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Wave Extraction in Numerical Relativity

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Gewidmet meinen Eltern, Gertrud und Peter, für all ihre Liebe und Unterstützung.

To my parents Gertrud and Peter, for all their love, encouragement and support.

Wave Extraction in Numerical Relativity

Abstract This work focuses on a fundamental problem in modern numerical relativity: Extracting gravitational waves in a coordinate and gauge independent way to nourish a unique and physically meaningful expression.

We adopt a new procedure to extract the physically relevant quantities from the numerically evolved space-time. We introduce a general canonical form for the Weyl scalars in terms of fundamental space-time invariants, and demonstrate how this approach supersedes the explicit definition of a particular null tetrad.

As a second objective, we further characterize a particular sub-class of tetrads in the Newman-Penrose formalism: the *transverse frames*. We establish a new connection between the two major frames for wave extraction: namely the *Gram-Schmidt frame*, and the *quasi-Kinnersley frame*. Finally, we study how the expressions for the Weyl scalars depend on the tetrad we choose, in a space-time containing distorted black holes. We apply our newly developed method and demonstrate the advantage of our approach, compared with methods commonly used in numerical relativity.

Abriss Diese Arbeit konzentriert sich auf eine fundamentale Problematik der numerischen Relativitätstheorie: Die Extraktion von Gravitationswellen in einer eich- und koordinateninvarianten Formulierung, um ein physikalisch interpretierbares Objekt zu erhalten.

Es wird eine neue Methodik entwickelt, um die physikalisch relevanten Größen aus einer numerisch erzeugten Raumzeit zu extrahieren. Wir präsentieren eine allgemeingültige kanonische Formulierung der Weyl Skalare im Newman-Penrose Formalismus

als eine Funktion von fundamentalen Raumzeit-Invarianten. Dadurch zeigt sich, dass mit Hilfe dieser Methodik die explizite Konstruktion eines Vierbeins vollständig redundant ist.

Als weiteren Schwerpunkt charakterisieren wir innerhalb des Newman-Penrose Formalismus eine spezielle Untergruppe von Tetraden, die *transversen Frames*. Es wird eine bisher unbekannte Verbindung zwischen den primär genutzen Vierbeinen für die Extraktion der Wellenform abgeleitet, dem *Gram-Schmidt* Vierbein und dem *quasi-Kinnersley* Vierbein. Abschliessend studieren wir die Abhängigkeit der Gravitationswellen eines gestörten Schwarzen Loches vom verwendeten Vierbein. Wir berechnen die Form der Gravitationswellen in dieser Raumzeit und demonstrieren inwieweit unsere neue Methodik robustere und exaktere Ergebnisse liefert, als die gewöhnlich verwendeten Ansätze zur Extraktion des Signals.

Full list of publications by the author

This thesis is mainly based upon the following publications:

- Nerozzi, Andrea; **Elbracht, Oliver** - Using curvature invariants for wave extraction in numerical relativity, *accepted by Physical Review D (2009)*.

- **Elbracht, Oliver**; Nerozzi, Andrea - Using curvature invariants for wave extraction in numerical relativity. II. Wave extraction in distorted black hole spacetimes, *submitted to Physical Review D*.

- **Elbracht, Oliver**; Nerozzi, Andrea - A new approach to wave extraction in numerical relativity, *submitted to Journal of Physics: Conference Series* (refereed).

Other publications by the author:

- **Elbracht, Oliver**; Nerozzi, Andrea; Matzner, Richard - Wave extraction in numerical evolutions of distorted black holes, *oclc/66137068 (2005)*.

- Burkart, Thomas; **Elbracht, Oliver**; Spanier, Felix - Simulation results of our newly developed PIC codes, *AN, Vol.328, Issue 7* (unrefereed).

- Rödig, Constanze; Burkart, Thomas; **Elbracht, Oliver**; Spanier, Felix - Multiwavelength periodicity study of Markarian 501, *Astronomy and Astrophysics, Volume 501, Issue 3, 2009, pp.925-932*.

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1. Introduction

In the beginning the Universe was created.
This has made a lot of people very angry
and has been widely regarded as a bad move.
Douglas Adams

The existence of gravitational radiation has become accepted as a hallmark prediction of Einstein's theory of General Relativity, but the problem of modeling astrophysical events such as binary black hole coalescence and extracting the gravitational wave signal is a very difficult task. One of the main difficulties lies in the structure of Einstein's equations: They are highly non-linear coupled partial differential equations, for which relatively few analytical solutions are known. The question of how to solve the initial data problem is still a very active subject of research (see e.g. [2]).

Nonetheless, the past decade has seen the birth of the field of gravitational wave phenomenology. Several ground-based detectors (GEO, TAMA, LIGO, VIRGO), using laser interferometry, have been constructed and operate around the world. The arrays have taken real data and are now gradually approaching their design sensitivities (see e.g. [3, 4, 5]) . There has also been numerous work done on LISA, a space-based antenna that will be able to achieve far better sensitivity than any ground-based detector, however, for the most part in a different frequency domain [6, 7]. The first indirect evidence for gravitational waves was reported by Russell Hulse and Joe Taylor in 1974 [8] . Their observations showed that the orbit of the pulsar PSR $1913+16$ is decaying, matching with extraordinary precision the prediction for such a decay, due to the loss of orbital energy and angular momentum by gravitational waves. Since the universe is most likely filled with a variety of signals from countless sources, such as super-massive black holes at the center of galaxies, neutron stars, massive stars undergoing supernova, and perhaps exotic matter sources we have not conceived of to date. The first direct detection of gravitational waves will open a new window to the universe and mark the beginning of an exciting new field: gravitational wave astronomy.

With the recent breakthroughs of long term numerical evolution of spiral infall and collisions of multiple black hole systems has come a demand for accurate waveform templates and algebraic expressions that describe the gravitational radiation. While the signal we will observe from gravitational waves will give rise to information that cannot be obtained by other means, it will be extremely weak with a bad signal-to-noise ratio at the same time. Thus it will be a major task to extract the radiation in form of gravitational waves from such astronomical important sources in a well-defined and highly accurate way and therefore being able to supply the community with templates that can distinguish the different sources in the universe from one another.

The original way to extract radiation from a numerically evolved space-time has been *black hole perturbation theory*, originally developed as a metric perturbation theory. Regge and Wheeler derived a single master equation for the metric perturbations of Schwarzschild black holes [9], the so-called *odd-parity* solution, and Zerilli [10] developed a similar formula for the *even-parity* solution. Afterwards, based on the null-tetrad formalism developed by Newman and Penrose [11, 12, 13], a master equation for the curvature perturbation was first developed by Bardeen and Press [14] for a Schwarzschild black hole without source ($T^{\mu\nu} = 0$), and by Teukolsky [15] for a Kerr black hole with source ($T^{\mu\nu} \neq 0$).

All these perturbative schemes, describing specific limits of source behavior, have been available for many years, but they all assume a particular knowledge of a specific background metric which, in a typical simulation of strong gravitational fields, is not known *a priori*. So, what we need is a non-perturbative scheme to extract the radiation signals solely from the physical metric. So far, the progress has been rather slow and the results have not been impressive.

A very promising way to perform wave extraction in numerical relativity is the usage of the Newman-Penrose formalism. In this formalism five complex scalars are defined, the Weyl scalars. These are computed by contracting the Weyl tensor on a set of four null vectors. With the right choice of the four null vectors the Weyl scalars obtain a precise physical meaning. In practice the right choice is dictated by linear theory, which states that in choosing a special frame (the Kinnersley tetrad [16]) for the background metric we end up with values for the scalars that can be associated with radiative degrees of freedom and furthermore obey the peeling-off theorem [17, 18].

Recent developments have shed some light on the theoretical background, introducing the so called *transverse frames* and the *quasi-Kinnersley frames* [19, 20, 21, 22]. These efforts helped to understand how it is possible to pick up the Kinnersley tetrad in a general space-time by introducing the notion of the quasi-Kinnersley tetrad, as a member of the quasi-Kinnersley frame. This particular set of null vectors will converge to the right tetrad chosen by linearized theory as soon as the space-time converges to an unperturbed Petrov type D space-time. This procedure has been applied successfully in e.g. [23]. Still, this approach is rather lengthy and complicated to apply in a numerical simulation. Even worse, it suffers from a crucial ambiguity in that it does not fix the so-called *spin-boost* parameter in a rigorous way. Therefore, what is still missing is a unique and simple approach to find an algebraic expression for the radiation quantities in the right tetrad.

In this work we develop a new approach for wave extraction and give a more rigorous physical explanation for *transverse tetrads*, by fully characterizing the spin coefficients and Weyl scalars related to this specific choice of tetrad, namely the one which is a member of the same equivalence class of transverse Newman-Penrose tetrads as the Kinnersley tetrad. This is a key step towards a full understanding of the properties of transverse tetrads and their potentiality for wave extraction. This method gives a rigorous expression for the spin-boost parameter, which was unknown before. By fixing the remaining degree of freedom of gravitational waves in the Newman-Penrose formalism we derive an expression for the Weyl scalars as functions of two fundamental curvature invariants, the first and second Kretschmann invariant, respectively.

As a second objective, we characterize the transverse frames in the Newman-Penrose formalism by establishing a new connection between the two major frames for wave extraction: the Gram-Schmidt frame and the quasi-Kinnersley frame. This connection facilitates to perform well-posed operations to both frames without any particular limitations.

Finally, we consider initial data of distorted black hole space-times constructed as a Cauchy problem, where we apply our newly developed method. We extract the waveform on the initial slice and compare the main approaches for wave extraction. The results are encouraging, clearly demonstrating the advantage of our approach compared with commonly used methods in numerical relativity.

This thesis is organized as follows:
In the remainder of this chapter we summarize the conventions and notations used in this thesis. In chapter 2 we give an overview of initial data in general relativity and how simulations of a four-dimensional space-time are commonly realized in numerical relativity, by employing the *ADM formalism*. In chapter 3 we give an introduction in the theory of gravitational waves within the linearized theory of general relativity. We describe the effect of space-time radiation, how it is measured and any information that is deducible. We close the chapter by describing what kind of modes we expect from a perturbed black hole, the *quasi-normal modes*. Chapter 4 details the concepts of wave extraction by introducing the Newman-Penrose formalism and the notion of the quasi-Kinnersley frame. In chapter 5 we present a new methodology for wave extraction making use of fundamental space-time invariants which emerge in a natural way from the theory of general relativity.
Finally, chapter 6 applies these concepts to a distorted black hole space-time, exploring the concepts of wave extraction in the Newman-Penrose formalism and clearly demonstrating the advantage of the method given in chapter 5.
In chapter 7 we draw some conclusions and give an outlook for further developments.

1.1. Notation and Units

Here we summarize the conventions and notations used in this work.
A space-like signature $(-,+,+,+)$ will be used, with Greek indices taken to run from 0 to 3, and Latin indices from 1 to 3. We adopt *relativistic units*, in which $G = c = 1$, thus mass, length and time have the same units in this system. The conversion is as follows: 1 second = $299,792,458$ meters $\simeq$ 3x10^8 meters, and thus 1 solar mass is the same as

$$1\ M_\odot = 1476.63 \text{ meters} \simeq 1.5 \text{ kilometers} = 4.92549 \text{ x } 10^{-6} \text{ seconds} \simeq 5\ \mu s.$$

When dealing with black holes it is also useful to normalize these units, not to the solar mass $M_\odot$, but to the mass of a black hole $M_\bullet$, commonly taken to be approximately twenty times the solar mass. Therefore, a unit of 1 $M_\bullet$ will be a length of about 30 km or a time of 100 μs.

We define the notion of a *tetrad* as a member of an equivalence class of Newman-Penrose tetrads, a so-called *frame*, differing only by a class III rotation (a spin-boost Lorentz transformation).

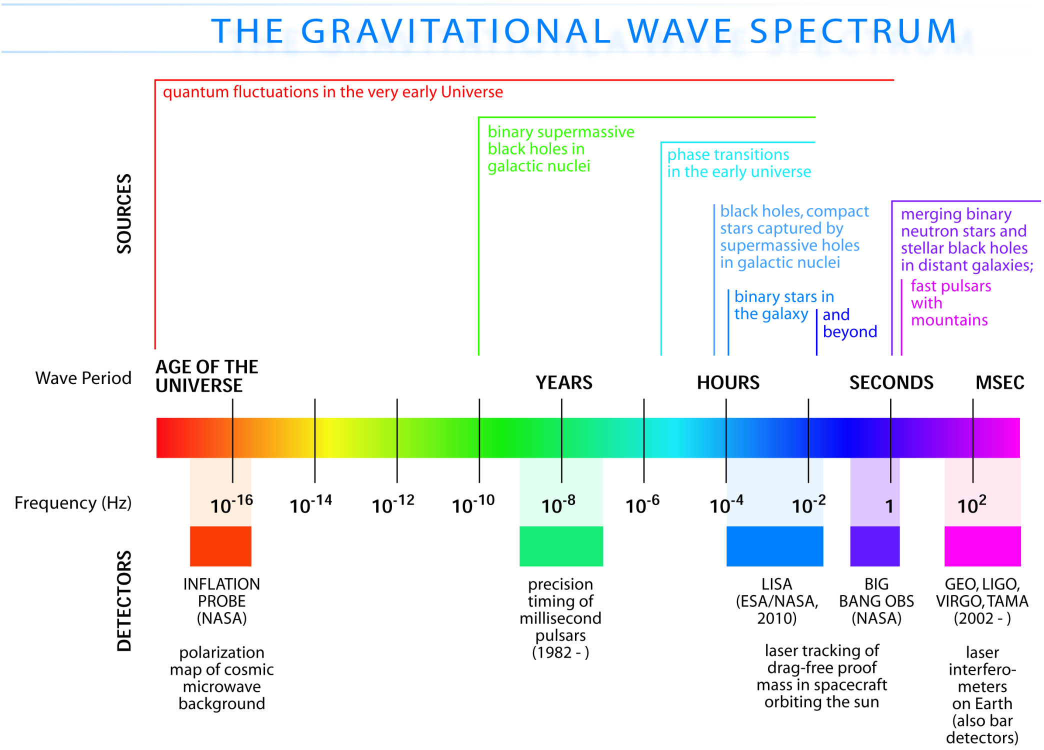

Figure 1.1.: This figure shows the sources that appear at various frequencies in the gravitational wave spectrum, together with the experiments that have either been carried out, or are planned with the intention of detecting them. (Image: Beyond Einstein roadmap)

2. The 3+1 Split and Initial Data

If I had only known, I would have been a locksmith.
Albert Einstein

In this chapter we introduce the fundamental structure of general relativity as well as mathematical formulations of the Einstein equations commonly used in numerical relativity. Furthermore, we give an introduction to Cauchy initial data for numerical evolution. The entire subject is covered comprehensively in literature, notably in review articles by Cook and York [2, 24]

2.1. Initial Value Problem

The fundamental structure in general relativity is a 4-dimensional space-time $(\mathfrak{M}, g_{\mu\nu})$ where $\mathfrak{M}$ is a four-dimensional space-time manifold with a metric $g_{\mu\nu}$ satisfying the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

with the energy-momentum tensor $T_{\mu\nu}$.

In general relativity physical events are described in a global, unified space-time manifold which is highly counterintuitive to how observers view the reality of localized phenomena. The observer naturally views events in a sequential, temporal manner to which we attribute the notion of causality.

To recover this causal description of the observed universe one can introduce an initial value formulation to re-examine the space-time manifold. There are two main features we wish to capture in such a formulation[1]: firstly, small changes in the initial data within a bounded region of the space-time S should lead to predictably bounded changes in the evolved solution: and secondly, changes in initial data within a space-time should not produce changes outside the causal future, as defined by the null vectors from the boundary of S.

Thus the question of interest from a perspective of a physicist is, can we reformulate the Einstein's equations as a Cauchy problem; that is, if we define a three-dimensional hypersurface within $\mathfrak{M}$ with an induced three-metric γ_{ij}, and a three momentum π_{ij} related to the rate of change of the three-metric, can we derive a subset of the Einstein equations which evolve γ_{ij} and π_{ij} on hypersurfaces in the causal future?

Hawking and Ellis demonstrated that the Cauchy problem for general relativity is in fact well-posed, and the causal development of Cauchy surfaces is unique and stable [25]. However, the Cauchy development has limitations; only globally hyperbolic space-times can be constructed by using a Cauchy ansatz. In particular, that means that nothing hidden behind a Cauchy horizon can be found with this approach. However, Penrose's strong cosmic censorship conjecture [26] suggests that all generic space-times are globally hyperbolic anyway.

With these results we can decompose $\mathfrak{M}$ into $\mathbf{R} \times \Sigma_t$ where $\Sigma_t : t \in \mathbf{R}$ are a set of space-like hypersurfaces that are level surfaces of a scalar function t. We call this collection of hypersurfaces $\{\Sigma_t\}$ a *foliation* of the space-time manifold.

Taking a close look at the reformulated Einstein equations we recognize that there are ten independent equations and ten independent components of the 4-dimensional metric $g_{\mu\nu}$. Writing the equations in their differential form we see that these ten equations are linear in the second derivatives and quadratic in the first derivatives of the metric. In fact, we find that the ten equations separate into a set of four constraint equations, and six evolution equations.

This decomposition raises a question, which has been still not fully answered, namely how to choose initial data which satisfy the constraint equations [2]. Analytically, once the constraints are satisfied on an arbitrary initial slice, the Bianchi

[1]Outlined by Wald in his textbook

identities

$$\nabla_\nu G_{\mu\nu} = 0, \tag{2.2}$$

ensures that they are satisfied on all successive hypersurfaces. Unfortunately this is not strictly true numerically, due to natural limitations in the accuracy of numerical codes.

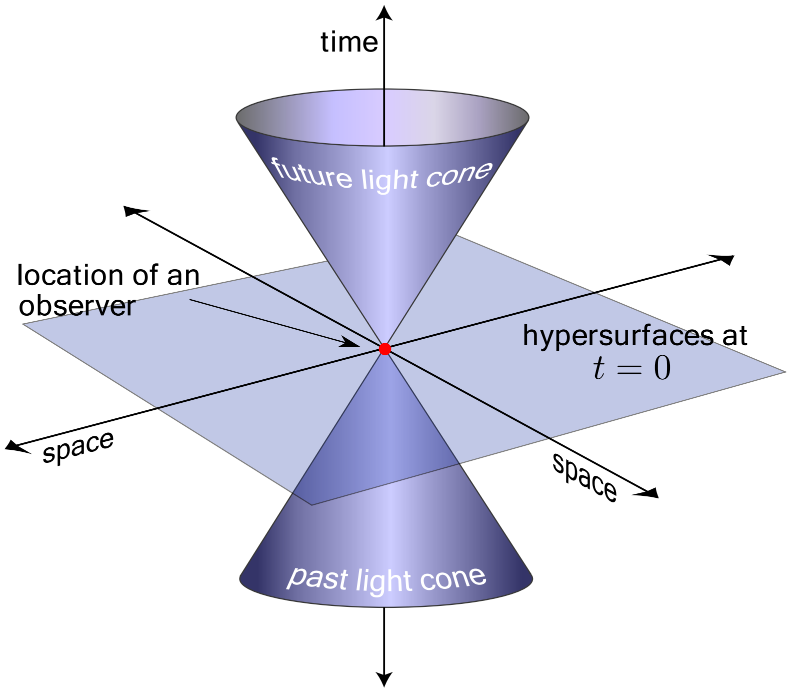

Figure 2.1.: The figure illustrates the notion of light cones in general relativity. The future light cone is the boundary of the causal future of a point in the hypersurface, and the past light cone is the boundary of its causal past.

2.2. The 3+1 Decomposition - Separating Space from Time

As outlined in the previous section the Einstein equations written in their usual form are manifestly covariant, time and space only appear as equal partners, i.e. as space-time. This is not only counterintuitive to how humans view the reality but it is also not a well suited form for numerical simulations, where we need to adopt some quantity as an evolution parameter. Therefore we will recast Einstein's equations into a more convenient form for such a task.

Among the formulations proposed for this purpose, by far the most frequently applied is the canonical "3+1" decomposition proposed by Arnowitt, Deser and Misner (ADM) in 1962 [27]. Alternatives such as null, 2+2 or (2+1)+1 (cf. e.g. [28, 29]) splits have also been studied, but in far less detail than the physically intuitive 3+1 decomposition. As pointed out in section 2.1 a suitable way to decompose space-time is the employment of an initial value problem. In the *ADM formalism* the space-time is disjoint into a 1-parameter family of 3-dimensional space-like hypersurfaces and constraints satisfying initial data are provided on one hypersurface in the form of the spatial 3-metric and its time derivative.

In fact, there have been many modifications to the original ADM formulation, but the main ideas of ADM still form the basis of standard approaches to numerical relativity. Our derivation of the evolution equations closely follows the textbook *Gravitation* by Misner, Thorne and Wheeler. An alternative derivation can be found in Appendix A.

2.3. The ADM Formalism

The field variable in General Relativity is the 4- dimensional space-time metric $g_{\mu\nu}$ defined on a Manifold $\mathfrak{M}$. Appropriate initial data can be determined via the well-known Hilbert variational principle. In general relativity we start from the Einstein-Hilbert action

$$S_{EH} = \int \mathrm{d}^4x\sqrt{-g}R, \tag{2.3}$$

where g is the determinant of the 4-dimensional metric and R is the Ricci scalar of an otherwise empty space. By varying the lagrangian density $\pounds = \sqrt{-g}R$ in Eq. (2.3) with respect to the space-time metric $g_{\mu\nu}$ we derive the covariant vacuum Einstein equations with the dynamics encoded in the set of differential equations. We separate the spatial degrees of freedom from the time-like degrees of freedom and introduce the ADM quantities,

$$\gamma_{ij} = g_{ij}, \tag{2.4a}$$
$$\alpha = (-g^{tt})^{\frac{1}{2}}, \tag{2.4b}$$
$$\beta_i = g_{0i}, \tag{2.4c}$$
$$\pi^{ij} = \sqrt{|g|}\left(\Gamma^0{}_{kl} - \gamma_{kl}\Gamma^0{}_{mn}\gamma^{mn}\right)\gamma^{ik}\gamma^{jl}. \tag{2.4d}$$

Arnowitt, Deser and Misner called these quantities the *spatial three-metric* γ_{ij}, the *lapse* function α, the *shift* vector β^i and the *conjugate momenta* π^{ij}, respectively. In the ADM formalism these quantities acquire a clear physical meaning, as illustrated in Fig. (2.2). In fact, we may choose a time-like vector t^μ to coincide with the normal vector n^μ to the hypersurfaces, but that might not be strictly true, therefore in general

$$t^\mu = \alpha n^\mu + \beta^\mu. \tag{2.5}$$

Here the lapse α encodes the proper distances of the slices as measured by an observer moving perpendicular to the slice, whereas the shift β^i lies in the surface Σ and describes the displacement away from the hypersurface. The three-metric γ_{ij} can be defined as the projection into Σ. We can invert this system of equations (2.4) and arrive at the following construction of the 4-metric out of the 3-metric and the lapse and shift functions

$$g_{00} = \beta_k\beta^k - \alpha^2, \tag{2.6a}$$
$$g_{0j} = \beta_j, \tag{2.6b}$$
$$g_{i0} = \beta_i, \tag{2.6c}$$
$$g_{ij} = \gamma_{ij}. \tag{2.6d}$$

The contravariant space-time metric reads

$$g^{00} = -1/\alpha^2, \tag{2.7a}$$
$$g^{0j} = \beta^j/\alpha^2, \tag{2.7b}$$
$$g^{i0} = \beta^i/\alpha^2, \tag{2.7c}$$
$$g^{ij} = \gamma^{ij} - \beta^i\beta^j/\alpha^2. \tag{2.7d}$$

If we substitute (2.4a) - (2.4d) into (2.3) we derive the Einstein-Hilbert action

$$S_{EH} = -\int \mathrm{d}x \left[\gamma_{ij}\partial_t \pi^{ij} + \alpha H + \beta_i P^i + 2\partial_i\left(\pi^{ij}\beta_j - \frac{1}{2}\pi\beta^i + \nabla^i\alpha\sqrt{\gamma}\right)\right], \tag{2.8}$$

where

$$H = -\sqrt{\gamma}\left(R + \frac{1}{\gamma}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right), \tag{2.9a}$$
$$P^i = -2\nabla_j \pi^{ij}, \tag{2.9b}$$

are the constraint equations, namely the hamiltonian and momentum constraint, respectively. The last term in Eq. (2.8) is a spatial divergence which does not contribute to the classical equations of motion. The dynamics are encoded in the resulting evolution equations

$$\partial_t\gamma_{ij} = 2\alpha g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}\gamma_{ij}\pi\right) + \nabla_j\beta_i + \nabla_i\beta_j, \tag{2.10a}$$
$$\begin{aligned}\partial_t\pi^{ij} = {} & -\alpha\sqrt{\gamma}\left(R^{ij} - \frac{1}{2}\gamma^{ij}R\right) + \frac{1}{2}\alpha\gamma^{-\frac{1}{2}}\gamma^{ij}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\right) \\ & -2\alpha\gamma^{-\frac{1}{2}}\left(\pi^{in}\pi_n{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{\gamma}\left(\nabla^i\nabla^i\alpha - \gamma^{ij}\Delta\alpha\right) \\ & +\nabla_n\left(\pi^{ij}\beta^n\right) - \pi^{nj}\nabla_n\beta^i - \pi^{ni}\nabla_n\beta^j,\end{aligned} \tag{2.10b}$$

where ∇ is the covariant derivative associated with γ_{ij} and R_{ij} is the Ricci tensor associated with γ_{ij}.

York performed a modification to the original ADM equations, introducing the *ex-*

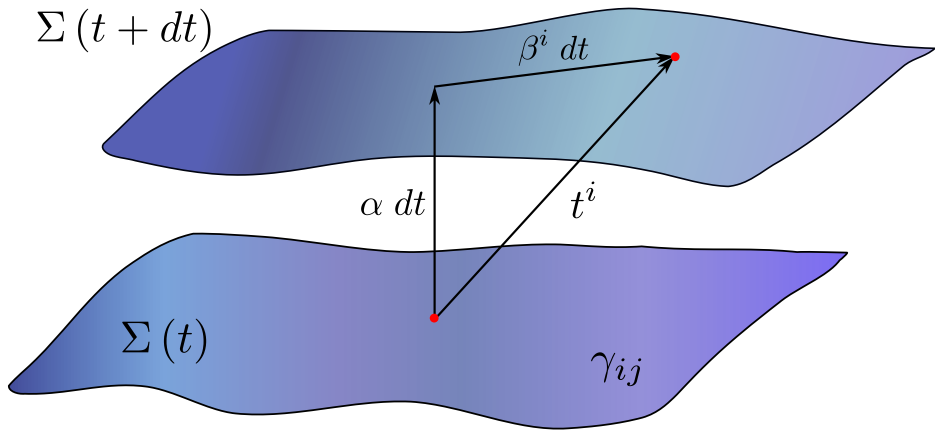

Figure 2.2.: The foliation of space-time in the ADM $3+1$ split showing the lapse function α and shift vector β^i for the displacement of a point embedded in successive hypersurfaces Σ labeled by a number t with the 3-dimensional metric γ_{ij}.

trinsic curvature or *second fundamental form* K_{ij}:

$$K_{ij} = -\gamma^{-1/2}\left(\pi_{ij} - \gamma_{ij}\pi\right), \tag{2.11}$$

where Eq. (2.11) shows the relation between π_{ij} and K_{ij}.

Rather than evolving the canonically conjugate momenta π_{ij}, York chose the extrinsic curvature K_{ij} of the three dimensional slices as an evolution variable. In pure geometrical terms, the extrinsic curvature quantifies roughly the "bend" of a hypersurface as measured from a higher dimensional space in which the hypersurface is embedded. In mathematical terms, we define the extrinsic curvature by applying the projection tensor on the covariant derivative of the normal vector $\nabla_\nu n^\mu$:

$$K_{ij} \equiv -\frac{1}{2}\pounds_n \gamma_{ij}, \tag{2.12}$$

where $\pounds_n$ denotes the Lie derivative along the n^μ direction. By combining the Gauss-

Codazzi relations, which define the extrinsic curvature on a sub-manifold, with the Einstein equations one can derive the vacuum evolution equations for K_{ij} and γ_{ij}, respectively (cf. Appendix A):

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i && \text{(2.13a)}\\
\partial_t K_{ij} &= \alpha\{R_{ij} - 2K_{il}K^l_j + KK_{ij}\} - \nabla_i \nabla_j \alpha + \pounds_\beta K_{ij}, && \text{(2.13b)}
\end{aligned}$$

where K is the trace of K_{ij}.

The Hamiltonian and momentum constraints in Eq. (2.9) turn out to be

$$\begin{aligned}
R + K^2 - K_{ij}K^{ij} &= 0, && \text{(2.14a)}\\
\nabla_j \left(K^{ij} - \gamma^{ij} K\right) &= 0, && \text{(2.14b)}
\end{aligned}$$

giving the well-known form of the constraint equations in numerical relativity.

2.4. BSSN - An Alternative

The ADM evolution equations introduced in the previous section are in fact highly non-unique. As long as we satisfy the constraints in Eqs. (2.14), there is no restriction at all that prohibits adding arbitrary multiples of the constraints to the equations. The physical solutions will not change, but the mathematical properties may alter seriously. As an example, we already demonstrated how the equations may be reformulated, as done by York. It has been shown that the reformulation by York behaved better mathematically concerning its constraints violating behavior.
But by the late 1990s', the community finally realized that even York's formulation was a rather unsuitable scheme for numerical black hole evolutions, due to its weak hyperbolicity [30]. An alternative to ADM was first suggested by Shibata and Nakamura in 1995 [31] and made popular by a subsequent paper by Baumgarte and Shapiro in 1999 [32]. Baumgarte and Shapiro investigated the stability properties of the Shibata-Nakamura formulation and showed the remarkable advantage compared with the standard ADM formulation. The scheme has since become known as the *BSSN formal-*

ism, and is nowadays almost exclusively used in numerical simulations. With these fundamental ingredients a major breakthrough concerning long-term binary black hole simulations was achieved in 2001 by two independent groups [33, 34]. Research in improving schemes continues and today there is no consensus, as to which formulation is the best for numerical purposes. Here we want to give a short overview of the fundamental equations of BSSN.

The first step to reformulate the ADM equations is the *conformal split* where we redefine the determinant of the physical three-metric

$$\phi = \frac{1}{12} \log g, \tag{2.15}$$

which enables us to rewrite the space-time metric by introducing the *conformal metric* $\tilde{g}_{ij}$:

$$\tilde{g}_{ij} = \mathrm{e}^{-4\phi} g_{ij}, \tag{2.16}$$

where now $det(\tilde{g}_{ij}) = 1$. In geometrical terms, the decomposition splits the geometry into "transverse" and "longitudinal" degrees of freedom (encapsulated by $\tilde{g}_{ij}$ and ϕ, respectively.) We extend this idea to the extrinsic curvature yielding

$$\tilde{K}_{ij} = \mathrm{e}^{-4\phi} K_{ij} \tag{2.17a}$$

$$K = g^{ij} K_{ij} = \tilde{g}^{ij} \tilde{K}_{ij}. \tag{2.17b}$$

As it is obvious from Eq. (2.17b) scalar quantities like the trace of the extrinsic curvature are not affected by the transformation.

The next step is to separate the trace-free part of the extrinsic curvature in the following manner

$$\tilde{A}_{ij} = \mathrm{e}^{-4\phi} \left(K_{ij} - \frac{1}{3} g_{ij} K \right), \tag{2.18}$$

where K_{ij} represents the "longitudinal" part and the trace-free tensor $\tilde{A}_{ij}$ the "transverse" part, respectively.

These two steps are meaningless unless the evolution equations are affected in a non-trivial manner. A crucial step is to introduce some variables representing spatial derivatives:

$$\tilde{\Gamma}^i = \tilde{g}^{ab}\tilde{\Gamma}^i_{ab}. \tag{2.19a}$$

The evolution equations for these new BSSN fields can then be worked out:

$$\begin{aligned}
\partial_t \tilde{g}_{ij} &= -2\alpha \tilde{A}_{ij} + \pounds_\beta g_{ij}, && (2.20a)\\
\partial_t \phi &= \pounds_\beta \phi - \frac{1}{6}\alpha K, && (2.20b)\\
\partial_t \tilde{A}_{ij} &= \pounds_\beta \tilde{A}_{ij} + \mathrm{e}^{-4\phi}\left[-\nabla_i \nabla_j \alpha + \alpha R_{ij}\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right), && (2.20c)\\
\partial_t K &= \pounds_\beta K - \nabla^i \nabla_j \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right), && (2.20d)\\
\partial_t \tilde{\Gamma}^i &= \tilde{g}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k + \beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j \\
&\quad -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j\phi - \frac{2}{3}\tilde{g}^{ij}\partial_j K\right). && (2.20e)
\end{aligned}$$

This decomposition has greatly extended evolution times in black hole simulations, in particular together with the *puncture approach* [35].

In fact, there are many other possible adjustments that can be made to the BSSN system that affect its stability properties. For example, the conformal connection has been modified in various formulation and the original expression in Eq. (2.20e) has been abandoned. A detailed derivation of possible formulations goes beyond the scope of this work, for which we like to recommend an article by Yoneda & Shinkai ([30], and subsequent work) about the stability properties of slicing formulations.

3. Gravitational Waves

If the universe is expanding, why can't I find a parking space?
Woody Allen

General relativity is consistent with special relativity in many respects, and in particular with the principle that nothing can travel faster than light in vacuum. As a consequence, space-time perturbation must propagate in a certain fashion. In general relativity, as demonstrated by Einstein, these perturbation propagate at exactly the speed of light and are denoted as *gravitational waves*.

The most natural starting point for any discussion of gravitational waves is the linearized theory, where one can keep only linear terms in Einstein's field equations.

3.1. The Linearized Theory of Gravity

This "linearized gravity" is an important theory in its own right and an adequate approximation to general relativity in the so-called "weak field", where the space-time metric $g_{\mu\nu}$ only slightly deviates from a flat metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathscr{O}\left(|h_{\mu\nu}|^2\right), \quad |h_{\mu\nu}| \ll 1. \tag{3.1}$$

Here $\eta_{\mu\nu}$ is defined to be the Minkowski metric $(-1,1,1,1)$ and $|h_{\mu\nu}|$ is the magnitude of a small perturbation to the flat space-time. In fact, it is precisely this "linearized theory of general relativity" that one obtains in classical field theory for particles of zero rest mass and spin two in flat space.

Fortunately, it is rather easy to find an example of a system where we deal with a small perturbation to an otherwise flat space-time. The conditions above are satisfied just by looking in our Solar system. Measuring the deviation away from flat space, for instance on the surface of the Sun, we yield

$$|h_{\mu\nu}| \approx |h_{00}| \lesssim \frac{M_{sun}}{R_{sun}} \approx 10^{-6}. \tag{3.2}$$

We might come to the misleading conclusion expecting a gravitational wave with an enormous amplitude, namely $|h| \approx 10^{-6}$. In fact, gravitational waves are only defined far away from the sources in an otherwise empty space-time. In the case of the earth-sun system the minimum distance to find waves is roughly $R \approx 1ly$, and since $|h_{\mu\nu}| \propto R^{-1}$ typical amplitudes will be $|h_{\mu\nu}| \approx 10^{-26}$.

By taking a closer look at the approximation we will see that the perturbation $h_{\mu\nu}$ encapsulates not only gravitational waves, but additional, non-radiative degrees of freedom as well.

As mentioned earlier we shall restrict our attention to linear terms in the perturbation and thus we shall neglect terms of second order or higher in $h_{\mu\nu}$. Moreover, we will impose a suitable outer boundary condition assuming the space-time is asymptotically flat (asymptotically "Minkowskian")

$$\lim_{r \to \infty} h_{\mu\nu} = 0, \tag{3.3}$$

where r denotes a radial parameter.

To derive linearized theory from general relativity we start by defining the contravariant form of the metric perturbation $h_{\mu\nu}$ via

$$h^{\mu\nu} \equiv \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}, \tag{3.4}$$

where it is true that

$$\left(\eta_{\mu\rho} + h_{\mu\rho}\right)\left(\eta^{\rho\nu} - h^{\rho\nu}\right) = \delta^{\nu}_{\mu}, \tag{3.5}$$

from what we derive the contravariant form of the full metric

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.6}$$

The connection coefficients (Christoffel symbols), when linearized in $h_{\mu\nu}$, read

$$\begin{aligned}
\Gamma^{\mu}{}_{\alpha\beta} &= g^{\mu\sigma}\Gamma_{\sigma\alpha\beta} = \frac{1}{2}g^{\mu\sigma}\left(g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} - g_{\mu\alpha,\sigma}\right) \\
&= \frac{1}{2}\left(\eta^{\mu\sigma} - h^{\mu\sigma}\right)\left((\eta_{\sigma\alpha} + h_{\sigma\alpha})_{,\beta} + (\eta_{\beta\sigma} + h_{\beta\sigma})_{,\alpha} - (\eta_{\mu\alpha} + h_{\mu\alpha})_{,\sigma}\right) \\
&\simeq \frac{1}{2}\eta^{\mu\sigma}\left(h_{\sigma\alpha,\beta} + h_{\beta\sigma,\alpha} - h_{\mu\alpha,\sigma}\right) + \mathscr{O}\left(\left[h_{\mu\nu}\right]^2\right).
\end{aligned} \tag{3.7}$$

The operation of raising and lowering the indices is performed by using $\eta_{\mu\rho}$ and $\eta^{\mu\rho}$, not the full metric, which is a consequence of linearization. Once the Christoffel symbols are computed we can calculate the Ricci tensor and Ricci scalar to linear order, yielding

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}{}_{\mu\sigma,\nu} \\
&= \frac{1}{2}\left(h_{\mu}^{\sigma}{}_{,\nu\sigma} + h_{\nu}^{\sigma}{}_{,\mu\sigma} - h_{\mu\nu,\sigma}^{\sigma} - h_{\mu\nu}\right),
\end{aligned} \tag{3.8}$$

and

$$R = \left(h^{\mu\sigma}{}_{,\mu\sigma} - h_{\sigma}^{\sigma}\right). \tag{3.9}$$

Finally, the linearized Einstein tensor turns out to be

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\
&= \frac{1}{2}\left(h_{\mu\sigma,\nu}{}^{\sigma} + h_{\nu\sigma,\mu}{}^{\sigma} - h_{\mu\nu,\sigma}{}^{\sigma} - h_{\mu\nu} - \eta_{\mu\nu}\left(h_{\sigma\rho,}{}^{\sigma\rho} - h_{,\sigma}{}^{\sigma}\right)\right) = 0
\end{aligned} \tag{3.10}$$

Note that the same result can be achieved by utilizing the variational principle as in chapter 2.

The expression in Eq. (3.10) is a bit unwieldy and does not seem yet to suggest any sort of wave-like behavior we would normally expected for a "wave". Somewhat remarkably, this behavior can be significantly unveiled by changing the notation: rather than working with the metric perturbation $h_{\mu\nu}$, we use the *trace-free* metric perturbation defined as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{3.11}$$

We can perform such a transformation without loss of generality since Eq. (3.11) merely presents a gauge transformation. With this new notation the field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ take the form

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi T_{\mu\nu}. \tag{3.12}$$

The first term on the left hand side of Eq. (3.12) is the usual d'Alembertian (or wave) operator

$$\Box\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu,\alpha}{}^{\alpha}, \tag{3.13}$$

whereas the other terms are merely pure gauge. Due to this fact we exploit the gauge freedom inherent to general relativity to recast (3.13) in a more accessible form. Without loss of generality we can impose a gauge condition in such a way to eliminate the terms that spoil the wave-like nature, in particular by choosing

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{3.14}$$

Making use of the gauge, which is mostly wrongly called the *Lorentz* gauge[1] , the linearized field equations then become

$$\Box\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0, \tag{3.15}$$

clearly showing the wave-like nature of the gravitational field if matter is absent (i.e. if $T_{\mu\nu} = 0$).

[1]The Lorenz gauge condition is named after Ludvig Lorenz and is frequently misspelled because of confusion with Hendrik Lorentz, after whom Lorentz invariance is named.

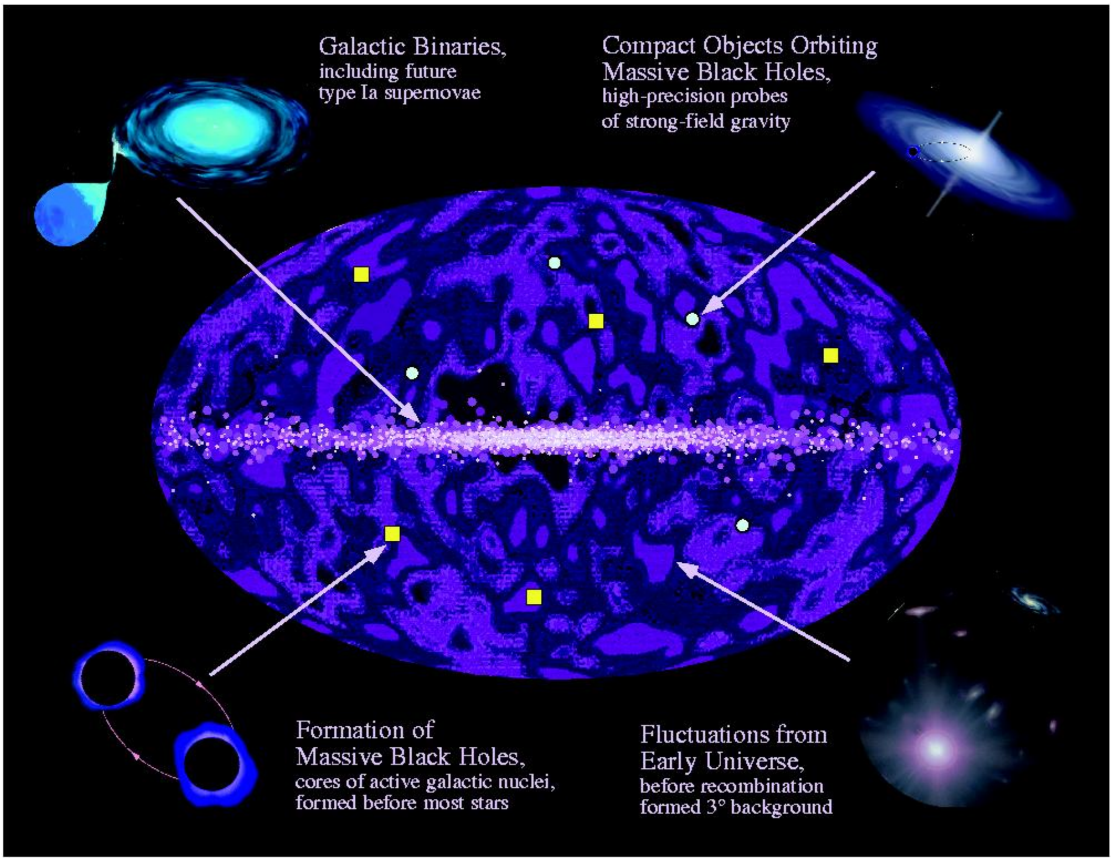

Figure 3.1.: This is an image of the sky as viewed by gravitational waves. The Milky Way galaxy forms the band in the middle of the image. LISA will see thousands of binary star systems in our galaxy, and will be able to determine the direction and distance to each binary, as well as the periods of the orbits and the masses of the stars. (Beyond Einstein Roadmap)

3.2. A Wave Solution and the Transverse-Traceless Gauge

The field equations of linearized theory bear a close analogy to the equations of electrodynamics, consequently we can infer much about linearized theory. Tracking this analogy, it is not surprising that the simplest solution to the linearized wave equation (3.15) is that of a monochromatic plane wave:

$$\bar{h}_{\mu\nu} = \Re\left[A_{\mu\nu} e^{i\kappa_\sigma x^\sigma}\right], \tag{3.16}$$

where $\Re[...]$ denotes the real part, $A_{\mu\nu}$ is the *amplitude tensor* and the wave-vector κ is light-like, $\kappa^\mu \kappa_\mu = 0$. The Lorenz gauge condition implies that the amplitude and the wave-vector are orthogonal $A_{\mu\nu}\kappa^\nu = 0$. Evidently, in such a solution, the plane wave in Eq. (3.16) travels in the spatial direction $\vec{k} = (\kappa_x, \kappa_y, \kappa_z)/\kappa^0$ with frequency $\omega = \kappa^0 = \sqrt{(\kappa^i \kappa_i)}$.

As mentioned before, linearized gravity can be described within classical field theory by a massless spin-2 field that propagates with the speed of light. We know from field theory that such a field has only two independent degrees of freedom ("helicities" in quantum theory, and "polarizations" in a classical description). On the other hand, one might come to the conclusion that the symmetric tensor $A_{\mu\nu}$ of this plane wave appears to have $16 - 6 = 10$ free components. But as we will demonstrate, there are in fact really just two dynamical degrees of freedom in linearized relativity. The "excess" is due to the arbitrariness tied up in the gauge freedom; by choosing a particular gauge, namely the *TT gauge*, one gets rid of the remaining unwanted degrees of freedom and one is only left with the two dynamical degrees. One can impose the following conditions:

(I) **Lorenz gauge conditions:** Since we impose the Lorenz gauge condition

$$A_{\mu\nu}\kappa^\nu = 0, \tag{3.17}$$

4 components of the amplitude tensor can be specified.

(II) **Global Lorentz Frame:** Just like in special relativity one can select a four-velocity

u - the same through all space-time and define a global Lorentz frame where one can impose the conditions:

$$A_{\mu\nu} u^{\nu} = 0. \tag{3.18}$$

These are only three constraints on $A_{\mu\nu}$ not four, because one of them,

$$\kappa^{\mu} A_{\mu\nu} u^{\nu} = 0, \tag{3.19}$$

is already fulfilled by the Lorenz gauge condition.

(III) **Diffeomorphism Condition:** We can impose an infinitesimal gauge transformation in such a way to set

$$A_{\mu}{}^{\mu} = 0. \tag{3.20}$$

We can translate these conditions in Eqs. (3.17 , 3.18 , 3.20) to constraints for the perturbation tensor h_{ij} by considering a reference Lorentz frame where $u^0 = 1$, $u^i = 0$ (globally at rest), and where κ^{μ} does not appear directly:

(I) $h_{ij,j} = 0$, i.e., the spatial components are divergence free,

(II) $h_{\mu 0} = 0$, i.e., only the spatial components h_{ij} are non-zero,

(III) $h_{ii} = 0$, i.e., the spatial components are trace-free.

Together these conditions define the so-called *Transverse Traceless gauge (TT)*.

Even if there is no need in general relativity to prefer one gauge over another, it is extremely convenient to choose the TT-gauge, since it fixes all the local gauge freedom, therefore eliminating unphysical degrees of freedom. Thus, the metric perturbation $h^{TT}_{\mu\nu}$ contains only physical, non-gauge information about the radiation.

To be able to interpret the effects of the metric perturbation $h^{TT}_{\mu\nu}$, we calculate the Riemann tensor in the transverse-traceless gauge, which encodes the curvature of the underlying space-time. It turns out that the only non-zero components of the Riemann tensor are

$$R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0}, \tag{3.21}$$

and the explicit expressions of the components of the linearized Riemann tensor read

$$R_{j0k0} = -\frac{1}{2} h^{TT}_{jk,00}. \tag{3.22}$$

These important relations between the metric perturbation and the components of the Riemann tensor in linearized general relativity facilitate to associate a traveling gravitational wave with a local oscillation of the space-time!

3.2.1. Interaction of Gravitational Waves with Test-Particles

With the results from the foregoing sections we are now able to calculate the effect of a gravitational wave on a freely falling particle following a geodesic in space-time. First, we will demonstrate how an unsuitable coordinate choice can lead to incorrect results, and therefore indicate how important it is to rely on coordinate independent quantities such as the Weyl scalars.

The motion of a particle is given by the usual geodesic equation without external forces

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0, \tag{3.23}$$

where τ is the proper time of the particle. We can rewrite the equation combining the time-like with the spatial part of the 4-vector x^μ yielding an equation for the *coordinate acceleration*:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\left(\Gamma^i{}_{tt} + 2\Gamma^i{}_{tj} v^j + \Gamma^i{}_{jk} v^j v^k\right) + v^i \left(\Gamma^t{}_{tt} + 2\Gamma^t{}_{tj} v^j \Gamma^t{}_{jk} v^j v^k\right). \tag{3.24}$$

Let us now restrict our attention to linearized theory written in TT-gauge and further assume the velocity of the test particle is rather slow ($v \ll 1$). As a valid approximation we can neglect the velocity dependent terms in Eq. (3.24), yielding the simplified equation:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\Gamma^i{}_{tt}, \tag{3.25}$$

where we now compute the Christoffel symbol $\Gamma^i{}_{tt}$ in the TT-gauge to lowest order yielding the surprising result

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\Gamma^i{}_{tt} = \frac{1}{2}\left(2\partial_t h^{TT}_{jt} - \partial_j h^{TT}_{tt}\right) = 0, \tag{3.26}$$

since $h^{TT}_{\mu t} = 0$ (global Lorentz frame condition). A naive interpretation of the result would be that the test particle is not influenced by a passing gravitational wave! This is certainly wrong, but is a clear example how important a careful coordinate choice in general relativity can be. Even more importantly is to focus upon coordinate invariant quantities like the Weyl scalars. We will introduce these scalar quantities in chapter 4. To return to our example, we will now show that in fact traveling gravitational waves produce oscillations in the separation between neighboring objects. As a gravitational wave passes, it perturbs the geodesic motion of the two particles and contributes to the geodesic deviation equation. To examine the action of the wave on the separation of freely falling test particles we start by introducing a locally flat coordinate system x^a, attached to the world line of a particle A. The line element takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \delta_{\hat{i}\hat{j}}\,\mathrm{d}x^{\hat{i}}\,\mathrm{d}x^{\hat{j}} + O(|x^{\hat{j}}|^2)\,\mathrm{d}x^{\hat{\alpha}}\,\mathrm{d}x^{\hat{\beta}}, \tag{3.27}$$

where the first and second term on the right-hand side correspond to Minkowski-flat space and the the last term on the right-hand site encodes the deviation from the geodesic motion.

We start by introducing the geodesic-deviation equation

$$u^\gamma u^\beta n^\alpha{}_{;\beta\gamma} = -R^\alpha{}_{\beta\gamma\delta} u^\beta u^\delta n^\gamma, \tag{3.28}$$

where $\mathbf{n}$ is the *separation four-vector* between two geodesic trajectories with tangent vector $\mathbf{u}$. Additionally, we define the separation vector as $n^{\hat{j}} \equiv x^{\hat{j}}_B - x^{\hat{j}}_A$, reaching from particle A to particle B. With this definition the geodesic-deviation equation can be expressed as

$$\frac{\mathrm{d}^2 n^{\hat{j}}}{\mathrm{d}\tau^2} = -R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} n^{\hat{k}}, \tag{3.29}$$

and with setting $x_A^{\hat{j}} = 0$ the geodesic-deviation equation simplifies to

$$\frac{\mathrm{d}^2 x_B^{\hat{j}}}{d\tau^2} = -R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} x_B^{\hat{k}}. \tag{3.30}$$

Since we want to carry out the results in the TT-gauge we use the definition of the Riemann tensor in Eq. (3.22) yielding

$$\frac{\mathrm{d}^2 x_B^{\hat{j}}}{d\tau^2} = \frac{1}{2}\frac{\partial^2 h^{TT}_{\hat{j}\hat{k}}}{\partial t^2} x_B^{\hat{k}}, \tag{3.31}$$

with the solution

$$x_B^{\hat{j}}(t) = x_B^{\hat{k}}(0)\left[\delta_{\hat{i}\hat{j}} + \frac{1}{2} h^{TT}_{\hat{j}\hat{k}}(t)\right]. \tag{3.32}$$

In contrast to the solution in Eq. (3.26) the result above has a straightforward and meaningful interpretation; particle B is seen oscillating with an amplitude proportional to the time-dependent metric perturbation $h^{TT}_{\hat{j}\hat{k}}(t)$.

3.2.2. Polarization of a Plane Wave

As discussed in the foregoing sections gravitational waves are transverse in linearized theory and the two remaining degrees of freedom can be associated with two different polarizations. To construct the possible polarizations of gravitational wave we start by considering a plane wave propagating with the speed of light along the positive x-axis. Thus, for the particular example the perturbation metric tensor in TT-gauge is defined as

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h^{TT}_{yy} & h^{TT}_{yz} \\ 0 & 0 & h^{TT}_{zy} & h^{TT}_{zz} \end{pmatrix}, \tag{3.33}$$

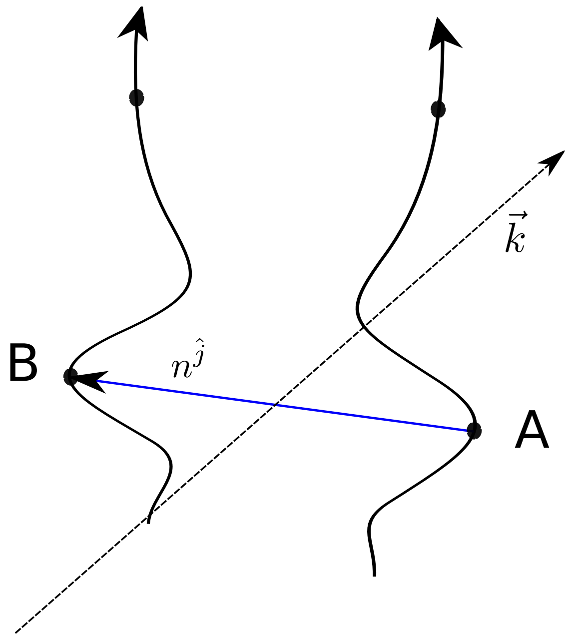

Figure 3.2.: The figure illustrates how the arrival of a gravitational wave propagating along the direction $\vec{k}$ perturbs the geodesic motion of two particles A and B.

with the only non-vanishing components

$$h_{yy}^{TT} = -h_{zz}^{TT} = \Re\left[A_+ e^{-i\omega(t-x)}\right] \tag{3.34a}$$

$$h_{yz}^{TT} = h_{zy}^{TT} = \Re\left[A_\times e^{-i\omega(t-x)}\right] \tag{3.34b}$$

where A_+ and $A_\times$ represent the amplitudes of two independent modes of polarization. As we already know from classical electrodynamics, we can recast such a planar wave into two linearly polarized plane waves or, by superposing the linear polarizations, into two circularly polarized ones. We call these linear polarizations "+" ("plus") and "×" ("cross") -polarizations. The unit linear-polarization tensors are called $\mathbf{e}_+$ and $\mathbf{e}_\times$, respectively, and may be written as

$$\mathbf{e}_+ \equiv \vec{e}_z \otimes \vec{e}_z - \vec{e}_y \otimes \vec{e}_y, \tag{3.35a}$$

$$\mathbf{e}_\times \equiv \vec{e}_z \otimes \vec{e}_y + \vec{e}_y \otimes \vec{e}_z. \tag{3.35b}$$

The deformation of a ring of test particles is shown in Figure (3.3). Note that the two linear polarized modes are simply rotated by $\pi/4$. In a similar manner, we can define two tensors describing circular polarizations $\mathbf{e}_R$ and $\mathbf{e}_L$ (clockwise and counterclockwise). A ring of test particles hit by a circular polarized wave gets deformed and rotates around either clockwise or counterclockwise:

$$\mathbf{e}_L \equiv \frac{1}{\sqrt{2}}\left(\mathbf{e}_+ + i\mathbf{e}_\times\right), \quad \mathbf{e}_R \equiv \frac{1}{\sqrt{2}}\left(\mathbf{e}_+ - i\mathbf{e}_\times\right). \tag{3.36}$$

The deformations associated with these two modes of polarization are also shown in Figure (3.3).

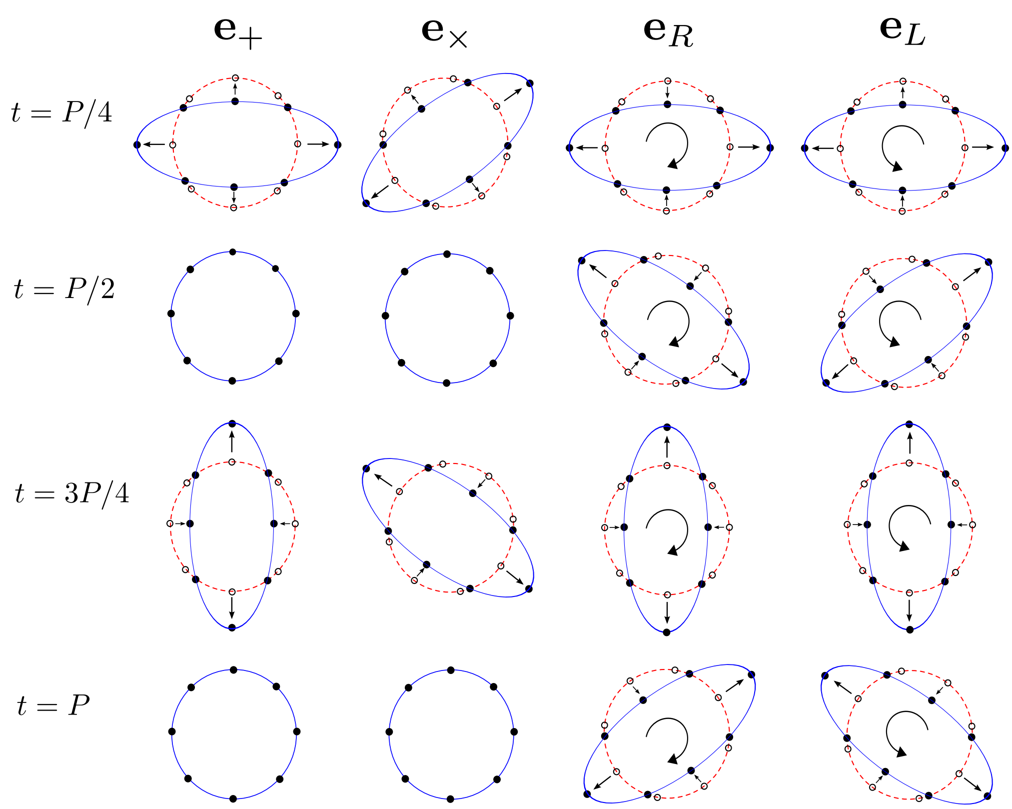

Figure 3.3.: The polarizations of a gravitational wave are illustrated by displaying their effect on a ring of particles arrayed in a plane perpendicular to the direction of the wave. The figure shows the distortions the wave produces if it carries plus/cross polarization or circular polarization, respectively.

3.3. Interaction of Gravitational Waves with Detectors

To detect a gravitational wave there are two basic and very different methods available. One is by measuring the energy deposited by the wave in a resonant-mass detector and is based on the pioneering work by Joseph Weber [36]. The other principle is by measuring the change in time it takes light to travel between two distinct locations. Here we want to concentrate on the beam detectors[2].

The measurement technique of beam detectors is based on an interferometric measurement with a Michelson interferometer operated with highly stabilized laser light. To have a reasonable detection rate of astronomical sources these detectors must be able to measure changes in its arm-length that are smaller than 1 part in 10^{-21} (cf. e.g. [37]). Currently four earth-based laser-interferometric detectors are taking real data. These are TAMA [38], GEO600 [39], LIGO [40], and VIRGO [41].

TAMA is a Japanese detector, with an arm-length of 300m, assembled near Tama, Tokyo. It was the first large scale laser-interferometric gravitational wave detector to have taken scientific data in September 1999. The current sensitivity is $10^{-21}/\sqrt{Hz}$ at 1 kHz. With such a sensitivity it is possible to detect gravitational waves from coalescing neutron-star binaries in our galaxy.

GEO600 is a British-German detector with 600m long arms constructed in Germany close to Hannover. The light is folded once in both arms increasing the light path to 1200m in each arm. The current sensitivity is $2 \times 10^{-22}/\sqrt{Hz}$ between 400 and 500 Hz.

LIGO is situated in the USA and consists of three long-baseline interferometers on two sites, one (4km/2km arm-length) at the Hanford Reservation near Washington and the other (4km arm-length) is situated at Baton Rouge, Louisiana. The current sensitivity is $2 \times 10^{-23}/\sqrt{Hz}$ between approximately 100 and 200 Hz. LIGO now moves into its next phase of progress, Enhanced LIGO. This consists of a set of upgrades and hardware improvements designed to extend the astrophysical reach.

VIRGO is a French-Italian detector with 3 km long arms situated close to Pisa in Italy. The design sensitivity is $3 \times 10^{-21}/\sqrt{Hz}$ at 1 Hz and $3 \times 10^{-23}/\sqrt{Hz}$ at 1 kHz.

[2]For an exhaustive overview over different detectors and related technical issues we would like to refer to a textbook by Ciufolini et al. [37]

Below about 1 Hz *gravity gradient noises* (i.e. tidal forces) are stronger than any gravitational wave from astrophysical objects we can expect in this frequency regime to detect on earth. This is one main reason why scientists proposed the LISA mission in the early nineties [42].

LISA (Laser Interferometer Space Antenna) will be a triangular array of spacecraft, with arm-lengths of 5×10^6 km. The three arms can be combined to form two independent interferometers. LISA will be sensitive in a range from 0.3 mHz to about 1 Hz and will be able, among other things, to detect supermassive binary black hole mergers almost anywhere in the universe. In the low-frequency window of LISA most sources will be observable during their merger for at least a few months.

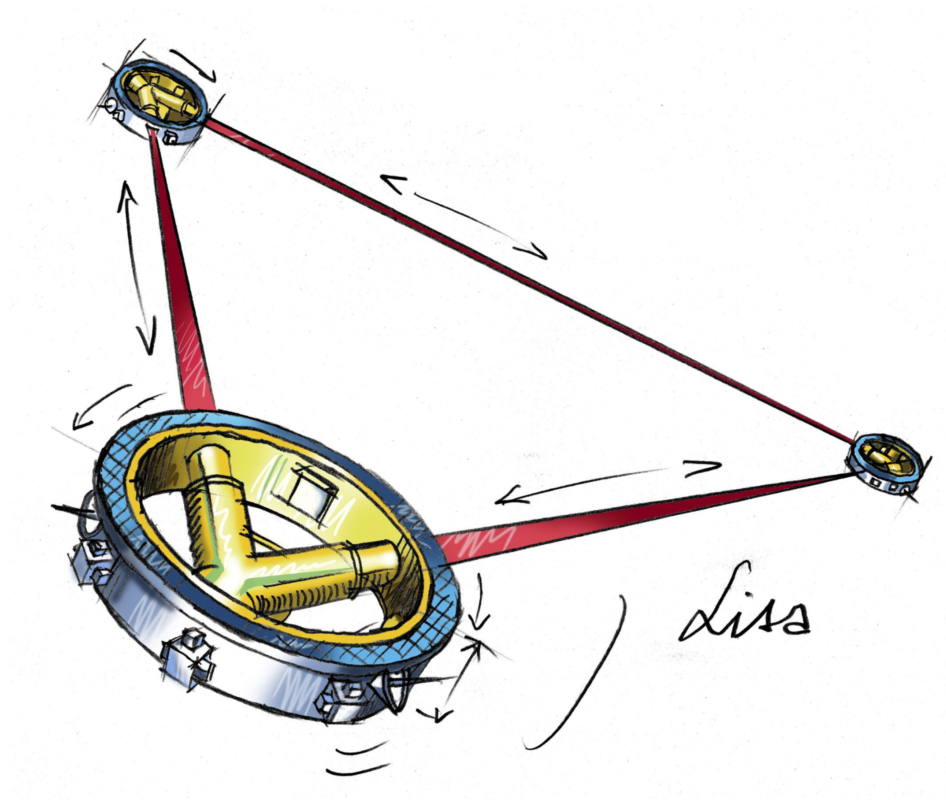

Figure 3.4.: This is an artist's impression showing the basic setup of the LISA spacecraft (Credit: ESA-C. Vijoux).

These facilities are not competing with each other but in contrast are forming a network of detectors. Apart from the fact that only a simultaneous detection in at least two detectors can be trusted, only a network of detectors is able to conduct full information of a gravitational wave. The information consists of five quantities; the amplitude of the wave, the phase between the two polarizations, and the position of the source, expressible in two angles. To derive these parameters at least three detectors need to measure a gravitational wave simultaneously.

To describe the way in which a interferometric detector works, suppose one arm of a beam detector, like GEO600, lies along the z-axis and the wave, for simplicity, is propagating down the x-axis with a "plus" polarization. Assume further that the two neighboring particles are located at $x = x_0 = 0$, and are separated on the z-axis by a distance L_D. The proper distance L between the two neighbored particles is then given by

$$L = \int_0^{L_D} \mathrm{d}x \sqrt{g_{zz}} = \int_0^{L_D} \mathrm{d}x \sqrt{1 + h_{zz}^{TT}(t, x_0)} \approx L_D \left[1 + \frac{1}{2} h_{zz}^{TT}(t, x_0)\right]. \tag{3.37}$$

The fractional length change $\delta L/L$ can be measured via interferometric instruments and is given by

$$\frac{\delta L}{L} \approx \frac{1}{2} h_{zz}^{TT}(t, x_0). \tag{3.38}$$

Even if this is a simple example it clearly shows the fundamental way a interferometric detector works. An obvious advantage of beam detectors is that the effect induced by a gravitational wave can be made larger simply by increasing the arm length as directly seen from Eq. (3.38). For example, assume a detector like LIGO with an arm-length L of approximately $L = 4$ km measures a gravitational wave with a strain amplitude of 10^{-21} and the directional dependences as described above. The measured fractional change will be as small as

$$\delta L \approx 2 \times 10^{-18} m, \tag{3.39}$$

which is less than $1/1000$ the diameter of a proton and, unfortunately, corresponds to the largest effects we can expect from astrophysical sources.

Figure 3.5.: An aerial view of the gravitational wave detector GEO600. In the bottom left corner the central building for the laser and the vacuum tanks can be seen. The tubes, 600m in length, run in covered trenches at the edge of the field upwards and to the right. Buildings for the mirrors are situated at the end of each tube (Credit: AEI Hannover/Deutsche Luftbild Hamburg).

3.4. The Energy of Gravitational Radiation

We now understand how gravitational waves emerge from the theory of general relativity and what kind of polarization a wave may have. But besides measuring the amplitude and phase of the polarizations, we can estimate the energy flux associated with a gravitational wave which, in general, may be extracted by a detector. Unfortunately, the energy is rather ill-defined in linearized theory and additionally the stress-energy cannot be localized inside a certain region of the wave package. To derive an expression for the energy flux it is necessary to assume being far from the emitting object, i.e. to reside in an otherwise flat space-time. We start by examining the form of the stress-energy tensor in the TT-gauge, the *Issacson tensor*

$$T_{\mu\nu} = \frac{1}{32\pi} \left\langle \partial_\mu h_{ij}^{TT} \, \partial_\nu h_{ij}^{TT} \right\rangle, \tag{3.40}$$

where $\langle ... \rangle$ denotes an average over the metric perturbations. The usual definition of the energy flux by solid angle is

$$\frac{\partial^2 E}{\partial t \partial \Omega} = \lim_{r\to\infty} r^2 T^r_{\ t}. \tag{3.41}$$

Combining Eq. (3.40) and Eq. (3.41) we yield a possible estimation of the energy flux of a gravitational wave in the TT-gauge

$$\frac{\partial^2 E}{\partial t \partial \Omega} = \lim_{r\to\infty} \frac{r^2}{16\pi} \left[\left(\frac{\partial h_{\hat{\theta}\hat{\theta}}^{TT}}{\partial t} \right)^2 + \left(\frac{\partial h_{\hat{\theta}\hat{\phi}}^{TT}}{\partial t} \right)^2 \right]. \tag{3.42}$$

3.5. Gravitational Waves from Perturbed Black Holes

One of the most interesting astrophysical sources of gravitational waves are black holes in the centers of galaxies. These are supermassive objects with up to 10^8 solar masses [43, 44, 45]. Such supermassive black holes are now believed to be common in centers of active galactic nuclei (AGN), and there is compelling evidence for at least one black hole of around three million solar masses in the center our own galaxy [46, 47, 48, 49].

Perhaps the most absorbing source involving massive black holes is their merger during a galactic merger process. Such an event from anywhere in the universe "must" be visible to LISA with very high signal-to-noise ratios. This will be a fundamentally important objective because if unseen by LISA, it would cause us to re-evaluate the very existence of gravitational waves.

The only remnant of such a merger allowed by general relativity is a more massive black hole with a perturbed event horizon. Gravitational waves from perturbed black holes are distinctive and reduce to a simple wave equation, which has been studied extensively [9, 10, 50, 51, 52, 53, 54]. They will carry a unique fingerprint which would lead to the direct identification of their existence.

3.5.1. Perturbation Theory and Quasi-Normal Modes

We will briefly review fundamental perturbations that characterize black holes without explicit derivation[3]. We have to restrict ourself to non-rotating black holes due to the fact that for the Kerr solution the analysis is highly complicated (beside being partly still unknown). Some of the main results date back in the 70's with first studies by Regge, Wheeler and Zerilli [9, 10]. In fact, a variety of perturbation schemes have been developed, but we want to focus our attention on the most important approaches. We will introduce a novel approach in chapter 4 which is based on the Newman-Penrose null-tetrad formalism, in which the tetrad components of the curvature tensor are the fundamental variables.

The results obtained in the middle of the nineteenth century raised considerable surprise and doubts at first. The idea that black holes oscillate and possess some

[3]For a good review we recommend an article by K. Kokkotas and B. Schmidt [55]

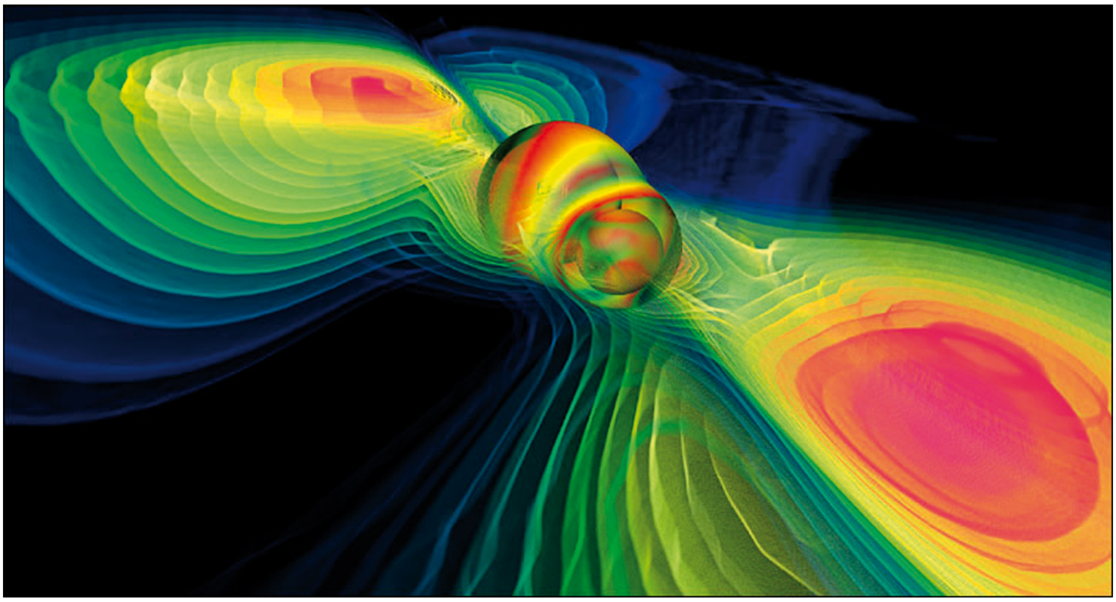

Figure 3.6.: One of the most violent astrophysical events: the merging of two black holes. (Image: MPI for Gravitational Physics/W.Benger-ZIB)

proper modes of vibration seemed rather awkward since it is not a material object, it is a singularity hidden by a horizon.

The procedure is very similar to the analysis carried out in linearized theory (cf. section 3.1); we deal with a static vacuum Schwarzschild space-time $g^0_{\mu\nu}$ superposed with a small perturbation $h_{\mu\nu}$ which encodes the deviation from spherical symmetry.

T. Regge and J. A. Wheeler showed that the equations describing the perturbations of a Schwarzschild black hole can be separated as (cf. section 2.2)

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \tag{3.43}$$

provided that the perturbed metric tensor can be expanded in tensorial spherical harmonics. This was possible since in the Schwarzschild case the perturbations naturally decouple due to spherical symmetry of the space-time. Regge and Wheeler called the result *odd-parity* and *even-parity* solution, respectively. The name *odd-parity* and *even-parity* emerges from the properties of the tensor spherical harmonics defined as

$$\mathrm{h}_{\mu\nu}(t,r,\theta,\phi) = \sum_{l,m} a_{lm}(t,r) A^{lm}_{\mu\nu}(\theta,\phi) + b_{lm}(t,r) B^{lm}_{\mu\nu}(\theta,\phi), \tag{3.44}$$

with distinctive transformation properties of the functions $A^{lm}_{\mu\nu}$ and $B^{lm}_{\mu\nu}$ under parity operations. Later, it was found that the odd perturbations represent really the angular perturbations to the metric, while the even ones are the radial perturbations to the metric [53, 56].

The perturbation equations are still commonly used in numerical relativity to extract the radiation quantities. That is partly due to a lack of serious investigations concerning the error of the method. We conclude this section by quoting Chandrasekhar from his book *The Mathematical Theory of Black Holes* and move to the next section where we will work out the details of the perturbations, also called *quasi-normal modes*:
..we may expect on general grounds that any initial perturbation will, during its last stages, decay in a manner characteristic of the black hole and independently of the original cause. In other words, we may expect that during the very last stages, the black hole will emit gravitational waves with frequencies and rates of damping, characteristic of itself, in the manner of a bell sounding its last dying pure note. These considerations underlie the formulation of the concept of the quasi-normal modes of a black hole.

3.5.2. The Regge-Wheeler and Zerilli Equation

The equation for the odd-parity perturbations are known as the Regge-Wheeler equation, describing the axial perturbations of the Schwarzschild metric in linear approximation, that is, we can decompose the perturbation $h_{\mu\nu}$ in Eq. (3.43) into tensor spherical harmonics according to Eq. (3.44) considering only odd terms. As in section (2.2) we can calculate the perturbed Einstein tensor, where we assume a time dependence for the Regge-Wheeler function $R(r,\omega)$ and Zerilli function $Z(r,\omega)$ of the form

$$R(r,\omega) \propto \mathrm{e}^{i\omega_n t}, \quad Z(r,\omega) \propto \mathrm{e}^{i\omega_n t}, \tag{3.45}$$

where ω_n is the oscillation frequency of the nth mode and is a complex number of the type

$$\omega_n = \omega_{r,n} + i\omega_{i,n}, \quad \text{with } n = 0, 1, 2, ... \tag{3.46}$$

The explicit derivation goes beyond the scope of this work and leads to no additional insight. Therefore we solely present and discuss the main results of black hole vibration modes.

Regge and Wheeler demonstrated that one ends up with three unknown variables, commonly called h_0, h_1 and h_2. We can set one of the three unknown variables to zero, namely $h_2 = 0$, by applying a particular gauge transformation, the *Regge-Wheeler gauge* [9]. Finally, we are left with the nontrivial Einstein equations to be determined:

$$\omega_n^2 R(r, \omega_n) + \partial_{r_*}^2 R(r, \omega_n) - V_s(r) R(r, \omega_n) = 0, \tag{3.47}$$

$$\partial_t h_0 - \partial_{r_*} [r_* R(r, \omega_n)] = 0 \tag{3.48}$$

where $R(r, \omega_n)$ is the master variable

$$R(r, \omega_n) = \frac{h_1}{r}\left(1 - \frac{2M}{r}\right), \tag{3.49}$$

and r_* is the *tortoise coordinate* $r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$. In general, the time derivative of $R(r, \omega_n)$ must also be calculated to provide full Cauchy data for an evolution. The function $V_s(r)$ is the so-called Regge-Wheeler potential defined as

$$V_s(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right], \tag{3.50}$$

where s is the spin of the particle and l is the angular momentum of the specific wave mode under consideration, with $l \geq s$. The spin can take the values $s = 0, \pm 1 \pm 2$ where the most important cases from astrophysical point of view are $s = \pm 1$ and $s = \pm 2$, which describe electromagnetic and gravitational waves, respectively. We can consider the function $V(r)$ as an effective, scattering potential barrier with a peak around

$r = 3.3M$, which is the location of the unstable photon orbit.

Next, we look at the even parity case where we yield a similar result for the Einstein equations commonly called the Zerilli equation

$$\omega_n^2 Z(r,\omega_n) + \partial_{r_*}^2 Z(r,\omega_n) - \tilde{V} Z(r,\omega_n) = 0, \tag{3.51}$$

where $Z(r,\omega_n)$ is the Zerilli master variable and $\tilde{V}_2(r)$ is the Zerilli potential

$$\tilde{V}_2(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{2n(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3}{r^3(nr+3M)^2}\right], \tag{3.52}$$

assuming $s = 2$ and $n = \frac{1}{2}(l-1)(l+2)$.

We may now calculate the response of a black hole to external perturbations as the solutions of Eq. (3.47, 3.53),

$$\begin{aligned} \omega_n^2 R(r,\omega_n) + \partial_{r_*}^2 R(r,\omega_n) - V_s(r) R(r,\omega_n) &= 0, & (3.53)\\ \omega_n^2 Z(r,\omega_n) + \partial_{r_*}^2 Z(r,\omega_n) - \tilde{V}_2(r) Z(r,\omega_n) &= 0. & (3.54) \end{aligned}$$

The approach to find the solution for the master variables $R(r,\omega_n)$ and $Z(r,\omega_n)$ is based on the standard WKB treatment of wave scattering at a potential barrier (cf. [57]).

Finally, having found the QNMs of a black hole via the Regge-Wheeler and Zerilli approach we can calculate the gravitational wave signal in terms of the master variables by the formula

$$\begin{aligned} h_+^{\mathrm{TT}}(t,r,\theta,\phi) &= \frac{1}{2\pi r}\int e^{i\omega_n(t-r_*)}\sum_{lm}\left[Z(r,\omega_n)\left({}_2\tilde{Y}_l^m - W_l^m\right) + \frac{1}{\omega_n}R(r,\omega_n)W_l^m\right]\mathrm{d}\omega_n, & (3.55a)\\ h_\times^{\mathrm{TT}}(t,r,\theta,\phi) &= -\frac{i}{2\pi r}\int e^{i\omega_n(t-r_*)}\sum_{lm}\left[Z(r,\omega_n)W_l^m - \frac{1}{\omega_n}R(r,\omega_n)\left({}_2\tilde{Y}_l^m - W_l^m\right)\right]\mathrm{d}\omega_n, & (3.55b) \end{aligned}$$

where Y_l^m $\left({}_sY_l^m\right)$ is the (spin-weighted) spherical harmonics and the functions W_l^m and

${}_2\tilde{Y}_l^m$ are defined as

$$
\begin{aligned}
{}_2\tilde{Y}_l^m &= \sqrt{\frac{(l+2)!}{(l-2)!}}\,{}_2Y_l^m, &\quad (3.56a)\\
W_l^m &= \frac{2i}{\sin\theta}\left(\partial_\theta - \cot\theta\right)\partial_\phi Y_l^m. &\quad (3.56b)
\end{aligned}
$$

n	$\omega_{r,n}$	$\omega_{i,n}$	$\omega_{r,n}$ (kHz) ($M = M_\odot$)	τ (ms) ($M = M_\odot$)
0	0.37367	$-0.08896i$	75.8695	5.5344×10^{-2}
1	0.34671	$-0.27391i$	70.3905	1.7983×10^{-2}
2	0.30105	$-0.47828i$	61.1297	1.0298×10^{-2}
3	0.25150	$-0.70514i$	51.0597	6.9856×10^{-3}

Table 3.1.: The first four frequencies for $l = 2$ are shown. The QNMs are given in geometrical units and hertz. In the third column the corresponding decaying times $\tau = 1/\omega_{i,n}$ are calculated. For conversion into kHz one should multiply by 2π (5.142 kHz)$\times M_\odot/M$.

Finally, let us briefly summarize what we have learned about the QNMs of a non-rotating black hole:

- Even if we now understand how to excite the QNMs, it is a nontrivial task to predict which ones will be excited, due to some arbitrariness in specifying the initial data of the space-time.

- The damping times of the QNMs depends linearly on the mass of the black hole, $\tau \propto 1/\omega_n \propto M$. As an implication, the detection of gravitational waves emitted by a perturbed black hole could provide a direct measure of its mass.

- Since the only parameter of a non-rotating black hole is its mass, it is the only variable the frequencies depend on. This explains why we expect different gravitational wave detectors to be sensitive to black holes with different masses. LIGO's sensitivity lies roughly between $10M_\odot$ to $10^3M_\odot$ whereas LISA will be sensitive to signals from black holes with masses from $10^5M_\odot$ to $10^8M_\odot$.

- The QNM frequencies of galactic size black holes, like the one at the center of our own galaxy with masses of $10^6 M_\odot$, will be in the mHz regime and therefore detectable only from LISA. Figure (3.7) gives an overview over various sources detectable by LIGO and LISA, respectively.

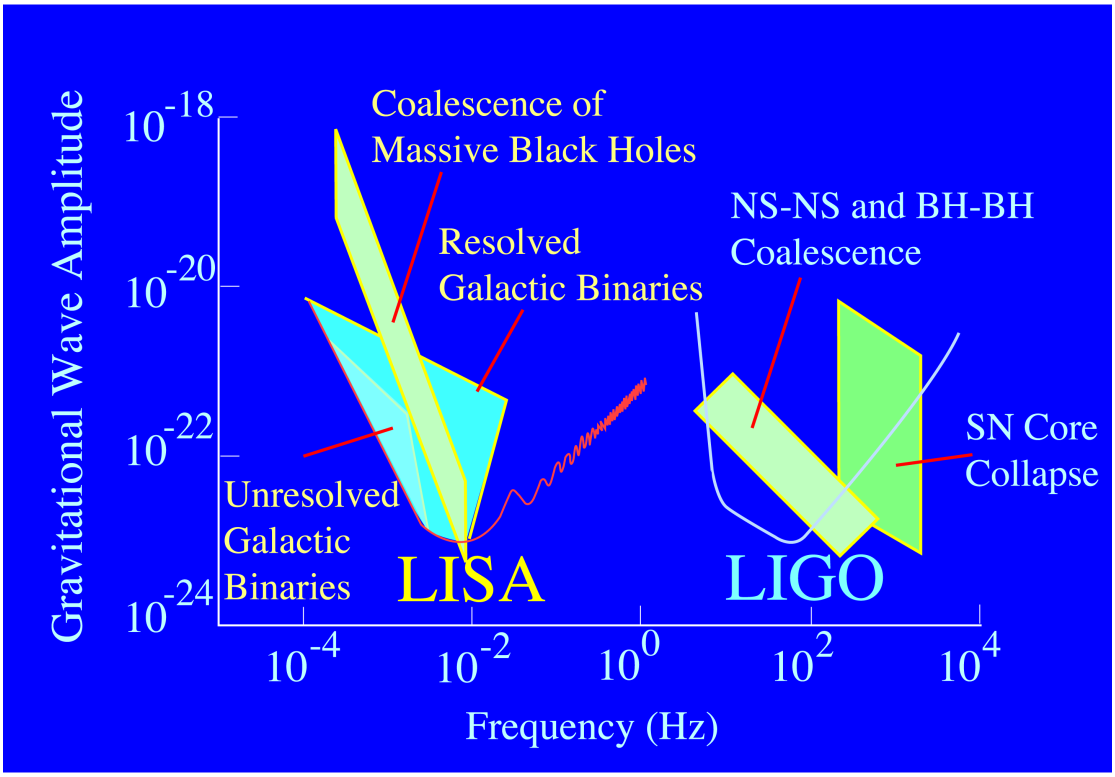

Figure 3.7.: The figure shows the strain sensitivity of LIGO and LISA, respectively. Regions where various sources are predicted to be are also shown. (Image: Beyond Einstein Roadmap)

4. The Newman-Penrose Formalism

If I have seen further it is by standing on the shoulders of giants.
Isaac Newton, February 5, 1675

In spite of the advantages of perturbation schemes like Regge-Wheeler [9] or Zerilli [10] there is a crucial drawback of linear perturbation theory: there is no information within linearized theory to determine its range of applicability, i.e. to determine which values are sufficiently small to be treated as a perturbation. These disadvantages may be resolved by extending the perturbation schemes to a still higher order. Unfortunately, the equations become extremely complicated and that makes it practically impossible to solve the problem analytically [58].

But even worse, all these perturbation approaches are fundamentally assuming knowledge of a background metric, in fact, they are well defined only for Schwarzschild background, formulated in a particular coordinate system. As a natural limitation of numerical simulations, such a specific knowledge is just not known *a-priori*. As a result there is a strong demand for a formalism that does not imply any knowledge of specific background structures in the first instance.

The Newman-Penrose formalism [11] is a fundamental contribution towards this demand. It has been shown that the introduced curvature quantities in this formalism, namely the Weyl scalars and the spin coefficients, acquire a direct physical relevance, carrying all information of the space-time under examination without the need of performing a linearization *a priori* [16, 59]. Despite its undeniable validity, the equations governing the formalism are rather complicated and their nature and the connection between all the equations is yet to be fully understood.

In this chapter we give a general introduction to the underlying mathematical techniques of the Newman-Penrose formalism, the fundamental variables and why it is regarded as a particular suitable approach to extract the gravitational wave signal in numerical simulations. In particular, we demonstrate how the transverse Weyl scalars Ψ_4 and Ψ_0 can be identified with the outgoing and ingoing gravitational radiation, respectively.

Afterwards, we discuss recent improvements in theoretical understanding of the Newman-Penrose formalism [1, 60, 19, 22]. We introduce the notion of the *quasi-Kinnersley frame* which assures that we recover the dynamics obeying Teukolsky's master equation in the limit of Petrov type D space-time [15]. This will be the basis to the next two chapters, where we propose a new formalism for wave extraction and apply our new method to a typical situation in numerical relativity.

4.1. Mathematical Preliminaries

The concept of the tetrad formalism is to introduce a suitable tetrad basis of four linearly independent vector-fields and project all relevant quantities of the problem under study on to the chosen basis. The choice of the tetrad basis depends on the underlying space-time symmetries we wish to exploit. To begin our discussion we introduce at each point of the Manifold a basis of four vector fields $e^{\mu}_{(i)}$, where i runs from 1 to 4 designating tetrad indices, and Greek indices denote tensor indices. We define the covariant form according to

$$e_{(i)\,\nu} = g_{\mu\nu} e^{\mu}_{(i)}, \tag{4.1}$$

where $g_{\mu\nu}$ is the metric tensor of the space-time under consideration. In addition, we can define the inverse of $e^{\mu}_{(i)}$ by

$$e^{\mu}_{(j)} e^{(i)}_{\mu} = \delta^{(i)}{}_{(j)} \quad \text{and} \quad e^{(i)}_{\mu} e^{\nu}_{(i)} = \delta_{\mu}{}^{\nu}. \tag{4.2}$$

To complete the definition of the basis vectors we define a symmetric matrix according to

$$e^{\mu}_{(i)} e_{(j)\,\mu} = \eta_{(i)(j)}. \tag{4.3}$$

Supposing a particular frame where the basic vectors are orthonormal we find that the matrix η turns out to be

$$\eta_{(i)(j)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

Therefore, as stated by the *Equivalence Principle of General Relativity*, starting from a general metric $g_{\mu\nu}$ on a Manifold we can always remove the gravitational field locally and thus end up locally with a Minkowski metric η.

As a simple example of this statement we consider a space-time Manifold with a single black hole. We define the line element in pseudo-spherical coordinates according to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{4.5}$$

To locally remove the space-time curvature we choose the tetrad vectors as

$$e_{\mu\,(1)} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} (dt)_{\mu}\,, \tag{4.6a}$$

$$e_{\mu\,(2)} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} (dr)_{\mu}\,, \tag{4.6b}$$

$$e_{\mu\,(3)} = r\,(d\theta)_{\mu}\,, \tag{4.6c}$$

$$e_{\mu\,(4)} = r\,sin\theta\,(d\phi)_{\mu}\,, \tag{4.6d}$$

thus yielding for the metric the usual Minkowski metric

$$ds^2 = -e^2_{(1)} + e^2_{(2)} + e^2_{(3)} + e^2_{(4)}. \tag{4.7}$$

To fully develop the tetrad formalism we need to define all space-time quantities in our new formalism.

4.1.1. Directional Derivatives and Ricci Rotation Coefficients

The directional derivatives in the tetrad frame are defined as

$$e_{(a)} = e^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}}, \tag{4.8}$$

where the contravariant vectors $e_{(a)}$ are considered as tangent vectors. Thus we shall write for the derivative of a scalar field

$$\Phi_{,(a)} = e^{\mu}_{(a)} \frac{\partial \Phi}{\partial x^{\mu}}, \tag{4.9}$$

and the action on a more general vector field is defined as

$$A_{(a),(b)} = e^{\mu}_{(b)} \frac{\partial}{\partial x^{\mu}} e^{\nu}_{(a)} A_{\nu} = e^{\nu}_{(a)} A_{\nu;\mu}\, e^{\mu}_{(b)} + \gamma_{(c)(a)(b)} A^{(c)}. \tag{4.10}$$

The *connection 1-forms* $\gamma_{(c)(a)(b)}$, which are defined in Eq. (4.10) by

$$\gamma_{(c)(a)(b)} = e^{\mu}_{(c)} e_{(a)\mu;\nu} e^{\nu}_{(b)}, \tag{4.11}$$

are called the Ricci rotation coefficients and satisfy

$$\gamma_{(b)(a)(c)} = -\gamma_{(a)(b)(c)}. \tag{4.12}$$

We shall emphasize that there are only 24 components due to the antisymmetry of the Ricci rotation coefficients compared with 40 components for the Christoffel symbols Γ. An alternative formulation of Eq. (4.10) is to define the first term on the right hand side as the *intrinsic derivative* $A_{(a)|(b)}$ of $A_{(a)}$ in the direction of $e_{(b)}$:

$$A_{(a),(b)} = A_{(a)|(b)} + \gamma_{(c)(a)(b)} A^{(c)}. \tag{4.13}$$

This procedure can be readily extended to the derivatives of tensor fields.

4.1.2. The Commutation Relation and Structure Constants

Starting from the torsion-free condition of the derivative operator $\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$ we are able to express this condition as the 24 *commutation relations* of the basis vector fields

$$\left[e_{(a)}, e_{(b)}\right] = C^{(c)}{}_{(a)(b)} e_{(c)}, \tag{4.14}$$

where the coefficients $C^{(c)}$ are the structure constants with

$$C^{(c)}{}_{(a)(b)} = \gamma^{(c)}{}_{(b)(a)} - \gamma^{(c)}{}_{(a)(b)}. \tag{4.15}$$

4.1.3. The Ricci Identities

From the viewpoint of the results obtained in this thesis the Ricci and the Bianchi identities take an extraordinary position, which we will demonstrate in the following chapters. Here we will introduce the relevant identities and their definition in a tetrad frame. The Ricci identities, often called the Newman-Penrose equations, are defined according to

$$e_{(i)\mu;\nu\rho} - e_{(i)\mu;\rho\nu} = R_{\sigma\mu\nu\rho} r^{\sigma}_{(i)}, \tag{4.16}$$

thus relating the Riemann tensor to the commutator of covariant derivatives. Projecting the Riemann tensor on to the tetrad frame it can be expressed in terms of the Ricci rotation coefficients in the following manner:

$$\begin{aligned} R_{(a)(b)(c)(d)} = & - \gamma_{(a)(b)(c),(d)} + \gamma_{(a)(b)(d),(c)} \\ & - \gamma_{(b)(a)(f)} \left[\gamma_{(c)}{}^{(f)}{}_{(d)} - \gamma_{(d)}{}^{(f)}{}_{(c)}\right] \\ & + \gamma_{(f)(a)(c)} \gamma_{(b)}{}^{(f)}{}_{(d)} - \gamma_{(f)(a)(d)} \gamma_{(b)}{}^{(f)}{}_{(c)}. \end{aligned} \tag{4.17}$$

Newman and Penrose identified 18 independent non-vanishing complex components of the Riemann tensor

$$R_{(1)(3)(1)(3)}, \quad R_{(1)(3)(1)(4)}, \quad \frac{1}{2}\left(R_{(3)(4)(1)(4)} - R_{(1)(2)(1)(4)}\right), \tag{4.18a}$$
$$R_{(1)(3)(1)(2)}, \quad R_{(2)(4)(4)(1)}, \quad \frac{1}{2}\left(R_{(1)(2)(1)(3)} - R_{(3)(4)(1)(3)}\right), \tag{4.18b}$$
$$R_{(2)(4)(3)(1)}, \quad R_{(2)(4)(2)(1)}, \quad \frac{1}{2}\left(R_{(1)(2)(1)(2)} - R_{(3)(4)(1)(2)}\right), \tag{4.18c}$$
$$R_{(2)(4)(4)(2)}, \quad R_{(3)(1)(4)(3)}, \quad \frac{1}{2}\left(R_{(1)(2)(3)(4)} - R_{(3)(4)(3)(4)}\right), \tag{4.18d}$$
$$R_{(2)(4)(4)(3)}, \quad R_{(2)(4)(2)(3)}, \quad \frac{1}{2}\left(R_{(1)(2)(3)(2)} - R_{(2)(4)(2)(3)}\right), \tag{4.18e}$$
$$R_{(1)(3)(3)(2)}, \quad R_{(1)(3)(2)(4)}, \quad \frac{1}{2}\left(R_{(1)(2)(4)(2)} - R_{(3)(4)(4)(2)}\right), \tag{4.18f}$$

and the additional complex-conjugate relations are obtained by replacing the index 3 with the index 4, and vice versa. Thus, in total we yield 36 linear independent *Ricci identities*. In chapter 5 the Ricci identities will acquire the role of evolution equations for the Weyl scalars.

4.1.4. The Bianchi Identities

Pursuing the same projection we can express the Bianchi identities

$$R_{\mu\nu[\rho\sigma;\tau]} = 0, \tag{4.19}$$

in terms of intrinsic derivatives and tetrad components, yielding

$$\begin{aligned} R_{(a)(b)[(c)(d)|(f)]} = \; & \frac{1}{6}\sum_{[(c)(d)(f)]}\Big\{R_{(a)(b)(c)(d),(f)} \\ & -\eta^{(n)(m)}\Big[\gamma_{(n)(a)(f)}R_{(m)(b)(c)(d)} + \gamma_{(n)(b)(f)}R_{(a)(m)(c)(d)} \\ & +\gamma_{(n)(c)(f)}R_{(a)(b)(m)(d)} + \gamma_{(n)(d)(f)}R_{(a)(b)(c)(m)}\Big]\Big\}. \end{aligned} \tag{4.20}$$

Due to the symmetries of the Riemann tensors only 20 linear independent equations can be derived. Written out explicitly, a complete set is provided by the eight following complex identities

$$R_{(1)(3)[(1)(3)|(4)]} = 0, \quad R_{(1)(3)[(2)(1)|(4)]} = 0, \tag{4.21a}$$

$$R_{(1)(3)[(1)(3)|(2)]} = 0, \quad R_{(1)(3)[(4)(3)|(2)]} = 0, \tag{4.21b}$$

$$R_{(4)(2)[(1)(3)|(4)]} = 0, \quad R_{(4)(2)[(2)(1)|(4)]} = 0, \tag{4.21c}$$

$$R_{(4)(2)[(1)(3)|(2)]} = 0, \quad R_{(4)(2)[(4)(3)|(2)]} = 0, \tag{4.21d}$$

and four real identities which follow from

$$\eta^{(b)(c)} \left(R_{(a)(b)} - \frac{1}{2} \eta_{(a)(b)} R \right)_{|c} = 0. \tag{4.21e}$$

Again, the additional complex-conjugate relations are obtained by replacing the index 3 with the index 4, and vice versa. In chapter 5 the Bianchi identities will provide a fundamental relation between the spin coefficients and Weyl scalars.

To summarize, the basic equations which must be satisfied in the tetrad formalism consists of 24 commutation relations, 36 Ricci identities and 20 Bianchi identities.

4.2. Null Tetrads and Null Frames

A particularly important tetrad choice has been made by Newman and Penrose, originally based on the idea of treating general relativity in terms of spinor fields [61]. Furthermore, their choice is motivated by symmetry reasons since it is particularly well adapted to the light-cone structure of radiation in four dimensional space-times. The vector basis chosen is a set of four null vectors; they consist of a pair real null vectors labeled as ℓ and n, completed by a pair of complex conjugate null vectors denoted by m and $\bar{m}$. The two real tetrad vectors point asymptotically radially inward and radially outward whereas the complex vectors are defined on a 2-sphere. The null tetrad satisfies the orthogonality condition

$$\ell \cdot m = \ell \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0, \tag{4.22a}$$

besides the requirements that the vectors be null

$$\ell \cdot \ell = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \tag{4.22b}$$

Additionally they satisfy the normalization conditions

$$\ell \cdot n = 1, \tag{4.22c}$$

$$m \cdot \bar{m} = -1, \tag{4.22d}$$

where this is not an essential requirement and may be dropped. Imposing these conditions the matrix η in Eq. (4.3) takes the form

$$\eta_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{4.23}$$

By utilizing the definition of the tetrad introduced in Eq. (4.1) to Eq. (4.3) we can deduce the following correspondence between the tetrad vectors and null vectors

$$e^{(1)} = e_{(2)} = n; \quad e^{(2)} = e_{(1)} = l, \tag{4.24a}$$

$$e^{(3)} = -e_{(4)} = -\bar{m}; \quad e^{(4)} = -e_{(3)} = -m. \tag{4.24b}$$

In this notation the metric tensor $g_{\mu\nu}$ of the 4-dimensional space-time takes the form

$$g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu = 2\ell_{(\mu} n_{\nu)} - 2m_{(\mu} \bar{m}_{\nu)}, \tag{4.25}$$

and the inverse is defined accordingly

$$g^{\mu\nu} = 2\ell^{(\mu}n^{\nu)} - 2m^{(\mu}\bar{m}^{\nu)}. \tag{4.26}$$

4.3. Spin Coefficients

The 24 Ricci rotation-coefficients $\gamma_{(a)(b)(c)}$ introduced in 4.1.1 are now 12 complex quantities and called *spin coefficients*, designated by a special notation:

$$\kappa = \gamma_{(3)(1)(1)}; \quad \rho = \gamma_{(3)(1)(4)}; \quad \varepsilon = \frac{1}{2}\left(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\right); \tag{4.27a}$$

$$\sigma = \gamma_{(3)(1)(3)}; \quad \mu = \gamma_{(2)(4)(3)}; \quad \gamma = \frac{1}{2}\left(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\right); \tag{4.27b}$$

$$\lambda = \gamma_{(2)(4)(4)}; \quad \tau = \gamma_{(3)(1)(2)}; \quad \alpha = \frac{1}{2}\left(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\right); \tag{4.27c}$$

$$\nu = \gamma_{(2)(4)(2)}; \quad \pi = \gamma_{(2)(4)(1)}; \quad \beta = \frac{1}{2}\left(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\right). \tag{4.27d}$$

An equivalent formulation in terms of tetrad components is

$$\kappa = m^\mu \ell^\nu \nabla_\nu \ell_\mu, \qquad \nu = n^\mu n^\nu \nabla_\nu \bar{m}_\mu, \tag{4.28a}$$

$$\sigma = m^\mu m^\nu \nabla_\nu \ell_\mu, \qquad \lambda = n^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu, \tag{4.28b}$$

$$\rho = m^\mu \bar{m}^\nu \nabla_\nu \ell_\mu, \qquad \mu = n^\mu m^\nu \nabla_\nu \bar{m}_\mu, \tag{4.28c}$$

$$\tau = m^\mu n^\nu \nabla_\nu \ell_\mu, \qquad \pi = n^\mu \ell^\nu \nabla_\nu \bar{m}_\mu, \tag{4.28d}$$

$$\varepsilon = \frac{1}{2}\ell^\nu \left(n^\mu \nabla_\nu \ell_\mu + m^\mu \nabla_\nu \bar{m}_\mu\right), \tag{4.28e}$$

$$\gamma = \frac{1}{2}n^\nu \left(n^\mu \nabla_\nu \ell_\mu + m^\mu \nabla_\nu \bar{m}_\mu\right), \tag{4.28f}$$

$$\alpha = \frac{1}{2}\bar{m}^\nu \left(n^\mu \nabla_\nu \ell_\mu + m^\mu \nabla_\nu \bar{m}_\mu\right), \tag{4.28g}$$

$$\beta = \frac{1}{2}m^\nu \left(n^\mu \nabla_\nu \ell_\mu + m^\mu \nabla_\nu \bar{m}_\mu\right). \tag{4.28h}$$

We want to stress here an important property related to the $\ell \leftrightarrow n$ exchange transformation. If we exchange the two real null vectors, then the spin coefficients become interchanged as follows

$$\kappa \leftrightarrow -\nu^*, \quad \rho \leftrightarrow -\mu^*, \quad \sigma \leftrightarrow -\lambda^*, \tag{4.29a}$$

$$\alpha \leftrightarrow -\beta^*, \quad \varepsilon \leftrightarrow -\gamma^*, \quad \pi \leftrightarrow -\tau^*. \tag{4.29b}$$

The spin coefficients can be related to physical properties of the tetrad vectors constituting the tetrad; utilizing the exchange operation we can immediately conclude that the physical property which connects $(\kappa,\rho,\sigma,\alpha,\varepsilon,\pi)$ to the null vector ℓ is the same that relates $(\nu,\mu,\lambda,\beta,\gamma,\tau)$ to the n vector. Therefore we can restrict our attention to the physical meaning of the spin coefficients which are related to the ℓ vector.

Assuming space-time contains a ray congruence (i.e. a foliation by null geodesics) that is singled out, we can relate the spin coefficients to the ℓ vector in the following manner:

- The null vector ℓ^μ may be chosen to point in the direction of the rays, whose geodicity can be stated as $\kappa = 0$.
- Affine normalization, i.e. parallel propagation of ℓ^μ, is stated as $\varepsilon + \bar{\varepsilon} = 0$.
- ρ as the "complex divergence", defined as $\rho = \frac{1}{2}\left(-\nabla_\mu \ell\mu + i\sqrt{\nabla_{[\mu}\,\ell_{\,\nu]}\nabla^{[\mu}\,\ell^{\,\nu]}}\right)$.
- σ measures the shear of the null congruence of geodesics dened by $\sigma\bar{\sigma} = \frac{1}{2}\left(\nabla_{(\mu}\,\ell_{\,\nu)}\nabla^{(\mu}\,\ell^{\,\nu)} - \frac{1}{2}\left(\nabla_\mu \ell^\mu\right)^2\right)$.
- When the ray congruence is a gradient field, i.e. $\ell_\mu = \nabla_\mu \Phi$, then $\tau = \bar{\alpha} + \beta$.

For a detailed derivation of such properties we suggest a very good textbook by Penrose and Rindler [62].

4.4. Weyl Tensor and Weyl Scalars

In addition to the spin coefficients the most relevant quantities in the Newman-Penrose formalism are the Weyl scalars, which comprise all the important information about the space-time under consideration. They are given by the contraction of the conformal Weyl tensor over a specific combination of the null tetrad $(\ell^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu})$. Since Weyl scalars are true scalars the choice of the coordinate system is irrelevant for their calculation; however, they do depend on the tetrad choice which constitutes the gauge freedom in this formalism.

But before we move to the calculation of these quantities we need to define the conformal Weyl tensor. It comprises of the conformally invariant part of the Riemann tensor, thus we introduce the Weyl tensor as the trace-free part of the Riemann tensor:

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \frac{1}{(n-2)}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho}\right) \\ &+ \frac{1}{(n-1)(n-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) R. \end{aligned} \tag{4.30}$$

It is a straightforward to show that the Weyl tensor possesses the same symmetries as the curvature tensor, namely

$$C_{\mu\nu\rho\sigma} = -C_{\mu\nu\sigma\rho} = -C_{\nu\mu\rho\sigma} = C_{\rho\sigma\mu\nu}, \tag{4.31a}$$

$$C_{\mu\nu\rho\sigma} + C_{\mu\sigma\nu\rho} + C_{\mu\rho\sigma\nu} = 0, \tag{4.31b}$$

with the additional symmetry $g^{\rho\mu}C_{\mu\nu\rho\sigma} = 0$. Secondly, to determine the Weyl tensor on a manifold, a connection and a metric must be defined.
Projecting the Weyl tensor on to the tetrad frame, we have in 4 dimensions

$$\begin{aligned} C_{(a)(b)(c)(d)} &= R_{(a)(b)(c)(d)} - \frac{1}{2}\left(\eta_{(a)(c)}R_{(b)(d)} + \eta_{(b)(d)}R_{(a)(c)} - \eta_{(b)(c)}R_{(a)(d)} - \eta_{(a)(d)}R_{(b)(c)}\right) \\ &+ \frac{1}{6}\left(\eta_{(a)(c)}\eta_{(b)(d)} - \eta_{(a)(d)}\eta_{(b)(c)}\right) R, \end{aligned} \tag{4.32}$$

where now we are raising and lowering quantities with $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$.

A detailed analysis of the Weyl tensor shows that it has 10 degrees of freedom, which we can express by introducing five complex self-dual scalar quantities, the *Weyl* *scalars*:

$$
\begin{aligned}
\Psi_0 &= -C_{(1)(3)(1)(3)} = -C_{\alpha\beta\gamma\delta}\, l^\alpha\, m^\beta\, l^\gamma\, m^\delta, && \text{(4.33a)}\\
\Psi_1 &= -C_{(1)(2)(1)(3)} = -C_{\alpha\beta\gamma\delta}\, l^\alpha\, n^\beta\, l^\gamma\, m^\delta, && \text{(4.33b)}\\
\Psi_2 &= -C_{(1)(3)(4)(2)} = -C_{\alpha\beta\gamma\delta}\, l^\alpha\, \bar{m}^\beta\, m^\gamma\, n^\delta, && \text{(4.33c)}\\
\Psi_3 &= -C_{(1)(2)(4)(2)} = -C_{\alpha\beta\gamma\delta}\, l^\alpha\, n^\beta\, \bar{m}^\gamma\, n^\delta, && \text{(4.33d)}\\
\Psi_4 &= -C_{(2)(4)(2)(4)} = -C_{\alpha\beta\gamma\delta}\, n^\alpha\, \bar{m}^\beta\, n^\gamma\, \bar{m}^\delta. && \text{(4.33e)}
\end{aligned}
$$

However, it is important to stress that the Weyl scalars still possess the inherit degrees of freedom of general relativity, even if the quantities do not depend on the coordinates any more. We can perform so-called tetrad *rotations* to change the value of the Weyl scalars. We will clarify these issues in the remaining part of the chapter.

4.5. Curvature Invariants

Still, it is possible to define quantities in the Newman-Penrose formalism which neither depend on the coordinate, nor are subject to tetrad transformations. Two well-known complex curvature invariants are (commonly) defined by

$$
\begin{aligned}
I &= \frac{1}{16}\left(C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\mu\nu} - iC_{\mu\nu}{}^{\rho\sigma}\bar{C}_{\rho\sigma}{}^{\mu\nu}\right), && \text{(4.34a)}\\
J &= \frac{1}{96}\left(C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\mu\nu}C_{\alpha\beta}{}^{\mu\nu} - iC_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}\bar{C}_{\alpha\beta}{}^{\mu\nu}\right), && \text{(4.34b)}
\end{aligned}
$$

where $\bar{C}_{\rho\sigma}{}^{\mu\nu} = \frac{1}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu}$ is the Hodge dual of the Weyl tensor $C_{\rho\sigma}{}^{\mu\nu}$. The scalars are also often called *Kretschmann invariants* after Erich Kretschmann [63]. They are essentially the square and cube of the self-dual part, $\tilde{C}_{\mu\nu}{}^{\rho\sigma} = C_{\mu\nu}{}^{\rho\sigma} + \frac{i}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}C_{\alpha\beta}{}^{\rho\sigma}$, of the Weyl tensor:

$$
\begin{aligned}
I &= \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma}, && \text{(4.35a)}\\
J &= \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\rho\sigma}{}_{\alpha\beta}\tilde{C}^{\alpha\beta\mu\nu}. && \text{(4.35b)}
\end{aligned}
$$

We may re-express the curvature invariants in terms of Weyl scalars as

$$I = 3\Psi_2^2 + \Psi_4\Psi_0 - 4\Psi_1\Psi_3, \quad (4.36a)$$
$$J = -\Psi_2^3 + 2\Psi_1\Psi_2\Psi_3 - \Psi_0\Psi_3^3 - \Psi_1^2\Psi_4 + \Psi_0\Psi_2\Psi_4. \quad (4.36b)$$

Additionally, we define the *speciality index* S in terms of curvature invariants

$$S = \frac{27J^2}{I^3}, \quad (4.37)$$

which plays an important role in the Petrov classification in section 4.7.

4.6. Tetrad Transformations

As mentioned in section 4.4 and section 4.5, Weyl scalars are not tetrad invariant in contrast to the curvature invariants. They are subject to transformations which we classify in this section. We can perform a Lorentz transformation at some point and extend it continuously through all of space-time.

The six degrees of freedom of the group of Lorentz transformations correspond to the following tetrad *rotations* in the Newman-Penrose formalism, which preserve the underlying orthogonality and normalization conditions:

- Type I $\quad \ell \to \ell;\ n \to n + a^*m + a\bar{m} + aa^*\ell;\ m \to m + a\ell;\ \bar{m} \to \bar{m} + a^*\ell,$
- Type II $\quad n \to n;\ \ell \to \ell + b^*m + b\bar{m} + bb^*n;\ m \to m + bn;\ \bar{m} \to \bar{m} + b^*n,$
- Type III $\quad \ell \to |\mathscr{B}|^{-1}\ell;\ n \to |\mathscr{B}|n;\ m \to e^{i\Theta}m;\ \bar{m} \to e^{-i\Theta}\bar{m}.$

Here a and b are two complex functions. The spin-boost parameter $\mathscr{B}$ is defined as $\mathscr{B} = Ae^{-i\Theta}$, where the modulus A and the phase Θ of the complex valued boost read

$$A = \sqrt{\Re[B]^2 + \Im[B]^2}, \quad (4.38a)$$
$$\Theta = \arctan\left(\frac{\Im[B]}{\Re[B]}\right). \quad (4.38b)$$

In defining the tetrad rotations we follow closely the textbook by Chandrashekar, *The Mathematical Theory of Black Holes*.

4.6.1. Type I Rotations

The definition of the type I rotations is that the null vector ℓ remains unchanged, while the other vectors are rotated according to

$$\ell \rightarrow \ell, \tag{4.39}$$
$$n \rightarrow n + a^* m + a\bar{m} + aa^* \ell, \tag{4.40}$$
$$m \rightarrow m + a\ell, \tag{4.41}$$
$$\bar{m} \rightarrow \bar{m} + a^* \ell. \tag{4.42}$$

The change in the Weyl scalars turns out to be

$$\Psi_0^I \rightarrow \Psi_0, \tag{4.43a}$$
$$\Psi_1^I \rightarrow \Psi_1 + a^* \Psi_0, \tag{4.43b}$$
$$\Psi_2^I \rightarrow \Psi_2 + 2a^* \Psi_1 + (a^*)^2 \Psi_0, \tag{4.43c}$$
$$\Psi_3^I \rightarrow \Psi_3 + 3a^* \Psi_2 + 3(a^*)^2 \Psi_1 + (a^*)^3 \Psi_0, \tag{4.43d}$$
$$\Psi_4^I \rightarrow \Psi_4 + 4a * \Psi_3 + 6(a^*)^2 \Psi_2 + 4(a^*)^3 \Psi_1 + (a^*)^4 \Psi_0. \tag{4.43e}$$

The spin coefficients transform as follows:

$$\kappa^I = \kappa, \tag{4.44a}$$
$$\sigma^I = \sigma + a\kappa, \tag{4.44b}$$
$$\rho^I = \rho + a^* \kappa, \tag{4.44c}$$
$$\varepsilon^I = \varepsilon + a^* \kappa, \tag{4.44d}$$
$$\tau^I = \tau + a\rho + a^* \sigma + aa^* \kappa, \tag{4.44e}$$

$$\pi^I = \pi + 2a^*\varepsilon + (a^*)^2\kappa + \ell^\mu\partial_\mu a^*, \tag{4.44f}$$
$$\alpha^I = \alpha a^*(\rho + \varepsilon) + (a^*)^2\kappa, \tag{4.44g}$$
$$\beta^I = \beta + a\varepsilon + a^*\sigma + aa^*\kappa, \tag{4.44h}$$
$$\gamma^I = \gamma + a\alpha + a^*(\beta + \tau) + aa^*(\rho + \varepsilon) + (a^*)^2\sigma + a(a^*)^2\kappa, \tag{4.44i}$$
$$\lambda^I = \lambda + a^*(2\alpha + \pi) + (a^*)^2(\rho + 2\varepsilon) + (a^*)^3\kappa + \bar{m}^\mu\partial_\mu a^* + a\ell^\mu\partial_\mu a^*, \tag{4.44j}$$
$$\mu^I = \mu + a\pi + 2a^*\beta + 2aa^*\varepsilon + (a^*)^2\sigma + a(a^*)^2\kappa + m^\mu\partial_\mu a^* + a\ell^\mu\partial_\mu a^*, \tag{4.44k}$$
$$\nu^I = \nu + a\lambda + a^*(\mu + 2\gamma) + (a^*)^2(\tau + 2\beta) + (a^*)^3\sigma + aa^*(\pi + 2\alpha). \tag{4.44l}$$

4.6.2. Type II Rotations

Instead of leaving the l null vector unchanged the n vector remains unchanged while the other vectors are rotated according to

$$l \rightarrow l + b^*m + b\bar{m} + bb^*n, \tag{4.45a}$$
$$n \rightarrow n, \tag{4.45b}$$
$$m \rightarrow m + bn, \tag{4.45c}$$
$$\bar{m} \rightarrow \bar{m} + b^*n. \tag{4.45d}$$

The transformation behavior of the Weyl scalars is the following:

$$\Psi_0^{\mathrm{II}} \rightarrow \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \tag{4.46a}$$
$$\Psi_1^{II} \rightarrow \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \tag{4.46b}$$
$$\Psi_2^{II} \rightarrow \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \tag{4.46c}$$
$$\Psi_3^{II} \rightarrow \Psi_3 + b\Psi_4, \tag{4.46d}$$
$$\Psi_4^{II} \rightarrow \Psi_4. \tag{4.46e}$$

To determine how the spin coefficients transform under a Type II rotation we utilize the result of a Type I rotation and the property of an exchange of the real vectors ℓ

and n yielding

$$\kappa^{II} \rightleftarrows -\left(\nu^{I}\right)^{*}, \tag{4.47a}$$

$$\sigma^{II} \rightleftarrows -\left(\lambda^{I}\right)^{*}, \tag{4.47b}$$

$$\rho^{II} \rightleftarrows -\left(\mu^{I}\right)^{*}, \tag{4.47c}$$

$$\varepsilon^{II} \rightleftarrows -\left(\gamma^{I}\right)^{*}, \tag{4.47d}$$

$$\pi^{II} \rightleftarrows -\left(\tau^{I}\right)^{*}, \tag{4.47e}$$

$$\alpha^{II} \rightleftarrows -\left(\beta^{I}\right)^{*}. \tag{4.47f}$$

4.6.3. Type III Rotations

Type III rotations consist of performing a null rotation in the m-$\bar{m}$ plane, and rescaling the l and n vectors in the following manner:

$$\ell \rightarrow A^{-1}\ell, \tag{4.48a}$$

$$n \rightarrow An, \tag{4.48b}$$

$$m \rightarrow e^{i\Theta}m, \tag{4.48c}$$

$$\bar{m} \rightarrow e^{-i\Theta}\bar{m}. \tag{4.48d}$$

The effect on the Weyl scalars is

$$\Psi_0^{\mathrm{III}} \rightarrow \mathscr{B}^{-2}\Psi_0, \tag{4.49a}$$

$$\Psi_1^{III} \rightarrow \mathscr{B}^{-1}\Psi_1, \tag{4.49b}$$

$$\Psi_2^{III} \rightarrow \Psi_2, \tag{4.49c}$$

$$\Psi_3^{III} \rightarrow \mathscr{B}\Psi_3, \tag{4.49d}$$

$$\Psi_4^{III} \rightarrow \mathscr{B}^{2}\Psi_4, \tag{4.49e}$$

and the corresponding transformations for the spin coefficients are

$$\rho^{III} = |\mathscr{B}|^{-1}\rho, \tag{4.50a}$$
$$\mu^{III} = |\mathscr{B}|\mu, \tag{4.50b}$$
$$\lambda^{III} = |\mathscr{B}|e^{-2i\Theta}\lambda, \tag{4.50c}$$
$$\sigma^{III} = |\mathscr{B}|^{-1}e^{2i\Theta}\sigma, \tag{4.50d}$$
$$\kappa^{III} = |\mathscr{B}|^{-1}\mathscr{B}^{-1}\kappa, \tag{4.50e}$$
$$\tau^{III} = e^{i\Theta}\tau, \tag{4.50f}$$
$$\pi^{III} = e^{-i\Theta}\pi, \tag{4.50g}$$
$$\nu^{III} = |\mathscr{B}|\mathscr{B}\nu, \tag{4.50h}$$
$$\gamma^{III} = |\mathscr{B}|\left(\gamma - \frac{1}{2}\Delta \ln \mathscr{B}\right), \tag{4.50i}$$
$$\varepsilon^{III} = |\mathscr{B}|^{-1}\left(\varepsilon - \frac{1}{2}D \ln \mathscr{B}\right), \tag{4.50j}$$
$$\alpha^{III} = e^{-i\Theta}\left(\alpha - \frac{1}{2}\delta^{*} \ln \mathscr{B}\right), \tag{4.50k}$$
$$\beta^{III} = e^{i\Theta}\left(\beta - \frac{1}{2}\delta \ln \mathscr{B}\right). \tag{4.50l}$$

4.7. Petrov Classification

From the trace-free property of the Weyl tensor $g^{\rho\mu}C_{\mu\nu\rho\sigma} = 0$ we can deduce important information regarding the underlying space-time by classifying its eigenvalues and eigenvectors. Since the Weyl tensor is anti-symmetric on each pair of indices and symmetric under their interchange we would expect a sixth-order polynomial, but, in fact, due to the additional symmetries in Eq. (4.31) the polynomial reduces to a quartic equation for the Weyl scalars, namely

$$\Psi_4 b^4 + 4\Psi_3 b^3 + 6\Psi_2 b^2 + 4\Psi_1 b + \Psi_0 = 0, \tag{4.51}$$

where b is the complex functions we introduced for the tetrad transformations. Obviously, the equation has (always) four roots and the corresponding new directions of

ℓ, namely, $\ell \rightarrow \ell + b^* m + b\bar{m} + bb^* n$, are called the principal null-directions of the Weyl tensor.

By performing tetrad rotations with the possible values for the function b as solution of Eq. (4.51) some of the Weyl scalars can be made to vanish. The question now is which of the Weyl scalars and how many of them can be made to vanish. Petrov classified the possible solutions and gave them a name or *type* as shown in Table (4.1).

Petrov Type	Weyl Scalars	Gauge Choice
I	$\Psi_0 = \Psi_4 = 0 \quad \Psi_1 \neq 0,\ \Psi_2 \neq 0,\ \Psi_3 \neq 0$	$\Psi_0 \wedge \Psi_4 \leftrightarrow \Psi_1 \wedge \Psi_3$
II	$\Psi_1 = \Psi_3 = \Psi_4 = 0 \quad \Psi_0 \neq 0,\ \Psi_2 \neq 0$	$\Psi_0 \leftrightarrow \Psi_4$
D	$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \quad \Psi_2 \neq 0$	-
III	$\Psi_0 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \quad \Psi_1 \neq 0$	$\Psi_1 \leftrightarrow \Psi_3$
N	$\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \quad \Psi_0 \neq 0$	$\Psi_0 \leftrightarrow \Psi_4$

Table 4.1.: Petrov types classified by Weyl scalars

If we add to this classification the completely degenerated case of conformally flat space-times in which the Weyl tensor vanishes (called type 0), then all types can be arranged in a triangular hierarchy as suggested by Penrose.

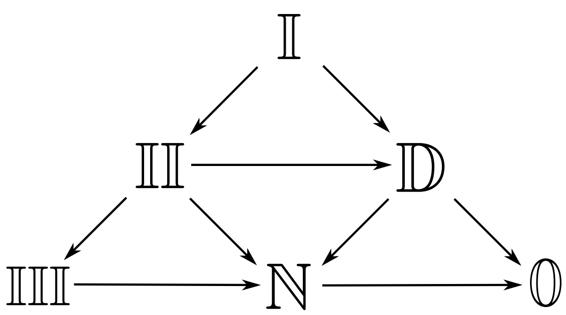

Figure 4.1.: A schematic classification of the different Petrov types suggested by R. Penrose. The arrows point in the direction of increasing specialization.

We want to consider an alternative method in classifying principal null directions by Debever. We will recall this classification in section () to understand the physical properties of the quasi-Kinnersley frame:

Theorem: *Every vacuum space-time admits at least one and at most four null directions*

$l^a \neq 0$, $l^a l_a = 0$, *which satisfy*

$$l_{[a}R_{b]ef[c}l_{d]}l^e l^f = 0. \tag{4.52}$$

If these principal null directions coincide, and the way they coincide leads to this classification. The details are shown in Table 4.2.

Petrov Type	Description
I	Four distinct principal null directions
II	Two principal null directions coincide
D	Principal null directions coincide in couples
III	Three principal null directions coincide
N	All four principal null directions coincide

Table 4.2.: Petrov types classified by the coincidence of the principal null directions

We will employ this classification to assign a physical meaning to the Weyl scalars in the next section.

4.8. Physical Interpretation of the Weyl Scalars & Peeling-off Theorem

In an excellent work Sachs demonstrated that the Riemann tensor of an asymptotically flat isolated radiative system can be expanded according to

$$R \propto \frac{N}{r} + \frac{III}{r^2} + \frac{II}{r^3} + \frac{I}{r^4} + \frac{I'}{r^5} + \mathscr{O}(r^{-6}), \tag{4.53}$$

in terms of an affine parameter r along each outward null ray. For simplicity we suppressed constants coefficients. He further demonstrated that if the tetrad is chosen appropriately, then the Weyl scalars satisfy the peeling theorem near infinity:

$$\lim_{r\to\infty} \Psi_n \propto \frac{1}{r^{5-n}}. \tag{4.54}$$

This indicates that, beside the strong contribution from the background, only Ψ_4 falls-off slowly enough to be non-zero when integrated over a large sphere near infinity.

Considering a radiative system where we start in the wave-zone (i.e. in the asymptotically flat regime) the Riemann tensor will be of type N according to Eq. (4.53), with a fourfold repeated principal null directions according to Table 4.2. The other principal null directions *peel of* as we "move closer" towards the source of radiation, where terms of less special nature predominate, as illustrated in Fig. 4.8. This is known as the *peeling-off theorem*.

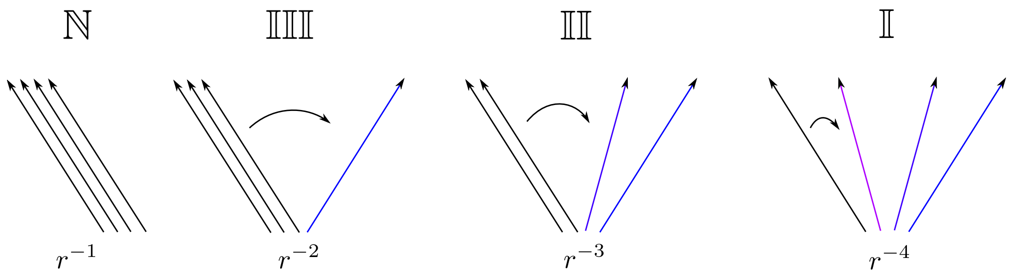

Figure 4.2.: The peeling-off theorem.

So far we have only mentioned that in general the Weyl scalars can be associated with physical observable quantities. In order to deduce this direct physical interpretation for the Weyl scalars, the natural approach is to consider their effect on the geodesic deviation equation similar to the derivation of the influence of the metric perturbation $h_{\mu\nu}$ on a ring of particles in chapter 3.

The pioneering work was done by Szekeres [59] where he investigated the effect of type N, III and D fields on a cloud of test particles.

We will outline his treatment and deduce the relevant physical properties of the Weyl scalars in different Petrov types.

Consider the geodesic worldline u^μ of an observer and let δx^μ be the displacement between neighboring geodesics, such that $u_\mu \delta x^\mu = 0$. The geodesic deviation equation in vacuum is similar to Eq. (3.30)

$$\delta \ddot{x}^\mu = R^\mu{}_{\nu\sigma\rho} u^\nu u^\sigma \delta x^\rho, \tag{4.55}$$

where we can substitute the Riemann tensor with the Weyl tensor in case of vacuum space-time. We set up a coordinate system (x^μ, y^μ, z^μ) to measure the imposed distortion on a ring of particles by the Weyl scalars in three dimensional space.

4.8.1. Petrov Type N

As outlined in the last section we end up having only Ψ_0 (or Ψ_4) non-zero in the most specialized Petrov type by making use of the tetrad rotations, see table 4.1. In this particular case the geodesic deviation equation (4.55) reads according to Szekeres [59]

$$\delta\ddot{x}^{\mu} = \Psi_0\left[\left(x^{\mu}x_{\nu} - y^{\mu}y_{\nu}\right) - i\left(x_{\nu}y^{\mu} - x^{\mu}y_{\nu}\right)\right]\delta x^{\nu}, \tag{4.56}$$

and considering only the real part we end up with

$$\delta\ddot{x}^{\mu} = \left[\Re\left(\Psi_0\right)\left(x^{\mu}x_{\nu} - y^{\mu}y_{\nu}\right) + \Im\left(\Psi_0\right)\left(x_{\nu}y^{\mu} - x^{\mu}y_{\nu}\right)\right]\delta x^{\nu}. \tag{4.57}$$

Since we have only performed type I and type II rotations so far we can make use of the remaining degree of freedom related to type III tetrad rotations to set the imaginary part $\Im\left(\Psi_0\right)$ to zero, yielding

$$\delta\ddot{x}^{\mu} = \Re\left(\Psi_0\right)\left(x^{\mu}x_{\nu} - y^{\mu}y_{\nu}\right)\delta x^{\nu}. \tag{4.58}$$

The resulting force on a ring of test particles located at $z = 0$ turns out being a transverse distortion. Szekeres terms this a *pure transverse* gravitational wave. This holds for Ψ_0 as well as for Ψ_4.

4.8.2. Petrov Type III

For a type III space-time all scalars vanish except Ψ_3 (or Ψ_1), see table 4.1. Szekeres demonstrated that the geodesic equation reads

$$\delta\ddot{x}^{\mu} = \Psi_1\left[\left(z^{\mu}x_{\nu} + x^{\mu}z_{\nu}\right) - i\left(z_{\nu}y^{\mu} + y^{\mu}z_{\nu}\right)\right]\delta x^{\nu}. \tag{4.59}$$

As in case of Petrov type N we can utilize spin transformations to set $\Im\left(\Psi_1\right) = 0$, thus ending up with

$$\delta\ddot{x}^{\mu} = \Re\left(\Psi_1\right)\left(z^{\mu}x_{\nu} + x^{\mu}z_{\nu}\right)\delta x^{\nu}. \tag{4.60}$$

Obviously, the effect on the ring of test particles located at $z=0$ in type III is still planar, but the wave contains a longitudinal directional contribution, i.e. along z^μ. Thus the two scalars Ψ_1 and Ψ_3 possess a longitudinal contribution which is the reason Szekeres terms this a *longitudinal wave* component.

4.8.3. Petrov Type D

Here the only non-vanishing Weyl scalar is Ψ_2 (table 4.1). The geodesic deviation equation turns out as

$$\delta\ddot{x}^\mu = \Psi_2 \left[z^\mu z_\nu - \frac{1}{2}\left(x^\mu y_\nu - y^\mu x_\nu\right) \right] \delta x^\nu . \tag{4.61}$$

The force on a ring of test particles results in a distortion to an ellipsoid with major contribution in the orthogonal direction. We will demonstrate in section 4.12 that an isolated black hole space-time is exactly Petrov type D. Furthermore, since Ψ_2 is the only non-vanishing scalar in type D it can be shown that the tidal force action on a radially in-falling observer can be calculated according to Eq. (4.61) in a Schwarzschild space-time. Szekeres terms this a *Coulomb-type* field.

4.8.4. Petrov Type II

According to the classification by Petrov the non-vanishing scalars for a type II field are Ψ_2 and Ψ_0 (or Ψ_4). Employing the results we have obtained for Petrov type D and type N we can relate Ψ_2 to the *Coulomb-type* contribution and Ψ_0 gives a *transverse contribution*, namely

$$\delta\ddot{x}^\mu = \left[\Re\left(\Psi_0\right)\left(x^\mu x_\nu - y^\mu y_\nu\right) - \Im\left(\Psi_0\right)\left(x_\nu y^\mu + x^\mu y_\nu\right)\right] \delta x^\nu , \tag{4.62}$$

where we perform a type III rotation to set $\Im\left(\Psi_0\right) = 0$. Thus the Weyl scalars in type II space-time are related to a superposition of a *Coulomb-type* field (Ψ_2) and a radiative wave component (Ψ_0).

4.8.5. Petrov Type I

For an algebraically general space-time we can find a particular tetrad where only Ψ_0, Ψ_2 and Ψ_4 are non-vanishing. With the results we have worked out for Petrov types N, D and II we can immediately conclude that the Weyl scalars in type I correspond to a *Coulomb-type* field (Ψ_2) superposed with two transverse radiative fields (Ψ_0 and Ψ_4).

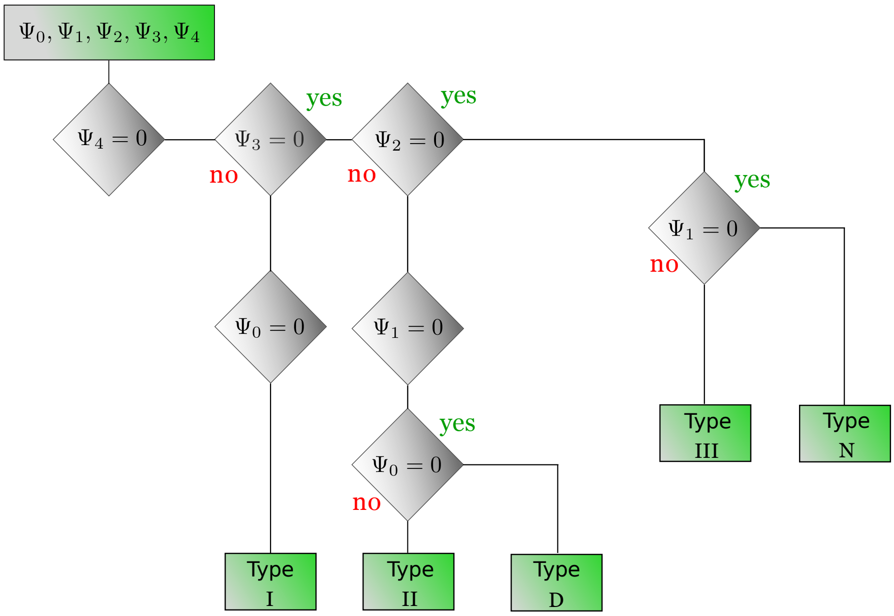

Figure 4.3.: A schematic classification of the different Petrov types.

4.9. Goldberg-Sachs Theorem

The Goldberg-Sachs theorem has been very useful in constructing algebraically special exact solutions of Einstein's vacuum equations. Most of the physically meaningful vacuum exact solutions, e.g. Kerr and Schwarzschild solution, are algebraically special. Its original formulation states that *a vacuum solution is algebraically special if and only if it contains a shear-free null geodesic congruence.*

According to the physical properties of the spin coefficients we derived in section

4.3, we can reformulate this statement to deduce the consequences of the Goldberg-Sachs theorem for the Weyl scalars and spin coefficients in the Newman-Penrose formalism. The proof of the theorem can be found in [64].

If a space-time is of Petrov type II and a null basis is chosen such that l is the repeated null direction and so $\Psi_0 = \Psi_1 = 0$, then the two spin coefficients κ and σ are vanishing; conversely if $\kappa = \sigma = 0$ the $\Psi_0 = \Psi_1 = 0$ and the space-time is of Petrov type II.

A consequence of this, which is commonly referred to as the Goldberg-Sachs theorem, states that

Theorem 1 *If a space-time is of Petrov type D and a null basis is chosen such that l is one repeated null direction and n is the other one, such that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, then the spin coefficients κ, σ, ν and λ are vanishing; conversely if $\kappa = \sigma = \nu = \lambda = 0$ then $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and the space-time is of Petrov type D.*

4.10. Bondi Frame

The calculations of gravitational radiation at future null infinity are based on the foundational works of Bondi [65] and Sachs [17] and have led to the extremely important conclusion that a gravitationally radiating system loses mass, and further to the identification of the Bondi-Metzner-Sachs asymptotic symmetry group [12, 11, 66, 67, 62, 65, 17]. The detailed derivation of these important but highly complex results goes beyond the scope of this work. Therefore we will only present the key results.

The identification of the Bondi frame relies upon the identification of a family of null surfaces labeled by a retarded time-coordinate u. In flat space-time a suitable family might be the null cones emanating from a time-like world line. Each of these surfaces is generated by a two-parameter family of null geodesics each labeled by sphere coordinates (θ, ϕ) or equivalently by complex stereographic coordinates $(\zeta, \bar{\zeta})$, where $\zeta = e^{i\phi} \cot(\theta/2)$. The "length" along the geodesics is given by the affine "radial" parameter r. In summary, in the standard literature a null coordinate system is commonly chosen as

$$\left(x^0, x^1, x^2, x^3\right) = \left(u, r, (\xi + \bar{\xi}), \frac{1}{i}(\xi - \bar{\xi})\right). \tag{4.63}$$

In fact, there is a large equivalence class of such coordinates as members of the Bondi-Metzner-Sachs group.

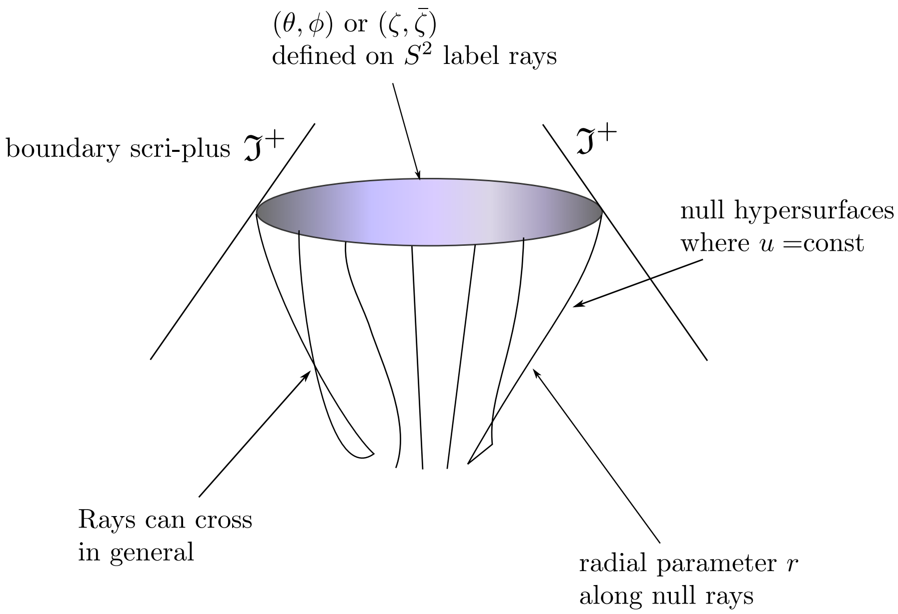

Figure 4.4.: Illustrating the idea of Bondi coordinates at future null infinity.

There is a natural choice of a null tetrad system: The ℓ null vector is taken as the tangent vector to the null geodesics, i.e. $\ell_a = \partial_a u = (1,0,0,0)$, while the $\left(\zeta, \bar{\zeta}\right)$ are tangent to the null 3-surface. Their remaining freedom is greatly limited by having them parallel propagated along the null geodesics. When these restrictions are translated to the tetrad we yield

$$D = \ell^a \nabla_a = \frac{\partial}{\partial r} \tag{4.64a}$$

$$\Delta = n^a \nabla_a = \frac{\partial}{\partial u} + U \left(\frac{\partial}{\partial r}\right) + X^i \left(\frac{\partial}{\partial x^i}\right) \tag{4.64b}$$

$$\delta = m^a \nabla_a = \omega \frac{\partial}{\partial r} + \xi^i \left(\frac{\partial}{\partial x^i}\right) \tag{4.64c}$$

where $X^i = (x^3, x^4) = (\zeta, \bar{\zeta})$. ξ^i, U and ω are arbitrary functions of the coordinates constituting the metric according to

$$g^{22} = 2(U - \omega\bar{\omega}), \tag{4.65}$$

$$g^{2A} = X^A - (\bar{\omega}\zeta^A + \omega\bar{\zeta}^A), \tag{4.66}$$

$$g^{AB} = -(\bar{\zeta}^A\zeta^B + \zeta^A\bar{\zeta}^B). \tag{4.67}$$

Keeping ($\xi = \text{const}$) and ($\bar{\xi} = \text{const}$) on the null hypersurface $u = \text{const}$, one moves along a null direction by increasing r.

4.11. Kinnersley Tetrad

A particular null frame in the Newman-Penrose formalism is constructed by making the following definition [16]:

Def. 1 *A Kinnersley frame for a type D space-time is a frame where the two real tetrad null vectors coincide with the two repeated principal null directions of the Weyl tensor.*

These conditions fix 4 of 6 degrees of freedom of the Lorentz group corresponding to type I and type II rotations.

Kinnersley enforced the additional condition $\varepsilon = 0$ to single out a particular tetrad, commonly known as the Kinnersley tetrad. The motivation for his choice is presumably related to the physical properties of the ℓ null vector: the geodesic equation for ℓ reads

$$\ell^\mu \nabla_\mu \ell^\nu = (\varepsilon + \varepsilon^*)\,\ell^\nu - \kappa\bar{m}^\nu - \kappa^* m^\nu, \tag{4.68}$$

which shows that if the two spin coefficients κ and ε vanish, the vector ℓ^μ is geodesic and affinely parametrized.

In the limit of type D the Goldberg-Sachs theorem (cf. section 4.9) guarantees that $\kappa = 0$, so the additional condition $\varepsilon = 0$ enforces the affine parameterization of ℓ^μ.

In the Kinnersley tetrad all scalars vanish except Ψ_2, i.e. it is a canonical frame [68] for Petrov type D, thus the speciality index S in Eq. (4.37) reduces to

$$S \to \frac{27\left(-\Psi_2^3\right)^2}{\left(3\Psi_2^2\right)^3} = 1. \tag{4.69}$$

4.12. Black Hole Space-Times in the NP Formalism

As an example of the Newman-Penrose formalism we calculate all relevant quantities of the Kerr solution. Since we expect every distorted black hole to relax to the Kerr (Schwarzschild) solution it plays a crucial role as the final state in every black hole simulation[1]. We start by defining the real null vectors ℓ and n in terms of radial null geodesics and adjoin orthogonally to them the complex conjugated null vector pair $(m, \bar{m})$, yielding

$$\ell^\mu = \frac{1}{\Delta}\left(r^2+a^2, \Delta, 0, a\right), \tag{4.70a}$$

$$n^\mu = \frac{1}{2\Sigma}\left(r^2+a^2, -\Delta, 0, a\right), \tag{4.70b}$$

$$m^\mu = \frac{1}{\sqrt{2}\bar{\Sigma}}\left(ia\sin\theta, 0, 1, i\,\mathrm{cosec}\,\theta\right), \tag{4.70c}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\bar{\Sigma} = r + ia\cos\theta$. Such a tetrad still satisfies the null vector conditions in Eqs. (4.22). We calculate the spin coefficients according to Eqs. (4.28) yielding

$$\rho = -\frac{1}{r - ia\cos\theta}, \qquad \beta = -\frac{\cot\theta}{2\sqrt{2}\bar{\Sigma}}, \tag{4.71}$$

$$\pi = \frac{ia\sin\theta}{\tilde{\Sigma}\sqrt{2}}, \qquad \tau = -\frac{ia\sin\theta}{\Sigma\sqrt{2}}, \tag{4.72}$$

$$\mu = -\frac{\Delta}{2\Sigma\tilde{\Sigma}}, \qquad \gamma = \mu + \frac{r-M}{2\Sigma}, \tag{4.73}$$

$$\alpha = \pi - \beta, \qquad \kappa = \sigma = \nu = \lambda = 0. \tag{4.74}$$

[1] This can be regarded as a consequence of Birkhoff's theorem in general relativity [69].

In agreement with the Goldberg-Sachs theorem in section 4.9 we find that the spin coefficients $\kappa, \sigma, \nu, \lambda$ vanish, leading us to the conclusion that the Kerr space-time is Petrov type D. Secondly, we can conclude that the only non-vanishing Weyl scalar is Ψ_2. Evaluating the Weyl scalars yields the expected result that

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{4.75}$$

and Ψ_2 reads

$$(\Psi_2)_K = -\frac{M}{(r - ia\cos\theta)^3}. \tag{4.76}$$

As usual in the limit of vanishing rotation, $a \to 0$, we recover the Schwarzschild solution.

4.13. Perturbation Approach in the NP Formalism

After introducing the basic concepts of the Newman-Penrose formalism in the forgoing sections we are now able to continue our analysis of black hole perturbations by presenting Teukolsky's results for the Kerr space-time. In addition, we will be able to enhance our understanding of the physical meaning of the Weyl scalars in the wave-zone within perturbation theory.

The perturbation equation for Schwarzschild (cf. Bardeen-Press equation [14]) can be easily recovered from Teukolsky's solution by letting the rotation parameter a go to zero. The problem is of considerable complexity; therefore we will only outline the vacuum solution and refer to the original paper by Teukolsky [15] for a detailed description.

4.13.1. The Perturbation Equations

The first step in Teukolsky's analysis is to consider a type D space-time, i.e. an isolated (Kerr) black hole which will be gravitationally perturbed later on. We start our analysis by defining a tetrad for wave-extraction where we can connect the transverse gravitational radiation with the Weyl scalars Ψ_0 and Ψ_4. As described in section 4.7 and section 4.8, a well-suited tetrad for such a task is the Kinnersley tetrad, where we know from the Goldberg-Sachs theorem that

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0; \quad \Psi_2 \neq 0; \tag{4.77a}$$

$$\kappa = \sigma = \nu = \lambda = 0; \quad \rho, \mu, \tau, \pi, \gamma, \beta, \alpha \neq 0. \tag{4.77b}$$

Teukolsky managed to combine the non-vanishing Bianchi identities and Ricci identities (cf. section 4.1) for the first-order perturbation of the radiation scalars Ψ_0^P and Ψ_4^P in such a way that he yielded the following two decoupled equations:

$$\begin{aligned} [(D-3\varepsilon+\varepsilon^*-4\rho-\rho^*)(\Delta-4\gamma+\mu) \quad & - \\ (\delta+\pi^*-\alpha^*-3\beta-4\tau)(\delta^*+\pi-4\alpha)-3\Psi_2^B\Big]\Psi_0^P \quad & = \quad 0 \end{aligned} \tag{4.78a}$$

$$\begin{aligned} [(D-3\varepsilon+\varepsilon^*-4\rho-\rho^*)(\Delta-4\gamma+\mu) \quad & - \\ (\delta+\pi^*-\alpha^*-3\beta-4\tau)(\delta^*+\pi-4\alpha)-3\Psi_2^B\Big]\Psi_4^P \quad & = \quad 0 \end{aligned} \tag{4.78b}$$

where D, Δ, δ and δ^* are the usual directional derivatives defined in Eqs. (4.8)

$$D = l^\mu \nabla_\mu,\ \Delta = n^\mu \nabla_\mu,\ \delta = m^\mu \nabla_\mu,\ \delta^* = \bar{m}^\mu \nabla_\mu. \tag{4.79}$$

These two decoupled equations (4.78a, 4.78b) carry all non-trivial features of the spacetime, i.e. they describe to linear order the dynamics of a gravitationally perturbed Kerr black hole.

4.13.2. Teukolsky Master equation

The next step is to write out the equations in a particular coordinate system. We make use of pseudo-spherical coordinates (Boyer-Lindquist coordinates), where the metric of a Kerr black hole reads

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \left(\frac{4Mar\sin^2\theta}{\Sigma}\right) dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \sin^2\theta \left(\frac{\tilde{\Delta}}{\Sigma}\right) d\phi^2, \tag{4.80}$$

with $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, $\tilde{\Delta} = r^2 + a^2 + 2Mar\sin^2\theta$, M is the mass of the black hole and Ma its angular momentum. The Kinnersley tetrad of a Kerr space-time in pseudo-spherical coordinates reads

$$\ell^\mu = \frac{1}{\Delta}\left(r^2 + a^2, \Delta, 0, a\right), \tag{4.81a}$$

$$n^\mu = \frac{1}{2\Sigma}\left(r^2 + a^2, -\Delta, 0, a\right), \tag{4.81b}$$

$$m^\mu = \frac{1}{\sqrt{2\bar{\Sigma}}}\left(ia\sin\theta, 0, 1, i\,\mathrm{cosec}\theta\right). \tag{4.81c}$$

The non-vanishing spin coefficients, which can be derived by using their definitions in Eq. (4.28), turn out as

$$\rho = -\frac{1}{r - ia\cos\theta}, \qquad \beta = -\frac{\cot\theta}{2\sqrt{2\bar{\Sigma}}}, \tag{4.82a}$$

$$\pi = \frac{ia\sin\theta}{\tilde{\Sigma}\sqrt{2}}, \qquad \tau = -\frac{ia\sin\theta}{\Sigma\sqrt{2}}, \tag{4.82b}$$

$$\mu = -\frac{\Delta}{2\Sigma\tilde{\Sigma}}, \qquad \gamma = \mu + \frac{r-M}{2\Sigma}, \tag{4.82c}$$

$$\alpha = \pi - \beta. \tag{4.82d}$$

By inserting the explicit form of the metric (4.80), null vectors (4.81) and spin coefficients (4.82) into Eqs. (4.78) Teukolsky managed to unify the perturbation equation for Ψ_0 and Ψ_4 to

$$\mathscr{P}_s\psi = 0, \tag{4.83}$$

where $\psi = \Psi_0$ or $\psi = \rho^{-4}\Psi_4$ and the operator $\mathscr{P}_s$ is given by

$$\begin{aligned}
\mathscr{P}_s = {} & \left[\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\phi^2} \\
& -\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a\left(r-M\right)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi} \\
& -2s\left[\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + \left(s^2\cot^2\theta - s\right)
\end{aligned} \tag{4.84}$$

with the spin weight s of the field (2 for Ψ_0, -2 for Ψ_4). This equation is valid equally well for a scalar field ($s=0$), a neutrino field ($s=\pm 1$) and an electromagnetic field ($s=\pm\frac{1}{2}$).

4.13.3. Asymptotic Behavior

Our particular interest lies in the behavior of the fields Ψ_0 and Ψ_4 in the regime where we will observe gravitational radiation. Teukolsky took advantage of the underlying symmetries of the space-time and managed to separate the solution to Eq. (4.83) in the following manner

$$\psi = e^{-i\omega t} e^{i\omega\phi} S(\theta) R(r) \tag{4.85}$$

where we focus our attention on the radial part $R(r)$ of the operator in Eq. (4.84) which reads

$$\Delta^{-s} \frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial R}{\partial r}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0, \tag{4.86}$$

with $K = \left(r^2+a^2\right)\omega - am$ and $\lambda = A + a^2\omega^2 - 2am\omega$. The unknown separation constant $A = {}_sA_l^m(a\omega)$ is given as a solution to the Sturm-Liouville eigenvalue problem for the angular equation $S(\theta)$. As pointed out, our main interest lies in the behavior of the quantities at future null infinity, i.e. when $r\to\infty$. Teukolsky introduced two additional quantities, namely

$$Y = \Delta^{s/2}\left(r^2+a^2\right)^{\frac{1}{2}} R(r), \tag{4.87a}$$

$$\frac{d\tilde{r}}{dr} = \frac{r^2+a^2}{\Delta}, \tag{4.87b}$$

and thus Eq. (4.86) can be rewritten as

$$\frac{\partial^2 Y}{\partial \tilde{r}^2} + \left[\frac{K^2 - 2is(r-M)K + \Delta(4ir\omega\omega s - \lambda)}{(r^2+a^2)^2} - G^2 - \frac{\partial G}{\partial \tilde{r}}\right] Y = 0, \tag{4.88}$$

where $G = \frac{s(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2}$. At null infinity (where $r \to \infty$ and consequently $\tilde{r} \to \infty$) the leading order in Eq. (4.86) reads

$$\frac{\partial^2 Y}{\partial \tilde{r}^2} + \left(\omega^2 + \frac{2i\omega s}{r}\right) Y \approx 0. \tag{4.89}$$

The solution to this equations can be easily worked out, yielding

$$Y \approx r^{\pm s} e^{\mp i\omega\tilde{r}}. \tag{4.90}$$

Consequently, the solutions for the radial part of Eq. (4.85) can be obtained from Eq. (4.89) and Eq. (4.87a)

$$R_1(\tilde{r}) \approx \frac{e^{-i\omega\tilde{r}}}{r}, \tag{4.91a}$$

$$R_2(\tilde{r}) \approx \frac{e^{i\omega\tilde{r}}}{r^{2s+1}}. \tag{4.91b}$$

From these results Teukolsky derived the radial behavior for the Weyl scalars Ψ_0 and Ψ_4 to linear order, namely

$$\begin{aligned} \Psi_0 &\approx \tfrac{e^{i\omega\tilde{r}}}{r^5}, \quad \Psi_4 \approx \tfrac{e^{i\omega\tilde{r}}}{r}, \quad \text{(outgoing waves)};\\ \Psi_0 &\approx \tfrac{e^{-i\omega\tilde{r}}}{r}, \quad \Psi_4 \approx \tfrac{e^{-i\omega\tilde{r}}}{r^5}, \quad \text{(ingoing waves)}. \end{aligned}$$

The main importance of the expressions for the Weyl scalars within the Teukolsky approach is related to the fact that the scalars are not only gauge invariant, but also independent of infinitesimal tetrad transformation (infinitesimal diffeomorphism invariance) and thus we ultimately deal with truly measurable physical quantities.

4.14. An Energy Measurement

Here we want to give an estimation of the energy flux through the Weyl scalars as we did in a similar fashion for the metric perturbation $h_{\mu\nu}$ in section 3.4.

In a vacuum we can write Ψ_4 in terms of the perturbatively relevant Riemann tensor components [64], which in the weak field approximation are only $R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}$ and $R_{\hat{t}\hat{\theta}\hat{t}\hat{\phi}}$:

$$\Psi_4 = -\left(R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} - iR_{\hat{t}\hat{\theta}\hat{t}\hat{\phi}}\right). \tag{4.92}$$

Thus we can relate Ψ_4 to the metric perturbation quantities $h^{TT}_{\mu\nu}$ in the transverse-traceless gauge by utilizing the definition of the linearized Riemann tensor in terms of the metric perturbation in Eq. (3.22), namely

$$R_{\hat{t}\hat{\mu}\hat{t}\hat{\nu}} = -\frac{1}{2}h^{TT}_{\hat{\mu}\hat{\nu},\hat{t}\hat{t}}. \tag{4.93}$$

Inserting Eq. (4.93) in Eq. (4.92) we yield for the Weyl scalar Ψ_4

$$\Psi_4 = \frac{1}{2}\left(\frac{\partial^2 h^{TT}_{\hat{\theta}\hat{\theta}}}{\partial t^2} - i\frac{\partial^2 h^{TT}_{\hat{\theta}\hat{\phi}}}{\partial t^2}\right). \tag{4.94}$$

We want to stress that Eq. (4.94) relates the real and imaginary part of Ψ_4 to the two independent polarization states e_+ and $e_\times$, respectively (cf. section 3.2.2).

Finally, we can estimate the energy flux per solid angle by rewriting Eq. (3.41) in terms of the radiation scalar

$$\frac{\partial^2 E}{\partial t \partial \Omega} = \lim_{r\to\infty}\frac{r^2}{4\pi}\left[\left(\int_0^T \Re\left(\Psi_4\right)dt\right)^2 + \left(\int_0^T \Im\left(\Psi_4\right)dt\right)^2\right]. \tag{4.95}$$

4.15. Selecting the Proper Frame for Wave Extraction

In this chapter we have so far introduced the main concepts of the Newman-Penrose approach which are nowadays important in gravitational wave theory. We have introduced all important quantities and demonstrated how the Weyl scalars, encoding the radiation degrees of freedom, can be extracted in a coordinate independent way. But of course, the inherit degrees of freedom in general relativity (diffeomorphism invariance) remain in the theory.

As pointed out, the physical quantities extracted from a numerically evolved spacetime depend strongly on the tetrad choice. But in a numerical simulation it is important, in order to recover Teukolsky's results, to ensure that the tetrad we choose converges to the Kinnersley tetrad when our space-time settles down to an unperturbed black-hole solution. The crucial problem here is that in a numerical simulation we do not know *a priori* the structure of the background metric, and therefore we are not able to define the background tetrad in a robust and general way. In this section we will exploit this dependence and will introduce the notion of the *quasi-Kinnersley frame* which has been shown to be a crucial element for wave extraction. By utilizing the properties of the quasi-Kinnersley frame it is possible to single out the right frame independently from the parameters of the final background metric[2].

4.15.1. Transverse Frames & Quasi-Kinnersley Frame

A *quasi-Kinnersley frame*, is a tetrad frame defined for a general Petrov type I space-time, which converges continuously into the asymptotic Kinnersley frame in the limit of Petrov type D space-time. From the definition of the Kinnersley tetrad in section 4.11 we can make the following two statements for the quasi-Kinnersley frame [60, 19]:

Def. 2 *A Kinnersley frame is a frame where the two real tetrad null vectors ℓ and n converge to the two repeated principal null directions of the Weyl tensor in the limit of Petrov type D.*

From Def. 2 we can further derive the following proposition:

Def. 3 *A quasi-Kinnersley frame for a Petrov type I space-time is a frame where $\Psi_0\Psi_4 \to 0$ for $S \to 1$.*

C. Beetle et al. [60, 19] enforced the additional condition $\Psi_1 = \Psi_3 = 0$, which they then call *transverse frames*. In fact, it is only required that a quasi-Kinnersley frame satisfies the criterion in Def. 3 and we may drop this more stringent condition of transversality. An example of a quasi-Kinnersley frame where this condition is dropped can be found in [70] for the Bondi-Sachs metric.

[2]We want to stress again that the quasi-Kinnersley frame still constitutes an infinite number of tetrads, only one of them being the Kinnersley tetrad.

Nevertheless, transverse tetrads, as a particular subset of all quasi-Kinnersley frames, have turned out as a very useful construction for wave extraction in numerical relativity [23, 1]. Since the Weyl scalars Ψ_1 and Ψ_3 are associated with longitudinal radiation degrees of freedom we can eliminate these non-physical effects *a priori* by restricting our attention to transverse quasi-Kinnersley frames without any loss of generality.

It has been shown in [60, 1] that transverse frames come in threefold in type I space-times, and only one of them is the transverse quasi-Kinnersley frames. It is unique for any generic Petrov type I space-time, and hence lies its importance. Members of this class are defined to be quasi-Kinnersley tetrads, and hence up to type III transformations, there is only one transverse quasi-Kinnersley tetrad.

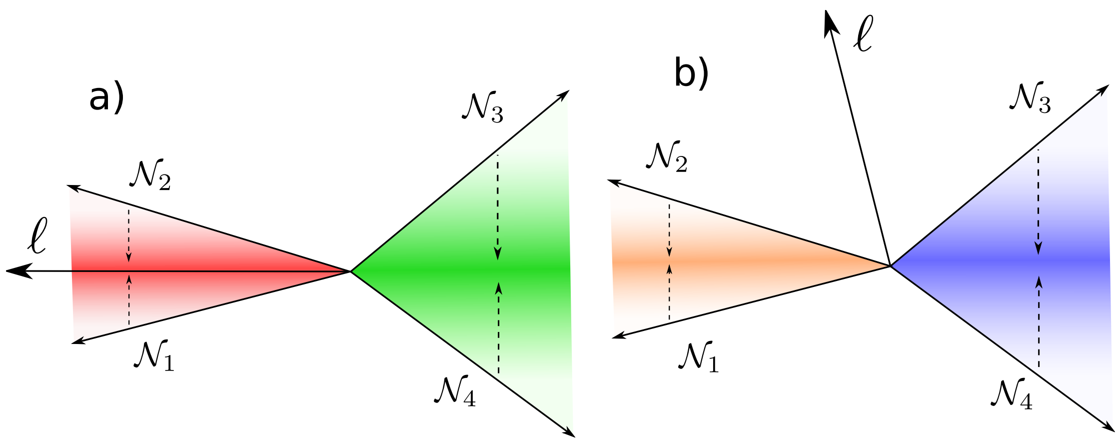

Figure 4.5.: (a) A transverse frame which is also a quasi-Kinnersley frame: in the limit of Petrov type D the principal null directions $\mathcal{N}_1$ and $\mathcal{N}_1$ will converge to ℓ. (b) A transverse frame which is not a quasi-Kinnersley frame: the ℓ vector of the frame sees the two principal null directions $\mathcal{N}_2$ and $\mathcal{N}_3$ as conjugate pair and they will not coincide with ℓ in the limit of type D [1].

4.15.2. Finding the Quasi-Kinnersley Frame

In [1] a mathematical procedure has been constructed to find the quasi-Kinnersley frame, while a method to single out the quasi-Kinnersley tetrad is still unknown. Before we will outline a method to break the remaining residual spin-boost symmetry, we review the procedure in [1] of finding the quasi-Kinnersley frame in a numerical simulation.

Assume a general situation having all five Weyl scalars non-vanishing in Petrov type I space-time; as demonstrated in section 4.6 we can utilize the definition of tetrad rotations to find a frame where Ψ_1 and Ψ_1 vanish. First, we perform a type I null rotation with parameter a and secondly a type II rotation with parameter b. Finally, we set the new values of Ψ_1 and Ψ_3 to zero therefore ending up with a system of two equations we wish to solve for parameters $\bar{a}$ and b:

$$\begin{aligned} 0 &= \left(\Psi_3+3\bar{a}\Psi_2+3\bar{a}^2\Psi_1+\bar{a}^3\Psi_3\right)b+\Psi_4+4\bar{a}\Psi_3+6\bar{a}^2\Psi_2+4\bar{a}^3\Psi_1+\bar{a}^4\Psi_0, \quad (4.96a)\\ 0 &= \Psi_1+\bar{a}\Psi_0+3b\left(\Psi_2+2\bar{a}\Psi_1+\bar{a}^2\Psi_0\right)+3b^2\left(\Psi_3+3\bar{a}\Psi_2+3\bar{a}^2\Psi_1+\bar{a}^3\Psi_0\right)\\ &\quad +b^3\left(\Psi_4+4\bar{a}\Psi_3+6\bar{a}^2\Psi_2+4\bar{a}^3\Psi_1+\bar{a}^4\Psi_0\right). \quad (4.96b)\end{aligned}$$

The equation for b is given by the explicit formula derived from Eqs. (4.96), namely

$$b=-\frac{\Psi_3+3\bar{a}\Psi_2+3\bar{a}^2\Psi_1+\bar{a}^3\Psi_0}{\Psi_4+4\bar{a}\Psi_3+6\bar{a}^2\Psi_2+4\bar{a}^3\Psi_1+\bar{a}^4\Psi_0}, \quad (4.97)$$

whereas we have to solve the following sixth order equation for the parameter $\bar{a}$

$$\mathscr{P}_1\bar{a}^6+\mathscr{P}_2\bar{a}^5+\mathscr{P}_3\bar{a}^4+\mathscr{P}_4\bar{a}^3+\mathscr{P}_5\bar{a}^2+\mathscr{P}_6\bar{a}+\mathscr{P}_7=0, \quad (4.98)$$

with

$$\begin{aligned} \mathscr{P}_1 &= -\Psi_3\Psi_0^2-2\Psi_1^3+3\Psi_2\Psi_1\Psi_0, & (4.99a)\\ \mathscr{P}_2 &= -2\Psi_3\Psi_1\Psi_0-\Psi_0^2\Psi_4+9\Psi_2^2\Psi_0-6\Psi_2\Psi_1^2, & (4.99b)\\ \mathscr{P}_3 &= -5\Psi_1\Psi_4\Psi_0-10\Psi_3\Psi_1^2+15\Psi_3\Psi_2\Psi_0, & (4.99c)\\ \mathscr{P}_4 &= -10\Psi_4\Psi_1^2+10\Psi_3^2\Psi_0, & (4.99d)\\ \mathscr{P}_5 &= 5\Psi_3\Psi_0\Psi_4+10\Psi_1\Psi_3^2-15\Psi_1\Psi_2\Psi_4, & (4.99e)\\ \mathscr{P}_6 &= 2\Psi_1\Psi_3\Psi_4+\Psi_4^2\Psi_0-9\Psi_2^2\Psi_4+6\Psi_2,\Psi_3^2 & (4.99f)\\ \mathscr{P}_7 &= \Psi_1\Psi_4^2+2\Psi_3^3-3\Psi_2\Psi_3\Psi_4. & (4.99g)\end{aligned}$$

Even though we have a sixth order polynomial there are only three independent solutions, corresponding to three transverse frames. This is due to a degeneracy of the transverse frames if we exchanged the null vectors ℓ with n and m with $\bar{m}$, respectively: the non-vanishing Weyl scalars would be exchanged as $\Psi_0 \rightleftarrows \Psi_4$.

Obviously, we can construct any tetrad in a numerical simulation as a starting point whereas then the main issue is to construct a solution to Eq. (4.96b); once we have obtained a solution of the polynomial, the parameter b is easily found. Since the Eq. (4.96, 4.96b) are well-posed (cf. [1]) it is always possible to find a transverse frame from a general Petrov type I space-time.

Motivated by a more geometrical ansatz it has been shown recently [60, 71] that the three different transverse frames correspond to the eigenvalues λ of a specific matrix $Q_{\mu\nu}$ built by contracting the Weyl tensor with the 4-velocity $\mathbf{u}$, $Q_{\mu\nu} = -C^*_{\mu\rho\nu\sigma} u^\rho u^\sigma$. The solutions of the characteristic polynomial[3]

$$\lambda^3 - 2I\lambda + 2J = 0, \tag{4.100}$$

are given by

$$\begin{aligned}
\lambda_1 &= -\left(P + \frac{I}{3P}\right), && (4.101a)\\
\lambda_2 &= -\left(e^{\frac{2\pi i}{3}} P + e^{\frac{4\pi i}{3}} \frac{I}{3P}\right), && (4.101b)\\
\lambda_3 &= -\left(e^{\frac{4\pi i}{3}} P + e^{\frac{2\pi i}{3}} \frac{I}{3P}\right), && (4.101c)
\end{aligned}$$

and P is defined as

$$P = \left[J + \sqrt{J^2 - (I/3)^3}\right]^{\frac{1}{3}}. \tag{4.102}$$

It is easy to see that Eq. (4.102) may lead to some ambiguity since the different solutions of the cubic root permute the definitions for the λ_i variables. Breaking this permutation symmetry is essential to the definition of the quasi-Kinnersley frame

[3]We like to refer for a comprehensive treatment of the subject to a book by Kramer et al., *Exact Solutions of Einstein's Field Equations* [72]

[60]. It can further be shown that the coulomb scalar Ψ_2 can be related to the three eigenvalues λ, namely

$$
\begin{aligned}
\Psi_2^{I} &= \frac{1}{2}\lambda_1, && (4.103a)\\
\Psi_2^{II} &= \frac{1}{2}\lambda_2, && (4.103b)\\
\Psi_2^{III} &= \frac{1}{2}\lambda_3, && (4.103c)
\end{aligned}
$$

The product of the transverse scalars can be specified accordingly

$$
\begin{aligned}
(\Psi_0\Psi_4)^{\mathrm{I}} &= \frac{(\lambda^{\mathrm{II}}-\lambda^{\mathrm{III}})^2}{4}, && (4.104)\\
(\Psi_0\Psi_4)^{\mathrm{II}} &= \frac{(\lambda^{\mathrm{I}}-\lambda^{\mathrm{III}})^2}{4}, && (4.105)\\
(\Psi_0\Psi_4)^{\mathrm{III}} &= \frac{(\lambda^{\mathrm{I}}-\lambda^{\mathrm{II}})^2}{4}. && (4.106)
\end{aligned}
$$

Thus, we are finally left with two methods to calculate all non-vanishing Weyl scalars in the three transverse frames. However, we are faced with a residual (type III) ambiguity in both approaches; We do not know the exact value of Ψ_0 and Ψ_4 but only the product $\Psi_0\Psi_4$, respectively.

5. Non-Perturbative Approach for Wave Extraction

I have noticed even people who claim everything is predestined,
and that we can do nothing to change it, look before they cross the road.
Stephen Hawking

In chapter 4 we presented the fundamental equations and physical results related to the Newman-Penrose formalism. We have introduced the Weyl scalars and demonstrated that they, extracted in a particular frame (the Kinnersley tetrad [16]), acquire a precise physical meaning. In fact, they carry all information about the space-time under consideration. In particular, we have demonstrated how Ψ_4 and Ψ_0 can be identified with the outgoing and ingoing gravitational radiation, respectively.

In section 4.15 we introduced the notion of *transverse tetrads*, satisfying the condition $\Psi_1 = \Psi_3 = 0$, and explained why those tetrads constitute a particular suitable choice for wave extraction. Furthermore, we described the most frequently applied method to find transverse frames in a numerical simulation; that is to calculate the Weyl scalars using an initial tetrad, and then calculate the rotation parameters for type I and type II rotations using the two methods given in [60, 1, 23]. This procedure is rather lengthy to apply in practice; moreover, the condition $\Psi_1 = \Psi_3 = 0$ itself does not fix the tetrad completely[1], leaving a spin-boost (type III rotation) ambiguity. As we discussed in section 4.11, Kinnersley imposed the additional condition $\varepsilon = 0$ to break the remaining symmetry.

Finally, we outlined a method to find the quasi-Kinnersley frame, as a tetrad frame defined for a general type I space-time. It turned out that the quasi-Kinnersley frame is

[1] only 4 of the 6 degrees of freedom of the Lorentz group of transformations are fixed

part of a general set of frames which satisfy the property $\Psi_0\Psi_4 \to 0$ when approaching the limit of type D (cf. Def. 3). By defining the Weyl scalars in this particular frame we assure that we recover Teukolsky's results in the limit of Type D. An equivalent statement is made by the peeling theorem [73, 18].

5.1. A new Formalism for Wave Extraction

In this chapter we will propose a new approach for wave extraction. We will present a general method in the Newman-Penrose formalism that relates the Weyl scalars to the connection coefficients (spin coefficients) when a specific choice of tetrad is performed, namely the one in which $\Psi_1 = \Psi_3 = 0$ and $\Psi_0 = \Psi_4$, which always exists in a general Petrov type I space-time. We use the approach to fix the optimal tetrad for gravitational wave extraction in numerical relativity, in particular by giving a canonical expression for the spin-boost parameter that was still unclear. The Weyl scalars Ψ_0, Ψ_2 and Ψ_4 are given as functions of the two space-time invariants I and J.

In fact, imposing the condition $\Psi_0 = \Psi_4$, what corresponds to $B = 1$ for the spin-boost degree of freedom, is not the best possible choice, since in this case the two transverse Weyl scalars have the radial fall-off of r^{-3} at future null infinity, which is contradictory to the prediction of the peeling-off theorem (cf. section 4.8). Nevertheless, we can use this choice as a starting point and reinsert the spin-boost degree of freedom into the expressions for the scalars.

The chapter is organized as follows: In section 5.2 we deduce a new expression for the three non-vanishing Weyl scalars Ψ_0, Ψ_2 and Ψ_4 in the transverse frames. In section 5.3 through section 5.5 we introduce the *directional derivatives* and analyze the Bianchi and Ricci identities in the transverse frames; moreover, we study the equations in the limit of Petrov type D. In section 5.6 we will show that the Bianchi identities provide a unique relation between spin coefficients in the limit of Petrov type D. An expression for ε is then obtained using the Ricci identities in section 5.7. Finally in section 5.8 we enforce the condition $\varepsilon = 0$ and obtain the corresponding spin-boost parameter. This result leads to the final expression for the Weyl scalars in section 5.9.

5.2. Redefining the Weyl Scalars in Transverse Frames

By requiring these two conditions, namely $\Psi_1 = \Psi_3 = 0$ and $\Psi_0 = \Psi_4$, the expressions for the two curvature invariants introduced in Eqs. (4.36) simplify to

$$
\begin{aligned}
I &= \Psi_4^2 + 3\Psi_2^2, && (5.1a)\\
J &= \Psi_4^2\Psi_2 - \Psi_2^3. && (5.1b)
\end{aligned}
$$

As discussed in detail in chapter 4 we recall that Ψ_2 is given by $\Psi_2 = \frac{1}{2}\lambda_i$, where λ_i represents the three different solutions of the characteristic polynomial

$$\lambda^3 - 2I\lambda + 2J = 0. \tag{5.2}$$

The three possible solutions are given by

$$
\begin{aligned}
\lambda_1 &= -\left(P + \frac{I}{3P}\right), && (5.3a)\\
\lambda_2 &= -\left(e^{\frac{2\pi i}{3}}P + e^{\frac{4\pi i}{3}}\frac{I}{3P}\right), && (5.3b)\\
\lambda_3 &= -\left(e^{\frac{4\pi i}{3}}P + e^{\frac{2\pi i}{3}}\frac{I}{3P}\right), && (5.3c)
\end{aligned}
$$

where P is defined as

$$P = \left[J + \sqrt{J^2 - (I/3)^3}\right]^{\frac{1}{3}}. \tag{5.4}$$

Eq. (5.1a) and Eq. (5.1b) can now be inverted to give not only Ψ_2 but also Ψ_4 as a function of the curvature invariants I and J. We start redefining the scalars by introducing the important variable $\Psi_\pm$, which unifies the three different solutions for the scalars in Eq. (5.3),

$$\Psi_\pm = I^{\frac{1}{2}}\left(e^{\frac{2\pi ik}{3}}\Theta \pm e^{-\frac{2\pi ik}{3}}\Theta^{-1}\right), \tag{5.5}$$

where k is an integer number assuming the values $\{0, 1, 2\}$ corresponding to the three different transverse frames and Θ is defined according to

$$\Theta = \sqrt{3} P I^{-\frac{1}{2}}. \tag{5.6}$$

It is worth noting that a key ingredient for establishing our new methodology is rewriting the Bianchi identities in terms of these newly introduced variables $\Psi_{\pm}$. We can now rewrite the three non-vanishing scalars in the transverse frames in the following manner

$$\Psi_0 = -\frac{i\mathscr{B}^{-2}}{2} \cdot \Psi_-, \tag{5.7a}$$
$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot \Psi_+, \tag{5.7b}$$
$$\Psi_4 = -\frac{i\mathscr{B}^{2}}{2} \cdot \Psi_-, \tag{5.7c}$$

where we have reinserted the spin-boost parameter B,

$$\mathscr{B} = \left(\frac{\Psi_4}{\Psi_0}\right)^{\frac{1}{4}}. \tag{5.8}$$

We want to stress the behavior of the introduced quantities in the limit of type D (cf. section 4.15). The speciality index in Eq. (4.37) reduces to $S \to 1$ from what we immediately deduce the behavior of the space-time invariants in the type D limit

$$J \to \sqrt{I^3/27}, \tag{5.9a}$$
$$P \to I^{1/2}/\sqrt{3}, \tag{5.9b}$$

and thus the quantity Θ in Eq. (5.6) reduces to

$$\Theta \to 1. \tag{5.10}$$

We now evaluate the quantities $\Psi_{\pm}$ in Eq. (5.5) for the frame with $k = 0$, yielding

$$\Psi_+ = I^{\frac{1}{2}}\left(\Theta + \Theta^{-1}\right) \to 2I^{\frac{1}{2}}, \tag{5.11a}$$
$$\Psi_- = I^{\frac{1}{2}}\left(\Theta - \Theta^{-1}\right) \to 0, \tag{5.11b}$$

and consequently, the original Weyl scalars simplify in Petrov type D according to

$$\begin{aligned}
\Psi_0 &= -\frac{i\mathscr{B}^{-2}}{2}\cdot\Psi_- \to 0, && (5.12a)\\
\Psi_2 &= -\frac{1}{2\sqrt{3}}\cdot\Psi_+ \to -I^{1/2}/\sqrt{3}, && (5.12b)\\
\Psi_4 &= -\frac{i\mathscr{B}^{2}}{2}\cdot\Psi_- \to 0. && (5.12c)
\end{aligned}$$

Since Ψ_0 and Ψ_4 tend to zero in Petrov type D we can conclude, by utilizing Def. 3, that the frame with $k = 0$ is the transverse frame which is also a quasi-Kinnersley frame.

5.3. The Bianchi Identities

In chapter 4 we introduced the Bianchi identities and defined the projection on to a tetrad frame. We now deduce the explicit expressions of all non-trivial terms in the Newman-Penrose formalism, given here in terms of the Weyl scalars and spin coefficients. As mentioned in section 4.1 the Bianchi identities can be expressed in the Newman-Penrose formalism according to

$$\begin{aligned}
R_{(a)(b)[(c)(d)|(f)]} = \frac{1}{6}\sum_{[(c)(d)(f)]}\Big\{ & R_{(a)(b)(c)(d),(f)} \\
& -\eta^{(n)(m)}\Big[\gamma_{(n)(a)(f)}R_{(m)(b)(c)(d)}+\gamma_{(n)(b)(f)}R_{(a)(m)(c)(d)} \\
& +\gamma_{(n)(c)(f)}R_{(a)(b)(m)(d)}+\gamma_{(n)(d)(f)}R_{(a)(b)(c)(m)}\Big]\Big\}. \qquad (5.13)
\end{aligned}$$

Written out explicitly, all non-vanishing identities in a vacuum space-time are given by the eight following complex identities

$$\begin{aligned}
-\delta^*\Psi_0 + \mathrm{D}\Psi_1 - 2(2\rho+\varepsilon)\Psi_1 + 3\kappa\Psi_2 + (4\alpha-\pi)\Psi_0 &= 0, && (5.14a)\\
-\delta^*\Psi_2 + D\Psi_3 + \kappa\Psi_4 - 3\pi\Psi_2 + 2\lambda\Psi_1 + 2(\varepsilon-\rho)\Psi_3 &= 0, && (5.14b)\\
-D\Psi_2 + \delta^* - \lambda\Psi_0 + 3\rho\Psi_2 + 2(\pi-\alpha)\Psi_1 - 2\kappa\Psi_3 &= 0, && (5.14c)\\
-\delta^*\Psi_3 + D\Psi_4 + 3\lambda\Psi_2 - 2(2\pi+\alpha)\Psi_3 + (4\varepsilon-\rho)\Psi_4 &= 0, && (5.14d)
\end{aligned}$$

$$
\begin{aligned}
-\Delta\Psi_0+3\sigma\Psi_2+(4\gamma-\mu)\Psi_0\delta\Psi_1-2(2\tau+\beta)\Psi_1 &= 0, && (5.14e)\\
-\Delta\Psi_1+\delta\Psi_2+\nu\Psi_0+2(\gamma-\mu)\Psi_1-3\tau\Psi_2+2\sigma\Psi_3 &= 0, && (5.14f)\\
-\Delta\Psi_2+\delta\Psi_3+2\nu\Psi_1+\sigma\Psi_4+2(\beta-\tau)\Psi_3-3\mu\Psi_2 &= 0, && (5.14g)\\
-\Delta\Psi_3+\delta\Psi_4+3\nu\Psi_2-2(\gamma+2\mu)\Psi_3-(\tau-4\beta)\Psi_4 &= 0. && (5.14h)
\end{aligned}
$$

There seems to be no obvious structure in the equations. But since we want to focus our attention on the expressions of the scalars in the transverse frames, we enforce the condition $\Psi_1=\Psi_3=0$, considerably simplifying the Bianchi identities:

$$
\begin{aligned}
D\Psi_4 &= -3\lambda\Psi_2-(4\varepsilon-\rho)\Psi_4, && (5.15a)\\
D\Psi_2 &= -\lambda\Psi_0+3\rho\Psi_2, && (5.15b)\\
\Delta\Psi_0 &= 3\sigma\Psi_2+(4\gamma-\mu)\Psi_0, && (5.15c)\\
\Delta\Psi_2 &= \sigma\Psi_4-3\mu\Psi_2, && (5.15d)\\
\delta\Psi_4 &= -3\nu\Psi_2+(\tau-4\beta)\Psi_4, && (5.15e)\\
\delta\Psi_2 &= -\nu\Psi_0+3\tau\Psi_2, && (5.15f)\\
\delta^*\Psi_0 &= 3\kappa\Psi_2+(4\alpha-\pi)\Psi_0, && (5.15g)\\
\delta^*\Psi_2 &= \kappa\Psi_4-3\pi\Psi_2. && (5.15h)
\end{aligned}
$$

We want to stress that we have recast the equations in a more convenient form to highlight the now appearing structure; we get a set of two coupled equations for every directional derivative.

5.4. The Ricci Identities

We will perform a similar procedure with the Ricci identities as done with the Bianchi identities. According to section 4.1 the Ricci identities are defined in the tetrad formalism as

$$
\begin{aligned}
R_{(a)(b)(c)(d)} = &- \gamma_{(a)(b)(c),(d)}+\gamma_{(a)(b)(d),(c)}-\gamma_{(b)(a)(f)}\left[\gamma_{(c)}{}^{(f)}{}_{(d)}-\gamma_{(d)}{}^{(f)}{}_{(c)}\right]\\
&+ \gamma_{(f)(a)(c)}\gamma_{(b)}{}^{(f)}{}_{(d)}-\gamma_{(f)(a)(d)}\gamma_{(b)}{}^{(f)}{}_{(c)}. \qquad (5.16)
\end{aligned}
$$

By projecting all non-vanishing Ricci identities on to the Newman-Penrose tetrad we yield a set of 18 complex equations:

$$
\begin{aligned}
D\rho - \delta^*\kappa &= \left(\rho^2 + \sigma\sigma^*\right) + \rho\,(\varepsilon + \varepsilon^*) - \kappa^*\tau - \kappa\,(3\alpha + \beta^* - \pi)\,, && (5.17a)\\
D\sigma - \delta\kappa &= \sigma\,(\rho + \rho^* + 3\varepsilon - \varepsilon^*) - \kappa\,(\tau - \pi^* + \alpha^* + 3\beta) + \Psi_0, && (5.17b)\\
D\tau - \Delta\kappa &= \rho\,(\tau + \pi^*) + \sigma\,(\tau^* + \pi) + \tau\,(\varepsilon - \varepsilon^*) - \kappa\,(3\gamma + \gamma^*) + \Psi_1, && (5.17c)\\
D\alpha - \delta^*\varepsilon &= \alpha\,(\rho + \varepsilon^* - 2\varepsilon) + \pi\,(\varepsilon + \rho) + \beta\sigma^* - \beta^*\varepsilon - \kappa\lambda - \kappa^*\gamma, && (5.17d)\\
D\beta - \delta\varepsilon &= \sigma\,(\alpha + \pi) + \beta\,(\rho^* - \varepsilon^*) - \kappa\,(\mu + \gamma) - \varepsilon\,(\alpha^* - \pi^*) + \Psi_1, && (5.17e)\\
D\gamma - \Delta\varepsilon &= \alpha\,(\tau + \pi^*) + \beta\,(\tau^* + \pi) + \tau\pi - \gamma(\varepsilon + \varepsilon^*) - \varepsilon\,(\gamma + \gamma^*) - \nu\kappa + \Psi_2, && (5.17f)\\
D\lambda - \delta^*\pi &= \rho\lambda + \sigma^*\mu + \pi\,(\pi + \alpha - \beta^*) - \lambda\,(3\varepsilon - \varepsilon^*) - \nu\kappa^*, && (5.17g)\\
D\mu - \delta\pi &= \rho^*\mu + \sigma\lambda + \pi\,(\pi + \alpha - \beta^*) - \lambda\,(3\varepsilon - \varepsilon^*) - \nu\kappa + \Psi_2, && (5.17h)\\
\delta\rho - \delta^*\sigma &= \rho\,(\alpha^* + \beta) - \sigma\,(3\alpha - \beta^*) + \tau\,(\rho - \rho^*) + \kappa\,(\mu - \mu^*) - \Psi_1. && (5.17i)
\end{aligned}
$$

The other nine Ricci identities can be obtained by swapping the tetrad vectors ℓ with n and m with $\bar{m}$.

In contrast to the Bianchi identities the Ricci identities do not simplify noticeable by enforcing the condition $\Psi_1 = \Psi_3 = 0$,

$$
\begin{aligned}
D\rho - \delta^*\kappa &= \left(\rho^2 + \sigma\sigma^*\right) + \rho\,(\varepsilon + \varepsilon^*) - \kappa^*\tau - \kappa\,(3\alpha + \beta^* - \pi)\,, && (5.18a)\\
D\sigma - \delta\kappa &= \sigma\,(\rho + \rho^* + 3\varepsilon - \varepsilon^*) - \kappa\,(\tau - \pi^* + \alpha^* + 3\beta) + \Psi_0, && (5.18b)\\
D\tau - \Delta\kappa &= \rho\,(\tau + \pi^*) + \sigma\,(\tau^* + \pi) + \tau\,(\varepsilon - \varepsilon^*) - \kappa\,(3\gamma + \gamma^*)\,, && (5.18c)\\
D\alpha - \delta^*\varepsilon &= \alpha\,(\rho + \varepsilon^* - 2\varepsilon) + \pi\,(\varepsilon + \rho) + \beta\sigma^* - \beta^*\varepsilon - \kappa\lambda - \kappa^*\gamma, && (5.18d)\\
D\beta - \delta\varepsilon &= \sigma\,(\alpha + \pi) + \beta\,(\rho^* - \varepsilon^*) - \kappa\,(\mu + \gamma) - \varepsilon\,(\alpha^* - \pi^*)\,, && (5.18e)\\
D\gamma - \Delta\varepsilon &= \alpha\,(\tau + \pi^*) + \beta\,(\tau^* + \pi) + \tau\pi - \gamma(\varepsilon + \varepsilon^*) - \varepsilon\,(\gamma + \gamma^*) - \nu\kappa + \Psi_2, && (5.18f)\\
D\lambda - \delta^*\pi &= \rho\lambda + \sigma^*\mu + \pi\,(\pi + \alpha - \beta^*) - \lambda\,(3\varepsilon - \varepsilon^*) - \nu\kappa^*, && (5.18g)\\
D\mu - \delta\pi &= \rho^*\mu + \sigma\lambda + \pi\,(\pi + \alpha - \beta^*) - \lambda\,(3\varepsilon - \varepsilon^*) - \nu\kappa + \Psi_2, && (5.18h)\\
\delta\rho - \delta^*\sigma &= \rho\,(\alpha^* + \beta) - \sigma\,(3\alpha - \beta^*) + \tau\,(\rho - \rho^*) + \kappa\,(\mu - \mu^*)\,. && (5.18i)
\end{aligned}
$$

5.5. Directional Derivatives

From the definition of the Lie bracket we derive a set of relations for the commutators and for the double derivatives in the Newman-Penrose formalism. As an example we calculate the Lie bracket of $[\Delta, D]$:

$$[\Delta, D] = [n, \ell] = -\gamma_{121}\Delta + \gamma_{212}D - (\gamma_{312} - \gamma_{321})\,\delta^* - (\gamma_{412} - \gamma_{421})\,\delta. \tag{5.19}$$

Introducing the designated symbols of the spin coefficients we derive the full set of commutation relations

$$\begin{aligned}
[\Delta, D] &= (\gamma + \gamma^*)\,D + (\varepsilon + \varepsilon^*)\,\Delta - (\tau^* + \pi)\,\delta - (\tau + \pi^*)\,\delta^*, && (5.20a)\\
[\delta, D] &= (\alpha + \beta - \pi^*)\,D + \kappa\Delta - (\rho^* + \varepsilon - \varepsilon^*)\,\delta - (\tau + \pi^*)\,\delta^*, && (5.20b)\\
[\delta, \Delta] &= -\nu^* D + (\tau - \alpha^* - \beta)\,\Delta + (\mu - \gamma + \gamma^*)\,\delta + \lambda^*\delta^*, && (5.20c)\\
[\delta, \delta^*] &= (\mu^* - \mu)\,D + (\rho^* - \rho)\,\Delta + (\alpha - \beta^*)\,\delta + (\beta - \alpha^*)\,\delta^*, && (5.20d)\\
[\delta^*, \Delta] &= -\nu D + (\tau^* - \alpha - \beta^*)\,\Delta + (\mu^* - \gamma^* + \gamma)\,\delta^* + \lambda\,\delta, && (5.20e)\\
[\delta^*, D] &= (\alpha^* + \beta^* - \pi)\,D + \kappa^*\Delta - (\rho + \varepsilon^* - \varepsilon)\,\delta^* - (\tau^* + \pi)\,\delta. && (5.20f)
\end{aligned}$$

Furthermore, we can calculate double derivatives in the Newman-Penrose formalism:

$$\begin{aligned}
DD &= (\varepsilon + \varepsilon^*)\,D - \kappa^*\delta - \kappa\delta^* + \ell^\mu \ell^\nu \nabla_\mu \nabla_\nu, && (5.21a)\\
\Delta\Delta &= -(\gamma + \gamma^*)\,\Delta + \nu\delta + \nu^*\delta^* + n^\mu n^\nu \nabla_\mu \nabla_\nu, && (5.21b)\\
\delta\delta &= \lambda^* D - \sigma\Delta + (\beta - \alpha^*)\,\delta + m^\mu m^\nu \nabla_\mu \nabla_\nu, && (5.21c)\\
\delta^*\delta^* &= \lambda D - \sigma^*\Delta - (\alpha - \beta^*)\,\delta^* + \bar{m}^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu, && (5.21d)\\
\Delta D &= (\gamma + \gamma^*)\,D - \tau^*\delta - \tau\delta^* + n^\mu \ell^\nu \nabla_\mu \nabla_\nu, && (5.21e)\\
D\Delta &= -(\varepsilon + \varepsilon^*)\,\Delta + \pi\delta + \pi^*\delta^* + \ell^\mu n^\nu \nabla_\mu \nabla_\nu, && (5.21f)\\
D\delta &= \pi^* D - \kappa\Delta + (\varepsilon - \varepsilon^*)\,\delta + \ell^\mu m^\nu \nabla_\mu \nabla_\nu, && (5.21g)\\
\delta D &= (\beta + \alpha^*)\,D - \rho^*\delta - \sigma\delta^* + m^\mu \ell^\nu \nabla_\mu \nabla_\nu, && (5.21h)
\end{aligned}$$

$$D\delta^* = \pi D - \kappa^*\Delta - (\varepsilon - \varepsilon^*)\,\delta^* + \ell^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu, \tag{5.21i}$$

$$\delta^* D = (\beta^* + \alpha)\,D - \sigma^*\delta - \rho\,\delta^* + \bar{m}^\mu \ell^\nu \nabla_\mu \nabla_\nu, \tag{5.21j}$$

$$\Delta\delta = \nu^* D - \tau\Delta + (\gamma - \gamma^*)\,\delta + n^\mu m^\nu \nabla_\mu \nabla_\nu, \tag{5.21k}$$

$$\delta\Delta = -(\beta + \alpha^*)\,\Delta + \mu\,\delta + \lambda^*\delta^* + m^\mu n^\nu \nabla_\mu \nabla_\nu, \tag{5.21l}$$

$$\Delta\delta^* = \nu D - \tau^*\Delta - (\gamma - \gamma^*)\,\delta^* + n^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu, \tag{5.21m}$$

$$\delta^*\Delta = -(\beta^* + \alpha)\,\Delta + \lambda\,\delta + \mu^*\delta^* + \bar{m}^\mu n^\nu \nabla_\mu \nabla_\nu, \tag{5.21n}$$

$$\delta\delta^* = \mu D - \rho^*\Delta - (\beta - \alpha^*)\,\delta^* + m^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu, \tag{5.21o}$$

$$\delta^*\delta = \mu^* D - \rho\Delta + (\alpha - \beta^*)\,\delta + \bar{m}^\mu m^\nu \nabla_\mu \nabla_\nu. \tag{5.21p}$$

We will make use of these directional derivatives and double derivatives in section 5.8 to determine the function H.

5.6. The Type D Spin Relation

As already mentioned, a key ingredient for the derivation of our new extraction formalism is rewriting the Bianchi identities in Eqs. (5.15, 5.18) in terms of the newly introduced variables $\Psi_{\pm}$. Since ε appears only in the first two Bianchi identities, namely Eqs. (5.15a, 5.15b), we will give the details of the calculation only for the derivative operator D; however, as the symmetry of the Bianchi identities suggests, the calculation for the other derivatives is analogous and trivial to perform, and we will use the symmetry properties at the end of this chapter to deduce the expressions for the spin coefficients γ, α and β.

We start by inserting Eq. (5.7), which relate the Weyl scalars Ψ_0, Ψ_2 and Ψ_4 to the new scalars Ψ_+ and Ψ_-, into the Bianchi identities yielding

$$D\Psi_+ = -\tilde{\lambda}\Psi_- + 3\rho\Psi_+, \tag{5.22a}$$

$$D\Psi_- = \tilde{\lambda}\Psi_+ - (4\tilde{\varepsilon} - \rho)\Psi_-, \tag{5.22b}$$

$$\Delta\Psi_+ = \tilde{\sigma}\Psi_- - 3\mu\Psi_+, \tag{5.22c}$$

$$\Delta\Psi_- = -\tilde{\sigma}\Psi_+ + (4\tilde{\gamma} - \mu)\Psi_-, \tag{5.22d}$$

$$\delta\Psi_+ = -\tilde{\nu}\Psi_- + 3\tau\Psi_+, \tag{5.22e}$$

$$\delta\Psi_- = \tilde{\nu}\Psi_+ - \left(4\tilde{\beta} - \tau\right)\Psi_-, \tag{5.22f}$$

$$\delta^*\Psi_+ = \tilde{\kappa}\Psi_- - 3\pi\Psi_+, \tag{5.22g}$$

$$\delta^*\Psi_- = -\tilde{\kappa}\Psi_+ + (4\tilde{\alpha} - \pi)\Psi_-, \tag{5.22h}$$

where we have additionally introduced the *rescaled spin coefficients*

$$\tilde{\lambda} = i\sqrt{3}\lambda\mathscr{B}^{-2}, \tag{5.23a}$$

$$\tilde{\sigma} = i\sqrt{3}\sigma\mathscr{B}^{2} \tag{5.23b}$$

$$\tilde{\nu} = i\sqrt{3}\nu\mathscr{B}^{-2}, \tag{5.23c}$$

$$\tilde{\kappa} = i\sqrt{3}\kappa\mathscr{B}^{2}, \tag{5.23d}$$

and

$$\tilde{\varepsilon} = \varepsilon + \frac{1}{2} D \ln \mathscr{B}, \tag{5.23e}$$

$$\tilde{\gamma} = \gamma + \frac{1}{2} \Delta \ln \mathscr{B}, \tag{5.23f}$$

$$\tilde{\beta} = \beta + \frac{1}{2} \delta \ln \mathscr{B}, \tag{5.23g}$$

$$\tilde{\alpha} = \alpha + \frac{1}{2} \delta^* \ln \mathscr{B}. \tag{5.23h}$$

This new set of rescaled spin coefficients now transforms in the same way under a spin-boost transformation (cf. section 4.6): for example the three spin coefficients $\left\{\rho, \tilde{\lambda}, \tilde{\varepsilon}\right\}$ transform according to

$$\rho \to |\mathscr{B}|^{-1} \rho, \tag{5.24a}$$

$$\tilde{\varepsilon} \to |\mathscr{B}|^{-1} \tilde{\varepsilon}, \tag{5.24b}$$

$$\tilde{\lambda} \to |\mathscr{B}|^{-1} \tilde{\lambda}, \tag{5.24c}$$

and analogous transformations for the other spin coefficients.

These transformation properties are not a surprising result when we consider how the individual quantities in Eq. (5.22a) and Eq. (5.22b) transform under a type III rotation:
Ψ_+ and Ψ_- are only functions of curvature invariants, therefore the only dependence on the spin-boost parameter on the left hand side comes from the ℓ^μ null vector of the D derivative operator which carries a $|\mathscr{B}|^{-1}$ factor. Obviously, the right hand side must be consistent and show the same spin-boost dependence in the rescaled spin coefficients, what is in fact the case as demonstrated in Eqs. (5.24).

We will now study the behavior of the Bianchi identities in the Petrov type D limit. By dividing Eq. (5.22a) by Ψ_+ and Eq. (5.22b) by Ψ_- the first two Bianchi identities become:

$$\frac{D\Psi_+}{\Psi_+} = -\tilde{\lambda}\frac{\Psi_-}{\Psi_+} + 3\rho, \tag{5.25a}$$

$$\frac{D\Psi_-}{\Psi_-} = \tilde{\lambda}\frac{\Psi_+}{\Psi_-} - (4\tilde{\varepsilon} - \rho). \tag{5.25b}$$

Employing the definition of $\Psi_\pm$ in Eq. (5.5) and applying the D operator to Ψ_+ and Ψ_- gives

$$D\Psi_+ = D\ln\Theta \cdot \Psi_- + D\ln\left(I^{\frac{1}{2}}\right)\Psi_+, \tag{5.26a}$$

$$D\Psi_- = D\ln\Theta \cdot \Psi_+ + D\ln\left(I^{\frac{1}{2}}\right)\Psi_-. \tag{5.26b}$$

In the limit of Petrov type D, what corresponds to $\Theta \to 1$ as demonstrated, these equations simplify to:

$$D\Psi_+ \to D\ln\left(I^{\frac{1}{2}}\right)\Psi_+, \tag{5.27a}$$

$$D\Psi_- \to D\ln\left(I^{\frac{1}{2}}\right)\Psi_-. \tag{5.27b}$$

This result implies that the left hand sides in Eq. (5.25) tend to the same value in type D, namely $D\ln\left(I^{\frac{1}{2}}\right)$, and therefore also the right hand sides can be set to be equal in this limit. Moreover, from the definition of $\Psi_\pm$ we can calculate the limit of the ratio $\frac{\Psi_-}{\Psi_+}$, in fact

$$\frac{\Psi_-}{\Psi_+} = \frac{I^{\frac{1}{2}}\left(e^{\frac{2\pi ik}{3}}\Theta - e^{-\frac{2\pi ik}{3}}\Theta^{-1}\right)}{I^{\frac{1}{2}}\left(e^{\frac{2\pi ik}{3}}\Theta + e^{-\frac{2\pi ik}{3}}\Theta^{-1}\right)} \to \frac{\left(e^{\frac{2\pi ik}{3}} - e^{-\frac{2\pi ik}{3}}\right)}{\left(e^{\frac{2\pi ik}{3}} + e^{-\frac{2\pi ik}{3}}\right)} = -i\tan\left(\frac{2\pi k}{3}\right). \tag{5.28}$$

Putting this all together, and subtracting Eq. (5.25b) from Eq. (5.25a) we find that the following relation between spin coefficients holds in the Petrov type D limit

$$(\rho + 2\tilde{\varepsilon})\sin\left(\frac{4\pi k}{3}\right) + i\tilde{\lambda}\cos\left(\frac{4\pi k}{3}\right) = 0, \tag{5.29}$$

which we call the *type D spin relation*. Eq. (5.29) is valid for all three transverse frames, depending on the value of k.

If we assume to be in the transverse frame that is also a quasi-Kinnersley frame, which corresponds to having $k = 0$, Eq. (5.29) reduces to $\tilde{\lambda} = 0$ consistently with the Goldberg-Sachs theorem (cf. section 4.9). Nevertheless, by combining Eq. (5.22a) and Eq. (5.27a) we can deduce the expression for one of the spin coefficients, namely ρ, in terms of curvature invariants

$$\rho = D \ln I^{\frac{1}{6}}. \tag{5.30}$$

But the key point to stress here is that in the quasi-Kinnersley frame the Bianchi identities leave the expression for $\tilde{\varepsilon}$ completely unresolved. However, it is really the expression for $\tilde{\varepsilon}$ we are interested in, as it is the one related to the spin-boost transformation. To obtain additional information on this spin coefficient, we therefore analyze the Ricci identities.

5.7. Spin Coefficients as Directional Derivatives

In this section, we will use the Ricci identities to understand how the spin coefficients ε, γ, α and β relate to the spin-boost parameter $\mathscr{B}$. We will first show that they can be expressed as directional derivatives of the same function, and then determine the equation that this function must satisfy in the limit of Petrov type D.

We start by assuming to be in the Petrov type D limit, where $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and also, as a consequence of the Goldberg-Sachs theorem, the four spin coefficients λ, σ, ν and κ are vanishing (cf. section 4.9). We begin with the following Ricci identity (5.18e),

$$D\beta - \delta\varepsilon = \sigma(\alpha + \pi) + \beta(\rho^* - \varepsilon^*) - \kappa(\mu + \gamma) - \varepsilon(\alpha^* - \pi^*), \tag{5.31}$$

simplifying in the limit of type D to

$$D\beta - \delta\varepsilon = \beta(\rho^* - \varepsilon^*) - \varepsilon(\alpha^* - \pi^*). \tag{5.32}$$

Here we obtain, after adding and subtracting the product $\beta\varepsilon$ on the right-hand side, the relation:

$$D\beta - \delta\varepsilon = \varepsilon\,(\pi^* - \alpha^* - \beta) + \beta\,(\rho^* + \varepsilon - \varepsilon^*)\,. \tag{5.33}$$

By inserting the definition of the rescaled spin coefficients $\tilde{\varepsilon}$ and $\tilde{\beta}$ in Eq. (5.23) we can re-express this Ricci identity in terms of $\tilde{\varepsilon}$ and $\tilde{\beta}$, leading to

$$D\tilde{\beta} - \delta\tilde{\varepsilon} = \tilde{\varepsilon}\,(\pi^* - \alpha^* - \beta) + \tilde{\beta}\,(\rho^* + \varepsilon - \varepsilon^*)\,. \tag{5.34}$$

Comparing Eq. (5.34) with the expression of the commutator $[D,\delta]$ (again assuming $\sigma = \kappa = 0$) in Eq. (5.20b), namely

$$[D,\delta] = D\delta - \delta D = (\pi^* - \alpha^* - \beta)\,D + (\rho^* + \varepsilon - \varepsilon^*)\,\delta\,, \tag{5.35}$$

it is possible to see that these equations are consistent by assuming $\tilde{\varepsilon} = D\mathscr{H}_1$ and $\tilde{\beta} = \delta\mathscr{H}_1$, where $\mathscr{H}_1$ is a function to be determined. Using the equivalent Ricci identity obtained after exchanging the tetrad vectors $\ell \leftrightarrow n$ and $m \leftrightarrow \bar{m}$,

$$\Delta\tilde{\alpha} - \delta^*\tilde{\gamma} = \tilde{\alpha}\,(\gamma^* - \gamma - \mu^*) + \tilde{\gamma}(\alpha + \beta^* - \tau^*)\,, \tag{5.36}$$

we obtain an equivalent result for the spin coefficients $\tilde{\gamma}$ and $\tilde{\alpha}$ when compared to the commutator $[\Delta,\delta^*]$ in Eq. (5.20e) and conclude that they also can be expressed as $\tilde{\gamma} = \Delta\mathscr{H}_2$ and $\tilde{\alpha} = \delta^*\mathscr{H}_2$, where $\mathscr{H}_2$ is a function to be determined.

Using the properties of transformation of the spin coefficients under the exchange operation $\ell^\mu \leftrightarrow n^\mu$ and $m^\mu \leftrightarrow \bar{m}^\mu$, which corresponds to exchanging $\tilde{\varepsilon} \leftrightarrow -\tilde{\gamma}$ and $\tilde{\alpha} \leftrightarrow -\tilde{\beta}$, we can immediately conclude that

$$\mathscr{H}_1 = -\mathscr{H}_2 = \mathscr{H}\,, \tag{5.37}$$

where $\mathscr{H}$ is still to be determined. Nevertheless, the four spin coefficients can then be written as

$$\tilde{\varepsilon} \;=\; D\mathscr{H}\,, \quad \tilde{\gamma} = -\Delta\mathscr{H}\,, \tag{5.38a}$$
$$\tilde{\beta} \;=\; \delta\mathscr{H}\,, \quad \tilde{\alpha} = -\delta^*\mathscr{H}\,, \tag{5.38b}$$

and the original spin coefficients are therefore given by

$$\varepsilon = D\mathscr{H} - \frac{1}{2}D\ln\mathscr{B} = D\mathscr{H}_-, \tag{5.39a}$$
$$\gamma = -\Delta\mathscr{H} - \frac{1}{2}\Delta\ln\mathscr{B} = -\Delta\mathscr{H}_+, \tag{5.39b}$$
$$\beta = \delta\mathscr{H} - \frac{1}{2}\delta\ln\mathscr{B} = \delta\mathscr{H}_-, \tag{5.39c}$$
$$\alpha = -\delta^*\mathscr{H} - \frac{1}{2}\delta^*\ln\mathscr{B} = -\delta^*\mathscr{H}_+, \tag{5.39d}$$

where we have introduced the additional quantity $\mathscr{H}_\pm = \mathscr{H} \pm \frac{1}{2}\ln\mathscr{B}$.

In the next section we make use of some other Ricci identities to find the explicit expression for $\mathscr{H}$.

5.8. The Function $\mathscr{H}$

To determine the function $\mathscr{H}$ we consider the two following Ricci identities

$$D\gamma - \Delta\varepsilon = \alpha(\tau+\pi^*) + \beta(\tau^*+\pi) + \tau\pi - \gamma(\varepsilon+\varepsilon^*) - \varepsilon(\gamma+\gamma^*) + \Psi_2, \tag{5.40a}$$
$$\delta\alpha - \delta^*\beta = \mu\rho + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho-\rho^*) + \varepsilon(\mu-\mu^*) - \Psi_2. \tag{5.40b}$$

Again, by utilizing the rescaled spin coefficients in Eq. (5.23) we can remove the spin-boost dependence in these identities, yielding

$$D\tilde{\gamma} - \Delta\tilde{\varepsilon} = \tilde{\alpha}(\tau+\pi^*) + \tilde{\beta}(\tau^*+\pi) + \tau\pi - \tilde{\gamma}(\varepsilon+\varepsilon^*) - \tilde{\varepsilon}(\gamma+\gamma^*) + \Psi_2, \tag{5.41a}$$
$$\delta\tilde{\alpha} - \delta^*\tilde{\beta} = \mu\rho + \tilde{\alpha}\alpha^* + \tilde{\beta}\beta^* - 2\tilde{\alpha}\tilde{\beta} + \tilde{\gamma}(\rho-\rho^*) + \tilde{\varepsilon}(\mu-\mu^*) - \Psi_2. \tag{5.41b}$$

Since we have just found that the reduced spin coefficients on the left-hand sides can be expressed as directional derivatives of $\mathscr{H}$, cf. Eq. (5.38), we yield terms of the form $D\Delta\mathscr{H}$ or $\delta\delta\mathscr{H}$ among others.

Thus, we can use the definition of double derivatives in Eq. (5.21) to find an equivalent form of Eq. (5.40) or Eq. (5.41), respectively. The particular equations we will make use of are the following

$$
\begin{aligned}
D\Delta &= -(\varepsilon+\varepsilon^*)\Delta+\pi\delta+\pi^*\delta^*+\ell^\mu n^\nu\nabla_\mu\nabla_\nu, && (5.42a)\\
\Delta D &= (\gamma+\gamma^*)D-\tau^*\delta-\tau\delta^*+n^\mu\ell^\nu\nabla_\mu\nabla_\nu, && (5.42b)\\
\delta\delta^* &= \mu D-\rho^*\Delta-(\beta-\alpha^*)\delta^*+m^\mu\bar{m}^\nu\nabla_\mu\nabla_\nu, && (5.42c)\\
\delta^*\delta &= \mu^*D-\rho\Delta+(\alpha-\beta^*)\delta+\bar{m}^\mu m^\nu\nabla_\mu\nabla_\nu. && (5.42d)
\end{aligned}
$$

As an example, we calculate the term $D\tilde{\gamma}$ on the left hand side of Eq. (5.41a). Using the property just found that in the Petrov type D limit $\tilde{\gamma}=-\Delta\mathscr{H}$, this term is given by

$$D\tilde{\gamma}=-D\Delta\mathscr{H}, \tag{5.43}$$

and using Eq. (5.42a) this corresponds to

$$D\tilde{\gamma}=-D\Delta\mathscr{H}=(\varepsilon+\varepsilon^*)\Delta\mathscr{H}-\pi\delta\mathscr{H}-\pi^*\delta^*\mathscr{H}-\ell^\mu n^\nu\nabla_\mu\nabla_\nu\mathscr{H}. \tag{5.44}$$

If we substitute $\tilde{\alpha}=-\delta^*\mathscr{H}$ and $\tilde{\beta}=\delta\mathscr{H}$ we yield

$$D\tilde{\gamma} = -(\varepsilon+\varepsilon^*)\tilde{\gamma}-\pi\tilde{\beta}+\pi^*\tilde{\alpha}-\ell^\mu n^\nu\nabla_\mu\nabla_\nu\mathscr{H}. \tag{5.45}$$

By repeating the same procedure for $\Delta\tilde{\varepsilon}$, $\delta\tilde{\alpha}$ and $\delta^*\tilde{\beta}$ and comparing the expressions with the Ricci identities in Eq. (5.41) we finally end up with the following two identities for the function $\mathscr{H}$:

$$
\begin{aligned}
2\ell^\mu n^\nu\nabla_\mu\nabla_\nu\mathscr{H} &= -2\pi\tilde{\beta}-2\tau\tilde{\alpha}-\pi\tau-\Psi_2, && (5.46a)\\
2m^\mu\bar{m}^\nu\nabla_\mu\nabla_\nu\mathscr{H} &= -2\mu\tilde{\varepsilon}-2\rho\tilde{\gamma}-\mu\rho+\Psi_2. && (5.46b)
\end{aligned}
$$

As a last step, we subtract Eq. (5.46b) from Eq. (5.46a) and utilize the definition of the metric in the tetrad, namely

$$g^{\mu\nu}=2\ell^{(\mu}n^{\nu)}-2m^{(\mu}\bar{m}^{\nu)}, \tag{5.47}$$

thus obtaining the *master equation* for $\mathscr{H}$:

$$\nabla^{\mu}\nabla_{\mu}\mathscr{H}+\nabla^{\mu}\ln\left(I^{\frac{1}{6}}\right)\nabla_{\mu}\left(2\mathscr{H}+\ln I^{\frac{1}{12}}\right)=-2\Psi_2, \tag{5.48}$$

where we have also used the fact that in the Petrov type D limit $\rho=D\ln I^{\frac{1}{6}}$, $\mu=-\Delta\ln I^{\frac{1}{6}}$, $\tau=\delta\ln I^{\frac{1}{6}}$ and $\pi=-\delta^{*}\ln I^{\frac{1}{6}}$.

In the next section we will solve Eq. (5.48) for the single black hole case to obtain the condition on the spin-boost parameter.

5.9. Weyl Scalars in Terms of Curvature Invariants

We can now apply the results we just found to the particular case of the Kerr solution using Boyer-Lindquist coordinates. The metric in this case reads (cf. section 4.11, 4.12)

$$ds^2=\left(1-\frac{2Mr}{\Sigma}\right)dt^2+\left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\phi-\left(\frac{\Sigma}{\Gamma}\right)dr^2-\Sigma d\theta^2-\sin^2\theta\left(\frac{\tilde{\Delta}}{\Sigma}\right)d\phi^2, \tag{5.49}$$

where $\Gamma=r^2-2Mr+a^2$, $\Sigma=r^2+a^2\cos^2\theta$, $\tilde{\Delta}=r^2+a^2+2Mar\sin^2\theta$, M is the black hole mass and a its rotation parameter. The Kinnersley tetrad in this coordinate system is given by

$$\ell^{\mu} = \frac{1}{\Gamma}\left[\left(r^2+a^2\right),\Gamma,0,a\right], \tag{5.50a}$$

$$n^{\mu} = \frac{1}{2\Sigma}\left[r^2+a^2,-\Gamma,0,a\right], \tag{5.50b}$$

$$m^{\mu} = \frac{1}{\sqrt{2}\bar{\rho}}\left[ia\sin\theta,0,1,i\sec\theta\right], \tag{5.50c}$$

where $\bar{\rho}=r+ia\cos\theta$.

The solution for Eq. (5.48) in this particular coordinate system can be straightforwardly carried out and reads

$$\mathscr{H}=\frac{1}{2}\ln\left(\Gamma^{\frac{1}{2}}I^{\frac{1}{6}}\sin\theta\right). \tag{5.51}$$

We now have all the elements to find the values of the spin coefficients ε, γ, β and α

in the limit of type D, and in particular the condition on the spin-boost parameter. As already shown, the four spin coefficients ε, γ, α and β can be written as follows

$$\varepsilon = D\mathscr{H} - \frac{1}{2}D\ln\mathscr{B}, \tag{5.52a}$$

$$\gamma = -\Delta\mathscr{H} - \frac{1}{2}\Delta\ln\mathscr{B}, \tag{5.52b}$$

$$\beta = \delta\mathscr{H} - \frac{1}{2}\delta\ln\mathscr{B}, \tag{5.52c}$$

$$\alpha = -\delta^*\mathscr{H} - \frac{1}{2}\delta^*\ln\mathscr{B}. \tag{5.52d}$$

This result can be compared with the expressions for the same spin coefficients in the Kinnersley tetrad, given by

$$\varepsilon = 0, \tag{5.53a}$$

$$\gamma = \mu + \rho\rho^*(r-M)/2, \tag{5.53b}$$

$$\beta = \cot\theta/(2\sqrt{2}\bar{\rho}), \tag{5.53c}$$

$$\alpha = \pi - \beta^*. \tag{5.53d}$$

Let us consider first the spin coefficient ε. Using Eq. (5.52a) and the solution for $\mathscr{H}$ found in Eq. (5.51) we can rewrite ε in the following way

$$\varepsilon = \frac{1}{2}D\ln\left(\Gamma^{\frac{1}{2}}I^{\frac{1}{6}}\mathscr{B}^{-1}\sin\theta\right). \tag{5.54}$$

In order for this expression to be zero, the function inside the logarithm must be constant with respect to the derivative operator D. Given the form of the Kinnersley tetrad in Eq. (5.50), one concludes that the D operator corresponds to the simple ∂_r derivative (assuming that the functions do not have a t or ϕ dependence, which is indeed the case, as the Kerr space-time is stationary and axisymmetric). As a consequence of this, ε vanishes if the function on the right hand side is a generic function only of the coordinate θ. This leads to the following condition on the spin-boost parameter

$$\mathscr{B} = \mathscr{B}_0 f(\theta) I^{\frac{1}{6}}\Gamma^{\frac{1}{2}}\sin\theta, \tag{5.55}$$

where $\mathscr{B}_0$ is an integration constant. By rewriting the other spin coefficients in a similar manner, namely

$$\gamma = -\frac{1}{2}\Delta \ln\left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \mathscr{B} \sin\theta\right), \tag{5.56}$$

$$\beta = -\frac{1}{2}\delta^* \ln\left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \mathscr{B} \sin\theta\right), \tag{5.57}$$

$$\alpha = \frac{1}{2}\delta \ln\left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \mathscr{B}^{-1} \sin\theta\right), \tag{5.58}$$

we can determine the function $f(\theta)$. In fact, it can be easily shown that the spin coefficient γ given in Eq. (5.53b) is consistent with Eq. (5.55), imposing no further condition on $f(\theta)$.

The spin coefficient β can instead be used to find the unknown function $f(\theta)$: the derivative operator δ is given by

$$\delta = m^\mu \nabla_\mu = \frac{1}{\sqrt{2}\bar{\rho}}\partial_\theta, \tag{5.59}$$

and is therefore related to the θ-dependence of the spin-boost parameter. A straightforward calculation yields $f(\theta) = \sin^{-1}\theta$, consistent also with the spin coefficient α. Thus, the final result for $\mathscr{B}$ reads

$$\mathscr{B} = \mathscr{B}_0 I^{\frac{1}{6}} \Gamma^{\frac{1}{2}}. \tag{5.60}$$

5.9.1. Final Expressions for the Weyl Scalars and Peeling Behavior

Finally, we want to apply these results to the Weyl scalars; by inserting the expression of the spin-boost parameter in Eq. (5.60) into the definition of the scalars in Eq. (5.7) we yield

$$\Psi_0 = -\frac{i}{2}\mathscr{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right), \tag{5.61a}$$

$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} \left(\Theta + \Theta^{-1}\right), \tag{5.61b}$$

$$\Psi_4 = -\frac{i}{2}\mathscr{B}_0^{2} \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right). \tag{5.61c}$$

It is remarkable how these expressions for the scalars immediately give the correct radial fall-offs at future null infinity once the peeling behavior of the Weyl tensor is assumed:

- The function Γ is only defined in the limit of Petrov type D and gives no radial contribution at future null infinity.
- Given under the peeling assumption that $I \propto r^{-6}$, we find the same result for Θ as it is the ratio of quantities that have the same radial behavior at future null infinity, namely

$$\Theta = \sqrt{3}PI^{-\frac{1}{2}} \propto \frac{r^{-3}}{r^{-3}}. \tag{5.62}$$

- In conclusion, the quantities that give a contribution at future null infinity are the factors $I^{\frac{1}{6}}$, $I^{\frac{1}{2}}$ and $I^{\frac{5}{6}}$, corresponding to

$$\Psi_0 \propto r^{-1}, \quad \Psi_2 \propto r^{-3} \quad \text{and} \quad \Psi_4 \propto r^{-5}. \tag{5.63}$$

The fact that we obtain radial fall-offs for Ψ_0 and Ψ_4 that are exchanged with respect to the normal assumption of outgoing radiation in the literature, where $\Psi_0 \propto r^{-5}$ and $\Psi_4 \propto r^{-1}$, is not surprising: this is due to the fact that in the Kinnersley tetrad the null vector ℓ^μ is ingoing while n^μ is outgoing. The standard notation requires instead the opposite situation where ℓ^μ is outgoing and n^μ is ingoing.

This means that one needs to exchange $\ell^\mu \leftrightarrow n^\mu$ to have the right convention, what in fact results in $\mathscr{B} \to \mathscr{B}^{-1}$ and the Weyl scalars are changed to

$$\Psi_0 = -\frac{i}{2}\mathscr{B}_0^2 \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right), \tag{5.64a}$$

$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} \left(\Theta + \Theta^{-1}\right), \tag{5.64b}$$

$$\Psi_4 = -\frac{i}{2}\mathscr{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right). \tag{5.64c}$$

giving this time, as expected, the correct radial fall-offs for Ψ_0 and Ψ_4.

5.9.2. Conclusion

Eqs. (5.64) are the main result that we propose for wave extraction in numerical relativity. As evident from the equations, the conditions on the spin coefficients do not completely fix the values of the Weyl scalars, leaving the constant $\mathscr{B}_0$ undetermined.

This is not surprising as such conditions involve the directional derivatives along the tetrad null vectors and are therefore independent of additional constant multiplication factors. The optimal value of this integration constant will have to be determined enforcing the values of the spin coefficients ρ, μ, τ and π. We will present a possible value for the integration constant in the following chapter 6.

We are also investigating the comparison of these expressions with the analogous quantities defined in the characteristic formulation of Einstein's equations [74, 75, 76]. As we expect, this should give us more insights on how to choose this integration constant from a more theoretical point of view. This is the subject of future work on this topic.

6. Distorted Black Hole Space-Times in the Newman-Penrose Formalism

I was born not knowing and have only had a little time to change that here and there.
Richard Feynman

In this chapter we study on an analytical level how the expressions for the Weyl scalars depend on the tetrad we choose in a space-time containing distorted black holes. We will calculate all relevant quantities in the transverse frames, and show how we can deduce the spin-boost parameter to find the quasi-Kinnersley tetrad. Finally and most importantly, we will extract the gravitational wave signal and we will demonstrate the advantage of our approach proposed in chapter 5 compared to commonly used methods in numerical relativity.
In spirit this work follows [77, 78, 79, 80, 81, 82, 83, 84, 85] in constructing a distorted black hole by superposing a Schwarzschild space-time and a pure Brill wave space-time.

In this chapter we will refer to a *symmetric tetrad* as a transverse tetrad ($\Psi_1 = \Psi_3 = 0$) obeying the additional symmetry $\Psi_0 = \Psi_4$. It further satisfies Def. 3, namely $\Psi_0 = \Psi_4 \to 0$ for $S \to 1$, therefore being a member of the *quasi-Kinnersley frame*. The *quasi-Kinnersley tetrad*, belonging to the same frame, obeys the additional condition of $\varepsilon \to 0$ in the limit of type D, thus being equivalent to the Kinnersley tetrad in Petrov type D. If it is at all necessary to impose the condition $\varepsilon = 0$ in Petrov type I has not been clarified as to yet. This is partly due to the fact that the Kinnersley tetrad is defined in Petrov type D, thus a more general definition of that particular tetrad just does not exist up to date. In this chapter we demonstrate an approach of how to impose this condition $\varepsilon = 0$ in Petrov type I in general, for the perturbed black hole space-time under consideration.

6.1. Brill Wave Initial Data

Brill waves have been used by the numerical relativity community from its earliest days since discovery by Brill. In his original work Brill gave the first positivity of energy result in General Relativity [77]. Brill waves are an excellent exploration tool for such purposes because the space-time contains only radiation, it *is* only radiation. Moreover, they have been interesting in their own right because they are a particularly simple solution to the vacuum equations of General Relativity, but rich in structure. Radiation and evolution of numerically constructed initial data of pure gravitational waves have already been studied (e.g. [80]) . Brill wave solutions have also shed light on the problem of accurately defining the mass of numerically generated initial data sets. They have been used as a test of the Cosmic Censorship (e.g. [26], [86]), moreover black hole interaction with gravitational waves (e.g. [85]) and gravitational collapse of Brill waves (e.g. [87]) have been investigated.

6.1.1. Pure Gravitational Waves

Brill originally considered axisymmetric, time symmetric, vacuum initial data for the Einstein equations of an asymptotically flat hypersurface with R^3 topology as a Cauchy problem, i.e. the initial data consists of a three metric γ_{ij} and the extrinsic curvature K_{ij}. These are vacuum solutions and they satisfy the Hamiltonian and Momentum constraints which reduce in a vacuum space-time to:

$$R + K^2 - K_{ij}K^{ij} = 0, \tag{6.1}$$

$$\nabla_i K^i_j - \nabla_j K^i_i = 0. \tag{6.2}$$

Here R is the scalar curvature and ∇ the covariant derivatives associated with γ_{ij}. By enforcing the condition of time symmetry of the initial slice the extrinsic curvature tensor K_{ij} vanishes and leaves only the condition $R = 0$ for the Hamiltonian constraint to be satisfied. Following York's Thin-Sandwich decomposition [88] the three metric can be written in conformal form

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}, \tag{6.3}$$

where ψ is the conformal factor. The axially symmetric orthogonal three-metric under consideration takes in polar-like coordinate the form

$$\mathrm{d}\bar{s}^2 = \bar{\gamma}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = e^q\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2\,\mathrm{d}\theta^2, \tag{6.4}$$

where $r^2 = \rho^2 + z^2$ and q is an (almost arbitrary) function of ρ and z. Brill imposed an equatorial symmetry condition across the $z = 0$ plane, that q decays fairly rapidly at infinity (faster than $1/r$) and that it is regular at $\rho = 0$:

$$\frac{\partial q}{\partial z}|_{z=0} = 0, \qquad \lim_{\rho\to\infty} q = \mathcal{O}\left(\rho^{-2}\right), \qquad \frac{\partial q}{\partial \rho}|_{\rho=0} = 0, \qquad q|_{\rho=0} = 0. \tag{6.5}$$

With these assumptions the Ricci scalar becomes

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{\nabla}^2\psi, \tag{6.6}$$

and thus the Hamiltonian constraint turns out to be

$$\bar{\nabla}^2\psi = \frac{1}{8}\bar{R}, \tag{6.7}$$

where $\bar{\nabla}^2$ is Laplacian associated with the conformal metric $\bar{\gamma}_{ij}$:

$$\bar{\nabla}^2\psi = e^{-q}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right), \tag{6.8}$$

and the scalar curvature $\bar{R}$ is, in case of a space-time consisting of radiation

$$\bar{R} = -e^{-q}\left(\frac{\partial^2 q}{\partial\rho^2} + \frac{\partial^2 q}{\partial z^2}\right). \tag{6.9}$$

From the assumption that we find a solution to Eq. (6.7) we can calculate the total energy of our space-time. In his thesis Brill gives the first positivity of mass result in General Relativity. His proof is valid for time-symmetric, axially symmetric and asymptotically Euclidian space-times [77][1].

The most simple solution satisfying the restrictions in Eq. (6.72) is the form first considered by Eppley [80]:

$$q(\rho, z) = \frac{a\rho^2}{\left(1 + \left(\frac{r}{\lambda}\right)^n\right)}, \tag{6.10}$$

where a and λ are constants, $r^2 = \rho^2 + z^2$ and $n \geq 4$. Another solution has been found by Holz [90]:

$$q(\rho, z) = 2a\rho^2 e^{-r^2}, \tag{6.11}$$

where again $r^2 = \rho^2 + z^2$ and a is a free choose-able parameter.

[1] An extension of this prove to maximal hypersurfaces and non trivial topologies was carried out by Sergio Dain [89]

6.1.2. Distorted Black Hole Initial Data

A distorted black hole creates a connection between pure gravitational waves and two black hole space-times because it contains an Einstein-Rosen bridge [91] and gravitational radiation, whereas interacting black holes and Brill waves are defined only in an otherwise empty space-time. Hence one may consider initial data that represents a black hole that is surrounded by a cloud of gravitational radiation, with a range of parameters from a weakly perturbed black hole to an interaction in which the wave has a mass many times the mass of the black hole. The space-time is a combination of conformally flat wormhole data sets (cf. Misner [92], Brill-Lindquist [93]) and Brill wave space-times. The 3-space topology of the Einstein-Rosen bridge is the one of a hypercylinder ($\mathbf{S}^2$ x $\mathbf{R}$), where two asymptotically flat sheets are connected through a 2-sphere.

If the amplitude of the Brill wave is equal to zero the resulting space-time is Minkowski-flat in the pure Brill wave case, while in the distorted black hole space-time we are left with a conformally-flat Schwarzschild space-time, logically. From this point of view and the mentioned construction as a combination of Misner Data and pure gravitational wave space-times, it is not surprising that the space-time is constructed similarly to the pure gravitational wave data sets.

The situation can be regarded as a scattering problem; incoming gravitational radiation from past null infinity "hits" a spherically symmetric hole, which is therefore deformed by the incoming radiation and emits radiation of its own. Together they form a state where the Bel-Robinson vector is momentarily zero[2] [85]. The initial-value problem is analogous to the case of the pure Brill wave space-time; It consists of finding a three-metric γ^{ij} and extrinsic curvature K^{ij} which satisfy the Hamiltonian and Momentum constraint of General Relativity in vacuum, cf. Eqs. (6.1, 6.2). As in the pure Brill wave space-time we enforce the initial slice to be time-symmetric. Thus the extrinsic curvature tensor vanishes and leaves only the Hamiltonian constraint, Eq.

[2]The Bel-Robinson tensor can be defined in terms of the Weyl tensor by $T_{\mu\nu\rho\sigma} = C_{\mu\lambda\rho\delta} C_{\nu\ \ \sigma}^{\ \lambda\ \ \delta} - \frac{3}{2} g_{\mu[\nu} C_{\kappa\gamma]\rho\delta} C^{\kappa\gamma\ \ \delta}_{\ \ \ \sigma}$. This construction is closely analogous to the definition of the stress tensor of the electromagnetic field and therefore can provide associations for the average gravitational stress (such as the pressure of gravitational radiation).

(6.1), $\bar{\nabla}^2\psi = \frac{1}{8}\,\psi\,\bar{R}$ to be satisfied. The Momentum constraint will be satisfied identically.

The way to proceed in the next section is to choose γ_{ij} and solve Eq. (6.1) for the conformal factor ψ. Conformal decomposition using a flat metric $\bar{\gamma}_{ij}$ leads only to trivial solutions thus we are forced to find another form for $\bar{\gamma}_{ij}$. We relax the flatness criteria and use a metric of the form

$$dl^2 = \psi^4\left[e^{2q}\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2\right) + \rho^2\,\sin\theta^2\,\mathrm{d}\phi^2\right], \tag{6.12}$$

where the scalar curvature turns out to be

$$\bar{R} = -2\,e^{-2q}\left[\frac{\partial^2 q}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 q}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial q}{\partial\rho}\right], \tag{6.13}$$

and $\bar{\nabla}^2 = e^{-2q}\times\bar{\nabla}^2_{flat}$.

Finally the Hamiltonian constraint $\bar{\nabla}^2\psi = \frac{1}{8}\,\psi\,\bar{R}$ becomes

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)\psi + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)\psi = -\frac{1}{4}\psi\left(\frac{\partial^2 q}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 q}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial q}{\partial\rho}\right). \tag{6.14}$$

In case of a vanishing amplitude of the Brill waves the perturbation $q\,(r,\theta)$ tends to zero.

Throughout this chapter we simplify the expressions by writing q and ψ while the functions always depend on r and θ.

6.2. Weyl Scalars and Spin Coefficients on the Initial Slice

We choose the metric for the space-time to Schwarzschild in isotropic coordinates:

$$ds^2 = \alpha^2 dt^2 - \psi^4\left[e^{2q}\left(dr^2 + r^2 d\theta^2\right) + r^2\sin^2\theta\, d\phi^2\right], \tag{6.15}$$

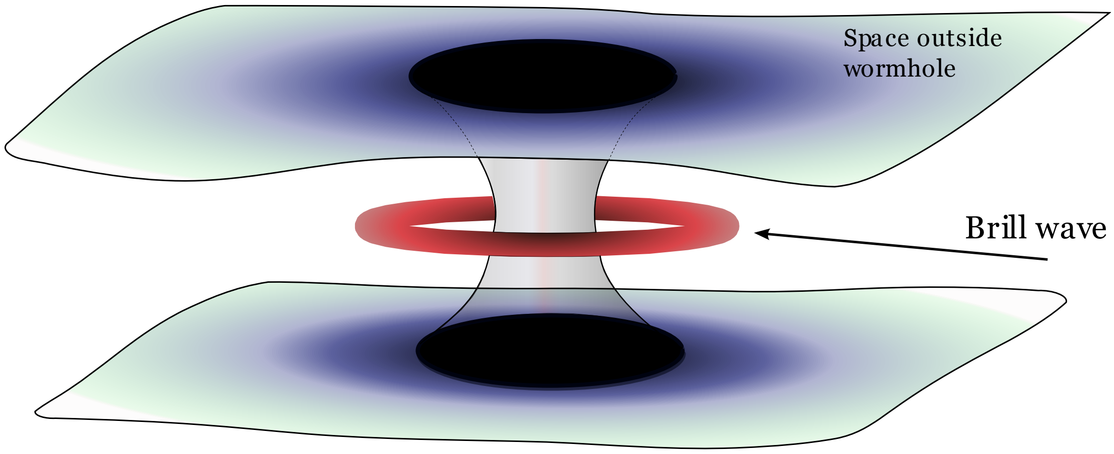

Figure 6.1.: The initial data of a "Brill wave plus black hole space-time" corresponds to a wormhole connecting two universes being surrounded by a cloud of gravitational waves.

where α is the analytic lapse, $\alpha = \left(\frac{2r-M}{2r+M}\right)$ of the space-time. We will follow the standard approach [94] in constructing an orthonormal set of null vectors: We define an extraction world-tube, $x^2+y^2+z^2=r^2$, and construct a triad of orthonormal spatial vectors by applying a Gram-Schmidt procedure in the following way:

$$u^i = [-y, x, 0], \tag{6.16a}$$

$$v^i = [x, y, z], \tag{6.16b}$$

$$u^i = \sqrt{g} g^{ia} \varepsilon_{abc} u^b v^c. \tag{6.16c}$$

Finally, by adjoining a time-component to the tetrad, four null vectors are given by

$$n^0 = \frac{1}{\sqrt{2}\alpha}, \quad n^i = \frac{1}{\sqrt{2}\alpha}\left(\frac{-\beta^i}{\alpha} - v^i\right), \tag{6.17a}$$

$$\ell^0 = \frac{1}{\sqrt{2}\alpha}, \quad \ell^i = \frac{1}{\sqrt{2}\alpha}\left(\frac{-\beta^i}{\alpha} + v^i\right), \tag{6.17b}$$

$$m^0 = 0, \quad m^i = \frac{1}{\sqrt{2}}\left(u^i + i w^i\right). \tag{6.17c}$$

The explicit expression of the tetrad is

$$\ell^{\mu}_{\rm B} = \frac{1}{\sqrt{2}}\left(\frac{(2r+M)}{(2r-M)}, \frac{e^{-q}}{\psi^2}, 0, 0\right), \tag{6.18a}$$

$$n^{\mu}_{\rm B} = \frac{1}{\sqrt{2}}\left(\frac{(2r+M)}{(2r-M)}, -\frac{e^{-q}}{\psi^2}, 0, 0\right), \tag{6.18b}$$

$$m^{\mu}_{\rm B} = \frac{1}{\sqrt{2}}\left(0, 0, -\frac{e^{-q}}{r\psi^2}, \frac{i}{r\psi^2 \sin\theta}\right), \tag{6.18c}$$

where the null vectors satisfy the null-vector conditions Eqs. (4.22, 4.22). The index B indicates quantities on the initial slice.

The vectors in such a tetrad will differ from the expressions of the Kinnersley tetrad; as a consequence, all quantities calculated in this frame will differ from the quantities calculated in the Kinnersley tetrad as well. Most importantly, the Weyl scalars will all be non-zero. For our particular choice of the metric these will result in

$$\begin{aligned}
\Psi^{\rm B}_2 &= \frac{e^{-2q(r,\theta)}}{S}\left(-6M_-\left(2r^2\left(\frac{\partial\psi}{\partial r}\right)^2 - \left(\frac{\partial\psi}{\partial\theta}\right)^2\right)M_+^2 - 2\psi\left(3M_\pm\left(\cot(\theta) - \frac{\partial q}{\partial\theta}\right)\frac{\partial\psi}{\partial\theta}\right.\right. \\
&+ \left. 3M_\pm\psi^{(0,2)} + r\left(6M^2 + 16rM - 24r^2 + 3rM_\pm\frac{\partial q}{\partial r}\right)\frac{\partial\psi}{\partial r}\right)M_+ \\
&- \left.\psi^2\left(8Mr(M+6r) - 3M_+\left(M_\pm\cot(\theta)\frac{\partial q}{\partial\theta} - r\left(M^2 + 4rM - 4r^2\right)\frac{\partial q}{\partial r}\right)\right)\right),
\end{aligned} \tag{6.19a}$$

$$\begin{aligned}
\Psi^{\rm B}_1 &= \frac{3e^{-2q(r,\theta)}M_+}{S}\left(\left(\left(M^2 + 4rM - 4r^2\right)\frac{\partial q}{\partial\theta} + rM_\pm\cot(\theta)\frac{\partial q}{\partial r}\right)\psi^2\right. \\
&+ \left. 2\left(\frac{\partial\psi}{\partial\theta}\left(M_\pm + 4rM + rM_\pm\frac{\partial q}{\partial r}\right) + rM_\pm\left(\frac{\partial q}{\partial\theta}\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial r\partial\theta}\right)\right)\psi + 6rM_\pm\frac{\partial\psi}{\partial\theta}\frac{\partial\psi}{\partial r}\right),
\end{aligned} \tag{6.19b}$$

$$\begin{aligned}
\Psi^{\rm B}_4 &= \frac{3e^{-2q(r,\theta)}M_+}{S}\left(\left(-M_\pm\cot(\theta)\frac{\partial q}{\partial\theta} + M_\pm\frac{\partial^2 q}{\partial\theta^2} + r\left(2\left(M_\pm + 2rM\right)\frac{\partial q}{\partial r} + rM_\pm\frac{\partial^2 q}{\partial r^2}\right)\right)\psi^2\right. \\
&+ \left. 2M_\pm\left(\frac{\partial q}{\partial r}\frac{\partial\psi}{\partial r}r^2 - \left(\cot(\theta) + \frac{\partial q}{\partial\theta}\right)\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta^2}\right)\psi - 6M_\pm\left(\frac{\partial\psi}{\partial\theta}\right)^2\right),
\end{aligned} \tag{6.19c}$$

where $S = -12r^2 M_- M_\mp^2 \psi(r,\theta)^6$ $\tilde{M}_\pm = M_+ M_-$, $M_+ = M + 2r$ and $M_- = M - 2r$. The Weyl scalars obey the additional symmetry

$$\Psi_4^{\mathrm{B}} = \Psi_0^{\mathrm{B}} \tag{6.20a}$$
$$\Psi_1^{\mathrm{B}} = -\Psi_3^{\mathrm{B}}. \tag{6.20b}$$

Beside computing the Weyl scalars we want to calculate other important quantities in the Newman-Penrose formalism. The spin coefficients are according to Eqs. (4.28):

$$\mu^{\mathrm{B}} = \rho^{\mathrm{B}}, \tag{6.21a}$$
$$\pi^{\mathrm{B}} = \kappa^{\mathrm{B}} = -\nu^{\mathrm{B}} = -\tau^{\mathrm{B}}, \tag{6.21b}$$
$$\sigma^{\mathrm{B}} = \lambda^{\mathrm{B}}, \tag{6.21c}$$
$$\gamma^{\mathrm{B}} = \varepsilon^{\mathrm{B}}, \tag{6.21d}$$
$$\beta^{\mathrm{B}} = -\alpha^{\mathrm{B}}, \tag{6.21e}$$
$$\rho^{\mathrm{B}} = -\frac{e^{-q}}{2\sqrt{2}r\psi^3}\left[\psi\left(2+r\frac{\partial q}{\partial r}\right)+4r\frac{\partial\psi}{\partial r}\right], \tag{6.21f}$$
$$\lambda^{\mathrm{B}} = -\frac{e^{-q}}{2\sqrt{2}\psi^2}\frac{\partial q}{\partial r}, \tag{6.21g}$$
$$\varepsilon^{\mathrm{B}} = -\frac{\sqrt{2}e^{-q}M}{(M^2-4r^2)\,\psi^2}, \tag{6.21h}$$
$$\pi^{\mathrm{B}} = \frac{e^{-q}}{2\sqrt{2}r\psi^3}\left(\psi\frac{\partial q}{\partial\theta}+2\frac{\partial\psi}{\partial\theta}\right), \tag{6.21i}$$
$$\alpha^{\mathrm{B}} = -\frac{e^{-q}}{2\sqrt{2}r\psi^3}\left(\psi\cot\theta+2\frac{\partial\psi}{\partial\theta}\right). \tag{6.21j}$$

The scalar curvature, encoded in the Ricci scalar, is non-zero and given by:

$$R = -\frac{16e^{-2q}M\left(M\psi+r\,(M+2r)\,\frac{\partial\psi}{\partial r}\right)}{(M-2r)\,(M+2r)^2\,r\psi^5}. \tag{6.22}$$

It vanishes for $q=0$ and $\psi = 1+\frac{M}{2r}$, corresponding to the vacuum solution of a Schwarzschild black hole.

6.3. Finding the Transverse Frames

We will now search for the transverse frames, which are three-fold in a Petrov type I space-time, one of them being the quasi-Kinnersley frame containing the quasi-Kinnersley tetrad. By applying the procedure in section 4.15.2 we calculate the transformation parameters a and b to perform a rotation into a transverse frame from the definition of the scalars on the initial slice. A solution is readily found as we will demonstrate now.

6.3.1. The First Transverse Frame

The equation for b is given by the explicit formula derived from Eqs. (4.96).

$$b = -\frac{\Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0}{\Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0}, \tag{6.23}$$

whereas we have to solve the following sixth order equation for the parameter $\bar{a}$

$$\mathscr{P}_1\bar{a}^6 + \mathscr{P}_2\bar{a}^5 + \mathscr{P}_3\bar{a}^4 + \mathscr{P}_4\bar{a}^3 + \mathscr{P}_5\bar{a}^2 + \mathscr{P}_6\bar{a} + \mathscr{P}_7 = 0, \tag{6.24}$$

where the $\mathscr{P}_n$ simplify in the case under study to

$$\mathscr{P}_1 = \mathscr{P}_7 = -\frac{1}{5}\mathscr{P}_3 = -\frac{1}{5}\mathscr{P}_5 \tag{6.25a}$$

$$\mathscr{P}_2 = -\mathscr{P}_6, \tag{6.25b}$$

$$\mathscr{P}_4 = 0. \tag{6.25c}$$

Therefore Eq. 6.24 reduces to

$$\mathscr{P}_1\left(\bar{a}^6 + 1 - \frac{1}{5}\bar{a}^4 - \frac{1}{5}\bar{a}^2\right) + \mathscr{P}_2\left(\bar{a}^5 - \bar{a}\right) = 0, \tag{6.26}$$

From this we can immediately find two solutions for $\bar{a}$, namely:

$$\bar{a} = \pm i. \tag{6.27}$$

The equation for the parameter b simplifies to

$$b = -\frac{3\bar{a}\Psi_2 + \left(3\bar{a}^2 - 1\right)\Psi_1 + \bar{a}^3\Psi_0}{6\bar{a}^2\Psi_2 + 4\bar{a}\left(\bar{a}^2 - 1\right)\Psi_1 + \left(1 + \bar{a}^4\right)\Psi_0}, \tag{6.28}$$

and the corresponding solution is

$$b = \pm\frac{i}{2}. \tag{6.29}$$

Performing a Type I and Type II rotation using the parameters $(a = -i, b = i/2)$ the tetrad vectors in the resulting transverse frame read:

$$\begin{aligned}
l^{\mu}_{TF} &= \left(-\frac{\sqrt{2}\,(M+2r)}{(M-2r)}, 0, 0, \frac{\sqrt{2}\csc\theta}{r\psi^2}\right), && (6.30a)\\
n^{\mu}_{TF} &= \left(-\frac{(M+2r)}{2\sqrt{2}\,(M-2r)}, 0, 0, -\frac{\csc\theta}{2\sqrt{2}r\psi^2}\right), && (6.30b)\\
m^{\mu}_{TF} &= \left(0, -\frac{ie^{-q}}{\sqrt{2}\psi^2}, \frac{e^{-q}}{\sqrt{2}r\psi^2}, 0\right), && (6.30c)
\end{aligned}$$

where quantities in the transverse frame are indicated by the sub- and superscript TF, respectively. Contracting the Weyl tensor with the null vectors the spin coefficients in this tetrad turn out to be:

$$\begin{aligned}
\rho^{TF} &= \mu^{TF} = \lambda^{TF} = \sigma^{TF} = \gamma^{TF} = \varepsilon^{TF} = 0, && (6.31a)\\
\tau^{TF} &= -\frac{e^{-q}}{2\sqrt{2}r\tilde{M}_{\pm}\psi^3}\left[\mathscr{X}\psi + 2\tilde{M}_{\pm}\left(\frac{\partial\psi}{\partial\theta} - ir\frac{\partial\psi}{\partial r}\right)\right], && (6.31b)\\
\pi^{TF} &= -\frac{e^{-q}}{2\sqrt{2}r\tilde{M}_{\pm}\psi^3}\left[\mathscr{T}\psi - 2\tilde{M}_{\pm}\left(\frac{\partial\psi}{\partial\theta} + ir\frac{\partial\psi}{\partial r}\right)\right], && (6.31c)\\
\nu^{TF} &= -\frac{e^{-q}}{8\sqrt{2}r\tilde{M}_{\pm}\psi^3}\left[\mathscr{U}\psi + 2\tilde{M}_{\pm}\left(\frac{\partial\psi}{\partial\theta} + ir\frac{\partial\psi}{\partial r}\right)\right], && (6.31d)\\
\kappa^{TF} &= -\frac{2e^{-q}}{\sqrt{2}r\tilde{M}_{\pm}\psi^3}\left[\mathscr{V}\psi - 2\tilde{M}_{\pm}\left(\frac{\partial\psi}{\partial\theta} - ir\frac{\partial\psi}{\partial r}\right)\right], && (6.31e)\\
\beta^{TF} &= -\frac{ie^{-q}}{2\sqrt{2}r\psi^3}\left[2i\frac{\partial\psi}{\partial\theta} + \psi\left(1 + i\frac{\partial q}{\partial\theta} + r\frac{\partial q}{\partial r}\right) + 2r\frac{\partial\psi}{\partial r}\right], && (6.31f)\\
\alpha^{TF} &= -\frac{ie^{-q}}{2\sqrt{2}r\psi^3}\left[-2i\frac{\partial\psi}{\partial\theta} + \psi\left(1 - i\frac{\partial q}{\partial\theta} + r\frac{\partial q}{\partial r}\right) + 2r\frac{\partial\psi}{\partial r}\right], && (6.31g)
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{X} &= \left(-i\left(\tilde{M}_{\pm}-4Mr\right)+\tilde{M}_{\pm}\cot\theta\right), &\text{(6.32a)}\\
\mathscr{T} &= \left(-i\left(\tilde{M}_{\pm}-4Mr\right)-\tilde{M}_{\pm}\cot\theta\right), &\text{(6.32b)}\\
\mathscr{U} &= \left(i\left(\tilde{M}_{\pm}+4Mr\right)+\tilde{M}_{\pm}\cot\theta\right), &\text{(6.32c)}\\
\mathscr{V} &= \left(i\left(\tilde{M}_{\pm}+4Mr\right)-\tilde{M}_{\pm}\cot\theta\right). &\text{(6.32d)}
\end{aligned}$$

Finally, we compute the Weyl scalars in this frame in terms of the scalars obtained from the Gram-Schmidt procedure, yielding:

$$\begin{aligned}
\Psi_0^{TF} &= 2\left(-3\Psi_2^B-4i\Psi_3^B+\Psi_4^B\right), &\text{(6.33a)}\\
\Psi_2^{TF} &= \frac{1}{2}\left(-\Psi_2^B-\Psi_4^B\right), &\text{(6.33b)}\\
\Psi_4^{TF} &= \frac{1}{8}\left(-3\Psi_2^B+4i\Psi_3^B+\Psi_4^B\right). &\text{(6.33c)}
\end{aligned}$$

Explicitly calculated the scalars read

$$\begin{aligned}
\Psi_0^{TF} &= \frac{e^{-2q(r,\theta)}}{S}\Bigg(\Bigg(8Mr(M+6r)+M_+\left(4\left(i\left(M_\pm+4rM\right)-M_\pm\cot(\theta)\right)\frac{\partial q}{\partial\theta}+M_\pm\frac{\partial^2 q}{\partial\theta^2}\right.\\
&+ r\left(\left(5M^2+16rM-20r^2+4iM_\pm\cot(\theta)\right)\frac{\partial q}{\partial r}+rM_\pm\frac{\partial^2 q}{\partial r^2}\right)\Bigg)\Bigg)\psi^2\\
&+ 4M_+\left(2M_\pm\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial\psi}{\partial\theta}\left(2i\left(M_\pm+4rM\right)+M_\pm\cot(\theta)-2M_\pm\left(\frac{\partial q}{\partial\theta}-ir\frac{\partial q}{\partial r}\right)\right)\right.\\
&+ r\left(\left(3M_\pm+8rM+2M_\pm\left(i\frac{\partial q}{\partial\theta}+r\frac{\partial q}{\partial r}\right)\right)\frac{\partial\psi}{\partial r}-2iM_\pm\frac{\partial^2\psi}{\partial r\partial\theta}\right)\Bigg)\psi \qquad (6.34a)\\
&+ 12M_-M_+^2\left(i\frac{\partial\psi}{\partial\theta}+r\frac{\partial\psi}{\partial r}\right)^2\Bigg),
\end{aligned}$$

$$\Psi_4^{TF} = \frac{1}{16}\left(\Psi_0^{TF}\right)^*, \qquad (6.34b)$$

$$\begin{aligned}
\Psi_2^{TF} &= \frac{e^{-2q(r,\theta)}}{12S}\Bigg(12M_-\left(\left(\frac{\partial\psi}{\partial\theta}\right)^2+r^2\left(\frac{\partial\psi}{\partial r}\right)^2\right)M_+^2\\
&+ 4\psi\left(3M_\pm\cot(\theta)\frac{\partial\psi}{\partial\theta}+r\left(3M_\pm+8rM\right)\frac{\partial\psi}{\partial r}\right)M_+\\
&+ \psi^2\left(8Mr(M+6r)-3M_-M_+^2\left(\frac{\partial^2 q}{\partial\theta^2}+r\left(\frac{\partial q}{\partial r}+r\frac{\partial^2 q}{\partial r^2}\right)\right)\right)\Bigg), \qquad (6.34c)
\end{aligned}$$

where $S=-2r^2M_\pm M_+\psi^6$, $M_\pm=M_+M_-$, $M_+=M+2r$ and $M_-=M-2r$. That we end up in a transverse frame is immediately recognized by the two longitudinal scalars being zero, $\Psi_1^{TF}=\Psi_3^{TF}=0$. Additionally, it is easy to see that we do not end up in the quasi-Kinnersley frame by going to the limit of future null infinity. Performing the limit of Petrov type D we yield for Ψ_0 and Ψ_4

$$\Psi_0^{TF} \rightarrow -6\Psi_2^B \qquad (6.35a)$$
$$\Psi_4^{TF} \rightarrow -\frac{3}{8}\Psi_2^B, \qquad (6.35b)$$

which is contradictory to Def. 3 for the quasi-Kinnersley frame.

6.3.2. The Quasi-Kinnersley Frame

To find the quasi-Kinnersley frame we first rescale the scalars in the transverse frame we just found to set $\Psi_0^{TF} = \Psi_4^{TF}$. Therefore, we perform a type III transformation, cf. Eqs. (4.48 - 4.50) with a boost parameter defined by

$$B = \left(\frac{\Psi_0^{TF}}{\Psi_4^{TF}}\right)^{1/4} = 2\left(\frac{\Psi_0^{TF}}{(\Psi_0^{TF}.)^*}\right)^{1/4} = \left(\frac{1}{16} + \frac{\Psi_3^B}{6i\Psi_2^B - 8\Psi_3^B - 2i\Psi_4^B}\right)^{-1/4} = Ae^{-i\Theta}, \quad (6.36)$$

where the modulus A and the phase Θ of the complex valued boost are defined as

$$A = \sqrt{\Re[B]^2 + \Im[B]^2}, \quad (6.37a)$$

$$\Theta = \arctan\left(\frac{\Im[B]}{\Re[B]}\right). \quad (6.37b)$$

The Weyl scalars are rescaled under this type III transformation according to

$$\Psi_0^{TF} = \frac{1}{2}\left(-3\Psi_2^B - 4i\Psi_3^B + \Psi_4^B\right)\sqrt{1 + \frac{8i\Psi_3^B}{-3\Psi_2^B - 4i\Psi_3^B + \Psi_4^B}}, \quad (6.38a)$$

$$\Psi_4^{TF} = \frac{\left(-3\Psi_2^B + 4i\Psi_3^B + \Psi_4^B\right)}{2\sqrt{1 + \frac{8i\Psi_3^B}{-3\Psi_2^B - 4i\Psi_3^B + \Psi_4^B}}}, \quad (6.38b)$$

$$\Psi_1^{TF} = \Psi_3^{TF} = 0, \quad (6.38c)$$

$$\Psi_2^{TF} = \frac{1}{2}\left(-\Psi_2^B - \Psi_4^B\right), \quad (6.38d)$$

and the spin coefficients become

$$\rho^{TF} = \mu^{TF} = \sigma^{TF} = \lambda^{TF} = \gamma^{TF} = \varepsilon^{TF} = 0, \quad (6.39a)$$

$$\kappa^{TF} = \frac{e^{i\Theta}(-4\alpha + 2i(2\varepsilon + \rho - \sigma))}{A^2}, \quad (6.39b)$$

$$\tau^{TF} = e^{i\Theta}\left(\alpha + \frac{1}{2}i(2\varepsilon - \rho + \sigma)\right), \quad (6.39c)$$

$$\pi^{TF} = \frac{1}{2}e^{-i\Theta}(-2\alpha + i(2\varepsilon - \rho + \sigma)), \quad (6.39d)$$

$$\nu^{TF} = \frac{1}{8}A^2 e^{-i\Theta}(2\alpha + i(2\varepsilon + \rho - \sigma)), \quad (6.39e)$$

$$\alpha^{TF} = \frac{e^{-i\Theta}\left(i\partial_\eta A\ell^\eta - \partial_\theta A m^\theta\right) + A\left(l^\eta \partial_\eta \Theta + i m^\theta \partial_\theta \Theta - 2\kappa + i(\rho+\sigma)\right)}{2A}, \tag{6.39f}$$

$$\beta^{TF} = \frac{e^{i\Theta}\left(-i\partial_\eta A\ell^\eta - \partial_\theta A m^\theta\right) + A\left(-l^\eta \partial_\eta \Theta + i m^\theta \partial_\theta \Theta + 2\kappa + i(\rho+\sigma)\right)}{2A}. \tag{6.39g}$$

Again, we want to solve Eqs. (4.96) to compute the parameters a and b. In particular, the $\mathscr{P}_n$ in Eq. (4.98) now simplify to

$$\mathscr{P}_1 = \mathscr{P}_7 = \mathscr{P}_3 = \mathscr{P}_5 = 0 \tag{6.40a}$$

$$\mathscr{P}_2 = -\mathscr{P}_6, \tag{6.40b}$$

$$\mathscr{P}_4 = 0. \tag{6.40c}$$

Thus, the sixth order polynomial in Eq. (4.98)

$$\mathscr{P}_1\bar{a}^6 + \mathscr{P}_2\bar{a}^5 + \mathscr{P}_3\bar{a}^4 + \mathscr{P}_4\bar{a}^3 + \mathscr{P}_5\bar{a}^2 + \mathscr{P}_6\bar{a} + \mathscr{P}_7 = 0, \tag{6.41}$$

reduces to

$$\mathscr{P}_2\left(\bar{a}^5 - \bar{a}\right) = 0. \tag{6.42}$$

Since $\mathscr{P}_2$ is non-zero we can immediately find the solutions for $\bar{a}$, namely:

$$\bar{a}^{\mathrm{I}} = 0, \tag{6.43a}$$

$$\bar{a}^{\mathrm{II}} = \pm i, \tag{6.43b}$$

$$\bar{a}^{\mathrm{III}} = \pm 1. \tag{6.43c}$$

The solution $\bar{a}^{\mathrm{I}} = 0$ reflects the fact that we are in a transverse frame already. The corresponding values for the parameter b we are interested in are

$$b^{\mathrm{II}} = \pm\frac{i}{2}, \tag{6.44a}$$

$$b^{\mathrm{III}} = \pm\frac{1}{2}. \tag{6.44b}$$

Finally, we perform a type I and type II transformation with $a = i$ and $b = -i/2$ which brings us in the quasi-Kinnersley frame. Again, for simplicity, we express the scalars in terms of the first transverse frame yielding

$$\begin{aligned}
\Psi_1^{QKF} &= \Psi_3^{QKF} = 0, && \text{(6.45a)}\\
\Psi_0^{QKF} &= \frac{1}{8}\left(-3\Psi_2^{TF} + \Psi_4^{TF}\right), && \text{(6.45b)}\\
\Psi_4^{QKF} &= -6\Psi_2^{TF} + 2\Psi_4^{TF}, && \text{(6.45c)}\\
\Psi_2^{QKF} &= \frac{1}{2}\left(-\Psi_2^{TF} - \Psi_4^{TF}\right), && \text{(6.45d)}
\end{aligned}$$

where the indices QKF indicate quantities in the tetrad we just found, which is a member of the quasi-Kinnersley frame. Finally, we perform a type III rotation with the spin-boost $B^{QKF} = 1/2$ to rescale the Weyl scalars to $\Psi_0^{QKF} = \Psi_4^{QKF}$. The corresponding tetrad is called *symmetric tetrad* and indicated by the sub and superscript S, respectively. The final result for the Weyl scalars in the symmetric tetrad turns out to be

$$\begin{aligned}
\Psi_1^{S} &= \Psi_3^{S} = 0, && \text{(6.46a)}\\
\Psi_0^{S} &= \Psi_4^{S} = \frac{1}{2}\left(3\Psi_2^{TF} - \Psi_4^{TF}\right), && \text{(6.46b)}\\
\Psi_2^{S} &= \frac{1}{2}\left(-\Psi_2^{TF} - \Psi_4^{TF}\right). && \text{(6.46c)}
\end{aligned}$$

Expressing the Weyl scalars in the symmetric tetrad in terms of the initial quantities we yield

$$\begin{aligned}
\Psi_1^{S} &= \Psi_3^{S} = 0, && \text{(6.47a)}\\
\Psi_0^{S} &= \Psi_4^{S} = \frac{1}{4}\left(3\Psi_2^{B} + 3\Psi_4^{B} + \frac{\left(-3\Psi_2^{B} - 4i\Psi_3^{B} + \Psi_4^{B}\right)}{\left(1 + \frac{8i\Psi_3^{B}}{-3\Psi_2^{B} - 4i\Psi_3^{B} + \Psi_4^{B}}\right)^{-1/2}}\right), && \text{(6.47b)}\\
\Psi_2^{S} &= \frac{1}{4}\left(\Psi_2^{B} + \Psi_4^{B} + \frac{\left(3\Psi_2^{B} + 4i\Psi_3^{B} - \Psi_4^{B}\right)}{\left(1 + \frac{8i\Psi_3^{B}}{-3\Psi_2^{B} - 4i\Psi_3^{B} + \Psi_4^{B}}\right)^{-1/2}}\right). && \text{(6.47c)}
\end{aligned}$$

Correspondingly, we express the spin coefficients in terms of the connection coefficients in the Gram-Schmidt tetrad, yielding

$$
\begin{aligned}
\rho^S &= \frac{\big(-8Ae^{i\Theta}\left(2\sin\Theta\left(A(\alpha-2\kappa)+\ell^\eta\left(A\partial_\eta\Theta+i\partial_\eta\Theta\right)\right)\right.}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{\cos\Theta\left(A(2\varepsilon-3\rho-\sigma)-2m^\theta\left(A\partial_\theta\Theta+i\partial_\theta\Theta\right)\right)\big)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{A^4(-2i\alpha+2\varepsilon+\rho-\sigma)+16e^{2i\Theta}(2i\alpha+2\varepsilon+\rho-\sigma)\big)}{32A^2e^{i\Theta}}, && (6.48a)\\
\mu^S &= \rho^S, && (6.48b)\\
\pi^S &= \frac{\big(16\mathrm{A}(\sin\Theta-i\cos\Theta)\left(\partial_\eta A\ell^\eta\cos\Theta\right)+\partial_\theta Am^\theta\sin\Theta\big)}{32\mathrm{A}^2e^{i\Theta}} \\
&- \frac{8\mathrm{A}^2e^{i\Theta}\left(2\cos\Theta\left(\ell^\eta\partial_\eta\Theta+\alpha-2\kappa\right)+\sin\Theta\left(2m^\theta\partial_\theta\Theta-2\varepsilon+3\rho+\sigma\right)\right)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{\mathrm{A}^4(2\alpha+i(2\varepsilon+\rho-\sigma))+16e^{2i\Theta}(2\alpha-i(2\varepsilon+\rho-\sigma))\big)}{32\mathrm{A}^2e^{i\Theta}}, && (6.48c)\\
\tau^S &= -\pi^S, && (6.48d)\\
\varepsilon &= \frac{\big(4e^{2i\Theta}\left(\mathrm{A}^2(-2i\alpha+2\varepsilon-\rho+\sigma)+4(2i\alpha+2\varepsilon+\rho-\sigma)\right)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{+\mathrm{A}^4(-2i\alpha+2\varepsilon+\rho-\sigma)+4\mathrm{A}^2(2i\alpha+2\varepsilon-\rho+\sigma)\big)}{32\mathrm{A}^2e^{i\Theta}}, && (6.48e)\\
\gamma^S &= \varepsilon^S, && (6.48f)\\
\kappa^S &= \frac{\big(16\mathrm{A}(\sin\Theta-i\cos\Theta)\left(\partial_\eta A\ell^\eta\cos\Theta+\partial_\theta Am^\theta\sin\Theta\right)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{8\mathrm{A}^2e^{i\Theta}\left(2\cos\Theta\left(-\ell^\eta\partial_\eta\Theta+\alpha+2\kappa\right)-\sin\Theta\left(2m^\theta\partial_\theta\Theta+2\varepsilon+\rho+3\sigma\right)\right)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{+\mathrm{A}^4(-(2\alpha+i(2\varepsilon+\rho-\sigma)))-16e^{2i\Theta}(2\alpha-i(2\varepsilon+\rho-\sigma))\big)}{32\mathrm{A}^2e^{i\Theta}}, && (6.48g)\\
\nu^S &= -\kappa^S, && (6.48h)\\
\sigma^S &= \frac{\big(8\mathrm{A}e^{i\Theta}\left(2\sin\Theta\left(\mathrm{A}\left(-\ell^\eta\partial_\eta\Theta+\alpha+2\kappa\right)-i\partial_\eta A\ell^\eta\right)\right.}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{+\cos(\Theta\left(\mathrm{A}\left(2m^\theta\partial_\theta\Theta+2\varepsilon+\rho+3\sigma\right)+2i\partial_\theta Am^\theta\right))}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{+\mathrm{A}^4(2i\alpha-2\varepsilon-\rho+\sigma)+16e^{2i\Theta}(-2i\alpha-2\varepsilon-\rho+\sigma)\big)}{32\mathrm{A}^2e^{i\Theta}}, && (6.48i)\\
\lambda^S &= \sigma^S, && (6.48j)\\
\alpha^S &= \frac{e^{-i\Theta}\left(8\mathrm{A}^2e^{i\Theta}(2\alpha\cos\Theta+(-2\varepsilon+\rho-\sigma)\sin\Theta\right)}{32\mathrm{A}^2e^{i\Theta}} \\
&+ \frac{+\mathrm{A}^4(2\alpha+i(2\varepsilon+\rho-\sigma))+16e^{2i\Theta}(2\alpha-i(2\varepsilon+\rho-\sigma))\big)}{32\mathrm{A}^2e^{i\Theta}}, && (6.48k)\\
\beta^S &= -\alpha^S.
\end{aligned}
$$

The symmetric null-tetrad reads

$$\ell_S^\mu = \left\{ \frac{\left(A^2+4\right)\ell^t}{4A}, -\ell^\eta \cos(\Theta), -m^\theta \sin(\Theta), \frac{i\left(A^2-4\right)m^\phi}{4A} \right\}, \tag{6.49a}$$

$$n_S^\mu = \left\{ \frac{\left(A^2+4\right)\ell^t}{4A}, \ell^\eta \cos(\Theta), m^\theta \sin(\Theta), \frac{i\left(A^2-4\right)m^\phi}{4A} \right\}, \tag{6.49b}$$

$$m_S^\mu = \left\{ -\frac{i\left(A^2-4\right)\ell^t}{4A}, -\ell^\eta \sin(\Theta), m^\theta \cos(\Theta), \frac{\left(A^2+4\right)m^\phi}{4A} \right\}. \tag{6.49c}$$

The tetrad obtained from this procedure will be transverse, and moreover, is a member of the same equivalence class of transverse Newman-Penrose tetrads as the Kinnersley tetrad, differing only by a class III rotation (a spin-boost Lorentz transformation).

Since we know the value of ε^S in the symmetric tetrad, we can easily compute the missing spin-boost parameter to break the remaining spin-boost degeneracy. From the definition of ε in the Kinnersley tetrad, $\varepsilon^{\mathrm{QKT}} = 0$, and definition of the type III rotation for the spin coefficient ε we can deduce

$$0 = \varepsilon^S - \frac{1}{2} D \ln B^{\mathrm{QKT}}, \tag{6.50}$$

where B^{QKT} refers to the boost to the Kinnersley tetrad. It is not possible analytically to simplify the expression for the spin-boost parameter significantly in Petrov type I. But, of course, numerically there is no major problem in calculating the value.

6.4. Connecting the Tetrads in Type D

Here we want to consider the behavior of the symmetric tetrad and the Gram-Schmidt tetrad in the limit of Petrov type D. So far no connection between the tetrads has been established in the literature. Actually, this is an important subject since we will gather fundamental information about the frames and we may classify the validity of the tetrads concerning wave extraction.

If we consider the limit of Petrov type D we know that the Weyl scalars $\Psi_3^B \to 0$ and therefore we know how the spin-boost parameter B^{TF} in Eq. (6.36) behaves in the limit, namely

$$B^{TF} = \left(\frac{1}{16} + \frac{\Psi_3^B}{6i\Psi_2^B - 8\Psi_3^B - 2i\Psi_4^B} \right)^{-1/4} \to 2, \tag{6.51}$$

and thus we know the values of the amplitude and modulus, respectively:

$$A^{TF} = \sqrt{\Re\left[B\right]^2 + \Im\left[B\right]^2} \to 2, \tag{6.52a}$$

$$\Theta^{TF} = \arctan\left(\frac{\Im\left[B\right]}{\Re\left[B\right]} \right) \to 0. \tag{6.52b}$$

Without further assumptions we can immediately deduce the relation between the symmetric transverse scalars and the Gram-Schmidt radiative scalars,

$$\Psi_0^S = \Psi_4^S \to \Psi_4^B = \Psi_0^B. \tag{6.53}$$

The background contribution encoded in Ψ_2 reads

$$\Psi_2^S \to \frac{1}{4}\left(\Psi_2^B + 3\Psi_2^B\right) = \Psi_2^B = \tilde{\Psi}_2, \tag{6.54}$$

where $\tilde{\Psi}_2 = -\frac{64Mr^3}{(M+2r)^6}$ indicates the unperturbed coulomb scalar.

It is worth noting there is also the possibility to set the phase to $\Theta = \pi/2$ yielding the desired result $\Psi_4 = \Psi_0$ in Eq. (6.46). This results in a sign change and/or multiplication by a complex number i in the scalars and spin coefficients like α (but not ε of course), $\alpha \to i\alpha$. This is a generic result due to the symmetry and degeneracy of the Weyl scalars in the transverse frame.

To compute the spin coefficients in terms of the initial quantities we only need to specify the modulus A and phase Θ, respectively. We find a similar result as for the Weyl scalars; the spin coefficients in the symmetric tetrad agree with the quantities in the Gram-Schmidt tetrad

$$(\rho,\mu,\sigma,\gamma,\varepsilon,\tau,\pi,\alpha,\beta,\kappa,\lambda,\nu)^S = (\rho,\mu,\sigma,\gamma,\varepsilon,\tau,\pi,\alpha,\beta,\kappa,\lambda,\nu)^B. \tag{6.55}$$

The symmetric null-tetrad reads in the limit of type D

$$l_S^\mu = \left(\ell_B^t, -\ell_B^r, 0, 0\right), \tag{6.56a}$$

$$n_S^\mu = \left(\ell_B^t, \ell_B^r, 0, 0\right), \tag{6.56b}$$

$$m_S^\mu = \left(0, 0, m_B^\theta, m_B^\phi\right), \tag{6.56c}$$

leading us to the final conclusion that the symmetric tetrad is equivalent to the Gram-Schmidt tetrad in the exact limit of type D. This is a fundamentally important result, since the transformation from the symmetric tetrad to the Kinnersley tetrad is a well-posed calculation in type D. Therefore, we can apply the same techniques to the Gram-Schmidt tetrad without any loss of generality.

We want to stress the fact that these results, namely the conformity of the Gram-Schmidt tetrad and the symmetric tetrad, are presumably related to the employed pseudo-spherical coordinates which are well adapted to a Bondi frame at future null infinity (cf. section 6.8). We are investigating how this relation modifies in a space-time with less symmetry and different coordinate systems. As we expect, this should lead us to a better understanding on the connection between Gram-Schmidt frames and quasi-Kinnersley frames.

6.5. Spin-Boost Degree of Freedom

The symmetric tetrad still deviates by a spin-boost transformation from the Kinnersley tetrad, indicated by the important fact that $\varepsilon^S \neq \varepsilon^{KT} = 0$, in particular it is

$$\varepsilon^S = \varepsilon^B = -\frac{4r^2\sqrt{2}M}{(M^2-4r^2)(M+2r)^2}. \tag{6.57}$$

By comparing the values of the spin coefficients in the two tetrads of interest it turns out that α^S, β^S, τ^S and π^S already agree with the value in the Kinnersley tetrad and therefore we can further constrain the unknown spin-boost parameter. Using Eqs. (4.50f, 4.50g, 4.50k, 4.50l)

$$
\begin{aligned}
\tau^{KT} &= e^{i\Theta_S}\tau^S, && (6.58a)\\
\pi^{KT} &= e^{-i\Theta_S}\pi^S, && (6.58b)\\
\alpha^{III} &= e^{-i\Theta_S}\alpha - \frac{1}{2}A_S^{-1}e^{-i\Theta_S}\delta^* A_S + \frac{1}{2}ie^{-i\Theta_S}\delta^*\Theta_S, && (6.58c)\\
\beta^{III} &= e^{i\Theta_S}\beta - \frac{1}{2}A_S^{-1}e^{i\Theta_S}\delta A_S + \frac{1}{2}ie^{i\Theta_S}\delta\Theta, && (6.58d)
\end{aligned}
$$

we can not only conclude that the phase Θ_S is constant but, in fact, is equal to zero. Additionally we can conclude that A_S does not depend on t and ϕ, thus

$$A_S = A_S(r,\theta)\,. \tag{6.59}$$

The next step is to determine the exact expression of the amplitude $A_S(r,\theta)$ to perform a type III transformation in the Kinnersley tetrad. The starting point will be the equation for the spin coefficient ε in the quasi-Kinnersley frame

$$0 = \frac{1}{A_S}\varepsilon^S - \frac{1}{2}(A_S)^{-2}DA_S, \tag{6.60}$$

where the directional derivative is defined as $D = \ell_S^\mu \partial_\mu$ and the left hand side corresponds to the value for the spin coefficient in the Kinnersley tetrad, $\varepsilon^{KT} = 0$. We may solve this equation for A_S:

$$\varepsilon^S = \frac{1}{2}D\log A_S, \tag{6.61}$$

where a straightforward calculation yields the desired result for the transformation:

$$A_S = B_S = \mathscr{D}\frac{(M-2r)}{(M+2r)}, \tag{6.62}$$

with a yet to be determined integration constant. Such a transformation will leave the scalar Ψ_2 unchanged, but will have in general a strong effect on the radiative quantities Ψ_0 and Ψ_4, scaling and mixing polarizations.

We can not constrain the spin-boost parameter from the spin coefficient ε any further. For this we make use of the spin coefficients γ. The type III transformation equation for γ is

$$\gamma^{KT} = A^S \gamma^S - \frac{1}{2}\Delta A^S, \tag{6.63}$$

fixing the integration constant to $\mathscr{D} = -1/\sqrt{2}$. The final result for the spin-boost parameter B^S reads

$$B_S = -\frac{(M-2r)}{\sqrt{2}\,(M+2r)}. \tag{6.64}$$

We compare the value we achieve for the boost with the more general definition we derived in chapter 5, Eq. (5.48), where the function $\mathscr{H}$ encodes all information of the Weyl scalars and spin coefficients in the transverse frames,

$$\nabla^\mu \nabla_\mu \mathscr{H} + \nabla^\mu \ln\left(I^{\frac{1}{6}}\right) \nabla_\mu \left(2\mathscr{H} + \ln I^{\frac{1}{12}}\right) = -2\Psi_2. \tag{6.65}$$

In the limit of a single black hole, Eq.(6.65) is immediately solvable, leading us to the following condition on the spin-boost parameter (cf. Eq. (5.60))

$$\mathscr{B} = \mathscr{B}_0 I^{\frac{1}{6}} \Gamma^{\frac{1}{2}}. \tag{6.66}$$

We perform a suitable transformation to adapt the result to our particular coordinate system yielding for Eq. (6.66)

$$\mathscr{B} = \mathscr{B}_0 \left(\sqrt{3}M\right)^{(1/3)} \frac{(M-2r)}{(M+2r)}. \tag{6.67}$$

Comparing Eq. (6.64) and Eq. (6.67) we can determine the remaining constant of integration $\mathscr{B}_0$, yielding

$$\mathscr{B}_0^{-1} = -\sqrt{2}\left(\sqrt{3}M\right)^{1/3}. \tag{6.68}$$

Whether this is a generic result or a particular value of the space-time under consideration is not yet fully determined. The calculation of $\mathscr{B}_0$ from a more theoretical point of view is still under investigation and will be published elsewhere.

Since we end up in the Kinnersley tetrad after performing a boost with $B_S = -\frac{(M-2r)}{\sqrt{2}(M+2r)}$ all spin coefficients have the correct value for the vacuum solution of a non-rotating black hole in isotropic coordinates:

$$\begin{aligned}
\tilde{\varepsilon} &= \tilde{\tau} = \tilde{\nu} = \tilde{\kappa} = \tilde{\pi} = \tilde{\sigma} = \tilde{\lambda} = 0, && (6.69a)\\
\tilde{\gamma} &= \frac{8Mr^2}{(M+2r)^4}, && (6.69b)\\
\tilde{\rho} &= -\frac{4r}{(M+2r)^2}, && (6.69c)\\
\tilde{\mu} &= \frac{1}{2}\tilde{\rho} + 2\tilde{\gamma}, && (6.69d)\\
\tilde{\beta} &= -\tilde{\alpha} = \frac{\sqrt{2}r\cot\theta}{(M+2r)^2}. && (6.69e)
\end{aligned}$$

The tetrad agrees with the well-known expressions of the Kinnersley tetrad:

$$\begin{aligned}
\tilde{\ell}^{\mu} &= \left(\frac{(M+2r)^2}{(M-2r)^2}, -\frac{4r^2}{(M^2-4r^2)}, 0, 0\right), && (6.70a)\\
\tilde{n}^{\mu} &= \left(\frac{1}{2}, \frac{2r^2(M-2r)}{(M+2r)^3}, 0, 0\right), && (6.70b)\\
\tilde{m}^{\mu} &= \left(0, 0, \frac{2\sqrt{2}r}{(M+2r)^2}, \frac{2\sqrt{2}ir\csc\theta}{(M+2r)^2}\right). && (6.70c)
\end{aligned}$$

6.6. Perturbation Theory for the Close Limit

In this section we provide the basic mathematical formalism for evolving distorted black holes as perturbative systems. We investigate the multipole moments of a general space-time as a linear perturbation about its background spherical part. Roughly speaking, this approach should be a valid way to describe black hole space-times, whose non-spherical departure from Schwarzschild is small. We are particularly interested in the radial fall-off of the Weyl scalars in the quasi-Kinnersley tetrad and the Gram-Schmidt tetrad in the limit of Type D, respectively. In the foregoing sections we have already deduced all expressions for the quantities in Petrov type I for both of the tetrads. To receive explicit expressions for the radiative scalars we need to specify the perturbation encoded in the function q. A suitable approach for such a task is perturbation theory. Our starting point for the perturbation analysis is the Hamiltonian constraint in Eq. (6.14):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\psi=-\frac{1}{4}\psi\left(\frac{\partial^2 q}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 q}{\partial\theta^2}+\frac{1}{r}\frac{\partial q}{\partial r}\right). \tag{6.71}$$

In the literature the Brill waves have been repeatedly invoked as a tool for wave extraction. The perturbation $q=q(r,\theta)$ has to satisfy a set of boundary conditions, but is an otherwise arbitrary function. We follow [77, 85] and assume that $q(r,\theta)$ decays fairly rapidly at infinity (at least as $1/r^2$) and that it is regular at $\theta=0$:

$$\left.\frac{\partial q(r,\theta)}{\partial\theta}\right|_{\theta=0}=0,\qquad \lim_{r\to\infty}q(r,\theta)=\mathcal{O}\left(r^{-2}\right),\qquad \left.q(r,\theta)\right|_{\theta=0}=0. \tag{6.72}$$

In case of a superposition of a black hole and Brill waves we impose an additional boundary condition at the throat of the Einstein-Rosen bridge. We choose the initial slice to be isometric at the throat. The inversion-through-the-sphere transformation from a point $r>\chi$ to a point inside the throat $r<\chi$ is given by [85]

$$r'=\chi^2/r. \tag{6.73}$$

From the usual tensor transformation rule for the metric component γ_{11},

$$\gamma'_{11}\left(r'=\chi\right)=\gamma_{11}\left(r=\chi\right), \tag{6.74}$$

we can determine the throat condition on the isometry surface (assuming continuity of the first derivate)

$$\left[\frac{\partial\gamma_{11}}{\partial r}+\frac{2\gamma_{11}}{\chi}\right]_{r=\chi}=0. \tag{6.75}$$

The most natural choice would be to choose the conformal factor to have the same condition as in the unperturbed Schwarzschild solution. It turned out that this conditions introduces numerical and analytical complications for the function $q(r,\theta)$. Therefore we will perform a coordinate transformation to make use of a widely used system of coordinates for the Brill waves, the η-coordinates [83, 84, 95, 85, 96] . The coordinate transformation of the form

$$r=\frac{M}{2}\mathrm{e}^{\eta} \tag{6.76}$$

was motivated by numerical reasons initially. The isometry constraint suffers from the fact that it is a so called "anti-Robin" condition: it has the wrong relative sign since the normal vector points in the wrong direction relative to the inner boundary. This behavior is not suitable for the standard proofs of uniqueness for the initial-value problem and may introduce problems with techniques for solving elliptic equations (see [88, 85]) . We have encountered the same issue in a preceding work (Elbracht et al. [97]).

In the new coordinates the line element, Eq. (6.15), now takes the form

$$ds^2=-\left(\frac{\mathrm{e}^{\eta}-1}{\mathrm{e}^{\eta}+1}\right)^2\mathrm{d}t^2+\psi(\eta)^4\left[\mathrm{e}^{2q}\left(\mathrm{d}\eta^2+\mathrm{d}\theta\right)+\sin^2\theta\,\mathrm{d}\phi\right], \tag{6.77}$$

and the Hamiltonian constraint, Eq. (6.14), is transformed into

$$\partial_\eta^2\psi+\nabla_\theta^2\psi+\cot\theta\,\partial_\theta\psi+\frac{1}{4}\psi\left(\partial_\eta^2 q+\partial_\theta^2 q-1\right)=0. \tag{6.78}$$

The inversion-through-the-sphere condition, Eq. (6.73), is now given by

$$\eta' = -\eta. \tag{6.79}$$

The throat condition on the isometry surface, Eq. (6.75), transforms into

$$\left[\frac{\partial \gamma_{11}}{\partial \eta}\right]_{\eta=0} = \left[\frac{\partial q}{\partial \eta} + \frac{2}{\psi}\frac{\partial \psi}{\partial \eta}\right]_{\eta=0} = 0. \tag{6.80}$$

We choose ψ to have the same boundary conditions as in the Schwarzschild solution, namely

$$\left.\frac{\partial \psi}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \psi}{\partial \theta}\right|_{\theta=\pi/2} = \left.\frac{\partial \psi}{\partial \eta}\right|_{\eta=0} = 0, \tag{6.81}$$

which determines the conditions for the perturbation q to

$$\left.\frac{\partial q}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial q}{\partial \theta}\right|_{\theta=\pi/2} = \left.\frac{\partial q}{\partial \eta}\right|_{\eta=0} = 0, \tag{6.82}$$

therefore decoupling the boundary conditions for the conformal factor and the perturbation. By imposing an outer boundary condition $\psi(\eta)$ is fixed completely.

The elliptic equation for the Hamiltonian constraint in Eq. (6.78) must in general be solved numerically. Once this is done, it can be used to compute initial data for the perturbation equations as described in the foregoing sections and can also be evolved with a numerical code (see e.g. Elbracht et al [97]). Here we focus on the extraction of initial data itself, not on the evolution. We expand the conformal factor and the function $q(r,\theta)$ appearing in this equation in terms of spherical harmonics $Y_{lm}(\theta,\phi)$ such that

$$\psi(\eta,\theta,\phi) = \tilde{\psi}(\eta) + a\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \psi_{lm}(\eta) Y_{lm}(\theta,\phi) + \mathscr{O}\left(a^2\right), \tag{6.83}$$

with $\tilde{\psi}(\eta) = \sqrt{2M}\cosh\left(\frac{\eta}{2}\right)$ being the conformal factor of the Schwarzschild solution.

The perturbation to linear order in a is

$$q(\eta,\theta,\phi) = a \sum_{l=2,4\ldots} q_{lm}(\eta) Y_{lm}(\theta,\phi)\,, \tag{6.84}$$

where we further assume $a<1$ which is a reasonable assumption considering the amplitude of the wave. We construct a perturbation q in the following manner

$$q(\eta,\theta) \;=\; \sqrt{\frac{\pi}{5}} 8\,a\,q(\eta)\left(-\frac{1}{3}Y_{20} + \frac{\sqrt{5}}{3}Y_{00}\right) = 2\,a\,\mathrm{e}^{-\eta^2}\sin^2\theta\,, \tag{6.85}$$

satisfying the boundary conditions in Eq. (6.82). We may now solve the Hamiltonian constraint with that particular perturbation. The only non-zero coefficients of the conformal factor in the first-order expansion are

$$\psi_{00} \;=\; \frac{1}{3}\sqrt{2}\sqrt{M}e^{-\eta^2}\left(\pi e^{\eta^2}\left(\mathrm{erf}(\eta)\sinh\left(\frac{\eta}{2}\right) - \cosh\left(\frac{\eta}{2}\right)\right) - 2\sqrt{\pi}\cosh\left(\frac{\eta}{2}\right)\right), \tag{6.86a}$$

$$\begin{aligned}\psi_{20} \;=&\; \frac{1}{3}\sqrt{\frac{2\pi}{5}}\sqrt{M}e^{-\frac{1}{2}\eta(2\eta+5)}\left(-\sqrt{\pi}e^{\eta^2+\frac{9}{4}} + e^{2\eta} + e^{3\eta}\right)\\ &+\frac{\pi\sqrt{M}e^{\frac{9}{4}-\frac{5\eta}{2}}\left(\mathrm{erfc}\left(\eta-\frac{3}{2}\right) - e^{5\eta}\mathrm{erfc}\left(\eta+\frac{3}{2}\right)\right)}{3\sqrt{10}}\,,\end{aligned} \tag{6.86b}$$

where $\mathrm{erf}(\eta)$ and $\mathrm{erfc}(\eta)$ are the error function and complementary error function, respectively. All terms with $m \neq 0$ vanish due to the imposed azimuthal symmetry. We end up not only with an angular contribution from the Y_{20} spherical harmonic but additionally we get a Y_{00}-term. This is a particular effect caused by the way we have constructed the perturbation. We have now all necessary information to calculate the radiative scalars and spin coefficients in the two tetrads under investigation.

6.6.1. Perturbed Newman-Penrose Quantities in the Gram-Schmidt Tetrad

We will use the exact analytic solution to the perturbative initial data equations to discuss a specific example of the extraction procedure applied to the axisymmetric black hole initial data sets discussed above. We construct the perturbed null vectors in the same way as in section 6.2, Eqs. (6.16, 6.17), still satisfying to linear order the null-vector conditions (cf. Eqs. (4.22, 4.22))

$$\ell^\mu \ell_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \tag{6.87a}$$

$$\ell^\mu n_\mu = m^\mu \bar{m}_\mu = 1. \tag{6.87b}$$

Contracting the Weyl tensor with the perturbed vectors yields for the spin coefficients

$$\rho^B = -\frac{2\sqrt{2}\sinh^4\left(\frac{\eta}{2}\right)\operatorname{csch}^3(\eta)}{M} - \frac{4\sqrt{2}\pi Q e^{-2\eta}}{3M}, \tag{6.88a}$$

$$\mu^B = \rho^B, \tag{6.88b}$$

$$\tau^B = -\frac{\sqrt{2}Q e^{-\eta^2}\sin(2\theta)}{M}, \tag{6.88c}$$

$$\pi^B = -\tau^B, \tag{6.88d}$$

$$\lambda^B = \frac{2\sqrt{2}Q e^{-\eta^2}\eta\sin^2(\theta)}{M}, \tag{6.88e}$$

$$\sigma^B = \lambda^B, \tag{6.88f}$$

$$\nu^B = \tau^B, \tag{6.88g}$$

$$\kappa^B = -\tau^B, \tag{6.88h}$$

$$\gamma^B = \frac{\sinh^2\left(\frac{\eta}{2}\right)\operatorname{csch}^3(\eta)}{\sqrt{2}M} + \frac{2\sqrt{2}\pi Q e^{-5\eta}}{3M}, \tag{6.88i}$$

$$\beta^B = \frac{\operatorname{sech}^2\left(\frac{\eta}{2}\right)\cot(\theta)}{4\sqrt{2}M} + \frac{\sqrt{2}\pi Q e^{-2\eta}\cot(\theta)}{3M}, \tag{6.88j}$$

$$\alpha^B = -\beta^B, \tag{6.88k}$$

$$\varepsilon^B = \gamma^B. \tag{6.88l}$$

The Weyl scalars to linear order in a read

$$\Psi_0 = \Psi_4 = \frac{2\sqrt{\pi}a e^{-5\eta}\sin^2\theta}{M^2}, \tag{6.89a}$$

$$\Psi_1 = -\Psi_3 = \frac{4\sqrt{\pi}a e^{-5\eta}\sin(2\theta)}{M^2}, \tag{6.89b}$$

and the correction in leading order to the background is

$$\Psi_2 \quad = \quad \tilde{\Psi}_2 + \frac{4\sqrt{\pi} a e^{-3\eta}}{3M^2}, \tag{6.90}$$

where $\tilde{\Psi}_2 = -\frac{1}{8M^2}\mathrm{sech}^6(\eta/2)$ is the Schwarzschild solution and the scalars still obey the symmetry

$$\Psi_0^B - \Psi_4^B \quad = \quad 0 + \mathcal{O}\left[a^2\right], \tag{6.91a}$$

$$\Psi_1^B + \Psi_3^B \quad = \quad 0 + \mathcal{O}\left[a^2\right]. \tag{6.91b}$$

6.6.2. Perturbed Newman-Penrose Quantities in the Symmetric Tetrad

Making use of Eqs. (6.47) the calculation of the Weyl scalars in the symmetric tetrad in Petrov type D is a straightforward task, yielding

$$\Psi_0^S \quad = \quad \Psi_0^B + \mathcal{O}\left(a^2\right), \tag{6.92a}$$

$$\Psi_4^S \quad = \quad \Psi_4^B + \mathcal{O}\left(a^2\right), \tag{6.92b}$$

and the correction in linear order to the background is

$$\Psi_2^S \quad = \quad \tilde{\Psi}_2 + \mathcal{O}\left(a^2\right). \tag{6.93}$$

The longitudinal scalars vanish in the symmetric tetrad even to linear order. As expected from the general definition of the quantities in Eq. (6.47) we yield that the transverse scalars and the coulomb contribution encoded in Ψ_2 in the symmetric tetrad agree exactly with the expressions in the Gram-Schmidt tetrad except Ψ_1 and Ψ_3, respectively. This is the main difference which appears when comparing the Weyl scalars in both frames. It positions the symmetric tetrad as a more rigorous frame for wave extraction since the unphysical wave components are eliminated by construction.

A similar result is obtained for the spin coefficients;

$$(\rho,\mu,\sigma,\gamma,\varepsilon,\tau,\pi,\alpha,\beta,\kappa,\lambda,\nu)^S = (\rho,\mu,\sigma,\gamma,\varepsilon,\tau,\pi,\alpha,\beta,\kappa,\lambda,\nu)^B + \mathscr{O}\left[a^2\right]. \qquad (6.94)$$

The components of the null vectors in the symmetric tetrad read to linear order in a

$$\begin{aligned} \ell_S^\mu &= \left\{\ell_B^t, -\ell_B^\eta, Am^\theta, 0\right\}, && (6.95a)\\ n_S^\mu &= \left\{\tilde{n}^t, \ell_B^\eta, -Am^\theta, 0\right\}, && (6.95b)\\ m_S^\mu &= \left\{0, A\ell^\eta, m_B^\theta, m_B^\phi\right\}, && (6.95c) \end{aligned}$$

where the perturbation is encoded in the function $A = -\frac{1}{6}e^{-\eta^2}\eta\sin(2\theta)a$.

6.6.3. Boost to the Kinnersley Tetrad

To find the correct spin-boost to perform a transformation from the symmetric tetrad in the Kinnersley tetrad we make use of an important property found by Teukolsky [15, 98]; the first order tetrad and gauge invariance in the linear regime, i.e. for infinitesimal tetrad transformation. Knowing this important property, we can perform the same boost as in section (6.5)

$$\mathscr{B} = \frac{\tanh(\eta/2)}{\sqrt{2}}, \qquad (6.96)$$

which brings us in the Kinnersley tetrad as expected. The final result for the null vectors is the well known form in the Kinnersley tetrad perturbed by an amplitude a of a Brill wave to linear order $\mathscr{O}[a]$.

The Weyl scalars in the Kinnersley tetrad are

$$\begin{aligned} \Psi_0 &= \frac{4\sqrt{\pi}ae^{-5\eta}\sin^2\theta}{M^2}, && (6.97a)\\ \Psi_4 &= = \frac{\sqrt{\pi}ae^{-5\eta}\sin^2\theta}{M^2}, && (6.97b)\\ \Psi_2 &= \tilde{\Psi}_2 + \frac{4\sqrt{\pi}ae^{-3\eta}}{3M^2}, && (6.97c) \end{aligned}$$

where the Weyl scalars Ψ_0 and Ψ_4 encode the gravitational wave signal (ingoing and outgoing signal, respectively) and Ψ_2 represents the background of a Schwarzschild black hole perturbed by a Brill wave. In case of a vanishing amplitude of the time symmetric gravitational wave, we end up with an unperturbed Schwarzschild spacetime without radiation.

Since we deal with scalars, we can perform a straightforward coordinate transformation to isotropic Schwarzschild to express the Weyl scalars in a more familiar way. The radial fall-off of the Weyl scalars in isotropic coordinates is

$$\Psi_0 = \frac{\sqrt{\pi} a M^3 \sin^2(\theta)}{8r^5}, \tag{6.98a}$$

$$\Psi_4 = \frac{\sqrt{\pi} a M^3 \sin^2(\theta)}{32r^5}, \tag{6.98b}$$

$$\Psi_2 = \tilde{\Psi}_2 + \frac{\sqrt{\pi} a M}{6r^3}. \tag{6.98c}$$

6.7. Extracting the Signal at Future Null Infinity

To study the dynamics we want to apply the results found by Teukolsky (cf. section 4.13) to the specific case of a Schwarzschild background. For this use we set the angular momentum in the Teukolsky equation to zero, therefore corresponding to the Bardeen & Press solution without source [14]. As deduced in section 4.13 the perturbation equation for ψ, is given by

$${}_s\mathscr{P}\psi = 4\pi\Sigma T, \tag{6.99}$$

where $\psi = \Psi_0$ or $\psi = \rho^{-4}\Psi_4$. The operator ${}_s\mathscr{P}$ is

$$\begin{aligned} \mathscr{P} = {} & \frac{r^4}{\Delta}\frac{\partial^2}{\partial t^2} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{2is\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} \\ & - 2s\left[\frac{Mr^2}{\Delta} - r\right]\frac{\partial}{\partial t} + \left(s^2\cot^2\theta - s\right), \end{aligned} \tag{6.100}$$

where s is the spin weight of the field (2 for Ψ_0, -2 for Ψ_4). Our particular interest lies in the behavior of the fields Ψ_0 and Ψ_4 at future null infinity where the quantities are observed. Our particular choice of a metric represents a vacuum space-time and thus has a vanishing energy momentum tensor $T = 0$. As shown by Teukolsky (cf. section 4.13) the radial behavior of the Weyl scalars Ψ_0 and Ψ_4 at future null infinity is given by

$$\begin{array}{lll} \Psi_0 \approx \frac{e^{i\omega\tilde{r}}}{r^5}, & \Psi_4 \approx \frac{e^{i\omega\tilde{r}}}{r}, & \text{(outgoing waves);} \\ \Psi_0 \approx \frac{e^{-i\omega\tilde{r}}}{r}, & \Psi_4 \approx \frac{e^{-i\omega\tilde{r}}}{r^5}, & \text{(ingoing waves).} \end{array}$$

Our solution for the scalars on the initial slice Eq. (6.98) is easily recognized as a solution of the Teukolsky equation, thus we can immediately conclude the asymptotic radial behavior for the outgoing waves, namely

$$\Psi_0 = \frac{\sqrt{\pi} a M^3 \sin^2(\theta)}{8r^5}, \tag{6.101a}$$

$$\Psi_4 = \frac{\sqrt{\pi} a M^3 \sin^2(\theta)}{32r}, \tag{6.101b}$$

Plugging Eq. (6.101) into Eq. (4.85) the quasi-normal gravitational modes of the distorted Schwarzschild space-time turn out to be

$$\Psi_0 \approx -\frac{1}{8r^5} e^{i\omega\tilde{r}} e^{-i\omega t} e^{i\omega\phi} \sqrt{\pi} a M^3 \sin^2\theta, \tag{6.102a}$$

$$\Psi_4 \approx -\frac{1}{32r} e^{i\omega\tilde{r}} e^{-i\omega t} e^{i\omega\phi} \sqrt{\pi} a M^3 \sin^2\theta. \tag{6.102b}$$

By using perturbative techniques to compute the quasi normal modes expected in the evolution of these data sets, we provide an important testbed for full non-linear codes that should evolve the same systems.

6.8. Limitations and Possible Issues of the Gram-Schmidt Approach

In regimes where the perturbations to the underlying Schwarzschild or Kerr geometries are considered moderately small, we can compare different methods to extract the radiation from the space-time. In this section we want to show how the freedom in the choice of the tetrad can cause serious problems, and may lead to a wrong spherical harmonic decomposition. We demonstrate that the difference in the waveform arises due to the tetrad adopted via the Gram-Schmidt procedure. As alluded in the foregoing chapters this is a fundamentally important issue, since it is crucial to be able to extract the waves in a robust and well-posed manner to define physically meaningful quantities.

We define an extraction world-tube, $x^2+y^2+z^2=r^2$, and construct a triad of orthonormal spatial vectors by applying a Gram-Schmidt procedure in the following way (cf. section 6.2):

$$u^i = [-y,x,0]\,, \tag{6.103a}$$

$$v^i = [x,y,z]\,, \tag{6.103b}$$

$$u^i = \sqrt{g}g^{ia}\varepsilon_{abc}u^b v^c, \tag{6.103c}$$

where four null vectors are then given by

$$n^0 = \frac{1}{\sqrt{2}\alpha}, \quad n^i = \frac{1}{\sqrt{2}\alpha}\left(\frac{-\beta^i}{\alpha} - v^i\right), \tag{6.104a}$$

$$\ell^0 = \frac{1}{\sqrt{2}\alpha}, \quad \ell^i = \frac{1}{\sqrt{2}\alpha}\left(\frac{-\beta^i}{\alpha} + v^i\right), \tag{6.104b}$$

$$m^0 = 0, \qquad m^i = \frac{1}{\sqrt{2}}\left(u^i + iw^i\right). \tag{6.104c}$$

To demonstrate how a particular choice of coordinate can lead to a wrong waveform extraction we considered a simple coordinate transformation of the form $\tilde{r} \to rg(t)$. This simple transformation of the radial coordinate will result in an incorrect extraction and therefore in an incorrect radiation scalar (see also [74]). First we will carry out the calculation of Ψ_4 using the standard pseudo-spherical coordinates (t, r, θ, ϕ) and afterwards performing the mentioned transformation of the form $\tilde{r} \to rg(t)$. As a final step we compare different wave extraction procedures and demonstrate the clear advantage of the method developed in chapter 5 .

We consider three different extraction methods:

1. Identifying the Bondi frame to compute the Weyl scalars (cf. section 4.10)
2. Performing a Gram-Schmidt orthogonalization to construct a tetrad
3. Using the extraction method developed in chapter 5 to calculate Ψ_4 as a function of space-time invariants

Our starting point is the space-time metric of the Brill wave and Black Hole superposition, Eq. (6.15):

$$ds^2 = \alpha^2 dt^2 - \psi^4 \left[e^{2q} \left(dr^2 + r^2 d\theta^2 \right) + r^2 \sin^2 \theta d\phi^2 \right]. \tag{6.105}$$

6.8.1. Case I - Pseudo-Spherical Coordinates

1. Bondi Frame

Since the metric is at leading order the flat metric in spherical coordinates, the Bondi frame is easily identified. A straightforward calculation yields

$$\Psi_4^{\mathrm{I}}(r, \theta) = \frac{\sqrt{\pi} a M^3 \sin^2 \theta}{32 r^5}. \tag{6.106}$$

We will use the radiative scalar Ψ_4 in the Bondi frame as a reference expression to compare the different solutions for Ψ_4.

2. Gram-Schmidt Tetrad

As already demonstrated in this chapter a Gram-Schmidt decomposition using the standard coordinates gives, cf. Eq. (6.98b),

$$\Psi_4^{\mathrm{II}}(r,\theta) = \frac{\sqrt{\pi} a M^3 \sin^2\theta}{16 r^5} = 2\Psi_4^{\mathrm{I}}, \tag{6.107}$$

yielding the same expression for the radiation scalar as in Eq. (6.110), except a well-known deviation by a factor of 2 introduced by a slightly different definition of the null coordinate u (see e.g. [74] and references within).

3. Ψ_4 from Space-Time Invariants

A different approach we have presented in chapter 5 calculates the Weyl scalars as functions of the two curvature invariants I and J, which are defined in Eqs. (4.34). The advantage in dealing with the Kretschmann scalars is the coordinate and gauge invariance, thus the calculation of Ψ_4^{III} does not rely on the identification of a particular tetrad, making it a more robust approach for wave extraction. The expression for Ψ_4 is according to Eq. (5.64c)

$$\Psi_4^{\mathrm{III}}(r,\theta) = -\frac{i}{2}\mathscr{B}_0^{-2}\Gamma^{-1} I^{\frac{1}{6}}\left(\Theta - \Theta^{-1}\right), \tag{6.108}$$

where $\mathscr{B}_0$ is an arbitrary constant of integration, we discussed earlier (c.f. section 6.5). In Eq. (6.68) we fixed the constant to $\mathscr{B}_0^{-1} = -\sqrt{2}\left(\sqrt{3}M\right)^{1/3}$. A straightforward calculation gives the expected result

$$\Psi_4^{\mathrm{III}}(r,\theta) = \frac{\sqrt{\pi} a M^3 \sin^2\theta}{32 r^5} = \Psi_4^{\mathrm{I}}(t,r,\theta). \tag{6.109}$$

In pseudo-spherical coordinates all three methods agree very well, except a negligible factor of 2. The question that may arise is: what happens in a different coordinate system which might not that well adapted to a Bondi frame. We will demonstrate how problems might be introduced by a simple coordinate transformation.

6.8.2. Case II - Transformation $\tilde{r} \to rg(t)$

We perform the described coordinate transformation, therefore rescaling the radial vector by $\tilde{r} \to rg(t)$ and recomputing the Weyl scalars through the different methods.

1. Bondi Frame

A simple computation yields for the Weyl scalar in the Bondi frame

$$\Psi_4^{\mathrm{I}}(r,\theta) = \frac{\sqrt{\pi} a M^3 \sin^2\theta}{32 g(t)^5 r^5}, \tag{6.110}$$

thus being essentially equivalent to a pure coordinate transformation of the radial vector.

2. Gram-Schmidt Tetrad

In the second case, the coordinate transformation induces an additional factor in the radiative quantities. The leading order of the tetrad that enters in computing Ψ_4 is

$$n^0 = \frac{1}{\sqrt{2}}, \qquad n^a = -\frac{1}{\sqrt{2}\,g(t)}\left[1 - r\dot{g}(t)\right]\partial_r^a, \tag{6.111a}$$

$$l^0 = \frac{1}{\sqrt{2}}, \qquad l^a = \frac{1}{\sqrt{2}\,g(t)}\left[1 - r\dot{g}(t)\right]\partial_r^a, \tag{6.111b}$$

$$m^0 = 0, \qquad m^a = \frac{1}{g(t)\sqrt{2}r}\left[\partial_\theta^a + i\partial_\phi^a,\right], \tag{6.111c}$$

where for $g = 1$ the tetrad reduced to the null vectors of case I. Contracting the Weyl tensor with this tetrad results in a different expression of the Weyl scalar Ψ_4, namely

$$\Psi_4^{\mathrm{II}}(t,r,\theta) = \frac{2}{g(t)}\Psi_4^{\mathrm{I}}(t,r,\theta). \tag{6.112}$$

note the additional factor of $g(t)$ in the denominator. Now, we can correct this dependence as demonstrated by Lehner & Moreschi [74]. This will be in a generic setup a nontrivial task and might introduce additional numerical errors.

3. Ψ_4 from Space-Time Invariants

Since this method does not rely on the identification of a particular tetrad and therefore does not depend in a crucial manner on a particular coordinate transformation the calculation is easily performed, yielding

$$\Psi_4^{\mathrm{III}}(t,r,\theta) = \frac{\sqrt{\pi} a M^3 \sin^2\theta}{32 g(t)^5 r^5} = \Psi_4^{\mathrm{I}}(t,r,\theta)\,. \tag{6.113}$$

The result obtained clearly demonstrates the advantage of the approach presented in chapter 5, since now additional correction has to be considered.

In recapitulation, in this section we have demonstrated how difficulties may arise in wave extraction of numerically generated space-times and how to circumvent the difficulties inherent in the choice of tetrad for wave extraction. We have provided, to our knowledge, the only definition of the Weyl scalars in numerical relativity which satisfies the expected physical properties *a priori* without the need of further correction.

Another point worth mentioning is the well known fact that the notion of total angular momentum in general relativity is sensitive to the so-called problem of *supertranslation ambiguities* [99, 100, 101, 102, 103, 104, 105, 106, 107]. This is an important issue since there is a clear need to possess a unique notion of total angular momentum to extract physical relevant information in numerically generated space-times. As we expect, our new method should aid in defining a unique notion of total angular moment in numerical relativity. This is subject of future work on this topic.

7. Conclusion and Outlook

They always say time changes things,
but you actually have to change them yourself.
Andy Warhol

The enormous developments in the last years finally permit well-resolved numerical simulations, and make it possible to address and evolve more complex and realistic astrophysical events such as binary black hole mergers. Since the accuracy of the numerical implementations are now at a stage where limitations of perturbation approaches have become more evident, the treatment of black hole space-times based on a non-perturbative approach is of increasing interest.

The main focus of this dissertation was to address the role of wave extraction in numerical relativity with the aim of having a generic and robust method without the need of perturbation theory. Secondly, we have studied a particular subset of frames in the Newman-Penrose formalism, the transverse frames, to some detail, gaining important insights in the mathematical properties of these families of tetrads.

The results we have presented in this work show that we are in fact able to extract the radiation quantities in a robust and unique way for various physical situations showing no limitation of applicability of our new approach. More encouraging is the realization that since our methods have been generic, therefore, the problems we can go on to address immediately are also generic.

Applying this methodology to space-times with strong directional dependences, for example black holes with different masses and/or spinning compact objects, will be of great importance, since the current used perturbative methods, mainly the Zerilli approach and the Gram-Schmidt procedure, have crucial limitations. In fact, the Zerilli equations are well defined only for the non-rotating case of an isolated Schwarzschild black hole, while the Gram-Schmidt approach to construct a null tetrad will not result in a Bondi tetrad at future null infinity for sure.

As a second objective, we have surveyed the mathematical properties of transverse frames by studying a distorted black hole space-time, using the "Brill wave plus black hole" family of initial data sets, which mimic the behavior of two black holes that have just collided head-on.

In this particular space-time we have been able to define the quasi-Kinnersley frame in the general case of Petrov type I space-time. We have found the quasi-Kinnersley tetrad with $\varepsilon = 0$ by breaking the residual spin-boost symmetry completely analytically. We have established a connection between the transverse quasi-Kinnersley tetrad with $\Psi_0 = \Psi_4$ and the tetrad constructed by the usual Gram-Schmidt procedure. It turned out that these two frames are the same in the limit of Petrov type D for the space-time under consideration. This has an important impact on the validity of the Gram-Schmidt tetrad as a wave extraction tool, endowing it with the properties of the quasi-Kinnersley tetrad with $\Psi_0 = \Psi_4$. These calculations can easily be extended to a rotating black hole superposed with a brill wave, yielding corresponding results. How and whether these results can be carried over to a space-time with less symmetry, deserves further studies. So far, the results are promising, and it seems likely that such a connection between the frames can be established in a completely generic space-time. These results will serve as a test configuration in an ongoing study to compare the different frames for wave extraction which have not been performed so far, therefore giving a greater knowledge in numerical errors induced in a simulation.

Finally, we have re-examined the issue of computing gravitational radiation effects through the use of Weyl scalars. In the case of a small amplitude of the Brill wave, we have managed to find a perturbative solution to the initial value problem. We have constructed a solution to the perturbative initial data problem to test the accuracy of the main extraction techniques.

In the particular case of "brill wave plus black holes" space-times the high degree of symmetry of the problem indicates that the different approaches might not deviate from one another significantly. But we have demonstrated that even here a simple coordinate transformation will lead to a wrong decomposition and thus wrong waveforms. Our analysis reveals the advantage of our approach compared to commonly used extraction methods, which will be even more apparent for space-times with strong directional dependences. Not only is the calculation enormously simplified, since the calculation of the Kretschmann scalars is straightforwardly carried out as soon as a metric is defined, but even more important, it makes the calculation of any corrections to the waveforms completely redundant. And still, since the radiation quantities extracted by our method are completely conform to a Bondi system, it will serve as an excellent tool to check consistency among different codes and implementations of Einstein's equations in numerical simulations.

A. Evolution Equations

To determine an alternative set of equations for the evolution system, we follow closely the determination by Arnowitt, Deser and Misner [27] outlined in the textbook by Wald.

Assume $(\mathfrak{M}, g_{\mu\nu})$ is a globally hyperbolic space-time, foliated by Cauchy surfaces and Σ_t, parameterized by a global time function t. Let n^μ be the unit normal vector field to the hypersurfaces Σ_t, then the space-time metric, $g_{\mu\nu}$, induces a spatial metric $\gamma_{\mu\nu}$ on each Σ_t by the formula

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{A.1}$$

The "flow of time" is encoded in a vector field t^μ on $\mathfrak{M}$, which satisfies

$$t^\mu \nabla_\mu t = 1. \tag{A.2}$$

Since the vector field t^μ is not necessarily orthogonal to Σ_t, it can be decomposed into its parts normal and tangential to the hypersurface by defining the *lapse function*, α, and the *shift vector*, β^μ according to

$$\begin{aligned} \alpha &= -t^\mu n_\mu = (n^\mu \nabla_\mu t)^{-1}, && \text{(A.3)} \\ \beta_\mu &= \gamma_{\mu\nu} t^\nu. && \text{(A.4)} \end{aligned}$$

To "move forward in time" we use the integral curves of t^μ to construct a diffeomorphism between two arbitrary hypersurfaces labeled as Σ_0 and Σ_t, respectively. We may view this effect of "moving forward in time" as changing the spatial metric on the hypersurface from $\gamma_{\mu\nu}(0)$ to $\gamma_{\mu\nu}(t)$. Thus, we can come to the conclusion that a globally hyperbolic space-time $(\mathfrak{M}, g_{\mu\nu})$ represents the time development of a Riemannian three metric $\gamma_{\mu\nu}$. Comparing these results with the Hamiltonian formulation (cf. chapter 1) we should expect the initial data to consist of the "dynamic variable" $\gamma_{\mu\nu}$ and its first time derivative. A natural candidate for the time derivative of $g_{\mu\nu}$ is the second fundamental form $K_{\mu\nu}$, which is defined as

$$K_{ij} = \gamma_i^{\mu} \gamma_j^{\nu} \nabla_\mu n_\nu, \tag{A.5}$$

where n^μ is an unit time-like vector field which is normal to the hypersurfaces Σ (cf. chapter 1). $K_{\mu\nu}$ is symmetric and spatial,

$$n^\mu K_{\mu\nu} = n^\nu K_{\mu\nu} = 0, \tag{A.6}$$

and can be rewritten as

$$K_{\mu\nu} \equiv \frac{1}{2} \pounds_n \gamma_{\mu\nu}. \tag{A.7}$$

We reformulate the Einstein equations in terms of our new evolution variables $(\gamma_{\mu\nu}, K_{\mu\nu})$ to nourish an appropriate set of equations we can evolve.

This section summarizes the derivation of the evolution equation of $\gamma_{\mu\nu}$, $K_{\mu\nu}$ and introduces the Hamiltonian and momentum equations, which are the governing equations and have to be satisfied on each hypersurface at all times.

We will make use of the so-called Gauss-Codazzi relations, relating the curvature of a manifold to that of a sub-manifold embedded in it by the following relations

$$
\begin{aligned}
{}^{(3)}R^{\sigma}_{\mu\nu\rho} &= \gamma^{\sigma}_{\alpha} \gamma^{\beta}_{\mu} \gamma^{\delta}_{\nu} \gamma^{\varepsilon}_{\rho} R^{\alpha}{}_{\beta\delta\varepsilon} - K_{\mu\rho} K^{\sigma}_{\nu} + K_{\nu\rho} K^{\sigma}_{\mu}, && \text{(A.8a)}\\
\gamma^{\mu}_{\rho} R_{\mu\nu} n^{\nu} &= \bar{\nabla}_{\sigma} K^{\sigma}_{\rho} - \bar{\nabla}_{\rho} K^{\sigma}_{\sigma}. && \text{(A.8b)}
\end{aligned}
$$

A.1. The Evolution Equation for the Metric $\gamma_{\mu\nu}$

To determine the evolution equation for $\gamma_{\mu\nu}$ we calculate its time derivative defined as the Lie derivate $\pounds_t \gamma_{\mu\nu}$:

$$\frac{\partial \gamma_{\mu\nu}}{\partial t} = \pounds_t \gamma_{\mu\nu} = \alpha \pounds_n \gamma_{\mu\nu} + \pounds_\beta \gamma_{\mu\nu} = 2\alpha K_{\mu\nu} + \pounds_\beta \gamma_{\mu\nu}, \tag{A.9}$$

where

$$\pounds_\beta \gamma_{\mu\nu} = \beta^\sigma \nabla_\sigma \gamma_{\mu\nu} + \gamma_{\mu\sigma} \nabla^\sigma \beta_\nu + \gamma_{\sigma\nu} \nabla^\sigma \beta_\mu = \bar{\nabla}_\mu \beta_\nu + \bar{\nabla}_\nu \beta_\mu. \tag{A.10}$$

The nabla operator $\bar{\nabla}^\mu$ is defined as the projection of the covariant derivative on the three dimensional hypersurface, namely $\bar{\nabla}_\mu = \gamma_\mu{}^\sigma \nabla_\sigma$. Finally, the time derivative of $\gamma_{\mu\nu}$ reads

$$\frac{\partial \gamma_{\mu\nu}}{\partial t} = 2\alpha K_{\mu\nu} + \bar{\nabla}^\mu \beta_\nu + \bar{\nabla}^\nu \beta_\mu. \tag{A.11}$$

A.2. The Evolution Equation for the Extrinsic Curvature $K_{\mu\nu}$

To derive the evolution equation for $K_{\mu\nu}$ we project the four-dimensional Ricci tensor $R_{\mu\nu}$ onto the three-dimensional sub-manifold according to

$$R_{ij} = \gamma_i^\mu \gamma_j^\nu R_{\mu\nu}, \tag{A.12}$$

where one possible convention for the Ricci tensor is

$$R_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{A.13}$$

and T is the trace of $T_{\mu\nu}$:

$$T = g^{\mu\nu} T_{\mu\nu} = \gamma^{\mu\nu} T_{\mu\nu} - n^\mu n^\nu T_{\mu\nu} = S - \rho, \tag{A.14}$$

with

$$S_{ij} = \gamma_i^{\mu}\gamma_j^{\nu}T_{\mu\nu}, \tag{A.15a}$$
$$\rho = T_{\mu\nu}n^{\mu}n^{\nu}. \tag{A.15b}$$

Now we can calculate the projection of $R_{\mu\nu}$ on Σ

$$R_{ij} = 8\pi\left[S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right]. \tag{A.16}$$

Another possibility to calculate the three dimensional Ricci tensor is to utilize the Gauss-Codazzi relations in Eqs. (A.8). We contract the Riemann tensor with the four-dimensional metric $g_{\mu\nu}$ yielding

$$R_{\mu\nu} = g^{\sigma\rho}R_{\mu\sigma\nu\rho} = \gamma^{\sigma\rho}R_{\mu\sigma\nu\rho} - n^{\sigma}n^{\rho}R_{\mu\sigma\nu\rho}. \tag{A.17}$$

By using the Gauss-Codazzi relations to project it onto the three dimensional manifold we yield for $\gamma^{\sigma\rho}R_{\mu\sigma\nu\rho}$

$$\gamma^{\sigma\rho}\gamma_i^{\mu}\gamma_j^{\nu}R_{\mu\sigma\nu\rho} = {}^{(3)}R_{ij} + KK_{ij} - K_i^{\ k}K_{kj}. \tag{A.18}$$

As a next step we want to project $n^{\sigma}n^{\rho}R_{\mu\sigma\nu\rho}$ onto the hypersurface, which consists of a more sophisticated calculation; With the definition of the Riemann tensor

$$\nabla_{\mu}\nabla\nu n_{\sigma} - \nabla_{\nu}\nabla\mu n_{\sigma} = R_{\mu\nu\sigma\rho}n^{\rho}, \tag{A.19}$$

it follows that

$$R_{\mu\sigma\nu\rho}n^{\sigma}n^{\rho} = n^{\sigma}(\nabla_{\mu}\nabla_{\sigma}n_{\nu} - \nabla_{\sigma}\nabla_{\mu}n_{\nu}). \tag{A.20}$$

Using Eq. (A.5 - A.7), we derive the final expression for the projection on the hypersurface which is given by

$$\gamma_i^{\ \mu}\gamma_j^{\ \nu}R_{\mu\sigma\nu\rho}n^{\sigma}n^{\rho} = \pounds_n K_{ij} + K_{ik}K^k_{\ j} + \alpha^{-1}\bar{\nabla}_i\bar{\nabla}_j\alpha. \tag{A.21}$$

Combining Eqs. (A.17, A.18, A.21) the Ricci tensor R_{ij} reads

$$R_{ij} = -\pounds_n K_{ij} - 2K_{ik}K^k{}_j - \alpha^{-1}\bar{\nabla}_i\bar{\nabla}_j\alpha + {}^{(3)}R_{ij} + KK_{ij}, \tag{A.22}$$

and we can finally write down the evolution equation for K_{ij}, defined as $\pounds_t K_{ij}$:

$$\pounds_t K_{ij} = \alpha \pounds_n K_{ij} + \pounds_\beta K_{ij}, \tag{A.23}$$

where $\pounds_\beta K_{ij} = \beta_k \bar{\nabla}^k K_{ij} + K_{ik}\bar{\nabla}^k \beta_j + K_{kj}\bar{\nabla}^k\beta_i$. and $\pounds_n K_{ij}$ is derived by combining Eq. (A.16) and Eq. (A.22), and its explicit expression is

$$\pounds_n K_{ij} = {}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j - 8\pi\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) - \alpha^{-1}\bar{\nabla}_i\bar{\nabla}_j\alpha. \tag{A.24}$$

The final result is obtained by combining Eq. (A.24) with (A.23):

$$\begin{aligned} \partial_t K_{ij} &= \alpha\left[{}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j - 8\pi\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right)\right] \\ &\quad -\bar{\nabla}_i\bar{\nabla}_j\alpha + \beta_k\bar{\nabla}^k K_{ij} + K_{ik}\bar{\nabla}^k\beta_j + K_{kj}\bar{\nabla}^k\beta_i. \end{aligned} \tag{A.25}$$

A.3. The Momentum Constraint

There are more governing equations then just the evolution equation. By computing

$$\gamma_\mu{}^\rho G_{\rho\nu}n^\nu = 8\pi\gamma_\mu{}^\rho T_{\rho\nu}n^\nu, \tag{A.26}$$

and defining a current field $j_\mu = -\gamma_\mu{}^\rho T_{\rho\nu}n^\nu$ the right hand side of Eq. (A.26) becomes $-8\pi j_\mu$. Rewriting the right hand side in terms of derivatives of the extrinsic curvature we get:

$$\gamma_\mu^\rho G_{\rho\nu}n^\nu = \gamma_\mu^\rho\left(R_{\rho\nu} - \frac{1}{2}Rg_{\rho\nu}\right)n^\nu = \bar{\nabla}_\sigma K^\sigma_\mu - \bar{\nabla}_\mu K^\sigma_\sigma, \tag{A.27}$$

where we used the relation $\gamma_\mu^\rho g_{\rho\nu}n^\nu = \gamma_\mu^\rho n_\rho = 0$.
Now, the final expression for the momentum constraint reads

$$\bar{\nabla}_\sigma K^\sigma_\mu - \bar{\nabla}_\mu K^\sigma_\sigma = -8\pi j_\mu. \tag{A.28}$$

A.4. The Hamiltonian Constraint

To complete the set of governing equations for the evolution system we project the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ on the hypersurface Σ_t we yield

$$G_{\mu\nu} n^\mu n^\nu = 8\pi\rho, \tag{A.29}$$

where the matter energy density ρ is defined as $\rho = T_{\mu\nu} n^\mu n^\nu$. To calculate the left hand side of Eq. (A.29) we need to reformulate the Einstein tensor $G_{\mu\nu}$: Contracting the Gauss-Codazzi equations in Eqs. (A.8) on σ and ν, the left hand side Eq. (A.29) reads

$$^{(3)}R_{\mu\rho} = \gamma^\beta_\mu \gamma^\varepsilon_\rho R_{\beta\varepsilon} - K K_{\mu\rho} + K_{\nu\rho} K^\nu_{\ \mu}. \tag{A.30}$$

Multiplying both sides with $\gamma^{\mu\rho}$, we derive the three-dimensional Ricci scalar:

$$^{(3)}R = \gamma^{\mu\nu} R_{\mu\nu} - K^2 + K_{\mu\nu} K^{\mu\nu} = R + 2R_{\mu\nu} n^\mu n^\nu - K^2 + K_{\mu\nu} K^{\mu\nu}. \tag{A.31}$$

Finally the left hand side of Eq. (A.29) is rewritten as

$$G_{\mu\nu} n^\mu n^\nu = R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R = \frac{1}{2} \left({}^{(3)}R + K^2 - K_{\mu\nu} K^{\mu\nu} \right). \tag{A.32}$$

Therefore we can write down the final expression for the Hamiltonian constraint

$$H = {}^{(3)}R + K^2 - K_{\mu\nu} K^{\mu\nu}. \tag{A.33}$$

B. Weyl Scalars in $3+1$ Form

For the purpose of calculating the Weyl scalars in our simulations we need a suitable way to derive them from the ADM quantities. We review in this section work done by Smarr [108], extended by results in the thesis by Bernard Kelly[1] .
We start by defining the three-dimensional Levi-Civita tensor according to

$$\varepsilon_{ijk} \equiv {}^4\varepsilon_{ijkl}\hat{n}^l = |\gamma|^{\frac{1}{2}}[123]_{ijk} \tag{B.1a}$$

$$\Rightarrow \varepsilon^{ijk} = |\gamma|^{-\frac{1}{2}}[123]^{ijk} \qquad , \tag{B.1b}$$

where $|\gamma| = [123]^{ijk}\gamma_{1i}\gamma_{2j}\gamma_{3k}$, $\hat{n}^i$ is a time-like unit normal vector and $[123]$ the pure alternating symbol, antisymmetric under odd permutations. We lower and raise the three-dimensional Levi-Civita tensor with the three-metric γ_{ij} and its inverse. Knowing these general definitions we can decompose the Weyl tensor C_{ijkl} into its *electric*, E, and *magnetic*, B, part:

$$C_{ijkl} = 4\hat{n}_{[i}E_{j]\,[k}\hat{n}_{l]} + 2\varepsilon_{ij}{}^{m}B_{m[k}\hat{n}_{l]} + 2\varepsilon_{kl}{}^{m}B_{m[i}\hat{n}_{j]} + \varepsilon_{ij}{}^{m}\varepsilon_{kl}{}^{n}E_{mn}. \tag{B.2}$$

To connect these quantities to the more familiar definitions in the ADM formalism, we can express E_{ij} and B_{ij} as

$$E_{ij} \equiv -C_{ijkl}\hat{n}^k\hat{n}^l = -R_{ij} + K_i{}^kK_{jk} - KK_{ij}, \tag{B.3a}$$

$$B_{ij} \equiv -{}^*C_{ijkl}\hat{n}^k\hat{n}^l = -\varepsilon_i{}^{kl}D_kK_{lj}, \tag{B.3b}$$

[1]The Maya project at Penn State

where ${}^*C_{ijkl} = \frac{1}{2}{}^4\varepsilon_{ij}{}^{mn}C_{mnkl}$ is the Hodge dual of the Weyl tensor. Smarr introduced a new complex tensor,

$$W_{ij} \equiv E_{ij} + iB_{ij}. \tag{B.4}$$

To deduce the Weyl scalars we need to define a null tetrad by

$$\begin{aligned} l^i &= \frac{1}{\sqrt{2}}\left(\hat{n}^i + \hat{u}^i\right), & \text{(B.5a)} \\ n^i &= \frac{1}{\sqrt{2}}\left(\hat{n}^i - \hat{u}^i\right), & \text{(B.5b)} \\ m^i &= \frac{1}{\sqrt{2}}\left(\hat{v}^i - i\hat{w}^i\right), & \text{(B.5c)} \end{aligned}$$

and take an arbitrary orthonormal triad $(\hat{u}^i, \hat{v}^i, \hat{w}^i)$, orthogonal to $\hat{n}^i$, satisfying $\varepsilon_{ijk}\hat{u}^i\hat{v}^j\hat{w}^k = 1$ and therefore

$$\varepsilon_{ij}{}^k\hat{u}^i\hat{v}^j = \hat{w}^k. \tag{B.6}$$

We now derive from these definitions

$$\begin{aligned} \varepsilon_{ij}{}^k\hat{u}^i m^j &= \frac{1}{\sqrt{2}}\varepsilon_{ij}{}^k\hat{u}^i\left(\hat{v}^j - i\hat{w}^j\right) = \frac{1}{\sqrt{2}}\left(\hat{w}^k + i\hat{v}^k\right) \\ &= im^i, & \text{(B.7a)} \\ \varepsilon_{ij}{}^k\hat{u}^i\bar{m}^j &= \frac{1}{\sqrt{2}}\varepsilon_{ij}{}^k\hat{u}^i\left(\hat{v}^j + i\hat{w}^j\right) = \frac{1}{\sqrt{2}}\left(\hat{w}^k - i\hat{v}^k\right) \\ &= -im^i. & \text{(B.7b)} \end{aligned}$$

With this choice of null vectors we decompose the Weyl tensor according to:

$$\begin{aligned} C_{ijkl} &= 4\hat{n}_{[i}E_{j][k}\hat{n}_{l]} + 2\varepsilon_{ij}{}^m B_{m[k}\hat{n}_{l]} + 2\varepsilon_{kl}{}^m B_{m[i}\hat{n}_{j]} + \varepsilon_{ij}{}^m\varepsilon_{kl}{}^n E_{mn} \\ &= \hat{n}_i E_{jk}\hat{n}_l - \hat{n}_j E_{ik}\hat{n}_l - \hat{n}_i E_{jl}\hat{n}_k + \hat{n}_j E_{il}\hat{n}_k \\ &\quad + \varepsilon_{ij}{}^m B_{mk}\hat{n}_l - \varepsilon_{ij}{}^m B_{ml}\hat{n}_k + \varepsilon_{kl}{}^m B_{mi}\hat{n}_j - \varepsilon_{kl}{}^m B_{mj}\hat{n}_i \\ &\quad + \varepsilon_{ij}{}^m\varepsilon_{kl}{}^n E_{mn}. \end{aligned} \tag{B.8}$$

Finally, to be able to calculate the Weyl scalars we compute the contraction of the Weyl tensor. First, contract C_{ijkl} with $\hat{n}_i$ yields :

$$C_{ijkl}\hat{n}^i = -E_{jk}\hat{n}_l + E_{jl}\hat{n}_k + \varepsilon_{kl}{}^m B_{mj}, \tag{B.9a}$$

$$C_{ijkl}\hat{n}^i\hat{n}^l = E_{jk}, \tag{B.9b}$$

contraction with $\hat{u}_i$ gives:

$$\begin{aligned} C_{ijkl}\hat{u}^i &= -\hat{n}_j E_{ik}\hat{u}^i\hat{n}_l + \hat{n}_j E_{il}\hat{u}^i\hat{n}_k + \varepsilon_{ij}{}^m \hat{u}^i B_{mk}\hat{n}_l \\ &\quad -\varepsilon_{ij}{}^m \hat{u}^i B_{ml}\hat{n}_k + \varepsilon_{kl}{}^m B_{mi}\hat{u}^i\hat{n}_j + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n E_{mn}, \end{aligned} \tag{B.10a}$$

$$C_{ijkl}\hat{u}^i\hat{u}^l = \hat{n}_j E_{ik}\hat{u}^i\hat{u}^l\hat{n}_k - \varepsilon_{ij}{}^m \hat{u}^i B_{ml}\hat{u}^l\hat{n}_k + \varepsilon_{kl}{}^m \hat{u}^l B_{mi}\hat{u}^i\hat{n}_j + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn}. \tag{B.10b}$$

And evaluating the contraction of C_{ijkl} with $\hat{n}_i$ and $\hat{u}_i$ is:

$$C_{ijkl}\hat{n}^i\hat{u}^l = E_{jl}\hat{u}^l\hat{n}_k + \varepsilon_{kl}{}^m \hat{u}^l B_{mi}, \tag{B.11a}$$

$$C_{ijkl}\hat{u}^i\hat{n}^l = \hat{n}_j E_{ik}\hat{u}^i - \varepsilon_{ij}{}^m \hat{u}^i B_{mk}. \tag{B.11b}$$

Now it is a straightforward procedure to express the Weyl scalars in terms of the electric and magnetic tensors. Following Kelly we first calculate Ψ_2:

$$
\begin{aligned}
\Psi_2 &\equiv C_{ijkl} l^i m^j \bar{m}^k n^l \\
&= \frac{1}{2} C_{ijkl} \left(\hat{n}^i \hat{n}^l - \hat{n}^i \hat{u}^l + \hat{u}^i \hat{n}^l - \hat{u}^i \hat{u}^l \right) m^j \bar{m}^k \\
&= \frac{1}{2} \Big[E_{jk} - E_{jl} \hat{u}^l \hat{n}_k - \varepsilon_{kl}{}^m \hat{u}^l B_{mj} + \hat{n}_j E_{ik} \hat{u}^i - \varepsilon_{ij}{}^m \hat{u}^i B_{mk} \\
&\quad - \hat{n}_j E_{il} \hat{u}^i \hat{u}^l \hat{n}_k + \varepsilon_{ij}{}^m \hat{u}^i B_{ml} \hat{u}^l \hat{n}_k - \varepsilon_{kl}{}^m \hat{u}^l B_{mi} \hat{u}^i \hat{n}_j - \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] m^j \bar{m}^k \\
&= \frac{1}{4} \Big[E_{jk} - \varepsilon_{kl}{}^m \hat{u}^l B_{mj} - \varepsilon_{ij}{}^m \hat{u}^i B_{mk} - \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] \Big[(\hat{v}^j \hat{v}^k + \hat{w}^j \hat{w}^k) + i(\hat{v}^j \hat{w}^j - \hat{w}^j \hat{v}^k) \Big] \\
&= \frac{1}{4} \Big(E_{jk} \Big[(\hat{v}^j \hat{v}^k + \hat{w}^j \hat{w}^k) + i(\hat{v}^j \hat{w}^j - \hat{w}^j \hat{v}^k) \Big] - B_{mj} \left[(-\hat{w}^m + i\hat{v}^m) \hat{v}^j + (\hat{v}^m + i\hat{w}^m) \hat{w}^j \right] \\
&\quad - B_{mk} \Big[(\hat{w}^m + i\hat{v}^m) \hat{v}^k + (-\hat{v}^m + i\hat{w}^m) \hat{w}^k \Big] - E_{mn} \left[(-\hat{w}^n + i\hat{v}^n) \hat{w}^m - (\hat{v}^n + i\hat{w}^n) \hat{v}^m \right] \Big) \\
&= \frac{1}{2} (E_{jk} - iB_{jk}) \Big[\hat{v}^j \hat{v}^k + \hat{w}^j \hat{v}^k \Big] \\
&= \frac{1}{2} (E_{jk} - iB_{jk}) \Big[g^{jk} + \hat{n}^j \hat{n}^k - \hat{u}^j \hat{u}^k \Big] \\
&= -\frac{1}{2} (E_{jk} - iB_{jk}) \hat{u}^j \hat{u}^k .
\end{aligned}
\tag{B.12}
$$

We can calculate the other scalars in a similar manner. The *transverse ingoing* scalar Ψ_0 reads:

$$
\begin{aligned}
\Psi_0 &\equiv C_{ijkl} l^i m^j l^k m^l = -C_{ijkl} l^i m^j m^k l^l \\
&= -\frac{1}{2} C_{ijkl} \left(\hat{n}^i \hat{n}^l + \hat{n}^i \hat{u}^l + \hat{u}^i \hat{n}^l + \hat{u}^i \hat{u}^l \right) m^j m^k \\
&= -\frac{1}{2} \Big[E_{jk} + E_{jl} \hat{u}^l \hat{n}_k + \varepsilon_{kl}{}^m \hat{u}^l B_{mj} + \hat{n}_j E_{ik} \hat{u}^i - \varepsilon_{ij}{}^m \hat{u}^i B_{mk} \\
&\quad - \hat{n}_j E_{il} \hat{u}^i \hat{u}^l \hat{n}_k + \varepsilon_{ij}{}^m \hat{u}^i B_{ml} \hat{u}^l \hat{n}_k + \varepsilon_{kl}{}^m \hat{u}^l B_{mi} \hat{u}^i \hat{n}_j + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] m^j m^k \\
&= -\frac{1}{2} \Big[E_{jk} + \varepsilon_{kl}{}^m \hat{u}^l \hat{u}^l B_{mj} - \varepsilon_{ij}{}^m \hat{u}^i B_{mk} + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] m^j m^k \\
&= -\frac{1}{2} \Big[E_{jk} + 2\varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] m^j m^k - \frac{1}{4} \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \left(\hat{v}^j \hat{v}^k - i\hat{v}^j \hat{w}^k - i\hat{w}^j \hat{v}^k - \hat{w}^j \hat{w}^k \right) \\
&= -\frac{1}{2} \Big[E_{jk} + 2\varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] m^j m^k - \frac{1}{4} E_{mn} \left(-\hat{w}^m \hat{w}^n - i\hat{w}^m \hat{v}^n - i\hat{v}^m \hat{w}^n - \hat{v}^m \hat{v}^n \right) \\
&= -\frac{1}{2} \Big[E_{jk} + 2\varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] m^j m^k - \frac{1}{2} E_{mn} m^m m^n \\
&= -\Big[E_{jk} + \varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] m^j m^k .
\end{aligned}
\tag{B.13}
$$

The *ingoing longitudinal* scalar, Ψ_1, becomes:

$$
\begin{aligned}
\Psi_1 &\equiv C_{ijkl}l^i n^j l^k m^l = C_{ijkl}l^i m^j l^k n^l \\
&= \frac{1}{2}C_{ijkl}\left(\hat{n}^i\hat{n}^l - \hat{n}^i\hat{u}^l + \hat{u}^i\hat{n}^l - \hat{u}^i\hat{u}^l\right)m^j l^k \\
&= -\frac{1}{2}\Big[E_{jk} - E_{jl}\hat{u}^l\hat{n}_k - \varepsilon_{kl}{}^m\hat{u}^l B_{mj} + \hat{n}_j E_{ik}\hat{u}^i - \varepsilon_{ij}{}^m\hat{u}^i B_{mk} \\
&\quad -\hat{n}_j E_{il}\hat{u}^i\hat{u}^l\hat{n}_k + \varepsilon_{ij}{}^m\hat{u}^i B_{ml}\hat{u}^l\hat{n}_k - \varepsilon_{kl}{}^m\hat{u}^l B_{mi}\hat{u}^i\hat{n}_j - \varepsilon_{ij}{}^m\hat{u}^i\varepsilon_{kl}{}^n\hat{u}^l E_{mn}\Big] m^j l^k \\
&= -\frac{1}{2\sqrt{2}}\Big[E_{jk} - E_{jl}\hat{u}^l\hat{n}_k - \varepsilon_{kl}{}^m\hat{u}^l B_{mj} - \varepsilon_{ij}{}^m\hat{u}^i B_{mk} \\
&\quad +\varepsilon_{ij}{}^m\hat{u}^i B_{ml}\hat{u}^l\hat{n}_k - \varepsilon_{ij}{}^m\hat{u}^i\varepsilon_{kl}{}^n\hat{u}^l E_{mn}\Big] m^j(\hat{n}^k + \hat{u}^k) \\
&= -\frac{1}{2\sqrt{2}}\Big[E_{jk}\hat{u}^k + E_{jl}\hat{u}^l - \varepsilon_{ij}{}^m\hat{u}^i B_{mk}\hat{u}^k - \varepsilon_{ij}{}^m\hat{u}^i B_{ml}\hat{u}^l\Big] m^j \\
&= -\frac{1}{\sqrt{2}}\Big[E_{jk}\hat{u}^k - \varepsilon_{ij}{}^m\hat{u}^i B_{mk}\hat{u}^k\Big] m^j. \qquad \text{(B.14)}
\end{aligned}
$$

The *outgoing longitudinal* scalar, Ψ_3, turns out to be:

$$
\begin{aligned}
\Psi_3 &\equiv C_{ijkl}l^i n^j \bar{m}^k n^l \\
&= \frac{1}{2}C_{ijkl}\left(\hat{n}^i\hat{n}^l - \hat{n}^i\hat{u}^l + \hat{u}^i\hat{n}^l - \hat{u}^i\hat{u}^l\right)n^j\bar{m}^k \\
&= \frac{1}{2}\Big[E_{jk} - E_{jl}\hat{u}^l\hat{n}_k - \varepsilon_{kl}{}^m\hat{u}^l B_{mj} + \hat{n}_j E_{ik}\hat{u}^i - \varepsilon_{ij}{}^m\hat{u}^i B_{mk} \\
&\quad -\hat{n}_j E_{il}\hat{u}^i\hat{u}^l\hat{n}_k + \varepsilon_{ij}{}^m\hat{u}^i B_{ml}\hat{u}^l\hat{n}_k - \varepsilon_{kl}{}^m\hat{u}^l B_{mi}\hat{u}^i\hat{n}_j - \varepsilon_{ij}{}^m\hat{u}^i\varepsilon_{kl}{}^n\hat{u}^l E_{mn}\Big] n^j\bar{n}^k \\
&= \frac{1}{2\sqrt{2}}\Big[E_{jk} - \varepsilon_{kl}{}^m\hat{u}^l B_{mj} + \hat{n}_j E_{ik}\hat{u}^i - \varepsilon_{ij}{}^m\hat{u}^i B_{mk} \\
&\quad -\varepsilon_{kl}{}^m\hat{u}^l B_{mi}\hat{u}^i\hat{n}_j - \varepsilon_{ij}{}^m\hat{u}^i\varepsilon_{kl}{}^n\hat{u}^l E_{mn}\Big](\hat{n}^j - \hat{u}^j)\bar{m}^k \\
&= \frac{1}{\sqrt{2}}\Big[-E_{jk}\hat{u}^j - \varepsilon_{kl}{}^m\hat{u}^l B_{mj}\hat{u}^j\Big]\bar{m}^k. \qquad \text{(B.15)}
\end{aligned}
$$

And finally, the *outgoing transverse* scalar, Ψ_4, becomes:

$$
\begin{aligned}
\Psi_4 &\equiv C_{ijkl} n^i \bar{m}^j n^k \bar{m}^l = -C_{ijkl} n^i \bar{m}^j \bar{m}^k n^l \\
&= -\frac{1}{2} C_{ijkl} \left(\hat{n}^i \hat{n}^l - \hat{n}^i \hat{u}^l - \hat{u}^i \hat{n}^l + \hat{u}^i \hat{u}^l \right) \bar{m}^j \bar{m}^k \\
&= -\frac{1}{2} \Big[E_{jk} - E_{jl} \hat{u}^l \hat{n}_k - \varepsilon_{kl}{}^m \hat{u}^l B_{mj} - \hat{n}_j E_{ik} \hat{u}^i + \varepsilon_{ij}{}^m \hat{u}^i B_{mk} + \hat{n}_j E_{il} \hat{u}^i \hat{u}^l \hat{n}_k \\
&\quad + \varepsilon_{ij}{}^m \hat{u}^i B_{ml} \hat{u}^l \hat{n}_k + \varepsilon_{kl}{}^m \hat{u}^l B_{mi} \hat{u}^i \hat{n}_j + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] \bar{m}^j \bar{m}^k \\
&= -\frac{1}{2} \Big[E_{jk} - \varepsilon_{kl}{}^m \hat{u}^l B_{mj} + \varepsilon_{ij}{}^m \hat{u}^i B_{mk} + \varepsilon_{ij}{}^m \hat{u}^i \varepsilon_{kl}{}^n \hat{u}^l E_{mn} \Big] \bar{m}^j \bar{m}^k \\
&= -\frac{1}{2} \Big[E_{jk} - 2\varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] \bar{m}^j \bar{m}^k - \frac{1}{4} E_{mn} \left(-\hat{w}^m \hat{w}^n + i\hat{w}^m \hat{v}^n + i\hat{v}^m \hat{w}^n + \hat{v}^m \hat{v}^n \right) \\
&= -\frac{1}{2} \Big[E_{jk} - 2\varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] \bar{m}^j \bar{m}^k - \frac{1}{2} E_{mn} \bar{m}^m \bar{m}^n \\
&= -\Big[E_{jk} - \varepsilon_{kl}{}^m \hat{u}^l B_{mj} \Big] \bar{m}^j \bar{m}^k. \qquad \text{(B.16)}
\end{aligned}
$$

If we project the electric and magnetic tensors along and perpendicular to the preferred spatial direction, $\hat{u}^i$, we can somewhat tie up the results. Defining

$$
\begin{aligned}
e &\equiv E_{ij} \hat{u}^i \hat{u}^j, && \text{(B.17a)} \\
b &\equiv B_{ij} \hat{u}^i \hat{u}^j, && \text{(B.17b)} \\
e_i &\equiv E_{jk} \hat{u}^j (\delta_i^k - \hat{u}_i \hat{u}^k) = E_{ij} \hat{u}^j - e \hat{u}_i, && \text{(B.17c)} \\
b_i &\equiv B_{jk} \hat{u}^j (\delta_i^k - \hat{u}_i \hat{u}^k) = B_{ij} \hat{u}^j - b \hat{u}_i, && \text{(B.17d)} \\
e_{ij} &\equiv E_{kl} (\delta_i^k - \hat{u}_i \hat{u}^k)(\delta_j^l - \hat{u}_j \hat{u}^l) + \frac{1}{2} e (\gamma_{ij} - \hat{u}_i \hat{u}_j), && \text{(B.17e)} \\
b_{ij} &\equiv B_{kl} (\delta_i^k - \hat{u}_i \hat{u}^k)(\delta_j^l - \hat{u}_j \hat{u}^l) + \frac{1}{2} b (\gamma_{ij} - \hat{u}_i \hat{u}_j), && \text{(B.17f)}
\end{aligned}
$$

and reconstruct the magnetic und electric part in these quantities

$$
\begin{aligned}
E_{ij} &= \frac{1}{2} (3\hat{u}_i \hat{u}_j - \gamma_{ij}) e + 2 e_{(i} \hat{u}_{j)} + e_{ij} && \text{(B.18a)} \\
B_{ij} &= \frac{1}{2} (3\hat{u}_i \hat{u}_j - \gamma_{ij}) b + 2 b_{(i} \hat{u}_{j)} + b_{ij}. && \text{(B.18b)}
\end{aligned}
$$

If we define the *rotation operator* $J_i^j \equiv \varepsilon_i{}^{jkl}\hat{u}_k\hat{n}_l = \varepsilon_i{}^{jk}\hat{u}_k$ we can write the five Weyl scalars as

$$\Psi_0 = -\left(e_{ij} - J_i^k b_{jk}\right) m^i m^j, \tag{B.19a}$$

$$\Psi_1 = \frac{1}{\sqrt{2}}\left(e_i + J_i^k b_k\right) m^i, \tag{B.19b}$$

$$\Psi_2 = -\frac{1}{2}(e - ib), \tag{B.19c}$$

$$\Psi_3 = -\frac{1}{\sqrt{2}}\left(e_i + J_i^k b_k\right) \bar{m}^i, \tag{B.19d}$$

$$\Psi_4 = -\left(e_{ij} + J_i^k b_{jk}\right) \bar{m}^i \bar{m}^j. \tag{B.19e}$$

We still need to specify a suitable tetrad to determine the Weyl scalars in a suitable way. This procedure is still under investigation and will be published elsewhere.

C. Spin-Weighted Spherical Harmonic Decomposition

We have introduced the Weyl scalars in Chapter 4 and identified them as coordinate independent quantities. Here we will demonstrate that they are really a tensor-like quantity. The rotation of the space-like null vectors $\Re(m^\mu)$ and $\Im(m^\mu)$ in their plane is given by

$$(m^\mu)' = e^{i\vartheta} m^\mu, \tag{C.1}$$

as defined in the original article by Newman and Penrose [13]. We may ask if such a rotation effects also the Weyl scalars. In particular, we consider the effects of a rotation on Ψ_4 and Ψ_0. Performing such a rotation for Ψ_0 yields

$$\Psi_0' = e^{2i\vartheta} \Psi_0, \tag{C.2}$$

and for the outgoing transverse scalar Ψ_4

$$\Psi_4' = e^{-2i\vartheta} \Psi_4. \tag{C.3}$$

Obviously, Ψ_0 and Ψ_4 transform differently what is an effect of the tetrad dependence as explained in Chapter 4. We can generalize that behavior for an arbitrary quantity η by introducing the *spin weight* s. A quantity η will have a spin weight if it behaves as

$$\eta' = e^{si\vartheta} \eta, \tag{C.4}$$

under the transformation in Eq. (C.1). With this definition it is obvious that Ψ_4 has spin weight -2 and Ψ_0 has spin weight +2, respectively. The spin-weighted spherical harmonics are constructed to transform as scalars of a given spin weight. Effectively, the concept of spin weight refers to the behavior of functions on the (θ, ϕ)-sphere at infinity only.

Indeed the concept can be applied to any two-dimensional abstract surface, with a Riemannian structure. Objects with spin weight correspond to irreducible tensor quantities on the two-dimensional surface. As a native choice, the vectors $\Re(m^\mu)$ and $\Im(m^\mu)$ may be regarded as tangential to the coordinate lines of the 2-sphere coordinates θ and ϕ, respectively. Thus it is not completely surprising that the appropriate description of the Weyl scalars are spin-weighted spherical harmonics rather than the usual scalar spherical harmonics. In fact, the decomposition of spin-weighted functions on a 2-sphere into spherical harmonics is mathematically a well-defined operation, but using spin-weighted spherical harmonics results in a more correct decomposition.

Since we do not only want to extract Ψ_4 from a numerical simulation, but also decompose the signal into the contributing modes, we define the spin-weighted spherical harmonics and the operator $\eth$ now, and list some key properties in the coming sections. We are interested in the (spin-weighted) spherical harmonic components of the gravitational waves, since there are a few noticeable advantages of this technique, we want to outline briefly:

- The main advantage of the decomposition process is the filtering of numerical noise, which tends to have higher angular frequency than genuine wave signals due to finite grid sizes.
- *A priori* knowledge about symmetries in the data or dominant modes associated with physical processes allow important checks on the plausibility of numerical solutions, in particular when exact solutions are not available.
- Some characteristics of gravitational radiation, for instance quasi-normal modes, are best understood in terms of spherical harmonic components (cf. section 3.5).

C.1. Definition and Properties of Spin-weighted Spherical Harmonics

Goldberg [109] defined the spin-weighted spherical harmonics in terms of rotation matrices D^l of the ordinary rotation group R^3, and related ð to an ordinary angular-momentum raising operator. His choice was motivated by the fact that the principal properties of the rotation group are familiar from the theory of angular momentum. He identified the spin-weighted spin harmonics in the following way:

$$\begin{aligned} {}_sY_l^m(\theta,\phi) &\equiv \sqrt{\frac{(2l+1)}{4\pi}}\, D^l_{-s,m}(\phi,\theta,0) \\ &\equiv \sqrt{\frac{(2l+1)(l+m)!(l-m)!}{4\pi(l+s)!(l-s)!}}\sin(\theta/2)^{2l}\sum_r \begin{pmatrix} l-s \\ r \end{pmatrix}\begin{pmatrix} l+s \\ r+s-m \end{pmatrix} \\ &\quad \times (-1)^{l-r-s}\, e^{im\phi}\, [\cot(\theta/2)]^{2r+s-m} \end{aligned} \tag{C.5}$$

,
where $D^l_{-s,m}(\phi,\theta,\vartheta)=\sqrt{\frac{4\pi}{2l+1}}\,{}_sY_l^m\, e^{-im\vartheta}$ is the rotation matrix representing rotations by the Euler angles (ϕ,θ,ϑ). We adopted a nowadays more commonly used definition for the relation between the Wigner D-matrices, and the spin-weighted spherical harmonics, where $D^l_{s,-m}(\phi,\theta,-\vartheta)=(-1)^m\sqrt{\frac{4\pi}{2l+1}}\,{}_sY_l^m\, e^{im\vartheta}$.

Another possible determination of the spin-weighted spherical harmonics can be carried out in the Newman-Penrose formalism [13]. We can determine a relation between the spherical harmonics and the spin-weighted spherical harmonics without explicit determination (cf. [13]) according to

$${}_sY_l^m(\theta,\phi)\equiv\sqrt{\frac{(l-|s|)!}{(l+|s|)!}}\,ð^s\,Y_l^m(\theta,\phi)\,, \tag{C.6}$$

where the operator ð is defined as

$$ð = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\left\{(\sin\theta)^{-s}\eta\right\}. \tag{C.7}$$

The operator $\eth$ is effectively a covariant differentiation operator on the surface. From this definition in Eqs. (C.1, C.4) it follows for the transformation of the quantity $\eth\eta$

$$(\eth\eta)' = e^{i(s+1)\vartheta}\,(\eth\eta)\,. \tag{C.8}$$

What can be seen from Eq (C.8) is that the operator $\eth$ has the important property of raising the spin weight by 1. We can define in a similar manner the quantity $\bar{\eth}$ by

$$\bar{\eth} = -\,(\sin\theta)^{-s}\left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\{(\sin\theta)^{s}\,\eta\}\,, \tag{C.9}$$

where the quantity $\bar{\eth}$ lowers the spin weight by 1 now. We can ask now what happens if we let the latter operators $\eth$ and $\bar{\eth}$ on the spin-weighted spherical harmonics on a sphere (see [13] for a more general treatment of $\eth$). Thus we apply the formalism on ${}_sY_l^m$, yielding

$${}_sY_l^m = \begin{cases} \left[\frac{(l-s)!}{(l+s)!}\right]^{\frac{1}{2}}\eth^s\,Y_l^m, & \text{if } 0\le s\le l \\ (-1)^s\left[\frac{(l+s)!}{(l-s)!}\right]^{\frac{1}{2}}\bar{\eth}^{-s}Y_l^m, & \text{if } -l\le s\le 0 \end{cases}, \tag{C.10}$$

where ${}_sY_l^m$ is only defined for $l \geq \|s\|$. Since we are mostly interested in the $(s=\pm 2)$ spin harmonics throughout this work we give the explicit derivation from the spherical harmonics: The $s=+2$ spin-weighted harmonics are calculated via the formula

$${}_2Y_l^m = \sqrt{\frac{(l-2)!}{(l+2)!}}\left[\partial_\theta^2 - \cot\theta\,\partial_\theta \pm \frac{2i}{\sin\theta}\,(\partial_\theta - \cot\theta)\,\partial_\phi - \frac{1}{\sin^2\theta}\,\partial_\phi^2\right]Y_l^m, \tag{C.11}$$

and the $s=-2$ spin harmonics are easily derived by making use of the parity relation:

$${}_sY_l^m(\pi-\theta,\phi+\pi) = (-1)^l\;{}_{-s}Y_l^m. \tag{C.12}$$

C.2. Decomposition on the Sphere

A general definition of a decomposition of a function into its spherical harmonic components is

$$f(r,\theta,\phi,t) = \sum_{m,l} a_l^m(r,t)\, Y_l^m(\theta,\phi), \tag{C.13}$$

where $ff(r,\theta,\phi,t)$ denotes an arbitrary function and $a_l^m(r,t)$ are the components of the spherical harmonic decomposition. Thus we calculate the coefficients for a given function $f(r,\theta,\phi,t)$ by recasting the relation:

$$a_l^m(r,t) = \int {}^*Y_l{}^m(\theta,\phi)\, f(r,\theta,\phi,t)\, d\Omega. \tag{C.14}$$

Equivalently, for a spin-weighted function of certain spin weight s we can write

$$_s a_l^m(r,t) = \int {}_s^*Y_l{}^m(\theta,\phi)\, f(r,\theta,\phi,t,s)\, d\Omega. \tag{C.15}$$

In a numerical simulation we replace the integrals with a sum since we are dealing with discretized values on a finite grid:

$$[{}_s a_l^m(r,t)]_{ij} = \sum_{\theta_0=0}^{\pi} \sum_{\phi_0=0}^{2\pi} {}_s^*Y_l{}^m(\theta_i,\phi_j)\, f(r,\theta_i,\phi_j,t,s)\, \sin\theta_i \,\triangle\,\theta_i \,\triangle\,\phi_j. \tag{C.16}$$

Consequently our equation for the Weyl scalars, in particular Ψ_4 becomes:

$$({}_{-2}[a_4]_l^m(r,t))_{ij} = \sum_{\theta_0=0}^{\pi} \sum_{\phi_0=0}^{2\pi} {}_{-2}{}^*Y_l{}^m(\theta_i,\phi_j)\, ({}_{-2}[\Psi_4]_l^m(r,\theta_i,\phi_j,t))_{ij}\, \sin\theta_i \,\triangle\,\theta_i \,\triangle\,\phi_j. \tag{C.17}$$

C.3. Group Properties

The spin-weighted spherical harmonics are elements of the irreducible matrix representation of SU(2). Because SU(2) is the covering group of SO(3), we can conclude that the spin-weighted spherical harmonics are a generalization of the scalar spherical harmonics $Y_l^m(\theta,\phi)$. From group theory the properties of the spin-weighted spherical harmonics follow immediately as special cases of the *compatibility relation* with spherical harmonics:

$$ {}_0Y_l^m = Y_l^m; \tag{C.18} $$

the *conjunction relation*:

$$ ({}_sY_l^m)^* = (-1)^{m+s}\; {}_{-s}Y_l^{-m}; \tag{C.19} $$

the *orthonormality relation*:

$$ \int d\Omega\; ({}_sY_{l'}^{m'})^*\; ({}_sY_l^m) = \delta_{ll'}\; \delta^{mm'}; \tag{C.20} $$

the *completeness relation*:

$$ \sum_{l,m} [{}_sY_l^m(\theta',\phi')]^*\; [{}_sY_l^m(\theta,\phi)] = \delta(\phi-\phi')\; \delta(\cos\theta - \cos\theta'); \tag{C.21} $$

and the *parity relation*:

$$ {}_sY_l^m(\pi-\theta,\phi+\pi) = (-1)^l\; {}_{-s}Y_l^m. \tag{C.22} $$

C.4. Explicit Derivation

The calculations have been carried out for Spin 2, the Spin -2 case can be derived by using the *parity relation* in Eq. (C.22). The spin-weighted spherical harmonics are calculated according to

$$ {}_2Y_l^m = \sqrt{\frac{(l-2)!}{(l+2)!}}\; \left[\partial_\theta^2 - \cot\theta\,\partial_\theta \pm \frac{2i}{\sin\theta}\,(\partial_\theta - \cot\theta)\,\partial_\phi - \frac{1}{\sin^2\theta}\,\partial_\phi^2\right] Y_l^m. \tag{C.23} $$

l	m	${}_2Y_l^m$
2	-2	$\frac{1}{2}e^{-2i\phi}\sqrt{\frac{5}{\pi}}\cos^4\left(\frac{\theta}{2}\right)$
2	-1	$-\frac{1}{4}e^{-i\phi}\sqrt{\frac{5}{\pi}}\left[1+\cos\theta\right]\sin\theta$
2	0	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta$
2	1	$\frac{1}{4}e^{i\phi}\sqrt{\frac{5}{\pi}}\left[-1+\cos\theta\right]\sin\theta)$
2	2	$\frac{1}{2}e^{2i\phi}\sqrt{\frac{5}{\pi}}\sin^4\left(\frac{\theta}{2}\right)$

Table C.1.: Spin-weighted spherical harmonics calculated for $l = 2$.

l	m	${}_2Y_l^m$
3	-3	$e^{-3i\phi}\sqrt{\frac{21}{2\pi}}\cos^5\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$
3	-2	$\frac{1}{2}e^{-2i\phi}\sqrt{\frac{7}{\pi}}\cos^4\left(\frac{\theta}{2}\right)\left[-2+3\cos\theta\right]$
3	-1	$\frac{1}{32}e^{-i\phi}\sqrt{\frac{35}{2\pi}}\left[\sin\theta-4\sin\left(2\theta\right)-3\sin\left(3\theta\right)\right]$
3	0	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\cos\left(\theta\right)\sin^2\left(\theta\right)$
3	1	$-\frac{1}{32}e^{i\phi}\sqrt{\frac{35}{2\pi}}\left[\sin\theta+4\sin\left(2\theta\right)-3\sin\left(3\theta\right)\right]$
3	2	$\frac{1}{2}e^{2i\phi}\sqrt{\frac{7}{\pi}}\sin^4\left(\frac{\theta}{2}\right)\left[2+3\cos\theta\right]$
3	3	$-e^{-3i\phi}\sqrt{\frac{21}{2\pi}}\cos\left(\frac{\theta}{2}\right)\sin^5\left(\frac{\theta}{2}\right)$

Table C.2.: Spin-weighted spherical harmonics calculated for $l = 3$.

l	m	${}_2Y_l^m$
4	-4	$3e^{-4i\phi}\sqrt{\frac{7}{\pi}}\cos^6\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right)$
4	-3	$3e^{-3i\phi}\sqrt{\frac{7}{2\pi}}\cos^5\left(\frac{\theta}{2}\right)\left[-1+2\cos\theta\right]\sin\left(\frac{\theta}{2}\right)$
4	-2	$\frac{3}{4\sqrt{\pi}}e^{-2i\phi}\cos^4\left(\frac{\theta}{2}\right)\left[9-14\cos\theta+7\cos(2\theta)\right]$
4	-1	$-\frac{3}{32\sqrt{2\pi}}e^{-i\phi}\left[3\sin\theta-2\sin(2\theta)+7\left(\sin(3\theta)+\sin(4\theta)\right)\right]$
4	0	$\frac{3}{16}\sqrt{\frac{5}{2\pi}}\left[5+7\cos(2\theta)\right]\sin^2\theta$
4	1	$-\frac{3}{32\sqrt{2\pi}}e^{i\phi}\left[3\sin\theta+2\sin(2\theta)+7\left(\sin(3\theta)-\sin(4\theta)\right)\right]$
4	2	$\frac{3}{4\sqrt{\pi}}e^{2i\phi}\left[9+14\cos\theta+7\cos(2\theta)\right]\sin^4\left(\frac{\theta}{2}\right)$
4	3	$-3e^{3i\phi}\sqrt{\frac{7}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[1+2\cos(\theta)\right]\sin^5\left(\frac{\theta}{2}\right)$
4	4	$3e^{4i\phi}\sqrt{\frac{7}{\pi}}\cos^2\left(\frac{\theta}{2}\right)\sin^6\left(\frac{\theta}{2}\right)$

Table C.3.: Spin-weighted spherical harmonics calculated for $l = 4$.

l	m	${}_2Y_l^m$
5	-5	$e^{-5i\phi}\sqrt{\frac{330}{\pi}}\cos^7\left(\frac{\theta}{2}\right)\sin^3\left(\frac{\theta}{2}\right)$
5	-4	$e^{-4i\phi}\sqrt{\frac{33}{\pi}}\cos^6\left(\frac{\theta}{2}\right)\left[-2+5\cos\theta\right]\sin^2\left(\frac{\theta}{2}\right)$
5	-3	$\frac{1}{4}e^{-3i\phi}\sqrt{\frac{33}{2\pi}}\cos^5\left(\frac{\theta}{2}\right)\left[17-24\cos\theta+15\cos(2\theta)\right]\sin\left(\frac{\theta}{2}\right)$
5	-2	$\frac{1}{8}e^{-2i\phi}\sqrt{\frac{11}{\pi}}\cos^4\left(\frac{\theta}{2}\right)\left[-32+57\cos\theta-36\cos(2\theta)+15\cos(3\theta)\right]$
5	-1	$\frac{1}{256}e^{-i\phi}\sqrt{\frac{77}{\pi}}\left[2\sin\theta-8\sin(2\theta)+3\left(\sin(3\theta)-4\sin(4\theta)-5\sin(5\theta)\right)\right]$
5	0	$\frac{1}{32}\sqrt{\frac{1155}{2\pi}}\left[5\cos\theta+3\cos(3\theta)\right]\sin^2\theta$
5	1	$-\frac{1}{256}e^{i\phi}\sqrt{\frac{77}{\pi}}\left[2\sin\theta+8\sin(2\theta)+3\left(\sin(3\theta)+4\sin(4\theta)-5\sin(5\theta)\right)\right]$
5	2	$\frac{1}{8}e^{2i\phi}\sqrt{\frac{11}{\pi}}\sin^4\left(\frac{\theta}{2}\right)\left[32+57\cos\theta+36\cos(2\theta)+15\cos(3\theta)\right]$
5	3	$-\frac{1}{4}e^{3i\phi}\sqrt{\frac{33}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[17+24\cos\theta+15\cos(2\theta)\right]\sin^5\left(\frac{\theta}{2}\right)$
5	4	$e^{4i\phi}\sqrt{\frac{33}{\pi}}\cos^2\left(\frac{\theta}{2}\right)\left[2+5\cos\theta\right]\sin^6\left(\frac{\theta}{2}\right)$
5	5	$-e^{5i\phi}\sqrt{\frac{330}{\pi}}\cos^3\left(\frac{\theta}{2}\right)\sin^7\left(\frac{\theta}{2}\right)$

Table C.4.: Spin-weighted spherical harmonics calculated for $l = 5$.

l	m	${}_2Y_l^m$
6	-6	$\frac{3}{2}e^{-6i\phi}\sqrt{\frac{715}{\pi}}\cos^8\left(\frac{\theta}{2}\right)\sin^4\left(\frac{\theta}{2}\right)$
6	-5	$\frac{1}{2}e^{-5i\phi}\sqrt{\frac{2145}{\pi}}\cos^7\left(\frac{\theta}{2}\right)[-1+3\cos\theta]\sin^3\left(\frac{\theta}{2}\right)$
6	-4	$\frac{1}{8}e^{-4i\phi}\sqrt{\frac{195}{2\pi}}\cos^6\left(\frac{\theta}{2}\right)[35-44\cos\theta+33\cos(2\theta)]\sin^2\left(\frac{\theta}{2}\right)$
6	-3	$\frac{3}{32}e^{-3i\phi}\sqrt{\frac{13}{\pi}}\cos^5\left(\frac{\theta}{2}\right)[-98+185\cos\theta-110\cos(2\theta)+55\cos(3\theta)]\sin\left(\frac{\theta}{2}\right)$
6	-2	$\frac{1}{256}e^{-2i\phi}\sqrt{\frac{13}{\pi}}\cos^4\left(\frac{\theta}{2}\right)[1709-3096\cos\theta+2340\cos(2\theta)-1320\cos(3\theta)+495\cos(4\theta)]$
6	-1	$-\frac{1}{1024}e^{-i\phi}\sqrt{\frac{65}{2\pi}}[20\sin\theta-17\sin(2\theta)+54\sin(3\theta)-12\sin(4\theta)+66\sin(5\theta)+99\sin(6\theta)]$
6	0	$\frac{1}{512}\sqrt{\frac{1365}{\pi}}[35+60\cos(2\theta)+33\cos(4\theta)]\sin^2\theta$
6	1	$-\frac{1}{1024}e^{i\phi}\sqrt{\frac{65}{2\pi}}[20\sin\theta+17\sin(2\theta)+54\sin(3\theta)+12\sin(4\theta)+66\sin(5\theta)-99\sin(6\theta)]$
6	2	$\frac{1}{256}e^{2i\phi}\sqrt{\frac{13}{\pi}}\left[1709+3096\cos\theta+2340\cos(2\theta)+1320\cos(3\theta)+495\cos(4\theta)\sin^4\left(\frac{\theta}{2}\right)\right]$
6	3	$-\frac{3}{2048}e^{3i\phi}\sqrt{\frac{13}{\pi}}[20\sin\theta+51\sin(2\theta)+6\sin(3\theta)+20\sin(4\theta)-110\sin(5\theta)+55\sin(6\theta)]$
6	4	$\frac{1}{8}e^{4i\phi}\sqrt{\frac{195}{2\pi}}\cos^2\left(\frac{\theta}{2}\right)[35+44\cos\theta+33\cos(2\theta)]\sin^6\left(\frac{\theta}{2}\right)$
6	5	$-\frac{1}{2}e^{5i\phi}\sqrt{\frac{2145}{\pi}}\cos^3\left(\frac{\theta}{2}\right)[1+3\cos\theta]\sin^7\left(\frac{\theta}{2}\right)$
6	6	$\frac{3}{2}e^{6i\phi}\sqrt{\frac{715}{\pi}}\cos^4\left(\frac{\theta}{2}\right)\sin^8\left(\frac{\theta}{2}\right)$

Table C.5.: Spin-weighted spherical harmonics calculated for $l=6$.

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there would be no history,
there could be no concept of humanity.
(Hermann Hesse)