Homogeneous spaces with the cohomology of sphere products and compact quadrangles

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Preface

The homogeneous spaces that have the same (singular) cohomology as spheres were classified by Borel, Bredon, Montgomery and Samelson and by Poncet. This was extended to homogeneous spaces which are simply connected and which have the same rational cohomology as spheres by Onishchik and also by Kramer. Furthermore, Kramer classified the simply connected homogeneous spaces with the rational cohomology of a sphere product $\mathbb{S}^k \times \mathbb{S}^m$, where $3 \leq k \leq m$ and m is odd; and Wolfrom classified this kind of spaces in the case $2 = k \leq m$ with odd m. By results of Cartan and Serre and Kramer it follows in both cases that the homogeneous spaces also have the same rational *homotopy* as a product $\mathbb{S}^k \times \mathbb{S}^m$. We treat here the case of homogeneous spaces G/H with the same rational homotopy as a product $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ with $m \in \mathbb{N}$. We show that these spaces have also the rational cohomology of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ if H is connected and if the quotient has dimension $m+2$. Furthermore, we prove that if additionally the fundamental group of G/H is cyclic, then $G/H \cong \mathrm{SO}(2) \times (A/H)$, where A/H is a simply connected rational cohomology $(m+1)$-sphere (and hence classified). If H fails to be connected, then the G-action on the covering space G/H^1 of G/H has connected stabilizers, and the results apply to G/H^1. To show that under the assumptions above every natural number may be realized as the order of the group H/H^1 we calculate the cohomology of certain homogeneous spaces.

We also determine the rational cohomology of the fibre bundle $H^1 \to G \to G/H^1$ if G/H meets the assumptions above. This is done by considering the respective Leray-Serre spectral sequence. The structure of the cohomology of $H^1 \to G \to G/H^1$ then gives a second proof for the structure of compact connected Lie groups acting transitively on spaces with the rational homotopy of a sphere product $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. Since a quotient of a homogeneous space with the same rational homotopy or cohomology as $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ is not simply connected, there often arises the question whether or not a considered fibre bundle or fibration is orientable. A large amount of space will therefore be given to the problem of showing that certain fibrations are orientable.

Assume that a Lie group acts transitively on a simply connected compact manifold. By a well known result of Montgomery, the maximal compact subgroup of the Lie group is also transitive on the manifold. This may be generalized to compact connected manifolds with finite fundamental groups. But if the fundamental group of the manifold in question fails to be finite the maximal compact subgroups need not act transitively . We will show that for compact connected $(m+2)$-manifolds with cyclic fundamental groups and with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ the situation is not too bad: if a connected Lie group acts transitively on the manifold, then the maximal compact subgroups are either transitive, or their orbits are simply connected rational cohomology spheres of codimension 1.

Homogeneous spaces with the same rational cohomology or homotopy as a sphere product $\mathbb{S}^k \times \mathbb{S}^m$ play a role in the study of different types of geometrical objects. They appear for example as focal manifolds of isoparametric hypersurfaces with four distinct principal curvatures. Further examples of such spaces are the point spaces and the line spaces of compact connected generalized quadrangles. In both cases, the case of isoparametric hypersurfaces and the case of generalized quadrangles, there is a space that is the total space of two fibrations, each of which has sphere-like fibres. We call this a double fibration. Using this double fibration Münzner proved many of his important results on isoparametric hypersurfaces. Kramer obtained similar results for generalized quadrangles. We extend the results of Münzner and Kramer for double fibrations to show that in certain circumstances there appears a base space with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. This holds in particular for the respective focal manifolds and the respective point spaces of generalized quadrangles. Hence, our results on homogeneous spaces apply.

Isoparametric hypersurfaces in spheres are closed submanifolds with constant principal curvatures. Hsiang and Lawson classified the isoparametric hypersurfaces that admit a transitive action of their isometry group. These so-called homogeneous hypersurfaces are exactly the principal orbits of isotropy representations of symmetric spaces of rank 2. To every isoparametric hypersurface belong two focal manifolds. Ferus, Karcher and Münzner constructed infinitely many examples of non-homogeneous isoparametric hypersurfaces such that their isometry groups act transitively on one of the focal manifolds. In most cases, the focal manifold have the cohomology of products of two spheres $\mathbb{S}^k \times \mathbb{S}^m$. Kramer and Wolfrom classified the cases where $1 < k \leq m$. We apply our results to determine the transitive actions in the case $k = 1$. The respective hypersurfaces are classified by a result of Takagi.

The second application of our results on homogeneous spaces will be a special kind of buildings. Buildings were introduced by Jacques Tits to give interpretations of simple groups of Lie type. They are a far-reaching generalization of projective spaces, in particular a generalization of projective planes. There is another generalization of projective planes called generalized polygons. A projective plane is the same as a generalized triangle. The generalized polygons are also contained in the class of buildings: they are the buildings of rank 2. Hence, the term building covers both generalizations of projective planes, the projective spaces and the generalized polygons.

The (irreducible, thick) spherical buildings of rank at least three were classified by Tits. Apart from not being classified, the buildings of rank 2 are of particular interest, because buildings are in a certain sense composed of buildings of rank two.

For infinite generalized polygons it is necessary to impose further reasonable assumptions on these point-line-geometries to obtain classification results. It is usually required that the point space and the line space of the geometry carry topologies that render the geometric operations of intersecting lines and joining points continuous. Grundhöfer, Knarr and Kramer classified the compact connected generalized polygons that admit automorphism groups which are transitive on the flag space. Here, a flag is a pair of a point and a line such that the point lies on the line. Furthermore, Kramer proved that point-homogeneous compact connected polygons are already flag-homogeneous provided that the generalized polygon is not a generalized quadrangle. There are indeed many point-homogeneous quadrangles which are not flag-homogeneous. Kramer started to classify compact connected quadrangles that admit an automorphism group acting transitively on the points or on the lines. By duality of the role of points and lines it suffices to consider point-homogeneous quadrangles. To these quadrangles one can assign a pair of natural numbers called the topological parameters of the quadrangles. Kramer classified the point-homogeneous quadrangles whose topological parameters (k, m) satisfy $k = m$. He obtained also far-reaching results in the cases $3 \leq k < m$. Biller classified the case of $m = 1$; and Wolfrom obtained a table of possible point-transitive groups in the case $k = 2$. We will treat the case $k = 1$ here. It turns out that there are no other point-transitive compact connected Lie groups for $(1, m)$-quadrangles than the ones for the real orthogonal quadrangles. In the case of special orthogonal groups the group action is unique and, hence, the one of the real orthogonal quadrangle. For the special unitary group we determine the structure of the line space. Furthermore, we solve the problem of three infinite series of group actions which Kramer left as open problems; there are no quadrangles with the homogeneous spaces in question as point spaces (up to maybe a finite number of small parameters in one of the three series). One may summarize the results in the following way.

Theorem *If a compact connected Lie group acts as an automorphism group transitively on the points of a compact connected quadrangle, then there is (up to maybe a finite number of exceptions) a point-transitive action of the group on a 'classical' quadrangle, or on an quadrangle with parameters* $(3, 4n)$ *or* $(8, 7)$ *due to Ferus, Karcher and Münzner, and in both actions, the given one and the classical one, the connected components of the stabilizers coincide.*

The chapters are organized as follows. In the first chapter we introduce the term double fibration in the way we will use it here. We quote some results already obtained by Kramer and Münzner. Then we show that in the cases we are interested in one of the two base spaces has the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. After that, three cases of group actions are excluded. This is done by applying results of the last chapter, in particular we use a certain $\mathbb{Q}$-orientable fibration. We treat these cases so early (even before the classification) because there is no further structure of the double fibration needed to exclude them.

The first sections of the second chapter consist of short introductions to (compact) Lie groups, actions and homogeneous spaces. There is an extra section for sphere-like spaces where we recall the classification of homogeneous cohomology spheres and the classification of the simply connected rational cohomology spheres. Here we use and extend also results of Biller on generalized spheres. In the last section we quote a theorem of Mostert on actions of Lie groups with codimension one in compact manifolds. This theorem will be an important tool throughout the following chapters. It plays also an important role for the description of homogeneous spaces that have the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. This description is worked out in the last part of the section.

We apply our classification of homogeneous spaces with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ to the two kinds of geometric objects mentioned above. We start with the application to focal manifolds of isoparametric hypersurfaces. After a short introduction to isoparametric hypersurfaces, we classify the transitive actions on the focal manifolds in question.

The following two chapters are devoted to generalized quadrangles. Their definition is given and some important properties are mentioned. We recall the definition of the most important example of generalized quadrangles for the following sections, the real orthogonal quadrangles. The point-homogeneous $(1, 2)$-quadrangles are discussed in the following section. Then each class of groups appearing in the classification is treated separately in a section. In the case of the orthogonal groups it turns out that the point-transitive action has to be the classical one on the real orthogonal quadrangle. In the case of the unitary groups we obtain no uniqueness result. But we describe the structure of the line space under the action of $\mathrm{SU}(n)$ and there are some geometric properties of orbits derived. For example, the existence of homogeneous spreads is proved. We close this chapter with a short summary. Here, most of the remaining singular cases are treated.

Also the following chapter is dedicated to point-homogeneous quadrangles. In Kramer's Habilitationsschrift appear three infinite classes of homogeneous spaces as candidates for point spaces of (k, m)-quadrangles. Here, we have $k > 1$ in each case. Kramer conjectured that none of these homogeneous spaces belongs to a quadrangle, but he left it as an open problem. We show that indeed these homogeneous spaces may not be realized as point spaces of quadrangles, except possibly for the smallest four cases in one of the three classes.

The last chapter addresses relations between the rational homotopy and the rational cohomology of homogeneous spaces. We might have put this chapter at the beginning as well (together with lemma 2.5.4). But as the concepts and the tools here are very different from the other chapters we put this chapter to the end. The chapter on double fibrations does not rely on new result of this last chapter.

After defining orientable fibrations we show in this chapter that for an $(m+2)$-dimensional homogeneous space X with connected stabilizers and the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ there is up to homotopy a $\mathbb{Q}$-orientable fibration $\widetilde{X} \to X \to B\pi$ where $\widetilde{X}$ is the universal covering space of X and $B\pi$ is the classifying space of the fundamental group of X. We recall the definition of spectral sequences and state a theorem on the Leray-Serre spectral sequence. Some simple conclusions are drawn. Applying the Leray-Serre spectral sequence to the $\mathbb{Q}$-orientable fibration above, we show that the space X has the same rational cohomology as $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. Finally, we determine the rational cohomology of $H^1 \to G \to G/H^1$ (where $X = G/H$ is as above). The structure of the cohomology of the fibre bundle leads to a second proof of the classification of homogeneous spaces with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. In the last section we calculate the cohomology of certain homogeneous spaces involving the special unitary and the symplectic groups. These homogeneous spaces are further examples of spaces having the homotopy and the cohomology of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. Furthermore, they show that in general it is not possible to derive further conclusions on the number of components of the stabilizers.

In all chapters we will always use singular homology and singular cohomology when we speak of homology and cohomology, respectively. If we define a notion, we will write it in **boldface** letters.

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Notation

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{H}$	quaternions
$\mathbb{Z}_n$	integers modulo n
$\pi_k(X)$	homotopy groups
$\pi_\bullet(X)$	complex of homotopy groups
$H_n(X;G)$	singular homology groups with coefficients in G
$H_\bullet(X;G)$	complex of singular homology groups with coefficients in G
$H^n(X;G)$	singular cohomology groups with coefficients in G
$H^\bullet(X;G)$	singular cohomology ring with coefficients in G
$\mathbb{S}^n$	unit sphere in $\mathbb{R}^{n+1}$
$\mathbb{D}^n$	unit disk in $\mathbb{R}^{n+1}$
$\mathbb{R}\mathrm{P}_n$	real projective n-space
$\mathbb{C}\mathrm{P}_n$	complex projective n-space
$\mathbb{H}\mathrm{P}_n$	quaternionic projective n-space
$\mathbb{V}_2(\mathbb{R}^n)$	Stiefel manifold of orthonormal pairs of vectors in $\mathbb{R}^n$
BG	classifying space of G
EG	total space of the classifying bundle of G
X/G	orbit space of X under the action of G
G/H	Quotient of (topological) groups
$\mathrm{Q_n}(\mathbb{R})$ *or* $\mathrm{Q}(1, \text{n-3})$	real orthogonal quadrangle in $\mathbb{R}^{n+1}$
$\mathcal{P}_l$	points on the line l
$\mathcal{L}_p$	lines through the point p
$p^\perp$	perp of the point p
$\simeq$	homotopy equivalence
$\approx$	homeomorphy

Chapter 1

Fibrations and double fibrations

Fibrations will play an important role in several chapters. In the first section we give the definition and recall some properties of fibrations. The notion of double fibrations is introduced in the second section. We cite results on the cohomology of the spaces involved and derive their rational homotopy. In particular, it is shown that under mild assumptions the base points of a double fibration have the same rational homotopy as a product of spheres.

1.1 Fibrations

There is an important generalization of fibre bundles. A continuous map $p : E \to B$ is called a **fibration**, if for every topological space X and every commutative diagram of continuous maps

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r] & B \end{tikzcd}$$

there is a continuous map $H : X \times [0,1] \to E$ such that the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r] \arrow[ur, dotted, "H"] & B \end{tikzcd}$$

is commutative. This is what sometimes is also called a **Hurewicz fibre space** or a **Hurewicz fibration**. If we restrict this property to all disks $X = \mathbb{D}^n$, then we call $p : E \to B$ a **Serre fibration**. This is the notion of, for example, tom Dieck [67] and [66], McCleary [40], Spanier [57], Whitehead [70], whereas some other authors call a Serre fibration simply a fibration, for example, Bredon [11], Mimura and Toda [42].

If $p : E \to B$ is a fibration, we call E the **total space**, B the **base space** of p, and $p^{-1}(b)$ the **fibre** over $b \in B$. If B is path-connected, then any two fibres are homotopy equivalent, see Spanier [57, 2.8.13]. We then write $F \to E \to B$ for the fibration, where $F = p^{-1}(b)$ for some $b \in B$. This is indeed a generalization of fibre bundles, because by Bredon [11, VII.6.12] every fibre bundle is a Serre fibration, and every fibre bundle with a paracompact Hausdorff base space is a (Hurewicz) fibration, see Spanier [57, 2.7.14].

An important property of a Serre fibration $p : E \to B$ is the exactness of the homotopy sequence: If we take a point $b \in B$ as a base point in B and some point in $F = p^{-1}(b)$ as base point in F and via $i : F \hookrightarrow E$ also as a base point in E, then there is an exact sequence, the **homotopy sequence** of the fibration $F \to E \to B$,

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_\#} \pi_n(E) \xrightarrow{p_\#} \pi_n(B) \longrightarrow \pi_{n-1}(F) \xrightarrow{i_\#} \cdots$$

$$\cdots \longrightarrow \pi_1(E) \xrightarrow{p_\#} \pi_1(B) \longrightarrow \pi_0(F) \xrightarrow{i_\#} \pi_0(E) \xrightarrow{p_\#} \pi_0(B),$$

see Spanier [57, 7.2.10] or Bredon [11, VII.6.7]. Later on, we will see some situations in which fibrations arise.

1.2 Double fibrations

Recall that the **mapping cylinder** of a continuous map $f : X \to Y$ between topological spaces X and Y is the quotient space that we get from the disjoint union $(X \times [0,1]) \,\dot\cup\, Y$ by identifying $(x,1)$ with $f(x)$ for every $x \in X$.

If we have two continuous mappings $f : X \to Y$ and $g : X \to Z$ with the same domain X, we consider the disjoint union $Z \,\dot\cup\, (X \times [0,1]) \,\dot\cup\, Y$. Identifying $(x,0)$ with $g(x)$ and $(x,1)$ with $f(x)$ for every $x \in X$, we get the **double mapping cylinder** D of f and g. Note that there are inclusions $Y \hookrightarrow D$, $Z \hookrightarrow D$ and homotopy equivalences $D \setminus Z \simeq Y$, $D \setminus Y \simeq Z$ and $(D \setminus Z) \cap (D \setminus Y) = D \setminus (Y \cup Z) \simeq X$. Therefore, there is a Mayer-Vietoris-sequence for the diagram

$$\begin{tikzcd}
 & Y \arrow[dr] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & D \\
 & Z \arrow[ur] &
\end{tikzcd}$$

see Bredon [11, V.8.3] or Spanier [57, 5.4.9]. The Seifert-Van Kampen theorem ([11, III.9.4]) also applies to the diagram and yields relations between the fundamental groups of the four spaces. The double mapping cylinder will be important later on for Mostert's theorem 2.5.1 and for the notion of double fibrations.

A subset A of a topological space X is a **retract** of X if there is a continuous map $r : X \to A$, a **retraction**, such that $r|_A = \mathrm{id}_A$. A metric space X is an **absolute neighbourhood retract** (**ANR**) if every homeomorphic image of X in a metric space Y is a retract of a neighbourhood of X in Y. For properties of ANRs see e.g. Hu [31].

We consider the situation that a space $\mathcal{F}$ is the total space of two fibrations simultaneously. Furthermore, we assume that the fibres 'look like' spheres. We are particularly interested in the case where one of these fibre types is a 1-sphere. This is the case in the situation we have in mind, and we will apply the results to the kind of isoparametric hypersurfaces and compact generalized quadrangles we investigate in the following chapters.

We start with the definition of double fibrations. For similar settings cf. Strauß [59] and Markert [39].

Definition 1.2.1 *Consider two fibrations $S^m \to \mathcal{F} \to \mathcal{P}$ and $S^k \to \mathcal{F} \to \mathcal{L}$ with the same total space $\mathcal{F}$. We call the diagram*

$$\begin{tikzcd}
S^k \arrow[dr, dashed] & & S^m \arrow[dl, dotted] \\
 & \mathcal{F} \arrow[dl, dotted] \arrow[dr, dashed] & \\
\mathcal{P} & & \mathcal{L}
\end{tikzcd}$$

*a **double fibration** of type (k, m) if the following conditions are satisfied.*

- *The double mapping cylinder of the two projections is a simply connected cohomology $(r + 1)$-sphere.*

- *$\mathcal{F}$ is homotopy equivalent to a CW-complex and a cohomology r-manifold. (See Bredon [10, V.16.7] for a definition of cohomology manifolds, he also uses $n - cm_{\mathbb{Z}}$ as notation for cohomology n-manifolds.)*
- *$\mathcal{P}$ and $\mathcal{L}$ are locally compact and finite-dimensional ANRs and cohomology manifolds of dimension $r - m$ and $r - k$, respectively.*
- *$S^k \simeq \mathbb{S}^k$ and $S^m \simeq \mathbb{S}^m$ are locally compact, finite-dimensional ANRs and homotopy spheres.*

Note the symmetry in the roles of $\mathcal{P}$ and $\mathcal{L}$ in the definition. In the situation above

$$\begin{tikzcd} S^m \arrow[dr, dashed] & & S^k \arrow[dl, dotted] \\ & \mathcal{F} \arrow[dl, dotted] \arrow[dr, dashed] & \\ \mathcal{L} & & \mathcal{P} \end{tikzcd}$$

is a double fibration of type (m, k); hence, we may always assume that $k \leq m$.

Examples of double fibrations of type (k, m) arise in compact (k, m)-quadrangles and in isoparametric hypersurfaces with multiplicities k and m, see chapters 3 and 4.

For the rest of this section we assume that we are given a double fibration of type (k, m), and we use the notation of the definition.

Münzner [49] obtained the following results on the cohomology of double fibrations under more specific assumptions. Kramer generalized these in [36, 6.4.1], see also Strauß [59].

Theorem 1.2.2 *For a double fibration of type (k, m) as above there is an $n \in \{1, 2, 3, 4, 6\}$ such that there are isomorphisms of groups*

$$\begin{array}{rcl} H^\bullet(\mathcal{P}; \mathbb{Z}_2) & \cong & {\mathbb{Z}_2}^n, \\ H^\bullet(\mathcal{L}; \mathbb{Z}_2) & \cong & {\mathbb{Z}_2}^n, \\ H^\bullet(\mathcal{F}; \mathbb{Z}_2) & \cong & {\mathbb{Z}_2}^{2n}. \end{array}$$

The cohomology manifold $\mathcal{F}$ is orientable. For $0 < l < r$ the maps of the fibrations induce isomorphisms

$$H^l(\mathcal{F}; \mathbb{Z}_2) \cong H^l(\mathcal{P}; \mathbb{Z}_2) \oplus H^l(\mathcal{L}; \mathbb{Z}_2).$$

Furthermore, we have $H^0(\mathcal{F};\mathbb{Z}_2)\cong\mathbb{Z}_2$; in particular, the spaces $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{L}$ are arcwise connected.

Suppose now that $n=4$. If $k+m$ is odd, then $\mathcal{P}$ and $\mathcal{L}$ are both orientable cohomology manifolds, and there are isomorphisms of graded rings

$$\begin{aligned} H^\bullet(\mathcal{P}) &\cong H^\bullet(\mathbb{S}^k\times\mathbb{S}^{k+m}),\\ H^\bullet(\mathcal{L}) &\cong H^\bullet(\mathbb{S}^m\times\mathbb{S}^{k+m}),\\ H^\bullet(\mathcal{F}) &\cong H^\bullet(\mathbb{S}^k\times\mathbb{S}^m\times\mathbb{S}^{k+m}).\end{aligned}$$

If $n=4$ and $k+m$ is even, then $k=m\in\{2,4\}$ or at least one of the parameters k, m is 1.

If $n=4$, $k=1$ and $m>2$ is odd, then $\mathcal{L}$ is orientable and $\mathcal{P}$ is not, and there are isomorphisms of graded rings

$$\begin{aligned} H^\bullet(\mathcal{P};\mathbb{Z}_2) &\cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^{1+m};\mathbb{Z}_2),\\ H^\bullet(\mathcal{L};\mathbb{Z}_2) &\cong H^\bullet(\mathbb{S}^m\times\mathbb{S}^{1+m};\mathbb{Z}_2),\\ H^\bullet(\mathcal{F};\mathbb{Z}_2) &\cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^m\times\mathbb{S}^{1+m};\mathbb{Z}_2)\end{aligned}$$

and

$$H^\bullet(\mathcal{L};\mathbb{Q}) \cong H^\bullet(\mathbb{S}^{2m+1};\mathbb{Q}).$$

Furthermore, for $n=4$, $k=1$ and $m>1$ we have $\pi_1(\mathcal{P})\cong\pi_1(\mathcal{F})\cong\mathbb{Z}$.

There is also much known about the structure of the cohomology rings in the other cases $n\in\{1,2,3,6\}$, see Kramer [36, 6.4.1]. We just quoted the cases of most interest for the following chapters.

If a space has the same cohomology (with coefficients in $\mathbb{Z}$) as a product of spheres, then this is true also for rational coefficients by the universal coefficient theorem. Note that the fundamental groups of $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{L}$ are trivial for k, $m>1$. Hence, the Cartan-Serre theorem, see 6.3.3, gives the following lemma. Note that for m even, the homotopy groups of $\mathbb{S}^m$ of rank 1 are $\pi_m(\mathbb{S}^m)$ and $\pi_{2m-1}(\mathbb{S}^m)$ in view of 6.3.3.

Lemma 1.2.3 *If in a double fibration of type (k,m) the sum $k+m$ is odd and if we have k, $m>1$, then*

$$\begin{aligned} \pi_\bullet(\mathcal{F})\otimes\mathbb{Q} &\cong \pi_\bullet(\mathbb{S}^k\times\mathbb{S}^m\times\mathbb{S}^{m+k})\otimes\mathbb{Q},\\ \pi_\bullet(\mathcal{L})\otimes\mathbb{Q} &\cong \pi_\bullet(\mathbb{S}^m\times\mathbb{S}^{m+k})\otimes\mathbb{Q} \quad \textit{and}\\ \pi_\bullet(\mathcal{P})\otimes\mathbb{Q} &\cong \pi_\bullet(\mathbb{S}^k\times\mathbb{S}^{m+k})\otimes\mathbb{Q}.\end{aligned}$$

Suppose that $k = 1$ and $m > 1$. If m is even, then

$$\pi_\bullet(\mathcal{L}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^m \times \mathbb{S}^{m+1}) \otimes \mathbb{Q},$$

and if m is odd, then

$$\pi_\bullet(\mathcal{L}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^{2m+1}) \otimes \mathbb{Q}.$$

We investigate now the rational homotopy of $\mathcal{F}$ and $\mathcal{P}$ in the case $k = 1$ and $m > 1$. Note that the fundamental groups of $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{L}$ are cyclic, in particular abelian. Hence, tensoring these groups with $\mathbb{Q}$ makes sense. By tensoring the homotopy sequence of the fibration $\mathbb{S}^1 \to \mathcal{F} \to \mathcal{L}$ with $\mathbb{Q}$ and by the last lemma we obtain the following result. For the case $m = 2$ note that $\pi_1(\mathcal{F}) \cong \mathbb{Z}$ by 1.2.2.

Lemma 1.2.4 *If m is even in a double fibration of type $(1, m)$, then*

$$\pi_\bullet(\mathcal{F}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^m \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}.$$

If $m > 1$ is odd, then

$$\pi_\bullet(\mathcal{F}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{2m+1}) \otimes \mathbb{Q}.$$

Now we will determine the rational homotopy of the base space $\mathcal{P}$ of the fibration with 1-spheres as fibres.

Lemma 1.2.5 *Assume $m > 2$ in a double fibration of type $(1, m)$. Then*

$$\pi_k(\mathcal{P}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } k \in \{1, m+1\} \\ 0 & \text{if } k \notin \{1, m+1, 2m-1, 2m\}. \end{cases}$$

If m is odd or if $\mathcal{P}$ has the homotopy type of an $(m+2)$-dimensional manifold then the space $\mathcal{P}$ has the same rational homotopy as a product of a 1-sphere and an $(m + 1)$-sphere:

$$\pi_\bullet(\mathcal{P}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}.$$

Proof We will distinguish two cases. First assume m to be odd, then $\pi_\bullet(\mathcal{F}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{2m+1}) \otimes \mathbb{Q}$ as we saw above. Then the homotopy sequence of $\mathbb{S}^m \to \mathcal{F} \to \mathcal{P}$ implies $\pi_\bullet(\mathcal{P}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$.

Now assume m to be even. Then by 1.2.4 the total space $\mathcal{F}$ has the same rational homotopy as a product of three spheres,

$$\pi_\bullet(\mathcal{F}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^m \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}.$$

As $\mathcal{F}$ is homotopy equivalent to a CW-complex by the definition of the double fibration, we get by Kramer [36] a cell decomposition of $\mathcal{F}$ and an isomorphism

$$H_m(\mathbb{S}^m) \xrightarrow{\cong} H_m(\mathcal{F})$$

induced by the fibre inclusion. From the Hurewicz homomorphisms, see Spanier [57, 7.4], we get a commutative diagram

$$\begin{tikzcd} \pi_m(\mathbb{S}^m) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & \pi_m(\mathcal{F}) \arrow[d] \\ H_m(\mathbb{S}^m) \arrow[r, "\cong"] & H_m(\mathcal{F}), \end{tikzcd}$$

where the vertical arrow on the left is in fact an isomorphism. Hence, the map φ in the upper row has trivial kernel. Therefore, tensoring the homotopy sequence of $\mathbb{S}^m \to \mathcal{F} \to \mathcal{P}$ with $\mathbb{Q}$ we get the following exact sequence, where we suppress the tensoring with $\mathbb{Q}$,

$$0 \longrightarrow \pi_{m+1}(\mathcal{F}) \longrightarrow \pi_{m+1}(\mathcal{P}) \xrightarrow{0} \pi_m(\mathbb{S}^m) \xrightarrow{\varphi} \pi_m(\mathcal{F}) \longrightarrow \pi_m(\mathcal{P}) \longrightarrow 0,$$

and it follows that $\pi_{m+1}(\mathcal{P}) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\pi_m(\mathcal{P}) \otimes \mathbb{Q} \cong 0$; furthermore, the homotopy sequence shows that $\pi_k(\mathcal{P}) \otimes \mathbb{Q} = 0$ for $k \notin \{1, m, 2m-1, 2m\}$ and that there is an exact sequence

$$0 \longrightarrow \pi_{2m}(\mathcal{P}) \longrightarrow \pi_{2m-1}(\mathbb{S}^m) \longrightarrow \pi_{2m-1}(\mathcal{F}) \longrightarrow \pi_{2m-1}(\mathcal{P}) \longrightarrow 0.$$

This shows the first claim. For the second, it suffices to show that $\pi_{2m}(\mathcal{P}) \otimes \mathbb{Q}$ is trivial in view of the exact sequence above involving $\pi_{2m-1}(\mathcal{P})$ and $\pi_{2m}(\mathcal{P})$. This follows if $\mathcal{P}$ has the homotopy type of a manifold, because then by Tatsuo Higa [25] the homotopy groups $\pi_{2l}(\mathcal{P})$ are finite for $2l > \dim \mathcal{P} = m + 2$. □

In 2.5.11 we determine the homogeneous spaces of compact connected Lie groups with infinite cyclic fundamental group and the same dimension and the same rational homotopy as $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. In view of lemma 1.2.5 these spaces may be one of the base spaces of a double fibration of type $(1, m)$. We are going to exclude some of the spaces that appear in 2.5.11 and in 2.4.2. We do this already here, because there is no further structure on the double fibration needed in contrast to the chapters on isoparametric hypersurfaces and generalized quadrangles. But we will refer sometimes to chapter 6. This is no problem, as we refer only to the statements of chapter 6 which are well-known.

As we will see in 2.4.2, the Stiefel manifold

$$\mathbb{V}_2(\mathbb{R}^{2n+1}) = \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)$$

of pairs of orthonormal vectors in $\mathbb{R}^{2n+1}$ is a simply connected rational cohomology $(m+1)$-sphere for $m = 4n-2$. It has also the rational homotopy of an $(m+1)$-sphere, see e.g. the rational Hurewicz theorem in Kramer [37, 2.2]. Hence, a quotient G/H of $G = \mathrm{SO}(2) \times \mathrm{SO}(2n+1)$ with a (closed) subgroup H with connected component $H^1 = \mathrm{SO}(2n-1)$ has the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ by covering theory, and this quotient space might therefore appear as the base space $\mathcal{P}$ of a double fibration of type $(1, m)$. We will show that this is actually not possible.

Proposition 1.2.6 *A quotient G/H with $G = \mathrm{SO}(2) \times \mathrm{SO}(2n+1)$, $n > 1$, and $H^1 = \mathrm{SO}(2n-1)$ does not appear as base space $\mathcal{P}$ of a double fibration of type $(1, 4n-2)$.*

Proof We show that the cohomology of such a quotient space is not the 'right' one. The cohomology groups of the Stiefel manifold $\mathbb{V}_2(\mathbb{R}^{2n+1})$ are given by

$$H^k(\mathbb{V}_2(\mathbb{R}^{2n+1})) \cong \begin{cases} \mathbb{Z} & \textit{if } k \in \{0, 4n-1\}, \\ \mathbb{Z}_2 & \textit{if } k = 2n \textit{ and} \\ 0 & \textit{otherwise}, \end{cases}$$

see Bredon [11, VI.13.5]. The Universal Coefficient Theorem (see Bredon [11, V.7.Ex.7] for an appropriate version) implies

$$H^k(\mathbb{V}_2(\mathbb{R}^{2n+1}); \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } k \in \{0, 2n-1, 2n, 4n-1\} \textit{ and} \\ 0 & \textit{otherwise}. \end{cases}$$

The universal covering of G/H is $\mathbb{R} \times \mathbb{V}_2(\mathbb{R}^{2n+1})$. As we will see in 6.1.8 there is a fibration

$$\mathbb{R} \times \mathbb{V}_2(\mathbb{R}^{2n+1}) \to \mathcal{P} \to B\mathbb{Z} \simeq \mathbb{S}^1$$

which is orientable over $\mathbb{Z}_2$, since every automorphism of the at most one-dimensional $\mathbb{Z}_2$-vector space $H^k(\mathbb{V}_2(\mathbb{R}^{2n+1}); \mathbb{Z}_2)$ is the identity, cf. 6.1.6. The spectral sequence collapses and one gets by a theorem of Leray and Hirsch [37, 1.7+p.5] an isomorphism of graded groups

$$H^\bullet(\mathcal{P}; \mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{V}_2(\mathbb{R}^{2n+1}); \mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{V}_2(\mathbb{R}^{2n+1}); \mathbb{Z}_2);$$

hence

$$H^k(\mathcal{P}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k \in \{0, 1, 2n-1, 2n+1, 4n-1, 4n\}, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k = 2n \textit{ and} \\ 0 & \textit{otherwise}, \end{cases}$$

see e.g. Bredon [11, VI.3.2]. This does not coincide with the cohomology given in 1.2.2 for base spaces of the respective double fibrations. Hence, there is no quotient of the form G/H with $G = \mathrm{SO}(2) \times \mathrm{SO}(2n+1)$ and $H^1 = \mathrm{SO}(2n-1)$ as base space of a double fibration of type $(1, 4n-2)$. $\square$

The same method as in the proof of the last theorem applies to the quotient with $\mathrm{SU}(3)/\mathrm{SO}(3)$ playing the role of $\mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)$.

Proposition 1.2.7 *A quotient G/H with $G = \mathrm{SO}(2) \times \mathrm{SU}(3)$ and $H^1 = \mathrm{SO}(3)$ does not appear as base space $\mathcal{P}$ of a double fibration of type $(1,4)$.*

Proof The cohomology of $\mathrm{SU}(3)/\mathrm{SO}(3)$ over $\mathbb{Z}_2$ is given by

$$H^\bullet(\mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2) \cong \Lambda(e_2, e_3),$$

where e_2 and e_3 are homogeneous elements of degree 2 and 3, respectively, see e.g. Mimura-Toda [42, III.6.7(3)]; hence, $H^k(\mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $k \in \{0, 2, 3, 5\}$ and $H^k(\mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2) = 0$ otherwise. Since $\mathrm{SU}(3)/\mathrm{SO}(3)$ is simply connected (see e.g. 2.4.2), $\mathbb{R} \times \mathrm{SU}(3)/\mathrm{SO}(3)$ is the universal covering space of $\mathcal{P} = (\mathrm{SO}(2) \times \mathrm{SU}(3))/H$. There is as in 6.1.8 a fibration

$$\mathbb{R} \times \mathrm{SU}(3)/\mathrm{SO}(3) \to \mathcal{P} \to B\mathbb{Z} \simeq \mathbb{S}^1$$

which is orientable over $\mathbb{Z}_2$, since every automorphism of the at most one-dimensional $\mathbb{Z}_2$-vector space $H^k(\mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2)$ is the identity, see 6.1.6. The spectral sequence collapses and one gets by a theorem of Leray and Hirsch [37, 1.7 and p.5] an isomorphism of graded groups

$$H^\bullet(\mathcal{P}; \mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{R} \times \mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathrm{SU}(3)/\mathrm{SO}(3); \mathbb{Z}_2),$$

and, hence, by the Künneth Theorem (see e.g. Bredon [11, VI.3.2])

$$H^k(\mathcal{P}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k \in \{0,1,2,4,5,6\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k = 3, \\ 0 & \textit{else}. \end{cases}$$

This does not coincide with the cohomology given in 1.2.2 for base spaces of the respective double fibrations. Hence, there is no quotient of the form G/H with $G = \mathrm{SO}(2) \times \mathrm{SU}(3)$ and $H^1 = \mathrm{SO}(3)$ that is a base space $\mathcal{P}$ of a double fibration of type $(1, 4)$. $\square$

We finally consider the pair $(\mathrm{Sp}(2), {}^{\mathbb{H}}\rho_{3\lambda_1})$ of 2.4.2, where ${}^{\mathbb{H}}\rho_{3\lambda_1}$ denotes a compact connected subgroup of $\mathrm{Sp}(2)$ that is isomorphic to $\mathrm{Sp}(1)$. The quotient $\mathrm{Sp}(2)/{}^{\mathbb{H}}\rho_{3\lambda_1}$ is a simply connected rational cohomology 7-sphere with $\pi_3(\mathrm{Sp}(2)/{}^{\mathbb{H}}\rho_{3\lambda_1}) \cong \mathbb{Z}_{10}$.

Proposition 1.2.8 *A quotient G/H with $G = \mathrm{SO}(2) \times \mathrm{Sp}(2)$ and $H^1 = {}^{\mathbb{H}}\rho_{3\lambda_1}$ does not appear as base space $\mathcal{P}$ of a double fibration of type $(1,6)$.*

Proof We determine first the cohomology of $X = \mathrm{Sp}(2)/H^1$ over $\mathbb{Z}_2$. Since $\mathrm{Sp}(2)$ and $H^1 \cong \mathrm{Sp}(1)$ are 2-connected, also the quotient $\mathrm{Sp}(2)/H^1$ is 2-connected by the homotopy sequence. The Hurewicz isomorphism theorem and the fact that $\pi_3(X) \cong \mathbb{Z}_{10}$, cf. Kramer [37, 6.A], imply that $H_1(X) = 0$, $H_2(X) = 0$ and $H_3(X) \cong \mathbb{Z}_{10}$.

The universal coefficient theorem implies that $\mathbb{Z}_2 \cong H_3(X;\mathbb{Z}_2)$ and that $H_k(X;\mathbb{Z}_2) \cong H^k(X;\mathbb{Z}_2)$, since $\mathrm{Ext}(A,\mathbb{Z}_p) = 0$ for finitely generated abelian groups and prime numbers p, see Bredon [11, V.6]. This and the Poincaré duality for the 7-dimensional manifold X yield that

$$\begin{aligned} \mathbb{Z}_2 &\cong H_3(X;\mathbb{Z}_2) \cong H^3(X;\mathbb{Z}_2) \\ &\cong H_4(X;\mathbb{Z}_2) \cong H^4(X;\mathbb{Z}_2) \end{aligned}$$

and also that

$$0 \cong H_2(X;\mathbb{Z}_2) \cong H^5(X;\mathbb{Z}_2)$$

and

$$0 \cong H_1(X;\mathbb{Z}_2) \cong H^6(X;\mathbb{Z}_2).$$

Hence,

$$H^k(\mathrm{Sp}(2)/H^1;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k \in \{0,3,4,7\} \\ 0 & \textit{else}. \end{cases}$$

This shows that the fibration

$$\mathbb{R} \times \mathrm{Sp}(2)/H^1 \to \mathcal{P} \to B\mathbb{Z} \simeq \mathbb{S}^1$$

is orientable over the field $\mathbb{Z}_2$, since every automorphism of the at most one-dimensional $\mathbb{Z}_2$-vector space $H^k(\mathrm{Sp}(2)/H^1;\mathbb{Z}_2)$ is the identity, see 6.1.6. The spectral sequence collapses, and one gets by the theorem of Leray and Hirsch [37, 1.7 and p.5] an isomorphism of graded groups

$$H^\bullet(\mathcal{P};\mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{R} \times \mathrm{Sp}(2)/H^1;\mathbb{Z}_2) \cong H^\bullet(\mathbb{S}^1 \times \mathrm{Sp}(2)/H^1;\mathbb{Z}_2),$$

and, hence, by the Künneth Theorem

$$H^k(\mathcal{P};\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k \in \{0,1,3,5,7,8\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k = 4, \\ 0 & \textit{else}. \end{cases}$$

This does not coincide with the cohomology given in 1.2.2 for base spaces of the respective double fibrations. Hence, there is no quotient of the form G/H with $G = \mathrm{SO}(2) \times \mathrm{SU}(3)$ and $H^1 = {}^{\mathbb{H}}\rho_{3\lambda_1}$ that is a base space $\mathcal{P}$ of a double fibration of type $(1,6)$. $\square$

Chapter 2

Lie group actions

In the first section we introduce topological groups and the notions we use in Lie theory. Then we state some results which we will use frequently. The focus of the representation is restricted mainly to Lie groups, but many of these facts can be generalized to (locally) compact groups. See the book of Hofmann and Morris [27] for results on compact groups, and Hilgert-Neeb [26] for Lie groups and Lie algebras. For later use we also extend work of Biller [2] in order to determine certain large subgroups of $\mathrm{SU}(n)$.

We give some essential facts for actions of Lie groups in the second section. A basic introduction is given in Kawakubo [34], where also deeper results for differential actions of Lie groups may be found. For non-differential actions see the classics of Montgomery-Zippin [47] or Bredon [9], or chapter 9 of the book of Salzmann et al. [54].

The third section deals with transitive actions, and in the following section we collect some results about homogeneous spheres. First the classification of homogeneous homology spheres is quoted, then the classification of homogeneous rational cohomology spheres. The term generalized sphere in the sense of Biller is introduced and some results on generalized spheres are mentioned.

Finally, in the last section we consider almost transitive actions on compact spaces that have the rational homotopy of a product of a 1-dimensional sphere and a higher-dimensional sphere. This section starts with a general result of Mostert. Then we apply Mostert's theorem to obtain a description of homogeneous spaces with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$.

2.1 Lie theory

Most of the following facts may be found in many books on Lie groups. Our sources were Chapter 9 of Salzmann et al. [54], the book of Gorbatsevich, Onishchik and Vinberg [20] and the book of Mimura and Toda [42].

A topological Hausdorff space is called a **topological group**, if it additionally carries a group structure with continuous group operations. Isomorphisms of topological groups are also supposed to be homeomorphisms.

We are mainly interested in topological groups that are also equipped with a differential structure.

A group is a **Lie group**, if it carries additionally the structure of a topological (Hausdorff) manifold that admits local coordinates near the identity such that the group operations are real analytic maps.

The Lie groups may be characterised among the topological groups in purely topological terms. This is known as Hilbert's 5th problem. The following solution may be found in Montgomery-Zippin [47, 4.10], see also Gorbatsevich, Onishchik and Vinberg [20, I, Ch. 4, 4.2]: a locally connected and locally compact topological group of finite (covering) dimension is a Lie group.

The connected component of (the identity of) a Lie group is a normal subgroup. It is a (closed and) open subset. If the connected component of a topological group is a topological manifold and an open subspace, then the topological group is a Lie group (see also [54, 93.3]).

Every (continuous) homomorphism between Lie groups is in fact analytic (see e.g. [54, 94.9] for some references); therefore, the analytic structure of a Lie group is unique.

If H is a closed subgroup of a Lie group G, then H is a Lie group and the coset space G/H is a differentiable manifold of dimension $\dim G - \dim H$. Moreover, the canonical projection $G \to G/H$ is a fibre bundle and in particular a fibration. More generally, if U and H are closed subgroups of a Lie group G with $U \subseteq H$, then $H/U \to G/U \to G/H$ is a fibre bundle. In particular, if $U = H^1$ is the connected component of H, there is a covering $G/H^1 \to G/H$ with fibre H/H^1.

If H is a closed normal subgroup of G, then the quotient G/H is again a Lie group. Hence, if we know the connected component G^1 of a Lie group G, then we know G up to the discrete group G/G^1.

This means that it is particularly important to describe the structure of connected Lie groups in order to describe the structure of Lie groups. Furthermore, for the topological structure it suffices to consider compact groups as the following facts show. In a connected Lie group G there is a maximal compact subgroup K. All the maximal compact subgroups of a connected Lie group are connected and conjugate to each other; therefore, each compact subgroup is contained in a maximal one. And by a theorem of Cartan, Malcev and Iwasawa (see [33]) there is a homeomorphism $G \approx \mathbb{R}^n \times K$, and in particular we have $G \simeq K$. The maximal compact subgroup of a simple Lie group is also maximal as an abstract subgroup.

We now give a short overview over the structure of compact connected Lie groups. For every connected Lie group G there is a simply connected Lie group $\widetilde{G}$ (unique up to an isomorphism) and a differentiable map $p : \widetilde{G} \to G$ such that p is a group homomorphism and a universal covering. We call $\widetilde{G}$ the **universal covering group** of G. The kernel N of p is a discrete subgroup in the centre of $\widetilde{G}$ which is isomorphic to the fundamental group $\pi_1(G)$ of G by the homotopy sequence. In particular, we have a group isomorphism $G \cong \widetilde{G}/N$ that in fact is also a diffeomorphism. Furthermore, it follows that Lie groups have finitely generated abelian fundamental groups.

We call a compact connected Lie group G **almost simple**, if it is not abelian and if it does not have a proper closed normal subgroup of positive dimension. A simple compact Lie group is of course almost simple, and an almost simple compact Lie group G may be described as a quotient $G \cong S/N$ of a simply connected almost simple Lie group S and a discrete finite subgroup $\pi_1(G) \cong N \subseteq \mathrm{Z}(S)$ in the centre of S. A non-trivial compact connected Lie group G is called **semi-simple**, if the universal covering group $\widetilde{G}$ is compact. Equivalent descriptions are that the fundamental group of G is finite, that the centre of G is finite, or that $G' = G$, where G' is the commutator group of G.

A typical example of a compact connected Lie group with infinite fundamental group is the **1-torus** $T = \mathbb{S}^1 \subseteq \mathbb{C}^\times$, which may also be described by $T \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$. The k-fold product of T with itself is called a ***k*-Torus**. We call a non-trivial compact connected abelian Lie group a **torus**. Every torus is a k-torus for some k. Closed tori in tori are complemented, i.e. if there is a closed torus T_1 in a torus T, then there exists a torus $T_2 \subseteq T$ such that $T \cong T_1 \times T_2$ as topological groups. The maximal tori of a compact connected Lie group G are maximal abelian subgroups. They are all conjugate to each other, and they cover G. Their dimension is the **rank** $\mathrm{rk}(G)$ of G.

Compact connected Lie groups are almost direct products of the almost simple compact connected Lie groups and tori, see Hoffmann and Morris [27, 9.24].

Theorem 2.1.1 *Let G be a compact connected Lie group. Then there are simply connected almost simple Lie groups $S_1, \ldots, S_n$, such that*

$$\widetilde{G} \cong T^k \times S_1 \times \cdots \times S_n$$

*is a covering group of G, where $k = \dim \mathrm{Z}(G)$. The simply connected Lie groups $S_1, \ldots, S_n$ are unique (up to ordering and isomorphisms, of course). They are called the **simple factors** of G.*

Note that $T^k \cong \mathrm{Z}(G)^1 \cong \mathrm{Z}(\widetilde{G})^1$ is the connected component of the centre of G and that $\widetilde{G}' \cong S_1 \times \cdots \times S_n$ is the commutator group of $\widetilde{G}$ in the theorem. If G is not trivial, it is semi-simple if and only if $k = 0$, and it is almost simple if and only if $k = 0$ and $n = 1$.

A Lie group G is the **locally direct product** of two closed normal subgroups G_1 and G_2 if $G_1 \cap G_2$ is discrete and $G = G_1 G_2 = \{g_1 \cdot g_2 \mid g_1 \in G_1, g_2 \in G_2\}$. For a compact connected Lie group this means that we have $\widetilde{G} \cong \widetilde{G}_1 \times \widetilde{G}_2$ for the respective covering groups in the theorem above.

There is a covariant functor L from the class of Lie groups to the class of Lie algebras, which assigns to a Lie group G the Lie algebra $\mathrm{L}(G)$ that may be identified with its tangent algebra. $\mathrm{L}(G)$ is called the Lie algebra of G, and we will call this functor the **Lie functor**. The dimension of a Lie group equals the vector space dimension of its Lie algebra. Every Lie algebra may be realized as the Lie algebra of some simply connected Lie group. Two compact Lie groups G_1 and G_2 have the same Lie algebra if and only if their universal covering groups are the same. They are then called **locally isomorphic**.The Lie functor assigns locally direct products to direct products of Lie algebras, (virtual Lie) subgroups to subalgebras, and normal subgroups to ideals. Hence, a compact connected group G is almost simple if and only if its Lie algebra $\mathrm{L}(G)$ is a **simple** Lie algebra, i.e. if every proper ideal of the Lie algebra $\mathrm{L}(G)$ is trivial. The Lie algebras of compact Lie groups are called **compact** Lie algebras.

Via their Lie algebras the compact connected Lie groups are classified. There are four series of local isomorphism types of compact Lie algebras

$$\begin{aligned} &\mathfrak{a}_r, \quad r \geq 1 \\ &\mathfrak{b}_r, \quad r \geq 2 \\ &\mathfrak{c}_r, \quad r \geq 3 \\ &\mathfrak{d}_r, \quad r \geq 4 \end{aligned}$$

and five exceptional types $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, and $\mathfrak{g}_2$. The subscripts indicate the rank of the Lie algebras, i.e. the rank of the corresponding compact Lie groups. These Lie groups are the complex unitary groups $\mathrm{SU}(r+1)$, the orthogonal groups in odd dimensions $\mathrm{SO}(2r+1)$, the quaternion unitary groups (also called the symplectic groups) $\mathrm{Sp}(r)$, and the orthogonal groups in even dimensions $\mathrm{SO}(2r)$, for the infinite series above, respectively. These are called the **classical** compact connected Lie groups. For the five exceptional types there are the **exceptional** simply connected compact Lie groups E_6, E_7, E_8, F_4, and G_2.

In order to emphasize the connection with the classical Lie groups, we will often also write $\mathfrak{su}_{r+1}$, $\mathfrak{so}_{2r+1}$ (or $\mathfrak{o}_{2r+1}$), $\mathfrak{sp}_r$, or $\mathfrak{so}_{2r}$ ($\mathfrak{o}_{2r}$) for the Lie algebras $\mathfrak{a}_r$, $\mathfrak{b}_r$, $\mathfrak{c}_r$, or $\mathfrak{d}_r$, respectively.

The complex and the quaternion unitary groups are simply connected. The simply connected covering groups of the orthogonal groups $\mathrm{SO}(n)$ are called the spinor groups $\mathrm{Spin}(n)$. For the types E_8, F_4, and G_2 there is only one compact connected Lie group of that type.

In low dimensions there are isomorphisms of the following simply connected groups

$$\begin{aligned} \mathrm{SU}(2) \cong \ & \mathrm{Spin}(3) && \cong \mathrm{Sp}(1), \\ & \mathrm{Spin}(5) && \cong \mathrm{Sp}(2), \\ \mathrm{SU}(4) \cong \ & \mathrm{Spin}(6), \end{aligned}$$

which are all almost simple, and, additionally, $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. The corresponding isomorphisms of the Lie algebras are $\mathfrak{a}_1 \cong \mathfrak{b}_1 \cong \mathfrak{c}_1$, $\mathfrak{b}_2 \cong \mathfrak{c}_2$, $\mathfrak{a}_3 \cong \mathfrak{d}_3$, and $\mathfrak{d}_2 \cong \mathfrak{a}_1 \times \mathfrak{a}_1$. (Here, we extend the notation given above to the smallest ranks in the natural way via the classical groups, so by $\mathfrak{b}_1$ we mean the Lie algebra of $\mathrm{SO}(3)$, and so on. But note that $\mathfrak{d}_2$, of course, is not simple.)

The rational cohomology of a compact connected almost simple Lie group G of rank r is an exterior algebra (over $\mathbb{Q}$) generated by r primitive elements $u_1, \ldots, u_r$, i.e. $H^\bullet(G;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(u_1, \ldots, u_r)$, see Spanier [57, 5.8.13]. The degrees of the primitive elements depend only on the local isomorphism type. They are listed in table 2.1.

Much more on the cohomology and homotopy invariants of the compact connected Lie groups can be found in Mimura's survey article [41].

The subalgebras of maximal dimensions of simple compact Lie algebras are given in the table of Mann in section 4 of [38]. We list them in table 2.2, which we took from Biller [2, p. 50]. The subalgebra of maximal dimension is in each case unique up to an inner automorphism, except in the case of $\mathfrak{b}_3 \hookrightarrow \mathfrak{d}_4$, where it is unique up to an automorphism of $\mathfrak{d}_4$, see Biller [2, 2.5.1].

Lie algebra	Degrees of generators
$\mathfrak{a}_r$	$(3,5,7,\cdots,2r+1)$
$\mathfrak{b}_r$	$(3,7,11,\cdots,4r-1)$
$\mathfrak{c}_r$	$(3,7,11,\cdots,4r-1)$
$\mathfrak{d}_r$	$(3,7,11,\cdots,4r-5,2r-1)$
$\mathfrak{e}_6$	$(3,9,11,15,17,23)$
$\mathfrak{e}_7$	$(3,11,15,19,23,27,35)$
$\mathfrak{e}_8$	$(3,15,23,27,35,39,47,59)$
$\mathfrak{f}_4$	$(3,11,15,23)$
$\mathfrak{g}_2$	$(3,11)$

Table 2.1: The compact simple Lie algebras with the degrees of primitive generators of their cohomology, which is the cohomology of the corresponding Lie groups. (Note that $r \geq 2$ for $\mathfrak{d}_r$.)

$\mathfrak{g}$		$\dim\mathfrak{g}$	$\mathfrak{h}$	$\dim\mathfrak{g}-\dim\mathfrak{h}$
$\mathfrak{a}_r$	$(r\neq 3)$	$r(r+2)$	$\mathbb{R}\oplus\mathfrak{a}_{r-1}$	$2r$
$\mathfrak{b}_r$	$(r\geq 3)$	$r(2r+1)$	$\mathfrak{d}_r$	$2r$
$\mathfrak{c}_r$	$(r\geq 2)$	$r(2r+1)$	$\mathfrak{a}_1\times\mathfrak{c}_{r-1}$	$4(r-1)$
$\mathfrak{d}_r$	$(r\geq 3)$	$r(2r-1)$	$\mathfrak{b}_{r-1}$	$2r-1$
$\mathfrak{e}_6$		78	$\mathfrak{f}_4$	26
$\mathfrak{e}_7$		133	$\mathbb{R}\times\mathfrak{e}_6$	54
$\mathfrak{e}_8$		248	$\mathfrak{a}_1\times\mathfrak{e}_7$	112
$\mathfrak{f}_4$		52	$\mathfrak{b}_4$	16
$\mathfrak{g}_2$		12	$\mathfrak{a}_2$	6

Table 2.2: The subalgebras of maximal dimensions in the compact simple Lie algebras

There are relations between the dimension and the rank of a compact Lie algebra. The following proposition was proved by Biller [2, 5.1.1]. It will be useful in 5.1.

Proposition 2.1.2 (Biller) *Let $\mathfrak{g}$ be a compact Lie algebra with $\mathrm{rk}(\mathfrak{g}) \leq r$ and $\dim\mathfrak{g} > \binom{2r+1}{2}$. Then either $\mathrm{rk}(\mathfrak{g}) = r$, and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2$, to $\mathfrak{f}_4$ or to $\mathfrak{e}_7$, or $8 \leq r \leq 11$ and some ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{e}_8$.*

In order to determine in 2.4.6 the actions of $\mathrm{SU}(k)$ on generalized $2k$-spheres we want to prove that for $k > 4$ every closed connected subgroup H of $\mathrm{SU}(k)$ with $\dim H = \dim\mathrm{SU}(k-1)$ is a conjugate of $\mathrm{SU}(k-1)$ (in the standard inclusion into $\mathrm{SU}(k)$). This was done by Biller [2, 3.2.6] for $k \in \{4,5,6,7\}$.

For $k = 2$ this is trivial, for $k = 3$ this is false as there is also the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(3)$.

Dynkin [18] determined the maximal closed connected subgroups of the classical groups over $\mathbb{C}$. Seitz [55] generalized this to algebraic groups over algebraically closed fields. We use the classification to obtain the following observation.

Lemma 2.1.3 *Let H be a maximal closed connected proper subgroup of $\mathrm{SU}(k)$ for $k > 4$. If $\dim H > \dim \mathrm{SU}(k-1)$, then H is a conjugate of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$ and its Lie algebra is isomorphic to $\mathbb{R} \times \mathfrak{a}_{k-2}$.*

Proof To use the results of Dynkin or Seitz we note that the complexification gives a one-to-one correspondence between simple compact Lie algebras and simple complex Lie algebras, see Onishchik-Vinberg [52, p. 136], and between compact connected Lie groups and connected reductive complex Lie groups ([52, Ch. 4, Th. 2.7]).

We consider for the Lie algebra $\mathfrak{h}$ of H the complexification $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{h} \oplus i\mathfrak{h}$ of $\mathfrak{h}$ as a complex subalgebra of the complex subalgebra $\mathfrak{sl}_k$, which has $\mathfrak{su}_k$ as compact real form, i.e. $\mathfrak{sl}_k = \mathfrak{su}_k \otimes \mathbb{C}$ where $\mathfrak{su}_k$ is considered as a Lie algebra over the reals. But $\mathfrak{h} \otimes \mathbb{C}$ may fail to be maximal in $\mathfrak{sl}_k$. Therefore, we consider a maximal subalgebra $\mathfrak{m}$ of $\mathfrak{sl}_k$ containing $\mathfrak{h} \otimes \mathbb{C}$. We first treat the case that $\mathfrak{m}$ is not semi-simple. Non-semi-simple maximal subalgebras of complex semi-simple Lie algebras are parabolic by Onishchik-Vinberg [52, Ch. 6, Th. 1.8], and the maximal parabolic subgroups of $\mathrm{SL}_k(\mathbb{C})$ are the conjugates of the subgroups

$$P_l = \{ \left(\begin{smallmatrix} A & * \\ & B \end{smallmatrix}\right) \in \mathrm{SL}_k(\mathbb{C}) \mid A \in \mathrm{GL}_l(\mathbb{C}), B \in \mathrm{GL}_{k-l}(\mathbb{C}) \}$$

with $1 \leq l \leq k$, see [52, Ch. 6, §1.4, Ex. 1]. Since $\mathrm{S}(\mathrm{U}(l) \times \mathrm{U}(m-l))$ is a maximal compact subgroup of P_l and since dim $\mathfrak{m} \geq \dim \mathfrak{h} \otimes \mathbb{C}$ and $\dim \mathfrak{h} \geq \dim \mathfrak{su}_{k-1}$ it follows that H is a conjugate of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$ and hence $\mathfrak{h} \cong \mathbb{R} \times \mathfrak{a}_{k-2}$.

We now consider the case that $\mathfrak{m}$ is semi-simple. Then $\mathfrak{m}$ has a (unique) compact real form $\mathfrak{m}_\mathbb{R}$, see [52, Ch. 4, §1.2, Th. 1.1]; in particular $\mathfrak{h} \subseteq \mathfrak{m}_\mathbb{R} \subsetneq \mathfrak{su}_k$ (up to conjugation) since $\mathfrak{su}_k$ is a real form of $\mathfrak{sl}_k$. Hence, $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{m}$ by the maximum property of H, and $\mathfrak{h} \otimes \mathbb{C}$ is one of the maximal subalgebras of the classification of Dynkin or Seitz. We now only have to check whether there appear Lie algebras in the classification other than the one of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$ with (complex) dimension bigger than $\dim \mathrm{SU}(k-1)$. Since this is not the case, the announced result follows. □

Thus we obtain the following corollary.

Corollary 2.1.4 *For $k \geq 4$ a closed connected subgroup H of* $\mathrm{SU}(k)$ *with* $\dim H = \dim \mathrm{SU}(k-1)$ *is a conjugate of* $\mathrm{SU}(k-1)$ *in the standard inclusion* $\mathrm{SU}(k-1) \hookrightarrow \mathrm{SU}(k)$.

Proof This was already obtained by Biller [2, 3.2.6] for $k \in \{4, 5, 6, 7\}$. Lemma 2.1.3 yields that the subgroup H is up to conjugation a subgroup of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$. Consider the projection of the Lie algebra $\mathbb{R} \times \mathfrak{a}_{k-2}$ of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$ to its first factor $\mathbb{R}$. The restriction of the projection to the Lie algebra $\mathfrak{h}$ of H has an image of dimension at most one. Therefore, the image of the projection of $\mathfrak{h}$ to the second factor $\mathfrak{a}_{k-2}$ has at least dimension $\dim \mathfrak{a}_{k-2} - 1$. Table 2.2 on page 16 shows that $\mathfrak{h} \cong \mathfrak{a}_{k-2}$ and the claim follows by representation theory, see table 4.10 of Kramer [37]. □

2.2 Group actions

If a group G acts on a set X, then we denote the **stabilizer** of an element $x \in X$ by $G_x = \{g \in G \mid g \cdot x = x\}$. Similarly, we write $G_{x,y} = G_x \cap G_y$ for the stabilizer of two points. G acts transitively on each of its **orbits**

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

and trivially on the **orbit space**

$$X/G = \{G \cdot x \mid x \in X\}.$$

The **kernel** $G_{[X]} = \bigcap \{G_x \mid x \in X\}$ of the action is the normal subgroup of elements acting on X as the identity. The action is **effective** if its kernel is trivial. G acts transitively if and only if X consists of a single orbit $X = G \cdot x$, i.e. if $X/G = \{X\}$.

If a topological group G acts on a topological space X such that the map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, is continuous, then we call X a ***G*-space** and G a **topological transformation group** on X. In this case we also call the action of G on X a **continuous action** or a **topological action**, but we will often suppress the words 'topological' or 'continuous' when it is clear in the context that we speak of topological groups or even of Lie groups. If the kernel has dimension 0 then we call the action **almost effective**.

Note that for continuous actions the **orbit map** $\pi : X \to X/G$, $x \mapsto G \cdot x$, is open with respect to the quotient topology on X/G; in fact, for an open subset U of X it holds that $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$ is open.

A closed subgroup H of G acts on G by right multiplication. The resulting orbit space is the quotient space G/H of residue classes modulo H in G. Furthermore, the quotient space G/H is a G-space where G acts by $g \cdot aH = gaH$.

Assume that G is a topological transformation group on X and on Y. Then the two actions are called **equivalent** (as actions of the topological group G), if there is a homeomorphism $\varphi : X \to Y$ and an isomorphism of topological groups $\Phi : G \to G$ such that $\varphi(g \cdot x) = \Phi(g) \cdot \varphi(x)$ for every $g \in G$ and every $x \in X$, i.e. such that there is a commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "\Phi\times\varphi"] \arrow[d] & G \times Y \arrow[d] \\ X \arrow[r, "\varphi"] & Y. \end{tikzcd}$$

We then write $X \overset{G}{\approx} Y$ and call this a G-equivalence.

If G is a group of homeomorphisms of a compact metric space, then the supremum metric induces on G a (metric) topology (which coincides in this case with the topology of uniform convergence and also with the compact-open topology) such that G is a topological transformation group on X with its natural action $g \cdot x = g(x)$ for $g \in G$ and $x \in X$, see [54, 96.7].

We will use the following lemma similar to Biller's lemma 4.1.8 in [2].

Lemma 2.2.1 *Let G be a compact group with a continuous action on $\mathbb{R}$. Then every orbit of G contains at most two elements. If G is connected, then G acts trivially. If G acts effectively and non-trivially, then $G \cong \mathbb{Z}_2$.*

Proof As G is compact, every orbit is compact, too. Since G acts by homeomorphisms, every $g \in G$ preserves or reverses the order of $\mathbb{R}$ and $g \cdot \{\min G \cdot x, \max G \cdot x\} = \{\min G \cdot x, \max G \cdot x\}$ for every $x \in \mathbb{R}$. But G acts also transitively on every orbit; therefore, every element in an orbit is mapped to the maximum of the orbit. It follows that every orbit $G \cdot x$ contains at most two elements, i.e. $G \cdot x = \{\min G \cdot x, \max G \cdot x\}$.

If G acts effectively on $\mathbb{R}$ and if there is a non-trivial element g, then g reverses the order in some orbit, and hence on the whole line, i.e. g interchanges maximum and minimum of every orbit. But then the homeomorphism g is uniquely determined and $G = \{id, g\}$.

For a connected group G every orbit is connected, too. Therefore, an action of a compact connected group on $\mathbb{R}$ is trivial. $\square$

We are particularly interested in (topological) actions of Lie groups.

Let G be a Lie group acting transitively as a topological transformation group on a locally compact space X. Then the map $G \to X$, $g \mapsto g \cdot x$, is open and induces for each $x \in X$ a G-equivalence of actions $G/G_x \overset{G}{\approx} X$. In particular, there is an exact homotopy sequence

$$\cdots \longrightarrow \pi_k(G_x) \longrightarrow \pi_k(G) \longrightarrow \pi_k(X) \longrightarrow \pi_{k-1}(G_x) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(X) \longrightarrow \pi_0(G_x) \longrightarrow \pi_0(G) \longrightarrow \pi_0(X),$$

where we chose 1 as the base point in G_x and G, and x as base point in X. Furthermore, we have

$$\dim G = \dim G_x + \dim X.$$

Because of $G_{g \cdot x} = gG_xg^{-1}$ the stabilizers of elements of an orbit $G \cdot x$ are conjugate to each other and $G \cdot x \overset{G}{\approx} G \cdot y$ for every $y \in G \cdot x$ by an inner automorphism of G. Therefore, we distinguish orbits only up to the conjugacy classes of the stabilizers, which we will call the **orbit type** of $G \cdot x$. But we will often, for short, call G/G_x the orbit type of $G \cdot x$. For actions of compact Lie groups on Hausdorff spaces there is a kind of 'biggest' orbit type. More precisely, there is a stabilizer of minimal dimension that has a minimal number of connected components under all other stabilizers of minimal dimension. Such stabilizers are called **principal stabilizers**, and the respective orbits are the **principal orbits** of the action. The principal orbits are all conjugate to each other. This and their existence was proved by Montgomery-Yang [46] for differentiable actions. For a generalization to topological actions see e.g. Biller [2, 2.1.6, 2.2.3]

Linear actions are very important in the theory of Lie group actions. An ***n*-dimensional representation** of a Lie group G is an action on $\mathbb{R}^n$ induced by a continuous homomorphism $G \to \mathrm{GL}_n(\mathbb{R})$. Such an induced action is also called a **linear action** of G, and $\mathbb{R}^n$ is called a ***G*-module**. An effective representation is called **faithful**.

Two representations $\varphi_1 : G \to \mathrm{GL}_{n_1}(\mathbb{R})$ and $\varphi_2 : G \to \mathrm{GL}_{n_2}(\mathbb{R})$ induce a representation $\varphi = \varphi_1 \oplus \varphi_2 : G \to \mathrm{GL}_{n_1+n_2}(\mathbb{R})$ on $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ by setting $\varphi(g)(v \oplus w) = \varphi_1(g)(v) \oplus \varphi_2(g)(w)$ for $v \in \mathbb{R}^{n_1}$, $w \in \mathbb{R}^{n_2}$ and $g \in G$.

A representation $\varphi : G \to \mathrm{GL}_n(\mathbb{R})$ is called irreducible and $\mathbb{R}^n$ is called a **simple *G*-module**, if there is no non-trivial invariant submodule. For compact connected groups this means that φ cannot be written as a sum

$\varphi = \varphi_1 \oplus \varphi_2$ of two representations of strictly smaller dimensions. To avoid confusion with the term 'irreducible action' of section 2.3 we will avoid the term 'irreducible representation'. Nevertheless, we will at some places use results from representation theory, especially the tables of Kramer [37, Ch.4] of low-dimensional irreducible representations of the almost simple compact connected Lie groups. For more details on representation theory see Bröcker-tom Dieck [12], chapter 95 of [54], Tits [64] or Bödi-Joswig [4].

2.3 Transitive actions

We now focus on transitive topological actions. If a (topological) action of a (topological) group G on X is not effective, we may consider the induced (topological) action of the quotient group $G/G_{[X]}$ of G and the kernel $G_{[X]}$. This action is effective and has the same orbits. Another possibility to reduce the 'size' of an acting group is the following. We call an action of G an **irreducible action** if there is no proper normal transitive subgroup of G. For compact connected Lie groups there always exists a connected normal subgroup that acts irreducibly. For regaining the original action or, more generally, for 'enlarging' a given action, see Onishchik [51], Gorbatsevich-Onishchik-Vinberg [20, II, Ch. 2, 4.2] or Kramer [37, Prop. 3.6].

If a Lie group G acts transitively on a connected space X, then the connected component G^1 of G still acts transitively. This often allows us to assume that a transitive Lie group is connected.

To recognize a topological transformation group as a Lie group the following three theorems are useful. The first one was obtained by Montgomery-Zippin [47], cf. also [20, I, Ch. 4.4, Th. 4.3].

Theorem 2.3.1 (Montgomery-Zippin) *Suppose that G is a locally compact topological group acting effectively and transitively on a finite-dimensional, compact and locally connected space X. Then G is a Lie group, X is a manifold and the action is differentiable.*

For the proof of the next criterion, the theorem of Szenthe [60], see also the remarks in Salzmann et al. [54, 96.14].

Theorem 2.3.2 (Szenthe) *If a locally compact topological group G with countable basis acts effectively and transitively on a connected and locally contractible space X, then G is a Lie group and X is a manifold.*

The next theorem is taken from Salzmann et. al [54, 96.24]. It does not assume a transitive action in contrast to the last two theorems, see also Biller's version 2.4.4 for generalized spheres. Biller [2, 2.2.2] also gives a version for metrizable cohomology manifolds.

Theorem 2.3.3 *If a locally compact topological group G acts effectively on a connected n-dimensional manifold such that a compact subgroup of G has an orbit of dimension at least $n-2$, then G is a Lie group.*

The following theorem often allows to restrict the considerations to compact Lie groups. The theorem was proved by Montgomery [43] for simply connected spaces. This may be generalized to spaces with finite fundamental groups, see e.g. the proof in [54, 96.19].

Theorem 2.3.4 *If a connected Lie group G acts transitively on a compact connected manifold X that has a finite fundamental group, then every maximal compact subgroup of G is transitive on X.*

In section 2.5 we treat the case of infinite cyclic fundamental groups under some additional assumptions.

The dimensions of the orbits give an easy criterion for the transitivity of a compact Lie group. If a compact Lie group G acts as a topological transformation group on a locally compact, locally contractible and connected space X and if there is an $x \in X$ with $\dim G \cdot x = \dim X$, then $G \cdot x = X$, i.e. the action of G is transitive. A space on which a compact Lie group acts transitively is also called a **homogeneous space**.

There is a well-known estimate of the dimension of a transitive and effective Lie group on a given manifold, see corollary 1 of theorem 6.2.5 in Montgomery-Zippin [47].

Proposition 2.3.5 *If a compact connected Lie group G acts effectively with an orbit of dimension k, then $\dim G \leq \binom{k+1}{2}$.*

2.4 Homogeneous spheres

Montgomery and Samelson [44] determined the structure of compact connected Lie groups that act effectively and transitively on spheres. Borel ([5] and [6]) gave an explicit list of these almost simple Lie groups. He also proved that if a **homogeneous (co)homology sphere**, i.e. a homogeneous space of a compact connected Lie group with the same (co)homology as a sphere, is additionally simply connected, then it is indeed an ordinary sphere. Note that a homology sphere is a cohomology sphere by the universal coefficient theorem, see Bredon [11, V.7.3], and a homogeneous cohomology sphere is a homology sphere by Spanier [57, 5.5.12]. (As a homogeneous space it is a manifold and the homology is of finite type, see Bredon [11, E.5]; hence Spanier [57, 5.5.12] applies.)

Poncet [53] proved that these transitive actions on spheres are all equivalent to actions of subgroups of the respective orthogonal group and hence linear. Finally, Bredon [8, 1.1, 1.2] showed that a homogeneous cohomology sphere is either a 1-sphere or simply connected (and hence a sphere) or the **Poincaré homology 3-sphere**, i.e. the homogeneous space $\mathrm{SO}(3)/I$ of $\mathrm{SO}(3)$ and the group I of rotational symmetries of an icosahedron (an **icosahedral subgroup**), see [11, VI.8.10].

This gives a classification of homogeneous cohomology spheres. We collect these results in the following theorem. For similar tables, see also Salzmann et al. [54, 95.4], Grundhöfer-Knarr-Kramer [23, 1.1] and Biller [2, 3.1.1].

Theorem 2.4.1 (Homogeneous cohomology spheres) *Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively on G/H and such that $H^\bullet(G/H) \cong H^\bullet(\mathbb{S}^n)$ for some $n \in \mathbb{N}$.*

If G/H is not a sphere, then $G = \mathrm{SO}(3)$ and H is an icosahedral subgroup of $\mathrm{SO}(3)$. Otherwise, the action of G is equivalent to the action of a subgroup of $\mathrm{SO}(n+1)$ on the standard sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

If G/H is a sphere and if G acts irreducibly (i.e. if there is no proper transitive normal subgroup), then there are (up to equivalence) only the following possibilities for (G, H, n).

$(\mathrm{SO}(n+1), \mathrm{SO}(n), n)$, $n \in \mathbb{N}$
$(\mathrm{SU}(k), \mathrm{SU}(k-1), 2k-1)$, $k \geq 2$
$(\mathrm{Sp}(k), \mathrm{Sp}(k-1), 4k-1)$, $k \geq 2$
$(G_2, \mathrm{SU}(3), 6)$
$(\mathrm{Spin}(7), G_2, 7)$
$(\mathrm{Spin}(9), \mathrm{Spin}(7), 15)$

If G/H is a sphere and if G has a proper transitive normal subgroup, then the possibilities for (G, H, n) are (up to equivalence) the following.

$(\mathrm{U}(k), \mathrm{U}(k-1), 2k-1)$, $k \geq 2$
$(\mathrm{U}(1) \cdot \mathrm{Sp}(k), \mathrm{U}(1) \cdot \mathrm{Sp}(k-1), 4k-1)$, $k \geq 2$
$(\mathrm{Sp}(1) \cdot \mathrm{Sp}(k), \mathrm{Sp}(1) \cdot \mathrm{Sp}(k-1), 4k-1)$, $k \geq 2$.

The homogeneous space also is known if it has the same *rational* cohomology as a sphere.

Theorem 2.4.2 (Homogeneous rational cohomology spheres) *Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively and irreducibly on G/H and such that $H^\bullet(G/H;\mathbb{Q}) \cong H^\bullet(\mathbb{S}^n;\mathbb{Q})$ for some $n \in \mathbb{N}$. Assume that G/H is simply connected, if $n > 1$.*

Then G/H is a sphere (and one of the irreducible homogeneous spheres in 2.4.1), or n is odd and (G, H) is one of the pairs in the following table.

G	H	$\mathrm{Cen}_G(H)^1$	n	
$\mathrm{SU}(3)$	$\mathrm{SO}(3)$	1	5	$\pi_3(G/H) = \mathbb{Z}_4$
$\mathrm{SO}(2k+1)$	$\mathrm{SO}(2k-1)$	$\mathrm{SO}(2)$	$4k-1$	$\pi_{2k-1}(G/H) = \mathbb{Z}_2$, $k \geq 2$
$\mathrm{Sp}(2)$	${}^{\mathbb{H}}\rho_{3\lambda_1}$	1	7	$\pi_3(G/H) = \mathbb{Z}_{10}$
G_2	$\mathrm{SU}(2)$	$\mathrm{SU}(2)$	11	$\pi_3(G/H) = \mathbb{Z}_2$
G_2	${}^{\mathbb{R}}\rho_{\lambda_1} + {}^{\mathbb{R}}\rho_{2\lambda_1}$	$\mathrm{SU}(2)$	11	$\pi_3(G/H) = \mathbb{Z}_3$
G_2	$2{}^{\mathbb{R}}\rho_{2\lambda_1}$	1	11	$\pi_3(G/H) = \mathbb{Z}_4$
G_2	${}^{\mathbb{R}}\rho_{6\lambda_1}$	1	11	$\pi_3(G/H) = \mathbb{Z}_{28}$

The symbols ${}^{\mathbb{H}}\rho_{3\lambda_1}$, ${}^{\mathbb{R}}\rho_{\lambda_1} + {}^{\mathbb{R}}\rho_{2\lambda_1}$, $2{}^{\mathbb{R}}\rho_{2\lambda_1}$ and ${}^{\mathbb{R}}\rho_{6\lambda_1}$ appearing in the table stand for the images of representations of Lie groups of type $\mathfrak{a}_1$, cf. Kramer [37, Ch. 4 and 6.A].

Proof If n is even, then the homogeneous space has Euler characteristic 2 and is indeed a sphere by Borel-De Siebenthal [7] and Borel [5].

A compact connected one-dimensional manifold is a 1-sphere. For odd $n \geq 3$ there is an almost simple transitive normal Lie subgroup by Kramer [37, 3.7] and Onishchik [51, §18, Prop. 2(i)]. We took the table from Kramer [37, 6.A], cf. also Onishchik [51, Ch. 5, §18, Table 10], where Sp(2) appears as SO(5). □

Later on we will use results for the following generalization of spheres due to Biller [2].

Definition 2.4.3 *A topological space S is called **locally homogeneous** if any two points have homeomorphic open neighbourhoods. The space S is called **pseudo-isotopically contractible relative to** $x \in S$ if there is a homotopy $F : S \times [0,1] \to S$ such that $F(\cdot, t)$ is a homeomorphism of S for all $t \in (0,1)$, such that the maps $F(x,\cdot)$ and $F(\cdot, 1)$ are the constant maps to x, and such that $F(\cdot, 0) = \mathrm{id}_S$ is the identity. S is an **euclidian neighbourhood retract** (**ENR**) if it is homeomorphic to a retract of an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.*

Finally, if S is a locally homogeneous n-dimensional ENR, such that every *point complement is non-empty and pseudo-isotopically contractible relative to one of its points, then it is called a* ***generalized n-sphere***.

A generalized n-sphere is compact, homotopy equivalent to $\mathbb{S}^n$, and an n-dimensional cohomology manifold, see [2, 1.3.2]. Note that every ordinary sphere $\mathbb{S}^n$ is indeed a generalized n-sphere. The Poincaré homology 3-sphere of 2.4.1 is not a generalized 3-sphere since it is not simply connected.

Generalized spheres arise for example as the line pencils and point rows of compact connected quadrangles, see 4.1.2, but in all known examples the line pencils and point rows are genuine spheres.

The next lemma says that a generalized sphere which admits an almost transitive action of a compact group is a sphere.

Lemma 2.4.4 (Biller) *Let G be a compact group acting effectively on a generalized n-sphere S^n. Then the action of G on every principal orbit is also effective.*
If there is an orbit of codimension at most 2, then $S^n \approx \mathbb{S}^n$ and G is a Lie group.
If $n \geq 2$ and every orbit has codimension at most 1, then G acts transitively.

Proof This is lemma 1.3.3 and lemma 1.3.5 of Biller [2] together with lemma 1.6 of Grundhöfer, Knarr and Kramer [23]. □

For a topological space X we call the quotient space of $X \times [0,1]$ under the identification of $X \times \{0\}$ and $X \times \{1\}$ to points $[0]$ and $[1]$ the (unreduced) **suspension** ΣX of X.

If X is a G-space, then G acts on $X \times [0,1]$ by $g \cdot (x,t) = (g \cdot x, t)$. This induces a G-action on ΣX which fixes $[0]$ and $[1]$. We call this action the **suspension** of the G-space X.

The suspension of an n-sphere is an $(n+1)$-sphere, $\Sigma\mathbb{S}^n \approx \mathbb{S}^{n+1}$; hence, a G-action on $\mathbb{S}^n$ induces a G-action on $\mathbb{S}^{n+1}$. We are going to show that a non-trivial action of $\mathrm{SU}(k)$ on a generalized $(2k)$-sphere is the suspension of the usual $\mathrm{SU}(k)$-action on $\mathbb{S}^{2k-1}$. The first step is the following lemma of Biller [2, 3.2.3].

Lemma 2.4.5 (Biller) *If $k \notin \{1,2,4\}$ and $\mathrm{SU}(k)$ acts almost effectively on a generalized $2k$-sphere S, then the principal orbits have codimension 1 and $S \approx \mathbb{S}^{2k}$.*

Note that there are counterexamples for $k \in \{1,2,4\}$. The lemma leads to a proof of the following proposition.

Proposition 2.4.6 (Biller) *If $4 \neq k \geq 3$ and $\mathrm{SU}(k)$ acts almost effectively on a generalized $2k$-sphere S, then the action is the suspension of the transitive action of $\mathrm{SU}(k)$ on $\mathbb{S}^{2k-1}$; in particular $S \approx \mathbb{S}^{2k}$.*

Proof This was proved by Biller [2, 3.2.8] under the additional assumption that

(*) the connected component of a stabilizer is conjugate to $\mathrm{SU}(k-1)$.

He also observed that one may drop the assumption if one knows that all subgroups of $\mathrm{SU}(k)$ of dimension $\dim \mathrm{SU}(k-1)$ are conjugate. But this is corollary 2.1.4. This indeed allows us to drop the assumption (*) above. By lemma 2.4.5 there is a principal orbit of codimension 1. Hence, there is a stabilizer of dimension $\dim \mathrm{SU}(k) - (2k-1) = \dim \mathrm{SU}(k-1)$, and corollary 2.1.4 assures that the connected component of the stabilizer is a conjugate of $\mathrm{SU}(k-1)$. Now we may apply the original result of Biller [2, 3.2.8]. □

2.5 Almost transitive actions

We will often apply the following theorem which was proved by Mostert in [48], confer also the errata and footnote 2 in Hofmann-Mostert [28].

Theorem 2.5.1 (Mostert) *Let M be a compact connected manifold with an action of a compact connected Lie group G such that there is a G-orbit of codimension 1 in M. Then $M/G \approx \mathbb{S}^1$ or $M/G \approx [0,1]$.*

If $M/G \approx \mathbb{S}^1$, then all G-orbits in M are of the same type, and there is a fibre bundle $N \to M \to \mathbb{S}^1$ where the typical fibre N is a G-orbit of the occuring type.

If $M/G \approx [0,1]$, then there are points l, r and m in M such that under the homeomorphism $M/G \approx [0,1]$ their orbits correspond to 0, 1 and an inner point of $[0,1]$, respectively, and such that $G_m \subseteq G_l \cap G_r$. Furthermore, the homogeneous spaces G_m/G_l and G_m/G_r are homology spheres, and there is a G-equivalence between M and the double mapping cylinder of the projections $G/G_m \to G/G_l$ and $G/G_m \to G/G_r$:

$$M \overset{G}{\approx} (G/G_m \times [0,1]) \Big/ (gG_m, 0) \sim (gG_l, 0), (gG_m, 1) \sim (gG_r, 1).$$

In particular, $G \cdot m$ is a principal orbit, and there are just two non-principal orbits $G \cdot l$ and $G \cdot r$, which are called the ***exceptional*** *or* ***singular*** *orbits.*

Mostert treated in [48] also the case of non-compact manifolds. But we will only need the theorem for compact manifolds, therefore the theorem is not stated in full generality. Biller gave an application of Mostert's theorem to generalized spheres in [2, 3.1.2].

A typical example for the case that the orbit space is an interval is the suspension of an action.

An easy criterion which of the two cases in Mostert's theorem applies in a concrete situation gives the following lemma of Montgomery-Yang [45, Cor. 2]. We give the formulation of the slight generalization in Bredon [9, II.6.3].

Lemma 2.5.2 *If a compact Lie group G acts on an arcwise connected Hausdorff space X such that there is a connected orbit, then the orbit map $X \to X/G$, $x \mapsto G \cdot x$, induces a surjective homomorphism from the fundamental group of X to the fundamental group of X/G, i.e.*

$$\pi_1(X) \to \pi_1(X/G) \to 0$$

is exact.

The existence of a connected orbit follows for example if G is connected or if there is a fixed point in X. For transitive actions the last lemma follows from the homotopy sequence. The following lemma is of the same spirit, see Bredon [9, II.6.5].

Lemma 2.5.3 *If a compact Lie group G acts on an arcwise connected Hausdorff space X, then the orbit map $X \to X/G$, $x \mapsto G \cdot x$, induces a surjective homomorphism from the first rational singular homology group of X to the the first rational singular homology group of X/G, i.e.*

$$H_1(X;\mathbb{Q}) \to H_1(X/G;\mathbb{Q}) \to 0$$

is exact.

Now we start examining the type of homogeneous spaces we are interested in, namely compact spaces with the same rational homotopy as a product of a 1-sphere and a higher-dimensional sphere such that the fundamental group is torsion-free. We will see later that there arise examples of such spaces as focal manifolds of certain isoparametric hypersurface, see chapter 3; another type of example are the point spaces of $(1, m)$-quadrangles as in chapter 4. Other examples than those arising from the mentioned geometries are given at the end of chapter 6.

First, we will see that the cohomology over the rationals of the universal covering is quite simple.

Lemma 2.5.4 *Let X be a compact connected topological $(m+2)$-manifold with infinite fundamental group such that $\pi_k(X)$ is finite for $2 \leq k \leq m$. If s is the rank of $\pi_{m+1}(X)$, then the rational cohomology groups of the universal covering space $\widetilde{X}$ are given by*

$$H^k(\widetilde{X};\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k=0, \\ \mathbb{Q}^s & \text{if } k=m+1, \\ 0 & \text{else} \end{cases}$$

Proof The cohomology groups of the $(m+2)$-dimensional manifold $\widetilde{X}$ are trivial in dimensions greater than $(m+2) = \dim \widetilde{X}$. As a covering space with infinite fibres, $\widetilde{X}$ is a non-compact manifold, and it follows that $H^{m+2}(\widetilde{X}) = 0$, see Bredon [11, VI.7.12 and 14].

Since $\widetilde{X}$ is the universal covering space of X, we have

$$\pi_k(\widetilde{X}) \otimes \mathbb{Q} \cong \pi_k(X) \otimes \mathbb{Q} = 0 \qquad \text{for } 2 \leq k \leq m$$

and $\pi_{m+1}(\widetilde{X}) \otimes \mathbb{Q} \cong \pi_{m+1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}^s$. There is a rational version of the Hurewicz theorem, see Kramer [37, 2.1, 2.2], which now implies that the rational cohomology groups in the 'middle' dimensions vanish, i.e. $H^k(\widetilde{X};\mathbb{Q}) = 0$ for $0 < k < m+1$, and that $H^{m+1}(\widetilde{X};\mathbb{Q}) \cong \mathbb{Q}^s$. □

The assumption on the homotopy in the last lemma is in particular fulfilled if $\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$. This gives the following corollary.

Corollary 2.5.5 *Let X be a compact connected topological $(m+2)$-manifold with an abelian fundamental group. Then*

$$\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q} \text{ implies } H^\bullet(\widetilde{X};\mathbb{Q}) \cong H^\bullet(\mathbb{S}^{m+1};\mathbb{Q})$$

for the universal covering space $\widetilde{X}$ of X.

We give a characterization of transitivity for compact Lie subgroups on homogeneous spaces as in 2.5.5.

Proposition 2.5.6 *Let X be an $(m+2)$-dimensional compact homogeneous space of a connected Lie group G with $\pi_1(X) \cong \mathbb{Z}$ and*

$$\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q},$$

and let K be a maximal compact subgroup of G. Then there is a point $x \in X$ such that the following statements are equivalent.

(i) *K is transitive.*

(ii) $\pi_1(K/{K_x}^1) \cong \mathbb{Z}$

(iii) $\pi_1(G/{G_x}^1) \cong \mathbb{Z}$

(iv) *$G_x/{G_x}^1$ is finite.*

Proof We begin the proof as in Salzmann et al. [54, 96.19]. We choose a point $x \in X$ such that ${K_x}^1$ is maximal compact in ${G_x}^1$. This is possible, because every compact subgroup lies in a maximal one and these are all conjugate.

Consider the commutative diagram

$$\begin{tikzcd} K/{K_x}^1 \arrow[r, "j"] \arrow[d, "p"] & G/{G_x}^1 \arrow[d, "q"] \\ K/K_x \arrow[r, "\alpha"] & G/G_x \end{tikzcd}$$

where j and α are the natural mappings and indeed embeddings by [54, 96.9]. As ${K_x}^1$ is maximal compact in ${G_x}^1$, the two inclusions $K \hookrightarrow G$ and ${K_x}^1 \hookrightarrow {G_x}^1$ are homotopy equivalences, and so they induce isomorphisms in homotopy. Therefore, the induced map j of the diagram also induces an isomorphism in homotopy by the five-lemma. But then j induces also an isomorphism in homology, see the Whitehead theorem [57, p. 399]. As p and q are coverings, α induces isomorphisms $\alpha_\# : \pi_n(K/K_x) \to \pi_n(G/G_x)$ for all $n \geq 2$.

In the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(K/{K_x}^1) \arrow[r] \arrow[d, "\cong"] & \pi_1(K/K_x) \arrow[r] \arrow[d] & K_x/{K_x}^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_1(G/{G_x}^1) \arrow[r] & \pi_1(G/G_x) \arrow[r] & G_x/{G_x}^1 \arrow[r] & 0 \end{tikzcd}$$

we have $\pi_1(G/G_x) \cong \pi_1(X) \cong \mathbb{Z}$ by assumption. This yields the equivalence of (ii), (iii) and (iv).

Now we will distinguish two cases. The first case is that $\pi_1(G/{G_x}^1)$ is not trivial, i.e. $\pi_1(G/{G_x}^1) \cong \mathbb{Z}$; thus in this case $G_x/{G_x}^1$ is finite and the covering space $G/{G_x}^1$ of X is compact. Then by the isomorphism induced by j the covering space $G/{G_x}^1$ of X has the same homology as the compact space $K/{K_x}^1$, and since homology over $\mathbb{Z}_2$ determines the dimension of compact manifolds, see Bredon [11, VI.7.12], the orbit $K \cdot x$ of K in $G/{G_x}^1$ has full dimension. Hence, in this case K is transitive on the covering and therefore also on X.

In the second case $G/{G_x}^1$ is simply connected; hence, $K/{K_x}^1$ is simply connected, too, and $G_x/{G_x}^1$ is infinite. This means that $G/{G_x}^1$ is the universal covering space of X and that it is not compact and not homeomorphic to the compact space $K/{K_x}^1$. Therefore, K is not transitive on X in this case, because if it were, the two universal coverings $G/{G_x}^1$ and $K/{K_x}^1$ had to be the same. $\square$

The next example shows that in the situation of the last proposition the maximal compact groups need not act transitively.

Example 2.5.7 The image of $\mathbb{R} \times \mathrm{SU}(n)$ in $\mathrm{GL}_{n+1}(\mathbb{C})$ under the injection

$$(t, A) \mapsto \begin{pmatrix} e^{2\pi i\sqrt{2}t} & \\ & e^{2\pi it} A \end{pmatrix}$$

acts effectively and transitively on $\mathbb{S}^1 \times \mathbb{S}^{2n-1} \subset \mathbb{C}^{n+1}$ and has no transitive compact subgroup, because the orbits of $\mathrm{SU}(n)$ are $(2n-1)$-spheres.

Next we determine the orbits of almost transitive compact subgroups.

Proposition 2.5.8 *Let X be an $(m+2)$-dimensional compact homogeneous space of a simply connected Lie group G with $\pi_1(X) \cong \mathbb{Z}$ and $\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$. Then the orbits of every maximal compact subgroup K of G are simply connected rational cohomology $(m+1)$-spheres and $\pi_\bullet(X) \cong \pi_\bullet(\mathbb{S}^1 \times (K \cdot x))$ for every $x \in X$.*

In particular, K is not transitive and G is not compact.

Proof Let K be a maximal compact subgroup of G and $x \in X$ a point such that ${K_x}^1$ is maximal compact in ${G_x}^1$ as in the proof of 2.5.6. (In the end we will see that all orbits are of the same type. Therefore, the choice of x is actually arbitrary.) As G is simply connected, K is semi-simple and $G/{G_x}^1$ is also simply connected. Furthermore, $G/{G_x}^1$ is the universal covering space of X. Then in view of 2.5.6 the action of K on X is not transitive.

In the proof of 2.5.6 we saw that $K/{K_x}^1 \hookrightarrow G/{G_x}^1$ induces isomorphisms in homotopy and (co)homology. It follows that

$$\pi_\bullet(K/{K_x}^1) \otimes \mathbb{Q} \cong \pi_\bullet(G_x/{G_x}^1) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^{m+1}) \otimes \mathbb{Q}$$

and by 2.5.5 that $H^\bullet(G/{G_x}^1; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^{m+1}; \mathbb{Q})$. Therefore $K/{K_x}^1$ is a simply connected rational cohomology $(m+1)$-sphere. As $K/{K_x}^1$ is compact, orientable, and has the rational cohomology of $\mathbb{S}^{m+1}$, one has $\dim K \cdot x = \dim K/K_x = \dim K/{K_x}^1 = m+1$.

We now apply Mostert's theorem 2.5.1 for orbits of codimension 1 in compact manifolds. There are just the two possibilities $X/K \approx \mathbb{S}^1$ or $X/K \approx [0,1]$. We will show that the second case does not occur. The compact connected Lie groups that act irreducibly on simply connected cohomology spheres are classified, see 2.4.2. They are always almost simple. Therefore, there is a connected almost simple normal subgroup A of K that still acts transitively on the cohomology sphere $K/{K_x}^1 = A/{A_x}^1$. As almost simple compact Lie groups have finite fundamental groups, the homotopy sequence shows that the orbits of almost simple compact Lie groups have also finite fundamental groups, see section 2.1. If $X/A \approx [0,1]$, then X may be described as a mapping cylinder of orbit projections of a principal orbit $A \cdot m$ onto two exceptional orbits $A \cdot l$ and $A \cdot r$ of the action of A, see Mostert's theorem 2.5.1. The injections of the two exceptional orbits into X induce mappings of their fundamental groups into the fundamental group of X. By the Seifert-Van Kampen theorem these images have to generate the fundamental group of X. But as $\pi_1(X) \cong \mathbb{Z}$ by assumption, the images of finite groups in $\pi_1(X)$ are trivial and cannot generate $\pi_1(X)$. Hence, $X/A \approx [0,1]$ is not possible.

Therefore, $X/K \approx \mathbb{S}^1 \approx X/A$. But then all orbits are of the same type A/A_x, and there is a fibre bundle $A/A_x \to X \to X/A \approx \mathbb{S}^1$, whose homotopy sequence shows that $\pi_1(A/A_x) \cong 0$, i.e. $A_x = {A_x}^1$ is connected. Hence, the orbits of A are simply connected rational cohomology $(m+1)$-spheres, and the homotopy sequence of $A/A_x \to X \to X/A \approx \mathbb{S}^1$ shows that $\pi_\bullet(X) \cong \pi_\bullet(\mathbb{S}^1 \times (A \cdot x)) \cong \pi_\bullet(\mathbb{S}^1 \times (K \cdot x))$. □

Remark 2.5.9 Note that the simply connected homogeneous rational cohomology spheres, which appear in the last proposition, are classified, see 2.4.2.

Remark 2.5.10 If $\pi_1(X)$ in the last lemma is abelian but fails to be torsion free, then there still is for every $x \in X$ a maximal compact subgroup of G such that its orbits are (not necessarily simply connected) rational cohomology $(m+1)$-spheres.

The preceding results lead to the following description of transitive actions of compact Lie groups on the kind of spaces we are considering. We give another proof of the following theorem in 6.3.7 where we restrict ourselves to irreducible actions.

Theorem 2.5.11 *Let X be an $(m+2)$-dimensional homogeneous space of a compact connected Lie group K with $\pi_1(X) \cong \mathbb{Z}$ and $\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$. Then there is a complementary one-dimensional torus group $T \cong \mathrm{SO}(2)$ for the commutator group A of K such that $T \cdot A$ is transitive on X and the orbits of A are simply connected rational cohomology $(m+1)$-spheres. For every $x \in X$ the quotient $K_x/{K_x}^1$ is cyclic and $A_x = A \cap K_x = {(T \cdot A)_x}^1$ is connected. $\mathbb{S}^1 \times A/A_x$ is a covering space of X.*

Proof The centre of K has positive dimension by 2.5.6. Furthermore, the connected component of the centre still acts transitively on X/A. Up to a finite covering, A is the maximal compact subgroup of the universal covering group of K; hence A is connected. By 2.5.8 the orbits of A in X are simply connected rational cohomology spheres and $X/A \approx \mathbb{S}^1$. It follows that there is a complementary torus group $T \cong \mathrm{SO}(2)$ in K such that $T \cdot A$ is transitive on X .

The homotopy sequence of $K_x \to K \to X$ shows that $\pi_0(K_x) \cong K_x/{K_x}^1$ is a quotient of the infinite cyclic group $\pi_1(X)$ and therefore cyclic. Furthermore, A_x is connected, since the orbit $A \cdot x$ is simply connected.

Since $T \cdot A$ is transitive, the intersection of the first factor T with K_x is finite, i.e. T_x is finite. Therefore, there is a covering space $\mathbb{S}^1 \times A/{A_x}^1$ of $X \approx T \cdot A/(T \cdot A)_x$. □

The rational cohomology of X and of ${K_x}^1 \to K \to K/{K_x}^1$ is determined in chapter 6.

Chapter 3

Isoparametric hypersurfaces

We apply our results, in particular 2.5.11, to determine the transitive and irreducible actions of compact connected isometry groups of isoparametric hypersurfaces on focal manifolds that have the rational homotopy of a sphere product $\mathbb{S}^1 \times \mathbb{S}^{m+1}$. First we give a survey on homogeneity results already known in the literature. In the second section we describe the transitive actions on the focal manifold.

3.1 Homogeneous hypersurfaces

An **isoparametric hypersurface** in a sphere is a closed hypersurface with constant principal curvatures. For a survey on isoparametric hypersurfaces see Thorbergsson [63]; many important results are due to Münzner [49], [50].

The sphere containing the isoparametric hypersurface M is partitioned by M, by parallel hypersurfaces that are diffeomorphic to M and by two submanifolds of smaller dimensions. This partition of the sphere is called an isoparametric foliation, and the two smaller manifolds are the **focal manifolds** of M.

For the number g of principal curvatures there are only the possibilities 1, 2, 3, 4 or 6, see Münzner [49], and the respective multiplicities $m_1, \ldots, m_g$ of the principal curvatures may be reordered such that $m_i = m_{i+2}$ (indices modulo g); in particular the multiplicities are all equal if $g = 3$. Hence, it suffices to know the multiplicities m_1 and m_2. If $g = 1$, then the hypersurface is a sphere, and if $g = 2$, then it is a product of spheres. (This is sometimes called a Clifford torus.) More interesting is the case $g = 3$. This case was classified by Cartan [14]. There are only 4 such hypersurfaces, namely embeddings of the classical projective planes over the reals, the complex numbers, Hamilton's quaternions and the Cayley numbers, see also Knarr and Kramer [35]. Here, the isoparametric hypersurface may be interpreted as the flag space of the geometry, and the focal manifolds are the point space and the line space, cf. chapter 4. All these four examples are homogeneous, i.e. the isometry group of the hypersurface acts transitively on the hypersurface.

In the case $g = 6$ one has that $m_1 = m_2 \in \{1, 2\}$, see Abresch [1]. Dorfmeister and Neher [16] showed that for $m_1 = m_2 = 1$ the hypersurface is uniquely determined. For $m_1 = m_2 = 2$ there is only one example known. Both examples for $g = 6$ are also homogeneous. They are related to geometries called generalized hexagons.

The case $g = 4$ is the most difficult case because there are a lot of examples. Infinitely many examples arise as principal orbits of isotropy representations. These examples are homogeneous. Ferus, Karcher and Münzner [19] constructed other infinite series of examples; most of them are not homogeneous. The homogeneous examples are classified by Hsiang and Lawson [29]; they arise all from the isotropy representations of the symmetric spaces of rank 2.

From an isoparametric hypersurface one can construct a simplicial complex. This complex is indeed a spherical building, as was proved by Thorbergsson [62] (for higher codimensions) and Immervoll [32] except in the case that $g = 6$ and $m_1 = m_2 = 2$. The hypersurfaces are therefore related to a special kind of buildings called generalized polygons. For generalized polygons and buildings see also chapter 4. Projective planes are the same as generalized triangles. This explains why there arise projective planes in the case $g = 3$, and generalized hexagons for $g = 6$.

For $g = 4$ an isoparametric hypersurface with multiplicities m_1 and m_2 may be interpreted as the flag space of a generalized quadrangle, more exactly of an (m_1, m_2)-quadrangle.

Under the non-homogeneous examples of isoparametric hypersurfaces due to Ferus, Karcher and Münzner there are infinitely many such that one of the focal manifolds is a homogeneous space of the isotropy group. Kramer [37], [36] classified the isoparametric hypersurfaces with a homogeneous focal manifold and four distinct principal curvatures such that the multiplicities m_1, m_2 are equal or such that $m_1 \geq 3$ and $m_1 + m_2$ is odd. Wolfrom [71] classified the isoparametric hypersurfaces with a homogeneous focal manifold such that $m_1 + m_2$ is odd and $m_1 = 2$. He also treated the case $m_1 = m_2$. The results of Münzner imply that for different multiplicities $m_1 \neq m_2$ the sum $m_1 + m_2$ is odd or $1 \in \{m_1, m_2\}$. Hence, up the exchanging the role of the focal manifolds the only case left for $g = 4$ is $m_1 = 1$. These isoparametric hypersurfaces were classified by Takagi [61]. For each value of $m \in \mathbb{N}$ there is a unique example of such a hypersurface, and all these examples are homogeneous.

We will apply our results to give the irreducible transitive actions of subgroups of the isometry group on the focal manifold with infinite cyclic fundamental group for $g = 4$ and $m_1 = 1$ in 3.2.1.

3.2 Transitive actions on focal manifolds

Isoparametric hypersurfaces yield double fibrations, see Münzner [49] and [50]. We use the same notation as in section 1.2. The total space $\mathcal{F}$ of the two fibrations is the isoparametric hypersurface, and the two base spaces $\mathcal{P}$ and $\mathcal{L}$ are the focal manifolds of $\mathcal{F}$.

Suppose now we are given an isoparametric hypersurfaces with exactly $g = 4$ principal curvatures such that we have $m_1 = 1$. Takagi [61] showed that for each $m \in \mathbb{N}$ there is a unique isoparametric hypersurface in a sphere with four distinct principal curvatures of multiplicities 1 and m. The isometry group of such an isoparametric hypersurface acts transitively on the isoparametric hypersurface $\mathcal{F}$ and also on the focal manifolds $\mathcal{P}$ and $\mathcal{L}$, see Cecil-Ryan [15, Ex. 7.4, pp 299-303] for a discussion of these examples. As a homogeneous space of the transitive isometry group we may write the focal manifold $\mathcal{P}$ with infinite cyclic subgroup $\pi_1(\mathcal{P}) \cong \mathbb{Z}$ as $\mathcal{P} = \mathrm{SO}(2) \times \mathrm{SO}(m+2)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(m+1))$; in particular, the focal manifold $\mathcal{P}$ has the same rational homotopy as a product of a 1-sphere and an $(m+1)$-sphere, i.e.

$$\pi_\bullet(\mathcal{P}) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q},$$

see also 1.2.5. Furthermore, we get by the classification of Takagi also that $\pi_3(\mathcal{P})$ is trivial for even $m > 2$.

We apply 2.5.11 and the classification of simply connected rational cohomology spheres 2.4.2 to determine the compact connected isometry groups which are transitive and irreducible on $\mathcal{P}$ together with their actions. In the proof we will use the Frattini argument (cf. Salzmann et al. [54, 91.2(a)]):

Suppose a group G acts transitively on a set X, then a subset H of G is transitive, if and only if $G = HG_x$ for the stabilizer G_x of (any) $x \in X$.

Theorem 3.2.1 *If a subgroup G of $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ is transitive and irreducible on the focal manifold $\mathcal{P}$, then there is an almost simple closed subgroup $A \subseteq \mathrm{SO}(m+2)$ acting transitively on $\mathbb{S}^{m+1}$ and a 1-torus $T \cong \mathrm{SO}(2)$ such that $G = T \cdot A$, and the induced homomorphism*

$$T \hookrightarrow T \cdot A \hookrightarrow \mathrm{SO}(2) \times \mathrm{SO}(m+2) \to \mathrm{SO}(2)$$

is an epimorphism, where the last map on the right is the projection of $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ to its first factor $\mathrm{SO}(2)$. Conversely, every group $T \cdot A$ of this form is transitive on $\mathcal{P}$.

Proof Since G acts irreducibly on the connected space $\mathcal{P}$, the group G is also connected. By 2.5.11 we know that G is locally isomorphic to $\mathrm{SO}(2) \times A$ with an almost simple compact connected Lie group A acting transitively on a simply connected rational cohomology $(m+1)$-sphere, and the connected component of the stabilizer of G is the stabilizer of the A-action on a cohomology $(m+1)$-sphere in $\mathcal{P}$. The possibilities for the A-action are listed in 2.4.2. We show that the rational cohomology sphere is actually a sphere. This is clear for $m = 1$ or $m = 2$ because there are no rational 2- or 3-spheres in the table of 2.4.2. The case of the Stiefel manifolds $\mathbb{V}_2(\mathbb{R}^{2n+1}) = \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)$ was excluded in 1.2.6. Finally, for $m > 2$ the third homotopy group of $\mathcal{P}$ is trivial as we remarked above. This excludes the remaining cases in the table of 2.4.2. Therefore, the orbit of A is an $(m+1)$-sphere, and A is a subgroup of $\mathrm{SO}(m+2)$ appearing in the table of 2.4.1.

As $\mathrm{SO}(m+2)$ is not transitive on $\mathcal{P}$, we have $G \not\subseteq \mathrm{SO}(m+2)$, and the projection of G to the first factor $\mathrm{SO}(2)$ of $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ has to be nontrivial, hence it is an epimorphism; its image is the image of the $T \cong \mathrm{SO}(2)$-factor of G because the factor A of G lies entirely in $\mathrm{SO}(m+2)$. This shows the first part.

Now we show that each image of such an injection

$$T \cdot A \hookrightarrow \mathrm{SO}(2) \times \mathrm{SO}(m+2)$$

is transitive. In fact, by the Frattini argument we only need to show (after identifying $\mathcal{P} = \mathrm{SO}(2) \times \mathrm{SO}(m+2)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(m+1))$) that

$$\mathrm{SO}(2) \times \mathrm{SO}(m+2) = T \cdot A \cdot \mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(m+1)).$$

We will even show that

$$\mathrm{SO}(2) \times \mathrm{SO}(m+2) = T \cdot A \cdot \mathrm{SO}(m+1).$$

Let $(t, s) \in \mathrm{SO}(2) \times \mathrm{SO}(m + 2)$ be arbitrary. Then there is by assumption an $r \in \mathrm{SO}(m + 2)$ with $(t, r) \in T$. As $A \subseteq \mathrm{SO}(m + 2)$ acts also transitively on $\mathbb{S}^{m+1} = \mathrm{SO}(m + 2)/\mathrm{SO}(m + 1)$, the Frattini argument gives $\mathrm{SO}(m+2) = A \cdot \mathrm{SO}(m + 1)$. Hence, there are $a \in A$ and $h \in \mathrm{SO}(m + 1)$ with $r^{-1}s = ah$, and it follows that

$$(t, s) = (t, ra)(1, h) \in \mathrm{SO}(2) \cdot A \cdot \mathrm{SO}(m + 1).$$

This shows that $\mathrm{SO}(2) \times \mathrm{SO}(m + 2) = \mathrm{SO}(2) \cdot A \cdot \mathrm{SO}(m + 1)$. $\square$

Note that 2.4.1 lists the subgroups of $\mathrm{SO}(m + 2)$ that act transitively on $\mathbb{S}^{m+1}$.

Chapter 4

Generalized quadrangles

In this chapter generalized quadrangles are introduced and some of their properties are stated. Then we apply our result 2.5.11 on transitive compact connected Lie groups to the point spaces of $(1, m)$-quadrangles. The case $m = 2$ is treated separately. After that we consider every series of possible groups case by case. In the case of the orthogonal groups the generalized quadrangle has to be the real orthogonal quadrangle. For the unitary groups we determine the structure of the line space as a $\mathrm{SU}(m/2+1)$-space. However, here the incidence structure is not completely determined. The case of the symplectic groups remains open. Most of the singular cases are solved.

4.1 Geometries and generalized quadrangles

Later on in this section we will give the definition of generalized quadrangles. These are special cases of a much wider class of geometries called buildings. Buildings were introduced by Tits in order to give geometric interpretations of the simple groups of Lie type. One type of examples of buildings are projective spaces. Another type of examples are generalized polygons, which are the buildings of rank 2. The (thick) spherical buildings of rank at least 3 were classified by Tits in [65]. The generalized triangles are nothing else than the projective planes. Therefore, projective spaces and generalized quadrangles may be considered as generalizations of projective planes in different ways. Both generalizations are contained in the even more general notion of buildings. Here, we are interested in generalized quadrangles, which are the buildings of type C_2.

We start with making precise what we mean by a geometry. A triple $\Gamma =$

$(\mathcal{P}, \mathcal{L}, \mathcal{F})$ of non-empty sets with $\mathcal{F} \subsetneq \mathcal{P} \times \mathcal{L}$ is called an **incidence geometry**, or simply a **geometry**. The elements of $\mathcal{P}$ are called **points**. The elements of $\mathcal{L}$ and of $\mathcal{F}$ are called **lines** and **flags**, respectively. For a flag $(p, l) \in \mathcal{F}$ we say that p is a point on l, or that l passes through p, or that p and $\mathcal{L}$ are **incident**.

For an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ one can build the **dual geometry** $\Gamma^{\text{dual}} = (\mathcal{L}, \mathcal{P}, \mathcal{F}^{-1})$, where $\mathcal{F}^{-1} = \{(l, p) \in \mathcal{L} \times \mathcal{P} \mid (p, l) \in \mathcal{F}\}$.

We will examine group actions on certain geometries such that the actions are compatible with the geometric structure. To make this precise, we call a bijective map $\varphi : \mathcal{P} \dot\cup \mathcal{L} \to \mathcal{P} \dot\cup \mathcal{L}$ an **automorphism** of the geometry, if $\varphi(\mathcal{P}) = \mathcal{P}$ and $\varphi(\mathcal{L}) = \mathcal{L}$, and if

$$\big(\varphi(p), \varphi(l)\big) \in \mathcal{F} \quad \Longleftrightarrow \quad \big(p, l\big) \in \mathcal{F}.$$

If a group acts on a geometry by automorphisms, then we say that the group acts as an automorphism group. If such a group action fixes a point, then it leaves the set of lines through that point invariant. Hence, there is an action on these lines. Of course, the dual statement—i.e. the statement in the dual geometry—is also valid; therefore, one can interchange the roles of the points and the lines in the statement.

Lemma 4.1.1 *If a group G acts as an automorphism group transitively on the point set of an incidence geometry and if there is a point p whose stabilizer G_p acts transitively on the lines through this point, then the group G also acts transitively on the set of flags of the geometry.*

Proof Given two flags (p_1, l_1) and (p_2, l_2), there are $g_1, g_2 \in G$ and $h \in G_p$ such that $g_1 \cdot p_1 = p$, $g_2 \cdot p = p_2$ and $h \cdot (g_1 \cdot l_1) = g_2^{-1} \cdot l_2$. Hence, $g_2 h g_1 \cdot (p_1, l_1) = (p_2, l_2)$. □

We introduce some more terminology. Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence geometry. For a point $p \in \mathcal{P}$ the set $\mathcal{L}_p = \{l \in \mathcal{L} \mid (p, l) \in \mathcal{F}\}$ of lines passing through p is called the **line pencil** of p. Similarly, for a line $l \in \mathcal{L}$ the set $\mathcal{P}_l = \{p \in \mathcal{P} \mid (p, l) \in \mathcal{F}\}$ of points on l is called the **point row** of l. The geometry Γ is **thick** if every line pencil and every point row contains at least three elements. The **perp** $p^\perp$ of a point $p \in \mathcal{P}$ is the set of points on the lines passing through p.

A **generalized quadrangle** is a thick incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ such that for every $(p, l) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$ there is a unique flag $(q, h) \in \mathcal{F}$ with (p, h), $(q, l) \in \mathcal{F}$, see figure 4.1.

We will give examples of generalized quadrangles later on. Note that for a generalized quadrangle Γ the dual geometry Γ^{dual} is also a generalized quad-

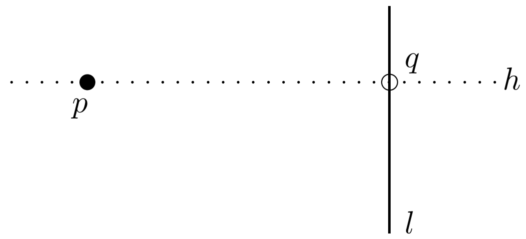

Figure 4.1: The projection of an anti-flag $(p, l) \notin \mathcal{F}$

rangle, called the **dual quadrangle** of Γ. By definition, in a generalized quadrangle two different lines have at most one point in common, and dually there is at most one line passing through two different given points; furthermore, the intersection of two perps is not empty.

Later on we will see orbits of group actions on generalized polygons with particularly nice geometric properties. For example, a subset $\mathcal{O} \subset \mathcal{P}$ is called an **ovoid**, if it contains exactly one point of each line. Dually, a subset $\mathcal{S} \subset \mathcal{L}$ is called a **spread**, if it contains exactly one line of every line pencil.

As we are interested in Lie group actions on generalized quadrangles, we will deal here only with generalized quadrangles with additional topological structure. For more combinatorial and geometric properties of generalized quadrangles, especially for finite ones, see Van Maldeghem [69].

A **compact connected generalized quadrangle**, or simply a **compact connected quadrangle** is a generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ with compact connected topological Hausdorff spaces $\mathcal{P}$, $\mathcal{L}$ such that $\mathcal{F}$ is closed in $\mathcal{P} \times \mathcal{L}$.

In a compact connected generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ the restriction of the projection $\mathcal{P} \times \mathcal{L} \to \mathcal{P}$ to $\mathcal{F}$ induces a continuous projection $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ from the flag space to the point space. Similarly, there is a projection $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$. We may describe the line pencils as $\mathcal{L}_p = \pi_{\mathcal{L}}(\pi_{\mathcal{P}}^{-1}(p))$ and, dually, the point rows as $\mathcal{P}_l = \pi_{\mathcal{P}}(\pi_{\mathcal{L}}^{-1}(l))$. It is well-known that any two line pencils and any two point rows are homeomorphic.

If k and m are the finite and positive covering dimensions of the point rows and the line pencils, respectively, then (k, m) is called the pair of **topological parameters** of the compact connected quadrangle. We also call such a compact connected quadrangle a **(k, m)-quadrangle**.

There are also the more general notions of a topological quadrangle and, even more general, of a topological polygon, which also admit totally disconnected compact quadrangles, see Kramer [36] and Grundhöfer and Van Maldeghem [24]. But we deal only with the notion of a compact quadrangles as it was introduced here, i.e. with connected ones.

The topology of compact connected quadrangles was examined by Grundhöfer and Knarr [21] and Kramer [36]. We summarize some of their results.

Proposition 4.1.2 (Topology of quadrangles) *Let* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a compact connected generalized quadrangle with topological parameters* (k, m)*.*

Then the line pencils and point rows are generalized k*- and* m*-spheres, respectively, in the sense of section 2.4; in particular, they are compact and connected.*

If k*,* $m \leq 2$*, then the point rows and line pencils are spheres. If* $k = 1$ *and* $m \geq 2$*, then the point rows are 1-spheres and* $\pi_1(\mathcal{P}) \cong \mathbb{Z}$*.*

The dimensions of the spaces are $\dim \mathcal{P} = 2k + m$*,* $\dim \mathcal{L} = k + 2m$ *and* $\dim \mathcal{F} = 2k + 2m$*. The point space* $\mathcal{P}$ *is* $(k-1)$*-connected and the line space* $\mathcal{L}$ *is* $(m-1)$*-connected.*

The line pencils and point rows are indeed genuine spheres in all known examples.

For every (k, m)-quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ the diagram

$$\begin{tikzcd}
\mathcal{L}_p \arrow[dr, dashed] & & \mathcal{P}_l \arrow[dl, dotted] \\
& \mathcal{F} \arrow[dl, dotted, "\pi_{\mathcal{P}}"'] \arrow[dr, dashed, "\pi_{\mathcal{L}}"] & \\
\mathcal{P} & & \mathcal{L}
\end{tikzcd}$$

is a double fibration of type (k, m) in the sense of section 1.2, see Kramer [36]. Hence, the cohomology of (k, m)-quadrangles is given in 1.2.2, and if $\mathcal{P}$ is a homogeneous space of a Lie group, then it has the same rational homotopy as a product of a 1-sphere and a higher dimensional sphere by 1.2.5.

4.2 Homogeneous quadrangles

For an **automorphism** of a (k, m)-quadrangle we require in addition that the automorphism of the generalized quadrangle is continuous. Similarly, an action of a topological group as an automorphism group on a compact connected quadrangle is required to be continuous. In particular, actions of Lie groups are supposed to be continuous (but not necessarily differentiable).

Now we give the most important examples of compact connected quadrangles for the following sections, the real orthogonal quadrangles. They turn out to be $(1, m)$-quadrangles.

Example 4.2.1 (The real orthogonal quadrangles) For $n \geq 4$ consider $\mathbb{R}^{n+1}$ equipped with the bilinear form induced by the matrix

$$Q = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

with respect to the canonical basis in $\mathbb{R}^{n+1}$.

Define $\mathcal{P}$ to be the set of one-dimensional totally isotropic vector subspaces

$$\mathcal{P} = \{p \subset \mathbb{R}^{n+1} \mid p = v\mathbb{R},\ v \neq 0,\ v^{\mathrm{T}}Qv = 0\},$$

and let $\mathcal{L}$ be the set of two-dimensional totally isotropic vector subspaces

$$\mathcal{L} = \{l \leq \mathbb{R}^{n+1} \mid \dim l = 2,\ \forall\, v \in l \quad v^{\mathrm{T}}Qv = 0\}.$$

As flag space $\mathcal{F}$ we take the pairs $(p, l) \in \mathcal{P} \times \mathcal{L}$ with $p \subset l$. Then $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a $(1, n-3)$-quadrangle called the **real orthogonal quadrangle** $\mathrm{Q_n}(\mathbb{R})$. We sometimes denote it also by $\mathrm{Q}(1, \text{n-3})$ to stress the values of the parameters. In other words we consider the quadratic form

$$\mathrm{f}(x) = -x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+1}^2,$$

which is induced by Q, and we select the (projective) points $v\mathbb{R}$ in the projective space $\mathbb{R}\mathrm{P}_n$ corresponding to vectors $v \in \mathbb{R}^{n+1}$ that are annulated by f. Then the lines are those projective lines which consist only of selected points.

Set $p = \begin{pmatrix} 1\\0\\1\\0\\0\\ \vdots \\ 0 \end{pmatrix}\mathbb{R}$ and $q = \begin{pmatrix} 0\\1\\0\\1\\0\\0\\ \vdots \\ 0 \end{pmatrix}\mathbb{R}$. Then $p, q \in \mathcal{P}$, $l = p \oplus q \in \mathcal{L}$ and hence $(p,l),(q,l) \in \mathcal{F}$. Furthermore, $\mathcal{P}_l = \{\begin{pmatrix} x\\y\\x\\y\\0\\ \vdots \\ 0 \end{pmatrix}\mathbb{R} \mid x, y \in \mathbb{R}\} = \mathbb{R}\mathrm{P}_1 \approx \mathbb{S}^1$ and $\mathcal{L}_p = \{p \oplus \begin{pmatrix} x_1\\x_2\\x_3\\ \vdots \\ x_{n+1} \end{pmatrix}\mathbb{R} \mid x_1 = x_3 = 0, x_2^2 = \sum_{k>3} x_k^2 = 1\} \approx \mathbb{S}^{n-3}$.

The group $G = \{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \mid A \in \mathrm{SO}(2), B \in \mathrm{SO}(n-1)\} \cong \mathrm{SO}(2) \times \mathrm{SO}(n-1)$ acts naturally on the quadrangle as a topological transformation group and by automorphisms of the quadrangle. The action of G is transitive on the flags and hence on the points and on the lines. The stabilizer of p,

$$G_p = \{\begin{pmatrix} \det B' & & & \\ & \det B' & & \\ & & \det B' & \\ & & & B' \end{pmatrix} \mid B' \in \mathrm{O}(n-2)\} \cong \mathrm{O}(n-2),$$

consists of two components, and its connected component $\mathrm{SO}(n-2)$ acts transitively on $\mathcal{L}_p \approx \mathbb{S}^{n-3}$.

The orbits of the second factor $\mathrm{SO}(n-1)$ of G in the point space are $(n-2)$-spheres and contain exactly one point of every point row. Hence these orbits are all ovoids. $\mathrm{SO}(n-1)$ even acts transitively on the lines. If $n-1$ is even, then there is an inclusion $\mathrm{SU}(\frac{n-1}{2}) \hookrightarrow \mathrm{SO}(n-1)$, which induces a point-transitive action of $\mathrm{SO}(2) \times \mathrm{SU}(\frac{n-1}{2})$ on the quadrangle where the orbits of $\mathrm{SU}(\frac{n-1}{2})$ in the point space are the same as those of $\mathrm{SO}(n-1)$. But $\mathrm{SU}(\frac{n-1}{2})$ (and even $\mathrm{SO}(2) \times \mathrm{SU}(\frac{n-1}{2})$) fails to be transitive on the lines. Similar observations hold for $(n-1) \in 4\mathbb{N}$ and the embedding $\mathrm{Sp}(\frac{n-1}{4}) \hookrightarrow \mathrm{SU}(\frac{n-1}{2}) \hookrightarrow \mathrm{SO}(n-1)$.

There are constructions similar to the one of example 4.2.1 for complex vector spaces and vector spaces over Hamilton's quaternions. These constructions yield $(2,m)$- and $(4,m)$-quadrangles, respectively. Van Maldeghem [69] gives the more general construction for arbitrary fields.

If $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a compact quadrangle, then the spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are metrizable and locally contractible, and the group of (topological) automorphisms is second countable, see Grundhöfer, Knarr and Kramer [23, 1.5, 1.8]. Burns and Spatzier [13, 2.1] proved that the automorphism group of a compact quadrangle with metrizable spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ is locally compact in the compact-open topology, see also Bletz [3]. Hence, the automorphism group of a compact quadrangle is also metrizable. The results of Montgomery-Zippin 2.3.1 or Szenthe 2.3.2 imply the following theorem.

Theorem 4.2.2 *If the automorphism group G of a compact connected quadrangle acts transitively on the point space, then G is a Lie group in the compact-open topology.*

Note that for compact metric spaces the compact-open topology may also be described by the metric, see Dugundji [17, XII, XIII]. Note also that the connected component of the automorphism group still acts transitively.

Grundöfer-Knarr-Kramer [22], [23] classified the compact connected polygons that admit transitive actions of automorphism groups on the flag space; in particular they classified the compact connected flag-homogeneous quadrangles. Biller [2, 4.2] determined the line-homogeneous $(1, m)$-quadrangles, i.e. the $(1, m)$-quadrangles admitting an automorphism group which is transitive on the line space.

We will consider point-homogeneous compact connected quadrangles. Assume that the automorphism group of a (k, m)-quadrangle is transitive on the point space. Recall from 4.1.2 that the point space of a (k, m)-quadrangle for $k > 1$ is simply connected. Therefore, 2.3.4 assures that the maximal compact connected subgroups of the automorphism group are transitive on the points of the quadrangle.

Kramer examined point-homogeneous (k, m)-quadrangles with $k = m$ in [36] and with $k \geq 3$ in [37]. Wolfrom [71] gave the list of the homogeneous point spaces with the 'right' cohomology for $k = 2$.

In the following sections we treat the case $k = 1$. The case $k = m = 1$ is covered by Kramer's work [36] and by duality also by Biller [2]. The case $k = 1$ and $m = 2$ is treated separately in the following section. For $m > 2$ the point space has the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ by 1.2.5. Hence, the orbits of the maximal compact connected subgroups of the automorphism group have at most codimension 1 in the point space, see 2.5.8. But there is no compact connected quadrangle known where the automorphism group acts transitively on the point space and where no compact subgroup of the automorphism group is transitive on the points. But our results do not allow us to conclude that there is always a *compact* subgroup of the automorphism group that acts transitively on the points. We therefore additionally assume that there is a point-transitive compact subgroup of the automorphism group.

By 2.5.11 we know the structure of an irreducible normal subgroup: an irreducible point-transitive compact connected subgroup of the automorphism group is locally isomorphic to a product $\mathrm{SO}(2) \times A$ with an almost simple compact Lie group A whose orbits in the point space are simply connected rational cohomology spheres. The simply connected homogeneous rational cohomology spheres are classified, see 2.4.2. We discuss each class of groups appearing in the classification separately in the following sections.

We often use the following two lemmas.

Lemma 4.2.3 *Let p be a point of a (k, m)-quadrangle on which the topological group G acts as an automorphism group. If G acts transitively on the points and if the connected component H of the stabilizer G_p of p acts trivially on $p^\perp$, then H is in the kernel of the action.*

Proof We show that H stabilizes each point of $\mathcal{P}$. Then H also acts trivially on $\mathcal{L}$, as H acts by automorphisms and fixes the points on the lines.

H fixes the points in $p^\perp$ by assumption. Let $r \in \mathcal{P} \setminus p^\perp$ be a point which is not on a line through p. As perps in generalized quadrangles always intersect, there is a point $q \in p^\perp$ which is also on a line through r. We choose a $g \in G$ with $g \cdot p = q$. As H acts trivially on $p^\perp$, its conjugate gHg^{-1} acts trivially on $(g \cdot p)^\perp = q^\perp$. In particular it fixes p, hence $gHg^{-1} \subseteq {G_p}^1 = H$ is an open and closed subgroup of ${G_p}^1$ and hence $gHg^{-1} = {G_p}^1$. As $gHg^{-1} = H$ fixes $q^\perp$ pointwise, H also fixes $r \in q^\perp$. □

The last lemma may be generalized to a bigger class of incidence geometries, cf. Kramer [37, 7.3]. For $(1, m)$-quadrangles one can use slightly different assumptions.

Lemma 4.2.4 *Let p be a point of a $(1, m)$-quadrangle on which the topological group G acts as an automorphism group. If G acts transitively on the points and if the connected component H of the stabilizer G_p of p is compact and acts trivially on $\mathcal{L}_p$, then H is contained in the kernel of the action.*

Proof Since H stabilizes each line in $\mathcal{L}_p$, there is an action on each of these point rows. As each of these point rows is a 1-sphere (by 4.1.2) and as p itself is fixed, there is an action of H on $\mathcal{P}_l \setminus \{p\} \approx \mathbb{R}$ for each $l \in \mathcal{L}_p$. Then 2.2.1 shows that this action of the compact connected group H has to be trivial. Therefore, H acts trivially on $p^\perp$, and hence on $\mathcal{P}$ and on $\mathcal{L}$ by 4.2.3. □

4.3 Point-homogeneous $(1,2)$-quadrangles

We now investigate more closely the point-transitive actions of compact automorphism groups of compact connected quadrangles. Because of the restriction $m > 2$ in 1.2.5 we treat the case $m = 2$ separately, i.e. we will first treat the case of $(1,2)$-quadrangles. The case of $(1,1)$-quadrangles is covered by Kramer [36, 5.2.7] and by duality also by Biller [2, 4.2.3].

The following proposition shows that the claim in 1.2.5 may be extended for generalized quadrangles to the case $m = 2$.

Proposition 4.3.1 *Let G be a compact connected Lie group acting as an automorphism group transitively and irreducibly on the point space $\mathcal{P}$ of a $(1,2)$-quadrangle. Then G is locally isomorphic to $\mathrm{SO}(2)\times\mathrm{SU}(2)$, and the stabilizer H is discrete and cyclic. Furthermore,*

$$\pi_\bullet\left(\frac{\mathrm{SO}(2)\times\mathrm{SU}(2)}{H}\right)\cong\pi_\bullet(\mathcal{P})\cong\pi_\bullet(\mathbb{S}^1\times\mathbb{S}^3).$$

Proof Biller [2, 4.3.7] proved that the Lie algebra $\mathfrak{g}$ of G embeds into $\mathbb{R}\times\mathfrak{o}_4\cong\mathbb{R}\times\mathfrak{su}_2\times\mathfrak{su}_2$ and in [2, 5.2.3] that the centre is at most three-dimensional. If H is the stabilizer of a point, then the homotopy sequence of $H\to G\to G/H$ shows that

$$1\le\dim\mathrm{Z}(G)\le 1+\dim\mathrm{Z}(H),$$

cf. [2, 4.3.7]. Therefore, by the classification of compact Lie algebras the only possibilities for $\mathfrak{g}$ are $\mathbb{R}\times\mathfrak{su}_2$, $\mathbb{R}^2\times\mathfrak{su}_2$, $\mathbb{R}^3\times\mathfrak{su}_2$, and $\mathbb{R}\times\mathfrak{su}_2\times\mathfrak{su}_2$. In all of these four cases there is an action of $\mathrm{SU}(2)$. We first show that the orbits of this action are three-dimensional.

Recall from table 2.2 on page 16 that there are no two-dimensional subalgebras of $\mathfrak{su}_2$. Hence, orbits of $\mathrm{SU}(2)$ have dimension 0, 2, or 3. As G acts transitively on the four-dimensional point space $\mathcal{P}$ of the quadrangle, the $\mathfrak{su}_2$-orbits have dimension 3 in the case $\mathfrak{g}\cong\mathbb{R}\times\mathfrak{su}_2$.

Denote by $\mathfrak{h}$ the Lie algebra of the point stabilizer H.

If $\mathfrak{g}\cong\mathbb{R}^2\times\mathfrak{su}_2$, then $\dim\mathfrak{h}=\dim G-\dim\mathcal{P}=1$, and also $\dim\mathfrak{z}(\mathfrak{h})\ge\dim\mathfrak{z}(\mathfrak{g})-1=1$, hence $\mathfrak{h}\cong\mathbb{R}$. If there were $\mathfrak{su}_2$-orbits of dimension two, then $\mathfrak{h}\subset\mathfrak{su}_2$ and $\mathcal{P}$ would be finitely covered by $\mathbb{S}^1\times\mathbb{S}^1\times\mathrm{SU}(2)/\mathrm{U}(1)$, a contradiction to $\pi_1(\mathcal{P})\cong\mathbb{Z}$.

Similarly, for $\mathfrak{g}\cong\mathbb{R}^3\times\mathfrak{su}_2$ we get $\dim\mathfrak{h}=2$ and $\dim\mathfrak{z}(\mathfrak{h})\ge\dim\mathfrak{z}(\mathfrak{g})-1=2$, hence $\mathfrak{h}\cong\mathbb{R}^2$. As in the last paragraph, $\mathfrak{h}\subset\mathfrak{su}_2$ is not possible. If

$\dim \mathfrak{h} \cap \mathfrak{su}_2 = 1$, then $\dim \mathfrak{h} \cap \mathbb{R}^3 = 1$, because the projection of $\mathfrak{h}$ into $\mathfrak{su}_2$ has at most dimension 1. But $\dim \mathfrak{h} \cap \mathbb{R}^3 = 1$ is not possible, since we assumed G to act irreducibly and therefore almost effectively. (Note that in the compact Lie group G there is a normal subgroup which is complementary to the kernel.) The same is true for $\dim \mathfrak{h} \cap \mathbb{R}^3 = 2$. Hence, there is no irreducible transitive action corresponding to $\mathfrak{g} \cong \mathbb{R}^3 \times \mathfrak{su}_2$.

Finally, suppose $\mathfrak{g} \cong \mathbb{R} \times \mathfrak{su}_2 \times \mathfrak{su}_2$. Here, $\dim \mathfrak{h} = 3$. As the semisimple group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ cannot act transitively on $\mathcal{P}$, we get for dimensional reasons that $\dim \mathfrak{h} \cap (\mathfrak{su}_2 \times \mathfrak{su}_2) = 3$, i.e. $\mathfrak{h} \subset \mathfrak{su}_2 \times \mathfrak{su}_2$. Consider the projection of $\mathfrak{h}$ into the two factors. As there are no two-dimensional subalgebras in $\mathfrak{su}_2$, the dimensions of the image of such a projection cannot be 2 or 1. It is not 0, either, because G acts almost effectively. Hence, it is 3 for both factors, and both factors belong to three-dimensional orbits. In particular, this action is not irreducible, but there are transitive actions related to $\mathbb{R} \times \mathfrak{su}_2$ in this case.

So far we have proved that the orbits of $\mathrm{SU}(2)$ are three-dimensional in the four-dimensional space $\mathcal{P}$. We now apply Mostert's theorem 2.5.1. If the orbit space of this action is a compact interval, then $\mathcal{P}$ is a double mapping cylinder, and the images of the fundamental groups in $\pi_1(\mathcal{P}) \cong \mathbb{Z}$ of the two exceptional orbits generate $\pi_1(\mathcal{P})$ by the Seifert–Van Kampen theorem. But orbits of $\mathrm{SU}(2)$ have finite fundamental groups, and therefore the images of these groups in $\pi_1(\mathcal{P})$ are trivial and cannot generate it. It follows that the orbit space is not an interval. But then it is $\mathbb{S}^1$ by Mostert's theorem, and all orbits are of the same type B and there is a fibre bundle $B \to \mathcal{P} \to \mathbb{S}^1$, whose homotopy sequence shows that the orbit B is simply connected, i.e. $B \approx \mathrm{SU}(2) \approx \mathbb{S}^3$, and that $\pi_\bullet(\mathcal{P}) \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^3)$. Since G acts irreducibly,

$$\mathcal{P} \approx (\mathrm{SO}(2) \times \mathrm{SU}(2))/H$$

with a finite stabilizer H, which is cyclic as a quotient group of the infinite cyclic group $\pi_1(\mathcal{P})$. □

In the following sections we will treat the case of a $(1, m)$-quadrangle, $m > 2$, whose point space is a homogeneous space of a compact Lie group in the automorphism group of the quadrangle. As a $(1, m)$-quadrangle yields a double fibration of type $(1, m)$, the point space has an infinite cyclic fundamental group and the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^{m+1}$ by 1.2.5. Hence, we may apply 2.5.11 to see that the transitive Lie group contains a transitive normal subgroup $\mathrm{SO}(2) \cdot A$ such that the connected component H of its stabilizer is a subgroup of A and such that A/H is a simply connected rational cohomology sphere. Hence, we may assume that (A, H) is one of the pairs (G, H) in the classification 2.4.2 of such rational cohomology spheres. We will go on by considering case by case the classes of pairs (A, H) appearing in 2.4.2.

4.4 Orthogonal actions

We now examine the case in 2.4.2 where the almost simple factor in a point-transitive group is a special orthogonal group. We will show that such an action is the well-known action on the real orthogonal quadrangle, cf. example 4.2.1.

Proposition 4.4.1 *If* $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ *acts as an automorphism group point-transitively on a generalized* $(1, m)$*-quadrangle, then the action is isomorphic to the classical action on the real orthogonal quadrangle* $\mathrm{Q_{m+3}}(\mathbb{R})$*.*

Proof Let $G = \mathrm{SO}(2)\times\mathrm{SO}(m+2)$ act almost effectively and transitively as an automorphism group on the points of a $(1, m)$-quadrangle. The connected component of the point stabilizer of a point p is isomorphic to the subgroup $\mathrm{SO}(m + 1)$ of $\mathrm{SO}(m + 2)$, see 2.5.11, and acts on the line pencil $\mathcal{L}_p$. As G acts almost effectively, $\mathrm{SO}(m + 1)$ cannot act trivially by 4.2.4.

We claim that $\mathrm{SO}(m + 1)$ acts transitively on $\mathcal{L}_p$. If $m \neq 3$ then $\mathrm{SO}(m + 1)$ is almost simple or a torus. Then the non-trivial orbits have dimension at least $m = \dim \mathcal{L}_p$, because this is the smallest codimension of proper subgroups, see table 2.2 on page 16. In the case $m = 3$, the stabilizer $\mathrm{SO}(4)$ is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Here, the non-trivial orbits have at least dimension 2. We will show that there is a three-dimensional orbit, i.e. that the action of $\mathrm{SO}(4)$ on $\mathcal{L}_p$ is transitive. Seeking a contradiction, suppose that there is a two-dimensional orbit of $\mathrm{SO}(4)$ in $\mathcal{L}_p$. This implies that one of the two factors of type $\mathrm{SU}(2)$ has to act transitively on that orbit, because there is no one-dimensional $\mathrm{SU}(2)$-orbit. We apply Mostert's theorem 2.5.1 on compact manifolds with orbits of codimension 1 to this $\mathrm{SU}(2)$-action on $\mathcal{L}_p$. Then $\mathcal{L}_p/\mathrm{SU}(2) \approx \mathbb{S}^1$ or $\mathcal{L}_p/\mathrm{SU}(2) \approx [0, 1]$. By Montgomery and Yang 2.5.2 the orbit projection

$$\mathcal{L}_p \to \mathcal{L}_p/\mathrm{SU}(2)$$

induces an epimorphism of the fundamental groups. As $\mathcal{L}_p$ is a generalized 3-sphere, and hence simply connected, $\mathcal{L}_p/\mathrm{SU}(2) \approx \mathbb{S}^1$ is not possible. Hence, $\mathcal{L}_p/\mathrm{SU}(2) \approx [0, 1]$. Then Mostert's theorem says that there are exactly two singular orbits $\mathrm{SU}(2)\cdot x$ and $\mathrm{SU}(2)\cdot y$, and one may choose stabilizers of lines x, y in the two singular orbits such that in their intersection is the stabilizer of a line l of a principal orbit

$$\mathrm{SU}(2)_l \subseteq \mathrm{SU}(2)_x \cap \mathrm{SU}(2)_y.$$

Furthermore, $\mathrm{SU}(2)_x/\mathrm{SU}(2)_l$ and $\mathrm{SU}(2)_y/\mathrm{SU}(2)_l$ are homology spheres. But $\mathrm{SU}(2)_l$ is a one-dimensional subgroup of $\mathrm{SU}(2)$, and the one-dimensional subgroups of $\mathrm{SU}(2)$ are of type $\mathrm{SO}(2)$ and $\mathbb{Z}_2 \cdot \mathrm{SO}(2)$. It follows that the principal orbit type is a 2-sphere

$$\mathrm{SU}(2)/\mathrm{SO}(2) \cong \mathrm{SU}(2)/\mathrm{U}(1) \cong \mathbb{C}\mathrm{P}_1 \approx \mathbb{S}^2,$$

and that the two exceptional orbits are real projective planes

$$\mathbb{S}^2/\mathbb{Z}_2 \approx \mathbb{R}\mathrm{P}_2\,.$$

The Mayer-Vietoris-sequence of the orbit decomposition induces an exact sequence

$$0 \longrightarrow H^2(\mathbb{R}\mathrm{P}_2) \oplus H^2(\mathbb{R}\mathrm{P}_2) \longrightarrow H^2(\mathbb{S}^2) \cong \mathbb{Z},$$

a contradiction. Therefore, even in the case $m = 3$, the point stabilizer $\mathrm{SO}(m)$ acts transitively on the generalized m-sphere $\mathcal{L}_p$, as we claimed above.

Since the point stabilizer of the point-transitive action of $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ acts transitively on the corresponding line pencil, it follows by 4.1.1 that $\mathrm{SO}(2) \times \mathrm{SO}(m+2)$ acts also transitively on the flag space $\mathcal{F}$ and therefore also on the line space $\mathcal{L}$.

As $\mathcal{L}_p \simeq \mathbb{S}^m$ is simply connected, the homotopy sequence allows us to conclude that $\mathrm{SO}(m+2)_{p,l} \cong \mathrm{SO}(m)$ is connected and that $\mathcal{L}_p \approx \mathbb{S}^m$ for every flag $(p,l) \in \mathcal{F}$ and every $m \in \mathbb{N}$, hence $\mathcal{F} = \mathrm{SO}(2) \times \mathrm{SO}(m+2)/\mathrm{SO}(m)$.

We will now show that even the simple factor $\mathrm{SO}(m+2)$ acts transitively on the line space. The connected group $\mathrm{SO}(m+2)_{p,l} \cong \mathrm{SO}(m)$ acts trivially on $\mathcal{P}_l \setminus \{p\} \approx \mathbb{S}^1 \setminus \{p\} \approx \mathbb{R}$, see 2.2.1, this implies that $\mathrm{SO}(m+2)_{p,l}$ is in the kernel of the action of $\mathrm{SO}(m+2)_l$ on $\mathcal{P}_l$. From

$$\dim \mathrm{SO}(m+2) \cdot l = \dim \mathrm{SO}(m+2) - \mathrm{SO}(m+2)_{p,l} - \dim \mathrm{SO}(m+2)_l \cdot p$$

for $p \in \mathcal{P}_l$ it follows that $\dim \mathrm{SO}(m+2) \cdot l \in \{\dim \mathcal{L}, \dim \mathcal{L} - 1\}$.

Assume that $\dim \mathrm{SO}(m+2) \cdot l = \dim \mathcal{L} - 1$, i.e. that $\mathrm{SO}(m+2)_l$ acts transitively on $\mathcal{P}_l \approx \mathbb{S}^1$ for every $l \in \mathcal{L}$. (Otherwise there would be an $\mathrm{SO}(m+2)$-orbit of full dimension in $\mathcal{L}$.) It follows that $\mathrm{SO}(m+2)_l \cong T \cdot \mathrm{SO}(m)$ with $\mathrm{SO}(2) \cong T \subseteq \mathrm{Cen}_{\mathrm{SO}(m+2)}(\mathrm{SO}(m+2)_{p,l})^1 \cong \mathrm{SO}(2)$. But all these stabilizers $\mathrm{SO}(m+2)_l \cong T \cdot \mathrm{SO}(m) = \mathrm{SO}(m+1)$ are conjugate in $\mathrm{SO}(m+2)$, and the orbits are all of dimension $\dim \mathrm{SO}(m+2) - \dim T \cdot \mathrm{SO}(m) = 2m = \dim \mathcal{L} - 1$

and equivalent. By Mostert's result 2.5.1 on actions with orbits of codimension one we get $\mathcal{L}/\mathrm{SO}(m+2) \approx \mathbb{S}^1$. But in view of Montgomery and Yang 2.5.2 we get also $\mathcal{L}/\mathrm{SO}(m+2) \approx [0,1]$, since $\mathcal{L}$ is simply connected. Therefore, $\dim \mathrm{SO}(m+2) \cdot l = \dim \mathcal{L} - 1$ is not possible.

It follows that $\dim \mathrm{SO}(m+2) \cdot l = \dim \mathcal{L}$ and that $\mathcal{L} = \mathrm{SO}(m+2)/\mathrm{SO}(m)$. The quadrangle is uniquely determined by Kramer [37, 7.15], or Biller [2]. The uniqueness follows also from the classification of flag-homogeneous compact connected generalized polygons by Grundhöfer, Knarr and Kramer [22], [23]. $\square$

4.5 Unitary actions

In this section we investigate point-transitive actions of $G = \mathrm{U}(1) \times \mathrm{SU}(n+1)$ for $m = 2n > 2$ on $(1,m)$-quadrangles. We get a description of $\mathcal{L}$ as a $\mathrm{SU}(n+1)$-space.

Proposition 4.5.1 *If $G = \mathrm{U}(1) \times \mathrm{SU}(n+1)$, $n > 1$, acts as an automorphism group transitive on the points of a $(1,2n)$-quadrangle $(\mathcal{P},\mathcal{L},\mathcal{F})$, then for $p \in \mathcal{P}$ there are lines $l,x,y \in \mathcal{L}_p$ such that $\mathrm{SU}(n+1)\cdot x \approx \mathrm{SU}(n+1)\cdot y \approx \mathbb{S}^{2n+1}$ are spreads, and all other $\mathrm{SU}(n+1)$-orbits in $\mathcal{L}$ are $\mathrm{SU}(n+1)$-equivalent to $\mathrm{SU}(n+1)\cdot l \overset{\mathrm{SU}(n+1)}{\approx} \mathrm{SU}(n+1)/\mathrm{SU}(n-1)$. Furthermore, $\mathrm{SU}(n+1)$ is transitive on every G-orbit in $\mathcal{L}$.*

The induced action of the stabilizer $\mathrm{SU}(n+1)_p \cong \mathrm{SU}(n)$ on the line pencil $\mathcal{L}_p \approx \mathbb{S}^{2n}$ of p is as described in 2.4.6, i.e. it is the suspension of the transitive $\mathrm{SU}(n)$-action on $\mathbb{S}^{2n-1}$.

Proof Let $p \in \mathcal{P}$ and identify $\mathcal{P} = G/G_p$. From 2.5.11 and 2.4.2 we know that $(G_p)^1 = \mathrm{SU}(n+1)_p = \mathrm{SU}(n)$. The stabilizer $\mathrm{SU}(n)$ acts on the generalized $2n$-sphere $\mathcal{L}_p$. This action is not trivial by 4.2.4, and 2.4.6 implies that the action of $\mathrm{SU}(n+1)_p = \mathrm{SU}(n)$ on $\mathcal{L}_p \approx \mathbb{S}^{2n}$ is the suspension of the natural action of $\mathrm{SU}(n)$ on $\mathbb{S}^{2n-1}$. This means that there are two types of orbits, two fixed lines x and y, and the stabilizers of the other lines are isomorphic to $\mathrm{SU}(n-1)$, i.e. we have

$$\begin{aligned} \mathrm{SU}(n+1)_{x,p} &= \mathrm{SU}(n+1)_p = \mathrm{SU}(n), \\ \mathrm{SU}(n+1)_{y,p} &= \mathrm{SU}(n+1)_p = \mathrm{SU}(n), \\ \forall l \in \mathcal{L}_p \setminus \{x,y\} \quad ({G_p}^1)_l &= \mathrm{SU}(n+1)_{l,p} \cong \mathrm{SU}(n-1). \end{aligned}$$

We claim that for a line $l \in \mathcal{L}_p \setminus \{x, y\}$ the connected component $(G_l)^1$ of the stabilizer (of the action of G on the line space) cannot act trivially on $\mathcal{P}_l$. Suppose on the contrary that $(G_l)^1$ acts trivially on $\mathcal{P}_l$. It follows that $(G_l)^1$ fixes p, which implies that $(G_l)^1 = (G_{p,l})^1 = \mathrm{SU}(n+1)_{p,l} \cong \mathrm{SU}(n-1)$; hence, $\dim G \cdot l = \dim \mathrm{U}(1) \times \mathrm{SU}(n+1) - \dim \mathrm{SU}(n-1) = 1 + 4n = \dim \mathcal{L}$, and the simply connected line space $\mathcal{L} = G \cdot l$ is finitely covered by $G/{G_l}^1 \approx \mathrm{U}(1) \times \mathrm{SU}(n+1)/\mathrm{SU}(n-1)$, a contradiction. Hence, the G-stabilizer of such a line l acts transitively on $\mathcal{P}_l$ as we claimed above.

The kernel of the transitive action of $(G_l)^1$ on $\mathcal{P}_l \approx \mathbb{S}^1$ stabilizes l and p; therefore ${G_l}^1 = T \cdot \mathrm{SU}(n+1)_{p,l}$, where $\mathrm{U}(1) \cong T \subseteq \mathrm{Cen}_G(\mathrm{SU}(n+1)_{l,p})^1 \cong \mathrm{U}(1) \times \mathrm{SU}(2)$ and $\dim G \cdot l = 4n = \dim \mathcal{L} - 1$.

As $\mathcal{L}$ is simply connected, $\mathcal{L}/G$ is simply connected, too, see Montgomery-Yang 2.5.2. By Mostert's theorem 2.5.1, the orbit space is $\mathcal{L}/G \approx [0, 1]$; furthermore, $\mathcal{L}$ is a double mapping cylinder, and there are exactly two singular orbits, one of these has to be $G \cdot x$, because of

$$\dim G \cdot x \leq \dim(\mathrm{U}(1) \times \mathrm{SU}(n+1)) - \dim \mathrm{SU}(n) = 2n + 2 < 4n.$$

Similarly, we get $\dim G \cdot y \leq 2n + 2 < 4n$. In other words, the two singular orbits (fixed points) of the ${G_p}^1 = \mathrm{SU}(n)$-action on $\mathcal{L}_p$ extend to singular orbits of the G-action on $\mathcal{L}$.

Mostert's result also implies that there is an $l \in \mathcal{L}$ such that $T \cdot \mathrm{SU}(n-1) \cong {G_l}^1 \subseteq G_l \subseteq G_x \cap G_y$ and such that G_x/G_l is a (homogeneous) homology sphere. It follows that G_x/G_l is a sphere, see 2.4.1. Because of $\mathrm{SU}(n) = \mathrm{SU}(n+1)_{x,p} \subseteq G_{x,p} \subseteq G_x$ and in view of the classification of homogeneous cohomology spheres 2.4.1 we get ${G_x}^1 \cong T_1 \cdot \mathrm{SU}(n)$ and $G_x/G_l \approx \mathbb{S}^{2n-1}$ with $\mathrm{U}(1) \cong T_1 \subseteq \mathrm{Cen}_G(\mathrm{SU}(n))^1 \cong \mathrm{U}(1) \times \mathrm{U}(1)$. Therefore,

$$\dim G \cdot x = \dim(\mathrm{U}(1) \times \mathrm{SU}(n+1)) - \dim(T \cdot \mathrm{SU}(n)) = 2n + 1,$$

and similarly, $\dim G \cdot y = 2n + 1$.

As the choice of $p \in \mathcal{P}$ was arbitrary, there are exactly two lines with different G-orbits of dimension less than $4n$ in every line pencil. It follows that the two singular orbits $G \cdot x$ and $G \cdot y$ are different, because $G \cdot x = G \cdot y$ would (by the point-transitivity of G) imply that there is only one singular orbit. Hence, $G \cdot x$ meets every line pencil at most once. Since a compact homogeneous $(m+1)$-dimensional set of lines containing at most one line of every line pencil (in a $(1, m)$-quadrangle) is a spread, see Biller [2, 5.2.2] (where the dual statement in the point space is shown), the orbit $G \cdot x$ is a spread and in particular $G \cdot x \approx \mathbb{S}^{2n+1}$ is a homogeneous sphere. It follows that

$\mathrm{SU}(n+1)\cdot x = G\cdot x \approx \mathbb{S}^{2n+1}$. (Note that the factor T_1 in ${G_x}^1 \cong T_1 \cdot \mathrm{SU}(n)$ cannot be contained in $\mathrm{SU}(n+1)$, since G/G_x is simply connected).

Moreover, $G_x \cong T_1 \cdot \mathrm{SU}(n)$ is connected. But then, the fibre bundle

$$G_l \to G_x \to G_x/G_l \approx \mathbb{S}^{2n-1}$$

shows that G_l is also connected; hence

$$G_l \cong T\cdot \mathrm{SU}(n-1) \text{ and } G\cdot l \approx \mathrm{U}(1)\times \mathrm{SU}(n+1)/T\cdot \mathrm{SU}(n-1).$$

From the fibration

$$\mathbb{S}^{2n-1} \approx G_x/G_l \hookrightarrow G\cdot l \approx G/G_l \to G\cdot x \approx G/G_x \approx \mathbb{S}^{2n+1}$$

one gets that $G\cdot l$ is simply connected.

For the factor T in $G_l \cong T\cdot\mathrm{SU}(n-1)$ we get as above that T is not contained in $\mathrm{SU}(n+1)$, since $G\cdot l$ is simply connected. It follows that $\dim \mathrm{SU}(n+1)\cdot l = \dim G\cdot l$; hence $\mathrm{SU}(n+1)\cdot l = G\cdot l$, and $\mathrm{SU}(n+1)\cdot l$ is covered by the Stiefel manifold $\mathrm{SU}(n+1)/\mathrm{SU}(n-1)$, but as $\mathrm{SU}(n+1)\cdot l$ is simply connected, the covering is trivial, i.e. $\mathrm{SU}(n+1)\cdot l = G\cdot l = \mathrm{SU}(n+1)/\mathrm{SU}(n-1)$. This all holds of course for x replaced by y. □

4.6 Summary

After treating the case $m = 2$ separately we examined for the infinite series of simply connected homogeneous rational cohomology spheres appearing in 2.4.2 and in 2.4.1 if they occur in point-homogeneous $(1, m)$-quadrangles as described in 2.5.11. The case of the Stiefel manifolds is excluded by 1.2.6. The cases of $(\mathrm{SU}(3), \mathrm{SO}(3))$ and $(\mathrm{Sp}(2), {}^{\mathbb{H}}\rho_{3\lambda_1})$ were excluded in 1.2.7 and in 1.2.8, respectively.

In the last two sections we examined the cases of the $\mathrm{SO}(m+1)$-spheres and of the $\mathrm{SU}(m/2)$-spheres. Some other of the singular cases may be solved.

The action of G_2 on $(1, 5)$-quadrangles is unique and, hence, the classical one on the real orthogonal quadrangle, see Biller [2, 4.2.13]. This excludes the 4 cases in 2.4.2 and settles the case of the G_2-homogeneous sphere in 2.4.1.

In the case of $\mathrm{SO}(2)\times\mathrm{Spin}(7)$ as a point-transitive automorphism group of a $(1, 6)$-quadrangle, the point stabilizer G_2 cannot act trivially on the line pencil by 4.2.4. Non-trivial G_2 orbits have at least dimension 6, see table 2.2 on page 16. Hence, G_2 acts transitively on the generalized 6-sphere. But then

$SO(2) \times Spin(7)$ acts transitively on the flags by 4.1.1. The classification of flag-homogeneous quadrangles by Grundhöfer-Knarr-Kramer [23] shows that it is the unique action on the real orthogonal $(1, 6)$-quadrangle.

We have seen up to now that the groups acting flag-transitively on the real orthogonal quadrangle have this 'classical' action as a unique point-transitive action on $(1, m)$-quadrangles.

From the homogeneous spaces in 2.4.2 and in 2.4.1 there remain the infinite series of pairs $(\mathrm{Sp}(n), \mathrm{Sp}(n-1))$ and the isolated case $(\mathrm{Spin}(9), \mathrm{Spin}(7))$. For these pairs there are indeed respective actions on the real orthogonal quadrangles which come from the injections of $\mathrm{Sp}(n)$ in $\mathrm{SO}(4n)$ and $\mathrm{Spin}(9)$ in $\mathrm{SO}(16)$. But a further examination of these actions seems to be hard, because in each case the action of the stabilizer on the line pencil may have principal orbits with high codimensions, and so these actions are far away from being transitive or almost transitive. This may be compared with the case of $\mathrm{Sp}(1) \times (\mathrm{Sp}(n)/\mathrm{Sp}(n-1))$ as point space of a $(3, 4n-4)$-quadrangle. This case occurs in Kramer [37, 7.H, p. 93.]. There is such an action on non-Moufang-quadrangles of FKM-type. But it remains an open problem if these actions are unique.

We summarize the results of the last sections as follows.

Theorem 4.6.1 *If a compact connected Lie group acts transitively and irreducibly as an automorphism group on the point space of an $(1, m)$-quadrangle, then there is an action of this group by automorphisms on the point space of the respective real orthogonal quadrangle, and the connected component of the point stabilizers are the same in both actions.*

If the action on the real orthogonal quadrangle is flag-transitive, then both point-transitive actions coincide.

As the isoparametric hypersurfaces of section 3.2 may be interpreted as real orthogonal quadrangles, theorem 3.2.1 describes also the point-transitive actions on the real orthogonal quadrangles.

Chapter 5

Three series of homogeneous spaces

In this chapter we investigate three other series of homogeneous spaces that are candidates of homogeneous point spaces of quadrangles, cf. Kramer [37, 7.33, 7.G and 3.15(B1)]. The homogeneous spaces are products of spheres and the respective quadrangles, if they existed, would have parameters $(5, 4n-6)$, $(7, 4n-8)$ and $(3, 2n-2)$, respectively, where $n \in \mathbb{N}$ is not too small. In his investigations of point-homogeneous quadrangles Kramer found these three infinite series of homogeneous spaces with the 'right' cohomology, i.e. with the cohomology of sphere products. For these examples no such actions on quadrangles were known and Kramer conjectured that these homogeneous spaces do not belong to point spaces of quadrangles with the respective parameters. But he left this problem open.

We will show that (up to some small parameters in the first series) there are indeed no quadrangles with these kinds of homogeneous point spaces. Each of the following three sections is dedicated to one of the series.

5.1 The $(5, 4n-6)$-series

In Kramer's Habilitationsschrift [37, 7.33 and 3.15(B1)] one finds the homogeneous space

$$\mathrm{SU}(3) \times \mathrm{Sp}(n)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1) \approx \mathbb{S}^5 \times \mathbb{S}^{4n-1}, \quad n \geq 2,$$

as a candidate for the point space of a $(5, 4n-6)$-quadrangle, where the group $\mathrm{SU}(3) \times \mathrm{Sp}(n)$ acts by automorphisms of the quadrangle.

We will show that there is no such quadrangle for $n \geq 6$. The idea is to apply 2.1.2 and the following proposition (which is a part of a proposition of Biller [2, 5.1.3]) to the kernel of the induced action of a line stabilizer on the points of the fixed line. Then the representation theory of compact Lie groups rules out the remaining cases. We will use the tables of Kramer [37, Ch. 4] for low-dimensional simple modules of the occuring Lie groups.

First we quote the mentioned result of Biller [2, 5.1.3].

Proposition 5.1.1 (Biller) *Suppose that a Lie group G acts effectively on a (k, m)-quadrangle with odd k and even m such that G fixes a point row elementwise. Then* $\mathrm{rk}(G) \leq \frac{m-2}{k-1} + 1$.

Now assume that there is a quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ with point space $\mathcal{P} = \mathrm{SU}(3) \times \mathrm{Sp}(n)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1) \approx \mathbb{S}^5 \times \mathbb{S}^{4n-1}$ where $\mathrm{SU}(3) \times \mathrm{Sp}(n)$ acts as a subgroup of the automorphism group. Then the parameters of the quadrangle are 5 and $4n - 6$, and $G = \mathrm{SU}(3) \times \mathrm{Sp}(n)$ acts effectively on the respective line space as a subgroup of the automorphism group.

For a line l in the line space we denote by $G_{[l]}$ the kernel of the induced action of G_l on the point row $\mathcal{P}_l$ of l. Then

$$\begin{aligned} 8 + 2n^2 + n &= \dim G = \dim G \cdot l + \dim G_l \\ &= \dim G \cdot l + \dim G_l/G_{[l]} + \dim G_{[l]}. \end{aligned}$$

Here, the dimension of the orbit of l is of course bounded by the dimension $8n-7$ of the line space, see 4.1.2. As $G_l/G_{[l]}$ acts effectively on the generalized 5-sphere $\mathcal{P}_l$, we get $\dim G_l/G_{[l]} \leq 15$ by 2.3.5. Hence, the equation above reads now

$$\dim G_{[l]} \geq 2n^2 - 7n. \tag{5.1.1}$$

On the other hand, Biller's result 5.1.1 yields

$$\mathrm{rk}(G_{[l]}) \leq \left\lfloor \frac{4n - 6 - 2}{5 + 1} + 1 \right\rfloor = \left\lfloor \frac{2n - 1}{3} \right\rfloor.$$

It follows that

$$\binom{2\,\mathrm{rk}(G_{[l]}) + 1}{2} \leq \frac{(2\frac{2n-1}{3} + 1)(2\frac{2n-1}{3})}{2} = \frac{1}{9}(4n+1)(2n-1) = \frac{1}{9}(8n^2-2n-1).$$

A short calculation shows that $\frac{1}{9}(8n^2 - 2n - 1) < 2n^2 - 7n$ for $n \geq 7$; hence $\binom{2\,\mathrm{rk}(G_{[l]})+1}{2} < \dim G_{[l]}$ for $n \geq 7$.

Now the relations 2.1.2 between the ranks and the dimensions of the Lie algebras imply that $n \leq 6$, or that $G_{[l]}$ is (up to a local isomorphism) one of the exceptional groups G_2, F_4, E_7, or $8 \leq \mathrm{rk}(G_{[l]}) \leq 11$ and E_8 is a normal subgroup of $G_{[l]}$. We show case by case that these possibilities do not occur.

We first consider the case $G_{[l]} = G_2$ and $2 = \lfloor \frac{2n-1}{3} \rfloor$, i.e. $n = 4$. Note that there is only one compact connected Lie group with Lie algebra $\mathfrak{g}_2$. As G_2 is almost simple and cannot inject into the first factor $\mathrm{SU}(3)$ of $G = \mathrm{SU}(3) \times \mathrm{Sp}(4)$ by dimensional reasons, the projections to the factors of G show that it has to inject into $\mathrm{Sp}(4)$. But there is no almost effective action of G_2 on $\mathbb{H}^4$ by representation theory, see Kramer [37, 4.26]. Therefore, this case can not occur.

The next case is $G_{[l]} = F_4$ and $4 = \lfloor \frac{2n-1}{3} \rfloor$, i.e. $n = 7$. Here we get by the same reasoning as above an injection of F_4 into $\mathrm{Sp}(7)$, but there is no almost effective action of F_4 on $\mathbb{H}^7$, cf. Kramer [37, 4.24].

If $G_{[l]} = E_7$ and $7 = \lfloor \frac{2n-1}{3} \rfloor$, then $n = 11$ or $n = 12$ and there would be an injection of E_7 into $\mathrm{Sp}(12)$, which by Kramer [37, 4.20] does not exist.

Finally, we have to consider the case that E_8 is a normal subgroup of $G_{[l]}$ and $8 \leq \mathrm{rk}(G_{[l]}) \leq \min\{11, \lfloor \frac{2n-1}{3} \rfloor\}$. That yields on the one hand $13 \leq n$. As the maximal dimension of a compact connected Lie group of rank at most 3 is $\dim \mathrm{SO}(7) = \dim \mathrm{Sp}(3) = 21$, we get on the other hand

$$2n^2 - 7n \leq \dim G_{[l]} \leq \dim E_8 + 21 = 269,$$

i.e. $13 \geq n$. Hence, $n = 13$, and there would be an injection of E_8 into $\mathrm{Sp}(13)$. But there is no almost effective representation of E_8 on $\mathbb{H}^{13}$, cf. Kramer [37, 4.22].

Therefore, the exceptional cases do not occur, and we are left with $n \leq 6$. For $n = 6$ the equation (5.1.1) still gives

$$\dim G_{[l]} \geq 2 \cdot 6^2 - 7 \cdot 6 = 30 > 21 = \binom{2 \cdot 6 + 1}{2} \geq \binom{2\,\mathrm{rk}(G_{[l]}) + 1}{2},$$

what we excluded above.

We have proved the following result.

Proposition 5.1.2 *For $n \geq 6$ there is no quadrangle with point space*

$$\mathrm{SU}(3) \times \mathrm{Sp}(n)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1) \approx \mathbb{S}^5 \times \mathbb{S}^{4n-1}$$

such that $\mathrm{SU}(3) \times \mathrm{Sp}(n)$ acts by automorphisms.

5.2 The $(7, 4n - 8)$-series

To rule out the next series, assume that $\mathrm{Sp}(2) \times \mathrm{Sp}(n)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1) \approx \mathbb{S}^7 \times \mathbb{S}^{4n-1}$ is the point space of a quadrangle where $\mathrm{Sp}(2) \times \mathrm{Sp}(n)$ acts by automorphisms of the quadrangle, cf. Kramer [37, 7.33 and 3.15(B1)]. Then $n \geq 3$ and the pair of topological parameters is $(7, 4n-8)$.

If p is a point which is stabilized by $\mathrm{Sp}(n-1)$, then $\mathrm{Sp}(n-1)$ acts on the generalized $(4n-8)$-sphere $\mathcal{L}_p$. Recall that non-trivial orbits of $\mathrm{Sp}(n-1)$ have at least dimension $\dim \mathrm{Sp}(n-1) - \dim \mathrm{Sp}(1) \times \mathrm{Sp}(n-2) = 4n-8 = \dim \mathcal{L}_p$ by table 2.2 of maximal-dimensional subalgebras of compact simple algebras on page 16, and for $n \geq 4$ a connected closed subgroup U of $\mathrm{Sp}(n-1)$ with the dimension of $\mathrm{Sp}(1) \times \mathrm{Sp}(n-2)$ is conjugate to $\mathrm{Sp}(1) \times \mathrm{Sp}(n-2)$: in fact, by table 2.2 on page 16 the group U is locally isomorphic to $\mathrm{Sp}(1) \times \mathrm{Sp}(n-2)$, and with representation theory it can be shown that the induced action of the $\mathfrak{sp}_{n-2}$-factor of U on $\mathbb{H}^{n-1}$ is the standard one, see Kramer [37, 4.14]. Hence, the corresponding $\mathfrak{sp}_{n-2}$-subgroup of U is a conjugate of $\mathrm{Sp}(n-2)$ in $\mathrm{Sp}(n-1)$. The $\mathfrak{sp}_1$-factor of U lies in the centralizer $\mathrm{Cen}_{\mathrm{Sp}(n-1)}(\mathrm{Sp}(n-2)) \cong \mathrm{Sp}(1)$ and is therefore uniquely determined. It follows for $n \geq 4$ that if there is a non-trivial orbit of $\mathrm{Sp}(n-1)$ on $\mathcal{L}_p$, then the action is transitive, and $\mathbb{H}\mathrm{P}_{n-2} \approx \mathrm{Sp}(n-1)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-2) \approx \mathcal{L}_p \approx \mathbb{S}^{4n-8}$, a contradiction.

Therefore, for $n \geq 4$ the action of $\mathrm{Sp}(n-1)$ on $\mathcal{L}_p$ is trivial and $\mathrm{Sp}(n-1)$ acts on the generalized 7-sphere $\mathcal{P}_l$ for every $l \in \mathcal{L}_p$. But this action on the smaller space is, of course, also trivial for $n \geq 4$, and for these values $\mathrm{Sp}(n-1)$ acts trivially on $p^\perp$. This is a contradiction to 4.2.3. Therefore, we are left only with the case $n = 3$.

For $n = 3$ the quadrangle is dual to a $(4, 7)$-quadrangle and $\dim G = 31 > 27 = \binom{2 \cdot 3+1}{2} + 6$. But Biller [2, 5.3.1] has shown that if a compact group of dimension bigger than 27 acts on a $(4, 4 \cdot 3 - 5)$-quadrangle then the action is a standard action on the quaternion hermitian quadrangle, which has a quaternion Stiefel manifold $\mathbb{V}_2(\mathbb{H}^3) \approx \mathrm{Sp}(3)/\mathrm{Sp}(1)$ as line space. Hence,

$$\mathrm{Sp}(2) \times \mathrm{Sp}(3)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(2) \approx \mathbb{S}^7 \times \mathbb{S}^{11}$$

had to be $\mathrm{Sp}(3)$-equivalent to

$$\mathbb{V}_2(\mathbb{H}^3) \approx \mathrm{Sp}(3)/\mathrm{Sp}(1).$$

But the structure of $\mathbb{V}_2(\mathbb{H}^3) \approx \mathrm{Sp}(3)/\mathrm{Sp}(1)$ as homogeneous space of a maximal compact connected and effective Lie group is

$$\mathrm{Nor}_{\mathrm{Sp}(3)}(\mathrm{Sp}(1)) \cdot \mathrm{Sp}(3)/\mathrm{Nor}_{\mathrm{Sp}(3)}(\mathrm{Sp}(1)) \cdot \mathrm{Sp}(1) = \mathrm{Sp}(2) \times \mathrm{Sp}(3)/\mathrm{Sp}(2) \times \mathrm{Sp}(1),$$

see Kramer [37, 3.6]. In the case of $\mathbb{S}^7 \times \mathbb{S}^{11}$, the intersection of the Sp(3)-factor with the stabilizer is Sp(2) and in the case of the quaternion Stiefel manifold it is Sp(1), a contradiction. Therefore, $\mathbb{S}^7 \times \mathbb{S}^{11}$ and $\mathbb{V}_2(\mathbb{H}^3)$ are not homeomorphic.

This yields the following result.

Proposition 5.2.1 *There is no quadrangle with point space*

$$\mathrm{Sp}(2) \times \mathrm{Sp}(n)/\mathrm{Sp}(1) \cdot \mathrm{Sp}(n-1) \approx \mathbb{S}^7 \times \mathbb{S}^{4n-1}$$

such that $\mathrm{Sp}(2) \times \mathrm{Sp}(n)$ *acts by automorphisms.*

5.3 The $(3, 2n-2)$-series

In this section assume that the point space of a $(3, 2n-2)$-quadrangle is a product of homogeneous spheres $\mathrm{Sp}(1) \times \mathrm{SU}(n+1)/\mathrm{SU}(n) \approx \mathbb{S}^3 \times \mathbb{S}^{2n+1}$, where $n \geq 2$ and $\mathrm{Sp}(1) \times \mathrm{SU}(n+1)$ acts by automorphisms, cf. Kramer [37, 7.G].

There is a point p which is stabilized by $\mathrm{SU}(n)$. We consider the action of $\mathrm{SU}(n)$ on the generalized $(2n-2)$-sphere $\mathcal{L}_p$. Non-trivial orbits of $\mathrm{SU}(n)$ have at least dimension $2n-2$ by table 2.2 (page 16) of maximal-dimensional subalgebras of simple compact Lie algebras. Hence, there cannot be a non-trivial orbit in $\mathcal{L}_p$ because, otherwise, the action would be transitive and we would have $\mathbb{C}\mathrm{P}_{n-1} \approx \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)) \approx \mathcal{L}_p \approx \mathbb{S}^{2n-2}$, cf. Biller [2, 3.2.2], a contradiction for $n \geq 3$. Since the action of $\mathrm{SU}(n)$ cannot be trivial either by 4.2.3, we are left with the case $n = 2$.

For $n = 2$ the stabilizer SU(2) of p acts on the generalized 2-sphere $\mathcal{L}_p$. If this action is not trivial, then it is transitive, and $G = \mathrm{Sp}(1) \times \mathrm{SU}(3)$ is flag transitive by 4.1.1. From the classification of compact connected flag homogeneous quadrangles, cf. Grundhöfer–Knarr–Kramer [23], we see that the quadrangle in question had to be the dual of the complex hermitian quadrangle with parameters $(2, 3)$. But this is not true, because then the normal factor SU(3) in $\mathrm{Sp}(1) \times \mathrm{SU}(3)$ would have to be transitive, which is not the case here.

Hence, $G_p = \mathrm{SU}(2)$ acts trivially on the line pencil $\mathcal{L}_p$. Consider the induced action of SU(2) on the point rows of each line of $\mathcal{L}_p$, which are generalized 3-spheres. This action cannot be trivial for all $l \in \mathcal{L}_p$, because then $G_p = \mathrm{SU}(2)$ would act trivially on $p^\perp$ contradicting 4.2.3. It follows that there is a line $l \in \mathcal{L}_p$ with a non-trivial SU(2)-orbit $\mathrm{SU}(2) \cdot q$ which has necessarily dimension

2; (note that there is no one-dimensional SU(2)-orbit, that the orbit of p is trivial and thus the action is not transitive). As $\mathrm{SU}(2) \cdot q$ has codimension 1 in the generalized 3-sphere $\mathcal{P}_l$, we may apply Mostert's theorem 2.5.1 and conclude that $\mathrm{SU}(2)_p/\mathrm{SU}(2)_q = \mathrm{SU}(2)/\mathrm{SU}(2)_q$ is a homology 2-sphere, a contradiction, see the classification 2.4.1 of homogeneous homology spheres.

In this section we have proved the following result.

Proposition 5.3.1 *There is no quadrangle with point space*

$$\mathrm{Sp}(1) \times \mathrm{SU}(n+1)/\mathrm{SU}(n) \approx \mathbb{S}^3 \times \mathbb{S}^{2n+1}$$

where $\mathrm{Sp}(1) \times \mathrm{SU}(n+1)$ *acts by automorphisms.*

If we combine the last three propositions with our classification results for $(1, m)$-quadrangles (see page 54) and with the classification results of Wolfrom [71] for $(2, m)$-quadrangles and of Kramer for (k, m)-quadrangles with $3 \le k < m$ in [37] or with $k = m$ in [36], then we get the following result.

Theorem 5.3.2 *If a compact connected Lie group acts as an automorphism group transitively and irreducibly on the points or on the lines of an* (k, m)*-quadrangle, then there is up to finitely many exceptions an action of this group by automorphisms on a classical quadrangle or on a FKM-quadrangle of type* $(3, 4n)$ *or* $(8, 7)$*, and the stabilizers of the two actions are the same for* $2 \le k \le m$*. For* $1 = k < m$ *the connected components of the stabilizers coincide.*

By the FKM-quadrangles we mean the quadrangles related to the constructions of Ferus, Karcher and Münzner [19] with Clifford algebras.

Chapter 6

Rational cohomology

This chapter does not require results from the other chapters (apart from Lemma 2.5.4), but we used some well-known results quoted in this chapter at the end of chapter 1. Nevertheless, we put this chapter at the end, because it deals with spectral sequences, which are very different objects from the topics involved in the previous chapters. Furthermore, we intend to give the reader an easy access to the main geometric results of this thesis.

We start this chapter by considering actions of fundamental groups arising from fibrations. These are important for orientable fibrations. Furthermore, we show that a certain kind of fibrations that we will use several times is orientable over the rationals.

After a short introduction to spectral sequences, one of the main tools used in the following sections, we recall some relations between rational homotopy and rational cohomology, e.g. the Cartan-Serre-theorem. Then we show that homogeneous spaces G/H with the same dimension and the same rational homotopy as $\mathbb{S}^1 \times \mathbb{S}^m$ also have the same rational cohomology as $\mathbb{S}^1 \times \mathbb{S}^m$ if H is connected. Furthermore, we determine the rational cohomology of the fibre bundle $H \to G \to G/H$. This leads to another proof of 2.5.11, the classification of such homogeneous spaces G/H.

Finally, we calculate the cohomology of certain homogeneous spaces. These homogeneous examples show that there is no hope of getting further restrictions in 2.5.11 on the number of connected components of the stabilizer.

6.1 Orientable fibrations

The following description of actions of the fundamental group is mainly taken from Whitehead [70] and Hu [30].

The first action we consider may be defined for every topological space.

Definition 6.1.1 *Let $g, h : X \to E$ be continuous maps between topological spaces X and E. We fix a base point $*$ in X. Then g is called **freely homotopic** to h **along** the path $v : [0,1] \to E$ if there is a homotopy $H : X \times [0,1] \to E$ such that $H_0 := H(\cdot, 0) = g$, $H_1 := H(\cdot, 1) = h$ and $H(*, \cdot) = v$. We denote this by $g \simeq_v h$.*

Now we choose a base point in E and denote it also by $*$. If g and h respect the base points, i.e. if $g(*) = * = h(*)$, and if we denote the constant path to $* \in E$ also by $*$, then we may express base point preserving homotopy by $g \simeq_* h$.

Let $[v] \in \pi_1(E, *)$ be an element of the fundamental group of E. By Whitehead [70, III.1-7] there is a group automorphism $\tau_{[v]}$ of $\pi_n(E, *)$ such that for $[g], [h] \in \pi_n(E, *)$ we have

$$\tau_{[v]}(g) = h \iff g \simeq_v h.$$

Furthermore, this group automorphism does not depend on the choices of v, g and h in the respective homotopy classes, and $[v] \mapsto \tau_{[v]}$ defines an action τ of $\pi_1(E, *)$ on $\pi_n(E, *)$ for every $n \in \mathbb{N}$.

This leads to the following kind of spaces.

Definition 6.1.2 *Let E be a topological space. Then E is called **n-simple** if the action τ of $\pi_1(E, *)$ on $\pi_n(E, *)$ in 6.1.1 is trivial for every base point $* \in E$. For a path-connected space this follows if the action is trivial for any base point. Therefore, we may suppress the base point in this case.*

*If E is n-simple for every n, then we call E **simple**.*

*To distinguish the simplicity of the topology of a topological group from the simplicity of the abstract group we will call the underlying topological space of the group also **homotopy simple**.*

The n-simplicity of a space may also be expressed as in Hu [30, III.16.9]:

Proposition 6.1.3 *A topological space E is n-simple if and only if for every point $* \in E$ and any two continuous maps $g, h : \mathbb{S}^n \to E$ with $g(s_0) = * =$*

$h(s_0)$, where $s_0 \in \mathbb{S}^n$ is a base point, we have

$$g \simeq h \implies g \simeq_{s_0} h.$$

Hu [30, III.16.10] shows that connected topological groups yield examples of simple spaces. We adapt the proof of Hu to show that also certain quotients of topological groups are simple spaces. Note that we may set $H = 1$ in the following lemma to get that the topological groups themselves are homotopy simple. This and the following result for Lie groups may also be found in Steenrod [58, 16.9 and 11].

Lemma 6.1.4 *If G is a locally compact path-connected topological group and H a connected closed subgroup, then the quotient space G/H is homotopy simple.*

Proof We apply 6.1.3. So assume there is a homotopy $h : \mathbb{S}^n \times [0,1] \to G/H$ such that $h_0(*) = gH = h_1(*)$ for some $g \in G$, where $h_0 = h(\cdot, 0)$ and $h_1 = h(\cdot, 1)$ are obtained by fixing the second argument of h and $*$ is a base point in $\mathbb{S}^n$. As multiplication by g is a homeomorphism of G and of G/H, we may assume that $g = 1$. We have to show that $h_0 \simeq_H h_1$ where H denotes also the constant path to H.

Consider the path $w = h(*, \cdot)$ in G/H. Since H is path-connected by assumption, the exact homotopy sequence of $H \hookrightarrow G \longrightarrow G/H$ shows that $\pi_1(G) \to \pi_1(G/H)$ is a surjective map, see Salzmann et al. [54, 96.12] for the homotopy sequence. Hence, there is a lift $\hat{w} : [0,1] \to G$ of w to G, i.e. we have $\hat{w}(0) = 1$, $\hat{w}(1) = 1$ and $w(t) = \hat{w}(t)H$ for all $t \in [0,1]$. Set $\bar{h}(x,t) = (\hat{w}(t))^{-1} \cdot h(x,t) = (\hat{w}(t)H)^{-1} \cdot h(x,t)$. Then $\bar{h}(x,0) = h(x,0) = h_0(x)$ and $\bar{h}(x,1) = h(x,1) = h_1(x)$ for all $x \in \mathbb{S}^n$, and $\bar{h}(*,t) = (\hat{w}(t))^{-1} \cdot h(*,t) = (\hat{w}(t)H)^{-1} \cdot h(*,t) = (w(t))^{-1} \cdot h(*,t) = H$ for all $t \in [0,1]$ since $w(t) = h(*,t)$.
□

The following lemma gives some kind of geometric meaning of simplicity, compare Hu [30, III.16.11].

Lemma 6.1.5 *If a topological space E is path-connected and n-simple, then there is a bijection from the base point preserving homotopy classes $\pi_n(E,*)$ of maps $g : \mathbb{S}^n \to E$, $g(*) = *$, to the set of free homotopy classes, sending $[g] \in \pi_n(E,*)$ to the set $\{h : \mathbb{S}^n \to E \mid g \simeq h\}$.*

Next we consider an action of the fundamental group of the base space of a fibration.

Definition 6.1.6 *Let* $F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$ *be a fibration and denote by* $*$ *the base point in* B *and also by the same symbol the base point in* E *(and in* $F = p^{-1}(*)$*). For a closed path* w, $[w] \in \pi_n(B, *)$*, define* $W : F \times [0,1] \to B$ *by* $W(f,t) = w(t)$ *and let* $F \times \{1\} \to E$*,* $(f,1) \mapsto f$ *be the inclusion. This gives a commutative diagram*

$$\begin{tikzcd} F \times \{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ F \times [0,1] \arrow[r, "W"] & B, \end{tikzcd}$$

which by the definition of fibrations on page 1 may be completed by a an extension $H : F \times [0,1] \to E$ *without affecting the commutativity of the diagram. The commutativity then gives* $H(F \times \{t\}) \subseteq p^{-1}(w(t))$ *for all* $t \in [0,1]$ *and* $H_0 = H(\cdot, 0)$ *is homotopic (in* E*) to the fibre inclusion* $H_1 = H(\cdot, 1)$*. One may also consider* H_0 *as a map to the fibre*

$$\Theta_w : F \to F.$$

Then the homotopy class $[\Theta_w]$ *depends only on the homotopy class of* w *and not really on* w *itself, see Whitehead* [70, IV.8.1-3]*, and*

$$[w] \mapsto (\Theta_w)^\bullet$$

defines an action of $\pi_1(B, *)$ *on the cohomology ring* $H^\bullet(F; R)$ *of* F *over any ring* R*.*

The fibration $F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$ *is called* ***R-orientable*** *for a ring* R*, if the action of* $\pi_1(B, *)$ *on* $H^\bullet(F; R)$ *is trivial. If the fibration is* $\mathbb{Z}$*-orientable then it is simply called* ***orientable****.*

Similarly, if the fibre F *is* n*-simple, then*

$$[w] \mapsto (\Theta_w)_\#$$

defines an action of the fundamental group $\pi_1(B, *)$ *of the base on the homotopy groups* $\pi_n(F, *)$ *of the fibre* F *for every* $n \in \mathbb{N}$*.*

Actually, the last action is an action of $\pi_1(B, *)$ on the orbits in $\pi_n(F, *)$ under the action of $\pi_1(F, *)$ described in 6.1.1. But we will consider this action only for simple fibres.

Note that fibrations over simply connected base spaces are of course R-simple for every ring R. The R-orientable fibrations are also called R-simple because then there is a simple system of local coefficients which may be identified with the singular cohomology groups. Spanier [57, p. 476] uses a similar construction of Θ_w and considers the induced maps in homology (and not in cohomology as we did). Whitehead [70, p. 649, 349] calls the fibration R-coorientable if the induced maps of Θ_w in cohomology are trivial.

We will see in 6.2.1 that R-orientable fibrations lead to spectral sequences that relate the (singular) cohomology of the base and the fibre to the cohomology of the total space.

We aim to show that certain fibrations are $\mathbb{Q}$-orientable. As one step we need the following lemma.

Lemma 6.1.7 *If Y is simply connected and $f : Y \longrightarrow Y$ induces the identity on the first infinite homotopy group of Y, then f also induces the identity on the first rational homology group and therefore also on the first rational cohomology group.*

Proof Let $m \in \mathbb{N}$ be the smallest value with $\pi_m(Y) \otimes \mathbb{Q} \neq 0$. By a version of the Hurewicz theorem over the rationals as in Kramer [37, 2.1, 2.2] or similarly in McCleary [40, 5.18], m is also the first value of non-trivial homology groups over $\mathbb{Q}$, and these vector spaces are isomorphic by the map induced by the Hurewicz map. Recall that the rational cohomology groups may be interpreted as the dual vector spaces of the rational homology groups, e.g. by the universal coefficient theorems.

From the Whitehead tower (or upside-down Postnikov tower) we get a fibration $\beta : Y\langle m\rangle \longrightarrow Y$, where $Y\langle m\rangle$ is $(m-1)$-connected and β induces an isomorphism between the m-th rational homology groups as its dual is an isomorphism between the m-th rational cohomology groups induced by β, see Kramer [37, 2.1]. But then the Hurewicz homomorphism of Y induces like that of $Y\langle m\rangle$ an isomorphism between the m-th rational homotopy groups and the rational homology groups of Y. So f induces in rational homology as in rational homotopy the identity, because the induced maps commute with the Hurewicz homomorphisms, see the commutative diagram 6.1 on page 66. Therefore, the dual map $f^\bullet$ is the identity on the rational cohomology groups of dimension m. □

Remark 6.1.8 We want to construct a fibration related to the universal covering of a space. We use the theory of classifying spaces and universal G-bundles as e.g. presented in Mimura-Toda [42, II.6], McCleary [40, Ch. 6.3] or tom Dieck [68, Ch. IX]. We denote the classifying space of a group G by BG.

$$\begin{tikzcd}
H_m(Y\langle m\rangle;\mathbb{Q}) \arrow[dd, "\beta_\bullet"] & \pi_m(Y\langle m\rangle)\otimes\mathbb{Q} \arrow[l, "\rho"'] \arrow[d, "\beta_\#"] & & \\
& \pi_m(Y)\otimes\mathbb{Q} \arrow[r, "f_\#"] \arrow[dl, "\rho"'] & \pi_m(Y)\otimes\mathbb{Q} \arrow[dr, "\rho"] & \\
H_m(Y;\mathbb{Q}) \arrow[rrr, "f_\bullet"'] & & & H_m(Y;\mathbb{Q})
\end{tikzcd}$$

Figure 6.1: The commutative diagram in the proof of 6.1.7. By ρ we mean the maps induced by the respective Hurewicz homomorphisms.

Let X be a 'nice' space, e.g. a manifold. Then there is a universal covering $p : \widetilde{X} \to X$, and the fundamental group $G = \pi_1(X)$ acts on $\widetilde{X}$ by deck transformations. Furthermore, G acts freely on the total space EG of the universal G-bundle $G \to EG \to BG$. Hence, for the principal G-bundle $G \to \widetilde{X} \to X$ there is the associated G-bundle $EG \to \widetilde{X} \times_G EG \to X$ with fibre EG.

The total space EG of the universal G-bundle is contractible, hence, the homotopy sequence of the universal G-bundle shows that the map $\widetilde{X} \times_G EG \to X$ in the associated G-bundle induces isomorphisms in homotopy, i.e. this map is a weak homotopy equivalence. But a weak homotopy equivalence between connected CW-complexes is a homotopy equivalence, see e.g. Bredon [11, VII.11.14], hence $\widetilde{X} \times_G EG \simeq X$.

There is also for the universal G-bundle $G \to EG \to BG$ an associated G-bundle $\widetilde{X} \to \widetilde{X} \times_G EG \to BG$ with fibre $\widetilde{X}$.

Therefore, there is up to homotopy a fibration $\widetilde{X} \to X \to BG$.

We will show that the fibration $\widetilde{X} \to X \to BG$ in 6.1.8 is $\mathbb{Q}$-orientable under certain additional assumptions.

Lemma 6.1.9 *Let X be an $(m+2)$-dimensional compact connected topological manifold with infinite fundamental group such that*

$$\pi_k(X) \otimes \mathbb{Q} \cong \pi_k(\mathbb{S}^{m+1}) \otimes \mathbb{Q} \qquad \textit{for all } k \geq 2.$$

If X is $(m+1)$-simple, then the fibration $\widetilde{X} \to X \to B\pi_1(X)$ in 6.1.8 is $\mathbb{Q}$-orientable.

Proof We will suppress the base points of the homotopy groups in the following, and we will shortly write B for the base space $B\pi_1(X)$ of the fibration. Now consider a $[w] \in \pi_1(B)$. Recall the definition of the actions in 6.1.6; there is a homotopy $H : F \times [0,1] \to E$ such that H_0 may also be considered as a map $\Theta_w : F \to F$, and $[w]$ acts on $H^\bullet(\widetilde{X};\mathbb{Q})$ and on $\pi_\bullet(\widetilde{X})$ by $(\Theta_w)^\bullet$ and $(\Theta_w)_\#$, respectively. Note for the action on the homotopy groups that $\widetilde{X}$ is simply connected and hence simple.

By 2.5.5 we have $H^\bullet(\widetilde{X};\mathbb{Q}) \cong H^\bullet(\mathbb{S}^{m+1};\mathbb{Q})$, i.e. the only non-trivial rational cohomology groups are $H^0(\widetilde{X};\mathbb{Q})$ and $H^{m+1}(\widetilde{X};\mathbb{Q})$. As the action of the fundamental group of B is induced by continuous maps and as continuous maps between path-connected spaces induce the identity on the 0-th cohomology groups, the action on $H^0(\widetilde{X};\mathbb{Q})$ is of course trivial. Hence, we have only to consider the action on $H^{m+1}(\widetilde{X};\mathbb{Q})$.

We have to show that $(\Theta_w)^\bullet = \mathrm{id} : H^{m+1}(\widetilde{X};\mathbb{Q}) \to H^{m+1}(\widetilde{X};\mathbb{Q})$ for every $[w] \in \pi_1(B)$. But in view of 6.1.7 it suffices to show that

$$(\Theta_w)_\# = \mathrm{id} : \pi_{m+1}(\widetilde{X}) \otimes \mathbb{Q} \to \pi_{m+1}(\widetilde{X}) \otimes \mathbb{Q}.$$

We will even show that $(\Theta_w)_\# = \mathrm{id} : \pi_{m+1}(\widetilde{X}) \to \pi_{m+1}(\widetilde{X})$.

Since the projection $\widetilde{X} \to X$ of the universal covering is up to homotopy the fibre inclusion of the fibration, the fibre inclusion induces an isomorphism $\pi_{m+1}(\widetilde{X}) \to \pi_{m+1}(X)$. Note that for the homotopy H we have $H_0 = i \circ \Theta_w$ and recall from 6.1.6 that $H_1 = i$. It follows that for every $[\alpha] \in \pi_{m+1}(\widetilde{X})$ the maps $i \circ \Theta_w \circ \alpha$ and $i \circ \alpha$ are homotopic in X by H. But X is by assumption $(m+1)$-simple; and therefore, by 6.1.5 two maps $\mathbb{S}^{m+1} \to X$ in the same homotopy class of X correspond to maps in the same base point preserving homotopy class, i.e. to the same element of $\pi_{m+1}(X)$. Therefore, $[i \circ \Theta_w \circ \alpha] = [i \circ \alpha] \in \pi_{m+1}(X)$. Since i induces an isomorphism $\pi_{m+1}(\widetilde{X}) \to \pi_{m+1}(X)$ it follows that $(\Theta_w)_\#([\alpha]) = [\alpha] \in \pi_{m+1}(\widetilde{X})$. This shows that $(\Theta_w)_\# = \mathrm{id} : \pi_{m+1}(\widetilde{X}) \to \pi_{m+1}(\widetilde{X})$ and that the fibration $\widetilde{X} \to X \to B\pi_1(X)$ is $\mathbb{Q}$-orientable. □

Quotient spaces of Lie groups and connected closed subgroups are simple by 6.1.4. This gives the following corollary to 6.1.9 which we state only for compact Lie groups. It would suffice to assume that the quotient space is compact.

Corollary 6.1.10 *Let H be a closed connected subgroup of a compact connected Lie group G such that the fundamental group of G/H is infinite,*

$$\pi_k(G/H) \otimes \mathbb{Q} \cong \pi_k(\mathbb{S}^{m+1}) \otimes \mathbb{Q} \qquad \textit{for all } k \geq 2$$

and $\dim G/H = m+2$. *Then there is up to homotopy a* $\mathbb{Q}$*-orientable fibration*

$$\widetilde{G/H} \to G/H \to B\pi_1(G/H)$$

where $\widetilde{G/H}$ *is the universal covering space of* G/H.

Recall from page 13 that connected Lie groups have abelian fundamental groups and by the exact homotopy sequence also their quotient spaces with closed connected subgroups. Hence, the assumption in the corollary on the homotopy of G/H is fulfilled if $\pi_\bullet(G/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$.

To determine the rational cohomology of G/H we will use a powerful tool, the Leray-Serre spectral sequence. We will therefore give a short introduction to spectral sequences in the next section.

6.2 Spectral sequences

The main tool in the next section will be the Leray-Serre spectral sequence for fibrations. Our source for the exposition was the book of McCleary [40]. We will sometimes also refer to Spanier [57].

Let R be a commutative ring with unit. A **graded module** over R is a module M over R together with a decomposition (called **gradation**) $M = \bigoplus_{n\in\mathbb{Z}} M^n$ in modules M^n over R. In the following all modules will be understood as modules over R, but we will suppress R most of the time.

For example, if X is a topological space and if we supplement the cohomology groups of X by setting $H^{-n}(X;R) = 0$ for $n \in \mathbb{N}$, then $\bigoplus_{n\in\mathbb{Z}} H^n(X;R)$ is a graded module.

A **differential graded module** is a graded module M additionally equipped with an R-linear map $\mathrm{d} : M \to M$, called the **differential**, such that $\mathrm{d}^2 = 0$ and d is compatible with the gradation, i.e. $\mathrm{d}(M^n) \subseteq M^{n+1}$ for $n \in \mathbb{Z}$. Of course, it would be more appropriate to call the pair (M, d) a differential graded module. But we will usually not mention the differential.

A module M is a **bigraded module** if there is a decomposition $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$ in modules $M^{p,q}$. It is a **differential bigraded module** (M, d) if there is a **differential** $\mathrm{d} : M \to M$ such that $\mathrm{d}(M^{p,q}) \subseteq M^{p+r,q+1-r}$ for some $r \in \mathbb{Z}$ and all $p, q \in \mathbb{Z}$. The pair $(r, 1-r)$ is called the **bidegree** of d. By definition there are restrictions $\mathrm{d}^{p,q} : M^{p,q} \to M^{p+r,q+1-r}$ of d. One may think of a bigraded module as a grid $\mathbb{Z}^2$ such that the module $M^{p,q}$ sits on the node $(p, q) \in \mathbb{Z}^2$, and the differentials $\mathrm{d}^{p,q} : M^{p,q} \to M^{p+r,q+1-r}$ may be visualised as arrows starting at the node (p, q) and ending at $(p+r, q+1-r)$. A differential bigraded module may be seen as a differential graded module $\bigoplus_{n\in\mathbb{Z}} \widetilde{M}^n$ by setting $\widetilde{M}^n = \bigoplus_{p+q=n} M^{p,q}$, i.e. by summing up the modules on the diagonals $p+q=n$ of the $\mathbb{Z}^2$-grid. An element $x \in M^{p,q}$ is said to have **bidegree** (p, q) and **total degree** $p+q$.

For topological spaces X, Y one may set $M^{p,q} = H^p(X; R) \otimes H^q(Y; R)$ to get an example of a bigraded module.

The **cohomology module** of a differential bigraded module M is the bigraded module

$$H(M) = \ker \mathrm{d} / \operatorname{im} \mathrm{d} \cong \bigoplus_{p,q\in\mathbb{Z}} \ker \mathrm{d}^{p,q} / \operatorname{im} \mathrm{d}^{p-r,q-1+r}.$$

This is well defined because of $\mathrm{d}^2 = 0$.

A **spectral sequence** is a sequence $(\mathrm{E}_k)_{k\geq 2}$ of bigraded differential modules $\mathrm{E}_k = \bigoplus_{p,q\in\mathbb{Z}} \mathrm{E}_k^{p,q}$ with differentials $(\mathrm{d}_k)_{k\geq 2}$ such that d_k is a differential of E_k of bidegree $(k, 1-k)$ and $\mathrm{E}_{k+1} = H(\mathrm{E}_k)$. Note that the differentials are not related to each other in an obvious way, for example we did not require that d_2 determines d_k for $k > 2$.

There are submodules $B_k \subseteq Z_k \subseteq \mathrm{E}_2$ for every $k \geq 2$ such that

$$B_2 \subseteq B_3 \subseteq \cdots \subseteq B_k \subseteq \cdots \subseteq \bigcup_{k\geq 2} B_k \subseteq \bigcap_{k\geq 2} Z_k \subseteq Z_k \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq \mathrm{E}_2$$

and $\mathrm{E}_k = \ker \mathrm{d}_k / \operatorname{im} \mathrm{d}_k \cong Z_k/B_k$ for all $k \in \mathbb{Z}$, see McCleary [40, 2.1]. We set $\mathrm{E}_\infty = \cap_{k\geq 2} Z_k / \cup_{k\geq 2} B_k$. Then also E_∞ is a bigraded module $\mathrm{E}_\infty \cong \bigoplus_{p,q\in\mathbb{Z}} \mathrm{E}_\infty^{p,q}$. The elements $x \in \mathrm{E}_2$ with $x \in Z_k$ represent elements of E_k. They are said to **survive to $\mathbf{E}_{\boldsymbol{k}}$**.

We will always deal with the case that for every pair (p, q) there is a k_0 such that for every $k \geq k_0$ the differential $\mathrm{d}_k^{p,q} : \mathrm{E}_k^{p,q} \to \mathrm{E}_k^{p+k,q+1-k}$ is trivial. Then $\mathrm{E}_{k+1}^{p,q}$ may be considered as a quotient of $\mathrm{E}_k^{p,q}$, and $\mathrm{E}_\infty^{p,q}$ is the direct limit of $(\mathrm{E}_k^{p,q})_{k\geq k_0}$, compare Spanier [57, p. 467]. This happens for example if $(\mathrm{E}_k)_{k\geq 2}$ is a **first-quadrant spectral sequence**, i.e. if $\mathrm{E}_2^{p,q}$ is trivial for $p < 0$ or $q < 0$, because then the arrows representing the differentials in the $\mathbb{Z}^2$ grid get longer and longer with growing k (approaching slope -1); finally the arrows stick out of the first quadrant of the grid, and then they map to trivial modules.

The spectral sequences that will arise from the Leray-Serre theorem 6.2.1 later on will always be first-quadrant spectral sequences.

A spectral sequence **collapses** at E_n if $\mathrm{d}_k = 0$ for all $k \geq n$. Then it follows that $\mathrm{E}_\infty^{p,q} \cong \mathrm{E}_n^{p,q}$ for all $p, q \in \mathbb{Z}$. If the spectral sequence collapses at E_2, then it is simply said to **collapse**.

We will also deal with multiplicative structures. A **differential bigraded algebra** (over R) is a differential bigraded module $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$ together with an R-bilinear and associative product

$$\cdot\ : M^{p,q} \times M^{r,s} \longrightarrow M^{p+r,q+s}$$

with unit such that the differential is a **derivation**, i.e. such that the differential satisfies the **Leibniz rule**

$$\mathrm{d}(x \cdot y) = d(x) \cdot y + (-1)^{p+q} x \cdot \mathrm{d}(y)$$

for $x \in M^{p,q}$ and $y \in M^{r,s}$. Note that $p + q$ is the total degree of x.

We define a **spectral sequence of algebras** to be a spectral sequence $(\mathrm{E}_k)_{k\geq 2}$ where E_2 is a differential bigraded algebra with a product $\cdot_2$ that induces products $\cdot_k$ on each cohomology module E_k such that E_k with $\cdot_k$ is a differential bigraded algebra for every $k > 2$ and for $k = \infty$.

Another important piece of structure in the Leray-Serre spectral sequence will be related to the following kind of subdivision of a module. By a (decreasing) **filtration** of a differential graded module $M = \bigoplus_{n\in\mathbb{Z}} M^n$ with differential d we mean a sequence of submodules $(\mathrm{F}^p M)_{p\in\mathbb{Z}}$ with $\mathrm{F}^p M \subseteq \mathrm{F}^{p-1} M \subseteq M$ and $\mathrm{d}(\mathrm{F}^p M) \subseteq \mathrm{F}^p M$ for all $p \in \mathbb{Z}$.

With the introduced terminology we are able to state the Leray-Serre-theorem, for proofs see McCleary [40, 2.1], Mimura-Toda [42, III.2.10], Spanier [57, 9.4.9] or Whitehead [70, VIII.4.9*], or see the paper of Serre [56].

Theorem 6.2.1 (Leray-Serre) *Let R be a commutative ring with unit and $F \rightarrow E \rightarrow B$ an R-orientable fibration with path-connected base B and connected fibre F. Then there is a spectral sequence $(\mathrm{E}_k)_{k\geq 2}$ of algebras and a filtration $(\mathrm{F}^p H^\bullet(E; R))_{p\in\mathbb{Z}}$ of $H^\bullet(E; R)$ such that the following is true.*

The E_2-term of the spectral sequence is

$$\mathrm{E}_2^{p,q} = H^p(B; H^q(F; R)),$$

and the multiplication $\cdot_2$ on the left hand side is up to sign defined by the cup product $\smile$ of $H^\bullet(B; H^\bullet(F; R))$, more precisely $x \cdot_2 y = (-1)^{qr} x \smile y$ for $x \in \mathrm{E}_2^{p,q}$ and $y \in \mathrm{E}_2^{r,s}$.

For the filtration $(\mathrm{F}^p H^\bullet(E; R))_{p\in\mathbb{Z}}$ we have $\mathrm{F}^0 H^\bullet(E; R) = H^\bullet(E; R)$, and it is compatible with the gradation of $H^\bullet(E; R)$, i.e. we get

$$\mathrm{F}^{p+1} H^k(E; R) \subseteq \mathrm{F}^p H^k(E; R) \subseteq \mathrm{F}^0 H^k(E; R)$$

for all p and all k. Furthermore, the filtration respects the cup product of $H^\bullet(E;R)$, i.e. for $x \in \mathrm{F}^p H^q(E;R)$ and $y \in \mathrm{F}^r H^s(E;R)$ we have $x \smile y \in \mathrm{F}^{p+r} H^{q+s}(E;R)$. There are product preserving isomorphisms

$$\mathrm{E}_\infty^{p,q} \cong \mathrm{F}^p H^{p+q}(E;R) \Big/ \mathrm{F}^{p+1} H^{p+q}(E;R)$$

for all $p, q \in \mathbb{Z}$.

The spectral sequence behaves naturally under fibre-preserving maps.

There is much more to say. First note that in the theorem above $\mathrm{E}_k^{p,q}$ is trivial if $p < 0$ or $q < 0$, i.e. $\mathrm{E}_k^{p,q}$ is a first-quadrant spectral sequence. For all $q \in \mathbb{Z}$ we have $\mathrm{E}_2^{0,q} \cong H^q(F;R)$, and the universal coefficient theorem gives $\mathrm{E}_2^{p,0} \cong H^p(B;R)$ for all $p \in \mathbb{Z}$. Furthermore, if $H^\bullet(F;R)$ is projective (for example if each cohomology group is trivial or isomorphic to $\mathbb{Z}^n$), then $\mathrm{E}_2^{p,q} \cong H^p(B;R) \otimes H^q(F;R)$. In particular, we get for fields $\mathbb{F}$ that $\mathrm{E}_2 \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$ as bigraded algebras. (But note that the multiplication is given by $(b_1 \otimes f_1) \cdot (b_2 \otimes f_2) = (-1)^{qr}(b_1 \smile b_2) \otimes (f_1 \smile f_2)$ for $f_2 \in H^q(F;\mathbb{F})$ and $b_2 \in H^r(B;\mathbb{F})$.)

The filtration also gives rise to short exact sequences

$$0 \longleftarrow \frac{\mathrm{F}^p H^{p+q}(E;R)}{\mathrm{F}^{p+1} H^{p+q}(E;R)} \longleftarrow \mathrm{F}^p H^{p+q}(E;R) \longleftarrow \mathrm{F}^{p+1} H^{p+q}(E;R) \longleftarrow 0\,.$$

These short exact sequences may be put together as in figure 6.2, where we suppressed the coefficient ring.

We will consider the relation between the filtration $(\mathrm{F}^p H^\bullet(E;R))_{p\in\mathbb{Z}}$ and the nth diagonal $\bigoplus_{p+q=n} \mathrm{E}_\infty^{p,q}$ in E_∞. Therefore, consider the diagram in figure 6.2. It can be seen in the diagram that if there is just one non-trivial module on the diagonal $p+q=n$ in (the $\mathbb{Z}^2$ grid of) E_∞, then this non-trivial module on the diagonal is isomorphic to $H^n(E)$.

If the coefficient ring is a field $\mathbb{F}$ and, hence, the modules are $\mathbb{F}$-vector spaces, then the diagram implies that $H^n(E;\mathbb{F}) \cong \bigoplus_{p+q=n} \mathrm{E}_\infty^{p,q}$ as vector spaces, in particular

$$\dim H^\bullet(E;\mathbb{F}) = \dim \mathrm{E}_\infty.$$

Now assume that $\mathrm{d}_2^{0,1} : \mathrm{E}_2^{0,1} \to \mathrm{E}_2^{2,0}$ is trivial, e.g. because $H^2(B;R) = 0$ is trivial. Then $\mathrm{E}_\infty^{0,1} \cong \mathrm{E}_2^{0,1} \cong H^1(F;R)$ and $\mathrm{E}_\infty^{1,0} \cong \mathrm{E}_2^{1,0} \cong H^1(B;R)$, and there is a short exact sequence

$$0 \longleftarrow H^1(F;R) \longleftarrow H^1(E;R) \longleftarrow H^1(B;R) \longleftarrow 0\,.$$

$$
\begin{tikzcd}[column sep=small]
0 & \mathrm{E}_\infty^{0,n} \arrow[l] & H^n(E) \arrow[l] & \mathrm{F}^1H^n(E) \arrow[l] \arrow[d, equal] & 0 \arrow[l] & & & & & \\
& 0 & \mathrm{E}_\infty^{1,n-1} \arrow[l] & \mathrm{F}^1H^n(E) \arrow[l] & \mathrm{F}^2H^n(E) \arrow[l] \arrow[d, equal] & 0 \arrow[l] & & & & \\
& & 0 & \mathrm{E}_\infty^{2,n-2} \arrow[l] & \mathrm{F}^2H^n(E) \arrow[l] & \mathrm{F}^3H^n(E) \arrow[l] \arrow[d, equal] & 0 \arrow[l] & & & \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
& & & & 0 & \mathrm{E}_\infty^{n-2,1} \arrow[l] & \mathrm{F}^{n-2}H^n(E) \arrow[l] & \mathrm{F}^{n-1}H^n(E) \arrow[l] \arrow[d, equal] & 0 \arrow[l] & \\
& & & & & 0 & \mathrm{E}_\infty^{n-1,1} \arrow[l] & \mathrm{F}^{n-1}H^n(E) \arrow[l] & \mathrm{F}^nH^n(E) \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\
& & & & & & 0 & \mathrm{E}_\infty^{n,0} \arrow[l] & \mathrm{F}^nH^n(E) \arrow[l, "\cong"'] & 0 \arrow[l]
\end{tikzcd}
$$

Figure 6.2: A diagram involving the diagonal of E_∞.

Hence, $H^1(E;\mathbb{F}) \cong H^1(B;\mathbb{F}) \oplus H^1(F;\mathbb{F})$ if $R = \mathbb{F}$ is a field and $\mathrm{d}_2^{0,1}$ is trivial.

There is a special case, in which one can easily recognize the multiplicative structure of $H^\bullet(E;\mathbb{F})$. Namely, consider again for $n \in \mathbb{N}$ the diagonal $\bigoplus_{p+q=n} \mathrm{E}_\infty^{p,q}$ in E_∞; assume that the spectral sequence collapses at E_2 and that there is on every diagonal in $\mathrm{E}_\infty \cong \mathrm{E}_2$ only at most one non-trivial entry. The compatibility of the filtration $(\mathrm{F}^pH^\bullet(E;\mathbb{F}))_{p\in\mathbb{Z}}$ with the cup product of $H^\bullet(E;\mathbb{F})$ implies that $H^\bullet(E;\mathbb{F}) \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$ as algebras. (Again be careful with the sign of the multiplication on the right.)

The last observations apply if the cohomology of the fibre is concentrated in one dimension and the cohomology of the base space vanishes above that dimension, or vice versa. This gives the following lemmas.

Lemma 6.2.2 *Let $\mathbb{F}$ be a field and $F \longrightarrow E \longrightarrow B$ an $\mathbb{F}$-orientable fibration with path-connected base B and connected fibre F. Assume there is some $n \in \mathbb{N}$ such that $H^k(F;\mathbb{F}) = 0$ for $k \in \mathbb{N}$, $k \neq n$, and $H^k(B;\mathbb{F}) = 0$ for $k \geq n$. Then $H^\bullet(E;\mathbb{F}) \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$ as algebras.*

Proof Because of the structure of $\mathrm{E}_2 \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$, the Leray-Serre spectral sequence of the fibration collapses at E_2. There is at most one non-trivial entry on each diagonal $\bigoplus_{p+q=n} \mathrm{E}_\infty^{p,q}$ of $\mathrm{E}_\infty \cong \mathrm{E}_2$. Hence,

$H^\bullet(E;\mathbb{F}) \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$ as algebras by the observations made above. $\square$

Exchanging the role of fibre and base gives the following lemma.

Lemma 6.2.3 *Let $\mathbb{F}$ be a field and $F \to E \to B$ an $\mathbb{F}$-orientable fibration with path-connected base B and connected fibre F. Assume there is some $n \in \mathbb{N}$ such that $H^k(F;\mathbb{F}) = 0$ for $k \geq n$ and $H^k(B;\mathbb{F}) = 0$ for $k \in \mathbb{N}$, $k \neq n+1$. Then $H^\bullet(E;\mathbb{F}) \cong H^\bullet(B;\mathbb{F}) \otimes H^\bullet(F;\mathbb{F})$ as algebras.*

We follow McCleary [40, Th. 5.9] to identify in the spectral sequence of $F \overset{i}{\hookrightarrow} E \xrightarrow{\mathrm{pr}} B$ the maps induced by i and pr in cohomology. First note that there are injections

$$\mathrm{E}_\infty^{0,p} = \mathrm{E}_{p+1}^{0,p} \subseteq \mathrm{E}_p^{0,p} \subseteq \cdots \subseteq \mathrm{E}_2^{0,p} \cong H^p(F;R)$$

which give rise to the commutative diagram

$$\begin{tikzcd}
 & \mathrm{E}_2^{0,p} & H^p(F;R) \arrow[l, "\cong"'] \\
0 & \mathrm{E}_\infty^{0,p} \arrow[l] \arrow[u, hook] & H^p(E;R) \arrow[l] \arrow[u, "i^\bullet"']
\end{tikzcd}$$

with exact bottom row. The composition of the surjective projections $\mathrm{E}_p^{q,0} \to \mathrm{E}_{p+1}^{q,0}$ gives the projection $\mathrm{E}_2^{p,0} \to \mathrm{E}_3^{p,0} \to \cdots \to \mathrm{E}_p^{p,0} \to \mathrm{E}_{p+1}^{p,0} = \mathrm{E}_\infty^{p,0}$ in the diagram

$$\begin{tikzcd}
0 & & \\
\mathrm{E}_\infty^{p,0} \arrow[u] \arrow[r, equal] & \mathrm{F}^p H^p(E) \arrow[r, hook] & H^p(E;R) \\
\mathrm{E}_2^{p,0} \arrow[u] \arrow[rr, "\cong"] & & H^p(B;R). \arrow[u, "\mathrm{pr}^\bullet"']
\end{tikzcd}$$

where the left column is exact.

Note that we find the isomorphism of the top row of the last diagram in the bottom row of figure 6.2. Similarly, the bottom row of the diagram before the last one is in the top row of figure 6.2. Therefore, the last two diagrams may be added to figure 6.2.

6.3 Rational homotopy and rational cohomology

We now aim to show that $\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$ implies also $H^\bullet(X;\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1};\mathbb{Q})$ for many spaces X, i.e. if the rational homotopy looks like $\mathbb{S}^1 \times \mathbb{S}^{m+1}$, then the same is true for the rational cohomology. We are mainly interested in homogeneous spaces of that kind and we will determine the rational cohomology of $H \to G \to G/H$ where $X = G/H$. We start with the rational cohomology of X.

Proposition 6.3.1 *Let X be a compact $(m+2)$-dimensional manifold with abelian fundamental group. If X is $(m+1)$-simple and*

$$\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q},$$

then $H^\bullet(X;\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1};\mathbb{Q})$ as an algebra.

Proof By 6.1.9 the universal covering space $\widetilde{X}$ of X is up to homotopy the fibre of a $\mathbb{Q}$-orientable fibration $\widetilde{X} \to X \to B\pi_1(X)$. The rational cohomology of $\widetilde{X}$ is $H^\bullet(\widetilde{X};\mathbb{Q}) \cong H^\bullet(\mathbb{S}^{m+1};\mathbb{Q})$, see 2.5.5.

Consider the universal G-bundle $G \to EG \to BG$ for $G = \pi_1(X)$. Since the total space EG is contractible, the homotopy sequence of the universal G-bundle shows that the first homotopy group of the classifying space BG is isomorphic to G and the higher homotopy groups are trivial. Hence, BG is an Eilenberg-MacLane-space of type $(G, 1)$. By assumption, $\pi_1(X)$ is an abelian group of rank 1. Spanier [57, 9.5.8 and 6] shows that the rational cohomology of BG is the one of the classifying space of the free part $\mathbb{Z}$ of G. Since $\mathbb{S}^1$ is of type $(\mathbb{Z}, 1)$, we have

$$H^\bullet(B\pi_1(X);\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1;\mathbb{Q}).$$

Now consider the Leray-Serre spectral sequence 6.2.1 of $\widetilde{X} \to X \to B\pi_1(X)$ over $\mathbb{Q}$. The only non-trivial entries in

$$\mathrm{E}_2 \cong H^\bullet(B\pi_1(X);\mathbb{Q}) \otimes_\mathbb{Q} H^\bullet(\widetilde{X};\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1;\mathbb{Q}) \otimes_\mathbb{Q} H^\bullet(\mathbb{S}^{m+1};\mathbb{Q})$$

are those with bidegrees $(0,0)$, $(0, m+1)$, $(1,0)$ and $(1, m+1)$, and they are all isomorphic to $\mathbb{Q}$. As the differential d_2 has bidegree $(2,-1)$ it has to be trivial; it follows that all the following differentials are also trivial; hence the spectral sequence collapses at $\mathrm{E}_2 \cong \mathrm{E}_3 \cong \ldots \cong \mathrm{E}_\infty$, and there is at most one non-trivial one-dimensional $\mathbb{Q}$-vector space on each of the diagonals in E_∞. Therefore, $H^\bullet(X;\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1};\mathbb{Q})$ as vector spaces, see page 71.

Now recall the multiplicative properties of the spectral sequence. The product of a non-trivial element of $E_\infty^{1,0} \cong E_2^{1,0}$ and a non-trivial element of $E_\infty^{0,m+1} \cong E_2^{0,m+1}$ is a non-trivial element of $E_\infty^{1,m+1} \cong E_2^{1,m+1}$. Hence, the multiplicative structure of $H^\bullet(X;\mathbb{Q})$ is the one of a sphere product. □

In view of 6.1.4 the last proposition may be applied to homogeneous spaces:

Corollary 6.3.2 *If H is a closed connected subgroup of a compact connected Lie group G such that $\pi_\bullet(G/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$ and $\dim G/H = m+2$, then $H^\bullet(G/H;\mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1};\mathbb{Q})$.*

The kind of relation between rational cohomology and rational homotopy as in 6.3.2 was already observed for certain simply connected spaces. We quote some of these results.

Proposition 6.3.3 1. *(Cartan-Serre): If X is simply connected with rational cohomology $H^\bullet(X;\mathbb{Q}) \cong \mathbb{Q}(a_1,\dots,a_r) \otimes_\mathbb{Q} \Lambda_\mathbb{Q}(u_1,\dots,u_s)$ where the degrees of $a_1,\dots,a_r$ are even and the degrees of $u_1,\dots,u_s$ are odd, then*

$$rk(\pi_k(X)) = |\{i|\deg(a_i)=k\}| + |\{i|\deg(u_i)=k\}|.$$

2. *(Kramer): If X is a simply connected space with rational cohomology $H^\bullet(X;\mathbb{Q}) \cong \mathbb{Q}(a)/(a^m) \otimes_\mathbb{Q} \Lambda_\mathbb{Q}(u_1,\dots,u_s)$ where the degree of a is even and the degrees of $u_1,\dots,u_s$ are odd, then*

$$\operatorname{rk}(\pi_k(X)) = \begin{cases} 1 & \text{if } k=\deg(a) \\ |\{i|\deg(u_i)=k\}|+1 & \text{if } k = m\cdot\deg(a)-1 \\ |\{i|\deg(u_i)=k\}| & \text{else.} \end{cases}$$

3. *If G is a connected Lie group, then its rational cohomology is of the form $H^\bullet(G;\mathbb{Q}) \cong \Lambda_\mathbb{Q}(u_1,\dots,u_s)$ where the degrees of $u_1,\dots,u_s$ are odd, and*

$$rk(\pi_k(G)) = |\{i|\deg(u_i)=k\}|.$$

Proof The first two assertions are 2.3, 2.4 of Kramer [37]. The third one is well-known: since Lie groups are H-spaces, their cohomology is of the mentioned form, see e.g. Spanier [57, 5.9.13] or Mimura-Toda [42, VI.5.3], and as Lie groups are homotopy equivalent to their maximal compact subgroups it suffices to consider compact Lie groups. If G is compact, then there is a connected covering (Lie) group $T^s \times H$, where H is a finite product of simply connected almost simple compact Lie groups and T^s is a torus, see 2.1.1. Now

$$H^\bullet(G;\mathbb{Q}) \cong H^\bullet(T^s \times H;\mathbb{Q}) \cong H^\bullet(T^s;\mathbb{Q}) \otimes_\mathbb{Q} H^\bullet(H;\mathbb{Q})$$

(see Mimura-Toda [42, VI.5.2] and Bredon [11, VI.3.2]), and

$$\pi_\bullet(G) \otimes \mathbb{Q} \cong \pi_\bullet(T^s \times H) \otimes \mathbb{Q} \cong (\pi_\bullet(T^s) \otimes \mathbb{Q}) \oplus (\pi_\bullet(H) \otimes \mathbb{Q}).$$

Therefore, as the claim is true for T^s and as we may apply the Cartan-Serre theorem 1 to H, the third assertion follows also for G. $\square$

Now we will concentrate on the cohomology of fibre bundles $H \to G \to G/H$. If the quotient space is simply connected, then the rational cohomology of the fibre bundle was determined by Kramer [37, 3.7, 3.11] in the following cases. Note that for connected G and simply connected G/H the exact homotopy sequence of the fibre bundle shows that H is connected.

Theorem 6.3.4 *Let G be a compact connected Lie group and H a closed subgroup such that G/H is simply connected.*

1. *(Onishchik, Kramer): If the rational cohomology of G/H is*

$$H^\bullet(G/H;\mathbb{Q}) \cong \Lambda_\mathbb{Q}(u_1,\dots,u_s)$$

where $1 < deg(u_i)$ is odd for all i, then $\mathrm{rk}(G) = \mathrm{rk}(H) + s$ and there is a commutative diagram

$$\begin{tikzcd}
H^\bullet(H;\mathbb{Q}) \arrow[d, "\cong"] & H^\bullet(G;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] & H^\bullet(G/H;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] \\
\Lambda_\mathbb{Q}(v_1,\dots,v_r) & \Lambda_\mathbb{Q}(v_1,\dots,v_r,u_1,\dots,u_s) \arrow[l] & \Lambda_\mathbb{Q}(u_1,\dots,u_s). \arrow[l]
\end{tikzcd}$$

2. *(Kramer): If the rational cohomology of G/H is*

$$H^\bullet(G/H;\mathbb{Q}) \cong \mathbb{Q}(a)/(a^2) \otimes_\mathbb{Q} \Lambda_\mathbb{Q}(u)$$

where $0 < \deg(a)$ is even, $\deg(u)$ is odd and $\deg(a) < \deg(u)$, then $\mathrm{rk}(G) = \mathrm{rk}(H) + 1$ and there is a commutative diagram

$$\begin{tikzcd}
H^\bullet(H;\mathbb{Q}) \arrow[d, "\cong"] & H^\bullet(G;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] & H^\bullet(G/H;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] \\
\Lambda_\mathbb{Q}(v_1,\dots,v_r) & \Lambda_\mathbb{Q}(v_1,\dots,v_r,w,u) \arrow[l] & \mathbb{Q}(a)/(a^2) \otimes_\mathbb{Q} \Lambda_\mathbb{Q}(u) \arrow[l]
\end{tikzcd}$$

with $\deg(w) = 2\deg(a) - 1$ and where a is mapped to 0.

We will now prove a similar result as 6.3.4 for certain quotients of Lie groups with fundamental groups of rank 1.

Proposition 6.3.5 *Let G be a compact connected Lie group and H a closed connected subgroup of G such that $\pi_\bullet(G/H)\otimes\mathbb{Q}\cong\pi_\bullet(\mathbb{S}^1\times\mathbb{S}^{m+1})\otimes\mathbb{Q}$ and $\dim G/H=m+2$ for some $m\geq 2$.*

1. *If m is even, i.e. if $H^\bullet(G/H;\mathbb{Q})\cong\Lambda_\mathbb{Q}(u,w)$ with $\deg(u)=1$ and $\deg(w)=m+1\geq 3$ by 6.3.2, then there is a commutative diagram*

$$\begin{tikzcd}
H^\bullet(H;\mathbb{Q}) \arrow[d, "\cong"] & H^\bullet(G;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] & H^\bullet(G/H;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] \\
\Lambda_\mathbb{Q}(v_1,\dots,v_r) & \Lambda_\mathbb{Q}(v_1,\dots,v_r,u,w) \arrow[l] & \Lambda_\mathbb{Q}(u,w). \arrow[l]
\end{tikzcd}$$

2. *If m is odd, i.e. if $H^\bullet(G/H;\mathbb{Q})\cong\Lambda_\mathbb{Q}(u)\otimes_\mathbb{Q}\mathbb{Q}(a)/(a^2)$ with $\deg(u)=1$ and $\deg(a)=m+1$ by 6.3.2, then there is a commutative diagram*

$$\begin{tikzcd}
H^\bullet(H;\mathbb{Q}) \arrow[d, "\cong"] & H^\bullet(G;\mathbb{Q}) \arrow[l] \arrow[d, "\cong"] & H^\bullet(G/H;\mathbb{Q}) \arrow[l, "p^\bullet"'] \arrow[d, "\cong"] \\
\Lambda_\mathbb{Q}(v_1,\dots,v_r) & \Lambda_\mathbb{Q}(v_1,\dots,v_r,w,u) \arrow[l] & \mathbb{Q}(a)/(a^2)\otimes_\mathbb{Q}\Lambda_\mathbb{Q}(u), \arrow[l]
\end{tikzcd}$$

   where $\deg(w)=2\deg(a)-1$ and where a is mapped to 0.

Proof Since H is connected the fibration $H\to G\to G/H$ is orientable by Mimura-Toda [42, III.2.9(2), II.2.12]. The homotopy groups of G/H are finite in even dimensions unequal to $m+1$ by assumption. But now exactly the same arguments as in Kramer [37, 3.7, 3.11] apply. □

This determines the structure of irreducible normal subgroups for the kind of homogeneous spaces we are considering.

Lemma 6.3.6 *Let G be a compact connected Lie group and H a closed connected subgroup of G such that $\pi_\bullet(G/H)\otimes\mathbb{Q}\cong\pi_\bullet(\mathbb{S}^1\times\mathbb{S}^{m+1})\otimes\mathbb{Q}$ and $\dim G/H=m+2$ for some $m\geq 2$. Then there exists a transitive normal subgroup in G that is locally isomorphic to $\mathrm{SO}(2)\times A$ with an almost simple compact connected Lie group A. The connected component U^1 of the stabilizer U of $\mathrm{SO}(2)\times A$ is also almost simple or trivial.*

Proof Kramer's result [37, 3.14] carries over to our case. Kramer used only the special form of the cohomology of $H\to G\to G/H$, which is similar to the one of 6.3.5.

Since the third cohomology group of a compact connected Lie group determines the number of its almost simple factors, the cohomology structure of the fibration as given in 6.3.5 shows that H^1 is almost simple or trivial. □

Using the results on the cohomology structure of $H \to G \to G/H$ we are able to give a second proof of 2.5.11.

Theorem 6.3.7 *Let G/H be an $(m+2)$-dimensional homogeneous space of a compact connected Lie group G such that the fundamental group of G/H is torsion-free and $\pi_\bullet(G/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}$ for an $m \geq 2$. Assume that G acts irreducibly on G/H. Then H/H^1 is finite and cyclic and there is an almost simple normal closed subgroup A of G such that G is locally isomorphic to $\mathrm{SO}(2) \times A$ and $H^1 = H \cap A$. The quotient A/H^1 is a simply connected rational cohomology $(m+1)$-sphere.*

Proof The homotopy sequence of the covering $G/H^1 \to G/H$ shows that the finite fibre H/H^1 is a quotient of $\mathbb{Z}$ and therefore cyclic. Furthermore,

$$\pi_\bullet(G/H^1) \otimes \mathbb{Q} \cong \pi_\bullet(G/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q}.$$

Proposition 6.3.1 shows that $H^\bullet(G/H^1; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}; \mathbb{Q})$, and it follows by 6.3.6 that up to a finite covering we may write $G = \mathrm{SO}(2) \times A$ with an almost simple Lie group A. Furthermore, H^1 is almost simple, as well, or trivial.

We may also assume that $\mathrm{SO}(2) \cap H = 1$, because otherwise we may factor out $\mathrm{SO}(2) \cap H$. Since H^1 is almost simple or trivial, the restriction to H^1 of the projection to the first factor $\mathrm{SO}(2)$ of $\mathrm{SO}(2) \times A$ cannot be surjective as the kernel is a normal subgroup of (the almost simple or trivial group) H^1. But as it has a connected image, this image has to be trivial.

It follows that $H^1 \subset A$, and as the covering

$$\mathrm{SO}(2) \times A/H^1 = (\mathrm{SO}(2) \times A)/H^1 \to (\mathrm{SO}(2) \times A)/H$$

has finite fibres H/H^1, we get

$$\pi_\bullet(\mathrm{SO}(2) \times A/H^1) \otimes \mathbb{Q} \cong \pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}) \otimes \mathbb{Q},$$

and hence by 6.3.2 also $H^\bullet(\mathrm{SO}(2) \times A/H^1; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^1 \times \mathbb{S}^{m+1}; \mathbb{Q})$.

For cohomology with coefficients in a field the cohomology of the product space of manifolds is the tensor product of the cohomology of the factors, see Bredon [11, VI.3.2], hence we get $H^\bullet(A/H^1; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^{m+1}; \mathbb{Q})$.

From the covering

$$\mathrm{SO}(2) \times A/A \cap H \to \mathrm{SO}(2) \times A/H$$

we get the exact sequence

$$0 \to \pi_1(\mathrm{SO}(2) \times A/A \cap H) \to \pi_1(\frac{\mathrm{SO}(2) \times A}{H}) \to H/A \cap H.$$

Since $\pi_1(\frac{\mathrm{SO}(2)\times A}{H})$ is by assumption torsion-free it follows that $\pi_1(A/A \cap H) = 1$. And as the exact sequence

$$1 \to \pi_1(A/H^1) \to \pi_1(A/A \cap H) \to (A \cap H)/H^1 \to 1$$

shows, also $1 = \pi_1(A/A \cap H) \cong A \cap H/H^1$, i.e. A/H^1 is simply connected and $H^1 = H \cap A$. □

Recall that the simply connected homogeneous rational cohomology spheres A/H where A acts effectively and irreducibly on A/H are given in 2.4.2.

6.4 Cohomology of some homogeneous spaces

In this section we determine the cohomology of certain homogeneous spaces. These spaces are covered by a sphere product $\mathbb{S}^1 \times \mathbb{S}^n$; therefore they have the homotopy of $\mathbb{S}^1 \times \mathbb{S}^n$, and it turns out that they have also the (integral) cohomology of $\mathbb{S}^1 \times \mathbb{S}^n$. But in these examples the number of connected components of the stabilizers may be arbitrarily chosen. This shows that there are no restrictions on the order of the connected component in 6.3.7 (or in 2.5.11).

We need some preparation.

Remark 6.4.1 Consider the cyclic subgroup $(m, n) \cdot \mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$ generated by a non-trivial element (m, n). First assume that m and n are coprime, i.e. that 1 is their greatest common divisor. Then there are $r, s \in \mathbb{Z}$ such that $rm + sn = 1$, and we may write $\mathbb{Z} \times \mathbb{Z}$ as a direct sum

$$\mathbb{Z} \times \mathbb{Z} = (m, n) \cdot \mathbb{Z} \oplus (s, -r) \cdot \mathbb{Z},$$

because of

$$\begin{aligned} (1, 0) &= (m, n) \cdot r + (s, -r) \cdot n \quad \text{and} \\ (0, 1) &= (m, n) \cdot s - (s, -r) \cdot m. \end{aligned}$$

It follows that

$$\frac{\mathbb{Z} \times \mathbb{Z}}{(m, n) \cdot \mathbb{Z}} = \frac{(m, n) \cdot \mathbb{Z} \oplus (s, -r) \cdot \mathbb{Z}}{(m, n) \cdot \mathbb{Z}} \cong \mathbb{Z}.$$

Let now (m,n) be an arbitrary non-trivial element of $\mathbb{Z}\times\mathbb{Z}$. If $d>0$ is the greatest common divisor of m and n, then there are $\widetilde{m}$ and $\widetilde{n}$ that are prime to each other such that $(m,n)=(\widetilde{m}\cdot d,\widetilde{n}\cdot d)$. Again there are r and s with $r\widetilde{m}+s\widetilde{n}=1$, and we get

$$\frac{\mathbb{Z}\times\mathbb{Z}}{(m,n)\cdot\mathbb{Z}}=\frac{(\widetilde{m},\widetilde{n})\cdot\mathbb{Z}\oplus(s,-r)\cdot\mathbb{Z}}{(\widetilde{m}\cdot d,\widetilde{n}\cdot d)\cdot\mathbb{Z}}\cong\mathbb{Z}_d\oplus\mathbb{Z}.$$

Consider a cyclic subgroup Z in a 2-torus $\mathrm{U}(1)\times\mathrm{U}(1)$. Then Z lies in a closed 1-torus $T\subseteq\mathrm{U}(1)\times\mathrm{U}(1)$, and there is a complementary 1-torus S of Z in $\mathrm{U}(1)\times\mathrm{U}(1)$ such that $\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z}\approx\frac{S\cdot T}{Z}\approx\mathbb{S}^1\times\mathbb{S}^1$. Hence, we get that $H^1(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z})\cong\mathbb{Z}\times\mathbb{Z}$.

Lemma 6.4.2 *Let $Z\subseteq\mathrm{U}(1)\times\mathrm{U}(1)$ be a cyclic subgroup of order $\mu\in\mathbb{N}$ with*

$$Z\cap(\mathrm{U}(1)\times 1)=1=Z\cap(1\times\mathrm{U}(1)).$$

Consider the basis in $H^1(\mathrm{U}(1)\times\mathrm{U}(1))$ that is induced by the injection of $\mathrm{U}(1)$ into $\mathrm{U}(1)\times\mathrm{U}(1)$ as first and second factor. Then there is a basis in $H^1(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z})$ such that relative to this basis and the basis in $H^1(\mathrm{U}(1)\times\mathrm{U}(1))$ the map induced by the projection $q:\mathrm{U}(1)\times\mathrm{U}(1)\to\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z}$ in cohomology is described by the matrix $\begin{pmatrix}1&0\\-\nu&\mu\end{pmatrix}\in\mathbb{Z}^{2\times 2}$, where $\nu=0$ and $\mu=1$ or where μ and ν are coprime natural numbers and $0<\nu<\mu$.

Proof If Z is trivial, then $\mu=1$ and we may set $\nu=0$ to get the identity matrix. Now assume $\mu>1$. The projections from $\mathrm{U}(1)\times\mathrm{U}(1)$ to the two factors yield isomorphic images of Z and project generators of Z to generators of the cyclic subgroup of order μ in $\mathrm{U}(1)$. Therefore, there is a generator z of Z which projects to $\xi=\exp(\frac{2\pi i}{\mu})$ in the second factor. Furthermore, z projects to a generator of the cyclic subgroup of order μ in the first factor, i.e. to ξ^ν with a $\nu\in\mathbb{N}$, $1\leq\nu<\mu$, that is coprime to μ. In other words, there is a generator $z=(\xi^\nu,\xi)$ of Z. Then Z lies in the torus $T=\{(t^\nu,t)|\ t\in\mathrm{U}(1)\}$ with complementary torus $S=\{(t,1)|\ t\in\mathrm{U}(1)\}$, so $\mathrm{U}(1)\times\mathrm{U}(1)=S\cdot T\cong S\times T$, and the factorisation $(t_1,t_2)=(t_1t_2^{-\nu},1)(t_2^\nu,t_2)$ yields an isomorphism

$$\varphi:\mathrm{U}(1)\times\mathrm{U}(1)\to S\times T.$$

We denote the injections of $\mathrm{U}(1)$ in $\mathrm{U}(1)\times\mathrm{U}(1)$ as first or second factor by i_1 and i_2 and the projections from $S\times T$ to S and T by pr_S and pr_T, respectively. The isomorphisms $\mathrm{U}(1) \to S$, $t \mapsto (t, 1)$, and $\mathrm{U}(1) \to T$, $t \mapsto (t^\nu, t)$, determine (via pr_S and pr_T) generators of $H^1(S \times T)$.

Then the induced maps

$$\mathrm{U}(1) \xrightarrow{i_1} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{\varphi} S \times T \xrightarrow{\mathrm{pr}_S} S$$

and

$$\mathrm{U}(1) \xrightarrow{i_2} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{\varphi} S \times T \xrightarrow{\mathrm{pr}_S} S$$

have mapping degrees 1 and $-\nu$, respectively. Similarly, we get mapping degrees 0 and 1 for

$$\mathrm{U}(1) \xrightarrow{i_1} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{\varphi} S \times T \xrightarrow{\mathrm{pr}_T} T$$

and

$$\mathrm{U}(1) \xrightarrow{i_2} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{\varphi} S \times T \xrightarrow{\mathrm{pr}_T} T\,.$$

Hence, by the Künneth theorem the map induced by φ between the first cohomology groups

$$\varphi^\bullet : H^1(S \times T) \to H^1(\mathrm{U}(1) \times \mathrm{U}(1))$$

may be described by the matrix $\begin{pmatrix} 1 & 0 \\ -\nu & 1 \end{pmatrix}$ with respect to the chosen generators, confer also Bredon [11, VI.4.14,13].

Next, consider the commutative diagram

$$\begin{tikzcd} \mathrm{U}(1) \times \mathrm{U}(1) \arrow[r, "q"] \arrow[d, "\varphi", "\cong"'] & \frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z} \arrow[d, "\cong"'] \\ S \times T \arrow[r] & S \times \frac{T}{Z} \end{tikzcd}$$

and the induced diagram in cohomology

$$\begin{tikzcd} H^1(\mathrm{U}(1) \times \mathrm{U}(1)) & H^1(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z}) \arrow[l, "q^\bullet"'] \\ H^1(S \times T) \arrow[u, "\varphi^\bullet"', "\cong"] & H^1(S \times \frac{T}{Z}). \arrow[l] \arrow[u, "\cong"] \end{tikzcd}$$

We may chose generators in $H^1(S \times \frac{T}{Z})$ such that the bottom map of the diagram is described by $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$.

If we take the images of these generators under the right vertical map in the last diagram above as basis in $H^1(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z})$, then the map $q^\bullet$ is by the commutativity of the diagram described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\nu & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\nu & \mu \end{pmatrix}.$$

$\square$

With these preparations we are able to calculate the cohomology of some homogeneous spaces.

Proposition 6.4.3 *Let C be a cyclic subgroup in the normaliser* $\mathrm{U}(1)\times S(\mathrm{U}(1)\times\mathrm{U}(n))$ *of* $\mathrm{SU}(n)$ *in* $\mathrm{U}(1)\times\mathrm{SU}(n+1)$. *Assume that* $\mathrm{SU}(n+1)\cap C = \mathrm{SU}(n)\cap C$. *Then there is an isomorphism (of graded groups)*

$$\pi_\bullet(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)}) \cong \pi_\bullet(\mathbb{S}^1\times\mathbb{S}^{2n+1})$$

and a ring isomorphism

$$H^\bullet(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)}) \cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^{2n+1}).$$

Proof As $\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{\mathrm{SU}(n)} \to \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)}$ is a covering with finite cyclic covering group C, the statement for the homotopy groups is clear.

We may assume that $\mathrm{U}(1)\cap C\cdot\mathrm{SU}(n) = 1$, because otherwise we may factor out $\mathrm{U}(1)\cap C\cdot\mathrm{SU}(n)$, and the homeomorphism

$$\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)} \approx \frac{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\cap C\cdot\mathrm{SU}(n)}\times\mathrm{SU}(n+1)}{\frac{C\cdot\mathrm{SU}(n)}{\mathrm{U}(1)\cap C\cdot\mathrm{SU}(n)}}$$

shows that the cohomology does not change. Here $\mathrm{U}(1)$ of course stands for the subgroup $\mathrm{U}(1)\times 1$ of $\mathrm{U}(1)\times\mathrm{SU}(n+1)$.

The complex projective space may be written as

$$\mathbb{C}\mathrm{P}_n = \frac{\mathrm{SU}(n+1)}{\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))} = \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}.$$

Then the tautological bundle $\mathbb{S}^1\to\mathbb{S}^{2n+1}\to\mathbb{C}\mathrm{P}_n$ may also be interpreted as

$$\frac{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{\mathrm{U}(1)\times\mathrm{SU}(n)} \to \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{\mathrm{U}(1)\times\mathrm{SU}(n)} \to \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}.$$

$$\begin{tikzcd}
\mathrm{U}(1) \times \frac{\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{\mathrm{SU}(n)} \arrow[r, "q"] \arrow[d] & \frac{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{C\cdot\mathrm{SU}(n)} \arrow[d] \\
\mathbb{S}^1 \times \mathbb{S}^{2n+1} \arrow[r] \arrow[d] & \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)} \arrow[d] \\
\mathbb{C}\mathrm{P}_n \arrow[r, equal] & \mathbb{C}\mathrm{P}_n .
\end{tikzcd}$$

Figure 6.3: A bundle morphism with $\mathbb{C}\mathrm{P}_n$ as base.

Without the factor U(1) in the nominators of the last bundle this gives the bundle $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}_n$, and there is the induced morphism of bundles of figure 6.3.

Our goal is to calculate the cohomology of the total space of the fibration on the right in figure 6.3. Therefore, we will calculate the differentials in the respective spectral sequence with help of the transgression. The transgression τ of the tautological bundle $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}_n$ is a homomorphism $H^1(\mathrm{U}(1)) \to H^2(\mathbb{C}\mathrm{P}_n)$ which may be viewed as part of the differential in the E_2-term of the respective spectral sequence, $\tau = \mathrm{d}_2^{0,1} : \mathrm{E}_2^{0,1} \to \mathrm{E}_2^{2,0}$. Here, the transgression is indeed an isomorphism. To see this consider for example the spectral sequence of $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}_n$.

We will call τ_l the transgression of the fibration

$$\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}_n$$

on the left in the diagram of figure 6.3, and by τ_r we will mean the transgression of the fibration

$$\frac{\mathrm{U}(1) \times \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}{C \cdot \mathrm{SU}(n)} \to \frac{\mathrm{U}(1) \times \mathrm{SU}(n+1)}{C \cdot \mathrm{SU}(n)} \to \mathbb{C}\mathrm{P}_n$$

on the right.

Choose an isomorphism

$$\rho : \frac{\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}{\mathrm{SU}(n)} \to \mathrm{U}(1).$$

The cyclic group

$$\frac{C}{C \cap \mathrm{SU}(n)} \cong \frac{C \cdot \mathrm{SU}(n)}{\mathrm{SU}(n)} \subseteq \mathrm{U}(1) \times \frac{\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}{\mathrm{SU}(n)}$$

is mapped by $\mathrm{id} \times \rho$ to $\mathrm{U}(1) \times \mathrm{U}(1)$. Denote by Z the image of this cyclic group in $\mathrm{U}(1) \times \mathrm{U}(1)$ and by μ the order of Z. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{U}(1) \times \mathrm{U}(1) \arrow[rr] & & \frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z} \\ \mathrm{U}(1) \times \frac{\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{\mathrm{SU}(n)} \arrow[u, "\mathrm{id}\times\rho", "\cong"'] \arrow[rr, "q"] & & \mathrm{U}(1) \times \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)} \arrow[u, "\cong"'] \end{tikzcd}$$

which is induced by ρ. (The vertical map on the right is the one induced by $\mathrm{id} \times \rho$.) By lemma 6.4.2 there are generators a, b of $H^1(\frac{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{C\cdot\mathrm{SU}(n)})$ such that with respect to these generators and to standard generators in $H^1(\mathrm{U}(1) \times \mathrm{U}(1))$ the map q looks in cohomology like $\begin{pmatrix} 1 & 0 \\ -\nu & \mu \end{pmatrix}$.

Now we choose a generator u of $H^2(\mathbb{C}\mathrm{P}_n)$ such that $H^\bullet(\mathbb{C}\mathrm{P}_n) \cong \mathbb{C}[u]$ and such that we may write the transgression τ_l of the fibration

$$\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}_n$$

as $(0, 1)$ with respect to u and the standard generators of $H^1(\mathrm{U}(1) \times \mathrm{U}(1))$ chosen above.

As transgressions are natural, the transgression τ_r of the fibration

$$\frac{\mathrm{U}(1) \times \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}{C \cdot \mathrm{SU}(n)} \to \frac{\mathrm{U}(1) \times \mathrm{SU}(n+1)}{C \cdot \mathrm{SU}(n)} \to \mathbb{C}\mathrm{P}_n$$

on the right-hand side of figure 6.3 is the composition of $q^\bullet$ and the transgression τ_l of the fibration on the left,

$$\tau_r = \tau_l \circ q^\bullet : H^1(\frac{\mathrm{U}(1) \times \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}{C \cdot \mathrm{SU}(n)}) \to H^2(\mathbb{C}\mathrm{P}_n).$$

Therefore, relative to the generators a, b of $H^1(\frac{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{C\cdot\mathrm{SU}(n)})$ and u of $H^2(\mathbb{C}\mathrm{P}_n)$, we may describe τ_r as

$$(-\nu, \mu) = (0, 1) \begin{pmatrix} 1 & 0 \\ -\nu & \mu \end{pmatrix} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},$$

where μ is the order of Z and ν is coprime to μ. So far we have determined the map induced in cohomology by the map between the fibres in the bundle morphism of figure 6.3.

The description of τ_r above allows us to calculate the rest of the differential in the E_2-term of the spectral sequence of

$$\frac{\mathrm{U}(1)\times\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))}{C\cdot\mathrm{SU}(n)} \to \frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)} \to \mathbb{C}\mathrm{P}_n.$$

Since the fibre $\mathbb{S}^1\times\mathbb{S}^1$ of the fibration has projective cohomology groups we know that

$$\mathrm{E}_2^{p,q} \cong H^p(\mathbb{C}\mathrm{P}_n)\otimes H^q(\mathbb{S}^1\times\mathbb{S}^1)$$

for all p and all q. Therefore, $\mathrm{E}_2^{p,q}$ is trivial if $q\notin\{0,1,2\}$ We have

$$\begin{aligned} \mathrm{E}_2^{0,1} &= (1\otimes a\cdot\mathbb{Z})\oplus(1\otimes b\cdot\mathbb{Z}) \qquad \text{and} \\ \mathrm{E}_2^{2,0} &= u\otimes 1\cdot\mathbb{Z}. \end{aligned}$$

The calculations for $\tau_r = \mathrm{d}_2^{0,1} : \mathrm{E}_2^{0,1} \to \mathrm{E}_2^{2,0}$ have shown that $\mathrm{d}_2(1\otimes a) = -\nu u\otimes 1$ and $\mathrm{d}_2(1\otimes b) = \mu u\otimes 1$. The Leibniz rule implies

$$\begin{aligned} \mathrm{d}_2(u\otimes a) &= \mathrm{d}_2(u\otimes 1\cdot 1\otimes a) = \mathrm{d}_2(u\otimes 1)\cdot 1\otimes a + (-1)^{0+2}u\otimes 1\cdot\mathrm{d}_2(1\otimes a) \\ &= 0 + u\otimes 1\cdot(-\nu)u\otimes 1 = -\nu u^2\otimes 1 \quad \text{and} \\ \mathrm{d}_2(u\otimes b) &= u\otimes 1\cdot\mu u\otimes 1 = \mu u^2\otimes 1, \end{aligned}$$

and we infer that $\mathrm{d}_2 : \mathrm{E}_2^{2,1} \to \mathrm{E}_2^{4,0}$ is surjective (remember that μ and ν are coprime) with kernel $(\mu u\otimes a + \nu u\otimes b)\mathbb{Z}$. The same calculations apply if we replace u by u^k for $1\le k\le n-1$. Of course, $\mathrm{d}_2^{2n,2}$ is trivial.

Recall that $\mathrm{E}_2^{0,2} = (1\otimes ab)\mathbb{Z}$. For $\mathrm{d}_2 : \mathrm{E}_2^{0,2} \to \mathrm{E}_2^{2,1}$ we get

$$\begin{aligned} \mathrm{d}_2(1\otimes ab) &= \mathrm{d}_2(1\otimes a\cdot 1\otimes b) = \mathrm{d}_2(1\otimes a)\cdot 1\otimes b + (-1)^{0+1}1\otimes a\cdot\mathrm{d}_2(1\otimes b) \\ &= -\nu u\otimes 1\cdot 1\otimes b - 1\otimes a\cdot\mu u\otimes 1 = -\nu u\otimes b - \mu u\otimes a, \end{aligned}$$

i.e. the map $\mathrm{d}_2^{0,2}$ is injective with image $(\mu u\otimes a + \nu u\otimes b)\mathbb{Z}$.

The map $\mathrm{d}_2 : \mathrm{E}_2^{2,2} \to \mathrm{E}_2^{4,1}$ is determined by

$$\mathrm{d}_2(u\otimes ab) = (-1)^{2+0}u\otimes 1\cdot\mathrm{d}_2(1\otimes ab) = -\nu u^2\otimes b - \mu u^2\otimes a.$$

The last calculation and the calculations above show also that for all $1\le k\le n-2$ the differential $\mathrm{d}_2^{2k,2}$ maps $u^k\otimes ab\mathbb{Z}$ isomorphically to the kernel $(\mu u^{k+1}\otimes a + \nu u^{k+1}\otimes b)\mathbb{Z}$ of $\mathrm{d}_2^{2k+2,1}$ in $\mathrm{E}_2^{2k+2,1}$. For $k = n-1$ we get

$$\mathrm{d}_2^{2n-2,2}(u^{n-1}\otimes ab) = -\nu u^n\otimes b - \mu u^n\otimes a,$$

and hence $\operatorname{im}\mathrm{d}_2^{2n-2,2} = (\mu u^n\otimes a + \nu u^n\otimes b)\mathbb{Z}$. But as $\mathrm{d}_2^{2n,1}$ is trivial, the kernel of $\mathrm{d}_2^{2n,1}$ is two-dimensional.

The calculations above determine the differential d_2. It follows that the homology algebra E_3 has non-trivial modules only in bidegree $(0,0)$, $(0,1)$, $(2n,1)$ and $(2n,2)$, and these non-trivial modules are all isomorphic to $\mathbb{Z}$. It follows that the differential d_3 is trivial, and so are all the following differentials, i.e. the spectral sequence collapses at the E_3-term and $\mathrm{E}_3 \cong \mathrm{E}_\infty$ as bigraded algebras.

Recall from page 71 that if there is just one non-trivial module in the diagonal $i+j=k$ in E_∞, then the non-trivial module is isomorphic to the kth cohomology group of the total space. Hence, there is an isomorphism of *graded groups*

$$H^\bullet(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)}) \cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^{2n+1}).$$

In order to get the multiplicative structure we first observe that the generator $\mu 1\otimes a+\nu 1\otimes b\in \mathrm{E}_2^{0,1}$ of $\ker \mathrm{d}_2^{0,1}$ projects to a generator of $\mathrm{E}_3^{0,1}$ (and hence to a generator of the group $\mathrm{E}_\infty^{0,1}\cong H^1(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})$).

We will construct an element of $\ker \mathrm{d}_2^{2n,1}$ that projects to a generator of $\mathrm{E}_3^{2n,1}\cong H^{2n+1}(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})$. Since μ and ν are coprime, there are $r,s\in\mathbb{Z}$ such that $r\mu+s\nu=1$. Hence, we may write

$$\begin{aligned}\ker \mathrm{d}_2^{2n+1,1} &= \mathrm{E}_2^{2n+1,1} = (u^n\otimes a)\mathbb{Z} \,\oplus\, (\nu u^n\otimes b)\mathbb{Z}\\ &= (\mu u^n\otimes a+\nu u^n\otimes b)\mathbb{Z}\oplus(su^n\otimes a-ru^n\otimes b)\mathbb{Z},\end{aligned}$$

cf. remark 6.4.1. This shows that $su^n\otimes a-ru^n\otimes b$ projects to a generator of $H^{2n+1}(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})\cong \mathrm{E}_3^{2n,1}=\ker \mathrm{d}_2^{2n+1,1}/\operatorname{im}\mathrm{d}_2^{2n-2,2}\cong\mathbb{Z}$, because we saw already that $\operatorname{im}\mathrm{d}_2^{2n-2,2}=(\mu u^n\otimes a+\nu u^n\otimes b)\mathbb{Z}$.

The product of the generators $\mu 1\otimes a+\nu 1\otimes b$ and $su^n\otimes a-ru^n\otimes b$ of $\mathrm{E}_3^{0,1}$ and $\mathrm{E}_3^{2n,1}$, respectively, is

$$\begin{aligned}(\mu 1\otimes a+\nu 1\otimes b)\cdot(su^n\otimes a-ru^n\otimes b) &= -s\nu u^n\otimes ab-r\mu u^n\otimes ab\\ &= -u^n\otimes ab\end{aligned}$$

because $a^2=0=b^2$ and $ab=-ba$. (Note that the product in the last equations is the cup-product in $H^\bullet(\mathbb{S}^1\times\mathbb{S}^1)$.)

Hence, the product of the mentioned generators projects to a generator of $\mathrm{E}_3^{2n,2}$, and then also to a generator of $\mathrm{E}_\infty^{2n,2}$ and finally to a generator of $H^{2n+2}(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})$.

But then the multiplicative structure of $H^\bullet(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})$ is the one of $H^\bullet(\mathbb{S}^1\times\mathbb{S}^{2n+1})$, i.e.

$$H^\bullet(\frac{\mathrm{U}(1)\times\mathrm{SU}(n+1)}{C\cdot\mathrm{SU}(n)})\cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^{2n+1}).$$

□

Similarly, we get the following proposition.

Proposition 6.4.4 *Let C be a cyclic subgroup in the normaliser* $\mathrm{U}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ *of* $\mathrm{Sp}(n)$ *in* $\mathrm{U}(1)\times\mathrm{Sp}(n+1)$*. Assume that* $\mathrm{Sp}(n+1)\cap C=\mathrm{Sp}(n)\cap C$*. Then there is an isomorphism of graded groups*

$$\pi_\bullet(\frac{\mathrm{U}(1)\times\mathrm{Sp}(n+1)}{C\cdot\mathrm{Sp}(n)})\cong\pi_\bullet(\mathbb{S}^1\times\mathbb{S}^{4n+3})$$

and a ring isomorphism

$$H^\bullet(\frac{\mathrm{U}(1)\times\mathrm{Sp}(n+1)}{C\cdot\mathrm{Sp}(n)})\cong H^\bullet(\mathbb{S}^1\times\mathbb{S}^{4n+3}).$$

Proof By covering theory the homotopy groups are as stated.

There is a fibration $\mathbb{S}^1\to\mathbb{S}^{4n+3}\to\mathbb{C}\mathrm{P}_{2n+1}$ from the transitive action of $\mathrm{Sp}(n+1)$ on $\mathbb{S}^{4n+3}$. The transgression of this fibration is an isomorphism. As $\mathrm{Sp}(1)$ is a group of rank 1, all 1-tori in $\mathrm{Sp}(1)$ are conjugate and we may assume that

$$Z=\frac{C}{\mathrm{Sp}(n)\cap C}\subseteq\mathrm{U}(1)\oplus\frac{\mathrm{U}(1)\times\mathrm{Sp}(n))}{\mathrm{Sp}(n)}\subseteq\mathrm{U}(1)\oplus\frac{\mathrm{Sp}(1)\times\mathrm{Sp}(n))}{\mathrm{Sp}(n)},$$

because the image of the cyclic group $Z\subset\mathrm{U}(1)\times\mathrm{Sp}(1)$ under the projection to $\mathrm{Sp}(1)$ is contained in a 1-torus $\mathrm{U}(1)\subset\mathrm{Sp}(1)$. The result follows then as in the proof of the last proposition. □

Remark 6.4.5 1. The homogeneous spaces in the last two propositions are examples of spaces as in 2.5.11 or 6.3.7; they have the same cohomology as a product $\mathbb{S}^1\times\mathbb{S}^{m+1}$ even with the coefficients in the *integers* (and not in the rationals), and they have exactly the same homotopy groups as $\mathbb{S}^1\times\mathbb{S}^{m+1}$ (and not only up to torsion).

Let μ be the order of the cyclic group $Z=\frac{C}{\mathrm{SU}(n)\cap C}$ or $Z=\frac{C}{\mathrm{Sp}(n)\cap C}$, respectively, in the last two propositions. Then every integer occurs as a possible value of μ. Hence, the homogeneous spaces in the last two propositions show that one cannot restrict the number of the connected components of the stabilizer in 2.5.11 or in 6.3.7.

2. Another interpretation of the last two propositions is that we have determined the integral cohomology of the orbit space $(\mathbb{S}^1 \times \mathbb{S}^k)/Z$ of the linear action of Z on $\mathbb{S}^1 \times \mathbb{S}^k$ with certain odd k.

3. The projection $\mathbb{S}^1 \times \mathbb{S}^k \to \mathbb{S}^k$ onto the second factor gives an action of Z on $\mathbb{S}^k$ induced by
$$x \mapsto \exp(\frac{2\pi i}{\mu}) \cdot x.$$
Therefore, the orbit space $(\mathbb{S}^1 \times \mathbb{S}^k)/Z$ may be viewed as a $\mathbb{S}^1$-fibre bundle over the lens space $L^k(\mu)$.

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