SYMMETRY RESOLUTION OF ENTANGLEMENT IN HOLOGRAPHY

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dedicated to my parents

ABSTRACT

This thesis investigates the charged moments and the symmetry-resolved entanglement entropy in the context of the $\text{AdS}_3/\text{CFT}_2$ duality. In the first part, I focus on the holographic $U(1)$ Chern-Simons-Einstein gravity, a toy model of $\text{AdS}_3/\text{CFT}_2$ with $U(1)$ Kac-Moody symmetry. I start with the vacuum background with a single entangling interval. I show that, apart from a partition function in the grand canonical ensemble, the charged moments can also be interpreted as the two-point function of vertex operators on the replica surface. For the holographic description, I propose a duality between the bulk $U(1)$ Wilson line and the boundary vertex operators. I verify this duality by deriving the effective action for the Chern-Simons fields and comparing the result with the vertex correlator. In the twist field approach, I show that the charged moments are given by the correlation function of the charged twist operators and the additional background operators. To solve the correlation functions involved, I prove the factorization of the $U(1)$ extended conformal block into a $U(1)$ block and a Virasoro block. The general expression for the $U(1)$ block is derived by directly summing over the current descendant states, and the result shows that it takes an identical form as the vertex correlators. This leads to the conclusion that the disjoint Wilson lines compute the neutral $U(1)$ block. The final result for the symmetry-resolved entanglement entropy shows that it is always charge-independent in this model. In the second part, I study charged moments in higher spin holography, where the boundary theory is a CFT with W_3 symmetry. I define the notion of the higher spin charged moments by introducing a spin-3 modular charge operator. Restricting to the vacuum background with a single entangling interval, I employ the grand canonical ensemble interpretation and calculate the charged moments via the known higher spin black hole solution. On the CFT side, I perform a perturbative expansion for the higher spin charged moments in terms of the connected correlation functions of the spin-3 modular charge operators. Using the recursion relation for the correlation functions of the W_3 currents, I evaluate the charged moments up to the quartic order of the chemical potential. The final expression matches with the holographic result. My results both for $U(1)$ Chern-Simons Einstein gravity and W_3 higher spin gravity constitute novel checks of the $\text{AdS}_3/\text{CFT}_2$ correspondence.

ZUSAMMENFASSUNG

Diese Arbeit untersucht die Symmetrie-aufgelöste Verschränkungsentropie im Kontext der $\mathrm{AdS}_3/\mathrm{CFT}_2$-Dualität. Im ersten Teil konzentriere ich mich auf die holographische $U(1)$ Chern-Simons-Einstein-Gravitations-Theorie, welches ein Spielzeugmodell für $\mathrm{AdS}_3/\mathrm{CFT}_2$ mit $U(1)$ Kac-Moody-Symmetrie ist. Ich beginne mit dem Vakuumhintergrund mit einem einzigen Verschränkungsintervall. Ich zeige, dass neben einer Partitionsfunktion im großen kanonischen Ensemble die geladenen Momente auch als Zweipunktfunktion von Vertex-Operatoren auf der Replikationsoberfläche interpretiert werden können. Für deren holographische Beschreibung wähle ich eine Dualität zwischen der Bulk $U(1)$ Wilson-Linie und den Randvertexoperatoren. Diese Dualität verifiziere ich, indem ich die effektive Wirkung für die Chern-Simons-Felder herleite und das Ergebnis mit dem Vertex-Korrelator vergleiche. Im Twist-Field-Ansatz zeige ich, dass die geladenen Momente durch die Korrelationsfunktion der geladenen Twist-Operatoren und der zusätzlichen Hintergrundoperatoren gegeben sind. Um die beteiligten Korrelationsfunktionen zu lösen, beweise ich die Faktorisierung des $U(1)$ erweiterten konformen Blocks in einen $U(1)$-Block und einen Virasoro-Block. Der allgemeine Ausdruck für den $U(1)$ Block wird direkt durch die Summierung über alle Absteigerzustände hergeleitet. Das erzielte Ergebnis hat tatsächlich die gleiche Form wie die Vertex-Korrelatoren hat. Dies führt zur Schlussfolgerung, dass die getrennten Wilson-Linien den neutralen $U(1)$ Block berechnen. Das Endergebnis für die Symmetrie-aufgelöste Verschränkungsentropie zeigt, dass sie in diesem Modell immer ladungsunabhängig ist. Im zweiten Teil untersuche ich geladene Momente in der Holographie höherer Spins, wobei die Randtheorie eine CFT mit W_3 Symmetrie ist. Ich definiere das Konzept der geladenen Momente höheren Spins, indem ich einen Spin-3-modularen Ladungsoperator einführe. Wenn ich mich auf den Vakuum-Hintergrund mit einem einzelnen Verschränkungsintervall beschränke, nutze ich die Interpretation des großkanonischen Ensembles und berechne die geladenen Momente mithilfe der bekannten Lösung für das schwarze Loch höheren Spins. Auf der CFT-Seite führe ich eine perturbative Expansion für die höheren spingeladenen Momente in Bezug auf die verbundenen Korrelationsfunktionen der modularen Spin-3-Ladungsoperatoren durch. Unter Verwendung der Rekursionsrelationen für die Korrelationsfunktionen der W_3-Ströme werte ich die geladenen Momente bis zur quartischen Ordnung des chemischen Potenzials aus. Das endgültige Ergebnis stimmt mit dem holographischen Ergebnis überein. Meine Ergebnisse für U(1) Chern-Simons-Einstein-Gravitation und W_3 höhere Spingravitation stellen neuartige Überprüfungen des $\mathrm{AdS}_3/\mathrm{CFT}_2$ dar Korrespondenz.

CONTENTS

1 INTRODUCTION

The discovery of quantum mechanics and general relativity in the early twentieth century greatly advanced our understanding of the laws of nature. While quantum mechanics reveals the probabilistic law of particles at the microscopic scale, general relativity shows that, at the macroscopic scale, matter curves the spacetime, and spacetime tells matters how to move. The field theory extension of the quantum mechanics, known as quantum field theory (QFT), assumes negligible quantum gravitational effects which are too small to be observable even at present day colliders. The currently accepted quantum field theory describing all known elementary particles and interactions except of gravity is the Standard Model [1], which has passed numerous experimental tests. For instance, the Higgs boson predicted by the Standard model has been successfully detected at Large Hadron Collider (LHC) [2, 3]. Despite all the successes, the exclusion of gravity in the Standard Model indicates that it is only an effective theory of nature, and must be incorporated into a larger theoretical framework incorporating both principles of general relativity and quantum mechanics.

What is quantum gravity?
Finding such a framework, i.e. the theory of quantum gravity, is one of the most outstanding problems in modern physics. At a first glance, it seems that there are following two possible ways to solve this problem. The first way is to simply couple quantum field theory with general relativity. While such a consideration do makes sense in the semi-classical regime, where the back-reaction of quantum fluctuations to the curved spacetime can be neglected, it can not be the final answer for quantum gravity. One easy way to see this is to consider a measurement in the quantum system. The collapse of the wave function will lead to the instant change of expectation value of energy-momentum tensor in the quantum system. Consequently, the metric of the spacetime sourced by the measured system will also change instantly, which violates the causality of general relativity. The second way is to directly quantize the general relativity around the flat background in the framework of quantum field theory. However, due to the negative mass dimension of the Newton constant in $d > 2$ dimensions, i.e. $[G_d] = 2 - d$, the Einstein-Hilbert action contains irrelevant perturbations to the Gaussian fixed point, indicating the perturbative nonrenormalizability of this approach.

To avoid the problem of perturbative nonrenormalizability in the naive quantization of general relativity, various alternative approaches to quantum gravity were developed in past decades, such as asymptotic safety program [4], loop quantum gravity [5], and string theory [6]. The diversity of these distinct approaches mainly arises from the lack of experimental data on quantum gravitational effects. Even so, at the purely theoretical level, these theories do provide us with a broader perspective on the mysteries of quantum gravity.

Asymptotic safety is a candidate for a quantum theory of the gravitational interactions, which does not require physics beyond the framework of relativistic QFT. The core idea of asymptotic safety was formulated by Weinberg in [4]: one attempts to seek for a non-Gaussian ultraviolet (UV) fixed point of the renormalization group (RG) equations, such that the general relativity can be regarded as the relevant perturbations to the UV theory. See [7] for a recent review on the development of asymptotic safety program.

In loop quantum gravity (LQG), the basic idea is to replace the Riemannian geometry of general relativity by the quantum Riemannian geometry, which by itself satisfies the uncertain principle of quantum mechanics [8]. Under this setup, the theory of LQG do not refer to any background fields and is formulated as a pure gauge theory, even for the gravitational sector. The canonical quantization of the quantum Riemannian geometry leads to the discreteness of space. Intuitively, one can think of the space as a fine fabric, made of very tiny loops of gravitational gauge field that are only visible at the Planck length scale $l_p \sim 10^{-35}m$. The network of loops is called the spin network, characterizing the quantum state of the gravitational field. The discreteness of the space in LQG naturally avoids the UV divergences associated with nonrenormalizablity of general relativity.

In string theory, the fundamental objects are strings instead of quantum point particles. Elementary particles in nature, including the graviton, are interpreted as particular oscillating modes of strings [6]. The consistence of the string theory requires that the background spacetime must consist of ten dimensions rather than four dimensions. The extra six dimensions are compactified in Planck scale and they are not visible at low energy. The Feynman diagrams of string theory are the two-dimensional string worldsheets and string theory as a theory of quantum gravity is completely free of any divergence.

The holographic principle for quantum gravity

In spite of the various approaches to quantum gravity introduced in above, there exists a universal principle that all theories of quantum gravity, which may or may not explain the real world, need to obey. This principle is known as the *holographic principle*, which originated from ideas of Gerard 't Hooft [9] and was further refined by Leonard Susskind [10]. The holographic principle states that a quantum gravity theory in a region of $(d+1)$-dimensional spacetime can be equivalently described by a quantum theory without gravity living on the boundary of that spacetime region. This dimensionality relationship between a

$(d+1)$-dimensional volume and a d-dimensional surface is very similar to optical holography, which encodes the image of three-dimensional space on a two-dimensional surface, hence the name "holography". As I will explain in below, the reasoning for the holographic principle is based on the black holes, whose existence in our universe has recently been confirmed by the Event Horizon Telescope (EHT) [11].

Black holes are classical solutions in the general relativity, but in fact they also manifest quantum gravitational effects such as Hawking radiation [12]. The reason such effects manifest themselves in general relativity is that there exist a semi-classical regime, where the effective theory of quantum gravity is described by the coupling between the quantum system and the classical gravity. Applying this regime to the black hole background requires the Compton wavelength of a quantum particle to be much smaller than the radius of the black hole horizon. Historically, it was first observed by Jacob Bekenstein [13] that the black hole behaves as a thermodynamical system, with its horizon area proportional to the thermal entropy. Shortly after, Stephen Hawking [12] gave a semi-classical interpretation on this phenomenon, and showed that black holes emit thermal radiations. This is similar to the Unruh effect in the flat spacetime – an accelerating observer feels thermal radiations in the unaccelerated vacuum [14, 15]. This development finally led to the *Bekenstein-Hawking (BH) entropy* formula,

$$S_{BH} = \frac{\mathbf{c}^3 k_B A_{horizon}}{4\hbar G} \ , \tag{1.1}$$

and the temperature of thermal radiation from the black hole,

$$T_H = \frac{\hbar \mathbf{c}^3}{8\pi G M k_B} \ . \tag{1.2}$$

Here I have employed the speed of light $\mathbf{c}$, the Planck constant $\hbar$, the Newton constant G, and the Boltzmann constant k_B, as well as the black hole mass M. The Bekenstein-Hawking entropy formula implies that, in a quantum gravity system, the maximal amount of entropy inside a given compact space region is bounded by the area law (1.1), since adding more matter into the region will increase the radius of the black hole horizon. This feature not only reveals why gravity is so special compared to the other three fundamental forces, leading to subextensive entropy growth, but also motivates the holographic principle, because in any quantum system without gravity, the maximal amount of entropy is proportional to the volume of the space. The Anti-de sitter/-conformal field theory (AdS/CFT) correspondence or gauge/gravity duality, discovered by Juan Maldacena in [16], is a concrete realization of the holographic principle. Roughly speaking, it conjectures that the physics of string theory on $AdS \times X$ background can be equivalently described by certain conformal field theory living on the asymptotic boundary of the AdS spacetime. Various examples of this duality have been discussed in [16]. In a particular limit explained in Chapter 2, the

AdS/CFT duality becomes a strong/weak coupling duality, where on the boundary side, the conformal field theory is strongly coupled, and on the bulk side, the string theory is approximated by a weakly-curved classical gravity theory on the asymptotically AdS space. In this respect, the AdS/CFT duality provides a nonperturbative approach to study various strongly interacting phenomenons in the real world, such as the quantum transport in strongly coupled fluids [17, 18] and the quark-gluon plasma in quantum chromodynamics [19].

Quantum information in holography: What and Why?

The traditional usage of quantum field theory, revolving around calculating the correlation functions of local operators and scattering amplitudes, does not leverage the full amount of quantum correlations inherent in composite quantum mechanical systems. Quantum information theory, in a complement approach to traditional QFT, eschews specific field content and observables, and instead studies the information content of the QFT wavefunctional. While applications of holography mostly focus on studying strongly coupled quantum systems via the dual classical gravity theories, a deeper understanding of the nature of quantum gravity requires us to reconstruct the bulk observables from the boundary quantum system. It turns out that in such a bulk reconstruction program, the study on the quantum information aspects of the boundary quantum system are necessary. The reason for this can be roughly explained by the optical hologram in the real world — the image of a local object in the three-dimensional space is encoded non-locally on the two-dimensional hologram. Thus, it is expected that any finite set of local operators on the boundary quantum system can not recover the bulk local observables. Instead, one has to first decode the hologram properly as a set of qubits, and then reconstruct the bulk qubits to obtain the bulk local observables. In the context of the AdS/CFT duality, the approach for such decoding and reconstruction processes is known as the *quantum error correction code* [20], a concept originating from the quantum information theory [21]. Besides this, other concepts in the quantum information theory, such as the entanglement entropy [22], the complexity [23], the quantum circuit [24] and the quantum teleportation [25], also play very important roles in the development of the AdS/CFT duality. One important result is the *Ryu-Takayanagi* (RT) formula [22], which relates the entanglement entropy in the boundary quantum system to the length of the minimal geodesic in the asymptotically AdS spacetime. This indicates that the spacetime may be understood as the emergent phenomenon of the entanglement among qubits [26].

The goals of the thesis

Recently, a new entanglement measure, the *symmetry-resolved entanglement entropy* (SREE) was proposed in [27]. The authors claimed that when an additional internal symmetry is present in a quantum field theory, the entanglement between two spatially separated subsystems is organized into different charge sectors. The SREE is defined as a measure for the amount of entanglement inside each charge sector, and a useful toolkit for calculating the SREE is the so-called *charged moments* [27]. Study of SREE not only helps us to understand global symmetries in QFTs from the quantum information point of views, but also motivates new ideas on detecting entanglement in the context of experimental physics [28]. This thesis is dedicated to study the SREE and the charged moments in the context of holography, in particular to understand their dual description in the bulk AdS spacetime. To build up the intuition, my discussion will focus on two instances of the AdS_3/CFT_2 correspondence, i.e. three-dimensional $U(1)$ Chern-Simons-Einstein gravity [29, 30], and the three-dimensional holographic $SL(3,R) \times SL(3,R)$ higher spin gravity [31]. These two theories contains non-trivial asymptotic symmetry algebras, which are necessary for the setup of SREE. Meanwhile, they are relatively simpler than other holographic models due to their topological property and gauge invariance.

Structure of the thesis

To keep the presentation self-contained, I will first review some prior materials in Chapter 2. Section 2.1 is a brief introduction into the AdS/CFT duality, starting from its top-down construction from string theory, and then passing to more general bottom-up constructions. Section 2.2 includes the introduction to entanglement entropy and the symmetry-resolved entanglement entropy in quantum field theories. The holographic description of the entanglement entropy, i.e. the RT formula, will also be briefly explained.

Chapter 3 is a detailed review on the holographic AdS_3 gravity. Section 3.1 reviews various aspects of the AdS_3 gravity as well as their connections to two-dimensional conformal field theory. Section 3.2 discuss how to derive the entanglement entropy in AdS_3/CFT_2 from both of the holographic and CFT point of views.

Chapter 4 studies the symmetry-resolved entanglement entropy and the charged moments in three-dimensional holographic $U(1)$ Chern-Simons-Einstein gravity. Section 4.1 introduces various aspects of $U(1)$ Chern-Simons fields in asymptotically AdS_3 space. Section 4.2 studies the SREE and the charged moments from various different approaches, i.e., holographic Wilson line approach, the replica approach and the twist field approach in the dual CFT. A general technique for solving the most general charged moments is presented at the end of Section 4.2.

Chapter 5 studies the higher spin charged moments in the context of higher spin holography. Section 5.1 is a brief introduction on higher spin holography. The Section 5.2 introduces the Chern-Simons formulation of the higher spin gravity, i.e. the bulk theory of higher spin holography. The Section 5.3 introduces the notion of higher spin charged moments and develops a new perturbative method to solve it.

Chapter 6 includes a summary of the thesis, as well as an outlook for the future research directions.

A short review on the construction of the entanglement W_3 algebra is presented in Appendix A.

Main results

Chapter 4: The first the main results in this chapter is that the SREE in the dual CFT is always independent of the charge sector. The universal charge-independent behavior of the SREE is in consistence with the results of [27]. The second result is that the bulk description of the charged moments is a set of disjoint $U(1)$ Wilson lines. Each of them is dual to a pair of vertex operators on the boundary replica surface. The third result is that any correlation function of current primary operators in the dual CFT factorizes into the product of a $U(1)$ part and a Virasoro part. This factorization property coincides with the decoupling between gravity and Chern-Simons fields in the bulk. In particular, the $U(1)$ part can be computed by the bulk disjoint Wilson lines, which further clarifies the above described Wilson line/vertex operators duality. Finally, a general framework for solving the charged moments via the null-state equation is developed. The first two results in above described have been published [29, 30], in collaboration with R. Meyer, C. Northe, and K. Weisenberger. The last three results are so-far unpublished work.

Chapter 5: In order to generalize the notion of charged moments to the higher spin case, a modular spin-3 charge operator is defined in the dual W_3 CFT. The holographic calculation for the higher spin charged moments is based on the topological black hole method [32] and the known higher spin black hole solutions [33]. In addition, a new independent calculation is performed in the dual W_3 CFT, in which the charged moments is expressed perturbatively in terms of the connected correlation functions of the spin-3 modular charge operators. The results of both calculations coincide. In particular, the higher spin charged moments is no longer a Gaussian function of the chemical potential, which implies a violation of the charge-independence of the SREE in the dual W_3 CFT. The results of this chapter have been published [31], in collaboration with R. Meyer, C. Northe, and K. Weisenberger.

Convention

In the rest part of this thesis, I will use natural units $\mathbf{c} = \hbar = k_B = 1$.

2 PRELIMINARIES

This chapter introduces the basics of gauge/gravity duality as well as the concept of entanglement entropy in QFT. I will start by explaining the top-down construction of gauge/gravity duality from string theory, and then pass to more general bottom-up constructions, which are more relevant to the main discussions in this thesis. Moreover, I will explain the formal path-integral formulation of entanglement entropy in a general QFT, as well as the Ryu-Takayanagi formula for holographic entanglement entropy.

2.1 GAUGE/GRAVITY DUALITY

Duality appears in various areas of the theoretical physics and mathematics, relating distinct theories and concepts together at either a fundamental or effective level. This section is a short introduction on the Anti-de sitter/Conformal field theory (AdS/CFT) duality, or more precisely gauge/gravity duality, which describes the dynamical equivalence between a gravity theory in a $(d+1)$-dimensional Anti-de Sitter (AdS) space, and a conformal field theory without gravity in d-dimensions. As this duality is analogous to optical holography in the real world, it is also referred as *holography*. In the following, I will first describe the overall picture of the AdS/CFT duality, and explain more details in the later parts of this section. The presentation in this section mainly follows Juan Maldacena's original paper [16], and the reviews [34, 35].

2.1.1 *A brief overview*

The conjecture of the AdS/CFT duality originated from ten-dimensional superstring theory, a prime candidate for the unified theory which unifies the four fundamental forces in nature. Roughly speaking, the strongest form of the AdS/CFT duality relates superstring theory on $AdS \times X$ background to certain superconformal field theory defined on the asymptotic boundary of the AdS spacetime. The manifold X is a closed manifold. In a particular limit, the AdS/CFT duality becomes an example of a strong-weak coupling duality, where the superconformal field theory becomes strongly coupled and the dual superstring theory is approximated by a weakly coupled classical supergravity (SUGRA) theory on $AdS \times X$. By the Kaluza-Klein reduction, the supergravity theory can be further reduced to a classical gravity coupled to extra fields on the AdS background. This eventually leads to the general concept of a duality between a weakly coupled classical gravity theory in asymptotically AdS spacetime and a strongly coupled conformal field theory on the boundary of AdS. Since the perturbative approach fails in the strongly coupled regime, the AdS/CFT duality provides a powerful way for understanding non-perturbative phenomena in a strongly coupled quantum system via perturbative calculations in the dual gravity theory.

There are several specific examples of the AdS/CFT duality which were considered in Juan Maldacena's original paper [16]. The ideas involved in those examples are similar. I will restrict to the most well-known example, the AdS$_5$/CFT$_4$ duality, which relates type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory in 3+1 dimensions. The strongest form of the AdS$_5$/CFT$_4$ duality states that

> $\mathcal{N} = 4$ Super Yang-Mills theory
> with gauge group $SU(N)$ and Yang-Mills coupling constant g_{YM}
> *is dynamically equivalent* to
> type IIB superstring theory
> with string length $l_s = \sqrt{\alpha'}$ and closed string coupling g_s
> on AdS$_5 \times S^5$ with radius of curvature L_5
> and N units of $F_{(5)}$ flux on S^5.

Here, the $(3+1)$-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory is conformally invariant even at the quantum level [36], and thus it is regarded as the "CFT side" of the duality. The parameter α' in string theory gives rise to the tension of the fundamental string, i.e. $T = 1/2\pi\alpha'$. Under the duality, the two dimensionless free parameters on the field theory side, i.e. g_{YM} and N, are mapped to the free parameters g_s and L_5/l_s on the string theory side via the relations

$$g_{\text{YM}}^2 = 2\pi g_s \ , \quad 2g_{\text{YM}}^2 N = \left(\frac{L_5}{l_s}\right)^4 \ . \tag{2.1}$$

The *dynamical equivalence* in the above statement means that the physical Hilbert spaces of two theories are isomorphic to each other. Although the strongest form of the AdS$_5$/CFT$_4$ duality described above is very interesting and generates new ideas, practically it is very difficult to perform explicit calculations for generic values of the parameters. Therefore, to gain deeper insights from the proposed AdS/CFT duality, it is necessary to weaken the statement by taking certain limits on both sides. A particular useful limit for understanding the SYM gauge theory is the so-called *'t Hooft limit* [37],

$$N \to \infty \ , \quad \lambda = g_{\text{YM}}^2 N \text{ fixed.} \tag{2.2}$$

The partition function of the large N SYM gauge theory admits a expansion in N,

$$Z_{gauge} = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \ , \tag{2.3}$$

where g is the genus of the Feynman diagrams in the double-line representation [37]. In the 't Hooft limit (2.2), the dominant contribution to the partition function scales as N^2, coming from all planar diagrams with $g = 0$. On the other hand, for constant dilaton field ϕ, the action of the superstring theory on $AdS_5 \times S^5$ involves a topological constant term,

$$\begin{aligned} S_{string} &= \tilde{S}_{string} + \frac{1}{4\pi\alpha'} \int d\sigma^2 \sqrt{-h}\ \alpha' R_h \phi \\ &= \tilde{S}_{string} + (2 - 2g)\phi \ , \end{aligned} \tag{2.4}$$

where g is the genus of the two-dimensional worldsheet surface and R_h is the Ricci curvature of the worldsheet metric $h_{\alpha\beta}$. The partition function of the string theory requires to sum over all topologies of two-dimensional surfaces. Thus, in the Euclidean signature, it takes the form of

$$Z_{string} = \sum_{g=0}^{\infty} \left(\frac{1}{g_s} \right)^{2-2g} Z_g \ . \tag{2.5}$$

where the closed string coupling constant $g_s = e^{\phi}$ is introduced. By the relations in (2.1), the 't Hooft limit (2.2) on the string theory side corresponds to

$$g_s \to 0 \ , \quad \frac{L_5}{l_s} \text{ fixed} \ , \tag{2.6}$$

and as a consequence, the $g = 0$ contribution to the partition function dominants. From the perspective of the string perturbation theory, genus zero two dimensional worldsheet surfaces are understood as the tree level diagrams of the closed string interactions, hence, in the limit (2.6), the AdS side is described by the classical type IIB superstring theory. This gives rise to the *strong form* of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality:

In the 't Hooft limit:
$\mathcal{N} = 4$ $SU(N)$ SYM $\cong$ Classical superstring on $AdS_5 \times S^5$

In this respect, the AdS/CFT duality is a concrete realization of 't Hooft's idea that the planar limit of a gauge theory is a string theory [37].

In the 't Hooft limit, the only free parameter in the SYM theory is the 't Hooft coupling constant λ, while on the string theory side it is the ratio L_5/l_s. We are mostly interested in the strongly coupled regime of the SYM theory where the usual perturbative approach becomes unreliable. Taking the limit $\lambda \gg 1$ on the field theory side yields the limit $l_s \ll L_5$ on the string theory side. Since the length scale of the string is much smaller than the radius of the curvature under such a limit, effectively we may treat a string as a point particle. And an effective theory of the superstring in this limit should be be classical gravitational theory, in which each field accounts for the corresponding string state. The energy scale of all massive string states is of the order of $\mathcal{O}(1/l_s)$, which is much larger than the typical energy scale $\omega \sim \mathcal{O}(1/L_5)$ of a wave propagating on $AdS_5 \times S^5$. So we further restrict to the low energy regime, i.e. $E \ll 1/l_s$, in which only massless superstring states survive. The effective theory of the superstring in this low energy regime is known as the classical type IIB supergravity (SUGRA) theory, which describes the propagation of those massless superstring modes in $AdS_5 \times S^5$. So we arrive at:

For $N \to \infty$ and large λ:
strongly coupled $\mathcal{N} = 4$ $SU(N)$ SYM $\cong$ SUGRA on $AdS_5 \times S^5$.

This is referred as the *weak form* of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality.

Matching the symmetries: The four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory is invariant under $PSU(2,2|4)$, which in particular includes the subgroups $SU(2,2) \simeq SO(4,2)$ and $SU(4) \simeq SO(6)$. The $SO(4,2)$ symmetry accounts for the conformal invariance of the theory in four dimensions, and the $SO(6)$ symmetry, which is often called the R-*symmetry*, arises from the rotation of the six scalar fields ϕ^I in the theory. On the other hand, the symmetry of the type IIB superstring theory on $AdS_5 \times S_5$ is also $PSU(2,2|4)$. The bosonic part of the symmetry comes from the spacetime isometries of AdS_5 and S_5, given by $SO(4,2)$ and $SO(6)$. Therefore, the symmetries in both theories coincide with each other. In particular, the conformal boundary of the AdS_5 spacetime defined at the spatial infinity is a four-dimensional flat spacetime, and the isometry group $SO(4,2)$ of AdS_5 acts on the conformal boundary as the conformal group of the boundary spacetime. Therefore, it is natural to think that the SYM gauge theory is defined on the conformal boundary of AdS_5. Then, the $SO(4,2)$ isometry of AdS_5 corresponds to the conformal symmetry of SYM, and the $SO(6)$ isometry of S^5 corresponds to the internal R-symmetry of SYM, which further lead to the identification of generators of those symmetries in both theories.

2.1.2 *From D-branes to the AdS/CFT*

The fundamental string is not the only dynamical object contained in superstring theory, there also exists Dirichlet-brane or D-brane for short. A D_p-brane is a $(p+1)$-dimensional surface where open strings end on. Oscillations of the open strings in the directions transverse to the brane will lead to fluctuations of the brane, hence the brane is dynamical. In type IIB superstring theory, the allowed D_p-branes need to preserve one half of the supersymmetry, and this can happen only if the number p is odd [38]. The $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality is motivated by considering the stack of N coincident D_3-branes in the type IIB superstring theory. But before explain how this works, let me first give an intuitive explanations on the physics of a single D_p-brane.

The physics of a D-brane. We start from the open string perspective of a D-brane, and we demonstrate the main idea by considering bosonic string theory in $(d+1)$-dimensional flat spacetime for simplicity. Consider a single D_p-brane, which extends in the time and p spatial directions at constant transverse spacetime coordinates $X^I = 0$, with $I = p+1, \cdots, d+1$. We use ξ^i with $i = 0, 1, \cdots, p$ to denote the coordinates on the brane. The presence of the D_p-brane breaks the spacetime Lorentz symmetry $SO(d,1)$ to $SO(d-p) \times SO(p,1)$. In the low energy regime $El_s \ll 1$, all the massive open string modes are integrated out, and we are left with only the massless string modes. $\alpha^I_{-1}|0,k\rangle$ and $\alpha^i_{-1}|0,k\rangle$. Here α^I_{-1} and α^i_{-1} are the bosonic creation operators and $|0,k\rangle$ denotes the string vacuum with the center of mass momentum k^i, which is tangent to the brane and satisfies the mass shell condition $M^2 = k^i k_i = 0$. We may associate those states with the polarization and nontrivial momentum profile to define more general states, i.e.

$$|A\rangle = \int d^{p+1}k A_i(k)\alpha^i_{-1}|0,k\rangle, \ |\phi\rangle = \int d^{p+1}k\phi_I(k)\alpha^I_{-1}|0,k\rangle. \tag{2.7}$$

Using the transition element between the momentum space and the position space of the brane, i.e. $\langle 0,\xi|0,k\rangle = e^{ik\cdot\xi}$, we are able to define functions associated with the those open string states,

$$A^i(\xi) = \langle 0,\xi|\alpha^i_1|A\rangle \ , \quad \phi^I(\xi) = \langle 0,\xi|\alpha^I_1|\phi\rangle \ . \tag{2.8}$$

Note that those functions are valued only in the brane coordinates ξ^i, since the conjugate of ξ^i, the center of mass momentum k^i, is tangent to the brane[1]. From the spacetime point of view, they characterize the shape of the D_p-brane under the quantum fluctuations. On the other hand, from the D_p-brane point of view, we may interpret those functions as the expectation values of quantum fields living on the brane, where $A_i(\xi)$ corresponds to a $U(1)$ gauge field and $\phi^I(\xi)$ are scalar fields. When the quantum fluctuations of the brane is small, the physics of the massless open string ending on the brane can be described by a quantum field theory of those fields localized on the brane. The action of this theory is known as the Dirac-Born-Infeld (DBI) action[2],

$$S_{\text{DBI}} = -\frac{T_p}{g_s}\int d^{p+1}\xi\sqrt{-\det(g_{ij} + 2\pi l_s^2 F_{ij} + \mathcal{O}(l_s^3))} \ . \tag{2.9}$$

1 The physical meaning of $A^i(\xi)$ and $\phi^I(\xi)$ is that they describe the translations and the fluctuations of the spacetime coordinates $X^\mu(\xi)$ along the longitudinal and transverse directions of the brane [6].

2 The DBI action here only includes the bosonic part and is valid in the ten-dimensional flat spacetime background with vanishing Kalb-Ramond B field and constant dilaton. See [6] for the general form.

Here $T_p = (2\pi)^{-p} l_s^{-p-1}$ is the brane tension, g_{ij} is the pull-back of the spacetime metric onto the brane, and $F = dA$ is the flux of the $U(1)$ gauge field. The $1/g_s$ factor appears in (2.9), because the open string coupling g_{open} is related to the closed string coupling[3] g_s via $g_s = g_{open}^2$, and the DBI action (2.9) only accounts for the tree-level physics of the open string, i.e. the self-interactions of open string and their couplings to the closed string, arise from the disk in the open string worldsheet. This tells us that the DBI action as an low energy effective theory of the open string only valid for small string coupling.

Note that, for small string length l_s, we are able to perform the perturbative expansion of the DBI action in the power of l_s^2. For convenience, let us choose the static gauge $X^i(\xi) = \xi^i$. The coordinates $X^I(\xi)$ of the brane in the transverse space fluctuate around $X^I = 0$, and they are described by the scalar fields ϕ^I,

$$X^I(\xi) = 2\pi l_s^2 \phi^I(\xi) + \cdots \ . \tag{2.10}$$

The prefactor $2\pi l_s^2$ makes sure ϕ^I has mass dimension one such that fluctuation of the brane in higher energy is described by the higher value of ϕ^I. The pull-back of the background spacetime metric onto the brane is expressed as

$$g_{ij} = \frac{\partial X^\mu}{\partial \xi^i}\frac{\partial X^\nu}{\partial \xi^j}\eta_{\mu\nu} = \eta_{ij} + (2\pi l_s^2)^2 \partial_i \phi^I \partial_j \phi_I \ . \tag{2.11}$$

Inserting (2.11) into the DBI action (2.9) and expanding the action in the power of l_s^2 yields

$$S_{\text{DBI}} \approx -\frac{T_p}{g_s}\int d^{p+1}\xi \left[1 + (2\pi l_s^2)^2 \left(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\partial_i \phi^I \partial^i \phi_I\right)\right] . \tag{2.12}$$

which is the well-known action for the $U(1)$ gauge field and free massless scalars. The Yang-Mills coupling for the gauge field can be read out from the action, given by

$$g_{\text{YM}}^2 = \frac{g_s}{T_p}(2\pi l_s^2)^{-2} = (2\pi)^{p-2} g_s l_s^{p-3} \ . \tag{2.13}$$

In the above, we discussed the physics of the D$_p$-brane from the open string point of view. There is also equivalent description of the D-brane from the closed string point of view. The equivalence between them is called the worldsheet duality. For instance, let us first Wick rotate the background spacetime to the Euclidean space and consider a worldsheet of the disk topology with its boundary anchored to the D$_p$-brane. From the open string point of view, the disk can be regarded

3 The one-loop diagram of the open string contains an emission and an absorption of the open string, which can be equivalently described by an emission of the closed string. This determines the relation between the couplings of the open and closed string, i.e. $g_s = g_{open}^2$.

as the propagation of an open string around the Euclidean time circle. On the other hand, from the closed string point of view, the disk is understood as the emission of a closed string from the D_p-brane to the interior of the spacetime. Due to this equivalence, the low energy physics of the D_p-brane can also be interpreted by the massless modes of the closed string close to the D_p brane. It should be noted that those modes are distinct from the massless modes of the closed string far away from the brane, since if they are not, we learn nothing about the D_p-brane. Or in other words, in the closed string perspective, we have to include the influence of the D_p-brane on the background spacetime. This is one of main differences with the open string perspective.

To analysis the D-brane from the closed string perspective, one first solves the closed string spectrum in flat spacetime. Analogous to the open string case, the massless modes of the closed string help us to identify the perturbative fields around the spacetime background. However, at the non-perturbative level, those fields should be backreact to the background. The low energy effective field theory associated with those fields may be formulated by some general requirements such as the general covariance and supersymmetry. After working out the effective field theory, one can find the non-perturbative solution of the effective theory, which incorporates with the influence of the D-brane on the background spacetime. Then, the physics of the D-brane may be extracted from the behavior of the fields closed to the D-brane.

In bosonic string theory, the massless modes of the closed string take the form of $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,k\rangle$ with $k^{\mu}k_{\mu}=0$. There are two differences to the open string modes. Firstly, the closed string contains both of the left- and right-mover parts, i.e. α^{μ}_{-1} and $\tilde{\alpha}^{\nu}_{-1}$. Secondly, the closed string can propagate in all directions of the spacetime, as we can see from the spacetime index μ of the center of mass momentum k^{μ}. Due to those two differences, the field associated the massless closed string modes is given by two-tensor field $\zeta_{\mu\nu}(x)$ over the whole spacetime, where x^{μ} is the center of mass position of the closed string in the spacetime, i.e. $x^{\mu}=\frac{1}{2\pi}\int d\sigma X^{\mu}(\tau=0,\sigma)$. The two-tensor field can be decomposed into a trace part, a symmetric traceless part and an antisymmetric part,

$$\zeta_{\mu\nu}(x)=h_{\mu\nu}(x)+B_{\mu\nu}(x)+\phi(x)\eta_{\mu\nu}\ , \tag{2.14}$$

The symmetric traceless tensor $h_{\mu\nu}$ is identified as the graviton, describing the fluctuation of the metric around the flat spacetime. The antisymmetric tensor $B_{\mu\nu}$ is known as the *Kalb-Ramond B*-field, and the scalar ϕ is the dilaton field. When we consider the type IIB superstring theory, the bosonic massless spectrum of the IIB superstring gives rise to not only the above three fields, but also extra differential form fields, denoted as $F_{(q)}=dC_{(q-1)}$, with the ranks of the form fields valued as $q=1,3,5$. The gauge potentials $C_{(q-1)}$ are called the *Ramond-Ramond* (R-R) fields. Those fields together with their fermionic partners give rise to the type IIB supergravity as the low energy effective field theory for the massless closed IIB superstring. The bosonic part of the type IIB supergravity action reads

$$S_{\text{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2}\Bigg[\int d^{10}x\sqrt{-G}\Big(e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_{(3)}|^2\right) - \frac{1}{2}|F_{(1)}|^2 - \frac{1}{2}|\tilde{F}_{(3)}|^2 - \frac{1}{4}|\tilde{F}_{(5)}|^2\Big) - \frac{1}{2}\int C_{(4)}\wedge H_{(3)}\wedge F_{(3)}\Bigg] , \tag{2.15}$$

where we use the notation

$$|F_{(q)}|^2 = \frac{1}{q!}G_{\mu_1\nu_1}\cdots G_{\mu_q\nu_q}\bar{F}^{\mu_1\cdots\mu_q}F^{\nu_1\cdots\nu_q} , \tag{2.16}$$

and $\bar{F}_q$ denotes the complex conjugate of $F_{(q)}$. The parameter $\tilde{\kappa}$ is the ten-dimensional gravitational constant, given via the relation $2\tilde{\kappa}_{10}^2 = (2\pi)^7 l_s^8$. For the constant dilaton field ϕ, the Newton constant G_{10} in ten dimensions reads[4]

$$16\pi G_{10} = 2\tilde{\kappa}_{10}^2 g_s^2 = (2\pi)^7 l_s^8 g_s^2 \tag{2.17}$$

The field strength tensors in the action are defined as

$$F_{(q)} = dC_{(q-1)} \ , \ H_{(3)} = dB \ , \ \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)} \ ,$$
$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)}\wedge H_{(3)} + \frac{1}{2}B\wedge F_{(3)} \ , \tag{2.18}$$

where the five-form flux $\tilde{F}_{(5)}$ needs to satisfy the self-duality constraint $\star\tilde{F}_{(5)} = \tilde{F}_{(5)}$, with $\star$ the Hodge star in the ten-dimensional (curved) spacetime.

The supergravity introduced in the above accounts for the tree-level physics of the massless closed string states, therefore it requires the string coupling to be small. In addition, the supergravity as a classical field theory approximation of the superstring also assumes the point particle limit of the closed string, which requires that the typical length scale L of a valid supergravity solution must be much larger than the string length l_s. The typical length scale of a supergravity solution is characterized by the inverse of the Ricci curvature, i.e. $L^2 \sim 1/R$. Therefore, we summarize that the supergravity is only valid in the weakly-curved regime, i.e. $Rl_s^2 \ll 1$, and small string coupling $g_s \to 0$.

What is the role of the D-brane from the supergravity point of view? Firstly, the emission of gravitons from the D-brane indicates that the D-brane is a massive object curving the spacetime. Secondly, a remarkable feature of the D-brane was found by Polchinski in [39], namely that a D$_p$-brane is also an electrically charged object under the $(p+1)$-form R-R field $C_{(p+1)}$. The idea in his calculations roughly goes as

4 This is obtained by computing the tree-level scattering amplitudes of corresponding closed string states in NS-NS sector.

followings. He first considered the exchange of closed strings between two D_p-branes and calculated the contribution to the amplitude from the Ramond-Ramond sector. Then, he turned to the supergravity and inserted defects at the locations of two Dp-branes into the supergravity action. At the perturbative level, the only relevant terms in the whole action are given by

$$S = -\frac{1}{2\tilde{\kappa}_{10}^2 g_s^2}\int d^{10}x\sqrt{-G}\frac{1}{2}|F_{(p+2)}|^2 + \mu_p \int_{branes} C_{(p+1)} \tag{2.19}$$

The first term in (2.19) comes from the original supergravity action and the second term in (2.19) denotes the coupling of brane defects to the R-R field $C_{(p+1)}$. In the next, he computed the amplitude of the exchange of the $(p+1)$-form between the two branes. The situation here is very similar to the case of QED, where we can consider two point particles carrying the same electric charge and discuss the exchange of photons between the two particles. Remarkably, the result he obtained can match with the R-R contribution to the exchange of closed strings between two D_p-branes, providing the free parameter μ_p identified as

$$\mu_p = (2\pi)^{-p} l_s^{-(p+1)} g_s^{-1} = \frac{T_p}{g_s} \ . \tag{2.20}$$

In summary, Polchinski's calculation in [39] showed that the D_p-branes can be identified as charged objects from the supergravity point of view, and the charge μ_p of a D_p-brane is determined from the R-R states contribution to the closed string amplitude.

Now we can go back to a single D_p-brane and ask how it backreacts to the background spacetime. D_p-brane defect in supergravity behaves as a delta-function source for the $C_{(p+1)}$ field in the transverse space, which can be seen from the equation of motion for $C_{(p+1)}$ in (2.19), given by

$$d\star F_{(p+2)} = 2\tilde{\kappa}_{10}^2 g_s^2 \mu_p \delta^{(9-p)}(x^I) \ , \tag{2.21}$$

Therefore, by Gauss's law, the total $\star F_{(p+2)}$ flux across a $(8-p)$-sphere S^{8-p} surrounding the D_p-brane is given by

$$\frac{1}{2\tilde{\kappa}_{10}^2 g_s^2}\int_{S^{8-p}} \star F_{(p+2)} = \mu_p \ . \tag{2.22}$$

On the other hand, the presence of the D_p-brane breaks the Lorentz symmetry $S(9,1)$ into $SO(p,1)\times SO(9-p)$. This symmetry should be satisfied by the configurations of each fields in the type IIB supergravity. There is a family of such solutions known as the *extremal black p-brane*, given by

$$\begin{aligned}
&ds^2 = H_p(r)^{-1/2}\eta_{ij}dx^i dx^j + H_p(r)^{1/2}\delta_{IJ}dx^I dx^J \ , \\
&e^\phi = g_s H_p(r)^{(3-p)/4} \ , \\
&C_{(p+1)} = \left(H_p(r)^{-1} - 1\right) dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p \ , \\
&B_{\mu\nu} = 0 \ ,
\end{aligned} \tag{2.23}$$

where r is the radial coordinate in the tranverse space of the brane, defined as $r^2 = \sum_{I=p+1}^{9} x_I^2$, and the function $H_p(r)$ reads

$$H_p = 1 + \left(\frac{L}{r}\right)^{7-p} . \tag{2.24}$$

The D$_p$-brane is located at $r = 0$. Computing the total $\star F_{p+2}$ flux across the sphere S^{8-p} for (2.23) and matching the results with (2.22) yields the value of the characteristic length L, given by

$$L^{7-p} = (4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right) g_s l_s^{7-p} . \tag{2.25}$$

Note that for a single D$_p$-brane, the characteristic length L of the resulting geometry is much smaller than the string length l_s in the limit $g_s \to 0$. This contradicts with the valid regime for the supergravity. Intuitively, we can also think of this contradiction in the following way. For the region with $r \gg L$, the metric tends to be flat and the influence of the D$_p$-brane to the spacetime can be neglected. In order to "see" gravitational effect of the D$_p$-brane, we need to set $r \sim L$ or $r \ll L$. However, the supergravity is only a low energy effective theory with the energy scale $El_s \ll 1$. Thus, in order to detect the $r \sim L$ region, the required energy scale $E_p \sim 1/L$ is much higher than the low energy scale. In other words, the region $r \sim L$ is not visible from the supergravity point of view. A simple resolution for this problem is to consider N D$_p$-branes sitting on top of each others, which will yield the total fluxes

$$\frac{1}{2\tilde{\kappa}_{10}^2 g_s^2}\int_{S^{8-p}} \star F_{(p+2)} = N\mu_p , \tag{2.26}$$

In this case, the relation between L and l_s becomes $L \sim (g_s N)^{1/(7-p)} l_s$. Therefore, in the regime $N \to \infty$, $g_s \to 0$, and $g_s N \gg 1$, the extremal black p-brane solution becomes a valid supergravity description of the N D$_p$-branes.

The AdS$_5$/CFT$_4$ duality from N D$_3$-branes. Now we are ready to analysis a stack of N D$_3$-branes in the type IIB string theory, which is relevant to the AdS$_5$/CFT$_4$ duality. From the open string perspective, when we consider N D$_3$-branes, both endpoints of an open string can attach on any one of the branes. In particular, when the branes sit on top of each others, the first excited states of the open string remain massless. Those two facts lead to the consequence that the gauge field on the stack of N D$_3$-branes becomes a $U(N)$ gauge field, $A_i = A_i^a T_a$, with T_a as the generators of $U(N)$ group. Similarly, $\phi_I = \phi_I^a T_a$. The generalization of the DBI action (2.12) for small string length l_s is a four-dimensional $U(N)$ gauge theory,

$$S_{\text{DBI}} \approx -\frac{T_3}{g_s}\int d^4\xi \Big(1 + (2\pi l_s^2)^2 \text{Tr}\Big[\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}D_i\phi^I D^i\phi_I - \frac{1}{4}\sum_{I,J}[\phi_I, \phi_J]^2\Big]\Big) , \tag{2.27}$$

where the field strength and the covariant derivative are defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \ , \quad D_i\phi^I = \partial_i\phi^I + i[A_i, \phi^I] \ . \tag{2.28}$$

The action (2.27) can be identified as the bosonic part of the four-dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory, with the dimensionless Yang-Mills coupling

$$g_{\text{YM}}^2 = 2\pi g_s \ . \tag{2.29}$$

As a remark, we have claimed that the DBI action only accounts for the tree-level amplitudes of massless open strings. When we consider the stack of N D_3-branes, the perturbation theory of open strings is not expanded in the power of g_s, but in the power of g_sN [40]. Therefore, the $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills description of the N D_3-branes is valid in the regime $g_sN \ll 1$ and $l_s \to 0$. The physics in the region away from the branes is described by the type IIB supergravity in the ten-dimensional flat spacetime for $g_s \to 0$ and $l_s \to 0$. There are of course non-trivial interactions between the fields on the branes and the supergravity fields in the bulk, however, those interactions can be neglected in the limit $l_s \to 0$ [35]. In other words, in this case the whole system is described by two decoupled theories, i.e. the SYM theory on branes and SUGRA on $\mathbb{R}^{9,1}$.

Now let us turn to the closed string perspective. As we mentioned before, the supergravity description of the stack of N D_p-branes is valid in the regime, $N \to \infty$, $g_s \to 0$, and $g_sN \gg 1$. For $p = 3$, the characteristic length L of the extremal black brane solutions (2.23) is given by

$$L^4 = 4\pi g_s N l_s^4 \ . \tag{2.30}$$

In the region $r \gg L$, the black brane tends to be a flat spacetime. But in the region $r \ll L$, the function $H_3(r)$ is approximated as $H_3(r) \approx (L/r)^4$, and the resulting metric is identified as $AdS_5 \times S_5$, with the radius of curvature $L_5 = L$,

$$ds^2 = \frac{L_5^2}{r^2} dr^2 + \frac{r^2}{L_5^2} \eta_{ij} dx^i dx^j + L_5^2 d\Omega_5^2 \ . \tag{2.31}$$

Alternatively, we can define a new radial coordinate $U = \frac{r}{l_s^2}$ in order to zoom in the $r \ll L$ region. The metric in the U coordinate becomes

$$l_s^{-2} ds^2 = l_s^2 \sqrt{4\pi g_s N} \left(\frac{dU^2}{U^2} + \frac{U^2}{4\pi g_s N} \eta_{ij} dx^i dx^j + d\Omega_5^2 \right) \ , \tag{2.32}$$

with $U \ll L/l_s^2 = \frac{(4\pi g_s N)^{1/4}}{l_s}$. Thus, in the limit $l_s \to 0$, we are still able to choose any value of U in the range $0 < U < \infty$. The conformal boundary of AdS$_5$ is located at $U = \infty$. Is there any physical motivation for defining this new coordinate? The answer is yes. Note that in the limit $l_s \to 0$, the energy scale $1/l_s$ characterized by the massive string modes must be measured by some reference energy scale. By definition, U has mass dimension one, so a fixed value of $U = U_0$ would gives rise to such a energy scale to the system. To be more concrete, let us consider an observer sitting at flat region $r \to \infty$. Due to the gravitational effect, a massive string excitation with the energy $E \sim \mathcal{O}(1/l_s)$ at constant $U = U_0$ position is redshifted at infinity as

$$E_\infty = \sqrt{-G_{00}(U_0)}E = (4\pi g_s N)^{-1/4} U_0 E l_s \sim \mathcal{O}(1) \ . \tag{2.33}$$

There are two facts we can learn from this result. First, U_0 indeed behaves as a running energy scale, going to the IR for small U_0 and going to the UV for large U_0. In particular, string modes with the same energy E but at two different positions $U = U_0$ and $U = \beta U_0$ are at the energy scales E_∞ and βE_∞, respectively. This reflects the conformal symmetry of the system. Second, for arbitrary massive string excitation $E \sim \mathcal{O}(1/l_s)$ on the $AdS_5 \times S^5$ background, we can always observe it at the finite energy $E_\infty \sim \mathcal{O}(1)$. Therefore, from the closed string perspective, an observer at infinity can detect two different kinds of finite energy modes, the redshifted modes of the superstring on the $AdS_5 \times S^5$ background and the supergravity modes in the ten-dimensional flat spacetime [16].

Now we can combine with the open string and closed string perspectives. The similarity between them is that the supergravity in ten-dimensional flat spacetime appears in both of them. The difference between them is that from the open string perspective, an observer at infinity can detect the spectrum of the SYM gauge theory on the D-branes without any redshift effects, while from the closed string perspective, the observer can detect the redshifted spectrum of the type IIB superstring on the $AdS_5 \times S^5$ background. **Since both perspectives describe the same object, the N D_3-branes, it is natural to conjecture that the type IIB superstring on $AdS_5 \times S^5$ and the four-dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory describe the same physics**.

2.1.3 *The AdS/CFT dictionary*

So far, we have explained the formal statements of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality. The duality provides the identification of the partition functions of superstring theory on $AdS_5 \times S^5$ and SYM gauge theory,

$$Z_{string} = Z_{CFT} \ . \tag{2.34}$$

If we work on the weak form of the duality, then the partition function of the gauge theory can be obtained by evaluating the classical supergravity action,

$$Z_{CFT} \approx Z_{\text{sugra}} = e^{-S_{\text{sugra}}} \ . \tag{2.35}$$

A natural question then is how to compute the correlation functions of the SYM gauge theory in terms of the superstring or supergravity on $AdS_5 \times S^5$. The answer was provided in [41, 42], shortly after the Juan Maldacena's discovery of the AdS/CFT duality [16]. The punchline is that there exist a precise one-to-one map between operators in the SYM gauge theory and the the spectrum of type IIB superstring on $AdS_5 \times S^5$. This map is usually referred as the *AdS/CFT dictionary* or the *holographic dictionary*. This dictionary essentially arises from the fact that the symmetries of the two theories coincide, which allow gauge theory operators lying in certain representations of $PSU(2,2|4)$ to be mapped to superstring states on $AdS_5 \times S^5$ in the same representations. Furthermore, in the weak form of the duality, this dictionary maps gauge theory operators to certain fields on the AdS_5 background [43]. The field content on the AdS side can be obtained from the Kaluza-Klein reduction of the type IIB supergravity fields.

At the perturbative level, the general rule for applying the dictionary is that adding a source ϕ_0 coupled to an operator O in the gauge theory corresponds to switching on a dual field ϕ on the AdS_5 background, with ϕ_0 encoding the boundary condition of ϕ at the asymptotically AdS boundary. This rule is known as the Gubser-Klebanov-Polyakov-Witten (GKPW) relation [41, 42], which identifies the generating functional of the both theories,

$$Z_{\mathrm{sugra}}[\phi_0] = \left\langle \exp\left(\int \phi_0 O\right)\right\rangle_{CFT} . \tag{2.36}$$

where $Z_{\mathrm{sugra}}[\phi_0]$ is regularized and renormalized. Using the GKPW relation, we are able to compute the connected correlation functions in the dual gauge theory via the functional variations of the supergravity action,

$$\langle O(x_1)\cdots O(x_n)\rangle_c = -\frac{\delta^n S_{\mathrm{sugra}}[\phi_0]}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)} , \tag{2.37}$$

Here the c index denotes the connected part of the correlator. From the D-brane point of view, the GKPW relation (2.36) can be thought of as adding perturbative interactions between the supergravity fields and the gauge fields on the D-branes.

Free scalar field on the AdS background. To explain the field-operator map as well as the GKPW relation in more detail, here I will consider the example of the scalar field, following from the discussion in [35]. Consider a special class of operators in the SYM gauge theory, which are the so-called 1/2 BPS or chiral primary operators. A scalar 1/2 BPS operator O_Δ with conformal dimension Δ is constructed from the fundamental scalars ϕ^I of the SYM gauge theory. It is $SU(N)$ gauge invariant and transforms in the representation of $SO(6)$ with Dykin labels $[\Delta, 0, 0]$. On the other hand, it can be show that the infinite tower of Kaluza-Klein scalar modes can be obtained by the reduction of the supergravity fields on S^5,

$$\varphi(x^\mu) = \sum_\Delta \varphi_\Delta(r, x^i) Y^\Delta \ , \tag{2.38}$$

where the scalar $\varphi(x^\mu)$ is constructed from the fluctuations of the metric and five-form $F_{(5)}$ on S^5. The function Y^Δ is the spherical harmonic on S^5 and lies in the representation of $SO(6)$ with same Dykin labels $[\Delta, 0, 0]$. The equation of motion for φ_Δ on the AdS_5 background can be deduced from the supergravity, which to quadratic approximation shows that the scalar modes φ_Δ is a free scalar field on the AdS_5 background with the mass obeying

$$m^2 L_5^2 = \Delta(\Delta - 4) \ . \tag{2.39}$$

Therefore, by matching the representations, we identify the chiral primary operators in the gauge theory with the scalar fields on AdS_5. In particular, from the relation (2.39), we see that marginal, relevant, and irrelevant scalar perturbations to the gauge theory corresponds to switching on massless, tachyonic, and massive scalar fields on the AdS_5 background.

The above discussions on the field-operator map in AdS_5/CFT_4 focus on the symmetry argument. But in fact, the duality may be formulated in general dimensions in terms of the AdS_{d+1}/CFT_d duality, and it can be show that analogous relation of (2.39) naturally arises from the asymptotic behavior of scalar field on the AdS_{d+1} background [42]. To see this, let us consider a free scalar field in the AdS_{d+1} spacetime, with the action

$$S = -\frac{C}{2} \int d^{d+1}x \sqrt{-g} \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) \ . \tag{2.40}$$

The coupling constant C may be derived from the reduction of the supergravity action, and it is not important here. The metric of AdS_{d+1} in the Poincaré coordinates is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\mu = \frac{L^2}{z^2} \left(dz^2 + \eta_{ij} dx^i dx^j \right) \ , \tag{2.41}$$

with $0 \leq i, j \leq (d-1)$ and $\eta_{ij} = \text{diag}(-, +, \cdots, +)$. The equation of motion for the scalar field reads,

$$\nabla^2 \phi - m^2 \phi = 0 \ , \tag{2.42}$$

with

$$\nabla^2 = \frac{1}{L^2} \left(z^2 \partial_z^2 - (d-1) z \partial_z + \eta^{ij} \partial_i \partial_j \right) \ . \tag{2.43}$$

To solve the equation of motion, we need to specify the boundary condition for the scalar field at $z = 0$. Since there is a second order singularity in the Laplacian, we make the ansatz that the scalar field behaves as $\phi(z,x) \sim \phi_0(x)z^{d-\Delta}$ as $z \to 0$. Insert this ansatz into the equation of motion, one find that at the leading order of z, the equation of motion gives rise to the analogous relation of (2.39),

$$m^2L^2 = \Delta(\Delta - d) \ . \tag{2.44}$$

The two roots of (2.44) are given by

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \ , \tag{2.45}$$

with $m^2L^2 \geq -\frac{d^2}{4}$ known as the *Breitenlohner–Freedman* (BF) bound [44]. Tachyonic scalar field with negative mass square above the BF bound is allowed here due to the curved AdS$_{d+1}$ spacetime. Since $\Delta_+ \geq \Delta_-$ under the BF bound, we set $\Delta \equiv \Delta_+$ and the boundary condition for the scalar field is $\phi(z,x) \sim \phi_0(x)z^{d-\Delta} = \phi_0(x)z^{\Delta_-}$ as $z \to 0$ with $\phi_0(x)$ fixed.

In order to understand the GKPW relation (2.36), it is necessary to evaluate the on-shell action for the scalar field. The standard method in the context of AdS/CFT is the so-called *holographic renormalization* procedure, which I will discuss in Chapter 3. In the holographic renormalization approach, one first expresses the field in the power series of z and solve them order by order. Due to the power series expansion, the action evaluated from the solution typically diverges as $z \to 0$. To remove those divergences, the regularization procedure and the additional counter terms are required. However, for the scalar field case, it is more straightforward to to obtain the full on-shell solution by using the integral kernel method in [42]. The idea is to first write the on-shell solution $\phi(z,x)$ as the following integral,

$$\phi(z,x) = \int d^dy K_\Delta(z;x,y)\phi_0(y) \ . \tag{2.46}$$

The function $K_\Delta(z;x,y)$ is the so-called *bulk-to-boundary propagator*, which needs to satisfy the $(\nabla^2 - m^2)K_\Delta(z;x,y) = 0$ and behave as a delta function closed to the AdS boundary,

$$\lim_{z\to 0}\left(z^{\Delta-d}K_\Delta(z;x,y)\right) = \delta^d(x-y) \ . \tag{2.47}$$

Such function is found to be [35]

$$K_\Delta(z;x,y) = C_\Delta\left(\frac{z}{z^2+|x-y|^2}\right)^\Delta \ , \tag{2.48}$$

where $|x-y|^2$ is the distance between two points and C_Δ is the normalization constant. Since (2.46) is the full solution, integrating by parts for the action yields the boundary term

$$\begin{aligned} S &= \frac{C}{2}\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon} d^dx\sqrt{\gamma}\phi n^\mu\partial_\mu\phi \ , \\ &= -\frac{C}{2}\lim_{\epsilon\to 0}\int d^dx\left(\frac{L}{\epsilon}\right)^{d-1}\phi(\epsilon,x)\partial_\epsilon\phi(\epsilon,x) \ , \end{aligned} \tag{2.49}$$

Here Σ_ϵ denotes the hypersurface at $z = \epsilon$ which will be pushed to the conformal boundary in the end. The normal vector to the hypersurface reads $n^\mu = \frac{z}{L}\left(\frac{\partial}{\partial_z}\right)^\mu$, and γ_{ij} is the induced metric. As $z \to \infty$, $\partial_\epsilon \phi(\epsilon, x)$ behaves as

$$\partial_\epsilon \phi(\epsilon, x) = \Delta C_\Delta \epsilon^{\Delta - 1} \int d^d y \frac{\phi_0(y)}{|x-y|^{2\Delta}} + \mathcal{O}(\epsilon^\Delta). \tag{2.50}$$

Using the property (2.47) and inserting (2.50) into (2.49) gives rise to the action

$$S = -\frac{\Delta C}{2} C_\Delta L^{d-1} \int d^d x \int d^d y \frac{\phi_0(x)\phi_0(y)}{|x-y|^{2\Delta}} \,. \tag{2.51}$$

To finally make contact with the GKPW relation (2.36), note that (2.50) is equivalent to

$$\phi_+(x) := \lim_{\epsilon \to 0} \left(\epsilon^{-\Delta} \phi(\epsilon, x) \right) = C_\Delta \int d^d y \frac{\phi_0(y)}{|x-y|^{2\Delta}} \,, \tag{2.52}$$

where the new field $\phi_+(x)$ can also be extracted from the functional variation of the action, i.e. $\phi_+(x) \sim -\frac{\delta S}{\delta \phi_0}$. If we think of ϕ_0 as a source in the dual gauge theory, then ϕ_+ should be interpreted as the expectation value of an operator O coupled with the source. In particular, the two-point function of the dual operator O behaves as

$$\langle O(x) O(y) \rangle = -\frac{\delta^2 S}{\delta \phi_0(x) \delta \phi_0(y)} \sim \frac{1}{|x-y|^{2\Delta}} \,. \tag{2.53}$$

This allows us to identify Δ as the conformal dimension of the dual operator. Hence, (2.44) is the relation between the mass of the scalar field in AdS_{d+1} with the conformal dimension of the dual operator in CFT_d.

Generalities of the AdS/CFT dictionary. In the above discussions on the scalar field, we have shown that the dynamics of a scalar field in AdS space indeed gives rise to the knowledge about the two-point function in the dual conformal field theory. In particular, we did not use any explicit information from the underling string theory or supergravity theory, except for the assumed coupling constant C. This gives us a first glance at the generalities of the AdS/CFT dictionary, which means that the duality may not rely on the string theory set-up. The procedure of "deriving" the AdS/CFT duality from the superstring theory by first principle is usually called the *top-down* approach. In the top-down approach, in principle we know all information about the two theories, such as the couplings and the operator/field contents. In contrast, there is also a *bottom-up* approach, in which one relates a gravity theory in the (asymptotically) AdS_{d+1} spacetime with a families of conformal field theories on the boundary. Such kind of CFTs are usually referred as the holographic CFTs. The cost for the generalities in the bottom-up approach is that we do not know the explicit information about the dual CFT, such as the Lagrangian and the full spectrum. In other words, we can say that in the bottom-up approach the AdS/CFT duality is not exact, and what we can learn about the dual holographic CFT from the gravity theory should be something universal. The scalar field we have discussed is just one example of the AdS/CFT dictionary in the bottom-up approach. There are other two important examples: the metric and the gauge fields in AdS are related to the stress tensor and conserved currents in the dual CFT. For a gravity theory in asymptotically AdS spacetime, the notion of isometry can be generalized to the asymptotic symmetry, which is defined via the invariance of asymptotic behavior of fields under certain diffeomorphism or gauge transformations. The asymptotic symmetry in the gravity theory is then identified with the global symmetry in the dual CFT, which I will discuss in detail in Chapter 3 and Chapter 5.

2.2 ENTANGLEMENT ENTROPY AND SYMMETRY RESOLUTION

In quantum field theory, we are usually mostly interested in calculating correlation functions of local observables and the associated scattering amplitudes. However, those quantities do not capture all interesting physical phenomenon in quantum system, such as entanglement. Entanglement is one of the key concepts in quantum information theory. It arises from the coherence of quantum states, and distinguishes quantum theories from classical ones. In a quantum system, the amount of the entanglement between two bipartite subsystems is measured by the *entanglement entropy*. Roughly speaking, this quantity describes the amount of the unknown information to the observers that only measure the domain of dependence of one subsystem. In the context of the AdS/CFT duality, entanglement plays a very special role. It was found by Ryu and Takayanagi in [22] that, in the semi-classical limit, the entanglement entropy in the boundary holographic CFT can be measured by the area of a particular minimal surface in the bulk AdS. This remarkable result relates the entanglement on the CFT side to the geometry of spacetime on AdS side, and motivates physicists to study the deeper mechanisms of the AdS/CFT duality from the quantum information perspective. More recently, there is another entanglement measure proposed in [27], the so-called *symmetry-resolved entanglement entropy* (SREE). It aims at characterizing the finer structure of the entanglement in a quantum field theory when additional internal symmetry is present. More precisely, the presence of the internal symmetry leads to an organization of the entanglement between two subsystems into different charge sectors. The SREE then quantifies the amount of entanglement encoded in these sectors. One of the goals of this thesis is to understand the SREE in the context of the AdS/CFT, and the purpose of this section is to introduce the above concepts.

2.2.1 *Entanglement entropy in QFTs*

Let me start by explaining entanglement entropy in a general quantum field theory. The Hilbert space of a quantum field theory is defined on a Cauchy slice Σ. The usual argument is that if we decompose the Cauchy slice into two disjoint regions, $\Sigma = A \cup B$, due to the locality of the quantum field theory, the Hilbert space will factorize into a tensor product, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The physics in the subsystem A is governed by the *reduced density matrix* ρ_A, obtained by taking trace for the density matrix ρ over the Hilbert space $\mathcal{H}_B$,

$$\rho_A = \mathrm{Tr}_B(\rho) \ . \tag{2.54}$$

Such a reduced density matrix can be mixed even for a pure state density matrix ρ. This indicates that the two disjoint regions A and B are entangled. The amount of entanglement between them is quantified by the von Neumann entropy of ρ_A, the so-called *entanglement entropy*,

$$S_{EE} = -\mathrm{Tr}(\rho_A \log \rho_A) \ , \tag{2.55}$$

where the trace here is taken over the Hilbert space $\mathcal{H}_A$ and the normalization $\mathrm{Tr}(\rho_A) = 1$ is assumed. In practice, it is hard to perform a straightforward calculation for the entanglement entropy in a quantum field theory. The difficulty arises from the $\log \rho_A$ term, which is generically non-local. The traditional way to solve this problem is that one can first consider the one-parameter family generalization of the entanglement entropy, the so-called *Rényi entropy*,

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n) \ , \tag{2.56}$$

and the entanglement entropy is obtained under the limit $n \to 1$,

$$S_{EE} = \lim_{n\to 1} S_n \ . \tag{2.57}$$

The technique for calculating the Rényi entropy in a quantum field theory is known as the *replica trick*. To briefly explain this trick, let us consider a vacuum state density matrix $\rho = |0\rangle\langle 0|$. In the configuration space, the elements of ρ can characterized by the wavefunctional Ψ on the Cauchy slice Σ,

$$\langle \phi_1|\rho|\phi_2\rangle = \frac{1}{Z_1}\Psi(\phi_1)\Psi(\phi_2)^* \ , \quad \Psi(\phi_i(x)) = \langle \phi_i(x)|0\rangle \ , \tag{2.58}$$

where $\phi_i(x)$ denotes the field configuration on Σ and Z_1 is a normalization constant such that $\mathrm{Tr}(\rho) = 1$. In the Euclidean path-integral formalism, if we consider Σ located at the time $\tau = 0$, then the wavefunctional $\Psi(\phi_1)$ is obtained by evaluating the path-intergral over the region $\tau < 0$ with the boundary condition $\phi(\tau = 0, x) = \phi_1(x)$,

$$\Psi(\phi_1(x)) = \int_{\tau=-\infty}^{\tau=0,\ \phi(\tau=0,x)=\phi_1(x)} \mathcal{D}\phi e^{-S[\phi]} \ . \tag{2.59}$$

Similarly, the complex conjugate $\Psi^*(\phi_2)$ is obtained from the path-integral over the region $\tau > 0$ with the boundary condition $\phi(\tau = 0, x) = \phi_2(x)$. However, applying the path-integral representation of the wavefunctional to the density matrix leads to inconsistency since it is invalid to have two different boundary conditions on the same Cauchy slice. The solution to this problem is to define the time coordinates of the two boundaries as $\tau = 0^\pm$ with the limit $0^\pm \to 0$. So in the end, we can express the elements of the density matrix as

$$\langle\phi_1|\rho|\phi_2\rangle = \frac{1}{Z_1}\int \mathcal{D}\phi e^{-S[\phi]} \prod_{x\in\Sigma} \delta(\phi(\tau = 0^+, x) - \phi_2(x)) \\ \times \prod_{x\in\Sigma} \delta(\phi(\tau = 0^-, x) - \phi_1(x)) \ . \tag{2.60}$$

Taking the trace for ρ over the Hilbert space $\mathcal{H}$ means identifying $\phi_1(x)$ with $\phi_2(x)$ and summing over the configuration space of the field on Σ. From the normalization condition $\mathrm{Tr}(\rho) = 1$, we hence identify Z_1 as the vacuum partition function, $Z_1 = \int \mathcal{D}\phi e^{S[\phi]}$.

Intuitively, one can think of the trace over the Hilbert space as gluing the two boundaries at $\tau = 0^\pm$ along the corresponding spatial region. Therefore, to obtain the reduced density matrix ρ_A, we need to glue the boundaries along the region B, but leave the region A as a cut in the whole spacetime. In terms of the path-integral, the elements of the reduced density matrix are then expressed as

$$\langle\phi_1^A|\rho_A|\phi_2^A\rangle = \frac{1}{Z_1}\int \mathcal{D}\phi e^{-S[\phi]} \prod_{x\in A} \delta(\phi(\tau = 0^+, x) - \phi_2^A(x)) \\ \times \prod_{x\in A} \delta(\phi(\tau = 0^-, x) - \phi_1^A(x)) \ . \tag{2.61}$$

Here $\phi_1^A(x)$ and $\phi_2^A(x)$ are the field configurations only supported on the region A. To obtain $\mathrm{Tr}(\rho_A^n)$, we take the product of n copies of (2.61),

$$\langle\phi_1^A|\rho_A|\phi_2^A\rangle \cdots \langle\phi_{2n-1}^A|\rho_A|\phi_{2n}^A\rangle \ , \tag{2.62}$$

and sum over the states under the boundary conditions,

$$\phi_{2n}^A(x) = \phi_1^A(x) \ , \quad \phi_{2l}^A(x) = \phi_{2l+1}^A(x) \ , \quad \text{for } x \in A \ . \tag{2.63}$$

Now the trick is that those boundary conditions in (2.63) can be automatically implemented in the path integral if we consider that the quantum field theory is not defined on the original spacetime, but on a replica manifold $\mathcal{R}_n$. The replica manifold $\mathcal{R}_n$ is constructed in the following way. We first take n copies of the original spacetime, and cut the region A in each of them. We may represent the cuts as

$$\mathcal{C}^{(l)} = A_+^{(l)} \cup A_-^{(l)} \ , \quad l = 1, \cdots, n \ , \tag{2.64}$$

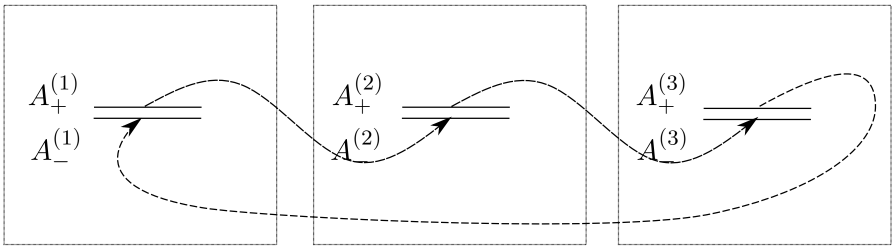

Figure 1: A replica surface with $n = 3$.

where $A_+^{(l)}$ and $A_-^{(l)}$ are the two boundaries located at $\tau = 0^\pm$ and they obey

$$A_+^{(l)} \cap A_-^{(l)} = \partial A \ . \tag{2.65}$$

Then the replica manifold is obtained by cyclically gluing the n cuts together, as shown in Figure 1,

$$A_+^{(l)} = A_-^{(l+1)} \ , \quad A_+^{(n)} = A_-^{(1)} \ . \tag{2.66}$$

In this way, a quantum field ϕ defined on $\mathcal{R}_n$ naturally fulfills the conditions in (2.63), and as a consequence, we arrive at

$$\mathrm{Tr}(\rho_A^n) = \frac{1}{Z_1^n} \int_{\mathcal{R}_n} \mathcal{D}\phi e^{-S[\phi]} = \frac{Z_n}{Z_1^n} \ , \tag{2.67}$$

where Z_n is the partition function of the quantum field theory on the replica manifold $\mathcal{R}_n$.

It is worth mentioning that although the replica trick provides a useful way for us to calculating the entanglement entropy, in practice, the application of this method is still very much restricted since calculating the partition function of a quantum field theory on the replica manifold is generically a hard problem. However, in the two-dimensional conformal field theories, which are mainly concerned in this thesis, the task is much simplified due to the power of conformal symmetry as well as our understanding of the theory of Riemann surfaces. For instance, consider the entangling region A as a single interval on the Cauchy slice of $\mathbb{R}^2$. In this case, the leading term in the cut-off expansion of entanglement entropy takes the universal form in any two-dimensional CFTs,

$$S_{EE} = \frac{c}{3} \log\left(\frac{L}{\epsilon}\right) \ , \tag{2.68}$$

where c is the central charge of the CFT and L is the length of A. ϵ is a short distance UV cut-off. Its appearance reflects the fact that the entanglement entropy in any quantum field theories diverges due to the infinite many local degrees of freedom. More details on the entanglement entropy in two-dimensional CFTs will be discussed in Section 3.2.

So far, I focused on the formal path-integral formulation of entanglement entropy in QFT. In the next section, I will discuss the gravity dual of entanglement entropy in holographic CFTs [22], and explain how to derive it based on the replica trick and the AdS/CFT duality.

2.2.2 *Entanglement entropy in holography*

Although the holographic principle was originally motivated from the quantum information perspective of a gravitational system, in the early days after the discovery of the AdS/CFT duality, rare attention was paid to this direction. The discussion of entanglement entropy in the context of AdS/CFT was first given by Ryu and Takayanagi (RT) in [22]. They started by an observation that the entanglement entropy in the boundary CFT_2 has a beautiful geometric description in the bulk AdS_3, i.e. the length of the minimal geodesic. This observation generalizes the relation between the Bekenstein-Hawking (BH) entropy and the area of the black hole horizon. Since BH entropy formula is universal in all dimensions, they proposed that an analogous relation between the entanglement entropy and the geometry holds in general $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$. This proposal, known as the *RT proposal*, states that, consider a spatial subregion A on the boundary holographic CFT_d, then the entanglement entropy between A and its complement in CFT_d is given by [22],

$$S_{EE}(A) = \frac{\mathrm{Area}(\gamma_A)}{4G_{d+1}} \ . \tag{2.69}$$

Here G_{d+1} denotes the Newton constant in the dual $(d+1)$-dimensional gravity theory, and γ_A is the so-called *Ryu-Takayanagi (RT) surface*, a minimal codimension-two surface in the dual (asymptotically) AdS_{d+1} spacetime, with the boundary condition, i.e. $\partial\gamma_A = \partial A$. The equation (2.69) is known as the *Ryu-Takayanagi (RT) formula*. In some cases, the bulk extremal codimension-two surface with the boundary ∂A may not be unique, and γ_A corresponds to the one with the smallest area [45]. Furthermore, if we consider an AdS_{d+1} black hole, the gravity dual of a thermal state in CFT_d, and choose A as the entire Cauchy slice on the boundary, then (2.69) recovers the BH entropy (1.1). This result is what we would expect, since the entanglement entropy in this case is the thermal entropy of the thermal state in CFT_d.

Generalizations. The RT formula (2.69) only applies to static cases, where the entanglement entropy in CFT_d does not evolve along time, and the bulk asymptotically AdS_{d+1} spacetime is time-reversal invariant. A dynamical generalization of the RT formula has been proposed in [46], and is known as the Hubeny-Rangamani-Takayanagi (HRT) formula. In addition, from the top-down perspective of the AdS/CFT duality, the RT formula (2.69) only gives the classical answer to the entanglement entropy. There are two sources of corrections to it, the stringy corrections of order of $1/\lambda$ and the quantum gravitational corrections of order of $1/N$. In $1/\lambda$ perturbation theory, the classical stringy corrections to the supergravity need to be included. This can be done by writing down the higher derivative modifications to the Einstein-Hilbert action that arise from the worldsheet stringy corrections [47]. Within the higher derivative gravity theories, modifications of the RT formula can be analyzed [48–51]. The results can be viewed as generalizations of the *Wald entropy* formula for the black hole in the higher derivative gravity theories [52]. The leading $1/N$ quantum gravitational correction to the RT formula was originally considered in [53], where the authors included the entanglement of bulk degrees of freedom separated by the RT surface γ_A. This idea eventually led to the quantum extremal surface proposal in [54], which played an important role in solving the black hole information paradox in particular setups [55–57].

Deriving the RT formula in $\mathbf{AdS}_{d+1}$ gravity. A proof for the RT formula (2.69) has been given in [58] by using the gravitational replica technique. The proof is relied on the assumption of the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality. To briefly explain the main idea of the proof, for simplicity, here I consider a holographic CFT_d, of which the dual theory is the pure AdS_{d+1} gravity, without other matter fields.

The starting point of the proof is to consider the gravity dual of the CFT partition function Z_n defined on the boundary replica manifold $\mathcal{R}_n$. By the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ dictionary, the metric of the dual bulk manifold $\mathcal{M}_n$ in principle can be obtained by solving the equation of motion under the boundary condition imposed by its conformal boundary $\mathcal{R}_n$[5]. Finding such a solution is a very hard problem in general $(d+1)$ dimensions, and currently it is only solvable in the case of $d=2$, as I will discuss in Chapter 3. In order to circumvent this problem, the authors in [58] turned to consider the quotient of $\mathcal{M}_n$. One notices that there is a discrete $\mathbb{Z}_n$ symmetry (isometry) on $\mathcal{R}_n$. It is then natural to assume that such a symmetry is also preserved by $\mathcal{M}_n$. We denote the quotient manifold as

$$\tilde{\mathcal{M}}_n = \mathcal{M}_n/\mathbb{Z}_n \ . \tag{2.70}$$

Consequently, the gravity partition functions on $\mathcal{M}_n$ and $\tilde{\mathcal{M}}_n$ are related as

$$Z_n \equiv Z[\mathcal{M}_n] = \left(Z[\tilde{\mathcal{M}}_n]\right)^n \ , \tag{2.71}$$

Hence, from the holographic point of view, the Rényi entropy in the boundary CFT_d can be expressed as

$$S_n = \frac{n}{1-n}\log\left(\frac{Z[\tilde{\mathcal{M}}_n]}{Z[\mathcal{M}_1]}\right) = \frac{n}{n-1}\left(I[\tilde{\mathcal{M}}_n] - I[\mathcal{M}_1]\right) \ . \tag{2.72}$$

where $I[\tilde{\mathcal{M}}_n]$ and $I[\mathcal{M}_1]$ are the Euclidean actions evaluated on those two manifolds. In $n\to 1$ limit, we express the entanglement entropy as

$$S_{EE} = \partial_n I[\tilde{\mathcal{M}}_n]|_{n=1} \ . \tag{2.73}$$

5 Note that such a bulk manifold may not be unique due to the additional spin structure of the AdS manifold [42]. In Chapter 3, I will review such issues in the AdS_3 gravity.

If we think of $\tilde{\mathcal{M}}_n$ as the one-parameter family of deformations of $\tilde{\mathcal{M}}_1 \equiv \mathcal{M}_1$, then (2.73) is just the first order variation of the action around the on-shell background $\mathcal{M}_1$.

Before calculating (2.73), we first need to clearly understand the relations between $\tilde{\mathcal{M}}_n$ and $\mathcal{M}_1$. Obviously, both of the two manifolds satisfy the same boundary condition, since $\mathcal{R}_n/\mathbb{Z}_n = \mathcal{R}_1$ by definition. However, inside the bulk, since $\mathcal{M}_n$ is a smooth on-shell solution to the equation of motion (Einstein's equation), the $\mathbb{Z}_n$ fixed points of $\mathcal{M}_n$ form a conical defect surface in the quotient space $\tilde{\mathcal{M}}_n$. Such a defect surface do not exist in the smooth manifold $\mathcal{M}_1$, and we may think of $\tilde{\mathcal{M}}_n$ as an one parameter family of deformations of $\mathcal{M}_1$, with $(n-1)$ as the deformation parameter.

The dimension of the conical defect surface is $(d-2)$. This can be understood in the following way. The set of the $\mathbb{Z}_n$ fixed points on $\mathcal{R}_n$ corresponds to the boundary ∂A of the entangling region. This is because that all components of the n cuts share the same boundary ∂A, as shown in (2.65). Under the gluing condition (2.66), only the points on ∂A are invariant under the $\mathbb{Z}_n$ transformation. Those fixed points need to extend into the bulk $\mathcal{M}_n$, hence form a codimension-two surface with the boundary ∂A. Thus, we conclude that the conical defect in the quotient space $\tilde{\mathcal{M}}_n$ is a codimension-two surface anchored on ∂A.

Now let us turn to calculate (2.73). When doing such a calculation, we must be very careful. A crucial statement made in [58] is that we should not include the contribution from the codimension-two conical defect into the action $I[\tilde{\mathcal{M}}_n]$. The reason is that the replica manifold $\mathcal{M}_n$ is entirely smooth and there is no contribution to $I[\mathcal{M}_n] = nI[\tilde{\mathcal{M}}_n]$ from the fixed points. To achieve this purpose, we can choose a codimension-one tube $\mathcal{B}_n$ encircling around the defect, and evaluate the action on the spacetime region outside $\mathcal{B}_n$. The radius of the tube, denoted as ϵ, can be sent to zero at the end. With this approach, the result for (2.73) can be written as [58]

$$S_{EE} = \frac{1}{16\pi G_{d+1}} \int_{\mathcal{B}_n} d^{d-1}x \sqrt{\gamma} n^{\mu} \left(\nabla^{\nu} \partial_n g_{\mu\nu} - g^{\nu\rho} \nabla_{\nu} \partial_n g_{\nu\rho} \right) \ , \tag{2.74}$$

where n^{μ} is the normal vector on $\mathcal{B}_n$ and γ_{ab} is the induced metric on it. Note that there is no bulk contribution to the entanglement entropy, because the first order variation of the action around the on-shell background $\mathcal{M}_1$ vanishes. The contribution from the conformal boundary was assumed to be vanishing, since far away from the defect, the deformation of the metric may be neglected.

To evaluate the boundary term in (2.74), we use Gaussian normal coordinates. In the region nearby the conical defect, the metric on $\tilde{\mathcal{M}}_n$ takes the following form in $n \to 1$ limit [58],

$$\begin{aligned} ds^2 &= e^{2\rho}(dr^2 + r^2 d\phi^2) + g_{ij} dy^i dy^j + \cdots \ , \\ g_{ij} &= h_{ij} + K^{(1)}_{ij} r \cos\phi + K^{(2)}_{ij} r \sin\phi + \mathcal{O}(r^2) \ , \\ e^{2\rho} &= \frac{1}{n^2} r^{\frac{2}{n}-2} \approx r^{2-2n} \ , \quad n \to 1 \ , \end{aligned} \tag{2.75}$$

Here (r, ϕ) are the local polar coordinates transverse to the conical defect, with the period $\phi \sim \phi + 2\pi$ and y^i are the coordinates on the conical defect. The defect is located at $r = 0$ and h_{ij} is the induced metric on it, independent of r and ϕ. $K^{(1)}_{ij}$ and $K^{(2)}_{ij}$ are the two extrinsic curvatures of the defect associated with the two orthogonal directions in the transverse space,

$$\begin{aligned} K^{(l)}_{ij} &= \mathcal{L}_{n_l} g_{ij} = \frac{\partial g_{ij}}{\partial x^l} \ , \quad l = 1, 2 \ , \quad \text{for } r \to 0 \ , \\ x^1 &= r\cos\phi \ , \quad x^2 = r \sin\phi \ . \end{aligned} \tag{2.76}$$

The conformal factor $e^{2\rho}$ is induced from the quotient procedure. In the transverse space of the fixed points in $\mathcal{M}_n$, the metric is flat in the region nearby the fixed points, i.e. $d\tilde{s}^2 = dz d\bar{z}$. The quotient procedure can be achieved by the conformal coordinate transformation $z = w^{1/n}$, with $w = re^{i\phi}$ and $\phi \sim \phi + 2\pi$. Thus, in (r, ϕ)-coordinates, the metric on the transverse space takes the form of

$$d\tilde{s}^2 = dz d\bar{z} = e^{2\rho}(dr^2 + r^2 d\phi^2) \ . \tag{2.77}$$

To determine the two extrinsic curvatures, we insert the metric (2.75) into the Einstein's equation, which in $r \to 0$ limit gives rise to two conditions on the extrinsic curvatures[6] [58],

$$K^{(l)}_{ij} h^{ij} = 0 \ , \quad l = 1, 2 \ . \tag{2.78}$$

These indicate that the defect is in fact an extremal surface in $\tilde{\mathcal{M}}_n$. Under the conditions (2.78), we can evaluate (2.74) by inserting (2.75) and taking the limit $r = \epsilon \to 0$ for the radius of the tube $\mathcal{B}_n$. The result gives the RT formula (2.69).

2.2.3 *Symmetry resolution of entanglement*

Now, let me turn to introduce the symmetry resolution of the entanglement entropy. This topic arises from a natural question: what does internal global symmetry in QFT tell us about the entanglement entropy? Or more generally, what is the role of internal global symmetry

6 Although the quotient spacetime is off-shell at the positions of conic defect. However, as we are working in the local polar coordinates, the Einstein's equation is expected to still hold in $r \to 0$ limit, similar to the case of Coulomb potential.

in QFT from quantum information perspective? To answer the former question, I will restrict to the $U(1)$ case in the following discussions.

Consider a d-dimensional quantum field theory with an internal $U(1)$ symmetry. The $U(1)$ charge operator Q is defined by integrating the time component of the $U(1)$ current over the Cauchy slice Σ. When decomposing the Cauchy slice into two joint subregions A and B, in order to measure the $U(1)$ charges supported on A and B, we can define the following subregion charge operators Q_A and Q_B,

$$Q_A = \int_A dx^{d-1} J^0 \ , \quad Q_B = \int_B dx^{d-1} J^0 \ . \tag{2.79}$$

where Q_A only acts on the Hilbert space $\mathcal{H}_A$ while Q_B acts on $\mathcal{H}_B$. So, in the matrix representation, the $U(1)$ charge operator Q is related to the subregion charge operators as

$$Q = Q_A \otimes \mathbb{1} + \mathbb{1} \otimes Q_B \ . \tag{2.80}$$

Now we consider a pure density matrix $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is an eigenstate of the charge operator Q. Taking the trace of $[\rho, Q]$ over the Hilbert space $\mathcal{H}_B$ yields

$$\begin{aligned} \mathrm{Tr}_B[\rho, Q] &= \sum_n \left(\langle b_n|\psi\rangle\langle\psi|Q|b_n\rangle - \langle b_n|Q|\psi\rangle\langle\psi|b_n\rangle\right) \\ &= \sum_n \left(\langle b_n|\psi\rangle\langle\psi|b_n\rangle Q_A - Q_A\langle b_n|\psi\rangle\langle\psi|b_n\rangle\right) \\ &= [\rho_A, Q_A] \end{aligned} \tag{2.81}$$

where $\{|b_n\rangle\}$ is the basis of eigenstates of Q_B in $\mathcal{H}_B$. Since $|\psi\rangle$ is an eigenstate of Q, we have $[\rho, Q] = 0$, and as a consequence,

$$[\rho_A, Q_A] = 0 \ . \tag{2.82}$$

This indicates that in the basis of eigenstates of Q_A, the reduced density matrix ρ_A is block diagonalized,

$$\rho_A = \bigoplus_q \rho_A(q) \ , \tag{2.83}$$

and each block $\rho_A(q)$ corresponds to the charge sector with the eigenvalue q of the subregion charge operator Q_A. The *symmetry-resolved entanglement entropy* (SREE) is defined as the von Neumann entropy of the q-sector [27],

$$S(q) = -\mathrm{Tr}\left[\frac{\rho_A(q)}{P(q)} \log\left(\frac{\rho_A(q)}{P(q)}\right)\right] \ . \tag{2.84}$$

Here $P(q) = \mathrm{Tr}[\rho_A(q)]$ is a normalization constant and it can be interpreted as the probability of detecting q units of $U(1)$ charge under a measurement operated in the subsystem A. The relation between the SREE and the entanglement entropy is described by the following identity,

$$S_{EE} = \sum_q P(q)S(q) - \sum_q P(q)\log P(q) = \langle S(q)\rangle_q + S_N \ , \quad (2.85)$$

which can be straightforwardly derived from the definition (2.84). The first term $\langle S(q)\rangle_q$ averages the SREE over all charge sectors, and it is known as the *configurational entropy*, which is closely related to the operationally accessible entanglement entropy [59–61]. The second term S_N characterizes the *Shannon entropy* of the charge distribution, and it is usually called the *number entropy* or *fluctuation entropy* [62].

To study the SREE in quantum field theories, it is necessary to also define the *symmetry-resolved Rényi entropy*, given by

$$S_n(q) = \frac{1}{1-n}\log \mathrm{Tr}\left[\left(\frac{\rho_A(q)}{P(q)}\right)^n\right] \ . \quad (2.86)$$

which in $n \to 1$ limit gives rise to the SREE. A useful decomposition of the symmetry-resolved Rényi entropy was pointed out in [62],

$$S_n(q) = S_n + \frac{1}{1-n}\log\frac{P_n(q)}{P(q)^n} \ . \quad (2.87)$$

where $P_n(q)$ is the probability distribution of the q-sector in ρ_A^n,

$$P_n(q) = \mathrm{Tr}[\rho_A(q)^n]/\mathrm{Tr}(\rho_A^n) \ . \quad (2.88)$$

Hence, taking the limit $n \to 1$, one can also expressed the SREE via the entanglement entropy and the distribution P_n,

$$S(q) = S_{EE} + \lim_{n\to 1}\frac{1}{1-n}\log\frac{P_n(q)}{P(q)^n} \ . \quad (2.89)$$

Projector and charged moments. In order to incorporate the SREE in the path-integral formulation, we first define a projector Π_q via a Fourier transformation of the subregion charge operator (2.79)[7],

$$\Pi_q = \delta(Q_A - q) = \int_{-\infty}^{\infty}\frac{d\mu}{2\pi}e^{i\mu(Q_A-q)} \ , \quad (2.90)$$

so that $\Pi_q\rho_A = \rho_A(q)$ and $\Pi_q\rho_A^n = \rho_A(q)^n$. We further define the so-called *charged moments* $Z_n[\mu]$ via the path-integral on the replica manifold $\mathcal{R}_n$,

$$Z_n[\mu] = \int_{\mathcal{R}_n}\mathcal{D}\phi e^{-S[\phi]}e^{i\mu Q_A} \ , \quad (2.91)$$

which is related to the reduced density as

$$\mathrm{Tr}[\rho_A^n e^{i\mu Q_A}] = Z_n[\mu]/Z_1^n \ . \quad (2.92)$$

7 Note that here we have assumed that the spectrum of Q_A is continuous. When the eigenvalues of Q_A take integers, one needs to change the range of the integration in (2.90) to be $\mu \in [-\pi, \pi]$

Then, by (2.67) and (2.92), the distribution P_n defined in (2.88) can be expressed as

$$P_n(q) = \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} e^{-i\mu q} \frac{Z_n[\mu]}{Z_n[0]} \ . \tag{2.93}$$

Using an idea presented in [29], one can further simplify the procedure for calculating the SREE by defining an effective action,

$$S_n[\mu] = -\log\left(Z_n[\mu]/Z_n[0]\right) \ , \tag{2.94}$$

so that

$$P_n(q) = \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} e^{-i\mu q - S_n[\mu]} \ . \tag{2.95}$$

The effective action captures the μ-dependence of the charged moments and can be obtained from the expectation value of the subregion charge operator Q_A,

$$\partial_\mu S_n[\mu] = -i\langle Q_A\rangle_{n,\mu} \ , \tag{2.96}$$

with the boundary condition $S_n[0] = 0$. Here $\langle \cdots \rangle_{n,\mu}$ means the expectation value of operators on the replica manifold $\mathcal{R}_n$, in presence of the insertion $e^{i\mu Q_A}$. Hence, by (2.89), one can obtain the SREE by calculating the entanglement entropy S_{EE} and the effective action separately. This method is called the *generating function method* in [29], and it is proved to be quite efficient in the context of holography. This is because, in general cases, calculating the $\langle Q_A\rangle_{n,\mu}$ is much easier than evaluating the charged moments $Z_n[\mu]$ on the replica manifold. Also, entanglement entropy in holography is directly given by the RT formula.

3 HOLOGRAPHIC AdS_3 GRAVITY

AdS_3 gravity as a toy holographic bottom-up model plays an important role in the context of AdS/CFT duality. While this model is much simpler than its higher dimensional counterparts due to its topological nature, it provides a perfect playground for studying the deeper mechanisms of the AdS/CFT duality. In this chapter, I will review various important aspects of AdS_3 gravity from the holographic point of view, as well as their applications to the holographic Rényi and entanglement entropy.

In Section 3.1, some important aspects of AdS_3 gravity will be introduced, aimed at building up its connections with the two-dimensional conformal field theory. Based on those pre-knowledges, in Section 3.2, I will review the calculations of the holographic entanglement entropy from both the holographic and CFT perspectives. As I will show later, the calculations in those two perspectives share some similar features, providing us the deeper connections between the AdS_3 gravity and the dual CFT_2. For instance, I will show that the projective structure on the conformal boundary of a AdS_3 space determines the dominate channel of vacuum conformal block that contributes to the partition function of the dual CFT_2. The lessons we learn in this section will be further explored in the holographic $U(1)$ Chern-Simons-Einstein gravity in Chapter 4, for calculating the holographical $U(1)$ symmetry-resolved entanglement.

3.1 ASPECTS OF AdS_3 GRAVITY

In this section, I will review some useful aspects of AdS_3 gravity, including the family of exact solutions to Einstein's equation of AdS_3 gravity, the Schottky uniformization, the effective action of AdS_3 gravity, and the asymptotic symmetry algebra. The discussions in this section are applicable for general higher genus handlebody AdS_3 solutions with a single conformal boundary.

3.1.1 *Generic vacuum solutions*

In Euclidean signature, the action for the AdS_3 gravity is given by

$$S_G = \frac{1}{16\pi G_3}\int_{\mathcal{M}} d^3x\sqrt{G}\left(R+\frac{2}{l^2}\right) - \frac{1}{8\pi G_3}\int_{\partial\mathcal{M}} d^2x\sqrt{h}K \ , \quad (3.1)$$

where G_3 is the Newton constant in three dimensions, and l is the AdS radius. The boundary term in (3.1) is the standard Gibbons-Hawking term. As first pointed out in [63], Einstein's gravity theories in three dimensions can also be formulated as Chern-Simons theories. This reveals that AdS_3 gravity in three dimensions is topological, and all the dynamical degrees of freedom at the theory are encoded in the boundary.

The study of the metric solutions to the Einstein equations is usually performed in the Fefferman-Graham gauge [64],

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} g_{ij}(\rho, x) dx^i dx^j \ , \tag{3.2}$$

where ρ is the radial coordinate, with $\rho = 0$ as the boundary $\partial\mathcal{M}$ at the spatial infinity. Inserting the above metric into the Einstein equation yields several differential equations for $g_{ij}(\rho, x)$, one of those being

$$\partial_\rho^3 g_{ij}(\rho, x) = 0 \ , \tag{3.3}$$

which implies that the Taylor expansion of the g_{ij} in ρ is truncated at ρ^2 order,

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} \left(g_{(0)ij} + \rho g_{(2)ij} + \rho^2 g_{(4)ij} \right) dx^i dx^j \ , \tag{3.4}$$

The metric on the conformal boundary of AdS_3 space is given by $d\hat{s}^2 = g_{(0)ij} dx^i dx^j$, which is called the *boundary metric*. Solving the remaining Einstein equations order by order in ρ, one can show that $g_{(4)}$ are determined by the first two terms, i.e. $g_{(0)}$ and $g_{(2)}$,

$$g_{(4)ij} = \frac{1}{4} g_{(2)ik} g_{(0)}^{kl} g_{(2)kj} \ , \tag{3.5}$$

where $g_{(0)}^{ij}$ is the inverse matrix of $g_{(0)ij}$. The equations of motion for $g_{(2)}$ are given by [65]

$$\mathrm{Tr}[g_{(2)} g_{(0)}^{-1}] = -\frac{R[g_{(0)}]}{2} \ , \ \nabla^i \left(g_{(2)ij} + \frac{g_{(0)ij} R[g_{(0)}]}{2} \right) = 0 \ , \tag{3.6}$$

where $R[g_{(0)}]$ denotes the Ricci curvature for the boundary metric $g_{(0)}$ and ∇_i is the covariant derivative associated with $g_{(0)}$. To solve these equations (3.6), we can locally express the boundary metric in the *isothermal coordinates*,

$$d\hat{s}^2 = e^{2\phi(z,\bar{z})} dz d\bar{z} \ , \tag{3.7}$$

where the Weyl factor $\phi(z, \bar{z})$ is a real function on the conformal boundary. Then, the non-vanishing Christoffel symbols are given by

$$\Gamma^z_{zz} = 2\partial_z \phi \ , \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}} \phi \ . \tag{3.8}$$

The Ricci scalar of the metric $g_{(0)}$ is given by

$$R[g_{(0)}] = -2\nabla_i \nabla^i \phi = -8e^{-2\phi} \partial_z \partial_{\bar{z}} \phi \ . \tag{3.9}$$

Then, expanding (3.6), one obtains

$$
\begin{aligned}
g_{(2)z\bar{z}} &= g_{(2)\bar{z}z} = \partial_z\partial_{\bar{z}}\phi \ , \\
\partial_{\bar{z}} g_{(2)zz} &= \partial_{\bar{z}}\partial_z^2\phi - 2\partial_z\phi\partial_z\partial_{\bar{z}}\phi \ , \\
\partial_z g_{(2)\bar{z}\bar{z}} &= \partial_z\partial_{\bar{z}}^2\phi - 2\partial_{\bar{z}}\phi\partial_z\partial_{\bar{z}}\phi \ .
\end{aligned} \tag{3.10}
$$

Solutions of $g_{(2)}$ to above equations can be written in a covariant form,

$$
g_{(2)ij} = \alpha_{ij} + t_{ij} \tag{3.11}
$$

where α_{ij} is the ϕ-dependent, given by

$$
\alpha_{ij} = \nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j\phi - \frac{1}{2}g_{(0)ij}\nabla_k\phi\nabla^k\phi \ , \tag{3.12}
$$

and t_{ij} is ϕ-independent and satisfies

$$
\partial_{\bar{z}} t_{zz} = 0 \ , \quad \partial_z t_{\bar{z}\bar{z}} = 0 \ , \quad t_{z\bar{z}} = t_{\bar{z}z} = 0 \ . \tag{3.13}
$$

Notice that (3.13) imply that $t_{zz}dz^2$ and $t_{\bar{z}\bar{z}}d\bar{z}^2$ are holomorphic (anti-holomorphic) quadratic differentials on the conformal boundary. As pointed out in [66, 67], while adding arbitrary quadratic differential to $g_{(2)}$ preserves the equation of motion locally, the corresponding three-dimensional geometry typically contains conical defects inside the bulk. We are only interested in the exact solutions to Einstein's equation, hence we simply set $t_{ij} = 0$. Then, $g_{(2)}$ are fixed by ϕ as

$$
g_{(2)ij} = \nabla_i\phi\nabla_j\phi + \nabla_i\nabla_j\phi - \frac{1}{2}g_{(0)ij}\nabla_k\phi\nabla^k\phi \ , \tag{3.14}
$$

which in components read

$$
\begin{aligned}
g_{(2)zz} &= \partial_z^2\phi - (\partial_z\phi)^2 = -T^\phi \ , \\
g_{(2)\bar{z}\bar{z}} &= \partial_{\bar{z}}^2\phi - (\partial_{\bar{z}}\phi)^2 = -\bar{T}^\phi \ , \\
g_{(2)z\bar{z}} &= \partial_z\partial_{\bar{z}}\phi = -R^\phi,
\end{aligned} \tag{3.15}
$$

Combining the above results, one obtains the full solution with given boundary metric $g_{(0)}$, given by [67]

$$
ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{l^2}{\rho}e^{2\phi}\left|dz - \rho e^{-2\phi}\left(R^\phi dz + \bar{T}^\phi d\bar{z}\right)\right|^2 \ . \tag{3.16}
$$

I call such a solution as a *vacuum AdS_3 solution*, where "vacuum" means that it is completely smooth in the bulk[1].

1 In fact, from the dual CFT point of view, t_{ij} can be understood as the stress tensor sourced by insertions of fields. When t_{ij} vanishes, the state in the dual CFT is the vacuum defined on the curved background with metric $g_{(0)}$. Thus, the vacuum AdS_3 really means the geometry dual to the vacuum state in the boundary CFT.

Relation to Bañados metric. The metric (3.16) takes a different form from the famous Bañados geometry [66], given by

$$ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{l^2}{\rho}\left|dw - \rho\bar{\mathcal{L}}d\bar{w}\right|^2 \ , \tag{3.17}$$

However, in the special case where $e^{2\phi} = |dw/dz|^2$ and w is a holomorphic function of z, we have

$$T^\phi = (\partial_z\phi)^2 - \partial_z^2\phi = -\frac{1}{2}\{w; z\} \ . \tag{3.18}$$

Here $\{\ ;\ \}$ is the Schwarzian derivative defined as

$$\{f; z\} = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right) \ , \quad \forall f = f(z, \bar{z}) \ . \tag{3.19}$$

There is a chain rule for the Schwarzian derivative, given by

$$\{f; z\} = \{f; h\}\left(\frac{\partial h}{\partial z}\right)^2 + \{h; z\} \ , \quad \text{iff } \frac{\partial f}{\partial \bar{h}}\frac{\partial \bar{h}}{\partial z} = 0 \ . \tag{3.20}$$

Then, it is straightforward to show that under a boundary coordinate transformation $z \to w$, (3.16) is transformed to the Bañados form (3.17), with

$$\mathcal{L} = \left(\frac{dz}{dw}\right)^2 T^\phi = \frac{1}{2}\{z; w\} \ , \tag{3.21}$$

where the chain rule (3.20) is used for $f = z$ and $h = w$.

Local transformation to Poincaré AdS$_3$. There is also a local coordinates transformation relating the metric (3.16) to the Euclidean Poincaré AdS$_3$,

$$ds^2 = \frac{l^2}{\xi^2}(d\xi^2 + dyd\bar{y}) \ , \tag{3.22}$$

given by [67],

$$\xi = \frac{\sqrt{\rho}e^{-\phi}}{1 + \rho e^{-2\phi}|\partial_z\phi|^2} \ , \quad y = z + \frac{\rho e^{-2\phi}\partial_{\bar{z}}\phi}{1 + \rho e^{-2\phi}|\partial_z\phi|^2} \ . \tag{3.23}$$

This transformation is regarded as a finite version of the well-known *Penrose-Brown-Henneaux (PBH) diffeomorphism* [35]. It acts on the boundary coordinates of Poincaré AdS$_3$ as an identity map $z = y$ in the limit $\rho \to 0$, and generates the Weyl factor in the boundary metric of (3.16) from Poincaré AdS$_3$.

Bulk dual of conformal transformation in CFT. In the special case $e^{2\phi} = |dw/dz|^2$, the combination of the finite PBH diffeomorphism and the boundary coordinate transformation locally maps the Poincaré AdS$_3$ (3.22) to the Bañados geometry (3.17). This is the bulk transformation dual to the local conformal transformation in the dual CFT, which consists of a Weyl transformation and a compensate boundary coordinates transformation.

3.1.2 *Global aspects from Schottky uniformization*

So far, I have discussed the exact solution (3.16) to Einstein's equation from the local point of view. In this section, I will discuss the global issue of the metric (3.16) by explaining its geometric origin. The fact that any solution to AdS_3 gravity is locally AdS_3, implies that it is always locally maximally symmetric, with six local Killing vectors. This locally indistinguishable feature motivates one to construct AdS_3 solutions with non-trivial topology by considering the quotient of the Euclidean Poincaré AdS_3 [68–70], denoted as $\mathbb{H}_3$ for shorthand. The metric of $\mathbb{H}_3$ is given in (3.22) and the boundary of $\mathbb{H}_3$ located at $\xi = 0$ is considered as a Riemann sphere $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$.

Notion of quotient: The idea of the quotient is as following. One considers a discrete subgroup of the global isometry group of $\mathbb{H}_3$, denoted as $\hat{\Gamma}$, which acts on the $\mathbb{H}_3$ properly. For points in $\mathbb{H}_3$, we say they are $\hat{\Gamma}$-equivalent if they are related by actions from elements of $\hat{\Gamma}$. The way how these elements act on $\mathbb{H}_3$ will be made clear later. The quotient of $\mathbb{H}_3$ by $\hat{\Gamma}$-equivalence provides a new hyperbolic 3-fold,

$$\mathcal{M} = \mathbb{H}_3/\hat{\Gamma} \ . \tag{3.24}$$

This quotient can be realized via a locally diffeomorphic projection map[2], $P : \mathbb{H}_3 \to \mathcal{M}$. We call P a covering map of $\mathcal{M}$, and $\mathbb{H}_3$ the corresponding covering space[3]. By construction (3.24), the covering map P is required to be $\hat{\Gamma}$-invariant, which means

$$P \circ x = P \circ \gamma \circ x \ , \quad \forall \gamma \in \hat{\Gamma} \ , \quad \forall x \in \mathbb{H}_3 \ . \tag{3.25}$$

We call $\gamma \in \hat{\Gamma}$ a covering transformation and $\hat{\Gamma}$ the covering group of P. The triple $(\mathbb{H}_3, \hat{\Gamma}, P : \mathbb{H}_3 \to \mathcal{M})$ defines a *uniformization* of $\mathcal{M}$.

Smoothness condition: We call a point x as a *fixed point* if there exists an element γ of $\hat{\Gamma}$ such that $x = \gamma \circ x$. If $\hat{\Gamma}$ acts on the region inside of $\mathbb{H}_3$ discontinuously, i.e., no fixed points inside of $\mathbb{H}_3$, then $\mathcal{M}$ is a smooth manifold satisfying Einstein's equation exactly. Otherwise, fixed points inside of $\mathbb{H}_3$ become conical defects in $\mathcal{M}$, which typically form a line.

Handlebody AdS_3: There is a large class of smooth solutions to Einstein's equation described by the handlebody AdS_3. By meant of "handlebody", the manifold contains a single conformal boundary, denoted as S_g, which is a compact Riemann surface with genus g. The Bañados geometry introduced before is a simplest example, in which

2 Although the existence of such a map is obvious, working out its explicit form in coordinate transformation is in general a very nontrivial problem.

3 In fact, $\mathbb{H}_3$ is the universal covering of $\mathcal{M}$. A covering is universal if and only if the covering space is simply connected.

the conformal boundary is in fact a torus $S_1 = \mathbb{T}^2$ when the stress tensor $\mathcal{L}$ in (3.17) is a constant. Higher genus cases have been studied in [70–73]. A handlebody AdS_3 can be obtained from a quotient of $\mathbb{H}_3$ with an appropriately chosen discrete subgroup $\hat{\Gamma}$. In general, the metric field of a handlebody AdS_3 can be put in the form of (3.16), where the Weyl factor ϕ encodes the geometric data, such as the generators of $\hat{\Gamma}$ and the *projective structure* on the conformal boundary S_g, which I will discuss later. For references, see [67, 70, 74].

Isometric action on $\mathbb{H}_3$. To understand the isometric action on $\mathbb{H}_3$, it is convenient to consider the embedding of $\mathbb{H}_3$ in the $(3+1)$-dimensional flat space $\mathbb{R}^{3,1}$, defined as

$$U^2 - V^2 + X^2 + Y^2 = -1 \ , \tag{3.26}$$

for $V > 0$, with the flat metric

$$ds^2 = l^2(dU^2 - dV^2 + dX^2 + dY^2) \ , \tag{3.27}$$

The Poincaré coordinates in (3.22) are related to the coordinates in $\mathbb{R}^{3,1}$ by

$$\xi = \frac{1}{V-X} \ , y = \frac{Y+iU}{V-X} \ , \bar{y} = \frac{Y-iU}{V-X} \ . \tag{3.28}$$

The space $\mathbb{H}_3$ can be mapped to the space of Hermitian matrices with unit determinate via the following combination,

$$\Lambda = \begin{pmatrix} V+X & Y+iU \\ Y-iU & V-X \end{pmatrix} = \begin{pmatrix} \xi + \xi^{-1}|y|^2 & \xi^{-1}y \\ \xi^{-1}\bar{y} & \xi^{-1} \end{pmatrix} \ . \tag{3.29}$$

Under the above construction, one identifies the orientation-preserving isometry group of $\mathbf{H}_3$, denoted as $\text{Isom}_+(\mathbb{H}_3)$, as the the group of linear fractional transformation $\text{PSL}(2,\mathbb{C}) = \text{SL}(2,\mathbb{C})/\{\pm\mathbb{1}\}$, in which the group element acts on Λ via the conjugation,

$$\Lambda \to h\Lambda h^\dagger = \begin{pmatrix} \xi' + \xi'^{-1}|y'|^2 & \xi'^{-1}y' \\ \xi'^{-1}\bar{y}' & \xi'^{-1} \end{pmatrix} \ . \tag{3.30}$$

with

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad a,b,c,d \in \mathbb{C} \ , \quad ab - cd = 1 \ , \tag{3.31}$$

We refer the isometric action of h on $\mathbb{H}_3$ as

$$h \circ (\xi, y, \bar{y}) = (\xi', y', \bar{y}') \ , \tag{3.32}$$

where set of new coordinates $(\xi', y', \bar{y}')$, by construction, maintains the form of the Poincaré metric in (3.22), and they are related to the old coordinates as,

$$y' = \frac{(ay+b)\overline{(cy+d)} + a\bar{c}\xi^2}{|cy+d|^2 + |c|^2\xi^2} \ , \quad \xi' = \frac{\xi}{|cy+d|^2 + |c|^2\xi^2} \ , \tag{3.33}$$

In the limit $\xi \to 0$, the transformation for y' degenerates to be the PSL(2,$\mathbb{C}$) fractional transformation on the boundary $\mathbb{CP}^1$,

$$y' = \frac{ay+b}{cy+d} \ . \tag{3.34}$$

A non-identity element h in PSL$(2,\mathbb{C})$ can classified through the norm of the matrix trace. It falls into one of the three classes, *elliptic*, *parabolic*, or *loxodromic*, corresponding to $|\mathrm{Tr}(h)| < 2$, $|\mathrm{Tr}(h)| = 2$, or $|\mathrm{Tr}(h)| > 2$, respectively. The type of fixed points associated to h is determined as,

- Elliptic: have fixed points inside of $\mathbb{H}_3$
- Parabolic: one fixed point on $\partial\mathbb{H}_3 = \mathbb{CP}^1$
- Loxodromic: two fixed points on $\partial\mathbb{H}_3 = \mathbb{CP}^1$

As I mentioned before, when fixed points inside of $\mathbb{H}_3$ exist under the actions from $\hat{\Gamma}$, they become conical defects in $\mathcal{M}$ under the quotient. Now, we can ready to see this through a simple example. We consider the discrete subgroup (modulo $\{\pm\mathbb{1}\}$) $\hat{\Gamma} = \mathbb{Z}_n$, generated by a single elliptic element,

$$t = h^{-1} p h \ , \tag{3.35}$$

with

$$h = \begin{pmatrix} \sqrt{\frac{1}{2L}} & -\sqrt{\frac{L}{2}} \\ \sqrt{\frac{1}{2L}} & \sqrt{\frac{L}{2}} \end{pmatrix} \ , \ p = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix} \ , \ n \in \mathbb{Z}_+ \ . \tag{3.36}$$

Inserting components of t into (3.33), one can show that the equations for fixed points, $(\xi', y') = (\xi, y)$, give rise to

$$\xi^2 + \mathrm{Re}(y)^2 = L^2 \ . \tag{3.37}$$

This half-circle is recognized as the geodesic path in Poincaré AdS_3, where the geodesic is anchored at the boundary points $y = \pm L$. Under the quotient $\mathbb{H}_3/\mathbb{Z}_n$, a line defect is generated, which extends along the geodesic path (3.37). It should be noted that this geodesic path in $\mathbb{H}_3$ is again a geodesic in the quotient space, since the underlying covering map (although I did not present it explicitly here) is locally diffeomorphic. This confirms the RT prescription of entanglement entropy discussed in Chapter 2.

Marked Schottky group. We are mostly interested in smooth handlebody AdS_3 solutions to Einstein's equation. The covering groups of such manifold lie in a special class of groups, called *Schottky groups*, which are defined as freely finitely generated discrete subgroups of PSL(2,$\mathbb{C}$) such that every non-identity elements are loxodromic [67]. A Schottky group $\hat{\Gamma}$ is called *marked* if its generators are ordered,

$$\hat{\Gamma} = \langle t_1, t_2, \cdots, t_g \rangle \ . \tag{3.38}$$

where the rank g of $\hat{\Gamma}$ is identical to the genus of the boundary S_g of the corresponding handlebody $\mathcal{M} = \mathbb{H}_3/\hat{\Gamma}$. A element of $\hat{\Gamma}$ is called a *word*, obtained by a combination of generators and their inverses. To understand the actions from the marked Schottky group on $\mathbb{H}_3$ in more detail, we recall that an isometric action on $\mathbb{H}_3$ degenerates to a PSL(2,$\mathbb{C}$) fractional transformation on the boundary $\mathbb{CP}^1$, as shown in (3.33) and (3.34). Thus, we can first investigate the actions from the boundary point of view, and then extend them into the bulk.

Schottky uniformization. Let me first give a punchline and then explain it in detail. From the boundary perspective, the action from the marked Schottky group on $\mathbb{CP}^1$ in fact provides a *Schottky uniformization* of the compact Riemann surface S_g, denoted by the triple

$$(\Omega, \Gamma, p : \Omega \to S_g) \ . \tag{3.39}$$

Here, p is a covering map of S_g. Γ is the marked Schottky group[4], the covering group of p. Ω is called the *domain of discontinuity* of Γ, on which elements of Γ act properly discontinuously. It is a non-simply connected planar region, and can be written as

$$\Omega = \mathbb{CP}^1 \backslash \Lambda(\Gamma) \ . \tag{3.40}$$

where $\Lambda(\Gamma)$ is called the *limit set*, i.e. the closure of the set of all fixed points on $\mathbb{CP}^1$ associated with Γ.

To explain these step by step, let me first discuss the loxodromic fractional transformation on $\mathbb{CP}^1$. In general, a loxodromic PSL(2,$\mathbb{C}$) matrix t_i can be written in the following form,

$$t_i = h_i^{-1} p_i h_i \ , \tag{3.41}$$

with

$$h_i = \frac{1}{\sqrt{e_i - f_i}} \begin{pmatrix} 1 & -e_i \\ 1 & -f_i \end{pmatrix} \ , \quad p_i = \begin{pmatrix} \sqrt{k_i} & 0 \\ 0 & 1/\sqrt{k_i} \end{pmatrix} \ , \tag{3.42}$$

where $|\sqrt{k_i} + 1/\sqrt{k_i}| > 2$ and $|k_i| > 1$. By $(h_i t_i) \circ y = (p_i h_i) \circ y$, we obtain the following equation,

$$\frac{t_i \circ y - e_i}{t_i \circ y - f_i} = k_i \frac{y - e_i}{y - f_i} \ . \tag{3.43}$$

which is called the fixed point equation of t_i. The dilation factor k_i is called the characteristic constant, and $y = e_i$ and $y = f_i$ are called the repelling and attracting fixed points, respectively. If we choose two non-intersecting closed curves C_i and C_i' on $\mathbb{CP}^1$, such that C_i is mapped to C_i' under the action of t_i and they encircle around e_i and f_i respectively, then any points outside of C_i is mapped to the region inside of C_i'. We

4 Here I denote Γ instead of $\hat{\Gamma}$, because the representations of them are different.

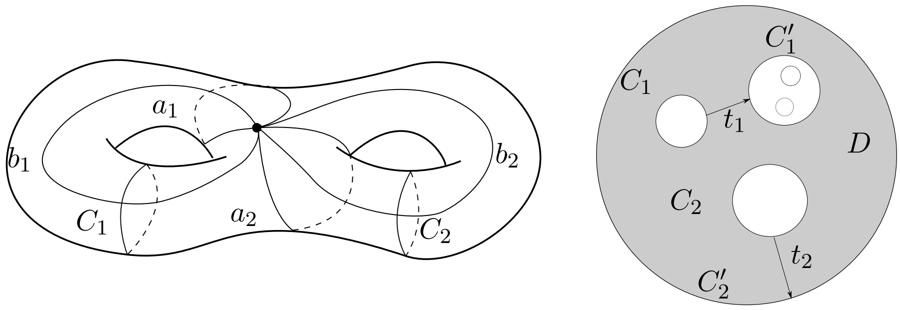

Figure 2: Schottky uniformization of Riemann surface with $g = 2$.

usually call such curves as *Jordan curves*. See Figure 2 for an example. The choices of Jordan curves are not unique. In order to explicitly show the above described features, let me consider the following case, where C_i and C_i' are chosen as

$$C_i : \left|\frac{y - e_i}{y - f_i}\right| = r \ , \quad C_i' : \left|\frac{y - e_i}{y - f_i}\right| = |k_i| r > r \ , \tag{3.44}$$

with r a positive real constant. The inside regions of C_i and C_i' are disconnected to each other, and include e_i and f_i respectively. Inserting $y = e_i$ and $y = f_i$ to the function $\rho(y) = |(y - e_i)/(y - f_i)|$, we get $\rho(e_i) = 0 < r$ and $\rho(f_i) = \infty > |k_i| r$. This implies that the inside regions of C_i and C_i' are given by

$$\text{inside of } C_i : \left|\frac{y - e_i}{y - f_i}\right| < r \ , \ \text{inside of } C_i' : \left|\frac{y - e_i}{y - f_i}\right| > |k_i| r \ . \tag{3.45}$$

Then, for a point y_0 outside C_i with $|(y_0 - e_i)/(y_0 - f_i)| = r_0 > r$, it gets mapped to a new point $t_i \circ y_0$ inside of C_i', since

$$\left|\frac{t_i \circ y_0 - e_i}{t_i \circ y_0 - f_i}\right| = \left|k_i \frac{y_0 - e_i}{y_0 - f_i}\right| = |k_i| r_0 > |k_i| r \ . \tag{3.46}$$

Coming back to the marked Schottky group, we can associate each generator t_i with a pair of Jordan curves, such that all curves in the set $\{C_1, \cdots C_g, C_1', \cdots C_g\}$ are non-intersecting. Under an arbitrary action from Γ, any point outside $2g$ Jordan curves is mapped to a point inside one of those closed curves, thus we conclude that the limit set $\Lambda(\Gamma)$ is a subset of the union of regions inside of the Jordan curves. The open set bounded outside of all Jordan curves, denoted as D, is a fundamental domain of Ω. This comes from the facts that any two distinct points inside of D are not Γ-equivalent, and any point outside D but inside Ω must be Γ-equivalent to a unique point inside of D. The compact Riemann surface S_g is obtained from the quotient

$$S_g = \Omega/\Gamma \ , \tag{3.47}$$

which is equivalent to pairwise gluing the $2g$ boundaries (Jordan curves) of the fundamental domain D. This quotient can be realized via a covering map p,

$$p : \Omega \to S_g = \Omega/\Gamma \ , \tag{3.48}$$

which is locally holomorphic. Since the covering space Ω is not simply connected, the covering group $\hat{\Gamma}$ of p is not isomorphic to the fundamental group $\pi_1(S_g)$. To make it clear, we can choose a canonical basis,

$$\pi_1(S_g) = \left\langle [a_1], [b_1], \cdots, [a_g], [b_g] : \prod_{i=1}^{g} [a_i, b_i] = \mathbb{I} \right\rangle \tag{3.49}$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$, such that each loop a_i on S_g is homologous to image $p(C_i)$. See Figure 2. Then, the covering group Γ of p is isomorphic to the following factor group,

$$\Gamma \simeq \pi_1(S_g)/\mathcal{N} \ , \tag{3.50}$$

where $\mathcal{N}$ is the smallest normal subgroup of $\pi_1(S_g)$ containing elements $[a_1], \cdots, [a_g]$ [75]. This will be used later when I discuss the origin of distinct gravity saddles.

Poincaré extension. Now we discuss the bulk picture. We use (3.33) to extend the action (3.34) on $\mathbb{CP}^1$ into the bulk. When doing so, one can find that the action of t_i from C_i to C_i' is extended to a map between two non-intersecting hemispheres, denoted by H_i and H_i', with

$$\partial H_i = C_i \ , \quad \partial H_i' = C_i' \ , \quad i = 1, \cdots g \ . \tag{3.51}$$

This implies that the fundamental domain of $\mathbb{H}_3$, denoted as $\hat{D}$, is a simply connected open set bounded by

$$\partial \hat{D} = D \cup H_1 \cup \cdots \cup H_g \ . \tag{3.52}$$

Thus, the quotient space $\mathcal{M}$ can be obtained by pairwise gluing these $2g$ hemispheres. The procedure described in above is known as the *Poincaré extension* [75]. One important fact we will use later is that while the homotopy classes $[a_i] = [p(C_i)]$ are nontrivial elements of $\pi_1(S_g)$, they are trivial in $\pi_1(\mathcal{M})$. This is because C_i is the boundary of the hemisphere H_i, which is simply connected and inside of $\mathcal{M}$. In other words, the $[a_i]$ loops on S_g are contractible in $\mathcal{M}$.

Normalization of Schottky group. We have known that, for a marked Schottky group $\hat{\Gamma}$ with rank g, one obtains a handlebody $\mathcal{M}$ with genus g by the quotient procedure. However, the space of genus g handlebodies and the space of $\hat{\Gamma}$ are not in one-to-one correspondence. Two marked Schottky group gives the same handlebody if the generators of them are related by the conjugation of an arbitrary PSL(2,$\mathbb{C}$) element. To see this, we consider

$$\hat{\Gamma} = \langle t_1, \cdots, t_g \rangle \ , \quad \hat{\Gamma}' = \langle q t_1 q^{-1}, \cdots, q t_g q^{-1} \rangle \ . \tag{3.53}$$

We denote P and P' as the covering map associated with $\hat{\Gamma}$ and $\hat{\Gamma}'$. By the required property

$$P \circ \gamma \circ x = P \circ x \ , \ P' \circ (q \gamma q^{-1}) \circ x = P' \circ x \ , \ \forall \gamma \in \hat{\Gamma} \tag{3.54}$$

where $x \in \mathbb{H}_3$, one finds that

$$P' = qPq^{-1} \ , \tag{3.55}$$

Thus, the two resulting handlebodies are globally diffeomorphic,

$$\mathcal{M}' = P'(\mathbb{H}_3) = q(P(\mathbb{H}_3)) = q(\mathcal{M}) \ . \tag{3.56}$$

The fixed points of element qt_iq^{-1} are given by $q \circ e_i$ and $q \circ f_i$, respectively. Therefore, we can normalize the marked Schottky group $\hat{\Gamma}$ by appropriately choosing q such that $q \circ e_1 = 0$, $q \circ f_1 = \infty$ and $q \circ e_2 = 1$. The *normalized marked Schottky group* contains $(3g-3)$ complex parameters,

$$e_3, \cdots e_g, f_2, \cdots f_g, k_1, \cdots k_g \ . \tag{3.57}$$

The space of normalized marked Schottky groups is called the *Schottky space*, denoted as $\mathfrak{S}_g$. It has real dimensions $(6g-6)$ and its universal covering space is the Teichmüller space $\mathcal{T}_g$ of genus g compact Riemann surfaces. The discussion of the covering map from $\mathcal{T}_g$ to $\mathfrak{S}_g$ can be found in [74, 75].

Discrete isometry of vacuum solutions. Now we turn to connect solutions in the form of (3.16) with Schottky uniformization. We consider a handlebody AdS_3 constructed via a given normalized marked Schottky group Γ. The boundary metric on its conformal boundary S_g can be locally written as

$$d\hat{s}^2 = e^{2\phi(z,\bar{z})} dz d\bar{z} \ . \tag{3.58}$$

As S_g is obtained by gluing the fundamental domain D on the boundary $\mathbb{CP}^1$ of $\mathbb{H}_3$, we can set $z = y$ when we approach to the conformal boundary S_g. Under such coordinates, z and $\gamma \circ z$ for any $\gamma \in \Gamma$ represent a same point on S_g. Thus, the boundary metric field at z and $\gamma \circ z$ is identical, i.e., $d\hat{s}^2|_z = d\hat{s}^2|_{\gamma\circ z}$, which implies the following *quasi-periodic boundary conditions* on ϕ,

$$\phi(\gamma \circ z, \overline{\gamma \circ z}) = \phi(z, \bar{z}) - \frac{1}{2} \log |\gamma'|^2 \ , \quad \forall \gamma \in \Gamma \ , \ z \in \Omega \ , \tag{3.59}$$

where $\gamma' = d(\gamma \circ z)/dz$ for shorthand. Notice that ϕ is a single-valued function of z on Ω. However, it is is multi-valued on S_g. By (3.15), it is straightforward to use (3.59) to check that T^ϕ and R^ϕ transforms as quadratic differentials with respect to Γ, given by

$$\begin{aligned} T^\phi(\gamma \circ z) &= \left(\frac{\partial \phi(\gamma \circ z)}{\partial(\gamma \circ z)} \right)^2 - \frac{\partial^2 \phi(\gamma \circ z)}{\partial(\gamma \circ z)^2} \\ &= (\gamma')^{-2} \left(T^\phi(z) - \frac{1}{2} \{\gamma \circ z; z\} \right) \\ &= (\gamma')^{-2} T^\phi(z) \ , \end{aligned} \tag{3.60}$$

and

$$R^{\phi}(\gamma \circ z) = \frac{\partial^2 \phi(\gamma \circ z)}{\partial(\gamma \circ z)\partial\overline{(\gamma \circ z)}} = |\gamma'|^{-2} R^{\phi}(z) \ . \tag{3.61}$$

Here we used the fact that Schwarizan derivative vanishes for fractional transformations, i.e., $\{\gamma \circ z; z\} = 0$. Inserting (3.59), (3.60) and (3.61) into the metric (3.16), one finds that the whole bulk metric is invariant under Γ,

$$ds^2|_{\rho,z} = ds^2|_{\rho,\gamma \circ z} \ , \quad \forall \gamma \in \Gamma \ . \tag{3.62}$$

In other words, Γ is the discrete isometry of (3.16) under the boundary conditions (3.59). Unlike the $\hat{\Gamma}$ action on $\mathbb{H}_3$ in (3.33), here Γ only acts on the boundary coordinates $(z, \bar{z})$. Under quotient, the metric (3.16) naturally descends to a metric on the manifold $\mathcal{M}$.

It should be noted that solutions to (3.59) are not unique. Given a solution ϕ, shifting $\phi \to \tilde{\phi} = \phi - f_\Gamma$ leads to a new solution if f_Γ is an automorphic function with respect to Γ,

$$f_\Gamma(\gamma \circ z) = f_\Gamma(z) \ , \quad \gamma \in \Gamma \ , \quad z \in \Omega \ . \tag{3.63}$$

Here, f_Γ is single-valued both on Ω and S_g. The degree of freedom of shifting ϕ by an automorphic function reflects the fact that a given normalized marked Schottky group Γ only fix the conformal class of the metric on S_g. From the bulk perspective, it tells that a metric (3.16) invariant under Γ is defined up to a finite PBH diffeomorphism with the relative Weyl factor f_Γ being automorphic with respect to Γ. This diffeomorphism can be deduced from (3.23) and I do not present it here. Intuitively, two manifolds $\mathcal{M}$ and $\mathcal{M}'$ related by such a diffeomorphism look almost the same from the point of view of the covering space $\mathbb{H}_3$, except that, in the limit $\rho \to 0$, their constant ρ slices approach to the fundamental domain D in different ways, which leads to different boundary metrics. In the dual CFT perspective, both manifolds $\mathcal{M}$ and $\mathcal{M}'$ are dual to the same vacuum state of the boundary CFT defined on S_g. The relative Weyl factor f_Γ in the boundary metrics is captured by the conformal anomaly, which leads to a universal shift in the CFT partition function, governed by the Liouville action of f_Γ [76]. In the next section, we will verify this property by calculating the semi-classical gravity partition function, and hence confirm the validity of the AdS_3/CFT_2 correspondence.

The degree of freedom in ϕ can be fixed if we provide additional data, such as the curvature R on S_g. By *Riemann uniformization theorem*[5], we have a canonical choice $R = \lambda$, with $\lambda = 0$ for genus $g = 1$ and $\lambda = -2$ for $g > 1$. By (3.9), this leads to the Liouville equation,

$$-8\partial_z \partial_{\bar{z}} \phi = \lambda e^{2\phi} \ . \tag{3.64}$$

5 The Riemann uniformization theorem states that any compact Riemann surface is conformally equivalent to a compact Riemann surface with a constant curvature and with the same complex structure. The curvature depends on the genus, i.e., $g = 0$, $g = 1$, and $g > 1$ correspond to $R = 2$, $R = 0$, and $R = -2$.

It is known that (3.64) and (3.59) determine a unique solution ϕ. However, it is hard to solve them in practice, except for the case of $g = 1$. We will discuss the $g = 1$ case as an explicit example later.

Origin of distinct gravity saddles. So far, we have discussed how to obtain a handlebody AdS_3 as well as its metric by a given normalized marked Schottky group. In the context of holography, such as the holographic Rényi entropy, we often encounter problems of a different type, that is, for a compact conformal boundary S_g with a given metric expressed in some coordinates, how to work out the whole bulk solution to Einstein's equation. This problem is essentially to find the Schottky uniformization of the conformal boundary, so that the bulk metric can be directly obtained via (3.16). An important point is that, for a compact Riemann surface with a nontrivial topology and a given metric, it admits different Schottky uniformizations. In fact, there are infinite many of them. This is the essential reason for existence of distinct gravity solutions with a same boundary metric. In the following, I will explain how different Schottky uniformizations arise for S_g with a given boundary metric.

Let me first give the general idea. Consider a Riemann surface S_g with a given metric in isothermal coordinates, $d\hat{s}^2 = e^{2\omega(w,\bar{w})}|dw|^2$. We can choose a canonical basis of $\pi_1(S_g)$ in the form of (3.49). Such choices are infinite many. With a fixed canonical basis, we can cut S_g along the g handle loops a_i, which leads to a planar surface with $2g$ boundaries. This planar surface, denoted as D, is the desired fundamental domain of Schottky uniformation, and boundaries of D are the desired Jordan curves. As the coordinate w on S_g are given, a base point w going once around a_i or b_i can be formally represented via a coordinate transformation, denoted as[6] $w \to A_i \circ w$ or $w \to B_i \circ w$. These actions A_i and B_i on w provide a representation of $\pi_1(S_g)$. Now, the problem of finding Schottky uniformization of S_g is to find a locally biholomorphic map, the so-called *developing map*,

$$J : w \mapsto z \ , \tag{3.65}$$

such that the following monodromy conditions hold,

$$J(A_i \circ w) = J(w) \ , \ J(B_i \circ w) = t_i \circ J(w) \ , \ i = 1, \cdots, g \ . \tag{3.66}$$

Here t_i's are PSL(2,$\mathbb{C}$) matrices and act on $J(w) = z$ as fractional transformations. They can be determined after we solve J. The required properties (3.66) come from the fact that the planar surface D as the desired fundamental domain of Schottky uniformization must be embedded in $\mathbb{CP}^1$, coordinated by z. When a point on S_g goes once around the a_i loop, its trajectory on $\mathbb{CP}^1$ is the corresponding Jordan

6 Note that, although w and $A_i \circ w$ represent the same point on S_g, the values of them can be different. This is also the case for B_i. In usual situations, the actions of A_i and B_i on coordinates are easy to work out.

curve, and hence the coordinate z of the point must be invariant. Similarly, when a point goes once around b_i, the coordinate z must be changed by a fractional transformation.

Mathematically speaking, the developing map J satisfying (3.66) defines a *projective structure*[7] on S_g, denoted as $[\sigma]$. As A_i and B_i provide a representation of $\pi_1(S_g)$, the properties (3.66) further provide a homomorphism,

$$\rho_\sigma : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{C}) \ , \quad \rho_\sigma : [a_i] \mapsto \mathbb{I} \ , \ \rho_\sigma : [b_i] \mapsto t_i \ . \tag{3.67}$$

We call ρ_σ a *monodromy representation* of the projective structure [77]. Furthermore, the image

$$\rho_\sigma(\pi_1(S_g)) = \langle t_1, \cdots, t_g \rangle = \Gamma \ , \tag{3.68}$$

is called the *monodromy group* of $[\sigma]$, which in present case is identical to the desired Schottky group.

After solving J, we can rewrite the metric in z-coordinates and using (3.16) to get the whole bulk solution. An important fact is that the global property of such a solution is characterized by the projective structure on S_g arising from the Schottky uniformization. As I mentioned before, the homotopy class $[a_i]$ is trivial in $\pi_1(\mathcal{M})$. This triviality also shows up in the monodromy representation (3.67).

What about other saddles to Einstein's equation? Notice that we fixed the canonical basis of $\pi_1(S_g)$ in previous discussion. A given canonical basis can be changed by acting with *Dehn twists*, which are the generators of the *Mapping Class Group* (MCG), denoted by $\mathrm{Mod}(S_g)$ [6]. In general, we may expect that a change of basis will lead to a different Schottky uniformization of S_g, and hence a distinct bulk geometry. This is almost true, except for some particular elements of $\mathrm{Mod}(S_g)$. I will not discuss the general cases of this issue, but only give the explicit example for $g = 1$ later. By the AdS/CFT duality, we conclude that the dual CFT partition function on a given compact Riemann surface can be written as a sum of gravity saddle points contributions over all possible projective structures on S_g arising from Schottky uniformizations,

$$Z_{CFT} = \sum_\sigma e^{-S^\sigma_{grav}} \ . \tag{3.69}$$

Example for $g = 1$. We consider a flat torus with metric $ds^2 = |dw|^2$. The periods of w are given by $w \sim w + w_1$ and $w \sim w + w_2$, with $w_1 = 2\pi$, $w_2 = 2\pi\tau$ and $\mathrm{Im}(\tau) > 0$. The fundamental group of torus is ablean and isomorphic to $\mathbb{Z}^2$, and the corresponding MCG is $\mathrm{PSL}(2, \mathbb{Z})$. We denote a "time" loop from w to $w + w_1$ as ω_1 and a spatial loop from

7 A projective structure $[\sigma]$ on a Riemann surface is a maximal analytic atlas, i.e., $[\sigma] = \{U_\alpha, \varphi_\alpha\}_{\alpha\in\sigma}$, such that all transition functions between charts are PSL(2,$\mathbb{C}$) fractional transformations [77].

w to $w + w_2$ as ω_2. Then a general basis of the fundamental group can be obtained via a PSL(2,$\mathbb{Z}$) action,

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \;, \quad ad - bc = 1 \;, \quad a, b, c, d \in \mathbb{Z} \;. \tag{3.70}$$

The integers a, b, c, d are winding numbers. The condition $ad - bc = 1$ ensures that the area of the torus is preserved. The two basic generators of PSL$(2, \mathbb{Z})$ are given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;. \tag{3.71}$$

The coordinate transformations correspond to a_1 and b_1 are transitions,

$$A_1 \circ w = w + aw_1 + bw_2 \;, \quad B_1 \circ w = w + cw_1 + dw_2 \;. \tag{3.72}$$

which by the conditions (3.66) give rise to

$$z(w + aw_1 + bw_2) = z \;, \quad z(w + cw_1 + dw_2) = t_1 \circ z \;. \tag{3.73}$$

The solution to (3.73) is easy to worked out, given by

$$z = e^{iw/(a\tau+b)} \;, \quad t_1 = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{pmatrix} \;, \quad k = e^{2\pi i \frac{c\tau+d}{a\tau+b}} \;. \tag{3.74}$$

Notice that developing map $z(w)$ only depends on a and b. There is a subgroup $\mathbb{Z}$ of PSL(2,$\mathbb{Z}$), generated by the element $K = S^{-1}TS$, which leave $a_1 = aw_1 + bw_2$ invariant,

$$K^n \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 - na_1 \end{pmatrix} \;, \quad n \in \mathbb{Z} \;. \tag{3.75}$$

The characteristic constant is also invariant under such transformations,

$$k \to k' = e^{2\pi i \frac{c\tau+d-n(a\tau+b)}{a\tau+b}} = k \;. \tag{3.76}$$

Thus, all possible projective structures arising from Schottky uniformizations are in one-to-one correspondence to elements in the following quotient [78],

$$\{[\sigma]\} \simeq \mathrm{PSL}(2, \mathbb{Z})/\mathbb{Z} \;. \tag{3.77}$$

The bulk metric can be obtained by directly inserting the Weyl factor

$$\phi = \frac{1}{2} \log |dw/dz|^2 = \frac{1}{2} |(a\tau + b)/z|^2 \;, \tag{3.78}$$

into (3.16). Under a boundary coordinate transformation, $z \to w$, one can show that the metric is in the Bañados form,

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} \left| d\bar{w} - \frac{\rho}{4(a\tau + b)^2} dw \right|^2 \;. \tag{3.79}$$

For $a = d = 1$ and $b = c = 0$, the metric solution is called the *black hole phase*. For $a = d = 0$ and $b = -c = 1$, it is called the *thermal* AdS_3 *phase*. The bulk contractible loops in two cases are given by the time loop and the spatial loop respectively.

3.1.3 *Boundary effective action*

In this section, I will review the derivation of the boundary effective action of the AdS_3 gravity by integrating out the radial direction of the on-shell action. The result was originally obtained in [79], where the authors worked in the Chern-Simons formalism, and by Gauss decomposition of the SL(2,$\mathbb{R}$) Chern-Simons fields they showed that the boundary effective action of the AdS_3 gravity is of Liouville type. Different ways of the derivation can also be found in [67, 80–83] (see also a recent review [84]). To incorporate the discussion of the previous sections, in the following, I will work in the metric formalism and derive the effective action through the holographic renormalization procedure. As examples, explicit evaluations of the effective action for genus one solutions as well as the replica solution will be given. I will also discuss various forms of the effective action, and clarify its relations with Zograf-Takhtajan Liouville action [85] as well as the Polyakov functional action in two-dimensional CFTs [76].

Before going into the details of the calculations, let me first clarify the choice of the cut-off surface here. In the usual holographic renormalization procedure, the cut-off surface is placed at a constant radial slice $\rho = \epsilon$, where the cut-off regulator ϵ is eventually sent to zero. However, as shown in [83, 84], a complete treatment that eventually yields the desired Liouville action at the boundary requires an unconventional cut-off surface, located at

$$\rho = \rho_0 = \epsilon e^{2\phi(z,\bar{z})} \ . \tag{3.80}$$

The meaning of (3.80) is clear from the diffeomorphism (3.23), which, at the leading order of ϵ, represents a constant radial slice $\xi = \sqrt{\epsilon}$ in the Poincaré AdS_3. Hence, in this way, one fixes the scheme of the holographic renormalization for on-shell solutions with different Weyl factors. Although the derivation of the boundary effective action is based on the on-shell solution given in (3.16), it is more convenient to express the metric $g_{ij}(x,\rho)$ as the following matrix product [65],

$$g = \left(\mathbb{1} + \frac{\rho}{2} g_{(2)} g_{(0)}^{-1}\right) g_{(0)} \left(\mathbb{1} + \frac{\rho}{2} g_{(0)}^{-1} g_{(2)}\right) \ , \tag{3.81}$$

which implies

$$\det[G] = \frac{l^6}{4\rho^4} \det[g_0] \det[\mathbb{1} + \frac{\rho}{2} g_{(2)} g_{(0)}^{-1}]^2 \ . \tag{3.82}$$

Then the on-shell Einstein-Hilbert action can be integrated as

$$\begin{aligned} S_{EH} &= \frac{1}{16\pi G_3} \int_{\rho \geq \rho_0} d^3x \sqrt{G} \left(-6l^{-2} + 2l^{-2}\right) \\ &= -\frac{l}{8\pi G_3} \int_{\rho \geq \rho_0} d\rho \int d^2z \sqrt{g_{(0)}} \left(\rho^{-2} + \rho^{-1} a_1 + \mathcal{O}(1)\right) \\ &= -\frac{l}{8\pi G_3} \int d^2z \sqrt{g_{(0)}} \left(a_1 \log \rho_0 - \rho_0^{-1} + \mathcal{O}(\epsilon)\right) \ , \end{aligned} \tag{3.83}$$

with

$$a_1 = \frac{1}{2}\text{Tr}[g_{(2)}g_{(0)}^{-1}] = -\frac{1}{4}R[g_{(0)}] \ . \tag{3.84}$$

The out-going normal covector n_μ to the cut-off surface is characterized by

$$n_\mu \propto \partial_\mu(\rho e^{-2\phi}) \ , \tag{3.85}$$

which yields the components of the out-going normal vector as

$$\begin{aligned} n^\rho &= -2\rho(l^2 + l^2\rho g^{ij}\partial_i\phi\partial_j\phi)^{-1/2} \ , \\ n^i &= \rho g^{ij}\partial_j\phi(l^2 + l^2\rho g^{mn}\partial_m\phi\partial_n\phi)^{-1/2} \ . \end{aligned} \tag{3.86}$$

Then the extrinsic curvature K on the cut-off surface can be worked out as

$$K = \nabla_\mu n^\mu|_{\rho=\rho_0} = 2l^{-1} + \mathcal{O}(\epsilon^2) \tag{3.87}$$

This is expected since $\rho = \rho_0$ corresponds to the constant radial slice in the Poincaré AdS_3 at the leading order of ϵ, which has constant extrinsic curvature $K = 2l^{-1}$. On the other hand, the induced metric on the cut-off surface is given by

$$\begin{aligned} h_{ij} &= l^2\left(\rho_0^{-1}g_{(0)ij} + g_{(2)ij} + \partial_i\phi\partial_j\phi\right) + \mathcal{O}(\epsilon) \ , \\ \sqrt{h} &= l^2\sqrt{g_{(0)}}\left(\rho_0^{-1} + a_1 + \frac{1}{2}(\partial\phi)^2\right) + \mathcal{O}(\epsilon) \ , \end{aligned} \tag{3.88}$$

Therefore, the Gibbons-Hawking term in (3.1) reads

$$S_{GH} = -\frac{l}{8\pi G_3}\int d^2z\sqrt{g_{(0)}}\left(2\rho_0^{-1} + 2a_1 + (\partial\phi)^2\right) \ . \tag{3.89}$$

Combining (3.83) and (3.89) yields the regularized on-shell action

$$S_{reg} = -\frac{l}{8\pi G_3}\int d^2z\sqrt{g_{(0)}}\left(\rho_0^{-1} + a_1(\log\rho_0 + 2) + (\partial\phi)^2\right), \tag{3.90}$$

The power law divergence associated with the first term in (3.90) can be eliminated by adding a covariant counter term on the cut-off surface [86],

$$S_{ct} = \frac{l}{8\pi G_3}\int d^2z\sqrt{h} \ . \tag{3.91}$$

Expressing ρ_0 explicitly and using the relation (3.84), one obtains the renormalized on-shell action as

$$\begin{aligned} S_{ren} &= \lim_{\epsilon\to 0}(S_{reg} + S_{ct}) \\ &= \frac{l}{16\pi G_3}\int d^2z\sqrt{g_{(0)}}\left(\phi R - (\partial\phi)^2 + \frac{1}{2}R\log\epsilon + \frac{1}{2}R\right), \end{aligned} \tag{3.92}$$

In two dimensions the integration of $\sqrt{g_{(0)}}R$ is the Euler characteristic if the cut-off surface is closed. Thus, the additional logarithmic divergent term in (3.92) can be removed by adding a second counter term,

$$S_{ct}^{(2)} = -\frac{l}{8G_3}\chi \log \epsilon \ . \tag{3.93}$$

Therefore, we arrive at

$$S_{ren} = \frac{l}{16\pi G_3}\int d^2z\sqrt{g_{(0)}}\left(\phi R - (\partial\phi)^2\right) + \frac{l}{8G_3}\chi \ , \tag{3.94}$$

Here, the Euler characteristic term is kept[8], and it turns out to be important for relating the above action to the Zograf-Takhtajan Liouville action [85]. The renormalized action (3.94) is of the Liouville type. In particular, by (3.9), the Weyl factor ϕ automatically satisfies the on-shell condition,

$$-2\nabla_i\nabla^i\phi = R \ , \tag{3.95}$$

In the non-compact boundary, the above action (3.94) needs to be modified by adding the boundary term [87]

$$S_{ren} = \frac{l}{16\pi G_3}\int d^2z\sqrt{g_{(0)}}\left(\phi R - (\partial\phi)^2\right) + \frac{l}{8\pi G_3}\int du\sqrt{\gamma}\phi K + \frac{l}{8G_3}\chi \ , \tag{3.96}$$

where γ denotes the induced metric on the boundary of the surface, and K is the associated extrinsic curvature. The derivation of (3.96) in the AdS$_3$ gravity requires us to include corner terms, such as corner terms at Jordan curves in the Schottky uniformization. Here I will not discuss them. To check the effective actions (3.94) and (3.96), in the following, I will consider two relevant examples.

Example for $g = 1$. Consider the effective action for genus one solutions (3.79). In this case, the cut-off surface in w-coordinate is a flat torus with area $4\pi^2\text{Im}(\tau)$. The Liouville field is given by

$$\phi = \frac{1}{2}\log|(a\tau+b)/z|^2 = -\text{Im}\left(\frac{iw}{a\tau+b}\right) + \log|a\tau+b| \ . \tag{3.97}$$

Evaluating the Liouville action (3.94) for ϕ in the w-coordinates yields

$$S_{ren} = -\frac{\pi l}{4G_3}\frac{\text{Im}(\tau)}{|a\tau+b|^2} = \frac{\pi l}{4G_3}\text{Im}\left(\frac{c\tau+d}{a\tau+b}\right) \ . \tag{3.98}$$

For the black hole phase, $a = d = 1$ and $b = c = 0$, we get

$$S_{BH}(\tau) = \frac{\pi l}{4G_3}\text{Im}(1/\tau) \ , \tag{3.99}$$

8 In [84], the Ricci scalar term dropped.

which coincides with the result in [88]. For the thermal AdS$_3$ phase, $a = d = 0$, $b = -c = 1$, we get

$$S_{thermal}(\tau) = -\frac{\pi l}{4G_3}\text{Im}(\tau) \ . \tag{3.100}$$

Although both of these two phases contribute to the gravity partition function (3.69), in the semi-classical limit $G_3 \to 0$, only one of them dominates. The Hawking-page phase transition between them happens at $|\tau| = 1$ [88], where for $|\tau| < 1$, the black hole phase dominates, and for $|\tau| > 1$, the thermal AdS$_3$ phase dominates. Another interesting fact is that, from (3.99) and (3.100), the action of the black hole phase with modular parameter $-1/\tau$ is identical to the action of the thermal AdS$_3$ phase with τ, i.e. $S_{BH}(-1/\tau) = S_{thermal}(\tau)$. At the classical level, this is because those two geometries are globally identical, i.e. both the conformal structure and the projective structure are the same. The diffeomorphism between those two geometries becomes $w_{BH} = -w_{thermal}/\tau$ on the boundary, which gives rise to a constant shift for the Liouville field $\phi_{thermal} \to \phi_{BH} - \log|\tau|$. This constant shift does not change the action since the torus is flat. At the quantum level, this feature is manifested by the modular invariance of the torus partition function, which for the S-transformation $\tau \to -1/\tau$, is expressed as $Z[\tau] = Z[-1/\tau]$.

Example for non-compact case. The replica manifold $\mathcal{M}_n$ in the AdS$_3$ gravity is usually discussed in the context of holographic Rényi entropy. Here I consider the simplest case, where the conformal boundary $\partial\mathcal{M}_n$ is a n-sheeted branched covering of $\mathbb{CP}^1$, with the branched points located at $w = 0$ and $w = \infty$. The boundary metric in w-coordinates takes the locally flat form,

$$d\hat{s}^2 = dwd\bar{w} \ . \tag{3.101}$$

The universal covering space of $\partial\mathcal{M}_n$ is still $\mathbb{CP}^1$, which means that the bulk geometry is $\mathbb{H}_3$ but with a deformed boundary. This deformation is encoded in the coordinate transformation from the replica surface to its universal covering, given by

$$z = w^{\frac{1}{n}} \ . \tag{3.102}$$

Expressing the metric in z-coordinates, we get the deformed boundary metric, $d\hat{s}^2 = e^{2\phi}|dz|^2$, with the Liouville field given by,

$$\phi = \frac{n-1}{2}\log|z|^2 + \log n = \frac{1}{2}(1-\frac{1}{n})\log|w|^2 + \log n \ . \tag{3.103}$$

The full metric solution can be obtained by inserting the Liouville field (3.103) into (3.16), but here we do not need it. Using (3.103), the Ricci curvature in w-coordinates can be worked out as

$$R = -2\nabla_i\nabla^i\phi = -4\pi(1-\frac{1}{n})\delta^{(2)}(w) \ , \tag{3.104}$$

which is singular at the branched point $w = 0$. In fact, by the inversion $w \to w' = 1/w$, one can show that the curvature is also singular at $w = \infty$. Those two singularities need to be removed from the region of integration, otherwise the renormalized action diverges. To do so, one introduces two cut-off boundaries circling the branched points, parametrized as[9],

$$C_1 : |w| = \epsilon \ , \quad C_2 : |w| = \epsilon^{-1} \ . \tag{3.105}$$

The action one needs to evaluate is as follows

$$S_{ren} = -\frac{l}{16\pi G_3} \int_{\epsilon<|w|<\frac{1}{\epsilon}} d^2w \sqrt{g_{(0)}} (\partial\phi)^2 + \frac{l}{8\pi G_3} \left(\oint_{C_1} r d\theta \phi K_{(1)} + \oint_{C_2} r d\theta \phi K_{(2)} \right) \ , \tag{3.106}$$

where the extrinsic curvature K_a is defined via the out-going normal vector, i.e. $K_{(a)} = \nabla_i n^i_{(a)}$, and (r, θ) is the polar coordinates, defined via $w = re^{i\theta}$, with the angle period $\theta \cong \theta + 2\pi n$. The Euler characteristic vanishes on this cylinder topology. The extrinsic curvatures are easily calculated in polar coordinates, given by

$$K_{(1)} = -\epsilon^{-1} \ , \quad K_{(2)} = \epsilon \ , \tag{3.107}$$

Inserting (3.103) and (3.107) into the action (3.106) yields

$$S_{ren} = \frac{l}{4G_3} \left(n - \frac{1}{n} \right) \log\left(1/\epsilon\right) \ , \tag{3.108}$$

which is indeed the correct result for the action associated with the Replica manifold. There are other methods to obtain this result, for instance, in [89], the topological black hole method is introduced and in [32], a different holographic renormalization procedure is performed, which also includes the infrared contribution to the renormalized action.

Effective action in different forms. One might notice that the renormalized actions (3.94) and (3.96) are not in the standard form of the Liouville action. The standard Liouville action with a reference metric $\tilde{g}$ takes the form of [87]

$$S_L[\phi, \tilde{g}] = \frac{l}{16\pi G_3} \int d^2z \sqrt{\tilde{g}} \left((\tilde{\partial}\phi)^2 + \phi \tilde{R} + \mu e^{2\phi} \right) + \frac{l}{8\pi G_3} \int du \sqrt{\tilde{\gamma}} \left(\phi \tilde{K} + \mu_B e^{\phi} \right) \ , \tag{3.109}$$

where the Liouville field ϕ is not necessary on-shell. In the following, I will show that by performing a Weyl transformation on the boundary metric, one can relate the renormalized action to this standard form.

9 Strictly speaking, those circles are the closed curves winding around the branched points n times.

The general idea is as following. We have knonw that a solution in the form of (3.16) can be obtained via quotient of $\mathbb{H}_3$, and defined up to a finite PHB transformation. Thus, for a given solution in the form of (3.16), we can choose a reference metric $\tilde{g} \in [g_{(0)}]$ such that the curvature of $\tilde{g}$ is a constant. For a non-compact conformal boundary with genus $g = 0$, we set $\tilde{R} = 0$. For a compact surface with $g = 1$ and $g > 1$, we set $\tilde{R} = 0$ and $\tilde{R} = -2$, respectively. Moreover, it should be kept in mind that in z-coordinates, the conformal boundary generically corresponds to a non-simply connected planar region on $\mathbb{CP}^1$. Thus, boundary terms are always needs to be included.

1. Flat reference metric. We first consider a solution in the form of (3.16), where the conformal boundary is non-compact with $g = 0$. We choose a flat reference metric $\tilde{g}$ as

$$g_{(0)} = e^{2\phi}\tilde{g} = e^{2\phi}dzd\bar{z} \ , \tag{3.110}$$

which is just the boundary metric of $\mathbb{H}_3$. The Ricci curvatures and the extrinsic curvatures of those two metrics are related as

$$\sqrt{g_{(0)}}R = -2\sqrt{\tilde{g}}\tilde{\nabla}^2\phi \ , \quad \sqrt{\gamma}K = \sqrt{\tilde{\gamma}}\tilde{K} - \tilde{n}^i\tilde{\partial}_i\phi \ . \tag{3.111}$$

Inserting (3.111) into (3.96) gives rise to

$$S_{ren} = \frac{l}{16\pi G_3}\int_D d^2z\sqrt{\tilde{g}}(\tilde{\partial}\phi)^2 + \frac{l}{8\pi G_3}\int_{\partial D} du\sqrt{\tilde{\gamma}}\phi\tilde{K} \ , \tag{3.112}$$

which takes the form of (3.109).

2. Reference metric with constant curvature. We consider a solution in the form of (3.16), where the conformal boundary is compact with $g = 1$. We choose the reference metric $\tilde{g}$ via the following ansatz[10]

$$g_{(0)} = e^{2\phi}dzd\bar{z} = e^{2\phi_F}\tilde{g} = e^{2\phi_F}\frac{dwd\bar{w}}{\mathrm{Im}(w)^2} \ , \tag{3.113}$$

so that $\tilde{R} = -2$. In principle, one can try to solve the Schottky uniformization problem to find the developing map, $J : w \to z$. But we do not need its explicit form here. We denote

$$\phi = \phi_F + \phi_S \ , \quad \phi_S = \frac{1}{2}\log\left|\frac{w'(z)}{\mathrm{Im}(w(z))}\right|^2 \tag{3.114}$$

In $\tilde{g}$ frame, the Ricci curvature and extrinsic curvature are given by

$$\sqrt{g_{(0)}}R = -2\sqrt{\tilde{g}}(1 + \tilde{\nabla}^2\phi_F) \ , \ \sqrt{\gamma}K = \sqrt{\tilde{\gamma}}\tilde{K} - \tilde{n}^i\tilde{\partial}_i\phi_F. \tag{3.115}$$

Using above relations, one can show the renormalized action decomposes into two pieces,

$$S_{ren}[\phi] = S_L[\phi_F, \tilde{g}] + S_{ren}[\phi_S] \ , \tag{3.116}$$

10 Notice that to compute ϕ_F for a given metric, one needs to solve the *Fuchsian uniformization problem.* But here I will not discuss them.

with

$$S_L[\phi_F,\tilde{g}] = \frac{l}{16\pi G_3}\int d^2w\sqrt{\tilde{g}}\left((\tilde{\partial}\phi_F)^2 - 2\phi_F\right) \ ,$$
$$S_{ren}[\phi_S] = \frac{l}{16\pi G_3}\int d^2w\sqrt{\tilde{g}}\left(\tilde{R}\phi_S - (\tilde{\partial}\phi_S)^2\right) + \frac{l}{8G_3}\chi \ . \quad (3.117)$$

The first action in (3.117) is the standard Liouville action on the compact Riemann surface with constant curvature, and the second one in (3.117) is again of the form (3.94).

Comments: Recall the well-known fact in two-dimensional CFTs, that a Weyl transformation on a background metric universally shifts the CFT partition function as [76]

$$Z_{CFT}[e^{2\phi_F}\tilde{g}] = e^{-S_L[\phi_F,\tilde{g}]}Z_{CFT}[\tilde{g}] \ . \quad (3.118)$$

By AdS/CFT duality, we expect that this relation should also hold for partition functions of gravity solutions which are related via a finite PBH transformation. Indeed, in the first case, the reference bulk solution is $\mathbb{H}_3$, of which the renormalized action vanishes. This just means that the partition function of the dual CFT defined on $\mathbb{CP}^1$ is normalized. Hence, $S_{ren}[\phi] = S_L[\phi,\tilde{g}]$ is consistent with the relation (3.118). In the second case, we get the similar result. The action $S_{ren}[\phi_S]$ should correspond to the semi-classical approximation of the dual CFT partition function defined on the constant curvature Riemann surface. If we write $S_{ren}[\phi_S]$ in z-coordinates, and express the Euler characteristic χ explicitly, then the action becomes

$$S_{ren}[\phi_S] = \frac{l}{16\pi G_3}\int_D d^2z\left((\partial\phi_S)^2 + e^{2\phi_S}\right) + S_{bdy} \ . \quad (3.119)$$

where D is the fundamental domain of corresponding Schottky group. The bulk piece of (3.119) is the same as the *Zograf-Takhtajan* (ZT) Liouville action [85], which was conjectured in [74] as the semi-classical approximation of the partition function of a holographic CFT defined on a constant curvature Riemann surface.

The renormalized action (3.94) can be viewed as the “localized” version of the non-local Polyakov functional action [76],

$$W_P[g] = -\frac{c}{96\pi}\iint d^2z d^2y\sqrt{g_{(0)}(z)}\sqrt{g_{(0)}(y)}R(z)G(z,y)R(y)$$
$$= -\frac{c}{96\pi}\int d^2z\sqrt{g_{(0)}}R\frac{1}{\nabla^2}R \ . \quad (3.120)$$

Here $G(z,y)$ denotes the Green's function of ∇^2, and c is the *Brown-Henneaux central charge*, encoded in the asymptotic symmetry of AdS_3 gravity [90],

$$c = \frac{3l}{2G_3} \ , \quad (3.121)$$

The equivalence between (3.94) and (3.120) may be obtained by considering ϕ as the functional of the metric via the Liouville equation, i.e. $-2\nabla^2\phi = R$, and then rewriting the renormalized action (3.94). On the other hand, it is well-known that the Polyakov functional action (3.120) is the generating functional for the connected stress tensor correlation functions, which is universal for any two-dimensional CFTs since the stress tensor correlators are fixed by the conformal symmetry and the central charge [76]. From the bottom-up perspective, these facts suggest a duality between the AdS_3 gravity and a CFT defined on the conformal boundary, with the CFT central charge being identical to the Brown-Henneaux central charge. As the gravity theory is semi-classical, $G_3 \to 0$, the central charge of the dual CFT must be large, $c \to \infty$. The top-down construction of the duality was conjectured in [16], where the type II string theory in an $AdS_3 \times S^3 \times T^4$ background corresponding to a certain two-dimensional orbifold conformal field theory on the boundary of AdS_3. This conjecture was explicitly verified by the recent studies of tensionless limit $l_s = l$ of the string theory, where equivalence between the spectrum and correlation functions of the tensionless string and the dual conformal field theory has been found [91–95]. However, without being embedded in the string theory, the questions of whether the pure AdS_3 gravity by itself is a consistent quantum gravity and what the holographic dual theory is, are still elusive. It was recently proposed in [96] that quantization of the AdS_3 gravity might not lead to a single CFT, but an ensemble of random CFTs. This proposal is the three-dimensional analog of the duality between the two-dimensional Jackiw-Teitelboim (JT) gravity and a random matrix model [97]. Although the precise description of the random CFTs are not given in [96], evidence for the proposal is provided by the spectral form factor obtained from wormhole partition functions of the AdS_3 gravity, It was found that the spectral form factor behaves similarly to the one in JT gravity.

3.1.4 *Stress tensor and asymptotic symmetry*

The stress tensor in AdS_3 gravity associated with the vacuum solution (3.16) is defined via the functional variation of the renormalized action (3.94) with respect to the boundary metric[11],

$$\delta S_{ren} = -\frac{1}{2}\int d^2z\sqrt{g_{(0)}}\; T_{ij}\delta g^{ij}_{(0)} \; . \tag{3.122}$$

The result is just the Liouville stress tensor [76],

$$T_{ij} = \frac{c}{12\pi}\left[\partial_i\phi\partial_j\phi + \nabla_i\nabla_j\phi - g_{(0)ij}\left(\nabla^2\phi + \frac{1}{2}(\partial\phi)^2\right)\right], \tag{3.123}$$

11 Although the metric $g_{(0)}$ depends on ϕ explicitly, here one should consider the metric as an independent variable when evaluating the variation. This procedure can be understood as fixing the location of the cut-off surface $\rho = \epsilon e^{2\phi}$ and allowing the boundary metric to vary.

which in terms of components can be expressed as

$$T_{zz} = -\frac{c}{12\pi}T^{\phi}, \quad T_{\bar{z}\bar{z}} = -\frac{c}{12\pi}\bar{T}^{\phi} , \quad T_{z\bar{z}} = -\frac{c}{12}R^{\phi} , \tag{3.124}$$

where $(T^{\phi}, \bar{T}^{\phi}, R^{\phi})$ are defined in (3.15). From (3.14), one can rewrite the stress tensor in a more familiar form [86],

$$T_{ij} = \frac{c}{12\pi}\left(g_{(2)ij} + \frac{1}{2}g_{(0)ij}R[g_{(0)}]\right) , \tag{3.125}$$

The equations of motion (3.6) imply the conservation law of the stress tensor as well as the holographic Weyl anomaly [65],

$$\nabla^i T_{ij} = 0 , \qquad T^i_i = \frac{c}{24\pi}R[g_{(0)}] . \tag{3.126}$$

which take the same form as the stress tensor conservation and the trace anomaly in CFTs. This leads to the assumption that a dual CFT is living on the boundary of AdS$_3$, providing the Gubser-Klebanov-Polyakov-Witten (GKPW) relations [41, 42],

$$e^{-S_{ren}} \approx e^{-W_{\text{CFT}}[g_{(0)}]} \tag{3.127}$$

Strictly speaking, the left-hand side of GKPW relation (3.127) should include all the saddle point contributions to the gravity partition function, with distinct projective structures, which I have discussed in (3.69). However, in the semi-classical limit $c \to \infty$, only one saddle point dominates since the renormalized action scales as c. This provides the validity of (3.127). By a functional variation of two side of (3.127) with respect to the boundary metric, we identify the gravitational stress tensor with the vacuum expectation value of the dual CFT stress tensor,

$$T_{ij} = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta}{\delta g^{ij}_{(0)}}W_{\text{CFT}}[g_{(0)}] = \langle T_{ij}\rangle . \tag{3.128}$$

This identification is consistent, since the Liouville stress tensor in (3.123) is indeed the one-point function of the stress tensor in a CFT with central charge c and curved metric $g_{(0)}$ [76].

Stress tensor in excited backgrounds. One can also generalize (3.128) to an excited state background in the dual CFT. For instance, we consider a set of local primary operators, denoted as $X = O_1O_2\cdots$, inserted in the path integral of the dual CFT. In such a background, the expectation value of the CFT stress tensor can be written as

$$\langle T_{ij}\rangle_X = T^{Liouville}_{ij} + t_{ij} , \tag{3.129}$$

where the additional term t_{ij} comes from the insertion of X. By conformal Ward identities [98], t_{ij} satisfies the following three equations[12],

$$\nabla^a t_{ab} = \sum_k \partial_b\delta^{(2)}(z - z_k) , \tag{3.130}$$

12 The conformal anomaly is encoded in the Liouville part.

and

$$\epsilon^{ab}t_{ab} = -i\sum_k s_k\delta^{(2)}(z-z_k),\ \ t^a{}_a = -\sum_k \Delta_k\delta^{(2)}(z-z_k). \quad (3.131)$$

Here, (Δ_k, s_k) are scaling dimension and spin of O_k, and delta function is defined as a scalar. Those three equations imply that if we replace the Liouville stress tensor by $\langle T_{ij}\rangle_X$, then (3.126), which are equivalent to the vacuum Einstein's equations (3.6), are violated at the insertions. Such violation is expected, because usually operators in the dual CFT are understood as additional matter fields coupled with gravity, which of course lead to the violation of vacuum Einstein's equation. However, within the pure AdS$_3$ gravity, we have another way to interpret such violations. We can interpret the boundary operators as conical defect lines in the bulk, which arise from quotient of a smooth AdS$_3$ solution by certain discrete subgroup of PSL(2,$\mathbb{C}$) containing elliptic elements. As a simplest example, we consider a $\mathbb{H}_3$ background. We insert $X = O(0)O(\infty)$ on $\mathbb{CP}^1$, with the conformal dimensions of the primary field O given by,

$$h = \bar{h} = \frac{c}{24}\left(1-\frac{1}{n^2}\right)\ , \quad n\in\mathbb{Z}_+\ . \quad (3.132)$$

Then, the gravitational stress tensor, identical to the CFT stress tensor by (3.128), is given by[13]

$$T_{zz} = -\frac{h}{2\pi}\frac{1}{z^2}\ , \quad T_{\bar{z}\bar{z}} = -\frac{\bar{h}}{2\pi}\frac{1}{\bar{z}^2}\ . \quad (3.133)$$

Using (3.124) for (3.133) and inserting it into (3.16), one finds that the bulk metric is deformed from $\mathbb{H}_3$ to a Bañados metric

$$ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{l^2}{\rho}|d\bar{z} - \rho\mathcal{L}dz|^2\ , \quad (3.134)$$

with

$$\mathcal{L} = -\frac{12\pi}{c}T_{zz} = \frac{1}{4}\left(1-\frac{1}{n^2}\right)\frac{1}{z^2} = \frac{1}{2}\{z^{1/n};z\}\ . \quad (3.135)$$

We can transform this geometry to $\mathbb{H}_3$ by a boundary coordinate transformation and a compensated finite PBH transformation. The boundary coordinate transformation $w = z^{1/n}$ is read out from the Schwarzian derivative in (3.135), and it tells us that (3.134) is a $\mathbb{Z}_n$ quotient of $\mathbb{H}_3$. This confirms the duality between X with a $\mathbb{Z}_n$ conical defect line in the bulk. In particular, the conformal weight h is encoded in the monodromy property of the covering map $w = z^{1/n}$. For a point z going once around the origin, $z \to e^{2\pi i}z$, it is transformed via an elliptic PSL(2,$\mathbb{C}$) transformation on the covering space,

$$w = z^{1/n} \to w' = (e^{2\pi i}z)^{1/n} = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n}\end{pmatrix}\circ w\ . \quad (3.136)$$

13 Here I adopt the usual convention in CFT, i.e., $T = -2\pi T_{zz}$ and $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$.

The above simple example reveals the close relation between geometries and operators in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, which essentially arise from the topological nature of AdS_3 gravity. We will use this relation frequently when discussing the holographic entanglement entropy. As an additional remark, for operators with conformal dimensions $h \sim \mathcal{O}(c)$, such as (3.132), we call them *heavy operators*, since the insertion of them will lead to a large deformation of the background geometry in the bulk. On the other hand, for operators with $h \sim \mathcal{O}(c^0)$, we call them *light operators*. This is because their backreaction to a background geometry is of order $\mathcal{O}(1/c)$, which can be neglect in the semi-classical limit $c \to \infty$.

Asymptotic symmetry in AdS$_3$ gravity. In the bottom-up approach to AdS/CFT duality, the gravity counterpart of global symmetry in the dual CFT is called the *asymptotic symmetry*. In general, asymptotic symmetry is defined as the group of gauge transformations on bulk gauge fields, which preserve a given set of boundary conditions and leave the global charges invariant. Infinitesimal asymptotic symmetry transformations form an algebra, called the *asymptotic symmetry algebra* (ASG), via the Possion bracket of the bulk gravity theory, which is dual to the Dirac bracket of the boundary CFT in the semi-classical limit. The set of all bulk configurations fulfilling the given boundary conditions and carrying same global charges defines a reduced phase space of the gravity theory. Asymptotic symmetry transformations generate transformations among bulk configurations in a same reduced phase space, in the same way as how global symmetry transformations in the dual CFT generate transformations among states in a same representation.

In pure AdS_3 gravity, the bulk field is just the metric, and bulk gauge transformations are diffeomorphisms. The global charges (mass and angular momentum) of a bulk configuration are related to the conformal family $(h, \bar{h})$ of the dual CFT state as

$$M \sim \Delta = \frac{h+\bar{h}}{2} \ , \quad J \sim s = \frac{h-\bar{h}}{2} \ . \tag{3.137}$$

Those charges are topological information, since they are encoded in the monodromy property of the underlying covering map, as I showed in the example (3.136). A boundary condition commonly used in the literature for deriving ASG of AdS_3 gravity is the *Brown-Henneaux boundary condition* [90]. It requires that infinitesimal asymptotic symmetry transformations need to keep the following Bañados metric invariant up to order of $1/\rho$,

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} |dz|^2 - l^2 (\mathcal{L} dz^2 + \bar{\mathcal{L}} d\bar{z}^2) + \mathcal{O}(\rho) \ . \tag{3.138}$$

In other words, the Brown-Henneaux boundary condition requires

$$\delta G_{\rho\rho} = \delta G_{\rho i} = 0 \ , \quad \delta G^{(0)}_{ij} = 0 \ , \tag{3.139}$$

but allows the stress tensors $\mathcal{L}$ and $\bar{\mathcal{L}}$ (up to a normalization) to vary. It can be worked out that an infinitesimal diffeomorphism fulfilling such conditions is a combination of a PBH transformation and a infinitesimal boundary diffeomorphism,

$$\rho \to \rho(1+2\sigma) \ , \ z \to z - \rho\partial_{\bar{z}}\sigma + \epsilon^z \ , \ \bar{z} \to \bar{z} - \rho\partial_z\sigma + \epsilon^{\bar{z}}. \quad (3.140)$$

Here σ corresponds to PBH and ϵ^z corresponds to the boundary diffeomorphism. These two infinitesimal parameters need to obey the following relations

$$2\sigma = \partial_z\epsilon^z + \partial_{\bar{z}}\epsilon^{\bar{z}} \ , \quad \epsilon^z = \epsilon(z) \ , \quad \epsilon^{\bar{z}} = \bar{\epsilon}(\bar{z}) \ . \quad (3.141)$$

One can replace the old coordinates in the metric by the new coordinates (3.140), and then expand the metric in the old coordinates again up to the first order of ϵ. The result reads

$$\begin{aligned} ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{l^2}{\rho}\left(|dz|^2 + \partial_{\bar{z}}\epsilon^z d\bar{z}^2 + \partial_z\epsilon^{\bar{z}}dz^2\right) \\ - l^2\left(\mathcal{L}'dz^2 + \bar{\mathcal{L}}'d\bar{z}^2\right) + \mathcal{O}(\rho,\epsilon^2) \ , \end{aligned} \quad (3.142)$$

where $\mathcal{L}'$ (similar for $\bar{\mathcal{L}}'$) reads

$$\mathcal{L}' = \mathcal{L} - 2\mathcal{L}\partial_z\epsilon^z - \epsilon^z\partial_z\mathcal{L} - \frac{1}{2}\partial_z^3\epsilon^z \ . \quad (3.143)$$

Notice that we keep $\partial_z\epsilon^{\bar{z}}$ and $\partial_{\bar{z}}\epsilon^z$ in (3.142) since they can not vanish on the whole boundary. We will use this to derive the algebra later. The normalized stress tensor T is related to $\mathcal{L}$ as

$$T = -2\pi T_{zz} = \frac{c}{6}\mathcal{L} \ . \quad (3.144)$$

Hence we have

$$\delta T = -2T\partial_z\epsilon^z - \epsilon^z\partial_z T - \frac{c}{12}\partial_z^3\epsilon^z \ , \quad (3.145)$$

which is identical to the variation law of CFT stress tensor under an infinitesimal conformal transformation. In the usual approach [90], one proceeds by performing the canonical analysis of AdS_3 gravity and rewriting δT as a Possion bracket, and the result will give rise to ASG. Here, I will not present this procedure, but derive ASG from the CFT perspective by assuming the validity of AdS_3/CFT_2 duality.

We first notice that (3.140) acts on the boundary metric as a conformal transformation. The PBH transformation shifts the boundary metric by a Weyl factor, which is canceled by a compensated boundary diffeomorphism. The non-trivial terms in $\delta g^{-1}_{(0)}$ read

$$\delta g^{zz}_{(0)} = -4\partial_{\bar{z}}\epsilon^z \ , \quad \delta g^{\bar{z}\bar{z}}_{(0)} = -4\partial_z\epsilon^{\bar{z}} \ , \quad (3.146)$$

which vanish except at poles of ϵ. Since two-dimensional CFT defined on a flat background satisfies the local scaling invariance. This means that the path integral under conformal transformation is invariant,

$$0 = \frac{\delta \int \mathcal{D}\Phi \; T X e^{-S[\Phi]}}{Z_0} = \langle \delta T X \rangle + \langle T \delta X \rangle - \langle T X \delta S \rangle \ , \tag{3.147}$$

with Z_0 denoting the vacuum partition function. Followed from (3.122) and (3.146), the variation of the CFT action reads

$$\delta S = \frac{1}{\pi} \int d^2 y \left(\partial_{\bar{y}} \epsilon^y T(y) + \partial_y \epsilon^{\bar{y}} \bar{T}(\bar{y}) \right) \ . \tag{3.148}$$

which allows us to identify

$$\langle \delta T(z) \rangle_X = \frac{1}{\pi} \int d^2 y \; \partial_{\bar{y}} \epsilon^y \langle T(z) T(y) \rangle_X \ . \tag{3.149}$$

A convenient way to proceed is to choose $\epsilon^y = \frac{1}{y-w}$. Then, using the identity $\partial_y \frac{1}{y-w} = \pi \delta^{(2)}(y-w)$, we write the above equation as

$$\langle \delta T(z) \rangle_X = \langle T(w) T(z) \rangle_X \ . \tag{3.150}$$

By the GKPW relation, we identify $T(z) = \langle T(z) \rangle_X$ and $\delta T(z) = \langle \delta T(z) \rangle_X$. Then, inserting (3.145) into the above equation and using $\epsilon^z = \frac{1}{z-w}$, we obtain the operator product expansion (OPE) of stress tensors,

$$T(w) T(z) \sim \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} \ . \tag{3.151}$$

Proceeding with the modes expansion

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \ , \tag{3.152}$$

one arrives at the Virasoro algebra in the dual CFT,

$$[L_n, L_m] = (n-m) L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \ . \tag{3.153}$$

Finally, I would like to mention that the Virasoro symmetry is not the unique asymptotic symmetry in AdS_3 gravity. Modifications of the Brown-Henneaux boundary condition (3.139), will lead to the change of the asymptotic symmetry algebra from Virasoro to something else, such as a warped conformal algebra [99], a centerless warped conformal algebra [100] or the Heisenberg algebra [101]. There is a comprehensive discussion for the most general boundary conditions in AdS_3 gravity in [102], where the asymptotic symmetry algebra appears to be the affine $\mathfrak{sl}(2)_k$ Kac-Moody algebra, and the Virasoro algebra can be realized in this system by considering the twisted Sugawara construction for the Kac-Moody currents.

3.2 RÉNYI AND ENTANGLEMENT ENTROPY

In this section, I will review the Rényi and entanglement entropy in the AdS_3/CFT_2 context, based on the discussion in Section 3.1. When discussing the entanglement entropy in a general two-dimensional CFT, we usually classify the entangling region as a single interval or a set of disjoint intervals. This is because the shape of a non-selfintersecting curve is unimportant due to the infinite conformal symmetries in two dimensions. Apart from the universal behavior mentioned in (2.68), in generic situations, calculating the Rényi and entanglement entropy in a two-dimensional CFT requires additional information about the theory, such as the full spectrum and the OPE coefficients. Those kinds of information are not readily accessible in the bottom-up AdS_3/CFT_2 models. Fortunately, due to the semi-classical limit $c \to \infty$ of a holographic CFT_2, this required information becomes unnecessary, and as a consequence, the Rényi and entanglement entropy are still under the control of the conformal symmetry and other properties such as monodromies and vacuum conformal block dominance. In the following, I will explain the details of this topic. The explanations will be given from both the holographic and CFT perspectives, which proceed in fact parallel to each other to a large extent. The main references of this review are [103] from the holographical point of view, and [104, 105] from the CFT point of view.

3.2.1 *Holographic perspective*

In pure AdS_3 gravity, the feature that any AdS_3 space can be obtained from a quotient of the Poincaré AdS_3 allows one to solve the holographic Rényi and entanglement entropy more rigorously, without using the cosmic string description proposed in [58, 106]. The problem of solving Rényi entropy on the gravity side is essentially to find the exact AdS_3 solutions with a given replica surface as its conformal boundary, or in other words, to find the Schottky uniformizations of the replica surface. From Section 3.1, we learned that Schottky uniformizations of a given compact Riemann surface with nontrivial topology are not unique. Each Schottky uniformization defines a projective structure on the Riemann surface, which is subordinated to the complex structure and contains the information about the global structure of the corresponding AdS_3 phase. As I explained in (3.69), the gravity partition function includes all saddle contributions with distinct projective structures, which however is dominated by the contribution from a single AdS_3 phase in the semi-classical limit $c \to \infty$. This statement also holds for the Rényi entropy,

$$S_n = \frac{1}{1-n} \log Z_n \approx \frac{1}{n-1} \min_{\{\sigma\}} \{S^{\sigma}_{ren}\} \ . \tag{3.154}$$

Analogous to the genus one case discussed in Section 3.1.3, the Rényi and entanglement entropy can also exhibit phase transitions. This is because the dominant AdS_3 phase is determined by the complex structure of the boundary replica surface, which relies on the positions of endpoints of entangling intervals. Thus, shifting the positions of those endpoints may lead to the change of the dominant phase.

In the following, I will introduce the *accessory parameter approach* to uniformizations of replica surfaces, which in this thesis are considered as different branched coverings of $\mathbb{CP}^1$. Then, I will explain the topological structures of such compact surfaces, such as the genus and the fundamental group. With a fixed canonical basis of the fundamental group, the monodromy conditions (3.66) required by Schottky uniformization will be used to solve the accessory parameters perturbatively, which implicitly determine the developing map associated with the Schottky uniformization. For branched coverings of a torus, they are related to the holographic entanglement entropy in a black hole background, which was discussed in [107].

Setup of problem. Let me first introduce the setup of the problem. We focus on the Rényi and entanglement entropy in the dual CFT defined on $\mathbb{CP}^1$, coordinated by w. In general, we can choose N disjoint intervals on the real axis of $\mathbb{CP}^1$ as the entangling region, denoted as,

$$A = [w_1, w_2] \cup [w_3, w_4] \cup \cdots \cup [w_{2N-1}, w_{zN}] \ , \tag{3.155}$$

with $w_1 < w_2 < \cdots < w_{2N}$. When we use the replica trick for the Rényi entropy S_n, the corresponding CFT partition function is defined on a new Riemann surface, called the replica surface. The replica surface, denoted as $R_{n,N}$, is a n-sheeted branched covering of $\mathbb{CP}^1$, with $2N$ branched points being the endpoints w_i's of the entangling region A. The relation between $R_{n,N}$ and $\mathbb{CP}^1$ is given via a projection map,

$$\pi : R_{n,N} \to \mathbb{CP}^1 = R_{n,N}/\mathbb{Z}_n \ , \tag{3.156}$$

where $\mathbb{Z}_n$ is a group of automorphisms of $R_{n,N}$. We denote the generator of $\mathbb{Z}_n$ as $\hat{\eta}$, which transforms a point w on one sheet of $R_{n,N}$ to another point w on the next sheet. Furthermore, since $R_{n,N}$ is compact due to the compactness of $\mathbb{CP}^1$, it admits Schottky uniformizations,

$$p : \Omega \to R_{n,N} = \Omega/\Gamma \ , \tag{3.157}$$

where Ω is an connected open subset of $\mathbb{CP}^1$, coordinated by z, and Γ is a Schottky group, which we do not know at present. The automorphism $\hat{\eta}$ of $R_{n,N}$ can be lifted to an automorphism η of Ω, denoted as $p \circ \eta = \hat{\eta} \circ p$, with η acting on Ω as a fractional transformation[14]. Then, the composition $J^{-1} = \pi \circ p$ is a surjective map,

$$J^{-1} : \Omega \to \mathbb{CP}^1 = \Omega/\langle \Gamma, \eta \rangle \ , \quad J^{-1} : z \to w \ , \tag{3.158}$$

14 In this case, the Schottky uniformization is called an *extended Schottky uniformization*. The group $\langle \Gamma, \eta \rangle$ is called the *extended Schottky group*, and the Schottky group Γ is a normal subgroup of it. See [108] for detailed explanations.

which is locally biholomorphic and needs to satisfy the monodromy conditions (3.66). In addition, near branched points, $z(w)$ must obey the following short distance behaviors,

$$z(w) - z(w_i) \sim (w - w_i)^{1/n} \ . \tag{3.159}$$

This is because the n-sheeted branched structure of a neighborhood of w_i is uniformized in z-coordinates. The Riemannian structure on $R_{n,N}$ is inherited from the original $\mathbb{CP}^1$. This means that in w-coordinates the metric on $R_{n,N}$ takes a flat form

$$d\hat{s}^2 = |dw|^2 \ . \tag{3.160}$$

However, this does not means $R_{n,N}$ is a flat surface. When we transform to z-coordinates, the metric becomes

$$d\hat{s}^2 = e^{2\phi}|dz|^2 \ , \quad \phi = -\frac{1}{2}\log|dz/dw|^2 \ . \tag{3.161}$$

Using (3.9) as well as the short distance behavior (3.159), one finds that the curvature on $R_{n,N}$ is delta-function-singular at each branched point,

$$\sqrt{g_0}R \sim R^{\phi} \sim (1-n)\delta^{(2)}(z - z(w_i)) \ , \quad z \to z(w_i) \ . \tag{3.162}$$

Our main task is to find the map $z(w)$ which obeys (3.66) and (3.159). The former conditions depend on the choice of the canonical basis of $\pi_1(R_{n,N})$. Different basis may lead to distinct solutions to the map. Assuming that $z(w)$ is worked out in a chosen basis, one can obtain the bulk metric by inserting the Liouville field $\phi(z,\bar{z})$ into (3.16). In fact, since $e^{2\phi} = |dw/dz|^2$, we can perform a boundary coordinate transformation $z \to w$ to transform the bulk metric into the Bañados form (3.17), which is uniquely characterized by the stress tensor

$$T(w) = \frac{c}{6}\mathcal{L} = \frac{c}{12}\{z;w\} \ . \tag{3.163}$$

In the case of $N = 1$, since the topology of $R_{n,1}$ is still a Riemann sphere for $n \in \mathbb{Z}_+$, it is easy to work out the map $z(w)$, given by

$$z = \left(\frac{w - w_1}{w_2 - w}\right)^{\frac{1}{n}} \ , \tag{3.164}$$

which is an analog of (3.102). The calculation for the renormalized action is analogous to the example in (3.108), and the resulting Rényi entropy reads

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left|\frac{w_1 - w_2}{\epsilon}\right| \ . \tag{3.165}$$

However, for $N \geq 2$, there is no obvious way to construct the map $z(w)$. Therefore, in the following discussion, I will introduce the accessory parameter approach to the uniformization problem.

Accessory parameter approach. Accessory parameter approach is very useful in solving the Schottky uniformizations of branched coverings of $\mathbb{CP}^1$. The idea of it goes as follows. We insert the required short distance behavior (3.159) of $z(w)$ into the stress tensor (3.163). The result shows that at each branched point w_i, the stress tensor has a second order pole,

$$T(w) \sim \frac{h}{(w-w_i)^2} + \mathcal{O}((w-w_i)^{-1}) \ , \quad w \to w_i \ , \tag{3.166}$$

with the weight h being determined by the replica index n,

$$h = \frac{c}{24}\left(1 - \frac{1}{n^2}\right) \ . \tag{3.167}$$

This motivates the following ansatz for $T(w)$, given by

$$T(w) = \sum_{i=1}^{2N}\left(\frac{h}{(w-w_i)^2} + \frac{p_i}{w-w_i}\right) \ . \tag{3.168}$$

where p_i's are called the *accessory parameters*. They are undetermined at present. To confirm the validity of this ansatz, we need to show the following two facts. First, $T(w)$ as a function of w is always single-valued, so that it does not contains any fractional power of $(w-w_i)$. Second, any non-negative power of $(w-w_i)$ is absent in $T(w)$. Then, those two facts will fix the form of $T(w)$ as (3.168).

The first one is true due to the $\mathbb{Z}_n$ symmetry of $R_{n,N}$. If $T(w) \sim \{z; w\}$ is single-valued on $R_{n,N}$, then we should have

$$\{\eta \circ z; \hat{\eta} \circ w\} \stackrel{!}{=} \{z; w\} \ . \tag{3.169}$$

By the symmetry $\hat{\eta} \circ w = w$, the above equation becomes

$$\{\eta \circ z; w\} = \{\eta \circ z; z\}\left(\frac{dz}{dw}\right)^2 + \{z; w\} \stackrel{!}{=} \{z; w\} \ , \tag{3.170}$$

which is indeed true, since the Schwarzian derivative vanishes for the fractional transformation, $\{\eta \circ z; z\} = 0$. For the second one, we assume that there is no branched point located at $w = \infty$. This implies that under a inversion $w \to \tilde{w} = 1/w$, the stress tensor $T(\tilde{w}) = (d\tilde{w}/dw)^{-2}T(w) = w^4 T(w)$ is finite at $\tilde{w} = 0$. In other words, when approaching to infinity, the stress tensor $T(w)$ needs to satisfy the fall-off condition,

$$T(w) \sim \frac{1}{w^4} + \mathcal{O}(w^{-5}) \ , \quad w \to \infty \ . \tag{3.171}$$

Thus, non-negative powers of $(w-w_i)$ can not exist in the stress tensor. In fact, (3.171) not only excludes the non-negative powers, but also imposes three constraints on the accessory parameters, given by

$$\sum_i^{2N} p_i = 0 \ , \ \sum_i^{2N}(h + p_i w_i) = 0 \ , \ \sum_{i=1}^{2N}(2hw_i + p_i w_i^2) = 0 \ . \tag{3.172}$$

These are the three coefficients associated with w^{-1}, w^{-2} and w^{-3} respectively in the $1/w$ expansion of the stress tensor. In the dual CFT, (3.172) are just the three global conformal Ward identities [98].

Under the ansatz (3.168) for the stress tensor, we can in principle solve Schottky uniformizations of $R_{n,N}$ by the following two steps. First, we solve the differential equation $T(w) = \frac{c}{12}\{z;w\}$ for the map $z(w)$, which depends on the accessory parameters p_i's. Second, we choose a canonical basis of $\pi_1(R_{n,N})$ and impose the corresponding monodromy conditions (3.66) on $z(w)$, which will fix the values of p_i's. After doing these, one may try to compute the renormalized action (3.96) associated with the resulting Bañados geometry. However, there is an alternative way to get the result. It turns out that the variations of the renormalized action of a Bañados geometry with respect to the positions of branched points precisely give rise to the accessory parameters [85, 103, 109, 110],

$$\frac{\partial S^\sigma_{ren}}{\partial w_i} = -np_i \ . \tag{3.173}$$

An analogous relation holds for the anti-holomorphic part. While the above formula is quite nontrivial from gravity perspective, on the CFT side, it is just the consequence of the conformal Ward identity. Therefore, the accessory parameter approach provides a very efficient way to compute the Rényi and entanglement entropy,

$$S_n = \min_{\{\sigma\}}\{S^\sigma_n\} \ , \quad \frac{\partial S^\sigma_n}{\partial w_i} = \frac{n}{1-n}p_i \ , \tag{3.174}$$

and

$$S_{EE} = \min_{\{\sigma\}}\{S^\sigma_{EE}\} \ , \quad \frac{\partial S^\sigma_{EE}}{\partial w_i} = -\partial_n p_i|_{n=1} \ . \tag{3.175}$$

In fact, since S^σ_n is directly determined by the accessory parameters, but not by the map $z(w)$, the two steps described in above can be further simplified by introducing the Wilson loop formalism, which I will discuss in below. The Wilson loop formalism allows us to express a monodromy matrix in terms of a path-ordered exponential of a $\mathfrak{sl}(2,\mathbb{C})$ matrix, which only depends on $T(w)$. By imposing the monodromy conditions (3.66) required by Schottky uniformization on monodromy matrices, we are able to determine the accessory parameters by calculating the Wilson loops.

Monodromy and Wilson loop. For the differential equation $T(w) = \frac{c}{12}\{z;w\}$, a generic solution $z(w)$ takes the following form,

$$z(w) = \frac{\varphi_1(w)}{\varphi_2(w)} \ , \tag{3.176}$$

where φ_1 and φ_2 are two linearly independent solutions to the Fuchsian differential equation [111],

$$\varphi'' + \frac{6}{c}T(w)\varphi = 0 \ . \tag{3.177}$$

In practice, solving Fuchsian differential equation is a very hard problem, except for the cases of $N = n = 2$ [104]. Essentially, this is because $R_{2,2}$ is topologically a torus, and one can use the Schwarz-Christoffel mapping to construct the solutions. For other cases, the genus of $R_{n,N}$ becomes higher than one, and exact solutions to Fuchsian differential equations are unknown. Nevertheless, I assume that $\varphi_{1,2}$ have been worked out, since the following discussion does not rely on the explicit forms of them.

In general, the solutions $\varphi_{1,2}(w)$ are multi-valued functions, which can be checked by taking the limit $w \to w_i$ for the Fuchsian differential equation. This implies the nontrivial monodromy properties of $\varphi_{1,2}$. Going around an arbitrary closed loop γ in w-coordinates, the functions $\varphi_{1,2}$ transform in the linear space spanned by $\{\varphi_1, \varphi_2\}$, which leads to a fractional transformation on z,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to M_\gamma \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \ , \quad z \to M_\gamma \circ z \ , \quad M_\gamma \in \mathrm{PSL}(2,\mathbb{C}) \ . \tag{3.178}$$

To incorporate the monodromy properties of $\varphi_{1,2}$, we can rewrite the Fuchsian differential equation as a first-order differential equation,

$$\partial_w W + W a_w = 0 \ , \tag{3.179}$$

by defining the Wronskian matrix W and a connection $a = a_w dw$,

$$W = \begin{pmatrix} \varphi_1' & \varphi_1 \\ \varphi_2' & \varphi_2 \end{pmatrix} \ , \quad a_w = \begin{pmatrix} 0 & -1 \\ \frac{6}{c}T(w) & 0 \end{pmatrix} \ . \tag{3.180}$$

Notice that the Wronskian $\det(W)$ is a constant, because there is no first derivative term in the Fuchsian differential equation. We can set $\det(W) = 1$ since an overall rescaling of φ_1 and φ_2 does not change the map $z(w)$. The solution to $W(w)$ can be formally written as

$$W(w) = u_0 \mathcal{P} e^{-\int_{w_0}^{w} a} \ , \tag{3.181}$$

where w_0 is an arbitrary base point and $u_0 = W(w_0)$ is a constant $\mathrm{PSL}(2,\mathbb{C})$ matrix. $\mathcal{P}$ denotes the path-ordering. Under the transformation (3.178), the Wronskian matrix transforms as $W \to M_\gamma W$. Thus, using (3.181), we can write the monodromy matrix as a Wilson loop,

$$M_\gamma = u_0 \mathcal{W}(\gamma) u_0^{-1} \ , \quad \mathcal{W}(\gamma) = \mathcal{P} e^{-\oint_\gamma a} \ . \tag{3.182}$$

As I will explain later, elements of $\pi_1(R_{n,N})$ correspond to closed loops in w-coordinates, which encircle even number of branched points. This means that for a given basis of $\pi_1(R_{n,N})$ with the base point w_0, denoted as

$$\pi_1(R_{n,N}) = \left\langle [a_1], [b_1], \cdots, [a_g], [b_g] : \prod_{i=1}^{g} [a_i, b_i] = \mathbb{1} \right\rangle \ , \tag{3.183}$$

where $g = (N-1)(n-1)$ is the genus of $R_{n,N}$, the actions A_i and B_i transform an arbitrary base point w_0 back to the its original position along the loops a_i and b_i. This allows us to express (3.66) via monodromy matrices introduced in above. The first ones in (3.66) reads

$$z(A_i \circ w_0) = M_{a_i} \circ z(w_0) \overset{!}{=} z(w_0) \ , \quad i = 1, \cdots, g \ . \tag{3.184}$$

which by (3.182) impose the triviality conditions on Wilson loops

$$M_{a_i} = \mathcal{W}(a_i) = \mathbb{1} \ , i = 1, \cdots, g \ . \tag{3.185}$$

These conditions will enable us to solve p_i's. The second ones in (3.66) read

$$z(B_i \circ w_0) = M_{b_i} \circ z(w_0) \ , \quad t_i = M_{b_i} \ , \tag{3.186}$$

which can be calculated after we solve p_i's. However, as we are only interested in the Rényi and entanglement entropy, it is not necessary to calculate them.

Basis of fundamental group. To represent basis of $\pi_1(R_{n,N})$, let me first define $\{\gamma_M | M = 1, \cdots, 2N\}$ as a set of simple closed curves on $\mathbb{CP}^1$ with the same base point w_0,

$$\gamma_M : [0,1] \to \mathbb{CP}^1 \ , \quad \gamma_M(0) = \gamma_M(1) = w_0 \ , \tag{3.187}$$

each of which goes once around the branched point w_M in the counter-clockwise direction. Under the inverse map $\pi^{-1} : \mathbb{CP}^1 \to R_{n,N}$, those closed curves are lifted to open curves on $R_{n,N}$, and endpoints of each curves are located at two adjacent sheets. More clearly, suppose the base point $\gamma_M(0)$ is on the k-th sheet of $R_{n,N}$. Then, for odd M the point $\gamma_M(1)$ is on the $(k+1)$-th sheet, while for even M it is on $(k-1)$-th sheet[15]. The endpoints of the product γ_M^n are identical, hence it is a loop on $R_{n,N}$, which however is a trivial element in the fundamental group, $[\gamma_M^n] = \mathbb{1} \in \pi_1(R_{n,N})$. A nontrivial element of the fundamental group is generically constructed by products of distinct γ_M's, such that endpoints of the curve are identical, for instance, $\gamma_2\gamma_1$ and $\gamma_3^{-1}\gamma_1$. A choice of canonical basis of the fundamental group can be given as following. We denote a subset of a-loops and b-loops as

$$a_m^{(0)} = \gamma_{2m}\gamma_{2m-1} \ , \quad b_m^{(0)} = \gamma_{2m+1}\gamma_{2m} \ , \tag{3.188}$$

with $m = 1, \cdots, N-1$. Each of those loops encircles two adjacent branched points. Notice that we did not include $a_N^{(0)} = \gamma_{2N}\gamma_{2N-1}$ in (3.188), because $a_N^{(0)} \cdots a_1^{(0)}$ is homotopically equivalent to a loop encircling all branched points, which hence is trivial in the fundamental group. Therefore, $[a_N^{(0)}]$ is identical to the inverse of $[a_{N-1}^{(0)} \cdots a_1^{(0)}]$. A

15 The $(n+1)$-th sheet is identical to the first sheet.

Similar statement holds for excluding $b_N^{(0)} = \gamma_{2N}\gamma_1$ in (3.188). The remaining generators can be constructed by using the n-sheeted branched structure of $R_{n,N}$, given by

$$a_m^{(k)} = \gamma_1^{-k} a_m \gamma_1^k \ , \quad b_m^{(k)} = \gamma_1^{-k} b_m \gamma_1^k \ , \tag{3.189}$$

with $k = 1, \cdots, n-2$. (3.188) and (3.189) together include $2(N-1)(n-1)$ generators and provide a basis of $\pi_1(R_{n,N})$. Different choices of the basis can be obtained by changing (3.188). For instance, given a partition of $2N$ branched into N pairs, one can associate each pair with an a-loop, which encircles the two branched points in the pair. A b-loop can be obtained by associating two branched points selected from two distinct pairs. For more detailed discussion, see [103].

Now, let me discuss the monodromies associated with a and b-loops defined in (3.188) and (3.189). For the a-loops, the triviality conditions (3.185) read

$$M(a_m^{(k)}) = u_0 \mathcal{W}(a_m^{(k)}) u_0^{-1} = \mathbb{1} \ , \tag{3.190}$$

which under the decomposition of the Wilson loop, yields

$$\mathcal{W}(\gamma_{2m-1})\mathcal{W}(\gamma_{2m}) = \mathbb{1} \ . \tag{3.191}$$

For the b-loops, the monodromy matrices, which are generators of the Schottky group Γ, are given by

$$M(b_m^{(k)}) = \eta^k M(b_m^{(0)}) \eta^{-k} \ , \ 0 \le k \le n-2, \ 0 < m < N \ , \tag{3.192}$$

where η as the generator of $\mathbb{Z}_n$ is associated with γ_1, defined as

$$\eta = u_0 \mathcal{W}(\gamma_1) u_0^{-1} \ , \quad \eta^n = \mathbb{1} \ . \tag{3.193}$$

Notice that for $0 \le k < n-2$, we have $\eta M(b_m^{(k)}) = M(b_m^{(k+1)})\eta$. A special case is that for $k = n-2$ we have $\eta M(b_m^{(n-2)}) = M(b_m^{(n-1)})\eta$, where $b_m^{(n-1)}$ is not the generator of the fundamental group. However, since the homotopy class is trivial[16], $[b_m^{(0)} \cdots b_m^{(n-1)}] = \mathbb{1}$, the following identity holds for the monodromy matrices,

$$M(b_m^{(0)}) \cdots M(b_m^{(n-2)}) = M(b_m^{(n-1)})^{-1} \ . \tag{3.194}$$

Combining the above results, one can conclude that $\eta\Gamma = \Gamma\eta$. Hence, the Schottky group is a normal subgroup of the extended Schottky group, which fulfills the requirement for an extended Schottky uniformization [108],

$$\Gamma = \langle M(b_m^{(k)}) | 0 \le k \le n-2, 0 < m < N \rangle \lhd \langle \Gamma, \eta \rangle \ . \tag{3.195}$$

16 A convenient way to see this is to continuously deform $b_m^{(k)}$ to a new loop which goes along the branched cut $[w_{2m}, w_{2m+1}]$ on $(k+1)$-th sheet, and then turn back on the $(k+2)$-th sheet. Under such a deformation, it is clear that the path $b_m^{(0)} \cdots b_m^{(n-1)}$ is equivalent a point.

Since the Rényi and entanglement entropy are determined by p_i's, I will only focus on a-loops and solving the triviality conditions for p_i's in the following discussion.

Perturbative calculation of accessory parameters. In principle, the Wilson loop can be calculated by expanding the path-ordered exponential,

$$\mathcal{W}(\gamma) = \mathbb{1} + \sum_{j=1}^{\infty} \oint dx_1 \int_{w_0}^{x_1} dx_2 \cdots \int_{w_0}^{x_{j-1}} dx_j \; a_{x_j} \cdots a_{x_1} \ , \quad (3.196)$$

where the connection $a = a_{x_j} dx_j$ is defined in (3.180). Components of $\mathcal{W}(\gamma)$ obtained in this way are polynomials of p_i's of infinite order. In practice, it is difficult to evaluate the expansions and convert the polynomials of p_i's into compact forms. This difficulty in solving p_i's via Wilson loop is the same as the one in solving the Fuchsian differential equation. However, as we are mostly interested in the replica limit[17] $n \to 1$, in which the accessory parameters should scale as $p_i \sim \mathcal{O}(\delta_n)$, with $\delta_n = (n-1)$, we can perturbatively expand the first-order differential equation (3.179) in δ_n, and solve them to the first order. More precisely, we assume

$$p_i = \frac{c}{12}\delta_n \rho_i + \mathcal{O}(\delta_n^2) \ , \quad n \to 1 \ , \quad (3.197)$$

then the stress tensor (3.168) reads

$$T(w) = \frac{c}{12}\delta_n t(w) + \mathcal{O}(\delta_n^2) \ , \quad (3.198)$$

with

$$t(w) = \sum_{i=1}^{2N} \left(\frac{1}{(w-w_i)^2} + \frac{\rho_i}{w-w_i} \right) \ . \quad (3.199)$$

We can perform decompositions for $W(w)$ and a_w in (3.180) as,

$$W(w) = W_1(w) W_0(w) \ , \quad a_w = a_0 + \delta_n a_1 + \mathcal{O}(\delta_n^2) \ , \quad (3.200)$$

with

$$a_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \ , \quad a_1 = \begin{pmatrix} 0 & 0 \\ t(w)/2 & 0 \end{pmatrix} \ , \quad (3.201)$$

where $W_0 \sim \mathcal{O}(1)$ and $W_1 = u_0 + \mathcal{O}(\delta_n)$. Then, (3.179) up to first order of δ_n becomes

$$\partial_w W_0 + a_0 W_0 = 0 \ , \quad \partial_w W_1 + \delta_n W_1 W_0 a_1 W_0^{-1} \approx 0 \ . \quad (3.202)$$

17 Seriously speaking, this limit does not make sense from the geometric point of view, since for non-integer n, the replica surface is ill-defined.

The solutions to W_0 and W_1 are given by

$$W_0(w) = \begin{pmatrix} 1 & w - w_0 \\ 0 & 1 \end{pmatrix} , \; W_1(w) \approx u_0 \mathcal{P} e^{-\delta_n \int_{w_0}^{w} dx \tilde{a}_1(x)} , \quad (3.203)$$

with $\tilde{a}_1 = W_0 a_1 W_0^{-1}$, where W_0 serves as a background and has trivial monodromy for any loops. These allow us to approximate the Wilson loop $\mathcal{W}(\gamma)$ up to the first order of δ_n as

$$\mathcal{W}(\gamma) \approx \mathcal{P} e^{-\delta_n \oint_\gamma dw \tilde{a}_1(w)} \approx \mathbb{1} - \delta_n \oint_\gamma dw \tilde{a}_1(w) \; . \quad (3.204)$$

Applying the above formula to the conditions (3.191) on a-loops, one can find that

$$\rho_{2m-1} = -\rho_{2m} = -\frac{2}{w_{2m-1} - w_{2m}} \; , \quad m = 1, \cdots, N-1 \; . \quad (3.205)$$

The remaining two parameters ρ_{2N-1} and ρ_{2N} can be solved by using the constraints (global conformal Ward identities) (3.172), which in fact are equivalent to

$$\mathcal{W}(\gamma_1) \cdots \mathcal{W}(\gamma_{2N}) = \mathbb{1} \implies \mathcal{W}(\gamma_{2N-1}) \mathcal{W}(\gamma_{2N}) = \mathbb{1} \; . \quad (3.206)$$

Therefore, the results of ρ_{2N-1} and ρ_{2N} are still in the form of (3.205).

Holographic entanglement entropy. By (3.175), one finds that S_{EE}^{σ} associated with the basis (3.188) and (3.189), labeled by the projective structure σ, is given by

$$S_{EE}^{\sigma} = \frac{c}{3} \sum_{m=1}^{N} \log \left(\frac{w_{2m} - w_{2m-1}}{\epsilon} \right) \; , \quad (3.207)$$

where a short distance cutoff ϵ is introduced in order to make the result dimensionless. For other choices of the basis of $\pi_1(R_{n,N})$, the results take the similar form as (3.207). The true entanglement entropy

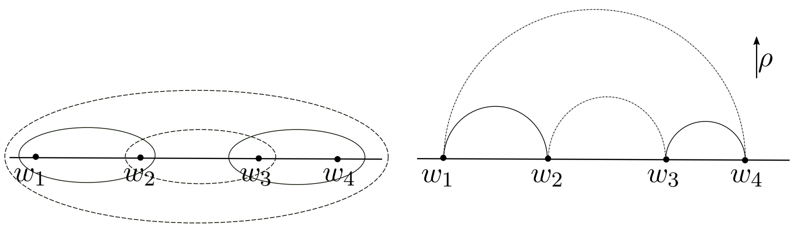

Figure 3: Entanglement phases in $N = 2$ case.

is given by the minimal value of them. As an example, consider the case of $N = 2$. We focus on the following two different partitions of the branched points, i.e. $\{(w_1, w_2), (w_3, w_4)\}$ and $\{(w_1, w_4), (w_2, w_3)\}$. The associated projective structures are labeled by σ_1 and σ_2, respectively. Then we have

$$S_{EE}^{\sigma_1} = \frac{c}{3} \log \left(\frac{w_{12} w_{34}}{\epsilon^2} \right) \; , \quad S_{EE}^{\sigma_2} = \frac{c}{3} \log \left(\frac{w_{14} w_{23}}{\epsilon^2} \right) \; . \quad (3.208)$$

which correspond two different geodesic phases in the AdS_3 space, as shown in Figure 3. The difference between two phase is given by

$$\Delta S = \frac{c}{3} \log \left(\frac{x}{1-x} \right) \ , \quad x = \frac{w_{12} w_{34}}{w_{13} w_{24}} \ , \tag{3.209}$$

where we denote $w_{ij} = w_i - w_j$. Since the true entanglement entropy is the minimal value of them, transition between these two phases happens when $\Delta S = 0$ with the cross ratio $x = \frac{1}{2}$. This entanglement transition is similar to the Hawking-Page phase transition between the thermal AdS_3 phase and the black hole phase, which has been discussed in Section 3.1.3.

3.2.2 *CFT perspective: replica & twist pictures*

There are two different approaches to calculating the Rényi and entanglement entropy in the dual CFT. The first approach is to work in the replica picture and compute the partition function of CFT defined on the replica surface $R_{n,N}$. And the second approach is to work in the twist picture and compute the correlation function of twist fields in the orbifold CFT defined on $\mathbb{CP}^1$ [112, 113]. In the following, I will review both approaches.

Replica picture

As we mentioned before, the replica surface $R_{n,N}$ is a genus $g = (N-1)(n-1)$ compact Riemann surface with delta-function singular curvature. To compute the CFT partition function on $R_{n,N}$, it is convenient to uniformize $R_{n,N}$ on a constant curvature Riemann surface, which is always possible due to the Riemann uniformization theorem. This uniformization is distinct with the extended Schottky uniformization of $R_{n,N}$, and we formally represent it as

$$(\Delta, \Gamma_F, \pi_F : \Delta \to R_{n,N} = \Delta / \Gamma_F) \ . \tag{3.210}$$

Here Δ is the universal covering of $R_{n,N}$, and it is simply connected. Γ_F is the covering group, which is isomorphic to $\pi_1(R_{n,N})$. π_F is a surjective and locally holomorphic map,

$$\pi_F : \tilde{w} \to w(\tilde{w}) \ , \tag{3.211}$$

and automorphic with respect to the covering group Γ_F,

$$w(\gamma \circ \tilde{w}) = w(\tilde{w}), \quad \forall \gamma \in \Gamma_F \ . \tag{3.212}$$

The universal covering space Δ depends on the genus of $R_{n,N}$. For $n = N = 2$, we have $g = 1$ and Δ is the complex plane $\mathbb{C}$. For other cases with $n \geq 2$, we have $g > 1$ and Δ is the hyperbolic half-plane $\mathbb{H}_2$. In practice, finding such uniformizations of replica surfaces, or the map $w(\tilde{w})$, is a very hard problem, except for the case of $N = n = 2$. Therefore, I will just briefly explain the idea in below, and assume that $w(\tilde{w})$ has been worked out in the later discussion. Essentially, this problem is similar to finding the Schottky uniformization. One needs to consider the Fuchsian differential equation,

$$\varphi'' + \frac{6}{c} T_F \varphi = 0 \ , \quad \tilde{w}(w) = \frac{\varphi_1(w)}{\varphi_2(w)} \ , \tag{3.213}$$

with a new projective connection $T_F = \frac{c}{12}\{\tilde{w}; w\}$ on $R_{n,N}$, given via the ansatz

$$T_F = \sum_{i=1}^{2N} \left(\frac{h}{(w - w_i)^2} + \frac{p_i^F}{w - w_i} \right) \ , \tag{3.214}$$

and find the new accessory parameters p_i^F, such that the resulting monodromy group of $\tilde{w}(w)$, identified as the covering group Γ_F, is isomorphic to the fundamental group $\pi_1(R_{n,N})$.

Assuming that the map $\tilde{w}(w)$ has been worked out, we can express the metric $g_{(0)} = d\hat{s}^2 = |dw|^2$ on $R_{n,N}$ in $\tilde{w}$-coordinates as

- $g = 1$, $d\hat{s}^2 = e^{2\phi_F} ds^2 = e^{2\phi_F} |d\tilde{w}|^2$,
- $g \geq 2$, $d\hat{s}^2 = e^{2\phi_F} ds^2 = e^{2\phi_F} \frac{|d\tilde{w}|^2}{\mathrm{Im}(\tilde{w})^2}$,

where $\tilde{g} = ds^2$ is the metric on the universal covering space Δ. Since $g_{(0)}$ and g_F are related by a Weyl transformation, the CFT partition functions defined on those two backgrounds are related by a Liouville action of ϕ_F on $\tilde{g}$ background,

$$Z_n = Z[g_{(0)}] = e^{-S_L[\phi_F, \tilde{g}]} Z[\tilde{g}] \ . \tag{3.215}$$

Moreover, due to the singular behavior of T_F at the branched points w_i, the Weyl factor ϕ_F is also singular, and as a consequence, the Liouville action $S_L[\phi_F, \tilde{g}]$ needs to be regularized. This can be done by introducing cutoff circles around the branched points, analogous to what we did in (3.106). The short distance cutoff term in Rényi and entanglement entropy essentially comes from the this Liouville action.

While the Liouville action is completely determined by the geometry of $R_{n,N}$, calculating the partition function $Z[\tilde{g}]$ requires us to know more explicit data of the dual CFT. For instance, for $g = 1$ we need to know the spectrum, and for $g > 1$ we need to know both of the spectrum and OPE coefficients. Those data are not readily accessible in the bottom-up AdS_3/CFT_2. However, being a holographic CFT, the large c limit as well as the sparseness condition on the density of states are usually required. These two requirements lead to the consequence that the CFT partition function is dominated by the contribution from a particular vacuum channel, or more generally, correlation functions in a holographic CFT are dominated by certain vacuum conformal blocks

[105, 114]. This feature allows us to approximately compute the partition function of a holographic CFT by only knowing the geometric data of the underlying Riemann surface.

Example of $N = n = 2$. To make the above statements more clear, here I consider the example of $N = n = 2$. The replica surface $R_{2,2}$ is topologically a torus, with its fundamental group isomorphic to the Abliean group $\mathbb{Z}^2$. It can be uniformized on $\mathbb{C}$ via the projection map, i.e. $\pi_F : \mathbb{C} \to R_{2,2} = \mathbb{C}/\mathbb{Z}^2$, where elements of $\mathbb{Z}^2$ act on $\mathbb{C}$ via translations. This means

$$w(\tilde{w} + 2\pi a + 2\pi b\tau) = w(\tilde{w}) \ , \quad \forall a, b \in \mathbb{Z} \ . \tag{3.216}$$

Thus, $w(\tilde{w})$ is a double periodic function on $\mathbb{C}$. To find the inverse map $\tilde{w}(w)$ as well as the modular parameter τ of $R_{2,2}$, instead of solving Fuchsian differential equation, it is more convenient to use the automorphic one-form[18] on $\mathbb{C}$ with respect to $\mathbb{Z}^2$, given by

$$\omega = d\tilde{w} \ . \tag{3.217}$$

Under the projection π_F, this automorphic one-form becomes a holomorphic one-form $\omega = \omega(w)dw$ on $R_{2,2}$, with $\omega(w) = d\tilde{w}/dw$. A holomorphic one-form on $R_{2,2}$ is non-singular except at the at the branched points. This requires the following asymptotic behavior,

$$\omega \sim w^{-2}dw + \mathcal{O}(w^{-3}) \ , \quad w \to \infty \ , \tag{3.218}$$

so that under the inversion $y = 1/w$, $\omega \sim dy$ is non-singular at $y = 0$. Furthermore, near each branched point w_i, the behavior of ω reads $\omega \sim (w - w_i)^{-k}dw$, with $k \in \frac{1}{2}\mathbb{Z}_+$. This is because when a point w winds around w_i twice, it goes back to the original position, and ω should be invariant in order to be well-defined on $R_{2,2}$. Imposing (3.218), we can fix the holomorphic one-form up to a constant factor as

$$\omega = \kappa\omega_0 = \frac{\kappa dw}{\sqrt{(w - w_1)(w - w_2)(w - w_3)(w - w_4)}} \ . \tag{3.219}$$

To fix the constant factor κ, we denote γ_{12} and γ_{23} as the two loops encircling the intervals $[w_1, w_2]$ and $[w_2, w_3]$, respectively. Identifying them with the lines $\tilde{w} \to \tilde{w} + 2\pi$ and $\tilde{w} \to \tilde{w} + 2\pi\tau$ on $\mathbb{C}$ yields

$$\kappa = \frac{2\pi}{K_{12}} \ , \quad \tau = \frac{K_{23}}{K_{12}} \ , \tag{3.220}$$

with

$$K_{12} = \oint_{\gamma_{12}} \omega_0 = 2\int_{w_1}^{w_2} \omega_0 \ , \ K_{23} = \oint_{\gamma_{23}} \omega_0 = 2\int_{w_2}^{w_3} \omega_0 \ . \tag{3.221}$$

18 An automorphic one-form $f(x)dx$ on Δ with respect to a discrete subgroup G of $Aut(\Delta)$ is defined by the property, $f(\gamma \circ x)d(\gamma \circ x) = f(x)dx$, $\forall \gamma \in G$. Here $Aut(\Delta)$ is the group of automorphisms of Δ, given by PSL(2,R) fractional transformations for $\Delta = \mathbb{H}_2$ and affine transformations for $\Delta = \mathbb{C}$.

In fact, the modular parameter in (3.220) only depends on the cross ratio x defined in (3.209). To show this, we consider a fractional transformation $z = \frac{(w_3-w_4)(w_1-w)}{(w_1-w_2)(w-w_4)}$, which sends (w_1, w_2, w_3, w_4) to $(0, x, 1, \infty)$. The holomorphic one-form ω_0 in z-coordinates reads

$$\omega_0 = \frac{i}{\sqrt{w_{13}w_{24}}} \frac{dz}{\sqrt{z(z-x)(z-1)}} \ . \tag{3.222}$$

Similarly, consider another fractional transformation $\xi = \frac{(w_4-w_1)(w_2-w)}{(w_2-w_3)(w-w_1)}$, which sends (w_2, w_3, w_4, w_1) to $(0, \tilde{x}, 1, \infty)$, with $\tilde{x} = 1 - x$. Then, the holomorphic one-form ω_0 reads

$$\omega_0 = \frac{i}{\sqrt{w_{24}w_{31}}} \frac{d\xi}{\sqrt{\xi(\xi-\tilde{x})(\xi-1)}} \ . \tag{3.223}$$

Using (3.222) and (3.223) for K_{12} and K_{23} respectively, we are able to express the modular parameter and κ as

$$\tau = i\frac{K(1-x)}{K(x)} \ , \quad \kappa = -i\pi \frac{\sqrt{w_{13}w_{24}}}{K(x)} \ , \tag{3.224}$$

where $K(x)$ is the complete elliptic integral of first kind,

$$K(x) = \int_0^x \frac{dz}{\sqrt{z(z-x)(z-1)}} \ . \tag{3.225}$$

Therefore, we finally obtain the map $\tilde{w}(w)$, given by

$$\tilde{w}(w) = \int^w \omega = -i\pi \frac{\sqrt{w_{13}w_{24}}}{K(x)} \int^w dw' \prod_{i=1}^{4} (w' - w_i)^{-1/2} \ , \tag{3.226}$$

which is known as the Schwarz-Christoffel mapping[19] [111].

We now turn to discuss the CFT partition function on $R_{2,2}$. Using (3.217) and (3.219), we obtain the Weyl factor ϕ_F as

$$\phi_F = \frac{1}{2} \log \left| \frac{dw}{d\tilde{w}} \right|^2 = \frac{1}{4} \sum_{i=1}^{4} \log |w - w_i|^2 - \log |\kappa| \ . \tag{3.227}$$

By inserting cutoff circles with radius ϵ around each branched points, the Liouville action can be calculated straightforwardly, analogous to (3.106). Since this contribution is universal, in the sense that it does not affect the transition of Rényi entropy, I will only focus on $Z[\tilde{g}]$ in the following discussion. The CFT partition function $Z[\tilde{g}]$ reads

$$Z[\tilde{g}] = (q\bar{q})^{-c/24} \mathrm{Tr}[q^{L_0} \bar{q}^{\bar{L}_0}] \approx \begin{cases} |q|^{-c/12} & |\tau| > 1 \ , \\ |\tilde{q}|^{-c/12} & |\tau| < 1 \ , \end{cases} \tag{3.228}$$

19 The inverse map $w(\tilde{w})$ can be constructed via the Weierstrass p-function $\wp(\tilde{w})$, which is double-periodic on torus. Hence, the condition (3.216) is fulfilled.

with $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. The two terms on the right hand side of (3.228) are the dominant contributions from two different vacuum channels. For $|\tau| > 1$, the vacuum state is defined on the spatial circle $\tilde{w} \to \tilde{w} + 2\pi$, and for $|\tau| < 1$ it is defined on the "time" circle $\tilde{w} \to \tilde{w} + 2\pi\tau$. The transition between those two channels of $Z[\tilde{g}]$ is the origin of the transition of Rényi entropy, which by (3.224) happens at $|\tau| = 1$ with the cross ratio $x = 1/2$. Moreover, those two channels are dual to the thermal AdS_3 phase and the black hole phase, respectively, which can be checked by comparing (3.100) and (3.99) with the two terms on the right hand side of (3.228). In particular, the circle on which the vacuum state is defined precisely corresponds to the contractible circle of the dual bulk solution, which is characterized by the associated projective structure σ. This tells us a hidden mechanism of $\mathrm{AdS}_3/\mathrm{CFT}_2$.

More generally, we can also consider the dual CFT defined on a compact Riemann surface S_g with genus $g > 1$ and a constant curvature $R = -2$. Although the exact CFT partition function should be $\mathrm{Sp}(2g, \mathbb{Z})$ modular invariant and depend on the complex structure on S_g, in the semi-classical limit, its saddle point approximation breaks the modular invariance. The dominant vacuum channel specifies the contractible loops in the dual handbody AdS_3, which is related to a particular choice of Schottky uniformizations of S_g, labeled by the associated projective structure σ. This provides an additional geometric data on S_g, since the projective structure is subordinated to the complex structure. We can summary these by the following one-to-one correspondence in $\mathrm{AdS}_3/\mathrm{CFT}_2$,

$$\text{projective structure on } S_g \Longleftrightarrow \text{vacuum channel on } S_g\ . \tag{3.229}$$

When we consider a correlation function in the dual CFT, the above correspondence can be generalized to the duality between projective structures and vacuum conformal blocks, which I will discuss later.

Twist picture

In the replica picture, the calculation of the Rényi entropy is rather complicated, since it requires us to solve the uniformization of $R_{n,N}$. Furthermore, the replica limit $n \to 1$ is not valid in that approach because $R_{n,N}$ is ill-defined for non-integer n. In this section, I will introduce the *twist field approach* to the Rényi and entanglement entropy [112, 113].

Basic idea. The twist field approach provides an alternative way to represent the partition function Z_n defined on replica surface. The basic idea is as follows. Instead of replicating the surface, we take n copies of the original CFT to get a new theory $\mathrm{CFT}^{\otimes n}$. This means that we formally take n-copies of the fundamental field and the action of the original theory, and sum over the actions to get a new action. The field configuration in k-th copy is identified with the field configuration on

k-th sheet of the replica surface. The branched structure of the replica surface then requires us to impose boundary conditions on those n fields, which lead to a restricted path integral of this new theory. For instance, if we consider the entangling region $A = [w_1, w_2]$, then the boundary conditions are given by [113],

$$\begin{aligned} \phi_k(e^{2\pi i}(w-w_1)+w_1) &= \phi_{k+1}(w) \ , \\ \phi_k(e^{2\pi i}(w-w_2)+w_2) &= \phi_{k-1}(w) \ , \end{aligned} \tag{3.230}$$

where ϕ_k with $k \mod n$ denotes the k-th copy of the fundamental field. One of the main ideas in [113] is that one can implement the boundary conditions (3.230) by inserting local fields $\sigma_n(w_1)$ and $\tilde{\sigma}_n(w_2)$ in the path integral, which act on n copies of the fundamental field as cyclic exchanges of copies,

$$\sigma_n : \phi_k \to \phi_{k+1} \ , \quad \tilde{\sigma}_n : \phi_k \to \phi_{k-1} \ . \tag{3.231}$$

These two local fields are called *twist field* and *anti-twist field*, respectively. In this approach, the partition function Z_n is then represented by a two-point function in the new CFT,

$$Z_n = \langle \sigma_n(w_1)\sigma_n(w_2) \rangle \ . \tag{3.232}$$

It was shown in [113] that the twist and anti-twist fields are conformal primaries, with conformal dimensions universally determined by the central charge,

$$h_n = \bar{h}_n = \frac{nc}{24}\left(1 - \frac{1}{n^2}\right) = nh \ . \tag{3.233}$$

To get this result, we first compute the vacuum expectation value of the stress tensor on $R_{n,1}$, which reads

$$T(w) = \frac{c}{12}\{z; w\} = h\left(\frac{1}{w-w_i} - \frac{1}{w-w_2}\right)^2 \ , \tag{3.234}$$

with $z(w)$ given in (3.164). It takes the same form on different sheets of $R_{n,1}$. The stress tensor on k-th sheet of $R_{n,N}$ is identified with the k-th copy of the stress tensor $T^{(k)}(w)$ in the new theory. The total stress tensor in the new theory is given by the sum over copies, so its expectation value reads,

$$T(w) = \sum_{k=1}^{n} T^{(k)}(w) = nh\left(\frac{1}{w-w_i} - \frac{1}{w-w_2}\right)^2 \ . \tag{3.235}$$

By comparing (3.235) with the conformal Ward identity for conformal primary fields, we can identify the twist and the anti-twist fields as conformal primaries with the same holomorphic conformal dimension h_n. The anti-holomorphic part can be analyzed analogously. The partition function Z_n is then given by

$$Z_n = \langle \sigma_n(w_1)\sigma_n(w_2) \rangle = \left|\frac{w_1 - w_2}{\epsilon}\right|^{-2h_n - 2\bar{h}_n} \ , \tag{3.236}$$

which leads to the $N = 1$ Rényi entropy,

$$S_n = \frac{1}{1-n} \log Z_n = \frac{c}{6}\left(1+\frac{1}{n}\right) \log \left|\frac{w_1 - w_2}{\epsilon}\right| \; . \tag{3.237}$$

Here we have assumed that the vacuum partition function Z_1 of the original CFT is normalized as $Z_1 = 1$. Essentially, twist and anti-twist fields encode the branched structure of the replica surface. For the case of $N > 1$, the partition function Z_n defined on $R_{n,N}$ can be expressed via the twist fields as

$$Z_n = \langle \sigma_n(w_1)\tilde{\sigma}_n(w_2) \cdots \sigma_n(w_{2N-1})\tilde{\sigma}_n(w_{2N})\rangle \; . \tag{3.238}$$

Furthermore, we can also generalize this to excited state backgrounds. For instance, we consider two conformal primary fields $O_1(x_1)$ and $O_2(x_2)$ inserted on $\mathbb{CP}^1$. Those fields lead to an excited state density matrix, since they appear in the path integral of the original CFT, $Z_1 = \langle O_1(x_1)O_2(x_2)\rangle$. When performing the replica trick, we need to insert these two fields on each sheet of $R_{n,N}$,

$$Z_n = \int_{R_{n,N}} \mathcal{D}\phi \; \prod_{k=1}^{n} O_1(x_1;k)O_2(x_2;k)e^{-S[\phi]} \; , \tag{3.239}$$

where $O_1(x_1;k)$ represents $O_1(x_1)$ on k-th sheet of $R_{n,N}$. Transforming to the twist picture, we take n copies of these two fields, denoted as $O_1^{(k)}(x_1)O_2^{(k)}(x_2)$ with $k = 1, \cdots n$, and each copy is associated with the fields on the corresponding sheet. The partition function Z_n in the twist picture is then given by

$$Z_n = \langle \tilde{O}_1(x_1)\tilde{O}_2(x_2)\sigma_n(w_1) \cdots \tilde{\sigma}_n(w_{2N})\rangle \; , \tag{3.240}$$

where $\tilde{O}_1$ and $\tilde{O}_2$ are the new fields in $\text{CFT}^{\otimes n}$, defined by the tensor products of n copies,

$$\tilde{O}_1 = \prod_{k=1}^{n} O_1^{(k)} \; , \quad \tilde{O}_2 = \prod_{k=1}^{n} O_2^{(k)} \; . \tag{3.241}$$

And $\text{Tr}[\rho_A^n]$ reads

$$\text{Tr}[\rho_A^n] = \frac{Z_n}{Z_1^n} = \frac{\langle \tilde{O}_1(x_1)\tilde{O}_2(x_2)\sigma_n(w_1) \cdots \tilde{\sigma}_n(w_{2N})\rangle}{\langle O_1(x_1)O_2(x_2)\rangle^n} \; . \tag{3.242}$$

To calculate the Rényi and entanglement entropy, our main task is to evaluate correlation functions (3.238) and (3.240). However, since OPE coefficients of those fields are not readily accessible, what we can do is to approximate those correlation functions by their dominant vacuum conformal blocks [105]. This is a valid approximation in a holographic CFT due to the semi-classical limit $c \to \infty$ and the sparseness condition on the spectrum [115]. As I will show later, calculation of the dominant vacuum conformal block essentially proceeds in parallel with the gravity analysis via the Schottky uniformization.

Discrete symmetry in CFT$^{\otimes n}$. Before calculating the semi-classical conformal blocks in CFT$^{\otimes n}$, let me first clarify the issue about the $\mathbb{Z}_n$-symmetric OPEs, which is necessary for the later discussion. As the new theory is obtained by n copies of the original CFT, the central charge of CFT$^{\otimes}$ becomes $c_n = nc$. This can be shown by considering the OPE between the total stress tensor in the theory,

$$T(w)T(0) \sim \sum_{k=1}^{n} T^{(k)}(w)T^{(k)}(0) \sim \frac{c_n/2}{w^4} + \cdots \ , \tag{3.243}$$

where OPEs between different copies $T^{(k)}$'s are trivial, due to the decoupling of ϕ_k's at the Lagrangian level. The total stress tensor $T(w)$ is the generator of conformal symmetry in CFT$^{\otimes n}$, which by definition is invariant under the cyclic exchanges of copies. However, since there is an additional discrete $\mathbb{Z}_n$ symmetry in CFT$^{\otimes n}$, an irreducible representation in the theory is not completely characterized by its conformal weights, but also by its eigenvalue of the generator of $\mathbb{Z}_n$, denoted as η. The action of η on the Virasoro mode in k-th copy reads

$$\eta L_m^{(k)} \eta^{-1} = L_m^{(k+1)} \ , \quad k \mod n \ , \tag{3.244}$$

so that it leaves the total Virasoro mode invariant,

$$\eta L_m \eta^{-1} = L_m \ , \quad L_m = \sum_{k=1}^{n} L_m^{(k)} \ . \tag{3.245}$$

To characterize states in CFT$^{\otimes n}$, let us consider the total Hilbert space of CFT$^{\otimes n}$,

$$\mathcal{H}_{total} = \otimes^n \mathcal{H} \ , \tag{3.246}$$

where $\mathcal{H}$ is the Hilbert space of the original theory. We can decompose the total Hilbert space via the discrete $\mathbb{Z}_n$ symmetry,

$$\mathcal{H}_{total} = \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(n)} \ , \tag{3.247}$$

where states in $\mathcal{H}^{(n)}$ are eigenstates of the $\mathbb{Z}_n$ generator η. The eigenvalues of them can be shown as follows. We consider a state $|\psi\rangle$,

$$|\psi\rangle = \otimes^n |\psi_k\rangle \in \mathcal{H}_{total}, \quad |\psi_k\rangle \in \mathcal{H} \ , \quad k = 1, \cdots, n \ . \tag{3.248}$$

It is useful to represent the state $|\psi\rangle$ as a vector, so that the $\mathbb{Z}_n$ generator η acts on it as a matrix,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} , \quad \eta = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ 1 & & & 0 \end{pmatrix} , \quad \eta \circ \psi = \begin{pmatrix} \psi_2 \\ \psi_3 \\ \vdots \\ \psi_1 \end{pmatrix} . \tag{3.249}$$

Then, the state $|\psi\rangle$ can be decomposed as a linear combination of eigenstates of η, denoted as

$$\psi = \sum_{k=1}^{n} \alpha_k \psi^{(k)} \ , \quad \eta \circ \psi^{(k)} = \lambda_k \psi^{(k)} \in \mathcal{H}^{(k)} \ , \tag{3.250}$$

with eigenvalues λ_k given by

$$\lambda_k = e^{2\pi i k/n} \ , \quad k = 1, \cdots, n \ . \tag{3.251}$$

We call $\mathcal{H}^{(n)}$ with $\lambda_n = 1$ as the *$\mathbb{Z}_n$-symmetric sector*, and the rest part of $\mathcal{H}_{total}$ as the *twist sector*. Suppose $|\psi\rangle$ is a conformal primary state in $\mathcal{H}^{(n)}$, the corresponding descendent states in $\mathcal{H}^{(n)}$ are then constructed by acting the total Virasoro modes L_{-m}'s on it, but not by acting arbitrary combination of $L^{(k)}_{-m}$'s,

$$|\psi^{\{m\}}\rangle = L_{-m_1} L_{-m_2} \cdots L_{-m_j} |\psi\rangle \in \mathcal{H}^{(n)}, \quad m_j \geq 1 \ . \tag{3.252}$$

where $\{m\} = \{m_1, m_2, \cdots m_j\}$ is a collection of indices, ordered via $m_i > m_{i+1}$. For the anti-holomorphic part, the definition is analogous.

The reason for claiming the structure of the total Hilbert space as well as the descendent states in the $\mathbb{Z}_n$-symmetric sector is as follows. In the context of the Rényi and entanglement entropy, we usually encounter the product operators defined in (3.241), which clearly lie in the $\mathbb{Z}_n$-symmetric sector $\mathcal{H}^{(n)}$. On the other hand, the twist and anti-twist fields are not lie in the $\mathbb{Z}_n$ symmetric-sector, but in the twist sector [112]. However, if we consider the OPE between twist and anti-twist fields, we could argue that all terms in their OPE should be $\mathbb{Z}_n$-symmetric operators, due to the actions (3.231) of (anti-) twist fields on the fundamental fields[20]. In other words, we should have

$$\begin{aligned}\sigma_n(w_1)\tilde{\sigma}_n(0)(w_2) = \sum_p \sum_{\{m,\bar{m}\}} & C^p_{\sigma\tilde{\sigma}} (w_1 - w_2)^{h_p - 2h_n + |m|} \\ & \times (\bar{w}_1 - \bar{w}_2)^{\bar{h}_p - 2h_n + |\bar{m}|} \beta^{p,k}_{\sigma\tilde{\sigma}} \bar{\beta}^{p,\bar{k}}_{\sigma\tilde{\sigma}} O^{\{m,\bar{m}\}}_p (w_1), \end{aligned} \tag{3.253}$$

with $O^{\{m,\bar{m}\}}_p \in \mathcal{H}^{(n)}$, where $(h_p, \bar{h}_p)$ are the conformal weights of the conformal primary O_p, and $|m| = \sum_j m_j$ and $|\bar{m}| = \sum_j \bar{m}_j$ are the levels of the descendent state $O^{\{m,\bar{m}\}}_p$. The coefficients $\beta^{p,k}_{\sigma\tilde{\sigma}}$ and $\bar{\beta}^{p,\bar{k}}_{\sigma\tilde{\sigma}}$ are constants, depending on conformal weights as well as the central charge c_n. They are fixed by the conformal symmetry [98]. $C^p_{\sigma\tilde{\sigma}}$'s are OPE coefficients defined by three-point functions,

$$\langle \sigma_n(w_1) \tilde{\sigma}_n(w_2) O_p(w_3) \rangle = \frac{C^p_{\sigma_1 \tilde{\sigma}_2}}{w_{12}^{2h_n - h_p} w_{23}^{h_p} w_{13}^{h_p} \times c.c.} \tag{3.254}$$

which however are unknown in our setup. The consequence of (3.253) is that, when calculating the Rényi and entanglement entropy, we can

20 Usually, this was implicitly assumed in the twist field approach to Rényi entropy [105].

always choose OPE channels with the twist and anti-twist fields being paired, such that all exchange states are in the form of (3.252). For convenience, I call such a channel as a $\mathbb{Z}_n$-symmetric channel.

Monodromy method for semi-classical conformal blocks. Now I turn to discuss the calculation of semi-classic conformal blocks by using the well-known *monodromy method* [105, 116]. The vacuum conformal blocks are relevant to the Rényi and entanglement entropy, and I will come back to these cases later. Eventually, we will see that the analysis for vacuum conformal blocks is parallel with the gravity analysis on Schottky uniformizations of replica surfaces. Vacuum conformal blocks in distinct $\mathbb{Z}_n$-symmetric channels are related to Schottky uniformizations labeled by distinct projective structures.

Let us start by considering a four-point function of primary fields O_i's in CFT$^{\otimes n}$, with conformal weights $h_i \sim \mathcal{O}(c)$,

$$Z = \langle O_1(w_1)O_2(w_2)O_3(w_3)O_4(w_4)\rangle \ . \tag{3.255}$$

We assume that the OPE between O_1 and O_2 is $\mathbb{Z}_n$-symmetric,

$$O_1(w_1)O_2(w_2) = \sum_p C^p_{12} g^p_{12}(w_1|w_2) \ , \tag{3.256}$$

with $g^p_{12}(w_1|w_2)$ defined as,

$$\begin{aligned} g^p_{12}(w_1|w_2) = \sum_{\{m,\bar{m}\}} &(w_1 - w_2)^{h_p - h_1 - h_2 + |m|} \\ &\times (\bar{w}_1 - \bar{w}_2)^{\bar{h}_p - \bar{h}_1 - \bar{h}_2 + |\bar{m}|} \beta^{p,k}_{12} \bar{\beta}^{p,\bar{k}}_{12} O^{\{m,\bar{m}\}}_p(w_1) \ , \end{aligned} \tag{3.257}$$

Insert this OPE into the four-point function yields the s-channel expansion,

$$Z = \sum_p C^p_{12} C^p_{34} A^{43}_{12}(p|w_i, \bar{w}_i) \ . \tag{3.258}$$

Here $A^{43}_{12}(p|w_i, \bar{w}_i)$ is called the *conformal partial wave* [98], defined as

$$A^{43}_{12}(p|w_i, \bar{w}_i) = (C^p_{34})^{-1} \langle g^p_{12}(w_1|w_2)O_3(w_3)O_4(w_4)\rangle \ . \tag{3.259}$$

and it admits a holomorphic factorization under the decomposition of the sum over $\{m, \bar{m}\}$,

$$A^{43}_{12}(p|w_i, \bar{w}_i) = \mathcal{F}^p_{12,34}(w_i)\bar{\mathcal{F}}^p_{12,34}(\bar{w}_i) \ , \tag{3.260}$$

where $\mathcal{F}^p_{12,34}$ and $\bar{\mathcal{F}}^p_{12,34}$ are the s-channel conformal blocks, labeled by the index p of the exchange conformal family $\{O_p\}$ [98]. The main task in the next is to compute the semi-classical approximation of the conformal partial wave $A^{43}_{12}(p|w_i, \bar{w}_i)$. Due to the factorization (3.260), I will focus on the holomorphic part, and the analysis for the anti-holomorphic part will be similar.

In the monodromy method for semi-classical conformal blocks, we start by considering a level-two descendent field χ in the theory, which generically takes the form of $\chi(w) = (L_{-2} + aL_{-1}^2)\hat{\varphi}(w)$, with $\hat{\varphi}$ a conformal primary field. In the special cases, where $L_n|\chi\rangle = 0$ for all $n > 0$, $|\chi\rangle$ by itself is also a primary state. We call such states as null states, since the norm of χ vanishes, i.e., $\langle\chi|\chi\rangle = \langle\hat{\varphi}|(L_2 + aL_1^2)|\chi\rangle = 0$. Null states need to be excluded from the physical Hilbert space of a unitary CFT. An analytic way to achieve this is to impose the null state equation $\chi(w) = 0$ [98], which means that the insertion of a null state χ into any correlation functions will lead to zero. In other words, we use the null state equation to constrain the correlation functions in a consistently defined CFT. In present case, the level-two null state χ in CFT$^{\otimes n}$ is constructed as

$$\chi(w) = \left(L_{-2} - \frac{3}{2(2h_\varphi + 1)}L_{-1}^2\right)\hat{\varphi}(w) = 0\ , \tag{3.261}$$

with the conformal weight h_φ of the primary field $\hat{\varphi}$ given by,

$$h_\varphi = \frac{1}{16}[5 - c_n \pm \sqrt{(c_n - 1)(c_n - 25)}]\ . \tag{3.262}$$

To use the monodromy method, we need to choose the $+$ sign for h_φ, which in the semi-classical limit $c \to \infty$ is of order $\mathcal{O}(c^0)$,

$$h_\varphi \approx -\frac{1}{2} - \frac{9}{2c_n} \sim \mathcal{O}(c^0)\ , \quad c \to \infty\ . \tag{3.263}$$

Then, the null state equation (or decoupling equation) becomes

$$\left(L_{-2} + \frac{c_n}{6}L_{-1}^2\right)\hat{\varphi}(w) = 0\ . \tag{3.264}$$

Inserting (3.264) into (3.255) and implementing the standard representation of Virasoro modes in correlation functions, $L_{-1}^{(w)} = \partial_w$ and

$$L_{-n}^{(w)} = \sum_{i=1}^{4}\left[\frac{(n-1)h_i}{(w_i - w)^n} - \frac{\partial_{w_i}}{(w_i - w)^{n-1}}\right]\ , \quad n \geq 2\ , \tag{3.265}$$

yields the following differential equation on the five-point function,

$$\left[\partial_w^2 + \frac{6}{c_n}\sum_{i=1}^{4}\left(\frac{h_i}{(w - w_i)^2} + \frac{\partial_{w_i}}{w - w_i}\right)\right]\Phi(w, w_i) = 0\ , \tag{3.266}$$

with

$$\Phi(w, w_i) = \langle O_1(w_1)O_2(w_2)\hat{\varphi}(w)O_3(w_3)O_4(w_4)\rangle\ . \tag{3.267}$$

Furthermore, by inserting the OPE (3.256) into this five-point function, one finds that each term in the sum involves a four-point function $\langle O_p^{\{m,\bar{m}\}}\hat{\varphi}O_3O_4\rangle$. For convenience, let me define a wave function

$$\varphi_p^{\{m,\bar{m}\}}(w, \bar{w}|w_i, \bar{w}_i) = \frac{\langle O_p^{\{m,\bar{m}\}}\hat{\varphi}O_3O_4\rangle}{\langle O_p^{\{m,\bar{m}\}}O_3O_4\rangle}\ . \tag{3.268}$$

If we assume $h_p \sim \mathcal{O}(c)$, then the semi-classical approximation of $\varphi_p^{\{m,\bar{m}\}}$ is given by [105]

$$\varphi_p^{\{m,\bar{m}\}} \approx \varphi_p(1+\mathcal{O}(1/c)) \ , \quad \varphi_p = \frac{\langle O_p \hat{\varphi} O_3 O_4 \rangle}{\langle O_p O_3 O_4 \rangle} \ . \tag{3.269}$$

This can be checked by acting a string of Virasoro modes $L_{-m_1}^{(w_1)} \cdots L_{-m_j}^{(w_1)}$ on the correlation functions. Using the approximation (3.269), we can rewrite the five-point function Φ as

$$\Phi \approx \sum_p C_{12}^p C_{34}^p A_{12}^{43}(p|w_i, \bar{w}_i) \varphi_p(w, \bar{w}|w_i, \bar{w}_i) = \sum_p \Phi_p \ . \tag{3.270}$$

Now, a crucial argument is that the differential equation (3.266) holds not only for $\Phi(w, w_i)$, but also for each contribution Φ_p in the sum (3.270) [105]. At this point, it is useful to know that the semi-classical conformal partial wave can be approximated as $A_{12}^{43}(p) \sim e^{-S_{cl}^p}$, with $S_{cl}^p \sim \mathcal{O}(c)$ [105]. This was recently proven in [114]. Meanwhile, since $h_\varphi \sim \mathcal{O}(c^0)$, φ_p as well as its derivatives are in order of $\mathcal{O}(e^{c^0})$. These two facts implies that when applying the differential equation (3.266) for Φ_p in (3.270), we can neglect $\partial_{w_i}\varphi_p$ terms. This leads to a Fuchsian differential equation for the wave function φ_p,

$$\partial_w^2 \varphi_p + \frac{6}{c} T(w) \varphi_p = 0 \ . \tag{3.271}$$

where the stress tensor reads

$$T(w) = \sum_{i=1}^{4} \left(\frac{h_i/n}{(w-w_i)^2} + \frac{f_i^p}{w-w_i} \right) \ , \tag{3.272}$$

with the accessory parameters f_i^p's encoding the desired conformal partial wave,

$$n f_i^p = \partial_{w_i} \log A_{12}^{43}(p|w_i, \bar{w}_i) = -\partial_{w_i} S_{cl}^p \ . \tag{3.273}$$

The smoothness condition for $T(w)$ at infinity imposes three constraints on the accessory parameters, analogous to (3.172). As I have discussed in Section 3.2.1, the accessory parameters can be determined by providing the monodromy matrices associated with the solutions to Fuchsian differential equation. In the present case, the w-dependence of φ_p defined in (3.269) is contained in the four-point function $\langle O_p \hat{\varphi}(w) O_3 O_4 \rangle$. The monodromy property of this four-point function can be derived by imposing the null-state equation on it. To show this, we consider the OPE between O_3 and O_4, which can be roughly written as

$$O_3(w_3) O_4(w_4) \sim \sum_r \sum_{\{m,\bar{m}\}} C_{34}^{r\{m,\bar{m}\}} O_r^{\{m,\bar{m}\}}(w_4) \ , \tag{3.274}$$

with $C_{34}^{r\{m,\bar{m}\}} = \beta_{34}^{r,m} \bar{\beta}_{34}^{r,\bar{m}}$. Inserting this OPE into $\langle O_p \hat{\varphi}(w) O_3 O_4 \rangle$, it is clear that w-dependence of this four-point function is controlled by a collection of three-point functions,

$$\varphi_p(w) \sim \sum_r \sum_{\{m,\bar{m}\}} \langle O_p \hat{\varphi}(w) O_r^{\{m,\bar{m}\}} \rangle \ . \tag{3.275}$$

However, most of these three-point functions vanish, because the null-state equation impose a strong constraint on the allowed conformal weights h_r in the three-point function $V_{p\varphi r} = \langle O_p \hat{\varphi} O_r \rangle$. More precisely, we have

$$\left[\partial_w^2 + \frac{6}{c_n} \sum_{i=q,r} \left(\frac{h_i}{(w-w_i)^2} + \frac{\partial_{w_i}}{w-w_i} \right)\right] V_{p\varphi r} = 0 \ , \tag{3.276}$$

with

$$V_{p\varphi r} \sim \frac{C^r_{p\varphi}}{(w-w_1)^{h_\varphi + h_p - h_r} (w-w_4)^{h_r + h_\varphi - h_p} w_{14}^{h_r + h_p - h_\varphi}} \ , \tag{3.277}$$

which in semi-classical limit fixes the allowed conformal weights h_r as

$$h_r = h_\varphi + h_p + \frac{1}{2}(1 \pm \Lambda_p) \ , \quad \Lambda_p = \sqrt{1 - \frac{24}{c_n} h_p} \ . \tag{3.278}$$

Then, the w-dependence of $V_{p\varphi r}$ reads

$$V_{p\varphi r} \sim (w-w_1)^{\frac{1}{2}(1 \pm \Lambda_p)} (w-w_4)^{\frac{1}{2}(1 \mp \Lambda_p)} \ , \tag{3.279}$$

where I have used $h_\varphi \approx -\frac{1}{2}$. For other terms $\langle O_p \hat{\varphi} O_r^{\{m,\bar{m}\}} \rangle$ in (3.275), their leading behavior in w is the same as $V_{p\varphi r}$. This statement is a similar to (3.269), for instance,

$$\langle O_p \hat{\varphi} (L_{-m} O_r) \rangle \sim (m-1) h_p w_{14}^{-m} V_{p\varphi r} (1 + \mathcal{O}(h_\varphi / h_p)) \ , \tag{3.280}$$

with $h_\varphi / h_p \sim \mathcal{O}(1/c)$. Therefore, the w-dependence of φ_p is approximately captured by $V_{p\varphi r}$. This allows us to identified the monodromy matrices of φ_p associated with the OPE channels. For instance, if $\hat{\varphi}$ once goes around O_p, the path should be a loop encircling w_1 and w_2, denoted as γ_{12}, because the OPE between $O_1(w_1)$ and $O_2(w_2)$ is defined within the radius of convergence. Then, the monodromy matrix of φ_p associated with γ_{12} is given by

$$M(\gamma_{12}) = \begin{pmatrix} e^{i\pi(1+\Lambda_q)} & 0 \\ 0 & e^{i\pi(1-\Lambda_q)} \end{pmatrix} \ . \tag{3.281}$$

Similarly, for a loop γ_{34} encircling w_3 and w_4, the monodromy matrix reads $M(\gamma_{34}) = M(\gamma_{12})^{-1}$. This is a consequence of the global Ward identity, which I have mentioned in the gravity analysis. The remaining task is to calculate the accessory parameters f_i^p with given monodromy matrices $M(\gamma_{12})$ and $M(\gamma_{34})$. This has been discussed in Section 3.2.1, so I will not repeat it here. There is also a t-channel expansion of the four-point function, in the case of which, one performs the OPE between O_2 and O_3, and the resulting monodromy matrices are associated with loops γ_{23} and γ_{14}.

The monodromy method can also be generalized to the higher-point correlation function, but only in particular channels, in which the operators are organized into pairs [104]. As an example, for a six-point function, the monodromy method is not applicable for the conformal blocks in the following channel,

$$\langle O_1(w_1)\cdots O_6(w_6)\rangle = \sum_{p,q,r} C_{12}^{p} C_{p3}^{q} C_{q4}^{r} C_{56}^{r} \mathcal{F}(w_i)\bar{\mathcal{F}}(\bar{w}_i) \ . \tag{3.282}$$

Instead, one should organize the OPE between fields in the correlation function into three pairs, for instance,

$$\langle O_1(w_1)\cdots O_6(w_6)\rangle = \sum_{p,q,r} C_{12}^{p} C_{34}^{q} C_{56}^{r} C_{pq}^{r} \mathcal{F}'(w_i)\bar{\mathcal{F}}'(\bar{w}_i) \ , \tag{3.283}$$

then the monodromy matrices are associated with the loops γ_{12}, γ_{34}, γ_{56}. This structure naturally incorporates the gravity analysis on the basis of fundamental group of the replica surface.

Entanglement entropy from vacuum conformal block. Now, we come back to the case of entanglement entropy. We first consider the case of $N = 2$ with vacuum background,

$$Z_n = \langle \sigma_n(w_1)\tilde{\sigma}_n(w_2)\sigma_n(w_3)\tilde{\sigma}_n(w_4)\rangle \ , \tag{3.284}$$

For the s-channel vacuum conformal block of this four-point function, the monodromy matrix (3.281) becomes the identity matrix, i.e. $M_{12} = M_{34} = \mathbb{1}$ with $h_p = 0$. This has been encountered in the gravity analysis on Schottky uniformizations of replica surfaces. In t-channel, it is similar. The true dominance of the four-point function is given by the larger one of those two channels,

$$Z_n \approx \max\{A_{12}^{43}(0|w_i,\bar{w}_i), A_{23}^{14}(0|w_i,\bar{w}_i)\} \ , \tag{3.285}$$

so that the Rényi and entanglement entropy are minimized. The results of the entanglement entropy in two different phases has been shown in (3.208) in the gravity analysis.

We can also consider the entanglement entropy in an excited state background. For simplicity, we insert two scalar fields O at $w = w_1$ and $w = w_4$, with conformal weights,

$$h_1 = \bar{h}_1 = \frac{c}{24}\left(1 - \frac{1}{\alpha^2}\right) \ , \quad \alpha \in \mathbb{Z}_+ \ . \tag{3.286}$$

On gravity side, this state is dual to the conical defect AdS$_3$, with α being related to the defect angle. The correlation function reads

$$Z_1 = \langle O(w_1)O(w_4)\rangle = |w_1 - w_4|^{-4h_1} = e^{-S} \ . \tag{3.287}$$

We consider a single interval entangling region $A = [w_2, w_3]$ in this background. Then the partition function Z_n reads,

$$Z_n = \langle \tilde{O}(w_1)\sigma_n(w_2)\tilde{\sigma}_n(w_3)\tilde{O}(w_4)\rangle \approx A_{23}^{14}(0|w_i,\bar{w}_i) \ , \tag{3.288}$$

where $\tilde{O} = \prod_{k=1}^{n} O^{(k)}$ is a product operator in CFT$^{\otimes n}$, with conformal weight (nh_1, nh_1). Notice that only the t-channel of (3.288) is $\mathbb{Z}_n$-symmetric. Thus, there is no entanglement transition happening in this case. The stress tensor (3.272) associated with the t-channel vacuum conformal block of (3.288) reads

$$T = \sum_{i=1}^{4} \left(\frac{h_i}{(w-w_i)^2} + \frac{f_i^0}{w-w_i} \right) \ , \tag{3.289}$$

with

$$n f_i^0 = \partial_{w_i} A_{23}^{14}(0|w_i, \bar{w}_i) = -\partial_{w_i} S_{cl}^0 \ . \tag{3.290}$$

Analogous to (3.198), in the replica limit $\delta_n = n - 1 \to 0$, we can decompose the stress tensor into two pieces,

$$T = T_0 + \delta_n \frac{c}{12} t_1 + \mathcal{O}(\delta_n^2), \tag{3.291}$$

where T_0 is associated with the original background, given by

$$T_0 = h_1 \left(\frac{1}{w-w_1} - \frac{1}{w-w_4} \right)^2 \ , \tag{3.292}$$

and t_1 is the nontrivial part generated by the twist fields,

$$t_1 = \sum_{i=2}^{3} \frac{1}{(w-w_i)^2} + \sum_{i=1}^{4} \frac{\rho_i}{w-w_i} \ . \tag{3.293}$$

Here ρ_i are related to the accessory parameters as

$$\begin{aligned} f_1^0 &= -\frac{2h_1}{w_1 - w_4} + \frac{c}{12}\delta_n \rho_1 + \mathcal{O}(\delta_n^2) \ , \\ f_4^0 &= -\frac{2h_1}{w_4 - w_1} + \frac{c}{12}\delta_n \rho_4 + \mathcal{O}(\delta_n^2) \ , \\ f_i^0 &= \frac{c}{12}\delta_n \rho_i + \mathcal{O}(\delta_n^2) \ , \quad i = 2, 3 \ . \end{aligned} \tag{3.294}$$

Imposing the smoothness condition for $T(w)$ at infinity, $T(w) \sim w^{-4}$, yields three constraints

$$\sum_{i=1}^{4} \rho_i = 0 \ , \ 2 + \sum_{i=1}^{4} w_i \rho_i = 0 \ , \ \ 2\sum_{i=2}^{3} w_i + \sum_{i=1}^{4} w_i^2 \rho_i = 0 \ . \tag{3.295}$$

As the background is not the vacuum state, the formula (3.175) for the entanglement entropy needs to be modified,

$$\frac{\partial S_{EE}}{\partial w_i} \approx \partial_{w_i} \partial_n (S_{cl}^0 - nS)|_{n=1} = -\frac{c}{12}\rho_i \ , \tag{3.296}$$

where S and S_{cl}^0 are defined in (3.287) and (3.290). Analogous to the discussion in Section 3.2.1, we can write the Fuchsian differential equation (3.271) as a first-order differential equation (3.179). In $n \to 1$ limit, the matrix W and the connection a are decomposed as

$$W(w) = W_1(w) W_0(w) \ , \quad a = a_0 + \delta_n a_1 + \mathcal{O}(\delta_n^2) \ , \tag{3.297}$$

with $W_0 \sim \mathcal{O}(1)$ and $W_1 = u_0 + \mathcal{O}(\delta_n)$, and

$$a_0 = \begin{pmatrix} 0 & -1 \\ 6T_0/c & 0 \end{pmatrix} \,, \quad a_1 = \begin{pmatrix} 0 & 0 \\ t_1/2 & 0 \end{pmatrix} \,, \tag{3.298}$$

and the differential equations for them are given by

$$\partial_w W_0 + a_0 W_0 = 0 \,, \quad \partial_w W_1 + \delta_n W_1 W_0 a_1 W_0^{-1} \approx 0 \,. \tag{3.299}$$

The solutions to W_0 and W_1 are given by

$$W_0 = \begin{pmatrix} \varphi_+^{(0)\prime} & \varphi_+^{(0)} \\ \varphi_-^{(0)\prime} & \varphi_-^{(0)} \end{pmatrix} \,, \quad W_1 = u_0 \mathcal{P} e^{-\delta_n \int_{w_0}^{w} \tilde{a}_1(x) dx} \,, \tag{3.300}$$

with

$$\varphi_\pm^{(0)} = \sqrt{\frac{\alpha}{w_1 - w_4}} (w - w_1)^{\frac{1}{2} \pm \frac{1}{2\alpha}} (w - w_4)^{\frac{1}{2} \mp \frac{1}{2\alpha}} \,, \tag{3.301}$$

and $\tilde{a}_1 = W_0 a_1 W_0^{-1}$. Notice that W_0 has trivial monodromy when w goes once around (w_1, w_4) or (w_2, w_3). Thus, the total monodromy of W around w_2 and w_3 is given by

$$M_{23} \approx u_0 \mathcal{P} e^{-\delta_n \oint_{w_{2,3}} \tilde{a}_1(w) dw} u_0^{-1} \,. \tag{3.302}$$

By the trivial monodromy condition on the t-channel vacuum conformal block, $M_{23} = \mathbb{1}$, we obtain the following condition,

$$\oint_{w_{2,3}} W_0 a_1(w) W_0^{-1} dw = 0 \,. \tag{3.303}$$

Solving (3.303) together with the constraints (3.295) yields

$$\begin{aligned}
\rho_1 &= \frac{w_{23}(w_{34} - w_{13})}{w_{12} w_{13} w_{14}} - \frac{w_{34}}{w_{12} w_{14}} \left(2 + \frac{x}{\alpha} + \frac{2(1-x)}{\alpha(x^{\frac{1}{\alpha}} - 1)} \right) , \\
\rho_4 &= \frac{w_{23}(w_{13} - w_{34})}{w_{14} w_{24} w_{34}} + \frac{w_{13}}{w_{24} w_{14}} \left(2 + \frac{x}{\alpha} + \frac{2(1-x)}{\alpha(x^{\frac{1}{\alpha}} - 1)} \right) , \\
\rho_2 &= \frac{w_{14}}{\alpha w_{12} w_{24}} + \frac{1}{w_{12}} - \frac{1}{w_{24}} + \frac{2 w_{14}}{\alpha w_{12} w_{24}} \frac{1}{x^{\frac{1}{\alpha}} - 1} \,, \\
\rho_3 &= \frac{w_{14}}{\alpha w_{13} w_{34}} + \frac{1}{w_{13}} - \frac{1}{w_{34}} + \frac{2 w_{14}}{\alpha w_{13} w_{34}} \frac{1}{x^{-\frac{1}{\alpha}} - 1} \,,
\end{aligned} \tag{3.304}$$

with $w_{ij} = w_i - w_j$ and $x = \frac{w_{12} w_{34}}{w_{13} w_{24}}$. Inserting (3.304) into (3.296) and taking the integration, it can be shown that the entanglement entropy in such an excited state background is given by

$$S_{EE} = \frac{c}{3} \left(\log |1 - x^{\frac{1}{\alpha}}| + \frac{\alpha - 1}{2\alpha} \log x + \log \left| \frac{w_{13} w_{24}}{w_{14} \epsilon} \right| \right) \,, \tag{3.305}$$

where I have introduced a short distance cutoff ϵ for the last term, in order to make the result dimensionless. For $\alpha = 1$, we have $h_1 = 0$, and (3.305) recovers the $N = 1$ entanglement entropy in the vacuum background, $S_{EE} = \frac{c}{3}\log|w_{23}/\epsilon|$. In the following, I will show the entanglement entropy (3.305) is precisely the RT formula (2.69) in conical defect AdS_3 background. Moreover, since the above calculation is rather involved, I will also introduce a formula for $N = 1$ entanglement entropy in a general background, by using the background wave functions $\varphi_{\pm}^{(0)}$ associated with a given stress tensor T_0.

An alternative form of $N = 1$ entanglement entropy. Let me start from the geodesic equation in $\mathbb{H}_3$ background, which is dual to the vacuum state of the boundary CFT. The metric of $\mathbb{H}_3$ is given in (3.22). A geodesic in $\mathbb{H}_3$ is a half-circle, which is uniquely determined by the positions of its endpoints on the conformal boundary. For convenience, we set the endpoints as $y_2 = -y_1 = r \in R$, then the geodesic equation reads

$$\xi^2 + y^2 = r^2 \ , \quad y \in R \ . \tag{3.306}$$

The length of this half-circle is calculated as

$$L = 2\int_{\epsilon}^{r} d\xi \sqrt{\frac{1+(dy/d\xi)^2}{\xi^2}} = 2\log(2r/\epsilon) \ . \tag{3.307}$$

where I have chosen a cutoff $\xi = \epsilon \to 0$ in order to regularize the length. This cutoff can be identified with the radius of the cutoff circles encircling endpoints of the entangling interval, when we compute the entanglement entropy from the renormalized action of AdS_3 gravity. It is clear that in this case, the RT formula $S_{EE} = \frac{c}{6}L$ reproduces the entanglement entropy in the vacuum background.

Now, we turn to calculate the geodesic length in the excited state background $Z_1 = \langle O(w_1)O(w_4)\rangle$ discussed in above. The dual bulk geometry is a conical defect AdS_3 in the Bañados form (3.17), with $\mathcal{L} = \frac{6}{c}T_0$, where the background stress tensor T_0 is given in (3.292). The idea is to transform this Bañados geometry back to $\mathbb{H}_3$, so that the geodesic is mapped to a geodesic in $\mathbb{H}_3$. As explained in Section 3.1.2, such a bulk diffeomorphism is characterized by the boundary map $w \to z(w)$, which satisfies $T_0 = \frac{c}{12}\{z; w\}$. The solution to $z(w)$ reads

$$z(w) = \frac{\varphi_+^{(0)}(w)}{\varphi_-^{(0)}(w)} \ , \tag{3.308}$$

where $\varphi_{\pm}^{(0)}$ are the solutions to the Fuchsian differential equation with stress tensor T_0, and they have been worked in (3.301). Under the mapping (3.308), the endpoints $w = w_2$ and $w = w_3$ of the entangling region are transformed to $z(w_2)$ and $z(w_3)$. Thus, the geodesic length can be written as

$$L = 2\log\left|\frac{z(w_2) - z(w_3)}{\delta}\right| \ . \tag{3.309}$$

Here δ is a new cutoff imposed as a lower bound of the radial coordinate of $\mathbb{H}_3$, and it should be related to the cutoff ϵ in the original Bañados geometry by the associated diffeomorphism. Here is a tricky way to derive their relation. Since the cutoff on the radial coordinate is identical to the radius of cutoff circles on the boundary, we can use the map $z(w)$ to identify the cutoff circles inserted on the boundaries of those two different geometries. More precisely, we have

$$\epsilon = |w - w_i| \quad , \delta_i = |z(w) - z(w_i)| \ , \quad i = 2, 3 \ , \tag{3.310}$$

and δ is the average of δ_2 and δ_3, which in the limit $\epsilon \to 0$ is given by

$$\delta = \sqrt{\delta_2 \delta_3} = \epsilon \sqrt{|z'(w_2) z'(w_3)|} \ . \tag{3.311}$$

Inserting (3.311) into (3.309) yields the entanglement entropy,

$$S_{EE} = \frac{c}{6} L = \frac{c}{3} \log \left| \frac{z(w_2) - z(w_3)}{\epsilon \sqrt{z'(w_2) z'(w_3)}} \right| \ . \tag{3.312}$$

As the Wronskian of the background wave functions (3.301) is normalized, $\det(W_0) = 1$, one can rewrite (3.312) in an alternative form,

$$S_{EE} = \frac{c}{3} \log |\det(U)/\epsilon| \ , \ U = \begin{pmatrix} \varphi_+^{(0)}(w_2) & \varphi_+^{(0)}(w_3) \\ \varphi_-^{(0)}(w_2) & \varphi_-^{(0)}(w_3) \end{pmatrix} \ , \tag{3.313}$$

which shows that the entanglement entropy is a *projective invariant*, i.e., $\det(U) = \det(\gamma \circ U)$, for any $\gamma \in \mathrm{PSL}(2, \mathbb{C})$. The formula (3.313) is a general result for the $N = 1$ entanglement entropy when the topology of the conformal boundary is trivial. The only required information is the stress tensor T_0 of the background state. One can check this by inserting (3.301) into (3.313), and the result is the same as (3.305). As a remark, when the topology of the background is nontrivial, such as the $N = 1$ entanglement entropy in a black hole background, the map $z(w)$ in general are multi-valued on the entangling interval, which does not happen in the case of (3.301). The consequence is that there are always different phases for the entanglement entropy. Concrete examples of this can be found in [45].

I have discussed many close relations between AdS_3 gravity and the dual CFT_2 by studying the holographic entanglement entropy. The generalizations of those relations in the holographic system of $U(1)$ Chern-Simon-Einstein gravity will be studied in the next chapter. This model, apart from the Virasoro symmetry, contains an additional $U(1)$ Kac-Moody symmetry. Hence, it provides a perfect background for investigating the $U(1)$ symmetry resolution of the holographic entanglement entropy. I will use the generalized relations between the bulk and the boundary theories to calculate the charged moments and the symmetry-resolved entanglement entropy.

4 U(1) CHERN-SIMONS-EINSTEIN GRAVITY

This chapter is dedicated to studying the $U(1)$ symmetry-resolved entanglement entropy (SREE) in a particular holographic model, three-dimensional $U(1)$ Chern-Simons Einstein gravity. The decoupling between gravity and the $U(1)$ Chern-Simons fields in this theory makes it a simple playground for studying the holographic SREE. Meanwhile, some features of the charged moments and the SREE in this toy model not only provide some hints on how to solve the SREE in more complicated holographic models, but also lead us to a deeper understanding of the mechanism of the AdS/CFT duality. The outline of this chapter is as follows.

In Section 4.1, I will review some relevant aspects of $U(1)$ Chern-Simons-Einstein gravity, including the asymptotic symmetry and the charged black hole. The partition function of the charged black hole will be used to solve the $N = 1$ symmetry-resolved entanglement in the vacuum background.

In Section 4.2, I will study the symmetry-resolved entanglement entropy and the charged moments in the $U(1)$ Chern-Simons Einstein gravity, based on my works [29, 30]. I will first review the topological black hole approach. Then, using the vertex operators description of the charged moments, I will discuss how to solve them on the replica surface. The gravity dual of those vertex operators turns out to be Wilson lines in the bulk. I provide a more general argument that the disjoint Wilson lines compute the neutral $U(1)$ block in the dual CFT. I will also perform an analysis in the twist picture. To compute the general correlation function involving the charged twist fields, I will first prove the factorization property of the $U(1)$ extended conformal block, which was originally argued in [117] as well as in our work [30]. Based on the factorization property, I will provide a general method for calculating the charged moments in the twist picture. The resulting SREE in all those calculations exhibits the same equipartition behavior, in the sense that it is always independent of the $U(1)$ subregion charge.

4.1 U(1) CHERN-SIMONS FIELDS IN ADS_3 SPACE

In this section, I will give an introduction to the classical $U(1)$ Chern-Simons theory in AdS_3 space. I will first review the derivation of the conserved current and the associated asymptotic symmetry in the theory, following from [29, 118]. The resulting $U(1)$ Kac-Moody symmetry is promoted as the symmetry of the dual CFT by the AdS/CFT correspondence. Then, starting from the torus partition of the dual CFT, I will discuss its saddle point approximation in the high temperature limit and show that the result can be identified as the partition function of the $U(1)$ charged black hole from the gravity perspective.

4.1.1 *Currents and asymptotic symmetry*

The action of the $U(1)$ Chern-Simons fields in AdS_3 space is given by two chiral sectors,

$$S[A,\tilde{A}] = S[A] - S[\tilde{A}] \ , \tag{4.1}$$

with

$$\begin{aligned} S[A] &= \frac{ik}{8k}\int_{\mathcal{M}} A\wedge dA - \frac{k}{16\pi}\int_{\partial\mathcal{M}} d^2x\sqrt{h}A^iA_i \ , \\ S[\tilde{A}] &= \frac{ik}{8k}\int_{\mathcal{M}} \tilde{A}\wedge d\tilde{A} + \frac{k}{16\pi}\int_{\partial\mathcal{M}} d^2x\sqrt{h}\tilde{A}^i\tilde{A}_i \ . \end{aligned} \tag{4.2}$$

Here k is the Chern-Simons level, which is not quantized in the $U(1)$ case. The boundary terms are included in (4.1) for a well-defined variational principle for the $U(1)$ Chern-Simons fields. This theory is topological, since the equations of motion $dA = d\tilde{A} = 0$ tell that the gauge fields are closed one-forms on the background manifold $\mathcal{M}$. Furthermore, since the Chern-Simons fields and the metric decouple in the bulk, by RT formula, one expects that the entanglement entropy is still the same as the case in pure AdS_3 gravity. I will confirm this in Section 4.2.2 by calculating the entanglement entropy in the dual CFT.

Analogous to the analysis in the pure gravity sector, the $U(1)$ Chern-Simons theory in the AdS_3 space also permits an asymptotic symmetry structure. In [29, 88], it was shown that under certain boundary conditions, the conserved currents of the system, together with the stress tensor from the gravity sector, furnish a $\hat{\mathfrak{u}}(1)_k$ Kac-Moody algebra, which, from the bottom-up perspective of the AdS/CFT correspondence, is identified with the symmetry algebra of the dual CFT. In the following, I shall briefly review the derivation of the conserved currents of the Chern-Simons theory, as well as their associated symmetry algebra. The derivation in principle can be done on the AdS_3 space with arbitrary genus. However, since the symmetry algebra only encodes the universal short distance behavior of the currents correlation function, I will restrict to the Poincaré AdS_3 background with the flat boundary metric, i.e. $g_{(0)} = dzd\bar{z}$. For readers who are interested in the Ward identity of currents on general Riemann surfaces, the reference [119] is recommended.

To discuss the asymptotic symmetry, let me first impose the following gauge choice for the Cherns-Simons field [88],

$$A = A^{(0)} + \rho^2 A^{(1)} + \mathcal{O}(\rho^4) \ , \quad A^{(0)}_\rho = 0 \ , \quad \rho\to 0 \ , \tag{4.3}$$

so that on the conformal boundary $\rho = 0$, the equation of motion reads

$$\partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)} = 0 \ . \tag{4.4}$$

For $\tilde{A}$, it is similar. The variation of the Chern-Simons action (4.1) gives rise to the boundary terms

$$\delta S[A,\tilde{A}] = -\frac{k}{2\pi}\int d^2z\sqrt{g_{(0)}}\left(A_z^{(0)}\delta A_{\bar{z}}^{(0)} + \tilde{A}_{\bar{z}}^{(0)}\delta\tilde{A}_z^{(0)}\right) \ , \tag{4.5}$$

so the boundary condition is to fix $A_{\bar{z}}^{(0)}$ and $\tilde{A}_z^{(0)}$, but allows $A_z^{(0)}$ and $\tilde{A}_{\bar{z}}^{(0)}$ to vary. Following from [88], one defines the boundary current J^i, which in components reads

$$J(z) = J_z = \frac{ik}{2}A_z^{(0)} \ , \quad \bar{J}(\bar{z}) = J_{\bar{z}} = -\frac{ik}{2}\tilde{A}_{\bar{z}}^{(0)} \ . \tag{4.6}$$

And $A_{\bar{z}}^{(0)}$ and $\tilde{A}_z^{(0)}$ are the conjugate source terms, which can be combined to form a new external gauge field, defined as

$$\mathcal{A} = i\tilde{A}_z^{(0)}dz - iA_{\bar{z}}^{(0)}d\bar{z} \ . \tag{4.7}$$

Then the variation of the action can be written in a covariant form,

$$\delta S[A,\tilde{A}] = -\frac{1}{2\pi}\int d^2z\sqrt{g_{(0)}}J^i\delta\mathcal{A}_i \ . \tag{4.8}$$

By GKPW relation (2.36), one identifies J as the expectation value of the current in the dual CFT, and $\mathcal{A}$ as the background gauge field. To derive the asymptotic symmetry algebra, we need to impose an additional boundary condition, $\mathcal{A} = 0$, which is called the *holomorphic boundary condition*. In this case, the equation of motion (4.4) implies that the current J is conserved, i.e. $\partial_i J^i = 0$. In addition, a gauge transformation for the $U(1)$ gauge field A, i.e. $A \to A + id\lambda(z)$, preserves the boundary condition $\delta A_{\bar{z}}^{(0)} = 0$, but it changes the value of the current $J_z \to J_z - \frac{k}{2}\partial_z\lambda(z)$. This implies an asymptotic symmetry structure of the Chern-Simons theory in AdS$_3$ space. To derive the algebra, one can perform the same trick as we did in the gravity case, namely, considering a singular gauge parameter as $\lambda = 1/(z-w)$, then the current and the background gauge field transform as

$$\delta J_z = \frac{k}{2}\frac{1}{(z-w)^2} \ , \quad \delta\mathcal{A}_{\bar{z}} \to \partial_{\bar{z}}\frac{1}{z-w} = \pi\delta^{(2)}(z-w) \ . \tag{4.9}$$

Inserting (4.9) into the Ward identity

$$\delta\langle J_z\rangle = \frac{1}{2\pi}\int d^2w\sqrt{g_{(0)}}g_{(0)}^{w\bar{w}}\delta\mathcal{A}_{\bar{w}}\langle J_w J_z\rangle \ , \tag{4.10}$$

gives rise to the currents OPE

$$J(w)J(z) \sim \frac{k/2}{(z-w)^2} \ . \tag{4.11}$$

Then the corresponding algebra can be read out by the modes of expansion, given by

$$[J_n, J_m] = \frac{k}{2} n \delta_{m+n,0} \ , \quad J(z) = \sum_{n=-\infty}^{\infty} J_n z^{-1-n} \ . \tag{4.12}$$

The algebra for the antiholomorphic part is similar. As a check for the AdS/CFT correspondence, consider a Lagrangian deformation for the CFT action by introducing a non-vanishing background gauge field,

$$S_{CFT} \to S_{CFT} + \Delta S \ , \ \Delta S = -\frac{1}{2\pi} \int d^2 w \sqrt{g_{(0)}} g_{(0)}^{w\bar{w}} J_w \mathcal{A}_{\bar{w}} \ . \tag{4.13}$$

Then, using the current OPE (4.11), one can compute the expectation value of $\partial_{\bar{z}} J(z)$ on this new background, given by

$$\langle \partial_{\bar{z}} J(z) e^{-\Delta S} \rangle = -\frac{k}{2} \partial_z \mathcal{A}_{\bar{z}} \langle e^{-\Delta S} \rangle \ . \tag{4.14}$$

This is identified as the chiral anomaly formula in the dual CFT, which on the Chern-Simons theory side, is nothing but the equation of motion (4.4). For the anti-holomorphic, the relation is analogous. Another useful example is to consider the insertions of current primary operators $X = \prod O_j(z_j, \bar{z}_j)$ on the boundary CFT, but keep the external gauge field vanishing $\mathcal{A} = 0$. The current primary fields are defined via

$$J_0 O_j(z_j, \bar{z}_j) = q_j O_j(z_j, \bar{z}_j) \ , \quad J_n O_j(z_j, \bar{z}_j) = 0 \ , \ \forall n > 0 \ , \tag{4.15}$$

where q_j is called the $U(1)$ *charge* of O_j. The OPE between the $U(1)$ current and O_j reads,

$$J(z) O_j(z_j, \bar{z}_j) \sim \frac{q_j O_j(z_j, \bar{z}_j)}{z - z_j} \ . \tag{4.16}$$

This leads to the local Ward identity for the $U(1)$ current,

$$\langle \partial_{\bar{z}} J(z) X \rangle = \pi \sum_j q_j \delta^{(2)}(z - z_j) \langle X \rangle \ . \tag{4.17}$$

On the bulk side, the insertion of operators leads to the breakdown of the equation of motion (4.4), given by

$$\partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)} = \frac{2\pi i}{k} \sum_j q_j \delta^{(2)}(z - z_j) \ . \tag{4.18}$$

The holonomy of A around each boundary point z_i is given by

$$\oint_{z_i} A = \oint_{z_i} A_z^{(0)} dz = \frac{4\pi q_i}{k} \ , \tag{4.19}$$

which is proportional to the $U(1)$ charge q_i of O_j. Intuitively, it is natural to suspect that the delta function singularities in (4.18) will extend from the boundary into the bulk. Indeed, as I will discuss in

Section 4.2.3, the bulk dual of those primary fields are Wilson line defects, anchored at the positions of the primary fields on the boundary. Those Wilson line defects generate the delta-function singularities for the Chern-Simons fields along their paths.

To complete this section, let me also introduce the stress tensor associated with the Chern-Simons fields. The original construction for the Chern-Simons stress tensor in [88] is given by the standard functional variation of the action (4.1) with respect to the metric. In absence of the background gauge field $\mathcal{A}$, it can be written as the standard Sugawara construction [29],

$$T^{gauge}(z) = \frac{1}{k}\left(JJ\right)(z)\ , \quad \bar{T}^{gauge}(\bar{z}) = \frac{1}{k}\left(\bar{J}\bar{J}\right)(\bar{z})\ . \tag{4.20}$$

where the bracket (JJ) denotes the normal ordering. The normal ordering removes the singular terms from the JJ contraction, and as a result, the expectation value of the stress tensor is equivalent to the square of the $U(1)$ current when acting on current primary states, i.e. $\langle T^{gauge}(z)\rangle = 1/k\langle J(z)\rangle^2$. Therefore, in terms of the Chern-Simons fields, the stress tensor can be expressed as

$$T^{gauge}(z) = -\frac{k}{4}A_z^{(0)}A_z^{(0)}\ , \quad \bar{T}^{gauge}(\bar{z}) = -\frac{k}{4}\tilde{A}_{\bar{z}}^{(0)}\tilde{A}_{\bar{z}}^{(0)}\ . \tag{4.21}$$

which may be combined with the gravitational stress tensor to give rise to the full stress tensor of the $U(1)$ Chern-Simons-Einstein gravity,

$$T = T^{gravity} + T^{gauge}\ . \tag{4.22}$$

To derive the full asymptotic algebra, one still needs to verify the transformation law of currents J and full stress tensor T under (3.140). It is straightforward to check that J transforms as

$$\delta J = -J\partial\xi - \xi\partial J\ , \tag{4.23}$$

and full stress T transforms in the same way as in (3.145). Therefore, by modes expansion for the full stress tensor, $L_n = L_n^{gravity} + L_n^{gauge}$, one arrives at the $\hat{\mathfrak{u}}(1)_k$ Kac-Moody algebra [29],

$$\begin{aligned}
[J_n, J_m] &= \frac{k}{2}n\delta_{m+n,0}\ ,\\
[L_n, J_m] &= -mJ_{n+m}\ ,\\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\ .
\end{aligned} \tag{4.24}$$

I would like to give some remarks on this section. Firstly, the above analysis of the asymptotic algebra is derived in the absence of the background gauge field $\mathcal{A}$. If $\mathcal{A}$ is non-vanishing, then in the Langrangian deformation formalism (4.13), the algebra will certainly be modified due to the redefinition of fields. For instance, the stress tensor of the deformed Lagrangian will be given as

$$T(z) \to T(z) + \frac{k}{4}\mathcal{A}_z^2 = -\frac{k}{4}\left((A_z^{(0)})^2 + (\tilde{A}_z^{(0)})^2\right)\ . \tag{4.25}$$

which implies a shift in the spectrum of the dual CFT. In contrast, one can also consider the canonical deformation for the torus partition function, namely the finite temperature theory of the dual CFT,

$$Z_{CFT} = (q\bar{q})^{-c/24}\mathrm{Tr}[q^{L_0}\bar{q}^{\bar{L}_0}y^{J_0}\tilde{y}^{\bar{J}_0}] \ , \tag{4.26}$$

with $q = e^{2\pi i\tau}$ and $y = e^{2\pi i\mu}$. In this case, although there are chemical potentials (or background gauge fields) μ and $\tilde{\mu}$ coupled to the zero modes J_0 and $\bar{J}_0$, the stress tensor and currents are still defined in terms of the original Lagrangian, or in other words, (4.26) is taken trace over the original spectrum of the dual CFT. We will encounter these two types of deformations when calculating the canonical partition function of a charged black hole in the next section. More discussions can be found in [120] from the CFT perspective, and in [121] from the holographic point of view. Secondly, the central charge c in (4.24) is still the Brown-Henneaux central charge. Apparently, this contradicts the fact that the $U(1)$ sector should also contribute to the central charge $c = 1$. However, since we are considering the semi-classical limit $c \to \infty$, it is valid to neglect the $c = 1$ contribution in the gravity theory.

4.1.2 *Charged black hole*

In this section, I will briefly review the $U(1)$ charged black hole solution as well as its canonical partition function, following the discussion in [118]. The result can be applied to solve the $N = 1$ symmetry-resolved entanglement in the vacuum background [29, 32].

Let me first consider a pure AdS_3 black hole introduced in (3.79),

$$ds^2 = \frac{l^2}{4\rho^2} + \frac{l^2}{\rho^2}|d\bar{w} - \frac{\rho}{4\tau^2}dw|^2 \ . \tag{4.27}$$

The period of the boundary torus reads $w \sim w + 2\pi \sim w + 2\pi\tau$. In $U(1)$ Chern-Simons Einstein gravity, a charged black hole is obtained by simply adding constant $U(1)$ gauge fields on the background (4.27), such that the holonomies of A and $\tilde{A}$ around the contractible "time" loop $w \to w + 2\pi\tau$ vanishes [118]. Thus, we have the conditions,

$$\tau A_w + \bar{\tau} A_{\bar{w}} = 0 \ , \quad \tau \tilde{A}_w + \bar{\tau}\tilde{A}_{\bar{w}} = 0 \ . \tag{4.28}$$

where the components of A and $\tilde{A}$ are constants. Since the source terms $A_{\bar{w}}$ and $\tilde{A}_w$ are non-vanishing, the path integral in the dual CFT is formally written as

$$Z_{PI} = \int \mathcal{D}\Phi \ e^{-S_{CFT}-\Delta S} \ . \tag{4.29}$$

where ΔS is a Lagrangian deformation mentioned in (4.13),

$$\Delta S = \frac{i}{\pi}\int dw^2 \sqrt{g_{(0)}}\,(J_w A_{\bar{w}} - J_{\bar{w}}\tilde{A}_w) \tag{4.30}$$

As the source terms are constants, we can further express the deformation via the modes of the currents, which on torus is defined as

$$J_n = \oint \frac{dw}{2\pi i} J(w) e^{-inw} \ , \quad \bar{J}_n = -\oint \frac{d\bar{w}}{2\pi i} \bar{J}(\bar{w}) e^{in\bar{w}} \ , \tag{4.31}$$

where the integrals are evaluated around the spatial circle. Thus, integrating (4.30) on torus yields

$$\Delta S = -4\pi \mathrm{Im}(\tau) \left(A_{\bar{w}} J_0 + \tilde{A}_w \bar{J}_0 \right) \ . \tag{4.32}$$

Following [118], we introduce two new variables μ and $\tilde{\mu}$, defined as

$$\mu = -2i \mathrm{Im}(\tau) A_{\bar{w}} \ , \quad \tilde{\mu} = 2i \mathrm{Im}(\tau) \tilde{A}_w \ , \tag{4.33}$$

and then the deformation reads

$$\Delta S = -2\pi i \mu J_0 + 2\pi i \tilde{\mu} \bar{J}_0 \ . \tag{4.34}$$

It was claimed in [118] that the saddle point approximation of Z_{PI} can be obtained by evaluating the on-shell action of the bulk theory,

$$Z_{PI} \approx e^{-S} \ , \quad S = S_{ren} + S[A, \tilde{A}] \ . \tag{4.35}$$

The gravity part S_{ren} has been computed in (3.99). The action (4.1) of the gauge fields only contains the boundary contributions, given by

$$S[A, \tilde{A}] = -k\pi \mathrm{Im}(\tau) \left(\tilde{A}_w \tilde{A}_{\bar{w}} + A_w A_{\bar{w}} \right) \ . \tag{4.36}$$

Using the holonomy conditions (4.28) as well as (4.33), we can express (4.36) in terms of the new variables. Then, the whole on-shell action is given by

$$S = \frac{c\pi}{6} \mathrm{Im}\left(\frac{1}{\tau}\right) + \frac{k\pi i}{2} \left(\frac{\mu^2}{\tau} - \frac{\tilde{\mu}^2}{\bar{\tau}} \right) - \frac{k\pi}{4\mathrm{Im}(\tau)} (\mu^2 + \tilde{\mu}^2) \ . \tag{4.37}$$

Canonical partition function. It is important to know that the path integral Z_{PI} we considered before is the partition function of a deformed CFT. The spectrum of the Hilbert space in the deformed CFT is changed due to the redefinition of the stress tensors [118],

$$T \to T' = T - \frac{k}{4} \tilde{A}_w^2 \ , \quad \bar{T} \to \bar{T}' = \bar{T}(\bar{w}) - \frac{k}{4} A_{\bar{w}}^2 \ . \tag{4.38}$$

In contrast, we can also consider the grand canonical partition function defined in the original CFT, given by

$$Z = \mathrm{Tr}\left[e^{-2\pi \mathrm{Im}(\tau) H + 2\pi i \mathrm{Re}(\tau) P - \Delta S} \right] \ . \tag{4.39}$$

Here, the trace is taken over the original Hilbert space. ΔS is the same as (4.34). H and P are the Hamiltonian and the momentum, given by

$$H = L_0 + \bar{L}_0 - \frac{c}{12} \ , \quad P = L_0 - \bar{L}_0 \ , \tag{4.40}$$

where the Virasoro zero modes on torus are defined as

$$L_0 - \frac{c}{24} = -\oint \frac{dw}{2\pi} T(w) \ , \quad \bar{L}_0 - \frac{c}{24} = -\oint \frac{d\bar{w}}{2\pi} \bar{T}(\bar{w}) \ . \tag{4.41}$$

Due to (4.38), the Hamiltonian H' in the deformed theory becomes

$$H \to H' = H + \Delta H \ , \quad \Delta H = -\frac{k}{8\mathrm{Im}(\tau)^2}(\mu^2 + \tilde{\mu}^2) \ , \tag{4.42}$$

where (4.33) has been used. Hence, Z_{PI} and Z are related by an overall factor as

$$Z_{PI} = e^{-2\pi \mathrm{Im}(\tau)\Delta H} Z \ . \tag{4.43}$$

Notice that the exponent $2\pi\mathrm{Im}(\tau)\Delta H$ exactly equals to the last term of the on-shell action (4.37). Thus, the saddle point approximation of (4.39) is given by [118],

$$\log Z = -\frac{c\pi}{6}\mathrm{Im}(1/\tau) - \frac{k\pi i}{2}\left(\frac{\mu^2}{\tau} - \frac{\tilde{\mu}^2}{\bar{\tau}}\right) \ , \tag{4.44}$$

and we identify it as the grand canonical partition function of the charged black hole. Following the standard formula in thermodynamics,

$$Z = e^{s - 2\pi\mathrm{Im}(\tau)\langle H\rangle + 2\pi i\langle P\rangle + 2\pi i\mu\langle J_0\rangle - 2\pi i\tilde{\mu}\langle \bar{J}_0\rangle} \ , \tag{4.45}$$

we can obtain the thermal entropy s by calculating the expectation values of $(H, P, J_0, \bar{J}_0)$. The final result of the thermal entropy reads

$$s = -\frac{c\pi}{3}\mathrm{Im}(1/\tau) \ , \tag{4.46}$$

which does not depend on the Chern-Simons fields and is identical to the thermal entropy of the pure AdS_3 black hole [118].

4.2 SYMMETRY-RESOLVED ENTANGLEMENT

This section is dedicated to studying the symmetry-resolved entanglement entropy (SREE) in the context of the holographic $U(1)$ Chern-Simons-Einstein gravity, based on my works [29, 30]. I will mainly focus on discussing the $U(1)$ charged moments, which by Fourier transformation gives rise to the SREE. A different approach to SREE without using the charged moments was formulated in the free boson theory in the recent work [122].

Starting from the vacuum background with $N = 1$, one can map the the replica surface $R_{n,1}$ to a cylinder and transform the $U(1)$ charged moments to a grand canonical partition function defined on the cylinder. This method was used in [32], where the authors concluded that in a holographic CFT with an addition $U(1)$ global symmetry, the gravity dual of the charged moments is a $U(1)$ charged topological black hole[1].

1 Here, “topological” just means the topology of the black hole is a solid cylinder, different from the usual solid torus topology.

A different CFT approach to this problem was original proposed in [27], where the authors suggested that one can replace $e^{i\mu Q}$ term in the charged moments by two $U(1)$ vertex operators inserted at branched points of $R_{n,1}$. This description of the charged moments allows one to go beyond the case of the vacuum background with $N = 1$, and study more general cases, such as excited state backgrounds with N entangling intervals. Furthermore, it was shown in [27] that when we transform from the replica picture to the twist picture, the presence of vertex operators on $R_{n,1}$ will lead to modifications of the original twist fields defined in $\mathrm{CFT}^{\otimes n}$. The modified twist field is called charged twist field, which was first investigated in [32] but from a different approach.

The motivation for our works in [29, 30] is to understand SREE and the holographic dual of the charged moments. The reason for choosing the $U(1)$ Chern-Simons-Einstein gravity is that the decoupling between Chern-Simons fields and gravity simplifies the problems a lot, but meanwhile, one can get some insights for more general holographic setups from this simple model. The proposal made in [29] is that the holographic dual of the charged moments can be realized by inserting $U(1)$ Wilson line defects along the $\mathbb{Z}_n$ fixed points of the bulk replica manifold. This is a generalization of the cosmic string prescription of holographic Rényi entropy [106]. Essentially, the proposal establishes a duality between the bulk $U(1)$ Wilson line with the boundary vertex operators. I will give a holographic derivation for this duality by explicitly evaluating the effective action of Chern-Simons fields. In particular, at the end of the derivation, I will show that the holographic computation for the charged moments finally reduces to the generating function method introduced in [29].

Furthermore, I will carry out independent calculations in the dual CFT, from both the replica picture and the twist picture. In the replica picture, I will first show that the vertex operator realization of charged moments can also be applied to the holographic CFT with $U(1)$ Kac-Moody symmetry. Several aspects of the vertex operator, such as the OPE and Knizhnik-Zamolodchikov (KZ) equation, will be discussed. Using those properties, I will also discuss the charged moments in charged background and multi-intervals cases, by investigating the current Ward identity on replica surface, based on my work [30]. The results in all those cases show that for the $U(1)$ symmetry resolved entanglement entropy is independent of the charge. This is the so-called *equipartition of entanglement* [27], and I will explain the origin of this equipartition behavior, based on our observation in [31].

A more systematic approach to this problem will be discussed in the twist picture. I will first introduce the charged twist fields, following from the same method in [27]. To calculate the charged moments in more general cases, I will turn to investigate the $U(1)$ extended conformal block and prove that it always factorizes into the product of a $U(1)$ block and a Virasoro block. This factorization property confirms the argument in our original work [30]. In particular, it turns out that the $U(1)$ block computes the effective action of the bulk Wilson line defects. This generalizes the duality between Wilson line and vertex operators proposed in my work [29]. Inspired by the factorization, I will also discuss a general approach to the charged moments by studying the null-state equation in the dual CFT.

At the end of the section, I will give a summary of those various approaches to charged moments. Analogous to the entanglement entropy case, the similarities and relations between different approaches to the charged moments reflect the mechanism of the AdS/CFT duality at the semi-classical level.

4.2.1 *Topological black hole approach*

In this section, I will review the charged topological black hole approach to the charged moments [32], which is restricted to the case of the vacuum background with $N = 1$.

Consider the vacuum state of the dual CFT defined on $\mathbb{CP}^1$. We denote a single interval $A = [w_1, w_2]$ as the entangling region. The $U(1)$ charged moments is defined as

$$Z_n[\mu] = \mathrm{Tr}[\rho_A^n e^{i\mu Q_A}] \ , \tag{4.47}$$

where the vacuum partition function on $\mathbb{CP}^1$ is normalized, $Z_1 = 1$. The reduced density matrix can be formally expressed as $\rho_A = e^{-2\pi H_A}$, where H_A is called the *modular Hamiltonian*. It is known that in a general excited state background, the modular Hamiltonian is non-local, in the sense that it can not be expressed by local fields [123]. However, in this vacuum case, it admits a local expression, given by [45, 124],

$$H_A = -\int_A \frac{dw}{2\pi}\frac{T(w)}{u'(w)} - \int_A \frac{d\bar{w}}{2\pi}\frac{\bar{T}(\bar{w})}{\bar{u}'(\bar{w})} + C \ , \tag{4.48}$$

where C is a normalization constant such that $\mathrm{Tr}[\rho_A] = 1$. The function $u(w)$, defined as

$$u(w) = \log\left(\frac{w - w_1}{w_2 - w}\right) \ , \tag{4.49}$$

is a map from $\mathbb{CP}^1$ to a infinite cylinder. Denote $u = x + it_E$. Then $w = w_1$ and $w = w_2$ are mapped to $x = -\infty$ and $x = \infty$. To make the cylinder finite, we insert two cut-off circles around the two endpoints, with radius ϵ. Under the map $u(w)$, those two circles become the boundaries of a finite cylinder. The width and the Euclidean time period of the finite cylinder can be worked out as $\Delta L = 2\log|\frac{w_1 - w_2}{\epsilon}|$ and $t_E \sim t_E + 2\pi$. Similarly, for $R_{n,1}$ with the same cutoff circles, $u(w)$ maps it to a cylinder with width ΔL and $t_E \sim t_E + 2\pi n$. Since the spatial direction of the cylinder is truncated, the classical conformal group on it is only given by a single SL(2,ℝ). Generators of the classical conformal group are vectors in the form of $(\xi_m)^i = e^{imt_E}\left(\frac{\partial}{\partial t_E}\right)^i$, and the Virasoro generators L_m are obtained by associating the stress tensor T_{ij} with the vectors $(\xi_m)^i$. The details of this are well-known in the context of boundary conformal field theory (BCFT) [125]. Here, what we need to know is that transforming $T(w)$ in (4.48) in u coordinates, one can show that H_A is linear in L_0, hence it generates the time translation on the cylinder [123]. The explicit relation between H_A and L_0 are given by [31, 123]

$$H_A = \frac{\pi}{\Delta L}\left(L_0 - \frac{c}{24}\right) + \frac{c\Delta L}{24\pi} \ , \tag{4.50}$$

with the normalization constant $C = \frac{c\Delta L}{12\pi}$. The $U(1)$ charge operator Q_A is defined on A as

$$Q_A = J_0 = \int_A \frac{dw}{2\pi i} J(w) + \int_A \frac{d\bar{w}}{2\pi i} \Omega(\bar{J})(\bar{w}) \ , \tag{4.51}$$

Here Ω is the automorphism of the $\mathfrak{u}(1)_k$ Kac-Moody algebra,

$$\Omega(\bar{J})(\bar{w}) = \pm\bar{J}(\bar{w}) \ , \quad \Omega(\bar{T})(\bar{w}) = \bar{T}(\bar{w}) \ , \tag{4.52}$$

and the choice for the automorphism Ω is related to the conformal boundary condition imposed in the BCFT. Without going into details, I now derive the conformal boundary condition from the $[H_A, Q_A]$ commutator. To make sense Q_A as a conserved charge operator on A, one must have $[H_A, Q_A] = 0$ (or $[L_0, J_0] = 0$). This commutator can be evaluated straightforwardly by using the OPE between the stress tensor and the current,

$$\begin{aligned}
[H_A, Q_A] &= \int_{A_+} \frac{dw}{2\pi} \int_A \frac{dz}{2\pi i} \frac{T(w)J(z)}{u'(w)} - \int_A \frac{dz}{2\pi i} \int_{A_-} \frac{dw}{2\pi} \frac{J(z)T(w)}{u'(w)} + c.c. \\
&= -\int_A \frac{dz}{2\pi i} \oint_z \frac{dw}{2\pi} \frac{1}{u'(w)} \left[\frac{J(z)}{(w-z)^2} + \frac{\partial J(z)}{w-z}\right] + c.c. \\
&= -\frac{1}{2\pi} \left[J(u) - \Omega(\bar{J})(\bar{u})\right]\Big|_{u_1}^{u_2} \ .
\end{aligned} \tag{4.53}$$

In the first step, A_+ and A_- are the two curves connecting the two boundary circles, and they enclose the interval A, representing the future and the past "time" slices respectively. In the second step, the contour integral is in the counter-clockwise direction. In the last step, $J(u) = J(w)/u'(w)$ is the $U(1)$ current in cylinder coordinates, and u_1 and u_2 represent the endpoints of A on the two boundaries of the cylinder. So the requirement $[H_A, Q_A] = 0$ gives rise to following condition on the two boundaries,

$$J(u) = \Omega(\bar{J})(\bar{u}) \ , \tag{4.54}$$

This condition implies that for "+" sign choice in (4.52), one has $J_{t_E} = 0$, while for "−" sign choice, $J_x = 0$.

With the above constructions, now we can rewrite the charged moments as

$$Z_n[\mu] = e^{-cn\Delta L/12} q^{-c/24} \mathrm{Tr}[q^{L_0} y^{J_0}] \ , \tag{4.55}$$

with $q = e^{2\pi i\tau}$, $y = e^{i\mu}$ and $\tau = \frac{in\pi}{\Delta L}$. The holographic interpretation for the charged moments is then the grand canonical partition function of a charged black hole with the cylinder topology. In particular, since in the long-distance limit $\Delta L \to \infty$ as $\epsilon \to 0$, the physics on the cylinder becomes the same as on the torus. Thus, one can directly use the charged black hole partition function (4.44) to obtain the charged moments, given by

$$Z_n[\mu] = \left| \frac{w_1 - w_2}{\epsilon} \right|^{-\frac{c}{6}\left(n-\frac{1}{n}\right)-\frac{k}{n}\left(\frac{\mu}{2\pi}\right)^2} \ . \tag{4.56}$$

Assume that the spectrum for the charge operator Q_A is continuous, then the probability distribution $P_n(q)$ can be worked out as

$$P_n(q) = \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} e^{-i\mu q} \frac{Z_n[\mu]}{Z_n[0]} = \sqrt{\frac{2\pi n}{k\Delta L}} e^{-\frac{2n\pi^2 q^2}{k\Delta L}} \ , \tag{4.57}$$

In $n \to 1$ limit, using (2.89) yields the SREE,

$$S(q) = \frac{c}{6}\Delta L - \frac{1}{2}\log\left(\frac{k\Delta L}{2\pi}\right) + \mathcal{O}(\epsilon^0) \ . \tag{4.58}$$

The above form of SREE in our holographic model matches the results in the original paper [27]. The q-independent behavior of SREE is called the equipartition of entanglement, which means the entanglement entropy distributes evenly among different charge sectors of the subsystem A.

Here are some remarks on the topological black hole approach to the $U(1)$ charged moments. Note that the automorphism Ω does not affect these semi-classical results for the charged moments since μ is quadratic in the exponent of $Z_n[\mu]$. However, it affects the $\mathcal{O}(\epsilon^0)$ quantum corrections in the exponent, which are known as the Affleck-Ludwig boundary entropy in the CFT context [40]. The boundary effects on the SREE were recently studied in [122, 126] via the BCFT approach. The possible way to study those boundary effects in the holographic model is to use the end-of-world brane construction in the AdS_3 space [127–129]. On the other hand, Ω does affect the Chern-Simons field configurations in the bulk. Writing down the variation of the charged moments with respect to μ in the cylinder coordinates explicitly,

$$\begin{aligned} \delta \log Z_n(\mu) &= \int_A \frac{dx}{2\pi} \left[J(u)\delta\mu + \Omega(\bar{J})(\bar{u})\delta\mu \right] \\ &= \frac{1}{2\pi} \int dt_E \int dx \frac{\delta\mu}{2\pi n} [J(u) \pm \bar{J}(\bar{u})] \ , \end{aligned} \tag{4.59}$$

and comparing with the variation of the Chern-Simons action in (4.8) yields

$$A_{\bar{u}} = \frac{\mu}{2\pi i n} \ , \quad \tilde{A}_u = \mp \frac{\mu}{2\pi i n} \ , \quad \text{for } \Omega = \pm 1 \ . \tag{4.60}$$

The A_u and $\tilde{A}_{\bar{u}}$ components are then determined by the trivial holonomy conditions around the t_E loop, i.e. $A_u = A_{\bar{u}}$, and $\tilde{A}_u = \tilde{A}_{\bar{\mu}}$. Transforming them back to the w-coordinates yields

$$\begin{aligned} A_w &= \pm \tilde{A}_w = \frac{\mu}{2\pi i n}\left(\frac{1}{w - w_2} - \frac{1}{w - w_1}\right) \ , \\ A_{\bar{w}} &= \pm \tilde{A}_{\bar{w}} = \frac{\mu}{2\pi i n}\left(\frac{1}{\bar{w} - \bar{w}_2} - \frac{1}{\bar{w} - \bar{w}_1}\right) \ , \end{aligned} \tag{4.61}$$

with the trivial holonomies around two endpoints, $\oint_{w_i} A = \oint_{w_i} \tilde{A} = 0$ for $i = 1, 2$. I emphasize the various aspects of the field configurations here because in the Wilson line approach to the charged moments, as I will introduce later, the bulk Chern-Simons fields A and $\tilde{A}$ will contain non-trivial holonomies. In addition, the fields have vanishing source terms $A_{\bar{w}}$ and $\tilde{A}_w$, namely they are in the pure holomorphic and anti-holomorphic gauges. Those two distinct approaches in the holographic calculations arise from two different interpretations for the $U(1)$ charged moments in the dual CFT. While in the above discussions the $e^{i\mu Q_A}$ is understood as introducing addition chemical potential μ coupled with the chiral currents in the thermal partition function, in [27], it was shown that one can equivalently treat $e^{i\mu Q_A}$ as two vertex operators inserted at the branched points of the replica surface. As I will show later, those vertex operators can also be applied in the holographic CFT with $U(1)$ Kac-Moody symmetry, and they generate the $U(1)$ Wilson line defect in the bulk [29].

4.2.2 *Vertex operators on replica surface*

In the following, I would like to introduce the vertex operators' description of the $U(1)$ charged moments in the dual CFT. By investigating the OPE structure and correlation functions of vertex operators, I will then discuss how to compute the $U(1)$ charged moments in the replica picture, and explain why SREE exhibits equipartition behavior by implementing the perturbative approach in [31]. The discussions include the cases of vacuum background and charged background as well as the multi-intervals, mainly based on our works in [29, 30].

Charged moments from local fields. Let me first explain the equivalence between charged moments and vertex operators. Consider $R_{n,1}$ with two branched points w_1 and w_2. We use the conformal transformation (3.164) map $R_{n,1}$ to $\mathbb{CP}^1$, coordinated by z. The transformation for the metric include a coordinate transformation and a compensated Weyl transformation,

$$d\hat{s}^2 = |dw|^2 \to d\tilde{s}^2 = |dz|^2 \ , \ d\hat{s}^2 = e^{2\phi} d\tilde{s}^2 \ , \ e^{2\phi} = \left|\frac{dw}{dz}\right|^2 \ , \tag{4.62}$$

so the vacuum partition function on those two surfaces are related as

$$Z_n[0] = Z[g_{(0)}] = e^{-S_L[\phi,\tilde{g}]} Z[\tilde{g}] \ . \tag{4.63}$$

Here the vacuuum partition function on $\mathbb{CP}^1$ is assumed to be normalized, $Z[\tilde{g}] = 1$.The Liouville action is similar to (3.108), given by

$$S[\phi, \tilde{g}] = \frac{c}{6}\left(n - \frac{1}{n}\right) \log\left|\frac{w_1 - w_2}{\epsilon}\right| \ . \tag{4.64}$$

The reason for working on $\mathbb{CP}^1$ is that the OPEs of the stress tensor and the $U(1)$ current take the usual form, which lead to two copies of the $\mathfrak{u}(1)_k$ Kac-Moody-Virasoro algebra (4.24) under the usual mode expansion. However, if we work on $R_{n,1}$, the OPEs and mode expansions will take different forms, due to the branched structure. By implementing the Sugawara construction

$$T_J(z) = \frac{1}{k}(JJ)(z) \ , \tag{4.65}$$

the stress tensor $T(z)$ can be decomposed into a pure Virasoro sector and a $U(1)$ sector, $T(z) = T_{vir}(z) + T_J(z)$. This is nothing but $T(z) = T^{gravity}(z) + T^{gauge}(z)$ on the gravity side. By $J(z)J(0) \sim \frac{k/2}{z^2}$, the OPEs between T_J and J can be checked directly, given by

$$\begin{aligned} T_J(z)J(0) &\sim \frac{J(0)}{z^2} + \frac{\partial J(0)}{z} \ , \\ T_J(z)T_J(0) &\sim \frac{1/2}{z^4} + \frac{2T_J(0)}{z^2} + \frac{\partial T_J(0)}{z} \ . \end{aligned} \tag{4.66}$$

The first one in (4.66) implies that the stress tensor T_{vir} decouples with the $U(1)$ sector,

$$T_{vir}(z)J(0) \sim 0 \ , \quad T_{vir}(z)T_J(0) \sim 0 \ . \tag{4.67}$$

The second one in (4.66) indicates that the central charge in the $U(1)$ sector is $c = 1$. Hence, the pure Virasoro sector has central charge $c' = c - 1$. Now, consider the $e^{i\mu Q_A}$ operator in the charged moments $Z_n[\mu]$. The branched points $w_{1,2}$ are mapped to $z = 0$ and $z = \infty$ respectively, so in z-coordinates one has

$$e^{i\mu Q_A} = \exp\left\{\frac{\mu}{2\pi}\int_0^\infty dz J(z) + \frac{\mu}{2\pi}\int_0^\infty d\bar{z}(\Omega\bar{J})(\bar{z})\right\} \ . \tag{4.68}$$

By Taylor expansion of $e^{i\mu Q_A}$ and using the JJ OPE, it is straightforward to check the following relation,

$$\langle J(z) e^{i\mu Q_A}\rangle = -\frac{k\mu}{4\pi}\frac{1}{z}\langle e^{i\mu Q_A}\rangle \ . \tag{4.69}$$

Similarly, we can also work out

$$\langle T(z)e^{i\mu Q_A}\rangle = \langle T_J(z)e^{i\mu Q_A}\rangle = \frac{k\mu^2}{16\pi^2}\frac{1}{z^2}\langle e^{i\mu Q_A}\rangle \ . \tag{4.70}$$

Comparing (4.69) and (4.70) with the Ward identities for the conformal and $U(1)$ symmetries, one then identifies $e^{i\mu Q_A}$ as two local primary fields inserted at $z = 0$ and $z = \infty$,

$$e^{i\mu Q_A} = V_{-a}(0)V_a(\infty) \ , \tag{4.71}$$

where the conformal weight and $U(1)$ charge of V_a are given by

$$h_a = \frac{a^2}{k} = \frac{k\mu^2}{16\pi^2} \ , \quad a = \frac{k\mu}{4\pi} \ . \tag{4.72}$$

and its OPEs with stress tensor and $U(1)$ current read

$$\begin{aligned} &T(z)V_a(0) = T_J(z)V_a(0) \sim \frac{h_a}{z^2} + \frac{\partial V_a(0)}{z} \ , \\ &J(z)V_a(0) \sim \frac{aV_a(0)}{z} \ . \end{aligned} \tag{4.73}$$

The antiholomorphic part is similar. For $\Omega = \pm 1$ and $\mu \in \mathbb{R}$, $\bar{a} = \pm a$ and $\bar{h}_{\bar{a}} = h_a$, and I will choose $\Omega = 1$ in the later discussion, since the conformal weights are not affected by the sign of Ω. With the identification (4.71), the charged moments is then expressed via the two-point function,

$$Z_n[\mu] = Z_n[0]\langle V_{-a}(0)V_a(\infty)\rangle = e^{-S_L[\phi,\tilde{g}]}\langle V_{-a}(0)V_a(\infty)\rangle \ . \tag{4.74}$$

For convenience, we define an effective action $S[\mu]$ to capture the μ-dependent part of $Z_n[\mu]$,

$$Z_n[\mu] = e^{-S_L[\phi,\tilde{g}]-S[\mu]} \ . \tag{4.75}$$

Similar to the Liouville action, this effective action $S[\mu]$ also needs to be regularized. This can be done by inserting cutoff circles $\delta = |z|$ and $\delta = |1/z|$ around the 0 and ∞, so that the two-point function becomes

$$e^{S[\mu]} = \langle V_{-a}(\delta)V_a(1/\delta)\rangle \approx e^{-4h_\alpha \log(1/\delta)} \ , \quad \delta \to 0 \ . \tag{4.76}$$

Under the inverse map of (3.164), those two cutoff circles can be transformed back to the w-coordinates, given by $\epsilon = |w - w_1|$ and $\epsilon = |w - w_2|$ respectively. The relation between δ and ϵ is determined by the map (3.164), given by

$$\delta = \left|\frac{\epsilon}{w_2 - w_1}\right|^{1/n} \ . \tag{4.77}$$

Inserting this into (4.76) and combining with (4.64) reproduces the result (4.56) derived in the black hole approach.

Universality of vertex operators correlator. The primary field V_a is in fact the standard vertex operator, which obeys the fusion rule $V_a \times V_b \to V_{a+b}$ [98]. To see this, we first notice that the global $U(1)$ symmetry requires charge conservation. Thus, in the OPE between V_a and V_b, only the fields with $U(1)$ charge $a+b$ can appear. A general descendent field in the theory is generated by acting a series of L_{-m_i} and J_{-l_j} on the primary field. Due to the decomposition of the stress tensor, the descendent field can also be expressed in the decoupled basis of $\{L^{vir}_{-m_i}, J_{-l_j}\}$. The relation between L^{vir}_m and L_m reads

$$L_m = L^{vir}_m + L^J_m \ , \quad L^J_m = \frac{1}{k}\sum_l : J_{m-l}J_l : \ , \tag{4.78}$$

where the normal ordering is defined as $: J_{m-l}J_l := J_{m-l}J_l$ for $m-l<l$, and $: J_{m-l}J_l := J_l J_{m-l}$ for $m-l \geq l$. Since $[J_{m-l}, J_l] = \frac{k}{2}(m-l)\delta_m$, the normal ordering can be neglected if $m \neq 0$. For $m=0$, we have

$$L^J_0 = \frac{1}{k}J_0^2 + \frac{2}{k}\sum_{m>0} J_{-m}J_m \ . \tag{4.79}$$

In the decoupled basis, the OPE between V_a and V_b can be formally written as

$$V_a(z,\bar z)|V_b\rangle = \sum_p \sum_{\{m,\bar m\},\{l,\bar l\}} C_{ab}^{p\{m,\bar m\}\{l,\bar l\}} z^{h_p-h_a-h_b+2ab/k+|m|+|l|} \times \bar z^{\bar h_p-h_a-h_b+2ab/k+|\bar m|+|\bar l|}|O^{\{m,\bar m\}\{l,\bar l\}}_{p,a+b}\rangle \ . \tag{4.80}$$

Here, we define a primary state $O_{p,a+b}$ as

$$\begin{aligned} L^{vir}_0|O_{p,a+b}\rangle &= h_p|O_{p,a+b}\rangle \ , \qquad L^{vir}_n|O_{p,a+b}\rangle = 0 \ , \\ J_0|O_{p,a+b}\rangle &= (a+b)|O_{p,a+b}\rangle \ , \quad J_n|O_{p,a+b}\rangle = 0 \ , \end{aligned} \tag{4.81}$$

for all $n>0$. The total conformal weight is obtained by acting L_0 on the primary state, given by

$$h_{total} = h_p + \frac{(a+b)^2}{k} \ . \tag{4.82}$$

The descendent states are defined as

$$|O^{\{m,\bar m\}\{l,\bar l\}}_{p,a+b}\rangle = L^{vir}_{-m_1}\cdots L^{vir}_{-m_i} J_{-l_1}\cdots J_{-l_j}|O_{p,a+b}\rangle \ . \tag{4.83}$$

with $|m| = \sum_i m_i$ and $|l| = \sum_j l_j$ denoting the levels. To analyze the structure of the OPE, we act L^{vir}_n on the both sides of (4.80). The non-singular OPE

$$T_{vir}(w)V_a(z) = \sum_{m\in\mathbb{Z}} \frac{(L^{vir}_m V_a)(z)}{(w-z)^{2+m}} \overset{!}{=} \text{regular terms} \ , \tag{4.84}$$

implies that the left hand side of (4.80) is annihilated by L^{vir}_n for $n \geq -1$, which then imposes the following condition on the descendent states on the right hand side of (4.80),

$$L^{vir}_n|O^{\{m,\bar m\}\{l,\bar l\}}_{p,a+b}\rangle = 0 \ , \quad \forall n \geq -1 \ . \tag{4.85}$$

This condition (4.85) implies two facts. First, there is no L^{vir}_{-m} modes in all those descendent states. Second, $h_p = 0$ for all those descendent states. Therefore, the OPE (4.80) is restricted to the following form,

$$V_a(z,\bar{z})|V_b\rangle = \sum_{\{l,\bar{l}\}} \gamma^{\{l\}}_{ab}\bar{\gamma}^{\{\bar{l}\}}_{ab} z^{2ab/k+|l|}\bar{z}^{2ab/k+|\bar{l}|}|V^{\{l,\bar{l}\}}_{a+b}\rangle \ , \tag{4.86}$$

where the OPE coefficient is normalized since only a single $U(1)$ family appears. Similar to the Virasoro case, the coefficients $\gamma^{\{l\}}_{ab}$ and $\bar{\gamma}^{\{\bar{l}\}}_{ab}$ are completely fixed by the $U(1)$ symmetry, and only depend on (a,b,k). The detailed calculations on those coefficients will be given later when I discuss the $U(1)$ extended conformal block.

The simple OPE structure of vertex operators in (4.86) implies that any higher-point correlation functions of them are fixed by the $U(1)$ symmetry, hence they are integrable. In fact, there is a simple way to determine them. Consider X as a product of vertex operators, i.e. $X = \prod_{i=1}^m V_{a_i}(z_i,\bar{z}_i)$, with charge conservation $\sum_{i=1}^m a_i = 0$. Using $L^{vir}_{-1}V_{a_i} = 0$, and $L^J_n = \sum_{l\in\mathbb{Z}} : J_{n-l}J_l :$, one obtains

$$L^{vir}_{-1}V_{a_i}(z_i,\bar{z}_i) = (\partial_{z_i} - \frac{2}{k}J_{-1}J_0)V_{a_i}(z_i,\bar{z}_i) = 0 \ . \tag{4.87}$$

On $\mathbb{CP}^1$, the action of J_{-n} on $V_{a_i}(z_i,\bar{z}_i)$ in the correlation function $\langle X\rangle$ is given by

$$J^{(z_i)}_{-n} = -\sum_{j\neq i}\frac{a_j}{(z_j-z_i)^n} \ , \quad J_0 = -\sum_{j\neq i} a_j = a_i \ . \tag{4.88}$$

Applying (4.87) and (4.88) on $\langle X\rangle$ yields the Knizhnik-Zamolodchikov (KZ) equations [130]

$$\left(\partial_{z_i} + \frac{2a_i}{k}\sum_{j\neq i}\frac{a_j}{z_j - z_i}\right)\langle X\rangle = 0 \ , \quad i = 1,\cdots,m \ . \tag{4.89}$$

Integrating the above differential equations gives rise to the holomorphic part of $\langle X\rangle$

$$\langle X\rangle = \prod_{i<j}(z_i - z_j)^{2a_ia_j/k} \times c.c. \ , \tag{4.90}$$

and the anti-holomorphic part is similar.

Application to $N = 1$ charged backgrounds. The correlation function (4.90) can be implemented to compute the charged moments in charged excited state background with $N = 1$ [30]. As an example, we consider the insertion of vertex operators $V_b(w_3)$ and $V_{-b}(w_4)$ on the original $\mathbb{CP}^1$. The background partition function reads,

$$Z_1[0] = \langle V_b(w_3)V_{-b}(w_4)\rangle = |w_3 - w_4|^{-4b^2/k} \ . \tag{4.91}$$

where the holomorphic and anti-holomorphic charges of $V_{\pm b}$ are assumed to be the same. On $R_{n,1}$ with branched points w_1 and w_2, there are n copies of $V_{\pm b}$, and each copy lives on the a single sheet of $R_{n,1}$. Using the map (3.164), one can work out the positions of the n pairs of local fields $\{V_b(z_{2m-1}), V_{-b}(z_{2m})\}$ in z-coordinates,

$$z_{2m-1} = e^{\frac{2\pi i m}{n}} \left(\frac{w_3 - w_1}{w_2 - w_3}\right)^{\frac{1}{n}}, \; z_{2m} = e^{\frac{2\pi i m}{n}} \left(\frac{w_4 - w_1}{w_2 - w_4}\right)^{\frac{1}{n}} . \quad (4.92)$$

with $m = 1, \cdots, n$. To calculate the charged moments, let us first analyze $Z_n[0]$ and the entanglement entropy, since the situation here is more complicated than the vacuum case. The partition function $Z_n[0]$ can be expressed in z-coordinates as

$$Z_n[0] = e^{-S_L[\phi,\tilde{g}]} \prod_{i=1}^{2n} \left|\frac{dw}{dz}\right|_{z_i}^{-2h_b} \langle V_b(z_1) \cdots V_{-b}(z_{2n})\rangle \; , \quad (4.93)$$

where we have used the transformation law of primary fields under the conformal transformation $z \to w$,

$$V_{\pm b}(z_i) \to V_{\pm b}(w(z_i)) = V_{\pm b}(z_i) \left|\frac{dw}{dz}\right|_{z_i}^{-2h_b} . \quad (4.94)$$

Since the vertex correlation function $\langle V_b(z_1) \cdots V_{-b}(z_{2n})\rangle$ is defined on $\mathbb{CP}^1$, we can use (4.90) to compute it. After a straightforward calculation of $Z_n[0]$ and rewriting the result in w-coordinates, one finds that

$$Z_n[0] = e^{-S_L[\phi,\tilde{g}]} |w_3 - w_4|^{-4nb^2/k} = e^{-S_L[\phi,\tilde{g}]} Z_1[0]^n \; . \quad (4.95)$$

Therefore, the $N = 1$ entanglement entropy in this excited background is the same as the case of the vacuum background. This result of course relies on the special structure of the vertex correlation function (4.90), and it looks a bit surprising from the CFT perspective[2]. However, from the bulk perspective, this is exactly what we would expect. The reason is as follows. Since the vertex operators $V_{\pm b}$ are annihilated by L_0^{vir}, but carry $U(1)$ charges, they should lead to excitation of $U(1)$ Chern-Simons fields, without deforming the AdS_3 geometry. By RT formula, the corresponding entanglement entropy should keep unchanged after adding such excitation. Therefore, the result (4.95) provides a nontrivial check for the consistence of this AdS_3/CFT_2 model.

Now, let me proceed with the charged moments. Similar as (4.75), we define an effective action $S[\mu, b]$ to capture the μ-dependence of the charged moments,

$$Z_n[\mu] = Z_n[0] e^{-S[\mu,b]} \; , \quad (4.96)$$

2 One might be confused about this result, since I have shown in (3.305) that the entanglement entropy is changed on a excited background. There is no contradiction here, because the derivation of (3.305) essentially relies on the nul-state equation in the dual CFT of pure AdS_3 gravity. However, the dual CFT here is a different one. It contains additional $U(1)$ symmetry, which leads to a modification of the null-state equation in the theory. I will explain this at the end of this chapter.

with

$$\begin{aligned} e^{-S[\mu,b]} &= \frac{\langle V_{-a}(0)V_b(z_1)\cdots V_{-b}(z_{2n})V_a(\infty)\rangle}{\langle V_b(z_1)\cdots V_{-b}(z_{2n})\rangle} \\ &= e^{-S[\mu]}\frac{\langle V_{-a}(0)V_b(z_1)\cdots V_{-b}(z_{2n})V_a(\infty)\rangle}{\langle V_{-a}(0)V_a(\infty)\rangle\langle V_b(z_1)\cdots V_{-b}(z_{2n})\rangle}\ , \end{aligned} \tag{4.97}$$

where $e^{-S[\mu]} = \langle V_{-a}(0)V_a(\infty)\rangle$ is regularized in (4.76). The remaining part (4.97) can be evaluated by using the general form (4.90). It is not hard to see that the numerator in (4.97) cancels out the denominator, leaving the terms which contain ab in exponents,

$$\begin{aligned} e^{-S[\mu,b]} &= e^{-S[\mu]}\prod_{m=1}^{n}\left|\frac{z_{2m}}{z_{2m-1}}\right|^{4ab/k} \\ &= \exp\left\{-\frac{k\mu^2}{4\pi^2 n}\log\left|\frac{w_1-w_2}{\epsilon}\right| + \frac{b\mu}{\pi}\log|1-x|\right\}\ , \end{aligned} \tag{4.98}$$

with the cross ratio $x = (w_{12}w_{34})/(w_{13}w_{24})$.

Generating function method. This μ-dependent part (4.98) of $Z_n[\mu]$ can be understood more clearly from the generating function method introduced in [29]. Note that the derivative of $S[\mu,b]$ with respect to μ should gives rise to the expectation value $\mathcal{Q}_A$ of the subregion charge operator Q_A,

$$\mathcal{Q}_A = -i\frac{\partial}{\partial\mu}\log Z_n[\mu] = i\partial_\mu S[\mu,b]\ , \tag{4.99}$$

On the other hand, one can also compute the expectation value of the $U(1)$ currents J and $\bar{J}$ in z-coordinates. We focus on the holomorphic part. By the OPE (4.16) between J and current primary fields, we have

$$\begin{aligned} \langle J(z)\rangle &= \frac{\langle J(z)V_{-a}(0)V_b(z_1)\cdots V_{-b}(z_{2n})V_a(\infty)\rangle}{\langle V_{-a}(0)V_b(z_1)\cdots V_{-b}(z_{2n})V_a(\infty)\rangle} \\ &= -\frac{a}{z} + \sum_{m=1}^{n}\left(\frac{b}{z-z_{2m-1}} - \frac{b}{z-z_{2m}}\right)\ . \end{aligned} \tag{4.100}$$

with a defined in (4.72). Using (3.164) to transform $J(z)$ back to w-coordinates, $J(w) = J(z)dz/dw$, yields a very simple form,

$$\langle J(w)\rangle = \frac{-a/n}{w-w_1} + \frac{a/n}{w-w_2} + \frac{b}{w-w_3} - \frac{b}{w-w_4}\ . \tag{4.101}$$

Introducing the cut-off ϵ around $w = w_1$ and $w = w_2$ and evaluating the expectation value of the charge by the definition (4.51), one finds the result exactly matches with (4.99),

$$\mathcal{Q}_A = \frac{ik\mu}{2\pi^2 n}\log\left|\frac{w_1-w_2}{\epsilon}\right| - \frac{ib}{\pi}\log|1-x| = q_a + q_b\ . \tag{4.102}$$

The first term q_a is universal and is related to the insertions of $V_{\pm a}$ at the branched points. The second term q_b expressed in terms of the cross ratio x is simply the $U(1)$ subregion charge contributed from the background state $V_{\pm b}$. Note that both a (or μ) and b are real quantum numbers, associated with $U(1)$ representations, so one might wonder why the $\mathcal{Q}_A$ is imaginary. The eigenvalue of the charge operator Q_A of course should be real, however, the expectation value of it can be complex in Euclidean signature. One can think of this from the reality condition on the action $S[\mu, b]$, which enforces $\mathcal{Q}_A$ to be pure imaginary here. Another example is the rotating AdS_3 black hole, where the expectation value of the angular momentum is complex in the Euclidean signature, but it becomes real in the Lorentzian case [88].

Origin of equipartition of entanglement. Let me complete discussion by calculating the SREE associated with the charged moments (4.96). Using (4.98), it is straightforward to show that the probability distribution $P_n(q)$, defined in (2.93), is still in type of Gaussian,

$$P_n(q) = \sqrt{\frac{2\pi n}{k\Delta L}} e^{-\frac{2n\pi^2(q-q_b)^2}{k\Delta L}} \;, \quad \Delta L = 2\log|\frac{w_1 - w_2}{\epsilon}| \;. \tag{4.103}$$

with $q_b = -\frac{ib}{\pi}\log|1-x|$ defined in (4.102). The only difference between the (4.103) with the vacuum case (4.57) is the shift of the saddle of q in the distribution. However, such a shift does not affect SREE, which in this case is still given by (4.58). Therefore, at the leading order of the expansion in ϵ, the equipartition of entanglement still holds in the charged background. But why does SREE behave in this way? Is there any counterexample? The answer to the first question was partially provided by the observation in [29]. One notices that the exponent in the distribution $P_n(q)$ in (4.103) is linear in n. Consequently, $P_n(q)/P_1(q)^n$ only contributes a constant factor in n, and the Δq dependence is removed in SREE. The Gaussian distribution can be traced back to the charged moments, which is a Gaussian function of μ, and a Fourier transformation of a Gaussian function is still Gaussian. Therefore, the key point is the quadratic behavior of μ in the exponent of the $U(1)$ charged moments. The answer was elaborated further in my work [31], by showing that the Gaussian type charged moments stem from the $U(1)$ symmetry. The way to show it is to use the perturbative expansion of $\log Z_n[\mu]$ in μ. I shall only focus on the μ dependent part, which is given by [31],

$$\log Z_n[\mu] = C_0 + \sum_{m=0}^{\infty} \frac{1}{m!}\langle (i\mu Q_A)^m\rangle_{b,c} \tag{4.104}$$

where the "c" index denotes the *connected correlation function* of $(Q_A)^m$ in the uniformized z-coordinates,

$$\langle Q_A^m\rangle_{b,c} = \left(\prod_{j=1}^{m}\int_A \frac{dx_j}{2\pi i}\right)\left(\frac{\langle J(x_1)\cdots J(x_m)V_b(z_1)\cdots V_{-b}(z_{2n})\rangle}{\langle V_b(z_1)\cdots V_{-b}(z_{2n})\rangle}\right)_c$$
$$+ \text{anti-holomorphic part} \;, \tag{4.105}$$

Being connected correlation function of Q_A means that one only includes the contributions, in which the contractions among J are all connected. Therefore, at the first order, the result of $\langle Q_A \rangle_{b,c}$ is simply given by the background charge q_b in (4.102),

$$\langle Q_A \rangle_{b,c} \sim \int_A \frac{dx_1}{2\pi i} \frac{\langle J V_{-b} \cdots V_b \rangle}{\langle V_b \cdots V_{-b} \rangle} + c.c. = q_b \ , \tag{4.106}$$

and the second order only comes from JJ and $\bar{J}\bar{J}$ contractions

$$\langle Q_A^2 \rangle_{b,c} = \int_A \frac{dx_1}{2\pi i} \int_A \frac{dx_2}{2\pi i} \frac{\langle JJ \rangle \langle V_b \cdots V_{-b} \rangle}{\langle V_b \cdots V_{-b} \rangle} + c.c. \ . \tag{4.107}$$

By choosing the cut-off δ in (4.77) around $z = 0$ and $z = \infty$, one can verify that (4.107) gives the correct μ^2 contribution in $S[\mu, b]$. It is because $J(z)J(0) \sim \frac{k/2}{z}$, all the higher-point correlation functions of J decompose into the sum of products of the first and quadratic order connected correlation functions. Thus, the connected correlation functions of J terminate at the quadratic order. Therefore, eventually, we see that it is due to the $U(1)$ symmetry expressed in the structure of the JJ OPE, that the charged moments is of Gaussian type. A direct expectation from this conclusion is that the equipartition of entanglement should also hold in the non-abelian symmetry case, such as SU(N) Wess-Zumino-Witten (WZW) model [131, 132], as long as one symmetry-resolves the entanglement with respect to the eigenvalues of the Cartan elements (global charges) of the algebra $\mathfrak{g}$. This is because the Cartan elements form the $U(1)$ subalgebras of $\mathfrak{g}$. Indeed, in [133], the authors studied the SREE in the WZW model with general compact simple Lie group G, and they found that

$$S(r) = S_{EE} - \frac{\dim(G)}{2} \log\left(\frac{k\Delta L}{4\pi^3}\right) + 2\log\dim(r) + \cdots \ . \tag{4.108}$$

This result shows that equipartition of entanglement holds at the leading order $\mathcal{O}(\log \Delta L)$, but breaks down at the sub-leading order $\mathcal{O}(1)$ by $2\log\dim(r)$, where r denotes a specific representation of G. On the other hand, the above answer to the first question also points out where to find the counter-example of the equipartition. One can study the symmetry resolution of entanglement with respect to the higher spin currents, such as the stress tensor and W_3 higher spin current studied in [31]. Because in those cases, the connected correlation function between currents will not terminate at the second order, hence the charged moments will not be the Gaussian type. As a consequence, a naive Fourier transformation will not yield a Gaussian distribution in q, so a nontrivial q-dependence of SREE is expected. Recently, in [126], SREE with respect to the conformal symmetry was studied, where instead of restricting to the charge sector with the same eigenvalue of L_0, the author turned to study the entanglement encoded in each conformal families. The results in [126] showed that in this case, equipartition holds for all minimal model CFTs. So, one can see that there are different ways to define the problem of equipartition, depending on the quantum number with respect to which the entanglement entropy are symmetry-resolved. Another meaningful question that one should ask about SREE is how it is related to experiments, and how much insight it provides in the quantum information theory.

Towards general N cases. The vertex operator prescription of the charged moments can also be applied to multi-interval cases, $N \geq 2$, in which, one inserts $V_{\pm a}$ at each pair of branched points on $R_{n,N}$. The $N = n = 2$ case has been studied in Appendix A of my work [30]. The SREE as well as the symmetry-resolved mutual information in general N intervals case have been extensively discussed in [134] for the compact free boson and free massless Dirac fermion. The basic idea in [30] is to first map $R_{2,2}$ to a flat torus by the Schwarz-Christoffel map (3.226), and then derive KZ-equations on the torus from the operator equation (4.87). Then, by solving KZ-equations for the vertex correlation function on torus, one can determine the μ-dependent part of the $N = n = 2$ charged moments. Moreover, it was shown in [30] that the generating function method provides a more convenient way to get the result, in which, the main task is to compute the expectation value of the $U(1)$ current on torus. However, both of those two approaches rely on the uniformization map of the replica surface, which is difficult to work out in the higher genus cases, as I commented in Section 3.2.2. A resolution to this problem is to calculate the expectation value of the $U(1)$ current on the replica surface directly, by using the Ward identity for $U(1)$ symmetry. A straightforward derivation of KZ-equations on the replica surface $R_{n,N}$ is also possible, which however has not been investigated in the literature. In the following, I will discuss how to determine the expectation value of $J(w)$ on $R_{n,N}$,

$$\langle J(w) \rangle = \frac{\langle J(w) V_{-a}(w_1) V_a(w_2) \cdots V_a(w_{2N}) \rangle_{R_{n,N}}}{\langle V_{-a}(w_1) V_a(w_2) \cdots V_a(w_{2N}) \rangle_{R_{n,N}}} \ , \tag{4.109}$$

where $V_{\pm a}$ with $a = \frac{k\mu}{4\pi}$ are vertex operators inserted at the branched points of $R_{n,N}$, and the original background is assumed to be the vacuum state.

Let me first explain the general structure of the $U(1)$ Ward identity on a compact genus g Riemann surface S_g. Consider X as a product of local current primary fields $X = O_1(x_1) \cdots O_m(x_m)$ on the Riemann surface S_g, with the $U(1)$ charge of O_j denoted as q_j. We choose a symplectic basis $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ of the first homology group $H_1(S_g, \mathbb{Z})$. Here α_i's and β_i's are non-contractible loops on S_g, with their intersection numbers, denoted by an inner product, satisfying

$$(\alpha_i, \beta_j) = -(\beta_j, \alpha_i) = \delta_{ij} \ , \quad i, j = 1, \cdots, g \ . \tag{4.110}$$

The space $H^+(S_g)$ of linearly independent holomorphic one-forms on S_g is g-dimensional. For a given symplectic basis, one can associate a basis $\{\omega_1, \cdots, \omega_g\}$ of $H^+(S_g)$, such that the holomorphic one-forms $\omega_i = \omega_i(w)dw$ are normalized as

$$\oint_{\alpha_j} \omega_i = \delta_{ij} \ , \quad \oint_{\beta_j} \omega_i = \mathbf{\Omega}_{ij} \ . \tag{4.111}$$

Here, $\mathbf{\Omega}_{ij}$ is called the *periodic matrix* of S_g [111]. Then, the Ward identity for the $U(1)$ Kac-Moody current on S_g reads [119],

$$\langle J(w)X\rangle = -\pi \sum_{j=1}^{m} q_j \tilde{G}(x_j, w)\langle X\rangle + \sum_{i=1}^{g} c_i\omega_i(w)\langle X\rangle \ . \tag{4.112}$$

Here $c_i\omega_i(w)$ are the zero mode contributions to $\langle J(w)\rangle$ on S_g,

$$c_i = \frac{\langle J_0^i X\rangle}{\langle X\rangle} = \langle J_0^i\rangle \ , \quad J_0^i = \oint_{\alpha_i} dx J(x) \ . \tag{4.113}$$

The zero modes contributions are non-vanishing, if there are chemical potentials θ_i on S_g, coupled with the zero modes J_0^i of the current, for instance [119],

$$Z = \mathrm{Tr}\left\{q^{L_0} e^{\theta_1 J_0^1 + \cdots \theta_g J_0^g} X\right\} \ , \quad \langle J_0^i\rangle = \partial_{\theta_i} \log Z \ . \tag{4.114}$$

However, there is not such chemical potential existing in our case, so we can neglect $c_i\omega_i$ terms in (4.112). $\tilde{G}(x, w)$ in (4.112) is the *modified Green's function* on S_g, defined as [119],

$$\partial_{\bar{x}} \tilde{G}(x, w) = \delta^{(2)}(x - w) \ . \tag{4.115}$$

With a fixed base point w_0, the holomorphic one-forms define the *Abel map* from the genus g Riemann surface to $\mathbb{C}^g$ [111],

$$y_i(w) = \int_{w_0}^{w} \omega_i(x)dx \ , \quad \mathbf{y} = \{y_1, \cdots, y_g\} \in \mathbb{C}^g \ . \tag{4.116}$$

The explicit form of $\tilde{G}(x, w)$ is related to the Jacobi-theta function for $g = 1$, and the Riemann-Siegel-theta function defined on $\mathbb{C}^g$ for $g > 1$, with the arguments of the theta function being the g-dimensional vector $\mathbf{y}(w)$. Rather than digging into too many details of those complicated functions, I would like to give some simple observations, which will make the result of (4.109) more transparent.

1. Current on replica surface. Now we consider the Riemann surface S_g as $R_{n,N}$, with branched coordinate w. The idea is to use the $\mathbb{Z}_n$ symmetry of $R_{n,N}$ as well as the short distance behavior of the current to fix $\langle J(w)\rangle$ defined in (4.109). For a general reference point x_j on a single sheet of $R_{n,N}$, the modified Green's function $\tilde{G}(x_j, w)$ should be a multi-valued function of w, because the delta-function singularity in (4.115) is localized at $w = x_j$, but not at $w = e^{2\pi i}x_j$ on the next sheet. However, in (4.109), the vertex operators are localized at the branched points w_i's, which are invariant under the $\mathbb{Z}_n$ transformation. Thus, we expect that (4.109), expressed as

$$\langle J(w)\rangle = \pi a \sum_{i=1}^{N} \left(\tilde{G}(w_{2i-1}, w) - \tilde{G}(w_{2i}, w)\right) \ , \tag{4.117}$$

should be a single-valued function of w. Therefore, the fractional powers of $(w - w_i)$ in $J(w)$ are excluded. Furthermore, in the limit $w \to w_i$, the neighborhood of each branched point w_i on $R_{n,N}$ can be uniformized by the map $z - z_i = (w - w_i)^{1/n}$. In z-coordinates, $\langle J(z)\rangle$ behaves as $\langle J(z)\rangle \sim \pm a(z - z_i)^{-1}$ for $z \to z_i$, where $\pm a$ is the $U(1)$ charge the corresponding vertex operator $V_{\pm a}(z_i)$. Performing the conformal transformation for the current, $J(w) = J(z)dz/dw$, yields following short distance behaviors,

$$\langle J(w)\rangle \sim \frac{-a/n}{w - w_{2i-1}} \ , \quad \langle J(w)\rangle \sim \frac{a/n}{w - w_{2i}} \ . \tag{4.118}$$

Since there is no vertex operator inserted at $w = \infty$, we further require the smoothness condition for the current, $\langle J(w)\rangle \sim w^{-2}$, for $w \to \infty$. The above three conditions uniquely fix $\langle J(w)\rangle$ as,

$$\langle J(w)\rangle = \frac{a}{n} \sum_{i=1}^{N} \left(\frac{1}{w - w_{2i}} - \frac{1}{w - w_{2i-1}}\right) \ . \tag{4.119}$$

The result (4.119) was explicitly verified in [30] for $N = n = 2$ case by using certain properties of the Jacobi-theta function as well as the Schwarz-Christoffel map (3.226). Notice that, this does not means that $\tilde{G}(w_i, w)$ in (4.117) is just the simple pole in (4.119), because it also contains a non-analytic part, which however cancels with each other in the sum (4.117). See [30, 119] for more detail.

2. Charged moments from generating function method. We may compare (4.119) with the stress tensor (3.168). The latter one contains additional accessory parameters which are not fixed at the first place. However, (4.119) is completely fixed by the charges of the insertions. This implies that the $U(1)$ part of this theory is integrable.

Now, I use the generating function method to calculate the charged moments. Evaluating the expectation value of the charge operators Q_A by (4.119) yields

$$\mathcal{Q}_A = \sum_{j=1}^{N} \int_{w_{2j-1}+\epsilon}^{w_{2j}-\epsilon} \frac{dw}{2\pi i} J(w) + c.c. = \frac{ik\mu}{4\pi^2 n} \log \Delta L_N \ , \tag{4.120}$$

where the cutoff ϵ was introduced in order to regularize the integration, and ΔL_N is given by

$$\Delta L_N = -N \log|\epsilon|^2 - \sum_{1\le i<j\le 2N} (-1)^{i-j} \log|w_i - w_j|^2 \ , \tag{4.121}$$

By inserting $\mathcal{Q}_A$ into (4.99) and taking integration, one can show that the charged moments can be written in the following form,

$$Z_n[\mu] = Z_n[0]|\epsilon|^{4Na^2/(nk)} \prod_{i<j} |w_i - w_j|^{4a_i a_j/(nk)} \ , \tag{4.122}$$

with $a_i = (-1)^i a$ and $a = \frac{k\mu}{4\pi}$. The ϵ term in (4.122) makes the charged moments to be dimensionless. Compare (4.122) with the vertex correlation function (4.90) defined on $\mathbb{CP}^1$. We see that the difference between them is just a shift of the Chern-Simons level $k \to nk$. This implies that the μ-dependent ($U(1)$) part of (4.122) naturally satisfies the crossing symmetry. The SREE in this case is given by,

$$S(q) = S_{EE} - \frac{1}{2}\log\left(\frac{k\Delta L_N}{2\pi}\right) + \mathcal{O}(\epsilon^0) \ . \tag{4.123}$$

The entanglement entropy S_{EE} with general N has been discussed in Section 3.2. It contains different phases, labeled by distinct vacuum channels or projective structures on S_g. The second term in (4.123) is an exact result and does not depend on the choice of channels. Hence, the transition of $S(q)$ is not affected by this term.

Finally, I would like to give some comments. Notice that the μ (or a) dependent part of (4.122) satisfies the KZ-equation (4.89) on $\mathbb{CP}^1$, with the level k being rescaled as $k \to nk$. Thus, even though I have not derived the KZ-equation on $R_{n,N}$, one can suspect that the answer is provided by simply rescaling the level k for (4.89). On the other hand, in the twist picture that I will discuss later, the same KZ-equation will appear, which constrains the $U(1)$ block part of the correlation function of charged twist fields. Therefore, a direct connection between the replica picture and the twist picture shows up here. A similar story should also happen in the entanglement and Rényi entropy, that is, if we try to directly express the null-state equation (3.264) in the original CFT defined on $R_{n,N}$, the resulting differential equation should be identical to the Fuchsian differential equation (3.266) in $\text{CFT}^{\otimes n}$. A detailed study on the Ward identity on general replica surfaces might be an interesting and useful topic.

4.2.3 *Wilson line/vertex operators duality*

This section is dedicated to studying the holographic dual of the boundary vertex operators, based on our work in [29]. The discussions are not restricted to the charged moments but rather aimed at providing a general duality between operator contents in the U(1) sector of our holographic model. The final result shows that the disjoint $U(1)$ Wilson lines compute the neutral $U(1)$ block in the dual CFT, where each Wilson line is dual to the pair of vertex operators with opposite charges. For general $U(1)$ blocks, such as exchanged states with non-zero $U(1)$ charge, the disjoint $U(1)$ Wilson lines construction fails. But this points out that a more general Witten diagram approach [42] is required, in order to reproduce general correlation functions of vertex operators from the $U(1)$ Chern-Simons theory. In fact, similar things happen to the Virasoro vacuum block and the W_3 vacuum block since they can be computed by the geodesics (SL(2,$\mathbb{R}$) Wilson lines) [105] and the higher spin Wilson lines [121, 135]. The deeper reason for those dualities should stem from the correspondence between the three-dimensional Chern-Simons theory and two-dimensional WZW model [43], which is known even before the discovery of the AdS/CFT correspondence. So in the following, I will first explain the Wilson line construction in the bulk and its relation with the boundary vertex operators. To confirm the duality between them, I will derive the effective action for the U(1) Chern-Simons fields in the presence of the Wilson line defects and show that the result matches the two-point function of the dual vertex operators. In particular, applying to the charged moments, I will show that the effective action $S_{eff}[A, \tilde{A}]$ satisfies a holographic version of the identity (4.99), hence it confirms the generating function methods introduced in [29]. Finally, I will give a discussion on the case of the higher-point correlation function and disjoint Wilson lines.

Construction of $U(1)$ Wilson line. We focus on the A sector or the holomorphic sector in the dual CFT. Consider two vertex operator insertions $V_{-q}(w_1)$ and $V_q(w_2)$ on the conformal boundary $\mathbb{CP}^1$ of an asymptotically AdS$_3$ space. By the identification (4.6), one reads out

$$A_w = \frac{2iq}{k}\left(\frac{1}{w-w_1} - \frac{1}{w-w_2}\right) \ , \quad A_{\bar{w}} = 0 \ , \quad \rho \to 0 \tag{4.124}$$

which implies the field strength $F = dA$ is delta-function singular at the boundary points w_1 and w_2. Since the delta-function singularity is different from the case of the Dirac monopole, where $B = dA \sim \mathbf{r}/r^3$, one expects that the singular behavior of A should not be localized at the boundary, but needs to extend from the boundary into the bulk. A natural candidate is a bulk Wilson line defect, anchored at the boundary points w_1 and w_2 [29]. More precisely, let me consider the following Wilson line defect action,

$$S_d = iP_q \int_{\mathcal{C}} A_s ds \ , \tag{4.125}$$

Here P_q is a constant that needs to be determined later, and $\mathcal{C}$ denotes a curve in the bulk, anchored at the boundary points w_1 and w_2. To make the orientation of $\mathcal{C}$ clear, I denote $\xi^\mu = (\partial/\partial s)^\mu$ as the tangent normal vector along the curve, and require

$$\xi^\mu|_{w_2} = -\left(\frac{\partial}{\partial\rho}\right)^\mu \ , \quad \xi^\mu|_{w_1} = \left(\frac{\partial}{\partial\rho}\right)^\mu \ , \quad \rho \to 0 \ . \tag{4.126}$$

This means that the directions of the Wilson line at two boundary points are orthogonal to the boundary, and it starts from the boundary

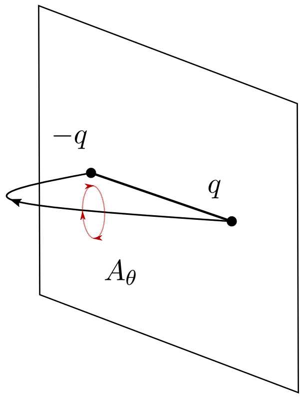

Figure 4: The construction of Wilson line in AdS_3.

point w_2 with positive charge q, and ends at w_1 with negative charge $-q$, as shown in Figure 4. To derive the equation of motion for the Chern-Simons field in the presence of the Wilson line defect, I define a two-form,

$$\mathbf{J} = \delta^{(2)}(x - x(s))\xi \cdot \varepsilon \ , \tag{4.127}$$

Here $\varepsilon = -\frac{i}{2} d\rho \wedge dw \wedge d\bar{w}$ is the "volume" form on the three-space, and $\xi \cdot \varepsilon$ denotes the contraction. We choose Σ_s as the family of hypersurfaces, each of which is locally orthogonal to the normal vector ξ^μ at $x = x(s)$. The delta function in (4.127) is defined with respect to the induced measure on Σ_s,

$$\int_{\Sigma_s} \xi \cdot \varepsilon \delta^{(2)}(x - x(s)) = \int_{\Sigma_s} \mathbf{J} = 1 \ . \tag{4.128}$$

Under the above construction, one can rewrite the Wilson line action as coupling between A and the two-form source,

$$S_d = iP \int A \wedge \mathbf{J} \ . \tag{4.129}$$

Combine (4.129) with the bulk Chern-Simons action (4.1), one can easily derive the equation of motion, given by

$$F = -\frac{4\pi P}{k} \mathbf{J} \ . \tag{4.130}$$

In other words, the presence of the Wilson line action generates a delta-function singular field strength F along the curve $\mathcal{C}$. In particular, the direction of $\varepsilon^{\mu\nu\sigma} F_{\nu\sigma}$ is the same as the tangent normal vector ξ^μ, up to $\pm$ sign depending on the sign of P. Now, from (4.124), the holonomies of A around w_2 on the boundary are given by

$$\oint_{w_2} A = \frac{4\pi q}{k} \ , \tag{4.131}$$

Combining (4.128) and (4.130), and using $\xi^\mu(w_2) = \left(\frac{\partial}{\partial\rho}\right)^\mu$, we can evaluate the integral on a neighborhood of w_2 on the boundary as[3]

$$\int_{\Sigma_s} F = \frac{4\pi P}{k}\int_{\Sigma_s} dz\wedge d\bar{z}\frac{i}{2}\delta^{(2)}(w-w_2) = \frac{4\pi P}{k}\ , \tag{4.132}$$

Matching (4.132) with (4.131) provides the identification $P = q$. So the desired Wilson line action associated with the vertex operators is

$$S_d = iq\int_{\mathcal{C}} A\ . \tag{4.133}$$

So far, I have not clarified what the precise trajectory $\mathcal{C}$ is. Since the Wilson line action is topological, the curve $\mathcal{C}$ can be in arbitrary shape, as long as it satisfies the requirements mentioned before. Therefore, if the duality between the Wilson line and vertex operators is true, then the answer for the effective action should also be topological. Indeed, as I will show later, the effective action does not depend on the explicit solution of A in the three-space, but only depends on its boundary value. Nevertheless, I would like to comment on what a preferred curve $\mathcal{C}$ should be if one wants to solve the equation of motion explicitly. Consider the special cases in which the vertex operators are located at the branched points of the replica manifold. Since the branched points are the fixed points on the replica surface, to preserve the bulk $\mathbb{Z}_n$ symmetry (generated by an elliptic element of SL(2,$\mathbb{C}$)), one can choose the curve $\mathcal{C}$ going along the trajectory of the fixed points in the bulk. As I have shown in the example (3.37), the trajectory of the fixed points in the bulk is typically parametrized by the geodesic equation of the AdS_3 space. So a good choice for $\mathcal{C}$ when we try to derive the exact solution to (4.130) should be the geodesic connecting the two boundary points. In fact, (4.133) is a $U(1)$ analog of the higher spin Wilson line that has been discussed in [135], in which the authors introduced a gauge invariant SL(3,$\mathbb{R}$) Wilson line action to probe the entanglement entropy in the holographic higher spin gravity. In particular, the authors showed that under certain gauge fixing, the Wilson line is localized along the geodesic [135].

Effective action for Chern-Simons fields To verify the duality between the bulk Wilson line and the boundary vertex operators, I would like to compute the effective action for the Chern-Simons field A and compare it with the two-point function of the vertex operators. Since I did not work out the explicit solution for A, evaluating $S[A] + S_d$ is impossible here. Instead, I will introduce a cutoff surface (a thin tube) $\mathcal{B}$ surrounding the Wilson line, and only evaluate the action outside the region of $\mathcal{B}$. This idea was implemented in the holographic proof of the RT-formula in [58]. The cost for doing so is that a proper boundary

3 Note that the induced measure $\xi\cdot\varepsilon$ has opposite orientations at the boundary points w_1 and w_2.

term on the cut-off surface is required in order to give well-defined dynamics in the bulk. For instance, the Wilson line imposes the constraint for A such that the holonomy of A around it is $4\pi a/k$. So a well-defined variational principle on $\mathcal{B}$ must be consistent with this constraint. The bulk action in (4.1) vanishes outside the surface $\mathcal{B}$, and the boundary term defined at $\rho = 0$ also vanishes due to the boundary condition $A^{(0)}_{\bar{w}} = 0$. So the effective action for the Chern-Simons field is just the boundary action on $\mathcal{B}$.

To work out the proper boundary term on $\mathcal{B}$, let me define the locally cylindrical coordinates, (s, r, θ), with the Wilson line following along the center $r = 0$. The surface $\mathcal{B}$ is chosen at a constant radius $r = \epsilon$, with $\epsilon \to 0$. The union $\mathcal{B} \cap \mathbb{CP}^1$ gives two circles on the conformal boundary,

$$|w - w_1| = \epsilon \ , \quad |w - w_2| = \epsilon \ , \quad \rho \to 0 \ . \tag{4.134}$$

Since the holonomy of A is a constant, in the limit $\epsilon \to 0$, it is reasonable to take A_θ as a constant on the cutoff surface $\mathcal{B}$. This can be checked from the boundary configuration (4.124), where around $w = w_2$, we have $w - w_2 = re^{i\theta}$, therefore,

$$A_\theta|_{r=\epsilon} = \frac{2iq}{k}\left(\frac{1}{\epsilon e^{i\theta} + w_2 - w_1} - \frac{1}{\epsilon e^{i\theta}}\right)(i\epsilon e^{i\theta}) \approx \frac{2q}{k} \ , \tag{4.135}$$

On the other hand, since the equation of motion $F = 0$ holds on $\mathcal{B}$, we have

$$F_{s\theta}|_{r=\epsilon} = \partial_s A_\theta|_{r=\epsilon} - \partial_\theta A_s|_{r=\epsilon} = 0 \ , \tag{4.136}$$

Combining (4.135) with (4.136) yields

$$\partial_\theta A_s|_{r=\epsilon} = 0 \ , \quad \epsilon \to 0 \ . \tag{4.137}$$

Therefore, we have completely determined the constraints for A on the cut-off surface imposed by the equation of motion and the holonomy condition. Since A_θ is fixed on the cut-off surface, the variation of the Chern-Simons action and the proper boundary term with respect to A should only contain δA_θ. This condition can be satisfied by considering

$$S_{bdy} = \frac{ik}{8\pi}\int_{\mathcal{B}} ds d\theta A_s A_\theta \ . \tag{4.138}$$

Note that boundary term (4.138) does not admit a covariant expression, but it has been used in [102] to discuss the most general asymptotic symmetry in SL(2,$\mathbb{R}$) Chern-Simons gravity. Now, using (4.135) and (4.137), and integrating out A_θ in (4.138) yields at the effective action, defined on the curve C_ϵ on $\mathcal{B}$ surface,

$$S_{eff}[A] = S[A] + S_{bdy} = \frac{ia}{2}\int_{C_\epsilon} ds A_s \ , \tag{4.139}$$

namely the holonomy or $U(1)$ charge a is coupled with the Wilson line along C_ϵ. Since the field strength F vanishes in the region outside the $\mathcal{B}$ surface, the above action is topologically invariant. So we can turn to evaluate the action (4.139) along an interval on the conformal boundary, which intersects with C_ϵ. The intersection points are located on the cut-off circles $|w-w_1|=\epsilon$ and $|w-w_2|=\epsilon$, so I denote them as $w=w_1+\epsilon$ and $w=w_2-\epsilon$. In this way, one finds that the effective action is determined by the boundary current,

$$S_{eff}[A] = -\frac{iq}{2}\int_{w_1+\epsilon}^{w_2-\epsilon} dw A_w^{(0)} = -\frac{q}{k}\int_{w_1-\epsilon}^{w_2-\epsilon} dw J(w) \ . \tag{4.140}$$

It is then easy to show that $e^{-S_{eff}[A]}$ matches with the holomorphic part of the two-point function of the vertex operators. One can perform the same analysis for the $\tilde{A}$ sector, and the result is similar, given by

$$S_{eff}[\tilde{A}] = -\frac{\bar{q}}{k}\int_{\bar{w}_1+\epsilon}^{\bar{w}_2-\epsilon} d\bar{w}\bar{J}(\bar{w}) \ . \tag{4.141}$$

In particular, applying to the $n=1$ charged moments $Z_1[\mu]$ with $U(1)$ charges $a=\bar{a}=\frac{k\mu}{4\pi}$ and using $J\propto\mu$, one can show the following identity

$$i\partial_\mu S_{eff}[A,\tilde{A}] = \mathcal{Q}_A \ , \tag{4.142}$$

where $S_{eff}[A,\tilde{A}] = S_{eff}[A]+S_{eff}[\tilde{A}]$ and Q_A is the charge expectation value on the boundary interval. The identity (4.142) matches with (4.99) introduced from the generating function method [29].

The above discussion can also be directly applied to $Z_n[\mu]$ charged moments, in which case the defect action S_d still takes the same form as (4.133) since the relation between the holonomy and $U(1)$ charge of the vertex operator is fixed. The main difference is that in this case, it is better to consider the Wilson line lying along the trajectory of the $\mathbb{Z}_n$ fixed points in the bulk. The advantage of this consideration is that in the local coordinates (s,r,θ), the period of θ is naturally changed to be $2\pi n$, meanwhile, A_θ is rescaled by a $1/n$ factor in order to preserve the holonomy. Approaching the conformal boundary $\rho=0$, one can reproduce the $J(w)\propto a/n$ behavior, and the final effective action is in the same form of (4.140).

Higher-point correlator from disjoint Wilson lines The generalization of (4.140) to higher-point correlation function of vertex operators is straightforward. However, since each Wilson line is associated with a unique $U(1)$ charge or holonomy, the above Wilson line construction can only be generalized to the case where the vertex operators are in pairs with opposite charges. Let us consider the four-point function,

$$\langle V_{-q_1}(w_1)V_{q_1}(w_2)V_{-q_2}(w_3)V_{q_2}(w_4)\rangle \ . \tag{4.143}$$

In the s-channel, this four-point is just the single vacuum $U(1)$ block

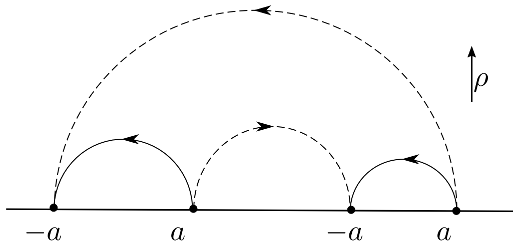

Figure 5: Two phases of Wilson lines dual to $N = 2$ charged moments.

due to the OPE expansion in (4.86). The exchanged states are the J-descendent states of the vacuum. Now, in the bulk, analogous to (4.133), one inserts two disjoint Wilson line defect actions,

$$S_d = i \sum_{j=1}^{2} q_j \int_{\mathcal{C}_j} A \ , \tag{4.144}$$

with C_j going from the position of the positive charge q_j to the position of the negative charge $-q_j$. Assume the two Wilson lines are not braided. Then the effective action is given by

$$S_{eff}[A] = -\frac{1}{k} \sum_{j=1}^{2} q_j \int_{w_{2j-1}+\epsilon}^{w_{2j}+\epsilon} dw J(w) \ . \tag{4.145}$$

Evaluating (4.145) explicitly, one finds that the result matches the holomorphic part of the four-point function,

$$e^{-S_{eff}[A]} = \left(\frac{w_{12}}{\epsilon}\right)^{-\frac{2q_1^2}{k}} \left(\frac{w_{34}}{\epsilon}\right)^{-\frac{2q_2^2}{k}} (1-x)^{\frac{2q_1 q_2}{k}} \ , \tag{4.146}$$

with the cross ratio $x = \frac{w_{12}w_{34}}{w_{13}w_{24}}$. Notice that for $q_1 = \pm q_2$, such as the $N = 2$ charged moments, the dual Wilson lines can take two different configurations in the bulk, as shown in Figure 5. The results of the effective action $S[A]$ in those two phases are identical, because (4.146) takes the form of the four-point function of vertex operators, which is crossing-symmetric. Another way to see this is by noticing that the curves in Figure 5 forms a closed loop in the bulk. The difference between the results of the effective action in those two phases is governed by the holonomy of A around this closed loop, which is clearly zero. However, if we consider a $U(1)$ charged black hole background, the result will be different. This is because the closed loop might wind around the spatial circle of the charged black hole, and leads to a nonvanishing holonomy of A.

We conclude that the disjoint $U(1)$ Wilson lines compute the correlation function of vertex operators or the vacuum $U(1)$ block in the dual CFT. This conclusion can be generalized to arbitrary correlation functions with neutral OPE channels. The reason is that for a CFT with $U(1)$ Kac-Moody symmetry, a general conformal block can be factorized as a product of a Virasoro block and a $U(1)$ block by considering the decomposition $T = T_{vir} + T_J$ [117],

$$\mathcal{F}^p(x_i|h_i, q_i, c, k) = \mathcal{F}^p_{vir}(x_i|h_i - \frac{q_i^2}{k}, c-1)\mathcal{V}_J(x_i|q_i, k), \qquad (4.147)$$

which I will prove in the next section. In particular, since the charge of the exchanged state in a fixed OPE channel is determined by the charge conservation, the $U(1)$ block becomes universal in the correlation function. Therefore, a general correlation function is always factorized and can be written as

$$\langle O_1 O_2 \cdots \rangle \sim \mathcal{V}_J \bar{\mathcal{V}}_{\bar{J}} \sum_p c_p \mathcal{F}^p_{vir} \bar{\mathcal{F}}^p_{vir} \ . \qquad (4.148)$$

The $\mathcal{V}_J$ part is nothing but the vertex correlation function (4.90), since if we can set $h_i = q_i^2/k$, then $\sum_p c_p \mathcal{F}^p_{vir} \bar{\mathcal{F}}^p_{vir}$ becomes the identity. Those properties reflect the fact that the $U(1)$ Chern-Simons fields decouple with gravity. Therefore, as long as the charges of the CFT operators are in pairs with opposite signs, one can always use the Wilson lines to compute the $\mathcal{V}_J$.

Finally, to generalize the Wilson lines construction for general correlator with non-neutral OPE channels, a more general Witten diagram [42] of Wilson lines is required. See [136–139] for applications of the Witten diagram in AdS/CFT. For instance, for a three-point function with the charges satisfying the charge conservation $q_1 + q_2 + q_3 = 0$, one can insert three Wilson lines that intersect with each other in the bulk. It will be interesting to understand the field configuration close to the intersection point, and hopefully to work out the effective action in that case.

4.2.4 *Charged twist field approach*

In this section, I will discuss the charged moments in the twist picture. I will first clarify the symmetry algebra in $\mathrm{CFT}^{\otimes n}$, obtained from the n copies of the original one. Then, I will introduce the charged twist field prescription of the charged moments, following from [27]. To solve the charged moments in general cases, I will discuss the $U(1)$ extended conformal block, and prove its factorization into the $U(1)$ block and the Virasoro block, which was argued in [117]. The derivation for the four-point $U(1)$ block in [117] is a bit tricky, following similar ideas as the generating function method introduced [29]. So to confirm the result, I will perform a derivation for the $U(1)$ block in the four-point function by directly summing over all J-descendent states. The conclusion for the $U(1)$ block is that it universally factorizes in the correlation function, and in particular, it acquires the form of the vertex correlation function. This conclusion consequently implies the $U(1)$ sector always satisfies the KZ-equation (with a rescaling of k in $\mathrm{CFT}^{\otimes n}$). Meanwhile, the Virasoro block satisfies the null state equation with central charge

$nc - 1$. Hence, this provides a general way to solve the charged moments in the twist picture, and at the same time confirms the assumption on the factorization of Hilbert space used in [30].

$\mathbb{Z}_n$-symmetric symmetry algebra in CFT$^{\otimes n}$. In the twist picture, one still considers the new theory CFT$^{\otimes n}$ defined on $\mathbb{CP}^1$. The $U(1)$ Kac-Moody-Virasoro symmetry in the original theory is enlarged to be n copies in CFT$^{\otimes n}$. However, similar to the pure Virasoro case discussed in Section 3.2.2, here we only focus on the total stress tensor and the total current in CFT$^{\otimes n}$, defined as

$$T(w) = \sum_{i=1}^{n} T^{(i)}(w) \ , \quad J(w) = \sum_{i=1}^{n} J^{(i)}(w) \ , \tag{4.149}$$

both of which are invariant under $\mathbb{Z}_n$. The reason for doing this is that the OPE channels in the charged moments can be chosen to be $\mathbb{Z}_n$ symmetric, similar to the pure Virasoro case discussed in Section 3.2.2. The modes of the total stress tensor and the total current are given by the sum over n copies,

$$L_m = \sum_{i=1}^{n} L_m^{(i)} \ , \quad J_m = \sum_{i=1}^{n} J_m^{(i)} \ , \tag{4.150}$$

where different copies of the modes commute. It is straightforward to check that (4.150) still furnishes a $U(1)$ Kac-Moody-Virasoro algebra,

$$\begin{aligned} &[L_m, J_r] = -rJ_{m+r} \ , \quad [J_m, J_r] = \frac{k_n}{2} m\delta_{m+r,0} \ , \\ &[L_m, L_r] = (m-r)L_{m+r} + \frac{c_n}{12}(m^3 - m)\delta_{m+r,0} \ , \end{aligned} \tag{4.151}$$

but the level and the central charge of the algebra are rescaled by n,

$$k \to k_n = nk \ , \quad c \to c_n = nc \ . \tag{4.152}$$

Analogous to the case in the original CFT, we can decompose the total stress tensor by defining a Sugawara stress tensor T_J associated with the total current,

$$T(w) = T_{vir}(w) + T_J(w) \ , \quad T_J(w) = \frac{1}{k_n}(JJ)(w) \ . \tag{4.153}$$

The modes L_m^J of T_J are expressed via J modes as

$$L_m = \frac{1}{k_m} \sum_{r\in\mathbb{Z}} : J_{m-r} J_r : , \tag{4.154}$$

where $: J_{m-r} J_r :$ are normal ordered, defined in the same way as in (4.78). The $1/k_n$ factor in (4.153) is chosen by requiring the OPEs,

$$\begin{aligned} &T_J(w)J(0) \sim \frac{J(0)}{w^2} + \frac{\partial J(0)}{w} \ , \\ &T_J(w)T_J(0) \sim \frac{1/2}{w^4} + \frac{2T_J(0)}{w^2} + \frac{\partial T_J(0)}{w} \ , \end{aligned} \tag{4.155}$$

from which, we see that the central charge c_J in the $\mathbb{Z}_n$-symmetric $U(1)$ sector of CFT$^{\otimes n}$ is $c_J = 1$. The first one in (4.155) leads to the non-singular OPE between T_{vir} and J,

$$T_{vir}(w)J(0) = 0 + \text{regular terms} \sim 0 \ , \tag{4.156}$$

which further implies

$$T_{vir}(w)T_{vir}(0) \sim \frac{(c_n-1)/2}{w^4} + \frac{2T_{vir}(0)}{w^2} + \frac{\partial T_{vir}(0)}{w} \ . \tag{4.157}$$

So the central charge c_{vir} in the pure Virasoro sector, associated with T_{vir}, is $c_{vir} = c_n - 1$. The modes of T_{vir} are denoted as L^{vir}_m, which commute with J modes.

General idea. The reason for clarifying the $\mathbb{Z}_n$-symmetric symmetry algebra is that the descendent states considered in the later discussion are defined by acting a string of $\mathbb{Z}_n$-symmetric generators L_{-m} and J_{-l} on primary fields, or equivalently by the decoupled basis L^{vir}_{-m} and J_{-l},

$$|O^{\{m,\bar{m}\},\{l,\bar{l}\}}\rangle = L^{vir}_{-m_1}\cdots L^{vir}_{-m_i}J_{l_1}\cdots J_{-l_j}|O\rangle \ . \tag{4.158}$$

This consideration is relevant to the calculation of the charged moments. As I will explain later, in the twist field approach, the most general form of the charged moments can be written as

$$Z_n[\mu] = \left\langle \tilde{O}_1(x_1)\tilde{O}_2(x_2)\prod_{m=1}^{N}\sigma_{n,-a}(w_{2m-1})\tilde{\sigma}_{n,a}(w_{2m})\right\rangle \ , \tag{4.159}$$

with $a = k\mu/(4\pi)$. This is similar to (3.240). The difference is that twist and anti-twist fields in (3.240) are replaced by the *charged twist field* $\sigma_{n,-a}$ and the *anti-charged twist field* $\tilde{\sigma}_{n,a}$ in (4.159), which I will explain in later discussion. The main point is that, similar to the twist and anti-twist fields, the OPE between charged and anti-charged twist fields is still $\mathbb{Z}_n$-symmetric. Therefore, we can always choose $\mathbb{Z}_n$ symmetric channels of the charged moments (4.159) by pairing the changed and anti-charged twist fields, so that all intermediate states in the conformal block expansion of (4.159) are in the form of (4.158). In particular, since the charged and anti-charged twist fields carry opposite $U(1)$ charges, which I will show later, all intermediate states in a $\mathbb{Z}_n$-symmetric channel are neutral, due to the charge conservation. Therefore, if we assume the vacuum block dominance for the charged moments (4.159), there will be no contradiction, since states in the vacuum family have zero $U(1)$ charge. This motivates the general idea in this section, that is to study the semi-classical conformal blocks in CFT$^{\otimes n}$, and approximate the charged moments (4.159) by the vacuum block contribution. The main difference with the discussion in Section 3.2.2 is that here the theory contains an additional $U(1)$ symmetry. As a consequence, the conformal blocks in the theory are $U(1)$ extended, and the null-state equation (3.264) in the pure Virasoro case needs to be modified.

Charged and anti-charged twist fields. In the twist picture, the existence of the vertex operators, located at the branched points of the replica surface, requires modifications for the boundary condition (3.230). Such modification can be derived explicitly when we consider a concrete CFT model [27, 32]. However, just as in the entanglement case, it turns out that the modified boundary conditions can be implemented implicitly by considering the insertions of primary fields at the branched points. Those primary fields are the so-called (anti) charged twist fields. In particular, the conformal weights and $U(1)$ charges of those operators turn out to be universal, in the sense that they only depend on the central charge, coupling constant, and the replica n index, but not on other details of the theory. The original derivation for the conformal weight of the charged twist field was performed in [32] for the free massless Dirac fermion theory, where the author used the diagonalization approach introduced in [113]. A more efficient derivation was performed in [27, 29], which I will explain in the following.

Consider the $N = 1$ case, with the entangling region $A = [w_1, w_2]$. We use (3.164) to map $R_{n,N}$ to $\mathbb{CP}^1$, coordinated by z. The expectation values of the stress tensor and the $U(1)$ current in z-coordinates read

$$\begin{aligned} \langle T(z)\rangle &= \frac{\langle T(z)V_{-a}(0)V_a(\infty)\rangle}{\langle V_{-a}(0)V_a(\infty)\rangle} = \frac{a^2/k}{z^2} \ , \\ \langle J(z)\rangle &= \frac{\langle J(z)V_{-a}(0)V_a(\infty)\rangle}{\langle V_{-a}(0)V_a(\infty)\rangle} = -\frac{a}{z} \ , \end{aligned} \tag{4.160}$$

with $a = k\mu/(4\pi)$ given in (4.72). By transforming back to the w-coordinates, one obtains

$$\langle T(w)\rangle = \frac{h(w_1 - w_2)^2}{(w-w_1)^2(w-w_2)^2}, \ h = \frac{c}{24}(1-\frac{1}{n^2}) + \frac{a^2}{kn^2} \ , \tag{4.161}$$

and

$$\langle J(w)\rangle = \frac{-a/n}{w-w_1} + \frac{a/n}{w-w_2} \ , \tag{4.162}$$

Since $T(w)$ and $J(w)$ are single-valued in w, they are identical in each sheet of the replica surface. In the twist picture, the stress tensor and the current in each sheet should be associated with each copies of the original CFT. Thus, one concludes that the total stress tensor and the total current in $\mathrm{CFT}^{\otimes n}$ should be given by n times of (4.161) and (4.162). Comparing with the Ward identities for the conformal symmetry and the $U(1)$ symmetry, one can identify the total stress tensor and the total current as generated by two primary fields, denoted as $\sigma_{n,-a}(w_1)$ and $\tilde{\sigma}_{n,a}(w_2)$, which are the so-called (anti) charged twist field. The conformal weights and $U(1)$ charges of the charged twist field $\sigma_{n,-a}$ are given by[4]

$$h_{n,a} = \bar{h}_{n,a} = \frac{c_n}{24}\left(1 - \frac{1}{n^2}\right) + \frac{a^2}{k_n} \ , \quad a_n = \bar{a}_n = -a \ , \tag{4.163}$$

For the anti-charged twist field, the conformal weights are the same as (4.163), but the $U(1)$ charges have a positive sign. Here is an intuitive explanation for the results (4.163). If we think carefully, the notion of $e^{i\mu Q_A}$ is not quite clear when we transform to the twist picture. The operator Q_A is defined via J and $\bar{J}$ in the original theory, however, there are n copies of currents $J^{(i)}$ in CFT$^{\otimes n}$, and it is unclear which copy is responsible to Q_A. A simple way to make the notion clear is to redefine Q_A in CFT$^{\otimes n}$ as following,

$$Q_A = \frac{1}{n}\int_A \frac{dw}{2\pi i} J(w) + c.c. \ , \tag{4.164}$$

where J is the total current, consisting of the n copies $J^{(i)}$. In this case, $e^{i\mu Q_A}$ can be understood as a pair of vertex operators in CFT$^{\otimes n}$,

$$e^{i\mu Q_A} = \tilde{V}_{-a}(w_1)\tilde{V}_a(w_2) \ , \tag{4.165}$$

and the conformal weight and $U(1)$ charge can be read out from (4.72), with the rescalings $k \to k_n = nk$ and $\mu \to \mu' = \mu/n$,

$$\tilde{h}_a = \frac{\tilde{a}^2}{k_n} = \frac{a^2}{k_n} \ , \quad \tilde{a} = \frac{k_n}{4\pi}\left(\frac{\mu}{n}\right) = a \ , \tag{4.166}$$

Now, notice that the conformal weight in (4.163) is the sum of $\tilde{h}_a$ and the conformal weight of the twist field. This means that we can think of $\sigma_{n,-a}$ as obtained from the fusion between the twist field and vertex operator in CFT$^{\otimes n}$,

$$\sigma_{n,-a} = (\sigma_n \tilde{V}_{-a}) \ . \tag{4.167}$$

Coming back to the charged moments, by (4.163), one finds that the two-point function of charged and anti-charged twist field indeed reproduces the $N = 1$ result in (4.56). Generalizing this to a general excited state background with N intervals leads to the charged moments in the form of (4.159), which in semi-classical limit $c \to \infty$ can be approximated by the vacuum conformal block in a particular $\mathbb{Z}_n$-symmetric channel. To study the conformal blocks in CFT$^{\otimes n}$, which are $U(1)$ extended, I will first prove a factorization property of them in the following discussion. This property does not rely on the semi-classical limit.

Factorization of $U(1)$ extended conformal block. The $U(1)$ extended conformal block has been partially studied in [117], where the

4 Here I still consider the automorphism $\Omega = +1$ for convenience so that the $U(1)$ charges $(a, \bar{a})$ for the vertex operator V_a are identical.

author argued that the $U(1)$ extended conformal block is factorized into a $U(1)$ block and a Virasoro block by analyzing the three-point function. However, an explicit proof of this argument is absent in [117]. In the following, I will prove the factorization property.

1. Setup and main tasks. Let me first explain the setup of the problem as well as the main tasks. I will consider current primary fields in CFT$^{\otimes n}$. The conformal weight of an operator O_i will always be parametrized as

$$h_i^{total} = h_i + q_i^2/k_n \ , \tag{4.168}$$

with

$$L_0^{vir}|O_i\rangle = h_i|O_i\rangle \ , \quad J_0|O_i\rangle = q_i|O_i\rangle \ . \tag{4.169}$$

I will assume the OPE considered in the later discussion is $\mathbb{Z}_n$-symmetric, so that descendent states in the OPE take the form of (4.158). To make the meaning of factorization clear, let me consider the following four-point function as an example,

$$Z = \langle O_4|O_3(1)O_2(x)|O_1\rangle \ , \quad \sum_{i=1}^{4} q_i = 0 \ . \tag{4.170}$$

Consider the OPE between O_1 and O_2,

$$\begin{aligned} O_2(x)|O_1\rangle = \sum_p \sum_{\{m,\bar{m}\}} \sum_{\{l,\bar{l}\}} C_{12}^p \beta_{12}^{p\{m\}\{l\}} \bar{\beta}_{12}^{p\{\bar{m}\}\{\bar{l}\}} f(x;|m|+|l|) \\ \times \bar{f}(\bar{x};|\bar{m}|+|\bar{l}|)|O_{p,q_1+q_2}^{\{m,\bar{m}\}\{l,\bar{l}\}}\rangle \ , \end{aligned} \tag{4.171}$$

where the primary field O_{p,q_1+q_2} is characterized by

$$L_0^{vir}|O_{p,q_1+q_2}\rangle = h_p|O_{p,q_1+q_2}\rangle \ , \quad J_0|O_{p,q_1+q_2}\rangle = (q_1+q_2)|O_{p,q_1+q_2}\rangle \ , \tag{4.172}$$

and for convenience I have defined a function f as

$$f(x;|m|+|l|) = x^{h_p-h_1-h_2+\frac{2q_1q_2}{k_n}+|m|+|l|} \ , \tag{4.173}$$

and the antiholomorphic part $\bar{f}$ is defined analogously. The sets $\{m\}$ and $\{l\}$ label the strings of actions from L_{-m_i} and J_{-l_j} respectively, with $|m| = \sum_i m_i$ and $|l| = \sum_j l_j$. The coefficients $\beta_{12}^{p\{m\}\{l\}}$ are fixed by the $U(1)$ Kac-Moody-Virasoro symmetry and depend on

$$\beta_{12}^{p\{m\}\{l\}} \sim (h_p, h_{1,2}, q_{1,2}, \{m\}, \{l\}, c_n, k_n) \tag{4.174}$$

For $\bar{\beta}_{12}^{p\{\bar{m}\}\{\bar{l}\}}$, it is similar. To show the factorization of (holomorphic) conformal blocks, the main task is to analyze those coefficients. More precisely, I will show that the following decomposition holds

$$\beta_{12}^{p\{m\}\{l\}} = \beta_{12}^{p\{m\}}\gamma_{12}^{\{l\}} \ , \quad \forall\{m\} \ , \{l\} \ , \tag{4.175}$$

with the dependence of them given by

$$\begin{aligned}\beta_{12}^{p\{m\}} &\sim (h_p, h_{1,2}, \{m\}, c_n - 1) \ ,\\ \gamma_{12}^{\{l\}} &\sim (q_{1,2}, \{l\}, k_n) \ .\end{aligned} \tag{4.176}$$

Using the decomposition (4.175) for the OPE (4.171) and inserting it into the four-point function yields the following factorization structure,

$$Z = \mathcal{V}_J \bar{\mathcal{V}}_J \sum_p C_{12}^p C_{34}^p \mathcal{F}_{vir}^p \bar{\mathcal{F}}_{vir}^p \ , \tag{4.177}$$

with

$$\mathcal{V}_J = \sum_{\{l\}} (C_{34}^p)^{-1} x^{\frac{2q_1 q_2}{k_n} + |l|} \gamma_{12}^{\{l\}} \langle O_4 | O_3(1) | O_{p,q_1+q_2}^{\{l\}} \rangle \ , \tag{4.178}$$

and

$$\mathcal{F}_{vir}^p = \sum_{\{m\}} (C_{34}^p)^{-1} x^{h_p - h_1 - h_2 + |m|} \beta_{12}^{p\{m\}} \langle O_4 | O_3(1) | O_{p,q_1+q_2}^{\{m\}} \rangle \tag{4.179}$$

where $C_{34}^p = \langle O_4 | O_3(1) | O_p \rangle$. For the anti-holomorphic part, it is similar. By (4.176), the dependence of $\mathcal{V}_J$ and $\mathcal{F}^p$ are given by

$$\mathcal{V}_J \sim \mathcal{V}_J(x|q_i, k_n) \ , \quad \mathcal{F}_{vir}^p \sim \mathcal{F}_{vir}^p(x|h_p, h_i, c_n - 1) \ . \tag{4.180}$$

The coefficients $\gamma_{12}^{\{l\}}$ as well as $\mathcal{V}_J$ will be derived explicitly, and I call $\mathcal{V}_J$ as the (holomorphic) $U(1)$ *block*. For $\beta_{12}^{p\{m\}}$, I will show that they obey the usual recursion relation in the pure Virasoro case [98]. This implies that $\mathcal{F}_{vir}^p$ are *Virasoro blocks* with central charge $c_n - 1$. Hence, they can be solved in the semi-classical limit by using the monodromy method discussed Section 3.2.2.

2. Recursion relation in $U(1)$ sector. Let me first focus on the $U(1)$ sector. By the standard approach in [98], we act J_r on both sides of (4.171), with $r > 0$. On the left hand side, the result reads

$$\begin{aligned} J_r O_2(x) | O_1 \rangle &= \oint_x \frac{dw}{2\pi i} w^r J(w) O_2(x) | O_1 \rangle \\ &= q_2 x^r O_2(x) | O_1 \rangle \ . \end{aligned} \tag{4.181}$$

where the fact that J_r annihilates $|O_1\rangle$ is used in the first step. Applying (4.181) for the right hand side of (4.171) yields

$$\begin{aligned} J_r O_2(x) | O_1 \rangle = &\sum_p \sum_{\{m,\bar{m}\}} \sum_{\{l,\bar{l}\}} C_{12}^p \beta_{12}^{p\{m\}\{l\}} \bar{\beta}_{12}^{p\{\bar{m}\}\{\bar{l}\}} \bar{f}(\bar{x}; |\bar{m}| + |\bar{l}|) \\ &\times q_2 f(x; |m| + |l| + r) | O_{p,q_1+q_2}^{\{m,\bar{m}\}\{l,\bar{l}\}} \rangle \ . \end{aligned} \tag{4.182}$$

To keep track of the relevant terms, I write (4.183) as

$$J_r O_2(x)|O_1\rangle \sim \sum \cdots q_2 \beta_{12}^{p\{m\}\{l\}} \times f(x; |m| + |l| + r)|h_p; \{m\}, \{l\}\rangle \ , \quad (4.183)$$

where I denote $|O_{p,q_1+q_2}^{p\{m,\bar{m}\}\{l,\bar{l}\}}\rangle = |h_p; \{m\}, \{l\}\rangle$, since I am focusing on the holomorphic part. I will keep using this shorthand writing in the following discussion. One can also act with J_r directly on the right-hand side, the non-vanishing contributions only come from the commutator between J_r and J_{-r}. Denote s_r as the number of J_{-r} in the set $\{l\}$, then by

$$[J_r, (J_{-r})^{s_l}] = \frac{k_n}{2} r s_r (J_{-r})^{s_r - 1} \ , \quad (4.184)$$

one obtains

$$J_r O_2(x)|O_1\rangle \sim \sum \cdots \frac{k_n}{2} r s_r \beta_{12}^{p\{m\}\{l\}} f(x; |m| + |l|) \times \ |h_p; \{m\}, \{l\} - r\rangle, \quad (4.185)$$

where $\{l\} - r$ represents removing one J_{-r} from the set. Comparing (4.183) with (4.185) gives rise to the recursion relation

$$q_2 \beta_{12}^{p\{m\}\{l\}-r} = \frac{k_n}{2} r s_r \beta_{12}^{p\{m\}\{l\}} \ , \quad (4.186)$$

Notice that the coefficients in the recursion relation do not depend on $h_{1,2,p}$ as well as the number s_{l_j} of other J_{-l_j} modes in $\{l\}$. The latter one is because negative modes J_{-l_j} commute with each other, so that the ordering of J_{-l_j} the set $\{l\}$ is not important. We can uniquely characterize $\{l\}$ by the set of numbers s_r,

$$\{l\} = J_{-1}^{s_1} J_{-2}^{s_2} \cdots \ , \quad \sum_{r=1}^{\infty} r s_r = |l| \ , \quad s_r \geq 0 \ . \quad (4.187)$$

Therefore, the coefficient $\beta_{12}^{p\{m\}\{l\}}$ can be factorized as a product,

$$\beta_{12}^{p\{m\}\{l\}} = \beta_{12}^{p\{m\}} \gamma_{12}^{\{l\}} \ , \quad (4.188)$$

with

$$\gamma_{12}^{\{l\}} = \prod_{r=1}^{\infty} \gamma_{12}^{(r,s_r)} \ , \quad \gamma_{12}^{(r,0)} = 1 \ , \quad (4.189)$$

where $\gamma_{12}^{(r,s_r)}$ is associated with $J_{-r}^{s_r}$. The recursion relation for the coefficient $\gamma_{12}^{(r,s_r)}$ can be deduced from (4.186), and is given by

$$\gamma_{12}^{(r,s_r)} = \frac{2q_2}{k_n r s_r} \gamma_{12}^{(r,s_r-1)} \ . \quad (4.190)$$

From this, it is easy to find the expression for $\gamma_{12}^{(r,s_r)}$,

$$\gamma_{12}^{(r,s_r)} = \frac{1}{s_r!}\left(\frac{2q_2}{k_n r}\right)^{s_r} . \tag{4.191}$$

This coefficient $\gamma_{12}^{(r,s_r)}$ will be used to evaluate the $U(1)$ block in the later. As I have shown the decomposition of $\beta_{12}^{\{m\}\{l\}}$, the next task in the following is to analyze the recursion relation for $\beta_{12}^{\{m\}}$.

3. Recursion relation in pure Virasoro sector. Analogous to the previous case, we act L_r^{vir} on both sides of the OPE (4.171), with

$$L_r^{vir} = L_r - \frac{1}{k_n}\sum_{t\in\mathbb{Z}} J_{r-t}J_t \ , \quad r>0 \ , \tag{4.192}$$

where the normal ordering is neglected, since $[J_{r-t}, J_t] = 0$ for $r \neq 0$. The situation becomes more complicated here because the action of L_r^{vir} on the fields is different from the usual differential action of the Virasoro generator. Let me first consider the action of L_r on the left hand side of the OPE (4.171), given by [98]

$$\begin{aligned} L_r O_2(x)|O_1\rangle &= \left[x^{r+1}\partial_x + (r+1)(h_2 + \frac{q_2^2}{k_n})x^r\right] O_2(x)|O_1\rangle \\ &\sim \sum \cdots B_r \beta_{12}^{p\{m\}} f(x;|m|+|l|+r)|h_p;\{m\},\{l\}\rangle \ , \end{aligned} \tag{4.193}$$

with the coefficient B_r defined as

$$B_r = h_p - h_1 + rh_2 + \frac{2q_1q_2}{k_n} + \frac{(r+1)q_2^2}{k_n} + |m| + |l| \ , \tag{4.194}$$

For the L_r^J piece in (4.192), we need to be more careful because there are negative J modes, which do not annihilate $|O_1\rangle$. We first consider the case of $0 \le t \le r$, so that both J_{r-t} and J_t are non-negative modes. Acting with $J_{r-t}J_t$ on the left hand side of the OPE (4.171) yields

$$\frac{1}{k_n} J_{r-t}J_t O_2(x)|O_1\rangle = \frac{q_2^2}{k_n} x^r O_2(x)|O_1\rangle \ , \tag{4.195}$$

with $0 < t < r$. Furthermore, for $t = r, 0$, one obtains

$$\begin{aligned} \frac{1}{k_n} J_0 J_r O_2(x)|O_1\rangle &= \frac{q_2}{k_n} x^r \left([J_0, O_2(x)] + O_2(x)J_0\right)|O_1\rangle \\ &= \frac{1}{k_n}(q_2^2 + q_1q_2)x^r O_2|O_1\rangle \ . \end{aligned} \tag{4.196}$$

Combining (4.193), (4.195) and (4.196) together gives rise to

$$\begin{aligned} &\left(L_r - \frac{1}{k_n}\sum_{t=0}^{r} J_{r-t}J_t\right) O_2(x)|O_1\rangle \\ &\sim \sum \cdots D_r \beta_{12}^{p\{m\}} f(x;|m|+|l|+r)|h_p;\{m\},\{l\}\rangle \ , \end{aligned} \tag{4.197}$$

with the coefficient D_r given by

$$D_r = h_p - h_1 + rh_2 + |m| + |l| \ . \tag{4.198}$$

Note that there is no $q_{1,2}$ dependence in D_r, and we need to verify that the $|l|$ term in D_r can be canceled out by including the remaining J modes in L_r^{vir}. For $t<0$, we have

$$\begin{aligned}\frac{1}{k_n} J_t J_{r-t} O_2(x)|O_1\rangle &= \frac{q_2}{k_n} x^{r-t} J_t O_2(x)|O_1\rangle \\ &\sim \sum \cdots \frac{q_2}{k_n} \beta_{12}^{p\{m\}\{l\}} f(x; |m|+|l|+r-t) \\ &\qquad \times |h_p; \{m\}, \{l\}+|t|\rangle \end{aligned} \tag{4.199}$$

Here I have written $\{l\}$ in $\beta_{12}^{p\{m\}\{l\}}$ explicitly in the above equation for later convenience, since the the level of each descendent state is lifted by $|t| = -t$. By the redefinition of the set $\{l\} \to \{l\} - |t|$, and consequently $|l| \to |l| - |t|$, then the above equation (4.199) can be rewritten as

$$\begin{aligned}&\sum \cdots \frac{q_2}{k_n} \beta_{12}^{p\{m\}\{l\}-|t|} f(x; |m|+|l|+r)|h_p; \{m\}, \{l\}\rangle \\ =&\sum \cdots \frac{t s_{-t}}{2} \beta_{12}^{p\{m\}\{l\}} f(x; |m|+|l|+r)|h_p; \{m\}, \{l\}\rangle,\end{aligned} \tag{4.200}$$

where in the second step, I have used the recursion relation (4.186). Note that if we include all $J_{r-t}J_t$ action with $t<0$ or $t>r$, then, for each descendent state, labeled by the set $\{l\}$, the coefficient in front of it should be given by

$$\sum_{t<0} \frac{t s_{-t}}{2} + \sum_{r-t<0} \frac{(r-t)s_{t-r}}{2} = -\sum_{r=1}^{\infty} r s_r = -|l| \ , \tag{4.201}$$

namely, we have

$$\begin{aligned}\frac{1}{k_n} \sum_{t<0,t>r} J_{r-t} J_t O_2(x)|O_1\rangle &\sim -\sum \cdots |l| \beta_{12}^{p\{m\}} f(x; |m|+|l|+r) \\ &\qquad \times |h_p; \{m\}, \{l\}\rangle \ . \end{aligned} \tag{4.202}$$

Combine (4.197) with (4.202), we finally arrive at the action of L_r^{vir} on the left hand side of the OPE (4.171),

$$\begin{aligned}L_r^{vir} O_2(x)|O_1\rangle &\sim \sum \cdots (h_p - h_1 + rh_2 + |m|) \beta_{12}^{p\{m\}} \\ &\qquad \times f(x; |m|+|l|+r)|h_p; \{m\}, \{l\}\rangle \ . \end{aligned} \tag{4.203}$$

Meanwhile, when L_r^{vir} acts on the right-hand side of (4.171), one has

$$\begin{aligned}L_r^{vir} O_2(x)|O_1\rangle &\sim \sum \cdots \beta_{12}^{p\{m\}} f(x; |m|+|l|) \\ &\qquad \times L_r^{vir}|h_p; \{m\}, \{l\}\rangle \ . \end{aligned} \tag{4.204}$$

We define the following state,

$$|h_p; M, \{l\}\rangle = \sum_{|m|=M} \beta_{12}^{p\{m\}} |h_p; \{m\}, \{l\}\rangle \ , \tag{4.205}$$

which is the linear combination of all descendent states at level $|m| = M$, with an arbitrary fixed $\{l\}$. Then, the identification between (4.203) and (4.204) leads to

$$L_r^{vir}|h_p; M+r, \{l\}\rangle = (h_p - h_1 + rh_2 + |m|)|h_p; M, \{l\}\rangle \ , \tag{4.206}$$

which is identified as the standard recursion relation in the pure Virasoro case [98]. This recursion relation relies on the central charge $c_n - 1$, which is encoded in the Virasoro algebra of L_r^{vir}. Since the recursion relation for β and γ are independent of each other, a general conformal block factorizes into a product of the Virasoro block and a $U(1)$ block,

$$\mathcal{F}(h_p, h_i, q_i, c_n, k_n) = \mathcal{F}_{vir}(h_p, h_i, c_n - 1)\mathcal{V}_J(q_i, k_n) \ . \tag{4.207}$$

Furthermore, since the charge of the exchanged state in any OPE channel is fixed by charged conservation, $V_J(q_i, k)$ is universal in the correlation function. So a general correlation function also factorizes as

$$\langle O_1(w_1)O_2(w_i)\cdots\rangle = \mathcal{V}_J\bar{\mathcal{V}}_J \sum_p C_p \mathcal{F}^p_{vir}\bar{\mathcal{F}}^p_{vir} \ . \tag{4.208}$$

This further implies that $\mathcal{V}_J$ must be identical the vertex correlation function (4.90) with $k \to k_n = nk$,

$$\mathcal{V}_J(q_i, k_n) = \langle \prod \tilde{V}_{q_i}(w_i)\rangle = \prod_{i<j}(w_i - w_j)^{2q_iq_j/(nk)} \ , \tag{4.209}$$

because when we choose $h_i = 0$ for all i, then the Virasoro piece in the correlation function becomes an identity. This suggests that we can effectively factorize the Hilbert space as a tensor product of the $U(1)$ and Virasora sectors, which decouple from each other. This confirms the validity of the factorization argument in our original work [30].

4. Deriving U(1) block. To confirm the validity of (4.209), here I consider the four-point $U(1)$ block defined in (4.178) as an example. Since $\gamma_{12}^{|l|}$ has been worked out explicitly in (4.191), we can directly sum over all J-descendent states in (4.178) to compute the $U(1)$ block. By using $\langle O_4|J_{-r} = 0$ and $[O_3(1), J_{-r}] = -q_3$ as well as (4.187), we have

$$\begin{aligned}\langle O_4|O_3(1)|O^{\{l\}}_{p,q_1+q_2}\rangle &= \langle O_4|[O_3, \prod_{r=1}^{\infty} J^{s_r}_{-r}]|O_{p,q_1+q_2}\rangle \\ &= C^p_{34}\prod_{r=1}^{\infty}(-q_3)^{s_r} \ . \end{aligned}\tag{4.210}$$

Thus, by using (4.191), we can express the four-point $U(1)$ block as

$$\mathcal{V}_J = x^{2q_1q_2/k_n}\sum_{\{l\}}\prod_{r=1}^{\infty}\frac{1}{s_r!}X_r^{s_r} \ , \quad X_r = -\frac{2q_2q_3x^r}{k_n r} \ . \tag{4.211}$$

Since the set $\{l\}$ is characterized by the set of numbers s_r, each of which can take any non-negative integer m, we can rewrite the above expression as

$$\mathcal{V}_J = x^{2q_1q_2/k_n} \prod_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} X_r^m = x^{2q_1q_2/k_n} \exp\left\{ \sum_{r=1}^{\infty} X_r \right\} , \tag{4.212}$$

By using the following Taylor series for the exponent in (4.212),

$$\log(1-x) = -\sum_{r=1}^{\infty} x^r/r \ , \tag{4.213}$$

we obtain

$$\mathcal{V}_J = x^{2q_1q_2/k_n}(1-x)^{2q_2q_3/k_n} \ , \tag{4.214}$$

which fulfills the vertex correlation function (4.90) under the rescaling $k \to k_n = nk$ in $\mathrm{CFT}^{\otimes n}$.

Charged moments and null state equation. The factorization of the correlation function in $\mathrm{CFT}^{\otimes n}$ provides a general way to solve the charged moments and SREE in the twist picture. Clearly, in the semi-classical limit $c \to \infty$, the vacuum Virasoro block with central charge $c_n - 1$ is responsible to the usual entanglement and Rényi entropy, and the $U(1)$ block contributes to the charge-dependent part of $Z_n[\mu]$,

$$Z_n[\mu] \approx \mathcal{V}_J \mathcal{F}^0_{vir} \times c.c. \ . \tag{4.215}$$

Meanwhile, to make the $U(1)$ block contribution survive in the semi-classical limit, we need to assume $k \sim \mathcal{O}(c)$, and $\mu \sim \mathcal{O}(c^0)$, so that the charge $a = \frac{k\mu}{4\pi}$ and the conformal weight $h_a = \frac{k\mu^2}{16\pi^2}$ are both in the order of c. While the $U(1)$ block is universally determined in (4.209), and in particular obeys the KZ-equation,

$$\left(\partial_{w_i} - \frac{2a_i}{k_n} \sum_{j\neq i} \frac{a_j}{w - w_i} \right) \mathcal{V}_J(w_i) = 0 \ , \tag{4.216}$$

we still need to solve the Virasoro sector. The Virasoro sector can be solved by applying the monodromy method discussed in Section 3.2.2. However, I would like to provide a bit more details here, because the theory now is of a different type and has more symmetry.

The null states in $\mathrm{CFT}^{\otimes n}$ with Kac-Moody symmetry should be annihilated by any positive modes L_n and J_m. At level two, the null state can be expressed in terms of $L^{vir}_{-1,-2}$, given by

$$\varphi^{(2)}(w) = \left(L^{vir}_{-2} - \frac{3}{2(2h_\varphi + 1)} (L^{vir}_{-1})^2 \right) \hat{\varphi}(w) = 0 \ , \tag{4.217}$$

with the conformal weight and $U(1)$ charge of primary field $\hat{\varphi}(w)$ as $h_\varphi \sim -\frac{1}{2} - \frac{9}{2(c_n-1)}$ and $a_\varphi = 0$. Since $\hat{\varphi}$ has zero $U(1)$ charge, inserting

$\hat{\varphi}$ into a correlation function has no influence on the $U(1)$ block. This can be checked from the definition (4.212) by using $[\hat{\varphi}(w), J_{-l}] = 0$. So the only influence from $\hat{\varphi}$ is on the Virasoro sector,

$$\langle\hat{\varphi}(w)O_1(w_1)O_2(w_2)\cdots\rangle = \mathcal{V}_J(w_i)\bar{\mathcal{V}}_J(\bar{w}_i)\Phi(w,w_i) \ , \qquad (4.218)$$

Now the question is what is the differential action of L^{vir}_{-n} on $\hat{\varphi}(w)$ in the correlation function if we want to apply (4.217) inside the correlation function (4.218). Let me first consider L^{vir}_{-1},

$$\begin{aligned}(L^{vir}_{-1}\hat{\varphi})(w) &= (L_{-1}\hat{\varphi})(w) - \frac{1}{k_n}\sum_{m\in\mathbb{Z}}(J_{-1-m}J_m\hat{\varphi})(w) \\ &= (L_{-1}\hat{\varphi})(w) \ . \end{aligned} \qquad (4.219)$$

where in the second step I used the fact that $J_n|\hat{\varphi}\rangle = 0$ for $n \geq 0$, since $a_\varphi = 0$. In this case, the differential action of L^{vir}_{-1} is given by ∂_w. Similarly, for L^{vir}_{-2}, one obtains

$$(L^{vir}_{-2}\hat{\varphi})(w) = (L_{-2}\hat{\varphi})(w) - \frac{1}{k_n}(J^2_{-1}\hat{\varphi})(w) \ . \qquad (4.220)$$

On the right-hand side of (4.220), the first term gives rise to

$$\begin{aligned}&\langle(L_{-2}\hat{\varphi})(w)O_1(w_1)\cdots\rangle \\ &= \sum_i\left(\frac{h_i + a_i^2/k_n}{(w-w_i)^2} + \frac{\partial_{w_i}}{w-w_i}\right)\mathcal{V}_J(w_i)\bar{\mathcal{V}}_J(\bar{w}_i)\Phi(w,w_i) \ , \end{aligned} \qquad (4.221)$$

and, by (4.88), the second term reads

$$\begin{aligned}&-\frac{1}{k_n}\langle(J^2_{-1}\hat{\varphi})(w)O_1(w_1)\cdots\rangle \\ &= -\frac{1}{k_n}\left(\sum_i\frac{a_i}{w-w_i}\right)^2\mathcal{V}_J(w_i)\bar{\mathcal{V}}_J(\bar{w}_i)\Phi(w,w_i) \ , \end{aligned} \qquad (4.222)$$

Using the fact that $\mathcal{V}_J$ is in the form of the vertex correlation function as shown in (4.209), one can derive the following identity,

$$\sum_i\left(\frac{a_i^2/k_n}{(w-w_i)} + \frac{\partial_{w_i}}{w-w_i}\right)\mathcal{V}_J(w_i) = \left(\sum_i\frac{a_i}{w-w_i}\right)^2\frac{\mathcal{V}_J(w_i)}{k_n} \ . \qquad (4.223)$$

Then, using the identity (4.223), and combining (4.221) and (4.222) yields the action of L^{vir}_{-2}, given by

$$\begin{aligned}&\langle(L^{vir}_{-2}\hat{\varphi})(w)O_1\cdots\rangle \\ &= \mathcal{V}_J(w_i)\bar{\mathcal{V}}_J(\bar{w}_i)\sum_i\left[\frac{h_i}{(w-w_i)^2} + \frac{\partial_{w_i}}{w-w_i}\right]\Phi(w,w_i) \ . \end{aligned} \qquad (4.224)$$

This implies that L^{vir}_{-2} is just the usual differential operator [98], but it only acts on the Virasoro sector of the correlation function. This statement is also true for general L^{vir}_{-n}, since all higher modes can be generated by the commutators of $L^{vir}_{-1,-2}$, and the differential operators by themselves obey the Virasoro algebra.

4.2.5 *Summary*

Let me summarize what I have discussed in this section. I have studied the SREE and charged moments in the holographic $U(1)$ Chern-Simons-Einstein gravity from several different approaches. The results from different approaches are consistent, and I show that, at leading order of the cut-off expansion, the SREE with respect to the $U(1)$ Kac-Moody symmetry is always independent of the $U(1)$ charge. This charge-independent behavior of the SREE is called equipartition of entanglement, and was originally found in [27]. It means that the entanglement entropies encoded in each charge sectors of the subsystem are identical. By using the perturbative method introduced in [31], I have explained that the equipartition of entanglement stems from the truncation of the connected correlation functions of $U(1)$ currents at the quadratic order. This observation provides a hint for where to find the counter-examples for the equipartition behavior, for instance, the SREE in the CFT with nonlinear W_3 algebra, which has been partially discussed in our work [31].

Apart from the SREE, the charged moments by itself are interesting topic in the context of the AdS/CFT. It can be viewed as a charged version of the Rényi entropy. The duality between the CFT charged moments and the charged topological black hole in holographic models was originally proposed in [32]. The authors in [32] developed a general method, the so-called topological black hole method, for solving the charged moments in any holographic models with an additional $U(1)$ symmetry. While this approach can be widely applied to many different holographic theories, such as $U(1)$ Chern-Simons Einstein gravity and Einstein-Maxwell theory, it is restricted to the case where the whole system is in the vacuum background, with only a single entangling region. The direct generalization of this topological black hole approach to multi-entangling regions requires to properly formulate the modular Hamiltonian associated with those regions. This is still an open problem, and it has only been partially investigated in [124]. Another advantage of the topological black hole approach is that it is related to the BCFT description of the quantum entanglement [124]. Although in this thesis I only focused on the leading contribution to the SREE, the subleading terms can be solved by considering the boundary states in the BCFT description [98, 126]. From the holographic perspective, this might require the insertions of cosmic branes on the spatial boundaries of the topological black hole, which are the so-called end-of-world branes of the construction in [127, 140]. It would be interesting to study whether the subleading corrections to SREE can be obtained by considering the corrections to the AdS_3 effective action from those cosmic branes.

On the other hand, I have also discussed the Wilson line approaches to the charged moments, based on our work [29]. The Wilson line approach is distinct from the topological black hole approach, in the sense that the configurations of the Chern-Simons fields in the bulk are different in those two cases. The Wilson line interpretations for the charged moments stem from the fact that the $e^{i\mu Q_A}$ in the CFT charged moments can also be regarded as two local vertex operators [27]. This vertex operator description happens in the case of $U(1)$ Kac-Moody symmetry, but for other symmetries, it is not ensured to be true. The Wilson lines can be viewed as the dual description of the boundary vertex operators. I have shown that the holonomy of the Wilson line is identified with the charge of the vertex operator up to a constant multiplier. In particular, the effective action of the Chern-Simons fields with the Wilson line defect gives rise to the correct answer for the charged moments. One interesting fact about the Wilson line approach is that it is no longer restricted to the vacuum background and the single interval case. Generalization of the charged moments to charged background as well as the multi-intervals case is straightforward in the Wilson approach. I have explained the deeper reason for the success of the Wilson line approach, that is the disjoint $U(1)$ Wilson lines in the bulk correspond to the neutral $U(1)$ block of any correlation function in the dual CFT. The disjoint condition on the Wilson lines is related to the neutral condition on the OPE channel of the correlation function. In the case of a non-neutral OPE channel, such as a general three-point function in the dual CFT, the disjoint Wilson lines interpretation fails, but meanwhile, it motivates one to study more general Witten-diagram of the Wilson lines in the bulk. This is an interesting topic for future work. Another interesting topic motivated by the duality between the $U(1)$ Wilson lines and $U(1)$ block is that, one can also consider non-abelian Chern-Simons fields in AdS_3 space. One example is the $SU(2)$ Chern-Simons field in AdS_3 gravity, which can be realized in the D1-D5 system in the type-IIB string theory [118, 141]. Relations between $SU(2)$ Wilson lines and the $SU(2)$ extended conformal block are expected. The study on the monodromy problem for the non-abelian KZ-equation might be useful for understanding the non-abelian structure of the $SU(2)$ Wilson lines in the bulk.

On the CFT side, I have discussed about the charged moments from both the replica picture and the twist picture. In the replica picture, I introduced the vertex operator description of the charged moments, based on the decomposition of the stress tensor into the Virasoro piece and the Sugawara piece. To solve the charged moments in a charged background, I first discussed the special OPE structure of the vertex operators and then showed that the vertex correlation functions on $\mathbb{CP}^1$ obey the KZ-equations. For the multi-interval case, I discussed the general idea for solving the vertex correlation function on the higher genus replica surface, by first deriving the KZ-equations on the surface and then solving them. By performing some analysis on the current Ward identity on the replica surface, I argued that the KZ-equation on the

Duality	
Vacuum channels on S_g	Projective structures on S_g
Factorization of correlators	Decoupling of CS and gravity
Vacuum Virasoro block	Geodesics in AdS_3
neutral $U(1)$ block	Disjoint Wilson lines
Null state eqs in $\mathrm{CFT}^{\otimes n}$	Null state eqs in CFT on $R_{n,N}$

Table 1: Duality in holographic $U(1)$ CS-Einstein gravity

replica surface should be in the same form as the one in $\mathbb{CP}^1$, with a rescaling of the level $k \to nk$. This argument was later confirmed from the calculations in the twist picture, since, whatever the picture we are working in, the constraints for the correlation functions should be identical in two cases. This observation provides a general connection between the twist picture and the replica picture. On the other hand, in the twist picture, I first clarified the $\mathbb{Z}_n$ symmetric symmetry algebra in $\mathrm{CFT}^{\otimes n}$, and then introduced the charged twist fields description of the charged moments. To compute the general charged moments, I turned to study the general structure of the $U(1)$ extended conformal block in $\mathrm{CFT}^{\otimes n}$. I proved that the $U(1)$ extended conformal block in the theory always factorizes as a product of a Virasoro block with central charge $nc-1$ and a $U(1)$ block with level nk. In particular, the neutrality condition enforces that the $U(1)$ block is always universal in all conformal blocks in the correlation functions. Hence, this implies the factorization of a general correlation function in the theory. I used the recursion relation to derive the four-point $U(1)$ block by directly summing over all J-descendent states and confirmed that the $U(1)$ block is in the form of the vertex correlation function. Consequently, the $U(1)$ block always obeys the KZ-equation with level $k_n = nk$. To complete the general procedure for solving the charged moments in twist picture, I discussed the (level-two null state) null state equation in $\mathrm{CFT}^{\otimes n}$ with $U(1)$ Kac-Moody symmetry. I showed that the null state equation effectively only acts on the Virasoro sector of the correlation function and takes an identical form as in the pure Virasoro case. Therefore, a general procedure for solving the charged moments in twist picture is to solve the null state equation for the Virasoro sector and the KZ-equation for the U(1) block. In particular, the vacuum block dominance imposed on the correlation function is consistent with the neutral condition on the $U(1)$ block, where operators with opposite $U(1)$ charges are always paired in the OPE channels. This incorporates the disjoint Wilson lines construction in the bulk.

The most important aspects of $\mathrm{AdS}_3/\mathrm{CFT}_2$ that we learned in Chapter 3 and Chapter 4 are summarized in Table 1, which provide a useful guideline for investigating the charged moments in more complicated holographic model.

5 HIGHER SPIN HOLOGRAPHY

As an extension of the previous chapter, it is natural to ask how the charged moments and the SREE behave when more complicated symmetries are involved in the holographic system. In this chapter, I will discuss the application of the charged moments in the bottom-up higher spin holographic model, based on our work in [31]. This chapter is outlined as follows. In Section 5.1, I will first review the top-down description of the higher spin holography. In Section 5.2, I will turn to the bottom-up perspective, and review some relevant aspects of the three-dimensional higher spin theory in AdS_3, in particular, the asymptotic symmetry and the higher spin black holes. In Section 5.3, the notion of higher spin charged moments will be introduced, The independent holographic and CFT calculations for the higher spin charged moments will be discussed. The results from the two independent approaches coincide, and show that the higher spin charged moments is no longer Gaussian in the chemical potential. As a consequence, the charge dependence of the higher spin SREE is expected, or in other words, the equipartition of entanglement breaks down in the higher spin symmetry case.

5.1 INTRODUCTION

Symmetry plays a key role on the way of finding the exact realizations of AdS/CFT duality. Since I will mainly focus on the higher spin holography in AdS_3/CFT_2, I would like to start the story with the higher spin symmetries in two-dimensional CFTs, which was discovered three decades ago [142]. In two-dimensional CFTs, the generalizations of the affine Kac-Moody extension of the Virasoro algebra are known as the $\mathcal{W}$-algebras. They are associated with a set of spin-s currents W^s, with the OPEs,

$$T(z)W^s(0) \sim \frac{sW^{(s)}(0)}{z^2} + \frac{\partial W^{(s)}(0)}{z} , \tag{5.1}$$

where the stress tensor $T(z)$, regarded as the spin-2 current, always exists in the algebras. The typical feature of all $\mathcal{W}$-algebras is that they are in general closed nonlinear associative algebras. The non-linearity implies that they do not belong to the class of Lie algebra, and the associativity means that they satisfy the Jacobi identity. Historically, the construction of the $\mathcal{W}$-algebra was first performed by Zamolodchikov in [142]. The author worked out the $s \leq 3$ case by directly solving the associativity condition, and the resulting algebra is known as the W_3 algebra. The complexity of the analysis of the associativity condition increases rapidly with the increasing spin s. More systematic methods are developed later, such as the coset construction [143, 144], and the quantum Drinfeld-Sokolov (DS) reduction [145–147] based on the early work in [148]. Many works on $\mathcal{W}$-algebra were done by physicists in the 1980s and 1990s, and a comprehensive review for various aspects of the $\mathcal{W}$-algebra can be found in [149]. While the structures of the $\mathcal{W}$-algebras are rather complicated, those early works showed that the $\mathcal{W}$-algebra plays a central role in many areas of the two-dimensional physics, such as the constrained WZW model [150], the Toda field theory [150–152], the Kadomtsev-Petviashvili (KP) hierarchy [153–155], the two-dimensional $\mathcal{W}$-quantum gravity [156, 157], and the quantum Hall effect [158].

Higher spin/vector model duality. After the discovery of the duality between Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) theory [16], some new ideas emerged on the manifestation of the infinite number of conservation laws that appear in the dual field theory [159–165]. Unlike the original Maldacena's consideration, the limit concerned here is the large N limit with the 't Hooft coupling $\lambda = N g_{YM}^2 \ll 1$, namely, the dual SYM theory tends to be free. Thus, an infinite number of conserved currents with the increasing spin-s can be constructed from this free theory, which schematically can be written as [166]

$$J_{\mu_1 \cdots \mu_s} = \sum_{I=1}^{6} \mathrm{Tr}\left(\phi^I \nabla_{(\mu_1} \cdots \nabla_{\mu_s)} \phi^I\right) + \cdots \ . \tag{5.2}$$

where ϕ^I are the six scalar fields transforming in the adjoint representation of $SU(N)$. From the string perspective, the above situation corresponds to the tensionless limit of the string,

$$g_s \to 0\ , \quad l_s \to \infty\ ; \quad N \gg 1\ , \quad L_5 \text{ fixed}\ . \tag{5.3}$$

where L_5 denotes the radius of AdS_5. Since higher spin modes in the string spectrum become massless in the tensionless limit, it is expected that at least the current sector of the boundary theory is effectively described by massless higher spin gauge theory on the AdS_5 background. One candidate of such a bulk theory is Vasiliev higher spin (HS) theory [167–169], which however is ruled out in [170]. The first observation provided in [170] is that: for adjoint fields Φ^i, there should exist an exponentially growing number of single-particle states in AdS_5 corresponding to the single trace operators in the form of

$$d_{I_0 I_1 \cdots I_k} \mathrm{Tr}\left(\phi^{I_0} \nabla^{l_1} \phi^{I_1} \cdots \nabla^{l_k} \phi^{I_k}\right)\ , \tag{5.4}$$

and Vasiliev HS theory does not contain enough fields in AdS to account for those boundary operators. So, to find out the classical gravity theory dual to the weakly coupled $\mathcal{N} = 4$ SYM theory, the appropriate generalization of the Vasiliev HS theory by adding infinite additional fields with consistent interactions is required. However, the story did not go in this direction. The second observation in [170] is that: if one instead considers the boundary theory containing vector-like fields ϕ^a, transforming in the fundamental representation of the gauge group, then the only possible class of the "single trace" operators is $\phi^a \nabla^l \phi^a$. In this case, the Vasiliev HS theory does have enough degrees of freedom to account for those operators. This observation motivates the Klebanov-Polyakov conjecture in [170], that the large N critical O(N) vector-model in three dimensions is dual to the Vasiliev HS theory in AdS_4 (see also the review [171]). More generally, it is suggested that theories of an infinite number of massless higher spin gauge fields in AdS_{d+1} may correspond to the large N conformal field theories in d-dimensions with N-components vector-like fields.

Large N coset model and Vasiliev theory in $\mathbf{AdS}_3$. An analog of the Klebanov-Polyakov conjecture in $\mathrm{AdS}_3/\mathrm{CFT}_2$ was proposed in [172], in which the authors argued that the Vasiliev HS theory in AdS_3 background with the gauge symmetry hs[λ] is dual to the two-dimensional large N coset model

$$\frac{SU(N)_k \otimes SU(N)_1}{SU(N)_{k+1}} \ , \tag{5.5}$$

with the finite parameter λ being identified as $\lambda = \frac{N}{N+k}$. Unlike the three-dimensional O(N) vector model, the coset model considered here is an interacting theory. In the following, I would like to briefly explain the duality through the symmetry perspective, following from the review [173].

Let me first focus on the boundary theory. The coset model (5.5) is based on the WZW models with gauge group $G = SU(N) \otimes SU(N)$, and the denominator of (5.5) denotes the (diagonal) subgroup H of G. This quotient G/H means that we are restricting to the $SU(N)_{k+1}$ invariant subsector of the Hilbert space, or in other words, gauging the affine $SU(N)_{k+1}$ symmetry. This requires that fields defined in the coset model decouple with the affine $\mathfrak{su}(N)_{k+1}$ Kac-Moody current so that they are invariant under the $SU(N)_{k+1}$ gauge transformation. Focusing on the vacuum module, the first of such fields is the stress tensor of the coset model, given by

$$T_{G/H} = T_G - T_H \ , \tag{5.6}$$

where stress tensors T_G and T_H are given via the standard Sugawara construction [172]. If we denote $J^a_{(1)}$ and $J^a_{(2)}$ as the Kac-Moody currents associated with $\mathfrak{su}(N)_k$ and $\mathfrak{su}(N)_1$, then the diagonal $\mathfrak{su}(N)_{k+1}$ Kac-Moody current is given by $J^a_{(3)} = J^a_{(1)} + J^a_{(2)}$. It is easy to see that the

OPE between $T_{G/H}$ and $J^a_{(3)}$ is non-singular, namely, $T_{G/H}$ decouples with $J^a_{(3)}$. The central charge associated with $T_{G/H}$ reads [172],

$$c_{N,k} = (N-1)\left[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right] \ , \tag{5.7}$$

which scales as $N(1-\lambda^2)$ in the large N limit with $\lambda = \frac{N}{N+k}$ fixed. The behavior $c \sim \mathcal{O}(N)$ confirms that the coset model is a vector-like model in the large N limit. Apart from the stress tensor, there also exist higher spin currents $W^{(s)}$ in the spectrum of the coset model. The constructions for the higher spin currents are similar to the Sugawara construction, in which we make use of the Killing form of $\mathfrak{su}(N)$. Recall that the rank of $\mathfrak{su}(N)$ is $(N-1)$, so there are additional $(N-2)$ independent Casimirs C_s in the universal enveloping algebra of $\mathfrak{su}(N)$. Those Casimirs are constructed in terms of the invariant symmetric tensors of $\mathfrak{su}(N)$, i.e. $C_s \sim d_{a_1\cdots a_s}T^{a_1}\cdots T^{a_s}$, with $s \leq N$. For instance, the cubic invariant symmetric tensor is given by

$$d_{abc} \propto \mathrm{Tr}_f\left[T_a\{T_b, T_c\}\right] \ , \quad d_{abc} = d_{(abc)} \ , \tag{5.8}$$

where the trace is taking over the fundamental representation of $\mathfrak{su}(N)$, and the indices are raised or lowered by the Killing form and its inverse. Using the cubic invariant symmetric tensor, we can write down the general ansatz for the spin-3 current [149]

$$\begin{aligned} W^{(3)} = d_{abc}\Big[&a_1(J^a_{(1)}J^b_{(1)}J^c_{(1)}) + a_2(J^a_{(2)}J^b_{(1)}J^c_{(1)}) \\ &+ a_3(J^a_{(2)}J^b_{(2)}J^c_{(1)}) + a_4(J^a_{(2)}J^b_{(2)}J^c_{(2)})\Big] \ . \end{aligned} \tag{5.9}$$

Note that there are four independent terms in the bracket of (5.9). Similarly, combinations of $(s+1)$ independent terms will appear in the general form of the spin-s current. Imposing the gauge invariant condition $W^{(s)}J^a_{(3)} \sim 0$ generically yields s independent constrains on the coefficients a_i. Therefore, at each spin s, one obtains a higher spin current $W^{(s)}$ from one particular combination of Kac-Moody currents. The remaining free parameter is just the normalization constant for the current, which can be fixed by adopting the convention in [174],

$$W^{(s)}(z)W^{(s)}(0) \sim \frac{c_{N,k}/s}{z^{2s}} + \cdots \ , \quad 2 \leq s \leq N \ . \tag{5.10}$$

Those higher spin currents together with the stress tensor furnish a closed associative $W_{N,k}$ algebra of the coset model, which belongs to a special class of the $\mathcal{W}$-algebra. There is a more general class of $\mathcal{W}$-algebra, called $W_\infty[\mu]$, which is controlled by the central charge c and the parameter μ. In the large N limit, the $W_{N,k}$ algebra will be identified with $W_\infty[N]$ with the central charge $c_{N,k}$ given in (5.7).

Now, on the bulk side, the Vasiliev HS theory in AdS_3 can be formulated in terms of the Chern-Simons theory with the gauge algebra $\mathrm{hs}[\lambda]$, coupled with one additional massive complex scalar field [168]. The Lie algebra hs[λ] is in general infinite-dimensional, and it is defined via the quotient of the universal enveloping algebra of $\mathfrak{sl}(2)$ by the element $\langle C_2(\mathfrak{sl}_2) - \frac{1}{4}(\lambda^2-1)\mathbb{1}\rangle$,

$$B[\lambda] := \frac{\mathcal{U}(\mathfrak{sl}(2))}{\langle C_2(\mathfrak{sl}_2) - \frac{1}{4}(\lambda^2-1)\mathbb{1}\rangle} = \mathbb{C} \oplus hs[\lambda] \ , \tag{5.11}$$

where $C_2(\mathfrak{sl}_2)$ is the quadratic Casimir of $\mathfrak{sl}(2)$. If we use the common basis $\{L_0, L_{\pm1}\}$ of $\mathfrak{sl}(2)$, with the commutation relations,

$$[L_1, L_{-1}] = 2L_0 \ , \quad [L_{\pm1}, L_0] = \pm L_{\pm1} \ , \tag{5.12}$$

then the quadratic Casimir is given by

$$C_2(\mathfrak{sl}_2) = L_0^2 - \frac{1}{2}\left(L_1L_{-1} + L_{-1}L_1\right) \ . \tag{5.13}$$

General elements in $\mathcal{U}(\mathfrak{sl}(2))$ are in arbitrary combinations of $L_{0,\pm1}$. Under the identification $C_2(\mathfrak{sl}_2) \cong \frac{1}{4}(\lambda^2-1)\mathbb{1}$, the number of independent elements are reduced at each order of the combinations. One convenient choice for the basis of $B[\lambda]$ reads as following [173],

$$V_n^s = (-1)^{s-1-n}\frac{(n+s-1)!}{(2s+2)!}(\mathrm{ad}_{L_{-1}})^{s-1-n}L_1^{s-1} \ , \tag{5.14}$$

where $\mathrm{ad}_{L_{-1}}$ represents the adjoint action of L_{-1}, and the range of s and n are given by $s \geq 1$ and $|n| \leq s-1$. So at each order s, there are $(2s-1)$ independent elements included in $B[\lambda]$. Explicit commutation relations of the algebra in a closed form can be found in [175]. However, here we do not need them. There is a simple observation on the structure of the algebra. If one applies the adjoint actions from $\mathfrak{sl}(2)$ on the subspace spanned by the set $\{V_{1-s}^s, \cdots V_{s-1}^s\}$, then by the Leibniz rule of the adjoint action, it is not hard to see that states in the subspace transform among themselves. More precisely, under the adjoint action of $\mathfrak{sl}(2)$, $B[\lambda]$ decomposes into the infinite sum of spin-$(s-1)$ representations of $\mathfrak{sl}(2)$,

$$B[\lambda] = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \cdots \ , \tag{5.15}$$

where in particular the triplet $\mathbf{3}$ is just the adjoint representation of $\mathfrak{sl}(2)$. This decomposition in terms of the adjoint action defines the so-called *principle embedding* of $\mathfrak{sl}(2)$ in $B[\lambda]$. Recall that the $SL(2,R)$ Chern-Simons theory describes the AdS_3 gravity under certain boundary conditions [63], thus in the principle embedding, one interprets the triplet as the gravity sector of the Vasiliev HS theory. The remaining multiplets $(\mathbf{2s-1})$ with $s \geq 3$ account for the massless higher spin-s gauge fields on AdS_3 background, and the singlet, corresponding to $\mathbb{C}$ in (5.11), accounts for the additional massive complex scalar field in the Vasiliev HS theory. Another important phenomenon in the hs[λ]

algebra is that for integer $\lambda = N \geq 2$, the set of infinite number of generators V_n^s with $s > N$ forms an ideal χ_N of the algebra [167, 176]. In particular, truncation of the hs[λ=N] algebra via the quotient over the ideal χ_N gives rise to the finite-dimensional Lie algebra $\mathfrak{sl}(N)$,

$$hs[\lambda = N]/\chi_N \cong \mathfrak{sl}(N) \ , \quad N \geq 2 \ . \tag{5.16}$$

thus, the $SL(N,R)$ Chern-Simons theory can be thought of as a χ_N constrained subsector of the full Vasilliev HS theory. Although some evidence has been found that for any finite $N > 2$ truncation the resulting $SL(N,R)$ Chern-Simons theories exhibit acausalities [177], it is nevertheless interesting to study them as a playground for understanding the full Vasiliev HS theory. Indeed, important progress on analyzing the asymptotic symmetry of $SL(3,R)$ Chern-Simons theory was made in [178]. The authors showed that under the Brown-Henneaux-like boundary condition, the asymptotic algebra of the theory is identified as the classical version of the Zamolodchikov W_3 algebra [142], denoted as W_3^{cl}. The "classical" means that the associativity of the algebra is fulfilled only in the semi-classical limit $c \to \infty$. Meanwhile, the analysis on the asymptotic symmetry of the hs[1,1] Chern-Simons theory was performed in [179], and the result showed that the algebra is of classical W_∞ type. So it became evident to believe that the hs[λ] Vasiliev HS theory should lead to the $W_\infty^{cl}[\lambda]$ symmetry, which was originally obtained in the context of the KP-hierarchy [154, 155],

$$hs[\lambda] \xrightarrow{\text{asymptotic symmetry}} W_\infty^{cl}[\lambda] \ . \tag{5.17}$$

Indeed, the above relation (5.17) was verified later in [180]. Quantization of the classical algebra $W_\infty^{cl}[\lambda]$ requires introducing $1/c$ corrections to the structure constants of the algebra, which finally leads to the $W_\infty[\lambda]$.

Matching the symmetries. A remarkable observation on the isomorphism between the $W_\infty[N]$ symmetry of the large N coset model (5.5) and the $W_\infty[\lambda]$ algebra of the hs[λ] Vasiliev HS theory with central charge $c = c_{N,k}$ and $\lambda = \frac{N}{N+k}$ was made in [181]. The key point is that what fix the $W_\infty[\lambda]$ algebra are the central charge and the OPE coefficient (structure constant) $\gamma = C_{33}^4$ between $W^{(3)}$ and $W^{(4)}$ currents[1], i.e.

$$W^{(3)} \times W^{(3)} \sim \frac{c}{3}\mathbf{1} + 2T + C_{33}^4 W^{(4)} + \cdots \ . \tag{5.18}$$

The exact relation between λ and γ has been found out in the early works [182–184], given by

$$\gamma^2 = \frac{64(c+2)(\lambda-3)\,(c(\lambda+3)+2(4\lambda+3)(\lambda-1))}{(5c+22)(\lambda-2)(c(\lambda+2)+(3\lambda+2)(\lambda-1))} \ . \tag{5.19}$$

1 It has been checked that a first few structure constants in the algebra depend directly on the central charge and γ. And in [181] the authors assumed that this is still true for all other higher order structure constants.

For fixed γ and central charge c, the above equation (5.19) is a cubic algebraic equation of λ. Therefore, the three roots of the cubic equation, denoted as $\lambda_{1,2,3}$, give rise to the *triality relation* on the following $\mathcal{W}$ algebras [181],

$$W_\infty[\lambda_1] \cong W_\infty[\lambda_2] \cong W_\infty[\lambda_3] \ , \tag{5.20}$$

For the coset model, insert $c = c_{N,k}$ in (5.7) and $\lambda_1 = N$ into (5.19), one can first determine the associated $\gamma(c_{N,k}, N)$. Then the remaining two roots of (5.19) can be worked out as $\lambda_2 = \frac{N}{N+k}$ and $\lambda_3 = -\frac{N}{N+k+1}$, providing the non-trivial identifications on the symmetries of the large N coset model and the hs[λ] Vasiliev HS theory,

$$W_{N,k} = W_\infty[N] \cong W_\infty[\lambda] \ , \quad N \to \infty, \ \lambda = \frac{N}{N+k} \text{fixed} \ . \tag{5.21}$$

Apart from matching the symmetries of the two theories, there are other tests for the duality which are reviewed in [173]. For instance, the one-loop partition function of the Vasiliev HS theory, accounting for the one-loop fluctuations of the massless higher spin fields in Poincaré AdS$_3$, matches with the vacuum character of the coset model. The partition function of the high temperature charged black hole in HS theory matches with the charged character of the coset model in the perturbative expansion of the chemical potential. Those two tests are universal, in the sense that they hold as long as the symmetries on the two sides coincide with each other. There are also non-universal tests, for instance, the matching between the spectrum of the two theories. Details on those tests can be found in the review [173] and reference therein. Further discussions on the linking between the Vasiliev HS theory and the large N coset model via the string theory can be found in [185–187].

5.2 HIGHER SPIN THEORY IN THREE DIMENSIONS

In this section, I would like to briefly review the higher spin gravity in AdS$_3$. I will first explain the $SL(2,R)$ Chern-Simons formulation of the AdS$_3$ gravity, and the generalization to $SL(N,R)$ or $hs[\lambda]$ higher spin theory is straightforward. The general procedure for finding asymptotic symmetry algebra will be demonstrated via the example of the higher spin-3 theory [178]. Moreover, the constructions for the charged black hole solutions in the higher spin-3 theory will also be discussed.

5.2.1 *AdS_3 gravity as a Chern-Simons theory*

The Chern-Simons formulation of the three-dimensional Einstein gravity with or without the cosmological constant was originally discovered in [63]. In the following, I would like to give a brief review of it. Consider the Einstein-Hilbert action in $(2+1)$-dimensional spacetime

$$S_{EH} = \frac{1}{16\pi G_3}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda) \ . \tag{5.22}$$

with the negative cosmological constant $\Lambda = -1/l^2$.

First-order formalism. To make contact with the Chern-Simon theory, we need to first introduce the first-order formalism of the gravity theory, in which, the basic variable is the vielbein field. The vielbein field is defined as an isomorphism between the tangent bundle $\mathcal{TM}$ and the orthonormal frame bundle (or the principle $SO(2,1)$-bundle),

$$e : \mathcal{TM} \to V \times_p \mathcal{M} \ , \quad \forall p \in \mathcal{M} \ . \tag{5.23}$$

Here the fiber V is a three-dimensional vector space with the structure group $SO(2,1)$, providing a natural metric η on it. If we choose an orthonormal basis P_a on V and define its dual P^a via $P^a P_b = \delta^a_b$, then the natural metric can be written as $\eta = \eta_{ab}P^a \otimes P^b$ with $\eta_{ab} = \mathrm{diag}(-1,1,1)$. The vielbein in terms of components reads

$$e = e^a P_a = e^a_\mu P_a dx^\mu \ , \quad a, \mu = 1,2,3 \ , \tag{5.24}$$

which can be thought of as a V-valued one-form, and the isomorphism (5.23) requires components e^a_μ to be invertible. Now, using the natural metric, we can define a Riemannian metric on $\mathcal{M}$ through the vielbein,

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \ . \tag{5.25}$$

Denoting e^μ_a as the inverse matrix of e^a_μ, with $e^\mu_a e^a_\nu = \delta^\mu_\nu$ and $e^\mu_a e^b_\mu = \delta^b_a$, then the inverse metric reads $g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$. To describe the gauge transformation (parallel transport) of a V-valued scalar field ϕ^a, we define the covariant derivative as

$$D_\mu \phi^a = \partial_\mu \phi^a + \omega_\mu{}^a{}_b \phi^b \ , \tag{5.26}$$

where $\omega_\mu{}^a{}_b$ is the spin connection. The gauge invariance of the natural metric requires $D_\mu \eta^{ab} = 0$, which by (5.26) yields the antisymmetric property of the spin connection $\omega_{\mu ab} = -\omega_{\mu ba}$. Therefore, the spin connection ω^{ab} is a $\wedge^2 V$-valued one-form. For V-valued tensor fields, for instance v^a_ν, we can define the general covariant derivative as

$$\nabla_\mu v^a_\nu = \partial_\mu v^a_\nu + \omega_\mu{}^a{}_b v^b_\nu - \Gamma^\sigma_{\mu\nu} v^a_\sigma \ , \tag{5.27}$$

where $\Gamma^\sigma_{\mu\nu}$ is the affine connection. Imposing the usual compatibility condition $\nabla_\mu g_{\nu\sigma} = 0$, or equivalently $\nabla_\mu e^a_\nu = 0$, yields the relation

$$\Gamma^\sigma_{\mu\nu} = e^\sigma_a D_\mu e^a_\nu \ . \tag{5.28}$$

The V-valued torsion one-form T^a is defined as

$$T^a = De^a = de^a + \omega^a{}_b \wedge e^b \ . \tag{5.29}$$

The Einstein-Hilbert theory requires the vanishing torsion, hence by using (5.28) one can show the symmetric property of the affine connection, i.e. $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$. Therefore, the affine connection in this case recovers the usual Christoffel symbol. On the other hand, the curvature two-form is defined as

$$R_{\mu\nu}{}^a{}_b \phi^b = [\nabla_\mu, \nabla_\nu]\phi^a \ , \quad \forall \phi^a \ . \tag{5.30}$$

Under the torsion-free condition, by (5.27), the $\wedge^2 V$-valued curvature two-form reads

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ , \tag{5.31}$$

and the Ricci scalar is given by the contraction $R = R_{\mu\nu}{}^{ab} e^\mu_a e^\nu_b$. Now, it is a simple exercise to show that the Einstein-Hilbert action (5.22) can be rewritten as

$$S_{EH} = \frac{1}{16\pi G_3} \int_{\mathcal{M}} \epsilon_{abc} \left(e^a \wedge R^{bc} + \frac{1}{3l^2} e^a \wedge e^b \wedge e^c \right) \tag{5.32}$$

where ϵ_{abc} is the epsilon tensor with $\epsilon_{123} = 1$. For instance, the curvature two-form term can be evaluated as

$$\begin{aligned}
\epsilon_{abc} e^a \wedge R^{bc} &= \frac{1}{2} \epsilon_{abc} e^a_\mu R_{\nu\sigma}{}^{bc} \varepsilon^{\mu\nu\sigma} \varepsilon_{\mu'\nu'\sigma'} dx^{\mu'} dx^{\nu'} dx^{\sigma'} \\
&= \frac{1}{2} \epsilon_{abc} e^a_\mu R_{\nu\sigma}{}^{bc} e^\mu_{a'} e^\nu_{b'} e^\sigma_{c'} \epsilon^{a'b'c'} \det(e^a_\mu) dx^3 \\
&= \frac{1}{2} \delta^{[a'}_a \delta^{b'}_b \delta^{c']}_c e^a_\mu R_{\nu\sigma}{}^{bc} e^\mu_{a'} e^\nu_{b'} e^\sigma_{c'} \det(e^a_\mu) dx^3 \\
&= \sqrt{-g} R dx^3
\end{aligned} \tag{5.33}$$

where $\varepsilon_{\mu\nu\sigma}$ is the component of the volume form, related with ϵ_{abc} via the vielbein as

$$\varepsilon_{\mu\nu\rho} = \epsilon_{abc} e^a_\mu e^b_\nu e^c_\sigma \ , \quad \varepsilon_{123} = \det(e^a_\mu) = \sqrt{-g} \ . \tag{5.34}$$

The equations of motion of the action (5.32) can be easily worked out, given by

$$T^a = 0 \ , \quad R^{ab} + \frac{1}{l^2} e^a \wedge e^b = 0 \ . \tag{5.35}$$

It is easy to see that the second equation gives rise to constant Ricci scalar $R = -6l^{-2}$.

$SL(2, R)$ **Chern-Simons theory.** The first-order formalism of the Einstein-Hilbert gravity introduced above can also be applied to other dimensions. The special thing in three dimensions is that using the epsilon tensor, we can define the dual spin-connection as $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$, and consequently $R^a = \frac{1}{2}\epsilon^{abc}R_{bc}$. Then the action (5.32) can be rewritten as

$$S_{EH} = \frac{1}{8\pi G_3}\int_{\mathcal{M}} e^a \wedge R_a + \frac{1}{6l^2}\epsilon_{abc}e^a \wedge e^b \wedge e^c \ . \tag{5.36}$$

and the equations of motion reads

$$T^a = 0 \ , \quad R^a + \frac{1}{2l^2}\epsilon^{abc}e_b \wedge e_c = 0 \ , \tag{5.37}$$

The idea provided in [43] is that the epsilon tensor in the above action can be treated as the structure constant,

$$[P_a, P_b] = \frac{1}{l^2}\epsilon_{abc}J^c \implies e \wedge e = \frac{1}{2l^2}\epsilon^{abc}e_b \wedge e_c J_a \ , \tag{5.38}$$

The basis J_a should be associated with the dual spin-connection, i.e. $\omega = \omega^a J_a$, as required by the equation of motion for ω^a. More precisely, J_a are the generators of the $SO(2,1)$ structure group, and they act on the fiber V, or in other words, V forms a three-dimensional representation of $SO(2,1)$. Therefore, the full algebra is given by

$$[P_a, P_b] = \frac{1}{l^2}\epsilon_{abc}J^c \ , \ [J_a, J_b] = \epsilon_{abc}J^c \ , \ [J_a, P_b] = \epsilon_{abc}P^c \ , \tag{5.39}$$

which is isomorphic to the local isometry of AdS$_3$, i.e. $\mathfrak{so}(2,2)$. One can think of P_a as the "translation" generators, and J_a as the local Lorentz generators of the spacetime. The reason that the "translation" P_a do not commute with each other is that in the embedding space $\mathbb{R}^{2,2}$ of AdS$_3$, the "translation" P_a represent the boost generators. If we take $l \to \infty$ such that the cosmological constant vanishes, then the above algebra reduces to the Poincaré algebra $\mathfrak{iso}(2,1)$, i.e. the isometry of the $(2+1)$-dimensional flat spacetime.

To finally formulate the Einstein-Hilbert theory as a gauge theory, we need to define the $\mathfrak{so}(2,2)$-invariant bilinear form. Since $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$ is semisimple, which implies the invariant bilinear form is not unique. The choice of the invariant bilinear form compatible with the action (5.32) is given by[2] [43]

$$\langle J_a, J_b\rangle = 0 \ , \ \langle P_a, P_b\rangle = 0, \ \langle J_a, P_b\rangle = \eta_{ab} \ , \tag{5.40}$$

The $\mathfrak{so}(2,2)$-invariance of (5.40) can be checked by the definition,

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle \ , \tag{5.41}$$

for arbitrary $\mathfrak{so}(2,2)$-valued X, Y and Z. Now we are ready to write the Einstein-Hilbert action (5.32) as a gauge theory, given by

$$S_{EH} = \frac{1}{8\pi G_3}\int_{\mathcal{M}} \langle e \wedge \mathbf{R} + \frac{1}{3}e \wedge e \wedge e\rangle \ . \tag{5.42}$$

2 Note that the bilinear form (5.40) is not the Killing form of $\mathfrak{so}(2,2)$.

with $\mathbf{R} = d\omega + \omega \wedge \omega$. If we define a Chern-Simons gauge field A as

$$A = e^a P_a + \omega^a J_a \ , \tag{5.43}$$

using (5.39) and (5.40), it is easy to show the following Chern-Simons action is equivalent to the (5.42) to a boundary term,

$$S[A] = \frac{1}{16\pi G_3} \int_{\mathcal{M}} \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \rangle \ , \tag{5.44}$$

Furthermore, if we define $J_a^{\pm} = \frac{1}{2}(J_a \pm l P_a)$, then the algebra (5.39) decomposes into two copies of $\mathfrak{sl}(2, R)$,

$$[J_a^+, J_b^+] = \epsilon_{ab}{}^c J_c^+ \ , \ [J_a^-, J_b^-] = \epsilon_{ab}{}^c J_c^- \ , \ [J_a^+, J_b^-] = 0 \ . \tag{5.45}$$

Consequently, the gauge field A can also decompose into two decoupled pieces,

$$A = A_+ + A_- = A_+^a J_a^+ + A_-^a J_a^- \ , \tag{5.46}$$

with $A_\pm^a = (\omega^a \pm e^a/l)$. The bilinear form (5.40) acts on $J_a^\pm$ as,

$$\langle J_a^+, J_b^+ \rangle = \frac{l}{2}\eta_{ab} \ , \ \langle J_a^-, J_b^- \rangle = -\frac{l}{2}\eta_{ab} \ , \ \langle J_a^+, J_b^- \rangle = 0 \ , \tag{5.47}$$

which together with the commutation relation (5.45) gives rise to the decomposition of the action $S[A] = S[A_+] + S[A_-]$. In the fundamental representation of $\mathfrak{so}(2, 2)$, $J_a^\pm$ take the form of

$$J_a^+ = \begin{pmatrix} T_a & 0 \\ 0 & 0 \end{pmatrix} \ , \quad J_a^- = \begin{pmatrix} 0 & 0 \\ 0 & T_a \end{pmatrix} \tag{5.48}$$

where T_a are the generators in the fundamental representation of $\mathfrak{sl}(2, R)$,

$$T_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ , \ T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \ , \ T_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ . \tag{5.49}$$

Taking the trace over the fundamental representation,

$$\mathrm{Tr}_f[J_a^\pm J_b^\pm] = \mathrm{Tr}[T_a T_b] = \frac{1}{2}\eta_{ab} \ , \tag{5.50}$$

and comparing with the bilinear form (5.47), we can write the action (5.44) as the difference between two Chern-Simons actions,

$$S[A] = S_{cs}[A_+] - S_{cs}[A_-] \tag{5.51}$$

with

$$S_{cs}[A_\pm] = \frac{k_{cs}}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}[A_\pm \wedge dA_\pm + \frac{2}{3} A_\pm \wedge A_\pm \wedge A_\pm] \ , \tag{5.52}$$

Here $A_\pm$ have been redefined as $A_\pm = A_\pm^a T_a$, namely they are valued in the fundamental representation of $\mathfrak{sl}(2, R)$, but not in $\mathfrak{so}(2, 2)$ any longer. And the Chern-Simons level is given by $k_{cs} = \frac{l}{4G_3}$. The Chern-Simons action (5.52) is invariant under the following gauge transformation up to a boundary term,

$$A_\pm \to g^{-1}A_\pm g + g^{-1}dg \xrightarrow{g\sim \mathbb{1}+\lambda} \delta A_\pm = d\lambda + [A_\pm, \lambda] \ , \tag{5.53}$$

The equations of motion for Chern-Simons fields are just the flat connection conditions,

$$F_\pm = dA_\pm + A_\pm \wedge A_\pm = 0 \ , \tag{5.54}$$

hence, locally one can express the Chern-Simons field as a gauge transformation from the "empty" solution, i.e. $A_\pm = h_\pm^{-1} dh_\pm$. As a remark, in the literature, the Riemannian metric is usually expressed as

$$g_{\mu\nu} = \frac{1}{\mathrm{Tr}[L_0 L_0]} \mathrm{Tr}[\tilde{e}_\mu \tilde{e}_\nu] \ . \tag{5.55}$$

where L_0 is the Cartan element of $\mathfrak{sl}(2)$, which in the fundamental representation is identical to T_2 in (5.49). The new vielbein $\tilde{e}$ here is defined as $\tilde{e} = \tilde{e}^a T_a$, which is distinct with the original vielbein $e = e^a P_a$. Nevertheless, since $\mathrm{Tr}[T_a T_b] = \mathrm{Tr}[L_0 L_0]\eta_{ab} = \frac{1}{2}\eta_{ab}$, the above formula (5.55) recovers the original definition of the metric in (5.25).

Remarks on Euclidean signature. The above discussions are in the Lorentz signature. In the Euclidean case, the isometry group of AdS_3 is $SO(3,1) \cong SL(2,\mathbb{C})$, which is the same as the isometry of the (2+1)-dimensional de Sitter space. So effectively, the cosmological constant in the algebra changed to be $-l^{-2} \to l^{-2}$. Therefore, the commutation relations for J_a and P_a are given by

$$[J_a, J_b] = \epsilon_{abc} J^c, \ \ [J_a, P_b] = \epsilon_{abc} P^c, \ \ [P_a, P_b] = -\frac{1}{l^2}\epsilon_{abc} J^c, \tag{5.56}$$

Here the indices are lowered and raised by Euclidean metric δ_{ab} and its inverse, reflecting the fact that the structure group on the fiber V is $SO(3)$, describing the local rotation symmetry of the Euclidean AdS_3. Unlike the Lorentzian case, the real form of $\mathfrak{so}(3,1)$ does not admit a direct sum decomposition. However, we can consider its complex form and define

$$J_a^\pm = \frac{1}{2}(J_a \mp i l P_a) \ , \tag{5.57}$$

which furnish two copies of complex $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ algebra. In the fundamental representation of $\mathfrak{so}(3,1)$, $J_a^\pm$ are matrices in the form of

$$J_a^+ = \begin{pmatrix} \tilde{T}_a & 0 \\ 0 & 0 \end{pmatrix} \ , \quad J_a^- = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{T}_a \end{pmatrix} \ , \tag{5.58}$$

where $\tilde{T}_a$ are traceless complex matrices, related to T_a in (5.49) as

$$\tilde{T}_1 = T_1 \ , \quad \tilde{T}_2 = iT_2 \ , \quad \tilde{T}_3 = iT_3 \ . \tag{5.59}$$

Thus, in the fundamental representation, the Chern-Simons connection $A = e^a P_a + \omega^a J_a$ can be redefined as

$$A = A_+ \oplus A_- = A_+^a \tilde{T}_a \oplus A_-^a \tilde{T}_a \ , \tag{5.60}$$

with the components

$$A_\pm^a = \omega^a \pm i e^a / l \ . \tag{5.61}$$

Notice that both e^a and ω^a are real, so we have $A_+^a = (A_-^a)^*$. Meanwhile, the matrices $\tilde{T}_a$ in (5.59) are traceless and anti-hermitian. Thus, we conclude that in the Euclidean case,

$$A_+ = -(A_-)^\dagger \ , \quad A_\pm \in \mathfrak{sl}(2, \mathbb{C}) \ . \tag{5.62}$$

Due to the flat connection conditions, i.e. $F_\pm = 0$, we can locally express the Chern-Simons fields as

$$A_+ = h^{-1} dh \ , \quad A_- = -dh^\dagger (h^\dagger)^{-1} \ , \quad h \in SL(2, \mathbb{C}) \ , \tag{5.63}$$

and the redefined vielbein reads

$$\tilde{e} = e^a \tilde{T}_a = -\frac{il}{2}(A_+ - A_-) = -\frac{il}{2} h^\dagger \Lambda^{-1} d\Lambda (h^\dagger)^{-1} \ , \tag{5.64}$$

with $\Lambda = hh^\dagger \in SL(2, \mathbb{C})$. Therefore, by (5.55), we have

$$g_{\mu\nu} \sim \mathrm{Tr}[\Lambda^{-1} \partial_\mu \Lambda \Lambda^{-1} \partial_\nu \Lambda] \ . \tag{5.65}$$

5.2.2 *Higher spin gravity*

A natural generalization of the $SL(2, R)$ Chern-Simons theory is to replace the gauge group by $SL(N, R) \times SL(N, R)$, and the resulting theory turns out to be a nonlinear interacting theory of massless higher spin fields. Though this generalization is rather straightforward, for a better understanding of the higher spin theory, I would like to start from the vielbein formalism. We consider a generalized vielbein e as a map from the tangent bundle $\mathcal{TM}$ to the principle $SL(N, R)$-bundle,

$$e : \mathcal{TM} \ \to \mathcal{F}_{\mathfrak{sl}_N}(\mathcal{M}) = V \times_p \mathcal{M} \ , \quad \forall p \in \mathcal{M} \ . \tag{5.66}$$

where V is a $(N^2 - 1)$-dimensional vector space acted by the $\mathfrak{sl}(N, R)$ action. The nature metric κ on V is given by the Killing metric of $\mathfrak{sl}(N, R)$ (up to a constant factor a_N),

$$\kappa_{ab} = a_N \mathrm{Tr}[\mathrm{ad}_{T_a} \mathrm{ad}_{T_b}] \ , \tag{5.67}$$

where $\{T_a\}$ denotes a basis of $\mathfrak{sl}(N, R)$. To fix the constant a_N, we need to first specify the pure gravity sector, namely the embedding of $\mathfrak{sl}(2, R)$ in $\mathfrak{sl}(N, R)$. Here we focus on the principle embedding, in which, $\mathfrak{sl}(N,R)$ decomposes into the following multiplets under the adjoint action of $\mathfrak{sl}(2,R)$,

$$\mathfrak{sl}(N,R) = \mathfrak{sl}(2,R) \oplus \mathbf{5} \oplus \cdots \oplus (\mathbf{2N-1}) \ , \tag{5.68}$$

Each $(2s-1)$ multiplet with $2 \leq s \leq N$ forms a spin-$(s-1)$ representation of $\mathfrak{sl}(2,R)$. In the Cartan-Weyl basis $\{V_n^s\}$ of $\mathfrak{sl}(N,R)$, with $2 \leq s \leq N$ and $1-s \leq n \leq s-1$, the above decomposition (5.68) can be seen from the following commutation relation

$$[V_n^2, V_m^s] = ((s-1)n - m)\, V_{m+n}^s \ . \tag{5.69}$$

Here $\{V_0^2, V_{\pm 1}^2\}$ is just the usual basis $\{L_0, L_{\pm 1}\}$ of $\mathfrak{sl}(2,R)$, and each set $\{V_n^s\}$ with fixed s is associated with the spin-s sector. The constant a_N then can be fixed by requiring the $\mathfrak{sl}(2,R)$ sector of κ to be identical to the natural metric η in the pure gravity case. As before, if we denote P_a as the basis of V, then the vielbein reads

$$e = e^a P_a = e^a_\mu P_a dx^\mu \ , \quad a = 1, \cdots, N^2-1 \ , \quad \mu = 1,2,3 \ , \tag{5.70}$$

and the spacetime metric is defined as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \kappa_{ab} \ . \tag{5.71}$$

Now, recall that in the pure gravity case, the vielbein e^a_μ is a 3×3 matrix with nine degrees of freedom (d.o.f). The metric manifests the local Lorentz invariance, hence vielbeins related by local Lorentz transformation are equivalent. Therefore, only six independent d.o.f are encoded in the vielbein, which matches with the d.o.f of the metric. However, in $SL(N,R)$ case, the vielbein is a $(N^2-1) \times 3$ matrix, so the metric can not account for all d.o.f of the vielbein. This motivates us to define the additional symmetric higher spin fields in the theory,

$$\phi_{\mu_1 \cdots \mu_s} \sim e^{a_1}_{\mu_1} \cdots e^{a_s}_{\mu_s} d_{a_1 \cdots a_s} \ , \quad s \leq N \ . \tag{5.72}$$

Here $d_{a_1 \cdots a_s}$ is the $\mathfrak{sl}(N,R)$-invariant symmetric tensor which can be constructed in the fundamental representation of $\mathfrak{sl}(N,R)$ as

$$d_{a_1 \cdots a_s} \sim \mathrm{Tr}[T_{(a_1} \cdots T_{a_s)}] \ . \tag{5.73}$$

Of course, the d.o.f of all higher spin fields including the metric are much larger than the d.o.f of the vielbein, hence different fields are not independent. This reflects the fact that the theory is an interacting theory of the higher spin fields. Apart from the vielbein, we should also introduce the generalized spin-connection associated with the fiber bundle $\mathcal{F}_{\mathfrak{sl}_N}(\mathcal{M})$. The $SL(N,R)$ structure group of the fiber V means there is a natural $\mathfrak{sl}(N,R)$ action on V. So we define the spin-connection as

$$\omega = \omega^a J_a \ , \tag{5.74}$$

where J_a form a $\mathfrak{sl}(N,R)$ subalgebra. Denote f_{abc} as the structure constant of $\mathfrak{sl}(N,R)$ in the J_a basis, then the full algebra of $\{J_a, P_a\}$ is given by

$$[J_a, J_b] = f_{abc} J_d \kappa^{cd} \,, [J_a, P_b] = f_{abc} P_d \kappa^{cd} \,,$$
$$[P_a, P_b] = \frac{1}{l^2} f_{abc} J_d \kappa^{cd} \,, \tag{5.75}$$

By defining $J_a^\pm = \frac{1}{2}(J_a \pm l P_a)$, the algebra (5.75) factorizes into two copies of $\mathfrak{sl}(N,R)$ algebra. In fact, in the fundamental representation, J_a and P_a take the form of

$$J_a = \begin{pmatrix} T_a & 0 \\ 0 & T_a \end{pmatrix} \,, \quad P_a = \frac{1}{l} \begin{pmatrix} T_a & 0 \\ 0 & -T_a \end{pmatrix} \,, \tag{5.76}$$

with T_a satisfying $[T_a, T_b] = f_{abc} T_d \kappa^{ab}$. So in the fundamental representation, the Chern-Simons gauge field can be written as

$$A = e^a P_a + \omega^a J_a = A_+^a T_a \oplus A_-^a T_a \,, \tag{5.77}$$

Now, if we choose the following invariant bilinear over the whole algebra,

$$\langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0 \,, \langle J_a, P_b \rangle = \kappa_{ab} \,, \tag{5.78}$$

then the construction for the Chern-Simons action will be the same as before. The only differences are that now the trace in $S_{cs}[A_\pm]$ is taking over the fundamental representation of $\mathfrak{sl}(N,R)$, and as a consequence, the Chern-Simons level needs to be modified as

$$k_{cs} = \frac{l}{8G_3 \mathrm{Tr}[L_0 L_0]} \,, \quad \mathrm{Tr}[L_0 L_0] = \frac{1}{12} N(N^2 - 1) \,. \tag{5.79}$$

This arises from the difference between the bilinear form and the trace,

$$\langle J_a^\pm, J_b^\pm \rangle = \frac{l}{2} \kappa_{ab} \,, \quad \mathrm{Tr}[T_a T_b] = \mathrm{Tr}[L_0 L_0] \kappa_{ab} \,. \tag{5.80}$$

If we redefine the vielbein as $\tilde{e} = e^a T_a$, then using the relation (5.80) one can express the spacetime metric (5.71) and the higher spin fields (5.72) as,

$$g_{\mu\nu} = \frac{1}{\mathrm{Tr}[L_0 L_0]} \mathrm{Tr}[\tilde{e}_\mu \tilde{e}_\nu] \,, \quad \phi_{\mu_1 \cdots \mu_s} \sim \mathrm{Tr}[\tilde{e}_{(\mu_1} \cdots \tilde{e}_{\mu_s)}] \,, \tag{5.81}$$

which are the usual expressions in the literature.

Why higher spin? Finally, we need to answer the question: why does the $SL(N,R)$ Chern-Simons theory describe the massless higher spin fields? The detailed answer can be found in [178], and here I shall just briefly explain the reason. The equations of motion of the generalized vielbein and spin connection are given by,

$$\begin{aligned} &de^a + f^{abc}\omega_b \wedge e_c = 0 \\ &d\omega^a + f^{abc}\left(\omega_b \wedge \omega_c + \frac{1}{l^2} e_b \wedge e_c\right) = 0 \ . \end{aligned} \tag{5.82}$$

where indices are raised and lowered by κ_{ab} and its inverse. In terms of the metric (5.71) and the higher spin fields (5.72), these equations are highly non-linearly coupled through the structure constant. The situation can be much simplified if we only switch on a single perturbative higher spin-s sector around a fixed AdS_3 background. This means that in the Cartan-Weyl basis, we consider the following form of the vielbein,

$$\tilde{e} = e^a T_a = \tilde{e}^{(2)} + \tilde{e}^{(s)} = e^n_{(2)} V^2_n + e^m_{(s)} V^s_m \ , \quad e^m_{(s)} \ll 1 \ . \tag{5.83}$$

where $\tilde{e}^{(2)}$ accounts for the AdS_3 background. At the linear order of $e^m_{(s)}$, all the higher spin-j fields become,

$$\phi_{\mu_1 \cdots \mu_j} \sim \mathrm{Tr}\left[\tilde{e}^{(s)}_{(\mu_1} \tilde{e}^{(2)}_{\mu_2} \cdots \tilde{e}^{(2)}_{\mu_j)}\right] \ . \tag{5.84}$$

However, due to the following property of the trace in the fundamental representation of $\mathfrak{sl}(N, R)$,

$$\mathrm{Tr}[V^s_{(m} V^2_{n_1} \cdots V^2_{n_{j-1})}] \neq 0 \ , \quad \text{iff } j = s \ , \ \sum_{i=1}^{j-1} n_i = -m \ , \tag{5.85}$$

only the spin $j = s$ field in (5.84) is non-vanishing. Therefore, we obtain the one-to-one correspondence between $\tilde{e}^{(s)}$ and the linearized higher spin-s field. By linearizing the equations of motion (5.82) for the vielbein and the spin-connection and transforming them to be a second-order linear differential equation of $\phi_{\mu_1 \cdots \mu_s}$, one can show that the resulting differential equation is the Fronsdal's equation on the fixed AdS_3 background [178]. Fronsdal's equation is the equation of motion which describes the free massless higher spin gauge field [188]. Therefore, at the linear level, the $SL(N, R)$ Chern-Simons theory describes the free massless higher spin gauge fields propagating on the AdS_3 background. One can also include the higher order perturbative expansions of the equation of motion in terms of the linearized higher spin fields, then the nonlinear higher order couplings such as $\phi^{(s)}\phi^{(s')}$ and $\phi^{(s)}\phi^{(s)}$ will appear. The above two facts together explain why the $SL(N, R)$ Chern-Simons theory is an interacting theory of massless higher spin gauge fields.

5.2.3 *Asymptotic symmetry*

Now, let me introduce the derivation of the asymptotic symmetry of the higher spin gravity. For simplicity, I will focus on the $SL(3, R)$ case and work in the Euclidean signature. For the general $SL(N,R)$ as well as the $hs[\lambda]$ cases, the procedure will be similar to this simple case. Let me first fix the convention. The basis of $\mathfrak{sl}(3,R)$ I will use later reads,

$$L_{-1}=\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix},\ L_0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\ L_1=\begin{pmatrix}0&0&0\\-\sqrt{2}&0&0\\0&-\sqrt{2}&0\end{pmatrix},$$
$$W_{-1}=\begin{pmatrix}0&\frac{1}{\sqrt{2}}&0\\0&0&-\frac{1}{\sqrt{2}}\\0&0&0\end{pmatrix},\ W_0=\begin{pmatrix}\frac{1}{3}&0&0\\0&-\frac{2}{3}&0\\0&0&\frac{1}{3}\end{pmatrix},\ W_1=\begin{pmatrix}0&0&0\\-\frac{1}{\sqrt{2}}&0&0\\0&\frac{1}{\sqrt{2}}&0\end{pmatrix},$$
$$W_{-2}=\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix},\ W_2=\begin{pmatrix}0&0&0\\0&0&0\\2&0&0\end{pmatrix}, \tag{5.86}$$

which satisfies $L_{-i}=(-1)^iL_i^\dagger$ and $W_{-n}=(-1)^nW_n^\dagger$ with $i=0,\pm1$ and $n=0,\pm1,\pm2$. The commutation relations read[3]

$$\begin{aligned}
&[L_i,L_j]=(i-j)L_{i+j}\ ,\\
&[L_i,W_n]=(2i-n)W_{i+n}\ ,\\
&[W_n,W_m]=-\frac{1}{12}(n-m)(2n^2+2m^2-nm-8)L_{m+n}\ .
\end{aligned}\tag{5.87}$$

The Chern-Simons level in (5.79) is given by

$$k_{cs}=\frac{l}{16G_3}\ . \tag{5.88}$$

In the later discussions, we will set the AdS$_3$ radius $l=1$ for convenience.

Highest weight gauge. Recall that when deriving the asymptotic symmetry algebra of pure AdS$_3$ gravity, we imposed the Brown-Henneaux boundary condition by requiring that the infinitesimal bulk diffeomorphisms preserve the Fefferman-Graham gauge (3.4) and keep the boundary metric $g^{(0)}$ invariant. In particular, when $g^{(0)}$ is flat, the resulting asymptotic symmetry algebra turns out to be two copies of the Virasoro algebra. In the Chern-Simons formalism, there is an analog of the Feffermann-Graham gauge for the Chern-Simons fields, called the *highest weight gauge*. On-shell solutions for Chern-Simons fields can be parametrized in the following way,

$$A_+=b^{-1}a_+b+b^{-1}db\ ,\quad A_-=ba_-b^{-1}+bdb^{-1}\ , \tag{5.89}$$

3 The normalization of W_n here corresponds to choose $\sigma=-\frac{1}{4}$ in [178].

where $b = e^{rL_0}$ is a Hermitian matrix, and the connections $a_\pm$ in the highest weight gauge read [178],

$$
\begin{aligned}
a_+ &= (L_1 + \frac{\mathcal{L}}{4k_{cs}} L_{-1} + \frac{\mathcal{W}}{4k_{cs}} W_{-2}) dw \ , \\
a_- &= (L_{-1} + \frac{\bar{\mathcal{L}}}{4k_{cs}} L_1 - \frac{\bar{\mathcal{W}}}{4k_{cs}} W_2) d\bar{w} \ .
\end{aligned} \tag{5.90}
$$

Here $\mathcal{L}$ and $\mathcal{W}$ are holomorphic functions of w, and the bar represents the complex conjugate, i.e. $\bar{\mathcal{L}} = \mathcal{L}^*$ and $\bar{\mathcal{W}} = \mathcal{W}^*$. It is easy to check $a_- = -a_+^\dagger$, so that $A_- = -A_+^\dagger$ is fulfilled in the Euclidean signature. Now, the metric associated with the Chern-Simons fields above turns out to be asymptotically AdS$_3$, in particular, with a flat boundary metric,

$$
ds^2 = dr^2 + e^{2r} \left| dw + e^{-2r} \frac{\bar{\mathcal{L}}}{4k_{cs}} d\bar{w} \right|^2 + e^{-4r} \left| \frac{\mathcal{W}}{4k_{cs}} \right|^2 dw d\bar{w} \ . \tag{5.91}
$$

Here the boundary is located at $r \to \infty$. Meanwhile, the spin-3 field can be worked out as

$$
\phi^{(3)} \propto \mathcal{W} dw^3 + \bar{\mathcal{W}} d\bar{w}^3 + \mathcal{O}(e^{-2r}) \ , \quad r \to \infty \ . \tag{5.92}
$$

For vanishing spin-3 field (or $\mathcal{W} = 0$), the metric (5.91) recovers the Bañados metric (3.17) under the redefinition $\rho = e^{-2r}$ for the radial coordinate.

Asymptotic symmetry algebra. Since on-shell Chern-Simons fields are flat connections, solutions with different values of $\mathcal{L}$ and $\mathcal{W}$ can always be related by gauge transformations. Therefore, we can obtain the asymptotic symmetry by analyzing the transformation laws of $\mathcal{L}$ and $\mathcal{W}$ under those infinitesimal gauge transformations. More precisely, for a gauge parameter Λ, we require the following Brown-Henneaux-like boundary condition,

$$
\begin{aligned}
&\delta_\Lambda A_r = \delta_\Lambda A_{\bar{w}} = 0 \ , \\
&\delta_\Lambda A_w = b^{-1} \left(\frac{\delta \mathcal{L}}{4kcs} L_{-1} + \frac{\delta \mathcal{W}}{4k_{cs}} W_{-2} \right) b \ ,
\end{aligned} \tag{5.93}
$$

The first condition in (5.93) requires Λ in the form of $\Lambda = b^{-1}\lambda(w) b$. And for the second condition in (5.93), we consider the general expansion of $\lambda(w)$

$$
\lambda(w) = \sum_{i=-1}^{1} \epsilon_i(w) L_i + \sum_{n=-2}^{2} \xi_n(w) W_n \ . \tag{5.94}
$$

where ϵ_i and ξ_n account for $SL(2,R)$ and higher spin transformation respectively. Inserting (5.94) into the second condition in (5.93) yields a set of differential equations for $\epsilon_i(w)$ and $\xi_n(w)$. If we denote $\epsilon_1 = \epsilon$ and $\xi_2 = \xi$, then the resulting differential equations are given by

$$
\epsilon_0 = -\epsilon' \ , \quad \epsilon_1 = \frac{\mathcal{L}\epsilon}{4k_{cs}} - \frac{\mathcal{W}\xi}{2k_{cs}} + \frac{1}{2}\epsilon'' \ , \tag{5.95}
$$

and

$$\begin{aligned}\xi_1 &= -\xi' \ ,\\ \xi_0 &= \frac{\mathcal{L}\xi}{2k_{cs}} + \frac{1}{2}\xi'' \ ,\\ \xi_{-1} &= -\frac{\mathcal{L}'\xi}{6k_{cs}} - \frac{5\mathcal{L}\xi'}{12k_{cs}} - \frac{1}{6}\xi^{(3)} \ ,\\ \xi_{-2} &= \frac{\mathcal{W}\epsilon}{4k_{cs}} + \frac{\mathcal{L}^2\xi}{16k_{cs}^2} + \frac{7\mathcal{L}'\xi'}{48k_{cs}} + \frac{\mathcal{L}''\xi}{24k_{cs}} + \frac{\mathcal{L}\xi''}{6k_{cs}} + \frac{1}{24}\xi^{(4)} \ . \end{aligned} \tag{5.96}$$

Inserting (5.95) and (5.96) back into the second equation in (5.93) yields the transformation laws of $\mathcal{L}$ and $\mathcal{W}$ associated with the parameters ϵ and ξ, given by

$$\begin{aligned}\delta_\epsilon\mathcal{L} &= \mathcal{L}'\epsilon + 2\mathcal{L}\epsilon' + 2k_{cs}\epsilon^{(3)} \ ,\\ \delta_\epsilon\mathcal{W} &= \mathcal{W}'\epsilon + 3\mathcal{W}\epsilon' \ ,\\ \delta_\xi\mathcal{W} &= \frac{(\mathcal{L}^2)'\xi}{3k_{cs}} + \frac{2\mathcal{L}^2\xi'}{3k_{cs}} + \frac{3\mathcal{L}''\xi'}{4}\\ &\qquad + \frac{5\mathcal{L}'\xi''}{4} + \frac{\mathcal{L}^{(3)}\xi}{6} + \frac{5\mathcal{L}\xi^{(3)}}{6} + \frac{k_{cs}\xi^{(5)}}{6} \ , \end{aligned} \tag{5.97}$$

The first equation in (5.97) can be identified as the transformation law of a CFT stress tensor under the conformal transformation, with the central charge

$$c = 24k_{cs} = \frac{3l}{2G_3} \ . \tag{5.98}$$

Hence, we can interpret the highest weight gauge (5.90) as the insertions of fields in the dual CFT, denoted as $X = O_1(w_1)O_2(w_2)\cdots$, and identify $\mathcal{L}$ as the expectation value of the stress tensor, i.e. $\mathcal{L}(w) = \langle T(w)\rangle_X = \langle T(w)X\rangle/\langle X\rangle$. Similarly, the second equation in (5.97) is the transformation law of a conformal primary field, denoted as $W(w)$, under the conformal transformation, with conformal weights $(3,0)$. So we identify $\mathcal{W}$ as the expectation value of $W(w)$ in the dual CFT. The last equation in (5.97) describes the transformation law of $W(w)$ under the spin-3 transformation. Since the three variations in (5.97) are closed, implying that $W(w)$ is a spin-3 conserved current in the dual CFT. The OPEs between the stress tensor and the spin-3 current can be extracted from the Ward identities,

$$\begin{aligned}\delta_\epsilon\mathcal{L} &= -\frac{1}{2\pi}\int dz^2\ \partial^z\epsilon(z)\langle T(z)T(w)\rangle_X \ ,\\ \delta_\epsilon\mathcal{W} &= -\frac{1}{2\pi}\int dz^2\ \partial^z\epsilon(z)\langle T(z)W(w)\rangle_X \ ,\\ \delta_\xi\mathcal{W} &= -\frac{1}{2\pi}\int dz^2\ \partial^z\xi(z)\langle W(z)W(w)\rangle_X \ . \end{aligned} \tag{5.99}$$

By choosing the parameters $\epsilon(z) = \xi(z) = \frac{1}{y-z}$ and inserting them into (5.97) and (5.99), one can work out the OPEs, given by

$$\begin{aligned} T(y)T(w) &\sim \frac{c/2}{(y-w)^4} + \frac{2T(w)}{(y-w)^2} + \frac{\partial T(w)}{y-w} \ , \\ T(y)W(w) &\sim \frac{3W(w)}{(y-w)^2} + \frac{\partial W(w)}{y-w} \ , \end{aligned} \tag{5.100}$$

and[4]

$$\begin{aligned} W(y)W(w) &\sim \frac{5c}{6(y-w)^6} + \frac{5T(w)}{(y-w)^4} + \frac{5\partial T(w)}{2(y-w)^3} \\ &+ \frac{5\beta_{cl}\Lambda_{cl}(w) + \frac{3}{4}\partial^2 T(w)}{(y-w)^2} + \frac{\frac{5}{2}\beta_{cl}\partial\Lambda_{cl}(w) + \frac{1}{6}\partial^3 T(w)}{y-w} \ , \end{aligned} \tag{5.101}$$

where the coefficient β_{cl} and the composite field $\Lambda_{cl}(w)$ are defined as

$$\beta_{cl} = \frac{16}{5c} \ , \quad \Lambda_{cl}(w) = T(w)^2 \ . \tag{5.102}$$

The above OPEs are "classical", in the sense that the associativity (crossing symmetry of current correlators) is only fulfilled in the semi-classical limit $c \to \infty$. The modes expansions of the currents furnish the W_3^{cl} asymptotic symmetry algebra for the $SL(3,R)$ higher spin gravity [178]. To quantize them, one needs to introduce the normal ordering for the composite field and modify the OPEs by $1/c$ corrections [142, 149],

$$\begin{aligned} \beta_{cl} &\to \beta = \frac{16}{22+5c} \ , \\ \Lambda_{cl}(w) &\to \Lambda(w) = (T(w)T(w)) - \frac{3}{10}\partial^2 T(w) \ . \end{aligned} \tag{5.103}$$

By the modes expansions,

$$\begin{aligned} T(w) &= \sum_n L_n w^{-2-n} \ , \ W(w) = \sum_n W_n w^{-3-n} \ , \\ \Lambda(w) &= \sum_n \lambda_n w^{-4-n} \end{aligned} \tag{5.104}$$

the symmetry algebra can be worked out in the standard way by evaluating the residues, given by

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \ , \\ [L_n, W_m] &= (2n-m)W_{n+m} \ , \\ [W_n, W_m] &= \frac{1}{12}(n-m)(2n^2 + 2m^2 - mn - 8)L_{n+m} \\ &+ \frac{5\beta}{2}(n-m)\lambda_{n+m} + \frac{5c}{6}\frac{1}{5!}n(n^2-1)(n^2-4)\delta_{n+m,0} \ , \end{aligned} \tag{5.105}$$

4 By the rescaling $W \to \sqrt{5/2}W$, the OPE becomes $WW \sim \frac{c}{3}\mathbb{1} + 2T + \cdots$, which matches with (5.18).

with

$$\lambda_n = \sum_m : L_{n-m} L_m : -\frac{3}{10}(n+3)(n+2)L_n \ . \tag{5.106}$$

This is the quantum W_3 algebra, i.e. the symmetry algebra in the dual CFT of the $SL(3,R)$ higher spin gravity.

Some comments. It is perhaps worth emphasizing that the dual CFT with W_3 symmetry will not be the coset model (5.5) any longer, since by choosing $N=3$, the central charge $c_{N,k}$ in (5.7) does not match with the Brown-Henneaux central charge in (5.98). To my knowledge, it is still unclear what is the precise W_3 CFT dual to the $SL(3,R)$ higher spin gravity. Nevertheless, there is still something universal but interesting that we can learn from this bottom-up model. For instance, the higher spin black hole originally discovered in [33], the higher spin entanglement entropy [135], and the higher spin wormhole from the modular bootstrap [189]. On the other hand, there are also some unsolved problems, towards a deeper understanding of the higher spin gravity. For example, like the pure AdS_3 gravity, it is expected that the effective action of the higher spin gravity might be the Toda action, of which, the Liouville action as the effective action of the AdS_3 gravity is a special case. This statement has not been proven yet, mainly due to the issue of the holographic renormalization in the Chern-Simons formalism. However, some evidence has been shown in [190], in which, the authors used the classical Toda theory to reproduce the results for the holographic Rényi entropy in the higher spin gravity. It will be interesting to work out the effective action of higher spin gravity explicitly in future work. In addition, the higher spin gravity is topological, thus one can study the quotient of higher spin solutions in a similar way as in the pure AdS_3 case. Detailed understandings on what are the analogs of the Schottky uniformization and the projective structure in the higher spin case, and how the higher spin transformations are related to the W_3 decoupling equation [190] in the dual CFT will also be interesting topics for the future works.

5.2.4 *Higher spin black hole*

In this subsection, I plan to introduce the higher spin black hole [33] and its partition function [191], which are relevant for the holographic calculation of the higher spin charged moments discussed in our work [31]. For simplicity, I will still focus on the $SL(3,R)$ case and work in the Euclidean signature.

What are the higher spin black holes? In a general diffeomorphism invariant gravity theory, a black hole is usually defined via the existence of the event horizon. The near horizon geometry of a black hole is always the same as a Rindle spacetime. And transforming to the

Euclidean signature, the near horizon geometry is required to satisfy the smoothness condition

$$ds^2 \approx d\tilde{r}^2 + \left(\frac{2\pi}{\beta}\right)^2 \tilde{r}^2 dt_E^2 + \cdots \implies t_E \in [0, \beta) \ . \tag{5.107}$$

where $\tilde{r} = 0$ denotes the location of the horizon and the Euclidean time period gives rise to the temperature of the black hole $T = 1/\beta$. However, in the higher spin gravity, for instance, the $SL(3,R)$ case, since the diffeomorphism only accounts for the gauge transformation in the $SL(2,R)$ subsector, the traditional definition of the black hole via the horizon will not manifest the full $SL(3,R)$ gauge invariance. In other words, one can always perform a $SL(3,R)$ gauge transformation to change the geometry and remove the horizon. Therefore, an alternative definition of the black hole with the manifestation of the full gauge invariance is required in the higher spin gravity theory.

Revisit on the $SL(2,R)$ case. To solve this problem, we first go back to the pure AdS$_3$ gravity or $SL(2,R)$ Chern-Simons theory. The AdS gravity in three-dimensional is more special than its higher dimensional analogs due to its topological nature, and it allows us to define a black hole via the quotient of the Poincaré AdS$_3$, as discussed in Section 3.1. In particular, from the quotient perspective, we do not pay attention to the event horizon but rather focus on the contractibility of the Euclidean time circle in the bulk. This contractibility requirement is equivalent to the smoothness condition of the near horizon geometry[5]. Now, how is the projective structure related to the $SL(2,R)$ Chern-Simons fields? To see the connection between them, we recall the general black hole solution (3.79) in pure AdS$_3$ gravity with the boundary period $w \cong w + 2\pi \cong w + 2\pi\tau$. This solution (3.79) can be realized by the $SL(2,R)$ Chern-Simons fields in the form of (5.89) with

$$a_+ = (L_1 + \frac{\mathcal{L}}{k_{cs}} L_{-1}) dw \ , \quad a_- = -(a_+)^\dagger \ , \quad \mathcal{L} = \frac{k_{cs}}{4\tau^2} \ . \tag{5.108}$$

Here, the $SL(2,R)$ generators reads

$$L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \ , \ L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \ , \tag{5.109}$$

and $k_{cs} = \frac{l}{4G_3} = \frac{c}{6}$. The constant $\mathcal{L}$ is the stress tensor of the black hole. The mapping from the black hole to the Poincaré AdS$_3$ is characterized by the boundary diffeomorphism $z(w) = e^{iw/\tau}$ (see Section 3.1.2), which is the solution to the following differential equation,

$$\mathcal{L} = \frac{c}{12}\{z, w\} \ . \tag{5.110}$$

5 Each Euclidean time slice of the black hole can be thought of as a hemisphere anchored at the boundary of the Poincaré AdS$_3$. The center of the hemisphere is the location of the black hole horizon, which is of course smooth by construction.

The information about the Chern-Simons connection is hidden in the above differential equation. To show it explicitly, we rewrite (5.110) as the Hill's equation (see (3.177))

$$\varphi'' + \frac{6}{c}\mathcal{L}\varphi = 0\ , \quad z(w) = \frac{\varphi_1(w)}{\varphi_2(w)}\ . \tag{5.111}$$

By defining a vector $\psi^{\mathrm{T}} = (\varphi', \varphi)$, we can further express Hill's equation as a matrix-valued differential equation,

$$\partial \begin{pmatrix} \varphi' \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & -\frac{6}{c}\mathcal{L} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi' \\ \varphi \end{pmatrix} \implies \partial\psi = (a_+)_w \psi\ . \tag{5.112}$$

where $(a_+)_w$ is just the w-component of the connection a_+. The solution for the vector ψ can be formally written as

$$\psi = \mathcal{P} e^{\int^w a_+} \psi_0\ . \tag{5.113}$$

Hence, the black hole is now characterized by the holonomies of the Chern-Simons connection around the two different loops,

$$\mathrm{Hol}[A_+, \alpha] \cong \mathcal{P} e^{\oint_\alpha a_+}\ , \quad \mathrm{Hol}[A_+, \beta] \cong \mathcal{P} e^{\oint_\beta a_+}\ . \tag{5.114}$$

Here "$\cong$" denotes the identification up to a $SL(2, R)$ conjugation, and α and β are the contractible and non-contractible loops, going around the time and spatial directions on the boundary, respectively. Those holonomies are the topological invariant quantities for the given Chern-Simon fields. Since $(a_+)_w$ is a constant here, the path ordering becomes unimportant for evaluating the holonomies. By diagonalizing $(a_+)_w$, with the eigenvalues $\lambda = \pm i/2\tau$, we obtain

$$\oint_\alpha a_+ = 2\pi\tau (a_+)_w \cong \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix}\ , \tag{5.115}$$

and similarly

$$\oint_\beta a_+ = 2\pi (a_+)_w \cong \begin{pmatrix} \frac{i\pi}{\tau} & 0 \\ 0 & -\frac{i\pi}{\tau} \end{pmatrix}\ . \tag{5.116}$$

Therefore, the holonomies are given by

$$\mathrm{Hol}[A_+, \alpha] \cong e^{2\pi i L_0} = -\mathbb{1}\ , \quad \mathrm{Hol}[A_+, \beta] \cong p\ , \tag{5.117}$$

where p is a loxodromic $\mathrm{PSL}(2, \mathbb{C})$ element. In terms of the $SL(2, R)$ action on the z coordinates, $-\mathbb{1}$ is regarded as the identity map on z,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d} \implies (-\mathbb{1}) \circ z = z\ . \tag{5.118}$$

Therefore, we conclude that the two holonomies of the Chern-Simons connection (5.108) are associated with the projective structure of the black hole. In particular, notice that $-\mathbb{1}$ belongs to the center $\{\pm\mathbb{1}\}$ of $SL(2,R)$, so it is invariant under arbitrary $SL(2,R)$ conjugation. Hence, we identify $\mathrm{Hol}[A_+,\alpha] = -\mathbb{1}$ in (5.117) as the gauge invariant notion of the black hole in the $SL(2,R)$ Chern-Simons theory. On the other hand, the holonomy around the non-contractible loop β specifies the modular parameter and the stress tensor of the black hole. The stress tensor can be evaluated from the trace,

$$\mathcal{L} = -\frac{k_{cs}}{2}\mathrm{Tr}[(a_+)_w^2] \ , \tag{5.119}$$

which is also a topological invariant quantity of the connection.

$SL(3,R)$ **higher spin black hole.** After understanding the gauge invariant notion of the pure AdS_3 black hole, we can straightforwardly generalize it to the higher spin case. For $SL(3,R)$ higher spin gravity, we define a black hole by requiring its holonomy around the contractible loop α to be

$$\mathrm{Hol}[A_+,\alpha] \cong e^{2\pi i L_0} = \mathbb{1}. \tag{5.120}$$

where $\mathbb{1}$ is the unique center element of $SL(3,R)$[6]. The explicit construction of the $SL(3,R)$ higher spin black hole via the condition (5.120) was first made in [191]. In the following, I would like to adopt its alternative form discussed in [193]. The idea is that for a general higher spin black hole, besides the modular parameter coupled to the stress tensor, there should also be a spin-3 chemical potential coupled to the spin-3 current. The modular parameter can be shown explicitly in the Chern-Simons connection by defining a new boundary coordinates system,

$$w = \sigma_1 + \tau\sigma_2 \ , \quad \sigma_1 \cong \sigma_1 + 2\pi \ , \quad \sigma_2 \cong \sigma_2 + 2\pi \ . \tag{5.121}$$

in terms of which, the connection a_+ of a pure AdS_3 black hole in the $SL(3,R)$ gravity reads

$$(a_+)_{\sigma_1} = L_1 + \frac{\mathcal{L}}{4k_{cs}}L_{-1} \ , \quad (a_+)_{\sigma_2} = \tau(a_+)_{\sigma_1} \ . \tag{5.122}$$

When adding the spin-3 current to the component $(a_+)_{\sigma_1}$, i.e.

$$(a_+)_{\sigma_1} \to L_1 + \frac{\mathcal{L}}{4k_{cs}}L_{-1} + \frac{\mathcal{W}}{4k_{cs}}W_{-2} \ , \tag{5.123}$$

one needs to modify $(a_+)_{\sigma_2}$ by the additional spin-3 chemical potential term coupled with $\mathcal{W}$. Since both $\mathcal{L}$ and $\mathcal{W}$ are constant in the black hole case, the equation of motion for the Chern-Simons field becomes

6 In general $SL(N,R)$ case, the condition reads $\mathrm{Hol}[A_+,\alpha] \cong e^{2\pi i L_0} = (-1)^{N-1}\mathbb{1}$ [192].

$[(a_+)_{\sigma_1}, (a_+)_{\sigma_2}] = 0$. Generic solutions for $(a_+)_{\sigma_2}$ can be written in the following compact form,

$$(a_+)_{\sigma_2} = \tau(a_+)_{\sigma_1} + \alpha\left[(a_+)^2_{\sigma_1} - \frac{1}{3}\mathrm{Tr}[(a_+)^2_{\sigma_1}]\mathbb{1}\right] , \quad (5.124)$$

which by construction is a traceless matrix and commute with $(a_+)_{\sigma_1}$. The free parameter α can be understood as the spin-3 chemical potential. The coupling between α and $\mathcal{W}$ can be seen from the variation of the Euclidean Chern-Simons action, which gives rise to the boundary term,

$$\begin{aligned}\delta I_{cs}[A_+] &= -\frac{ik_{cs}}{4\pi}\int d\sigma_1 d\sigma_2 \mathrm{Tr}\left[a_1\delta a_2 - a_2\delta a_1\right] \\ &= 2\pi i\mathcal{L}\delta\tau - 3\pi i\mathcal{W}\delta\alpha - i\pi\alpha\delta\mathcal{W} ,\end{aligned} \quad (5.125)$$

where I denote $a_1 = (a_+)_{\sigma_1}$ and $a_2 = (a_+)_{\sigma_2}$ for short. How to fix $\mathcal{L}$ and $\mathcal{W}$ as functions of τ and α? Recall the holonomy condition (5.120), which in the new coordinates reads

$$\mathrm{Hol}[A,\alpha] \cong e^{2\pi a_2} \cong e^{2\pi i L_0} . \quad (5.126)$$

Therefore, we need to require the eigenvalues of $2\pi a_2$ to be the same as the ones of $2\pi i L_0$. Since the eigenvalues of a 3×3 traceless matrix are characterized by its determinant and the square trace, we obtain two equivalent conditions of (5.126), given by

$$\mathrm{Det}[a_2] = \mathrm{Det}[iL_0] = 0 , \quad \mathrm{Tr}[(a_2)^2] = \mathrm{Tr}[(iL_0)^2] = -2 . \quad (5.127)$$

Inserting (5.124) into (5.127) yields two algebraic equations,

$$\begin{aligned}0 &= 2\mathcal{L}^3\alpha^3 - 27k_{cs}\mathcal{L}\mathcal{W}\alpha^2\tau + 18k_{cs}\mathcal{L}^2\alpha\tau^2 + 27k_{cs}\mathcal{W}^2\alpha^3 + 27k_{cs}^2\mathcal{W}\tau^3 , \\ 0 &= \mathcal{L}^2\alpha^2 + 3k_{cs}^2 + 9k_{cs}\mathcal{W}\alpha\tau - 3k_{cs}\mathcal{L}\tau^2 .\end{aligned} \quad (5.128)$$

Those two algebraic equations (5.128) can be regarded as the state equations characterizing this thermodynamical system. To solve them, let us first make some simplifications. Recall that the length dimensions of $(\tau, \mathcal{L}, \mathcal{W})$ are given by $(1, -2, -3)$. By the dimensional analysis on (5.125), we conclude that α has length dimension two. The Chern-Simons level is dimensionless since it is related to the central charge by $k_{cs} = c/24$ in the $SL(3,R)$ case. In addition, both the stress tensor and spin-3 current of the black hole should scale in c. So now we can define the following three dimensionless quantities,

$$L = \frac{\mathcal{L}\tau^2}{k_{cs}} , \quad W = \frac{\mathcal{W}\tau^3}{k_{cs}} , \quad \gamma = \frac{\alpha}{\tau^2} . \quad (5.129)$$

In terms of (L, W, γ), the algebraic equations (5.128) are expressed as

$$\begin{aligned}0 &= W + W^2\gamma^3 + \frac{2}{27}L^2\gamma(9 + L\gamma^2) - LW\gamma^2 , \\ 0 &= 3 + 9W\gamma + L^2\gamma^2 - 3L .\end{aligned} \quad (5.130)$$

There are four different branches of solutions to (5.130), which has been discussed in detail in [194]. Here I focus on the so-called *BTZ branch*, in which, the vanishing of γ leads to $L = 1$ and $W = 0$. In other words, we recover the pure AdS$_3$ black hole with stress tensor $\mathcal{L} = \frac{c}{24\tau^2} = \frac{k_{cs}}{\tau^2}$ when the spin-3 chemical potential vanishes, i.e. $\alpha = 0$. The expressions of L and W as functions of γ are rather complicated, so here I would like to provide their perturbative expansions in the region $\gamma \ll 1$. In the BTZ branch, they are given by

$$L = 1 - \frac{5}{3}\gamma^2 + \frac{10}{3}\gamma^4 - \frac{221}{27}\gamma^6 + \mathcal{O}(\gamma^8) \ ,$$
$$W = -\frac{2}{3}\gamma + \frac{40}{27}\gamma^3 - \frac{34}{9}\gamma^5 + \mathcal{O}(\gamma^7) \ . \tag{5.131}$$

To this end, we have fully determined the Chern-Simons connections describing the higher spin black hole. And in the following, I would like to briefly review the entropy formula and the partition function of the higher spin black hole.

Black hole entropy and partition function. From the dual CFT perspective, the higher spin black hole should correspond to the grand canonical ensemble with the partition function,

$$Z[\tau, \bar{\tau}, \alpha, \bar{\alpha}] = (q\bar{q})^{-c/24}\mathrm{Tr}[q^{L_0}\bar{q}^{\bar{L}_0}y^{W_0}\bar{y}^{\bar{W}_0}] \tag{5.132}$$

Here the parameter q and y are defined as $q = e^{2\pi i\tau}$ and $y = e^{2\pi i\alpha}$. The constant stress tensor $\mathcal{L}$ and the higher spin current $\mathcal{W}$ are identified with the expectation values of the zero modes L_0 and W_0 as[7],

$$L_0 - \frac{c}{24} := -\oint_\beta \frac{dw}{2\pi} T(w) = -\mathcal{L} \ ,$$
$$W_0 := \oint_\beta \frac{dw}{2\pi} W(w) = \mathcal{W} \ . \tag{5.133}$$

Under the saddle point approximation, the partition function reads

$$\log Z[\tau, \bar{\tau}, \alpha, \bar{\alpha}] = S - 2\pi i\tau\mathcal{L} - \overline{2\pi i\tau\mathcal{L}} + 2\pi i\alpha\mathcal{W} + \overline{2\pi i\alpha\mathcal{W}} \ . \tag{5.134}$$

Here S is the entropy of the higher spin black hole, which as a function of $(\mathcal{L}, \bar{\mathcal{L}}, \mathcal{W}, \bar{\mathcal{W}})$ should satisfy the first law,

$$\delta S = 2\pi i\tau\delta\mathcal{L} + \overline{2\pi i\tau\delta\mathcal{L}} - 2\pi i\alpha\delta\mathcal{W} - \overline{2\pi i\alpha\delta\mathcal{W}} \ . \tag{5.135}$$

It is perhaps worth emphasizing that there exist different ways to compute the entropy of higher spin black holes, and all of them give the same result. In the original work [191], the entropy was derived by first expressing τ and α as functions of the charges $\mathcal{L}$ and $\mathcal{W}$ and then integrating the first law (5.135). For a Hamiltonian derivation of the

7 The modes expansions on torus here match with the convention in [178].

entropy, see, e.g., [193, 195, 196]. The entropy can also be understood as the on-shell value of the appropriate action functional in a micro-canonical ensemble, where the charges $\mathcal{L}$ and $\mathcal{W}$ are held fixed [197]. In [198], the entropy was derived from the symplectic two-form of the Chern-Simons theory by using the *Wald formalism* [52]. The punchline is that the entropy of a higher spin black hole is given by

$$S = -2\pi i k_{cs}\mathrm{Tr}[a_1 a_2] + 2\pi i k_{cs}\mathrm{Tr}[\bar{a}_1\bar{a}_2] \ . \tag{5.136}$$

where $\bar{a}_i = -(a_i)^\dagger$ with $i = 1, 2$. Inserting (5.123) and (5.124) into the expression (5.136) yields,

$$S = 4\pi i\tau\mathcal{L} - 6\pi i\alpha\mathcal{W} + \overline{4\pi i\tau\mathcal{L}} - \overline{6\pi i\alpha\mathcal{W}} \ , \tag{5.137}$$

which is a real number. The saddle point approximation of the partition function now reads

$$\begin{aligned}\log Z[\tau,\bar{\tau},\alpha,\bar{\alpha}] &= S - 2\pi i\tau\mathcal{L} + 2\pi i\alpha\mathcal{W} - \overline{2\pi i\tau\mathcal{L}} + \overline{2\pi i\alpha\mathcal{W}} \\ &= 2\pi i\tau\mathcal{L} - 4\pi i\alpha\mathcal{W} + \text{complex conjugate} \\ &= \frac{ic\pi}{12\tau}\left(1 - \frac{1}{3}\gamma^2 + \frac{10}{27}\gamma^4 + \cdots\right) + c.c. \ .\end{aligned} \tag{5.138}$$

where solutions for L and W in (5.131) have been inserted. Remarkably, the above perturbative expansion (5.138) of the partition function was also obtained from the CFT calculation performed in [199], where the author only used the correlation function of spin-3 currents on the torus. This implies that the partition function of the higher spin black hole in (5.138) is universal, or in other words, it is determined by the W_3 symmetry. On the other hand, this also implies that the holonomy condition (5.120) on the gravity side is closely related to the W_3 symmetry on the CFT side.

5.3 HIGHER SPIN CHARGED MOMENTS

In this subsection, I will discuss the charged moments in $SL(3,\mathbb{R})$ higher spin gravity, as well as in the dual CFT with W_3 symmetry. The discussion is based on our work in [31]. Focusing on the simplest case, i.e., the vacuum background and a single entangling interval $\mathcal{A}$, I will first show that the W_3 algebra of the dual CFT induces an entanglement W_3 algebra acting on the quantum state in the entangling interval. The algebra contains a spin-3 modular charge $Q_\mathcal{A}$ which commutes with the modular Hamiltonian $H_\mathcal{A}$, so the quantum numbers of the states in the subsystem $\mathcal{A}$ are characterized by the eigenvalues of those two modular charges. Hence, this provides us with a reasonable notion of the higher spin charged moments,

$$Z_n[\mu] = \mathrm{Tr}[\rho_\mathcal{A}^n e^{2\pi i\mu Q_\mathcal{A}}] \ , \quad \mu \in \mathbb{R} \ , \tag{5.139}$$

which is potentially useful for the study of SREE concerning the W_3 symmetry[8]. The holographic calculation of the charged moments can be done in a rather straightforward way, by applying the topological black hole method [32]. On the CFT side, the charged moments is calculated perturbatively in the spin 3 chemical potential μ. By evaluating the corresponding connected correlation functions of the spin-3 modular charge operator up to quartic order in the chemical potential, I will show that the result exactly matches the holographic result. Since the higher spin charged moments is not Gaussian in the chemical potential any longer, as readily seen from (5.138), one expects that the dual W_3 CFT must feature the breakdown of equipartition of entanglement to the leading order in the large c expansion. On the other hand, compared with the discussion on the $U(1)$ charged moments, we can naturally ask whether a Wilson line approach and the charged twist fields description of the higher spin charged moments are also available. This is still an open question and I will leave it for future work.

5.3.1 *Spin-3 modular charge*

The subregion charge operator in the study of the SREE and the charged moments in two-dimensional CFTs is usually defined via the integration of the current over the subregion [27, 32], i.e. $Q_{\mathcal{A}} \sim \int_{\mathcal{A}} dx J^0$. However, this notion of the subregion charge operator only makes sense for the vector-like current, i.e., the current transforming as a vector under the conformal transformation. The simplest counter-example of this notion is the modular Hamiltonian $H_{\mathcal{A}}$ (4.48) in the vacuum background with the single interval $\mathcal{A}$, which can be regarded as the subregion charge operator associated with the stress tensor. As first pointed out in [200], it is possible to construct a full Virasoro algebra on a single entangling interval $\mathcal{A}$, called *entanglement Virasoro algebra*. In particular, its Virasoro zero mode L_0 is proportional to the modular Hamiltonian (4.48) up to an additive constant [124] (see equation (4.50)). The basic idea in [200] is that by choosing cut-off circles around the endpoints of the entangling interval, the complex plane becomes conformally equivalent to a semi-annulus embedded on the up-half plane. A conformal field theory defined on the up-half plane is known as the boundary conformal field theory (BCFT), in which only a single copy of Virasoro algebra can be constructed under appropriate conformal boundary conditions [125]. Using the Virasoro modes defined in the BCFT, one can derive their expressions in the original complex plane, which gives rise to the entanglement Virasoro algebra on the interval $\mathcal{A}$. To study the higher spin charged moments, in [31], we implemented the above method in the dual holographic CFT with W_3 symmetry. By

8 Unlike the $U(1)$ case, there is no reason to assume the spectrum of the higher spin charge to be either integer or continuous. Therefore, a naive Fourier transformation for the charged moments will not give rise to the SREE in the higher spin case.

modifying the conformal boundary conditions, we constructed a single copy of W_3 algebra on the interval $\mathcal{A}$, called the entanglement W_3 algebra. The details are reviewed in Appendix A. The punchline is that we can define a spin-3 analog of the modular Hamiltonian (4.48), called the spin-3 modular charge operator [31],

$$Q_{\mathcal{A}} = \int_{\mathcal{A}} \frac{dw}{2\pi i} \frac{W(w)}{u'(w)^2} - \int_{\mathcal{A}} \frac{d\bar{w}}{2\pi i} \frac{(\Omega \bar{W})(\bar{w})}{\bar{u}'(\bar{w})^2} \ . \tag{5.140}$$

with

$$u(w) = \log\left(\frac{w - w_1}{w_2 - w}\right) \ . \tag{5.141}$$

Here Ω is the automorphism of the W_3 algebra, and two choices for Ω are given by $\Omega = \pm\mathbb{1}$ when it acts on the spin-3 current. In particular, the spin-3 modular charge defined above is related to the zero mode of the W_3 entanglement algebra on $\mathcal{A}$ as

$$Q_{\mathcal{A}} = \left(\frac{\pi}{\Delta L}\right)^2 W_0 \ , \tag{5.142}$$

where $\Delta L = 2\log|\frac{w_1 - w_2}{\epsilon}|$ and ϵ is the radius of the cut-off circles around the endpoints $w_{1,2}$ of the single interval $\mathcal{A}$. The commutation relation $[L_0, W_0] = 0$ implies that $Q_{\mathcal{A}}$ automatically commutes with the modular Hamiltonian $H_{\mathcal{A}}$. This convinces us that $Q_{\mathcal{A}}$ is indeed a conserved charge in the subsystem $\mathcal{A}$. Therefore, the higher spin charged moments defined in (5.139) is nothing but the grand canonical thermal partition for the subsystem. Using (4.50) and (5.142), one can rewrite it as[9]

$$Z_n[\mu] = e^{-cn\Delta L/12} q^{-c/24} \mathrm{Tr}[q^{L_0} y^{W_0}] \ , \tag{5.143}$$

with

$$q = e^{2\pi i\tau} \ , \quad y = e^{2\pi i\alpha} \ , \quad \tau = \frac{in\pi}{\Delta L} \ , \quad \alpha = \left(\frac{\pi}{\Delta L}\right)^2 \mu \ . \tag{5.144}$$

Notice that while the temperature $T = 1/\beta = 1/2\pi\mathrm{Im}(\tau)$ tends to infinity as the cut-off $\epsilon \to 0$, the ratio $\gamma = \alpha/\tau^2 = -\mu/n^2$ maintains finite. From the holographic perspective, the dual of the higher spin charged moments is given by a higher temperature topological higher spin black hole. Using the holomorphic sector of (5.138), it is then straightforward to obtain the charged moments as

$$\begin{aligned} \log Z_n[\mu] &= -\frac{cn\Delta L}{12} + \frac{ic\pi}{12\tau}\left(1 - \frac{1}{3}\gamma^2 + \frac{10}{27}\gamma^4 + \cdots\right) \\ &= \frac{c\log|\frac{w_1 - w_2}{\epsilon}|}{6n}\left(1 - n^2 - \frac{1}{3}\frac{\mu^2}{n^4} + \frac{10}{27}\frac{\mu^4}{n^8} + \cdots\right) \ . \end{aligned} \tag{5.145}$$

9 Strictly speaking, the trace here is taken over the states which fulfill the boundary conditions in the BCFT, so it is distinct with the trace in the torus partition function (5.132). Nevertheless, in the higher temperature limit $\tau \to 0$, their difference becomes unimportant since the higher temperature physics is not sensible to the topology of the system.

To confirm this holographic result, in the following, I would like to present an independent CFT calculation for the higher spin charged moments, based on the perturbative method introduced in [31].

5.3.2 *CFT calculation*

Unlike the $U(1)$ case, it is not obvious whether the higher spin charged moments allows a charged twist field description. So, for the calculation of it in the dual CFT, we have to work in the replica picture. In [31], we computed the higher spin charged moments via a direct perturbative method, which however only works in the region with small chemical potential μ. In the following, I would like to introduce this method in details. Let us first consider the perturbative expansion of $\log Z_n[\mu]$ in the chemical potential μ,

$$\begin{aligned}\log Z_n[\mu] &= \log Z_n[0] + \frac{Z_n'[0]}{Z_n[0]}\mu + \frac{Z_n[0]Z_n''[0] - Z_n'[0]^2}{2Z_n[0]^2}\mu^2 \\ &+ \frac{Z_n[0]^2 Z_n^{(3)}[0] - 3Z_n[0]Z_n'[0]Z_n''[0] + 2Z_n'[0]^3}{6Z_n[0]^3}\mu^3 + \cdots\ , \end{aligned} \tag{5.146}$$

where the prime represents the derivative with respect to μ. The general term $Z_n^{(m)}[0]/Z_n[0]$ involved in (5.146) can be understood as the expectation value $(2\pi i Q_{\mathcal{A}})^m$ on the n-replica surface, i.e.,

$$\langle (2\pi i Q_{\mathcal{A}})^m \rangle_n = \frac{\mathrm{Tr}[\rho_{\mathcal{A}}^n (2\pi i Q_{\mathcal{A}})^m]}{\mathrm{Tr}[\rho_{\mathcal{A}}^n]} = \frac{Z_n^{(m)}[0]}{Z_n[0]}\ . \tag{5.147}$$

The OPE structure between W currents, i.e., $WW \sim \frac{5c}{6}\mathbb{1} + 5T + \cdots$ allow us to construct $\langle Q_{\mathcal{A}}^m \rangle_n$ from the sum of the products of its connected pieces. In terms of the recursion relation, we can write it as

$$\langle Q_{\mathcal{A}}^m \rangle_n = \sum_{r=0}^{m} C_m^r \langle Q_{\mathcal{A}}^{m-r} \rangle_n \langle Q_{\mathcal{A}}^r \rangle_{n,c}\ , \tag{5.148}$$

where the contractions among W and $\bar{W}$ currents in $\langle Q_{\mathcal{A}}^r \rangle_{n,c}$ are all connected, or more precisely, no identity exchange states exist in its OPE channel. Now, by inserting (5.147) and (5.148) into (5.146), one can verify that the charged moments can be expressed via the connected correlations as

$$\log Z_n[\mu] = \log Z_n[0] + \sum_{m=1}^{\infty} \frac{\mu^m}{m!} \langle (2\pi i Q_{\mathcal{A}})^m \rangle_{n,c}\ . \tag{5.149}$$

Therefore, the higher spin charged moments can be obtained perturbatively by evaluating the connected correlation functions of the spin-3 modular charges in the n-replica surface. In actual calculations, it is convenient to transform the system to the flat complex plane by the uniformization map, i.e. $z = \left(\frac{w-w_1}{w_2-w}\right)^{1/n}$. Using the tensor transformation law of the spin-3 current, one find that the spin-3 modular charge (5.140) in the z-coordinates reads

$$Q_{\mathcal{A}} = \frac{1}{n^2}\int_{\delta}^{\frac{1}{\delta}} \frac{dz}{2\pi i} z^2 W(z) - \frac{1}{n^2}\int_{\bar{\delta}}^{\frac{1}{\bar{\delta}}} \frac{d\bar{z}}{2\pi i}\bar{z}^2(\Omega\bar{W})(z)\ , \tag{5.150}$$

Here the cut-offs δ and $1/\delta$ have been introduced around the endpoints of $\mathcal{A}$ at $z = 0$ and $z = \infty$ to regularize the integrals. Those cut-offs are related to the radius ϵ of the cut-off circles on the original replica surface through the uniformization map, given by

$$\delta = \left(\frac{w_1+\epsilon-w_1}{w_2-w_1-\epsilon}\right)^{\frac{1}{n}} \approx \left(\frac{\epsilon}{w_2-w_1}\right)^{\frac{1}{n}}\ . \tag{5.151}$$

Perturbative result. To evaluate the connected correlation functions of $Q_{\mathcal{A}}$ in the z coordinates, we first recall that all contractions in $\langle Q^m_{\mathcal{A}}\rangle_{n,c}$ are required to be connected. Therefore, the vanishing $W\bar{W}$ OPE implies that $\langle Q^m_{\mathcal{A}}\rangle_{n,c}$ comes from the sum of the two chiral sectors. The WW OPE in (5.101) allows us to determine the recursion relation for the connected correlation functions of the spin-3 currents. In the semi-classical limit $c \to \infty$, we neglect the nonlinear term Λ and its derivatives in (5.101), since those terms contribute to the $1/c$ corrections to the recursion relation. It is then straightforward to show that the connected m-point function of spin-3 currents is determined by the $(m-2)$-point function as

$$\begin{aligned}
&\langle W(z)W(z_1)\cdots W(z_{m-1})\rangle_c \\
&= \sum_{i=1}^{m-1} F(z,z_i;\partial_{z_i})\langle W(z_1)\cdots T(z_i)\cdots W(z_{m-1})\rangle_c \\
&= \sum_{i=1}^{m-1}\sum_{j\neq i} F(z,z_i;\partial_{z_i})G(z_i,z_j;\partial_{z_j}) \\
&\qquad\qquad \times \langle W(z_1)\cdots W(z_{i-1})W(z_{i+1})\cdots W(z_{m-1})\rangle_c\ ,
\end{aligned} \tag{5.152}$$

with

$$\begin{aligned}
F(z,z_i;\partial_{z_i}) &= \frac{5}{(z-z_i)^4} + \frac{5\partial_{z_i}}{2(z-z_i)^3} + \frac{3\partial^2_{z_i}}{4(z-z_i)^2} + \frac{\partial^3_{z_i}}{6(z-z_i)}\ , \\
G(z_i,z_j;\partial_{z_j}) &= \frac{3}{(z_i-z_j)^2} + \frac{\partial_{z_j}}{z_i-z_j}\ .
\end{aligned} \tag{5.153}$$

In the first step of (5.152), contractions are taken over the W currents, and in particular, the most singular term in the WW OPE, corresponding to the identity field, is dropped out in each contraction. In the second step of (5.152), contractions between the stress tensor and the remaining spin-3 currents are considered. An important consequence of the recursion relation (5.152) is that all the odd-order connected correlation functions of W currents vanish, due to the vanishing of the one-point function in the z-complex plane. Therefore, we conclude that only the even orders of $Q_{\mathcal{A}}$ contribute to the charged moments,

$$\log Z_n[\mu] = \log Z_n[0] + \sum_{m=1}^{\infty} \frac{\mu^{2m}}{(2m)!} \langle (2\pi i Q_{\mathcal{A}})^{2m} \rangle_{n,c} \ . \tag{5.154}$$

It is ready to see that the expansion in (5.154) has the same structure with the holographic result (5.145), so the next step is to check them order by order. The complexity of the correlation function grows rapidly as the increasing order, so in [31], we only computed the higher spin charged moments up to the quartic order of the chemical potential μ. The zeroth order term is the vacuum partition function on the replica surface, given by the μ-independent part of (5.145). For the quadratic order, using (5.150), we obtain

$$\begin{aligned} \langle (2\pi i Q_{\mathcal{A}})^2 \rangle_{n,c} &= \frac{1}{n^4} \int_{\delta}^{\frac{1}{\delta}} dz_1 \int_0^{\infty} dz_2 \frac{5c z_1^2 z_2^2}{6(z_1 - z_2)^2} + c.c. \\ &\approx -\frac{c}{9n^4} \log \left| \frac{1}{\delta} \right| = -\frac{c}{9n^5} \log \left| \frac{w_2 - w_1}{\epsilon} \right| \ . \end{aligned} \tag{5.155}$$

As a remark, notice that the above integral (5.155) is singular when the two points collide, i.e. $z_1 = z_2$, so in the actual calculation, we need to choose the two integral paths in (5.155) to be two non-intersecting curves anchored at $z = 0$ and $z = \infty$ in the complex plane. This choice does not contradict the definition of the spin-3 modular charge, since (5.140) and (5.150) are path-independent. For the quartic order, we can first use the recursion relation (5.152) to obtain the four-point function of the spin-3 currents, and then perform similar integrals. The result was showed in [31], given by

$$\langle (2\pi i Q_{\mathcal{A}})^4 \rangle_{n,c} \approx \frac{40c}{27n^9} \log \left| \frac{w_2 - w_1}{\epsilon} \right| \ . \tag{5.156}$$

By inserting (5.155) and (5.156) into (5.154), we recover the expansion of the higher spin charged moments in the holographic result (5.145). It is interesting to compare the above CFT calculations with the holographic calculations. In the holographic calculations, we implemented the holonomy condition for the higher spin black hole, while in the CFT calculations, we only use the W_3 symmetry to determine the charged moments. This indicates that there is a deeper relation between the holonomy of the bulk $SL(3,R)$ Chern-Simons fields with the W_3 symmetry in the boundary CFT. Investigating such a relation in future works may help us to understand the hidden mechanisms of the higher spin holography, and hopefully to generalize it to the higher dimensional holographic models. Finally, since the charged moments is not of Gaussian type, we expect a breakdown of equipartition of entanglement in the W_3 CFT.

6 CONCLUSION AND OUTLOOK

In this chapter, I give a summary of the results obtained in this thesis, as well as an outlook on possible future research directions.

6.1 THE U(1) CASE

The goal of Chapter 4 is to investigate the symmetry-resolved entanglement entropy (SREE) and the charged moments in three-dimensional holographic $U(1)$ Chern-Simons-Einstein gravity, a bottom-up model of $\mathrm{AdS}_3/\mathrm{CFT}_2$ with $U(1)$ Kac-Moody symmetry. The final results for the SREE show that it is always charge-independent, whenever the background is in the vacuum or in an excited state, as well as the entangling region being a single interval or N intervals. This universal charge independent behavior of the SREE is called equipartition of entanglement [27], which was also found in various CFT and QFT examples with an internal $U(1)$ symmetry.

Besides the above conclusion on the SREE, at the technical level, there are other interesting results obtained in this chapter. In fact, since the SREE can be easily derived from the charged moments, the main focus in this chapter was put on solving the charged moments from both the CFT and holographic perspectives. In my original works [29, 30], we developed the generating function method, by which the charged moments can be easily obtained via the expectation value of the subregion charge operator. Such a method has proven to be quite efficient in various cases, rendering the problems of computing SREE and the charged moments rather straightforward. I also performed the analysis for the charged moments from first principle, and showed why the generating function method works. In particular, some hidden mechanisms of this $\mathrm{AdS}_3/\mathrm{CFT}_2$ model show up in the discussions, which may inspire us for solving the SREE and the charged moments in more complicated holographic models. In the following, I would like to summarize those findings in Chapter 4.

1. Duality between the bulk Wilson line and boundary vertex operators

I started by considering the simplest case, i.e., the charged moments for a single entangling interval ($N = 1$) on the vacuum background. In the CFT calculations, I explained the two different interpretations for the charged moments. The first one relies on the replica picture, in which one thinks of the charged moments as the partition function of the CFT defined on the replica surface, with two vertex operators inserted at the branched points (or $\mathbb{Z}_n$ fixed points) of the replica surface. The second one relies on the twist picture, in which one takes n copies of the original CFT and defines a new CFT$^{\otimes n}$ on $\mathbb{CP}^1$. The charged moments is then understood as the two-point function of the charged twist operators in the new CFT$^{\otimes n}$. In the holographic calculations, I introduced the $U(1)$ Wilson line description of the charged moments, which was originally proposed in our work [29]. This description is distinct from the topological charged black hole interpretation for the charged moments proposed in [32], in the sense that the bulk $U(1)$ Chern-Simons fields in these two cases take different configurations. The basic picture is that there is a $U(1)$ Wilson line following the trajectory of the $\mathbb{Z}_n$ fixed points of the bulk replica manifold. The endpoints of the Wilson line are the two fixed points of the boundary replica surface, which are exactly the locations of the two vertex operators in the dual CFT. The charge carried by the Wilson line was related to the $U(1)$ charge of the boundary vertex operators by implementing the AdS/CFT dictionary. To fully confirm the validity of the Wilson line description of the charged moments, I evaluated the bulk action for the Chern-Simons fields and showed that the result coincides with the CFT results of the charged moments. From this simplest example, essentially what we learned is a new entry to the AdS/CFT dictionary, the duality between the bulk Wilson line and the boundary vertex operators.

2. Vertex correlators and Knizhnik-Zamolodchikov equation

Next, I turned to the case of $N = 1$ in certain excited state background. The excited state was chosen to be generated by inserting charged heavy vertex operators at the origin and the point at infinity of $\mathbb{CP}^1$. Such a choice leads to a simple interpretation for the charged moments in the replica picture of the boundary CFT: It is the $(2n + 2)$-point function of vertex operators on the replica surface $R_{n,1}$. By analysing the OPE structure between the vertex operators, I showed that the correlation functions of the vertex operators are completely constrained by the $U(1)$ Kac-Moody symmetry, and obey the Knizhnik-Zamolodchikov (KZ) equation. On the other hand, by implementing the Wilson line/vertex operators duality, the AdS dual of the charged moments is obtained by adding additional n disjoint Wilson lines. Each Wilson line attaches a pair of vertex operators with opposite charges on the boundary replica surface. It turned out that the results obtained in those two approaches exactly coincide with each other. However, the situation becomes a bit subtle if we study the charged moments in the twist picture of the dual CFT. In the twist picture, one needs to evaluate the four-point function of two vertex operators and two charged twist operators. The problem is that the theory considered here is a bottom-up holographic toy model, so we do not have access to the OPE between those two kinds of operators. In fact, this problem also happens if we choose the background to be a general excited state. Fortunately, in the semi-classical limit $c \to \infty$, with c being the central charge, the vacuum block contribution is assumed to be dominant in any correlation function in a holographic CFT. Hence, the next goal in this chapter was to find out a method for computing the semi-classical conformal blocks in the most general cases, i.e., general excited state and multi-intervals.

3. Factorization, neutral U(1) block and disjoint Wilson lines

In the most general cases, i.e., general excited state background with multi-interval entangling region, it is inconvenient to work in the replica picture of the dual CFT any longer, since the corresponding replica surface is of higher genus. Calculating general correlation functions on a higher genus replica surface is a hard problem, though for the special cases, such as vertex correlators, it is still solvable and has been discussed in [30, 134]. Instead, I turned to work in the twist picture, and the task was to study the conformal blocks in $\mathrm{CFT}^{\otimes}n$ defined on a $\mathbb{CP}^1$. The symmetry in $\mathrm{CFT}^{\otimes n}$ get enhanced due to the n-copy replicated construction, for instance, $U(1) \to U(1)^{\otimes n}$. However, I claimed that in the context of entanglement, only the $\mathbb{Z}_n$ symmetric currents are relevant for defining descendant states that appear in the OPEs. Those $\mathbb{Z}_n$ symmetric currents still furnish a $U(1)$ Kac-Moody algebra, but with the central charge and the level being rescaled as $c \to nc$ and $k \to nk$. In [117], it was argued that for a CFT with $U(1)$ Kac-Moody symmetry, the $U(1)$ extended conformal block factorized into the product of a Virasoro block and a $U(1)$ block. A similar factorization property of the Hilbert space was also assumed in our original work [30], due to the observation of the decoupling between the bulk Chern-Simons fields and the metric. In this chapter, I gave a proof for the factorization of the $U(1)$ extended conformal block and showed that effectively the Hilbert space indeed factorizes into a $U(1)$ sector and Virasoro sector. The states in the $U(1)$ sector are dual to the corresponding configurations of the Chern-Simons fields, and the states in the Virasoro sector are responsible for different asymptotically AdS_3 geometries in the bulk. The result for the $U(1)$ block was explicitly derived in (4.209). It obeys the KZ-equations with $U(1)$ level nk, and hence takes the same form as the vertex correlation function. In particular, due to the charge conservation, the $U(1)$ block is universally factorized in any correlation function of current primary fields, i.e.,

$$\langle O_1 O_2 \cdots \rangle = \mathcal{V}_J \bar{\mathcal{V}}_{\bar{J}} \sum_p C_p \mathcal{F}^p_{vir} \bar{\mathcal{F}}^p_{vir} \ .$$

A more convenient way to see the above factorization is to consider the level-two null-state equation (4.217) in the theory, which get modified from the usual case, due to the presence of the $\mathbb{Z}_n$ symmetric $U(1)$ current. By factorizing the function $\mathcal{V}_J \bar{\mathcal{V}}_{\bar{J}}$ out of the conformal partial wave function, I showed that the rest part of the conformal partial wave function obeys the usual null state equation (Fuchsian differential equation) for the pure Virasoro correlators. Hence, a general method for computing the charged moments is to first impose trivial monodromy conditions on each pairs of operators with opposite $U(1)$ charges. Then, we solve the KZ-equations to obtain the $U(1)$ part, and solve the Fuchsian differential equation under the trivial monodromy conditions to obtain the vacuum Virasoro block. A trivial monodromy condition imposed on a pair of operators projects their OPE onto the $U(1)$ extended vacuum module. Due to the charge conservation, this condition is valid only if the pair of operators carry opposite charges. From the AdS point of view, since the $U(1)$ block contribution is identical to the correlation function of vertex operators, by the Wilson line/vertex operators duality we interpret the $U(1)$ block as the insertions of disjoint Wilson lines inside the bulk. The generating function method developed in our original works [29, 30] computes the $U(1)$ block from the bulk disjoint Wilson lines.

6.2 THE HIGHER SPIN CASE

In Chapter 5, I investigated the simplest case of the charged moments in the $SL(3, R)$ higher spin holography, in which the background state was assumed to be the vacuum with a single interval entangling region. To give a valid definition for the subregion charge operator in the higher spin charged moments, I first claimed that there exists a single copy of the W_3 algebra, the so-called entanglement W_3 algebra, acting on the entangling interval. The construction of this algebra was briefly reviewed in Appendix A, based on my work [31]. The existence of the entanglement W_3 algebra implies that the states in the Hilbert space defined on the entangling interval are the representations of the W_3 symmetry. Those representations are labeled by the eigenvalues of the zero modes (L_0, W_0) of the entanglement W_3 algebra. The zero mode L_0 is linearly related to the modular Hamiltonian. Analogously, I defined the subregion charge operator, the so-called spin-3 modular charge, via the zero mode W_0. Under such a definition, the higher spin charged moments can be understood as a grand canonical partition function of the dual W_3 CFT, with a chemical potential coupled to the spin-3 modular charge. Hence, by the argument in [32], the holographic dual of the higher spin charged moments is the topological charged higher spin black hole. Using the known results of the partition functions of the higher spin black holes [191], the higher spin charged moments was readily obtained by appropriately identifying certain parameters.

On the other hand, an independent calculation for the higher spin charged moments was also performed in the dual W_3 CFT. Unlike the $U(1)$ case, so far it is unclear that whether or not the higher spin charged moments is equivalent to certain vertex operators in the dual W_3 CFT. Hence, the methods used in the $U(1)$ case can not be simply applied to the higher spin case. To solve the problem, I turned to implement the perturbative approach developed in our work [31], and showed that the perturbative expansions of the higher spin charged moments in the chemical potential give rise to the series of connected correlation functions of the spin-3 modular charges. Using the OPE of the higher spin currents, those connected correlation functions can be calculated recursively. The result showed a perfect match with the perturbative expansion of the holographic result.

The perturbative method implemented in this chapter can be used to explain the origin of the equipartition of entanglement in the $U(1)$ case. I have discussed this issue in Chapter 4. The point is that to have the equipartition behavior, the corresponding charged moments must be a Gaussian function of the chemical potential. This happens in the $U(1)$ case, because the connected correlation functions of the $U(1)$ subregion charge operators terminate at quadratic order in the $U(1)$ chemical potential. In contrast, in the higher spin case, all the even order connection correlations of the spin-3 modular charges are nonvanishing. Hence, a breakdown of the equipartition of entanglement is expected for the dual W_3 CFT.

6.3 OUTLOOK

There are many interesting questions raised from the discussions in this thesis. Concerning the results in Chapter 4, the following topics may be interesting for the future research:

1. The $U(1)$ SREE in other holographic models

Since the $U(1)$ Chern-Simons-Einstein gravity is the simplest holographic toy model with an internal $U(1)$ symmetry in the boundary CFT, it will be interesting to study the $U(1)$ SREE in more complicated holographic models, in order to better understand the model dependence or independence of these results. For instance, at high energies, we need to include the Maxwell term in the bulk action, which couples the Chern-Simons gauge field to gravity. Such a theory still admits a conserved $U(1)$ current on the conformal boundary of AdS_3. The difference is that, due to the coupling between the metric and the Chern-Simons field, the asymptotic symmetry algebra will no longer be the Kac-Moody extension of the Virasoro algebra. A full study on the $U(1)$ SREE in this model of course involves a lot of tasks, such as performing the holographic renormalization, analyzing the asymptotic symmetry, as well as investigating the conformal blocks and the constraints for them. However, even without the detailed study, in general we expect that the $U(1)$ SREE in this model may not exhibit the equipartition behavior. The reason is that, from the quantum information point of views, if the SREE does reflect the influence of the symmetry algebra on the finer structure of the entanglement entropy, then its behavior should distinguish the two systems with different symmetry algebras.

2. Boundary entropy corrections to the SREE in holography

In a general quantum field theory, the tensor factorization of the Hilbert space under the bipartition of the Cauchy slice is not simply established [201]. Well-defined quantum fields on the entangling region A require us to impose appropriate conditions at ∂A. Under the fixed boundary conditions, labeled by α, the Hilbert space on the Cauchy slice can then be mapped to the tensor product of the Hilbert spaces on the subregions, i.e. $\iota_\alpha : \mathcal{H} \to \mathcal{H}_{A,\alpha} \otimes \mathcal{H}_{B,\alpha}$. Due to the boundary conditions, the entanglement entropy calculated in this set-up generically contains additional contributions, which in the context of the two-dimensional conformal field theories are known as the Affleck-Ludwig boundary entropy [40]. The boundary entropy corrections to the SREE has been discussed in [122] for the two-dimensional massless free boson theory, in which the authors implemented the BCFT techniques. In the AdS/CFT context, the BCFT is closely related to the end-of-world brane construction in the AdS space [127–129]. Using this construction, we might be able to find out the dual holographic description of the boundary entropy corrections to the SREE.

3. Reconstruction of bulk gauge fields from the SREE

An importance of the RT formula (2.69) is that it illuminates the way towards understanding the bulk gravity theories in terms of the quantum information aspects of the boundary field theories. At the current stage, physicists tend to think of the bulk spacetime as an emergent phenomenon of the entanglement in the boundary world [26]. Fruitful achievements have been made in this research direction. For instance, it was shown in [202, 203] that by considering the small perturbations to the vacuum state in a holographic CFT and imposing certain constraints for the ball-shaped entangling region, the linearized Einstein's equation can be derived from the variation of the entanglement entropy. Since the SREE characterizes a finer structure than the entanglement entropy, associated with the internal symmetry, it is natural to ask whether the behavior of the SREE determines the equation of motions for the bulk gauge fields. In the following sense, this question might have a positive answer. Recall that the entanglement entropy in holographic CFTs takes a universal form as long as the background state is the vacuum. This universality stemmed from the conformal symmetry of the CFT and may also be related to the dynamics of the AdS gravity. This is because the conformal symmetry, as the asymptotic symmetry of pure AdS gravity, is derived from the on-shell deformation of the metric. Similarly, one may relate the dynamics of the bulk gauge fields to the SREE. In particular, if the behavior of SREE can distinguish two holographic CFTs with distinct symmetry algebras, then the dynamics of the gauge fields in the dual gravity theories are expected to be different.

4. SREE in higher dimensions, the role of higher form symmetries in holography

In this thesis, the SREE has only been discussed in the two-dimensional conformal field theories and the AdS_3/CFT_2. Hence, studying the generalizations of the SREE in higher dimensions is a very interesting topic. In higher dimensional AdS/CFT models, the charged topological black hole method developed in [32] can still be used to compute the SREE in the vacuum background with a ball-shaped entangling region. However, in more general cases, no technique tool is currently available. So here I would like to make some educated guesses on the higher dimensional cases.

We first come back to the holographic $U(1)$ Chern-Simons-Einstein gravity. On the bulk side, a Wilosn line defect in the Chern-Simons theory can be understood as a one-form charged object, with its charge detected by a Wilson loop encircling around it. Formally, the charge of the Wilson line defect is related to the 't Hooft anomaly of the one-form symmetry $\mathbb{Z}_k$ in the $U(1)_k$ Chern-Simons theory. On the boundary side, a vertex operator is understood as a zero-form (point-like) charged object, detected by the codimension-two charge operator $\oint \star J$. In this respect, we can understand the Wilson line/vertex duality as a duality between one-form symmetry in the Chern-Simons theory with the zero-form symmetry in the boundary CFT. The duality between one-form and zero-form symmetries in AdS_3/CFT_2 suggests us to solve the charged moments in higher dimensions by seeking for the duality between the higher form symmetries on both theories. For instance, in AdS_{d+1}/CFT_d, the charged moments in the dual CFT may be characterized by a $(d-2)$ dimensional defect operator placed at the boundary ∂A of the entangling region A. The $(d-2)$-form charge of the defect operator may be detected by $(d-(d-2)-1)$-dimensional charged operator, which is a one-dimensional 't Hooft loop operator. On the AdS side, we expect that the charged moments would be described by a $(d-1)$-dimensional brane defect, anchored to the boundary defect operator, and carrying $(d-1)$-form charge. The $(d-1)$-form charge is still detected by a 't Hooft loop in AdS_{d+1}. By the AdS/CFT dictionary, we might be able to build up relations between the bulk and the boundary 't Hooft loop operators, and hence to establish the duality between the $(d-1)$-form symmetry in AdS_{d+1} and the $(d-2)$-form symmetry in the CFT_d. If the procedure works successfully, we are able to solve the charged moments by studying the back-reactions of the brane to the AdS_{d+1} spacetime.

Concerning the results in Chapter 5, the following topics may be interesting for the future research:

In Chapter 5, I only discussed the simplest case of the higher spin charged moments, i.e., the vacuum background with a single entangling interval. A detailed study on the higher spin charged moments in more general cases is an interesting research direction. However, to attack this topic, one needs to first figure out some more fundamental questions about the higher spin holography. Consider the generating functional of connected correlation functions of spin-3 chiral currents in the dual CFT,

$$Z_n[\mu] = \int \mathcal{D}\Phi e^{-S-\int d^2z\mu(z,\bar{z})W(z)+\bar{\mu}(z,\bar{z})\bar{W}(\bar{z})} , \tag{6.1}$$

where $\mu(z,\bar{z})$ and $\bar{\mu}(z,\bar{z})$ are the local sources coupled to the spin-3 chiral currents. The charged moments discussed in Chapter 5 can be understood as a special case of this generational functional, with the local source terms being of specific forms. Although the result of the general form of (6.1) has not been derived in the literature, it is believed that the effective action of (6.1) might be of Toda type [190]. Toda theories are generalizations of the Liouville theory, and the action of the latter one can be regarded as the generating functional of connected correlation functions of the stress tensor, as mentioned in Chapter 3. In particular, the source term coupled to the stress tensor in the generating functional can be understood as the Beltrami differential, characterizing the deformation of the complex structure on the Riemann surface, on which the CFT is defined. Similarly, the source terms in (6.1) can be understood as the deformation of the generalized complex structure on the Riemann surface. A detailed study on the relation between (6.1) and Toda actions may also require us to have a better understanding on the spin-3 transformation in the dual CFT. This is analogous to the relation between the conformal transformation and the Liouville action in two-dimensional CFTs. The conformal transformation of the stress tensor is characterized by the anomalous term, the Schwarzian derivative. Analogously, the spin-3 transformation of the spin-3 current is characterized by a generalization of Schwarzian derivative [204], which was originated from the context of the linear ordinary differential equation (ODE), and is known as the *Halphen invariant* [205]. Mathematically speaking, the Schwarzian derivative, defined via a map from a given Riemann surface to $\mathbb{CP}^1$, characterizes the deformation of the projective structure on the Riemann surface. Similarly, the Halphen invariant characterizes the deformation of the *generalized projective structure* on a Riemann surface, which is defined via a map from a given Riemann surface to $\mathbb{CP}^2$. This is related to the fact that the problem of immersions of Riemann surfaces into higher dimensional spaces is essential to the classical Toda theory, which should also be true for the case of higher spin gravity. The study of generalized complex structures as well as generalized projective structures in higher spin gravity might provide us a better understanding for the global structures of the higher spin solutions, and hence help us for deriving the effective action of higher spin gravity.

A

ENTANGLEMENT ALGEBRA

In this appendix, I will review the construction of the single copy W_3 algebra on a single entangling interval, following from our work in [31]. The construction of the algebra is based on the BCFT description of entanglement [124, 200, 201]. The basic idea is as followings. The cut-off circles inserted around the two endpoints of the entangling interval are the boundaries of the system, so, for the theory to be well-defined, appropriate boundary conditions need to be imposed on the cut-off circles. For a CFT with a concrete Lagrangian, the appropriate boundary conditions can be derived by requiring a well-defined variation principle. For instance, in the free boson theory, the boundary conditions are of the Dirichlet or Neumann types. However, since a CFT is usually formulated independently of a particular set of fundamental fields and a Lagrangian, one must be able to impose the boundary conditions in a more general manner. As pointed out in [125], a natural requirement is that the off-diagonal component $T_{\parallel\perp}$ of the stress tensor parallel/perpendicular to each boundary should vanish locally. This is the so-called conformal boundary condition, which ensures no energy or momentum flow across the boundaries. Furthermore, for a CFT with a extended symmetry (Kac-Moody or $\mathcal{W}$ symmetry), the boundary conditions can be generalized in a straightforward manner by imposing additional constraints on the extra conserved currents. Those constraints on the conserved currents eventually reduce the two chiral sectors of the symmetry algebra into a single sector in the BCFT.

More explicitly, we consider a single interval $\mathcal{A}$ on the w-complex plane and locate its two endpoints at $w = w_{1,2}$, with $|w_1| < |w_2|$. To regularize the endpoints, we impose cut-off circles around each endpoints so that the domain we considered here satisfies $|w - w_1| \geq |\epsilon|$ and $|w - w_2| \geq \epsilon$. This domain can be mapped to a semi-annulus on a upper half z-complex plane by the following conformal transformation

$$z = e^{i\theta(w)}, \tag{A.1}$$

with

$$\theta(w) = \frac{\pi}{2} - \frac{\pi}{\Delta L} u(w) \ , \tag{A.2}$$

and

$$u(w) = \log\left(\frac{w - w_1}{w_2 - w}\right) \ , \quad \Delta L = 2\log\left|\frac{w_2 - w_1}{\epsilon}\right| \ . \tag{A.3}$$

The inner and outer semi-circles of the semi-annulus locate at the constant radius $r = 1$ and $r = e^{\frac{2\pi^2}{\Delta L}}$. Both of them correspond to the interval $\mathcal{A}$ in the original w-complex plane, so they are identified. The cut-off circles around the endpoints $w_{1,2}$ are now mapped to the two boundaries of the semi-annulus located along the real axis of the upper half plane. Now, in the z-coordinates, the conformal boundary condition $T_{\|\perp} = 0$ becomes

$$T(z) = \bar{T}(\bar{z}) \ , \quad \forall z = \bar{z} \in \mathbb{R} \ . \tag{A.4}$$

Since $T(z)$ and $\bar{T}(\bar{z})$ are holomorphic and anti-holomorphic, the condition (A.4) leads to the consequence that correlators of $\bar{T}(\bar{z})$ are those of $T(z)$ analytically continued into the lower half plane, i.e. $\bar{T}(\bar{z}) = T(\bar{z})$. Therefore, the Virasoro modes in the BCFT can be written as [125],

$$\begin{aligned} L_n &= \oint \frac{dz}{2\pi i} z^{1+n} T(z) \\ &= \int_{\mathcal{S}} \frac{dz}{2\pi i} z^{1+n} T(z) - \int_{\mathcal{S}} \frac{d\bar{z}}{2\pi i} \bar{z}^{1+n} \bar{T}(\bar{z}) \ , \end{aligned} \tag{A.5}$$

where the integral path $\mathcal{S}$ is an arbitrary semi-circle centered around the origin, going along the counterclockwise direction.

In presence of the additional higher spin-3 currents in the theory, we can further impose the following maximally symmetric boundary condition on the spin-3 currents such that it is compatible with (A.4),

$$W(z) = (\Omega \bar{W})(\bar{z}) \ , \quad \forall z = \bar{z} \in \mathbb{R} \ . \tag{A.6}$$

Here the map Ω is the automorphism of the W_3 algebra, given by $\Omega = \pm \mathbb{1}$. This can be easily seen from the fact that the redefinition $W \to \pm W$ does not change the WW and TW OPEs. Similar to the Virasoro case, the higher spin modes in the BCFT can be written as

$$W_n = \int_{\mathcal{S}} \frac{dz}{2\pi i} z^{2+n} W(z) - \int_{\mathcal{S}} \frac{d\bar{z}}{2\pi i} \bar{z}^{2+n} (\Omega \bar{W})(\bar{z}) \ . \tag{A.7}$$

Using the transformation laws,

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z; w\} \ , W(w) = \left(\frac{dz}{dw}\right)^3 W(z), \tag{A.8}$$

we can derive the modes of the currents in terms of the original w-coordinates, given by

$$\begin{aligned} L_n &= -\int_{\mathcal{A}} \frac{dw}{2\pi} \frac{e^{in\theta(w)}}{u'(w)} T(w) - \int_{\mathcal{A}} \frac{d\bar{w}}{2\pi} \frac{e^{-in\bar{\theta}(\bar{w})}}{\bar{u}'(\bar{w})} \bar{T}(\bar{w}) + \frac{c}{24}\left(1 + \frac{\Delta L^2}{\pi^2}\right) \ , \\ W_n &= \int_{\mathcal{A}} \frac{dw}{2\pi i} \frac{e^{in\theta(w)}}{(u'(w))^2} W(w) - \int_{\mathcal{A}} \frac{d\bar{w}}{2\pi i} \frac{e^{-in\bar{\theta}(\bar{w})}}{(\bar{u}'(\bar{w}))^2} (\Omega \bar{W})(\bar{w}) \ , \end{aligned} \tag{A.9}$$

where the direction of the integral path $\mathcal{A}$ coincides with $\mathcal{S}$, going from w_1 to w_2. The modes in (A.9) furnishes the single copy W_3 algebra on the entangling interval $\mathcal{A}$, which in fact can also be checked in the w-coordinates directly by using the currents OPEs and the boundary conditions. The detailed calculation on the check can be found in the appendix B of [31].

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