Augmented Lagrangian Methods invoking (Proximal) Gradient-type Methods for (Composite) Structured Optimization Problems

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Dissertation

Abstract

This thesis, first, is devoted to the theoretical and numerical investigation of an augmented Lagrangian method for the solution of optimization problems with geometric constraints, subsequently, as well as constrained structured optimization problems featuring a composite objective function and set-membership constraints. It is then concerned to convergence and rate-of-convergence analysis of proximal gradient methods for the composite optimization problems in the presence of the Kurdyka–Łojasiewicz property without global Lipschitz assumption.

We employ an (safeguarded) augmented Lagrangian approach for the optimization problems with structured geometric constraints, which, to the best of my knowledge, is at the first time applied to such programs with general constraints. Specifically, we study the situations where parts of the constraints are nonconvex and possibly complicated. The key idea behind our method is to keep those complicated constraints explicitly in the constraints and to penalize only the remaining constraints by an (safeguarded) augmented Lagrangian function. The resulting subproblems are then solved by the spectral gradient method, a problem-tailored projected gradient-type method, which generates an inexact Mordukhovich-stationary point of these subproblems rather than Clarke-stationary point like most works. Furthermore, spectral gradient method is first here generalized to solve the programs with nonconvex constraint set. Nevertheless, the overall algorithm computes so called M-stationary points of the original problem under a mild asymptotic regularity condition. Some illustrative numerical examples visualize the power of our approach.

We then generalize such optimization problem to the composite one where the sum of a continuously differentiable function and a merely lower semicontinuous function needs to be minimized. Inspired by the above work, we study the stationarity and regularity concepts, note that the latter is not a constraint qualification in the narrow sense since it is relevant to the nonsmooth part of the objective function. Meanwhile, an (safeguarded) augmented Lagrangian scheme is used to penalize the constraints which have been reformulated by the slack variables, the resulting subproblems are solved approximately by PANOC$^+$ which is a kind of proximal gradient method. Then M-stationarity of the original problem is derived eventually under a mild asymptotic regularity. Some numerical examples demonstrate the effectiveness of the algorithm and the versatility of the constrained composite programs, furthermore the accelerated PANOC$^+$ has a good performance on the bad-scaling and ill-conditioning.

It is well known that proximal gradient algorithms are the classical methods to solve the composite optimization problems. Most existing papers show the convergence of the entire sequence by means of the global Lipschitz gradient of the differentiable part of and Kurdyka–Łojasiewicz property of the objective function. Note that the requirement of the global Lipschitz gradient is very restrictive except if the objective function is guaranteed to be quadratic. Therefore some recent contributions try to overcome the global Lipschitz condition by replacing it with a local one, where, however, the convergence of the entire sequence fails to be obtained, despite every accumulation point of the generated sequence being M-stationary. We, in this work, recover the convergence of the entire sequence and hence the rate-of-convergence results of proximal gradient method only with the aid of the local Lipschitz condition as well as the Kurdyka–Łojasiewicz property, neither the global Lipschitz condition nor the boundedness of iterates and stepsizes.

Zusammenfassung

Diese Diplomarbeit widmet sich zunächst der theoretischen und numerischen Untersuchung eines erweiterten Lagrange-Verfahrens zur Lösung von Optimierungsproblemen mit geometrischen Nebenbedingungen, in weiterer Folge, sowie eingeschränkten strukturierten Optimierungsproblemen mit einer zusammengesetzten Zielfunktion und Mengenzugehörigkeitsbeschränkungen. Es befasst sich dann mit der Konvergenz- und Konvergenzanalyse von Proximalgradientenverfahren für zusammengesetzte Optimierungsprobleme in Gegenwart der Kurdyka–Łojasiewicz-Eigenschaft ohne globale Lipschitz-Annahme.

Für die Optimierungsprobleme mit strukturierten geometrischen Nebenbedingungen verwenden wir einen (abgesicherten) erweiterten Lagrange-Ansatz, der meines Wissens zum ersten Mal auf solche Programme mit allgemeinen Nebenbedingungen angewendet wird. Insbesondere untersuchen wir die Situationen, in denen Teile der Beschränkungen nicht konvex und möglicherweise kompliziert sind. Die Schlüsselidee hinter unserer Methode besteht darin, diese komplizierten Einschränkungen explizit in den Einschränkungen zu halten und nur die verbleibenden Einschränkungen durch eine (abgesicherte) erweiterte Lagrange-Funktion zu bestrafen. Die resultierenden Teilprobleme werden dann durch die Spektralgradientenmethode gelöst, eine auf Probleme zugeschnittene Methode des projizierten Gradiententyps, die einen ungenauen Mordukhovich-stationären Punkt dieser Teilprobleme erzeugt und nicht wie die meisten Arbeiten einen Clarke-stationären Punkt. Darüber hinaus wird hier zunächst das Spektralgradientenverfahren verallgemeinert, um die Programme mit nichtkonvexem Beschränkungssatz zu lösen. Trotzdem berechnet der Gesamtalgorithmus sogenannte M-stationäre Punkte des ursprünglichen Problems unter einer milden asymptotischen Regelmäßigkeitsbedingung. Einige anschauliche Zahlenbeispiele veranschaulichen die Leistungsfähigkeit unseres Ansatzes.

Wir verallgemeinern dann ein solches Optimierungsproblem auf das zusammengesetzte Problem, bei dem die Summe einer stetig differenzierbaren Funktion und einer lediglich niedrigeren halbstetigen Funktion minimiert werden muss. Inspiriert von der obigen Arbeit untersuchen wir die Konzepte Stationarität und Regularität, beachten Sie, dass letzteres keine Einschränkungsqualifikation im engeren Sinne ist, da es für den nicht glatten Teil der Zielfunktion relevant ist. Währenddessen wird ein (abgesichertes) erweitertes Lagrange-Schema verwendet, um die durch die Schlupfvariablen neu formulierten Einschränkungen zu bestrafen, die resultierenden Teilprobleme werden ungefähr durch PANOC $^+$ gelöst, was eine Art proximale Gradientenmethode ist. Dann wird schließlich die M-Stationarität des ursprünglichen Problems unter einer milden asymptotischen Regelmäßigkeit abgeleitet. Einige numerische Beispiele demonstrieren die Effektivität des Algorithmus und die Vielseitigkeit der eingeschränkten zusammengesetzten Programme, außerdem hat das beschleunigte PANOC$^+$ eine gute Leistung bei schlechter Skalierung und schlechter Konditionierung.

Es ist allgemein bekannt, dass proximale Gradientenalgorithmen klassische Methoden sind, um die zusammengesetzten Optimierungsprobleme zu lösen. Die meisten existierenden Arbeiten zeigen die Konvergenz der gesamten Folge mittels des globalen Lipschitz-Gradienten des differenzierbaren Teils von und der Kurdyka–Łojasiewicz-Eigenschaft der Zielfunktion. Beachten Sie, dass die Anforderung des globalen Lipschitz-Gradienten sehr restriktiv ist, es sei denn, die Zielfunktion ist garantiert quadratisch. Einige neuere Beiträge versuchen daher, die globale Lipschitz-Bedingung zu überwinden, indem sie sie durch eine lokale ersetzen, wobei jedoch die Konvergenz der gesamten Folge nicht erreicht wird, obwohl jeder Häufungspunkt der erzeugten Folge M-stationär ist. Wir gewinnen in dieser Arbeit die Konvergenz der gesamten Sequenz und damit die Konvergenzratenergebnisse der proximalen Gradientenmethode nur mit Hilfe der lokalen Lipschitz-Bedingung sowie

der Kurdyka–Łojasiewicz-Eigenschaft, auch nicht die globale Lipschitz-Bedingung noch die Beschränktheit von Iterationen und Schrittweiten.

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Contents

Symbols and Abbreviations

Abbreviation

a.e.	almost everywhere
e.g.	for example
etc.	and so forth
i.e.	that is
w.r.t.	with respect to
s.t.	subject to
lsc	lower semicontinuous
usc	upper semicontinuous
MPCC	mathematical programs with complementarity constraints

Sets and Relations

$\emptyset$	empty set
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+, \mathbb{R}_{++}$	set of nonnegative and positive real numbers
$\overline{\mathbb{R}}$	set of extended real numbers, i.e. $\mathbb{R} \cup \{+\infty\}$
$\mathbb{R}^n$	space of n-dimensional real vectors
$\mathbb{N}$	set of natural numbers
$B_\varepsilon(x)$	open ball with radius $\varepsilon > 0$ around x
$(a, b), [a, b]$	open and closed intervals, respectively
$A \cup B$	union of A and B
$A \cap B$	intersection of A and B
$A \subset B$	A is a subset of B
$\lesssim$	partial ordering

Matrices and Vectors

I	identity matrix in $\mathbb{R}^{n\times n}$
$\mathbb{R}^{m\times n}$	set of real $m \times n$ matrices
$\mathbb{R}^{n\times n}_{\text{sym}}$	set of symmetric, real, semipositive definite $n \times n$ matrices
$A \succeq 0$	positive semidefinite matrix A

$\lambda_{\min}(A), \lambda_{\max}(A)$	smallest and largest eigenvalue of a symmetric matrix A
$\operatorname{diag}(x)$	diagonal matrix with entries $\operatorname{diag}(x)_{[ii]} = x_i$ for $i = 1, \ldots, n$
$\langle \cdot, \cdot \rangle$	Euclidean inner product, i.e. $\langle x, y \rangle = x^T y$
$\\| \cdot \\|, \\| \cdot \\|_2$	Euclidean norm, i.e. $\\|x\\|^2 = \\|x\\|_2^2 = \langle x, x \rangle$
$\\|x\\|_0$	number of nonzero entries of the vector x
$\\| \cdot \\|_1$	ℓ_1-norm, i.e. $\\|x\\|_1 = \sum_{i=1}^n \lvert x_i \rvert$
$\\| \cdot \\|_p$ $(0 < p < 1)$	ℓ_p-norm, i.e. $\\|x\\|_p = \sum_{i=1}^n \lvert x_i \rvert^p$
$\\| \cdot \\|_\infty$	ℓ_∞-norm, i.e. $\\|x\\|_\infty = \max_{i=1,\ldots,n} \lvert x_i \rvert$
$\langle \cdot, \cdot \rangle_H$	inner product induced by $H \in \mathbb{S}^n_{++}$, i.e. $\langle x, y \rangle_H = x^T H y$
$\\| \cdot \\|_H$	norm induced by $H \in \mathbb{S}^n_{++}$, i.e. $\\|x\\|_H^2 = \langle x, x \rangle_H$

Sequences

$\{x^k\}$	sequence of points $(k = 1, 2, \ldots)$
$\{x^k\}_{\mathcal{K}}$	subsequence of $\{x^k\}$ with $k \in \mathcal{K}$
$x^k = \mathcal{O}(y^k)$,	Landau-Symbols for sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{y^k\} \subset \mathbb{R}$,
$x^k = o(y^k)$	i.e. $\limsup_{k \to \infty} \frac{\\|x^k\\|}{\lvert y^k \rvert} < +\infty$, $\lim_{k \to \infty} \frac{x^k}{y^k} = 0$, respectively.
$t_k \downarrow 0$	convergence of the sequence $\{t_k\} \subset \mathbb{R}_+$ to 0 from above

Functions and operations on functions

$\operatorname{dom} \phi$	domain of the function $\phi : \mathbb{X} \to \overline{\mathbb{R}}$
Id	identity mapping
$\operatorname{lev}_{\leq \alpha} \phi$	lower level set of ϕ at the level $\alpha \in \mathbb{R}$, i.e. $\operatorname{lev}_{\leq \alpha} \phi = \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\}$
$I_C(\cdot)$	indicator function of the set C
$P_C(\cdot)$ or $\Pi_C(\cdot)$	projection(s) on the set C
$\partial \phi(x)$	limiting subdifferential or Clarke's generalized gradient of ϕ in x
$\nabla \phi(x)$	gradient of function $\phi : \mathbb{X} \to \mathbb{R}$ in x

Cones and Subdifferentials

$\mathcal{T}_D(\cdot)$	Bouligand tangent cone of set D
$\mathcal{T}_D^C(\cdot)$	Clarke tangent cone of set D
$\mathcal{N}_D^F(\cdot)$	regular normal cone of set D
$\mathcal{N}_D^{\lim}(\cdot)$	limiting normal cone of set D
$\mathcal{N}_D^C(\cdot)$	Clarke normal cone of set D
$\partial^F g(\cdot)$	regular subdifferential of g
$\partial^{\lim} g(\cdot)$	limiting subdifferential of g
$\partial^\infty g(\cdot)$	horizon subdifferential of g

Stationarities and Constraint Qualifications

B-stationarity	Bouligand-stationarity
C-stationarity	Clarke stationarity
M-stationarity	Mordukhovich stationarity
AM-stationarity	asymptotic Mordukhovich stationarity
S-stationarity	strong stationarity
ACQ	Abadie constraint qualification
AM-regularity	asymptotic Mordukhovich regularity
CPLD	constraint positive linear dependence condition
RCPLD	relaxed constraint positive linear dependence condition
∂^∞-RCPLD	relaxed constraint positive linear dependence condition w.r.t. horizontal subdifferential
KKT	Karush-Kuhn-Tucker conditions
AKKT	approximate Karush-Kuhn-Tucker conditions
LICQ	linear independence constraint qualification
MFCQ	Mangasarian-Fromovitz constraint qualification
GMFCQ	generalized Mangasarian-Fromovitz constraint qualification
NNAMCQ	no nonzero abnormal multiplier constraint qualification

1. Introduction

Constrained optimization problems have varieties of applications in the practical areas of physics, engineering, economics, medicine, information science, and so on. Augmented Lagrangian or multiplier penalty methods are the most classical methods for tracing the solution(s) of the constrained nonlinear programs, see [31] as well as the other references on this topic [31, 34, 62]. In particular, the augmented Lagrangian framework can deal with large-scale nonconvex constrained problems, enjoying good warm-starting capabilities and avoiding ill-conditioning, which is often superior to pure penalty methods, because of a pure penalty approach to deal with constraints without softening them, [145, 148]. As the augmented Lagrangian scheme constitutes a framework, rather than a single algorithm, several augmented Lagrangian-type methods have been presented in the past decades, expressing the foundational ideas in different flavors. Some prominent contributions are those in [31, 34, 62, 79, 101, 145], and for primal-dual methods [76]. The recent book [34] proposed a slight modification of this classical augmented Lagrangian method, which uses a safeguarded update of the Lagrange multipliers and has stronger global convergence properties compared to the classical augmented Lagrangian or multiplier penalty approaches. Moreover, the so-called safeguarded augmented Lagrangian method has been applied to many kinds of optimization problems with disjunctive constraints, see [7, 81, 92, 97, 135].

In this thesis, a safeguarded augmented Lagrangian method is allowed to trace the suitable stationary points of the (composite) nonlinear nonconvex optimization problems with generally structured geometric constraints. The core idea behind the approach is to penalize the set-membership constraints by the augmented Lagrangian function and keep the remaining simple-to-project constraints (if they exist) explicitly in the constraints. Then some approaches need to be sought for the solutions of the resulting subproblems.

We in this thesis take two cases of the objective function of original problem (as well as the corresponding subproblems) into account, on the one hand, the objective function is single and continuous, then the solution methods for the resulting (constraint or unconstraint) subproblems are reviewed in the books [31, 34]. On the other hand, the optimization problem has a generally composite objective function with a nonsmooth part, it can naturally fall into the above one by easily setting the nonsmooth operator as zero mapping. Conversely, a classical program with continuously single objective function can be interpreted as a composite one if the nonsmooth part is regarded as the indicator function of some set constraints of this problem. Therefore it seems a nearby idea to exploit the composite form, i.e., specifically the sum of a continuously differentiable function and a lower semicontinuous (typically nonsmooth) function, so that the underlying problems are solved successfully. More precisely, the so-called proximal mapping of the nonsmooth objective function must be available. The idea behind the definition of proximal mappings is to interrelate the search for minimizers (or at least stationary points) with a fixed-point problem, and to apply a fixed-point iteration to the proximal mapping in order to tackle the minimization of the underlying function. Combining the available oracles for the composite objective functions in order to construct an algorithm to minimize the composite programs leads to the development of so-called proximal methods, inaugurated by Moreau [125], which can handle some nonsmooth, nonconvex and extended real-valued cost functions, see [61, 95, 130, 149].

The relationship between augmented Lagrangian and proximal methods could be traced back to Rockafellar [139]. These approaches have been included in [69], where some unconstrained, composite optimization problems whose nonsmooth term is convex are considered. Inspired by this, the proximal augmented Lagrangian method has been considered for constrained composite programs in [63, Chapter 1], however lacking of theoretical guarantee and convergence analysis. A first step to overcome these drawbacks is constituted by proximal gradient method, which dates back to [75]. It is worth to note that proximal gradient algorithms can be interpreted as so-called forward-backward splitting methods, see [45, 131] for their origins and [19] for a modern review. Meanwhile, proximal gradient methods can cope with local Lipschitz continuous gradient of the part of smooth cost function in the Euclidean setting, see [67, 95].

Inspired by the above works, proximal gradient-type algorithms can be adopted as inner solvers for augmented Lagrangian subproblems arising from the composite problems with general nonlinear constraints. When the objective function is continuously single, they are reduced as gradient descent-type methods.

This thesis first consider the optimization problem

$$\min_x \; f(x) \qquad \text{s.t.} \quad G(x) \in C, \; x \in D, \tag{P}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are Euclidean spaces, i.e., real and finite-dimensional Hilbert spaces, $f\colon \mathbb{X} \to \mathbb{R}$ and $G\colon \mathbb{X} \to \mathbb{Y}$ are continuously differentiable, $C \subset \mathbb{Y}$ is nonempty, closed, and convex, whereas the set $D \subset \mathbb{X}$ is only assumed to be nonempty and closed (not necessarily convex). Later, a more general composite optimization problem is considered

$$\min_x \; q(x) := f(x) + g(x) \qquad \text{s.t.} \quad c(x) \in K, \tag{CP}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are Euclidean spaces, $f : \mathbb{X} \to \mathbb{R}$ and $c : \mathbb{X} \to \mathbb{Y}$ are smooth functions, $g : \mathbb{X} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is proper and lower semicontinuous, and $K \subset \mathbb{Y}$ is a nonempty closed set. (CP) is called a constrained composite optimization problem because it contains set-membership constraints and a composite objective function $q := f + g$. Obviously, (CP) covers (P) totally by setting $g := 0$, $c := (G; \mathrm{Id})$, and $K := (C; D)$ (misuse of spaces). Notice that the above f, g, G, c, D, and K, can be nonconvex, then (P) and (CP) are fully nonconvex optimization problem. These settings are pretty general and cover, amongst others, standard nonlinear programs, second-order cone and, more generally, conic optimization problems [26, 53], as well as many so-called disjunctive programming problems like mathematical programs with vanishing, cardinality, complementarity, or switching constraints, see [28, 29, 72, 119] for an overview and suitable references. Since $\mathbb{X}$ and $\mathbb{Y}$ are finite Hilbert spaces, our models also cover matrix optimization problems like semidefinite programs, low-rank approximation problems, or matrix completion problems [116].

One of the aims of the thesis is to apply safeguarded augmented Lagrangian methods for finding the suitable stationary points of (P) and (CP), the resulting subproblems could then be solved by gradient-type methods and proximal gradient-type methods, respectively, which once return the approximate stationary points, the overall approaches trace finally so-called Mordukhovich-stationary points of the original problems with the aid of a mild asymptotic regularity condition. The more specific processes are described in Chapter 3 and Chapter 4.

We subsequently consider the following unconstrained composite optimization problem

$$\min_{x} \; q(x) := f(x) + g(x) \qquad \text{s.t.} \quad x \in \mathbb{X}, \tag{Q}$$

where $f\colon \mathbb{X} \to \mathbb{R}$ is continuously differentiable, $g\colon \mathbb{X} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is lower semicontinuous (possibly infinite-valued and nondifferentiable), and $\mathbb{X}$ denotes an Euclidean space. It is evidently a general version of (CP). In order to minimize the function $q\colon \mathbb{X} \to \overline{\mathbb{R}}$ in (Q), as mentioned above, its composite structure which allows for gradient steps with respect to the continuously differentiable function f on the one hand and so-called proximal steps with respect to g on the other hand, will be exploited, i.e., a splitting approach will be applied. Throughout the last decades, experiments on numerous practically relevant optimization problems have shown that splitting methods are superior to the direct applications of standard methods from nonsmooth optimization to the function g. Most of the relevant works about such type of approaches assume that g is convex [21,35,151,157]. Subsequently, the nonconvex setting g currently has gained more attention [67,95,112,161] like the general setting (Q). It is worth to note that the requirement of global Lipschitz ∇f is indispensable for the convergence theories in the majority of associated works, which recently has been reduced in [95] by a local Lipschitz one. However, the authors just obtained the convergence result of the generated subsequence, rather than the entire sequence. Motivated by [95], in Chapter 5, we recall a proximal gradient method proposed by [95] for (Q) and continue to exploit the convergence results with respect to the whole sequence. The contributions in Chapter 5 are that the entire sequence generated by the proximal gradient method converges to a limit point with a suitable rate, provided that this point satisfies the Kurdyka–Łojasiewicz property with respect to the objective function. The underlying convergence theory is still based on a merely local Lipschitz assumption on ∇f, neither its global Lipschitzness nor the (a priori) boundedness of the iterates and stepsizes is presumed. To this end, it is stressed that the convergence analysis is independent on any kind of (global) descent lemma and any additional convexity assumptions.

This thesis is virtually a summary of two papers [66, 94] and one preprint [93], which will be presented in a more unified way as soon as possible.

This thesis is organized as follows: Chapter 2 is devoted to some useful notations, definitions, and results that are required to better understand the following chapters. In this context, Section 2.1 recalls some common notes and basic preliminaries, and some properties of local Lipschitz function, rate-of-convergence theory, and many types of cones and subdifferentials which play fundamental roles through this thesis. Section 2.2 introduces some basic concepts about nonlinear optimization problems, including stationary points as well as some well-known constraint qualifications. We give some discussions about Kurdyka–Łojasiewicz function in Section 2.3.

In Chapter 3, an augmented Lagrangian approach is proposed to solve the optimization problems with structured geometric constraints (P), which is based on [94]. Therefore, Section 3.1 tells the motivation that we use the (safeguarded) augmented Lagrangian methods to address the optimization problems with the general forms, which are always used to solve some specific optimization problems with special constraints, such as equality and (or) inequality constraints. In Section 3.3, we generalize the spectral gradient method, which is initially proposed for customizing the problems with convex constraints, here in order to solve constrained optimization problems with nonconvex sets and furthermore find the approximate M-stationary points of the original problem. The safeguarded version of the augmented Lagrangian method and convergence analysis are elaborated in Section 3.4, the resulting subproblem is evidently a optimization problem with nonconvex constraint set, which hence can be solved by the spectral gradient method. Section 3.5 introduces some kinds of optimization problems with different cases of nonconvex set D, for instance, disjunctive optimization problems, sparse optimization problems, and rank-constrained matrix optimization problems, and gives the corresponding projection formulas on D. Section 3.6 implements the proposed safeguarded augmented Lagrangian approach based on the general spectral gradient method as the underlying subproblem solver, and illustrates its effectiveness by some manual examples and three test practical examples, they are MPCCs, recommender system problems, cardinality-constrained optimization problems, as well as the famous MAXCUT problems.

Inspired by Chapter 3, we consider the more generally composite optimization problems with set-membership constraints in Chapter 4, and use an augmented Lagrangian method to find the underlying stationaries, the results are mainly from [66]. Since the objective function is composite with a lower semicontinuous part, one has to adjust the standard definitions of M-, AM-stationarity, as well as AM-regularity according to the definition of subdifferential in order to ensure the convergence of proposed algorithm, the more details are presented in Section 4.1. Note that the original problem can be reformulated by the slack variables to avoid the infeasibility of projecting on set-membership constraints, then Section 4.2 provides the theoretical guarantee that the reformulated problem has the same M- and AM-stationary points with the original one. In Section 4.3, the safeguarded augmented Lagrangian method is proposed in details and the corresponding convergence results are given. The resulting subproblems and subproblem solvers are introduced in Section 4.4, where PANOC$^+$ are recalled as the subproblem solver which can generate an approximate M-stationary point eventually. Section 4.5 gives some numerical examples to illustrate the problem model is more flexible and our approach is powerful.

Chapter 5 is concerned to the convergence results of proximal gradient methods, which are always used to solve the composite optimization problems, the work is inspired by [95] where the algorithm is proposed and some convergence results of subsequence are given under the requirement of local Lipschitz gradient of the smooth part of objective function, and mainly based the preprint [93]. Note that the underlying global Lipschitz condition has been widely used for proximal gradient methods, Section 5.1 presents the necessity of weakening it into the local one. The specific algorithm and some underlying known results from [95] are recalled in Section 5.2. Then Section 5.3 is eventually devoted to the convergence of the entire sequence generated by the proposed algorithm as well as the rate-of-convergence results with the mild requirement of Kurdyka–Łojasiewicz property and local Lipschitz condition, which are necessary for the desired results. To ensure that the algorithm is smoothly applied, we in Section 5.4, introduce how its corresponding subproblems with a class of nonconvex nonsmooth regularization functions are realized. Section 5.5 first illustrates that these nonconvex regularizers can generate sparser solution than the convex one by some random academic problem and then demonstrates that our method is better than Gurobi optimizer in generating the sparse solutions by some testproblems including the image recovery problem and the portfolio problem.

We will close this thesis with some final conclusions and future works in Chapter 6.

2. Background

This chapter aims to provide some basic concepts and fundamental results in Euclidean spaces which are supportive for the remaining chapters. Most of the material is a careful collection of results mainly from the references [122, 123, 138, 140], which will be provided in a structured and clear way. Hence, we skip the proofs of most results and please refer to associated references for the interested readers.

2.1 Basics about Variational Analysis

2.1.1 Notations and Preliminaries

Set $\mathbb{X}$ and $\mathbb{Y}$ are two Euclidean spaces, we use $\langle \cdot, \cdot \rangle$ to denote the inner product and $\|\cdot\|$ to denote the associated norm. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N}$ denote the set of nonnegative integers, the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, and the n-dimensional nonnegative orthant is denoted by $\mathbb{R}^n_+$. Given a set $A \subset \mathbb{X}$ and an element $x \in \mathbb{X}$, we use $A + x := x + A := \{x\} + A := \{x + a \,|\, a \in A\}$ for brevity. Furthermore,

$$\operatorname{dist}(x, A) := \inf\{\|y - x\| \,|\, y \in A\}$$

denotes the distance of the point x to the set A with $\operatorname{dist}(x, \emptyset) := \infty$. For given $\varepsilon > 0$, $B_\varepsilon(x) := \{y \in \mathbb{X} \,|\, \|y - x\| \leq \varepsilon\}$ denotes the closed ε-ball around x.

The continuous linear operator $f'(x) \colon \mathbb{X} \to \mathbb{R}$ denotes the derivative of the continuously differentiable function $f \colon \mathbb{X} \to \mathbb{R}$ at $x \in \mathbb{X}$, and we will make use of $\nabla f(x) := f'(x)^* 1$ where $f'(x)^* \colon \mathbb{R} \to \mathbb{X}$ is the adjoint of $f'(x)$. This way, ∇f is a mapping from $\mathbb{X}$ to $\mathbb{X}$.

We will in this thesis consider the sequential properties of sets and mappings (or functions), so the following definitions are recalled.

Definition 2.1. We say $\{x^k\} \subset \mathbb{X}$ is *convergent* to $\bar{x}$ and write as $x^k \to \bar{x}$, if $\lim_{k\to\infty} x^k = \bar{x}$ is satisfied.

Definition 2.2. An arbitrary subset $C \subset \mathbb{X}$ is called

(i) *convex*, if for all $x, y \in C$ and $\lambda \in (0, 1)$, it holds $\lambda x + (1 - \lambda) y \in C$;

(ii) *bounded*, if there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in C$;

(iii) *closed*, if for all sequence $\{x^k\} \subset C$ with $x^k \to \bar{x}$, it holds $\bar{x} \in C$;

(iv) *compact*, if it is bounded and closed;

(v) *a cone*, if for all $x \in C$ and $\alpha > 0$, it holds $\alpha x \in C$.

We say that an extended real-valued function $g : \mathbb{X} \to \overline{\mathbb{R}} := (-\infty, \infty]$ is proper if its domain $\operatorname{dom} g := \{x \in \mathbb{X} \,|\, g(x) < \infty\} \neq \emptyset$. Moreover, we use $\operatorname{epi} g := \{(x, t) \in \mathbb{X} \times \mathbb{R} \,|\, g(x) \leq t\}$ to denote the epigraph of g.

Definition 2.3. A function $f : \mathbb{X} \to \overline{\mathbb{R}}$ is *convex* on a convex set $C \subset \mathbb{X}$, if for all $x, y \in C$ and $\lambda \in (0, 1)$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds. We say f is convex, if it is convex on $\mathbb{X}$. The function f is concave if $-f$ is convex.

We now introduce the definitions of lower and upper semicontinuity of some mapping.

Definition 2.4. Let $g : \mathbb{X} \to \overline{\mathbb{R}}$ a set-valued mapping and $\bar{x} \in \mathbb{X}$, then g is called

(i) *lower/inner semicontinuous* at $\bar{x}$ if every sequence $\{x^k\}$ converging $\bar{x}$ satisfying

$$g(\bar{x}) \leq \liminf_{k\to\infty} g(x^k);$$

(ii) *upper/outer semicontinuous* at $\bar{x}$ if every sequence $\{x^k\}$ converging $\bar{x}$ satisfying

$$g(\bar{x}) \geq \limsup_{k\to\infty} g(x^k);$$

It is called *continuous* at x if both conditions hold, i.e., $\lim_{x^k \to \bar{x}} g(x^k) \to g(\bar{x})$.

A natural consequence of lower semicontinuity is as follows.

Proposition 2.5. *For any function $g : \mathbb{X} \to \overline{\mathbb{R}}$, then the following properties are equivalent*

(a) *g is lower semicontinuous;*

(b) $\operatorname{epi} g$ *is closed;*

(c) *all level sets are closed, i.e., for all $\alpha \in \mathbb{R}$, the level sets* $\operatorname{lev}_{\leq\alpha} g := \{x \in \mathbb{X} \mid g(x) \leq \alpha\}$ *are closed.*

We now introduce the concept of coercivity, which avoids in some sense the empty of level sets.

Definition 2.6. A function $g : \mathbb{X} \to \overline{\mathbb{R}}$ is coercive if

$$\lim_{\|x\|\to\infty} g(x) = +\infty.$$

Note that continuously coercive functions can be specified by the boundedness of their level sets.

Proposition 2.7. *Let $g : \mathbb{X} \to \overline{\mathbb{R}}$ be continuous, g is coercive if and only if all the underlying level sets are bounded.*

Keeping the convergence of a sequence and lower semicontinuity of a function in mind, we now introduce attentive convergence of a sequence in terms of some lower semicontinuous function, which plays an essential role in Chapter 4.

Definition 2.8. Let $g : \mathbb{X} \to \overline{\mathbb{R}}$ and a point $\bar{x} \in \operatorname{dom} g$ a proper and lsc function, we call *g-attentive* convergence of a sequence $\{x^k\}$:

$$x^k \xrightarrow{g} \bar{x} \quad :\Longleftrightarrow \quad x^k \to \bar{x} \quad \text{with} \quad g(x^k) \to g(\bar{x}). \tag{2.1}$$

Given a parameter value $\gamma > 0$, the *proximal* mapping $\operatorname{prox}_{\gamma g}$ is defined by

$$\operatorname{prox}_{\gamma g}(x) := \operatorname{argmin}_z \left\{ g(z) + \frac{1}{2\gamma}\|z - x\|^2 \right\},$$

and we say that g is *prox-bounded* if it is proper and $g + \|\cdot\|^2/(2\gamma)$ is bounded below on $\mathbb{X}$ for some $\gamma > 0$. The supremum of all such γ is the threshold γ_g of prox-boundedness for g. In particular, if g is bounded below by an affine function, then $\gamma_g = \infty$. When g is lsc, for any $\gamma \in (0, \gamma_g)$ the proximal mapping $\text{prox}_{\gamma g}$ is locally bounded, nonempty and compact-valued [140, Theorem 1.25].

2.1.2 Local Lipschitz Continuity

In optimization and variational analysis, Lipschitz continuity property, used to guarantee a mapping or function varies at some point in a bounded range in order to prevent it from changing too rapidly, plays a very fundamental role, which supports a host of applications where such Lipschitz constants serve to quantify the stability of a problem's solutions or the rate of convergence in a numerical method for determining a solution [140]. This thesis is more concerned to the local Lipschitz continuity, which is more weaker than the global one. Let us start with their definitions.

Definition 2.9. Let $S \subset \mathbb{X}$ be a nonempty set and $\phi : S \to \mathbb{Y}$ be a continuous mapping. Then

a) ϕ is (*globally*) *Lipschitz continuous* over S with parameter $L \geq 0$ if the following holds

$$\|\phi(x) - \phi(y)\| \leq L\,\|x - y\| \quad \forall x, y \in S;$$

b) ϕ is *locally Lipschitz continuous* over S if for every $y \in S$ there exist $\varepsilon(y) > 0$, $L_\varepsilon(y) \geq 0$, and a neighborhood $N_\varepsilon(y) := \{x \in S : \|x - y\| < \varepsilon(y)\}$, such that ϕ is $L_\varepsilon(y)$ (-globally) Lipschitz continuous over $N_\varepsilon(y)$, i.e.,

$$\|\phi(x) - \phi(y)\| \leq L_\varepsilon(y)\,\|x - y\| \quad \forall x, y \in N_\varepsilon(y).$$

When $S := \mathbb{X}$, we call ∇f is globally or locally Lipschitz continuous.

We next recall the properties of local Lipschitz continuity on some compact set.

Proposition 2.10. *[59] Let $S \subset \mathbb{X}$ be nonempty set. Then, a mapping $\phi : \mathbb{X} \to \mathbb{Y}$ is locally Lipschitz continuous over S if and only if for every nonempty and compact set $C \subset S$, there exists $L_C > 0$ such that ϕ is L_C-Lipschitz continuous over C, i.e.,*

$$\|\phi(x) - \phi(y)\| \leq L_C\|x - y\| \quad \forall x, y \in C.$$

[60, Proposition B.1] concluded a proposition which deals with local Lipschitz gradient.

Proposition 2.11. *Let $\phi : \mathbb{X} \to \mathbb{Y}$ be continuously differentiable, then the following claims hold:*

(i) *ϕ is locally Lipschitz continuous;*

(ii) *Let $B \subset \mathbb{X}$ be a closed ball, i.e., $B = \{x \in \mathbb{X} \,|\, \|x - z\| \leq r\}$, for some $z \in \mathbb{X}$ and $r \in (0, \infty]$, and assume that ϕ is L_B-Lipschitz continuous over B with $L_B \geq 0$. Then*

$$\|\nabla\phi(x)\| \leq L_B \quad \forall x \in B.$$

2.1.3 Rate of Convergence

We are now concerned to the rate of convergence of the sequence, which can evaluate in some sense the effectiveness of the numerical algorithms. We summarize the definitions of different types of convergence rate as well as the corresponding methods.

Definition 2.12. We say that a sequence $\{x^k\} \subset \mathbb{X}$ converges to $\bar{x} \in \mathbb{X}$

a) *linearly* or *Q-linearly*, if there exists $c \in (0,1)$ such that

$$\|x^{k+1} - \bar{x}\| \le c\|x^k - \bar{x}\| \quad \text{for all sufficiently large } k \in \mathbb{N};$$

b) *R-linearly*, if one has

$$\limsup_{k\to\infty} \|x^k - \bar{x}\|^{1/k} < 1.$$

Note that this R-linear convergence holds if there exist constants $\omega > 0$ and $\mu \in (0,1)$ such that $\|x^k - \bar{x}\| \le \omega\mu^k$ holds for all sufficiently large $k \in \mathbb{N}$, i.e., if the expression $\|x^k - \bar{x}\|$ is dominated by a Q-linearly convergent null sequence.

The following lemmas are practical tools to obtain bounds on the rate of convergence of the sequence. We start with introducing the widely-used one.

Lemma 2.13. *[13, Lemma 1] Let $\{s^k\}$ be a sequence in $\mathbb{R}_+$ and let α, β be some nonnegative and positive constants, respectively. Suppose that $s_k \to 0$ and that the sequence satisfies*

$$(s^k)^\alpha \le \beta(s^k - s^{k+1}) \tag{2.2}$$

for all sufficiently large k. Then

(i) *if $\alpha = 0$, the sequence $\{s^k\}$ converges to 0 in a finite number of steps;*
(ii) *if $\alpha \in (0,1]$, the sequence $\{s^k\}$ converges linearly to 0 with rate $\beta/(1+\beta)$;*
(iii) *if $\alpha > 1$, then there exists $\eta > 0$ such that*

$$s^k \le \eta k^{-\frac{1}{\alpha-1}} \quad \text{for all } k \text{ sufficiently large.}$$

This property is very prevalent and broadly utilised, e.g., [13, 104, 108]. However, it does not suit our cases in Chapter 5, therefore the following one is illustrated.

Lemma 2.14. *Let $\{s^k\}$ be a sequence in $\mathbb{R}_+$ and let α, β be some nonnegative and positive constants, respectively. Suppose that $s_k \to 0$ and that the sequence satisfies*

$$(s^{k+1})^\alpha \le \beta(s^k - s^{k+1}) \tag{2.3}$$

for all sufficiently large k. Then

(i) *if $\alpha = 0$, the sequence $\{s^k\}$ converges to 0 in a finite number of steps;*
(ii) *if $\alpha \in (0,1]$, the sequence $\{s^k\}$ converges linearly to 0 with rate $\beta/(1+\beta)$;*
(iii) *if $\alpha > 1$, assuming that $\{s^k\}$ is decreasing sequence, then there exists $\eta > 0$ such that*

$$s^k \le \eta k^{-\frac{1}{\alpha-1}} \quad \text{for all } k \text{ sufficiently large.} \tag{2.4}$$

Proof. If $\alpha = 0$, then (2.3) implies

$$0 \le s^{k+1} \le s^k - \frac{1}{\beta},$$

and (i) follows.

Assume that $\alpha \in (0,1]$. Since $s^k \to 0$, then one has $s^{k+1} < 1$ for some $k \in \mathbb{N}$. Hence, (2.3) implies

$$s^{k+1} \le (s^{k+1})^\alpha \le \beta(s^k - s^{k+1})$$

holds for sufficiently large k. Therefore, one has

$$s^{k+1} \le \frac{\beta}{1+\beta} s^k$$

holds for sufficiently large k, which says $\{s^k\}$ converges linearly to 0 with rate $\frac{\beta}{1+\beta}$.

Assume now that $\alpha > 1$. Since $\{s^k\}$ is decreasing, then one has $s^{k+1} \le s^k$ for all $k \in \mathbb{N}$, which implies from $\alpha > 1$ that $(s^{k+1})^{1-\alpha} \ge (s^k)^{1-\alpha}$ for all $k \in \mathbb{N}$. Then (2.3) implies

$$\begin{aligned}\frac{1}{\beta} &\le (s^k - s^{k+1})(s^{k+1})^{-\alpha} = s^k (s^{k+1})^{-\alpha} - (s^{k+1})^{1-\alpha} \le s^k (s^{k+1})^{-\alpha} - (s^k)^{1-\alpha} \\ &= s^k \left((s^{k+1})^{-\alpha} - (s^k)^{-\alpha} \right) \le s^0 \left((s^{k+1})^{-\alpha} - (s^k)^{-\alpha} \right)\end{aligned}$$

for sufficiently large k. Then there exists positive integer N such that

$$\frac{1}{\beta s^0} \le (s^{k+1})^{-\alpha} - (s^k)^{-\alpha}$$

holds for all $k \ge N$. Summing it for k from N to $j-1 \ge N$, one has

$$(s^j)^{-\alpha} - (s^N)^{-\alpha} \ge \frac{1}{\beta s^0}(j - N),$$

which gives, for all $j \ge N+1$,

$$s^j \le \left((s^N)^{-\alpha} + \frac{1}{\beta s^0}(j-N) \right)^{-\frac{1}{\alpha}}.$$

Therefore, there exists some $\eta > 0$ such that

$$s^j \le \eta j^{-\frac{1}{\alpha}} \quad \text{for all sufficiently large } j,$$

which completes the proof. □

Note that regardless of which (2.2) or (2.3) is satisfied, the first two cases can be achieved, however the requirement of decreasing $\{s^k\}$ is mandatory for the desired (iii) in Lemma 2.14. Provided that (2.2) holds and $\{s^k\}$ is decreasing, Lemma 2.14 holds automatically deduced by Lemma 2.13. However, such requirement may not be satisfied in some sense, such as the case in Theorem 5.12 as mentioned above, it becomes valuable to introduce the assumption (2.3).

2.1.4 Normal Cones and Differentiations

All kinds of cones and differentiations play an essential role in variational analysis and optimization theory, which provide some theoretical guarantees when seeking for optimality conditions. This section is first devoted to some types of tangent and normal cones in order to describe the geometric structure of the closed, convex set $C \subset \mathbb{Y}$ and the closed (not necessarily convex) set $D \subset \mathbb{X}$ which appear in (P). Subsequently, we introduce the concepts and properties of some types of subdifferential of a proper and lower semicontinuous function g mentioned in (CP). Let us start with the introduction of tangent cones as well as polar cones.

Definition 2.15. Let $D \subset \mathbb{X}$ and $x \in D$, then we denote

a) *Bouligand* tangent cone of a subset $D \subset \mathbb{X}$ at a point $\bar{x} \in D$, i.e.,

$$\mathcal{T}_D(\bar{x}) := \left\{ d \in \mathbb{X} \,\middle|\, \exists \{x^k\} \subset D,\ \exists \{t_k\} \downarrow 0 : x^k \to \bar{x}, \frac{x^k - \bar{x}}{t_k} \to d \right\};$$

b) *Clarke* tangent cone of a subset $D \subset \mathbb{X}$ at a point $\bar{x} \in D$, i.e.,

$$\mathcal{T}_D^C(\bar{x}) := \left\{ d \in \mathbb{X} \,\middle|\, \forall \{x^k\} \subset D,\ \forall \{t_k\} \downarrow 0,\ \exists \{d_k\} \subset \mathbb{X} : x^k \to \bar{x}, d_k \to d, x^k + t_k d_k \in D \right\}.$$

Definition 2.16. Let $P \subset \mathbb{X}$ a nonempty cone, then we call

$$P^\circ := \{y^* \in \mathbb{X} \,|\, \langle y^*, y\rangle \leq 0 \quad \forall y \in P\}$$

the polar cone of P.

Let us turn to the normal cone, it is well known that Fréchet normal cone is defined as the polar cone of Bouligand tangent cone, we then give the specific representations of Fréchet normal cone, limiting and Clarke normal cones.

Definition 2.17. Let $D \subset \mathbb{X}$ and $x \in D$, then we denote

a) *regular* (or *Fréchet*) *normal cone* of D at $\bar{x} \in D$, i.e.,

$$\mathcal{N}_D^F(\bar{x}) := \{v \,|\, \langle v, x - \bar{x}\rangle \leq o(\|x - \bar{x}\|) \quad \forall x \in D\};$$

b) *limiting* (or *Mordukhovich*) *normal cone* of D at $\bar{x} \in D$, i.e.,

$$\mathcal{N}_D^{\lim}(\bar{x}) := \left\{ v \,|\, \exists \{x^k\} \subset D,\ \exists \{v^k\} :\ x^k \to \bar{x}, v^k \to v, v^k \in \mathcal{N}_D^F(x^k) \quad \forall k \in \mathbb{N} \right\};$$

c) *Clarke normal cone* of D at $\bar{x} \in D$, if

$$\mathcal{N}_D^C(\bar{x}) = \left(\mathcal{T}_D^C(\bar{x})\right)^\circ.$$

For $x \notin D$, we set $\mathcal{N}_D^F(x) := \varnothing$, $\mathcal{N}_D^{\lim}(x) := \varnothing$, and $\mathcal{N}_D^C(x) := \varnothing$. Applying [122, Proposition 2.45], for all $x \in D$, one has the following inclusions

$$\mathcal{N}_D^F(x) \subset \mathcal{N}_D^{\lim}(x) \subset \mathcal{N}_D^C(x). \tag{2.5}$$

Note that the limiting normal cone is stable in the sense that

$$\limsup_{x \to \bar{x}} \mathcal{N}_D^{\lim}(x) = \mathcal{N}_D^{\lim}(\bar{x}) \qquad \forall \bar{x} \in \mathbb{X} \tag{2.6}$$

holds. This stability property, which might be referred to as outer semicontinuity of the set-valued operator $\mathcal{N}_D^{\lim} \colon \mathbb{X} \rightrightarrows \mathbb{X}$, will play an essential role in our subsequent analysis. The limiting normal cone to the convex set A coincides with the standard normal cone from convex analysis, i.e., for $\bar{y} \in A$, we have

$$\mathcal{N}_A^F(\bar{y}) = \mathcal{N}_A^{\lim}(\bar{y}) = \mathcal{N}_A^C(\bar{y}) = \mathcal{N}_A(\bar{y}) := \{\lambda \in \mathbb{Y} \mid \langle \lambda, y - \bar{y}\rangle \leq 0 \quad \forall y \in A\}.$$

For points $y \notin A$, we set $\mathcal{N}_A(y) := \varnothing$ for formal completeness. Note that the stability property (2.6) is also satisfied by the set-valued operator $\mathcal{N}_A \colon \mathbb{Y} \rightrightarrows \mathbb{Y}$.

The relationship between the normal cones and the projections on the underlying convex set is very closed.

Proposition 2.18. *[147, Proposition 2.36] Let $C \subset \mathbb{X}$ be a nonempty, closed, convex set. Then, for any $x \in \mathbb{X}$, one has $y = P_C(x)$ is and only if $x - y \in \mathcal{N}_C(y)$.*

Let us fix a merely lower semicontinuous function $g\colon \mathbb{X} \to \overline{\mathbb{R}}$ and pick $x \in \operatorname{dom} g$ where $\operatorname{dom} g := \{x \in \mathbb{X} \,|\, g(x) < \infty\}$ denotes the domain of g. We now introduce the concepts of regular, limiting, and horizon subdifferential of g.

Definition 2.19. Let $g\colon \mathbb{X} \to \overline{\mathbb{R}}$ be a merely lower semicontinuous function, we define

(a) *regular* (or *Fréchet*) *subdifferential* of g at x, i.e.,

$$\partial^F g(x) := \left\{\eta \in \mathbb{X} \,\middle|\, \liminf_{y \to x,\, y \neq x} \frac{g(y) - g(x) - \langle \eta, y - x\rangle}{\|y - x\|} \geq 0\right\};$$

(b) *limiting* (or *Mordukhovich*) *subdifferential* of g at x, i.e.,

$$\partial g(x) := \left\{\eta \in \mathbb{X} \,\middle|\, \exists \{x^k\}, \{\eta^k\} \subset \mathbb{X}\colon\; x^k \xrightarrow{g} x,\, \eta^k \to \eta,\, \eta^k \in \partial^F g(x^k)\, \forall k \in \mathbb{N}\right\};$$

(c) *horizon* (or *singular limiting*) *subdifferential* of g at x, i.e.,

$$\partial^\infty g(x) := \left\{\eta \in \mathbb{X} \,\middle|\, \exists \{x^k\}, \{\eta^k\} \subset \mathbb{X}, t_k \downarrow 0\colon\; x^k \xrightarrow{g} x,\, \eta^k \in \partial^F g(x^k)\ \forall k \in \mathbb{N},\, t_k\eta^k \to \eta\right\}.$$

Clearly, one always has $\partial^F g(x) \subset \partial g(x)$ and $\partial^F g(x) \subset \partial^\infty g(x)$ by construction of these sets.

Moreover, subdifferentials have some connections to normal vectors through the variational geometry of epigraphs.

Theorem 2.20. *[140, Theorem 8.9] For $g : \mathbb{X} \to \overline{\mathbb{R}}$, and any point x at which g is finite, one has*

$$\begin{aligned}
\partial^F g(x) &= \left\{\eta \,|\, (\eta, -1) \in \mathcal{N}^F_{\operatorname{epi} g}(x, g(x))\right\};\\
\partial g(x) &= \left\{\eta \,|\, (\eta, -1) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(x, g(x))\right\};\\
\partial^\infty g(x) &= \left\{\eta \,|\, (\eta, 0) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(x, g(x))\right\}.
\end{aligned}$$

The above characterization can be used to prove the following result.

Lemma 2.21. *Let g be lower semicontinuous, $\bar{x} \in \operatorname{dom} g$, $\{x^k\} \xrightarrow{g} \bar{x}$, and $\xi^k \in \partial g(x^k)$ a possibly unbounded sequence such that $t_k\xi^k \to s$ for some sequence $t_k \downarrow 0$. Then $s \in \partial^\infty g(\bar{x})$.*

Proof. Since $\xi^k \in \partial g(x^k)$, one has $(\xi^k, -1) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(x^k, g(x^k))$. Then cone property implies

$$(t_k\xi^k, -t_k) = t_k(\xi^k, -1) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(x^k, g(x^k))$$

holds for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$, using $x^k \xrightarrow{g} \bar{x}$ and the upper semicontinuity of the limiting normal cone yield

$$(s, 0) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(\bar{x}, g(\bar{x})).$$

In view of Theorem 2.20, one gets $s \in \partial^\infty g(\bar{x})$. □

In general, for lower semicontinuous functions g, neither the limiting nor the horizon subdifferentials are upper semicontinuous, it is illustrated by the following counterexample.

Example 2.22. Consider the one-dimensional function

$$g(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ 1 - x, & \text{if } x > 0. \end{cases}$$

The g is evidently lower semicontinuous, and an elementary calculation shows that $\partial g(0) = [0, \infty)$, whereas one has

$$\limsup_{x \to 0} \partial g(x) = \{-1\} \cup \partial g(0) = \{-1\} \cup [0, \infty),$$

which says the desired inclusion $\limsup_{x \to 0} \partial g(x) \subset \partial g(0)$ does not hold.

On the other hand, the limiting and horizon subdifferentials are robust in the sense that

$$\limsup_{x \xrightarrow{g} \bar{x}} \partial g(x) \subset \partial g(\bar{x}) \quad \text{and} \quad \limsup_{x \xrightarrow{g} \bar{x}} \partial^\infty g(x) \subset \partial^\infty g(\bar{x}) \tag{2.7}$$

hold, cf. [123, Proposition 1.20], where

$$\limsup_{x \xrightarrow{g} \bar{x}} \partial g(x) := \left\{ \eta \mid \exists \{x^k\} \xrightarrow{g} \bar{x}, \exists \eta^k \in \partial g(x^k) \ \forall k \in \mathbb{N} \text{ such that } \eta^k \to \eta \right\}$$

and $\limsup_{x \xrightarrow{g} \bar{x}} \partial^\infty g(x)$ is defined similarly. Note that, for continuous functions g, this robust property is the same as the ordinary upper semicontinuity of the limiting and the horizon subdifferentials, whereas Example 2.22 illustrates that these are, in general, not equivalent for discontinuous functions.

We now introduce some calculus rules of subdifferentials for the composite function.

Proposition 2.23. *(Calculus rules)*

(i) *[123, Proposition 1.30(ii)] Let $f : \mathbb{X} \to \mathbb{R}$ be continuously differentiable, and $g : \mathbb{X} \to \overline{\mathbb{R}}$ be a merely lower semicontinuous, for any $x \in \operatorname{dom} g$, one has the following sum rule*
$$\partial(f + g)(x) = \nabla f(x) + \partial g(x);$$

(ii) *[158, Proposition 2] Let $g : \mathbb{X} \to \overline{\mathbb{R}}$ be a merely lower semicontinuous, for any $x \in \operatorname{dom} g$, if $f : \mathbb{X} \to \mathbb{R}$ is Lipschitz continuous near x, then one has*
$$\partial(\alpha f + \beta g)(x) \subset \alpha \nabla f(x) + \beta \partial g(x)$$
with some nonnegative scalars α and β.

Lemma 2.24. *Let $C \subset \mathbb{X}$ be nonempty, closed and convex. Furthermore, let $c : \mathbb{X} \to \mathbb{Y}$ be continuously differentiable. We consider the function $\vartheta : \mathbb{X} \to \mathbb{R}$ given by $\vartheta(x) := \frac{1}{2} \operatorname{dist}_C^2(c(x))$ for all $x \in \mathbb{X}$. Then, ϑ is continuously differentiable, and for each $\bar{x} \in \mathbb{X}$, one has*

$$\nabla \vartheta(\bar{x}) = c'(\bar{x})^* (c(\bar{x}) - P_C(c(\bar{x}))).$$

Proof. We define $\psi : \mathbb{Y} \to \mathbb{R}$ by means of $\psi(y) := \frac{1}{2} \operatorname{dist}_C^2(y)$ for all $y \in \mathbb{Y}$ and observe that $\vartheta = \psi \circ c$. Since C is assumed to be convex, ψ is continuously differentiable with gradient $\nabla \psi(\bar{y}) = \bar{y} - P_C(\bar{y})$ for each $\bar{y} \in \mathbb{Y}$, see [19, Corollary 12.30], and the statements of the lemma follow trivially from the standard chain rule. $\square$

2.2 Basic Concepts about Nonlinear Programming

This section is concerned with the optimality conditions of nonlinear optimization problems in Euclidean spaces. We consider the following generic minimization problem

$$\min_x \varphi(x) \qquad \text{s.t.} \quad G(x) \in K \tag{2.8}$$

with a functional $\varphi : \mathbb{X} \to \overline{\mathbb{R}}$ (or $\mathbb{R}$ by situations), a continuously differentiable mapping $G : \mathbb{X} \to \mathbb{Y}$, nonempty closed subset $K \subset \mathbb{Y}$. We denote

$$\mathcal{F} := \{x \in \mathbb{X} \,|\, G(x) \in K\}$$

as the feasible set of (2.8) and say $\bar{x}$ is feasible for (2.8) if and only if $\bar{x} \in \mathcal{F}$. We now give a classical example with inequality and equality constraints.

Example 2.25. Consider the optimization problem

$$\begin{array}{ll} \min\limits_x & \varphi(x) \\ \text{s.t.} & \begin{array}{ll} g_i(x) \le 0 & \forall i = 1, \dots, m, \\ h_j(x) = 0 & \forall j = 1, \dots, p \end{array} \end{array}$$

with differentiable function $\varphi, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, setting

$$\mathbb{Y} := \mathbb{R}^m \times \mathbb{R}^p, \quad G := \begin{pmatrix} g \\ h \end{pmatrix}, \quad K := (-\infty, 0]^m \times \{0\}^p$$

yields an optimization problem of type (2.8).

Definition 2.26. Let $\bar{x} \in \mathcal{F}$, then $\bar{x}$ is called

- a local solution of (2.8), if there exists $\varepsilon > 0$ such that

$$\varphi(\bar{x}) \le \varphi(x) \quad \forall x \in \mathcal{F} \cap B_\varepsilon(\bar{x});$$

- a global solution of (2.8), if

$$\varphi(\bar{x}) \le \varphi(x) \quad \forall x \in \mathcal{F}.$$

In order to ensure the existence of solution of the constrained optimization problems, we first recall the famous Weierstrass existence theorem.

Theorem 2.27 (Weierstrass Extreme Value Theorem)**.** *Every continuous function on a compact set attains its extreme values on that set.*

Based on this and the properties of coercive function, the existence result of solution(s) of unconstrained programs can be implied.

Theorem 2.28. *Let $\varphi : \mathbb{X} \to \mathbb{R}$ be continuous. If φ is coercive, then φ has at least one global minimizer.*

In fact, the objective function to be dealt with may be far from continuous, then we recall the following theorem to introduce the attainment of a minimum for the noncontinuous function.

Theorem 2.29. *Let $\varphi : \mathbb{X} \to \overline{\mathbb{R}}$ be lower semicontinuous, level-bounded (or coercive) and proper. Then the value* $\inf \varphi$ *is finite and the set* $\operatorname{argmin} \varphi$ *is nonempty and compact.*

We have already built the existence theory of solution(s), which are quite useful, however they can not provide us some constructive tests for optimality. In order to describe and find the solution(s), we always refer to the first-order optimality conditions. The following results provide the close connections between variational geometry and necessary optimality conditions.

Theorem 2.30. *Let $\bar{x}$ be a local minimum of the program* (2.8) *with the continuously differentiable function φ, then one has*

$$\langle \nabla \varphi(\bar{x}), d \rangle \geq 0 \quad \forall d \in \mathcal{T}_{\mathcal{F}}(\bar{x}).$$

This thesis is mainly concerned to the normal cones, hence we give the following equivalent conclusion with the help of Definition 2.16.

Theorem 2.31. *Let $\bar{x}$ be a local minimum of the programming* (2.8) *with the continuously differentiable function φ, then one has*

$$-\nabla \varphi(\bar{x}) \in \mathcal{N}_{\mathcal{F}}(\bar{x}),$$

which, if $\mathcal{F}$ is convex, can be written in the form

$$\langle \nabla \varphi(\bar{x}), x - \bar{x} \rangle \geq 0 \qquad \forall x \in \mathcal{F}.$$

When φ too is convex, the equivalent conditions are sufficient for $\bar{x}$ to be globally optimal.

We, keeping Proposition 2.23 (i) in mind, deduce the following optimality condition where the objective function is composite.

Theorem 2.32. *Consider the programs* (2.8)*, where $\varphi := f + g$ with continuously differentiable $f : \mathbb{X} \to \mathbb{R}$ and proper, lower semicontinuous $g : \mathbb{X} \to \overline{\mathbb{R}}$. Let $\bar{x}$ is a local minimum of such program, then one has*

$$-\nabla f(\bar{x}) \in \mathcal{N}_{\mathcal{F}}(\bar{x}) + \partial g(\bar{x}).$$

Note that it can be reduced as generalized Fermat's rule if $\mathcal{F} := \mathbb{X}$ (equivalently $K := \mathbb{Y}$).

2.2.1 Stationarity Theory

In this section, we recall the definitions of some types of stationary point of the nonlinear programming (2.8).

Definition 2.33. Let $\bar{x}$ is a local minimum of (2.8) and φ is continuously differentiable, then we say $\bar{x}$ is

- B-stationary (*Bouligand* stationary) if

$$0 \in \nabla \varphi(\bar{x}) + \mathcal{N}^F_{\mathcal{F}}(\bar{x});$$

- S-stationary (strongly stationary)/a KKT point if

$$0 \in \nabla \varphi(\bar{x}) + G(\bar{x})^* \mathcal{N}^F_K(G(\bar{x}));$$

- M-stationary (Mordukhovich stationary) if

$$0 \in \nabla\varphi(\bar{x}) + G(\bar{x})^* \mathcal{N}_K^{lim}(G(\bar{x}));$$

- C-stationary (Clarke stationary) if

$$0 \in \nabla\varphi(\bar{x}) + G(\bar{x})^* \mathcal{N}_K^C(G(\bar{x})).$$

The concepts of S- and M-stationary points have been introduced in [27], every local minimum is evidently also B-stationary. We here introduce C-stationarity based on the definition of Clarke normal cone, in fact, it is mainly used to characterize the classical MPCCs as well as second-order cone MPCCs [7,10,142,159]. From the relationship of normal cones (2.5), one evidently has S-stationarity is also M- and C-stationary, M-stationarity is C-stationary as well. It is deduced that every S-stationarity is also B-stationary from the inclusion of $G(\bar{x})^* \mathcal{N}_K^F(G(\bar{x})) \subset \mathcal{N}_{\mathcal{F}}^F(\bar{x})$. If C is convex, then S-stationarity corresponds with M-stationarity. Note that in order to ensure that an B-stationary point $\bar{x}$ is also M-stationary, one needs the requirement

$$\mathcal{N}_{\mathcal{F}}^F(\bar{x}) \subset G(\bar{x})^* \mathcal{N}_K(G(\bar{x})),$$

the similar requirements are appropriate for the other stationarities, such requirements can be regarded as constraint qualifications.

2.2.2 Constraint Qualification

In optimization theory, constraint qualifications are proposed in order to establish optimality conditions, where a local minimizer is guaranteed to be stationary. Generally speaking, constraint qualifications are actually properties of set presented by the constraint functions around a given feasible point, which are very crucial in the constrained optimization problems. We start this section with some fundamental constraint qualifications.

Definition 2.34. Let $\bar{x}$ be feasible of (2.8) in Example 2.25, and define the set of indices of active inequality constraints at $\bar{x}$ by

$$I^g := \{i = 1, \dots, m \,|\, g_i(\bar{x}) = 0\}.$$

Then we say

- *Linear independence constraint qualification* (LICQ) holds at $\bar{x}$, if the gradients

$$\begin{aligned} &\nabla g_i(\bar{x}), \quad i \in I^g, \\ &\nabla h_j(\bar{x}), \quad j = 1, \dots, p, \end{aligned}$$

  are linearly independent.
- *Mangasarian-Fromovitz constraint qualification* (MFCQ) holds at $\bar{x}$, if the gradients

$$\nabla h_j(\bar{x}), \quad j = 1, \dots, p,$$

  are linearly independent and there exists a $s \in \mathbb{R}^n$ satisfying

$$\begin{aligned} &\nabla g_i(\bar{x}) s < 0, \quad i \in I^g, \\ &\nabla h_j(\bar{x}) d = 0, \quad j = 1, \dots, p. \end{aligned}$$

- [11, 134] *Constant positive linear dependence condition* (CPLD) holds at $\bar{x}$, if for any $I_0 \subset I^g, J_0 \subset \{1, \cdots, p\}$ such that the set of gradients

$$\{\nabla g_i(\bar{x})\}_{i \in I_0} \cup \{\nabla h_j(\bar{x})\}_{j \in J_0}$$

is positive linearly dependent, then there exists a neighborhood $B_\alpha(\bar{x})$ of $\bar{x}$ with $\alpha > 0$ such that for any $x \in B_\alpha(\bar{x})$, the set

$$\{\nabla g_i(x)\}_{i \in I_0} \cup \{\nabla h_j(x)\}_{j \in J_0}$$

is linearly dependent.

In order to introduce more constraint qualifications, we now denote *linearized tangent cone* of $\mathcal{T}_\mathcal{F}(\bar{x})$ as

$$\mathcal{T}_\mathcal{F}^{\lim}(\bar{x}) := \{d \in \mathbb{X} \mid G'(x)^* d \in \mathcal{T}_\mathcal{F}(G(\bar{x}))\}.$$

Definition 2.35. Let $\bar{x}$ be feasible of (2.8), i.e., $x \in \mathcal{F}$. Then we say

- *Abadie constraint qualification* (ACQ) holds at $\bar{x}$, if

$$\mathcal{T}_\mathcal{F}(\bar{x}) = \mathcal{T}_\mathcal{F}^{\lim}(\bar{x}).$$

- *Guignard constraint qualification* (GCQ) holds at $\bar{x}$, if

$$(\mathcal{T}_\mathcal{F}(\bar{x}))^\circ = (\mathcal{T}_\mathcal{F}^{\lim}(\bar{x}))^\circ.$$

One evidently has ACQ is stronger than GCQ from the inclusion $\mathcal{T}_\mathcal{F}(\bar{x}) \subset \mathcal{T}_\mathcal{F}^{\lim}(\bar{x})$ and then $(\mathcal{T}_\mathcal{F}^{\lim}(\bar{x}))^\circ \subset (\mathcal{T}_\mathcal{F}(\bar{x}))^\circ$. Therefore, from [132, 144], the following implication holds

$$\text{LICQ} \Rightarrow \text{MFCQ} \Rightarrow \text{CPLD} \Rightarrow \text{ACQ} \Rightarrow \text{GCQ},$$

however the converse implications do not hold, please see [132] for counterexamples. Note that LICQ, MFCQ, and CPLD are easy to verify, but they are very strong requirements. On the other hand, ACQ and GCQ are very weak qualifications, however they are pretty difficult to verify. As a result, one aims at finding some weaker and easily-to-verify constraint qualifications.

After introducing those constraint qualifications, we now recall the following important relationship between constraint qualifications, local minimizers, and KKT points.

Theorem 2.36. *Let $\bar{x}$ be a local minimizer for* (2.8) *with convex C, and assume that GCQ holds at $\bar{x}$, then $\bar{x}$ is a KKT point for* (2.8)*.*

Since GCQ is the weakest constraint qualification till now, it then can be replaced by all the mentioned ones in the above theorem. Note that some other constraint qualifications between CPLD and ACQ have attracted more attentions, where a local minimizer maybe fail to be a KKT point, but be some stationary point.

2.3 Kurdyka–Łojasiewicz Property

It is well known that proximal gradient-type method is a good candidate for solving the composite programs, where the objective function is the sum of a continuously differential one and a lower semicontinuous one. At the beginning of development of this method, the continuously differentiable function is always required to be convex and have global Lipschitz gradient, which have been gradually weakened, to be honest, we get rid of both in

Chapter 5. Then in order to guarantee the underlying convergence, the requirement of error bound or Kurdyka–Łojasiewicz property is necessary, see [15,15,36,38,109,124], hereas [38] illustrates the relationship between error bound and Kurdyka–Łojasiewicz property. This section is concerned to Kurdyka–Łojasiewicz property which will be used in Chapter 5.

Lechner Theresa introduced the historical process of Kurdyka–Łojasiewicz function in her PhD thesis [105], we here recall the process roughly. Łojasiewicz [113] gave the continuously differential function φ a powerful condition in order to obtain the convergence results for gradient-type methods, i.e., for such $\varphi : \mathbb{R}^n \to \mathbb{R}$, there exists $\rho \in [1/2, 1)$ such that

$$\frac{|\varphi(x) - \varphi(\bar{x})|^\rho}{\|\nabla \varphi(x)\|} \tag{2.9}$$

remains locally bounded around any critical point $\bar{x}$ of φ, which was proven by him to hold for any real-analytic function. However, it fails obviously for some smooth functions, such as twice- or more times differentiable functions. Later, Kurdyka [103] generalized this idea and introduced the following property: there exists $\eta > 0$, a neighbourhood U of a critical point $\bar{x}$, and a continuous function $\chi : [0, \eta) \to \mathbb{R}_+$, which is continuously differentiable on $(0, \eta)$, satisfies $\chi(0) = 0$ as well as $\chi'(t) > 0$ for all $t \in (0, \eta)$, and

$$\|\nabla(\chi \circ \varphi)\| \geq 1 \tag{2.10}$$

for all $U \cap \{x \in \mathbb{R}^n \,|\, \varphi(\bar{x}) < \varphi < \varphi(\bar{x}) + \eta\}$. Note that (2.9) is a special case of (2.10) with $\chi(s) = (s - \varphi(\bar{x}))^{1-\rho}$. Recently, a nonsmooth generation about φ was proposed in [14,37]. We now recall the definition from [14].

Definition 2.37. Let $\varphi\colon \mathbb{X} \to \overline{\mathbb{R}}$ be lower semicontinuous. We say that φ has the *KL property*, where KL abbreviates *Kurdyka–Łojasiewicz*, at $\bar{x} \in \{x \in \mathbb{X} \,|\, \partial\varphi(x) \neq \emptyset\}$ if there exist a constant $\eta > 0$, a neighborhood $U \subset \mathbb{X}$ of $\bar{x}$, and a continuous concave function $\chi\colon [0, \eta] \to [0, \infty)$ which is continuously differentiable on $(0, \eta)$ and satisfies $\chi(0) = 0$ as well as $\chi'(t) > 0$ for all $t \in (0, \eta)$ such that the so-called *KL inequality*

$$\chi'(\varphi(x) - \varphi(\bar{x}))\,\mathrm{dist}\,(0, \partial\varphi(x)) \geq 1$$

holds for all $x \in U \cap \{x \in \mathbb{X} \,|\, \varphi(\bar{x}) < \varphi(x) < \varphi(\bar{x}) + \eta\}$. The function χ from above is referred to as the *desingularization function*. Furthermore, if g satisfies the above KL property at each point of $\mathrm{dom}\,\partial\varphi$, then φ is called a KL function.

Besides generalizing the requirement of smooth function to the nonsmooth one, the point $\bar{x}$ is not required to be stationary any more, as it is pointed out in [14] that for any proper and lower semicontinuous $\varphi : \mathbb{X} \to \overline{\mathbb{R}}$, it has KL property in any nonstationary point. Those make Kurdyka–Łojasiewicz more superfluous.

The proofs of convergence and rate-of-convergence of the sequence where KL property holds are usually technical and with similar structures. As mentioned in [60, Definition 4.1], assuming that $\{x^k\}$ is a sequence generated by some approach in order to minimize φ, the following assertions must be satisfied:

- (Sufficient decrease of objective function) There exists $a > 0$ such that

$$\varphi(x^{k+1}) - \varphi(x^k) \leq -a\|x^{k+1} - x^k\| \quad \forall k \in \mathbb{N}; \tag{2.11}$$

- (Relative error of subdifferential) There exist $c > 0$ and $s^{k+1} \in \partial\varphi(x^{k+1})$ such that

$$\mathrm{dist}(0, s^{k+1}) \leq c\|x^{k+1} - x^k\| \quad \forall k \in \mathbb{N};$$

- (Continuity condition) Let $\{x^k\}_{\mathcal{K}}$ be a subsequence converging to some $\bar{x}$. Then

$$\limsup_{k\in\mathcal{K}} \varphi(x^k) \leq \varphi(\bar{x}).$$

With the aid of such requirements and KL property, we can obtain the sequence is bounded and therefore converges to some stationary point.

We note that there exist some classes of functions where the KL property holds with the corresponding desingularization function given by $\chi(t) := ct^{\kappa}$ for $\kappa \in (0, 1]$ and some constant $c > 0$, where the parameter κ is called the *KL exponent*, see [37, 103].

3. Augmented Lagrangian Methods invoking Spectral Gradient Methods for Structured Optimization Problems

This chapter is dedicated to a detailed discussion of a (safeguarded) augmented Lagrangian method for the optimization problems with structured geometric constraints of the form mentioned in (P), the results in this chapter are fundamentally based on the publication [94]. Let us start with recalling again the problem

$$\min_x \; f(x) \quad \text{s.t.} \quad G(x) \in C,\; x \in D, \tag{P}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are Euclidean spaces, $f\colon \mathbb{X} \to \mathbb{R}$ and $G\colon \mathbb{X} \to \mathbb{Y}$ are continuously differentiable, $C \subset \mathbb{Y}$ is nonempty, closed, and convex, whereas the set $D \subset \mathbb{X}$ is only assumed to be nonempty, closed, and nonconvex. Note again that the augmented Lagrangian scheme is applied to penalize the general constraints $G(x) \in C$, but leave the comparably complicated $x \in D$ explicitly in the constraints, which leads to the resulting subproblems are constrained programs. With the aid of projected gradient-type method as the subproblem solver (using general spectral gradient method in this manuscript), we need to assume that the nonconvex D is sufficiently simple in the sense that the projections on D (not necessarily unique due to nonconvexity of D) are allowed for a fast computation. In particular, we show some cases of D, as well as the theoretical computation of the associated projections on D in Section 3.5.

This chapter is organized as follows. In Section 3.1, we demonstrate how augmented Lagrangian approaches can be motivated to solve the optimization problems with abstract constraints. Section 3.2 introduces the so-called stationaries and constraint qualification of (P), and some underlying properties. We then present the spectral gradient method for the optimization problems over nonconvex sets in Section 3.3. This method is used to solve the resulting subproblems of the safeguarded augmented Lagrangian method whose details are given in Section 3.4. Global convergence to M-type stationary points is also shown in this section. Since, as mentioned above, in our augmented Lagrangian approach, we penalize the constraints $G(x) \in C$, but keep the condition $x \in D$ explicitly in the constraints, we have to compute projections onto D. Section 3.5 therefore considers a couple of situations where the corresponding projections can be calculated in a very efficient way. Extensive computational experiments for some of these situations are documented in Section 3.6, this includes MPCCs, either-or-constrained optimization problems, cardinality-constrained (sparse) optimization problems, and a rank-constrained reformulation of the famous MAXCUT problem.

3.1 Motivation

Since, to the best of my knowledge, augmented Lagrangian methods have not yet been applied to the general problem (P) with general nonconvex D and arbitrary convex sets C in the setting of Euclidean spaces, and in order to get a better understanding of the contributions, at the first step, some comments regarding the existing results for the probably most prominent non-standard optimization problem will be added, namely the class of mathematical programs with complementarity constraints (MPCCs). Due to the particular structure of the feasible set, the usual KKT conditions are typically not satisfied at a local minimum. Hence, other (weaker) stationarity concepts have been proposed, like C- and M-stationarity, with M-stationarity being the stronger concept. Most algorithms, such as regularization, penalty, augmented Lagrangian methods etc., for the solution of MPCCs need to solve a sequence of standard nonlinear programs, and their limit points are typically C-stationary points only. Some approaches can identify M-stationary points if the underlying nonlinear programs are solved exactly, but they loose this desirable property if these programs are solved only inexactly, see the discussion in [99] for more details.

Only three approaches currently are awared due to my limited knowledge where convergence to M-stationary points for a general (nonlinear) MPCC is shown using inexact solutions of the corresponding subproblems, namely [12, 81, 135]. All three papers deal with suitable modifications of the (safeguarded) augmented Lagrangian method. The basic idea of reference [12] is to solve the subproblems such that both a first- and a second-order necessary optimality condition hold inexactly at each iteration, i.e., satisfaction of the second-order condition is the key point which, obviously, causes some extra costs for the subproblem solver and usually excludes the application of this approach to large-scale problems. The paper [135] proves convergence to M-stationary points by solving some complicated subproblems, but for the latter no method is specified. Finally, the recent method described in [81] provides an augmented Lagrangian technique for the solution of MPCCs where the complementarity constraints are kept as constraints, whereas the standard constraints are penalized. The authors present a technique which computes a suitable stationary point of these subproblems in such a way that the entire method generates M-stationary accumulation points for the original MPCC. In addition, [86] suggests to solve (a discontinuous reformulation of) the M-stationarity system associated with an MPCC by means of a semismooth Newton-type method. Naturally, this approach should be robust with respect to an inexact solution of the appearing Newton-type equations although this issue is not discussed in [86].

The ideas from [81] are naturally generalized to the structured optimization problem (P). In fact, a closer look at the corresponding proofs shows that the techniques from [81] can also be generalized using some relatively small modifications.

3.2 Stationarities and Constraint Qualification

For optimization problem (P), noting that the abstract set D is generally nonconvex in the exemplary settings we have in mind, the so-called concept of Mordukhovich-stationarity, which exploits limiting normals to D, is a reasonable concept of stationarity which addresses (P).

Definition 3.1 (M-stationarity of (P))**.** Let $\bar{x} \in \mathbb{X}$ be feasible for the optimization problem (P). Then $\bar{x}$ is called an *M-stationary point* (Mordukhovich-stationary point) of (P) if there exists a multiplier $\lambda \in \mathbb{Y}$ such that

$$0 \in \nabla f(\bar{x}) + G'(\bar{x})^* \lambda + \mathcal{N}_D^{\lim}(\bar{x}), \quad \lambda \in \mathcal{N}_C(G(\bar{x})).$$

Note that this definition coincides with the usual KKT conditions of (P) if the set D is convex. An asymptotic counterpart of this definition is the following one, see [117].

Definition 3.2 (AM-stationarity of (P))**.** Let $\bar{x} \in \mathbb{X}$ be feasible for the optimization problem (P). Then $\bar{x}$ is called an *AM-stationary point* (asymptotically M-stationary point) of (P) if there exist sequences $\{x^k\}, \{\varepsilon^k\} \subset \mathbb{X}$ and $\{\lambda^k\}, \{z^k\} \subset \mathbb{Y}$ such that $x^k \to \bar{x}$, $\varepsilon^k \to 0$, $z^k \to 0$, as well as

$$\varepsilon^k \in \nabla f(x^k) + G'(x^k)^* \lambda^k + \mathcal{N}_D^{\lim}(x^k), \quad \lambda^k \in \mathcal{N}_C(G(x^k) - z^k) \qquad \forall k \in \mathbb{N}.$$

Note that the definition of an AM-stationary point is similar to the notion of an AKKT (asymptotic or approximate KKT) point in standard nonlinear programming, see [34], but requires some explanation: The meanings of the iterates x^k and the Lagrange multiplier estimates λ^k should be clear. The vector ε^k measures the inexactness by which the stationary conditions are satisfied at x^k and λ^k. The vector z^k does not occur (at least not explicitly) in the context of standard nonlinear programs, but is required here for the following reason: the augmented Lagrangian method to be considered in this paper generates a sequence $\{x^k\}$ satisfying $x^k \in D$, while the constraint $G(x) \in C$ gets penalized, hence, the condition $G(x^k) \in C$ will typically be violated. Consequently, the corresponding normal cone $\mathcal{N}_C(G(x^k))$ would be empty which is why we cannot expect to have $\lambda^k \in \mathcal{N}_C(G(x^k))$, though we hope that this holds asymptotically. In order to deal with this situation, we therefore have to introduce the sequence $\{z^k\}$. Let us note that AM-stationarity corresponds to so-called AKKT stationarity for conic optimization problems, i.e., where C is a closed, convex cone and $D := \mathbb{X}$, see [4, Section 5]. The more general situation where C and D are closed, convex sets and the overall problem is stated in arbitrary Banach spaces is investigated in [41]. Asymptotic notions of stationarity addressing situations where D is a nonconvex set of special type can be found, e.g., in [7, 98, 135]. As shown in [117], the overall concept of asymptotic stationarity can be further generalized to feasible sets which are given as the kernel of a set-valued mapping. Let us mention that the theory in this section is still valid in situations where C is merely closed. In this case, one may replace the normal cone to C in the sense of convex analysis by the limiting normal cone everywhere. However, the nonconvex set C causes the convergence analysis of the proposed augmented Lagrangian approach fails.

Apart from the aforementioned difference, the motivation of AM-stationarity is similar to the one of AKKT-stationarity: Suppose that the sequence $\{\lambda^k\}$ is bounded and, therefore, convergent along a subsequence. Then, taking the limit on this subsequence in the definition of an AM-stationary point while using the stability property (2.6) of the limiting normal cone shows that the corresponding limit point satisfies the M-stationarity conditions from Definition 3.1. In general, however, the Lagrange multiplier estimates $\{\lambda^k\}$ in the definition of AM-stationarity might be unbounded. Though this boundedness can be guaranteed under suitable (relatively strong) assumptions, the resulting convergence theory works under significantly weaker conditions.

Here, we would like to mention the price of a slack variable $x_{\mathrm{s}} \in \mathbb{Y}$, we can transfer the given constraint system into

$$G(x) - x_{\mathrm{s}} = 0, \ (x_{\mathrm{s}}, x) \in C \times D$$

where the right-hand side of the nonlinear constraint is trivially convex. In order to apply the algorithmic framework of this paper to this reformulation, projections onto C have to be computed efficiently. Moreover, there might be a difference between the asymptotic notions of stationarity and regularity discussed here when applied to this reformulation or the original formulation of the constraints, the more details will be exploited in Chapter 4 in a more general setting.

The following result shows that each local minimizer of (P) is AM-stationary.

Theorem 3.3. *If $\bar{x}$ is a local minimizer of* (P)*, then $\bar{x}$ is an AM-stationary point.*

Proof. Since $\bar{x}$ is local minimizer of (P), then for any $\delta > 0$, one has

$$f(\bar{x}) \leq f(x) \quad \forall x \in B_\delta(\bar{x}) \cap \mathcal{F},$$

where $B_\delta(\bar{x})$ is the closed interval around $\bar{x}$ with radius δ, $\mathcal{F} := \{x \in D \,|\, G(x) \in C\}$ is the feasible set of (P). Then obviously, $\bar{x}$ is the unique global minimizer of

$$\min_x f(x) + \frac{1}{2}\|x - \bar{x}\|^2 \quad \text{s.t. } x \in B_\delta(\bar{x}) \cap \mathcal{F}. \tag{3.1}$$

Now for each $k \in \mathbb{N}$, we consider the following problem

$$\min_{x,u} f(x) + \frac{k}{2}\|G(x) - u\|^2 + \frac{1}{2}\|x - \bar{x}\|^2 \quad \text{s.t. } (x, u) \in B_\delta(\bar{x}, G(\bar{x})) \cap (D \times C), \tag{3.2}$$

where $B_\delta(\bar{x}, G(\bar{x}))$ is the closed ball around $(\bar{x}, G(\bar{x}))$ with radius $\delta > 0$. The objective function of (3.2) is continuous and feasible set is compact, hence each of the problems (3.2) attains a global minimum $(x^k, u^k) \in B_\delta(\bar{x}, G(\bar{x})) \cap (D \times C)$, without loss of generality, we assume $(x^k, u^k) \to (x^*, u^*)$. Now, we want to show that $(x^*, u^*) = (\bar{x}, G(\bar{x}))$. As $(\bar{x}, G(\bar{x})) \in B_\delta(\bar{x}, G(\bar{x})) \cap (D \times C)$, it is feasible for (3.2) for each $k \in \mathbb{N}$. Thus, we obtain for each $k \in \mathbb{N}$ that

$$f(x^k) + \frac{k}{2}\left\|G(x^k) - u^k\right\|^2 + \frac{1}{2}\|x^k - \bar{x}\|^2 \leq f(\bar{x}).$$

Taking the limit $k \to \infty$ and using continuity arguments, it follows that $G(x^*) = u^* \in C$ and hence

$$f(x^*) + \frac{1}{2}\|x^* - \bar{x}\|^2 \leq f(\bar{x}) + \frac{1}{2}\|\bar{x} - \bar{x}\|^2.$$

Since $\bar{x}$ is the unique global solution of (3.1), we then have $\bar{x} = x^*$, which deduces $(x^*, u^*) = (x^*, G(x^*)) = (\bar{x}, G(\bar{x}))$ by the continuity of G. This shows $(x^k, u^k) \to (\bar{x}, G(\bar{x}))$, hence $(x^k, u^k) \in B_\delta(\bar{x}, G(\bar{x}))$ for all $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, (x^k, u^k) is a local minimizer of

$$\min_{x,u} f(x) + \frac{k}{2}\|G(x) - u\|^2 + \frac{1}{2}\|x - \bar{x}\|^2 \quad \text{s.t. } (x, u) \in D \times C.$$

Theorem 2.31 implies

$$-\left(\nabla f(x^k) + kG'(x^k)^*(G(x^k) - u^k) + x^k - \bar{x}\right) \in \mathcal{N}_D(x^k) \subset \mathcal{N}_D^{\lim}(x^k) \tag{3.3}$$

and

$$k(G(x^k) - u^k) \in \mathcal{N}_C(u^k).$$

Setting $\varepsilon^k = \bar{x} - x^k$, $\lambda^k = k(G(x^k) - u^k)$, $z^k = G(x^k) - u^k$, one obviously has $\varepsilon^k \to 0$, $\lambda^k \in \mathcal{N}_C(G(x^k) - z^k)$, and $z^k \to 0$. Moreover, (3.3) can be reformulated as

$$\varepsilon^k \in \nabla f(x^k) + G'(x^k)^* \lambda^k + \mathcal{N}_D^{\lim}(x^k), \tag{3.4}$$

hence one has $\bar{x}$ is an AM-stationary point of (P). □

In order to infer that an AM-stationary point is already M-stationary, the presence of so-called asymptotic regularity is necessary, see [117, Definition 4.4].

Definition 3.4 (AM-regularity of (P))**.** A feasible point $\bar{x} \in \mathbb{X}$ of (P) is called *AM-regular* (asymptotically Mordukhovich-regular) whenever the condition

$$\limsup_{x \to \bar{x},\, z \to 0} \mathcal{M}(x, z) \subset \mathcal{M}(\bar{x}, 0)$$

holds, where $\mathcal{M} \colon \mathbb{X} \times \mathbb{Y} \rightrightarrows \mathbb{X}$ is the set-valued mapping defined via

$$\mathcal{M}(x, z) := G'(x)^* \mathcal{N}_C(G(x) - z) + \mathcal{N}_D^{\lim}(x).$$

The concept of AM-regularity has been inspired by the notion of AKKT-regularity (sometimes referred to as cone continuity property), which became popular as one of the weakest constraint qualifications for standard nonlinear programs or MPCCs, see e.g. [9, 10, 135], and can be generalized to a much higher level of abstractness. In this regard, we would like to point the reader's attention to the fact that AM-stationarity and -regularity from Definitions 3.2 and 3.4 are referred to as *decoupled* asymptotic Mordukhovich-stationarity and -regularity in [117] since these are already refinements of more general concepts. For the sake of a concise notation, however, we omit the term *decoupled* here.

It has been shown in [117, Section 5.1] that validity of AM-regularity at a feasible point $\bar{x} \in \mathbb{X}$ of (P) is implied by

$$0 \in G'(\bar{x})^* \lambda + \mathcal{N}_D^{\lim}(\bar{x}), \quad \lambda \in \mathcal{N}_C(G(\bar{x})) \quad \Longrightarrow \quad \lambda = 0. \tag{3.5}$$

The latter is known as NNAMCQ (no nonzero abnormal multiplier constraint qualification) or GMFCQ (generalized Mangasarian–Fromovitz constraint qualification) in the literature. Indeed, in the setting where we fix $C := \mathbb{R}^{m_1}_- \times \{0\}^{m_2}$ and $D := \mathbb{X}$, (3.5) boils down to the classical Mangasarian–Fromovitz constraint qualification from standard nonlinear programming. The latter choice for C will be of particular interest, which is why we formalize this setting below.

Setting 3.5. *Given $m_1, m_2 \in \mathbb{N}$, we set $m := m_1 + m_2$, $\mathbb{Y} := \mathbb{R}^m$, and $C := \mathbb{R}^{m_1}_- \times \{0\}^{m_2}$. No additional assumptions are postulated on the set D. We denote the component functions of G by $G_1, \ldots, G_m \colon \mathbb{X} \to \mathbb{R}$. Thus, the constraint $G(x) \in C$ encodes the constraint system*

$$G_i(x) \leq 0 \quad i = 1, \ldots, m_1, \qquad G_i(x) = 0 \quad i = m_1 + 1, \ldots, m$$

of standard nonlinear programming. For our analysis, we exploit the index sets

$$I(\bar{x}) := \{i \in \{1, \ldots, m_1\} \mid G_i(\bar{x}) = 0\}, \qquad J := \{m_1 + 1, \ldots, m\},$$

whenever $\bar{x} \in D$ satisfies $G(\bar{x}) \in C$ in the present situation.

RCPLD has been introduced for standard nonlinear programs (i.e., $D := \mathbb{X} = \mathbb{R}^n$ in Setting 3.5) in [6]. Some extensions to complementarity-constrained programs can be found in [58, 82]. A more restrictive RCPLD-type constraint qualification which is capable of handling an abstract constraint set can be found in [83, Definition 1]. Constraint regions as characterized in Setting 3.5 can be tackled with the following version of RCPLD.

Definition 3.6. [156, Definition 1.1] Let $\bar{x} \in \mathbb{X}$ be a feasible point of the optimization problem (P) in Setting 3.5. Then $\bar{x}$ is said to satisfy RCPLD whenever the following conditions hold:

(i) the family $(\nabla G_i(x))_{i \in J}$ has constant rank on a neighborhood of $\bar{x}$,

(ii) there exists an index set $S \subset J$ such that the family $(\nabla G_i(\bar{x}))_{i \in S}$ is a basis of the subspace $\operatorname{span}\{\nabla G_i(\bar{x}) \mid i \in J\}$, and

(iii) for each index set $I \subset I(\bar{x})$, each set of multipliers $\lambda_i \geq 0$ $(i \in I)$ and $\lambda_i \in \mathbb{R}$ $(i \in S)$, not all vanishing at the same time, and each vector $\eta \in \mathcal{N}_D^{\lim}(\bar{x})$ which satisfy

$$0 \in \sum_{i \in I \cup S} \lambda_i \nabla G_i(\bar{x}) + \eta,$$

we find neighborhoods U of $\bar{x}$ and V of η such that for all $x \in U$ and $\tilde{\eta} \in \mathcal{N}_D^{\lim}(x) \cap V$, the vectors from

$$\begin{cases} (\nabla G_i(x))_{i \in I \cup S},\ \tilde{\eta} & \text{if } \tilde{\eta} \neq 0, \\ (\nabla G_i(x))_{i \in I \cup S} & \text{if } \tilde{\eta} = 0 \end{cases}$$

are linearly dependent.

In case where D is a set of product structure, condition (iii) in Definition 3.6 can be slightly weakened in order to obtain a reasonable generalization of the classical relaxed constant positive linear dependence constraint qualification, see [156, Remark 1.1] for details. Observing that GMFCQ from (3.5) takes the particular form

$$0 \in \sum_{i \in I(\bar{x}) \cup J} \lambda_i \nabla G_i(\bar{x}) + \mathcal{N}_D^{\lim}(\bar{x}), \quad \lambda_i \geq 0\,(i \in I) \quad \Longrightarrow \quad \lambda_i = 0\,(i \in I(\bar{x}) \cup J)$$

in Setting 3.5, it is obviously sufficient for RCPLD. The subsequently stated result generalizes related observations from [9, 135].

Lemma 3.7. *Let $\bar{x} \in \mathbb{X}$ be a feasible point for the optimization problem* (P) *in Setting 3.5 where RCPLD holds. Then $\bar{x}$ is AM-regular.*

Proof. Fix some $\xi \in \limsup_{x \to \bar{x},\, z \to 0} \mathcal{M}(x, z)$. Then we find $\{x^k\}, \{\xi^k\} \subset \mathbb{X}$ and $\{z^k\} \subset \mathbb{R}^m$ which satisfy $x^k \to \bar{x}$, $\xi^k \to \xi$, $z^k \to 0$, and $\xi^k \in \mathcal{M}(x^k, z^k)$ for all $k \in \mathbb{N}$. Particularly, there are sequences $\{\lambda^k\}$ and $\{\eta^k\}$ satisfying $\lambda^k \in \mathcal{N}_C(G(x^k) - z^k)$, $\eta^k \in \mathcal{N}_D^{\lim}(x^k)$, and $\xi^k = G'(x^k)^* \lambda^k + \eta_k$ for each $k \in \mathbb{N}$. From $G(x^k) - z^k \to G(\bar{x})$ and the special structure of C, we find $G_i(x^k) - z_i^k < 0$ for all $i \in \{1, \ldots, m_1\} \setminus I(\bar{x})$ and all sufficiently large $k \in \mathbb{N}$, i.e.,

$$\lambda_i^k \begin{cases} = 0 & i \in \{1, \ldots, m_1\} \setminus I(\bar{x}), \\ \geq 0 & i \in I(\bar{x}) \end{cases}$$

for sufficiently large $k \in \mathbb{N}$. Thus, we may assume without loss of generality that

$$\xi^k = \sum_{i \in I(\bar{x}) \cup J} \lambda_i^k \nabla G_i(x^k) + \eta^k$$

holds for all $k \in \mathbb{N}$. By definition of RCPLD, $(\nabla G_i(x^k))_{i\in S}$ is a basis of the subspace $\operatorname{span}\{\nabla G_i(x^k) \mid i \in J\}$ for all sufficiently large $k \in \mathbb{N}$. Hence, there exist scalars μ_i^k $(i \in S)$ such that

$$\xi^k = \sum_{i\in I(\bar{x})} \lambda_i^k \nabla G_i(x^k) + \sum_{i\in S} \mu_i^k \nabla G_i(x^k) + \eta^k$$

holds for all sufficiently large $k \in \mathbb{N}$. On the other hand, [6, Lemma 1] yields the existence of an index set $I^k \subset I(\bar{x})$ and multipliers $\hat{\mu}_i^k > 0$ $(i \in I^k)$, $\hat{\mu}_i^k \in \mathbb{R}$ $(i \in S)$, and $\sigma_k \geq 0$ such that

$$\xi^k = \sum_{i\in I^k\cup S} \hat{\mu}_i^k \nabla G_i(x^k) + \sigma_k\, \eta^k$$

and

$$\begin{aligned} \sigma_k > 0 \quad &\Longrightarrow \quad (\nabla G_i(x^k))_{i\in I^k\cup S},\ \eta^k \text{ linearly independent},\\ \sigma_k = 0 \quad &\Longrightarrow \quad (\nabla G_i(x^k))_{i\in I^k\cup S} \text{ linearly independent}. \end{aligned}$$

Since there are only finitely many subsets of $I(\bar{x})$, there needs to exist $I \subset I(\bar{x})$ such that $I^k = I$ holds along a whole subsequence. Along such a particular subsequence (without relabeling), we furthermore may assume $\sigma_k > 0$ (otherwise, the proof will be easier) and, thus, may set $\hat{\eta}^k := \sigma_k \eta^k \in \mathcal{N}_D^{\lim}(x^k) \setminus \{0\}$. From above, we find linear independence of

$$(\nabla G_i(x^k))_{i\in I\cup S},\ \hat{\eta}^k.$$

Furthermore, one has

$$\xi^k = \sum_{i\in I\cup S} \hat{\mu}_i^k \nabla G_i(x^k) + \hat{\eta}^k. \tag{3.6}$$

Suppose that the sequence $\{((\hat{\mu}_i^k)_{i\in I\cup S}, \hat{\eta}^k)\}$ is not bounded. Dividing (3.6) by the norm of $((\hat{\mu}_i^k)_{i\in I\cup S}, \hat{\eta}^k)$, taking the limit $k \to \infty$, and respecting boundedness of $\{\xi^k\}$, continuity of G', and outer semicontinuity of the limiting normal cone yield the existence of a non-vanishing multiplier $((\hat{\mu}_i)_{i\in I\cup S}, \hat{\eta})$ which satisfies $\hat{\mu}_i \geq 0$ $(i \in I)$, $\hat{\eta} \in \mathcal{N}_D^{\lim}(\bar{x})$, and

$$0 = \sum_{i\in I\cup S} \hat{\mu}_i \nabla G_i(\bar{x}) + \hat{\eta}.$$

Obviously, the multipliers $\hat{\mu}_i$ $(i \in I \cup S)$ do not vanish at the same time since, otherwise, $\hat{\eta} = 0$ would follow from above which yields a contradiction. Now, validity of RCPLD guarantees that the vectors

$$(\nabla G_i(x^k))_{i\in I\cup S},\ \hat{\eta}^k$$

need to be linearly dependent for sufficiently large $k \in \mathbb{N}$. However, we already have shown above that these vectors are linearly independent, a contradiction.

Thus, the sequence $\{((\hat{\mu}_i^k)_{i\in I\cup S}, \hat{\eta}^k)\}$ is bounded and, therefore, possesses a convergent subsequence with limit $((\bar{\mu}_i)_{i\in I\cup S}, \bar{\eta})$. Taking the limit in (3.6) while respecting $\xi^k \to \xi$, the continuity of G', and the outer semicontinuity of the limiting normal cone, we come up with $\bar{\mu}_i \geq 0$ $(i \in I)$, $\bar{\eta} \in \mathcal{N}_D^{\lim}(\bar{x})$, and

$$\xi = \sum_{i\in I\cup S} \bar{\mu}_i \nabla G_i(\bar{x}) + \bar{\eta}.$$

Finally, we set $\bar{\mu}_i := 0$ for all $i \in \{1,\ldots,m\} \setminus (I \cup S)$. Then we have $(\bar{\mu}_i)_{i=1,\ldots,m} \in \mathcal{N}_C(G(\bar{x}))$ from $I \subset I(\bar{x})$, i.e.,

$$\xi \in G'(\bar{x})^* \mathcal{N}_C(G(\bar{x})) + \mathcal{N}_D^{\lim}(\bar{x}) = \mathcal{M}(\bar{x}, 0).$$

This shows that $\bar{x}$ is AM-regular. □

A popular situation, where AM-regularity simplifies and, thus, becomes easier to verify, is described in the following lemma.

Lemma 3.8. *[117, Theorems 3.10, 5.2] Let $\bar{x} \in \mathbb{X}$ be a feasible point for the optimization problem* (P) *where C is a polyhedron and D is the union of finitely many polyhedrons. Then $\bar{x}$ is AM-regular if any only if*

$$\limsup_{x\to\bar{x}}\Big(G'(x)^*\mathcal{N}_C(G(\bar{x}))+\mathcal{N}_D^{\lim}(\bar{x})\Big)\subset G'(\bar{x})^*\mathcal{N}_C(G(\bar{x}))+\mathcal{N}_D^{\lim}(\bar{x}).$$

Particularly, in case where G is an affine function, $\bar{x}$ is AM-regular.

Let us consider the situation where (P) is given as described in Setting 3.5, and assume in addition that $D := \mathbb{X}$ holds, i.e., that (P) is a standard nonlinear optimization problem with finitely many equality and inequality constraints. Then Lemma 3.8 shows that AM-regularity corresponds to the cone continuity property from [9, Definition 3.1], and the latter has been shown to be weaker than most of the established constraint qualifications which can be checked in terms of initial problem data.

For the so-called disjunctive programs of special type, such as (box) switching constraints, complementarity constraints, and relaxed reformulated cardinality constraints, they can be addressed in the setting mentioned below which provides a refinement of Setting 3.5.

Setting 3.9. *Let $\mathbb{W}$ be another Euclidean space, let $W \subset \mathbb{W}$ be the union of finitely many convex, polyhedral sets, and let $T \subset \mathbb{R}^2$ be the union of two polyhedrons $T_1, T_2 \subset \mathbb{R}^2$. For functions $g\colon \mathbb{W} \to \mathbb{R}^{m_1}$, $h\colon \mathbb{W} \to \mathbb{R}^{m_2}$, and $p, q\colon \mathbb{W} \to \mathbb{R}^{m_3}$, we consider the constraint system given by*

$$\begin{aligned} g_i(w) &\le 0 \qquad && i=1,\dots,m_1,\\ h_i(w) &= 0 && i=1,\dots,m_2,\\ (p_i(w),q_i(w)) &\in T && i=1,\dots,m_3,\\ w &\in W. \end{aligned}$$

Setting $\mathbb{X} := \mathbb{W}\times\mathbb{R}^{m_3}\times\mathbb{R}^{m_3}$, $\mathbb{Y} := \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{m_3}\times\mathbb{R}^{m_3}$,

$$G(w,u,v) := \big(g(w),h(w),p(w)-u,q(w)-v\big),$$

and

$$C := \mathbb{R}^{m_1}_{-}\times\{0\}^{m_2+2m_3},\qquad D := W\times\widetilde{T},$$

where we used $\widetilde{T} := \{(u,v)\mid (u_i,v_i)\in T\ \forall i\in\{1,\dots,m_3\}\}$, we can handle this situation in the framework of (P)*.*

Lemma 3.8 also helps us to find a tangible representation of AM-regularity in Setting 3.9.

Lemma 3.10. *Let $\bar{w} \in \mathbb{W}$ be a feasible point of the optimization problem from Setting 3.9. Furthermore, define a set-valued mapping $\widetilde{\mathcal{M}}\colon \mathbb{W} \rightrightarrows \mathbb{W}$ by*

$$\widetilde{\mathcal{M}}(w) := \left\{\mathfrak{L}(w,\lambda,\rho,\mu,\nu,\xi)\ \middle|\ \begin{array}{l} 0\le\lambda\perp g(\bar{w}),\\ (\mu,\nu)\in\mathcal{N}^{\lim}_{\widetilde{T}}(p(\bar{w}),q(\bar{w})),\\ \xi\in\mathcal{N}^{\lim}_{W}(\bar{w}) \end{array}\right\}$$

where $\mathfrak{L}\colon \mathbb{W}\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{m_3}\times\mathbb{R}^{m_3}\times\mathbb{W}\to\mathbb{W}$ is the function given by

$$\mathfrak{L}(w,\lambda,\rho,\mu,\nu,\xi) := g'(w)^*\lambda+h'(w)^*\rho+p'(w)^*\mu+q'(w)^*\nu+\xi.$$

Then the feasible point $(\bar{w}, p(\bar{w}), q(\bar{w}))$ *of the associated problem* (P) *is AM-regular if and only if*

$$\limsup_{w\to\bar{w}} \widetilde{\mathcal{M}}(w) \subset \widetilde{\mathcal{M}}(\bar{w}). \tag{3.7}$$

Proof. First, observe that transferring the constraint region from Setting 3.9 into the form used in (P) and keeping Lemma 3.8 in mind shows that AM-regularity of $(\bar{w}, p(\bar{w}), q(\bar{w}))$ is equivalent to

$$\limsup_{w\to\bar{w}} \widehat{\mathcal{M}}(w) \subset \widehat{\mathcal{M}}(\bar{w}) \tag{3.8}$$

where $\widehat{\mathcal{M}}\colon \mathbb{W} \rightrightarrows \mathbb{W} \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_3}$ is given by

$$\widehat{\mathcal{M}}(w) := \left\{ (\mathfrak{L}(w,\lambda,\rho,\tilde{\mu},\tilde{\nu},\xi), -\tilde{\mu}+\mu, -\tilde{\nu}+\nu) \,\middle|\, \begin{array}{l} 0 \le \lambda \perp g(\bar{w}), \\ (\mu,\nu) \in \mathcal{N}^{\lim}_{\widetilde{T}}(p(\bar{w}), q(\bar{w})), \\ \xi \in \mathcal{N}^{\lim}_{W}(\bar{w}) \end{array} \right\}.$$

Observing that $\eta \in \widetilde{\mathcal{M}}(x)$ is equivalent to $(\eta, 0, 0) \in \widehat{\mathcal{M}}(x)$, (3.8) obviously implies (3.7). In order to show the converse relation, we assume that (3.7) holds and fix $(\eta, \alpha, \beta) \in \limsup_{w\to\bar{w}} \widehat{\mathcal{M}}(w)$. Then we find sequences $\{w^k\}, \{\xi^k\}, \{\eta^k\} \subset \mathbb{W}$, $\{\lambda^k\} \subset \mathbb{R}^{m_1}$, $\{\rho^k\} \subset \mathbb{R}^{m_2}$, and $\{\mu^k\}, \{\tilde{\mu}^k\}, \{\nu^k\}, \{\tilde{\nu}^k\} \subset \mathbb{R}^{m_3}$ such that $w^k \to \bar{w}$, $\eta^k \to \eta$, $-\tilde{\mu}^k + \mu^k \to \alpha$, $-\tilde{\nu}^k + \nu^k \to \beta$, and $\eta^k = \mathfrak{L}(w^k, \lambda^k, \rho^k, \tilde{\mu}^k, \tilde{\nu}^k, \xi^k)$, $0 \le \lambda^k \perp g(\bar{w})$, $(\mu^k, \nu^k) \in \mathcal{N}^{\lim}_{\widetilde{T}}(p(\bar{w}), q(\bar{w}))$, as well as $\xi^k \in \mathcal{N}^{\lim}_{w}(\bar{w})$ for all $k \in \mathbb{N}$. Setting $\alpha^k := -\tilde{\mu}^k + \mu^k$ and $\beta^k := -\tilde{\nu}^k + \nu^k$, we find $\eta^k + p'(w^k)^*\alpha^k + q'(w^k)^*\beta^k = \mathfrak{L}(w^k, \lambda^k, \rho^k, \mu^k, \nu^k, \xi^k)$ for each $k \in \mathbb{N}$, and due to $\alpha^k \to \alpha$ and $\beta^k \to \beta$, validity of (3.7) yields $\eta + p'(\bar{w})^*\alpha + q'(\bar{w})^*\beta \in \widetilde{\mathcal{M}}(\bar{w})$, i.e., the existence of $\lambda \in \mathbb{R}^{m_1}$, $\rho \in \mathbb{R}^{m_2}$, $\mu, \nu \in \mathbb{R}^{m_3}$, and $\xi \in \mathbb{W}$ such that $\eta + p'(\bar{w})^*\alpha + q'(\bar{w})^*\beta = \mathfrak{L}(\bar{w}, \lambda, \rho, \mu, \nu, \xi)$, $0 \le \lambda \perp g(\bar{w})$, $(\mu, \nu) \in \mathcal{N}^{\lim}_{\widetilde{T}}(p(\bar{w}), q(\bar{w}))$, and $\xi \in \mathcal{N}^{\lim}_{W}(\bar{w})$. Thus, setting $\tilde{\mu} := \mu - \alpha$ and $\tilde{\nu} := \nu - \beta$, we find $(\eta, \alpha, \beta) \in \widehat{\mathcal{M}}(\bar{w})$ showing (3.8). □

Let us specify these findings for MPCCs which can be stated in the form (P) via Setting 3.9. Taking lemmas 3.8 and 3.10 into account, AM-regularity corresponds to the so-called MPCC cone continuity property from [135, Definition 3.9]. The latter has been shown to be strictly weaker than MPCC-RCPLD, see [135, Definition 4.1, Theorem 4.2, Example 4.3] for a definition and this result. A similar reasoning can be used in order to show that problem-tailored versions of RCPLD associated with other classes of disjunctive programs are sufficient for the respective AM-regularity. This, to some extend, recovers our result from Lemma 3.7 although we need to admit that, exemplary, RCPLD from Definition 3.6 applied to MPCC in Setting 3.9 does not correspond to MPCC-RCPLD.

The above considerations underline that AM-regularity is a comparatively weak constraint qualification for (P). Exemplary, for standard nonlinear problems and for MPCCs, this follows from the above comments and the considerations in [9, 135]. For other types of disjunctive programs, the situation is likely to be similar, see e.g. [110, Figure 3] for the setting of switching-constrained optimization. It remains a topic of future research to find further sufficient conditions for AM-regularity which can be checked in terms of initial problem data, particularly, in situations where C and D are of particular structure like in semidefinite or second-order cone programming, see e.g. [8, Section 6]. Let us mention that the provably weakest constraint qualification which guarantees that local minimizers of a geometrically constrained program are M-stationary is slightly weaker than validity of the pre-image rule for the computation of the limiting normal cone to the constraint region of (P), see [82, Section 3] for a discussion, but the latter cannot be checked in practice. Due to [117, Theorem 3.16], AM-regularity indeed implies validity of this pre-image rule.

3.3 Spectral Gradient Methods as Subproblem Solver

When augmented Lagrangian methods are applied for (P), structurally, the constraints $G(x) \in C$ will be penalised by the augmented Lagrangian scheme, however the comparatively easy constraints $x \in D$ will be left, which causes the resulting subproblems are constrained optimization problems. In this section, an general Spectral Gradient method is applied to solve such a problem whose constraint set is nonconvex, and generates a approximately M-stationarity of the corresponding optimization problem, whereas Section 3.3.1 shows the approach exactly and tells the reason why it is employed, the analysis about convergence results and termination criterion is shown in Section 3.3.2.

3.3.1 Motivation and Statement of the Algorithm

First the following unconstrained optimization problem is considered

$$\min_x \ \varphi(x) \quad \text{s.t.} \quad x \in \mathbb{X}$$

where $\varphi\colon \mathbb{X} \to \mathbb{R}$ is a continuously differentiable objective function, and set x^j as a current estimate for a solution of this problem. Then the unique minimizer of the local quadratic model

$$\min_x \ \varphi(x^j) + \nabla\varphi(x^j)^\top(x - x^j) + \frac{\gamma_j}{2}\|x - x^j\|^2$$

for some $\gamma_j > 0$ is defined as the next iterate x^{j+1}, which could be computed explicitly as

$$x^{j+1} := x^j - \frac{1}{\gamma_j}\nabla\varphi(x^j),$$

i.e., a steepest descent method is obtained with stepsize $t_j := 1/\gamma_j$. Classical approaches compute t_j by using a suitable stepsize rule such that $\varphi(x^{j+1}) < \varphi(x^j)$. On the other hand, the update formula can be viewed as a special instance of a quasi-Newton scheme

$$x^{j+1} := x^j - B_j^{-1}\nabla\varphi(x^j)$$

with the very simple quasi-Newton matrix $B_j := \gamma_j I$ as an estimate of the (not necessarily existing) Hessian $\nabla^2\varphi(x^j)$. Then the corresponding quasi-Newton equation

$$B_{j+1}s^j = y^j \quad \text{with } s^j := x^{j+1} - x^j,\ y^j := \nabla\varphi(x^{j+1}) - \nabla\varphi(x^j),$$

see [68], reduces to the linear system $\gamma_{j+1}s^j = y^j$. Solving this overdetermined system in a least squares sense, one then obtains the stepsize

$$\gamma_{j+1} := (s^j)^\top y^j/(s^j)^\top s^j,$$

which has been introduced by Barzilai and Borwein [17]. This stepsize always leads to very good numerical results, but may not yield a monotone decrease in the function value. The convergence analysis for general nonlinear programs is therefore difficult, even if the choice of γ_j is safeguarded in the sense that it is projected onto some box $[\gamma_{\min}, \gamma_{\max}]$ for suitable constants $0 < \gamma_{\min} < \gamma_{\max}$.

Raydan [136] then suggested to control this nonmonotone behavior by combining the Barzilai–Borwein stepsize with the nonmonotone linesearch technique introduced by Grippo et al. [80], which, in particular, promotes a global convergence theory for general unconstrained optimization problems.

This idea was then generalized by Birgin et al. [35] to constrained optimization problems

$$\min_x \ \varphi(x) \quad \text{s.t.} \quad x \in X$$

with a nonempty, closed, and convex set $X \subset \mathbb{X}$ and is called the *nonmonotone spectral gradient method*. Note that the constraint set X should be convex. Then, one idea that this approach may be popularized to solved the following optimization problems with nonconvex constraint set is certainly arised,

$$\min_x \ \varphi(x) \quad \text{s.t.} \quad x \in D \tag{3.9}$$

with a continuously differentiable function $\varphi\colon \mathbb{X} \to \mathbb{R}$ and some nonempty, closed set $D \subset \mathbb{X}$, where $\mathbb{X}$ is an arbitrary Euclidean space. Note that neither φ nor D need to be convex in the subsequent considerations.

A detailed description of the corresponding generalized spectral gradient is given in Algorithm 3.3.1.

Algorithm 3.3.1: General Spectral Gradient Method

Data: $\tau > 1, \sigma \in (0,1), 0 < \gamma_{\min} \leq \gamma_{\max} < \infty, n \in \mathbb{N}, x^0 \in D$

1 **for** $j \leftarrow 0$ **to** ∞ **do**
2   Set $m_j := \min(j, n)$, $i \leftarrow 0$ and choose $\gamma_j^0 \in [\gamma_{\min}, \gamma_{\max}]$;
3   **repeat**
4   Set $i \leftarrow i+1$, $\gamma_{j,i} := \tau^{i-1}\gamma_j^0$ and compute a solution $x^{j,i}$ of

$$\min_x \ \varphi(x^j) + \langle \nabla\varphi(x^j), x - x^j \rangle + \frac{\gamma_{j,i}}{2}\|x - x^j\|^2 \quad \text{s.t.} \quad x \in D; \tag{Q(j,i)}$$

5   **if** $x^{j,i}$ *satisfies some termination criterion* **then**
6   **return** $x^{j,i}$;
7   **end**
8   **until** $\varphi(x^{j,i}) \leq \max_{r=0,1,\dots,m_j} \varphi(x^{j-r}) + \sigma\langle \nabla\varphi(x^j), x^{j,i} - x^j \rangle$;
9   Set $i_j := i$, $\gamma_j := \gamma_{j,i}$, and $x^{j+1} := x^{j,i}$;
10 **end**

Particular instances of the method with nonconvex sets D have already been found in [22, 55, 56, 81]. Note that all iterates from Algorithm 3.3.1 belong to the set D, that the subproblems $(\mathrm{Q}(j,i))$ are always solvable, and that only one solution needs to be computed, although their solutions are not necessarily unique.

It is also emphasized that $\nabla\varphi(x^j)$ was used in the formulation of $(\mathrm{Q}(j,i))$ in order to underline that Algorithm 3.3.1 is actually a projected gradient method. Indeed, simple calculations reveal that the global solutions of $(\mathrm{Q}(j,i))$ correspond to the projections of $x^j - \gamma_{j,i}^{-1}\nabla\varphi(x^j)$ onto D. Note also that the acceptance criterion in Line 8 is the nonmonotone Armijo rule introduced by Grippo et al. [80]. In particular, the parameter $m_j := \min(j, n)$ controls the nonmonotonicity. The choice $n = 0$ corresponds to the standard monotone method, whereas $n > 0$ typically allows larger stepsizes and then often leads to faster convergence of the method.

3.3.2 Convergence Analysis

The goal of Algorithm 3.3.1 is that the computation of a point is approximately M-stationary for (3.9). Recall that x is an M-stationary point of (3.9) if

$$0 \in \nabla\varphi(x) + \mathcal{N}_D^{\lim}(x)$$

holds, and that each locally optimal solution of (3.9) satisfying some assertions is M-stationary by [123, Theorem 6.1]. Similarly, since $x^{j,i}$ solves the subproblem ($\mathrm{Q}(j,i)$), it is a candicate for the corresponding M-stationarity condition

$$0 \in \nabla\varphi(x^j) + \gamma_{j,i}(x^{j,i} - x^j) + \mathcal{N}_D^{\lim}(x^{j,i}). \tag{3.10}$$

Recall the fact that strong stationarity, where the limiting normal cone is replaced by the smaller regular normal cone in the stationarity system, provides a more restrictive necessary optimality condition for (Q) and the surrogate ($\mathrm{Q}(j,i)$), see [140, Definition 6.3, Theorem 6.12]. It is well known that the limiting normal cone is the outer limit of the regular normal cone. In contrast to the limiting normal cone, the regular one is not robust in the sense of (2.6), and since one is interested in taking limits later on, one either way ends up with a stationarity systems in terms of limiting normals at the end. Thus, one relies on the limiting normal cone and the associated concept of M-stationarity.

For the following theoretical results, the termination criterion in Line 5 will be neglected temporally. This means that Algorithm 3.3.1 does not terminate and performs either infinitely many inner or infinitely many outer iterations. The first result analyzes the inner loop.

Proposition 3.11. *Consider a fixed (outer) iteration j in Algorithm 3.3.1. Then the inner loop terminates (due to Line 8) or*

$$\|\gamma_{j,i}(x^j - x^{j,i}) + \nabla\varphi(x^{j,i}) - \nabla\varphi(x^j)\| \to 0 \qquad \textit{as} \quad i \to \infty. \tag{3.11}$$

If the inner loop does not terminate, one gets $x^{j,i} \to x^j$ and x^j is M-stationary.

Proof. Since $x^{j,i}$ is a solution of ($\mathrm{Q}(j,i)$) with $\gamma_{j,i} = \tau^{i-1}\gamma_j^0$. From $x^j \in D$, the optimality condition of $x^{j,i}$ for ($\mathrm{Q}(j,i)$) yields

$$\varphi(x^j) + \langle\nabla\varphi(x^j), x^{j,i} - x^j\rangle + \frac{\gamma_{j,i}}{2}\|x^{j,i} - x^j\|^2 \le \varphi(x^j) + \langle\nabla\varphi(x^j), x^j - x^j\rangle + \frac{\gamma_{j,i}}{2}\|x^j - x^j\|^2$$

holds for all $i \in \mathbb{N}$, which equals to

$$\langle\nabla\varphi(x^j), x^{j,i} - x^j\rangle + \frac{\gamma_{j,i}}{2}\|x^{j,i} - x^j\|^2 \le 0 \qquad \forall i \in \mathbb{N}. \tag{3.12}$$

(3.12) with Cauchy–Schwarz inequality therefore gives

$$\frac{\gamma_{j,i}}{2}\|x^{j,i} - x^j\| \le \|\nabla\varphi(x^j)\| \qquad \forall i \in \mathbb{N}.$$

This implies that $x^{j,i} \to x^j$ for $i \to \infty$. Now, there exist two cases. First, if

$$\limsup_{i\to\infty} \gamma_{j,i}\|x^{j,i} - x^j\| > 0. \tag{3.13}$$

Hence, there exist a (sub)sequence $i_l \to \infty$ and a constant $\rho > 0$ such that

$$\gamma_{j,i_l} \|x^{j,i_l} - x^j\| \geq \rho \qquad \forall l \in \mathbb{N}.$$

Consequently, one obtains from (3.12) that

$$\frac{\rho}{2} \|x^{j,i_l} - x^j\| \leq \frac{\gamma_{j,i_l}}{2} \|x^{j,i_l} - x^j\|^2 \leq -\langle \nabla \varphi(x^j), x^{j,i_l} - x^j \rangle.$$

It, together with a Taylor expansion, therefore implies

$$\begin{aligned} \varphi(x^{j,i_l}) - \max_{r=0,1,\ldots,m_j} \varphi(x^{j-r}) &\leq \varphi(x^{j,i_l}) - \varphi(x^j) \\ &= \langle \nabla \varphi(x^j), x^{j,i_l} - x^j \rangle + o(\|x^{j,i_l} - x^j\|) \\ &\leq \sigma \langle \nabla \varphi(x^j), x^{j,i_l} - x^j \rangle \end{aligned}$$

for all l sufficiently large, i.e., the inner loop terminates.

In the second case, (3.13) is not satisfied, i.e., $\gamma_{j,i} \|x^{j,i} - x^j\| \to 0$. Because $\nabla \varphi$ is continuous, this yields (3.11). Together with $x^{j,i} \to x^j$, the continuity of $\nabla \varphi$, and (2.6), one can pass to the limit $i \to \infty$ in (3.10) and obtain that x^j is M-stationary. □

It remains to analyze the situation where the inner loop always terminates. Let $x^0 \in D$ be the starting point from Algorithm 3.3.1, and let

$$\mathcal{S}_\varphi(x^0) := \{x \in D \mid \varphi(x) \leq \varphi(x^0)\}$$

denote the corresponding (feasible) sublevel set. With the aid of these assumptions, then the following result holds, also see [80, 154].

Proposition 3.12. *Assume that the inner loop in Algorithm 3.3.1 always terminates (due to Line 8) and denote $\{x^j\}$ as the infinite sequence of (outer) iterates. Assume that φ is bounded from below and uniformly continuous on $\mathcal{S}_\varphi(x^0)$. Then one has $\|x^{j+1} - x^j\| \to 0$ as $j \to \infty$.*

Proof. Let us assume that $l(j) \in \{j - m_j, \ldots, j\}$ is an index satisfying

$$\varphi(x^{l(j)}) = \max_{r=0,1,\ldots,m_j} \varphi(x^{j-r}) \qquad \forall j \in \mathbb{N}.$$

One can rewrite the nonmonotone Armijo rule from Line 8 in Algorithm 3.3.1 as

$$\varphi(x^{j+1}) \leq \varphi(x^{l(j)}) + \sigma \langle \nabla \varphi(x^j), x^{j+1} - x^j \rangle. \tag{3.14}$$

Since x^{j+1} solves

$$\min_x \ \varphi(x^j) + \langle \nabla \varphi(x^j), x - x^j \rangle + \frac{\gamma_j}{2} \|x - x^j\|^2 \quad \text{s.t.} \quad x \in D, \tag{3.15}$$

one has

$$\langle \nabla \varphi(x^j), x^{j+1} - x^j \rangle + \frac{\gamma_j}{2} \|x^{j+1} - x^j\|^2 \leq 0,$$

i.e.,

$$\langle \nabla \varphi(x^j), x^{j+1} - x^j \rangle \leq -\frac{\gamma_j}{2} \|x^{j+1} - x^j\|^2.$$

Hence, (3.14) implies

$$\varphi(x^{j+1}) \leq \varphi(x^{l(j)}) - \frac{\sigma\gamma_j}{2}\|x^{j+1} - x^j\|^2. \tag{3.16}$$

One first notes that the sequence $\{\varphi(x^{l(j)})\}$ is monotonically decreasing. Then the fact that $m_{j+1} \leq m_j + 1$ implies

$$\begin{aligned}
\varphi(x^{l(j+1)}) &= \max_{r=0,1,\dots,m_{j+1}} \varphi(x^{j+1-r}) \\
&\leq \max_{r=0,1,\dots,m_j+1} \varphi(x^{j+1-r}) \\
&= \max\Big(\max_{r=0,1,\dots,m_j} \varphi(x^{j-r}), \varphi(x^{j+1})\Big) \\
&= \max\big(\varphi(x^{l(j)}), \varphi(x^{j+1})\big) \\
&= \varphi(x^{l(j)}),
\end{aligned}$$

where the last equality follows from (3.16). Since φ is bounded from below, this deduces

$$\lim_{j\to\infty} \varphi(x^{l(j)}) = \varphi^* \tag{3.17}$$

for some finite $\varphi^* \in \mathbb{R}$. Applying (3.16) with j replaced by $l(j) - 1$ and rearranging terms yields

$$\varphi(x^{l(j)}) - \varphi(x^{l(l(j)-1)}) \leq -\frac{\sigma\gamma_{l(j)-1}}{2}\left\|x^{l(j)} - x^{l(j)-1}\right\|^2 \leq 0.$$

Taking the limit $j \to \infty$ and using (3.17) therefore implies

$$\lim_{j\to\infty} \gamma_{l(j)-1}\left\|x^{l(j)} - x^{l(j)-1}\right\|^2 = 0.$$

Since $\gamma_j \geq \gamma_{\min} > 0$ for all $j \in \mathbb{N}$, one has

$$\lim_{j\to\infty} d^{l(j)-1} = 0, \tag{3.18}$$

where, for simplicity, one has $d^j := x^{j+1} - x^j$ for all $j \in \mathbb{N}$. From (3.17) and (3.18), one then obtains

$$\varphi^* = \lim_{j\to\infty} \varphi(x^{l(j)}) = \lim_{j\to\infty} \varphi(x^{l(j)-1} + d^{l(j)-1}) = \lim_{j\to\infty} \varphi(x^{l(j)-1}), \tag{3.19}$$

where the last equality is due to the uniform continuity of φ.

Let us now prove, by induction, that

$$\lim_{j\to\infty} d^{l(j)-r} = 0 \quad \text{and} \quad \lim_{j\to\infty} \varphi(x^{l(j)-r}) = \varphi^* \quad \forall r \in \mathbb{N}. \tag{3.20}$$

It has been implied from (3.18) and (3.19) that (3.20) holds for $r = 1$. Suppose that (3.20) holds for some $r \geq 1$. One has to show that it holds for $r + 1$. Using (3.16) with j replaced by $l(j) - r - 1$, one has

$$\varphi(x^{l(j)-r}) \leq \varphi(x^{l(l(j)-r-1)}) - \frac{\sigma\gamma_{l(j)-r-1}}{2}\left\|d^{l(j)-r-1}\right\|^2$$

(it is assumed implicitly that j is large enough such that no negative indices $l(j) - r - 1$

occur). Rearranging this expression and using $\gamma_j \geq \gamma_{\min}$ for all j yields

$$\|d^{l(j)-r-1}\|^2 \leq \frac{2}{\gamma_{\min}\sigma}\big(\varphi(x^{l(l(j)-r-1)}) - \varphi(x^{l(j)-r})\big).$$

Taking the limit $j \to \infty$ in (3.17) as well as the induction hypothesis, it follows that

$$\lim_{j\to\infty} d^{l(j)-r-1} = 0, \tag{3.21}$$

which proves the induction step for the first limit in (3.20). The second limit is from

$$\lim_{j\to\infty} \varphi(x^{l(j)-(r+1)}) = \lim_{j\to\infty} \varphi(x^{l(j)-(r+1)} + d^{l(j)-(r+1)}) = \lim_{j\to\infty} \varphi(x^{l(j)-r}) = \varphi^*,$$

where the first equation follows from (3.21) and the uniform continuity of φ, whereas the final equation is the induction hypothesis.

The final step of the proof aims to show that $\lim_{j\to\infty} d^j = 0$. To be contrary, then there exists a (suitably shifted, for notational simplicity) subsequence $\{d^{j-n-1}\}_{\mathcal{K}}$ and a constant $\rho > 0$ such that

$$\|d^{j-n-1}\| \geq \rho \qquad \forall j \in \mathcal{K}. \tag{3.22}$$

Now, for every $j \in \mathcal{K}$, the corresponding index $l(j)$ is one of the indices $j-n, j-n+1, \ldots, j$. Hence, one can write $j - n - 1 = l(j) - r_j$ for some index $r_j \in \{1, 2, \ldots, n+1\}$. Since there are only finitely many possible indices r_j, one may assume without loss of generality that $r_j = r$ holds for some fixed index r. Then (3.20) implies

$$\lim_{j\to_{\mathcal{K}}\infty} d^{j-n-1} = \lim_{j\to_{\mathcal{K}}\infty} d^{l(j)-r} = 0,$$

which contradicts (3.22) and therefore completes the proof. □

The following gives the main convergence result of Algorithm 3.3.1.

Proposition 3.13. *Assume that the inner loop in Algorithm 3.3.1 always terminates (due to Line 8) and denote $\{x^j\}$ as the infinite sequence of (outer) iterates. Assume that φ is bounded from below and uniformly continuous on $\mathcal{S}_\varphi(x^0)$. Suppose that $\bar{x}$ is an accumulation point of $\{x^j\}$, i.e., $x^j \to_{\mathcal{K}} \bar{x}$ along a subsequence $\mathcal{K}$. Then $\bar{x}$ is an M-stationary point of the optimization problem* (3.9)*, and one has $\gamma_j\,(x^{j+1} - x^j) \to_{\mathcal{K}} 0$.*

Proof. Let $\bar{x}$ be an arbitrary accumulation point, and $\{x^j\}_{\mathcal{K}}$ be a subsequence such that $x^j \to_{\mathcal{K}} \bar{x}$.

We claim that $\gamma_j\,(x^{j+1} - x^j) \to_{\mathcal{K}} 0$. If $\{\gamma_j\}_{\mathcal{K}}$ is bounded, this follows directly from Proposition 3.12. Otherwise, in the case that $\{\gamma_j\}_{\mathcal{K}}$ is unbounded, one can find a subsequence $\mathcal{K}' \subset \mathcal{K}$ with $\gamma_j \to_{\mathcal{K}'} \infty$ and $\gamma_j > \gamma_{\max}$ for all $j \in \mathcal{K}'$. Then $\hat{\gamma}_j := \gamma_j/\tau = \tau^{i_j-1}\gamma_j^0 = \gamma_{j,i_j-1}$ also converges to infinity. Due to $\gamma_j > \gamma_{\max}$, one obtains $i_j > 0$. Therefore, $\hat{x}^{j+1} := x^{j,i_j-1}$ solving (Q(j, i_j-1)), however violates the nonmonotone Armijo-type condition from Line 8 in Algorithm 3.3.1, i.e., one has

$$\varphi(\hat{x}^{j+1}) > \max_{r=0,1,\ldots,m_j} \varphi(x^{j-r}) + \sigma\langle\nabla\varphi(x^j), \hat{x}^{j+1} - x^j\rangle \tag{3.23}$$

for all $j \in \mathcal{K}'$ sufficiently large. The following analysis is highly similar to the proof of Proposition 3.11 (except that j is not fixed now). Since $\hat{x}^{j+1}$ is the solution of the subproblem (Q$(j, i_j - 1)$), one has

$$\langle \nabla\varphi(x^j), \hat{x}^{j+1} - x^j \rangle + \frac{\hat{\gamma}_j}{2} \|\hat{x}^{j+1} - x^j\|^2 \leq 0, \tag{3.24}$$

which implies that

$$\frac{\hat{\gamma}_j}{2} \|\hat{x}^{j+1} - x^j\| \leq \|\nabla\varphi(x^j)\|.$$

Since $x^j \to_{\mathcal{K}'} \bar{x}$, this yields $\hat{x}^{j+1} - x^j \to_{\mathcal{K}'} 0$. Hence, one also gets $\hat{x}^{j+1} \to_{\mathcal{K}'} \bar{x}$. For each $j \in \mathcal{K}'$, the mean value theorem implies that there exists ξ^j on the line segment between $\hat{x}^{j+1}$ and x^j such that

$$\varphi(\hat{x}^{j+1}) - \varphi(x^j) = \langle \nabla\varphi(\xi^j), \hat{x}^{j+1} - x^j \rangle.$$

Because of $\hat{x}^{j+1}, x^j \to_{\mathcal{K}'} \bar{x}$, one gets $\nabla\varphi(\xi^j) - \nabla\varphi(x^j) \to_{\mathcal{K}'} 0$. (3.23) deduces that

$$\begin{aligned}
\sigma \langle \nabla\varphi(x^j), \hat{x}^{j+1} - x^j \rangle &< \varphi(\hat{x}^{j+1}) - \max_{r=0,1,\ldots,m_j} \varphi(x^{j-r}) \\
&\leq \varphi(\hat{x}^{j+1}) - \varphi(x^j) \\
&\leq \langle \nabla\varphi(x^j), \hat{x}^{j+1} - x^j \rangle + \|\nabla\varphi(\xi^j) - \nabla\varphi(x^j)\| \, \|\hat{x}^{j+1} - x^j\|.
\end{aligned}$$

Together with (3.24), one achieves

$$\frac{\hat{\gamma}_j}{2} \|\hat{x}^{j+1} - x^j\|^2 \leq -\langle \nabla\varphi(x^j), \hat{x}^{j+1} - x^j \rangle \leq \frac{\|\nabla\varphi(\xi^j) - \nabla\varphi(x^j)\|}{1 - \sigma} \|\hat{x}^{j+1} - x^j\|.$$

Hence, $\hat{\gamma}_j \|\hat{x}^{j+1} - x^j\| \to_{\mathcal{K}'} 0$. Using the optimality conditions of $\hat{x}^{j+1}$ and x^{j+1} for (Q$(j, i_j - 1)$) and (Q(j, i_j)), respectively, one obtains

$$\gamma_j \|x^{j+1} - x^j\| = \tau\hat{\gamma}_j \|x^{j+1} - x^j\| \leq \tau\hat{\gamma}_j \|\hat{x}^{j+1} - x^j\| \to_{\mathcal{K}'} 0.$$

Now, one can use a standard subsequence-subsequence argument to conclude that $\gamma_j \|x^{j+1} - x^j\| \to_{\mathcal{K}} 0$ holds along the entire subsequence $\mathcal{K}$.

It remains to prove M-stationarity of $\bar{x}$. Since x^{j+1} solves the subproblem (3.15), the corresponding optimality condition yields

$$0 \in \nabla\varphi(x^j) + \gamma_j(x^{j+1} - x^j) + \mathcal{N}_D^{\lim}(x^{j+1}).$$

Due to Proposition 3.12, one also has $x^{j+1} \to_{\mathcal{K}} \bar{x}$. Hence, taking the limit $j \to_{\mathcal{K}} \infty$ and exploiting once again the upper semicontinuity of the limiting normal cone, one can obtain

$$0 \in \nabla\varphi(\bar{x}) + \mathcal{N}_D^{\lim}(\bar{x}),$$

i.e., $\bar{x}$ is an M-stationary point of (3.9). □

Proposition 3.12 elaborates that the iterates of Algorithm 3.3.1 belong to the sublevel set $\mathcal{S}_\varphi(x^0)$ although the associated sequence of function values is not necessary monotonically decreasing. Hence, whenever this sublevel set is bounded, e.g., if φ is coercive or if D is bounded, the existence of an accumulation point as in Proposition 3.13 is ensured. Moreover, the boundedness of $\mathcal{S}_\varphi(x^0)$ implies that this set is compact. Hence, φ is automatically bounded from below and uniformly continuous on $\mathcal{S}_\varphi(x^0)$ in this situation.

By combining Propositions 3.11 and 3.13, one gets the following convergence result, which shows that the infinite sequence of (inner or outer) iterates of Algorithm 3.3.1 always converges towards M-stationary points (along subsequences).

Theorem 3.14. *Assume that Algorithm 3.3.1 without termination in Line 5 and* $\mathcal{S}_\varphi(x^0)$ *is bounded, Then exactly one of the following situations occurs.*

(i) *The inner loop does not terminate in the outer iteration* j, $x^{j,i} \to x^j$ *as* $i \to \infty$, x^j *is M-stationary, and* (3.11) *holds.*

(ii) *The inner loop always terminates, the infinite sequence* $\{x^j\}$ *of outer iterates possesses convergent subsequences* $\{x^j\}_\mathcal{K}$ *and every convergent subsequence satisfies* $x^j \to_\mathcal{K} \bar{x}$, $\bar{x}$ *is M-stationary, and* $\gamma_j\,(x^{j+1} - x^j) \to_\mathcal{K} 0$.

Note that $\mathcal{S}_\varphi(x^0)$ could be replaced by the assumptions on φ of Proposition 3.13, but then the outer iterates $\{x^j\}$ might fail to possess accumulation points.

In what follows, these theoretical results also give rise to a reasonable and applicable termination criterion which could be used in Line 5. To this end, we note that the optimality condition (3.10) is equivalent to

$$\gamma_{j,i}(x^j - x^{j,i}) + \nabla\varphi(x^{j,i}) - \nabla\varphi(x^j) \in \nabla\varphi(x^{j,i}) + \mathcal{N}_D^{\lim}(x^{j,i}),$$

which motivates us to use

$$\|\gamma_{j,i}(x^j - x^{j,i}) + \nabla\varphi(x^{j,i}) - \nabla\varphi(x^j)\| \leq \varepsilon_{\text{tol}} \tag{3.25}$$

with $\varepsilon_{\text{tol}} > 0$, as a termination criterion in Line 5. Indeed, Proposition 3.11 implies that the inner loop always terminates when (3.25) is satisfied. Moreover, the termination criterion (3.25) directly encodes that $x^{j,i}$ is approximately M-stationary for (3.9). This is very desirable since the goal of Algorithm 3.3.1 is the computation of approximately M-stationary points.

Furthermore, one can check that (3.25) always ensures the finite termination of Algorithm 3.3.1 if the mild assumptions of Theorem 3.14 (or the even weaker assumptions of Proposition 3.13) are satisfied. Indeed, due to $\gamma_j = \gamma_{j,i_j}$ and $x^{j+1} = x^{j,i_j}$, one has $\gamma_{j,i_j}(x^j - x^{j,i_j}) = \gamma_j(x^j - x^{j+1}) \to_\mathcal{K} 0$. Using $x^{j+1}, x^j \to_\mathcal{K} \bar{x}$ and the continuity of $\nabla\varphi\colon \mathbb{X} \to \mathbb{X}$ show that $\nabla\varphi(x^{j,i_j}) - \nabla\varphi(x^j) = \nabla\varphi(x^{j+1}) - \nabla\varphi(x^j) \to_\mathcal{K} 0$. Thus, the left-hand side of (3.25) with $i = i_j$ is arbitrarily small if $j \in \mathcal{K}$ is large enough. Thus, Algorithm 3.3.1 with the termination criterion (3.25) terminates in finitely many steps.

Let us mention that the above convergence theory differs from the one provided in [55,56], since no Lipschitzness of $\nabla\varphi\colon \mathbb{X} \to \mathbb{X}$ is needed. In the particular setting of complementarity-constrained optimization, related results have been obtained in [81, Section 4]. The findings substantially generalize the theory from [81] to arbitrary set constraints.

3.4 Augmented Lagrangian Methods for Structured Optimization Problems

In this section, we are devoted to the safeguarded augmented Lagrangian method in detail and some further analysis. In particular, Section 3.4.1 contains a detailed statement of our augmented Lagrangian method applied to the general class of problems (P) together with several explanations. The convergence theory is then presented in Section 3.4.2.

3.4.1 Statement of the Algorithm

We now consider the optimization problem (P) under the given smoothness of f, convexity assumption of C, and not necessarily convex assumption of D stated there. A safeguarded augmented Lagrangian approach is presented here for the solution of (P). The method penalizes the constraints $G(x) \in C$, but leaves the condition $x \in D$ explicitly in the constraints. Hence, the resulting subproblems that have to be solved in the augmented Lagrangian framework have exactly the structure of the (simplified) optimization problems discussed in Section 3.3.

To be specific, we consider the (partially) augmented Lagrangian of (P)

$$\mathcal{L}_\rho(x, \lambda) := f(x) + \frac{\rho}{2} d_C^2\Big(G(x) + \frac{\lambda}{\rho}\Big), \tag{3.26}$$

where $\rho > 0$ denotes the penalty parameter. Note that the squared distance function of a nonempty, closed, and convex set is always continuously differentiable, see e.g. [19, Corollary 12.30], which yields that $\mathcal{L}_\rho(\cdot, \lambda)$ is a continuously differentiable mapping. From the definition of the distance, one can alternatively write (3.26) as

$$\mathcal{L}_\rho(x, \lambda) = f(x) + \frac{\rho}{2}\Big\|G(x) + \frac{\lambda}{\rho} - P_C\Big(G(x) + \frac{\lambda}{\rho}\Big)\Big\|^2.$$

In order to control the update of the penalty parameter ρ, we introduce the auxiliary function

$$V_\rho(x, u) := \Big\|G(x) - P_C\Big(G(x) + \frac{u}{\rho}\Big)\Big\|, \tag{3.27}$$

which can also be used to obtain a meaningful termination criterion, see the discussion after (3.29) below. The overall method is stated in Algorithm 3.4.1.

Algorithm 3.4.1: Safeguarded Augmented Lagrangian Method for Geometric Constraints

Data: $\rho_0 > 0$, $\beta > 1$, $\eta \in (0, 1)$, $x^0 \in D$, nonempty and bounded set $U \subset \mathbb{Y}$

```
1  for k ← 0 to ∞ do
2      if x^k satisfies some termination criterion then
3          return x^k;
4      end
5      Choose u^k ∈ U;
6      Compute an approximately M-stationary point x^{k+1} of the subproblem
               min_x  L_{ρ_k}(x, u^k)   s.t.   x ∈ D,
7      i.e., for some suitable (sufficiently small) vector ε^{k+1} ∈ X, x^{k+1} needs to satisfy
               ε^{k+1} ∈ ∇_x L_{ρ_k}(x^{k+1}, u^k) + N_D^lim(x^{k+1});
8      Set λ^{k+1} := ρ_k [G(x^{k+1}) + u^k/ρ_k − P_C(G(x^{k+1}) + u^k/ρ_k)];
9      if k = 0 or V_{ρ_k}(x^{k+1}, u^k) ≤ η V_{ρ_{k−1}}(x^k, u^{k−1}) then
10         ρ_{k+1} := ρ_k;
11     else
12         ρ_{k+1} := β ρ_k;
13     end
14 end
```

Line 6 of Algorithm 3.4.1, in general, contains the main computational effort since one has to "solve" a constrained nonlinear program at each iteration. Due to the nonconvexity of this subproblem, we only require to compute an M-stationary point of this program. In fact, we allow the computation of an approximately M-stationary point, with the vector ε^{k+1} measuring the degree of inexactness. The choice of $\varepsilon^{k+1} = 0$ corresponds to an exact M-stationary point. As mentioned above, the subproblems arising in Line 6 have precisely the structure of the problem investigated in Section 3.3, hence, the spectral gradient method discussed there is a candidate for the solution of these subproblems.

Note that Algorithm 3.4.1 is called a safeguarded augmented Lagrangian method due to the appearance of the auxiliary sequence $\{u^k\} \subset U$ where U is a bounded set. In fact, if we replace u^k by λ^k in Line 6 of Algorithm 3.4.1 (and the corresponding subsequent formulas), then the classical augmented Lagrangian method will be obtained. However, the safeguarded version has superior global convergence properties, see [34] for a general discussion and [100] for an explicit (counter-) example. In practice, u^k is typically chosen to be equal to λ^k as long as this vector belongs to the set U, otherwise u^k is taken as the projection of λ^k onto this set. In situations where $\mathbb{Y}$ is equipped with some (partial) order relation $\lesssim$, a typical choice for U is given by the box $[u_{\min}, u_{\max}] := \{u \in \mathbb{Y} \mid u_{\min} \lesssim u \lesssim u_{\max}\}$ where $u_{\min}, u_{\max} \in \mathbb{Y}$ are given bounds satisfying $u_{\min} \lesssim u_{\max}$.

In order to understand the update of the Lagrange multiplier estimate in Line 8 of Algorithm 3.4.1, recall that the augmented Lagrangian is differentiable, with its derivative given by

$$\nabla_x \mathcal{L}_\rho(x, \lambda) = \nabla f(x) + \rho G'(x)^* \left[G(x) + \frac{\lambda}{\rho} - P_C\left(G(x) + \frac{\lambda}{\rho} \right) \right],$$

see [19, Corollary 12.30] again. Hence, if we denote the usual (partial) Lagrangian of (P) by

$$\mathcal{L}(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle,$$

then we obtain from Line 8 that

$$\nabla_x \mathcal{L}_{\rho_k}(x^{k+1}, u^k) = \nabla f(x^{k+1}) + G'(x^{k+1})^* \lambda^{k+1} = \nabla_x \mathcal{L}(x^{k+1}, \lambda^{k+1}). \tag{3.28}$$

This formula is actually the motivation for the precise update used in Line 8.

The updating rule in Lines 9 to 13 of Algorithm 3.4.1 is quite common, but other formulas might also be possible. In particular, one can choose a different norm of V_ρ in the definition (3.27). Exemplary, we exploited the maximum-norm for our experiments in Section 3.6 where $\mathbb{X}$ is a space of real vectors or matrices. Let us emphasize that increasing the penalty parameter ρ_k based on a pure infeasibility measure does not usually work in Algorithm 3.4.1. One usually has to take into account both the infeasibility of the current iterate (w.r.t. the constraint $G(x) \in C$) and a kind of complementarity condition (i.e., $\lambda \in \mathcal{N}_C(G(x))$).

For the better discussion of a suitable termination criterion, one first defines

$$z^k := G(x^k) - P_C\left(G(x^k) + \frac{u^{k-1}}{\rho_{k-1}} \right).$$

Using (3.28) and the update formula for λ^k, Algorithm 3.4.1 ensures

$$\varepsilon^k \in \nabla f(x^k) + G'(x^k)^* \lambda^k + \mathcal{N}_D^{\lim}(x^k), \tag{3.29a}$$

$$\lambda^k \in \mathcal{N}_C(G(x^k) - z^k), \tag{3.29b}$$

which is AM-stationary points from Definition 3.2. Thus, it is reasonable to require $\varepsilon^k \to 0$

and to use

$$\|z^k\| = V_{\rho_{k-1}}(x^k, u^{k-1}) \le \varepsilon_{\text{tol}} \tag{3.30}$$

for some $\varepsilon_{\text{tol}} > 0$ as a termination criterion. By the way, in practical implementations of Algorithm 3.4.1, a maximum number of iterations should also be incorporated into the termination criterion.

3.4.2 Convergence Analysis

In this section, we explore the convergence results of the proposed safeguarded augmented Lagrangian method. It is first assumed that Algorithm 3.4.1 does not stop after finitely many iterations.

Like all penalty-type methods in the setting of nonconvex programming, augmented Lagrangian methods suffer from the drawback that they generate accumulation points which are not necessarily feasible for the given optimization problem (P). The following (standard) result therefore presents some requirements under which it is guaranteed that limit points are feasible.

Proposition 3.15. *Each accumulation point $\bar{x}$ of a sequence $\{x^k\}$ generated by Algorithm 3.4.1 is feasible for the optimization problem* (P) *if one of the following conditions holds:*

(a) *$\{\rho_k\}$ is bounded, or*
(b) *there exists some $B \in \mathbb{R}$ such that $\mathcal{L}_{\rho_k}(x^{k+1}, u^k) \le B$ holds for all $k \in \mathbb{N}$.*

Proof. Let $\bar{x}$ be an arbitrary accumulation point of $\{x^k\}$ and $\{x^{k+1}\}_{\mathcal{K}}$ be a corresponding subsequence with $x^{k+1} \to_{\mathcal{K}} \bar{x}$.

We start to prove the case (a). Since $\{\rho_k\}$ is bounded, Lines 9 to 13 of Algorithm 3.4.1 imply that $V_{\rho_k}(x^{k+1}, u^k) \to 0$ for $k \to \infty$, which means

$$d_C(G(x^{k+1})) \le \left\| G(x^{k+1}) - P_C\left(G(x^{k+1}) + \frac{u^k}{\rho_k} \right) \right\| = V_{\rho_k}(x^{k+1}, u^k) \to 0.$$

A continuity argument yields $d_C(G(\bar{x})) = 0$. Since C is a closed set, this implies $G(\bar{x}) \in C$. Furthermore, by construction, one has $x^{k+1} \in D$ for all $k \in \mathbb{N}$, so that the closedness of D also yields $\bar{x} \in D$. Altogether, this shows that $\bar{x}$ is feasible for the optimization problem (P).

Let us now prove the result in presence of (b). In view of (a), it suffices to consider the situation where $\rho_k \to \infty$. By assumption, one has

$$\mathcal{L}_{\rho_k}(x^{k+1}, u^k) = f(x^{k+1}) + \frac{\rho_k}{2} d_C^2\left(G(x^{k+1}) + \frac{u^k}{\rho_k} \right) \le B \qquad \forall k \in \mathbb{N},$$

which could be rewritten as

$$d_C^2\left(G(x^{k+1}) + \frac{u^k}{\rho_k} \right) \le \frac{2(B - f(x^{k+1}))}{\rho_k} \qquad \forall k \in \mathbb{N}. \tag{3.31}$$

Taking the limit $k \to_{\mathcal{K}} \infty$ in (3.31) and using the boundedness of $\{u^k\}$, one has

$$d_C^2(G(\bar{x})) = \lim_{k \to_{\mathcal{K}} \infty} d_C^2\left(G(x^{k+1}) + \frac{u^k}{\rho_k} \right) = 0$$

by a continuity argument. Similar to part (a), this implies feasibility of $\bar{x}$. □

The two conditions (a) and (b) of Proposition 3.15 are, of course, difficult to check a priori. Nevertheless, recall that each iterate x^{k+1} is actually a global minimizer of the subproblem in Line 6 of Algorithm 3.4.1, if x denotes any feasible point of the optimization problem (P), then from the boundedness of the sequence $\{u^k\}$, there exists some suitable constant B such that

$$\mathcal{L}_{\rho_k}(x^{k+1}, u^k) \leq \mathcal{L}_{\rho_k}(x, u^k) \leq f(x) + \frac{\|u^k\|^2}{2\rho_k} \leq f(x) + \frac{\|u^k\|^2}{2\rho_0} \leq B$$

holds. The same argument also works if x^{k+1} is only an inexact global minimizer.

The following result shows that, even in the case where a limit point is not feasible, it still contains some useful information in the sense that it is at least a stationary point for the constraint violation. To be honest, this is the best that one can expect.

Proposition 3.16. *Assume that the sequence $\{\varepsilon^k\}$ in Algorithm 3.4.1 is bounded. Then each accumulation point $\bar{x}$ of a sequence $\{x^k\}$ generated by Algorithm 3.4.1 is an M-stationary point of the so-called feasibility problem*

$$\min_x \tfrac{1}{2} d_C^2(G(x)) \quad s.t. \quad x \in D. \tag{3.32}$$

Proof. If $\{\rho_k\}$ is bounded, in view of Proposition 3.15, then each accumulation point is a global minimum of the feasibility problem (3.32) and, therefore, an M-stationary point of this problem.

Hence, it remains to consider the case where $\{\rho_k\}$ is unbounded, i.e., we have $\rho_k \to \infty$ as $k \to \infty$. In view of Lines 6 and 8 of Algorithm 3.4.1, see also (3.28), one has

$$\varepsilon^{k+1} \in \nabla f(x^{k+1}) + G'(x^{k+1})^* \lambda^{k+1} + \mathcal{N}_D^{\lim}(x^{k+1})$$

with λ^{k+1} as in Line 8. Dividing this inclusion by ρ_k and from the fact that $\mathcal{N}_D^{\lim}(x^{k+1})$ is a cone, one therefore gets

$$\frac{\varepsilon^{k+1}}{\rho_k} \in \frac{\nabla f(x^{k+1})}{\rho_k} + G'(x^{k+1})^* \left[G(x^{k+1}) + \frac{u^k}{\rho_k} - P_C\left(G(x^{k+1}) + \frac{u^k}{\rho_k} \right) \right] + \mathcal{N}_D^{\lim}(x^{k+1}).$$

Now, let $\bar{x}$ be an accumulation point and $\{x^{k+1}\}_\mathcal{K}$ be a subsequence satisfying $x^{k+1} \to_\mathcal{K} \bar{x}$. Then the sequences $\{\varepsilon^{k+1}\}_\mathcal{K}$, $\{u^k\}_\mathcal{K}$, and $\{\nabla f(x^{k+1})\}_\mathcal{K}$ are bounded. Thus, taking the limit $k \to_\mathcal{K} \infty$ yields

$$0 \in G'(\bar{x})^* \big[G(\bar{x}) - P_C(G(\bar{x})) \big] + \mathcal{N}_D^{\lim}(\bar{x})$$

by the outer semicontinuity of the limiting normal cone. Since one also has $\bar{x} \in D$ and due to

$$\nabla\big(\tfrac{1}{2} d_C^2 \circ G\big)(\bar{x}) = G'(\bar{x})^* \big[G(\bar{x}) - P_C(G(\bar{x})) \big],$$

see, once more, [19, Corollary 12.30], it follows that $\bar{x}$ is an M-stationary point of the feasibility problem (3.32). □

We next investigate suitable properties of feasible limit points, which shows that any such accumulation point is automatically an AM-stationary point in the sense of Definition 3.2.

Theorem 3.17. *Assume that the sequence $\{\varepsilon^k\}$ in Algorithm 3.4.1 satisfies $\varepsilon^k \to 0$. Then each feasible accumulation point $\bar{x}$ of a sequence $\{x^k\}$ generated by Algorithm 3.4.1 is an AM-stationary point.*

Proof. Let $\{x^{k+1}\}_{\mathcal{K}}$ be the subsequence such that $x^{k+1} \to_{\mathcal{K}} \bar{x}$. Define

$$s^{k+1} := P_C\left(G(x^{k+1}) + \frac{u^k}{\rho_k}\right) \quad \text{and} \quad z^{k+1} := G(x^{k+1}) - s^{k+1}$$

for each $k \in \mathbb{N}$. We claim that the four (sub-) sequences $\{x^{k+1}\}_{\mathcal{K}}, \{z^{k+1}\}_{\mathcal{K}}$, $\{\varepsilon^{k+1}\}_{\mathcal{K}}$, and $\{\lambda^{k+1}\}_{\mathcal{K}}$ generated by Algorithm 3.4.1 or defined in the above way satisfy the properties from Definition 3.2 and therefore show that $\bar{x}$ is an AM-stationary point. By construction, one has $x^{k+1} \to_{\mathcal{K}} \bar{x}$ and $\varepsilon^{k+1} \to_{\mathcal{K}} 0$. Further, it implies from Line 6 of Algorithm 3.4.1 and (3.28) that

$$\varepsilon^{k+1} \in \nabla_x \mathcal{L}_{\rho_k}(x^{k+1}, u^k) + \mathcal{N}_D^{\lim}(x^{k+1}) = \nabla f(x^{k+1}) + G'(x^{k+1})^* \lambda^{k+1} + \mathcal{N}_D^{\lim}(x^{k+1}).$$

Since $\mathcal{N}_C(s^{k+1})$ is a cone, the relation between P_C and $\mathcal{N}_C$ together with the definitions of s^{k+1}, λ^{k+1}, and z^{k+1} deduce

$$\lambda^{k+1} = \rho_k\left[G(x^{k+1}) + \frac{u^k}{\rho_k} - s^{k+1}\right] \in \mathcal{N}_C(s^{k+1}) = \mathcal{N}_C(G(x^{k+1}) - z^{k+1}).$$

Hence, it remains to show $z^{k+1} \to_{\mathcal{K}} 0$. To this end, we consider two cases, namely whether the sequence $\{\rho_k\}$ stays bounded or is unbounded. In the bounded case, Lines 9 to 13 of Algorithm 3.4.1 imply that $V_{\rho_k}(x^{k+1}, u^k) \to 0$ for $k \to \infty$. The corresponding definitions therefore yield

$$\|z^{k+1}\| = \|G(x^{k+1}) - s^{k+1}\| = V_{\rho_k}(x^{k+1}, u^k) \to 0 \quad \text{as } k \to_{\mathcal{K}} \infty. \tag{3.33}$$

On the other hand, if $\{\rho_k\}$ is unbounded, one has $\rho_k \to \infty$. Since $\{u^k\}$ is bounded by construction, the continuity of the projection operator together with the assumed feasibility of $\bar{x}$ implies

$$s^{k+1} = P_C\left(G(x^{k+1}) + \frac{u^k}{\rho_k}\right) \to P_C(G(\bar{x})) = G(\bar{x}) \quad \text{as } k \to_{\mathcal{K}} \infty.$$

Consequently, one obtains $z^{k+1} = G(x^{k+1}) - s^{k+1} \to_{\mathcal{K}} 0$. Altogether, this implies that $\bar{x}$ is AM-stationary. □

Note that (3.33) implies that stopping criterion (3.30) will be satisfied after finitely many steps.

By definition, each AM-stationary point of (P) which is AM-regular must already be M-stationary, then we obtain the following corollary.

Corollary 3.18. *Suppose that the sequence $\{\varepsilon^k\}$ in Algorithm 3.4.1 satisfies $\varepsilon^k \to 0$. Then each feasible AM-regular accumulation point $\bar{x}$ of a sequence $\{x^k\}$ generated by Algorithm 3.4.1 is an M-stationary point.*

Keeping the discussions after Lemma 3.10 in mind, this result generalizes [81, Theorem 3] which addresses a similar MPCC-tailored augmented Lagrangian method and exploits MPCC-RCPLD.

3.5 Realizations

Let k be a fixed iteration of Algorithm 3.4.1, we use Algorithm 3.3.1 to obtain the approximate solution of the ALM-subproblem in Line 6 of Algorithm 3.4.1. Recall that, given an outer iteration j of Algorithm 3.3.1, we need to solve the subproblem

$$\min_x \ \mathcal{L}_{\rho_k}(x^j, u^k) + \langle \nabla_x \mathcal{L}_{\rho_k}(x^j, u^k), x - x^j \rangle + \frac{\gamma_{j,i}}{2} \|x - x^j\|^2 \quad \text{s.t.} \quad x \in D$$

with some given x^j and $\gamma_{j,i} > 0$ in the inner iteration i of Algorithm 3.3.1, which possesses the same solutions as

$$\min_x \left\| x - \left(x^j - \frac{1}{\gamma_{j,i}} \nabla_x \mathcal{L}_{\rho_k}(x^j, u^k) \right) \right\|^2 \quad \text{s.t.} \quad x \in D.$$

Then it is required that D is simple in the sense that projections (possibly multi-valued) onto D, i.e., $\Pi_D(x^j - \frac{1}{\gamma_{j,i}} \nabla_x \mathcal{L}_{\rho_k}(x^j, u^k))$ are easy to compute. In other words, one has to be in position to find projections of arbitrary points onto the set D in an efficient way. In particular, we show that this is the case for MPCCs, optimization problems with cardinality constraints, and some rank-constrained matrix optimization problems.

3.5.1 The Disjunctive Programming Case

We consider (P) in the special Setting 3.9 with $\mathbb{W} := \mathbb{R}^n$ and $W := [\ell, u]$ where $\ell, u \in \mathbb{R}^n$ satisfy $-\infty \le \ell_i < u_i \le \infty$ for $i = 1, \dots, n$. Recall that the set D is given by

$$D = \{(w, y, z) \in \mathbb{R}^n \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_3} \mid w \in [\ell, u],\ (y_i, z_i) \in T \quad \forall i \in \{1, \dots, m_3\}\} \tag{3.34}$$

in this situation. For given $\bar{x} = (\bar{w}, \bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_3}$, we would like to characterize the elements of $\Pi_D(\bar{x})$ explicitly. Therefore, we first consider the optimization problem

$$\min_x \ \tfrac{1}{2}\|x - \bar{x}\|^2 \quad \text{s.t.} \quad x = (w, y, z) \in D, \tag{3.35}$$

which can be decomposed into the n one-dimensional optimization problems

$$\min_{w_i} \ \tfrac{1}{2}(w_i - \bar{w}_i)^2 \quad \text{s.t.} \quad w_i \in [\ell_i, u_i],$$

$i = 1, \dots, n$, possessing the respective solution $P_{[\ell_i, u_i]}(\bar{w}_i)$, as well as into m_3 two-dimensional optimization problems

$$\min_{y_i, z_i} \ \tfrac{1}{2}(y_i - \bar{y}_i)^2 + \tfrac{1}{2}(z_i - \bar{z}_i)^2 \quad \text{s.t.} \quad (y_i, z_i) \in T, \tag{3.36}$$

$i = 1, \dots, m_3$. Due to $T = T_1 \cup T_2$, each of these problems can be decomposed into the two two-dimensional subproblems

$$\min_{y_i, z_i} \ \tfrac{1}{2}(y_i - \bar{y}_i)^2 + \tfrac{1}{2}(z_i - \bar{z}_i)^2 \quad \text{s.t.} \quad (y_i, z_i) \in T_j, \tag{$R(i,j)$}$$

$j = 1, 2$. In most of the popular settings from disjunctive programming, $(R(i, j))$ can be solved with ease. By a simple comparison of the associated objective function values, we find the solutions of (3.36). Putting the solutions of the subproblems together, we obtain the solutions of (3.35), i.e., the elements of $\Pi_D(\bar{x})$.

We now consider a particularly interesting instance of this setting where T is given by

$$T := \{(s,t) \mid s \in [\sigma_1, \sigma_2],\ t \in [\tau_1, \tau_2],\ st = 0\}. \tag{3.37}$$

Here, $-\infty \le \sigma_1, \tau_1 \le 0$ and $0 < \sigma_2, \tau_2 \le \infty$ are given constants. Particularly, we find the decomposition

$$T_1 := [\sigma_1, \sigma_2] \times \{0\}, \qquad T_2 := \{0\} \times [\tau_1, \tau_2]$$

of T in this case. Due to the geometrical shape of the set T, one might be tempted to refer to this setting as "box-switching constraints". Note that it particularly covers

- switching constraints ($\sigma_1 = \tau_1 := -\infty$, $\sigma_2 = \tau_2 := \infty$), see [96, 120],
- complementarity constraints ($\sigma_1 = \tau_1 := 0$, $\sigma_2 = \tau_2 := \infty$), see [115, 129], and
- relaxed reformulated cardinality constraints ($\sigma_1 := -\infty$, $\sigma_2 := \infty$, $\tau_1 := 0$, $\tau_2 := 1$), see [47, 51].

We refer the reader to Figure 3.1 for a visualization of these types of constraints.

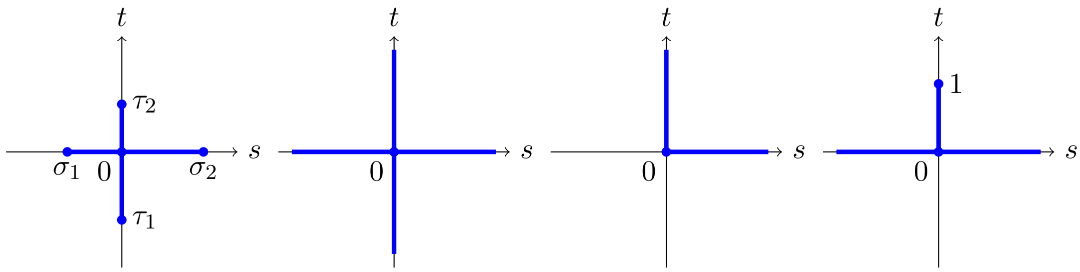

Figure 3.1: Geometric illustrations of box-switching, switching, complementarity, and relaxed reformulated cardinality constraints (from left to right), respectively.

It remains to consider the solutions of (R$(i,1)$) and (R$(i,2)$) in the setting of (3.37), which can be easily given by $(P_{[\sigma_1,\sigma_2]}(\bar{y}_i), 0)$ and $(0, P_{[\tau_1,\tau_2]}(\bar{z}_i))$, respectively. This yields the following result.

Proposition 3.19. *Consider the set D from* (3.34) *where T is given as in* (3.37)*. For given $\bar{x} = (\bar{w}, \bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_3}$, one has $\hat{x} := (\hat{w}, \hat{y}, \hat{z}) \in \Pi_D(\bar{x})$ if and only if $\hat{w} = P_{[\ell,u]}(\bar{w})$ and*

$$(\hat{y}_i, \hat{z}_i) \in \begin{cases} \{(P_{[\sigma_1,\sigma_2]}(\bar{y}_i), 0)\} & \text{if } \phi_s(\bar{y}_i, \bar{z}_i) < \phi_t(\bar{y}_i, \bar{z}_i), \\ \{(0, P_{[\tau_1,\tau_2]}(\bar{z}_i))\} & \text{if } \phi_s(\bar{y}_i, \bar{z}_i) > \phi_t(\bar{y}_i, \bar{z}_i), \\ \{(P_{[\sigma_1,\sigma_2]}(\bar{y}_i), 0), (0, P_{[\tau_1,\tau_2]}(\bar{z}_i))\} & \text{if } \phi_s(\bar{y}_i, \bar{z}_i) = \phi_t(\bar{y}_i, \bar{z}_i) \end{cases}$$

for all $i = 1, \ldots, m_3$, where we denote

$$\phi_s(a,b) := (P_{[\sigma_1,\sigma_2]}(a) - a)^2 + b^2, \qquad \phi_t(a,b) := a^2 + (P_{[\tau_1,\tau_2]}(b) - b)^2.$$

Particularly, it turns out that in order to compute the projections onto the set D under consideration, one basically needs to compute $n + 2m_3$ projections onto real intervals. In the specific setting of complementarity-constrained programming, this already has been observed in [81, Section 4].

Let us briefly mention that other popular instances of disjunctive programs like vanishing- and or-constrained optimization problems, see e.g. [2, 118], where T is given by

$$T := \{(s,t) \mid st \le 0,\ t \ge 0\} \quad \text{or} \quad T := \{(s,t) \mid \min(s,t) \le 0\},$$

respectively, can be treated in an analogous fashion. Furthermore, an analogous procedure applies to more general situations where T is the union of finitely many convex, polyhedral sets.

3.5.2 The Sparsity-Constrained Case

We fix $\mathbb{X} := \mathbb{R}^n$ and some $\kappa \in \mathbb{N}$ with $1 \le \kappa \le n-1$. Consider the set

$$S_\kappa := \{x \in \mathbb{R}^n \mid \|x\|_0 \le \kappa\}$$

with $\|x\|_0$ being the number of nonzero entries of the vector x. This set plays a prominent role in sparse optimization and for problems with cardinality constraints. Since S_κ is nonempty and closed, projections of some vector $x \in \mathbb{R}^n$ (w.r.t. the Euclidean norm) onto this set exist (may not be unique), and are known to consist of those vectors $y \in \mathbb{R}^n$ such that the nonzero entries of y are precisely the κ largest (in absolute value) components of x (which may not be unique), see e.g. [20, Proposition 3.6].

Hence, within our augmented Lagrangian framework, it is possible to take $D := S_\kappa$ and then get an explicit formula for the solutions of the corresponding subproblems arising within the spectral gradient method. However, typical implementations of augmented Lagrangian methods (like `ALGENCAN`, see [3]) do not penalize box constraints, i.e., they leave the box constraints explicitly as constraints when solving the corresponding subproblems. Hence, let us assume that we have some lower and upper bounds satisfying $-\infty \le \ell_i < u_i \le \infty$ for all $i = 1, \ldots, n$, we are then forced to compute projections onto the set

$$D := S_\kappa \cap [\ell, u]. \tag{3.38}$$

It turns out that there exists an explicit formula for this projection. Before presenting the result, let us first assume, for notational simplicity, that

$$0 \in [\ell_i, u_i] \qquad \forall i = 1, \ldots, n. \tag{3.39}$$

We mention that this assumption is not restrictive, which in the meantime ensures D is nonempty and we then can trace the elements of projections on D. Indeed, let us assume that, e.g., $0 \notin [\ell_1, u_1]$. Then the first component of $x \in D$ cannot be zero, and this shows

$$D = S_\kappa \cap [\ell, u] = [\ell_1, u_1] \times \Big(\hat{S}_{\kappa-1} \cap [\hat{\ell}, \hat{u}]\Big), \tag{3.40}$$

where $\hat{S}_{\kappa-1} := \{x \in \mathbb{R}^{n-1} \mid \|x\|_0 \le \kappa - 1\}$ and the vectors $\hat{\ell}, \hat{u} \in \mathbb{R}^{n-1}$ are obtained from ℓ, u by dropping the first component, respectively. For the computation of the projection onto S_κ, we can now exploit the product structure (3.40). Similarly, we can remove all remaining components $i = 2, \ldots, n$ with $0 \notin [\ell_i, u_i]$ from D. Thus, we can assume (3.39) without loss of generality. We now give the following simple observation.

Lemma 3.20. *Let $x \in \mathbb{R}^n$ be arbitrary. Then, for each $y \in \Pi_D(x)$, where D is the set from* (3.38)*, one has*

$$y_i \in \Big\{0, P_{[\ell_i, u_i]}(x_i)\Big\} \qquad \forall i = 1, \ldots, n.$$

Proof. To the contrary, assume that $y_i \ne 0$ and $y_i \ne P_{[\ell_i,u_i]}(x_i)$ hold for some index $i \in \{1, \ldots, n\}$. Define the vector $q \in \mathbb{R}^n$ by $q_j := y_j$ for $j \ne i$ and $q_i := P_{[\ell_i,u_i]}(x_i)$. Due to $y_i \ne 0$, one has $\|q\|_0 \le \|y\|_0 \le \kappa$, i.e., $q \in S_\kappa$. Additionally, $q \in [\ell, u]$ is clear from $y \in [\ell, u]$ and $q_i = P_{[\ell_i,u_i]}(x_i)$. Thus, one finds $q \in D$. Furthermore, $\|q - x\| < \|y - x\|$ since $q_i = P_{[\ell_i,u_i]}(x_i) \ne y_i$. This contradicts the fact that y is a projection of x onto D. $\square$

Due to the above lemma, one only has two candidates for the value of the components associated with projections to D from (3.38). Thus, for an arbitrary index set $I \subset \{1, \dots, n\}$ and an arbitrary vector $x \in \mathbb{R}^n$, we define $p^I(x) \in \mathbb{R}^n$ via

$$p_i^I(x) := \begin{cases} P_{[\ell_i,u_i]}(x_i) & \text{if } i \in I, \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \dots, n.$$

It remains to characterize those index sets I which ensure that $p^I(x)$ is a projection of x onto D. To this end, we define an auxiliary vector $d(x) \in \mathbb{R}^n$ via

$$d_i(x) := x_i^2 - \big(P_{[\ell_i,u_i]}(x_i) - x_i\big)^2 \qquad \forall i = 1, \dots, n.$$

Note that this definition directly yields

$$\|p^I(x) - x\|^2 = \|x\|^2 - \sum_{i \in I} d_i(x). \tag{3.41}$$

We state the following simple observation.

Lemma 3.21. *Fix $x \in \mathbb{R}^n$ and assume that* (3.39) *is valid. Then the following statements hold:*

(a) $d_i(x) \geq 0$ *for all* $i = 1, \dots, n$,
(b) $d_i(x) = 0 \iff P_{[\ell_i,u_i]}(x_i) = 0$.

Proof. (a) Since $0 \in [\ell_i, u_i]$, one gets

$$d_i(x) = (x_i - 0)^2 - \big(x_i - P_{[\ell_i,u_i]}(x_i)\big)^2 \geq 0$$

by definition of the (one-dimensional) projection.
(b) If $P_{[\ell_i,u_i]}(x_i) = 0$ holds, one immediately obtains $d_i(x) = 0$. Conversely, let $d_i(x) = 0$, then

$$0 = x_i^2 - \big(x_i - P_{[\ell_i,u_i]}(x_i)\big)^2 = P_{[\ell_i,u_i]}(x_i)\big(2x_i - P_{[\ell_i,u_i]}(x_i)\big).$$

Hence, one finds $P_{[\ell_i,u_i]}(x_i) = 0$ or $P_{[\ell_i,u_i]}(x_i) = 2x_i$. In the first case, we are done. In the second case, we have $\{0, 2x_i\} \subset [\ell_i, u_i]$. By convexity, this gives $x_i \in [\ell_i, u_i]$. Consequently, $x_i = P_{[\ell_i,u_i]}(x_i) = 2x_i$. This implies $P_{[\ell_i,u_i]}(x_i) = 0$. □

Observe that the second assertion of the above lemma implies

$$\|p^I(x)\|_0 = \big|\{i \in I \mid P_{[\ell_i,u_i]}(x_i) \neq 0\}\big| = \big|\{i \in I \mid d_i(x) \neq 0\}\big| \qquad \forall x \in \mathbb{R}^n. \tag{3.42}$$

This can be used to characterize the set of projections onto the set D from (3.38).

Proposition 3.22. *Let D be the set from* (3.38) *and assume that* (3.39) *holds. Then, for each $x \in \mathbb{R}^n$, $y \in \Pi_D(x)$ holds if and only if there exists an index set $I \subset \{1, \dots, n\}$ with $|I| = \kappa$ such that*

$$d_i(x) \geq d_j(x) \qquad \forall i \in I,\ \forall j \notin I \tag{3.43}$$

and $y = p^I(x)$ hold.

Proof. If $y \in \Pi_D(x)$ holds, then $y = p^J(x)$ is valid for some index set J, see Lemma 3.20. Thus, it remains to check that $p^J(x)$ is a projection onto D if and only if $p^J(x) = p^I(x)$ holds for some index set I satisfying $|I| = \kappa$ and (3.43).

Note that $p^J(x)$ is a projection if and only if J minimizes $\|p^I(x) - x\|$ over all $I \subset \{1, \ldots, n\}$ satisfying $\|p^I(x)\|_0 \le \kappa$. This can be reformulated via $d(x)$ by using (3.41) and (3.42). In particular, $p^J(x)$ is a projection if and only if J solves

$$\max_I \quad \sum_{i \in I} d_i(x) \quad \text{s.t.} \quad I \subset \{1, \ldots, n\}, \quad |\{i \in I \mid d_i(x) \neq 0\}| \le \kappa. \tag{3.44}$$

It is clear that index sets I with $|I| = \kappa$ and (3.43) are solutions of this problem. This shows the direction $\Longleftarrow$.

To prove the converse direction $\Longrightarrow$, let $p^J(x)$ be a projection. Thus, J solves (3.44). We note that the solutions of this problem are invariant under addition and removal of indices i with $d_i(x) = 0$. Due to Lemma 3.21 (b), these operations also do not alter the associated $p^I(x)$. Thus, for each projection $p^J(x)$, we can add or remove indices i with $d_i(x) = 0$, to obtain a set I with $p^I(x) = p^J(x)$ and $|I| = \kappa$. It is also clear that (3.43) holds for such a choice of I. □

Let us give some comment on the result of Proposition 3.22.

Remark 3.23. (a) Let $y = p^I(x)$ be a projection of $x \in \mathbb{R}^n$ onto D from (3.38) such that (3.39) holds. Observe that $y_i = 0$ may also hold for some indices $i \in I$.

(b) In the unconstrained case $[\ell, u] = \mathbb{R}^n$, we find $d_i(x) = x_i^2$ for each $x \in \mathbb{R}^n$ and all $i = 1, \ldots, n$. Thus, Proposition 3.22 recovers the well-known characterization of the projection onto the set S_κ which can be found in [20, Proposition 3.6].

For the variational geometry of $D = S_k \cap [\ell, u]$ from (3.38), observing that the sets S_κ and $[\ell, u]$ are both polyhedral in the sense that they can be represented as the union of finitely many polyhedrons, the normal cone intersection rule

$$\mathcal{N}_D^{\lim}(x) = \mathcal{N}_{S_\kappa \cap [\ell,u]}^{\lim}(x) \subset \mathcal{N}_{S_\kappa}^{\lim}(x) + \mathcal{N}_{[\ell,u]}^{\lim}(x) = \mathcal{N}_{S_\kappa}^{\lim}(x) + \mathcal{N}_{[\ell,u]}(x)$$

applies for each $x \in D$ by means of [88, Corollary 4.2] and [137, Proposition 1]. While the evaluation of $\mathcal{N}_{[\ell,u]}(x)$ is standard, a formula for $\mathcal{N}_{S_\kappa}^{\lim}(x)$ can be found in [20, Theorem 3.9].

3.5.3 Low-Rank Approximation

3.5.3.1 General Low-Rank Approximations

For natural numbers $m, n \in \mathbb{N}$ with $m, n \ge 2$, we fix $\mathbb{X} := \mathbb{R}^{m \times n}$. Equipped with the standard Frobenius inner product, $\mathbb{X}$ indeed is a Euclidean space. Now, for fixed $\kappa \in \mathbb{N}$ satisfying $1 \le \kappa \le \min(m, n) - 1$, let us investigate the set

$$D := \{X \in \mathbb{X} \mid \operatorname{rank} X \le \kappa\}.$$

Constraint systems involving rank constraints of type $X \in D$ can be applied to model numerous practically relevant problems in computer vision, machine learning, computer algebra, signal processing, or model order reduction, see [116, Section 1.3] for an overview. Nowadays, one of the most popular applications behind low-rank constraints is the so-called low-rank matrix completion, particularly, the "Netflix-problem", see [48] for details.

Observe that the variational geometry of D has been explored recently in [90]. Particularly, a formula for the limiting normal cone to this set can be found in [90, Theorem 3.1]. Using the singular value decomposition of a given matrix $\widetilde{X} \in \mathbb{X}$, one can easily construct an element of $\Pi_D(\widetilde{X})$ by means of the so-called Eckart–Young–Mirsky theorem, see e.g. [116, Theorem 2.23].

Proposition 3.24. *For a given matrix $\widetilde{X} \in \mathbb{X}$, let $\widetilde{X} = U\Sigma V^\top$ be its singular value decomposition with orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ as well as a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ whose diagonal entries are in non-increasing order. Let $\widehat{U} \in \mathbb{R}^{m\times\kappa}$ and $\widehat{V} \in \mathbb{R}^{n\times\kappa}$ be the matrices resulting from U and V by deleting the last $m-\kappa$ and $n-\kappa$ columns, respectively. Furthermore, let $\widehat{\Sigma} \in \mathbb{R}^{\kappa\times\kappa}$ be the top left $\kappa\times\kappa$ block of Σ. Then we have $\widehat{U}\widehat{\Sigma}\widehat{V}^\top \in \Pi_D(\widetilde{X})$.*

Note that the projection formulas from the previous sections allow a very efficient computation of the corresponding projections, which is in contrast to the projection provided by Proposition 3.24. Though the formula given there is conceptually very simple, its realization requires to compute the singular value decomposition of the given matrix.

3.5.3.2 Symmetric Low-Rank Approximation

Given $n \in \mathbb{N}$ with $n \geq 2$, we consider the set of symmetric matrices $\mathbb{X} := \mathbb{R}^{n\times n}_{\text{sym}}$, still equipped with the Frobenius inner product. Now, for fixed $\kappa \in \mathbb{N}$ satisfying $1 \leq \kappa \leq n$, let us investigate the set

$$D := \{X \in \mathbb{X} \mid X \succeq 0,\ \operatorname{rank} X \leq \kappa\}.$$

Above, the constraint $X \succeq 0$ is used to abbreviate that X has to be positive semidefinite. Constraint systems involving rank constraints of type $X \in D$ arise frequently in several different mathematical models of data science, see [106] for an overview, and Section 3.6.4 for an application. Note that $\kappa := n$ covers the setting of pure semidefiniteness constraints.

Exploiting the eigenvalue decomposition of a given matrix $\widetilde{X} \in \mathbb{X}$, one can easily construct an element of $\Pi_D(\widetilde{X})$.

Proposition 3.25. *For a given matrix $\widetilde{X} \in \mathbb{X}$, we denote by $\widetilde{X} = \sum_{i=1}^n \lambda_i v_i v_i^\top$ its (orthonormal) eigenvalue decomposition with non-increasingly ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and associated pairwise orthonormal eigenvectors $v_1, \ldots, v_n$. Then we have $\widehat{X} := \sum_{i=1}^\kappa \max(\lambda_i, 0) v_i v_i^\top \in \Pi_D(\widetilde{X})$.*

Proof. We define the positive and negative part $\widetilde{X}^\pm := \sum_{i=1}^n \max(\pm\lambda_i, 0) v_i v_i^\top$. This yields $\widetilde{X} = \widetilde{X}^+ - \widetilde{X}^-$ and $\langle \widetilde{X}^+, \widetilde{X}^-\rangle = \operatorname{trace}(\widetilde{X}^+\widetilde{X}^-) = 0$. Thus, for each positive semidefinite $B \in \mathbb{X}$, one has

$$\|\widetilde{X} - B\|^2 = \|\widetilde{X}^+ - B\|^2 + \|\widetilde{X}^-\|^2 + 2\langle \widetilde{X}^-, B\rangle \geq \|\widetilde{X}^+ - B\|^2 + \|\widetilde{X}^-\|^2.$$

Since the singular value decomposition of $\widetilde{X}^+$ coincides with the eigenvalue decomposition, the right-hand side is minimized by $B = \widehat{X}$, see Proposition 3.24 while noting that we have $\widehat{X} = \widetilde{X}^+$ in case $\kappa = n$. Due to $\langle \widetilde{X}^-, \widehat{X}\rangle = 0$, $B = \widehat{X}$ also minimizes the left-hand side. □

It is clear that the computation of the κ largest eigenvalues of $\widetilde{X} \in \mathbb{X}$ is sufficient to compute an element from the projection $\Pi_D(\widetilde{X})$. This can be done particularly efficient for small κ (note that $\kappa = 1$ holds in our application from Section 3.6.4).

3.5.4 Extension to Nonsmooth Objectives

For some lower semicontinuous functional $g\colon \mathbb{X} \to \mathbb{R}$, we consider the optimization problem

$$\min_x \ f(x) + g(x) \quad \text{s.t.} \quad G(x) \in C. \tag{3.45}$$

Particularly, we do not assume that g is continuous. Actually, (3.45) has the same form with (CP), the following is our motivation to discuss (CP) in Chapter 4. Exemplary, let us mention the special cases where g is the indicator function of a closed set, counts the nonzero entries of the argument vector (in case $\mathbb{X} := \mathbb{R}^n$), or encodes the rank of the argument matrix (in case $\mathbb{X} := \mathbb{R}^{m\times n}$). In this regard, (3.45) can be used to model real-world applications from e.g. image restoration or signal processing. Necessary optimality conditions and qualification conditions addressing (3.45) can be found in [83]. In [55], the authors suggest to handle (3.45) numerically with the aid of an augmented Lagrangian method (without safeguarding) based on the (partially) augmented Lagrangian function (3.26) and the subproblems

$$\min_x \ \mathcal{L}_{\rho_k}(x, \lambda^k) + g(x) \quad \text{s.t.} \quad x \in \mathbb{X}$$

which are solved with a nonmonotone proximal gradient method inspired by [154]. In this regard, the solution approach to (3.45) described in [55] possesses some parallels to our strategy for the numerical solution of (P). The authors in [55] were able to prove convergence of their method to reasonable stationary points of (3.45) under a variant of the basic qualification condition and RCPLD. Let us mention that the authors in [55, 83] only considered standard inequality and equality constraints, but the theory in these papers can be easily extended to the more general constraints considered in (3.45) doing some nearby adjustments.

We note that (P) can be interpreted as a special instance of (3.45) where g plays the role of the indicator function of the set D. Then the nonmonotone proximal gradient method from [55] reduces to the spectral gradient method from Section 3.3. However, the authors in [55] did not challenge their method with discontinuous functionals g and, thus, cut away some of the more reasonable applications behind the model (P). Furthermore, we would like to mention that (3.45) can be reformulated (by using the epigraph $\operatorname{epi} g := \{(x, \alpha) \mid g(x) \le \alpha\}$ of g) as

$$\min_{x,\alpha} \ f(x) + \alpha \quad \text{s.t.} \quad G(x) \in C, \ (x, \alpha) \in \operatorname{epi} g \tag{3.46}$$

which is a problem of type (P). One can easily check that (3.45) and (3.46) are equivalent in the sense that $\bar{x} \in \mathbb{X}$ is a local/global minimizer of (3.45) if and only if $(\bar{x}, g(\bar{x}))$ is a local/global minimizer of (3.46). Problem (3.46) can be handled with Algorithm 3.4.1 as soon as the computation of projections onto $D := \operatorname{epi} g$ is possible in an efficient way. Our result from Corollary 3.18 shows that Algorithm 3.4.1 applied to (3.46) computes M-stationary points of (3.45) under AM-regularity (associated with (3.46)) at $(\bar{x}, g(\bar{x}))$, i.e., we are in position to find points satisfying

$$0 \in \nabla f(\bar{x}) + \partial g(\bar{x}) + G'(\bar{x})^* \mathcal{N}_C(G(\bar{x}))$$

under a very mild condition which enhances [55, Theorem 3.1]. Here, we used the limiting subdifferential of g given by

$$\partial g(x) := \{\xi \in \mathbb{X} \mid (\xi, -1) \in \mathcal{N}^{\lim}_{\operatorname{epi} g}(x, g(x))\}.$$

3.6 Numerical Results

In this section, we aim to implement Algorithm 3.4.1, based on the underlying subproblem solver Algorithm 3.3.1, in MATLAB (R2021b) and tested it on four classes of difficult problems which are discussed in Sections 3.6.1 to 3.6.4. All test runs use the following parameters:

$$\tau := 2,\ \sigma := 10^{-4},\ \beta := 10,\ \eta := 0.8,\ m := 10,\ \gamma_{\min} := 10^{-10},\ \gamma_{\max} := 10^{10}.$$

In iteration k of Algorithm 3.4.1, we terminate Algorithm 3.3.1 if the inner iterates $x^{j,i}$ satisfy

$$\left\|\gamma_{j,i}(x^j - x^{j,i}) + \nabla\varphi(x^{j,i}) - \nabla\varphi(x^j)\right\|_\infty \le \frac{10^{-4}}{\sqrt{k+1}}, \tag{3.47}$$

where $\|\cdot\|_\infty$ stands for the maximum-norm for the cases $\mathbb{X}$ equal to $\mathbb{R}^n$, equal to $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\times n}_{\text{sym}}$ (other Euclidean spaces do not occur in the subsequent applications), see (3.25). Similarly, we use the infinity norm in the definition (3.27) of V_ρ. Algorithm 3.4.1 is terminated as soon as (3.30) is satisfied with $\varepsilon_{\text{tol}} := 10^{-4}$. These two termination criteria ensure that the final iterate x^k together with the multiplier λ^k is approximately M-stationarity, see (3.29).

Given an arbitrary (possibly random) starting point x^0, note that we first project this point onto the set D and then use this projected point as the true starting point, so that all iterates x^k generated by Algorithm 3.4.1 belong to D. The choice of the initial penalty parameter is similar to the rule in [34, p. 153] and given by

$$\rho_0 := P_{[10^{-3},10^3]}\left(10\frac{\max(1, f(x^0))}{\max(1, \frac{1}{2}d_C^2(G(x^0)))}\right).$$

In all our examples, the space $\mathbb{Y}$ is given by $\mathbb{R}^m$ as in Setting 3.5. This allows us to choose the safeguarded multiplier estimate u^k as the projection of the current value λ^k onto a given box $[u_{\min}, u_{\max}]$, where this box is (in componentwise fashion) chosen to be $[-10^{20}, 10^{20}]$ for all equality constraints and $[0, 10^{20}]$ for all inequality constraints. In this way, we basically guarantee that the safeguarded augmented Lagrangian method from Algorithm 3.4.1 coincides with the classical approach as long as bounded multiplier estimates λ^k are generated.

3.6.1 MPCC Examples

The specification of Algorithm 3.4.1 to MPCCs is essentially the method discussed in [81], where extensive numerical results (including comparisons with other methods) are presented. We therefore keep this section short and consider only two particular examples in order to illustrate certain aspects of our method.

Example 3.26. Here, for $x := (y, z) \in \mathbb{R}^2$, we consider the two-dimensional MPCC given by

$$\min_x \tfrac{1}{2}(y-1)^2 + \tfrac{1}{2}(z-1)^2 \quad \text{s.t.} \quad y + z \le 2,\ y \ge 0,\ z \ge 0, yz = 0,$$

which is essentially the example from [141] with an additional (inactive) inequality constraint in order to have at least one standard constraint, so that Algorithm 3.4.1 does not automatically reduce to the spectral gradient method. The problem possesses two global minimizers at $(0, 1)$ and $(1, 0)$ which are M-stationary (in fact, they are even strongly stationary in the MPCC-terminology). Moreover, it has a local maximizer at $(0, 0)$ which is a point of attraction for many MPCC solvers since it can be shown to be C-stationary, see e.g. [89] for the corresponding definitions and some convergence results to C- and M-stationary points. Due to Lemma 3.8, each feasible point of the problem is AM-regular. In view of our convergence theory, Algorithm 3.4.1 should not converge to the origin. To verify this statement numerically, we generated 1000 random starting points (uniformly distributed) from the box $[-10, 10]^2$ and then applied Algorithm 3.4.1 to the above example. As expected, the method converges for all 1000 starting points to one of the two minima. Moreover, we can even start our method at the origin, and the method still converges to the point $(1, 0)$ or $(0, 1)$. The limit point itself depends on our choice of the projection which is not unique for iterates (y^k, z^k) with $y^k = z^k > 0$.

The next example is used to show the spectral gradient method as a subproblem problem has some limitations for ill-conditioned problems in details. There are examples where this spectral gradient method reduces the number of iterations even for two-dimensional problems from more than 100000 to just a few iterations. Nevertheless, in the end, the spectral gradient method is a projected gradient method, which exploits a different stepsize selection, but which eventually reduces to a standard projected gradient method if there are a number of consecutive iterations with very small progress, i.e., with almost identical function values during the last few iterations so that the maximum term in the nonmonotone line search is almost identical to the current function value used in the monotone version. This situation typically happens for problems which are ill-conditioned, and we illustrate this observation by the following example.

Example 3.27. We consider the optimal control of a discretized obstacle problem as investigated in [86, Section 7.4]. Using $x := (w, y, z)$, in our notation, the problem is given by

$$\begin{aligned} \min_{x} \quad & f(x) := \tfrac{1}{2}\|w\|^2 - e^\top y + \tfrac{1}{2}\|y\|^2 \\ \text{s.t.} \quad & w \geq 0,\ -Ay - w + z = 0,\ y \geq 0,\ z \geq 0,\ y^\top z = 0. \end{aligned}$$

Here, A is a tridiagonal matrix which arises from a discretization of the negative Laplace operator in one dimension, i.e., $a_{ii} = 2$ for all i and $a_{ij} = -1$ for all $i = j \pm 1$. Furthermore, e denotes the all-one vector of appropriate size. We note that $\bar{x} := 0$ is the global minimizer as well as an M-stationary point of this program. Again, Lemma 3.8 shows that each feasible point is AM-regular. Viewing the constraint $x \geq 0$ as a box constraint, taking a moderate discretization with $A \in \mathbb{R}^{64\times 64}$, and using the all-one vector as a starting point, we obtain the results from Table 3.1. The number of (outer) iterations is denoted by k, j is the number of inner iterations, j_{cum} the accumulated number of inner iterations, f-ev. provides the number of function evaluations (note that, due to the stepsize rule, we might have several function evaluations in a single inner iteration, hence, f-ev. is always an upper bound for j_{cum}), $f(x^k)$ denotes the current function value, the column titled "V_k" contains $V_{\rho_{k-1}}(x^k, u^{k-1})$, $t_j := 1/\gamma_j$ is the stepsize, and ρ_k denotes the penalty parameter at iteration k.

The method terminates after 12 outer iterations, which is a reasonable number, especially taking into account that the final penalty parameter ρ_k is relatively large, so that several subproblems with different values of ρ_k have to be solved in the intermediate steps. On the other hand, the number of inner iterations j (at each outer iteration k) is very large. In the final step, the method requires more than one million inner iterations. This is a typical behavior of gradient-type methods and indicates that the underlying subproblems are ill-conditioned. This is also reflected by the fact that the stepsize t_j tends to zero.

There are two types of difficulties in Example 3.27: there are challenging constraints (the complementarity constraints), and there is an ill-conditioning. The difficult constraints are treated by Algorithm 3.4.1 successfully, but the ill-conditioning causes some problems when solving the resulting subproblems. In principle, this difficulty can be circumvented

k	j	j_{cum}	f-ev.	$f(x^k)$	V_k	t_j	ρ_k
0	0	0	1	32.0000000	—	—	320
1	4889	4889	8561	-30.2322093	0.017885	0.00019214	320
2	2765	7654	13171	-29.5693079	0.010772	0.00019553	320
3	2959	10613	18148	-29.1713687	0.008367	0.00019264	320
4	2734	13347	23001	-28.8787629	0.007077	0.00020241	3200
5	16380	29727	51233	-27.6160751	0.003845	0.00001961	3200
6	16412	46139	80229	-26.8702076	0.002675	0.00001967	3200
7	17708	63847	111596	-26.4929700	0.002437	0.00003231	32000
8	128146	191993	333580	-25.3129057	0.002357	0.00000196	320000
9	596930	788923	1364773	-13.1312431	0.000868	0.00000021	320000
10	756029	1544952	2686144	-5.3024263	0.000316	0.00000020	320000
11	911019	2455971	4320526	-2.0002217	0.000115	0.00000020	320000
12	1084340	3540311	6367887	-0.7376656	0.000042	0.00000020	320000

Table 3.1: Numerical results for Example 3.27.

by using another subproblem solver (like a semismooth Newton method, see [86]), but then it is no longer guaranteed that we obtain M-stationary points at the limit.

Despite the fact that the ill-conditioning causes some difficulties, we stress again that each iteration of the spectral gradient method is extremely cheap. Moreover, for all test problems in the subsequent sections, we put an upper bound of 50000 inner iterations (as a safeguard), and this upper bound was not reached in any of these examples.

3.6.2 Cardinality-Constrained Problems

We first consider an artificial example to illustrate the convergence behavior of Algorithm 3.4.1 for cardinality-constrained problems.

Example 3.28. Consider the example

$$\min_x \; f(x) := \tfrac{1}{2}x^\top Qx + c^\top x \quad \text{s.t.} \quad e^\top x \leq 8, \; \|x\|_0 \leq 2,$$

where $Q := E + I$ with $E \in \mathbb{R}^{5\times5}$ being the all one matrix, $I \in \mathbb{R}^{5\times5}$ the identity matrix, and $c := -(3,2,3,12,5)^\top \in \mathbb{R}^5$. Clearly, by Lemma 3.8, all feasible points are AM-regular. This is a minor modification of an example from [22], to which we added an (inactive) inequality constraint for the same reason as in Example 3.26. Taking into account that there are $\binom{5}{2}$ possibilities to choose two possibly nonzero components of x, an elementary calculation shows that there are exactly 10 M-stationary points $\bar{x}^1, \ldots, \bar{x}^{10}$ which are given in Table 3.2 together with the corresponding function values. It follows that $\bar{x}^6$ is the global minimizer. The points $\bar{x}^3, \bar{x}^8$, and $\bar{x}^{10}$ have function values which are not too far away from $f(\bar{x}^6)$, whereas all other M-stationary points have significantly larger function values. We then took 1000 random starting points from the box $[-10, 10]^5$ (uniformly distributed) and applied Algorithm 3.4.1 to this example. Surprisingly, the method converged, for all 1000 starting points, to the global minimizer $\bar{x}^6$. We then changed the example by putting an upper bound $x_4 \leq 0$ to the fourth component. This excludes the four most interesting points $\bar{x}^3, \bar{x}^6, \bar{x}^8$, and $\bar{x}^{10}$. Among the remaining points, the three vectors $\bar{x}^4, \bar{x}^7$, and $\bar{x}^9$ have identical function values. Running our program again using 1000 randomly generated starting points, we obtain convergence to $\bar{x}^4$ in 589 cases, convergence to $\bar{x}^7$ in 350 situations, whereas in 61 instances only we observe convergence to the non-optimal point $\bar{x}^2$.

$\bar{x}^i$	$f(\bar{x}^i)$	$\bar{x}^i$	$f(\bar{x}^i)$
$\bar{x}^1 := (4/3, 1/3, 0, 0, 0)^\top$	-2.33	$\bar{x}^6 := (0, -8/3, 0, 22/3, 0)^\top$	-41.33
$\bar{x}^2 := (1, 0, 1, 0, 0)^\top$	-3.00	$\bar{x}^7 := (0, -1/3, 0, 0, 8/3)^\top$	-6.33
$\bar{x}^3 := (-2, 0, 0, 7, 0)^\top$	-39.00	$\bar{x}^8 := (0, 0, -2, 7, 0)^\top$	-39.00
$\bar{x}^4 := (1/3, 0, 0, 0, 7/3)^\top$	-6.33	$\bar{x}^9 := (0, 0, 1/3, 0, 7/3)^\top$	-6.33
$\bar{x}^5 := (0, 1/3, 4/3, 0, 0)^\top$	-2.33	$\bar{x}^{10} := (0, 0, 0, 19/3, -2/3)^\top$	-36.33

Table 3.2: M-stationary points and corresponding function values for Example 3.28.

We next consider a class of cardinality-constrained problems of the form

$$\min_x \ \tfrac{1}{2} x^\top Q x \quad \text{s.t.} \quad \mu^\top x \geq \varrho, \ e^\top x = 1, \ 0 \leq x \leq u, \ \|x\|_0 \leq \kappa. \tag{3.48}$$

This is a classical portfolio optimization problem, where Q and μ denote the covariance matrix and the mean of n possible assets, respectively, while ϱ is some lower bound for the expected return. Furthermore, u provides an upper bound for the individual assets within the portfolio. The affine structure of the constraints in (3.48) implies that all feasible points are AM-regular, see Lemma 3.8. The data Q, μ, ϱ, u were randomly created by the test problem collection [74], which is available from the webpage `https://commalab.di.unipi.it/datasets/MV/`. Here, we used all 30 test instances of dimension $n := 200$ and three different values $\kappa \in \{5, 10, 20\}$ for each problem. We apply three different methods:

(a) Algorithm 3.4.1 with starting point $x^0 := 0$,
(b) a boosted version of Algorithm 3.4.1, and
(c) a CPLEX solver [91] to a reformulation of the portfolio optimization problem as a mixed integer quadratic program.

The CPLEX solver is used to (hopefully) identify the global optimum of the optimization problem (3.48). Note that we put a time limit of 0.5 hours for each test problem. Method (a) applies our augmented Lagrangian method to (3.48) using the set $D := \{x \in [0, u] \mid \|x\|_0 \leq \kappa\}$. Projections onto D are computed using the analytic formula from Proposition 3.22. Finally, the boosted version of Algorithm 3.4.1 is the following: We first delete the cardinality constraint from the portfolio optimization problem. The resulting quadratic program is then convex and can therefore be solved easily. Afterwards, we apply Algorithm 3.4.1 to a sequence of relaxations of (3.48) in which the cardinality is recursively decreased by 10 in each step (starting with $n - 10$) as long as the desired value $\kappa \in \{5, 10, 20\}$ is not undercut. For $\kappa = 5$, a final call of Algorithm 3.4.1 with the correct cardinality is necessary since, otherwise, the procedure would terminate with cardinality level 10. In each outer iteration, the projection of the solution of the previous iteration onto the set D is used as a starting point.

The corresponding results are summarized in Figure 3.2 for the three different values $\kappa \in \{5, 10, 20\}$. This figure compares the optimal function values obtained by the above three methods for each of the 30 test problems. The optimal function values produced by CPLEX are used here as a reference value in order to judge the quality of the results obtained by the other approaches. The main observations are the following: The optimal

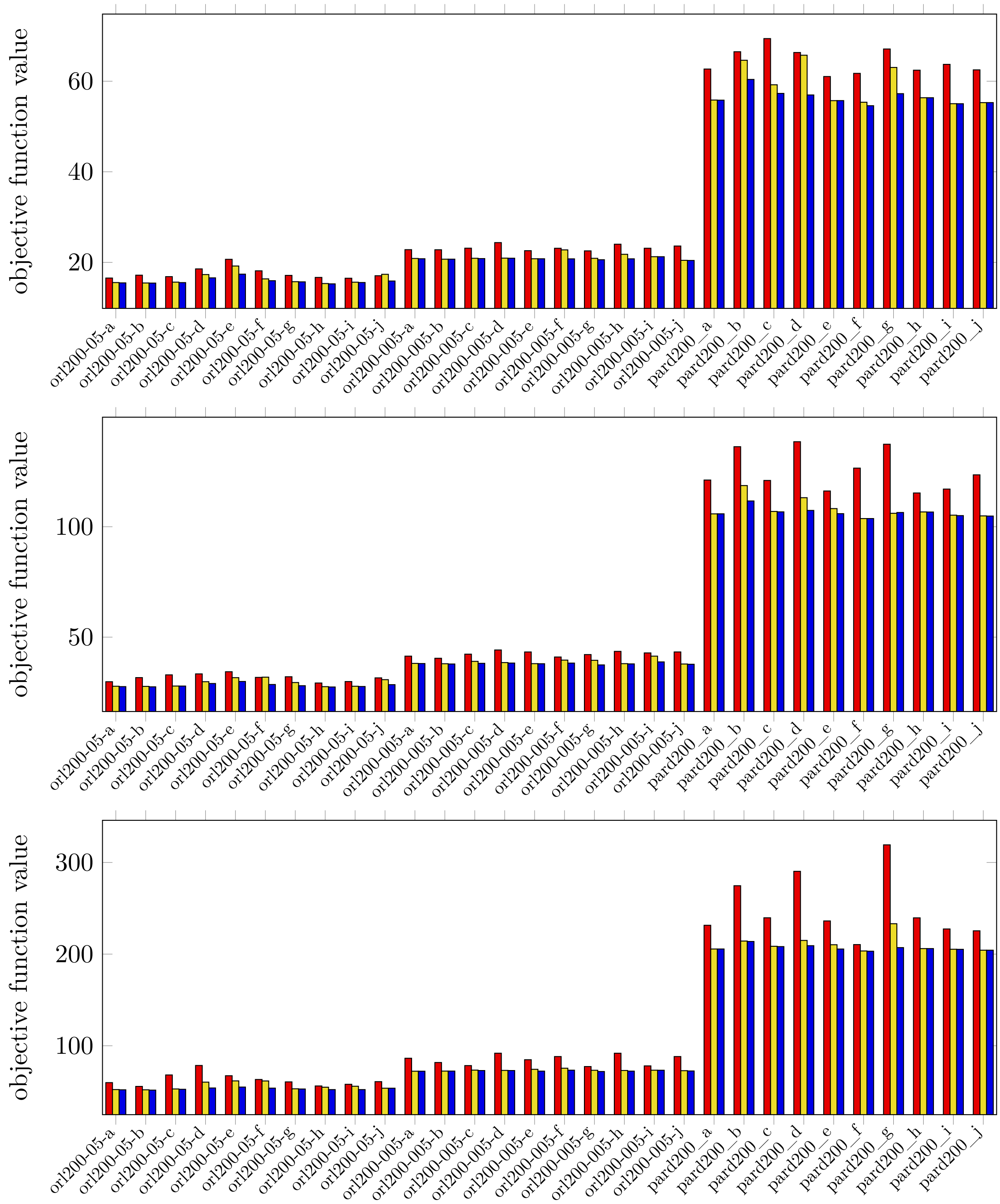

Figure 3.2: Optimal function values obtained by Algorithm 3.4.1 (red), Algorithm 3.4.1 with boosting technique (yellow), and CPLEX (blue), applied to the portfolio optimization problem (3.48) with cardinality $\kappa = 20$, $\kappa = 10$, and $\kappa = 5$ (top to bottom).

function value computed by CPLEX is (not surprisingly) always the best one. On the other hand, the corresponding values computed by method (a) are usually not too far away from the optimal ones. Moreover, for all test problems, the boosted version (b) generates even better function values which are usually very close to the ones computed by CPLEX. Of course, if κ is taken smaller, the problems are getting more demanding and are therefore more difficult to solve (in general). Nevertheless, also for $\kappa = 5$, especially the boosted algorithm still computes rather good points. In this context, one should also note that our methods always terminate with a (numerically) feasible point, hence, the final iterate computed by our method can actually be used as a (good) approximation of the global minimizer. We also would like to mention that our MATLAB implementation of Algorithm 3.4.1 typically requires, on an Intel Core i7-8700 processor, only a CPU time of about 0.1 seconds for each of the test problems, whereas the boosted version requires roughly two seconds CPU time in average.

3.6.3 Recommender Systems

Recommender systems are used to establish direct connections between users and items by matching their interests and preferences in order to help users discover new items and reduce the overwhelming amount of choices available on the web. The main problem of recommender systems is the missing data, by the given ratings of some item from the users, one needs to infer the missing ratings, which is certainly not unique. In most recommender systems, the underlying assumption is that the complete matrix of ratings is low-rank [116, Section 1.3].

Let us assume there are m users and n items in a recommender system, set $A \in \mathbb{R}^{m\times n}$ being a matrix with observed entries indexed by the set Ω, i.e., $\Omega = \{(i,j) \mid A_{ij} \text{ is observed}\}$. Then, such recommender system problem can be transferred into the following rank-constrained optimization problem

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \ & \frac{1}{2}\|X - A\|_F^2 \\ \text{s.t. } & X_{ij} = A_{ij} \quad \forall (i,j) \in \Omega \\ & \operatorname{rank} X \le \kappa, \end{aligned} \tag{3.49}$$

where $\|\cdot\|_F$ is the Frobenius norm, X is the to-be-recovered matrix, $\kappa \le \min(m,n) - 1$ is some certain constrained rank. Note that for the given matrix A, among its all underlying completions, there exists a minimal rank. If κ is smaller than this minimal rank, then (3.49) has no feasible set. In order to overcome such shortcomings, we first solve the following optimization problem

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \ & \|X\|_q \\ \text{s.t. } & X_{ij} = A_{ij} \quad \forall (i,j) \in \Omega, \end{aligned} \tag{3.50}$$

with the Schatten-quasi-q norm $\|\cdot\|_q := (\sum_i \sigma_i(\cdot)^q)^{\frac{1}{q}} (q \in (0,1))$, where $\sigma(X)$ denotes the vector of singular value of a matrix X. Note that (3.50) is always used to formulate the recommender system problems [116, Section 1.3]. We then solved (3.50) by augmented Lagrangian method invoking monotone proximal gradient method (see Algorithm 5.1 or [95]), the rank of solution would be regarded as the upper bound of κ. Set $D := \{X \in \mathbb{R}^{m\times n} \mid \operatorname{rank} X \le \kappa\}$, we applied Algorithm 3.4.1 and its boosted version to solve (3.49), where the projections on D can be calculated via Proposition 3.24. We chose the solution of (3.50) as the starting point of Algorithm 3.4.1 and $X^0 := A$ as the starting point of boosted Algorithm 3.4.1, respectively. Note that the boosted Algorithm 3.4.1 is highly

Algorithm			Algorithm 3.4.1				Boosted Algorithm 3.4.1			
q	f_q	κ	k	j	f	max	k	j	f	max
0.1	26.16	911	2	63	22.96	$9.8e-5$	4	211	35.17	$5.2e-5$
0.3	72.75	895	2	253	50.45	$6.9e-5$	6	391	61.65	$9.9e-5$
0.5	291.43	867	2	356	133.82	$4.1e-5$	9	1090	164.09	$8.9e-5$
0.7	1556.64	813	2	795	474.75	$6.1e-5$	20	2747	474.05	$9.1e-5$
0.9	7601.44	740	3	1620	1563.82	$4.8e-5$	27	7393	1710.37	$8.2e-5$

Table 3.3: Numerical results of Algorithm 3.4.1 and its boosted version for MovieLens 100K w.r.t. different κ.

like that in Section 3.6.2: We first delete the rank constraint from (3.49). We then apply Algorithm 3.4.1 to solve a sequence of relaxations of (3.49) where the rank is recursively decreased by 10 in each step starting with $\min\{m, n\}$ as long as the desired value κ is less than the presolved one (note that undercutting is permitted here). In each outer iteration, the projection of the solution of the previous iteration onto the set D is used as a starting point.

In this section, we tested Algorithm 3.4.1 and its boosted version on MovieLens 100K dataset [87], which is available from website `https://grouplens.org/datasets/movielens/100k/`. Here there are 943 users, 1682 movies, and a total of 100000 ratings from 1 to 5, making the resulting matrix 93 % missing. Since MovieLens 100K dataset is large-scaled, We weakened the inner residual error (3.47) as $\frac{10^{-2}}{\sqrt{k+1}}$. We chose $q \in \{0.1, 0.3, \ldots, 0.9\}$ and hence induced 5 different problems (3.49) (we here set κ is equal to the rank of the solution of (3.50)). The results are listed in Table 3.3, where $f_q := \frac{1}{2}\|X - A\|_F^2$ where X is the solution of (3.50) with different q, k means the total number of outer iterations and j means the total number of inner iterations, f represents the optimal function value at the final iterate, and $\max := \max_{(i,j)\in\Omega}\{\|X_{ij} - A_{ij}\|\}$ means the max of the differences between the predicted values and observed values where X is the final iterate.

Table 3.3 illustrates that with the increasement of q, κ becomes smaller in general, which causes the problem becomes more demanding and hence more difficult to be solved. Note that Algorithm 3.4.1 and its boosted version can recover matrix A successfully, though the boosted Algorithm 3.4.1 costed more including the (outer and inner) iterations and CPU time. Meanwhile, both of them are more robust since they generated the lower objective functions than f_q. In principle, the optimal objective function generated by boosted Algorithm 3.4.1 should be larger than that of Algorithm 3.4.1 since overcutting is permitted, which always occurs in Table 3.3. Let us take a closer look at $\kappa = 813$, where the desired and presolved κ are equal, the value objective function generated by boosted Algorithm 3.4.1 is lower than that of Algorithm 3.4.1, as indicated by the results obtained by boosted Algorithm 3.4.1 in Section 3.6.2.

3.6.4 MAXCUT Problems

This section considers the famous MAXCUT problem as an application of our algorithm to problems with rank constraints. To this end, let $G = (V, E)$ be an undirected graph with vertex set $V = \{1, \ldots, n\}$ and edges e_{ij} between vertices $i, j \in V$. We assume that we have a weighted graph, with $a_{ij} = a_{ji}$ denoting the nonnegative weights of the edge e_{ij}. Since we allow zero weights, we can assume without loss of generality that G is a complete graph. Now, given a subset $S \subset V$ with complement S^c, the *cut* defined by S is the set $\delta(S) := \{e_{ij} \mid i \in S, j \in S^c\}$ of all edges such that one end point belongs to S and the other one to S^c. The corresponding weight of this cut is defined by

$$w(S) := \sum_{e_{ij} \in \delta(S)} a_{ij}.$$

The MAXCUT problem looks for the maximum cut, i.e., a cut with maximum weight. This graph-theoretical problem is known to be NP-hard, thus very difficult to solve.

Let $A := (a_{ij})$ and define $L := \operatorname{diag}(Ae) - A$. Then it is well known, see e.g. [77], that the MAXCUT problem can be reformulated as

$$\max_X \ \tfrac{1}{4}\operatorname{trace}(LX) \quad \text{s.t.} \quad \operatorname{diag} X = e,\ X \succeq 0,\ \operatorname{rank} X = 1, \tag{3.51}$$

where the variable X is chosen from the space $\mathbb{X} := \mathbb{R}^{n\times n}_{\text{sym}}$. Due to the linear constraint $\operatorname{diag} X = e$, it follows that this problem is equivalent to

$$\max_X \ \tfrac{1}{4}\operatorname{trace}(LX) \quad \text{s.t.} \quad \operatorname{diag} X = e,\ X \succeq 0,\ \operatorname{rank} X \le 1. \tag{3.52}$$

Deleting the difficult rank constraint, one gets the (convex) relaxation

$$\max_X \ \tfrac{1}{4}\operatorname{trace}(LX) \quad \text{s.t.} \quad \operatorname{diag} X = e,\ X \succeq 0, \tag{3.53}$$

which is a famous test problem for semidefinite programs.

Here, we directly deal with (3.52) by taking $D := \{X \in \mathbb{X} \mid X \succeq 0,\ \operatorname{rank} X \le 1\}$. Projections onto D can be calculated via Proposition 3.25: Let $X \in \mathbb{X}$ denote an arbitrary symmetric matrix with maximum eigenvalue λ and corresponding (normalized) eigenvector v (note that λ and v are not necessarily unique), then $\max(\lambda, 0)vv^\top$ is a projection of X onto D. In particular, the computation of this projection does not require the full spectral decomposition. However, it is not clear whether a projection onto the feasible set of (3.52) can be computed efficiently. Consequently, we penalize the linear constraint $\operatorname{diag} X = e$ by the augmented Lagrangian approach.

Throughout this section, we take the zero matrix as the starting point. In order to illustrate the performance of our method, we begin with the simple graph from Figure 3.3. Algorithm 3.4.1 applied to this example using the reformulation (3.52) (more precisely, the corresponding minimization problem) together with the previous specifications yields the iterations shown in Table 3.4. The meaning of the columns is the same as for Table 3.1.

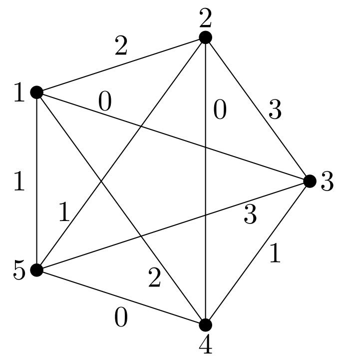

Figure 3.3: Example of a complete graph for the MAXCUT problem.

Note that the penalty parameter stays constant for this example. The feasibility measure tends to zero, and we terminate at iteration $k = 6$ since this measure becomes less

k	j	j_{cum}	f-ev.	$f(X^k)$	V_k	t_j	ρ_j
0	0	0	1	0.0000000	—	—	4
1	11	11	16	19.6691638	0.839210	1.25237254	4
2	9	20	27	12.0395829	0.027365	0.63395340	4
3	5	25	34	12.0097591	0.006361	1.25001275	4
4	3	28	38	12.0023821	0.001553	0.62522386	4
5	3	31	42	12.0005415	0.000382	0.62504390	4
6	3	34	46	12.0001534	0.000097	0.62502107	4

Table 3.4: Numerical results for MAXCUT associated to the graph from Figure 3.3.

than 10^{-4}, i.e., we stop successfully. The associated function value is (approximately) 12 which actually corresponds to the maximum cut $S := \{1, 3\}$ for the graph from Figure 3.3, i.e., our method is able to solve the MAXCUT problem for this particular instance.

We next apply our method to two test problem collections that can be downloaded from `http://biqmac.aau.at/biqmaclib.html`, namely the `rudy` and the `ising` collection. The first class of problems consists of 130 instances, whereas the second one includes 48 problems. The optimal function value f_{opt} of all these examples is known. The details of the corresponding results obtained by our method are given in Appendix A.1. Let us summarize the main observations.

All $130 + 48$ test problems were solved successfully by our method since the standard termination criterion was satisfied after finitely many iterations, i.e., we stop with an iterate X^k which is feasible (within the given tolerance). Hence, the corresponding optimal function value f_{ALM} is a lower bound for the optimal value f_{opt}. For the sake of completeness, we also solved the (convex) relaxed problem from (3.53), using again our augmented Lagrangian method with $D := \{X \in \mathbb{X} \mid X \succeq 0\}$. The corresponding function value is denoted by f_{SDP}. Since the feasible set of (3.53) is larger than the one of (3.52), we have the inequalities $f_{\text{ALM}} \leq f_{\text{opt}} \leq f_{\text{SDP}}$. The corresponding details for the solution of the SDP-relaxation are provided in Appendix A.1 for the `rudy` collection.

The bar charts from Figures 3.4 and 3.5 summarize the results for the `rudy` and `ising` collections, respectively, in a very condensed way. They basically show that the function value f_{ALM} obtained by our method is very close to the optimal value f_{opt}. More precisely, the interpretation is as follows: For each test problem, we take the quotient $f_{\text{ALM}}/f_{\text{opt}} \in [0, 1]$. If this quotient is equal to, say, 0.91, we count this example as one where we reach 91% of the optimal function value. Figure 3.4 then says that all 130 test problems were solved with at least 88% of the optimal function value. There are still 106 test examples which are solved with a precision of at least 95%. Almost one third of the test examples, namely 43 problems, are even solved with an accuracy of at least 99%. For two examples (pm1d_80.9, and pw01_100.8), we actually get the exact global maximum.

Figure 3.5 has a similar meaning for the `ising` collection: Though there is no example which is solved exactly, almost one half of the problems reaches an accuracy of at least 99%, and even in the worst case, we obtain a precision of 94%.

Altogether, this shows that we obtain a very good lower bound for the optimal function value. Moreover, since we are always feasible (in particular, all iterates are matrices of rank one), the final matrix can be used to create a cut through the given graph, i.e., the method provides a constructive way to create cuts which seem to be close to the optimal cuts. Note that this is in contrast to the semidefinite relaxation (3.53) which gives an upper bound, but the solution associated with this upper bound is usually not feasible for the MAXCUT

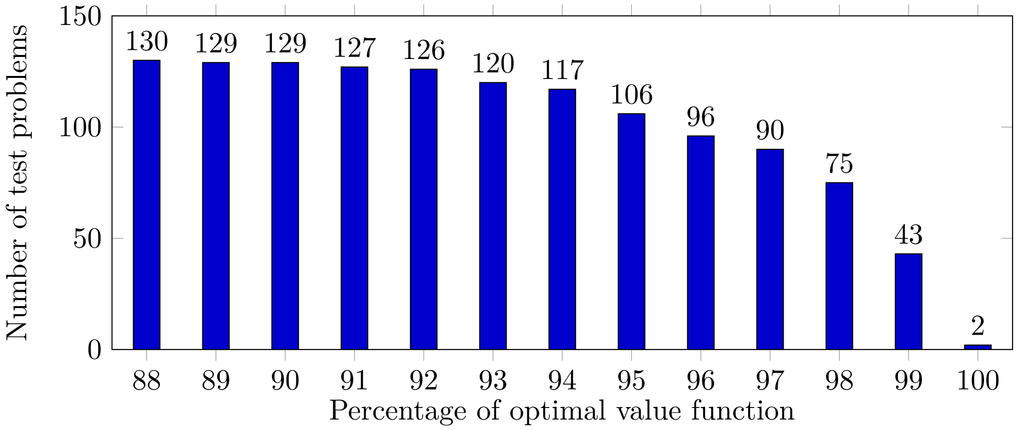

Figure 3.4: Summary of the results from the `rudy` collection.

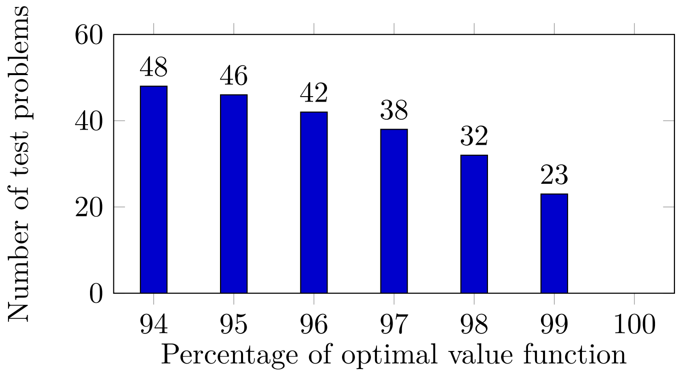

Figure 3.5: Summary of the results from the `ising` collection.

problem since the rank constraint is violated (the results in Appendix A.1 show that the solutions of the relaxed programs for the `rudy` collection are matrices of rank between 4 and 7). In particular, these matrices can, in general, not be used to compute a cut for the graph and, therefore, are less constructive than the outputs of our method. Moreover, it is interesting to observe that f_{ALM} is usually much closer to f_{opt} than f_{SDP}. In any case, both techniques together might be useful tools in a branch-and-bound-type method for solving MAXCUT problems.

4. Augmented Lagrangian Methods invoking Proximal Gradient-type Methods for Composite Structured Optimization Problems

This chapter, inspired by Section 3.5.4, aims at a detailed discussion of augmented Lagrangian methods used for the composite optimization problems with set-membership constraints, whose objective function is the sum of a continuously differentiable function and a lower semicontinuous function, of the form as in (CP), the subsequent results are generally based on the publication [66]. Let us recall the program again

$$\min_{x} \; q(x) := f(x) + g(x) \qquad \text{s.t.} \quad c(x) \in K, \tag{CP}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are Euclidean spaces, the following blanket assumptions are considered throughout, without further mention.

Assumption 4.1. The following hold in (CP):

(a) $f : \mathbb{X} \to \mathbb{R}$ and $c : \mathbb{X} \to \mathbb{Y}$ are continuously differentiable with locally Lipschitz continuous derivatives;

(b) $g : \mathbb{X} \to \overline{\mathbb{R}}$ is proper, lower semicontinuous and prox-bounded;

(c) $K \subset \mathbb{Y}$ is a nonempty and closed set.

By (a) and (b), the cost function $q := f + g$ has nonempty domain, that is, $\operatorname{dom} q \neq \emptyset$. Similarly, (c) guarantees that it is always possible to project onto the constraint set K. Nevertheless, these conditions do not imply the existence of feasible points for (CP). Let us look back the set-membership constraints $c(x) \in K$, which do provide much flexible and compact expressions. However, in fact, the projections onto the set $\{x \in \mathbb{X} \,|\, c(x) \in K\}$ induced by the constraints $c(x) \in K$ are not simple to calculate, even if K is convex and variationally simple (like the nonnegative orthant). In order to overcome such difficulty, in this chapter, the original problem needs to be reformulated by introducing slack variables $s \in K$ such that $c(x) - s = 0$, we consequently need to consider the reformulated program equipped with the slack variables whose constraints now become $c(x) - s = 0$ to be penalized by the augmented Lagrangian scheme, as well as the set $\{(x, s) \in \mathbb{X} \times K\}$. As a result, we alternatively seek for the projections of s on to set K, which are in some sense available according to Section 3.5, where the projection formulas about some different cases of set constraints were illustrated. It is carved out that, apart from the higher number of decision variables, this reformulation is nonhazardous.

For the actual implementation, we work under the practical assumption that (only) the following computational oracles are available or simple to evaluate:

- cost function value $f(x)$ and gradient $\nabla f(x)$, given $x \in \operatorname{dom} q$;
- (arbitrary) proximal point $z \in \operatorname{prox}_{\gamma g}(x)$ and function value $g(z)$ therein, given $x \in \mathbb{X}$ and $\gamma \in (0, \gamma_g)$, γ_g being the prox-boundedness threshold of g;
- constraint function value $c(x)$ and Jacobian-vector product $c'(x)^* v$, given $x \in \operatorname{dom} q$ and $v \in \mathbb{Y}$;
- (arbitrary) projected point $z \in \Pi_K(v)$, given $v \in \mathbb{Y}$.

Relying only on these oracles, the method considered for our numerical examples is first-order and matrix-free by construction, it involves only simple operations and has low memory footprint.

The chapter is organised as follows. Based on Section 3.2, we begin with the modified definitions of M-, AM-stationary points, as well as AM-regularity customized for the program (CP) in Section 4.1. Section 4.2 provides us some theoretical guarantees about the feasibility of using slack variables to reformulate (CP). The associated optimization problems with slack variables can be solved by a (safeguarded) augmented Lagrangian method, which is proposed in Section 4.3 and its underlying convergence analysis is included there. Section 4.4 is devoted to the solution solvers of the resulting subproblems, more specifically, PANOC$^+$, as a kind of proximal gradient-type method, is employed to solve such subproblem both theoretically and numerically. Some computational experiments are documented in Section 4.5 including nonsmooth signal recovery problems, Rosenbrock problems, sparse portfolio optimization problems, and matrix completion problems.

4.1 Stationarities and Qualification Condition

This section aims to discuss the stationary points and the corresponding qualification condition of (CP). Notice that its objective function is lower semicontinuous and extended-valued, then feasibility of a point must account for its domain.

Definition 4.2 (Feasibility of (CP))**.** A point $\bar{x} \in \mathbb{X}$ is called *feasible* for (CP) if $\bar{x} \in \operatorname{dom} q$ and $c(\bar{x}) \in K$.

Working under the assumption that the constraint set K is nonconvex, a plausible stationarity concept for addressing (CP) is that of Mordukhovich-stationarity, which exploits limiting normals to K, cf. [117, Section 3] and [122, Theorem 5.48].

Definition 4.3 (M-stationarity of (CP))**.** Let $\bar{x} \in \mathbb{X}$ be a feasible point for (CP). Then, $\bar{x}$ is called a *Mordukhovich-stationary* point of (CP) if there exists a multiplier $\bar{\lambda} \in \mathbb{Y}$ such that

$$-c'(\bar{x})^* \bar{\lambda} \in \partial q(\bar{x}) \tag{4.1a}$$

$$\bar{\lambda} \in \mathcal{N}_K^{\lim}(c(\bar{x})). \tag{4.1b}$$

Notice that these conditions implicitly require the feasibility of $\bar{x}$, for otherwise the subdifferential and limiting normal cone would be empty. Note that this definition coincides with the usual KKT conditions of (CP) if g is smooth and K is a convex set.

Subsequently, we study an asymptotic counterpart of this definition. In case where q is locally Lipschitz continuous, one could apply the notions from [94, Section 2.2] and [117, Section 5.1] for that purpose. However, since g is assumed to be merely lsc, these concepts need to be at least slightly adjusted.

Definition 4.4 (AM-stationarity of (CP))**.** Let $\bar{x} \in \mathbb{X}$ be a feasible point for (CP). Then, $\bar{x}$ is called an *asymptotically M-stationary* point of (CP) if there exist sequences $\{x^k\}, \{\eta^k\} \subset \mathbb{X}$ and $\{\lambda^k\}, \{\zeta^k\} \subset \mathbb{Y}$ such that $x^k \xrightarrow{q} \bar{x}$, $\eta^k \to 0$, $\zeta^k \to 0$ and

$$-c'(x^k)^*\lambda^k + \eta^k \in \partial q(x^k) \tag{4.2a}$$

$$\lambda^k \in \mathcal{N}_K^{\lim}(c(x^k) - \zeta^k) \tag{4.2b}$$

for all $k \in \mathbb{N}$.

The detailed explanations are highly similar with that of Definition 3.2. Note that the convergence $x^k \xrightarrow{q} \bar{x}$ will be important later on when taking the limit in (4.2a) since we aim to recover the limiting subdifferential of the objective function as stated in Definition 2.19, also see for more explanations.

Next, we give that each local minimizer of (CP) is always AM-stationary. The proof is very similar to Theorem 3.3, see Appendix A.2 for more details. Related results can be also found in [102, Theorem 6.2] and [117, Section 5.1].

Theorem 4.5. *Let $\bar{x} \in \mathbb{X}$ be a local minimizer for* (CP)*. Then, $\bar{x}$ is an AM-stationary point for* (CP)*.*

Note that, by Definition 4.3, one immediately obtain

$$\bar{x} \text{ is M-stationary} \Longleftrightarrow -\nabla f(\bar{x}) \in \mathcal{M}(\bar{x}, 0), \tag{4.3}$$

where the set-valued mapping $\mathcal{M} : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is defined by

$$\mathcal{M}(x, z) \coloneqq \partial g(x) + c'(x)^*\mathcal{N}_K^{\lim}(c(x) - z).$$

A similar characterization holds for AM-stationary points.

Proposition 4.6. *Let $\bar{x} \in \mathbb{X}$ be feasible for* (CP)*, then $\bar{x}$ is AM-stationary of* (CP) *if and only if $-\nabla f(\bar{x}) \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$.*

Proof. First assume $\bar{x}$ is an AM-stationary point of (CP), then there exist sequences $\{x^k\}, \{\eta^k\} \subset \mathbb{X}$, and λ^k, $\zeta^k \subset \mathbb{Y}$, such that $x^k \xrightarrow{q} \bar{x}$, $\eta^k \to 0$, $\zeta^k \to 0$, as well as

$$-c'(x^k)^*\lambda^k + \eta^k \in \partial q(x^k), \quad \lambda^k \in \mathcal{N}_K^{\lim}(c(x^k) - \zeta^k) \quad \forall k \in \mathbb{N}.$$

Setting $w^k := \eta^k - \nabla f(x^k)$, then one has

$$w^k \in c'(x^k)^*\mathcal{N}_K^{\lim}(c(x^k) - \zeta^k) + \partial g(x^k) \quad \forall k \in \mathbb{N}.$$

Taking the limit $k \to \infty$ and using the continuity of ∇f, one therefore obtain $-\nabla f(\bar{x}) \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$.

Conversely, assume that $-\nabla f(\bar{x}) \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$ holds. Then there exist sequences $\{x^k\}, \{w^k\} \subset \mathbb{X}$ and $\{\zeta^k\} \subset \mathbb{Y}$ satisfying $x^k \xrightarrow{q} \bar{x}$, $\zeta^k \to 0$, $w^k \to -\nabla f(\bar{x})$, and $w^k \in \mathcal{M}(x^k, \zeta^k)$ for all $k \in \mathbb{N}$. By the definition of $\mathcal{M}(x^k, \zeta^k)$, there exists $\lambda^k \in \mathcal{N}_K^{\lim}(c(x^k) - \zeta^k)$ such that $w^k \in c'(x^k)^*\lambda^k + \partial g(x^k)$. Setting $\eta^k := w^k + \nabla f(x^k)$, then $\eta^k \to 0$, the statement follows. $\square$

In order to guarantee that local minimizers for (CP) are not only AM- but already M-stationary, the presence of a qualification condition is necessary. The subsequent definition generalizes the constraint qualification from [117, Section 3.2] to the non-Lipschitzian setting and is closely related to the so-called *uniform qualification condition* introduced in [102, Definition 6.8].

Definition 4.7 (AM-regularity of (CP))**.** Let $\bar{x} \in \mathbb{X}$ be a feasible point for (CP), then, $\bar{x}$ is called *asymptotically M-regular* for (CP) if

$$\limsup_{\substack{x \xrightarrow{g} \bar{x} \\ z \to 0}} \mathcal{M}(x, z) \subset \mathcal{M}(\bar{x}, 0).$$

Let us point the reader's attention to the fact that AM-regularity is not a constraint qualification for (CP) in the narrower sense since it depends explicitly on the objective function. However, note that AM-regularity of some feasible point $\bar{x} \in \mathbb{X}$ for (CP) reduces to

$$\limsup_{\substack{x \to \bar{x} \\ z \to 0}} c'(x)^* \mathcal{N}_K^{\lim}(c(x) - z) \subset c'(\bar{x})^* \mathcal{N}_K^{\lim}(c(\bar{x})) \tag{4.4}$$

whenever g is locally Lipschitz continuous around $\bar{x}$ since $x \rightrightarrows \partial g(x)$ is locally bounded at $\bar{x}$ in this case, see [122, Cor. 1.81].

From Definition 4.3, Definition 4.4, Definition 4.7, Theorem 4.5, we obviously obtain the following corollary.

Corollary 4.8. *Let $\bar{x} \in \mathbb{X}$ be an AM-regular AM-stationary point for* (CP)*. Then, $\bar{x}$ is an M-stationary point for* (CP)*. Particularly, each AM-regular local minimizer for* (CP) *is M-stationary.*

It turns out that AM-regularity guarantees that any AM-stationary point of the program (CP) is already an M-stationary point of this program. Moreover, in some sense, AM-regularity is the weakest strict qualification condition associated with AM-stationarity based on the terminology coined in [9], the statement is formalized in the following result from the lines of the proofs of [9, Theorem 3.2] or [41, Theorem 4.6].

Corollary 4.9. *Let $\bar{x} \in \mathbb{X}$ be feasible for* (CP)*, if for every continuously differentiable function f, the implication*

$$\bar{x} \textit{ is AM-stationary point of } (\mathrm{CP}) \Longrightarrow \bar{x} \textit{ is M-stationary point of } (\mathrm{CP})$$

holds, then $\bar{x}$ satisfies AM-regularity.

Proof. Now assume that AM-stationarity of $\bar{x}$ implies M-stationarity for every continuously differentiable f. One then has to verify that $\limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z) \subset \mathcal{M}(\bar{x}, 0)$ holds. To this end, take an arbitrary element $w \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$. Then define the particular function $f(x) := -\langle w, x \rangle$. Since f is continuously differentiable with $\nabla f = -w$ for all $x \in \mathbb{X}$, one has $-\nabla f(\bar{x}) = w \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$. Proposition 4.6 then tells us that $\bar{x}$ is an AM-stationary point. By assumption, this implies that $\bar{x}$ is already an M-stationary point of (CP). In view of (4.3), this is equivalent to $-\nabla f(\bar{x}) \in \mathcal{M}(\bar{x}, 0)$. The definition of f therefore implies that $w \in \mathcal{M}(\bar{x}, 0)$. $\square$

There exist a couple of qualification conditions that can be formulated for the general program (CP). We next introduce one of these qualification conditions which is, in fact, viewed as a very weak condition, namely generalized RCPLD original introduced by [6] for standard nonlinear programs, and later generalized in [83] and Definition 3.6. In order to state this generalized RCPLD, one needs a particular setting and therefore consider the optimization problem

$$\begin{aligned} \min_x \quad & f(x) + g(x) \\ \text{s.t.} \quad & \theta_i(x) \le 0 \quad \forall i = 1, \ldots, m, \\ & h_j(x) = 0 \quad \forall j = 1, \ldots, p, \end{aligned} \tag{4.5}$$

which corresponds to the general setting (CP) with

$$c := \begin{pmatrix} \theta \\ h \end{pmatrix} \quad \text{and} \quad K := (-\infty, 0]^m \times \{0\}^p,$$

where $\theta : \mathbb{X} \to \mathbb{R}^m$ and $h : \mathbb{X} \to \mathbb{R}^p$ are continuously differentiable. For this particular setting, the following definition is taken motivated by [83].

Definition 4.10. Let $\bar{x} \in \mathbb{X}$ be a feasible point of the optimization problem (4.5). We say that $\bar{x}$ satisfies ∂^∞-RCPLD holds if the following conditions hold:

(a) the vectors $\{\nabla h_j(x)\}_{j=1}^p$ have constant rank for all x in a neighbourhood of $\bar{x}$.

(b) there exists an index set $J \subset \{1, \ldots, p\}$ such that the gradients $\{\nabla h_j(\bar{x})\}_{j \in J}$ from a basis of the subspace $\operatorname{span}\{\nabla h_j(\bar{x}) \mid j = 1, \ldots, p\}$.

(c) for each $I \subset I(\bar{x}) := \{i \mid \theta_i(\bar{x}) = 0\}$, each set of multipliers $\lambda_i \ge 0$ $(i \in I)$ and $\mu_j \in \mathbb{R}$ $(j \in J)$, not all vanishing at the same time, such that

$$0 \in \partial^\infty g(\bar{x}) + \sum_{i \in I} \lambda_i \nabla \theta_i(\bar{x}) + \sum_{j \in J} \nu_j \nabla h_j(\bar{x}), \tag{4.6}$$

then there exists a neighbourhood U of $\bar{x}$ such that the vectors

$$\{\nabla \theta_i(x)\}_{i \in I} \cup \{\nabla h_j(x)\}_{j \in J}$$

are linearly dependent for all $x \in U$.

For standard nonlinear programs, it is known that RCPLD is a very weak constraint qualification. The following result shows that the above generalized version of RCPLD, which covers the case of nonsmooth term in the objective function, still implies AM-regularity. The proof is highly similar to Lemma 3.7, see Appendix A.2 for more details.

Lemma 4.11. *Let $\bar{x}$ be a feasible point for* (CP) *such that ∂^∞-RCPLD holds at $\bar{x}$, then $\bar{x}$ is AM-regular.*

4.2 Theoretical Guarantees of Using Slack Variables

Constrained optimization problems such as (CP) are amenable to be addressed by means of augmented Lagrangian methods. Here, by introducing the slack variable $s \in \mathbb{Y}$, (CP) can be rewritten as

$$\min_{x,s} \; q(x) := f(x) + g(x) \qquad \text{s.t.} \quad c(x) - s = 0, \; s \in K. \tag{CP$_\text{S}$}$$

Notice that (CP$_\text{S}$) is a particular problem in the form of (CP). Moreover, if g is smooth, and thus so is q, then (CP$_\text{S}$) falls into the problem class analyzed in Chapter 3, or see [94]. Note that $\bar{x} \in \mathbb{X}$ is a global (local) minimizer of (CP) if and only if $(\bar{x}, c(\bar{x}))$ is a global (local) minimizer of ($\mathrm{CP_S}$). Similarly, the M-stationary points of (CP) and ($\mathrm{CP_S}$) correspond to each other. An elementary calculation additionally reveals that even the AM-stationary points of (CP) and ($\mathrm{CP_S}$) can be identified with each other.

Lemma 4.12. *A feasible point $\bar{x} \in \mathbb{X}$ of* (CP) *is AM-stationary for* (CP) *if and only if* $(\bar{x}, c(\bar{x}))$ *is AM-stationary for* ($\mathrm{CP_S}$)*.*

Proof. We know the implication $\Rightarrow$ holds obviously, so it remains to prove the converse one. If $(\bar{x}, c(\bar{x}))$ is AM-stationary for ($\mathrm{CP_S}$), then we can find sequences $\{x^k\}, \{\eta_1^k\} \subset \mathbb{X}$ and $\{s^k\}, \{\lambda_1^k\}, \{\lambda_2^k\}, \{\eta_2^k\}, \{\zeta_1^k\}, \{\zeta_2^k\} \subset \mathbb{Y}$ such that $x^k \xrightarrow{q} \bar{x}$, $s^k \to c(\bar{x})$, $\eta_i^k \to 0$, $\zeta_i^k \to 0$, $i = 1, 2$, and

$$-c'(x^k)^*\lambda_1^k + \eta_1^k \in \partial q(x^k), \tag{4.7a}$$

$$\lambda_1^k - \lambda_2^k + \eta_2^k = 0, \tag{4.7b}$$

$$c(x^k) - s^k - \zeta_1^k = 0, \tag{4.7c}$$

$$\lambda_2^k \in \mathcal{N}_K^{\lim}(s^k - \zeta_2^k) \tag{4.7d}$$

for all $k \in \mathbb{N}$, where we already used the Cartesian product rule for the limiting normal cone, cf. [122, Proposition 1.2], in order to split

$$(\lambda_1^k, \lambda_2^k) \in \mathcal{N}_{\{0\}\times K}^{\lim}(c(x^k) - s^k - \zeta_1^k, s^k - \zeta_2^k)$$

into (4.7c) and (4.7d). Now, for each $k \in \mathbb{N}$, set $\lambda^k := \lambda_2^k$, $\eta^k := c'(x^k)^*\eta_2^k + \eta_1^k$ and $\zeta^k := c(x^k) - s^k + \zeta_2^k$. Then, (4.2a) follows from (4.7a) and (4.7b). Furthermore, (4.2b) can be distilled from (4.7d). The convergence $\eta^k \to 0$ is clear from continuous differentiability of c, and $\zeta^k \to 0$ follows from $c(x^k) - s^k \to 0$ which is a consequence of the continuity of c (or (4.7c)). ☐

Though we incorporated the slack variable in ($\mathrm{CP_S}$), it does not change the solution and the stationarity behavior when compared with (CP). In light of [30], where similar issues are discussed in a much broader context. As a result, we use the lifted reformulation ($\mathrm{CP_S}$) as a theoretical tool to develop our approach for solving (CP) and investigate its properties. For some penalty parameter $\rho > 0$, let us first denote the (single-valued) augmented Lagrangian function $\mathcal{L}_\rho : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ associated to (CP) by

$$\mathcal{L}_\rho(x, \lambda) = q(x) + \frac{1}{2\rho}\operatorname{dist}_K^2(c(x) + \lambda\rho) - \frac{\rho}{2}\|\lambda\|^2, \tag{4.8}$$

as well as the augmented Lagrangian function $\mathcal{L}_\rho^S : \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \to \overline{\mathbb{R}}$ associated to ($\mathrm{CP_S}$) as

$$\begin{aligned}\mathcal{L}_\rho^S(x, s, \lambda) &:= q(x) + I_K(s) + \langle \lambda, c(x) - s\rangle + \frac{1}{2\rho}\|c(x) - s\|^2 \\ &= q(x) + I_K(s) + \frac{1}{2\rho}\|c(x) + \lambda\rho - s\|^2 - \frac{\rho}{2}\|\lambda\|^2.\end{aligned} \tag{4.9}$$

Observe that, by adopting the indicator I_K, the constraint $s \in K$ is considered hard, in the sense that it must be satisfied exactly. These simple, nonrelaxable lower-level constraints have been discussed, e.g., in [3, 34, 62, 94]. Let us compute the subdifferential of $\mathcal{L}_\rho^S$ w.r.t.

the variables x and s:

$$\partial_x \mathcal{L}_\rho^S(x,s,\lambda) = \partial q(x) + \frac{1}{\rho} c'(x)^* \left(c(x) + \lambda\rho - s\right), \tag{4.10a}$$

$$\partial_s \mathcal{L}_\rho^S(x,s,\lambda) = \mathcal{N}_K^{\lim}(s) - \frac{1}{\rho}\left(c(x) + \lambda\rho - s\right). \tag{4.10b}$$

The algorithm we are about to present requires, at each inner iteration, the (approximate) minimization of $\mathcal{L}_\rho^S(\cdot,\cdot,\lambda)$, given some $\rho > 0$ and $\lambda \in \mathbb{Y}$, while in each outer iteration, ρ and λ are updated. This nested-loops structure naturally arises in the augmented Lagrangian framework, as it does more generally in nonlinear programming.

In the following, we discuss why we reformulate the original problem (CP) by introducing slack variables $s \in K$, rather than consider (CP) directly. Given some $\rho > 0$, $x \in \mathbb{X}$, and $\lambda \in \mathbb{Y}$, in order to eliminate the slack variables, we first obtain the explicit minimization of $\mathcal{L}_\rho^S(x,\cdot,\lambda)$ by the set-valued mapping (due to the nonconvexity of K):

$$\operatorname{argmin}_s \mathcal{L}_\rho^S(x,s,\lambda) = \Pi_K\left(c(x) + \lambda\rho\right). \tag{4.11}$$

It turns out that the evaluation of the augmented Lagrangian (4.9) on the set corresponding to the explicit minimization over the slack variable s, is equal to the augmented Lagrangian function of (CP) denoted in (4.8):

$$\mathcal{L}_\rho(x,\lambda) = \min_s \mathcal{L}_\rho^S(x,s,\lambda) = q(x) + \frac{1}{2\rho}\operatorname{dist}_K^2\left(c(x) + \lambda\rho\right) - \frac{\rho}{2}\|\lambda\|^2. \tag{4.12}$$

One, from [30, Section 4.1], can easily check that the problems $\min L_\rho(\cdot,\lambda)$ and $\min \mathcal{L}_\rho^S(\cdot,\cdot,\lambda)$ are equivalent in the sense that $\bar{x}$ is a local (global) minimizer of $L_\rho(\cdot,\lambda)$ if and only if $(\bar{x},\bar{s})$, for each $\bar{s} \in \operatorname{argmin} \mathcal{L}_\rho^S(\bar{x},\cdot,\lambda)$, is a local (global) minimizer of $\mathcal{L}_\rho^S(\cdot,\cdot,\lambda)$, cf. (4.11). However, we highlight that the term $\operatorname{dist}_K^2 : \mathbb{Y} \to \mathbb{R}$ is not continuously differentiable in general, as the projection operator onto K is a set-valued mapping, thus making this approach difficult in practice. Therefore, we focus on the lifted programming ($\mathrm{CP_S}$) and need to find the approximate minimization of $\mathcal{L}_\rho^S(\cdot,\cdot,\lambda)$, cf. (4.9).

Remark 4.13. Whenever K is a convex set, the augmented Lagrangian function L_ρ from (4.8) or (4.12) is a continuously differentiable function with a locally Lipschitz continuous gradient, cf. Lemma 2.24. Following the literature, see e.g. [2, 34, 69], one can directly augment the corresponding set-membership constraints within the corresponding augmented Lagrangian framework without the need of additional slack variables. In practical implementations of an augmented Lagrangian framework addressing (CP), it is, thus, recommendable to treat only the difficult set-membership constraints with a nonconvex right-hand side with the aid of the lifting approach discussed here. The remaining set-membership constraints can either be augmented without slacks or remain explicitly in the constraint set of the augmented Lagrangian subproblems if simple enough (like box constraints).

4.3 Augmented Lagrangian Methods for Composite Structured Optimization Problems

This section presents an augmented Lagrangian method for the solution of composite programs with general nonconvex constraints in Section 4.3.1, and analyze the corresponding convergence results in Section 4.3.2.

4.3.1 Statement of the Algorithm

With the aid of a safeguarded augmented Lagrangian method under Assumption 4.1, we aim to solve (CP). The overall method is stated in Algorithm 4.3.1 and corresponds to the popular augmented Lagrangian solver Algencan from [3] applied to ($\mathrm{CP_S}$). Let us mention, however, that the analysis in [3] does neither cover composite objective functions $q := f + g$ nor constraints of the form $c(x) \in K$ with potentially nonconvex constraint set K.

Let us first give some comments on the starting point of Algorithm 4.3.1. In [70], a merely

Algorithm 4.3.1: Safeguarded Augmented Lagrangian Method for Composite Optimization Problems

Data: $\rho_0 > 0$, $\theta, \kappa \in (0, 1)$, and nonempty and bounded set $U \subset \mathbb{Y}$

1 **for** $k \leftarrow 0$ **to** ∞ **do**

2 Choose $u^k \in U$ and $\varepsilon_k \geq 0$;

3 Compute an ε_k-M-stationary point $(x^k, s^k) \in \mathbb{X} \times K$ of the subproblem

$$\min_{x,s} \ \mathcal{L}^S_{\rho_k}(\cdot, \cdot, u^k) \qquad \text{s.t.} \quad (x, s) \in \mathbb{X} \times K;$$

4 Set $\lambda^k := u^k + (c(x^k) - s^k)/\rho_k$;

5 **if** $k = 0$ **or** $\|c(x^k) - s^k\| \leq \theta \|c(x^{k-1}) - s^{k-1}\|$ **then**

6 $\rho_{k+1} := \rho_k$;

7 **else**

8 $\rho_{k+1} \in (0, \kappa \rho_k]$;

9 **end**

10 **end**

lower semicontinuous cost function has been considered and an augmented Lagrange method is employed to solve the resulting problems. Inspired by [79, Algorithm 1] and leveraging the idea behind [34, Example 4.12], the convergence properties of [70, Algorithm 1] hinge on the upper boundedness of the augmented Lagrangian along the iterates ensured by the initialization at a feasible point. Although possible in some cases, in general, finding a feasible starting point can be as hard as the original problem. Therefore, we deviate in this respect, seeking instead a method able to start from any $x^0 \in \mathbb{X}$, but the assumption of lower boundedness of the cost function q and the employment of slack variables are necessary, which can guarantee that the subproblems Line 3 generate approximate stationary points. Section 4.2 gave some theoretical statements that introducing slack variables could not change the solution of the original problem, furthermore contribute the problem to be handled much effectively, though the increased dimension of programs leads to a little time-consuming in some sense, but controllable.

As a result, the primal starting point is not necessarily feasible. Note that a primal-dual starting point is not explicitly required. In practice, however, the subproblems at Line 3 should be solved starting from the current primal estimate x^{k-1} paired with some s^{k-1}, preferably an element of $\Pi_K(c(x^{k-1}) + \rho_k u^k)$ as suggested by (4.11), thus exploiting initial guesses. As for the update rules of safeguarded dual multipliers λ^k and penalty parameters ρ_k, they are highly similar with that in Algorithm 3.4.1, hence we omit the corresponding analysis.

The augmented Lagrangian functions and subproblems discussed above appear at Line 3. Section 4.4 is devoted to the numerical solution of the subproblems, which are usually solved only approximately, in some sense, for the sake of computational efficiency. More precisely, the subproblem solver needs to be able to find ε-M-stationary points of $\mathcal{L}_\rho^S(\cdot,\cdot,u)$ for arbitrarily small $\varepsilon > 0$, $\rho > 0$, and $u \in U$.

Before proceeding to the convergence analysis, we highlight a different interpretation of the method. As first observed in [139], the augmented Lagrangian method on the primal problem has an associated proximal point method on the dual problem. Introducing the auxiliary variable $r \in \mathbb{Y}$, we rewrite the augmented Lagrangian subproblem $\min \mathcal{L}_\rho^S(\cdot,\cdot,\lambda)$ as

$$\min_{x,s,r} \quad q(x) + I_K(s) + \frac{1}{2\rho}\|r - \lambda\rho\|^2 \qquad \text{s.t.} \quad c(x) - s + r = 0$$

and then, by eliminating the slack variable s, as

$$\min_{x,r} \quad q(x) + \frac{1}{2\rho}\|r - \lambda\rho\|^2 \qquad \text{s.t.} \quad c(x) + r \in K.$$

The latter reformulation amounts to a proximal dual regularization of (CP) and corresponds to a lifted representation of $\min \mathcal{L}_\rho(\cdot,\lambda)$, where $\mathcal{L}_\rho$ is given in (4.8), thus showing that the approach effectively consists in solving a sequence of subproblems, each one being a proximally regularized version of (CP). Yielding feasible and more regular subproblems, this (proximal) regularization strategy has been explored in different contexts; some recent works are, e.g., [64, 133].

4.3.2 Convergence Analysis

Throughout our convergence analysis, we assume that Algorithm 4.3.1 is well-defined, thus requiring that each subproblem at Line 3 admits an approximate M-stationary point. Moreover, the following statements assume the existence of some accumulation point $\bar{x}$ or $(\bar{x},\bar{s})$ for a sequence $\{x^k\}$ or $\{(x^k,s^k)\}$, respectively, generated by Algorithm 4.3.1. In general, coercivity or (level) boundedness arguments should be adopted to verify this precondition, cf. Proposition 4.14 as well.

Due to their practical importance, we focus on affordable, or *local*, solvers, which return merely stationary points, for the subproblems at Line 3. Instead, we do not present results on the case where the subproblems are solved to *global* optimality. The analysis would follow the classical results in [34, Chapter 5] and [101], see [102, Section 6.2] as well. In summary, feasible problems would lead to feasible accumulation points that are global minima, in case of existence. For infeasible problems, infeasibility would be minimized and the objective cost would be minimized for the minimal infeasibility.

Like all penalty-type methods in the nonconvex setting, Algorithm 4.3.1 may generate accumulation points that are infeasible for (CP). Patterning standard arguments, the following result gives conditions that guarantee feasibility of limit points, cf. [34, Example 4.12], [94, Proposition 4.1].

Proposition 4.14. *Let Assumption 4.1 hold and consider a sequence $\{(x^k,s^k)\}$ of iterates generated by Algorithm 4.3.1. Then, each accumulation point $\bar{x}$ of $\{x^k\}$ is feasible for* (CP) *if one of the following conditions holds:*

a) *$\{\rho_k\}$ is bounded away from zero, or*
b) *there exists some $B \in \mathbb{R}$ such that $\mathcal{L}_{\rho_k}^S(x^k,s^k,u^k) \le B$ for all $k \in \mathbb{N}$.*

In both situations, $(\bar{x},c(\bar{x}))$ is an accumulation point of $\{(x^k,s^k)\}$ which is feasible to (CP_S)*.*

Proof. Let $\bar{x} \in \mathbb{X}$ be an arbitrary accumulation point of $\{x^k\}$ and $\{x^k\}_K$ a subsequence such that $x^k \to_K \bar{x}$. We need to show $c(\bar{x}) \in K$ under two circumstances.

a) If $\{\rho_k\}$ is bounded away from zero, the conditions at Lines 5 and 9 of Algorithm 4.3.1 imply that $\|c(x^k) - s^k\| \to 0$ for $k \to \infty$. Due to $s^k \in K$, one has $\|c(x^k) - s^k\| \geq \operatorname{dist}_K(c(x^k))$ for all $k \in \mathbb{N}$, taking the limit $k \to_{\mathcal{K}} \infty$ and continuity yield $\operatorname{dist}_K(c(\bar{x})) = 0$, hence $c(\bar{x}) \in K$, i.e., $\bar{x}$ is feasible to (CP). Further, $s^k \to_{\mathcal{K}} c(\bar{x})$ holds.
b) In case where $\{\rho_k\}$ is bounded away from zero, we have obtained the first statement. Thus, it remains to consider the case $\rho_k \to 0$. By assumption, we have

$$B \geq \mathcal{L}^S_{\rho_k}(x^k, s^k, u^k) = q(x^k) + \frac{1}{2\rho_k}\left\|c(x^k) + u^k\rho_k - s^k\right\|^2 - \frac{\rho_k}{2}\left\|u^k\right\|^2 \tag{4.13}$$

and $s^k \in K$ for all $k \in \mathbb{N}$. Rearranging terms yields the inequality

$$q(x^k) + \frac{1}{2\rho_k}\left\|c(x^k) + u^k\rho_k - s^k\right\|^2 \leq B + \frac{\rho_k}{2}\left\|u^k\right\|^2$$

for all $k \in \mathbb{N}$. Taking the lower limit $k \to_{\mathcal{K}} \infty$ while respecting that q is lsc and $\{u^k\}$ is bounded gives $\bar{x} \in \operatorname{dom} q$. Particularly, $\{q(x^k)\}_{\mathcal{K}}$ is bounded from below. Rearranging (4.13) yields

$$\left\|c(x^k) + u^k\rho_k - s^k\right\|^2 \leq 2\rho_k\left(B - q(x^k)\right) + \left\|u^k\rho_k\right\|^2,$$

and taking the upper limit $k \to_{\mathcal{K}} \infty$ yields $\|c(x^k) - s^k\| \to_{\mathcal{K}} 0$, again by boundedness of $\{u^k\}$ and $\rho_k \to 0$. On the other hand, $c(x^k) \to_{\mathcal{K}} c(\bar{x})$ follows by continuity, and this gives $s^k \to_{\mathcal{K}} c(\bar{x})$, since K is closed and $s^k \in K$ for all $k \in \mathbb{N}$. Hence, $(\bar{x}, c(\bar{x}))$ is feasible to (CP$_\text{S}$), i.e., $\bar{x}$ is feasible to (CP).

The final statement of the lemma follows from the above arguments. □

Constrained optimization algorithms aim at finding feasible points and minimizing the objective function subject to constraints. Employing affordable local optimization techniques, one cannot expect to find global minimizers of any infeasibility measure. Nevertheless, the next result proves that Algorithm 4.3.1 with bounded $\{\varepsilon_k\}$ finds stationary points of an infeasibility measure. Notice that this property does not require $\varepsilon_k \to 0$, but only boundedness, cf. [34, Theorem 6.3].

Proposition 4.15. *Let Assumption 4.1 hold and consider a sequence $\{(x^k, s^k)\}$ of iterates generated by Algorithm 4.3.1 with $\{\varepsilon_k\}$ bounded. Let $(\bar{x}, \bar{s})$ be an accumulation point of $\{(x^k, s^k)\}$ and $\{(x^k, s^k)\}_{\mathcal{K}}$ a subsequence such that $x^k \xrightarrow{q}_{\mathcal{K}} \bar{x}$ and $s^k \to_{\mathcal{K}} \bar{s}$. Then, $(\bar{x}, q(\bar{x}), \bar{s})$ is an M-stationary point of the feasibility problem*

$$\min_{(x,\alpha,s)\in \operatorname{epi} q \times K} \quad \tfrac{1}{2}\|c(x) - s\|^2. \tag{4.14}$$

If q is locally Lipschitz continuous at $\bar{x}$, then $\bar{x}$ is an M-stationary point of the constraint violation

$$\min_{(x,s)\in \mathbb{X}\times K} \quad \tfrac{1}{2}\|c(x) - s\|^2. \tag{4.15}$$

Proof. By Proposition 4.14 a), if $\{\rho_k\}$ is bounded away from zero, $\bar{x}$ is feasible for (CP) and $\bar{s} = c(\bar{x}) \in K$. Thus, $(\bar{x}, q(\bar{x}), c(\bar{x}))$ is a global minimizer of (4.14) and $(\bar{x}, c(\bar{x}))$ is a global minimizer of (4.15). By continuous differentiability of the objective function, M-stationarity w.r.t. both problems follows, see [122, Proposition 5.1]. Hence, it remains to consider the case $\rho_k \to 0$.

Owing to Line 3 of Algorithm 4.3.1, for all $k \in \mathbb{N}$ one has

$$\xi^k \in \partial q(x^k) + c'(x^k)^* \left(u^k + (c(x^k) - s^k)/\rho_k\right), \tag{4.16a}$$

$$\nu^k \in -\left(u^k + (c(x^k) - s^k)/\rho_k\right) + \mathcal{N}_K^{\lim}(s^k) \tag{4.16b}$$

for some $\xi^k \in \mathbb{X}$, $\|\xi^k\| \le \varepsilon_k$, and $\nu^k \in \mathbb{Y}$, $\|\nu^k\| \le \varepsilon_k$; cf. (4.10). Particularly, (4.16a) gives us

$$\left(\xi^k - c'(x^k)^* \left(u^k + (c(x^k) - s^k)/\rho_k\right), -1\right) \in \mathcal{N}_{\operatorname{epi} q}^{\lim}(x^k, q(x^k)).$$

Multiplying by $\rho_k > 0$ and exploiting that $\mathcal{N}_{\operatorname{epi} q}^{\lim}(x^k, q(x^k))$ is a cone, we have

$$\left(\rho_k \xi^k - c'(x^k)^* \left(c(x^k) + u^k \rho_k - s^k\right), -\rho_k\right) \in \mathcal{N}_{\operatorname{epi} q}^{\lim}(x^k, q(x^k)). \tag{4.17}$$

Furthermore, (4.16b) yields

$$\rho_k(\nu^k + u^k) + c(x^k) - s^k \in \mathcal{N}_K^{\lim}(s^k) \tag{4.18}$$

since $\mathcal{N}_K^{\lim}(s^k)$ is a cone. Taking the limit $k \to_{\mathcal{K}} \infty$ in (4.17) and (4.18), the robustness of the limiting normal cone, $x^k \xrightarrow{q}_{\mathcal{K}} \bar{x}$ and boundedness of $\{u^k\}$, $\{\xi^k\}$ and $\{\nu^k\}$ yield

$$\begin{aligned} (-c'(\bar{x})^*(c(\bar{x}) - \bar{s}), 0) &\in \mathcal{N}_{\operatorname{epi} q}^{\lim}(\bar{x}, q(\bar{x})), \\ c(\bar{x}) - \bar{s} &\in \mathcal{N}_K^{\lim}(\bar{s}). \end{aligned} \tag{4.19}$$

Keeping the Cartesian product rule for the computation of limiting normals in mind, see [122, Proposition 1.2], $(\bar{x}, q(\bar{x}), \bar{s})$ is an M-stationary point of (4.14).

Finally, assume that q is locally Lipschitz continuous at $\bar{x}$. Then, due to [122, Corollary 1.81], we have

$$(\bar{\lambda}, 0) \in \mathcal{N}_{\operatorname{epi} q}^{\lim}(\bar{x}, q(\bar{x})) \quad \Longrightarrow \quad \bar{\lambda} = 0,$$

so that the above arguments already show M-stationarity of $(\bar{x}, \bar{s})$ for (4.15). $\square$

In case where K is convex, the assertion of Proposition 4.15 can be slightly strengthened.

Corollary 4.16. *Let K be convex, let Assumption 4.1 hold and consider a sequence $\{(x^k, s^k)\}$ of iterates generated by Algorithm 4.3.1 with $\{\varepsilon_k\}$ bounded. Let $(\bar{x}, \bar{s})$ be an accumulation point of $\{(x^k, s^k)\}$ and $\{(x^k, s^k)\}_{\mathcal{K}}$ a subsequence such that $x^k \xrightarrow{q}_{\mathcal{K}} \bar{x}$ and $s^k \to_{\mathcal{K}} \bar{s}$. Then, $(\bar{x}, q(\bar{x}))$ is an M-stationary point of the feasibility problem*

$$\min_{(x,\alpha) \in \operatorname{epi} q} \quad \tfrac{1}{2} \operatorname{dist}_K^2(c(x)).$$

If q is locally Lipschitz continuous at $\bar{x}$, then $\bar{x}$ is an M-stationary point of the constraint violation

$$\min_{x \in \mathbb{X}} \quad \tfrac{1}{2} \operatorname{dist}_K^2(c(x)).$$

Proof. We proceed as in the proof of Proposition 4.15 in order to come up with (4.19). By convexity of K, $c(\bar{x}) - \bar{s} \in \mathcal{N}_K^{\lim}(\bar{s})$ is equivalent to $\bar{s} \in P_K(c(\bar{x}))$. Thus, the assertion follows from Lemma 2.24. $\square$

The following convergence result provides fundamental theoretical support to Algorithm 4.3.1. It shows that, under subsequential attentive convergence, any feasible accumulation point is an AM-stationary point for (CP).

Theorem 4.17. *Let Assumption 4.1 hold and consider a sequence* $\{(x^k, s^k)\}$ *of iterates generated by Algorithm 4.3.1 with* $\varepsilon_k \to 0$. *Let* $(\bar{x}, c(\bar{x}))$ *be an accumulation point of* $\{(x^k, s^k)\}$ *feasible to* (CP$_\mathrm{S}$) *and* $\{(x^k, s^k)\}_\mathcal{K}$ *a subsequence such that* $x^k \xrightarrow{q}_\mathcal{K} \bar{x}$ *and* $s^k \to_\mathcal{K} c(\bar{x})$. *Then,* $\bar{x}$ *is an AM-stationary point for* (CP).

Proof. Define $\zeta^k := c(x^k) - s^k$ for all $k \in \mathbb{N}$. Then, from Lines 3 and 4 of Algorithm 4.3.1, one has that

$$-c'(x^k)^*\lambda^k + \xi^k \in \partial q(x^k), \tag{4.20}$$

$$\lambda^k + \nu^k \in \mathcal{N}_K^{\lim}(c(x^k) - \zeta^k) \tag{4.21}$$

for some $\xi^k \in \mathbb{X}$, $\|\xi^k\| \leq \varepsilon_k$, and $\nu^k \in \mathbb{Y}$, $\|\nu^k\| \leq \varepsilon_k$. Set $y^k := \lambda^k + \nu^k$ and $\eta^k := c'(x^k)^*\nu^k + \xi^k$ for all $k \in \mathbb{N}$.

We claim that the four subsequences $\{x^k\}_\mathcal{K}$, $\{\eta^k\}_\mathcal{K}$, $\{\lambda^k\}_\mathcal{K}$ and $\{\zeta^k\}_\mathcal{K}$ satisfy the properties in Definition 4.4 and therefore show that $\bar{x}$ is an AM-stationary point for (CP).

By construction, one has $x^k \xrightarrow{q}_\mathcal{K} \bar{x}$ as well as $-c'(x^k)^*y^k + \eta^k \in \partial q(x^k)$ and $y^k \in \mathcal{N}_K^{\lim}(c(x^k) - \zeta^k)$ for each $k \in \mathbb{N}$. From the fact that c is continuously differentiable and $\|\xi^k\|, \|\nu^k\| \leq \varepsilon_k$, one has $\|\eta^k\| \to_\mathcal{K} 0$. Finally, $\zeta^k \to_\mathcal{K} 0$ follows from $s^k \to_\mathcal{K} c(\bar{x})$, $x^k \to_\mathcal{K} \bar{x}$ and continuity of c.

Overall, $\bar{x}$ is an AM-stationary point for (CP). □

The additional assumption $x^k \xrightarrow{q}_\mathcal{K} \bar{x}$ in Theorem 4.17 is trivially satisfied if g is continuous on its domain since all iterates of Algorithm 4.3.1 belong to $\operatorname{dom} g$. However, the following one-dimensional example illustrates how this additional requirement appears to be indispensable in a discontinuous setting.

Example 4.18. We consider $n := m := 1$ and set $K := (-\infty, 0]$,

$$\forall x \in \mathbb{R}\colon \quad f(x) := 0, \qquad g(x) := \begin{cases} x & \text{if } x \leq 0, \\ 1 - x & \text{otherwise,} \end{cases} \qquad c(x) := x.$$

Note that g is merely lsc at $\bar{x} := 0$, and that $\partial g(\bar{x}) = [1, \infty)$, cf. Figure 4.1. Although $\bar{x}$ is the global maximizer of the associated problem (CP), $\bar{x}$ is not an M-stationary point. Since $\nabla f(\bar{x}) = 0$, $\nabla c(\bar{x}) = 1$ and $\mathcal{N}_K^{\lim}(c(\bar{x})) = \mathbb{R}_+$, there is no $\bar{\lambda} \in \mathcal{N}_K^{\lim}(c(\bar{x}))$ such that $0 \in \nabla f(\bar{x}) + \partial g(\bar{x}) + c'(\bar{x})^*\bar{\lambda}$. Indeed, $\bar{x}$ is not even AM-stationary. Possibly discarding early iterates, any sequence $\{x^k\}$ such that $x^k \xrightarrow{q} \bar{x}$ satisfies $x^k \leq 0$ for each $k \in \mathbb{N}$. Hence, we find $\partial q(x^k) \subset [1, \infty)$, $\nabla c(x^k) = 1$ and $\mathcal{N}_K^{\lim}(c(x^k) - \zeta^k) \subset \mathbb{R}_+$ for each $\zeta^k \in \mathbb{Y}$ and $k \in \mathbb{N}$, showing that the distance between 0 and the set $\partial q(x^k) + c'(x^k)^*\mathcal{N}_K^{\lim}(c(x^k) - \zeta^k)$ is at least 1.

We apply Algorithm 4.3.1 with $U := \{0\}$, $\rho_0 := 1$, $\theta := 1/4$, and $\kappa := 1/2$. This may yield sequences $\{x^k\}$, $\{s^k\}$ and $\{\rho_k\}$ given by $x^0 := \rho_0$, $s^0 := 0$, $x^k := \rho_k := 2^{1-k}$ and $s^k := 0$ for each $k \in \mathbb{N}$, $k \geq 1$, cf. Figure 4.2. Hence, one has $x^k \to \bar{x}$ and, crucially, not $x^k \xrightarrow{q} \bar{x}$.

The next result readily follows from Corollary 4.8 and Theorem 4.17.

Corollary 4.19. *Let Assumption 4.1 hold and consider a sequence* $\{(x^k, s^k)\}$ *of iterates generated by Algorithm 4.3.1 with* $\varepsilon_k \to 0$. *Let* $(\bar{x}, c(\bar{x}))$ *be an accumulation point of* $\{(x^k, s^k)\}$ *feasible to* (CP$_\mathrm{S}$) *and* $\{(x^k, s^k)\}_\mathcal{K}$ *a subsequence such that* $x^k \xrightarrow{q}_\mathcal{K} \bar{x}$ *and* $s^k \to_\mathcal{K} c(\bar{x})$. *Furthermore, assume that* $\bar{x}$ *is AM-regular for* (CP). *Then,* $\bar{x}$ *is an M-stationary point for* (CP).

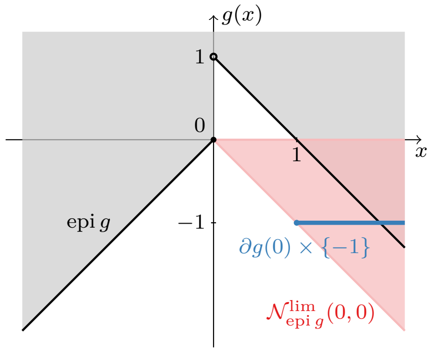

Figure 4.1: Computation of $\partial g(0)$.

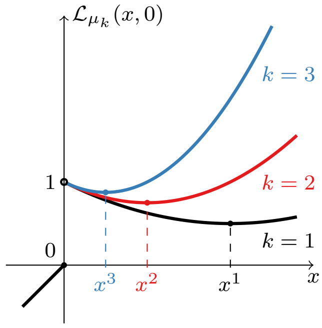

Figure 4.2: Iterates x^k for $k \in \{1,2,3\}$.

Figure 4.3: Visualizations for Example 4.18.

4.4 Subproblem Solvers

In this section we elaborate upon Line 3 of Algorithm 4.3.1 that aims at minimizing the augmented Lagrangian function $\mathcal{L}^S_\rho(\cdot,\cdot,u)$ defined in (4.9) (setting $\lambda := u$). To this end, let us take a closer look at the structure of this subproblem.

Using the decomposition $\mathcal{L}^S_\rho(\cdot,\cdot,u) = f^{\mathrm{S}}(\cdot,\cdot) + g^{\mathrm{S}}(\cdot,\cdot)$ with component functions $f^{\mathrm{S}} : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ and $g^{\mathrm{S}} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ given by

$$f^{\mathrm{S}}(x,s) := f(x) + \frac{1}{2\rho}\|c(x) + u\rho - s\|^2 - \frac{\rho}{2}\|u\|^2, \tag{4.22}$$

$$g^{\mathrm{S}}(x,s) := g(x) + I_K(s), \tag{4.23}$$

one immediately sees that this split recovers the classical setting of an *unconstrained* composite optimization problem with f^{S} being continuously differentiable, while g^{S} is merely lsc, but of a particular structure. In principle, proximal gradient-type methods can therefore be applied as approximate solvers for our subproblems, see [21] for an introduction of this class of methods. Let us also mention that, at least in [95], it has been verified that accumulation points of sequence generated by proximal gradient-type methods are M-stationary while along the associated subsequence, the iterates are ε_k-M-stationary for a null sequence $\{\varepsilon_k\}$. This requirement is essential in Algorithm 4.3.1. A standing assumption of the corresponding convergence theory in [21] and all previous works on proximal gradient-type methods, is a global Lipschitz condition regarding the gradient of the smooth part f^{S}. Note that this gradient is given by

$$\nabla f^{\mathrm{S}}(x,s) = \begin{bmatrix} \nabla f(x) + \frac{1}{\rho}c'(x)^*\left(c(x) + u\rho - s\right) \\ -\frac{1}{\rho}\left(c(x) + u\rho - s\right) \end{bmatrix}.$$

some recent contributions on proximal gradient-type methods show that these methods also work under suitable assumptions if the smooth term has a locally Lipschitz gradient only; cf. [18, 67, 93, 95] for more details. Hence, the standing assumptions from Assumption 4.1 is enough for the associated convergence analysis, which implies that the gradient is locally (not globally) Lipschitz continuous. For a practical implementation of these proximal methods, it is advantageous to exploit the nonsmooth term g^{S}. In fact, due to the separability of g^{S} with respect to x and s, it follows that the corresponding proximal mapping is easily computable. More precisely, one obtains

$$\operatorname{prox}_{\gamma g^{\mathrm{S}}}(x,s) = \begin{bmatrix} \operatorname{prox}_{\gamma g}(x) \\ \Pi_K(s) \end{bmatrix}$$

for any $\gamma \in (0, \gamma_g)$.

A special proximal gradient-type approach from [67], called PANOC$^+$, is applied to solve the resulting subproblems and used in our numerical setting (see Section 4.5 for more details). In Section 4.4.1, it is recalled and the corresponding convergence results are given. In addition, Section 4.4.2 discusses more about the other solution solvers for the subproblems.

4.4.1 PANOC$^+$ as Subproblem Solver

For simplicity of notation, let us consider the abstract unconstrained, composite optimization problem

$$\min_{z\in\mathbb{Z}} \quad \omega(z) \coloneqq \varphi(z) + \psi(z) \tag{Q}$$

where $\mathbb{Z}$ is an Euclidean space, φ, ψ, and ω satisfy the following standing assumptions.

Assumption 4.20. The following hold in (Q):

a) $\varphi : \mathbb{Z} \to \mathbb{R}$ is continuously differentiable with locally Lipschitz continuous gradient;
b) $\psi : \mathbb{Z} \to \overline{\mathbb{R}}$ is proper, lower semicontinuous and prox-bounded with threshold $\gamma_\psi > 0$;
c) $\inf_{z\in\mathbb{Z}} \omega(z) > -\infty$.

In order to make the optimization problem (Q) and Line 3 of Algorithm 4.3.1 equivalent, the following settings are necessary: $z := (x,s)$, $\mathbb{Z} := \mathbb{X} \times \mathbb{Y}$, $\omega := \mathcal{L}^S_\rho$, $\varphi := f^S$, and $\phi := g^S$ as in (4.22) and (4.23). Moreover, we introduce a set-valued mapping $\mathcal{T}_\gamma : \mathbb{Z} \to \mathbb{Z}$ for arbitrary $\gamma \in (0, \gamma_\psi)$ by means of

$$\mathcal{T}_\gamma(z) \coloneqq \operatorname{prox}_{\gamma\psi}(z - \gamma\nabla\varphi(z)). \tag{4.24}$$

Furthermore, the algorithm makes use of the so-called *forward-backward envelope* (FBE) relative to (Q) with stepsize $\gamma \in (0, \gamma_\psi)$ given by

$$\omega^{\mathrm{FB}}_\gamma(z) \coloneqq \min_{w\in\mathbb{Z}} \varphi(z) + \langle \nabla\varphi(z), w - z\rangle + \psi(w) + \tfrac{1}{2\gamma}\|w - z\|^2.$$

Clearly, for any $\bar z \in \mathcal{T}_\gamma(z)$, one has

$$\omega^{\mathrm{FB}}_\gamma(z) = \varphi(z) + \langle \nabla\varphi(z), \bar z - z\rangle + \psi(\bar z) + \tfrac{1}{2\gamma}\|\bar z - z\|^2. \tag{4.25}$$

The pseudo code for PANOC$^+$ will be provided in Algorithm 4.4.1, whose peculiarity is the intricate structure emerging at Lines 6 and 12. The two backtracking linesearches are entangled, concurrently affecting both the direction stepsize τ_k and the proximal stepsize γ_k. These persistent adjustments allow PANOC$^+$ to construct a tighter merit function $\omega^{\mathrm{FB}}_\gamma$ that better captures the (local) landscape of ω, obviating the need for global Lipschitz gradient continuity for the smooth term in (Q).

The analysis in [67] provides global convergence guarantees for PANOC$^+$ under Assumption 4.20. Let us recall the basic result associated with Algorithm 4.4.1 that is important in the context of Algorithm 4.3.1. For the reader's convenience, we present a brief proof of the result as it is not explicitly stated in [67].

Algorithm 4.4.1: PANOC$^+$ [67]

Data: $z^0 \in \mathbb{Z}$, $\gamma_0 \in (0, \gamma_\psi)$, $\Delta \geq 0$, $\alpha, \beta \in (0,1)$, $\varepsilon > 0$

```
1  k ← 0, and start from Line 5;
2  γ_k ← γ_{k−1};
3  Select an update direction d^k ∈ Z with ‖d^k‖ ≤ Δ‖z̄^{k−1} − z^{k−1}‖ and set τ_k = 1;
4  Set z^k = (1 − τ_k) z̄^{k−1} + τ_k (z^{k−1} + d^k);
5  Compute z̄^k ∈ T_{γ_k}(z^k) and set Φ_k := ω^{FB}_{γ_k}(z^k) as in (4.25);
6  if φ(z̄^k) > φ(z^k) + ⟨∇φ(z^k), z̄^k − z^k⟩ + α/(2γ_k) ‖z̄^k − z^k‖² then
7  |  γ_k ← γ_k/2, and go back to Line 3 if k > 0, or Line 5 if k = 0;
8  end
9  if ‖(1/γ_k)(z̄^k − z^k) − ∇φ(z̄^k) + ∇φ(z^k)‖ ≤ ε then
10 |  return z̄^k;
11 end
12 if k > 0 AND Φ_k > Φ_{k−1} − β (1−α)/(2γ_{k−1}) ‖z̄^{k−1} − z^{k−1}‖² then
13 |  τ_k ← τ_k/2 and go back to Line 4;
14 end
15 k ← k + 1 and start the next iteration at Line 2;
```

Proposition 4.21. *Let $\{z^k\}$ and $\{\bar{z}^k\}$ be sequences generated by Algorithm 4.4.1. Furthermore, let z^* be an accumulation point of $\{z^k\}$ and $\{z^k\}_\mathcal{K}$ a subsequence such that $z^k \to_\mathcal{K} z^*$. Then, z^* is an M-stationary point of ω. Additionally, $\bar{z}^k \to_\mathcal{K} z^*$ holds, and for each $\varepsilon > 0$ and any large enough $k \in \mathcal{K}$, $\bar{z}^k$ is an ε-M-stationary point of ω.*

Proof. Owing to [67, Thm 4.3], one has $\bar{z}^k \to_\mathcal{K} z^*$, and $\gamma_k = \gamma$ holds for some $\gamma > 0$ and large enough $k \in \mathcal{K}$. Furthermore, this result gives

$$\Phi_k = \varphi(z^k) + \left\langle \nabla\varphi(z^k), \bar{z}^k - z^k \right\rangle + \psi(\bar{z}^k) + \tfrac{1}{2\gamma_k}\|\bar{z}^k - z^k\|^2,$$

so that taking the lower limit $k \to_\mathcal{K} \infty$ yields $z^* \in \operatorname{dom}\psi$ due to Line 12, Line 13, and Line 14 of Algorithm 4.4.1. Next, Line 5 of Algorithm 4.4.1 yields from $z^k \to_\mathcal{K} z^*$ and continuity of ϕ that

$$\begin{aligned}
\omega(z^*) &\leq \liminf_{k \to_\mathcal{K} \infty} \Phi_k \\
&\leq \liminf_{k \to_\mathcal{K} \infty} \left(\varphi(z^k) + \left\langle \nabla\varphi(z^k), z^* - z^k \right\rangle + \psi(z^*) + \tfrac{1}{2\gamma_k}\|z^* - z^k\|^2 \right) \\
&\leq \limsup_{k \to_\mathcal{K} \infty} \left(\varphi(z^k) + \left\langle \nabla\varphi(z^k), z^* - z^k \right\rangle + \psi(z^*) + \tfrac{1}{2\gamma_k}\|z^* - z^k\|^2 \right) \\
&= \omega(z^*),
\end{aligned}$$

which gives $\bar{z}^k \overset{\omega}{\to}_\mathcal{K} z^*$ by continuity of φ. Considering the stationarity condition resulting from evaluation of the proximal map $\mathcal{T}_{\gamma_k}$,

$$0 \in \nabla\varphi(z^k) + \partial\psi(\bar{z}^k) + \tfrac{1}{\gamma_k}(\bar{z}^k - z^k)$$

holds for each $k \in \mathcal{K}$, giving

$$\tfrac{1}{\gamma_k}(z^k - \bar{z}^k) + \nabla\varphi(\bar{z}^k) - \nabla\varphi(z^k) \in \nabla\varphi(\bar{z}^k) + \partial\psi(\bar{z}^k) = \partial\omega(\bar{z}^k).$$

Taking the limit $k \to_{\mathcal{K}} \infty$ while respecting continuous differentiability of φ, the result follows. □

Let us recall again the resulting subproblem Line 3 of Algorithm 4.3.1, Proposition 4.21 implies PANOC$^+$ is available to obtain the corresponding approximate M-stationary point, which gives exactly what we want.

Finally, in order to serve for Section 4.5, we shall comment on the acceleration mechanism in PANOC$^+$. Although robust to arbitrary choices of (bounded) directions d^k, the practical performance of Algorithm 4.4.1 is strongly affected by the specific selection; we refer to [149, Section 4.3] for an overview on some potential update directions. In the numerical experiments, we consider two strategies for executing Line 3 of Algorithm 4.4.1. First, we may select $d^k := \bar{z}^{k-1} - z^{k-1}$, so that $z^k = \bar{z}^{k-1}$ holds, effectively reducing the algorithm to an adaptive proximal gradient method, without any acceleration [67, Section 4.4]. Second, as a baseline, we use the default acceleration strategy in ProximalAlgorithms.jl, namely LBFGS directions with memory 5. Inspired by quasi-Newton methods, these are recursively constructed by keeping memory of pairs $z^{k+1} - z^k$ and $r^{k+1} - r^k$, with $r^k := z^k - \bar{z}^k$, and retrieving $d^k := -H^k r^k$ by simply performing scalar products [111]. Herein, the linear operator H_k mimics the (inverse) fixed-point residual mapping associated to the splitting scheme in a neighborhood of z^k [148, 150]. Notice that, as the geometry of the residual mapping depends on the proximal stepsize, (the memory of) the LBFGS approximation is reset every time the stepsize is adapted [67, Section 3.1].

4.4.2 Comments on other subproblem solvers

We stress that there exist other candidates for the numerical solution of the resulting augmented Lagrangian subproblems. To this end, recall that the previous discussion looked at these subproblems as an unconstrained composite optimization problem. Alternatively, we may view these subproblems from the point of view of machine learning, where (essentially) the same class of optimization problems is solved by (possibly) different techniques. We refer the interested reader to [146,155] for a survey of optimization methods for machine learning and data analysis problems. These techniques might be applicable very successfully at least in certain situations. For example, if the smooth term f^{S} is convex (the gradient does not have to be globally Lipschitz), whereas the nonsmooth term g^{S} is still just assumed to be lsc (and not necessarily convex), it is possible to adapt the idea of cutting plane methods to this setting by applying the cutting plane technique to f^{S} only, whereas one does not change the nonsmooth term. The resulting subproblems then use a piecewise affine lower bound for the function f^{S} and add the (possibly complicated) function g^{S}. Of course, and similar to the proximal gradient-type approaches, these subproblems need to be easily solvable for the overall augmented Lagrangian method to be efficient, and this, in general, is true only for particular classes of problems.

4.5 Numerical Results

This section is concerned to implement Algorithm 4.3.1, based on the resulting subproblem solver PANOC$^+$ as well as its accelerated version (see Section 4.4.1) and test it on some classes of different problems. In all examples, the space $\mathbb{X}$ is given by $\mathbb{R}^n$ and $\mathbb{Y}$ is given by $\mathbb{R}^m$.

Algorithm 4.3.1 runs by requiring the data functions f, g, c and constraint set K specified as objects returning the oracles discussed at the beginning of this chapter. As mentioned in Section 4.3.1, the primal starting points are not necessarily feasible for (CP).

Hence, we choose the initialization by requiring a primal starting point $x^{\text{init}} \in \mathbb{X}$ with an arbitrary element of $\operatorname{prox}_{\gamma g}(x^{\text{init}}) \subset \operatorname{dom} q$, where $\gamma = \epsilon_M$ and ϵ_M denotes the machine epsilon of a given floating-point system. The examples presented in the following are in double precision (Float64), so $\epsilon_M \approx 2.22 \cdot 10^{-16}$. Moreover, the safeguarded estimate $u^k (k > 0)$ is chosen as the projection of λ^{k-1} onto a given box $U := [u_{\min}, u_{\max}]$, where this box is chosen to be $[-10^{20}, 10^{20}]$ for all equality constraints and $[0, 10^{20}]$ for all inequality constraints. Note that u^0 is chosen as the projection of dual starting point λ^{init} on U, where $\lambda^{\text{init}} \in \mathbb{R}^m$ is chosen arbitrarily.

The inner tolerances ε_k at Line 2 of Algorithm 4.3.1 are constructed as a sequence of decreasing values, defined by

$$\varepsilon_{k+1} = \max\{\kappa_\varepsilon \varepsilon_k, \varepsilon^{\text{dual}}\},$$

starting from $\varepsilon_0 := (\varepsilon^{\text{dual}})^{\frac{1}{3}}$ and given some $\varepsilon^{\text{dual}}, \kappa_\varepsilon \in (0, 1)$ [33]. The initial penalty parameter ρ_0 is automatically chosen by default, similarly to [34, Equation 12.1]. Given $x^{\text{init}} \in \operatorname{dom} q$, we evaluate the constraints $c^{\text{init}} := c(x^{\text{init}})$, select an arbitrary element $s^{\text{init}} \in \Pi_K(c^{\text{init}})$ and compute the vector $\Delta^{\text{init}} := c^{\text{init}} - s^{\text{init}}$. Then, the vector $\rho_0 \in \mathbb{Y}$ of penalty parameters is selected componentwise as follows:

$$(\rho_0)_i := \max\left\{10^{-8}, \min\left\{\frac{1}{10}\frac{\max\{1, (\Delta_i^{\text{init}})^2/2\}}{\max\{1, q(x^{\text{init}})\}}, 10^8\right\}\right\},$$

effectively scaling the contribution of each constraint [34,62]. Then, according to the overall feasibility-complementarity of the iterate, the penalty parameters are updated in unison at Line 9, since using a different penalty parameter for each constraint is theoretically worse than using a common parameter [5, Section 3.4]. We set $\rho_{k+1} := \kappa\rho_k$, for some fixed $\kappa \in (0, 1)$. At the kth iteration, the subsolver at Line 3 is warm-started from the previous estimate $(x^{k-1}, s^{k-1}) \in \operatorname{dom} q \times K$; from $(x^{\text{init}}, s^{\text{init}})$ for $k = 0$. The default parameters in Algorithm 4.3.1 are $\theta = 0.8$, $\kappa = 0.5$ and $\kappa_\varepsilon = 0.1$, termination tolerances $\varepsilon^{\text{prim}} = \varepsilon^{\text{dual}} = 10^{-6}$ and a maximum number of (outer) iterations is 100.

In the following, we consider some challenging problems where the cost function is nonsmooth and nonconvex or where the constraints are inherently nonconvex by a disjunctive structure of the respective set K. Section 4.5.1 deals with some collections of signal recovery problems with different classes of measurement matrices and signals, where the minimization of zero norm is considered. The results show that PANOC$^+$ has difficulty in dealing with bad scaling and ill-conditioning. Subsequently, Section 4.5.2 demonstrates the benefit of accelerated PANOC$^+$ for solving the subproblems by means of a simple two-dimensional problem where a nonsmooth variant of the Rosenbrock function is minimized over a set of combinatorial structure. Section 4.5.3 is devoted to a test collection of portfolio optimization problems from [74] which are equipped with a nonconvex sparsity-promoting term in the objective function. Finally, in Section 4.5.4 we address a class of matrix recovery problems discussed e.g. in [143] where the rank of the unknown matrix has to be minimized.

I claim that the numerical experiments in Section 4.5.2 is independently done by my co-worker Alberto De Marchi, thank him for allowing me to show this part for the completeness of the numerical results.

4.5.1 Sparse Signal Recovery

Sparse signal recovery problems are concerned to recover a r-sparse signal $x^* \in \mathbb{R}^n$ from relatively few incomplete measurements $b = Ax^*$ with carefully chosen $A \in \mathbb{R}^{m\times n}$, and solve the following optimization problem

$$\min_{x\in\mathbb{R}^n} \ \|x\|_0 \quad \text{s.t. } Ax = b, \tag{4.26}$$

where $\|x\|_0$ is the number of nonzero x, $r \leq m \leq n$ (always $r \ll m \ll n$). It can be reformulated into (CP) by setting $f := 0$ and $K := \{0\}^m$, which means slack variables s vanish.

Next, we implement Algorithm 4.3.1 invoking PANOC$^+$ by the following randomly generated problems. Given the dimension n of a signal, the number of observations m and the number of nonzero r, we generated a random measurement matrix $A \in \mathbb{R}^{m\times n}$ and a random signal $x^* \in \mathbb{R}^n$ in the way as [32, 152]. More specifically, we first generated the matrix A by one of the following types:

Type 1: Gaussian matrix whose elements are generated independently and identically distributed from the normal distribution $N(0, 1)$;

Type 2: Orthogonalized Gaussian matrix whose rows are orthogonalized using a QR decomposition;

Type 3: Bernouli matrix whose elements are ± 1 independently with equal probability.

We then randomly selected m rows from this matrix to construct the matrix A. In order to generate the signal x^*, we first generated randomly r between 1 and n, and then assigned a value to the component x_i^* for each $i \in \{1, \cdots, r\}$ by one of the following methods:

Type 1: A normally distributed random variable (Gaussian signal);

Type 2: A uniformly distributed random variable in $(-1, 1)$;

Type 3: One (zero-one signal);

Type 4: A signal x with power-law decaying entries (known as compressible sparse signals) whose components satisfy

$$|x_i| \leq c_x i^{-p} \quad \text{with} \quad c_x = 10^5,\ p = 1.5;$$

Type 5: A signal x with exponential decaying entries whose components satisfy

$$|x_i| \leq c_x e^{-pi} \quad \text{with} \quad c_x = 1,\ p = 0.005.$$

Then, the observation b was computed as $b = Ax^*$. Note that all types of matrix were stored explicitly. We call a signal recovered successfully by a solver if the relative error between the solution x^s generated and the original signal x^* is less than 5×10^{-5}.

In order to detect whether or not Algorithm 4.3.1 solves (4.26) successfully, as well as the influence of the number of measurements m on the recovery of Algorithm 4.3.1 (since too small m is easier to lead to the ill-conditioning which causes Algorithm 4.3.1 not effective, even failed), we choose all types of matrix to match all types of signal by setting $r = 40$, $n = 600$, $m \in \{80, 90, 100, \cdots, 260\}$. For each m, we generated 50 problems randomly, and test the frequency of successful recovery for Algorithm 4.3.1 invoking PANOC$^+$. The results are depicted in Figure 4.4. Let us mention that Algorithm 4.3.1 solved all problem instances around average 45 iterations, however subproblem solver PANOC$^+$ always terminated at the maximum number limit of inner iterations (here we set 1000) for the problems where $m < 170$ and terminated within 100 inner iterations when $m \geq 170$. Figure 4.4 shows that the signal with different types is sensitive with different matrices, more specifically, for

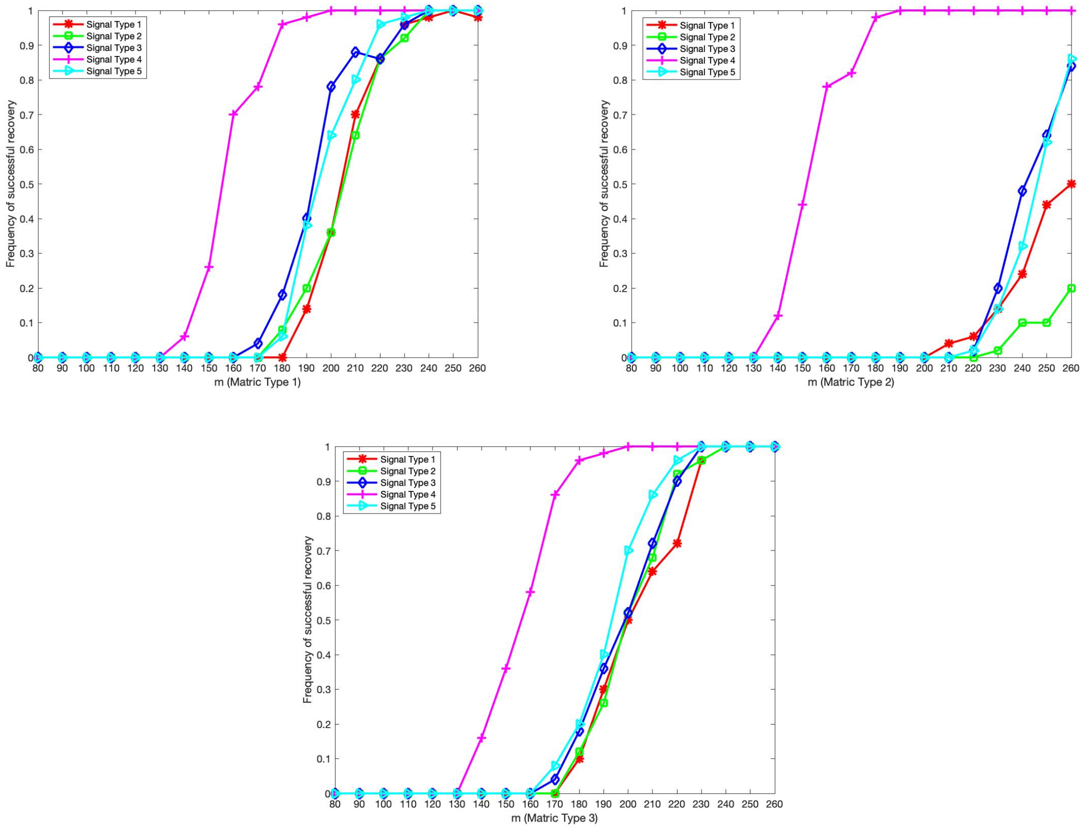

Figure 4.4: Frequency of successful recovery for all types of matrices matched with all types of signals.

Matrix 1 and Matrix 3, our algorithm can recover every type of signals effectively when $m \geq 180$ and $m \geq 170$, respectively, which, by contrast, has a bad performance on Matrix 2. Generally speaking, Signal 4 is easier to be recovered by our algorithm.

From [44, Lemma 2], (4.26) is equivalent to the following mixed integer optimization problem

$$\min_{\substack{x \in \mathbb{R}^n \\ v \in \{0,1\}^n}} \sum_{i=1}^{n} v_i \quad \text{s.t.} \quad Ax = b, \ -Mv \leq x \leq Mv$$

with some sufficiently large pre-defined value $M > 0$ satisfying $\|x^*\|_\infty \leq M$, which can be solved by CPLEX optimizer [91]. Hence, we now compare Algorithm 4.3.1 invoking PANOC$^+$ with CPLEX by some random problems, where Matrix 1 and 3 were employed and we chose $n = 600, m \in \{250, 260\}, r = 40$ in order to ensure that those random problems can be solved Algorithm 4.3.1 from the result of Figure 4.4. Note that the termination time of CPLEX is set as 30 minutes for each problems, the results are depicted in Table 4.1. The testproblems are named as $M_i S_j$ with $i \in \{1, 3\}$ and $j \in \{1, \ldots, 5\}$, for example, $M_1 S_1$ represents that Matrix 1 and Signal 1 are combined for a testproblem. f_{ALM} and f_{CPLEX} denote the optimal function value generated by Algorithm 4.3.1 and CPLEX, respectively, which we know should be equal to $r = 40$ if the signal is recovered successfully. i means the number of (outer) iterations and i_{cum} denotes total accumulated number of iterations (sum of the outer and inner iterations), which can evaluate the performance of PANOC$^+$ in some sense.

We claim that Algorithm 4.3.1 recovered signals successfully for all testproblems,

m	problem	f_{CPLEX}	f_{ALM}	i	i_{cum}
250	M_1S_1	246	40	22	1732
	M_1S_2	40	40	16	1732
	M_1S_3	40	40	23	1932
	M_1S_4	40	40	42	6239
	M_1S_5	40	40	15	1570
	M_3S_1	246	40	11	1444
	M_3S_2	40	40	24	2272
	M_3S_3	40	40	22	1682
	M_3S_4	40	40	50	7155
	M_3S_2	40	40	1	790
260	M_1S_1	252	40	17	1944
	M_1S_2	40	40	11	1457
	M_1S_3	40	40	1	901
	M_1S_4	40	40	48	8157
	M_1S_5	40	40	23	1899
	M_3S_1	256	40	17	1549
	M_3S_2	40	40	20	1375
	M_3S_3	40	40	52	6822
	M_3S_4	40	40	43	6686
	M_3S_5	40	40	1	461

Table 4.1: Numerical results generated by Algorithm 4.3.1 and CPLEX, respectively.

however CPLEX can not recover Signal 1. The number of outer iteration is average 23, and the accumulated number of iteration shows that PANOC$^+$ did not terminate at the maximum for every inner iteration, even performed well, which is better than the case $m < 170$ as mentioned above. Furthermore, CPLEX always terminated in 30 seconds for Signal 3 and 5.

All results illustrate that Algorithm 4.3.1 can solve (4.26) where different types of measurement matrix and signal are chosen, however we have to admit that PANOC$^+$ has difficulty in dealing with the bad-scaled and ill-conditioned programs. If PANOC$^+$ solves the subproblems effectively, then Algorithm 4.3.1 invoking PANOC$^+$ performs well, even better than CPLEX optimizer sometimes. The next example will compare PANOC$^+$ with its accelerated version to demonstrate that the later can deal with the bad-scaling and ill-conditioning in some sense.

4.5.2 Nonsmooth Rosenbrock and Either-Or Constraints

Let us consider a two-dimensional optimization problem involving a nonsmooth Rosenbrock-like objective function and either-or constraints, namely set-membership constraints entailing an inclusive disjunction. It reads

$$\min_x \; 10\left(x_2 + 1 - (x_1+1)^2\right)^2 + |x_1| \qquad \text{s.t.} \quad x_2 \le -x_1 \;\vee\; x_2 \ge x_1 \tag{4.27}$$

and admits a unique (global) minimizer $x^* = (0,0)$. The feasible set is nonconvex and connected; see Figure 4.5. One casts (4.27) into the form of (CP) by defining the data

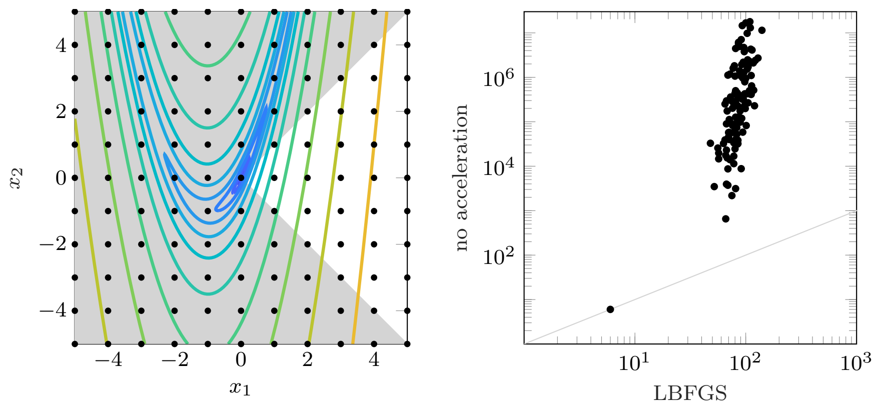

Figure 4.5: Setup and results for the illustrative problem (4.27). Left: Feasible region (gray background), objective contour lines, global minimizer $x^* = (0, 0)$ and grid of starting points. Right: Comparison of inner iterations needed without acceleration against LBFGS acceleration; each mark corresponds to a starting point and the gray line has unitary slope.

functions as

$$f(x) \coloneqq 10\left(x_2 + 1 - (x_1 + 1)^2\right)^2, \qquad g(x) \coloneqq |x_1|, \qquad c(x) \coloneqq \begin{pmatrix} -x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix},$$

and let the constraint set be $K \coloneqq K_{EO}$, where the (nonconvex) set

$$K_{EO} \coloneqq \{(a, b) \mid a \geq 0 \vee b \geq 0\} = \{(a, b) \mid a \geq 0\} \cup \{(a, b) \mid b \geq 0\}$$

describes the either-or constraint.

We consider a uniform grid of $11^2 = 121$ starting points x^0 in $[-5, 5]^2$ and let the initial multiplier estimate be $\lambda^0 = 0$. Also, we compare the performance of Algorithm 4.3.1 by solving the subproblems using PANOC$^+$ without or with (LBFGS) acceleration, see the last paragraph of Section 4.4.1 for more details.

Algorithm 4.3.1 solves all the problem instances, approximately (tolerance 10^{-3} in Euclidean distance) reaching $\bar{x} = (0, 0)$ in all cases. Figure 4.5 depicts the feasible region of (4.27), some contour lines of its objective function and the grid of starting points x^0. Over all problems, Algorithm 4.3.1 with no acceleration takes at most 17 870 346 (cumulative) inner iterations to find a solution (median 291 756), whereas with LBFGS directions only 140 inner iterations are needed at most (median 86). A closer look at Figure 4.5 indicates that not only the accelerated PANOC$^+$ usually requires far less iterations, but also that its behavior is more consistent, as the majority of cases spread over a narrow interval. These results support the claim that (quasi-Newton) acceleration techniques can give a mean to cope with bad scaling and ill-conditioning [148, 149], meanwhile can generate approximate M-stationary point. Hence, we in the following use the accelerated PANOC$^+$ rather than the original one.

4.5.3 Sparse Portfolio Optimization

Inspired by Section 3.6.2, we consider the following portfolio optimization problems

$$\begin{aligned} \min_{x} \quad & \frac{1}{2} x^\top Q x + \alpha \|x\|_0 \\ \text{s.t.} \quad & \mu^\top x \geq \varrho, \quad 1_n^\top x = 1, \quad 0 \leq x \leq u, \end{aligned} \tag{4.28}$$

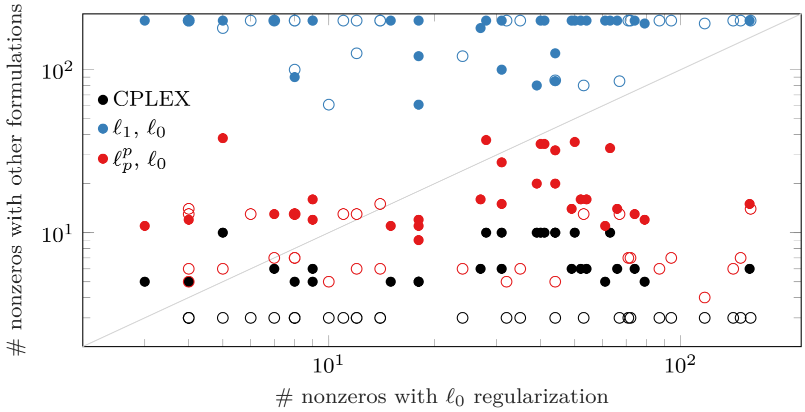

Figure 4.6: Results for the portfolio problem (4.28): Comparison of the solutions found with ℓ_0 regularization against those obtained with CPLEX and ℓ_0 warm-started with ℓ_1 or ℓ_p^p, with $p = 0.5$. We depict the number of nonzero entries of the solutions returned for $\alpha = 10$ (dot) and $\alpha = 100$ (circle). The gray line has unitary slope.

where ℓ_0 quasi-norm is used as a regularization term that penalizes the number of chosen assets within the portfolio. Recall that $Q \in \mathbb{R}^{n\times n}$ and $\mu \in \mathbb{R}^n$ denote the covariance matrix and the mean of $n \in \mathbb{N}$ possible assets, respectively, while $\varrho \in \mathbb{R}$ is a lower bound for the expected return. Furthermore, $u \in \mathbb{R}^n$ provides an upper bound for the individual assets within the portfolio. All the problem data are taken from the test problem collection [74].

We reformulate the model in the form of (CP) by letting f be the quadratic cost, g the nonsmooth cost and indicator of the bounds, $c : \mathbb{R}^n \to \mathbb{R}^m$, $m := 2$, defined by $c(x) := [\mu, 1_n]^\top x$ and $K := [\varrho, \infty) \times \{1\}$.

Through a mixed-integer quadratic program formulation of (4.28), which can be obtained via the theory provided in [71], we compute a solution using CPLEX [91], for comparison. We also solve (4.28) using a continuation procedure: the ℓ_0 minimization is warm-started at a primal-dual point found by replacing the discontinuous ℓ_0 function with either the norm $\ell_1 := \|\cdot\|_1$ or the p-th power of the ℓ_p quasi-norm, i.e., $\ell_p^p := \|\cdot\|_p^p$ $(p = 0.5)$ and solving the corresponding problems. Notice that (4.28) with the ℓ_0- replaced by the ℓ_1-term boils down to a convex quadratic program; in fact, it is $\|x\|_1 = 1$ for each feasible point of (4.28) by the nonnegativity and equality constraints. Here, we used all 30 test instances of dimension $n := 200$ and the two different values $\alpha \in \{10, 100\}$ for each problem.

The results of our experiments are depicted in Figure 4.6. Let us mention that Algorithm 4.3.1 with PANOC$^+$ as the subproblem solver solved all problem instances. Below, we comment on some median values for our experiments with parameters $\alpha = 10/100$: a direct use of ℓ_0 minimization resulted in 10/13 outer and 908/1633 inner iterations, while warm-starting with the continuous ℓ_p^p $(p = 0.5)$ function required 13/9 outer and 686/1830 inner iterations. Let us point the reader's attention to the fact that the ℓ_p^p-warm-started ℓ_0 minimization did not affect the solution sparsity. In other words, the numbers of nonzero components of the obtained solutions were the same with and without an additional round of ℓ_0 minimization after the ℓ_p^p $(p = 0.5)$ warm-start. Although one cannot expect to find a global minimum in general, we recall that the standard ℓ_1 regularization does not work in this example, as confirmed by the poor performance depicted in Figure 4.6, whereas the nonconvex ℓ_p^p $(p = 0.5)$ penalty already leads to very sparse solutions.

4.5.4 Matrix Completion with Minimum Rank

For some $\ell \in \mathbb{N}$, $\ell \geq 2$, let us consider $N \in \mathbb{N}$ points $x_1, \ldots, x_N \in \mathbb{R}^\ell$ and define a block matrix $X \in \mathbb{R}^{N \times \ell}$ by means of $X := [x_1, x_2, \ldots, x_N]^\top$. Let $\Delta \in \mathbb{R}^{N \times N}$ denote the Euclidean distance matrix associated with these points, given by $\Delta_{ij} := \|x_i - x_j\|^2 = (x_i - x_j)^\top (x_i - x_j)$ for all $i, j \in \mathcal{I} := \{1, \ldots, N\}$. We aim at recovering X based on a partial knowledge of Δ. In particular, we assume that $\Omega \subset \mathcal{I}^2$ is a set of pairs such that only the entries Δ_{ij}, $(i,j) \in \Omega$, of Δ are known.

Following [143], we lift the problem by introducing a symmetric matrix $B := XX^\top$ whose rank is, by construction, smaller than or equal to ℓ. Hence, we seek a matrix $B \in \mathbb{R}^{N \times N}$ that satisfies the symmetry constraint $B = B^\top$ and the distance constraints associated with the observations, i.e., $B_{ii} + B_{jj} - B_{ij} - B_{ji} = \Delta_{ij}$ has to hold for all $(i,j) \in \Omega$. Among these admissible matrices, those with minimum rank are preferred.

Let us consider the following problem

$$\begin{aligned} \min_B \quad & g(B) \\ \text{s.t.} \quad & B_{ii} + B_{jj} - B_{ij} - B_{ji} = \Delta_{ij} \quad && \forall (i,j) \in \Omega, \\ & B_{ij} = B_{ji} && \forall i, j \in \mathcal{I}, j < i \end{aligned} \tag{4.29}$$

where the function $g : \mathbb{R}^{N \times N} \to \mathbb{R}$ encodes a matrix regularization term. In the following, we consider $g := \operatorname{rank} := \|\sigma(\cdot)\|_0$, the nuclear norm $g := \|\cdot\|_* := \sum_i \sigma_i(\cdot)$ or the p-powered Schatten p-quasi-norm $g := \|\cdot\|_p^p := \sum_i \sigma_i(\cdot)^p$, $p \in (0,1)$, where $\sigma(A)$ denotes the vector of singular values of a matrix A. Denoting $m_o := |\Omega|$ and $m_s := N(N-1)/2$ the number of observation and symmetry constraints, respectively, there are $n := N^2$ variables and $m := m_o + m_s$ constraints in (4.29). We reformulate the model in the form of (CP) by setting $f := 0$, $K := \{0\}$ and a constraint function $c : \mathbb{R}^{N \times N} \to \mathbb{R}^m$ returning the observation and symmetry constraints stacked in vector form.

For our experiments, we chose $N \in \{10, 20\}$, $\ell = 5$, $m_o = \lfloor (n - m_s)/3 \rfloor$, $p = 0.5$ and consider 30 randomly generated instances for each value of N. We generate $X \in \mathbb{R}^{N \times \ell}$ by sampling the standard normal distribution, i.e., $X_{ij} \sim \mathcal{N}(0,1)$, $(i,j) \in \mathcal{I}^2$, and then compute Δ. Finally, we sample observations by selecting m_o different entries of Δ with uniform probability.

We run our solver Algorithm 4.3.1 with default options, and abstain from setting an iteration limit for the subproblem solver. The initial guess $B^0 \in \mathbb{R}^{N \times N}$ is chosen randomly based on $B^0_{ij} \sim \mathcal{N}(0,1)$, $(i,j) \in \mathcal{I}^2$, whereas the safeguarded dual initial guess is fixed to $u^0 := 0$. We invoke Algorithm 4.3.1 directly for solving (4.29) with the different cost functions mentioned above. Additionally, the solutions obtained with nuclear norm and Schatten quasi-norm as cost functions, which are at least continuous, are used as initial guesses for another round of minimization exploiting the discontinuous rank functional.

We depict the results of our experiments in Figure 4.7. Minimization based on the (convex) nuclear norm produces matrices with rank between 3 and 8, while the use of the Schatten quasi-norm culminates in solutions having rank between 2 and 5. These findings outperform the direct minimization of the rank which results in matrices of rank between 9 and 20. This behavior is not surprising since (4.29) possesses plenty of non-global minimizers in case where minimization of the discontinuous rank is considered, and Algorithm 4.3.1 can terminate in such solutions. Let us mention that, out of 60 instances, the warm-started rank minimization yields further reduction of the rank in one case after minimization of the Schatten quasi-norm and 11 cases after minimization of the nuclear norm; in all other cases, no deterioration has been observed. In summary, Algorithm 4.3.1

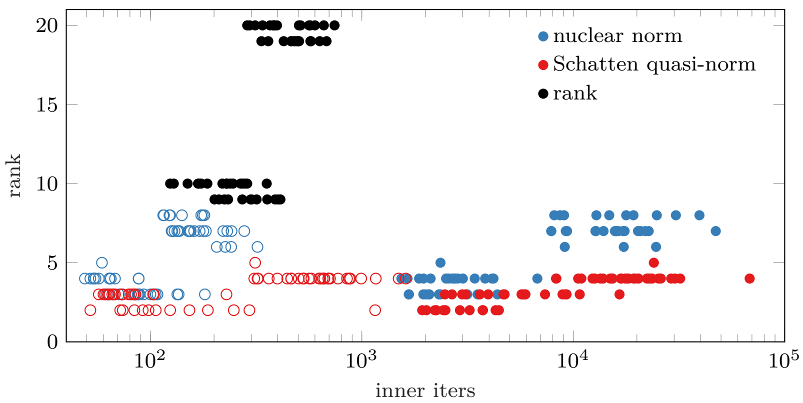

Figure 4.7: Results for the matrix recovery problem (4.29): Comparison of (accumulated) inner iteration numbers and rank of the solutions found with different formulations, including warm-started rank minimization (circle).

manages to find feasible solutions of (4.29) in all cases, and with adequate objective value in cases where we minimize the nuclear norm or the Schatten quasi-norm. These solutions can be used as initial guesses for a warm-started minimization of the rank via Algorithm 4.3.1 or tailored mixed-integer numerical methods.

5. Convergence Analysis of Proximal Gradient Methods

This chapter is concerned to the global convergence and the rate of convergence of the entire sequence generated by a proximal gradient method, proposed in [95], which is a good candidate for solving the composite programs (Q), where no any accelerated techniques, e.g., involving inertial terms or Bregman distances (see [18, 40, 42, 43] and the references therein), are used. The following results are mainly from the preprint [93].

Let us recall again (Q),

$$\min_x \; q(x) := f(x) + g(x) \qquad \text{s.t.} \quad x \in \mathbb{X}, \tag{Q}$$

where $f : \mathbb{X} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{X} \to \overline{\mathbb{R}}$ is assumed merely lower semicontinuous. We claim that the unconstrained composite optimization problem (Q) is in totally nonconvex settings.

Note that proximal gradient methods for solving (Q) can reduce to other well-known algorithms in some special cases. For example, when $g := I_D$ where I_D is the indicator function on some set constraint D, they reduce to projected gradient methods [78, 107]. When $g := 0$, they boil down to gradient descent methods [49]. When $f := 0$, they become proximal minimization methods [50, 121].

In [95], the authors show global convergence results for proximal gradient methods in the sense that every accumulation point is shown to be a suitable stationary point of the composite optimization problem. The analysis in [95] is based on the local Lipschitz continuity of ∇f, and does not require the iterates to be bounded. To be honest, this is a great breakthrough for the convergence results of proximal gradient methods by weakening the global Lipschitz condition into a local one, and can be carried over to more general versions. However, the convergence of entire sequence is not addressed in [95], hence no associated rate-of-convergence results are given. This chapter is devoted to filling the gaps, more specifically, the desired global convergence of the entire sequence and rate-of-convergence results can be achieved by just requiring the merely local Lipschitz continuity of ∇f and together with the Kurdyka–Łojasiewicz property of q, without the global one nor the boundedness of the iterates and stepsizes.

This chapter is organised as follows. Section 5.1 illustrates the necessity of reducing the global Lipschitz condition into the local one. Section 5.2 recalls again the algorithm and some known conclusions from [95]. Section 5.3 is the most core part, where the convergence of the entire iterates and therefore the rate-of-convergence results are presented in the presence of Kurdyka–Łojasiewicz property. The main key using the algorithm is that the corresponding subproblems can be solved successfully, so in Section 5.4, we introduce a class of nonconvex nonsmooth regularization functions for the realization. Section 5.5 shows some numerical results about the image recovery problem and the portfolio problem by comparing our method with Gurobi optimizer.

5.1 Motivation

For the proximal gradient method, the popular instances are the iterative shrinkage/threshold algorithm (ISTA) and its accelerated version (FISTA = fast ISTA), see [24], where g has to be convex. The monograph [21] presents a nice overview of existing results addressing proximal gradient methods where the nonsmooth part enjoys convexity. Later, the seminal works [16, 39] pointed out that the convergence theory can be extended to situations where the nonsmooth part g is merely lower semicontinuous and not necessarily convex. In both aforementioned papers, the analysis, which covers both (global) convergence and rate-of-convergence results, requires a so-called descent lemma as well as the celebrated Kurdyka–Łojasiewicz property, originating from [103, 113, 114]. The majority of available convergence results regarding proximal gradient methods seems to indicate that the price one has to pay for allowing g to be nonsmooth is that the gradient ∇f of the smooth part has to be globally Lipschitz continuous. This requirement, which holds naturally when f is a (convex) quadratic function, turns out to be rather restrictive in the non-quadratic situation which also is of practical interest. See the following examples where the standard global Lipschitz assumption on the gradient of f is typically violated, whereas a local Lipschitz condition is often satisfied.

Example 5.1. (Augmented Lagrangian Methods)
Let us consider the following programs

$$\min_x \; f(x) + g(x) \quad \text{s.t.} \quad c(x) \in C,$$

where $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \overline{\mathbb{R}}$ are as in (Q). In addition, $c : \mathbb{X} \to \mathbb{Y}$ is continuously differentiable, $C \in \mathbb{Y}$ is assumed a nonempty, closed, and convex set. Given a current iterate $x^k \in \mathbb{X}$ and a corresponding Lagrange multiplier estimate $\lambda^k \in \mathbb{Y}$, augmented Lagrangian techniques then compute the next iterate x^{k+1} by solving (approximately) the subproblem

$$\min_x f(x) + g(x) + \frac{\rho_k}{2} \operatorname{dist}^2\left(c(x) + \frac{\lambda^k}{\rho_k}, C\right) \quad \text{s.t.} \quad x \in \mathbb{X}$$

for some penalty parameter $\rho_k > 0$. From Lemma 2.24, this subproblem has exactly the structure of the composite optimization problem (Q) and can therefore, in principle, be solved by a proximal gradient method, see [55, 66, 81, 94] for suitable realizations of this approach.

Assuming that the gradient of the smooth part of this objective function (with respect to the variable x) is globally Lipschitz continuous, however, is pretty strong in this setting and, basically, requires the constraint function c to be linear and the set C to be polyhedral, whereas local Lipschitzness of this gradient holds under mild conditions on the smoothness of f and c.

The following example makes use of conjugate functions, see [19, Definition 3.1]. Since, within this chapter, they only occur in this particular application, we refrain from stating their precise definitions and properties, and refer the interested reader to the excellent monographs [19, 21, 140] for more details.

Example 5.2. (Dual Proximal Gradient Methods)
Consider the (primal) optimization problem

$$\min_x \; g(x) + h(Ax), \qquad x \in \mathbb{X} \tag{5.1}$$

where both functions $g\colon \mathbb{X} \to \overline{\mathbb{R}}$ and $h\colon \mathbb{Y} \to \overline{\mathbb{R}}$ are lower semicontinuous and convex while possessing nonempty domains, and $A\colon \mathbb{X} \to \mathbb{Y}$ is a linear operator. Above, $\mathbb{Y}$ is another Euclidean space. Note that none of the functions g or h is assumed to be (continuously) differentiable.

The (Fenchel) dual problem of (5.1) is given by

$$\min_y \; g^*(A^*y) + h^*(-y), \qquad y \in \mathbb{Y} \tag{5.2}$$

with the two conjugate functions $g^*: \mathbb{X} \to \overline{\mathbb{R}}$ and $h^*: \mathbb{Y} \to \overline{\mathbb{R}}$ being lower semicontinuous and convex, and $A^*: \mathbb{Y} \to \mathbb{X}$ being the adjoint of A. Under suitable assumptions, the pair (5.1), (5.2) enjoys strong duality, i.e., the optimal objective function values of these problems coincide, see [138], which motivates to solve (5.2) instead of (5.1) in some applications where the conjugate functions are explicitly available.

Assuming, in addition, that g is uniformly convex, it is known that g^* is real-valued everywhere and continuously differentiable with a globally Lipschitz continuous gradient, see [140, Proposition 12.60]. Consequently, as promoted in [25], a standard proximal gradient algorithm can be applied to the dual problem (5.2). On the other hand, if g is only strictly convex, then the domain of g^* is, in general, no longer the entire space, but g^* can still be shown to be continuously differentiable on the interior of its domain. Its gradient, however, is no longer guaranteed to be globally Lipschitz continuous on the domain.

The current work is based on [95] where the global convergence results for proximal gradient methods are presented in the sense that every accumulation point is shown to be M-stationary point of (Q). The analysis in [95] is based on the local Lipschitz continuity of ∇f, and does not require the iterates to be bounded. An extension of this work, using a nonmonotone line search, is given in [65]. In contrast to most existing papers on proximal gradient methods, however, convergence of the entire sequence is not addressed in [65, 95]. Hence, no associated rate-of-convergence results could be given ([65] presents some standard worst-case rate-of-convergence results addressing the difference of two subsequent iterates along convergent subsequences). The aim of this work is to fill this gap.

Provided that such accumulation point satisfies the KL property and a local Lipschitz assumption on ∇f, without any boundedness assumption of whole iterative sequence, we aim at the convergence of entire iterative sequence and associated rate-of-convergence result. Based on some recent contributions in the area of proximal gradient and related first-order methods, it seems reasonable to expect such a result to hold. For example, [39,128] consider a class of first-order methods and investigate their (essentially local) convergence showing, in particular, that the entire sequence $\{x^k\}$ generated by their methods stays within a certain neighborhood of a solution provided that the KL property holds at this solution. Their approach is not directly applicable to our situation since, on the one hand, we do not use the a priori assumption that our iterates are bounded, and, on the other hand, because the adaption of the methods considered in [39,128] to the proximal gradient setting would result in an algorithm with a constant stepsize. However, we know from the local Lipschitz assumption on ∇f that a respective global Lipschitz condition holds in a suitable neighborhood of the accumulation point, which then can be used to verify that the stepsizes computed by proposed algorithm remain bounded in case that such accumulation point satisfies KL property. This – more or less heuristic – idea fortifies us to believe that one can also get convergence and rate-of-convergence results under the KL property in the presence of the local Lipschitz gradient of f.

5.2 Algorithm and Known Results

This section begins with a formal description of a proximal gradient method for the composite optimization problem (Q), and then summarizes the associated global convergence properties established in [95]. Note that the proximal gradient method uses a line search which is important to get global convergence properties without a global Lipschitz assumption. We start with a precise statement of the algorithm.

Algorithm 5.1 Proximal Gradient Method [95]

Require: $\tau > 1$, $0 < \gamma_{\min} \le \gamma_{\max} < \infty$, $\delta \in (0,1)$, $x^0 \in \operatorname{dom} \phi$

1: Set $k := 0$.
2: **while** A suitable termination criterion is violated at iteration k **do**
3: Choose $\gamma_k^0 \in [\gamma_{\min}, \gamma_{\max}]$.
4: For $i = 0, 1, 2, \ldots$, compute a solution $x^{k,i}$ of

$$\min_x \; f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + \frac{\gamma_{k,i}}{2} \|x - x^k\|^2 + g(x), \quad x \in \mathbb{X} \tag{5.3}$$

with $\gamma_{k,i} := \tau^i \gamma_k^0$, until the acceptance criterion

$$q(x^{k,i}) \le q(x^k) - \delta \frac{\gamma_{k,i}}{2} \|x^{k,i} - x^k\|^2 \tag{5.4}$$

holds.
5: Denote by $i_k := i$ the terminal value, and set $\gamma_k := \gamma_{k,i_k}$ and $x^{k+1} := x^{k,i_k}$.
6: Set $k \leftarrow k + 1$.
7: **end while**
8: **return** x^k

Note that the convergence analysis requires some technical assumptions as well as a local Lipschitz condition on the gradient of the continuously differentiable function f.

Assumption 5.3.
(a) The function q is bounded from below on $\operatorname{dom} g$.
(b) The function g is bounded from below by an affine function.
(c) The function $\nabla f \colon \mathbb{X} \to \mathbb{X}$ is locally Lipschitz continuous.

Keeping in mind that our goal is to minimize the function q in (Q), Assumption 5.3 (a) is reasonable. Furthermore, Assumption 5.3 (b) is employed to guarantee existence of solutions for the appearing subproblems (5.3). To be precise, Assumption 5.3 (b) implies that the objective function of the subproblem (5.3) is, for fixed $k, i \in \mathbb{N}$, coercive, and therefore always attains a global minimizer $x^{k,i}$ (which does not need to be unique). Finally, the local Lipschitz condition for ∇f from Assumption 5.3 (c) will play a crucial role especially in Section 5.3 where we consider situations where a sequence generated by Algorithm 5.1 converges as a whole and present associated rate-of-convergence results.

We now recall that the stepsize rule in 4 of Algorithm 5.1 is always finite if the current iterate is not already stationary. Hence, the overall method is well-defined.

Lemma 5.4. *[95] Consider a fixed iteration $k \in \mathbb{N}$ of Algorithm 5.1, assume that x^k is not an M-stationary point of* (Q)*, and suppose that Assumption 5.3 (b) holds. Then the inner loop in 4 of Algorithm 5.1 is finite, i.e., one has $\gamma_k = \gamma_{k,i_k}$ for some finite index $i_k \in \{0, 1, 2, \ldots\}$.*

The following results summarize some of the properties and global convergence of Algorithm 5.1 that will later be used in Section 5.3.

Proposition 5.5. *Let Assumption 5.3 (a) and (b) hold, and let $\{x^k\}$ be a sequence generated by Algorithm 5.1. Then the following statements hold:*

(i) $\|x^{k+1} - x^k\| \to 0$ *as* $k \to \infty$,

(ii) *for any convergent subsequence $\{x^k\}_\mathcal{K}$, $\gamma_k \|x^{k+1} - x^k\| \to_\mathcal{K} 0$ holds as $k \to_\mathcal{K} \infty$,*

(iii) *if, additionally, Assumption 5.3 (c) is valid, then for any convergent subsequence $\{x^k\}_\mathcal{K}$, $\{\gamma_k\}_\mathcal{K}$ is bounded.*

Theorem 5.6. *Let Assumption 5.3 be satisfied. Then each accumulation point of a sequence $\{x^k\}$ generated by Algorithm 5.1 is an M-stationary point of* (Q)*.*

5.3 Convergence Analysis in the Presence of the KL Property

The aim of this section is to show convergence of the entire sequence $\{x^k\}$ generated by Algorithm 5.1 provided that there exists an accumulation point $\bar{x}$ which, in addition, satisfies the KL property, and to present associated rate-of-convergence results. The proofs of these results are based on a local Lipschitz assumption on ∇f only, without the a priori assumption that the whole sequence $\{x^k\}$ is bounded.

We begin with a result which shows that, locally around an accumulation point of the sequence $\{x^k\}$, the associated stepsizes γ_k remain bounded. This observation and its proof are related to [95, Corollary 3.1]. Note that this statement is essentially different from the boundedness of stepsizes along convergent subsequences of iterates which is inherent in the presence of Assumption 5.3, see Proposition 5.5 (iii).

Lemma 5.7. *Let Assumption 5.3 hold, let $\{x^k\}$ be any sequence generated by Algorithm 5.1, and let $\bar{x}$ be an accumulation point of this sequence. Then, for any $\rho > 0$, there is a constant $\bar{\gamma}_\rho > 0$ (usually depending on ρ) such that $\gamma_k \leq \bar{\gamma}_\rho$ holds for all $k \in \mathbb{N}$ such that $x^k \in B_\rho(\bar{x})$.*

Proof. First, recall from Lemma 5.4 that the stepsize γ_k is well-defined for each $k \in \mathbb{N}$. Let $\rho > 0$ be fixed, and recall that the assumed local Lipschitz continuity of ∇f implies that this gradient mapping is (globally) Lipschitz continuous on the compact set $B_{2\rho}(\bar{x})$ (note that we took 2ρ as the radius of this ball here). Let us denote the corresponding Lipschitz constant by $L_{2\rho}$. Since $\bar{x}$ is an accumulation point of the sequence $\{x^k\}$, there are infinitely many iterates of this sequence belonging to $B_\rho(\bar{x})$.

Now, assume, by contradiction, that there is a subsequence $\{\gamma_k\}_\mathcal{K}$ with $x^k \in B_\rho(\bar{x})$ for all $k \in \mathcal{K}$ such that $\{\gamma_k\}_\mathcal{K}$ is unbounded. Without loss of generality, we may assume that $\gamma_k \to_\mathcal{K} \infty$, that the subsequence of iterates $\{x^k\}_\mathcal{K}$ converges to some point $\tilde{x}$ (not necessarily equal to $\bar{x}$), and that, for each $k \in \mathcal{K}$, the acceptance criterion (5.4) is violated in the first iteration of the inner loop. Then, for the trial stepsize $\hat{\gamma}_k := \gamma_k/\tau = \tau^{i_k-1}\gamma_k^0$, one also has $\hat{\gamma}_k \to_\mathcal{K} \infty$, whereas the corresponding trial vector $\hat{x}^k := x^{k,i_k-1}$ does not satisfy the acceptance criterion from (5.4), i.e., one has

$$q(\hat{x}^k) > q(x^k) - \delta\frac{\hat{\gamma}_k}{2}\|\hat{x}^k - x^k\|^2 \quad \forall k \in \mathcal{K}. \tag{5.5}$$

On the other hand, since $\hat{x}^k$ solves the corresponding subproblem (5.3) with $\hat{\gamma}_k$, one has

$$\langle \nabla f(x^k), \hat{x}^k - x^k \rangle + \frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\|^2 + g(\hat{x}^k) - g(x^k) \leq 0. \tag{5.6}$$

We claim that this, in particular, implies $\hat{x}^k \to_{\mathcal{K}} \tilde{x}$. In fact, using (5.6), the Cauchy-Schwarz inequality, and the fact that $\{\psi(x^k)\}$ is monotonically decreasing by construction of Algorithm 5.1, we obtain

$$\begin{aligned}
\frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\|^2 &\leq \|\nabla f(x^k)\| \|\hat{x}^k - x^k\| + g(x^k) - g(\hat{x}^k) \\
&= \|\nabla f(x^k)\| \|\hat{x}^k - x^k\| + q(x^k) - f(x^k) - g(\hat{x}^k) \\
&\leq \|\nabla f(x^k)\| \|\hat{x}^k - x^k\| + q(x^0) - f(x^k) - g(\hat{x}^k).
\end{aligned}$$

Since f is continuously differentiable and $-g$ is bounded from above by an affine function in view of Assumption 5.3 (b), the above estimate implies $\|\hat{x}^k - x^k\| \to_{\mathcal{K}} 0$. In fact, if $\{\|\hat{x}^k - x^k\|\}_{\mathcal{K}}$ would be unbounded, then the left-hand side would grow more rapidly than the right-hand side, and if $\{\|\hat{x}^k - x^k\|\}_{\mathcal{K}}$ would be bounded, but staying away, at least on a subsequence, from zero by a positive number, the right-hand side would be bounded, whereas the left-hand side would be unbounded on the corresponding subsequence. Consequently, we have $\|\hat{x}^k - x^k\| \to_{\mathcal{K}} 0$, and since $x^k \to_{\mathcal{K}} \tilde{x}$, this implies $\hat{x}^k \to_{\mathcal{K}} \tilde{x}$. In particular, since $\tilde{x} \in B_\rho(\bar{x})$, this implies that, for all sufficiently large $k \in \mathcal{K}$, we have both $x^k \in B_{2\rho}(\bar{x})$ and $\hat{x}^k \in B_{2\rho}(\bar{x})$.

Let us fix some $k \in \mathcal{K}$. Using the mean-value theorem yields the existence of a point ξ^k on the line segment connecting x^k with $\hat{x}^k$ such that

$$\begin{aligned}
q(\hat{x}^k) - q(x^k) &= f(\hat{x}^k) + g(\hat{x}^k) - f(x^k) - g(x^k) \\
&= \langle \nabla f(\xi^k), \hat{x}^k - x^k \rangle + g(\hat{x}^k) - g(x^k).
\end{aligned}$$

Substituting the resulting expression for $g(\hat{x}^k) - g(x^k)$ into (5.6) yields

$$\langle \nabla f(x^k) - \nabla f(\xi^k), \hat{x}^k - x^k \rangle + \frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\|^2 + q(\hat{x}^k) - q(x^k) \leq 0. \tag{5.7}$$

Exploiting (5.5), one therefore obtains

$$\begin{aligned}
\frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\|^2 &\leq -\langle \nabla f(x^k) - \nabla f(\xi^k), \hat{x}^k - x^k \rangle + q(x^k) - q(\hat{x}^k) \\
&\leq \|\nabla f(x^k) - \nabla f(\xi^k)\| \|\hat{x}^k - x^k\| + \delta \frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\|^2
\end{aligned}$$

which can be rewritten as

$$(1 - \delta) \frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\| \leq \|\nabla f(x^k) - \nabla f(\xi^k)\|.$$

Since ξ^k in an element from the line connecting x^k and $\hat{x}^k$, it follows that $\xi^k \in B_{2\rho}(\bar{x})$ for all $k \in \mathcal{K}$ sufficiently large. Hence, the Lipschitz continuity of ∇f on this ball yields

$$(1 - \delta) \frac{\hat{\gamma}_k}{2} \|\hat{x}^k - x^k\| \leq L_{2\rho} \|x^k - \xi^k\| \leq L_{2\rho} \|x^k - \hat{x}^k\|$$

for all sufficiently large $k \in \mathcal{K}$. Since $\hat{x}^k \neq x^k$ in view of (5.5), this implies that $\{\hat{\gamma}_k\}_{\mathcal{K}}$ is bounded which, in turn, yields the boundedness of the subsequence $\{\gamma_k\}_\mathcal{K}$, contradicting our assumption. This completes the proof. $\square$

We next show that the entire sequence $\{q(x^k)\}$ converges to $q(\bar{x})$, where $\bar{x}$ is an arbitrary accumulation point of a sequence $\{x^k\}$ generated by Algorithm 5.1. Note that this result is not completely obvious since q is only lower semicontinuous but not continuous in general. Indeed, this property results from the construction of the iterates x^{k+1} of Algorithm 5.1.

Lemma 5.8. *Let Assumption 5.3 hold, and let $\bar{x}$ be an accumulation point of a sequence $\{x^k\}$ generated by Algorithm 5.1. Then the entire sequence $\{q(x^k)\}$ converges to $q(\bar{x})$.*

Proof. Let $\{x^k\}_\mathcal{K}$ be a subsequence converging to $\bar{x}$. By means of Proposition 5.5 (i), one also has $x^{k+1} \to_\mathcal{K} \bar{x}$. Since q is lower semicontinuous, one then has

$$q(\bar{x}) \leq \liminf_{k \to_\mathcal{K} \infty} q(x^{k+1}). \tag{5.8}$$

On the other hand, by construction, the entire sequence $\{q(x^k)\}$ is monotonically decreasing. Since it is also bounded from below by $q(\bar{x})$ as a consequence of (5.8), it follows that the whole sequence $\{q(x^k)\}$ converges. It remains to show that its limit is equal to (the lower bound) $q(\bar{x})$.

To this end, we first note that x^{k+1} solves the subproblem (5.3) with stepsize γ_k. Hence, one has

$$\begin{aligned}\langle \nabla f(x^k), x^{k+1} - x^k \rangle &+ \frac{\gamma_k}{2}\|x^{k+1} - x^k\|^2 + g(x^{k+1}) \\ &\leq \langle \nabla f(x^k), \bar{x} - x^k \rangle + \frac{\gamma_k}{2}\|\bar{x} - x^k\|^2 + g(\bar{x})\end{aligned}$$

for each $k \in \mathbb{N}$. Taking the upper limit as $k \to_\mathcal{K} \infty$, and using the continuity of ∇f as well as Proposition 5.5, one obtains

$$\limsup_{k \to_\mathcal{K} \infty} g(x^{k+1}) \leq g(\bar{x}).$$

Combining this with (5.8) and using the continuity of f yields $q(x^{k+1}) \to_\mathcal{K} q(\bar{x})$. Since $\{q(x^k)\}$ converges, the assertion follows. $\square$

Note that all results stated so far are independent of the KL property. The remaining part of the analysis, however, is heavily based on the assumption that objective function q satisfies the KL property at a given accumulation point $\bar{x}$ of a sequence $\{x^k\}$ generated by Algorithm 5.1. In particular, let $\eta > 0$ be the corresponding constant from the definition of the associated desingularization function χ. Furthermore, we will assume that Assumption 5.3 is valid. In view of Proposition 5.5 (i), one can find a sufficiently large index $\hat{k} \in \mathbb{N}$ such that

$$\sup_{k \geq \hat{k}} \|x^{k+1} - x^k\| \leq \eta. \tag{5.9}$$

We then define

$$\rho := \eta + \frac{1}{2} \tag{5.10}$$

as well as the compact set

$$C_\rho := B_\rho(\bar{x}) \cap \mathcal{L}_q(x^0), \tag{5.11}$$

where $\mathcal{L}_q(x^0) := \{x \in \mathbb{X} \,|\, q(x) \leq q(x^0)\}$ is the sublevel set of q with respect to x^0, the starting point exploited in Algorithm 5.1. By monotonicity of $\{q(x^k)\}$, one has

$\{x^k\} \subset \mathcal{L}_q(x^0)$. Finally, throughout the section, let $L_\rho > 0$ be a (global) Lipschitz constant of ∇f on C_ρ from (5.11). Finally, in view of Lemma 5.7, one has

$$\gamma_k \leq \bar{\gamma}_\rho \quad \forall x^k \in C_\rho \tag{5.12}$$

with some suitable upper bound $\bar{\gamma}_\rho > 0$ (depending on our choice of ρ from (5.10)). Using this notation, one can formulate the following result.

Lemma 5.9. *Let Assumption 5.3 hold, and let $\{x^k\}$ be any sequence generated by Algorithm 5.1. Suppose that $\{x^k\}_\mathcal{K}$ is a subsequence converging to some limit point $\bar{x}$, and that q has the KL property at $\bar{x}$ with desingularization function χ. Then there is a sufficiently large constant $k_0 \in \mathcal{K}$ such that the corresponding constant*

$$\alpha := \|x^{k_0} - \bar{x}\| + \sqrt{\frac{8\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{2(\bar{\gamma}_\rho + L_\rho)}{\delta\gamma_{\min}}\chi\big(q(x^{k_0}) - q(\bar{x})\big) \tag{5.13}$$

satisfies $\alpha < \frac{1}{2}$, where $\rho > 0$ and $\bar{\gamma}_\rho > 0$ are the constants defined in (5.10) *and* (5.12)*, respectively, while $L_\rho > 0$ is a Lipschitz constant of ∇f on C_ρ from* (5.11)*, and $\delta > 0$ as well as $\gamma_{\min} > 0$ are the parameters from Algorithm 5.1.*

Proof. The statement follows from the fact that each part on the right-hand side of (5.13) can be made arbitrarily small. This is clear for the first one since the subsequence $\{x^k\}_\mathcal{K}$ converges to $\bar{x}$. This is also true for the second part as a consequence of Lemma 5.8. Finally, the third one can be made arbitrarily small since we have $q(x^k) \to q(\bar{x})$ by Lemma 5.8, taking into account that the desingularization function χ is continuous at the origin. Hence, the statement follows by taking an index $k_0 \in \mathcal{K}$ sufficiently large. $\square$

We next state another technical result.

Lemma 5.10. *Let Assumption 5.3 hold, and let $\{x^k\}$ be any sequence generated by Algorithm 5.1. Suppose that $\{x^k\}_\mathcal{K}$ is a subsequence converging to some limit point $\bar{x}$, and that q has the KL property at $\bar{x}$ with desingularization function χ. Then*

$$\operatorname{dist}\big(0, \partial q(x^{k+1})\big) \leq (\bar{\gamma}_\rho + L_\rho)\|x^{k+1} - x^k\|$$

holds for all sufficiently large $k \in \mathbb{N}$ such that $x^k \in B_\alpha(\bar{x})$, where $\alpha < \frac{1}{2}$ denotes the constant from (5.13)*, $\bar{\gamma}_\rho > 0$ is the constant from* (5.12)*, and $L_\rho > 0$ is the Lipschitz constant of ∇f on C_ρ from* (5.11)*.*

Proof. For any $k \in \mathbb{N}$, since x^{k+1} is a solution of (5.3), one obtains

$$0 \in \nabla f(x^k) + \gamma_k(x^{k+1} - x^k) + \partial g(x^{k+1})$$

from the corresponding M-stationary condition. This implies

$$\gamma_k(x^k - x^{k+1}) + \nabla f(x^{k+1}) - \nabla f(x^k) \in \nabla f(x^{k+1}) + \partial g(x^{k+1}) = \partial q(x^{k+1}) \tag{5.14}$$

for all $k \in \mathbb{N}$, where we used the sum rule Proposition 2.23 (i) for the limiting subdifferential.

Now, take an arbitrary index $k \in \mathbb{N}$ sufficiently large such that $x^k \in B_\alpha(\bar{x})$ and $k \geq \hat{k}$, where $\hat{k}$ is the index from (5.9). In view of (5.10) and Lemma 5.9, one has $\alpha \leq \rho$. Therefore, Lemma 5.7 shows that

$$\gamma_k \leq \bar{\gamma}_\rho. \tag{5.15}$$

Moreover, using (5.9), (5.10), and Lemma 5.9, one obtains

$$\|x^{k+1} - \bar{x}\| \le \|x^{k+1} - x^k\| + \|x^k - \bar{x}\| \le \eta + \alpha \le \rho.$$

Hence, $x^k, x^{k+1} \in C_\rho$ holds with the compact set C_ρ from (5.11). Therefore, one has

$$\|\nabla f(x^{k+1}) - \nabla f(x^k)\| \le L_\rho \|x^{k+1} - x^k\|$$

by definition of L_ρ. Together with (5.14) and (5.15), we thus obtain

$$\begin{aligned} \operatorname{dist}\left(0, \partial q(x^{k+1})\right) &\le \|\gamma_k(x^k - x^{k+1}) + \nabla f(x^{k+1}) - \nabla f(x^k)\| \\ &\le \gamma_k \|x^{k+1} - x^k\| + L_\rho \|x^{k+1} - x^k\| \\ &\le (\bar{\gamma}_\rho + L_\rho)\|x^{k+1} - x^k\| \end{aligned}$$

for all $k \in \mathbb{N}$ satisfying $k \ge \hat{k}$ and $x^k \in B_\alpha(\bar{x})$. □

The following result shows that the entire sequence $\{x^k\}$, generated by Algorithm 5.1, already converges to one of its accumulation points $\bar{x}$ provided that the objective function q satisfies the KL property at this point. The proof combines our previous results with a technique used in [39].

Theorem 5.11. *Let Assumption 5.3 hold, and let $\{x^k\}$ be any sequence generated by Algorithm 5.1. Suppose that $\{x^k\}_\mathcal{K}$ is a subsequence converging to some limit point $\bar{x}$, and that q has the KL property at $\bar{x}$. Then the entire sequence $\{x^k\}$ converges to $\bar{x}$.*

Proof. In view of Lemma 5.8, we know that the whole sequence $\{q(x^k)\}$ is monotonically decreasing and converging to $q(\bar{x})$. This implies that $q(x^k) \ge q(\bar{x})$ holds for all $k \in \mathbb{N}$.

Now, suppose one has $q(x^k) = q(\bar{x})$ for some index $k \in \mathbb{N}$. Then, by monotonicity, one also gets $q(x^{k+1}) = q(\bar{x})$. Consequently, one obtains from (5.4) that

$$0 \le \frac{\delta \gamma_{\min}}{2} \|x^{k+1} - x^k\|^2 \le q(x^k) - q(x^{k+1}) = 0$$

and, thus, $x^{k+1} = x^k$. Since, by assumption, the subsequence $\{x^k\}_\mathcal{K}$ converges to $\bar{x}$, this implies that $x^k = \bar{x}$ for all $k \in \mathbb{N}$ sufficiently large. In particular, one has convergence of the entire (eventually constant) sequence $\{x^k\}$ to $\bar{x}$ in this situation.

For the remainder of this proof, one can therefore assume that $q(x^k) > q(\bar{x})$ holds for all $k \in \mathbb{N}$. Let $\alpha \in (0, 1/2)$ be the constant from (5.13), and $k_0 \in \mathcal{K}$ be the corresponding iteration index which is used in the definition of α, see Lemma 5.9. One then has $0 < q(x^k) - q(\bar{x}) \le q(x^{k_0}) - q(\bar{x})$ for all $k \ge k_0$. Without loss of generality, we may also assume that $k_0 \ge \hat{k}$ (the latter being the index defined by (5.9)) and that k_0 is sufficiently large to satisfy

$$q(x^{k_0}) < q(\bar{x}) + \eta. \tag{5.16}$$

Let $\chi\colon [0, \eta] \to [0, \infty)$ be the desingularization function which comes along with the validity of the KL property at $\bar{x}$. Due to $\chi(0) = 0$ and $\chi'(t) > 0$ for all $t \in (0, \eta)$, one obtains

$$\chi\big(q(x^k) - q(\bar{x})\big) \ge 0 \quad \forall k \ge k_0. \tag{5.17}$$

We now claim that the following two statements hold for all $k \ge k_0$:

(i) $x^k \in B_\alpha(\bar{x})$,

(ii) $\|x^{k_0} - \bar{x}\| + \sum_{i=k_0}^{k} \|x^{i+1} - x^i\| \leq \alpha$, which is equivalent to

$$\sum_{i=k_0}^{k} \|x^{i+1} - x^i\| \leq \sqrt{\frac{8\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{2(\bar{\gamma}_\rho + L_\rho)}{\delta\gamma_{\min}} \chi\big(q(x^{k_0}) - q(\bar{x})\big). \tag{5.18}$$

We verify these two statements jointly by induction. For $k = k_0$, statement (i) holds simply by the definition of α in (5.13). Furthermore, the acceptance criterion (5.4) together with the monotonicity of $\{q(x^k)\}$ implies

$$\|x^{k_0+1} - x^{k_0}\| \leq \sqrt{\frac{2\big(q(x^{k_0}) - q(x^{k_0+1})\big)}{\delta\gamma_{\min}}} \leq \sqrt{\frac{2\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}}. \tag{5.19}$$

In particular, this shows that (5.18) holds for $k = k_0$. Suppose that both statements hold for some $k \geq k_0$. Using the triangle inequality, the induction hypothesis, and the definition of α, one obtains

$$\|x^{k+1} - \bar{x}\| \leq \sum_{i=k_0}^{k} \|x^{i+1} - x^i\| + \|x^{k_0} - \bar{x}\| \leq \alpha,$$

i.e., statement (i) holds for $k+1$ in place of k. The verification of the induction step for (ii) is more involved.

To this end, first note that (5.16) implies

$$q(\bar{x}) < q(x^i) < q(\bar{x}) + \eta \quad \forall i \geq k_0. \tag{5.20}$$

Since q has the KL property at $\bar{x}$, one has

$$\chi'\big(q(x^i) - q(\bar{x})\big) \operatorname{dist}\big(0, \partial q(x^i)\big) \geq 1 \quad \forall i \geq k_0. \tag{5.21}$$

Since $x^i \in B_\alpha(\bar{x})$ for all $i \in \{k_0, k_0+1, \ldots, k\}$ by the induction hypothesis, one can apply Lemma 5.10 and obtain (after a simple index shift)

$$\operatorname{dist}\big(0, \partial q(x^i)\big) \leq (\bar{\gamma}_\rho + L_\rho)\|x^i - x^{i-1}\| \quad \forall i \in \{k_0+1, k_0+2, \ldots, k+1\}.$$

In view of (5.21), one therefore obtains

$$\chi'\big(q(x^i) - q(\bar{x})\big) \geq \frac{1}{(\bar{\gamma}_\rho + L_\rho)\|x^i - x^{i-1}\|} \quad \forall i \in \{k_0+1, k_0+2, \ldots, k+1\}. \tag{5.22}$$

To simplify some of the subsequent formulas, we follow [39] and introduce the short-hand notation

$$\Delta_{i,j} := \chi\big(q(x^i) - q(\bar{x})\big) - \chi\big(q(x^j) - q(\bar{x})\big)$$

for $i, j \in \mathbb{N}$. The assumed concavity of χ then implies

$$\Delta_{i,i+1} \geq \chi'\big(q(x^i) - q(\bar{x})\big)\big(q(x^i) - q(x^{i+1})\big). \tag{5.23}$$

Using (5.22), (5.23), and the acceptance criterion (5.4), one therefore gets

$$\begin{aligned}\Delta_{i,i+1} &\geq \chi'\big(q(x^i)-q(\bar{x})\big)\big(q(x^i)-q(x^{i+1})\big)\\ &\geq \frac{q(x^i)-q(x^{i+1})}{(\bar{\gamma}_\rho+L_\rho)\|x^i-x^{i-1}\|} \geq \frac{\delta\gamma_{\min}}{2(\bar{\gamma}_\rho+L_\rho)}\frac{\|x^{i+1}-x^i\|^2}{\|x^i-x^{i-1}\|} = \beta\frac{\|x^{i+1}-x^i\|^2}{\|x^i-x^{i-1}\|}\end{aligned}$$

for all $i \in \{k_0+1, k_0+2, \dots, k+1\}$, with the constant $\beta := \frac{\delta\gamma_{\min}}{2(\bar{\gamma}_\rho+L_\rho)}$. Noting that $a+b \geq 2\sqrt{ab}$ holds for all real numbers $a, b \geq 0$, one therefore obtains

$$\frac{1}{\beta}\Delta_{i,i+1} + \|x^i-x^{i-1}\| \geq 2\sqrt{\frac{1}{\beta}\Delta_{i,i+1}\|x^i-x^{i-1}\|} \geq 2\|x^{i+1}-x^i\|$$

for all $i \in \{k_0+1, k_0+2, \dots, k+1\}$. Summation yields

$$\begin{aligned}2\sum_{i=k_0+1}^{k+1}\|x^{i+1}-x^i\| &\leq \sum_{i=k_0+1}^{k+1}\|x^i-x^{i-1}\| + \frac{1}{\beta}\sum_{i=k_0+1}^{k+1}\Delta_{i,i+1}\\ &= \sum_{i=k_0+1}^{k}\|x^{i+1}-x^i\| + \|x^{k_0+1}-x^{k_0}\| + \frac{1}{\beta}\Delta_{k_0+1,k+2}\\ &\leq \sum_{i=k_0+1}^{k+1}\|x^{i+1}-x^i\| + \|x^{k_0+1}-x^{k_0}\| + \frac{1}{\beta}\Delta_{k_0+1,k+2}.\end{aligned}$$

Subtracting the first summand from the right-hand side, exploiting the estimate (5.19), and using the nonnegativity as well as monotonicity of the desingularization function χ, we obtain

$$\sum_{i=k_0+1}^{k+1}\|x^{i+1}-x^i\| \leq \sqrt{\frac{2\big(q(x^{k_0})-q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{1}{\beta}\chi\big(q(x^{k_0})-q(\bar{x})\big).$$

Adding the term $\|x^{k_0+1}-x^{k_0}\|$ to both sides and using (5.19) once again, one gets

$$\sum_{i=k_0}^{k+1}\|x^{i+1}-x^i\| \leq \sqrt{\frac{8\big(q(x^{k_0})-q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{1}{\beta}\chi\big(q(x^{k_0})-q(\bar{x})\big).$$

Hence, statement (ii) holds for $k+1$ in place of k, and this completes the induction.

In particular, it follows from (i) that $x^k \in B_\alpha(\bar{x})$ for all $k \geq k_0$. Taking $k \to \infty$ in (5.18) therefore shows that $\{x^k\}$ is a Cauchy sequence and, thus, convergent. Since one already knows that $\bar{x}$ is an accumulation point, it follows that the entire sequence $\{x^k\}$ converges to $\bar{x}$. □

Note that Theorem 5.11 says that, in the presence of Assumption 5.3 and the KL property (on the overall domain of g), any sequence $\{x^k\}$ generated by Algorithm 5.1 either satisfies $\|x^k\| \to \infty$ or converges to a limit point (which is an M-stationary point of (Q) by Theorem 5.6). This alternative behavior, which typically comes along with the KL property, see e.g. [15, Theorem 3.2], first has been observed in [1, Theorem 3.2] in the context of descent methods for analytic functions.

We finally state our rate-of-convergence result for one general class of desingularization functions.

Theorem 5.12. *Let Assumption 5.3 hold, and let $\{x^k\}$ be any sequence generated by Algorithm 5.1. Suppose that $\{x^k\}_{\mathcal{K}}$ is a subsequence converging to some limit point $\bar{x}$, and that q has the KL property at $\bar{x}$. Then the entire sequence $\{x^k\}$ converges to $\bar{x}$, and if the corresponding desingularization function has the form $\chi(t) = ct^\theta$ for some $c > 0$ and $\theta \in (0, 1]$, the following statements hold:*

(i) *if $\theta \in (0, \frac{1}{2}]$, then the sequence $\{q(x^k)\}$ converges Q-linearly to $q(\bar{x})$, and the sequence $\{x^k\}$ converges R-linearly to $\bar{x}$.*

(ii) *if $\theta \in (\frac{1}{2}, 1)$, then there exist some positive constants η_1 and η_2 such that*

$$
\begin{aligned}
q(x^k) - q(\bar{x}) &\le \eta_1 k^{-\frac{1}{2\theta-1}}, \\
\|x^k - \bar{x}\| &\le \eta_2 k^{-\frac{1}{2(2\theta-1)}}
\end{aligned}
$$

for sufficiently large k.

(iii) *if $\theta = 1$, then the sequences $\{q(x^k)\}$ and $\{x^k\}$ converge in a finite number of steps to $q(\bar{x})$ and $\bar{x}$, respectively.*

Proof. In view of Theorem 5.11, we only need to verify the quantitative statements (i), (ii), and (iii) of the theorem. As noted at the beginning of the proof of Theorem 5.11, one may assume, without loss of generality, that $q(x^k) > q(\bar{x})$ holds for all $k \in \mathbb{N}$. In view of Lemma 5.8, one then has

$$
x^k \in B_\alpha(\bar{x}) \cap \{x \in \operatorname{dom} g \mid q(\bar{x}) < q(x) < q(\bar{x}) + \eta\}
$$

for all $k \in \mathbb{N}$ sufficiently large, where $\alpha > 0$ is the constant from (5.13) and $\eta > 0$ denotes the constant from the definition of the desingularization function χ. Since q satisfies the KL property at $\bar{x}$ with $\chi(t) = ct^\theta$, we have

$$
\begin{aligned}
1 &\le \chi'\big(q(x^{k+1}) - q(\bar{x})\big) \operatorname{dist}\big(0, \partial q(x^{k+1})\big) \\
&= c\theta\big(q(x^{k+1}) - q(\bar{x})\big)^{\theta-1} \operatorname{dist}\big(0, \partial q(x^{k+1})\big)
\end{aligned}
$$

for all sufficiently large $k \in \mathbb{N}$. Taking into account Lemma 5.10, this yields

$$
1 \le c\theta(\bar{\gamma}_\rho + L_\rho)\big(q(x^{k+1}) - q(\bar{x})\big)^{\theta-1} \|x^{k+1} - x^k\|
$$

for all $k \in \mathbb{N}$ sufficiently large, where $\bar{\gamma}_\rho > 0$ is the constant from (5.12) and $L_\rho > 0$ is the global Lipschitz constant of ∇f on C_ρ from (5.11). Rearranging this expression yields

$$
\|x^{k+1} - x^k\| \ge \frac{1}{c\theta(\bar{\gamma}_\rho + L_\rho)}\big(q(x^{k+1}) - q(\bar{x})\big)^{1-\theta}. \tag{5.24}
$$

On the other hand, by the acceptance criterion (5.4) and $\gamma_k \ge \gamma_{\min}$, one has

$$
q(x^{k+1}) - q(x^k) \le -\delta \frac{\gamma_{\min}}{2} \|x^{k+1} - x^k\|^2. \tag{5.25}
$$

Combining (5.24) and (5.25) implies

$$\begin{aligned}\big(q(x^{k+1}) - q(\bar{x})\big) - \big(q(x^k) - q(\bar{x})\big) &= q(x^{k+1}) - q(x^k)\\ &\le -\delta\frac{\gamma_{\min}}{2}\|x^{k+1} - x^k\|^2\\ &\le -\frac{\delta\gamma_{\min}}{2c^2\theta^2(\bar{\gamma}_\rho + L_\rho)^2}\big(q(x^{k+1}) - q(\bar{x})\big)^{2(1-\theta)}\\ &= -\sigma\big(q(x^{k+1}) - q(\bar{x})\big)^{2(1-\theta)}\end{aligned} \tag{5.26}$$

for all $k \in \mathbb{N}$ sufficiently large, with the constant $\sigma := \frac{\delta\gamma_{\min}}{2c^2\theta^2(\bar{\gamma}_\rho+L_\rho)^2}$ for brevity. Set $q_k := q(x^k) - q(\bar{x})$ for short, rearranging these terms yields

$$q_{k+1}^{2(1-\theta)} \le \frac{1}{\sigma}\left(q_k - q_{k+1}\right) \tag{5.27}$$

for all $k \in \mathbb{N}$ large enough. Since q_k is deceasing, $\theta \in (0,1]$, and $\sigma > 0$, then the statements (i), (ii), and (iii) regarding the sequence $\{q(x^k)\}$ follow from Lemma 2.14. More specifically, when $\theta \in (0, 1/2]$, the sequence $\{q(x^k)\}$ converges Q-linearly to $q(\bar{x})$ with rate $1/(1 + \sigma)$.

It remains to verify the convergence rate with respect to the sequence $\{x^k\}$. We now consider the different cases of θ.

- $\theta = 1$: then from (5.18), for all $k \ge k_0$, one has

$$\|x^k - \bar{x}\| \le \sum_{i=k_0}^{\infty}\|x^{i+1} - x^i\| \le \sqrt{\frac{8\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{2c\big(\bar{\gamma}_\rho + L_\rho\big)}{\delta\gamma_{\min}}\big(q(x^{k_0}) - q(\bar{x})\big).$$

  Since $\{q(x^k)\}$ converges to $q(\bar{x})$ in finite steps when $\theta = 1$, then one has $\{x^k\}$ also converges to $\bar{x}$ in finite steps.
- $\theta \in (1/2, 1)$: from the general setting that $q(x^k) > q(\bar{x})$ for all $k \in \mathbb{N}$, $\chi(t) = ct^\theta$ is continuous, and $t > 1/2$, as well as the fact $q(x^k) \to q(\bar{x})$, then for sufficiently large k_0, one has

$$(q(x^{k_0}) - q(\bar{x}))^\theta \le (q(x^{k_0}) - q(\bar{x}))^{\frac{1}{2}},$$

  which, together with (5.18) implies

$$\begin{aligned}\|x^k - \bar{x}\| &\le \sum_{i=k_0}^{\infty}\|x^{i+1} - x^i\| \le \sqrt{\frac{8\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{2c\big(\bar{\gamma}_\rho + L_\rho\big)}{\delta\gamma_{\min}}\big(q(x^{k_0}) - q(\bar{x})\big)^\theta\\ &\le \sqrt{\frac{8\big(q(x^{k_0}) - q(\bar{x})\big)}{\delta\gamma_{\min}}} + \frac{2c\big(\bar{\gamma}_\rho + L_\rho\big)}{\delta\gamma_{\min}}\sqrt{q(x^{k_0}) - q(\bar{x})}\\ &= \tau\sqrt{q(x^{k_0}) - q(\bar{x})}\end{aligned}$$

  holds for all $k \ge k_0$, with $\tau := \sqrt{\frac{8}{\delta\gamma_{\min}}} + \frac{2c\left(\bar{\gamma}_\rho+L_\rho\right)}{\delta\gamma_{\min}}$. From the conclusion of (ii) regarding $\{q(x^k)\}$, there exists some $\eta_1 > 0$ satisfying $q(x^k) - q(\bar{x}) \le \eta_1 k^{-\frac{1}{2\theta-1}}$ for k sufficiently large, then one has

$$\|x^k - \bar{x}\| \le \tau(\eta_1 k^{-\frac{1}{2\theta-1}})^{\frac{1}{2}} = \tau{\eta_1}^{\frac{1}{2}}k^{-\frac{1}{2(2\theta-1)}}$$

  for sufficiently large k. Setting $\eta_2 := \tau{\eta_1}^{1/2}$ completes the proof of (ii).

- $\theta \in (0, 1/2]$: observe that the descent test (5.4) and the monotonicity of the sequence $\{q(x^k)\}$ yield

$$\frac{\delta\gamma_{\min}}{2}\|x^{k+1} - x^k\|^2 \leq q(x^k) - q(x^{k+1}) \leq q(x^k) - q(\bar{x}) = q_k,$$

and that the sequence $\{q_k\}$ is Q-linearly convergent. Taking this into account, it is not difficult to see that there exist constants $\omega > 0$ and $\mu \in (0, 1)$ such that

$$\|x^{k+1} - x^k\| \leq \omega\mu^k$$

holds for all sufficiently large $k \in \mathbb{N}$. Hence, for given integers $\ell > k > 0$ large enough, one therefore obtains

$$\|x^{\ell+1} - x^k\| \leq \sum_{j=k}^{\ell} \|x^{j+1} - x^j\| \leq \omega \sum_{j=k}^{\ell} \mu^j \leq \omega\mu^k \sum_{j=0}^{\infty} \mu^j = \frac{\omega}{1-\mu}\mu^k.$$

Taking the limit $\ell \to \infty$ yields

$$\|x^k - \bar{x}\| \leq \frac{\omega}{1-\mu}\mu^k$$

for all large enough $k \in \mathbb{N}$. This completes the proof of the (local) R-linear convergence of $\{x^k\}$ to its limit $\bar{x}$.

□

So far, all the convergence and rate-of-convergence results of Algorithm 5.1 have been done. Note that Algorithm 5.1 is one of the most simple but fundamental proximal gradient methods, then keeping these findings in mind, it might be promising to check whether the technique of proof can be applied in several generalizations of the this method.

5.4 Realization for a Class of Nonconvex Regularizers

In fact, (Q) has many practical applications, such as signal processing [46, 52, 61], machine learning [73], compressed sensing [160], and image processing [23, 57] et al., where, typically, f models a tracking-type term and g is used to promote sparse structures of solutions, which is usually called the regularization function, penalty function or regularizer. Note that g is usually nonsmooth and possibly also nonconvex, since such function are advantageous in that they usually yield sparser solutions [54].

As mentioned above, Algorithm 5.1 requires the global minimum of (5.3) at each iteration, this section is concerned to the solution of (5.3) resulting from Algorithm 5.1. By adding and subtracting constant terms (constant with respect to the variable x), it follows that (5.3) can be rewritten as

$$\min_{x \in \mathbb{X}} \frac{\gamma_k}{2}\left\| x - \left(x^k - \frac{1}{\gamma_k}\nabla f(x^k)\right)\right\|^2 + g(x), \tag{5.28}$$

which obviously has the unique solution if g is convex. In this section, we consider a useful class of nonconvex, nonsmooth regularization function g [127] and therefore set $\mathbb{X} := \mathbb{R}^n$:

$$g(x) := \sum_{i=1}^{n} \phi(x_i),$$

where ϕ is always called a potential function. Then (5.28) can be reformulated to

$$\min_{x\in\mathbb{R}^n} \frac{\gamma_k}{2}\left\|x-\left(x^k-\frac{1}{\gamma_k}\nabla f(x^k)\right)\right\|^2+\sum_{i=1}^{n}\phi(x_i),$$

which is totally separable, therefore reduces to the n one-dimensional subproblems

$$\min_{x_i\in\mathbb{R}} \frac{\gamma_k}{2}\left\|x_i-p_i^k\right\|^2+\phi(x_i)$$

for $i=1,\dots,n$, where we set $p^k := x^k-\frac{1}{\gamma_k}\nabla f(x^k)$. Multiplying the objective function by $\lambda_k := \frac{1}{\gamma_k}$, we have

$$\min_{x_i\in\mathbb{R}} \frac{1}{2}\left\|x_i-p_i^k\right\|^2+\lambda_k\phi(x_i).$$

To simplify the notation, we therefore consider the problem

$$\min_{t\in\mathbb{R}} \psi(t) := \frac{1}{2}(t-p)^2+\lambda\phi(t) \tag{5.29}$$

with $\phi(t)$ satisfying the following assumption:

Assumption 5.13. We assume that $\phi:\mathbb{R}\to\mathbb{R}_+$ is continuous function and satisfies

(a) $\phi(t)\geq 0$, $\phi(t)=\phi(-t)$; in other words, ϕ is nonnegative and symmetric.

(b) ϕ is continuously differentiable on $\mathbb{R}$ except at 0, $\phi'(t)t>0$ holds for all $t\neq 0$, and

$$\lim_{t\downarrow 0}\phi'(t)=\phi'(0^+)=-\lim_{t\uparrow 0}\phi'(t)=-\phi'(0^-)\in(0,\infty].$$

(c) ϕ is twice differentiable on $\mathbb{R}$ except at 0 with $\phi''(t)<0$ for all $t\neq 0$, and $\phi''(t)$ is monotonically increasing for $t>0$ and monotonically decreasing for $t<0$, respectively.

Obviously, the basic requirements are satisfied by the following functions:

$$\phi_0(t)=|t|^q,\quad \phi_1(t)=\frac{\alpha|t|}{1+\alpha|t|},\quad \phi_2(t)=\log(1+\alpha|t|), \tag{5.30}$$

with $\alpha>0$ and $q\in(0,1)$ satisfying the assumptions on ϕ. [54] also considered a family of penalty functions, where ϕ_2 is satisfied, however ϕ_0 and ϕ_1 can not be included. ϕ_0 ($q=0.1,0.5,0.9$), ϕ_1, and ϕ_2 with $\alpha=0.8$ are illustrated in Figure 5.1, which have an almost fast growth beyond an interval surrounding the origin. As for the motivation of Assumption 5.13 (a) and (b), discontinuity of ϕ' at the origin plays a very important role in generating sparser solution, which is proposed in the following lemma using the technique from [126, Lemma 3.1].

Lemma 5.14 (Discontinuity of $\phi'(t)$ at $t=0$ implies existence of global minimum of (5.29))**.** *If $\phi'(0^-)\neq\phi'(0^+)\neq 0$, there exists a nonzero neighborhood of $p=0$ such that 0 is the global minimum of* (5.29).

Proof. Since the objective function of (5.29) is coercive, then it has a global minimum by Weierstrass' theorem. From Assumption 5.13 (a), then the solution of (5.29) is either equal to 0 or satisfies

$$t-p+\lambda\phi'(t)=0. \tag{5.31}$$

From Assumption 5.13 (b), t and $\phi'(t)$ have the same sign, hence one has $|t|<|p|$. Due to $\phi'(0^+)\neq 0$, there exists some neighborhood $[-p_0,p_0]$ of $p=0$ such that (5.31) has no

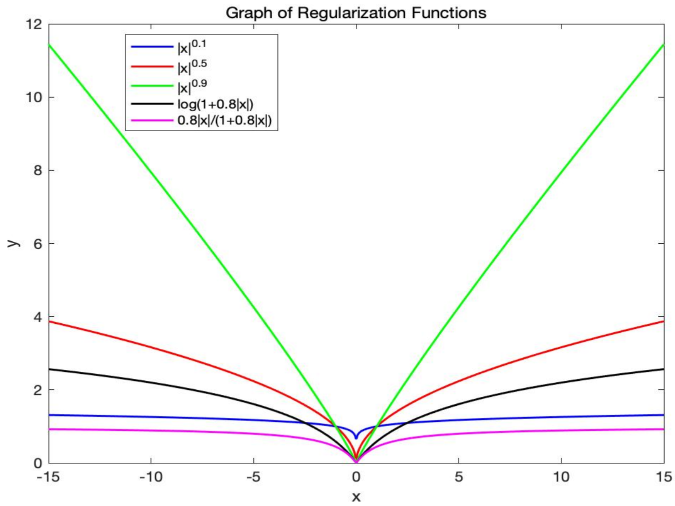

Figure 5.1: Several sparsity promoting regularizations satisfying Assumption 5.13.

solution. So in the case, the only possible solution for $|p| \le p_0$ is 0. □

In other words, Assumption 5.13 (a) and (b) aim to generate the sparser solution, at least than the case of convex regularization functions. However, the countless local mimimizers causes much difficulty in seeking for the global one, as a result, Assumption 5.13 (c) is introduced for the analytical solution. We in the following present the solution analysis of (5.29).

Due to Assumption 5.13 (a) and (b), we have $\operatorname{argmin}_{t\in\mathbb{R}} \phi(t) = 0$. As a result, if $p = 0$, then $t_* := 0$ is the unique global minimum of (5.29).

We next consider the case where $p > 0$. Since ψ is obviously coercive, then (5.29) has a global minimum by Weierstrass' theorem. Moreover, since it has an isolated point at the origin, then we know 0 is always a local minimum (not necessarily a global one). In order to check whether the origin is a global minimum, due to the decrement of ψ on $(-\infty, 0)$ from Assumption 5.13 (b), we only need to take a closer look at the function ψ on the open interval $(0, \infty)$. Observe that ψ is continuous, whose derivative on $(0, \infty)$ is given by

$$\psi'(t) = t - p + \lambda\phi'(t), \quad t > 0. \tag{5.32}$$

Meanwhile ψ' and ψ'' are continuous except at 0 from Assumption 5.13. By again Assumption 5.13 (b), $p, \lambda > 0$, we get

$$\psi'(t) > 0, \quad t \ge p. \tag{5.33}$$

We next have to explore whether the nonlinear equation $\psi'(t) = 0$ has solution(s) on the open interval $(0, p)$. The twice derivative of ψ over $(0, p)$ is given by

$$\psi''(t) = 1 + \lambda\phi''(t), \quad 0 < t < p.$$

Now, with the aid of the monotone increasement of $\phi''(t)$ for $t > 0$, we need to consider whether $\psi''(t) = 0$ has a solution, i.e., $\phi''(t) \stackrel{!}{=} -\frac{1}{\lambda}$ on $(0, p)$.

Case 1: $\phi''(t) < -\frac{1}{\lambda}, \forall t \in (0, p)$ (or $\phi''(p) < -\frac{1}{\lambda}$). Then ψ' is monotonically decreasing over $(0, p)$, since ψ' is continuous on $(0, p)$ and $\psi'(p) > 0$, hence $\psi'(t) > 0, \forall t \in (0, p)$. It, together with (5.33), implies $t_* := 0$ is the unique global minimum of (5.29) with $p > 0$ by

continuity of ψ.
Case 2: $\phi''(t) > -\frac{1}{\lambda}, \forall t \in (0,p)$ (or $\phi''(0^+) > -\frac{1}{\lambda}$). Then ψ' is monotonically increasing over $(0,p)$, in this case, we have to distinguish whether $\psi'(t)$=0 has a solution over $(0,p)$.

- If $\psi'(0^+) \geq 0$, which says $\psi'(t)$=0 has no solution on $(0,p)$, so that $t_* := 0$ is the unique global minimum of (5.29) with $p > 0$ by continuity of ψ.
- Otherwise, we set $\psi'(\tilde{t})$=0 with $\tilde{t} \in (0,p)$. Then from (5.33), we have

$$\psi'(t) \begin{cases} < 0, & \text{if } 0 < t < \tilde{t} \\ = 0, & \text{if } t = \tilde{t} \\ > 0, & \text{if } t > \tilde{t}, \end{cases} \tag{5.34}$$

  by continuity of ψ, we know $t_* := \tilde{t}$ is the unique global minimum of (5.29) with $p > 0$, though it has no analytic formula. One can apply Newton's methods to obtain a very good approximation of $\tilde{t}$, by choosing a suitable starting point $t_0 > 0$.

Case 3: There exists the unique $\bar{t} \in (0,p)$ satisfying $\phi''(\bar{t}) = -\frac{1}{\lambda}$ (or $\phi''(0^+) < -\frac{1}{\lambda} < \phi''(p)$). Since ϕ'' is increasing on $(0,p)$, and consequently ψ'' is increasing on $(0,p)$, then ψ' is decreasing first till at $t = \bar{t}$ and then increasing to $\psi'(p)(> 0)$ at $t = p$.

- If $\psi'(\bar{t}) \geq 0$, then from (5.33), we have

$$\psi'(t) \geq 0, \quad t > 0.$$

  Hence, $t_* := 0$ is the global minimum of (5.29) with $p > 0$ by continuity of ψ.
- Otherwise, $\psi'(t) = 0$ has one or two solutions.
  - If $\psi'(0^+) \leq 0$, then $\psi'(t) = 0$ has one solution on $(0,p)$, setting as $\tilde{t}$ again, then from $\tilde{t} > \bar{t}$. From (5.33), we also have (5.34), which says $t_* := \tilde{t} \in (\bar{t},p)$ is the global minimum, and could be obtained by Newton's method where starting point satisfies $t_0 > \bar{t}$.
  - If $\psi'(0^+) > 0$, then $\psi'(t) = 0$ has two solutions on $(0,p)$, one is smaller than $\bar{t}$, which is the local maximum of ψ, and the other one is larger than $\bar{t}$, which is the local minimum of ψ, setting the larger solution as $\hat{t}$. We use Newton's method aims to find pretty approximate $\hat{t}$ by choosing any starting point $t_0 > \bar{t}$. By comparing the corresponding function values of $\psi(\hat{t})$ and $\psi(0)$, one then decides which candidate is the global minimum.

We now finish the solution analysis of (5.29). For $p < 0$, the analysis is highly similar with the case $p > 0$, so we just list the solution of different cases.
Case 1: $\phi''(t) < -\frac{1}{\lambda}, \forall t \in (p,0)$, $t_* := 0$ is the global minimum of f.
Case 2: $\phi''(t) > -\frac{1}{\lambda}, \forall t \in (p,0)$, if $\psi'(0^-) \leq 0$, $t_* := 0$ is the global minimum of ψ. Otherwise, the point satisfying $\psi'(t) = 0$ with $t \in (p,0)$ is the global minimum of ψ.
Case 3: There exists the unique $\bar{t} \in (p,0)$ satisfying $\phi''(\bar{t}) = -\frac{1}{\lambda}$, since ϕ'', and therefore ψ'' are decreasing on $(-\infty,0)$, we have $\psi'(t)$ is increasing till $t = \bar{t}$ and then decreasing. Hence, if $\psi'(\bar{t}) \leq 0$, then $t_* := 0$ is the global minimum of ψ. Otherwise,

- if $\psi'(0^-) \geq 0$, the solution $\psi'(t) = 0$ with $t \in (p,0)$ is the global minimum of ψ.
- if $\psi'(0^-) < 0$, calculate the solution of $\psi'(t) = 0$ with $t \in (p,\bar{t})$, setting as $\hat{t}$. By comparing $\psi(\hat{t})$ and $\psi(0)$ to obtain the global minimum.

In order to give an unified framework about the solution of the subproblem (5.29) for both cases $p > 0$ and $p < 0$, we set

$$\psi'(c) := \begin{cases} \psi'(0^+), & \text{if } p > 0 \\ \psi'(0^-), & \text{if } p < 0, \end{cases} \tag{5.35}$$

and

$$\phi''(c) := \begin{cases} \phi''(0^+), & \text{if } p > 0 \\ \phi''(0^-), & \text{if } p < 0. \end{cases} \tag{5.36}$$

We now summarize the method Algorithm 5.2 for the computation of a global minimum t_* of the subproblem (5.29). In order to guarantee the feasibility of (S.3), we maybe need to

Algorithm 5.2 (Newton-type Method for solving (5.29))

(S.0) Input p, if $p = 0$, set $t_* := 0$ and STOP. Otherwise, compute $\psi'(c)$ and $\phi''(c)$ as (5.35) and (5.36), respectively.

(S.1) If $\phi''(p) < -\frac{1}{\lambda}$, then $t_* := 0$ and STOP. Else if $\phi''(c) > -\frac{1}{\lambda}$, go to (S.2). Otherwise, go to (S.3).

(S.2)

  (S.2.1) If $\psi'(c)p \geq 0$, then $t_* := 0$ and STOP, else go to (S.2.2).

  (S.2.2) Apply Newton's Method for minimization the unconstrained function ψ with starting point t_0 satisfying $t_0 p > 0$ until it converges to some point $\tilde{t}$, set $t_* := \tilde{t}$ and STOP.

(S.3) Compute the solution $\bar{t}$ of equation

$$\phi''(t) = -\frac{1}{\lambda}.$$

If $\psi'(\bar{t})p \geq 0$, set $t_* := 0$ and STOP. Otherwise

  (S.3.1) If $\psi'(c)p \leq 0$, go to (S.2.2) by replacing $t_0 p > 0$ with $t_0 := p$. Otherwise go to (S.3.2) .

  (S.3.2) Apply Newton's Method for minimization the unconstrained function ψ with starting point t_0 satisfying $(t_0 - \bar{t})p > 0$ until it converges to some point $\bar{t}_*$, if $\psi(0) \leq \psi(\bar{t}_*)$, set $t_* := 0$, else set $t_* := \bar{t}_*$.

assume that ϕ'' is differential except at the origin if Newton method is employed for the solution of $\psi'' = 0$. The following result illustrates the sequence (or point 0) generated by Algorithm 5.2 converges to the global minimum of (5.29).

Theorem 5.15. *Let the sequence $\{t^k\}$ generated by Algorithm 5.2, then its limit point is the global minimum of subproblem* (5.29).

Proof. It is obvious that 0 is the global minimimum when $p = 0$. We only need to consider the case where $p \neq 0$, let us recall Algorithm 5.2 again, $\{t^k\}$ is either directly equal to 0 (here $k = 0$), or generated by the Newton method. If $t^k = 0$ and $k = 0$, then the underlying cases in Algorithm 5.2 imply that $\psi'(t) \leq 0, \forall t > 0$ when $p > 0$ and $\psi'(t) \geq 0, \forall t < 0$ when $p < 0$. Both situations imply from Assumption 5.13 (a) that 0 is global minimum of (5.29). It remains to consider the case where $\{t^k\}$ is generated from the Newton method, due to the choice of starting points in Algorithm 5.2, then its limit point satisfies evidently the corresponding equation, we now give the following specific analysis.

Case 1: If $\{t^k\}$ is generated by (S.2.2), then its limit $\tilde{t}$ satisfies the equation $\psi'(t) = 0$ for

$t \in (0, p)$ with $p > 0$ or $t \in (p, 0)$ with $p < 0$. Due to $\psi'(c)p < 0$ and $\phi''(c) > 0$, where $\psi'(c)$ and $\phi''(c)$ are as in (5.35) and (5.36), respectively, then one has

$$\psi'(t) \begin{cases} < 0, & \text{if } t < \tilde{t} \\ = 0, & \text{if } t = \tilde{t} \\ > 0, & \text{if } t > \tilde{t}, \end{cases} \tag{5.37}$$

which says that $\tilde{t}$ is the global mimimum of (5.29) from Assumption 5.13 (b).
Case 2: If $\{t^k\}$ is generated by (S.3.1), then its limit $\tilde{t}$ satisfies the equation $\psi'(t) = 0$ for $t \in (\bar{t}, p)$ with $p > 0$ or $t \in (p, \bar{t})$ with $p < 0$, where $\bar{t}$ satisfies $\phi''(t) = -1/\lambda$. Due to $\psi'(\bar{t})p < 0$ and $\psi'(c)p \le 0$, one has (5.37) again, hence $\tilde{t}$ is the global mimimum of (5.29).
Case 3: If $\{t^k\}$ is generated by (S.3.2), then its limit $\bar{t}_*$ satisfies the equation $\psi'(t) = 0$ for $t \in (\bar{t}, p)$ with $p > 0$ or $t \in (p, \bar{t})$ with $p < 0$, where $\bar{t}$ satisfies $\phi''(t) = -1/\lambda$. Due to $\psi'(\bar{t})p < 0$ and $\psi'(c)p > 0$, one has $\bar{t}_*$ is a local minimum on $(0, \infty)$ with $p > 0$ or $(-\infty, 0)$ with $p < 0$. Since 0 is a local minimum too, we have to compare $\psi(\bar{t}_*)$ with $\psi(0)$, the smaller is the minimizer of (5.29). □

The following example shows a specific analysis with the aid of ϕ_1.

Example 5.16. We now set $\phi = \phi_1$, then (5.29) becomes

$$\min_{t\in\mathbb{R}} \psi(t) := \frac{1}{2}(t-p)^2 + \lambda \frac{\alpha|t|}{1+\alpha|t|}, \tag{5.38}$$

where $\alpha > 0$ and $\lambda > 0$. To easily apply the approach, we need to calculate the first and second derivatives of ψ except at the origin:

$$\psi'(t) = \begin{cases} t - p + \lambda \dfrac{\alpha}{(1+\alpha t)^2}, & \text{if } t > 0 \\ t - p + \lambda \dfrac{-\alpha}{(1-\alpha t)^2}, & \text{if } t < 0 \\ -p + \lambda\alpha, & \text{if } t = 0^+ \\ -p - \lambda\alpha, & \text{if } t = 0^-, \end{cases}$$

and

$$\psi''(t) = \begin{cases} 1 + \lambda \dfrac{-2\alpha^2}{(1+\alpha t)^3}, & \text{if } t > 0 \\ 1 + \lambda \dfrac{2\alpha^2}{(1-\alpha t)^3}, & \text{if } t < 0 \\ 1 - 2\lambda\alpha^2, & \text{if } t = 0^+ \\ 1 + 2\lambda\alpha^2, & \text{if } t = 0^-. \end{cases}$$

Evidently, $\psi''(t) > 0, \forall t < 0$ always holds, which says the approach will be easier for the case $p < 0$. As a result, we will be careful for $p > 0$. We first notice that $\psi''(t)$ is increasing when $t > 0$. So, it is easy to check

$$\psi''(t) \overset{!}{=} 0, \quad t \in (0, p), \tag{5.39}$$

i.e., if $\psi''(0^+) > 0$, then $\psi''(t) > 0, \forall t \in (0, p)$, if $\psi''(p) < 0$, then $\psi''(t) < 0, \forall t \in (0, p)$, which says (5.39) has no solution. Otherwise, (5.39) has an unique solution, which is obviously given by

$$\bar{t} := \frac{1 - \sqrt[3]{2\lambda\alpha^2}}{\alpha} > 0. \tag{5.40}$$

Then we know $\psi''(t)$ is strictly monotonically increasing for both $t > 0$ and $t < 0$, so if $\phi''(a) > -\frac{1}{\lambda}$, then $\phi''(t) > -\frac{1}{\lambda}, \forall t \in (a, b)$, similarly $\phi''(b) < -\frac{1}{\lambda}$, then $\phi''(t) < -\frac{1}{\lambda}, \forall t \in (a, b)$. Meanwhile, $\psi''(t) > 0, \forall t < 0$ and ψ has third-order derivative except at the origin, so we can apply Newton's method for the computation of the solution of $\psi''(t) = 0$ and the solution furthermore is positive. In total, we conclude the solution method Algorithm 5.3.

Algorithm 5.3 (Newton-type Method for Solving (5.38))

(S.0) Input p, λ and α. If $p = 0$, then $t_* = 0$ and STOP. If $p < 0$, go to (S.1). Otherwise, go to (S.2)

(S.1)

- (S.1.1) If $p + \lambda\alpha \geq 0$, then $t_* := 0$ and STOP. Otherwise, go to (S.1.2),
- (S.1.2) Apply Newton's Method for minimization the unconstrained function ψ with starting point t_0 satisfying $t_0 p > 0$ until it converges to some point $\tilde{t}$, set $t_0 := \tilde{t}$ and STOP.

(S.2) If $1 - \lambda\frac{2\alpha^2}{(1+\alpha p)^3} < 0$, then $t_* := 0$ and STOP, else if $1 - 2\lambda\alpha^2 > 0$, then go to (S.2.1), otherwise, go to (S.2.2).

- (S.2.1) If $\lambda\alpha \geq p$, then $t_* := 0$ and STOP, else go to (S.1.2).
- (S.2.2) Calculate $\bar{t}$ as in (5.40). If $\psi'(\bar{t}) \geq 0$, then $t_* := 0$ and STOP. Otherwise,
  - (S.2.2.1) If $\lambda\alpha \leq p$, go to (S.1.2) by replacing $t_0 p > 0$ with $t_0 := p$, otherwise go to (S.2.2.2).
  - (S.2.2.2) Apply Newton's Method for minimization the unconstrained function ψ with starting point $t_0 > \bar{t}$ until it converges to some point $\bar{t}_*$, if $\psi(0) \leq \psi(\bar{t}_*)$, set $t_* := 0$, else set $t_* := \bar{t}_*$.

The following example aims to obtain the sparse solution of the famous portfolio problem.

Example 5.17. We first recall the portfolio optimization problem

$$\min_x \frac{1}{2} x^T Q x \quad \text{s.t.} \quad r^T x \geq \rho, \; e^T x = 1, \; x \geq 0 \tag{5.41}$$

by leaving the nonnegative box constraints of the augmented Lagrangian approach and using a regularization based on the l_q-quasi-norm, possibly with different value of $q \in (0, 1)$. Finally we consider the one-dimensional subproblem of this approach

$$\min_t \psi(t) := \frac{1}{2}(t - p)^2 + \lambda |t|^q \quad \text{s.t.} \quad t \geq 0, \tag{5.42}$$

where $u > 0$ denotes a given upper bound. The idea of solving this modified subproblem is very much the same as the one for solving the previous one, it is evidently shown that

$$\bar{t} = \left(-\frac{1}{\lambda q(q-1)} \right)^{1/(q-2)} > 0 \tag{5.43}$$

is the unique solution of twice derivative of the objective function of (5.42) on $(0, +\infty)$. Now, the solution method can be described in the following way

Algorithm 5.4 (Newton-type Method for solving (5.42))

(S.0) If $p \leq 0$, then $t_* := 0$ and STOP. Otherwise, go to (S.1).
(S.1) Compute $\bar{t}$ as in (5.43).
(S.2) If $\psi'(\bar{t}) \geq 0$, set $t_* := 0$ and STOP. Otherwise go to (S.3) .
(S.3) Apply Newton's Method for minimization the unconstrained function ψ with starting point $t_0 > \bar{t}$ until it converges to some point $\bar{t}_*$, if $\psi(0) \leq \psi(\bar{t}_*)$, set $t_* := 0$, else set $t_* := \bar{t}_*$.

Note that all realizations and results in this section can be applied in Chapter 4 by penalizing the constraint $c(x) \in K$.

5.5 Numerical Results

In this section, we use a series of test problems to demonstrate the power of Algorithm 5.1, where the subproblem (5.3) is solved by Algorithm 5.2. For the constrained problems, we first employ augmented Lagrangian function to penalize the constraints and use the classical augmented Lagrangian method (Algorithm 4.3.1 without slack variables) to solve it, the corresponding subproblem is solved by Algorithm 5.1.

We terminate Algorithm 5.1 if the iterates $x^{k,i}$ satisfy

$$\left\| \gamma_{k,i}(x^k - x^{k,i}) + \nabla f(x^{k,i}) - \nabla f(x^k) \right\|_\infty \leq \varepsilon, \tag{5.44}$$

where $\|\cdot\|_\infty$ stands for the maximum-norm. The parameters are chosen as $\tau := 2$, $\delta := 10^{-6}$, $\gamma_{\min} = 10^{-10}$, and $\gamma_{\max} = 10^{10}$. Moreover, we terminate Algorithm 5.2 for the case where Newton's method is used if the underlying error is less than $1e-6$. If Algorithm 4.3.1 is used, the termination criteria and parameters are the same as the rules in Section 4.5.

To show the effectiveness of our method, we compare it with Gurobi optimizer by the following numerical examples, which hence must be reformulated into some forms recognized by Gurobi solver. For the convenience of our readers, we here summarize useful information from the Gurobi optimizer reference manual [84]: "We first refer to the class of an optimization problem model. If the objective is quadratic, the model is quadratic program (QP). If any of constraints are quadratic, the model is a quadratically-constraint program (QCP). We sometimes refer to a few special cases of QCP: QCPs with convex constraints, QCPs with nonconvex constraints, bilinear programs, and second-order cone programs. Gurobi solver handles all of these model class, as well as linear programs and many kinds of mixed integer programs (MIPs). In addition, Gurobi accepts a number of additional constraints to the existing models, which are designed to allow you to define certain variable relationships. In particular, max, min, abs, and, or, indicator, and piecewise-linear constraints are permitted by Gurobi, which will be transferred into a MIP eventually. What's more, Gurobi also supports the following function constraints, like polynomial, (natural) exponential, (natural) logarithm, power, sine, cosine, and tangent." All problems are implemented in MATLAB (R2020a).

We start with a random testproblem in order to illustrate that the nonconvex regularizarion functions satisfying Assumption 5.13 generate sparser solution than the convex one.

	$\lambda := 0.1$	$\lambda := 0.5$
$g(x)$	nnz	nnz
$\|x\|_1$	92	51
$\|x\|_{0.1}^{0.1}$	8	0
$\|x\|_{0.5}^{0.5}$	12	47
$\|x\|_{0.9}^{0.9}$	14	39
$\frac{\alpha\|x\|}{1+\alpha\|x\|}$	12	12
$log(1+\alpha\|x\|)$	12	12

Table 5.1: Numerical results generated by Algorithm 5.1 with different regularizations.

5.5.1 Random Testproblems

This section aims to solve the following optimization problem

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2 + \lambda\sum_{i=1}^{n}\phi(x_i) \\ &\text{s.t. } x \geq 0, \end{aligned} \tag{5.45}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are random uniformly distributed matrix and vector, respectively, $\lambda > 0$ is the penalty parameter. In order to compare the sparsity of solution between the nonconvex and convex ϕ, we here set $n := 100$ and employ nonconvex ϕ_0 with $q \in \{0.1, 0.5, 0.9\}$, ϕ_1, ϕ_2 with arbitrary $\alpha \in \{0.01, 0.1, 0.5, 0.8, 1\}$, as well as convex $|\cdot|$ as ϕ to obtain the corresponding solutions. Note that λ should not be too large, otherwise the solution of (5.45) with the above ϕ is all 0 from Figure 5.1 and the graph of $|\cdot|$, hence we here chose $\lambda = 0.1$ and $\lambda = 0.5$, and employed Algorithm 5.1 with Algorithm 5.2 as the subproblem solver to generate the corresponding solutions, the results are listed in Table 5.1, where $g(x) := \sum_{i=1}^{n}\phi(x_i)$, nnz means the number of nonzero components of the solution. Note that for both ϕ_1 and ϕ_2 with different $\alpha \in \{0.01, 0.1, 0.5, 0.8, 1\}$, nnz always equals to 12 for both case $\lambda := 0.1$ and $\lambda := 0.5$. It demonstrates that the solution of (5.45) with ϕ_1 and ϕ_2 are not sensitive with the λ and α, however we can not provide theoretical guarantee till now. Meanwhile, ϕ_1 and ϕ_2 generate more stable solution than ϕ_0. Table 5.1 illustrates that the nonconvex regularizations do generate sparser solutions than the convex ones.

5.5.2 Image Restoration

The image restoration aims to restore the blurred or degraded image into the original one. The most classical image degradation model is

$$b = Ax + \eta,$$

where $\eta \in \mathbb{R}^m$ is the noise, $x \in \mathbb{R}^n$ is the undermined image, $b \in \mathbb{R}^m$ is the observed image, respectively, and A is an $m \times n$ blurring matrix, for details on its different kinds of choices, please see the book [85]. In order to obtain the underlying image x, since η is always unknown, we alternatively solve

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|.$$

However we still could not get a satisfactory solution because the system is very sensitive to the noise and lack of information [57]. To overcome the difficulty, some regularization functions are used. In this numerical experiment, we consider the following optimization problem

$$\min_{x\in\mathbb{R}^n_+} \ \frac{1}{2}\|Ax-b\|^2+\beta\sum_{i=1}^{n}\phi_1(x_i), \tag{5.46}$$

where ϕ_1 is stated in (5.30). Note that a nonnegative constraint is given in (5.46), which promotes Algorithm 5.2 (or Algorithm 5.3) more easier, since in this case, 0 is the global minimimum of subproblem (5.3) when $p^k := x^k - \frac{1}{\gamma_k}\nabla f(x^k) \leq 0$.

Employing the auxiliary variables $y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, (5.46) is equivalent to

$$\begin{aligned}
&\min_{x,y,z} \ \frac{1}{2}(Ax)^\top Ax - b^\top Ax + \frac{1}{2}\|b\|^2 + 0_{1\times n}y + \beta z \\
&\text{s.t. } z = \sum_{i=1}^{n} y_i, \ y_i = \frac{\alpha x_i}{1+\alpha x_i} \ \forall i = 1,\ldots,n, \ (x,y,z) \in \mathbb{R}^{2n+1}_+,
\end{aligned}$$

where $0_{1\times n}$ defines all-0 $1\times n$-row vector. By introducing $r := (-b^\top A, 0_{1\times n}, \beta)$, $s := (x,y,z)^T$, and Q is a $2n+1\times 2n+1$ dimensional matrix whose diagonal entries are $\frac{1}{2}A^\top A$, $0_{n\times n}$, and 0, the remaining entries are 0, it can be rewritten as

$$\begin{aligned}
&\min_{s\in\mathbb{R}^{2n+1}_+} \ s^\top Qs - rs + \frac{1}{2}\|b\|^2 \\
&\quad \text{s.t. } (0_{1\times n}, I_{1\times n}, -1)s = 0, \ s_{n+j} = \frac{\alpha s_j}{1+\alpha s_j} \ \forall j = 1\ldots n,
\end{aligned} \tag{5.47}$$

with $I_{1\times n}$ as all-1 $1\times n$-row vector and $0_{n\times n}$ as $n\times n$ dimensional zero matrix. By rewriting the last nonlinear constraints, (5.47) is equivalent to

$$\begin{aligned}
&\min_{s\in\mathbb{R}^{2n+1}_+} \ s^\top Qs - rs + \frac{1}{2}\|b\|^2 \\
&\quad \text{s.t. } (0_{1\times n}, I_{1\times n}, -1)s = 0, \\
&\qquad\quad \alpha s_{n+j}s_j + s_{n+j} - \alpha s_j = 0 \ \forall j = 1\ldots n,
\end{aligned} \tag{5.48}$$

such (5.48), with a quadratic objective function and linear plus quadratic constraints, is a QCP, and thus could be solved by Gurobi. Note that the increased dimension maybe cause some difficulties, such as time consumption, especially for the large-scale problems.

For the image restoration problem, the corresponding optimization problem (5.46) is always large-scale, because the discretized scenes have a large number $n = l\times l$ of pixels, which probably causes Gurobi solver is pretty time-consuming, even fails to generate the solution. As a result, we first consider 8 different $n-$dimensional random test instances with $n := 10, 20, 50, 100, 200, 500, 1000, 2000$. We stop the iteration if (5.44) is satisfied with $\varepsilon = 10^{-3}$. Meanwhile, the termination time of Gurobi is set as 30 minutes. The results are listed in Table 5.2, where

n: dimension of random (5.46),
It.: iterations of Algorithm 5.1,
$f_{\rm opt}$: optimal function value generated by Algorithm 5.1 and Gurobi,
t(s): cost time taken from Algorithm 5.1 and Gurobi.

Table 5.2: Numerical results for n-dimensional random problems.

	Proximal Gradient Method			Gurobi	
n	It.	f_{opt}	t(s)	f_{opt}	t(s)
10	99	0.3792	0.30	0.3868	3.32
20	200	0.8635	0.53	0.8807	4.18
50	182	1.9497	0.35	1.9889	14.33
100	335	4.3875	0.64	4.4754	1800
200	469	8.4859	1.29	8.65712	1800
500	534	19.8123	2.33	20.2100	1800
1000	729	40.3168	31.45	171.4790	1800
2000	752	78.8754	36.95	334.3280	1800

The main observations are as follows: The optimal function value computed by Algorithm 5.1 is always superior than Gurobi, and the CPU time is completely shorter. Meanwhile, Gurobi has much difficulty in dealing with large-scale optimization problems.

Let us look back the image restoration problem. We here test the Cameraman images and the Phantom image. The pixels of the observed image are contaminated by Gaussian white noise with signal-to-noise ratios of 60 dB with blurring, the blurring function is chosen to be a two-dimensional Gaussian:

$$a(i,j) = e^{-2(i/3)^2-2(j/3)^2}$$

truncated such that the function has a support of 7×7. The corresponding blurring matrix is chosen as the Kronecker product of Toeplitz-plus-Hankel matrices.

We use Algorithm 5.1 and Gurobi to recover the Cameraman images by solving (5.46). Due to our limited understanding of the Gurobi solver's code, we represent the matrix A explicitly (i.e., not as a operator), which results in only low-pixel images being processed when MATLAB interface is invoked. The parameters are chosen as: $\alpha := 1$ and $\beta := 0.001$. For Algorithm 5.1 with Algorithm 5.2 as the subproblem solver, we stop the algorithm if (5.44) is satisfied with $\varepsilon := 10^{-3}$ and Algorithm 5.2 terminates if the corresponding residual is less than 10^{-3}. The termination time of Gurobi is set as 30 minutes. We first choose very small 36×36 Cameraman image, the corresponding observed image is pretty blurred due to the too low pixels, Gurobi eventually generated a totally grey image, which seems reasonable from the observation of Table 5.2. Then we decided to recover the 128×128 Cameraman image (we did not detect more big image, since Gurobi costs pretty much time on presolution), Gurobi failed too, and we then used Algorithm 5.1 by choosing the observed image as the initial point, then the Cameraman image was restored after 31 iteration, which costed about 31.54s. For comparison, we also chose quasi-l_1 norm as the regularization function, the numerical results are listed in Figure 5.2, from which, nonconvex ϕ_1 does achieve a better recovery of Cameraman image than the convex quasi-l_1 norm. However, to be honest, the smoothly varying regions are a little deficient to be recovered. The reason is given in the following: Let us consider again (5.46) with a more general class of regularization function, which is addressed as

$$\min_{x \in \mathbb{R}^n_+} \frac{1}{2}\|Ax - b\|^2 + \beta \sum_{i=1}^{n} \phi(w_i^\top x), \tag{5.49}$$

where $w_i \in \mathbb{R}^n$ are difference operators. [127] mentioned: "Edges in images and breaking points in signals concentrate critically important information. Hence we have the require-

Figure 5.2: First line: the original (left), observed (right) Cameraman Image; Second line: restored Cameraman image with ϕ_1 (left) and quasi-l_1 norm (right) as regularization using Algorithm 5.1.

ment that ϕ leads to minimizers x involving large differences $|w_i^\top x|$ at the location of edges in the original signal or image and smooth differences elsewhere. However, $\phi'(0^+) > 0$ gives rise to local minimum $\hat{x}$ such that $w_i^\top \hat{x} = 0$ for some $i \in \{1, \dots, n\}$." The numerical results from [127] also showed the same deficiency in the smoothly varying regions. As a result, we also tested the 128×128 Phantom image, which has fewer smooth varying regions, whose results are depicted in Figure 5.3. Note that quasi-l_1 norm as the regularization function hardly achieves a recovery from the observed images, and the blurred diaphragms in the restored images seem caused by the low pixels from the numerical results of [57].

5.5.3 Portfolio Problems

Example 5.17 aimed to find the sparse solution of the classical portfolio optimization problem, by using the augmented Lagrangian approach and then regularizing it by l_q-quasi-norm. In this section, l_q-quasi-norm is directly applied as a regularization of the objective function in order that Gurobi can generate the sparse solution which provides a reference of solution. We now introduce the problem:

$$\begin{aligned} \min_x \quad & \frac{1}{2} x^T Q x + \alpha \|x\|_q^q \\ \text{s.t.} \quad & \mu^T x \geq \varrho, \quad e^T x = 1, \quad 0 \leq x \leq u, \end{aligned} \tag{5.50}$$

where $Q \in \mathbb{R}^{n \times n}$ and $\mu \in \mathbb{R}^n$ denote the covariance matrix and the mean of $n \in \mathbb{N}$ possible assets, respectively. $\varrho \in \mathbb{R}$ is some lower bound for the expected return, and $u \in \mathbb{R}$ provides an upper bound for the individual assets with the portfolio, α is a constant as the regularization parameter. The data Q, μ, ϱ, u are created by the test problem collection [74]. which is available from the webpage `https://commalab.di.unipi.it/datasets/MV/`. Here, we used all 30 test instances of dimension $n := 200$, three different value $q \in \{0.1, 0.5, 0.9\}$, and $\alpha := 1$ for each problem. We apply Algorithm 4.3.1 without slack

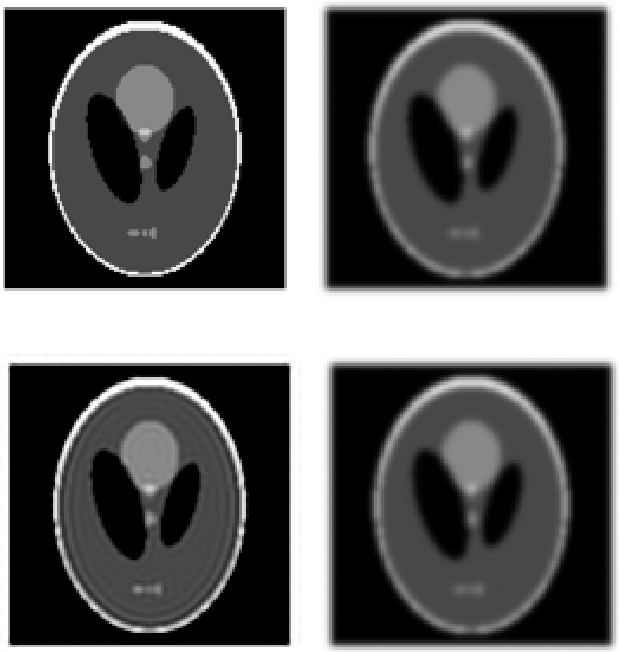

Figure 5.3: First line: the original (left), observed (right) Phantom Image; Second line: restored Phantom image with ϕ_1 (left) and quasi-l_1 norm (right) as regularization using Algorithm 5.1.

variables and Gurobi for the test problems. Highly like the analysis of Section 5.5.2, we need to give more comments about how to reformulate (5.50) into some form recognized by Gurobi.

By introducing two auxiliary variables $y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, which then naturally causes the dimension of (5.50) increased, to avoid abuse of notations we need to express all matrix (or vector)'s dimension clearly. Denoting $0_{p\times q}$ as all-0 $p \times q$ matrix, then (5.50) is equivalent to

$$\begin{aligned} \min_{x,y,z} \quad & \frac{1}{2} x^T Q x + 0_{1\times n} y + \alpha z \\ \text{s.t.} \quad & \mu^T x \geq \varrho, \quad e^T x = 1, \quad 0_{n\times 1} \leq x \leq u, \\ & z = \sum_{i=1}^{n} y_i, \quad y_i = |x_i|^q \ \forall i = 1, \dots, n, \quad (x, y, z) \in \mathbb{R}^{2n+1}, \end{aligned}$$

with $0_{1\times n}$ as all-0 n-row vectors. Setting $s := (x, y, z)^T$, $r := (0_{1\times n}, 0_{1\times n}, \alpha)$, $\hat{\mu} := (\mu; 0_{n\times 1}; 0)$, $\hat{e} := (e; 0_{n\times 1}; 0)$, $\hat{Q}$ as a $2n+1 \times 2n+1$ dimensional matrix whose diagonal entries are $0.5Q$, $0_{n\times n}$ ($n \times n$ dimensional zero matrix), 0 and the remaining entries are 0, the problem can be transferred as

$$\begin{aligned} \min_{s\in\mathbb{R}^{2n+1}} \quad & s^\top \hat{Q} s + rs \\ \text{s.t.} \quad & \hat{\mu}^\top s \geq \varrho, \quad \hat{e}^\top s = 1, \quad 0_{(2n+1)\times 1} \leq s \leq (u; u; \infty) \\ & (0_{1\times n}, I_{1\times n}, -1) s = 0, \quad s_{n+j} = |s_j|^q \ \forall j = 1, \dots, n. \end{aligned} \tag{5.51}$$

where $I_{1\times n}$ is all-1 n-row vector (Please ignore the overuse of $s \leq (u; u; \infty)$). It, rewriting the constraints by $c := (0_{1\times n}, I_{1\times n}, -1)$, $\hat{u} := [u; u; \infty)$, is then equivalent to

$$\begin{aligned} \min_{s\in\mathbb{R}^{2n+1}} \quad & s^\top \hat{Q} s + rs \\ \text{s.t.} \quad & \hat{\mu}^\top s \geq \varrho, \quad \hat{e}^\top s = 1, \quad cs = 0, \\ & s_{n+j} = |s_j|^q \ \forall j = 1, \ldots, n, \quad 0_{(2n+1)\times 1} \leq s \leq \hat{u}. \end{aligned} \tag{5.52}$$

The objective function of (5.52) is obviously quadratic, some parts of whose constraints are linear and additive power functional constraints are included, which, from the mentioned above, can be solved by Gurobi. Because of the equivalence of (5.50) and (5.52), (5.50) can be traced a global minimizer. We put 0.5 hours as the time termination of Gurobi for each test problem.

Note that an extra box constraint from (5.52) needs to consider in Algorithm 4.3.1 and Algorithm 5.1 invoking Algorithm 5.2, which makes Algorithm 5.2 more easier since 0 is the global minimum of subproblem (5.3) only if $p^k := x^k - \frac{1}{\gamma_k}\nabla f(x^k) \leq 0$. As a result, we just consider Algorithm 5.2 for the case where $p^k > 0$, and project the solution on the box set. The corresponding results are summarized in Figure 5.4 for the three different $q \in \{0.1, 0.5, 0.9\}$. This figure compares the optimal function values obtained by the above two methods for each of the thirty test problems. It evidently shows that the function value generated by our method is lower than Gurobi solver (within 30 minutes) for the vast majority of the test problems, the latter provides a reference for optimal function value of (5.50). On the other hand, like the analysis in Section 3.6.2 (or see [94]), when q is taken smaller, even near to 0, the optimization problem (5.50) is getting more demanding and is therefore difficult to solve, hence the gap between the optimal function values obtained by the two methods are larger. In addition, our method finished the whole 90 optimization problems (three different q with 30 test problems) in 150 seconds. By above, it is well-found that our method computes better sparse solution effectively than Gurobi (within 30 minutes).

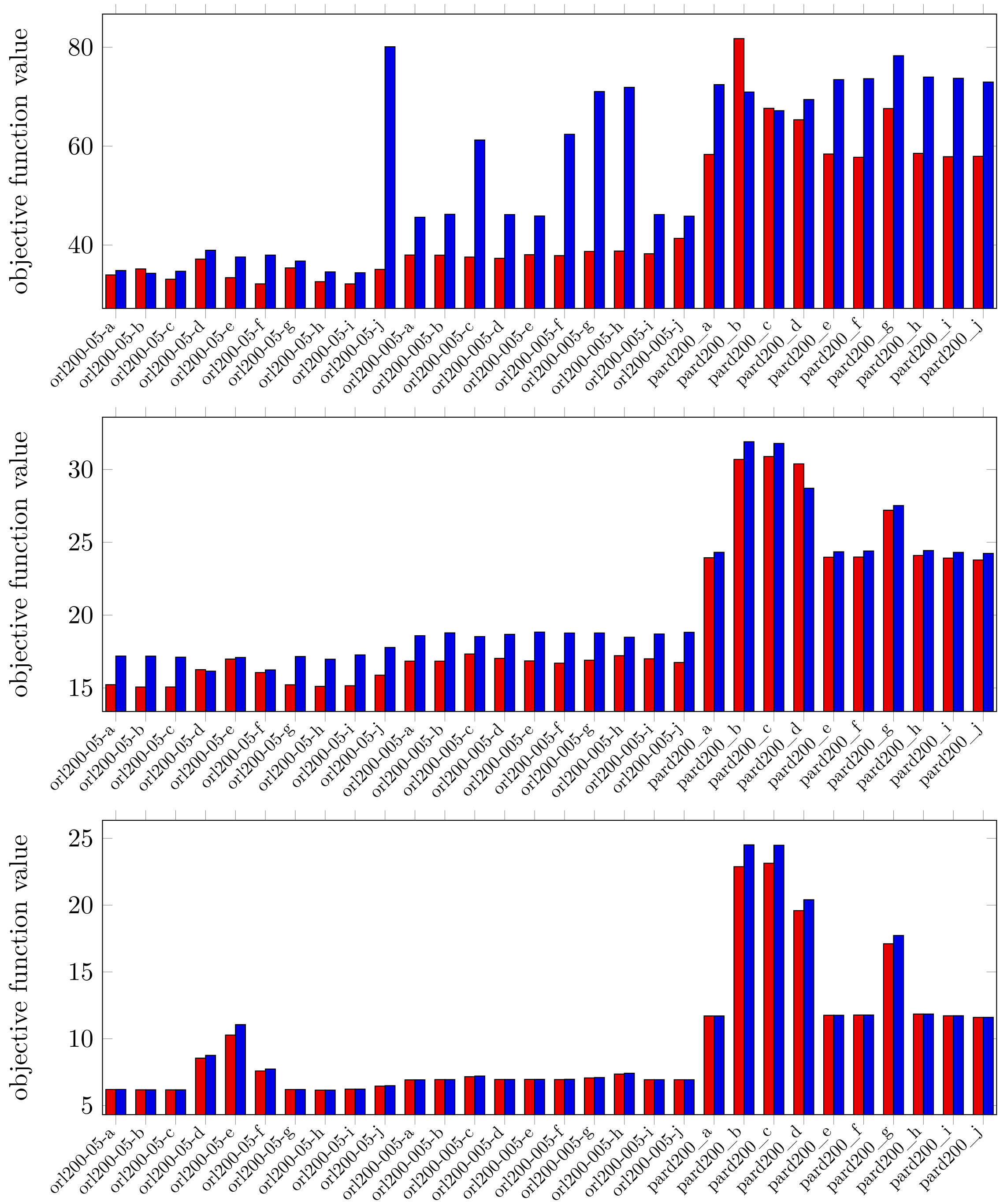

Figure 5.4: Optimal function values obtained by Algorithm 4.3.1 (red) and Gurobi(blue), applied to the portfolio optimization problem (5.50) regularized by q-norm with $q = 0.1$, $q = 0.5$, and $q = 0.9$ (top to bottom).

6. Conclusions and Outlooks

I will close my thesis with the final conclusions, as well as some future works.

Conclusions

This thesis concluded my research works [66,93,94], where the general optimization problem was considered

$$\min_{x\in\mathbb{X}}\ f(x)+g(x) \qquad \text{s.t.} \quad G(x)\in C,\ x\in D, \tag{F}$$

where $\mathbb{X}$ and $\mathbb{Z}$ are two Euclidean spaces, $f:\mathbb{X}\to\mathbb{R}$ is continuously differentiable, $g:\mathbb{X}\to\overline{\mathbb{R}}$ is lower semicontinuous, $G:\mathbb{X}\to\mathbb{Z}$ is also continuously differentiable, $C\subset\mathbb{Z}$ is nonempty closed convex, and $D\subset\mathbb{X}$ is nonempty closed (possibly nonconvex). This model is very general, in totally nonconvex setting, and matrix-free, which hence can be encoded into many types of well-known programs, e.g., norm or rank minimization programs, problems with complementarity or cardinality constraints, as well as matrix optimization problems.

This thesis exploited the stationarity and associated regularity concepts of (F), the algorithms (including subproblem solvers) for solving (F) numerically and theoretically, as well as the convergence and rate-of-convergence results of of entire sequence generated by proximal gradient methods for solving the unconstrained (F).

Note that (F) reduces to (P) discussed in Chapter 3 by setting

$$g:=0,\ \mathbb{Z}:=\mathbb{Y}, \tag{6.1}$$

boils down to (CP) stated in Chapter 4 by setting

$$c:=\begin{pmatrix}G\\ \mathrm{Id}\end{pmatrix},\quad K:=\begin{pmatrix}C\\ D\end{pmatrix},\ \text{with identity mapping Id}, \tag{6.2}$$

as well as $\mathbb{Z}:=\mathbb{Y}\times\mathbb{X}$, and becomes (Q) described in Chapter 5 by setting

$$C:=\mathbb{Z}:=\mathbb{Y},\quad D:=\mathbb{X}. \tag{6.3}$$

Stationarities and Regularities

In Chapter 3, Definition 3.1 and Definition 3.2 gave the definitions of M- and AM-stationary point of (F) with setting (6.1), respectively. They are well-understood from the respective of KKT and AKKT, and M-stationarity (AM-stationarity) corresponds with KKT (AKKT) if D is convex. A constraint qualification called AM-regularity was recalled in Definition 3.4, which ensures that an AM-stationary point is also M-stationary. To be honest, AM-regularity is comparatively weak constraint qualification, there Lemma 3.7, Lemma 3.8, Lemma 3.10 gave the sufficient condition (also necessary condition in Lemma 3.10) of the attainment of AM-regularity.

In order to make the above stationary points suitable well to the composite optimization problem (F) with setting (6.2), Definition 3.1 and Definition 3.2 need to be adjusted in some sense where the nonsmooth objective function has to be taken into consideration for the convergence analysis. Therefore, Definition 4.3 and Definition 4.4 in Chapter 4 denoted the (another) M- and AM-stationary point of (F) with setting (6.2), respectively. In order to guarantee that underlying AM-stationary point is already M-stationary, some regularity condition in Definition 4.7 was introduced, which has been proved in Corollary 4.9 as the weakest qualification condition associated with AM-stationarity based on the terminology coined in [9]. Note that it is not a real constraint qualification in a narrow sense since it is relevant with the (partial) objective function.

Note that the AM-regularity of (F) with setting (6.2) can be reduced into the one with setting (6.1) provided that the nonsmooth part of function g has local Lipschitz gradient and the constraints are slightly adjusted in some sense.

Augmented Lagrangian Methods

This thesis is concerned to safeguarded augmented Lagrangian methods (Algorithm 3.4.1 and Algorithm 4.3.1) to solve (F), which have been never applied to solve the programs with generally structured nonconvex constraints. The augmented Lagrangian scheme is employed to penalize the constraints, more specifically, which, in Chapter 3 penalizes the constraint $G(x) \in C$ and leaves $x \in D$ explicitly, in Chapter 4 penalizes the constraint $c(x) - s = 0$ with added slack variable s which can avoid the difficulties caused by the discontinuity of distance operator on the nonconvex set D and the hard calculation of projections directly on the set $\{x \in \mathbb{X} \mid c(x) \in K\}$, cf. Remark 4.13. We claim that the underlying subproblems in this thesis need to be addressed inexactly and their limit point should be hence approximate M-stationary point, which guaranteed that every feasible accumulation point of the sequence generated by the safeguarded augmented Lagrangian method is AM-stationary and furthermore M-stationary point under AM-regularity, cf. Corollary 3.18 and Corollary 4.19.

Subproblem solvers

As mentioned above, the resulting subproblems deduced from the augmented Lagrangian methods need to generate an approximate M-stationary limiting point, which is a little challenging for (F) with the general constraints since some subproblem solvers normally achieve C-stationary point in some special cases, like MPCCs.

In Chapter 3, we employ the spectral gradient method Algorithm 3.3.1 for solving the constrained problems $(\mathrm{Q}(j,i))$ with constraint $x \in D$, which is first applied to solve the programs with nonconvex constraint. Theorem 3.14 illustrated that the infinite sequence of iterates of Algorithm 3.3.1 converges towards M-stationary point along some subsequences. Note that $(\mathrm{Q}(j,i))$ of Algorithm 3.3.1 in every iteration has to be solved effectively, which actually is a mathematical programming with a quadratic objective function and a comparatively complicated constraint $x \in D$, in other words, the projections on D must be shown explicitly. Section 3.5 analyzed the projection principles for some cases of D, in particular, complementarity constraints, cardinality constraints, and rank constraints.

In Chapter 4, a proximal gradient method, called PANOC$^+$, was used for the composite subproblems with slack variables, whose pseudocodes were shown in Algorithm 4.4.1.

Proposition 4.21 demonstrated that the accumulation point of the sequence generated by Algorithm 4.4.1 is an M-stationary point of this subproblem. Meanwhile, accelerated PANOC$^+$ can get rid of the bad-scaling and ill-conditioning in some sense from the numerical results of Section 4.5.2.

Convergence analysis for proximal gradient method

Proximal gradient methods are always used for the unconstrained composite optimization problems (F) with setting (6.3). In Chapter 5, Algorithm 5.1 and its some known results were recalled in Section 5.2, whose every accumulation point is M-stationary with the aid of local Lipschitz ∇f, which is the weaker version of widely-used (global) Lipschitz gradient, however no convergence of entire sequence was proposed there. Hence, Section 5.3 filled the gaps by showing that the entire sequence generated by Algorithm 5.1 converges to a limit with a suitable rate, provided that this point satisfies the Kurdyka–Łojasiewicz property. The underlying convergence theory is only based on a merely local Lipschitz assumption on ∇f, no more requirements were used, e.g., the boundedness of iterates and stepsizes. More specifically, Lemma 5.7 showed that when the iterates are locally around the accumulation point, then the corresponding stepsizes are bounded. Lemma 5.8 stated that the entire sequence of objective function converges globally. They are together to generate a sufficiently small constant (5.13) which was then used as to characterize a closed ball centered at the accumulation point with this constant as the radium, subsequently the famous assertion of error bound of the subdifferential was obtained in Lemma 5.10 only if iterates belong to such closed ball. Based on these, a technique proof in Theorem 5.11 was given to obtain the convergence of the entire sequence, hence some rate-of-convergence results were following, cf. Theorem 5.12.

Future Works

I hope the theory and practical results presented through this thesis will be useful and helpful to the other researchers. There are still some ideas for the future research.

- **Projections of D**: In Chapter 3, in order to ensure that $(\mathrm{Q}(j,i))$ of Algorithm 3.3.1 can be solved successfully, it is necessary to find the projections of any points onto the set D. Though Section 3.5 discussed some cases of D, it still needs to exploit more interesting and meaningful ones in order our model and algorithm becoming more practical and applicable.
- **Exploitation of g**: Highly similar with the above idea, we are devoted to find the element of proximality operator of g at any point as in (4.24) $(g := \psi)$, in other words, solve the following problems

$$\min_x \ \phi(z) + \langle \nabla\phi(z), x - z\rangle + \frac{1}{2\gamma}\|x - z\|^2 + g(x)$$

  with some $z \in \mathbb{X}$, then the structure of g plays more important role for the solution of this problem.
- **Subproblem Solvers**: As Example 3.27 and Section 4.5.1 are shown, the corresponding (proximal) gradient-type solver has difficulty in dealing with ill-conditioned or bad-scale problems, we will therefore exploit the other algorithms as subproblem solvers in order to avoid such drawbacks and ensure that limit point (or accumulation point) of the generated sequence is M-stationary.

- **Convergence of nonmonotone proximal gradient methods**: In [65], the author showed that the global convergence properties of Algorithm 5.1 from Theorem 5.6 remain valid if, instead of the exploited monotone line search, a nonmonotone scheme is used to determine the step sizes. In the future, it should be clarified whether the results of Theorems 5.11 and 5.12 can be carried over to nonmonotone proximal gradient methods.

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A. Appendix

A.1 Complete Numerical Results for MAXCUT Problems

Tables A.1 and A.2 present the details of Algorithm 3.4.1 applied to the MAXCUT problem as described in Section 3.6.4. The columns have the following meanings:

probl.: name of the test problem,
vert.: number of vertices of the underlying graph,
edges: number of edges with positive weights of the underlying graph,
k: number of (outer) iterations of Algorithm 3.4.1,
j: number of inner iterations of the spectral gradient method (accumulated),
f-ev.: number of function evaluations (accumulated),
f_{ALM}: function value at the final iterate generated by Algorithm 3.4.1,
feas.: feasibility measure at the final iterate generated by Algorithm 3.4.1,
ρ: penalty parameter at the final iterate generated by Algorithm 3.4.1,
f_{opt}: optimal function value taken from the report [153],
f_{SDP}: function value obtained by the SDP relaxation, and
rk: rank of the final matrix obtained by solving the SDP relaxation.

Recall that the SDP relaxation was also solved by Algorithm 3.4.1 with the semidefiniteness constraint taken as the complicated constraint, whereas the remaining linear equality constraint was penalized by the augmented Lagrangian approach.

Table A.1: Numerical results for MAXCUT problems, `rudy` collection.

probl.	vert.	edges	k	j	f-ev.	f_{ALM}	feas.	ρ	f_{opt}	f_{SDP}	rk
g05_100.0	100	2475	14	310	349	1420	2.8e-5	20.0	1430	1463.52	6
g05_100.1	100	2475	32	396	448	1420	9.4e-5	2.0	1425	1464.05	6
g05_100.2	100	2475	21	339	378	1431	9.9e-5	2.0	1432	1461.65	5
g05_100.3	100	2475	28	500	572	1411	1.0e-4	2.0	1424	1456.68	7
g05_100.4	100	2475	24	475	550	1430	9.7e-5	2.0	1440	1468.80	6
g05_100.5	100	2475	36	409	507	1415	9.8e-5	2.0	1436	1464.66	6
g05_100.6	100	2475	26	476	535	1429	8.8e-5	2.0	1434	1463.17	6
g05_100.7	100	2475	12	292	328	1428	5.5e-5	20.0	1431	1464.27	5
g05_100.8	100	2475	30	325	361	1425	8.3e-5	2.0	1432	1464.75	5
g05_100.9	100	2475	35	344	391	1415	8.2e-5	2.0	1430	1462.39	5
g05_60.0	60	885	20	286	322	530	7.6e-5	3.3	536	550.05	5
g05_60.1	60	885	18	290	323	524	9.5e-5	3.3	532	543.11	5
g05_60.2	60	885	19	283	313	524	6.3e-5	3.3	529	543.18	4
g05_60.3	60	885	18	253	302	523	6.2e-5	3.3	538	548.65	4
g05_60.4	60	885	18	326	413	526	9.6e-5	3.3	527	541.39	5
g05_60.5	60	885	18	219	252	523	6.5e-5	3.3	533	542.59	6
g05_60.6	60	885	18	278	311	520	9.7e-5	3.3	531	544.72	5
g05_60.7	60	885	20	277	306	530	6.8e-5	3.3	535	550.42	5
g05_60.8	60	885	16	338	381	520	6.1e-5	3.3	530	543.98	5

Table A.1: Numerical results for MAXCUT problems, rudy collection (continued).

probl.	vert.	edges	k	j	f-ev.	f_{ALM}	feas.	ρ	f_{opt}	f_{SDP}	rk
g05_60.9	60	885	20	216	245	529	6.4e-5	3.3	533	549.89	5
g05_80.0	80	1580	17	241	263	918	6.9e-5	2.5	929	950.92	5
g05_80.1	80	1580	22	380	465	929	8.3e-5	2.5	941	957.25	4
g05_80.2	80	1580	24	307	341	923	9.2e-5	2.5	934	955.55	5
g05_80.3	80	1580	28	324	362	906	7.9e-5	2.5	923	947.59	5
g05_80.4	80	1580	26	303	346	923	8.2e-5	2.5	932	955.32	5
g05_80.5	80	1580	22	299	343	915	7.5e-5	2.5	926	947.51	6
g05_80.6	80	1580	27	257	293	920	7.8e-5	2.5	929	948.68	5
g05_80.7	80	1580	22	289	321	915	9.0e-5	2.5	929	949.86	5
g05_80.8	80	1580	27	436	504	918	7.6e-5	2.5	925	946.67	5
g05_80.9	80	1580	25	266	296	918	8.7e-5	2.5	923	943.66	6
pm1d_100.0	100	4901	13	373	475	338	2.5e-5	20.0	340	405.39	6
pm1d_100.1	100	4901	13	517	612	310	3.2e-5	20.0	324	396.09	6
pm1d_100.2	100	4901	12	489	571	370	3.0e-5	20.0	389	453.98	7
pm1d_100.3	100	4901	15	419	492	396	2.8e-5	20.0	400	459.03	5
pm1d_100.4	100	4901	14	450	526	349	6.0e-5	20.0	363	430.32	6
pm1d_100.5	100	4901	16	345	383	440	2.6e-5	20.0	441	510.72	5
pm1d_100.6	100	4901	14	511	623	360	4.0e-5	20.0	367	431.92	5
pm1d_100.7	100	4901	12	246	278	348	3.8e-5	20.0	361	421.52	5
pm1d_100.8	100	4901	14	414	494	365	2.3e-5	20.0	385	438.03	6
pm1d_100.9	100	4901	13	355	408	404	7.8e-5	20.0	405	470.66	5
pm1d_80.0	80	3128	16	395	445	214	2.0e-5	25.0	227	269.97	5
pm1d_80.1	80	3128	25	431	497	239	7.0e-5	2.5	245	292.60	5
pm1d_80.2	80	3128	25	320	354	270	9.4e-5	2.5	284	325.99	6
pm1d_80.3	80	3128	13	405	475	283	4.4e-5	25.0	291	331.31	5
pm1d_80.4	80	3128	11	307	355	248	2.0e-5	25.0	251	290.75	5
pm1d_80.5	80	3128	28	342	381	233	8.9e-5	2.5	242	290.50	5
pm1d_80.6	80	3128	15	376	430	192	5.5e-5	25.0	205	253.56	5
pm1d_80.7	80	3128	11	684	895	244	5.0e-5	25.0	249	292.52	4
pm1d_80.8	80	3128	27	343	373	288	8.5e-5	2.5	293	329.98	5
pm1d_80.9	80	3128	24	310	346	258	9.0e-5	2.5	258	294.31	5
pm1s_100.0	100	495	16	252	289	118	2.0e-6	20.0	127	143.23	6
pm1s_100.1	100	495	17	329	376	119	1.9e-6	20.0	126	144.61	5
pm1s_100.2	100	495	17	281	312	116	9.5e-5	20.0	125	140.23	5
pm1s_100.3	100	495	17	357	429	100	8.8e-5	2.0	111	130.09	6
pm1s_100.4	100	495	16	298	369	127	9.0e-5	2.0	128	145.61	6
pm1s_100.5	100	495	17	335	367	117	9.0e-5	2.0	128	144.66	5
pm1s_100.6	100	495	16	252	274	118	9.0e-5	20.0	122	139.91	6
pm1s_100.7	100	495	17	422	491	105	6.6e-5	2.0	112	126.80	5
pm1s_100.8	100	495	16	288	329	118	7.9e-5	2.0	120	135.86	6
pm1s_100.9	100	495	17	248	273	125	7.0e-7	20.0	127	143.52	5
pm1s_80.0	80	316	14	291	342	73	4.6e-5	2.5	79	90.29	4
pm1s_80.1	80	316	14	255	287	81	6.2e-5	2.5	85	96.18	4
pm1s_80.2	80	316	14	416	492	80	7.6e-5	2.5	82	94.02	6
pm1s_80.3	80	316	13	231	265	77	4.8e-5	2.5	81	92.14	5
pm1s_80.4	80	316	15	298	374	62	5.4e-5	2.5	70	82.06	5
pm1s_80.5	80	316	13	219	243	86	7.9e-5	2.5	87	98.69	4
pm1s_80.6	80	316	13	246	297	70	6.1e-5	2.5	73	85.69	5
pm1s_80.7	80	316	15	346	411	81	4.4e-5	2.5	83	95.45	5
pm1s_80.8	80	316	13	341	394	79	7.5e-5	2.5	81	95.47	5
pm1s_80.9	80	316	15	198	230	66	4.3e-5	2.5	70	82.00	5
pw01_100.0	100	495	16	352	421	1963	9.7e-5	20.0	2019	2125.43	5
pw01_100.1	100	495	16	438	538	2025	5.9e-5	20.0	2060	2161.61	5
pw01_100.2	100	495	17	365	432	2009	3.4e-5	20.0	2032	2135.62	5
pw01_100.3	100	495	16	357	443	2053	6.7e-5	20.0	2067	2167.93	5
pw01_100.4	100	495	15	361	432	1990	4.5e-5	20.0	2039	2116.66	5
pw01_100.5	100	495	16	371	420	2068	8.9e-5	20.0	2108	2195.59	5
pw01_100.6	100	495	17	387	442	2010	5.8e-5	20.0	2032	2135.28	5
pw01_100.7	100	495	17	260	304	2068	6.0e-5	20.0	2074	2182.48	5
pw01_100.8	100	495	15	253	293	2022	7.0e-5	20.0	2022	2102.02	6
pw01_100.9	100	495	17	612	748	1986	5.5e-5	20.0	2005	2114.21	5
pw05_100.0	100	2475	24	334	405	8118	8.9e-5	20.0	8190	8427.71	6
pw05_100.1	100	2475	12	299	360	7954	5.3e-5	200.0	8045	8260.33	5
pw05_100.2	100	2475	27	403	519	7915	7.6e-5	20.0	8039	8271.30	6
pw05_100.3	100	2475	29	446	576	8002	6.7e-5	20.0	8139	8320.32	6
pw05_100.4	100	2475	23	472	600	8024	7.3e-5	20.0	8125	8350.81	6
pw05_100.5	100	2475	24	939	1193	8149	6.7e-5	20.0	8169	8373.47	5
pw05_100.6	100	2475	20	349	424	8133	6.4e-5	20.0	8217	8467.12	6

Table A.1: Numerical results for MAXCUT problems, rudy collection (continued).

probl.	vert.	edges	k	j	f-ev.	f_{ALM}	feas.	ρ	f_{opt}	f_{SDP}	rk
pw05_100.7	100	2475	21	365	445	8186	7.3e-5	20.0	8249	8487.57	5
pw05_100.8	100	2475	23	371	466	8051	7.8e-5	20.0	8199	8382.98	5
pw05_100.9	100	2475	22	289	358	8076	9.4e-5	20.0	8099	8304.87	5
pw09_100.0	100	4455	30	398	504	13497	7.9e-5	20.0	13585	13805.97	6
pw09_100.1	100	4455	31	491	603	13357	9.8e-5	20.0	13417	13643.51	6
pw09_100.2	100	4455	38	485	665	13324	8.7e-5	20.0	13461	13645.66	6
pw09_100.3	100	4455	32	432	543	13554	9.8e-5	20.0	13656	13842.17	6
pw09_100.4	100	4455	14	350	440	13480	3.1e-5	200.0	13514	13712.77	5
pw09_100.5	100	4455	34	386	505	13487	9.1e-5	20.0	13574	13790.21	5
pw09_100.6	100	4455	31	353	448	13587	8.4e-5	20.0	13640	13835.51	5
pw09_100.7	100	4455	36	385	486	13451	9.3e-5	20.0	13501	13712.94	6
pw09_100.8	100	4455	35	401	516	13516	8.3e-5	20.0	13593	13804.64	6
pw09_100.9	100	4455	24	358	438	13523	8.8e-5	20.0	13658	13864.02	5
w01_100.0	100	495	17	335	389	624	4.2e-5	20.0	651	740.89	5
w01_100.1	100	495	17	467	560	671	5.4e-5	20.0	719	811.83	5
w01_100.2	100	495	17	345	406	642	4.0e-5	20.0	676	781.40	6
w01_100.3	100	495	18	365	449	793	6.8e-5	20.0	813	910.42	5
w01_100.4	100	495	15	222	259	618	5.3e-5	20.0	668	747.01	5
w01_100.5	100	495	16	349	405	610	7.9e-5	20.0	643	737.14	5
w01_100.6	100	495	17	373	435	627	4.2e-5	20.0	654	740.11	5
w01_100.7	100	495	17	451	531	667	5.4e-5	20.0	725	828.69	4
w01_100.8	100	495	16	331	395	713	7.2e-5	20.0	721	792.74	4
w01_100.9	100	495	16	302	369	721	7.9e-5	20.0	729	816.08	5
w05_100.0	100	2475	23	424	537	1612	5.7e-5	20.0	1646	1918.05	7
w05_100.1	100	2475	23	502	605	1524	6.0e-5	20.0	1606	1857.11	5
w05_100.2	100	2475	22	385	476	1815	5.9e-5	20.0	1902	2182.08	5
w05_100.3	100	2475	21	362	429	1617	7.1e-5	20.0	1627	1893.10	5
w05_100.4	100	2475	22	534	654	1512	5.7e-5	20.0	1546	1838.11	5
w05_100.5	100	2475	22	391	492	1491	5.7e-5	20.0	1581	1871.73	5
w05_100.6	100	2475	22	476	595	1367	8.7e-5	20.0	1479	1747.94	6
w05_100.7	100	2475	22	402	501	1896	8.4e-5	20.0	1987	2248.94	5
w05_100.8	100	2475	21	386	468	1263	6.9e-5	20.0	1311	1598.22	5
w05_100.9	100	2475	20	304	382	1747	7.9e-5	20.0	1752	2017.39	4
w09_100.0	100	4455	27	402	525	2011	6.6e-5	20.0	2121	2500.30	5
w09_100.1	100	4455	25	392	512	2085	8.5e-5	20.0	2096	2511.47	5
w09_100.2	100	4455	26	439	547	2675	9.5e-5	20.0	2738	3130.01	6
w09_100.3	100	4455	13	372	464	1958	1.8e-5	200.0	1990	2333.06	6
w09_100.4	100	4455	26	477	606	1921	8.4e-5	20.0	2033	2424.99	6
w09_100.5	100	4455	25	429	558	2357	9.5e-5	20.0	2433	2733.65	4
w09_100.6	100	4455	26	385	503	2172	6.6e-5	20.0	2220	2552.12	6
w09_100.7	100	4455	28	636	823	2122	7.0e-5	20.0	2252	2639.74	6
w09_100.8	100	4455	15	447	575	1665	7.8e-5	200.0	1843	2213.13	6
w09_100.9	100	4455	24	367	446	2041	1.0e-4	20.0	2043	2409.78	6

Table A.2: Numerical results for MAXCUT problems, ising collection.

probl.	vert.	edges	k	j	f-ev.	f_{ALM}	feas.	ρ	f_{opt}
ising2.5-100_5555	100	4950	40	687	1619	2406360.19	9.8e-5	20000.0	2460049
ising2.5-100_6666	100	4950	17	472	794	1978286.63	3.3e-5	200000.0	2031217
ising2.5-100_7777	100	4950	36	590	1370	3333168.11	8.0e-5	20000.0	3363230
ising2.5-150_5555	150	11175	18	739	1198	4315079.10	9.2e-5	133333.3	4363532
ising2.5-150_6666	150	11175	25	959	1648	4041057.48	7.2e-5	133333.3	4057153
ising2.5-150_7777	150	11175	31	734	1389	4224911.34	5.5e-5	133333.3	4243269
ising2.5-200_5555	200	19900	24	651	1148	6267758.47	4.5e-5	100000.0	6294701
ising2.5-200_6666	200	19900	19	630	1005	6752676.03	6.9e-5	100000.0	6795365
ising2.5-200_7777	200	19900	19	586	948	5506984.49	9.4e-5	100000.0	5568272
ising2.5-250_5555	250	31125	22	692	1154	7864741.24	8.1e-5	80000.0	7919449
ising2.5-250_6666	250	31125	19	896	1437	6852662.07	8.8e-5	80000.0	6925717
ising2.5-250_7777	250	31125	22	624	1030	6462343.01	6.9e-5	80000.0	6596797
ising2.5-300_5555	300	44850	22	1076	1705	8523955.32	5.9e-5	66666.7	8579363
ising2.5-300_6666	300	44850	21	1304	2075	9058514.43	6.7e-5	66666.7	9102033
ising2.5-300_7777	300	44850	25	887	1420	8168651.28	6.0e-5	66666.7	8323804
ising3.0-100_5555	100	4950	38	711	1690	2431408.96	9.8e-5	20000.0	2448189
ising3.0-100_6666	100	4950	35	510	1238	1975552.41	9.2e-5	20000.0	1984099
ising3.0-100_7777	100	4950	36	694	1505	3327994.14	8.0e-5	20000.0	3335814

Table A.2: Numerical results for MAXCUT problems, `ising` collection (continued).

probl.	vert.	edges	k	j	f-ev.	f_{ALM}	feas.	ρ	f_{opt}
ising3.0-150_5555	150	11175	15	456	761	4246560.88	8.2e-5	133333.3	4279261
ising3.0-150_6666	150	11175	26	647	1189	3935446.60	6.7e-5	133333.3	3949317
ising3.0-150_7777	150	11175	31	840	1582	4205069.88	5.3e-5	133333.3	4211158
ising3.0-200_5555	200	19900	20	592	1014	6209779.63	8.1e-5	100000.0	6215531
ising3.0-200_6666	200	19900	18	599	973	6697966.13	7.0e-5	100000.0	6756263
ising3.0-200_7777	200	19900	18	721	1120	5529450.26	8.1e-5	100000.0	5560824
ising3.0-250_5555	250	31125	22	753	1247	7790361.34	5.3e-5	80000.0	7823791
ising3.0-250_6666	250	31125	20	1003	1642	6879016.15	7.2e-5	80000.0	6903351
ising3.0-250_7777	250	31125	20	1670	2935	6287504.49	6.3e-5	80000.0	6418276
ising3.0-300_5555	300	44850	20	942	1598	8426148.11	5.6e-5	66666.7	8493173
ising3.0-300_6666	300	44850	23	928	1515	8907934.89	4.9e-5	66666.7	8915110
ising3.0-300_7777	300	44850	23	816	1375	8169591.67	5.4e-5	66666.7	8242904
t2g10_5555	100	200	18	411	775	5778570.15	5.4e-5	200000.0	6049461
t2g10_6666	100	200	18	463	820	5503220.56	6.9e-5	200000.0	5757868
t2g10_7777	100	200	19	492	882	6261175.46	4.1e-5	200000.0	6509837
t2g15_5555	225	450	25	601	1108	14446186.29	7.6e-5	88888.9	15051133
t2g15_6666	225	450	27	605	1191	15454604.55	6.0e-5	88888.9	15763716
t2g15_7777	225	450	27	1351	2153	14798901.82	7.9e-5	88888.9	15269399
t2g20_5555	400	800	22	919	1408	24487271.71	4.6e-5	500000.0	24838942
t2g20_6666	400	800	19	1131	1782	28725534.39	4.2e-5	500000.0	29290570
t2g20_7777	400	800	19	1026	1515	27294253.39	7.4e-5	500000.0	28349398
t3g5_5555	125	375	22	496	951	10843693.83	9.4e-5	160000.0	10933215
t3g5_6666	125	375	22	724	1187	11358698.15	8.4e-5	160000.0	11582216
t3g5_7777	125	375	25	668	1225	11196295.63	9.1e-5	160000.0	11552046
t3g6_5555	216	648	30	1092	1819	17046996.57	7.2e-5	92592.6	17434469
t3g6_6666	216	648	26	671	1252	20014468.29	8.9e-5	92592.6	20217380
t3g6_7777	216	648	32	1403	2282	18487004.21	8.3e-5	92592.6	19475011
t3g7_5555	343	1029	20	1201	1665	26773833.26	9.4e-5	583090.4	28302918
t3g7_6666	343	1029	19	912	1342	32934614.99	8.0e-5	583090.4	33611981
t3g7_7777	343	1029	21	871	1332	27953083.17	1.9e-5	583090.4	29118445

A.2 Proofs

Proof of Theorem 4.5. By local optimality of x^* for (CP), one finds some $\varepsilon > 0$ such that $q(x) \geq q(x^*)$ is valid for all $x \in B_\varepsilon(x^*) := \{x \in \mathbb{X} \mid \|x - x^*\| \leq \varepsilon\}$ which are feasible for (CP). Consequently, x^* is the uniquely determined global minimizer of

$$\begin{aligned} \min_x \quad & q(x) + \frac{1}{2}\|x - x^*\|^2 \\ \text{s.t.} \quad & c(x) \in K, \quad x \in B_\varepsilon(x^*). \end{aligned} \tag{A.1}$$

Let us now consider the penalized surrogate problem

$$\begin{aligned} \min_{x,s} \quad & q(x) + \frac{k}{2}\|c(x) - s\|^2 + \frac{1}{2}\|x - x^*\|^2 \\ \text{s.t.} \quad & x \in B_\varepsilon(x^*), \quad s \in K \cap B_1(c(x^*)) \end{aligned} \tag{P(k)}$$

where $k \in \mathbb{N}$ is arbitrary. Noting that the objective function of this optimization problem is lsc, while its feasible set is nonempty and compact, it possesses a global minimizer $(x^k, s^k) \in \mathbb{X} \times \mathbb{Y}$ for each $k \in \mathbb{N}$. Without loss of generality, we assume $x^k \to \tilde{x}$ and $s^k \to \tilde{s}$ for some $\tilde{x} \in B_\varepsilon(x^*)$ and $\tilde{s} \in K \cap B_1(c(x^*))$.

We claim that $\tilde{x} = x^*$ and $\tilde{s} = c(x^*)$. To this end, we note that $(x^*, c(x^*))$ is feasible to (P(k)) which yields the estimate

$$q(x^k) + \frac{k}{2}\|c(x^k) - s^k\|^2 + \frac{1}{2}\|x^k - x^*\|^2 \leq q(x^*) \tag{A.2}$$

for each $k \in \mathbb{N}$. Using lower semicontinuity of q as well as the convergences $c(x^k) \to c(\tilde{x})$ and $s^k \to \tilde{s}$, taking the limit $k \to \infty$ in (A.2) gives $c(\tilde{x}) = \tilde{s} \in K$. Particularly, $\tilde{x}$ is feasible for (A.1). Therefore, the local optimality of x^* implies $q(x^*) \leq q(\tilde{x})$. Furthermore, we find

$$q(\tilde{x}) + \frac{1}{2}\|\tilde{x} - x^*\|^2 \leq \liminf_{k\to\infty} \left(q(x^k) + \frac{k}{2}\|c(x^k) - s^k\|^2 + \frac{1}{2}\|x^k - x^*\|^2 \right) \leq q(x^*) \leq q(\tilde{x}).$$

Hence, $\tilde{x} = x^*$, and noting that (A.2) gives $q(x^k) \leq q(x^*)$ for each $k \in \mathbb{N}$,

$$q(x^*) \leq \liminf_{k\to\infty} q(x^k) \leq \limsup_{k\to\infty} q(x^k) \leq q(x^*),$$

i.e., $x^k \xrightarrow{q} x^*$ follows.

Due to $x^k \to x^*$ and $s^k \to c(x^*)$, one may assume without loss of generality that $\{x^k\}$ and $\{s^k\}$ are taken from the interior of $B_\varepsilon(x^*)$ and $B_1(c(x^*))$, respectively. Thus, for each $k \in \mathbb{N}$, (x^k, s^k) is an unconstrained local minimizer of

$$(x, s) \mapsto q(x) + \frac{k}{2}\|c(x) - s\|^2 + \frac{1}{2}\|x - x^*\|^2 + I_K(s).$$

Let us introduce $\theta : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ by means of $\theta(x, s) := g(x) + I_K(s)$ for each pair $(x, s) \in \mathbb{X} \times \mathbb{Y}$. Applying [122, Proposition 1.107 and 1.114], one finds

$$(0, 0) \in \Big(\nabla f(x^k) + kc'(x^k)^*(c(x^k) - s^k) + x^k - x^*, k(s^k - c(x^k)) + \partial\theta(x^k, s^k) \Big)$$

for each $k \in \mathbb{N}$. The decoupled structure of θ and Proposition 2.23 yield the inclusion $\partial\theta(x^k, s^k) \subset \partial g(x^k) \times \mathcal{N}_K^{\lim}(s^k)$ for each $k \in \mathbb{N}$. Thus, setting $\eta^k := x^* - x^k$, $\lambda^k := k(c(x^k) - s^k)$ and $\zeta^k := s^k - c(x^k)$ for each $k \in \mathbb{N}$ while observing that $\partial q(x^k) = \nabla f(x^k) + \partial g(x^k)$ holds, one has shown that x^* is AM-stationary for (CP). □

Proof of Lemma 4.11. Fix some $w \in \limsup_{x \xrightarrow{q} \bar{x}, z \to 0} \mathcal{M}(x, z)$, then there exist sequences $\{x^k\}$, $\{\zeta^k\}$, $\{w^k\}$ satisfying $x^k \xrightarrow{q} \bar{x}$, $\zeta^k \to 0$, $w^k \in w$, and $w^k \in \mathcal{M}(x^k, \zeta^k)$ for all $k \in \mathbb{N}$. Hence, due to the special structure of the optimization problem (4.5), each w^k can be represented as

$$w^k = \sum_{i=1}^{m} \lambda_i^k \nabla\theta_i(x^k) + \sum_{j=1}^{p} \mu_j^k \nabla h_j(x^k) + \xi^k$$

with $\xi^k \in \partial g(x^k)$, $\mu_j^k \in \mathbb{R}$, and $\lambda_i^k \in \mathbb{R}$ satisfying

$$\lambda_i^k \begin{cases} = 0 & \text{if } i \notin I(\bar{x}), \\ \geq 0 & \text{if } i \in I(\bar{x}) \end{cases}$$

for all $k \in \mathbb{N}$ sufficiently large. Thus, without loss of generality, we assume that

$$w^k - \xi^k = \sum_{i \in I(\bar{x})} \lambda_i^k \nabla\theta_i(x^k) + \sum_{j=1}^{p} \mu_j^k \nabla h_j(x^k)$$

with $\lambda_i^k \geq 0$ for all $i \in I(\bar{x})$, $\mu_j^k \in \mathbb{R}$ for all $j = 1, \ldots, p$, and $\xi^k \in \partial g(x^k)$ for all $k \in \mathbb{N}$. Due to Definition 4.10 (a), (b), there exists an index set $J \subset \{1, \ldots, p\}$ such that this representation can be rewritten as

$$w^k - \xi^k = \sum_{i\in I(\bar{x})} \lambda_i^k \nabla\theta_i(x^k) + \sum_{j\in J} \hat{\mu}_j^k \nabla h_j(x^k)$$

with suitable scalars $\hat{\mu}_j^k \in \mathbb{R}$ $(j \in J)$ and the gradient $\{\nabla h_j(x^k)\}_{j\in J}$ being linearly independent. In view of [6, Lemma 1], for each $k \in \mathbb{N}$, there exist an index set $I_k \subset I(\bar{x})$ and multipliers $\bar{\lambda}_i^k$ $(i \in I_k)$, $\bar{u}_j^k \in \mathbb{R}$ $(i \in J)$ such that

$$w^k - \xi^k = \sum_{i\in I_k} \bar{\lambda}_i^k \nabla\theta_i(x^k) + \sum_{j\in J} \bar{\mu}_j^k \nabla h_j(x^k)$$

and the gradients $\{\nabla\theta_i(x^k)\}_{i\in I_k} \cup \{\nabla h_j(x^k)\}_{j\in J}$ being linearly independent. Since there are only finitely many index sets $I_k \subset I(\bar{x})$, we may assume, subsequencing if necessary, that

$$w^k - \xi^k = \sum_{i\in I} \bar{\lambda}_i^k \nabla\theta_i(x^k) + \sum_{j\in J} \bar{\mu}_j^k \nabla h_j(x^k) \tag{A.3}$$

holds for all $k \in \mathbb{N}$ with some fixed index set $I \subset I(\bar{x})$.

We claim that the sequence $\{t_k\}$ defined by

$$t_k := \left\| \left(\bar{\lambda}_i^k \ (i \in I), \bar{\mu}_j^k \ (j \in J) \right) \right\|$$

is bounded. Assume that it is unbounded, say, $t_k \to \infty$. Then, dividing the expression (A.3) by t_k and then taking the limit $k \to \infty$, one has

$$0 - \bar{\xi} = \sum_{i\in I} \bar{u}_i \nabla\theta_i(\bar{x}) + \sum_{j\in J} \bar{\mu}_j \nabla h_j(\bar{x}) \tag{A.4}$$

for some suitable limits $\bar{\lambda}_i \geq 0$ $(i \in I)$, $\bar{\mu}_j \in \mathbb{R}$ $(j \in J)$, not vanishing at the same time, and the (obviously existing) limit $\bar{\xi} := \lim_{k\to\infty} \xi^k / t_k$. Since $\xi^k \in \partial g(x^k)$, $x^k \xrightarrow{g} \bar{x}$ and $t_k \to \infty$, it follows from Lemma 2.21 that $\bar{\eta} \in \partial^\infty g(\bar{x})$. In view of (A.4) and Definition 4.10 (c), it then follows that the gradients $\{\nabla\theta_i(\bar{x})\}_{i\in I} \cup \{\nabla h_j(\bar{x})\}_{j\in J}$ are linearly dependent for all $k \in \mathbb{N}$ sufficiently large. This, however, contradicts the choice of the index sets $I = I_k$.

Consequently, the sequences $\{\bar{\lambda}_i^k\}_{i\in I}$ and $\{\bar{\mu}_j^k\}_{j\in J}$ are bounded, which in view of (A.3), also deduces that $\{\xi^k\}$ is bounded. Subsequencing if necessary, we may therefore assume that these sequences converge to suitable limits $\lambda_i^* \geq 0$ $(i \in I)$, μ_j^* $(j \in J)$, and ξ^*, respectively. Then, taking the limit $k \to \infty$ in (A.3) yields

$$w = \sum_{i\in I} \lambda_i^* \nabla\theta_i(\bar{x}) + \sum_{j\in J} \mu_j^* \nabla h_j(\bar{x}) + \xi^*.$$

Since $\xi^k \in \partial g(x^k)$ for all $k \in \mathbb{N}$, $x^k \xrightarrow{g} \bar{x}$, and $\xi^k \to \xi^*$, it follows from the robustness property (2.7) of the limiting subdifferential that $\xi^* \in \partial g(\bar{x})$. Setting $\lambda_i^* := 0$ $(i \notin I)$ and $\mu_j^* := 0$ $(j \notin J)$, it follows that $w \in \mathcal{M}(\bar{x}, 0)$. Hence, $\bar{x}$ satisfies AM-regularity.

$\square$