]$, any smooth function $J : [0,1] \to \mathbb{R}$ and any $\delta > 0$

$$\lim_{N\to\infty} \nu_t^N \left[\left| \frac{1}{N}\sum_{i=1}^N J(\frac{i}{N})\zeta_{\alpha,i} - \int_0^1 J(x)u_\alpha(x,t)dx \right| > \delta \right] = 0, \tag{2.5.8}$$

for $\alpha = 1,2$ and $\boldsymbol{\zeta}_i := (\zeta_{1,i},\zeta_{2,i}) = (r_i,p_i)$.

The main tool to prove the Hydrodynamic limit is the relative entropy method. It states that:

Theorem 2.5.2 (Relative entropy). *Under the same assumptios as in Theorem 2.5.1, for any time $t \in [0,T]$*

$$\lim_{N\to\infty} \frac{1}{N} H_N\left(\nu_t^N \mid \nu^N_{\mathfrak{u}(\cdot,t)}\right) := \lim_{N\to\infty} \frac{1}{N}\int f_t^N(\mathbf{r},\mathbf{p}) \log \frac{f_t^N(\mathbf{r},\mathbf{p})}{g_{\mathfrak{u}(\cdot,t)}(\mathbf{r},\mathbf{p})} d\nu_\star(\mathbf{r},\mathbf{p}) = 0.$$

Here $H_N\left(\nu_t^N \mid \nu^N_{\mathfrak{u}(\cdot,t)}\right)$ is called the relative entropy of ν_t^N with respect to the reference measure $\nu^N_{\mathfrak{u}(\cdot,t)}$.

Before giving the sketch of the proof how this Theorem implies the hydrodynamic limit, it may be useful to state some properties and results concerning the relative entropy:

Notice that the relative entropy $H(\alpha|\beta)$ of a probability measure α with respect to a reference measure β and having densities f and g respectively with respect to the measure $\nu_\star$ which is absolutely continuous with respect to the Lebesgue measure, can be rewritten as

$$H(\alpha|\beta) = \sup_\varphi \{\int \varphi d\alpha - \log \int e^\varphi d\beta\} \tag{2.5.9}$$

where the supremum is taken over all bounded functions. To see this let us introduce the functional $\Psi : \Omega^N \to \mathbb{R}$:

$$\Psi(\varphi) := \int \varphi d\alpha - \log \int e^\varphi d\beta$$

which is concave and takes its maximum if its gradient vanishes, that means for each function $h : \Omega^N \to \mathbb{R}$ such that

$$\int h \cdot f\, \nu_\star d\mathbf{r}d\mathbf{p} - \frac{\int h \cdot e^\varphi g\, \nu_\star d\mathbf{r}d\mathbf{p}}{\int e^\varphi g\, \nu_\star d\mathbf{r}d\mathbf{p}} = 0.$$

Since ψ is invariant under the addition of a constant, we may chose

$$\int e^\varphi g\, d\mathbf{r}d\mathbf{p} = 1 \Rightarrow e^\varphi = \frac{f}{g} \Rightarrow \varphi = \log\frac{f}{g}.$$

Then we obtain the expression for the relative entropy from Theorem 2.5.2 by the following argument:

$$H(\alpha|\beta) = \sup_\varphi \psi(\varphi) = \psi\left(\log\frac{f}{g}\right) = \int \log\frac{f}{g} f\, d\mathbf{r}d\mathbf{p} - \log\int \frac{f}{g} g\, d\mathbf{r}d\mathbf{p} = \int \log\frac{f}{g} f\, d\mathbf{r}d\mathbf{p},$$

where the last equality is true since in the second term the integral is equal to one. Now it is easy to see that the relative entropy has the following properties: $H(\alpha|\beta)$ is

$(H1)$: positive,

$(H2)$: convex,

$(H3)$: lower semicontinuous.

These properties imply the following useful inequality called entropy inequality. It dictates, that for any measurable function F, any positive constant σ and some probability measures α and β:

$$E_\alpha[F] \leq \frac{1}{\sigma} \log E_\beta[\exp(\sigma F)] + \frac{1}{\sigma} H(\alpha|\beta). \tag{2.5.10}$$

Now we will see how Theorem 2.5.2 implies the Main Theorem:

*P*roof of Theorem 2.5.1 A useful special case of the entropy inequality can be stated if we set $F := \mathbf{1}_{[A]}$ to be the indicator function on a set A. With the choice $\sigma = \log\left(1 + \frac{1}{\beta[A]}\right)$, we obtain the inequality

$$\begin{aligned} E_\alpha[\mathbf{1}_{[A]}] = \alpha[A] &\leq \frac{1}{\sigma} \log \beta[\exp(\sigma \mathbf{1}_{[A]})] + \frac{1}{\sigma} H(\alpha|\beta) \\ &= \frac{1}{\sigma} \log E_\beta[\mathbf{1}_{[A]} e^\sigma + (1 - \mathbf{1}_{[A]})] + \frac{1}{\sigma} H(\alpha|\beta) \\ &= \frac{1}{\sigma} \log\left(\beta[A](e^\sigma - 1) + 1\right) + \frac{1}{\sigma} H(\alpha|\beta) \end{aligned}$$

$$\Rightarrow \alpha[A] \leq \frac{\log 2 + H(\alpha|\beta)}{\log\left(1 + \frac{1}{\beta[A]}\right)} \tag{2.5.11}$$

This inequality means that any set which has exponentially small probability with respect to β, also has small probability with respect to α if $H(\alpha|\beta) = o(N)$.

We now define the set A_δ to be

$$A_\delta := \left\{ \left| \frac{1}{N} \sum_{i=1}^{N} J(\frac{i}{N}) \zeta_{\alpha,i} - \int_0^1 J(x) u_\alpha(x,t) dx \right| > \delta \right\}.$$

By Theorem 2.5.2 we know that $H_N\left(\nu_t^N \,|\, \nu_{\mathrm{u}(\cdot,t)}^N\right) = o(N)$. Then with inequality (2.5.11), to prove that $\lim_{N\to\infty} \nu_t^N[A_\delta] = 0$, it is enough to show that for each $\delta > 0$,

$$\log\left(1 + \frac{1}{\nu_{\mathrm{u}(\cdot,t)}^N}\right) \geq C(\delta) N$$

for some constant C not depending on N. But this is satisfied if $\nu_{\mathrm{u}(\cdot,t)}^N[A_\delta]$ is exponentially small, i.e

$$\nu_{\mathrm{u}(\cdot,t)}^N[A_\delta] \leq \frac{1}{e^{C(\delta)N}}. \tag{2.5.12}$$

This is a result of the large deviation theory which we will prove in Corollary 2.5.26.

2.5.2 The relative entropy method

In this Section we will prove Theorem 2.5.2. To simplify the notation we set

$$H_N\left(\nu_t^N \mid \nu_{\mathrm{u}(\cdot,t)}^N\right) := H_N(t).$$

Notice that by the choice of our initial distribution the relative entopy at time 0 is equal to zero:

$$H_N(0) = 0$$

The strategy is to show that for some constant C

$$\frac{d}{dt}H_N(t) \le CH_N(t) + R_N(t). \tag{2.5.13}$$

If we can show that the third term is such that

$$\lim_{N\to\infty}\frac{1}{N}\int_0^t R_N(s)ds = 0, \tag{2.5.14}$$

then it follows by Gronwall's inequality that $\lim_{N\to\infty}\frac{H_N(t)}{N} = 0$ which concludes the proof of Theorem 2.5.2. We start by establishing the following differential inequality:

Lemma 2.5.3 (Differential inequality).

$$\frac{d}{dt}H_N(t) \le \int N\tau(t)p_N f d\nu_\star(\mathbf{r},\mathbf{p}) - \int\left[\left(NA_N + \frac{\partial}{\partial_t}\right)\log g_{\mathrm{u}(\cdot,t)}^N\right] f_t^N d\nu_\star(\mathbf{r},\mathbf{p})$$

Proof.

$$\begin{aligned}
\frac{d}{dt}H_N(t) \quad &= \quad \int \frac{\partial f_t^N}{\partial t}\log\frac{f_t^N}{g_{\mathrm{u}(\cdot,t)}^N}d\nu_\star(\mathbf{r},\mathbf{p}) + \int\frac{\partial f_t^N}{\partial t}d\nu_\star(\mathbf{r},\mathbf{p}) \\
&\qquad - \int\frac{\partial}{\partial t}\log\left(g_{\mathrm{u}(\cdot,t)}^N\right) f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) \\
&\overset{(2.5.3)}{=} \quad \int \log\frac{f_t^N}{g_{\mathrm{u}(\cdot,t)}^N}NA_N^* f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) + \frac{\partial}{\partial t}\int f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) \\
&\qquad - \int\frac{\partial}{\partial t}\log\left(g_{\mathrm{u}(\cdot,t)}^N\right) f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) \\
&= \quad \int f_t^N NA_N\log\frac{f_t^N}{g_{\mathrm{u}(\cdot,t)}^N}d\nu_\star(\mathbf{r},\mathbf{p}) + 0 - \int\frac{\partial}{\partial t}\log\left(g_{\mathrm{u}(\cdot,t)}^N\right) f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) \\
&\le \quad \int NA_N f_t^N d\nu_\star(\mathbf{r},\mathbf{p}) - \int f_t^N NA_N\log g_{\mathrm{u}(\cdot,t)}^N d\nu_\star(\mathbf{r},\mathbf{p}) \\
&\qquad - \int\frac{\partial}{\partial t}\log\left(g_{\mathrm{u}(\cdot,t)}^N\right) f_t^N d\nu_\star(\mathbf{r},\mathbf{p})
\end{aligned}$$

For the inequality we used the fact that

$$A_N \log f_t^N \leq \frac{A_N f_t^N}{f_t^N}.$$

This comes from the maximum principle for A_N which says that for any convex function ϕ,

$$A_N \phi(f) \leq \phi'(f) A_N(f).$$

The first term can be computed explicitely: Applying integration by parts on $\int S_N f_t^N d\nu_\star$ and on $\int L_N^0 f_t^N d\nu_\star$, we obtain that both terms are equal to zero. Then the remaining term gives

$$\tau(t) \int \frac{\partial}{\partial p_N} f_t^N d\nu_\star = \tau(t) \int p_N f_t^N d\nu_\star,$$

which concludes the proof. □

In the following lemmas we calculate $\left(N A_N + \frac{\partial}{\partial_t}\right) \log g^N_{\mathrm{u}(\cdot,t)}$

Lemma 2.5.4. *Recall the definition of the Liouville generator given by* (2.3.1),

$$N L_N^\tau \log g^N_{\mathrm{u}(\cdot,t)} =$$

$$-\sum_{i=1}^N \left(\frac{\partial \lambda_2}{\partial x}\left(\frac{i}{N}, t\right) V'(r_i) - \frac{\partial \lambda_1}{\partial x}\left(\frac{i}{N}, t\right) p_{i-1} \right) + N(\lambda_2(1,t) + p_N)\tau(t) + a_N(t)$$

where $a_N(t)$ is such that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^t \int a_N(s) d\nu_s^N ds = 0$$

Proof. We start with calculating $L_N^\tau \log g^N_{\mathrm{u}(\cdot,t)}(\mathbf{r}, \mathbf{p})$ which is equal to

$$\begin{aligned}
& L_N^\tau \left(\sum_{i=1}^N \lambda_1(\frac{i}{N}, t) r_i + \sum_{i=1}^N \lambda_2(\frac{i}{N}, t p_i \right) \\
= \; & \sum_{i=1}^N (p_i - p_{i-1}) \lambda_1(\frac{i}{N}, t) + \sum_{i=1}^{N-1} (V'(r_{i+1}) - V'(r_i)) \lambda_2(t, \frac{i}{N}) \\
& + (\tau(t) - V'(r_N)) \lambda_2(1,t) \\
= \; & \sum_{i=1}^N \lambda_1(\frac{i}{N}, t) p_i - \sum_{i=1}^{N-1} \lambda_1(\frac{i+1}{N}, t) p_i + \sum_{i=2}^N \lambda_2(\frac{i-1}{N}, t) V'(r_i) - \sum_{i=1}^{N-1} \lambda_2(t, \frac{i}{N}) V'(r_i) \\
& + \lambda_2(1,t)(\tau(t) - V'(r_N)) \\
= \; & \sum_{i=1}^{N-1} \left(\lambda_1(\frac{i}{N}, t) - \lambda_1(\frac{i+1}{N}, t) \right) p_i + \sum_{i=1}^N \left(\lambda_2(\frac{i-1}{N}, t) - \lambda_2(t, \frac{i}{N}) \right) V'(r_i) \\
& + \lambda_2(1,t)\tau(t) + \lambda_1(1,t) p_N \\
= \; & -\frac{1}{N} \sum_{i=1}^{N-1} \frac{\partial \lambda_1(\frac{i}{N}, t)}{\partial x} p_i - \frac{1}{N} \sum_{i=1}^N \frac{\partial \lambda_1(\frac{i}{N}, t)}{\partial x} V'(r_i) + (\lambda_2(1,t) + p_N)\tau(t) \\
& + \mathcal{O}\left(\frac{1}{N^2}\right) \sum_{i=1}^{N-1} (V'(r_i) + p_{i-1})
\end{aligned}$$

This is the expression we wanted to have with

$$a_N(t) = \mathcal{O}\left(\frac{1}{N}\right) \sum_{i=1}^{N-1} (V'(r_i) + p_{i-1})$$

It remains to show, that $\lim_{N\to\infty} \int_0^t \int \frac{a_N(s)}{N} d\nu_s^N ds = 0$. This will be done in Lemma 2.5.8 $\square$

Lemma 2.5.5.

$$\frac{\partial}{\partial t} \log g^N_{\mathfrak{u}(\cdot,t)} = \sum_{i=1}^{N} (D\mathbf{A})^T (\mathfrak{u}(\frac{i}{N},t)) \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},t) \left(\boldsymbol{\zeta}_i - \mathfrak{u}(\frac{i}{N},t)\right)$$

where $\boldsymbol{\zeta}_i = (\zeta_{1,i}, \zeta_{2,i}) := (r_i, p_i)$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t} \log g^N_{\mathfrak{u}(\cdot,t)} &= \frac{\partial}{\partial t}\left[\sum_{i=1}^{N}\left(\lambda_1(\frac{i}{N},t) r_i + \lambda_2(\frac{i}{N},t) p_i - h_i - \log Z\left(\boldsymbol{\lambda}(\frac{i}{N},t)\right)\right)\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \lambda_1}{\partial t}(\frac{i}{N},t) r_i + \frac{\partial \lambda_2}{\partial t}(\frac{i}{N},t) p_i - \frac{\frac{\partial Z}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial t}(\frac{i}{N},t) - \frac{\partial Z}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial t}(\frac{i}{N},t)}{Z\left(\boldsymbol{\lambda}(\frac{i}{N},t)\right)}\right] \\
&= \sum_{i=1}^{N}\left[\frac{\partial \lambda_1}{\partial t}(\frac{i}{N},t)\left(r_i - \frac{\partial \log Z}{\partial \lambda_1}\right) + \frac{\partial \lambda_2}{\partial t}(\frac{i}{N},t)\left(p_i - \frac{\partial \log Z}{\partial \lambda_2}\right)\right].
\end{aligned}$$

Since we assumed that the solution $\mathfrak{u}$ of the p-system is the dual of the parameter $\boldsymbol{\lambda}$, by relation (A.6) the last expression is equal to

$$\sum_{i=1}^{N}\left[\frac{\partial \lambda_1}{\partial t}(\frac{i}{N},t)\left(r_i - \mathfrak{r}(\frac{i}{N},t)\right) + \frac{\partial \lambda_2}{\partial t}(\frac{i}{N},t)\left(p_i - \mathfrak{p}(\frac{i}{N},t)\right)\right]. \tag{2.5.15}$$

Then together with (2.5.7) we obtain the result. $\square$

Lemma 2.5.6. *Recall the definition of the symmetric operator given by* (2.4.3)*.*

$$\int_0^t \int N S_N \log g^N_{\mathfrak{u}(s,\cdot)} d\nu_s^N ds = o(N).$$

Proof. $S_N\left(\sum_{j=0}^N \lambda_2(\frac{j}{N},t)p_j\right)$ is equal to:

$$
\begin{aligned}
& S_N\left(\lambda_2\left(\frac{1}{N},t\right)p_1\right) + S_N\left(\sum_{i=2}^{N-1}\lambda_2(\frac{i}{N},t)p_i\right) + S_N\left(\lambda_2\left(1,t\right)p_N\right) \\
= \quad & \frac{1}{2}\lambda_2(\frac{1}{N},t)(p_2-p_1) + \sum_{j=2}^{N-1}\lambda_2(\frac{j}{N},t)\Delta p_j - \frac{1}{2}\lambda_2(1,t)(p_N-p_{N-1}) \\
= \quad & \frac{1}{2}\Bigg(-\lambda_2(\frac{1}{N},t)p_1 - \sum_{i=2}^{N-1}2\lambda_2(\frac{i}{N},t)p_i + \sum_{i=1}^{N-1}\lambda_2(\frac{i+1}{N},t)p_i \\
& + \sum_{i=2}^{N}\lambda_2(\frac{i-1}{N},t)p_i - \lambda_2(1,t)p_N\Bigg) \\
= \quad & -\frac{1}{2}\lambda_2(\frac{1}{N},t)p_1 - \frac{1}{2}\sum_{i=2}^{N-1}p_i\Delta\lambda_2(\frac{i}{N},t) + \frac{1}{2}\lambda_2(\frac{2}{N},t)p_1 + \frac{1}{2}\lambda_2(\frac{N-1}{N},t)p_N + \frac{1}{2}\lambda_2(1,t)p_N \\
= \quad & \frac{1}{2}\sum_{i=2}^{N-1}p_j\Delta\lambda_2(\frac{j}{N},t) - \frac{1}{2}p_N\left(\lambda_2(1,t)-\lambda_2(\frac{N-1}{N},t)\right) - \frac{1}{2}p_1\left(\lambda_2(\frac{2}{N},t)-\lambda_2(\frac{1}{N},t)\right)
\end{aligned}
$$

These terms are all of order $\mathcal{O}(\frac{1}{N})$ if we can show that the expectation with respect to ν_t^N of $\frac{1}{N}\sum_i p_i$ is uniformly bounded for all N. This will be done in Lemma 2.5.8. This concludes the proof. □

To conclude the proofs of Lemma 2.5.4 and Lemma 2.5.6, it remains to find a uniform bound on the expected values of the densities. In this context we first figure out some more properties of the relative entropy:

In the following we denote for constant $\boldsymbol{\lambda}$ by $g_{\boldsymbol{\lambda}}^N$ the density of the measure $\nu_{\boldsymbol{\lambda}}^N$ with respect to the reference measure $\nu_\star^N$. Observe that if we consider the relative entropy of ν_t^N with respect to a reference measure $\nu_{\boldsymbol{\lambda}}^N$ we have:

$$
\begin{aligned}
& \frac{\partial}{\partial t}H_N\left(\nu_t^N \,|\, \nu_{\boldsymbol{\lambda}}^N\right) = \int \frac{\partial f_t^N}{\partial t}\log\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}d\nu_\star^N + \frac{\partial}{\partial t}\int f_t^N d\nu_\star^N \\
= \quad & \int \log\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}NA_N^* f_t^N d\nu_\star^N = \int f_t^N NA_N\log\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}d\nu_\star^N. \\
= \quad & \int \frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}N\left(L_N^\tau\log\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N} + \gamma S_N\log\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\right)d\nu_{\boldsymbol{\lambda}}^N \\
= \quad & \int NL_N^\tau\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}d\nu_{\boldsymbol{\lambda}}^N + \gamma N\int S_N\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}d\nu_{\boldsymbol{\lambda}}^N \\
& +\gamma\frac{1}{2}N\sum_{i=2}^N\int\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\left(\frac{\partial}{\partial p_i}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\cdot\frac{\partial}{\partial p_i}\frac{g_{\boldsymbol{\lambda}}^N}{f_t^N} + \frac{\partial}{\partial p_{i-1}}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\cdot\frac{\partial}{\partial p_{i-1}}\frac{g_{\boldsymbol{\lambda}}^N}{f_t^N}\right)d\nu_{\boldsymbol{\lambda}}^N \\
& -\gamma\frac{1}{2}N\sum_{i=2}^N\int\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\left(\frac{\partial}{\partial p_{i-1}}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\cdot\frac{\partial}{\partial p_i}\frac{g_{\boldsymbol{\lambda}}^N}{f_t^N} + \frac{\partial}{\partial p_i}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\cdot\frac{\partial}{\partial p_{i-1}}\frac{g_{\boldsymbol{\lambda}}^N}{f_t^N}\right)d\nu_{\boldsymbol{\lambda}}^N
\end{aligned}
$$

If we further assume that $\boldsymbol{\lambda} = (\tau, 0)$, then the reference measure is invariant with respect to A_N and thus the first two terms of the last expression are equal to zero. For the last two terms, by integration by parts they are equal to

$$-\gamma\frac{1}{2}N\left(\sum_{i=2}^{N}\int\frac{g_{(\tau,0)}^N}{f_t^N}\left(\frac{\partial}{\partial p_i}\frac{f_t^N}{g_{(\tau,0)}^N}-\frac{\partial}{\partial p_{i-1}}\frac{f_t^N}{g_{(\tau,0)}^N}\right)^2 d\nu_{(\tau,0)}^N-\int S_N\frac{f_t^N}{g_{(\tau,0)}^N}d\nu_{(\tau,0)}^N\right)$$

where again the second term is equal to zero. We will denote the first term by

$$\begin{aligned}-\gamma N\mathcal{D}_N\left(\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\right) &:= -\gamma\frac{1}{2}N\sum_{i=2}^{N}\int\frac{g_{\boldsymbol{\lambda}}^N}{f_t^N}\left(\frac{\partial}{\partial p_i}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}-\frac{\partial}{\partial p_{i-1}}\frac{f_t^N}{g_{\boldsymbol{\lambda}}^N}\right)^2 d\nu_{\boldsymbol{\lambda}}^N\\ &= -\gamma\frac{1}{2}N\sum_{i=2}^{N}\int\frac{1}{f_t^N}\left(\frac{\partial}{\partial p_i}f_t^N-\frac{\partial}{\partial p_{i-1}}f_t^N\right)^2 d\nu_\star^N. \qquad (2.5.16)\end{aligned}$$

where we used the fact that $\frac{\partial g_{(\tau,0)}}{\partial p_i}=0$. This term is called Dirichlet form. It is easy to see, that $\mathcal{D}_N$ is

($\mathcal{D}1$): positive,

($\mathcal{D}1$): convex,

($\mathcal{D}1$): lower semi continuous.

Thus if we integrate in time we obtained that for an invariant reference measure $\nu_{(\tau,0)}$:

$$H_N\left(\nu_t^N\,|\,\nu_{(\tau,0)}^N\right)+\gamma N\int_0^t\mathcal{D}_N\left(\frac{f_s^N}{g_{(\tau,0)}^N}\right)ds=H_N\left(\nu_0^N\,|\,\nu_{(\tau,0)}^N\right)=H_N\left(\nu_{\boldsymbol{\lambda}(\cdot,0)}^N\,|\,\nu_{(\tau,0)}^N\right)\le CN.$$

Since the Dirichlet form is positive, by

$$H_N\left(\nu_t^N\,|\,\nu_{(\tau,0)}^N\right)-H_N\left(\nu_0^N\,|\,\nu_{(\tau,0)}^N\right)\le-\gamma N\int_0^t\mathcal{D}_N\left(\frac{f_s^N}{g_{(\tau,0)}^N}\right)ds$$

we can see that the relative entropy is decreasing in time.

Further, since for any parameter $\boldsymbol{\lambda}(\cdot,t)$ there exists a constant C such that

$$H_N\left(\nu_{\boldsymbol{\lambda}(\cdot,0)}^N\,|\,\nu_{(\tau,0)}^N\right)\le CN$$

with the positivity and the convexity of $\mathcal{D}$ we have:

$$H_N\left(\nu_t^N\,|\,\nu_{(\tau,0)}^N\right)+t\gamma N\mathcal{D}_N\left(\frac{1}{t}\int_0^t\frac{f_s^N}{g_{(\tau,0)}^N}ds\right)\le CN.$$

We proved the following Lemma:

Lemma 2.5.7. *Let ν_t^N be the probability measure with density satisfying* (2.5.3). *Then the relative entropy with respect to $\nu_{(\tau,0)}^N$ is decreasing in time.*

If furthermore ν_t^N satisfies

$$H_N\left(\nu_0^N\,|\,\nu_{(\tau,0)}^N\right)\le CN$$

for some uniform constants $C > 0$, then for any $N \in \mathbb{N}$

$$H_N\left(\nu_t^N \mid \nu_{(\tau,0)}^N\right) \leq CN \quad \text{and} \quad \mathcal{D}_N\left(\frac{1}{t}\int_0^t \frac{f_s^N}{g_{(\tau,0)}^N} ds\right) \leq \frac{C}{t\gamma}$$

Using the bound we obtained for the relative entropy, we can now prove the following Lemma:

Lemma 2.5.8.

$$\int \frac{1}{N}\sum_{i=1}^N |r_i| d\nu_t^N \leq C \tag{2.5.17}$$

$$\int \frac{1}{N}\sum_{i=1}^N |p_i| d\nu_t^N \leq C \tag{2.5.18}$$

$$\int \frac{1}{N}\sum_{i=1}^N |V'(r_i)| d\nu_t^N \leq C, \tag{2.5.19}$$

where C is a constant not depending on N.

Proof. To prove the first three conditions we apply the entropy inequality on ν_t^N with invariant reference measure $\nu_{(\tau,0)}^N$. This gives us for any measurable function F and any $\sigma > 0$:

$$E_{\nu_t^N}[F] \leq \frac{1}{\sigma}\log E_{\nu_{(\tau,0)}^N}[\exp(\sigma F)] + \frac{1}{\sigma} H(\nu_t^N | \nu_{(\tau,0)}).$$

Since by assumption on the initial distribution $H_N\left(\nu_0^N \mid \nu_{(\tau,0)}^N\right) \leq CN$ for some positive constant C, with Lemma 2.5.7 we obtain the upper bound

$$E_{\nu_t^N}[F] \leq \frac{1}{\sigma}\log E_{\nu_{(\tau,0)}^N}[\exp(\sigma F)] + \frac{1}{\sigma} CN. \tag{2.5.20}$$

Furthermore notice that the logarithmic moment generating function for any λ_1, $\lambda_2 \in \mathbb{R}$

$$\begin{aligned} \log \int e^{\alpha r + \beta p - h} d\nu_{(\tau,0)} &= \log \int \frac{e^{(\lambda_1+\tau)r + (\lambda_2+0)p - h}}{Z((\tau,0))} dr dp \\ &= \log \frac{Z(\lambda_1+\tau, \lambda_2)}{Z(\tau,0)} < \infty \end{aligned} \tag{2.5.21}$$

by assumption (2.3.4).

To prove (2.5.17), we set

$$F(\mathbf{r}) := \frac{1}{N}\sum_{i=1}^N |r_i|, \quad \sigma = \sigma' N$$

for some σ' independent of N. Then

$$\begin{aligned} \int \frac{1}{N}\sum_{i=1}^N |r_i| d\nu_t^N &\leq \frac{1}{\sigma' N}\log \int e^{\sigma' \sum_{i=1}^N |r_i|} d\nu_{(\tau,0)}^N + \frac{C}{\sigma'} \\ &= \frac{1}{\sigma' N}\sum_{i=1}^N \log\left(\frac{1}{Z_1(\tau)}\int e^{(\sigma'+\tau)r_i - V(r_i)} dr_i\right) + \frac{C}{\sigma'} \end{aligned}$$

the expression inside the sum, by (2.5.21) is bounded by a constant independant of N. This concludes the proof of (2.5.17).

To prove (2.5.18) we proceed in a similar way: We now chose

$$F(\mathbf{p}) := \frac{1}{N}\sum_{i=1}^{N}|p_i|, \quad \sigma = \sigma' N$$

Then the entropy inequality leads to

$$\begin{aligned}\int \frac{1}{N}\sum_{i=1}^{N}|p_i| d\nu_t^N &\leq \frac{1}{\sigma' N}\log\int e^{\sigma'\sum_{i=1}^N |p_i|} d\nu_{(\tau,0)}^N + \frac{C}{\sigma'} \\ &= \frac{1}{\sigma' N}\sum_{i=1}^{N}\log\left(\frac{1}{Z((\tau,0))}\int e^{\tau r_i + \sigma'|p_i| - h_i} dr_i dp_i\right) + \frac{C}{\sigma'}\end{aligned}$$

which is bounded uniformly by the same arguments as before. This proves (2.5.18)

For the proof of (2.5.19), by setting

$$F(\mathbf{p}) := \frac{1}{N}\sum_{i=1}^{N}|V'(r_i)|, \quad \sigma = \sigma' N,$$

it remains to prove that

$$\frac{1}{\sigma' N}\sum_{i=1}^{N}\log\left(\frac{1}{Z_1(\tau)}\int e^{\tau r_i - V(r_i) + \sigma'|V'(r_i)|} dr_i\right) \leq C$$

Here we use assumptions (2.1.1) on the potential. This makes sure that $|V'(r)| \leq |V(r)|$. Then again by (2.5.21) we can conclude the proof for (2.5.19) □

This Lemma is enough to conclude the proofs of Lemmas 2.5.4 and 2.5.6.

So far we have from Lemma 2.5.4, 2.5.5 and 2.5.6

$$\begin{aligned}\frac{d}{dt}H_N(t) &\leq \int \sum_{i=1}^{N}\left\{\frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},t)\left[\mathbf{A}_i - (D\mathbf{A})^T(\mathfrak{u})\left(\boldsymbol{\zeta}_i - \mathfrak{u}(\frac{i}{N},t)\right)\right]\right\} f_t^N \nu_\star(d\mathbf{r},d\mathbf{p}) \\ &\quad + \int \left(-N(\lambda_2(1,t)+p_N)\tau(t) + N\tau(t)p_N\right) f_t^N \nu_\star(d\mathbf{r},d\mathbf{p}) + R_N(t) \\ &= \int \sum_{i=1}^{N}\frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},t)\left[\mathbf{A}_i - (D\mathbf{A})^T(\mathfrak{u})\left(\boldsymbol{\zeta}_i - \mathfrak{u}(\frac{i}{N},t)\right)\right] f_t^N \nu_\star(d\mathbf{r},d\mathbf{p}) \\ &\quad - \int N\tau(t)\lambda_2(1,t) f_t^N \nu_\star(d\mathbf{r},d\mathbf{p}) + R_N(t)\end{aligned} \tag{2.5.22}$$

where $\mathbf{A}_i := (p_{i-1}, V'(r_i))$ and $R_N(t)$ is such that (2.5.14) holds.

With (2.5.5) we furthermore have

$$\int_0^1 \frac{\partial \boldsymbol{\lambda}(x,t)}{\partial x}\cdot \mathbf{A}(\mathfrak{u}(x,t))dx = \int \frac{\partial}{\partial x}(\lambda_1(x,t)\lambda_2(x,t))dx = \tau(t)\lambda_2(1,t).$$

This means that in the limit as $N \to \infty$ we can replace the last term of (2.5.22) by

$$-\int \sum_{i=1}^{N} \frac{\partial \boldsymbol{\lambda}(\frac{i}{N}, t)}{\partial x} \cdot \mathbf{A}(\mathfrak{u}(\frac{i}{N}, t)) f_t^N d\nu_\star(d\mathbf{r}, d\mathbf{p}) + R_N(t)$$

and consequently from (2.5.22) we have

$$\frac{1}{N} H_N(t) \leq \frac{1}{N} \int_0^t R_N(s) ds + \frac{1}{N} \int_0^t \int \sum_{i=1}^{N} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s) \times \\ \left[\mathbf{A}_i - \mathbf{A}\left(\mathfrak{u}(\frac{i}{N}, s)\right) - (D\mathbf{A})^T(\mathfrak{u}(x,s))\left(\boldsymbol{\zeta}_i - \mathfrak{u}(\frac{i}{N}, s)\right)\right] f_s^N \nu_\star(d\mathbf{r}, d\mathbf{p}) ds \quad (2.5.23)$$

where we used the fact that $H(0) = 0$.

Our next goal is to prove a weak form of local equilibrium. In view of this we introduce microscopic averages over blocks of size $k+1$. Let

$$\boldsymbol{\zeta}_i^k = (\zeta_{1,i}^k, \zeta_{2,i}^k) := (r_i^k, \zeta_i^k) := \frac{1}{k+1} \sum_{|i-l| \leq \frac{k}{2}} \boldsymbol{\zeta}_l. \quad (2.5.24)$$

denote the emprical densities of (r_i, p_i) over the blocks. These blocks are microscopically large but on the macroscopic scale they are small, thus we let go N to infinity first and then k goes to infinity.

The introduction of the block averages will be obtained by a summation by parts. Of course this has to be done carefully near the boundaries. But notice that for any smooth function $J : [0,1] \to \mathbb{R}$ and any bounded function $\psi \to \mathbb{R}$, a block of the form

$$\sum_{|j-i|| \leq \frac{k}{2}} J(\frac{j}{N}) \psi(\mathbf{r}, \mathbf{p})$$

is bounded by kC, where C is a constant not depending on k and on N. Therefore, as $N \to \infty$, such terms devided by N vanish in the limit. In this way we can ignore finitely many blocks at the boundaries. In *Step I* of the proof of Corollary 2.5.26, we will show the following summation by parts formula: For any smooth and bounded function $J : [0,1] \to \mathbb{R}$ and any bounded function $\Omega \to \mathbb{R}$, we have:

$$\frac{1}{N} \sum_{i=1}^{N} J(\frac{i}{N}) \psi(\boldsymbol{\zeta}_i) = \frac{1}{N} \sum_{i=\frac{k}{2}+1}^{N-\frac{k}{2}} J(\frac{i}{N}) \frac{1}{k+1} \sum_{|j-i| \leq \frac{k}{2}} \psi(\boldsymbol{\zeta}_j) + \mathcal{O}(\frac{k}{N}).$$

To apply the summation by parts formula to (2.5.23), we therefore first need to do some cut off in order to have only bounded variables: Let $\mathcal{C}_{i,b} := \{|h_i| \leq b\}$, and define

$$\mathbf{A}_{i,b} := \mathbf{A}_i \mathbf{1}_{\mathcal{C}_{i,b}} \qquad \text{and} \qquad \boldsymbol{\zeta}_{i,b} := \boldsymbol{\zeta}_i \mathbf{1}_{\mathcal{C}_{i,b}}$$

Since assumption (2.1.1) asserts that $|V'(r)| \leq V(r)$ and $r \leq V(r)$ for large r, the cut offs just defined are bounded. The entropy inequality (2.5.10) with reference measure $g^N_{\mathfrak{u}(\cdot,t)} d\nu_\star$

shows that the error we make by the replacement of $\mathbf{A}_i$ and $\boldsymbol{\zeta}_i$ by $\mathbf{A}_{i,b}$ and $\zeta_{i,b}$ respectively is small in N if we can show that $\frac{1}{N}H_N(s) \to 0$ as $N \to 0$: For any $\sigma > 0$ small enough

$$\begin{aligned}
&\int \sum_{i=1}^{N} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s)\mathbf{A}_i \mathbf{1}_{\mathcal{C}^c_{i,b}} f_s^N d\nu_\star \\
\leq\ & \frac{1}{\sigma}\sum_{i=1}^{N} \log\left(\int e^{\sigma \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)\mathbf{A}_i \mathbf{1}_{\mathcal{C}^c_{i,b}} + \boldsymbol{\lambda}(\frac{i}{N},s)\boldsymbol{\zeta}_i - \log Z(\boldsymbol{\lambda}(\frac{i}{N},s))} d\nu_\star\right) + \frac{H_N(s)}{\sigma} \\
\leq\ & \frac{1}{\sigma}\sum_{i=1}^{N} \log\left(1 + \int_{\mathcal{C}^c_{i,b}} e^{\sigma \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)\mathbf{A}_i + \boldsymbol{\lambda}(\frac{i}{N},s)\boldsymbol{\zeta}_i - \log Z(\boldsymbol{\lambda}(\frac{i}{N},s))} d\nu_\star\right) + \frac{H_N(s)}{\sigma} \\
=\ & \frac{NC(b)}{\sigma} + \frac{H_N(s)}{\sigma}
\end{aligned}$$

where $\lim_{b\to\infty} C(b) = 0$

Now we are allowed to apply summation by parts formula (2.5.2) on (2.5.23) using the smoothness of $\boldsymbol{\lambda}$ and $\mathfrak{u}$ and we arrive at

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s)\left[\mathbf{A}_{i,b} - \mathbf{A}\left(\mathfrak{u}(\frac{i}{N}, s)\right) - (D\mathbf{A})^T\left(\mathfrak{u}(\frac{i}{N}, s)\right)\left(\boldsymbol{\zeta}_{i,b} - \mathfrak{u}(\frac{i}{N}, s)\right)\right] \\
=\ & \sum_{i=\frac{k}{2}+1}^{N-\frac{k}{2}} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s) \times \\
&\quad \left[\frac{1}{k+1}\sum_{|l-i|\leq \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}\left(\mathfrak{u}(\frac{i}{N}, s)\right) - (D\mathbf{A})^T\left(\mathfrak{u}(\frac{i}{N}, s)\right)\left(\boldsymbol{\zeta}^k_{i,b} - \mathfrak{u}(\frac{i}{N}, s)\right)\right] \\
&+\mathcal{O}\left(\frac{k}{N}\right)\sum_{i=0}^{N}\left[\mathbf{A}_{i,b} - \boldsymbol{\zeta}_{i,b}\right].
\end{aligned}$$

The last term is of order $\mathcal{O}(k)$ since by the cut off, $|\mathbf{A}_{i,b} - \boldsymbol{\zeta}_{i,b}|$ is bounded. As $N \to \infty$ first and then $k \to \infty$, the last term devided by N converges to 0.

We now introduce further block averages:

For some small $\ell > 0$, such that $\ell \to 0$ after $N \to \infty$ thus $\ell N >> k$, we restrict the sum in the last expression as follows:

$$\begin{aligned}
&\sum_{i=[N\ell]}^{[N(1-\ell)]} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s) \times \\
&\left[\frac{1}{k+1}\sum_{|l-i|\leq \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}\left(\mathfrak{u}(\frac{i}{N}, s)\right) - (D\mathbf{A})^T\left(\mathfrak{u}(\frac{i}{N}, s)\right)\left(\boldsymbol{\zeta}^k_{i,b} - \mathfrak{u}(\frac{i}{N}, s)\right)\right] \\
&\qquad\qquad + C(k + N\ell),
\end{aligned}$$

where C is a constant not depending on N and k. The error we made devided by N will vanish in the limit as well since $\ell \to 0$. Let us denote by $\Lambda_i^{2\varepsilon N+k}$ the box $\{i-\varepsilon N-\frac{k}{2},\dots,i+\varepsilon N+\frac{k}{2}\}$. Performing again a summation by parts over blocks of size $2\varepsilon N$ for small $\varepsilon > \varepsilon > 0$ and $\varepsilon N \to \infty$ as $N \to \infty$, we get

$$\frac{1}{N}\int_0^t \int \sum_{i=[N\ell]}^{[N(1-\ell)]} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)\tau_i \left(\frac{1}{k+1}\sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b}\right) f_s^N|_{\Lambda_i^{2\varepsilon N+k}} d\nu_\star^{N(1-2\ell)+k} ds$$

$$= \frac{1}{N}\int_0^t \sum_{i=[N\ell]}^{[N(1-\ell)]} \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)$$
$$\times \int \frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N} \tau_j \left(\frac{1}{(k+1)}\sum_{|l|\le\frac{k}{2}} \mathbf{A}_{l,b}\right) f_s^N|_{\Lambda_i^{2\varepsilon N+k}} d\nu_\star^{2\varepsilon N+k} ds + C\varepsilon$$
$$= \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]} \int_0^t \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)$$
$$\times \left(\int \frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} \mathbf{A}_{l,b}\frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N} \tau_j f_s^N|_{\Lambda_i^{2\varepsilon N+k}} d\nu_\star^{2\varepsilon N+k}\right) ds + C\varepsilon$$
$$= t\int \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},s)\left(\frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} \mathbf{A}_{l,b}\right)$$
$$\times \left(\frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\frac{1}{t}\int_0^t \frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N} \tau_j f_s^N|_{\Lambda_i^{2\varepsilon N+k}} ds\right) d\nu_\star^{N-2N(\ell-2\varepsilon)+2k+1} + C\varepsilon$$

and also here the prize is small in N because of the cut off.

Let $\hat{\nu}_t^{N,\varepsilon}(d\mathbf{r},d\mathbf{p}) := \hat{f}_t^{N,\varepsilon} d\nu_\star^{N-2N(\ell-2\varepsilon)+2k+1}$ denote the measure corresponding to the density

$$\hat{f}_t^{N,\varepsilon}(\mathbf{r},\mathbf{p}) := \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\frac{1}{t}\int_0^t \frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N} \tau_j f_s^N|_{\Lambda_i^{2\varepsilon N+k}} ds, \tag{2.5.25}$$

which is a function of $(\boldsymbol{\zeta}_{[N\ell]l-2N\varepsilon-k},\dots,\boldsymbol{\zeta}_{[N(1-\ell)]+2N\varepsilon+k})$, then the last expression reads as

$$t\int \frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N},t)\int \frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} \mathbf{A}_{l,b}\hat{f}_{t,i}^{N\varepsilon}(\mathbf{r},\mathbf{p}) d\nu_\star^{N-2N(\ell-2\varepsilon)+2k+1}(d\mathbf{r},d\mathbf{p}).$$

Notice also that

$$\mathbf{A}_b(\boldsymbol{\zeta}_i^k) := \mathbf{A}(\boldsymbol{\zeta}_i^k)\mathbf{1}_{\mathcal{C}_{i,b}} = E_{\nu_{\boldsymbol{\lambda}(\boldsymbol{\zeta}_i^k)}}[\mathbf{A}_i\mathbf{1}_{\mathcal{C}_{i,b}}] = \mathbf{A}(\boldsymbol{\zeta}_{i,b}^k) = E_{\nu_{\boldsymbol{\lambda}(\boldsymbol{\zeta}_{i,b}^k)}}[\mathbf{A}_i] = \mathbf{A}(\boldsymbol{\zeta}_{i,b}^k)$$

The next Theorem will be proved in section 2.5.4. Known as the one-block estimate, it allows to replace the averages of the functions over the microscopic blocks by a function of the average and it is a crucial step towards the proof of the hydrodynamic limit:

Theorem 2.5.9 (The one-block estimate). *Let $J : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^2$ be a function with continuous first derivative. Then*

$$\lim_{b\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} \int J(\frac{i}{N}, t) \left| \frac{1}{k+1} \sum_{|l|\leq \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k\right) \right| \hat{f}_t^{N,\varepsilon} \nu_\star^N(d\mathbf{r}, d\mathbf{p}) = 0 \qquad (2.5.26)$$

Observe that the blocks appearing here are of size of order k with $k \to \infty$ after $N \to \infty$. Since by Lemma 2.5.8 the integral is uniformly bounded we are allowed to neglect a finite number of blocks near the boundaries.

With this Theorem we obtain:

$$\lim_{N\to\infty} H_N(t) \leq \lim_{k\to\infty} \lim_{N\to\infty} \Big(\int_0^t R_N(s) ds + \int_0^t \frac{H_N(s)}{\sigma'} ds$$
$$\sum_{i=[N\ell]}^{[N(1-\ell)]} \int_0^t \int \left[\frac{\partial \boldsymbol{\lambda}}{\partial x}(\frac{i}{N}, s) \cdot \boldsymbol{\omega}(\boldsymbol{\zeta}_i^k, \mathfrak{u}(\frac{i}{N}, s)) - \frac{\partial \boldsymbol{\lambda}}{\partial s}(\frac{i}{N}, s) \cdot \mathfrak{w}(\boldsymbol{\zeta}_i^k, \mathfrak{u}(\frac{i}{N}, s)) \right] f_s^N d\nu_\star^N ds \Big)$$
$$(2.5.27)$$

Here the entropy term came from the introduction of the cut off functions and we defined

$$\boldsymbol{\omega}(\mathfrak{z}, \mathfrak{u}) := \mathbf{A}(\mathfrak{z}) - \mathbf{A}(\mathfrak{u}) \quad \text{and} \quad \mathfrak{w}(\mathfrak{z}, \mathfrak{u}) := (\mathfrak{z} - \mathfrak{u}).$$

To simplify the notation further, let $\boldsymbol{\Omega}$ be as follows:

$$\boldsymbol{\Omega}(\mathfrak{z}, \mathfrak{u}) := \frac{\partial \boldsymbol{\lambda}}{\partial x} \cdot \boldsymbol{\omega}(\mathfrak{z}, \mathfrak{u}) - \frac{\partial \boldsymbol{\lambda}}{\partial s} \cdot \mathfrak{w}(\mathfrak{z}, \mathfrak{u})$$

Notice that

$$D_{\mathfrak{z}} \boldsymbol{\Omega}(\mathfrak{z}, \mathfrak{u}) = (D\mathbf{A})^T(\mathfrak{z}) \cdot \frac{\partial \boldsymbol{\lambda}}{\partial x} - \frac{\partial \boldsymbol{\lambda}}{\partial s} \qquad (2.5.28)$$

is equal to zero if $\mathfrak{z}$ is a solution of (2.5.7). Consequently some properties of $\boldsymbol{\Omega}$ are:

($\boldsymbol{\Omega}$1): $\boldsymbol{\Omega}(\mathfrak{u}, \mathfrak{u}) = 0$,

($\boldsymbol{\Omega}$1): $D_{\mathfrak{z}} \boldsymbol{\Omega}(\mathfrak{u}, \mathfrak{u}) = 0$.

Lets go back to the right hand side of expression (2.5.27). Rewritten in terms of $\boldsymbol{\Omega}$ the sum is equal to

$$\int_0^t \int \sum_{i=[N\ell]}^{[N(1-\ell)]} \left(\boldsymbol{\Omega}(\boldsymbol{\zeta}_i^k, \mathfrak{u}(\frac{i}{N}, s)) \right) f_s^N \nu_\star^N(d\mathbf{r}, d\mathbf{p}) ds$$

Applying the entropy inequality (2.5.10) again, we obtain that for $\sigma > 0$ this is bounded above by

$$\frac{1}{\sigma} \int_0^t \log \int \exp \left\{ \sigma \sum_{i=[N\ell]}^{[N(1-\ell)]} \boldsymbol{\Omega}(\boldsymbol{\zeta}_i^k, \mathfrak{u}(\frac{i}{N}, s)) \right\} g_{\mathfrak{u}(\cdot,t)}^N d\nu_\star^N ds + \frac{1}{\sigma} \int_0^t H_N(s) ds. \qquad (2.5.29)$$

It thus remains to prove, that the first term of this expression is of order $o(N)$.

In section 2.5.5 we will prove the following special case of Varadhan's Lemma:

Theorem 2.5.10 (Varadhan's Lemma). *Let $\nu^n_{\boldsymbol{\lambda}}$ be the product homogenuous measure with marginals $\nu_{\boldsymbol{\lambda}}$ given by* (2.3.3) *and with rate function $I : \Omega \to \mathbb{R}$ defined in Appendix A.*

Then for any bounded continuous function on Ω

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{nF(\boldsymbol{\zeta})} d\nu^n_{\boldsymbol{\lambda}} = \sup_{\mathbf{x}} \{F(\mathbf{x}) - I(\mathbf{x})\}$$

To apply this theorem to (2.5.29), we need the homogenuous product measures to get involved. In order to do this it is better to arrange the sum as sums over disjoint blocks: By the same procedure as in *Step III* of the proof of Corollary 2.5.26, we can rewrite the first term of (2.5.29) as follows:

$$\frac{1}{\sigma} \int_0^t \log \int \exp\left\{ \sigma \sum_{j\in\{-\frac{k}{2},\dots,\frac{k}{2}\}} \sum_{i\in B_{[N\ell]}} \tau_j \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s)) \right\} g^N_{\mathfrak{u}(\cdot,s)} d\nu^N_\star ds. \tag{2.5.30}$$

where we assume without loss of generality, that k devides $[N(1-2\ell)]+1$ and $B_{[N\ell]}$ is the set $B_{[N\ell]} := \{q(k+1) + [N\ell] : q = 0, \dots, \frac{[N(1-2\ell)]+1}{k+1} - 1\}$. By τ_r we denoted the spacial shift on the configurations by r.
For a fixed $j \in \{-\frac{k}{2}, \dots, \frac{k}{2}\}$, the sum over $i \in B_{[N\ell]}$ is a sum over disjoint blocks and thus the random variables

$$\tau_j \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s))$$

are independent under $g^N_{\mathfrak{u}(\cdot,s)}$ which is product. Therefore, if we apply the Hölder inequality we obtain

$$\begin{aligned} & \frac{1}{\sigma} \int_0^t \log \int \prod_{j\in\{-\frac{k}{2},\dots,\frac{k}{2}\}} \exp\left\{ \sigma \sum_{i\in B_{[N\ell]}} \tau_j \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s)) \right\} g^N_{\mathfrak{u}(\cdot,s)} d\nu^N_\star ds \\ \leq\; & \frac{1}{\sigma(k+1)} \int_0^t \sum_{j\in\{-\frac{k}{2},\dots,\frac{k}{2}\}} \log \int \exp\left\{ \sigma(k+1) \sum_{i\in B_{[N\ell]}} \tau_j \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s)) \right\} g^N_{\mathfrak{u}(\cdot,s)} d\nu^N_\star ds \\ =\; & \frac{1}{\sigma(k+1)} \sum_{i=[N\ell]}^{[N(1-\ell)]} \int_0^t \log \int \exp\left\{ \sigma(k+1) \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s)) \right\} g^N_{\mathfrak{u}(\cdot,s)}|_{\{-\frac{k}{2},\dots,\frac{k}{2}\}} d\nu^{k+1}_\star ds. \end{aligned}$$

Where the last equality is true by the independence of the random variables in the exponent.

Then, since all the functions in this expression are smooth and the family of local Gibbs measures converges weakly, we obtain that

$$\begin{aligned} & \lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{(k+1)N\sigma} \sum_{i=[N\ell]}^{[N(1-\ell)]} \int_0^t \log \int \exp\left\{ \sigma(k+1) \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(\frac{i}{N}, s)) \right\} g^N_{\mathfrak{u}(\cdot,s)} d\nu^k_\star ds \\ & \quad = \lim_{k\to\infty} \frac{1}{\sigma(k+1)} \int_0^t \int_0^1 \log \int \exp\left\{ (k+1)\sigma \boldsymbol{\Omega}(\boldsymbol{\zeta}^k_i, \mathfrak{u}(x, s)) \right\} g^k_{\mathfrak{u}(x,s)} d\nu^k_\star dx ds. \end{aligned}$$

So now for each $x \in [0,1]$, the distribution of the particles in a box of size k is given by the invariant Gibbs measure with average $\mathfrak{u}(x,s)$. Then we can apply Theorem 2.5.10 on this product measure and obtain that the last expression is equal to

$$\frac{1}{\sigma}\int_0^t \int_0^1 \sup_{\mathfrak{z}}\{\sigma \mathbf{\Omega}\left(\mathfrak{z}, \mathfrak{u}(x,s)\right) - I(\mathfrak{z})\}dx. \tag{2.5.31}$$

To prove Theorem 2.5.2 it thus remains to show that this is equal to zero. Since I and $\mathbf{\Omega}$ are both convex, since both functions and their derivatives are vanishing at $\mathfrak{z} = \mathfrak{u}$, it follows from assumption (2.1.1) on the potential that $\sigma\mathbf{\Omega}(\mathfrak{z}, \mathfrak{u}) \leq I(\mathfrak{z})$ for σ small enough. Hence there exists a σ such that the last expression is equal to zero.

This concludes the proof of Theorem 2.5.2 since now we have proved that:

$$\frac{1}{N}H_N(t) \leq \frac{1}{\sigma}\int_0^t \frac{H_N(s)}{N}ds + \int_0^t \frac{R_N(s)}{N}ds.$$

Then the claim follows by the Gronwall inequality since the second term in the right and side vanishes if $N \to \infty$.

2.5.3 The one block estimate

Tightness

For a fixed $k > 0$ let

$$\hat{\nu}_{t,i}^{N,\varepsilon,k}(d\mathbf{r}, d\mathbf{p}) := \hat{f}_{t,i}^{N,\varepsilon,k} \prod_{i=l-\frac{k}{2}-1}^{l+\frac{k}{2}} d\nu_\star(r_l, p_l)$$

be the projection on the configurations in a block of size k around site i of the measure $\hat{\nu}_t^{N,\varepsilon}$ whose density is given by (2.5.25). Thus the density corresponding to $\hat{\nu}_{t,i}^{N,\varepsilon,k}$ is given by

$$\hat{f}_{t,i}^{N,\varepsilon,k} := \hat{f}_t^{N,\varepsilon}|_{\{i-\frac{k}{2}-1,\dots,i+\frac{k}{2}\}}. \tag{2.5.32}$$

We have the following

Lemma 2.5.11 (Tightness). *For each $k \geq 2$ fixed , the sequence $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N\geq 1}$ of probability measures is tight.*

Proof. We need to prove that for each $n > 0$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \hat{\nu}_{t,i}^{N,\varepsilon,k}\{\frac{1}{k+1}\sum_{|i-l|\leq \frac{k}{2}} |h_l| > n\} = 0 \tag{2.5.33}$$

Then, by assumption (2.1.1) on the potential, the tightness of the sequence $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N\geq 1}$ follows. Notice that by the Markov inequality

$$\hat{\nu}_{t,i}^{N,\varepsilon,k}\{\frac{1}{k+1}\sum_{|i-l|\leq \frac{k}{2}} |h_l| > n\} \leq \frac{E_{\hat{\nu}_{t,i}^{N,\varepsilon,k}}\left[\frac{1}{k+1}\sum_{|i-l|\leq\frac{k}{2}} h_l\right]}{n}$$

it suffices to prove that the expectations are bounded by a finite constant C independent of N (and n).

Since the expectation in the last expression depends on configurations only through $l \in \{i-\frac{k}{2},\ldots,i+\frac{k}{2}\}$, we can write

$$\int \frac{1}{k+1}\sum_{|i-l|\le\frac{k}{2}} h_l \hat{f}^{N,\varepsilon,k}_{t,i}\, d\nu_\star^{k+1} = \int \frac{1}{k+1}\sum_{|i-l|\le\frac{k}{2}} h_l \hat{f}^{N,\varepsilon}_{t}\, d\nu_\star^{k+1}. \tag{2.5.34}$$

In the following we define by $\bar{\nu}_t^N := \bar{f}_t^N \prod_{i=1}^N d\nu_\star(r_i,p_i)$ the probability measure with density $\bar{f}_t^N$ defined by the time average

$$\bar{f}_t^N(d\mathbf{r},d\mathbf{p}) := \frac{1}{t}\int_0^t f_s^N ds \tag{2.5.35}$$

Using the definition (2.5.25) of $\hat{f}_t^{N,\varepsilon}$, the right hand side of (2.5.34) is equal to

$$\begin{aligned}
&\int \left(\frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} h_i\right)\left(\frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N}\tau_j \bar{f}_t^N|_{\Lambda_i^{2\varepsilon N+k}}\right) d\nu_\star^{N-2N(\ell-2\varepsilon)+2k+1}\\
=\;& \int \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N}\tau_j\left(\frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} h_i\right)\bar{f}_t^N|_{\Lambda_i^{2\varepsilon N+k}} d\nu_\star^{N-2N(\ell-2\varepsilon)+2k+1}\\
=\;& \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\int \frac{1}{2\varepsilon N}\sum_{|j-i|\le\varepsilon N}\tau_j\left(\frac{1}{k+1}\sum_{|l|\le\frac{k}{2}} h_i\right)\bar{f}_t^N d\nu_\star^N\\
\le\;& C_1\int\Big(\frac{1}{N}\sum_{i=[N(l-\varepsilon)]-\frac{k}{2}}^{[N(1-l+\varepsilon)]+\frac{k}{2}} h_i\Big)\bar{f}_t^N d\nu_\star^N
\end{aligned}$$

where C_1 is a constant independent of N. This inequality is true, since the last expession averages out all the h_i from for $i\in\{[N(\ell-\varepsilon)]-\frac{k}{2}],\ldots,[N(1-\ell+\varepsilon)]+\frac{k}{2}]\}$. Then by entropy inequality (2.5.10), choosing as reference measure the Gibbs equilibrium measure $\nu^N_{(\tau,0)}$, for any $\sigma>0$

$$\frac{1}{N}\int\sum_{i=[N(l-\varepsilon)]-\frac{k}{2}}^{[N(1-l+\varepsilon)]+\frac{k}{2}} h_i\bar{f}_t^N d\nu_\star^N \le \frac{1}{\sigma N}\left(\log\int e^{\sigma\sum_{i=1}^N|h_l|}d\nu^N_{(\tau,0)} + H_N(\bar{\nu}_t^N|\nu^N_{(\tau,0)})\right).$$

The first term is equal to

$$\frac{1}{\sigma N}\log\int\prod_{i=[N(l-\varepsilon)]-\frac{k}{2}}^{[N(1-l+\varepsilon)]+\frac{k}{2}}\frac{1}{Z(\tau,0)}e^{\tau r_l-(1-\sigma)h_l}dr_i dp_i$$

and for $\sigma<1$ is therefore bounded by $C_2\frac{1}{N}([N(1-2l+2\varepsilon)]+k)$ with a constant C_2 not depending on N by assumption (2.3.4). Hence this term converges to 0 since $N\to\infty$ first, and ε and ℓ go to 0 after k. On the other hand, by Lemma (2.5.7), the entropy $H_N(\bar{\nu}_t^N|\nu^N_{(\tau,0)})$ is bounded above by C_3N for a uniform constant in N, and thus (2.5.33) follows.

□

Lemma 2.5.11 asserts that for each fixed $k \geq 2$ and each fixed i there exists a limit point $\nu_{t,i}^k$ of the sequence $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N\geq 1}$. On the other hand, since the sequence $(\nu_{t,i}^k)_{k\geq 2}$ forms a consistent family of measures, by Kolmogorov's Theorem, for $k \to \infty$, there exists a unique probability measure $\nu_{t,i}$ on the configuration space $\{(\boldsymbol{\zeta}_i)_{i\in\mathbb{Z}} \in \Omega^\infty\}$, such that the restriction of $\nu_{t,i}$ on $\{(\boldsymbol{\zeta}_j)_{j\in\{i-\frac{k}{2}\ldots,i+\frac{k}{2}]} \in \Omega^{k+1}\}$ is $\nu_{t,i}^k$.

Proof of the one block estimate

Let us define the formal generator $\mathcal{A}$ of the infinite dynamics by

$$\mathcal{A} := \mathcal{L} + \gamma \mathcal{S}, \tag{2.5.36}$$

with the antisymmetric part

$$\mathcal{L} := \sum_{j\in\mathbb{Z}} \left\{ p_j \left(\frac{\partial}{\partial r_j} - \frac{\partial}{\partial r_{j+1}} \right) + (V'(r_{j+1}) - V'(r_j)) \frac{\partial}{\partial p_j} \right\} \tag{2.5.37}$$

and the symmetric part

$$\mathcal{S} := \frac{1}{2} \sum_{j\in\mathbb{Z}} \left\{ \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_{j-1}} \right)^2 - (p_j - p_{j-1}) \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_{j-1}} \right) \right\}. \tag{2.5.38}$$

In section 2.5.4 we will prove the following ergodic Theorem:

Theorem 2.5.12 (Ergodicity). *Any limit point ν of $\hat{\nu}_{t,i}^{N,k}(d\mathbf{r}, d\mathbf{p})$ is a convex combination of Gibbs i.e there exists a probability measure $\alpha(d\boldsymbol{\lambda})$ on $\mathbb{R}^2$ such that*

$$\nu(d\mathbf{r}, d\mathbf{p}) = \int \alpha(d\boldsymbol{\lambda}) \prod_{i\in\mathbb{Z}} \frac{1}{Z(\boldsymbol{\lambda})} e^{\lambda_1 r_i + \lambda_2 p_i - h_i} dr_i dp_i.$$

With Lemma 2.5.11 and Theorem 2.5.12 it will turn out, that 2.5.9 is an application of the law of large numbers:

Proof of Theorem 2.5.9:

Since J is a bounded function the left hand side of (2.5.26) is bounded above by

$$\lim_{b\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{N} \|J\|_\infty \int \left| \frac{1}{k+1} \sum_{|l|\leq \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k \right) \right| \hat{f}_t^{N,\varepsilon} \nu_\star^N (d\mathbf{r}, d\mathbf{p})$$

and thus it is enough to prove that for each i

$$\lim_{b\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} \int \left| \frac{1}{k+1} \sum_{|l|\leq \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k \right) \right| \hat{f}_t^{N,\varepsilon} \nu_\star^N (d\mathbf{r}, d\mathbf{p}) = 0 \tag{2.5.39}$$

$\hat{\nu}_{t,i}^{N,\varepsilon}$ can be replaced by $\hat{\nu}_{t,i}^{N,\varepsilon,k}$ since the configurations inside the integral depend on the configurations only through $i - \frac{k}{2} - 1, \ldots, i + \frac{k}{2}$.

$$\lim_{b\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} \int \left| \frac{1}{k+1} \sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k\right) \right| \hat{\nu}_{t,i}^{N,\varepsilon,k}(d\mathbf{r}, d\mathbf{p}) \le 0.$$

By Lemma 2.5.11 there exists a limit point ν_i^k of the sequence $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N\ge 1}$ and in particular there exists a unique probability measure ν, such that $\nu_i^k = \nu|_{\{i-\frac{k}{2}-1,\ldots,i+\frac{k}{2}\}}$, where ν is a limit point of the sequence ν_i^k. Hence the left hand side of the last expression can be rewritten as

$$\lim_{b\to\infty} \lim_{k\to\infty} \int \left| \frac{1}{k+1} \sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k\right) \right| d\nu^k = \lim_{b\to\infty} \lim_{k\to\infty} \int \left| \frac{1}{k+1} \sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b \left(\boldsymbol{\zeta}_i^k\right) \right| d\nu$$

By Theorem 2.5.12 ν is a convex combination of Gibbs measures. Taking the conditional expectation with respect to ζ_i^k we are left to prove

$$\lim_{b\to\infty} \lim_{k\to\infty} \int_{\mathbb{R}^2} \left(\int \left| \frac{1}{k+1} \sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b\left(\mathbf{z}\right) \right| \prod_{i\in\mathbb{Z}} e^{\boldsymbol{\lambda}(\mathbf{z}) - h_i} dr_i dp_i \right) \tilde{\alpha}(d\boldsymbol{\lambda}) = 0.$$

Because of the cut off the expression inside the integral is bounded hand hence applying the dominated convergence Theorem it follows that it is enough to prove

$$\lim_{k\to\infty} \int \left| \frac{1}{k+1} \sum_{|l|\le \frac{k}{2}} \mathbf{A}_{l,b} - \mathbf{A}_b\left(\mathbf{z}\right) \right| \prod_{i\in\mathbb{Z}} e^{\boldsymbol{\lambda}(\mathbf{z}) - h_i} dr_i dp_i = 0.$$

But this is just the law of large numbers and converges to 0 if $k \to \infty$.

□

The translation invariant stationary measures

In view of Theorem 2.5.12, we first identify the limit point ν of the probability measure $\hat{\nu}_{t,i}^{N,\varepsilon,k}$ as translation invariant and stationary measure with respect to $\mathcal{A}$ in the limit as k goes to infinity after N.

We start with some notations:

- By $\nu_{t,i}^{N,n} := f_{t,i}^{N,n} \prod_{|i-l|\le \frac{n}{2}} d\nu_\star(r_l, p_l)$ we denote the reduction of the measure ν_t^N to the box $\{i - \frac{n}{2} - 1, \ldots, i + \frac{n}{2}\}$ around site i and of size n. Thus its density $f_{t,i}^{N,n}$ is given by
$$f_{t,i}^{N,n} := f_t^N|_{\{i-\frac{n}{2}-1,\ldots,i+\frac{n}{2}\}}$$
- By $\bar{\nu}_t^N := \bar{f}_t^N \prod_{i=1}^N d\nu_\star(r_i, p_i)$ we denote the measure with density $\bar{f}_t^N$ defined by the time average
$$\bar{f}_t^N := \frac{1}{t} \int f_s^N ds.$$

and by $\bar{\nu}_{t,i}^{N,n} := \bar{f}_{t,i}^{N,n} \prod_{|i-l|\le \frac{n}{2}} d\nu_\star(r_l,p_l)$ we denote its reduction to a box around site i and of size n.Then its density $\bar{f}_{t,i}^{N,n}$ is defined by

$$\bar{f}_{t,i}^{N,n} := \frac{1}{t}\int f_{s,i}^{N,n} ds.$$

- Finally we define the density

$$\tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k}(\mathbf{r},\mathbf{p}) := \frac{1}{\varepsilon N}\sum_{|j-i|\le\varepsilon N} \tau_j \bar{f}_{t,i}^{N,2\varepsilon N+k}(\boldsymbol{\zeta}_{i-\varepsilon N-\frac{k}{2}},\dots,\boldsymbol{\zeta}_{i+\varepsilon N+\frac{k}{2}}).$$

Then with the notations above the $\hat{f}_{t,i}^{N,\varepsilon}$ reads as:

$$\hat{f}_t^{N,\varepsilon}(\mathbf{r},\mathbf{p}) = \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]} \tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k}(d\mathbf{r},d\mathbf{p})$$

With the introduction of the local averages it is easy to show that the limit points $\nu_{t,i}^k$ and $\nu_{t,i}$ are translation invariant in space:

Lemma 2.5.13. *Let*

$$\lim_{k\to\infty}\lim_{N\to\infty}\int \psi(\mathbf{r},\mathbf{p})d\hat{\nu}_{t,i}^{N,\varepsilon,k} = \lim_{k\to\infty}\int \psi(\mathbf{r},\mathbf{p})d\nu_{t,i}^k = \int \psi(\mathbf{r},\mathbf{p})\nu_t.$$

Then the measure ν_t and $\nu_{t,i}^k$ are translation invariant in space and we write $\nu_{t,i}^k := \nu_t$

Proof. Let ψ be a local function depending on configurations only through $\{i-\frac{k}{2},\dots,i+\frac{k}{2}\}$. Since $(\hat{f}_{t,i}^{N,\varepsilon,k})_N$ is tight, we only need to prove that for each z we have

$$\lim_{N\to\infty}\int(\psi-\tau_z\psi)\hat{f}_{t,i}^{N,\varepsilon,k}\nu_\star(d\mathbf{r},d\mathbf{p}) = 0$$

For a fixed i the integral is equal to:

$$\begin{aligned}\int(\psi-\tau_z\psi)\hat{f}_{t,i}^{N,\varepsilon,k}\nu_\star(d\mathbf{r},d\mathbf{p}) &= \int(\psi-\tau_z\psi)\hat{f}_{t}^{N,\varepsilon}\nu_\star(d\mathbf{r},d\mathbf{p}) \\ &= \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]}\int(\tau_i\psi-\tau_{z+i}\psi)\tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k}(d\mathbf{r},d\mathbf{p}).\end{aligned}$$

Then since

$$\begin{aligned}\int(\tau\psi-\tau_z\psi)\tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k}\nu_\star(d\mathbf{r},d\mathbf{p}) &= \frac{1}{2\varepsilon N}\int(\psi-\tau_z\psi)\sum_{|i-j|\le\varepsilon N}\bar{f}_{t,i+j}^{N,\varepsilon,2\varepsilon N+k}\nu_\star(d\mathbf{r},d\mathbf{p}) \\ &= \frac{1}{2\varepsilon N}\int\sum_{|j|\le\varepsilon N}(\tau_j\psi-\tau_{j+z}\psi)\bar{f}_{t,i}^{N,\varepsilon,2\varepsilon N+k}\nu_\star(d\mathbf{r},d\mathbf{p}) \\ &= \frac{1}{2\varepsilon N}\int\left(\sum_{|j|\le N\varepsilon}\tau_j\psi-\sum_{|j-z|\le N\varepsilon}\tau_j\psi\right)\bar{f}_t^N\nu_\star(d\mathbf{r},d\mathbf{p}) \\ &= \mathcal{O}\left(\frac{z}{\varepsilon N}\right)\end{aligned}$$

converges to 0 as N approaches ∞, ν_t and thereby its projection ν_t^k are translation invariant. □

Next we prove that ν_t is invariant with respect to the formal generator $\mathcal{A}$ defined by (2.5.36):

Lemma 2.5.14. *For some fixed i and any local smooth bounded function ψ, let*

$$\lim_{k\to\infty}\lim_{N\to\infty}\int \psi(\mathbf{r},\mathbf{p})d\hat{\nu}_{t,i}^{N,\varepsilon,k} = \int \psi(\mathbf{r},\mathbf{p})\nu_t.$$

Then the measure ν_t and ν_t^k are stationary in time with respect to the generator $\mathcal{A} = \mathcal{L}+\gamma\mathcal{S}$, that means for any bounded smooth local function $\psi(\mathbf{r},\mathbf{p})$

$$\int \mathcal{A}\psi \, d\nu_t = 0. \tag{2.5.40}$$

and we write $\nu_t := \nu$, and $\nu_t^k := \nu^k$

Proof. Since ψ is a local function there exists some $k \in \mathbb{N}$ and some i, such that ψ depends on the configurations (r_j, p_j) only through $j \in \{i - \frac{k}{2}, \dots, i + \frac{k}{2}\}$. Define $\psi := \psi_i$. Proving (2.5.40) is equivalent to prove

$$\int \mathcal{A}\psi_i \, d\nu_t^k = 0 \tag{2.5.41}$$

where ν_t^k is a limit point of the sequence $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_N$. Recall that

$$\begin{aligned}\int \mathcal{A}\psi_i \, d\hat{\nu}_{t,i}^{N,\varepsilon,k} &= \int (\mathcal{A}\psi_i)\hat{f}_{t,i}^{N,\varepsilon,k} d\nu_\star(\mathbf{r},\mathbf{p}) \\ &= \int (\mathcal{A}\psi_i)\hat{f}_t^{N,\varepsilon} d\nu_\star(\mathbf{r},\mathbf{p}).\end{aligned}$$

Therefore, by the same arguments as in the proof of Lemma (2.5.13) it is enough to show that

$$\lim_{N\to\infty}\frac{1}{2tN\varepsilon}\int\int_0^t \sum_{|j|\le N\varepsilon} \tau_j(\mathcal{A}\psi_i) f_{s,i}^{N,2\varepsilon N+k} d\nu_\star(\mathbf{r},\mathbf{p})ds = 0. \tag{2.5.42}$$

Now define the spacial average $\bar{\psi} := \frac{1}{2tN\varepsilon}\sum_{|j|\le N\varepsilon}\tau_j\psi_i$, then with

$$\mathcal{A}\bar{\psi} = \frac{1}{2tN\varepsilon}\sum_{|j|\le N\varepsilon}\tau_j(\mathcal{A}\psi_i),$$

we can rewrite the integral of (2.5.42) as

$$\begin{aligned}\frac{1}{t}\int\int_0^t (\mathcal{A}\bar{\psi}_i) f_{s,i}^{N,2\varepsilon N+k} d\nu_\star(\mathbf{r},\mathbf{p})ds &= \frac{1}{t}\int\int_0^t (\mathcal{A}\bar{\psi}) f_s^N d\nu_\star(\mathbf{r},\mathbf{p})ds \\ &= \frac{1}{tN}\int_0^t E_{\nu_s^N}\left[\frac{\partial\bar{\psi}}{\partial s}\right] ds.\end{aligned}$$

Then by Itô's formula the right hand side is equal to

$$\frac{1}{Nt}\left\{E_{\nu_t^N}\left[\bar{\psi}\right] - E_{\nu_0^N}\left[\bar{\psi}\right]\right\}.$$

We conclude the proof by observing that this expression converges to 0 if $N \to \infty$, since ψ and hence $\bar{\psi}$ is a bounded function. □

2.5.4 Ergodicity

It remains to show the ergodic Theorem (2.5.12) of the infinite volume dynamics. We begin the section with a definition of what ergodicity means for the infinite stochastic system:

Definition 2.5.15 (Ergodicity). *The infinite stochastic dynamics defined through the generator $\mathcal{A}$ is said to be ergodic if any measure μ on the configuration space Ω^∞ that*

(i) has finite density entropy: there exists a constant $C > 0$ such that for all subsets $\Lambda \subset \mathbb{Z}$

$$H_{|\Lambda|}\left(\mu|_\Lambda \middle| \nu^{|\Lambda|}_{(\tau,0)}\right) \leq C|\Lambda|,$$

(ii) is translation invariant: For any local function ψ and any $j \in \mathbb{Z}$,

$$\int \psi \, d\mu = \int (\tau_j \psi) \, d\mu$$

where τ_j denotes the spatial shift by j on the configurations.

(iii) is stationary with respect to the operator $\mathcal{A}$: For any smooth bounded local function ψ

$$\int (\mathcal{A}\psi) d\mu = 0.$$

is a convex combination of Gibbs measures, i.e. there exists a probability measure $\alpha(d\boldsymbol{\lambda})$ on $\mathbb{R} \times \mathbb{R}$ such that

$$\nu(d\mathbf{r}, d\mathbf{p}) = \int \alpha(d\boldsymbol{\lambda}) \prod_{i \in \mathbb{Z}} \frac{1}{Z(\boldsymbol{\lambda})} e^{\lambda_1 r_i + \lambda_2 p_i - h_i} dr_i dp_i$$

The Entropy density

From section 2.5.3 we know that properties (*ii*) and (*iii*) for the limit point ν of $\hat{\nu}^{N,\varepsilon,k}_{t,i}$ of the Definition 2.5.15 are satisfied. In this section we prove that ν has finite entropy density

Lemma 2.5.16. *The limit point ν of the sequence $(\nu^k)_{k \geq 2}$ has finite entropy density, that means there exists a constant $C > 0$ such that for all subsets $\Lambda_i^k := \{i - \frac{k}{2}, \dots, i + \frac{k}{2}\} \subset \mathbb{Z}$*

$$H_{\Lambda_i^k}(\nu | \nu^\infty_{(\tau,0)}) \leq C|\Lambda_i^k|.$$

where we define $\nu^\infty_{(\tau,0)}(d\mathbf{r}, d\mathbf{p}) := \prod_{i \in \mathbb{Z}} \nu_{(\tau,0)}(dr_i, dp_i)$ and

$$H_{\Lambda_i^k}(\nu | \nu^\infty_{(\tau,0)}) := H(\nu^k | \nu^k_{(\tau,0)}).$$

In particular there exists the limit

$$\bar{H}(\nu | \nu^\infty_{0,\tau}) = \lim_{k \to \infty} \sup_i \frac{1}{k+1} H_{\Lambda_i^k}(\nu | \nu^\infty_{(\tau,0)}) = \sup_k H_k(\nu | \nu^\infty_{(\tau,0)}).$$

Proof. By Lemma 2.5.11 the sequence $(\hat{\nu}_{t,i}^{N,\epsilon,k})_N$ is tight. By the lower semicontinuity of the relative entropy we then have

$$\lim_{N\to\infty} H_k\left(\hat{\nu}_{t,i}^{N,\varepsilon,k}\middle|\nu_{(\tau,0)}^k\right) = H_k\left(\nu_i^k\middle|\nu_{(\tau,0)}^k\right),$$

where for each i, the limit point ν_i^k is the restriction of ν to the box Λ_i^k. Consequently $\nu_i^k = \nu|_{\Lambda_i^k}$ is translation invariant since by Lemma 2.5.13 ν is translation invariant. Hence

$$H_k\left(\nu_i^k\middle|\nu_{(\tau,0)}^k\right) = H_k\left(\nu|_{\Lambda_i^k}\middle|\nu_{(\tau,0)}^k\right) = H_{\Lambda_i^k}\left(\nu\middle|\nu_{(\tau,0)}\right) = H_k\left(\nu\middle|\nu_{(\tau,0)}\right).$$

We will prove in 2.5.17 that H_N is superadditive in the following sense:

$$H_N(\hat{\nu}_t^{N,\varepsilon}|\nu_{(\tau,0)}^{N-2N(\ell-2\varepsilon)+2k+1}) \geq \frac{N-2N(\ell-2\varepsilon)+2k+1}{k} H_k\left(\nu\middle|\nu_{(\tau,0)}\right).$$

On the other hand by the convexity of H and Lemma 2.5.7 we have

$$H_N(\hat{\nu}_t^{N,\varepsilon}|\nu_{(\tau,0)}) \leq \frac{1}{N}\sum_{i=[N\ell]}^{[N(1-\ell)]} \frac{1}{\varepsilon N}\sum_{|j-i|\leq\varepsilon N} H_{\Lambda_{i+j}^{2\varepsilon N}}(\bar{f}_t^N|\nu_{(\tau,0)}) \leq CN$$

By these two results the Lemma follows since ε and ℓ are sent to zero after $k\to\infty$. □

To complete the proof of Lemma 2.5.16 it remains to show the superadditivity.

Lemma 2.5.17 (Superadditivity). *The relative entropy of any measure μ^N on Ω^N with respect to the measure $\nu_{(\tau,0)}^N$ is superadditive in the sense that for each $j\in\{0,\dots,k-1\}$*

$$H_N(\mu^N|\nu_{(\tau,0)}^N) \geq \sum_{i=-\frac{j}{k}}^{\frac{N-j}{k}-1} H_{k+1}\left(\mu^N|_{\{ki+j+1,\dots,k(i+1)+j\}}\middle|\nu_{(\tau,0)}^{k+1}\right).$$

Proof. To prove the Lemma we assume without any loss of generality that $j=0$ and we consider arbitrary continuous and bounded functions ψ_{ki} for $i\in\{0,\dots,\frac{N}{k}-1\}$, depending on the configurations only through sites in $\{ki+1,\dots,k(i+1)\}$. Then we can write for each fixed i

$$\int \psi_{ki}d\mu^N|_{\{ki+1,\dots,k(i+1)\}} - \log\left(\int e^{\psi_{ki}}d\nu_{(\tau,0)}^k\right) = \int \psi_{ki}d\mu^N - \log\left(\int e^{\psi_{ki}}d\nu_{(\tau,0)}^N\right)$$

Summing up this over all i and using that $\nu_{(\tau,0)}^N$ is a product measure and thereby the fact that the ψ_{ki} are independent under $\nu_{(\tau,0)}^N$, we obtain by the definition of the relative entropy

$$\begin{aligned}
&\sum_{i=0}^{\frac{N}{k}-1}\left[\int \psi_{ki}d\mu^N|_{\{ki+1,\dots,k(i+1)\}} - \log\left(\int e^{\psi_{ki}}d\nu_{(\tau,0)}^k\right)\right]\\
=\;& \int\sum_{i=0}^{\frac{N}{k}-1}\psi_{ki}d\mu^N - \log\prod_{i=0}^{\frac{N}{k}-1}\left(\int e^{\psi_{ki}}d\nu_{(\tau,0)}^N\right)\\
=\;& \int\sum_{i=0}^{\frac{N}{k}-1}\psi_{ki}d\mu^N - \log\left(\int e^{\sum_{i=0}^{\frac{N}{k}-1}\psi_{ki}}d\nu_{(\tau,0)}^N\right) \leq H_N(\mu^N|\nu_{(\tau,0)}^N).
\end{aligned}$$

Since this is true for any function $\sum_{i=0}^{\frac{N}{k}-1} \psi_{ki}$, we can chose the ψ_{ki} such that

$$\begin{aligned}
&\sum_{i=0}^{\frac{N}{k}-1} \left[\int \psi_{ki} d\mu^N|_{\{ki+1,\dots,k(i+1)\}} - \log\left(\int e^{\psi_{ki}} d\nu^k_{(\tau,0)} \right) \right] \\
=& \sup_{(\phi_0,\dots,\phi_{\frac{N}{k}-1})} \sum_{i=0}^{\frac{N}{k}-1} \left[\int \phi_{ki} d\mu^N|_{\{ki+1,\dots,k(i+1)\}} - \log\left(\int e^{\phi_{ki}} d\nu^k_{(\tau,0)} \right) \right] \\
\geq& \sum_{i=0}^{\frac{N}{k}-1} \sup_{\phi_{ki}} \left[\int \phi_{ki} d\mu^N_t|_{\{ki+1,\dots,k(i+1)\}} - \log\left(\int e^{\phi_{ki}} d\nu^k_{(\tau,0)} \right) \right] \\
=& \sum_{i=0}^{\frac{N}{k}-1} H_k\left(\mu^N|_{\{ki+1,\dots,k(i+1)\}} | \nu^k_{(\tau,0)} \right).
\end{aligned}$$

□

The convex combinations of Gibbs measures

It remains to prove that ν is a convex combination of Gibbs measures.

Observe that by the entropy inequality (2.5.10), for the Gibbs measure $\nu^N_{(\tau,0)}$ restricted to configurations on Λ_i^{k+1}, we obtain for $\alpha = 1, 2$

$$\int \zeta^k_{0,\alpha} d\nu^k \leq \log \int_{\Omega^k} e^{\zeta^k_{0,\alpha}} d\nu^k_{(\tau,0)} + \frac{1}{k+1} H(\nu^k | \nu^k_{(\tau,0)}).$$

By Lemma 2.5.16, the entropy term is bounded by a constant not depending on k, while the first term is bounded uniformly by assumption (2.3.4). This allows us to define ν a.s. the following quantities:

$$\begin{aligned}
z_1 &:= \lim_{k\to\infty} z^k_{i,1} := \lim_{k\to\infty} \frac{1}{k+1} \sum_{l\in\Lambda_i^k} r_l, \\
z_2 &:= \lim_{k\to\infty} z^k_{i,2} := \lim_{k\to\infty} \frac{1}{k+1} \sum_{l\in\Lambda_i^k} p_l,
\end{aligned}$$

and we define $\mathbf{z} := (z_1, z_2)$ and $\mathbf{z}_i^k := (z^k_{i,1}, z^k_{i,2})$.

With this notation we are ready to state the main Theorem of this section. It characterizes ν as a convex combination of Gibbs measures:

Theorem 2.5.18. *Assume $\nu(d\mathbf{r}, d\mathbf{p})$ is stationary with respect to the Liouville operator $\mathcal{L}$. Assume furthermore that the distribution of the velocities conditioned to the position $\pi(d\mathbf{p}|\mathbf{r})$ is a convex combination of Gaussian measures.*
Then ν is a convex combination of Gibbs measures with parameters given by $\lambda_1(\mathbf{z}) = z_1$ and $\lambda_2(\mathbf{z}) = z_2$, i.e

$$\nu(d\mathbf{r}, d\mathbf{p}|\mathbf{z}) = \prod_{i\in\mathbb{Z}} \frac{1}{Z(\boldsymbol{\lambda})} e^{\lambda_1(\mathbf{z}) r_i + \lambda_2(\mathbf{z}) p_i - h_i} dr_i dp_i$$

To prove this theorem, we first need to prove that the densities $\mathbf{z}(\mathbf{r}, \mathbf{p})$ are constants of motion defined by $\mathcal{L}$, that means that functions depending only on $\mathbf{z}$ can be considered as constants under $\nu(d\mathbf{r}, d\mathbf{p}|\mathbf{z})$. This is stated in the following Lemma:

Lemma 2.5.19. *Under the same assumptions as in Theorem 2.5.18, for any local smooth bounded functional ψ and any smooth function h with compact support, the $\mathbf{z}(\boldsymbol{r},\boldsymbol{p})$ are constants of motion in the sense that*

$$\int h(\mathbf{z}(\mathbf{r},\mathbf{p}))(\mathcal{L}\psi)(\mathbf{r},\mathbf{p})d\nu = 0. \tag{2.5.43}$$

In particular

$$\int (\mathcal{L}\psi(\mathbf{r},\mathbf{p}))\mu(d\mathbf{r}|\mathbf{z})\pi(d\mathbf{p}|\mathbf{z}) = 0. \tag{2.5.44}$$

where $\mu(d\mathbf{r}|\mathbf{z})$ denotes the distribution of $\mathbf{r}$ conditioned on the values of $\mathbf{z}$.

Proof. By the assumption that ν is stationary with respect to the Liouville operator, we have for any i

$$\int \mathcal{L}\big(h(\mathbf{z}_i^k(\mathbf{r},\mathbf{p}))\psi(\mathbf{r},\mathbf{p})\big)d\nu = \int h(\mathbf{z}_i^k(\mathbf{r},\mathbf{p}))\mathcal{L}\big(\psi(\mathbf{r},\mathbf{p})\big)d\nu + \int \psi(\mathbf{r},\mathbf{p})\mathcal{L}\big(h(\mathbf{z}_i^k(\mathbf{r},\mathbf{p}))\big)d\nu = 0.$$

Then it is enough to prove that the second term on the right hand side converges to 0 as $k \to \infty$. Furthermore since ψ and the partial derivatives of h are bounded, it remains to show that

$$\lim_{k\to\infty}\int \big|\mathcal{L}z_{i,1}^k(\mathbf{r},\mathbf{p})\big|\,d\nu = 0, \quad \text{and} \lim_{k\to\infty}\int \big|\mathcal{L}z_{i,2}^k(\mathbf{r},\mathbf{p})\big|\,d\nu = 0.$$

With the definition (2.5.37) of the Liouville operator on the infinite volume, for any i, we arrive at

$$\mathcal{L}\left(\frac{1}{k+1}\sum_{l\in\Lambda_i^k} r_l\right) = \frac{p_{i+k}-p_{i-k-1}}{k+1} \quad \text{and} \quad \mathcal{L}\left(\frac{1}{k+1}\sum_{l\in\Lambda_i^k} p_l\right) = \frac{V'(r_{i+k+1})-V'(r_{i-k})}{k+1}.$$

Since $\int p_i d\nu$ and $\int |V'(r_i)|d\nu$ are bounded by constants independent of k, (2.5.43) follows immediately.

To see (2.5.44), notice that since by assumption $\pi(d\mathbf{p}|\mathbf{r})$ is is a convex combination of Gaussian product measures i.e. there exists a measure $\beta(d\hat{\lambda}_2, d\hat{\lambda}_3|\mathbf{r})$ on $\mathbb{R}$ such that

$$\pi(d\mathbf{p}|\mathbf{r}) = \frac{1}{\tilde{Z}(\hat{\lambda}_2,\hat{\lambda}_3)}\int \beta(d\hat{\lambda}_2, d\hat{\lambda}_3|\mathbf{r})\prod_{i\in\mathbb{Z}} e^{-\frac{\hat{\lambda}_3}{2}(p_i-\frac{\hat{\lambda}_2}{\hat{\lambda}_3})^2}dp_i,$$

Here $\tilde{Z}(\hat{\lambda}_2,\hat{\lambda}_3)$ is a normalization and $\hat{\lambda}_2,\ \hat{\lambda}_3$ are parameters determined by the values of $\mathbf{z}$. Consequently, if we condition the measure further on $\mathbf{z}$, it becomes Gaussian

$$\pi(d\mathbf{p}|\mathbf{r},\mathbf{z}) = \prod_{i\in\mathbb{Z}}\sqrt{\frac{\hat{\lambda}_2(\mathbf{z})}{2\pi}}e^{-\frac{\hat{\lambda}_3(\mathbf{z})}{2}(p_i-\frac{\hat{\lambda}_2(\mathbf{z})}{\hat{\lambda}_3(\mathbf{z})})^2}dp_i.$$

Since in our system the temperature is fixed to be 1, the variance of momenta is identically 1 and hence $\hat{\lambda}_3 \equiv 1$. Then the parameter $\hat{\lambda}_2$ is given by $\hat{\lambda}_2(\mathbf{z}) = z_2$.
Futhermore the conditional measure defined above is independent of $\mathbf{r}$, so we have $\pi(d\mathbf{p}|\mathbf{r},\mathbf{z}) = \pi(d\mathbf{p}|\mathbf{z})$. Now we can represent ν as

$$\nu(d\mathbf{r},d\mathbf{p}) = \int \pi(d\mathbf{p}|\mathbf{z})\mu(d\mathbf{r}|\mathbf{z})\alpha(d\mathbf{z}), \tag{2.5.45}$$

where by α we denote a measure on the space of possible densities and $\mathbf{z}$ ranges over that space. Then with (2.5.43)

$$\int h(\mathbf{z})\Big(\int \mathcal{L}\psi(\mathbf{r},\mathbf{p})\mu(d\mathbf{r}|\mathbf{z})\pi(d\mathbf{p}|\mathbf{z})\Big)\alpha(d\mathbf{z}) = 0.$$

and (2.5.44) follows. □

We are now ready to proof that ν is a convex combination of Gibbs measures:

Proof of Theorem 2.5.18.

With the definition of the operator $\mathcal{L}$ and (2.5.44), for any smooth bounded local function ψ, we have

$$\int \mathcal{L}\psi\nu(d\mathbf{r},d\mathbf{p}|\mathbf{z}) = \int \sum_{j\in\mathbb{Z}} \left\{ p_j \left(\frac{\partial\psi}{\partial r_j} - \frac{\partial\psi}{\partial r_{j+1}} \right) + \left(V'(r_{j+1}) - V'(r_j)\right) \frac{\partial\psi}{\partial p_j} \right\} \nu(d\mathbf{r},d\mathbf{p}|\mathbf{z}) = 0.$$

Now we chose for the local function

$$\psi := \phi(\mathbf{r})(p_i - z_2),$$

where ϕ is a local function which we will define later. Then we obtain

$$\int \left\{ \sum_{j\in\mathbb{Z}} p_j(p_i - z_2) \left(\frac{\partial\phi}{\partial r_j} - \frac{\partial\phi}{\partial r_{j+1}} \right) + \left(V'(r_{i+1}) - V'(r_i)\right)\phi(\mathbf{r}) \right\} \nu(d\mathbf{r},d\mathbf{p}|\mathbf{z}) = 0. \quad (2.5.46)$$

Since $\pi(d\mathbf{p}|\mathbf{z})$ is gaussian,

$$\int p_j(p_i - z_2)\pi(d\mathbf{p}|\mathbf{z}) = \delta_{ij}.$$

Hence with the representation (2.5.45) of $\nu(d\mathbf{r},d\mathbf{p})$, instead of (2.5.46) we can write

$$\int \left\{ \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} p_j(p_i - z_2)\pi(d\mathbf{p}|\mathbf{z}) \left(\frac{\partial\phi}{\partial r_j} - \frac{\partial\phi}{\partial r_{j+1}} \right) + \left(V'(r_{i+1}) - V'(r_i)\right)\phi(\mathbf{r}) \right\} \mu(d\mathbf{r}|\mathbf{z})$$

$$= \int \left\{ \left(\frac{\partial\phi}{\partial r_i} - \frac{\partial\phi}{\partial r_{i+1}} \right) + \left(V'(r_{i+1}) - V'(r_i)\right)\phi(\mathbf{r}) \right\} \mu(d\mathbf{r}|\mathbf{z}) = 0 \quad (2.5.47)$$

Next we choose ϕ to be of the following form:

$$\phi(\mathbf{r}) := \chi_j^r(\mathbf{r})\psi_j^k(\mathbf{r}),$$

where for any $j \in \mathbb{Z}$, χ_j^r is a smooth bounded function with compact support, depending on $\{r_{j-r},\dots,r_{j+r}\}$ with $r > k$ and

$$\psi_j^k(\mathbf{r}) := \prod_{l\in\Lambda_j^k} e^{-\lambda_1(\mathbf{z})r_l + V(r_l)},$$

where $\lambda_1(\mathbf{z})$ will be determined later. Plugging in this in (2.5.47), we arrive at

$$\int \left\{ \left(\frac{\partial\chi_j^r}{\partial r_i} - \frac{\partial\chi_j^r}{\partial r_{i+1}} \right) \psi_j^k(\mathbf{r}) \right\} \mu(d\mathbf{r}|\mathbf{z}) = 0 \qquad \text{for each} \quad i \in \{j - \frac{k}{2},\dots,j + \frac{k}{2} - 1\}, \quad (2.5.48)$$

and

$$\int\left\{\left(\frac{\partial\chi_j^r}{\partial r_i}\psi_j^r(\mathbf{r})+V'(r_{i+1})\chi_j^r(\mathbf{r})\right)\psi_j^k(\mathbf{r})\right\}\mu(d\mathbf{r}|\mathbf{z})=0 \qquad \text{if} \quad i=j+\frac{r}{2}. \tag{2.5.49}$$

In the first equation we now set

$$\chi_j^k(\mathbf{r}):=\chi^c(\mathbf{r})g(\sum_{l\in\Lambda_j^k} r_i)\chi_j^k(\mathbf{r})$$

where χ^c depends only on r_l with $l\in(\Lambda_j^r\setminus\Lambda_j^k)$ and g is a continuous function on $\mathbb{R}$. Then, for each $i\in\{j-\frac{k}{2},\dots,j+\frac{k}{2}-1\}$

$$\frac{\partial\chi_j^r}{\partial r_i}-\frac{\partial\chi_j^r}{\partial r_{i+1}}=\chi^c(\mathbf{r})g\left(\sum_{l\in\Lambda_j^k} r_i\right)\left(\frac{\partial\chi_j^k}{\partial r_i}-\frac{\partial\chi_j^k}{\partial r_{i+1}}\right).$$

Plugging this in (2.5.48), we can condition the measure $\mu(d\mathbf{r}|\mathbf{z})$ further on the values $\sum_{l\in\Lambda_j^k} r_i=(k+1)z_{i,1}^k$ and on the configurations r_i for $i\notin\Lambda_j^k$. Then we obtain for $i\in\{j-\frac{k}{2},\dots,j+\frac{k}{2}-1\}$

$$\int\left(\frac{\partial\chi_j^k}{\partial r_i}-\frac{\partial\chi_j^k}{\partial r_{i+1}}\right)\psi_j^k(\mathbf{r})\mu\left(dr_{j-\frac{k}{2}},\dots,dr_{j+\frac{k}{2}}\Big|\mathbf{z},\sum_{l\in\Lambda_j^k} r_l,\ r_i,i\neq j-\frac{k}{2},\dots,j+\frac{k}{2}\right)=0.$$

But this is enough to characterize the measure

$$\prod_{l\in\Lambda_j^k} e^{-\lambda_1(\mathbf{z})r_l+V(r_l)}\mu\left(dr_{j-\frac{k}{2}},\dots,dr_{j+\frac{k}{2}}\Big|\mathbf{z},\sum_{l\in\Lambda_j^k} r_l,\ r_i,i\neq j-\frac{k}{2},\dots,j+\frac{k}{2}\right)$$

up to a multiplicative constant as a Lebesgue measure

$$\mathbf{1}\{\sum_{l\in\Lambda_j^k} r_l=(k+1)z_{i,1}^k\}dr_{j-\frac{k}{2}},\dots,dr_{j+\frac{k}{2}}$$

on the hyperplane $\{r_{j-\frac{k}{2}},\dots,r_{j+\frac{k}{2}}:\sum_{l\in\Lambda_j^k} r_i=(k+1)z_{i,1}^k\}$. With this we finally obtain

$$\mu\left(dr_{j-\frac{k}{2}},\dots,dr_{j+\frac{k}{2}}\Big|\mathbf{z},\ r_i,i\neq j-\frac{k}{2},\dots,j+\frac{k}{2}\right)=\prod_{l\in\Lambda_j^k}\frac{e^{\lambda_1(\mathbf{z})r_l-V(r_l)}}{Z_1(\lambda_1(\mathbf{z}))}dr_l$$

where Z_1 is a normalizing constant in dependent on the outside configurations and therefore

$$\mu\left(d\mathbf{r}|\mathbf{z}\right)=\prod_{l\in\mathbb{Z}}\frac{e^{\lambda_1(\mathbf{z})r_l-V(r_l)}}{Z_1(\lambda_1(\mathbf{z}))}dr_l.$$

It remains to determine $\lambda_1(\mathbf{z})$. This can be done using (2.5.49): Applying the result just obtained for $\mu\left(d\mathbf{r}|\mathbf{z}\right)$ (2.5.49) gives us for $i=j+\frac{r}{2}$:

$$\int V'(r_{i+1})\chi_j^r(\mathbf{r})\prod_{l\in\mathbb{Z}}\frac{e^{\lambda_1(\mathbf{z})r_l-V(r_l)}}{Z_1(\lambda_1(\mathbf{z}))}dr_l=-\int\frac{\partial\chi_j^r}{\partial r_i}\psi_j^k(\mathbf{r})\mu(d\mathbf{r}|\mathbf{z}).$$

The right hand side integrates to zero since χ_j^r has compact support and because of the structure of $\psi_j^k(\mathbf{r})\mu(d\mathbf{r}|\mathbf{z})$. The left hand side can be computed with integration by parts as

$$\lambda_1(\mathbf{z})\int \chi_j^r(\mathbf{r})\mu(d\mathbf{r}|\mathbf{z}) = \int \chi_j^r(\mathbf{r})V'(r_i)\mu(d\mathbf{r}|\mathbf{z})$$

and we obtain the relation

$$\lambda_1(\mathbf{z}) := \lambda_1(z_1) = \int V'(r_i)\mu(d\mathbf{r}|\mathbf{z}).$$

In summary we obtained:

$$\nu(d\mathbf{r}, d\mathbf{p}|\mathbf{z}) = \pi(d\mathbf{p}|\mathbf{z})\mu(d\mathbf{r}|\mathbf{z}) = \prod_{i\in\mathbb{Z}} \frac{1}{Z(\lambda_1(z_1), z_2)} e^{\lambda_1(z_1)r_i + z_2 p_i - h_i} dr_i dp_i.$$

□

The distribution of momenta

To conclude the proof of Theorem 2.5.12, we are left to show that the assumptions in Theorem 2.5.18 are verified. It is here where we need the noise.

Denote by

$$\mathcal{D}_{\Lambda_i^n}\left(\frac{f^n}{g^n_{(\tau,0)}}\right) := \mathbf{D}_{\Lambda_i^n}(\nu^n) := \sum_{j=i-\frac{n}{2}+1}^{i+\frac{n}{2}} D_i^n(\nu^n),$$

where f^n denotes the density corresponding to the limit point ν.

$$D_i^n(\nu^n) := \frac{1}{2}\int \frac{g^n_{(\tau,0)}}{f^n}\left(\frac{\partial}{\partial p_i}\frac{f^n}{g^n_{(\tau,0)}} - \frac{\partial}{\partial p_{i-1}}\frac{f^n}{g^n_{(\tau,0)}}\right)^2 d\nu^n_{(\tau,0)} = \frac{1}{2}\int \frac{1}{f^n}\left(\Upsilon_{i,i-1}f^n\right)^2 d\nu^n_*,$$

with $\Upsilon_{i,i-1}$ defined by (2.4.2). We furthermore denote by $\mathcal{S}_{\Lambda_i^n}$ the symmetric operator $\mathcal{S}$ reduced to the box Λ_i^n that means

$$\mathcal{S}_{\Lambda_i^n} := \frac{1}{2}\sum_{j=i-\frac{n}{2}+1}^{i+\frac{n}{2}} \left\{\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_{j-1}}\right)^2 - (p_j - p_{j-1})\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_{j-1}}\right)\right\}.$$

By a Theorem of Donsker and Varadhan it is known that for any finite subset $\Lambda \in \mathbb{Z}$

$$\mathbf{D}_\Lambda(f) = \sup_\psi \left\{-\int \frac{\mathcal{S}_\Lambda(\psi)}{\psi} d\nu\right\} \tag{2.5.50}$$

if $\mathbf{D}(f) \leq \infty$, and where the supremum is taken over all positive functions ψ belonging to the domain of $\mathcal{S}_\Lambda$.

Lemma 2.5.20. *For each fixed $k \geq 2$ a limit point ν^k of the sequence $(\hat{\nu}_{t,i}^{N,k})_N$ is stationary with respect to the symmetric operator $\gamma\mathcal{S}_{\Lambda_i^n}$ for each $n = -k, \dots, k$.*

Proof. Since the limit points ν and hence ν^k are translation invariant, we can omit the index which specifies the center of the blocks.

The problem of the proof lies in the fact that we do not know whether $H(\nu|\nu_{(\tau,0)})$ is bounded. Therefor denote by μ_t the probability measure having density g_t, generated by $\mathcal{A}$ and starting from some generic probability measure μ. Similar we denote for any k by μ^k its projection on the box Λ^k.

Using the operator $\mathcal{A}$ we obtain the following inequality in a similar way as the results from Lemma (2.5.7) have been obtained: We get

$$H(\mu_t|\nu_{(\tau,0)}) + t\mathbf{D}(\bar{\mu}_t) \leq H(\mu|\nu_{(\tau,0)})$$

where $\bar{\mu}_t$ is the measure corresponding to the density $\bar{g}_t := \frac{1}{t}\int_0^t g_s ds$, and hence

$$H(\mu_t^k|\nu_{(\tau,0)}^k) + t\mathbf{D}(\bar{\mu}^k) = H(\mu_t^k|\nu_{(\tau,0)}^k) + t\sum_{j=-\frac{k}{2}+1}^{\frac{k}{2}}\int D_j^k(\bar{\mu}_t^n) \leq H(\mu|\nu_{(\tau,0)})$$

with $\bar{\mu}_t^k$ corresponding to $\bar{g}_t^k := \frac{1}{t}\int_0^t g_s^k ds$.

Since the Dirichlet form is a sum of positive terms, for any fixed $n \leq k$, we can rewrite the last inequality as

$$H(\mu_t^k|\nu_{(\tau,0)}^k) + t\sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}}\int D_j^k(\bar{\mu}_t^k) \leq H(\mu|\nu_{(\tau,0)}).$$

With the the definition of the relative entopy and (2.5.50), we obtain for any local function ϕ and any bounded local function ψ on $\Omega^{|\Lambda^k|}$

$$\int \phi d\mu_t^k - \log\int e^{\phi} d\nu_{(\tau,0)}^k - t\int \frac{\mathcal{S}_{\Lambda^n}\psi}{\psi} d\bar{\mu}_t^k \leq H(\mu|\nu_{(\tau,0)}).$$

Now we can let go k to ∞: Since ψ and ϕ are local functions we obtain that

$$\int \phi d\mu_t - \log\int e^{\phi} d\nu_{(\tau,0)} - t\int \frac{\mathcal{S}_{\Lambda^n}\psi}{\psi} d\bar{\mu}_t \leq H(\mu|\nu_{(\tau,0)}). \tag{2.5.51}$$

In the next step we choose μ to be of the special form

$$\mu := \mu^{(n)} := \nu|_{\Lambda^n} \otimes \nu_{(\tau,0)}|_{(\Lambda^n)^c}.$$

In this way we get that

$$H(\mu^{(n)}|\nu_{(\tau,0)}) = H_{\Lambda^n}(\nu|\nu_{(\tau,0)})$$

which by Lemma (2.5.16) is bounded by $C|\Lambda^n|$. Consequently the limit $\bar{H}(\nu|\nu_{(\tau,0)})$ exists and

$$\lim_{n\to\infty}\frac{1}{n+1}H(\mu^{(n)}|\nu_{(\tau,0)}) = \bar{H}(\nu|\nu_{(\tau,0)}).$$

We next chose for any $j \in \{-\frac{n}{2}+1, \ldots, \frac{n}{2}\}$ the functions ϕ and ψ to be of the form

$$\phi := \sum_{i=-\frac{n}{2}}^{\frac{n}{2}} \tau_i\phi_j \quad \text{and} \quad \psi := \tau_i\psi_j,$$

where ϕ_j and ψ_j are local measurable bounded functions. Then from (2.5.51) and since $\nu_{(\tau,0)}$ is translation invariant we obtain

$$\frac{1}{n+1}\left(\sum_{i=-\frac{n}{2}}^{\frac{n}{2}}\int \tau_i\phi_j d\mu_t - \log\int e^{(n+1)\phi_j}d\nu_{(\tau,0)} - t\int \tau_i \frac{\mathcal{S}_{\Lambda^n}\psi_j}{\psi_j}d\bar{\mu}_t\right)$$
$$\leq \frac{1}{n+1}\sum_{i=-\frac{n}{2}}^{\frac{n}{2}} H(\mu|\nu_{(\tau,0)}).$$

By Hölder inequality we have

$$\frac{1}{n+1}\log\int e^{(n+1)\phi_j}d\nu_{(\tau,0)} = \log\int e^{\phi_j}d\nu_{(\tau,0)},$$

and thus the limit

$$\lim_{n\to\infty}\frac{1}{n+1}\log\int e^{(n+1)\phi_j}d\nu_{(\tau,0)} = \log\int e^{\phi_j}d\nu_{(\tau,0)}$$

exists.

Now let us assume that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=-\frac{n}{2}}^{\frac{n}{2}}\int \tau_i\phi_j d\mu_t^{(n)} = \int \phi_j d\nu \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n+1}\int \tau_i\frac{\mathcal{S}_{\Lambda^n}\psi_j}{\psi_j}d\bar{\mu}(n)_t = \int \frac{S_j\psi_j}{\psi_j}d\nu, \tag{2.5.52}$$

with

$$\int \frac{\mathcal{S}\psi_j}{\psi_j}d\nu := \frac{1}{2}\int\frac{1}{\psi_j}\left[(\Upsilon_{j,j-1}\psi)^2 - (p_i - p_{i-1})\Upsilon_{j,j-1}\psi_j\right]d\nu.$$

With this assumption we are done: Taking the supremum of

$$\int \phi_j d\nu - \log\int e^{\phi_j}d\nu_{(\tau,0)} - t\int\frac{\mathcal{S}_j\psi_j}{\psi_j}d\bar{\mu}_t \leq \bar{H}(\mu|\nu_{(\tau,0)}).$$

we obtain

$$\sup_k H_{\Lambda^k}(\nu|\nu_{(\tau,0)}) - t\inf_\psi\int\frac{\mathcal{S}_j\psi_j}{\psi_j}d\nu \leq \sup_k H_{\Lambda^k}(\nu|\nu_{(\tau,0)})$$

and hence for each $j \in -\frac{n}{2}+1,\ldots,\frac{n}{2}$

$$-\inf_\psi\int\frac{\mathcal{S}_j\psi_j}{\psi_j}d\nu \leq 0.$$

Summing up over all j we get

$$\mathbf{D}_{\Lambda^n}(\nu) \leq -\inf_\psi\left\{\sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}}\int\frac{\mathcal{S}_j\psi_j}{\psi_j}d\nu\right\} \leq -\sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}}\inf_\psi\left\{\int\frac{\mathcal{S}_j\psi_j}{\psi_j}d\nu\right\} \leq 0$$
$$\Rightarrow \mathbf{D}_{\Lambda^n}(\nu) = 0, \quad \forall n \in \mathbb{N}$$

With this result the invariance of ν^k with respect to $\mathcal{S}_{\Lambda^n}$ for each $m = \{2,\ldots,k\}$ follows:

Since ν^k is translation invariant we can omit the index i. Performing an integration by parts on $\mathbf{D}^k_{\Lambda^k}(f^k)$, we get:

$$\begin{aligned} 0 &= \frac{1}{2}\sum_{j=-\frac{k}{2}+1}^{\frac{k}{2}} \int \frac{1}{f^k}\left(\frac{\partial f^k}{\partial p_j} - \frac{\partial f^k}{\partial p_{j-1}}\right)^2 \nu_{(\tau,0)}(d\mathbf{r}, d\mathbf{p}) \\ &= \frac{1}{2}\sum_{j=-\frac{k}{2}+1}^{+\frac{k}{2}} \int \left(\frac{\partial \log f^k}{\partial p_j} - \frac{\partial \log f^k}{\partial p_{j-1}}\right)\cdot\left(\frac{\partial f^k}{\partial p_j} - \frac{\partial f^k}{\partial p_{j-1}}\right) d\nu_{(\tau,0)} \\ &= -\frac{1}{2}\int \log f^k \left(\mathcal{S}_{\Lambda^k} f^k\right) \nu_{(\tau,0)}(d\mathbf{r}, d\mathbf{p}) \Rightarrow \mathcal{S}_{\Lambda^n} f^k = 0 \quad \forall n \in \{2,\dots,k\} \quad (2.5.53) \end{aligned}$$

It remains to prove assumption (2.5.52). Here the difficulty comes from the fact that ϕ is not local.

For the marginals of $\mu^{(n)}$ we have

$$\int h_i d\nu^n \le C_1 \quad \text{and} \quad \int h_i d\nu_{(\tau,0)} \le C_2$$

Thus for any j it follows that

$$\int h_i d(\tau_j \mu_t^{(n)}) \le C_3$$

with a uniform constant C_3 in n. This tells us that the sequence is weakly compact.

Then we have

$$\lim_{n\to\infty} \int \phi \; d(\tau_j \mu_t^{(n)}) = \lim_{l\to\infty}\lim_{n\to\infty} \int \phi \; d(\tau_j \mu_t^{(n)})|_{\Lambda_j^l}.$$

Now ϕ is a local function and thus by translation invariance of ν

$$\lim_{l\to\infty}\lim_{n\to\infty} \int \phi \; d\mu_{t,j}^{(n)}|_{\Lambda_j^l} = \lim_{l\to\infty} \int \phi \, d\nu^l = \int \phi \, d\nu = \int \phi_0 \, d\nu.$$

The second assumption can be obtained in the same way. □

With Lemma 2.5.20 we can prove the following two Corollaries :

Corollary 2.5.21. *Any limit point ν of the sequence $(\nu^k)_k$ is stationary with respect to the genrator $\gamma\mathcal{S}$.*

Proof. We have to show that for any smooth bounded local function ψ we have

$$\int \mathcal{S}\psi(d\mathbf{r}, d\mathbf{p}) d\nu = 0.$$

Since ψ is a local function there exists some $k \ge 1$, such that ψ depends on configurations only through $j \in \Lambda_i^k$. Thus we have:

$$\int \mathcal{S}\psi(d\mathbf{r}, d\mathbf{p}) d\nu = \int \mathcal{S}_{\Lambda^k}\psi(d\mathbf{r}, d\mathbf{p}) d\nu \quad = \quad \int (S_{\Lambda^k}\psi)(d\mathbf{r}, d\mathbf{p}) d\nu^k$$

But by Lemma 2.5.20 ν^k is stationary with respect to S_{Λ^k} and hence the last expression is equal to 0 □

Notice that since we already proved that ν is stationary with respect to $\mathcal{A}$, we proved with this Lemma that ν is stationary separately for $\mathcal{L}$ and for $\mathcal{S}$. Hence the first assumption of Theorem 2.5.18 is satisfied.

Corollary 2.5.22. *The distribution of the momenta conditioned to the position $\pi(d\mathbf{p}|\mathbf{r})$ is a convex combination of Gaussian measures.*

Proof. From Lemma (2.5.20), we obtained that for each $j \in \{-\frac{n}{2}+1, \ldots, \frac{n}{2}\}$ and each $n \in \{2, \ldots, k\}$

$$\Upsilon_{j,j-1} f^k \left(r_j, p_j;\ j, j-1 \in \Lambda^k\right) = 0.$$

Then since $\Upsilon_{j,j-1}\left(\sum_{j=-\frac{k}{2}+1}^{\frac{k}{2}} p_j\right) = 0$ it is tangent to the hypersurface

$$\mathbb{S}_m\{(p_{-\frac{n}{2}}, \ldots, p_{\frac{n}{2}}) \in \mathbb{R}^k;\ \sum_{-\frac{n}{2}+1}^{\frac{n}{2}} p_i = m,\, m \in \mathbb{R}\} \subset \mathbb{R}^{n-1}$$

for each $m \in \mathbb{R}$. Thus we can consider the density f^k as a function of the form

$$f^k\left(r_j, p_j;\ j, j-1 \in \Lambda^n\right) := f^k\left(r_j;\ j, j-1 \in \Lambda^n\ ,\ \sum_{-\frac{n}{2}+1}^{\frac{n}{2}} p_i\right).$$

Consequently

$$\pi^k\left(\left\{p_{-\frac{n}{2}+1}, \ldots, p_{\frac{n}{2}}\right\} \Big| \left\{r_{-\frac{n}{2}+1}, \ldots, r_{\frac{n}{2}}\right\}\right) = \pi^k\left(\sum_{-\frac{n}{2}+1}^{\frac{n}{2}} p_i \Big| \left\{r_{-\frac{n}{2}+1}, \ldots, r_{\frac{n}{2}}\right\}\right).$$

is an exchangable measure. Hence for each marginal,

$$\pi^k(dp_j, j \in \Lambda^k|\mathbf{r}) \overset{k\to\infty}{\to} \pi(dp_j, j \in \mathbb{Z}|\mathbf{r}).$$

in the weak sense and thus $\pi(d\mathbf{p}|\mathbf{r})$ is exchangeable. Furthermore by the Hewitt-Savage Theorem it is a convex combination of probability product measure and more precisely with our choice of the noise it is a convex combination of product Gaussian measures i.e. there exists a measure $\beta(d\hat{\lambda}_2, d\hat{\lambda}_2|\mathbf{r})$ on $\mathbb{R}^2$ such that

$$\pi(d\mathbf{p}|\mathbf{r}) = \frac{1}{\tilde{Z}(\hat{\lambda}_2, \hat{\lambda}_3)} \int \beta(d\hat{\lambda}_2, d\hat{\lambda}_3|\mathbf{r}) \prod_{i\in\mathbb{Z}} e^{-\frac{\hat{\lambda}_3}{2}(p_i - \frac{\hat{\lambda}_2}{\hat{\lambda}_3})^2} dp_i,$$

where $\tilde{Z}(\hat{\lambda}_2, \hat{\lambda}_3)$ denotes a normalization. Then we can proceed as in the proof of Theorem 2.5.19 to fix the parameters by the relations

$$\hat{\lambda}_3 \equiv 1 \quad \text{and} \quad \hat{\lambda}_2(\mathbf{z}) = z_2$$

□

2.5.5 Large deviation

In order to prove Theorem 2.5.10 and inequality (2.5.12), in this section we will give an introduction to the large deviation theory applied to the local Gibbs measures. We refer to [37] for the general overview. We first give a definition of a family of probability measures satisfying the large deviation principle:

Definition 2.5.23 (Large Deviation principle). *Let P_n be a family of probability measures on the Borel subsets of the finite dimensional, complete separable metric space X. We say that that $\{P_n\}$ obeys the large deviation principle with a rate function $I(\cdot)$ if there exists a function $I : X \to [0, \infty]$ satisfying:*

(i) $0 \leq I(\mathbf{x}) \leq \infty \quad \forall \mathbf{x} \in X$,

(ii) *$I(\cdot)$ is lower semicontinuous,*

(iii) *For each $l < \infty$ the set $\{\mathbf{x} : I(\mathbf{x}) \leq l\}$ is a compact set in X,*

(iv) *For each closed set $C \subset X$:* $\limsup_{n\to\infty} \frac{1}{n} \log P_n(C) \leq -\inf I(\mathbf{x})$,

(v) *For each open set $G \subset X$:* $\liminf_{n\to\infty} \frac{1}{n} \log P_n(G) \geq -\inf I(\mathbf{x})$.

To prove Varadhan's Lemma 2.5.10, we first have to make sure that the rate function $I : \Omega \to \infty$ we defined for $\nu_{\boldsymbol{\lambda}}$ in the Appendix A by

$$I(\mathbf{x}) = \sup_{\boldsymbol{\theta}}\{\Phi(\mathbf{x}) - \Lambda(\boldsymbol{\theta})\} = \Phi(\mathbf{x}) - \mathbf{x} \cdot \boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}) \tag{2.5.54}$$

indeed satisfies the the properties for $\nu_{\boldsymbol{\lambda}}$ we enumerated in Definition 2.5.23. For this let us distribute the 2-dimensional random vectors $\boldsymbol{\zeta}_j = (r_j, p_j)$ i.i.d according to the Gibbs measure $\nu_{\boldsymbol{\lambda}}$ defined on $X = \Omega$, then we can define the empirical means $\boldsymbol{\zeta}^n$ by

$$\boldsymbol{\zeta}^n = (\zeta_1^n, \zeta_2^n) := (r^n, p^n) := \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\zeta}_i.$$

Then we obtain the following special case of Cramér's Theorem:

Theorem 2.5.24 (Multidimensional Version of Cramér's Theorem). *Denote by $\nu_{\boldsymbol{\lambda}}^n$ the common law of $\boldsymbol{\zeta}^n$. Then the sequence $\{\nu_{\boldsymbol{\lambda}}^n\}$ satisfies the large deviation principle with rate function $I(\cdot)$ given by* (2.5.54).

Proof. To prove the theorem, it satisfies to check the lower and upper bounds (iv) and (v) of Definition 2.5.23, since $(i) - (iii)$ has already been checked in Appendix A.

Lower bound: To prove the lower bound, it is enough to prove that for each open disk

$$D_\delta^n := \{(\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n) : |\boldsymbol{\zeta}^n - \mathbf{y}| < \delta\} \subset \mathbb{R}^2$$

with radius, $\delta > 0$ around each point $\mathbf{y} \in \Omega$ we have:

$$\lim_{n\to\infty} \frac{1}{n} \log \nu_{\boldsymbol{\lambda}}^n (D_\delta^n(\mathbf{y})) \geq -I(\mathbf{y}). \tag{2.5.55}$$

Since in our case for any $\boldsymbol{\theta} \in \mathbb{R}^2$ we have that $M(\boldsymbol{\theta})$ is finite, we also have that $I(\mathbf{y}) < \infty \, \forall \mathbf{y} \in \Omega$. It follows that for each $\mathbf{y} \in \Omega$ the supremum

$$I(\mathbf{y}) = \sup_{\theta} \{\boldsymbol{\theta} \cdot \mathbf{y} - \Lambda(\boldsymbol{\theta})\}$$

is attained for some $\boldsymbol{\theta} := \mathbf{y}^\star$, thus to each $\mathbf{y} \in \Omega$ there exists a unique $\mathbf{y}^\star$ such that

$$I(\mathbf{y}) = \mathbf{y} \cdot \mathbf{y}^\star - \Lambda(\mathbf{y}^\star) \quad \text{with} \quad \mathbf{y} = \frac{DM(\mathbf{y}^\star)}{M(\mathbf{y}^\star} = D\Lambda(\mathbf{y}^\star). \tag{2.5.56}$$

Where the definition and properties of the logarithmic moment generating function $\Lambda(\cdot)$ and the moment generating function $M(\cdot)$ can be found in Appendix A.
In view of the structure of the moment generating function $M(\cdot)$, for $\boldsymbol{\theta} \in \mathbb{R}^2$ let us introduce a new probability measure $\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}$ on Ω^n being absolutely continuous with respect to the Lebesgue measure and with marginals given by

$$\nu_{\boldsymbol{\lambda}+\mathbf{y}^\star}(d\boldsymbol{\zeta}) = \frac{1}{M(\mathbf{y}^\star)} e^{\boldsymbol{\zeta}(\boldsymbol{\lambda}+\mathbf{y}^\star)-h} d\boldsymbol{\zeta}. \tag{2.5.57}$$

Observe that then the mean with respect to this measure is equal to

$$\int \boldsymbol{\zeta} d\nu_{\boldsymbol{\lambda}+\mathbf{y}^\star} = \frac{\int \boldsymbol{\zeta} e^{\boldsymbol{\zeta}\cdot(\mathbf{y}^\star+\boldsymbol{\lambda})-h} d\boldsymbol{\zeta}}{\int e^{\boldsymbol{\zeta}\cdot(\mathbf{y}^\star+\boldsymbol{\lambda})-h} d\boldsymbol{\zeta}} = \frac{\int \boldsymbol{\zeta} e^{\boldsymbol{\zeta}\cdot\mathbf{y}^\star} d\nu_{\boldsymbol{\lambda}}}{\int e^{\boldsymbol{\zeta}\cdot\mathbf{y}^\star} d\nu_{\boldsymbol{\lambda}}} = \frac{DM(\mathbf{y}^\star)}{M(\mathbf{y}^\star)} = D\Lambda(\mathbf{y}^\star) = \mathbf{y},$$

because of (2.5.56). In particular we obtain by the law of large numbers:

$$\lim_{n\to\infty} \nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star} \left[|\boldsymbol{\zeta}^n - \mathbf{y}| < \delta\right] = 1. \tag{2.5.58}$$

Furthermore, for any $\delta_1 < \delta$

$$\begin{aligned}
\nu^n_{\boldsymbol{\lambda}} \left[|\boldsymbol{\zeta}^n - \mathbf{y}| < \delta\right] &= \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta} d\nu^n_{\boldsymbol{\lambda}} = \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta} \prod_{i=1}^n e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_i - h_i} d\mathbf{r} d\mathbf{p} \\
&= M^n(\mathbf{y}^\star) \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta} \prod_{i=1}^n e^{-\mathbf{y}^\star\cdot\boldsymbol{\zeta}_i} \cdot \frac{\prod_{i=1}^n e^{\boldsymbol{\zeta}_i\cdot(\boldsymbol{\lambda}+\mathbf{y}^\star)-h_i}}{\left(\int e^{\boldsymbol{\zeta}\cdot(\boldsymbol{\lambda}+\mathbf{y}^\star)-h} d\boldsymbol{\zeta}\right)^n} d\mathbf{r} d\mathbf{p} \\
&= M^n(\mathbf{y}^\star) \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta} e^{-\mathbf{y}^\star\cdot\sum_{i=1}^n \boldsymbol{\zeta}_i} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star} \\
&\geq M^n(\mathbf{y}^\star) \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} e^{-\mathbf{y}^\star\cdot\sum_{i=1}^n \boldsymbol{\zeta}_i} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star} \\
&= M^n(\mathbf{y}^\star) e^{-n\mathbf{y}\cdot\mathbf{y}^\star} \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} e^{-\mathbf{y}^\star\cdot(\sum_{i=1}^n \boldsymbol{\zeta}_i - n\mathbf{y})} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}.
\end{aligned}$$

And then we have:

$$\begin{aligned}
\frac{1}{n} \log \nu^n_{\boldsymbol{\lambda}} \left[|\boldsymbol{\zeta}^n - \mathbf{y}| < \delta\right] &\geq \log M\mathbf{y}^\star - \mathbf{y}\cdot\mathbf{y}^\star + \frac{1}{n} \log \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} e^{-\mathbf{y}^\star\cdot(\sum_{i=1}^n \boldsymbol{\zeta}_i - n\mathbf{y})} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star} \\
&= -I(\mathbf{y}) + \frac{1}{n} \log \int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} e^{-\mathbf{y}^\star\cdot(\sum_{i=1}^n \boldsymbol{\zeta}_i - n\mathbf{y})} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}, \qquad (2.5.59)
\end{aligned}$$

which is true by (2.5.56). The integral of the second term of the last expression, by Jensen inequality can be bounded below by

$$e^{-\int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} \mathbf{y}^\star\cdot(\sum_{i=1}^n \boldsymbol{\zeta}_i - n\mathbf{y})} d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}.$$

Consequently,

$$\begin{aligned}
\frac{1}{n}\log\int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1} e^{-\mathbf{y}^\star\cdot(\sum_{i=1}^n\boldsymbol{\zeta}_i-n\mathbf{y})}d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star} &\geq -\int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1}\mathbf{y}^\star\cdot(\frac{1}{n}\sum_{i=1}^n\boldsymbol{\zeta}_i-\mathbf{y})d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}\\
&\geq -|\mathbf{y}^\star|\int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1}|\frac{1}{n}\sum_{i=1}^n\boldsymbol{\zeta}_i-\mathbf{y}|d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}\\
&\geq -\delta_1|\mathbf{y}^\star|\int_{|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1}d\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}\\
&= -\delta_1|\mathbf{y}^\star|\nu^n_{\boldsymbol{\lambda}+\mathbf{y}^\star}\left[|\boldsymbol{\zeta}^n-\mathbf{y}|<\delta_1\right].
\end{aligned}$$

By (2.5.58) this converges to $-\delta_1|\mathbf{y}^\star|$ as $n\to\infty$ and since δ_1 is arbitrary we can let it go to 0. This together with the right hand side of (2.5.59) concludes the proof for the lower bound (v) of Definition 2.5.23.

Upper bound: To Prove the upper bound we will need the minimax Theorem. It says that:

Lemma 2.5.25 (The minimax Theorem). *Let $g(\boldsymbol{\zeta},\boldsymbol{\theta}):\Omega\times\mathbb{R}^2\to\mathbb{R}$ be a function which is*

- *convex and lower semicontinuous in $\boldsymbol{\zeta}$,*
- *concave and upper semicontinuous in $\boldsymbol{\theta}$.*

Let C be any convex and compact set in Ω, then

$$\inf_{u\in C}\sup_{\boldsymbol{\theta}} g(\boldsymbol{\zeta},\boldsymbol{\theta})=\sup_{\boldsymbol{\theta}}\inf_{u\in C} g(\boldsymbol{\zeta},\boldsymbol{\theta}).$$

Proof. See [37] □

The proof of the upper bound is devided in three steps:

Step I: We claim that for any compact and convex set $C\subset\Omega$ the inequality (iv) of Definition 2.5.23 holds. That means, we need

$$\lim_{n\to\infty}\frac{1}{n}\log\nu^n_{\boldsymbol{\lambda}}[\boldsymbol{\zeta}^n\in C]\leq-\inf_{\mathbf{x}\in C}I(\mathbf{x}). \tag{2.5.60}$$

To prove this consider any Borel subset C of Ω. Then

$$\begin{aligned}
\nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n\in C) &= \int_{\boldsymbol{\zeta}^n\in C}\prod_{i=1}^n e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_i-h_i}d\mathbf{r}d\mathbf{p}\\
&= \int_{\boldsymbol{\zeta}^n\in C}e^{-\sum_{i=1}^n\boldsymbol{\zeta}_i\boldsymbol{\theta}}\prod_{i=1}^n e^{(\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{\zeta}_i-h_i}d\mathbf{r}d\mathbf{p}\\
&\leq e^{-\inf_{x\in C}(n\mathbf{x}\cdot\boldsymbol{\theta})}\int_{\boldsymbol{\zeta}^n\in C}e^{n\boldsymbol{\zeta}^n\cdot\boldsymbol{\theta}}d\nu^n_{\boldsymbol{\lambda}}\\
&\leq e^{-\inf_{x\in C}(n\mathbf{x}\cdot\boldsymbol{\theta})}\int_{\Omega^n}e^{n\boldsymbol{\zeta}^n\cdot\boldsymbol{\theta}}d\nu^n_{\boldsymbol{\lambda}}
\end{aligned}$$

Thus we have so far

$$\begin{aligned}\frac{1}{n}\log \nu_\lambda^n[\boldsymbol{\zeta}^n \in C] &\leq -\inf_{x\in C}(\boldsymbol{\theta}\cdot\mathbf{x}) + \frac{1}{n}\log\left(\int_\Omega e^{\boldsymbol{\zeta}\boldsymbol{\theta}}d\nu_{\boldsymbol{\lambda}}\right)^n \\ &= -\inf_{x\in C}\{\boldsymbol{\theta}\cdot\mathbf{x} + \frac{1}{n}\log M^n(\boldsymbol{\theta})\}\end{aligned}$$

and since this is true for any $\theta \in \mathbb{R}^2$,

$$\frac{1}{n}\log \nu_\lambda^n[\boldsymbol{\zeta}^n \in C] \leq -\sup_{\boldsymbol{\theta}}\inf_{x\in C}\{\boldsymbol{\theta}\cdot\mathbf{y} + \log M(\boldsymbol{\theta})\}.$$

With property ($\Lambda 4$) of $\Lambda(\cdot)$ it follows that $g(\mathbf{x},\boldsymbol{\theta}) := \mathbf{x}\cdot\theta - \log M(\boldsymbol{\theta})$ is concave and upper semicontinuous in $\boldsymbol{\theta}$ while it is convex and lower semicontinuous in $\mathbf{x}$. Thus we can apply Lemma 2.5.25 for compact and convex sets C which concludes the proof of the claim, since then the right hand side of the last expression is equal to

$$-\inf_{x\in C}\sup_{\boldsymbol{\theta}}\{\boldsymbol{\theta}\cdot\mathbf{x} + \log M(\boldsymbol{\theta})\} = -\inf_{x\in C} I(\mathbf{x}).$$

Step II: Here we extend the proof from compact and convex sets to compact sets:

Let $K \subset \Omega$ be any compact set. By property ($I5$) of $I(\cdot)$ we can choose some $l > 0$ such that $I(\mathbf{x}) \geq 0$. We set:

$$\inf_{\mathbf{x}} I(\mathbf{x}) = l. \tag{2.5.61}$$

Then, since $I(\cdot)$ is lower semicontinuous, for every $\varepsilon > 0$, there exists a small disc $D(\tilde{\mathbf{x}})$ around each $\tilde{\mathbf{x}} \in K$ such that

$$I(\mathbf{x}) \geq l - \varepsilon, \quad \forall \mathbf{x} \in D(\tilde{\mathbf{x}}). \tag{2.5.62}$$

Furthermore, since we assume that K is compact, there exists a finite subcover, $\cup_{i=1}^M D(\tilde{\mathbf{x}}_i) \supset K$ extracted from these discs. Consequently,

$$\nu_{\boldsymbol{\lambda}}^n[\boldsymbol{\zeta}^n \in K] \leq \nu_{\boldsymbol{\lambda}}^n\left[\boldsymbol{\zeta}^n \in \cup_{i=1}^N D(\tilde{\mathbf{x}}_i)\right] \leq \sum_{i=1}^M \nu_{\boldsymbol{\lambda}}^n[\boldsymbol{\zeta}^n \in D(\tilde{\mathbf{x}}_i)]$$

Now we can apply the result of Step I on the discs to conclude the proof for compact sets. By (2.5.60) we we obtain

$$\begin{aligned}&\sum_{i=1}^M \nu_{\boldsymbol{\lambda}}^n[\boldsymbol{\zeta}^n \in D(\tilde{\mathbf{x}}_i)] \leq e^{-n\inf_{\mathbf{x}\in D(\tilde{\mathbf{x}}_i)} I(\mathbf{x})} \\ \Rightarrow \quad &\frac{1}{n}\log \nu_{\boldsymbol{\lambda}}^n[\boldsymbol{\zeta}^n \in K] \leq -\inf_{\mathbf{x}\in D(\tilde{\mathbf{x}}_i)} I(\mathbf{x}).\end{aligned}$$

By (2.5.62) this is bounded above by $-(l-\varepsilon)$, and since ε is arbitrary, we can let it go to zero. Then we obtain the upper bound for compact sets with (2.5.61).

Step III: It remains to extend the proof to arbitrary closed sets in Ω.

For this let $C \subset \Omega$ be an arbitrary closed set with

$$\inf_{\mathbf{x}\in C} I(\mathbf{x}) = k.$$

Denote furthermore by R_ρ the compact set defined by

$$S_\rho := \{\mathbf{x} := (x_1, x_2) \,:\, \mathbf{x} \in [0,1] \times [-\rho, \rho]\},$$

which is the rectangle of width 1 and of length 2ρ and denote by $\tilde{R}_\rho$ the set

$$\tilde{R}_\rho := \Omega \setminus R_\rho.$$

Thus $C = (C \cap R_\rho) \cup (C \cap \tilde{R}_\rho)$, where $(C \cap R_\rho)$ and $(R \cap \tilde{S}_\rho)$ are disjoint. Therefore it is enough to compute

$$\begin{aligned} \nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n \in C) &= \nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n \in (C \cap R_\rho)) + \nu^n_{\boldsymbol{\lambda}}\left(\boldsymbol{\zeta}^n \in (C \cap \tilde{R}_\rho)\right) \\ &\le \nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n \in (C \cap R_\rho)) + \nu^n_{\boldsymbol{\lambda}}\left(\boldsymbol{\zeta}^n \in \tilde{R}_\rho\right). \end{aligned} \tag{2.5.63}$$

Since $(C \cap R_\rho)$ is compact, for the first term we can apply step II which asserts that

$$\nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n \in (C \cap R_\rho)) \le e^{-n \inf_{\mathbf{x} \in (C \cap R_\rho)} I(\mathbf{x})}. \tag{2.5.64}$$

It thus remains to find an upper bound for the second term of the right hand side of expression (2.5.63). But this is equal to 0 since

$$\nu^n_{\boldsymbol{\lambda}}\left(\boldsymbol{\zeta}^n \in \tilde{R}_\rho\right) = \nu^n_{\boldsymbol{\lambda}}(r^n \notin [0,1] \text{ and } p^n \notin [-\rho, \rho]) = 0.$$

Thereby with (2.5.63) and (2.5.64), we obtain

$$\frac{1}{n} \log \nu^n_{\boldsymbol{\lambda}}(\boldsymbol{\zeta}^n \in C) \le -\inf_{\mathbf{x} \in (C \cap R_\rho)} I(\mathbf{x}) \le -\inf_{\mathbf{x} \in C} I(\mathbf{x}).$$

for any closed subset of Ω. □

Proof of Varadhan's Lemma

Proof. The proof of Theorem 2.5.10 is divided in two parts:

Upper bound: We claim that

$$\limsup_{n \to \infty} \frac{1}{n} \log \int_{\Omega^n} e^{nF(\boldsymbol{\zeta})} d\nu^n_{\boldsymbol{\lambda}} \le \sup_{\mathbf{x}} \{F(\mathbf{x}) - I(\mathbf{x})\}. \tag{2.5.65}$$

Since F is a bounded continuous function, for each $\delta > 0$, we can find a finite number of closed sets C_j, $j \in \{1, \ldots M\}$ covering Ω:

$$\cup_{j=1}^M C_j \supset \Omega$$

and such that the oscillation of F on each on these closed sets is at most δ, that means:

$$\sup_{\mathbf{x} \in C_j} F(\mathbf{x}) - \inf_{\mathbf{x} \in C_j} F(\mathbf{x}) \le \delta \quad \forall\, j \in \{1, \ldots, M\}.$$

Then we obtain by applying Cramér's Theorem to the closed sets,

$$\begin{aligned}\int_{\Omega^n} e^{nF} d\nu_{\boldsymbol{\lambda}}^n &\leq \sum_{j=1}^{M} \int_{C_j} e^{nF(\boldsymbol{\zeta})} d\nu_{\boldsymbol{\lambda}}^n \\ &\leq \sum_{j=1}^{M} e^{n \sup_{\mathbf{x}\in C_j} F(\mathbf{x})} \int_{C_j} d\nu_{\boldsymbol{\lambda}}^n \\ &\leq \sum_{j=1}^{M} e^{n(\inf_{\mathbf{x}\in C_j} F(\mathbf{x})-\delta)} \nu_{\boldsymbol{\lambda}}^n(\boldsymbol{\zeta}\in C_j) \\ &\leq \sum_{j=1}^{M} e^{n(\inf_{\mathbf{x}\in C_j} F(\mathbf{x})-\delta)} e^{-n\inf_{\mathbf{x}\in C_j} I(\mathbf{x})}.\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \int_{\Omega^n} e^{nF} d\nu_{\boldsymbol{\lambda}}^n &\leq \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{j=1}^{M} e^{n(\inf_{\mathbf{x}\in C_j} F(\mathbf{x})-\delta)} e^{-n\inf_{\mathbf{x}\in C_j} I(\mathbf{x})} \right) \\ &\leq \sup_{j\in\{1,\dots,M\}} \{ \inf_{\mathbf{x}\in C_j} F(\mathbf{x}) - \inf_{\mathbf{x}\in C_j} I(\mathbf{x})\} - \delta \\ &\leq \sup_{j\in\{1,\dots,M\}} \inf_{\mathbf{x}\in C_j} \{F(\mathbf{x}) - I(\mathbf{x})\} - \delta \\ &\leq \sup_{j\in\{1,\dots,M\}} \sup_{\mathbf{x}\in C_j} \{F(\mathbf{x}) - I(\mathbf{x})\} - \delta \\ &= \sup_{\mathbf{x}\in\Omega} \{F(\mathbf{x}) - I(\mathbf{x})\} - \delta\end{aligned}$$

.

Since δ is arbitrary, we can conclude the proof of (2.5.65) by letting δ go to 0.

Lower bound: We now claim that

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_{\Omega^n} e^{nF(\boldsymbol{\zeta})} d\nu_{\boldsymbol{\lambda}}^n \leq \sup_{\mathbf{x}\in\Omega} \{F(\mathbf{x}) - I(\mathbf{x})\}. \tag{2.5.66}$$

To prove this, recall that $F(\cdot)$ is a continuous and $I(\cdot)$ is a lower semi continuous function on Ω. Thus we can find for each $\delta > 0$ a

- $\mathbf{y} \in \Omega$ such that
$$\sup_{x\in\Omega} \{F(\mathbf{x}) - I(\mathbf{x})\} - \delta \leq F(\mathbf{y}) - I(\mathbf{y}),$$
- neighborhood $U_\delta(\mathbf{y})$ of $\mathbf{y}$, such that
$$F(\mathbf{y}) - \delta \leq F(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in U_\delta(\mathbf{y}) \tag{2.5.67}$$

Thereby we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \int_{\Omega^n} e^{nF(\boldsymbol{\zeta})} d\nu_{\boldsymbol{\lambda}} &\geq \liminf_{n\to\infty} \frac{1}{n} \int_{U_\delta(y)} e^{nF(\boldsymbol{\zeta})} d\nu_{\boldsymbol{\lambda}} \\ &\geq \liminf_{n\to\infty} \frac{1}{n} \int_{U_\delta(y)} e^{n(F(\mathbf{y})-\delta)} d\nu_{\boldsymbol{\lambda}} \\ &= F(\mathbf{y}) - \delta + \liminf_{n\to\infty} \frac{1}{n} \nu_{\boldsymbol{\lambda}}(\boldsymbol{\zeta} \in U_\delta(\mathbf{y}))\end{aligned}$$

Applying Cramér's Theorem now for open sets, this is bounded below by

$$F(\mathbf{y}) - \delta - \inf_{\mathbf{x}\in U_\delta(\mathbf{y})} I(\mathbf{x}) \geq F(\mathbf{y}) - I(\mathbf{x}) - \delta.$$

And then by (2.5.67), we obtain the lower bound

$$\sup_{\mathbf{x}\in\Omega}\{F(\mathbf{x}) - I(\mathbf{x})\} - 2\delta.$$

Again we can conclude the proof of claim (2.5.65) with the arbitrariness of δ, by letting it tend to 0. □

Recall that to conclude the proof of the Hydrodynamic limit, we need to show (2.5.12), i.e. that the local Gibbs measures converge exponentially fast. This is stated in the following Corollary to Varadhan's Lemma:

Corollary 2.5.26. *Let $J : [0,1] \to \mathbb{R}$ be any continuous function. Then*

$$\lim_{N\to\infty} \frac{1}{N} \log \nu^N_{\mathtt{u}(\cdot,t)} \left[\left| \frac{1}{N}\sum_{i=1}^N J(\frac{i}{N})\zeta_{\alpha,i} - \int_0^1 J(x)u_\alpha(x,t)dx \right| > \delta \right] \leq -C(\delta). \tag{2.5.68}$$

Here $\nu^N_{\mathtt{u}(\cdot,t)}$ denotes the local Gibbs measures with $\frac{u}{(}x,t) := (u_1(x,t), u_2(x,t))$ and C is some constant depending on δ.

Proof. The Proof is devided in several steps:

Step I: Introduction of block averages over large microscopic boxes.

To prove the Corollary we will need to introduce block averages over size $2k+1$. This will be done by performing a summation by parts, but since we have to handle with boundaries, this must be done carefully. Our claim is that for any smooth and bounded function $J : \mathbb{R} \to \mathbb{R}$ and any bounded function $\psi : \Omega \to \mathbb{R}$, we have:

$$\frac{1}{N}\sum_{i=1}^N J(\frac{i}{N})\psi(\boldsymbol{\zeta}_i) = \frac{1}{N}\sum_{i=k+1}^{N-k} J(\frac{i}{N})\frac{1}{2k+1}\sum_{|j-i|\leq k}\psi(\boldsymbol{\zeta}_j) + \frac{o(N)}{N}. \tag{2.5.69}$$

To prove this, observe that

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N J(\frac{1}{N})\psi(\boldsymbol{\zeta}_i) &= \frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k} J(\frac{i}{N})\sum_{|j-i|\leq k}\psi(\boldsymbol{\zeta}_i) + \frac{1}{N}\sum_{i=1}^{k} J(\frac{i}{N})\frac{1}{i+k}\sum_{j=1}^{i+k}\psi(\boldsymbol{\zeta}_i) \\ &\quad + \frac{1}{N}\sum_{i=N-k+1}^{N} J(\frac{i}{N})\frac{1}{N+k+1-i}\sum_{j=i-k}^{N}\psi(\boldsymbol{\zeta}_i)\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k} J(\frac{i}{N}) \sum_{|j-i|\leq k} \big(\psi(\boldsymbol{\zeta}_i)-\psi(\boldsymbol{\zeta}_j)\big) + \frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k} J(\frac{i}{N}) \sum_{|j-i|\leq k} \psi(\boldsymbol{\zeta}_j) \\
& + \frac{1}{N}\sum_{i=1}^{k} J(\frac{i}{N})\frac{1}{i+k}\sum_{j=1}^{i+k} J(\frac{i}{N}) \big(\psi(\boldsymbol{\zeta}_i)-\psi(\boldsymbol{\zeta}_j)\big) + \frac{1}{N}\sum_{i=1}^{k} J(\frac{i}{N})\frac{1}{i+k}\sum_{j=1}^{i+k}\psi(\boldsymbol{\zeta}_j) \\
& + \frac{1}{N}\sum_{i=N-k+1}^{N} J(\frac{i}{N})\frac{1}{N+k+1-i}\sum_{j=i-k}^{N} J(\frac{i}{N}) \big(\psi(\boldsymbol{\zeta}_i)-\psi(\boldsymbol{\zeta}_j)\big) \qquad (2.5.70) \\
& \qquad + \frac{1}{N}\sum_{i=N-k+1}^{N} J(\frac{i}{N})\frac{1}{N+k+1-i}\sum_{j=i-k}^{N}\psi(\boldsymbol{\zeta}_j).
\end{aligned}
$$

It thus remains to prove that all the terms except the second one of this expression are of order $o(N)$. For the second, third and fourth line this is immediate, since these are sums of k^2 terms at most. Then since $J(\cdot)$ and $\psi(\cdot)$ are bounded functions, the terms are of order $\mathcal{O}(\frac{k}{N})$ and thus they vanish in the limit if k is sent to infinity after N. It then only remains to show that the first term goes to zero in the limit. For this we look at

$$
\frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k}\sum_{j=i-k}^{i+k} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_j)
$$

and change the order of summation. Then arrive at

$$
\begin{aligned}
& \frac{1}{N}\frac{1}{2k+1}\sum_{j=2k}^{N-2k}\sum_{i=j-k}^{j+k} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_j) \\
& \qquad + \frac{1}{N}\frac{1}{2k+1}\sum_{j=1}^{2k-1}\sum_{i=k+1}^{j+k} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_j) + \frac{1}{N}\frac{1}{2k+1}\sum_{j=N-2k+1}^{N}\sum_{i=j-k}^{N} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_j)
\end{aligned}
$$

Again the second and the third terms are sums of the order of k^2 terms. By the same arguments as above, they vanish in the limit as N goes to infinity faster than k. In the first term we make a change of variables and then it is equal to

$$
\frac{1}{N}\frac{1}{2k+1}\sum_{i=2k}^{N-2k}\sum_{j=i-k}^{i+k} J(\frac{j}{N})\psi(\boldsymbol{\zeta}_i)
$$

now going back to the first term of (2.5.70), it can be rewritten as

$$
\begin{aligned}
& \frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k} J(\frac{i}{N}) \sum_{|j-i|\leq k} \big(\psi(\boldsymbol{\zeta}_i)-\psi(\boldsymbol{\zeta}_j)\big) \\
& \quad = \frac{1}{N}\frac{1}{2k+1}\sum_{i=2k}^{N-2k}\sum_{j=i-k}^{i+k}\left(J(\frac{i}{N})-J(\frac{j}{N})\right)\psi(\boldsymbol{\zeta}_i) \\
& \qquad + \frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{2k-1} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_i) + \frac{1}{N}\frac{1}{2k+1}\sum_{i=N-2k+1}^{N-k} J(\frac{i}{N})\psi(\boldsymbol{\zeta}_i) + \mathcal{O}(\frac{k}{N})
\end{aligned}
$$

Here the second and the third terms are sums of $k-1$ terms and thereby are of order $\mathcal{O}(\frac{1}{N})$. The first term on the right hand side is equal to

$$\frac{1}{N}\frac{1}{2k+1}\sum_{i=2k}^{N-2k}\sum_{j=i-k}^{i+k}\left(\frac{i-j}{N}J'(\frac{j}{N})+\mathcal{O}\left(\frac{(i-j)^2}{N^2}\right)\right)\psi(\boldsymbol{\zeta}_i)$$

Then since $i-j\le k$ for all j, by the smoothness of J and since ψ is a bounded function, this term is of order $\mathcal{O}(\frac{k}{N})$. This proves our claim (2.5.69) and thus in the limit as $N\to\infty$ first and then $k\to\infty$, we can replace local bounded functions by their average over large microscopic boxes :

$$\left|\frac{1}{N}\sum_{i=1}^{N}J(\frac{i}{N})\psi(\boldsymbol{\zeta}_i)-\frac{1}{N}\sum_{i=k+1}^{N-k}J(\frac{i}{N})\frac{1}{2k+1}\sum_{|j-i|\le k}\psi(\boldsymbol{\zeta}_j)\right|\le\frac{C_{N,k}}{N}.$$

Where $C_{N,k}$ is a constant small in N and l and depending on $\|J\|_\infty$ and $\|\psi\|_\infty$.

Step II: The exponential Chebychev inequality.

For $\alpha=1,2$ we have that

$$\begin{aligned}&\left|\frac{1}{N}\sum_{i=1}^{N}J(\frac{i}{N})\zeta_{\alpha,i}-\int_0^1 J(x)u_\alpha(x,t)dx\right|\\ \le\;&\left|\frac{1}{N}\sum_{i=1}^{N}J(\frac{i}{N})\left(\zeta_{\alpha,i}-u_\alpha(\frac{i}{N},t)\right)\right|+\left|\int_0^1 J(x)u_{\alpha(x,t)}dx-\frac{1}{N}\sum_{i=1}^{N}J(\frac{i}{N})u_\alpha(\frac{i}{N},t)\right|.\end{aligned}$$

By smoothness of $J(\cdot)$ and $u_\alpha(\cdot,t)$ the second term on the right hand side. In the sequel will be denoted by R_N terms converging to zero as $n\to\infty$.

Recall from Section 2.5.2, that if we replace $\boldsymbol{\zeta}_i$ by the the cut off $\boldsymbol{\zeta}_{i,b}$ the error we make is of order $\frac{C(b)}{\sigma}+\frac{H_N(t)}{N\sigma}$ where $\lim_{b\to\infty}C(b)=0$. By Theorem 2.5.2 this converges to 0 as $N\to\infty$. With this replacement all the functions involved are bounded an we can apply the summation by parts formula (2.5.69): Recall that by $\boldsymbol{\zeta}_i^{2k}:=(\zeta_{1,i}^{2k},\zeta_{1,i}^{2k}):=(r_i^{2k},p_i^{2k})$ we denoted the block average over a box of size $2k+1$ and centered at i with $i\in\{k+1,\dots,N-k\}$:

$$\boldsymbol{\zeta}_{i,b}^{2k}:=\frac{1}{2k+1}\sum_{|i-j|\le k}\boldsymbol{\zeta}_{j,b}.$$

Then by step I:

$$\left|\frac{1}{N}\sum_{i=1}^{N}J(\frac{i}{N})\left(\zeta_{\alpha,i,b}-u_\alpha(\frac{i}{N},t)\right)\right|\le\left|\frac{1}{N}\sum_{i=k+1}^{N-k}\zeta_{\alpha,i,b}^{2k}-\frac{1}{N}\sum_{i=1}^{N}u_\alpha(\frac{i}{N},t)\right|+\frac{C_{k,N}}{N}$$

Then the probability in expression (2.5.68) is bounded above by

$$\nu_{\boldsymbol{\lambda}(\cdot,t)}^N\left[\left|\frac{1}{N}\sum_{i=k+1}^{N-k}\zeta_{\alpha,i,b}^{2k}-\frac{1}{N}\sum_{i=1}^{N}u_\alpha(\frac{i}{N},t)\right|+\frac{C_{k,N}}{N}+R_N>\delta\right].$$

By the exponential Chebychev inequality, we obtain the following upper bound for any $a > 0$:

$$e^{-\delta a} E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}} \left[\exp\left\{ a \left| \frac{1}{N} \sum_{i=k+1}^{N-k} \zeta^{2k}_{\alpha,i,b} - \frac{1}{N} \sum_{i=1}^{N} u_\alpha(\frac{i}{N},t) \right| + a\frac{C_{k,N}}{N} + aR_N \right\} \right]$$
$$= \quad e^{-\delta^2 N} e^{\delta N R_N} e^{\delta C_{k,N}} E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}} \left[\exp\left\{ \delta \left| \sum_{i=k+1}^{N-k} \zeta^{2k}_{\alpha,i,b} - \sum_{i=1}^{N} u_\alpha(\frac{i}{N},t) \right| \right\} \right],$$

where in the second line we chose $a = N\delta$. Furthermore by the inequaliy $e^{|x|} \le e^x + e^{-x}$, we can drop the absolute value for the this proof and then it is enough to prove that:

$$\lim_{N\to\infty} \frac{1}{N} \log \left(e^{-\delta^2 N} e^{\delta N R_N} e^{\delta C_{k,N}} E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}} \left[\exp\left\{ \delta \sum_{i=k+1}^{N-k} \zeta^{2k}_{\alpha,i,b} - \delta \sum_{i=1}^{N} u_\alpha(\frac{i}{N},t) \right\} \right] \right)$$
$$= \lim_{N\to\infty} \left(-\delta^2 + \delta R_N + \frac{\delta C_{k,N}}{N} \right)$$
$$+ \lim_{N\to\infty} \left(\frac{1}{N} \log E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}} \left[\exp\left\{ \delta \sum_{i=k+1}^{N-k} \zeta^{2k}_{\alpha,i,b} - \delta \sum_{i=1}^{N} u_\alpha(\frac{i}{N},t) \right\} \right] \right) \le -C(\delta),$$

Since $\delta^2 > 0$ and

$$\lim_{N\to\infty} R_N = 0, \quad \text{and} \quad \lim_{N\to\infty} \frac{C_{k,N}}{N} = 0,$$

It remains to prove that

$$\lim_{b\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{N} \log E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}} \left[\exp\left\{ \delta \sum_{i=k+1}^{N-k} \zeta^{2k}_{\alpha,i,b} - \delta \sum_{i=1}^{N} u_\alpha(\frac{i}{N},t) \right\} \right] = 0 \tag{2.5.71}$$

Step III: Introduction of disjoint blocks

Recall that we want to apply Lemma 2.5.10. For this we need to replace the local Gibbs measure $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$ by homogeneous ones. To do so, we next regroup the sum such that we can see disjoint blocks and then take advantage of the fact that the local Gibbs measures are product measures: We assume without loss of generality that $\frac{N-2k}{2k+1} \in \mathbb{N}$. Observe that for $r \in \{-k, \cdots, k\}$ and $\alpha = 1, 2$ we can rewrite the sum

$$\sum_{i=k+1+r}^{N-k+r} (\zeta_{\alpha,i,b} - u_\alpha(\frac{i}{N},t)) = (2k+1) \sum_{i=0}^{\frac{N-2k}{2k+1}-1} \frac{1}{2k+1} \sum_{l=i(2k+1)+k+1+r}^{(i+1)(2k+1)+k+r} (\zeta_{\alpha,l,b} - u_\alpha(\frac{l}{N},t))$$

This is a sum of disjoint blocks of size $2k+1$ and thus for a fixed r, the block averages

$$X^r_q := \frac{1}{2k+1} \sum_{l=q+r}^{q+r+2k} (\zeta_{\alpha,l,b} - u_\alpha(\frac{l}{N},t))$$

with $q \in B_{k+1} := \{i(2k+1) + k + 1 : i = 0, \ldots, \frac{N-2k}{2k+1} - 1\}$ are independent under the product measure $\nu^N_{\boldsymbol{\lambda}(\cdot,t)}$. Then, if we shift the r from $-k$ to k and sum these up, we obtain all the blocks which are present in (2.5.71). In other words we can rewrite (2.5.71) as

$$E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}\left[\exp\left\{\delta\sum_{r\in\{-k,\dots,k\}}\sum_{q\in B_{k+1}}X_q^r\right\}\right]=E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}\left[\prod_{r\in\{-k,\dots,k\}}\exp\left\{\delta\sum_{q\in B_{k+1}}X_q^r\right\}\right].$$

Now applying the Hölder inequality we obtain that this is bounded above by

$$\prod_{r\in\{-k,\dots,k\}}\left(E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}\left[\exp\left\{\delta(2k+1)\sum_{q\in B_{k+1}}X_q^r\right\}\right]\right)^{\frac{1}{2k+1}}.$$

Then it remains to estimate

$$\lim_{N\to\infty}\frac{1}{N}\frac{1}{2k+1}\sum_{r\in\{-k,\dots,k\}}\log E_{\nu^N_{\boldsymbol{\lambda}(\cdot,t)}}\left[\prod_{q\in B_{k+1}}\exp\left\{\delta(2k+1)X_q^r\right\}\right].$$

But recall that now in the exponent we have independent variables with respect to the Gibbs measure for each fixed r. Thereby, using the fact that it is a product measure, in the last expression the expectation of the product is equal to the product of the expectation:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\frac{1}{2k+1}\sum_{r\in\{-k,\dots,k\}}\sum_{q\in B_{k+1}}\log E_{\nu^{2k+1}_{\boldsymbol{\lambda}(\cdot,t)}}\left[\exp\left\{\delta(2k+1)X_q^r\right\}\right]\\=&\lim_{N\to\infty}\frac{1}{N}\frac{1}{2k+1}\sum_{i=k+1}^{N-k}\log E_{\nu^{2k+1}_{\boldsymbol{\lambda}(\cdot,t)}}\left[e^{\delta(2k+1)\left(\zeta_{\alpha,i,b}-\frac{1}{2k+1}\sum_{|i-l|\le k}u_\alpha(\frac{l}{N},t)\right)}\right].\end{aligned}$$

In any case, since our parameter $\boldsymbol{\lambda}$ and the solution $\mathfrak{u}$ to the p-system (2.1.3) are smooth, as $N\to\infty$ this converges to:

$$\int_0^1\frac{1}{2k+1}\log E_{\nu^{2k}_{\boldsymbol{\lambda}(x,t)}}\left[\exp\left\{\delta(2k+1)\left(\zeta_{\alpha,i,b}-u_\alpha(x,t)\right)\right\}\right].$$

Step IV: Application of Varadhans Lemma.

Now our parameter for each $x\in[0,1]$ is fixed, consequently the expected value is with respect to a homogenous Gibbs measure. Furthermore since the logarithmic generating moment is finite the quantity inside the integral is uniformly bounded. This means that we are allowed to exchange the integral with the limits sending k and b to infinity by the dominated convergence Theorem. Hence we finally can apply Varadhans Lemma:

$$\int_0^1\lim_{k\to\infty}\frac{1}{2k+1}\log E_{\nu^{2k}_{\boldsymbol{\lambda}(x,t)}}\left[e^{\delta(2k+1)(\zeta_{\alpha,i}-u_\alpha(x,t))}\right]=\int_0^1\sup_{\boldsymbol{\zeta}}\{\delta\left(\boldsymbol{\zeta}-\mathfrak{u}(x,t)\right)-I(\boldsymbol{\zeta})\}dx.$$

To conclude the Theorem it thus remains to show that this is less than zero. For this we just have to chose the $\delta>0$ small enough: Since $\mathfrak{u}$ and $\boldsymbol{\lambda}$ satisfy the dual relation (A.6), we know by property $(I6)$ that $I(\mathfrak{u})=0$. Therefore $\mathbf{F}_\delta(\mathfrak{u},\mathfrak{u})=0$ for each $\delta\in[0,\infty)$, where

$$\mathbf{F}_\delta(\boldsymbol{\zeta},\mathfrak{u}):=\delta\left(\boldsymbol{\zeta}(x,t)-\mathfrak{u}(x,t)\right)-I(\boldsymbol{\zeta}(x,t)).$$

On the other hand $\mathbf{F}_0(\boldsymbol{\zeta},\mathfrak{u})<0$ for each $\boldsymbol{\zeta}\neq\mathfrak{u}$ and $\mathbf{F}_0(\mathfrak{u},\mathfrak{u})=0$. Then we have

$$\sup_{\boldsymbol{\zeta}=\mathfrak{u}}F_\delta(\boldsymbol{\zeta},\mathfrak{u})=0\quad\text{and}\quad\sup_{\boldsymbol{\zeta}\neq\mathfrak{u}}F_0(\boldsymbol{\zeta},\mathfrak{u})<0.$$

Then by the lower semicontinuity there exists a $\delta>0$, such that $\sup_{\boldsymbol{\zeta}}F_\delta(\boldsymbol{\zeta},\mathfrak{u})\le 0$. □

Appendix A

Some useful Functions and their Properties

Here are some frequently used functions and their properties

- **The free energy function:** For $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbb{R} \times \mathbb{R}$ and $\boldsymbol{\zeta} := (r, p) \in \Omega$ we define the free energy function $\Theta : \mathbb{R}^2 \to \mathbb{R}$ as

$$\Theta(\boldsymbol{\eta}) := \log Z(\boldsymbol{\eta}) = \log \int_\Omega e^{\boldsymbol{\eta}\cdot\boldsymbol{\zeta}-h} d\boldsymbol{\zeta} \tag{A.1}$$

By Hölder inequality, it is easy to tee that it is convex and thereby lower semi continuous: for any $\alpha \in [0, 1]$ and for $\boldsymbol{\eta}$, $\tilde{\boldsymbol{\eta}}$

$$\begin{aligned}
\Theta(\alpha\boldsymbol{\eta} + (1-\alpha)\tilde{\boldsymbol{\eta}}) &= \log \int e^{\alpha(\boldsymbol{\eta}\cdot\boldsymbol{\zeta}-h)+(1-\alpha)(\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\zeta}-h)} d\boldsymbol{\zeta} \\
&\leq \log \left[\left(\int e^{\boldsymbol{\eta}\cdot\boldsymbol{\zeta}-h} d\boldsymbol{\zeta} \right)^{\alpha} \cdot \left(\int e^{\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{\zeta}-h} d\boldsymbol{\zeta} \right)^{1-\alpha} \right] \\
&= \alpha\Theta(\boldsymbol{\eta}) + (1-\alpha)\Theta(\tilde{\boldsymbol{\eta}}).
\end{aligned}$$

In summary Θ has the following properties:

(Θ1): lower semi continuous,

(Θ2): convex,

(Θ1): $\Theta(\mathbf{0}) = 0$.

- **The thermodynamic entropy function:** For $\boldsymbol{\xi} := (\xi_1, \xi_2) \in \Omega$ we define the thermodynamic entropy function $\Phi : \Omega \to \mathbb{R}$ by the Legendre transform of Θ

$$\Phi(\boldsymbol{\xi}) := \sup_{\boldsymbol{\lambda}} \{\boldsymbol{\lambda}\cdot\boldsymbol{\xi} - \Theta(\boldsymbol{\lambda})\}. \tag{A.2}$$

Here the supremum is taken over all $\boldsymbol{\lambda} \in \mathbb{R}^2$ It is positive, convex and lower semicontinuous. Convexity can easily be seen by the following:

$$\begin{aligned}
\Phi(\alpha\boldsymbol{\xi}\cdot\boldsymbol{\lambda} + (1-\alpha)\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\lambda}) &= \sup_{\boldsymbol{\lambda}} \left\{ \alpha\left(\boldsymbol{\lambda}\cdot\boldsymbol{\xi} - \Theta(\boldsymbol{\lambda})\right) + (1-\alpha)\left(\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{\xi}} - \Theta(\boldsymbol{\lambda})\right) \right\} \\
&\leq \alpha \sup_{\boldsymbol{\lambda}} \{\boldsymbol{\lambda}\cdot\boldsymbol{\xi} - \Theta(\boldsymbol{\lambda})\} + (1-\alpha) \sup_{\boldsymbol{\lambda}} \left\{\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{\xi}} - \Theta(\boldsymbol{\lambda})\right\}
\end{aligned}$$

To see that it is nonnegative, just notice, that for any $\boldsymbol{\xi}$, we obtain with the definition of Θ that $\mathbf{0}\cdot\boldsymbol{\xi} - \Theta(\mathbf{0}) = 0$, thus

$$\sup_{\boldsymbol{\lambda}} \{\boldsymbol{\lambda}\cdot\boldsymbol{\xi} - \Theta(\boldsymbol{\lambda})\} \geq 0$$

.

Now we want to see the relations between these two functions. First, since $\Theta(\cdot)$ is convex, lower semi continuous and not identically equal to infinity, Θ is the Legendre transform of the thermodynamic entropy. Thus we have

$$\Theta(\boldsymbol{\eta}) = \sup_{\boldsymbol{\zeta}} \{\boldsymbol{\eta}\cdot\boldsymbol{\zeta} - \Phi(\boldsymbol{\zeta})\} \tag{A.3}$$

Since

$$D_{\boldsymbol{\lambda}}[\boldsymbol{\lambda}\cdot\boldsymbol{\xi} - \Theta(\boldsymbol{\lambda})] = \boldsymbol{\xi} - D\Theta(\boldsymbol{\lambda}) = 0 \Leftrightarrow \boldsymbol{\xi} = D\Theta(\boldsymbol{\lambda}) = \int \boldsymbol{\zeta}\, d\nu_{\boldsymbol{\lambda}} := \bar{\boldsymbol{\zeta}},$$

where the last equality is true because of (A.1), we obtain that

$$\Phi(\bar{\boldsymbol{\zeta}}) = \boldsymbol{\lambda}\cdot\bar{\boldsymbol{\zeta}} - \Theta(\boldsymbol{\lambda}). \tag{A.4}$$

On the other hand, with $\bar{\boldsymbol{\zeta}} = \int \boldsymbol{\zeta}\, d\nu_{\boldsymbol{\lambda}}$

$$D_{\bar{\boldsymbol{\zeta}}}[\boldsymbol{\eta}\cdot\bar{\boldsymbol{\zeta}} - \Phi(\bar{\boldsymbol{\zeta}})] = \boldsymbol{\eta} - D\Phi(\bar{\boldsymbol{\zeta}}) = 0 \Leftrightarrow \boldsymbol{\eta} = D\Phi(\bar{\boldsymbol{\zeta}}) = \boldsymbol{\lambda},$$

we obtain that

$$\Theta(\boldsymbol{\lambda}) = \boldsymbol{\lambda}\cdot\bar{\boldsymbol{\zeta}} - \Phi(\bar{\boldsymbol{\zeta}}). \tag{A.5}$$

Therefore in the sequel, we say that $\boldsymbol{\lambda} \in \mathbb{R}^2$ and $\bar{\boldsymbol{\zeta}} \in \Omega$ are in duality if they are related by the formulae:

$$\frac{\partial\Phi}{\partial\bar{\zeta}_\alpha} = \lambda_\alpha \quad \text{and} \quad \frac{\partial\Theta}{\partial\lambda_\alpha} = \bar{\zeta}_\alpha, \quad \alpha = 1, 2. \tag{A.6}$$

If this is the case, then with (A.4) and (A.5) it is then immediate that

$$\Theta(\boldsymbol{\lambda}) + \Phi(\bar{\boldsymbol{\zeta}}) = \boldsymbol{\lambda}\cdot\bar{\boldsymbol{\zeta}}. \tag{A.7}$$

- **The logarithmic moment generating function:** The moment generating function with respect to the probability measure $\nu_{\boldsymbol{\lambda}}$, for $\boldsymbol{\theta} \in \mathbb{R}^2$ is given by

$$M(\boldsymbol{\theta}) := \int_\Omega e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}} d\nu_{\boldsymbol{\lambda}} = \frac{1}{Z(\boldsymbol{\lambda})} \int_\Omega e^{\boldsymbol{\zeta}\cdot(\boldsymbol{\lambda}+\boldsymbol{\theta})-h} d\boldsymbol{\zeta}.$$

Then its logarithm is denoted by $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ with

$$\Lambda(\boldsymbol{\theta}) := \log M(\boldsymbol{\theta}).$$

Notice that with (A.1) the logarithmic moment generating function here is equal to

$$\Lambda(\boldsymbol{\theta}) = \log \int e^{\boldsymbol{\zeta}\cdot(\boldsymbol{\lambda}+\boldsymbol{\theta})-h} d\boldsymbol{\zeta} - \log \int e^{\boldsymbol{\zeta}\cdot\boldsymbol{\lambda}-h} d\boldsymbol{\zeta} = \Theta(\boldsymbol{\lambda}+\boldsymbol{\theta}) - \Theta(\boldsymbol{\lambda}). \tag{A.8}$$

Some important properties of $\Lambda(\cdot)$ are:

(Λ1): $\Lambda(\cdot)$ is convex. This follows by convexity of Θ.

($\Lambda 2$): $\Lambda(\cdot)$ is continuously differentiable with

$$D\Lambda(\boldsymbol{\theta}) = \frac{E_{\nu_{\boldsymbol{\lambda}}}[\boldsymbol{\zeta} e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]}{M(\boldsymbol{\theta})} = E_{\nu_{\boldsymbol{\lambda}+\boldsymbol{\theta}}}[\boldsymbol{\zeta}].$$

This can be deduced with the dominated convergence Theorem: For the first component we have

$$\lim_{\varepsilon\to 0} f_\varepsilon(r) := \lim_{\epsilon\to 0} \frac{e^{(\theta_1+\epsilon)r} - e^{\theta_1 r}}{\varepsilon} = re^{\theta_1 r}$$

pointwise on one hand and on the other hand for any $|\varepsilon| \leq \delta$, $|f_\varepsilon|$ is bounded in the following way: Since for any $x \in \mathbb{R}$, $e^{|x|} \leq e^x + e^{-x}$

$$|f_\varepsilon(r)| \leq \frac{e^{\theta_1 r}\left(e^{\delta|r|} - 1\right)}{\varepsilon} \leq \frac{e^{(\theta_1+\delta)r} + e^{(\theta_1-\delta)r}}{\varepsilon} = h_\varepsilon(r).$$

Now, for any $\theta_1 \in \mathbb{R}$, we can chose $\delta > 0$ small enough, such that

$$E_{\nu_{\boldsymbol{\lambda}}}[h_\varepsilon(r)] \leq E_{\nu_{\boldsymbol{\lambda}}}[e^{\theta_1+\delta}] + E_{\nu_{\boldsymbol{\lambda}}}[e^{\theta_1-\delta})] = M(\theta_1+\delta,\theta_2) + M(\theta_1-\delta,\theta_2) < \infty.$$

Then by the dominated convergence Theorem we obtain:

$$\lim_{\epsilon\to 0} E_{\nu_{\boldsymbol{\lambda}}}\left[\frac{e^{(\theta_1+\epsilon)r} - e^{\theta_1 r}}{\epsilon} e^{\theta_2 p}\right] = E_{\nu_{\boldsymbol{\lambda}}}[re^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}] = \frac{\partial M(\boldsymbol{\theta})}{\partial\theta_1}$$

In the similar way we can deduce that

$$\frac{\partial M(\boldsymbol{\theta})}{\partial\theta_2} = E_{\nu_{\boldsymbol{\lambda}}}[pe^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]..$$

Thus it follows that $D\Lambda(\boldsymbol{\theta}) = \frac{DM(\boldsymbol{\theta})}{M(\boldsymbol{\theta})} = \frac{E_{\nu_{\boldsymbol{\lambda}}}[\boldsymbol{\zeta} e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]}{M(\boldsymbol{\theta})}$

($\Lambda 3$): Λ is in $C^2(\mathbb{R}^2)$. This can be obtained by the same arguments as for ($\Lambda 2$). One obtains that the second partial derivatives are for α and $\beta = 1, 2$:

$$\frac{\partial^2\Lambda(\boldsymbol{\theta})}{\partial\theta_\alpha\partial\theta_\beta} = \frac{E_{\nu_{\boldsymbol{\lambda}}}[\zeta_\alpha\zeta_\beta e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]}{M(\boldsymbol{\theta})} - \frac{E_{\nu_{\boldsymbol{\lambda}}}[\zeta_\alpha e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]\cdot E_{\nu_{\boldsymbol{\lambda}}}[\zeta_\beta e^{\boldsymbol{\theta}\cdot\boldsymbol{\zeta}}]}{M^2(\boldsymbol{\theta})}$$

($\Lambda 4$): $\Lambda(\cdot)$ is strictly convex: Computing the Hessian Matrix explicitely, we obtain:

$$\begin{aligned} D^2\Lambda(\boldsymbol{\theta}) &= \begin{pmatrix} E_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r^2] & (\lambda_2+\theta_2)E_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r] \\ (\lambda_2+\theta_2)E_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r] & 1+(\lambda_2+\theta_2)^2 \end{pmatrix} \\ \Rightarrow \left|D^2\Lambda(\boldsymbol{\theta})\right| &= E_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r^2](1+(\lambda_2+\theta_2)^2) - (\lambda_2+\theta_2)^2 E^2_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r] \\ &\geq E^2_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r]\left(1+(\lambda_2+\theta_2)^2 - (\lambda_2+\theta_2)^2\right) = E^2_{\nu_{\boldsymbol{\theta}+\boldsymbol{\lambda}}}[r] > 0. \end{aligned}$$

($\Lambda 5$): $\Lambda(\mathbf{0}) = 0$. This follows immediately from (A.8).

- **The rate function:** We will need the rate function which is defined by the Legendre transform of the logarithmic generating function Λ. For each $\mathbf{x} \in \Omega$, $I : \Omega \to \mathbb{R}$ is given by :

$$I(\mathbf{x}) := \sup_{\boldsymbol{\theta}} \{\mathbf{x} \cdot \boldsymbol{\theta} - \Lambda(\boldsymbol{\theta})\}$$

Notice that with (A.8) and (A.2), and the definition of the thermodynamic entropy Φ, the function I has the form :

$$\begin{aligned} I(\mathbf{x}) &= \sup_{\boldsymbol{\theta}} \{\mathbf{x} \cdot \boldsymbol{\theta} - \Theta(\boldsymbol{\lambda} + \boldsymbol{\theta}) + \Theta(\boldsymbol{\lambda})\} \\ &= \sup_{\boldsymbol{\theta}} \{\mathbf{x} \cdot (\boldsymbol{\lambda} + \boldsymbol{\theta}) - \Theta(\boldsymbol{\lambda} + \boldsymbol{\theta})\} - \mathbf{x} \cdot \boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}) \\ &= \Phi(\mathbf{x}) - \mathbf{x} \cdot \boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}). \end{aligned}$$

Some properties of the rate function are the following:

($I1$): $I(\cdot)$ is non negative. This follows with the definition of the entropy Φ:

$$I(\mathbf{x}) = \sup_{\boldsymbol{\theta}} \{\mathbf{x} \cdot \boldsymbol{\theta} - \Theta(\boldsymbol{\theta})\} - (\mathbf{x} \cdot \boldsymbol{\lambda} - \Theta(\boldsymbol{\lambda})) \geq 0.$$

($I2$): It is convex: For any $\alpha \in [0, 1]$, and $\mathbf{x}$, $\tilde{\mathbf{x}} \in \Omega$, by convexity of Φ,

$$\begin{aligned} I(\alpha\mathbf{x} + (1-\alpha)\tilde{\mathbf{x}}) &= \Phi(\alpha\mathbf{x} + (1-\alpha)\tilde{\mathbf{x}}) - \alpha\mathbf{x} \cdot \boldsymbol{\lambda} - (1-\alpha)\tilde{\mathbf{x}}\boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}) \\ &\leq \alpha\Phi(\mathbf{x}) - \alpha(\mathbf{x}) \cdot \boldsymbol{\lambda} + (1-\alpha)\Phi(\tilde{\mathbf{x}}) - (1-\alpha)\tilde{\mathbf{x}}\boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}) \\ &= \alpha I(\mathbf{x}) + (1-\alpha) I(\tilde{\mathbf{x}}) \end{aligned}$$

($I3$): $I(\cdot)$ is lower semi continuous.

($I4$): $I(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$

($I5$): For each $l < \infty$ the set $\{\mathbf{x} : I(\mathbf{x}) \leq l\}$ is a compact set in Ω, By Property ($I4$) the level sets are bounded and closed by continuity of I.

Furthermore, if $\mathbf{x}$ and $\boldsymbol{\lambda}$ satisfy the dual relation (A.6), that means if $\mathbf{x} = \bar{\boldsymbol{\zeta}} := D\Theta(\boldsymbol{\lambda}) = \int \boldsymbol{\zeta} d\nu_\lambda$, then we obtain

($I6$): $I(\bar{\boldsymbol{\zeta}}) = 0$ and thus $\bar{\boldsymbol{\zeta}}$ is a minimum: By Jensen's inequality, for all $\boldsymbol{\theta}$

$$M(\boldsymbol{\theta}) = \int_\Omega e^{\boldsymbol{\theta} \cdot \boldsymbol{\zeta}} d\nu \boldsymbol{\lambda} \geq e^{\int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\zeta} d\nu_{\boldsymbol{\lambda}}} = e^{\boldsymbol{\theta} \bar{\boldsymbol{\zeta}}}$$

and thereby

$$\bar{\boldsymbol{\zeta}} \cdot \boldsymbol{\theta} - \log M(\boldsymbol{\theta}) = \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{\theta} - \Lambda(\boldsymbol{\theta}) \leq 0.$$

on the other hand, with property ($\Lambda5$), $\Lambda(0) = 0$. But since we know from ($I1$) that $I(\cdot)$ is nonnegative, it must be true that $I(\bar{\boldsymbol{\zeta}}) = 0$.

($I7$): $DI(\bar{\boldsymbol{\zeta}}) = 0$ since we have

$$DI(\bar{\boldsymbol{\zeta}}) = D\Phi(\bar{\boldsymbol{\zeta}}) - \boldsymbol{\lambda} = D\Phi(\bar{\boldsymbol{\zeta}}) - D\Phi(\bar{\boldsymbol{\zeta}}) = 0$$

Now again we want to see the relation between the logarithmic moment generating function and the rate function:

Since we have:

$$D_\theta\{\mathbf{x}\cdot\boldsymbol{\theta} - \Theta(\boldsymbol{\lambda}+\boldsymbol{\theta}) + \Theta(\boldsymbol{\lambda})\} = \mathbf{x} - D\Theta(\boldsymbol{\lambda}+\boldsymbol{\theta}) = 0 \Leftrightarrow \mathbf{x} = D\Theta(\boldsymbol{\lambda}+\boldsymbol{\theta}) = D\Lambda(\boldsymbol{\theta}),$$

where by property ($\Lambda 2$), we obtained that

$$D\Lambda(\boldsymbol{\theta}) = E_{\nu_{\boldsymbol{\lambda}+\boldsymbol{\theta}}}[\boldsymbol{\zeta}] := \mathbf{y},$$

it follows that the $\mathbf{y}$ the supremum is attained at

$$I(\mathbf{y}) = \mathbf{y}\cdot\boldsymbol{\theta} - \Lambda(\boldsymbol{\theta}) \tag{A.9}$$

Furthermore since I is convex lower semicontinuous and not identically equal to infinity, Fenchel-Moreau's Theorem implies that Λ is the Legendre transform of I, that is

$$\Lambda(\boldsymbol{\theta}) = \sup_{\mathbf{x}}\{\mathbf{x}\cdot\boldsymbol{\theta} - I(\mathbf{x})\}.$$

And then we obtain

$$\begin{aligned} & D_{\mathbf{x}}\{\mathbf{x}\cdot\boldsymbol{\theta} - I(\mathbf{x})\} = 0 \\ \Leftrightarrow\quad & \boldsymbol{\theta} = DI(\mathbf{x}) := \mathbf{y}^* \\ \Rightarrow\quad & \Lambda(\mathbf{y}^*) = \mathbf{y}^*\cdot\mathbf{x} - I(\mathbf{x}) \end{aligned}$$

Furthermore since we proved in property ($\Lambda 4$) that Λ is strictly convex, together with (A.9) there exists a diffeomorphism between Ω and $\mathbb{R}^2$

$$(D^2\Lambda)(\mathbf{y}) = [D^2 I(\mathbf{y}^*)]^{-1}, \quad \text{with} \quad \mathbf{y} := D\Lambda(\mathbf{y}^*) \quad \text{and} \quad \mathbf{y}^* := DI(\mathbf{y}). \tag{A.10}$$

Index of frequently used notations

CHAPTER 2:

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