On the Geometry and Parametrization of Almost Invariant Subspaces and Observer Theory

Dissertation zur Erlangung des
naturwissenschaftlichen Doktorgrades
der Bayerischen Julius-Maximilians-Universität Würzburg

vorgelegt von
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aus
Buchen

Würzburg 2002

Acknowledgements

I want to thank Prof. Dr. U. Helmke for introducing me to the exciting field of geometric control theory as well as for his always open ear during his supervision of the last five years of my work in the field. Special thanks go to Prof. Dr. P.A. Fuhrmann for our valuable discussions during my stays in Beer Sheva and San Diego. I feel indebted to the people of the Math Department of the University of Würzburg, who created a pleasant and family like working atmosphere. Especially, I want to mention I. Böhm, Dr. G. Dirr, Prof. Dr. R. Göb, Dr. R. Greiner, Prof. Dr. T. Grundhöfer, Dr. K. Hüper, M. Kleinsteuber, Dr. L. Kramer, K. Krumpholz, Dr. N. Rosehr and Dr. O. Roth.

This work has been partially supported by a grant from the German-Israeli Foundation GIF I-526-034.06/97.

Contents

Chapter 1

Introduction

The concept of an invariant subspace is the basic tool in the structure theory of linear dynamical systems. Consider for example a linear finite-dimensional time-invariant ODE

$$\dot{x} = Ax \ , \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. An invariant subspace of this system is a linear subspace $\mathcal{V} \subset \mathbb{R}^n$ with the property that every trajectory starting in $\mathcal{V}$ will stay in $\mathcal{V}$ in the future. Apparently the system can be restricted to such a subspace. As is well known, the invariant subspaces for the system (1.1) are exactly the invariant subspaces of the linear operator $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, i.e. the subspaces $\mathcal{V} \subset \mathbb{R}^n$ with $A\mathcal{V} \subset \mathcal{V}$.

What is called the geometric approach to linear systems theory is the attempt to apply the same idea to linear control systems. Consider, for example, a linear finite-dimensional time-invariant control systems in state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \\ y &= Cx \ . \end{aligned} \tag{1.2}$$

A subspace $\mathcal{V}$ of the state space is called (A, B)-invariant or controlled invariant, if for every starting point in $\mathcal{V}$ there exists a control input such that the corresponding trajectory stays in $\mathcal{V}$ in the future. One of the most important results of linear systems theory says that such a control input can always be chosen in a special form, namely constant state feedback $u(t) := Fx(t)$. In algebraic terms this means that $\mathcal{V}$ is (A, B)-invariant if and only if there exists F such that $\mathcal{V}$ is $(A + BF)$-invariant, i.e. $(A + BF)\mathcal{V} \subset \mathcal{V}$ holds. Applying constant state feedback makes the (A, B)-invariant subspace dynamically invariant, such that the control system can be restricted to the subspace. An equivalent condition is $A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B$, i.e. that $\mathcal{V}$ is A-invariant up to $\operatorname{Im} B$.

While (A, B)-invariant subspaces address the input side of a control system an analogous formation of a concept on the output side results in the notion of (C, A)-invariant or conditioned invariant subspaces. A subspace $\mathcal{V}$ of the state space is called (C, A)-invariant, if $A(\mathcal{V} \cap \operatorname{Ker} C) \subset \mathcal{V}$ holds. This is the case if and only if there exists J such that $\mathcal{V}$ is $(A - JC)$-invariant, i.e. $(A - JC)\mathcal{V} \subset \mathcal{V}$ holds. Apparently, a (C, A)-invariant subspace can be made dynamically invariant by applying constant output injection. Then the control system can be restricted to the subspace.

From the point of view of linear algebra the concepts of (A, B)-invariance and (C, A)-invariance are dual to each other. A subspace $\mathcal{V}$ is (A, B)-invariant (i.e. controlled invariant) if and only if its orthogonal complement $\mathcal{V}^{\perp}$ with respect to the standard inner product is (B^*, A^*)-invariant (i.e. conditioned invariant). In system theoretic terms this means that a subspace is (A, B)-invariant with respect to the system (1.2) if and only if its orthogonal complement is (C, A)-invariant with respect to the dual system. In view of duality, restrictions to (A, B)-invariant subspaces translate to corestrictions to (C, A)-invariant subspaces, i.e. restrictions to the factor space with respect to the given subspace.

A natural question now is what can be said about the set of all invariant subspaces for a given fixed system. In the case of a linear ODE of the form (1.1), i.e. in the case of a linear operator A, the geometry of the set of A-invariant subspaces is well understood. It can easily be seen to be a compact algebraic subvariety of the Grassmann manifold it is part of. Most of the literature in the field treats the more general case of A-invariant partial flags, which includes the case of A-invariant subspaces as the special case of flag length one. Steinberg [Ste76] obtained some results about the dimension of the variety and the number of irreducible components. His results imply that the variety is connected when A is nilpotent. For the case of flag length one, connectedness was already proved by Douglas and Pearcy [DP68]. Shimomura [Shi80] constructed a stratification of the variety into a finite number of affine spaces indexed by a Young diagram. In [Shi85] he studied the irreducible components of these strata. In the case of flag length one Shayman [Sha82] constructed a stratification of the variety in a finite number of smooth submanifolds of the Grassmannian. The strata consist of the subspaces with fixed cyclic structure, i.e. fixed Jordan type of the restriction of A to the subspace. Shayman showed that these strata in general are neither unions nor intersections of Schubert varieties. Helmke and Shayman [HS87] generalized these results to arbitrary flag length. Furthermore, they showed that each Jordan stratum has a biflag manifold as a strong deformation retract, which allows to identify the homology type of the strata as that of a product of Grassmannians. In the case of flag length one, the biflag manifold is a generalized partial flag manifold. Helmke and Shayman used the Schubert cell decomposition of that flag manifold to calculate the Betti numbers of the Jordan strata in this case.

The structure of the set of all (C, A)-invariant subspaces for a given fixed control system is considerably more complicated and not fully understood, yet. In a remarkable paper Hinrichsen, Münzner and Prätzel-Wolters [HMP81] associated to each (C, A)-invariant subspace a module of Laurent series and parametrized these modules using their so-called Kronecker-Hermite bases. This leads to a parametrization of (C, A)-invariant subspaces, which depends on the restriction indices, i.e. the observability indices of the restriction of the pair (C, A) to the subspace. This paper also contains a rudimentary structure theory for (C, A)-invariant subspaces. In unpublished work based on these ideas Münzner constructed a stratification of the set of (C, A)-invariant subspaces into smooth submanifolds of the Grassmannian. He also constructed cell decompositions of these strata.

Polynomial descriptions of (A, B)-invariant and (C, A)-invariant subspaces have also been derived by Fuhrmann and Willems [FW79, FW80], Emre and Hautus [EH80] and Fuhrmann [Fuh81], focussing rather on characterizations, not so much on parametrization.

During the last five years there has been a sort of a revival of interest in parametrization issues in geometric control theory. Fuhrmann and Helmke [FH97, FH00, FH01] constructed a smooth map from the manifold of similarity classes of μ-regular controllable pairs of the appropriate size onto the set $\mathcal{V}_k(C, A)$ of all (C, A)-invariant subspaces of codimension k of a fixed control system of the form (1.2). They showed that this map restricts to a diffeomorphism between the submanifold of μ-tight pairs and the subset $\mathcal{T}_k(C, A)$ of tight subspaces. The map is given in terms of kernels of truncated and permuted reachability matrices (hence the name kernel representation), whose entries are directly related to corestrictions of the given control system to the (C, A)-invariant subspaces under consideration. The exact relation is stated and proved in this thesis, see the discussion below Proposition 5.30. Furthermore, analogous mappings between sets of restricted system equivalence classes of controllable triples and sets of almost (C, A)-invariant subspaces are constructed (Sections 5.2 and 5.3). Almost (C, A)-invariant subspaces were introduced in a series of papers by Willems [Wil80, Wil81, Wil82]. They satisfy the invariance condition only up to an arbitrarily small ε (in the metric of the state space).

Ferrer, F. Puerta and X. Puerta [FPP98, FPP00] constructed a stratification of $\mathcal{V}_k(C, A)$ into smooth submanifolds of the Grassmann manifold of codimension k subspaces. These so called Brunovsky strata consist of the subspaces with fixed restriction indices. F. Puerta, X. Puerta and Zaballa [PPZ00] explicitely described coordinate atlases for the strata, in [PPZ01] they constructed cell decompositions. While kernel representations are related to corestrictions, these are not used in [FPP98, FPP00, PPZ00, PPZ01]. Instead, the stratification and cell decomposition results of Ferrer, F. Puerta, X. Puerta and Zaballa make use of restrictions of control systems to (C, A)-invariant subspaces. They lead to image representations, i.e. the subspaces are parametrized via columnspans of block Toeplitz type matrices. Such descriptions have also been derived by Fuhrmann and Helmke [FH00, FH01], using submodules of polynomial models to describe (C, A)-invariant subspaces.

X. Puerta and Helmke [PH00] showed that the Brunovsky strata have generalized flag manifolds as strong deformation retracts, thus deriving effective formulas for the Betti numbers of the strata. Inspired by this paper, the stratification and cell decomposition results of Ferrer, F. Puerta, X. Puerta and Zaballa are re-derived and simplified in this thesis in the special case of tight subspaces using a different approach involving Kronecker forms and unipotent transformations (Chapter 6). In this way the known cell decomposition of $\mathcal{T}_k(C, A)$ is shown to be induced by a Bruhat decomposition of a generalized flag manifold. Furthermore, the adherence order of the cell decomposition is identified.

One of the difficulties arising in the study of the geometry and the topology of $\mathcal{V}_k(C, A)$ is that this set is in general not compact, i.e. not closed in the Grassmannian. A first attempt at obtaining a compactification might be to extend the set by including all almost (C, A)-invariant subspaces. An example due to Özveren, Verghese and Willsky [ÖVW88] shows that this attempt fails by constructing a subspace in the closure of all (C, A)-invariant subspaces which is not almost (C, A)-invariant. With a different approach it is however possible to construct a compactification, at least in some special cases. The corresponding result is not part of this thesis, but will appear in a joint paper with Helmke and Fuhrmann [THF02].

While one of the most important applications of (almost) (A, B)-invariant subspaces is the solution of the (almost) disturbance decoupling problem, in the dual setting this role is played by partial observers. A partial observer is a control system which reconstructs part of the state of the observed system from the input and output data of that system. The literature on partial observers splits into two lines of research. One part is mainly concerned with existence results, therefore the theory is formulated in the context of factor spaces with respect to (almost) invariant subpaces. The second approach is more application-oriented, the main focus lies on characterization results in terms of matrix equations. Since characterizations of partial observers is what is needed for parametrization issues, the second approach is chosen in this work. Unfortunately the literature in this field is a kind of fragmentary and contains a number of flaws and misunderstandings. Hence the whole theory is redeveloped from scratch, here. Furthermore, the characterization results are extended to the almost invariant case using descriptor systems as observers (Section 3.2.1). Concerning parametrization issues this new approach seems to be more fruitful than the usual PID-observer approach (cf. Section 5.7).

This thesis is organized as follows.

In Chapter 2 the definitions and basic properties of various kinds of (almost) invariant subspaces are presented. This includes decomposition results, algorithms for computing such subspaces and (co)restrictions of control systems to such subspaces. The (dual) Brunovsky form and the (dual) Kronecker form of a control system is explained.

In Chapter 3 dynamic characterizations of (almost) (C, A)-invariant subspaces in terms of observers are derived. A structure theory for observers is developed. New results in this chapter are the characterization of almost (C, A)-invariant subspaces in terms of singular observers (Section 3.2.1), the characterization of tracking output observers (Section 3.2.3) and the characterization of asymptotic observers for functions of the state of non controllable systems (Theorem 3.57).

Chapter 4 contains some basic topological facts about moduli spaces of linear systems. New results in this chapter are the μ-partial Kalman embedding (Proposition 4.4) and the vector bundle structure of the set of (C, A)-invariant subspaces plus friends over the moduli space of tracking observer parameters (Section 4.3).

In Chapter 5 kernel representations of almost (C, A)-invariant subspaces are used for parametrization issues. The link to partial realization theory and to the restriction pencil (see Jaffe and Karcanias [JK81]) is explained. New results in this chapter are the kernel representations of almost observability and almost (C, A)-invariant subspaces (Sections 5.2 and 5.3), the characterization of observability subspaces in this context (Section 5.4), the link to the restriction pencil (Section 5.5) and the connection between kernel representations, corestrictions and observers (Section 5.7).

The aim of Chapter 6 is to construct a Bruhat type cell decomposition of certain sets (Brunovsky strata) of (C, A)-invariant subspaces. The construction uses image representations of these subspaces. New results in this chapter are the construction of Kronecker strata of tight (C, A)-invariant subspaces (Theorem 6.12), the equivalence of the reverse Bruhat order and the Kronecker order on any $S(m)$-orbit of combinations (Section 6.2) and the identification of the Kronecker cell decomposition of a Brunovsky stratum as being induced by the Bruhat decomposition of a generalized flag manifold (Proposition 6.33 and Theorem 6.34).

In Appendix A some facts about orbit spaces of Lie group actions are presented. A somehow best possible criterion for the orbit space to be a manifold is given. The meaning of "best possible" is explained by an example.

Appendix B contains a sufficient criterion for a (topological) vector bundle to be a differentiable vector bundle. It can also be seen as a sufficient criterion for the preimage of a smooth submanifold to be again a smooth submanifold. Quotients of vector bundles with respect to Lie group actions are considered.

Chapter 2

Almost invariant subspaces

In this chapter, the definitions and basic properties of various kinds of invariant subspaces are presented. These invariant subspaces are linear subspaces of the *state space* $\mathcal{F}^n$, $\mathcal{F} = \mathbb{R}, \mathbb{C}$, of the linear finite-dimensional time-invariant *control system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \ , \end{aligned} \tag{2.1}$$

where $A \in \mathcal{F}^{n\times n}$, $B \in \mathcal{F}^{n\times m}$ and $C \in \mathcal{F}^{p\times n}$. Here $x \in \mathcal{F}^n$ is referred to as the *state*, $u \in \mathcal{F}^m$ as the *control* or *input* and $y \in \mathcal{F}^p$ as the *output* of system (2.1), respectively.

2.1 Almost (A, B)-invariant subspaces

The concept of almost (A, B)-invariant subspaces was introduced by Willems [Wil80, Wil81]. He generalized the concept of (A, B)-invariant subspaces introduced by Basile and Marro [BM69] and by Wonham and Morse [WM70]. An extensive treatment of (A, B)-invariant subspaces can be found in the books by Wonham [Won74] and Basile and Marro [BM92]. A good source for further results on almost (A, B)-invariant subspaces is Trentelman's thesis [Tre85]. All the results presented in this section, except for some of the results on feedback transformations (subsection 2.1.2) and (co)restrictions (subsection 2.1.3), can be found in that work. Further references will be given on the spot.

The definitions in this section are formulated from the behavioral point of view, i.e. using solution trajectories rather than state space vectors (cf. Polderman and Willems [PW98]). In the behavioral approach to system theory a system is regarded as a subset of a function space, the *behavior*, containing the input/state/output-trajectories defined by equations of the form (2.1). The state part of the behavior defined by (2.1) is the set

$$\Sigma_x(A,B) := \{x \in \mathrm{C}_{\mathrm{abs}}(\mathbb{R},\mathcal{F}^n) \mid \exists x_0 \in \mathcal{F}^n\ \exists u \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathcal{F}^m)\ \forall t \in \mathbb{R}$$
$$x(t) = \mathrm{e}^{At}\, x_0 + \int_0^t \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\mathrm{d}\tau\}\ ,$$

where $\mathrm{C}_{\mathrm{abs}}(\mathbb{R},\mathcal{F}^i)$ and $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathcal{F}^i)$ denote the spaces of absolutely continuous and locally integrable functions $f : \mathbb{R} \longrightarrow \mathcal{F}^i$, respectively.

Remark 2.1. It would not change the subsequent notion of invariance if x was required to lie in $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathcal{F}^n)$, with the defining variation of constant equation of $\Sigma_x(A,B)$ assumed to hold almost everywhere. Since any linear subspace $\mathcal{V}$ of $\mathcal{F}^n$ is closed, any absolutely continuous trajectory which does not lie in $\mathcal{V}$ in time t_0 in fact does not lie in $\mathcal{V}$ for a time set of measure greater than zero (this time set necessarily contains a non empty interval around t_0).

Definition 2.2. A linear subspace $\mathcal{V}$ of the state space $\mathcal{F}^n$ of system (2.1) is called (A,B)*-invariant*, if for every starting point $x_0 \in \mathcal{V}$ there exists at least one trajectory $x \in \Sigma_x(A,B)$ satisfying $x(0) = x_0$ and $x(t) \in \mathcal{V}$ for all $t \in \mathbb{R}$.

A linear subspace $\mathcal{V}_a$ of the state space $\mathcal{F}^n$ of system (2.1) is called *almost* (A,B)*-invariant*, if for every starting point $x_0 \in \mathcal{V}_a$ and every $\varepsilon > 0$ there exists at least one trajectory $x \in \Sigma_x(A,B)$ satisfying $x(0) = x_0$ and $\mathrm{dist}(x(t),\mathcal{V}_a) < \varepsilon$ for all $t \in \mathbb{R}$.

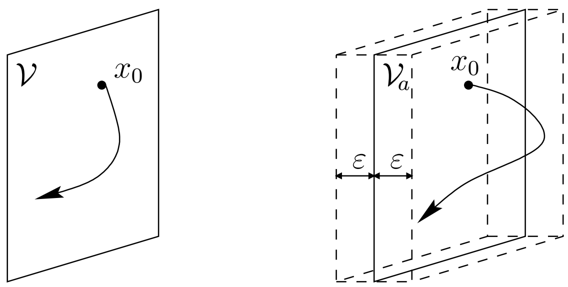

Figure 2.1: (almost) (A,B)-invariant subspaces

The (A, B)-invariant subspaces are exactly those subspaces, to which the control system can be restricted (cf. Section 2.1.3). Thus (A, B)-invariant subspaces correspond uniquely to linear subspaces of the behavior $\Sigma_x(A, B)$. There are the following geometric and *feedback* characterizations.

Proposition 2.3. *A subspace $\mathcal{V}$ is (A, B)-invariant if and only if $A\mathcal{V} \subset \mathcal{V} +$ $\operatorname{Im} B$ or, equivalently, if and only if there exists a feedback matrix $F \in \mathcal{F}^{m\times n}$ such that $A_F\mathcal{V} \subset \mathcal{V}$, where $A_F = A + BF$. Such an F is called a* friend of $\mathcal{V}$.

To obtain similar characterizations for almost (A, B)-invariant subspaces, it is convenient to introduce another concept, first.

Definition 2.4. A linear subspace $\mathcal{R}$ of the state space $\mathcal{F}^n$ of system (2.1) is called a *controllability subspace*, if for every pair of points $x_0, x_1 \in \mathcal{R}$ there exists a time $T > 0$ and at least one trajectory $x \in \Sigma_x(A, B)$ satisfying $x(0) = x_0$, $x(T) = x_1$ and $x(t) \in \mathcal{R}$ for all $t \in \mathbb{R}$.

A linear subspace $\mathcal{R}_a$ of the state space $\mathcal{F}^n$ of system (2.1) is called an *almost controllability subspace*, if for every pair of points $x_0, x_1 \in \mathcal{R}$ there exists a time $T > 0$ such that for every $\varepsilon > 0$ there exists at least one trajectory $x \in \Sigma_x(A, B)$ satisfying $x(0) = x_0$, $x(T) = x_1$ and $\operatorname{dist}(x(t), \mathcal{R}_a) < \varepsilon$ for all $t \in \mathbb{R}$.

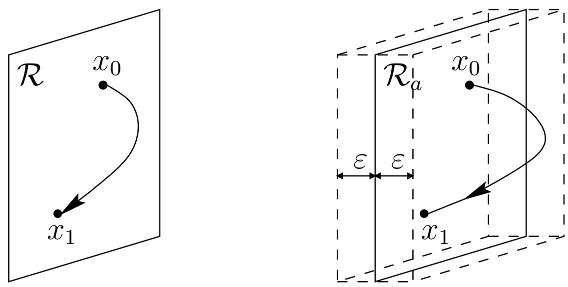

Figure 2.2: (almost) controllability subspaces

Controllability subspaces were introduced by Wonham and Morse [WM70], almost controllability subspaces by Willems [Wil80, Wil81]. Note, that T depends on x_0 and x_1 but not on ε in the definition of almost controllability subspaces. The controllability subspaces are exactly those (A, B)-invariant subspaces, such that the restricted system is controllable (cf. Section 2.1.3). Note further, that every almost controllability subspace is almost (A, B)-invariant. There are the following feedback characterizations.

Proposition 2.5. *A subspace $\mathcal{R}$ is a controllability subspace if and only if there exist a feedback matrix $F \in \mathcal{F}^{m\times n}$ and a subspace $\mathcal{B}_1 \subset \operatorname{Im} B$ such that*

$$\mathcal{R} = \mathcal{B}_1 + A_F\mathcal{B}_1 + \cdots + A_F^{n-1}\mathcal{B}_1 \ .$$

A subspace $\mathcal{R}_a$ is an almost controllability subspace if and only if there exist a feedback matrix $F \in \mathcal{F}^{m\times n}$ and a chain of subspaces $\operatorname{Im} B \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_r$ such that

$$\mathcal{R}_a = \mathcal{B}_1 + A_F\mathcal{B}_2 + \cdots + A_F^{r-1}\mathcal{B}_r \ .$$

The chain can be chosen such that the sums are direct sums and $\dim \mathcal{B}_i = \dim A_F^{i-1}\mathcal{B}_i$ for $i = 1, \dots, r$.

Let

$$\mathcal{R}(A,B) := \operatorname{Im} R_n(A,B) := \operatorname{Im}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

denote the *reachable subspace* of system (2.1). Since $\mathcal{R}(A+BF,B) = \mathcal{R}(A,B)$ for every feedback matrix $F \in \mathcal{F}^{m\times n}$, any (almost) controllability subspace is contained in the reachable subspace.

To generalize the well known spectral assignability property of controllability subspaces to the almost controllability case, the notion of $\mathbb{C}_g \subset \mathbb{C}$, a 'good' part of the complex plane, is needed. See Trentelman's thesis [Tre85, Section 2.7] for a reasoning on this. $\mathbb{C}_g$ denotes any subset of $\mathbb{C}$ which is symmetric (i.e. $\lambda \in \mathbb{C}_g \Leftrightarrow \bar{\lambda} \in \mathbb{C}_g$, where $\bar{\lambda}$ denotes the complex conjugate of λ) and contains a left semi infinite real interval (i.e. there exists a number $c \in \mathbb{R}$ such that $]-\infty, c] \subset \mathbb{C}_g$). The prototype of such a 'good' part is $\mathbb{C}^- = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$.

Proposition 2.6. *A subspace $\mathcal{R}$ is a controllability subspace if and only if for every monic polynomial p of degree $\dim \mathcal{R}$ there exists a feedback matrix $F \in \mathcal{F}^{m\times n}$ such that $A_F\mathcal{R} \subset \mathcal{R}$ and the characteristic polynomial of $A_F|_{\mathcal{R}}$ is equal to p.*

A subspace $\mathcal{R}_a$ is an almost controllability subspace if and only if for every set $\mathbb{C}_g$ and every $\varepsilon > 0$ there exists a feedback matrix $F \in \mathcal{F}^{m\times n}$ such that $\operatorname{dist}(\mathrm{e}^{A_F t}x_0, \mathcal{R}_a) < \varepsilon$ for all $t \geq 0$ and $x_0 \in \mathcal{R}_a$, while the spectrum of $A_F|_{\mathcal{R}(A,B)}$ lies in $\mathbb{C}_g$.

The use of $\mathcal{R}(A,B)$ rather than $\mathcal{R}_a$ in the almost part of this characterization is due to the fact that a restriction of A_F to $\mathcal{R}_a$ is not well definable since $\mathcal{R}_a$ is not necessarily (A,B)-invariant. Now it is possible to formulate a geometric and a feedback characterization of almost (A,B)-invariant subspaces.

Proposition 2.7. *A subspace $\mathcal{V}_a$ is almost (A, B)-invariant if and only if there exist an (A, B)-invariant subspace $\mathcal{V}$ and an almost controllability subspace $\mathcal{R}_a$ such that $\mathcal{V}_a = \mathcal{V} + \mathcal{R}_a$ or, equivalently, if and only if for every $\varepsilon > 0$ there exists a feedback matrix $F \in \mathcal{F}^{m\times n}$ such that* $\operatorname{dist}(\mathrm{e}^{A_F t} x_0, \mathcal{V}_a) < \varepsilon$ *for all $t \geq 0$ and $x_0 \in \mathcal{V}_a$.*

Further insight into the structure of almost (A, B)-invariant subspaces is achieved by introducing the concepts of coasting and sliding subspaces due to Willems [Wil80]. They provide a tool to eliminate the ambiguity resulting of the fact that controllability subspaces are both (A, B)-invariant and almost controllability subspaces.

Definition 2.8. A linear subspace $\mathcal{C}$ of the state space $\mathcal{F}^n$ of system (2.1) is called *coasting*, if for every starting point $x_0 \in \mathcal{C}$ there exists exactly one trajectory $x \in \Sigma_x(A, B)$ satisfying $x(0) = x_0$ and $x(t) \in \mathcal{C}$ for all $t \in \mathbb{R}$, i.e. if $\mathcal{C}$ is (A, B)-invariant and $x_1, x_2 \in \Sigma_x(A, B)$, $x_1(0) = x_2(0)$, $x_1(t) \in \mathcal{C}$ and $x_2(t) \in \mathcal{C}$ for all $t \in \mathbb{R}$ imply $x_1 = x_2$.

A linear subspace $\mathcal{S}$ of the state space $\mathcal{F}^n$ of system (2.1) is called *sliding*, if $\mathcal{S}$ is an almost controllability subspace and $x \in \Sigma_x(A, B)$ and $x(t) \in \mathcal{S}$ for all $t \in \mathbb{R}$ imply $x(t) = 0$ for all $t \in \mathbb{R}$.

Coasting and sliding subspaces can be understood as (A, B)-invariant subspaces or almost controllability subspaces, respectively, which contain no nontrivial controllability subspace. To formulate this property in mathematically correct terms, the upper *semilattice structure* of the set of (A, B)-invariant subspaces contained in a given subspace is presented next.

Proposition 2.9. *The sum of any two (A, B)-invariant subspaces is an (A, B)-invariant subspace. Let any linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$ of system (2.1) be given. The set of all (A, B)-invariant subspaces contained in $\mathcal{U}$ forms an upper semilattice with respect to $\subset, +$, hence it admits a supremum, the maximal (A, B)-invariant subspace contained in $\mathcal{U}$, which will be denoted by $\mathcal{V}^*(\mathcal{U})$.*

The analogous result holds for almost (A, B)-invariant subspaces, controllability subspaces and almost controllability subspaces. The corresponding suprema will be denoted by ${\mathcal{V}_a}^*(\mathcal{U})$, $\mathcal{R}^*(\mathcal{U})$ and $\mathcal{R}_a^*(\mathcal{U})$, respectively.

Now it is possible to formulate the following charaterization of coasting and sliding subspaces.

Proposition 2.10. *A subspace $\mathcal{C}$ is coasting if and only if $\mathcal{C}$ is (A,B)-invariant and $\mathcal{R}^*(\mathcal{C}) = \{0\}$ holds.*

A subspace $\mathcal{S}$ is sliding if and only if $\mathcal{S}$ is an almost controllability subspace and $\mathcal{R}^*(\mathcal{S}) = \{0\}$ holds.

Using these concepts, Proposition 2.7 can be refined as follows.

Proposition 2.11. *For every almost (A,B)-invariant subspace $\mathcal{V}_a$ there exist a coasting subspace $\mathcal{C}$ and a sliding subspace $\mathcal{S}$ such that*

$$\begin{aligned} \mathcal{V}_a &= \mathcal{C} \oplus \mathcal{R} \oplus \mathcal{S} \ , \\ \mathcal{R} &= \mathcal{R}^*(\mathcal{V}_a) \ , \\ \mathcal{C} \oplus \mathcal{R} &= \mathcal{V}^*(\mathcal{V}_a) \ \textit{and} \\ \mathcal{R} \oplus \mathcal{S} &= \mathcal{R}_a^*(\mathcal{V}_a) \ . \end{aligned}$$

If $\mathcal{R}^*(\mathcal{V}_a) = \{0\}$ then $\mathcal{C}$ and $\mathcal{S}$ are uniquely determined.

Note the following consequence of this decomposition result: If an almost (A,B)-invariant subspace is both (A,B)-invariant and an almost controllability subspace, then it is a controllability subspace.

Remark 2.12. Apparently the notion of (almost) (A,B)-invariance depends on the function space the behavior defined by (2.1) lives in. It has been shown by Willems [Wil81] that (A,B)-invariance with respect to the space of *distributions* is the same as almost (A,B)-invariance as it is presented here. For (almost) controllability subspaces a similar result holds.

Computations done by the author indicate that using *hyperfunctions* with compact support (see Morimoto [Mor93]) instead of distributions does not change this picture. Whether the use of hyperfunctions with non compact support would generate a new notion of invariance is an open problem.

2.1.1 Subspace algorithms I

In this section two algorithms for computing some types of maximal invariant subspaces are presented. Both algorithms (ISA and ACSA) were already used by Wonham [Won74]. The meaning of the limit of ACSA was discovered by Willems [Wil80].

Proposition 2.13. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then $\mathcal{V}^*(\mathcal{U})$ coincides with the limit $\mathcal{V}^\infty(\mathcal{U})$ of the* invariant subspace algorithm*:*

$$\begin{aligned} \mathcal{V}^1(\mathcal{U}) &= \mathcal{U} \ , \\ \mathcal{V}^{i+1}(\mathcal{U}) &= \mathcal{U} \cap A^{-1}(\mathcal{V}^i(\mathcal{U}) + \operatorname{Im} B) \ . \end{aligned} \tag{ISA}$$

Note, that the sequence $(\mathcal{V}^i(\mathcal{U}))_{i\in\mathbb{N}}$ is nonincreasing. Hence the limit $\mathcal{V}^\infty(\mathcal{U})$ exists.

Proposition 2.14. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then $\mathcal{R}_a^*(\mathcal{U})$ coincides with the limit $\mathcal{R}^\infty(\mathcal{U})$ of the* almost controllability subspace algorithm*:*

$$\begin{aligned} \mathcal{R}^0(\mathcal{U}) &= \{0\} \ , \\ \mathcal{R}^{i+1}(\mathcal{U}) &= \mathcal{U} \cap (A\mathcal{R}^i(\mathcal{U}) + \operatorname{Im} B) \ . \end{aligned} \tag{ACSA}$$

Here the sequence $(\mathcal{R}^i(\mathcal{U}))_{i\in\mathbb{N}_0}$ is nondecreasing and contained in $\mathcal{U}$. Hence the limit $\mathcal{R}^\infty(\mathcal{U})$ exists. Using Proposition 2.11, similar results for controllability subspaces and almost (A, B)-invariant subspaces are achieved.

Proposition 2.15. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then*

(1) $\mathcal{R}^*(\mathcal{U}) = \mathcal{V}^\infty(\mathcal{U}) \cap \mathcal{R}^\infty(\mathcal{U}) = \mathcal{R}^\infty(\mathcal{V}^\infty(\mathcal{U})) = \mathcal{V}^\infty(\mathcal{R}^\infty(\mathcal{U}))$ *and*

(2) $\mathcal{V}_a{}^*(\mathcal{U}) = \mathcal{V}^\infty(\mathcal{U}) + \mathcal{R}^\infty(\mathcal{U})$.

The preceding results can be used to obtain the following geometric characterisations of coasting and sliding subspaces, respectively.

Proposition 2.16. *A subspace $\mathcal{C}$ is coasting if and only if $\mathcal{C}$ is (A, B)-invariant and $\mathcal{R}^\infty(\mathcal{C}) = \{0\}$ or, equivalently, $\mathcal{C} \cap \operatorname{Im} B = \{0\}$ holds.*

A subspace $\mathcal{S}$ is sliding if and only if $\mathcal{S}$ is an almost controllability subspace and $\mathcal{V}^\infty(\mathcal{S}) = \{0\}$ holds.

Proof. Let $\mathcal{C}$ be (A, B)-invariant, then $\mathcal{V}^*(\mathcal{C}) = \mathcal{C}$ and $\mathcal{C}$ is coasting if and only if $\mathcal{R}^*(\mathcal{C}) = \{0\}$ (Proposition 2.10). According to Propositions 2.15 and 2.13 this is the case if and only if $\mathcal{R}^\infty(\mathcal{C}) = \{0\}$. Since the sequence $\mathcal{R}^i(\mathcal{C})$ is nondecreasing this is the case if and only if $\mathcal{R}^1(\mathcal{C}) = \mathcal{C} \cap \operatorname{Im} B = \{0\}$.

Let $\mathcal{S}$ be an almost controllability subspace with respect to the pair (A, B), then $\mathcal{R}_a^*(\mathcal{S}) = \mathcal{S}$ and $\mathcal{S}$ is sliding if and only if $\mathcal{R}^*(\mathcal{S}) = \{0\}$ (Proposition 2.10). According to Propositions 2.15 and 2.14 this is the case if and only if $\mathcal{V}^\infty(\mathcal{S}) = \{0\}$. □

2.1.2 Feedback transformations

The results on controllability indices presented in the first part of this section are due to Brunovsky [Bru70] and Wonham and Morse [WM72]. A nice treatment of the subject can be found in the book by Wonham [Won74, Chapter 5.7]. The results on Kronecker indices presented in the second part of this section are due to Helmke [Hel82, Hel85].

Note that all types of invariant subspaces introduced in the first part of this chapter depend only on the linear maps represented by the pair of matrices (A, B) and have feedback characterizations. That is to say, these concepts are invariant under *feedback transformations*

$$((T, F, S), (A, B)) \mapsto (T(A + BF)T^{-1}, TBS^{-1}) ,$$

where $F \in \mathcal{F}^{m\times n}$ is an arbitrary matrix and $T \in \mathcal{F}^{n\times n}$, $S \in \mathcal{F}^{m\times m}$ are invertible. In fact, T describes the choice of a basis in the state space $\mathcal{F}^n$, S describes the choice of a basis in the input space $\mathcal{F}^m$, F describes constant state feedback. All such transformations form a group, the *feedback transformation group*, which acts on the set of controllable pairs (A, B). Let Ω be the orbit space of this action. Two pairs belonging to the same orbit are called *feedback equivalent*.

Proposition 2.17. *There is a bijection from Ω onto the set of all lists of integers $\kappa = (\kappa_1, \dots, \kappa_m)$ with $\kappa_1 \geq \cdots \geq \kappa_m \geq 0$ and $\kappa_1 + \cdots + \kappa_m = n$.*

Given a controllable pair (A, B), the corresponding numbers $(\kappa_1, \dots, \kappa_m)$ are called the *controllability indices* of the pair.

Proposition 2.18. *A controllable pair (A, B) with controllability indices $(\kappa_1, \dots, \kappa_m)$ is feedback equivalent to the pair*

$$\mathsf{A} = \begin{array}{c}\left(\begin{array}{c|c|c}
\begin{matrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{matrix} & & \\ \hline
 & \ddots & \\ \hline
 & & \begin{matrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{matrix}
\end{array}\right)\\
\underbrace{\hphantom{0\;1\;\ddots\;1\;0}}_{\kappa_1}\quad \cdots \quad \underbrace{\hphantom{0\;1\;\ddots\;1\;0}}_{\kappa_m}\end{array}
\quad \textit{and} \quad
\mathsf{B} = \left(\begin{array}{c|c|c}
\begin{matrix}0\\\vdots\\0\\1\end{matrix} & & \\ \hline
 & \ddots & \\ \hline
 & & \begin{matrix}0\\\vdots\\0\\1\end{matrix}
\end{array}\right)
\begin{matrix}\left.\vphantom{\begin{matrix}0\\\vdots\\0\\1\end{matrix}}\right\}\kappa_1\\ \vdots \\ \left.\vphantom{\begin{matrix}0\\\vdots\\0\\1\end{matrix}}\right\}\kappa_m\end{matrix} .$$

This pair is said to be in Brunovsky canonical form. *Actually, the Brunovsky canonical form is a normal form for the feedback equivalence relation.*

For the pupose of parametrization or classification of invariant subspaces (of any of the above types) of a given controllable system (2.1), it is sometimes convenient to assume that the system, i.e. the matrix pair (A, B), is in Brunovsky canonical form. This can be done without loss of generality. Note that the previous proposition implies that the number of nonzero controllability indices is equal to $\operatorname{rk} B$.

Helmke [Hel82] considered the action of the following subgroup of the feedback transformation group. A *restricted feedback transformation* is defined by

$$((T, F, U), (A, B)) \mapsto (T(A + BF)T^{-1}, TBU^{-1}) \ ,$$

where $F \in \mathcal{F}^{m\times n}$ is an arbitrary matrix, $T \in \mathcal{F}^{n\times n}$ is invertible and $U \in \mathcal{F}^{m\times m}$ is *unipotent* upper triangular (i.e. U is invertible and upper triangular with all diagonal entries equal to 1). The choice of a basis in the input space $\mathcal{F}^m$ is thus restricted to unipotent transformations of the standard basis. All such transformations form a group, the *restricted feedback transformation group*, which acts on the set of all controllable pairs (A, B). Let $\hat{\Omega}$ be the orbit space of this action. Two pairs belonging to the same orbit are called *restricted feedback equivalent*.

Proposition 2.19. *There is a bijection from $\hat{\Omega}$ onto the set of all unordered lists of nonnegative integers $K = (K_1, \ldots, K_m)$ with $K_1 + \cdots + K_m = n$.*

Given a controllable pair (A, B), the corresponding numbers are called the *Kronecker indices* of the pair.

Proposition 2.20. *A controllable pair (A, B) with Kronecker indices $(K_1, \ldots, K_m)$ is restricted feedback equivalent to a pair of the form shown in Proposition* 2.18 *but with block sizes $K_1, \ldots, K_m$. This pair is said to be in* Kronecker canonical form. *Actually, the Kronecker canonical form is a normal form for the restricted feedback equivalence relation.*

Apparently the only difference between controllability indices and Kronecker indices is that the former are ordered while the latter need not to be ordered. The precise connection is as follows.

Proposition 2.21. *Let the controllable pair (A, B) have controllability indices $(\kappa_1, \ldots, \kappa_m)$ and Kronecker indices $(K_1, \ldots, K_m)$. Then there exists a permutation $\pi \in S(m)$ such that*

$$(K_1, \ldots, K_m) = (\kappa_{\pi(1)}, \ldots, \kappa_{\pi(m)}) \ .$$

Proof. Let (A_K, B_K) be the Kronecker canonical form of (A, B). Then (A_K, B_K) is restricted feedback equivalent, hence feedback equivalent to (A, B). Therefore (A_K, B_K) is feedback equivalent to the Brunovsky canonical form (A_B, B_B) of (A, B).

Let $\pi^{-1} \in S(m)$ be a permutation that orders $(K_1, \ldots, K_m)$, i.e. $K_{\pi^{-1}(1)} \geq \cdots \geq K_{\pi^{-1}(m)}$. Let P^{-1} be a permutation matrix that represents π^{-1} and let $\hat{P}^{-1}$ be a permutation matrix that permutes the blocks in (A_K, B_K) in the same way. Then $(\hat{A}, \hat{B}) = (\hat{P} A_K \hat{P}^{-1}, \hat{P} B_K P^{-1})$ is in Brunovsky canonical form and feedback equivalent (in general *not* restricted feedback equivalent, since P is in general not unipotent) to (A_K, B_K).

It follows that $(\hat{A}, \hat{B}) = (A_B, B_B)$ and $(K_1, \ldots, K_m) = (\kappa_{\pi(1)}, \ldots, \kappa_{\pi(m)})$. $\square$

As a consequence of this statement the orbits of the feedback transformation group action (*Brunovsky strata*) decompose into orbits of the restricted feedback transformation group action (*Kronecker strata*) and this decomposition is parametrized by the permutations $\pi \in S(m)$. In the dual setting of Section 6.3 this fact will be used to identify a certain cell decomposition as a Bruhat decomposition.

Remark 2.22. The (restricted) feedback transformation group also acts on the set of all matrix pairs $(A, B) \in \mathcal{F}^{n\times n} \times \mathcal{F}^{n\times m}$. The notion of (restricted) feedback equivalence will also be used in the general case. A complete set of feedback invariants is provided by the controllability indices together with the *invariant factors* (*uncontrollable modes*) of (A, B), see e.g. Wolovich [Wol74] for the details. For restricted feedback equivalence a corresponding result seems to be unknown.

2.1.3 Restricted and corestricted systems I

In this section the meaning of "restricting" and "corestricting" system (2.1) to a given (A, B)-invariant subspace $\mathcal{V}$ is clarified.

The notion of (co)restriction can be easily understood if one considers the linear maps represented by the matrices A, B and so on. Given an (A, B)-invariant subspace $\mathcal{V}$, Proposition 2.3 asserts that there exists a feedback matrix F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$, i.e. $\mathcal{V}$ is an invariant subspace of the linear map represented by $A_F = A + BF$. But then the commutative diagram of linear maps

$$\begin{tikzcd}
\mathcal{F}^m / B^{-1}(\mathcal{V}) \arrow[r, "\tilde{B}"] & \mathcal{F}^n/\mathcal{V} \arrow[r, "\tilde{A}"] & \mathcal{F}^n/\mathcal{V} \\
\mathcal{F}^m \arrow[u, two heads] \arrow[r, "B"] & \mathcal{F}^n \arrow[u, two heads] \arrow[r, "A_F"] & \mathcal{F}^n \arrow[u, two heads] \\
B^{-1}(\mathcal{V}) \arrow[u, hook] \arrow[r, "\bar{B}"] & \mathcal{V} \arrow[u, hook] \arrow[r, "\bar{A}"] & \mathcal{V} \arrow[u, hook]
\end{tikzcd}$$

defines a *restriction* $(\bar{A}, \bar{B})$ and a *corestriction* $(\tilde{A}, \tilde{B})$ of the pair (A, B) to $\mathcal{V}$ in terms of linear maps. Of course, both of them depend on the choice of F. Let $k = \dim \mathcal{V}$ and $l = \dim B^{-1}(\mathcal{V})$. The matrix representations $(\bar{A}, \bar{B}) \in \mathcal{F}^{k \times k} \times \mathcal{F}^{k \times l}$ and $(\tilde{A}, \tilde{B}) \in \mathcal{F}^{(n-k)\times(n-k)} \times \mathcal{F}^{(n-k)\times(m-l)}$ of the (co)restriction depend on the choice of a basis in $\mathcal{V}$ and $B^{-1}(\mathcal{V})$, also. More precisely, the following holds.

Proposition 2.23. *Let $\mathcal{V}$ be an (A, B)-invariant subspace. Any two matrix representations of (co)restrictions of the pair (A, B) to $\mathcal{V}$ are feedback equivalent.*

Proof. Let $(\bar{A}_1, \bar{B}_1)$ and $(\bar{A}_2, \bar{B}_2)$ be two matrix representations of restrictions of (A, B) to $\mathcal{V}$ with corresponding feedback matrices F_1 and F_2, respectively. Since $\bar{B}_1$ and $\bar{B}_2$ do not depend on the choice of feedback, they represent the same map $\bar{B} : B^{-1}(\mathcal{V}) \longrightarrow \mathcal{V}$. Hence there exist invertible $T \in \mathcal{F}^{k\times k}$ and $S \in \mathcal{F}^{l \times l}$ such that $\bar{B}_2 = T\bar{B}_1 S^{-1}$.

Furthermore, $(A+BF_1)\mathcal{V} \subset \mathcal{V}$ and $(A+BF_2)\mathcal{V} \subset \mathcal{V}$, hence $B(F_2 - F_1)\mathcal{V} \subset \mathcal{V}$ and

$$(F_2 - F_1)\mathcal{V} \subset B^{-1}(\mathcal{V}) \ .$$

It follows that

$$(A + BF_2)|_{\mathcal{V}} = (A + BF_1)|_{\mathcal{V}} + \bar{B}(F_2 - F_1)|_{\mathcal{V}} \ . \tag{2.2}$$

Let $\bar{F} \in \mathcal{F}^{l\times k}$ represent the map $(F_2 - F_1)|_{\mathcal{V}} : \mathcal{V} \longrightarrow B^{-1}(\mathcal{V})$ in the basis corresponding to $(\bar{A}_1, \bar{B}_1)$. Then the map on the right side of equation (2.2) is represented by $\bar{A}_1 + \bar{B}_1\bar{F}$ in the basis corresponding to $(\bar{A}_1, \bar{B}_1)$. On the other hand the map on the left side of equation (2.2) is represented by $\bar{A}_2$ in the basis corresponding to $(\bar{A}_2, \bar{B}_2)$. It follows that $\bar{A}_2 = T(\bar{A}_1 + \bar{B}_1\bar{F})T^{-1}$.

The result on corestrictions follows along the same lines. □

Remark 2.24. Note that the restriction $(\bar{A}, \bar{B})$ of a controllable pair (A, B) to $\mathcal{V}$ is controllable if and only if $\mathcal{V}$ is a controllability subspace, while the corestriction $(\tilde{A}, \tilde{B})$ is always controllable. Note further that $\bar{B}$ and $\tilde{B}$ are both of full column rank if B has full column rank.

Corollary 2.25. *The controllability indices of all matrix representations of controllable restrictions (resp. corestrictions) coincide. They are called the* (co)restriction indices *of (A, B) with respect to $\mathcal{V}$.*

The situation is even more convenient since there is a canonical choice for the matrix representation of a controllable (co)restriction of the pair (A, B) to $\mathcal{V}$.

Proposition 2.26. *Given an (A, B)-invariant subspace $\mathcal{V}$ and a matrix representation of a (co)restriction of (A, B) to $\mathcal{V}$, every feedback equivalent matrix pair is also a matrix representation of a (co)restriction of (A, B) to $\mathcal{V}$.*

Proof. Let $(\bar{A}_0, \bar{B}_0)$ be a matrix representation of a restriction of (A, B) to $\mathcal{V}$ with corresponding feedback matrix F_0. Given invertible $T \in \mathcal{F}^{k\times k}$ and $S \in \mathcal{F}^{l\times l}$ as well as any $\bar{F} \in \mathcal{F}^{l\times k}$, it has to be shown that the pair

$$(\bar{A}, \bar{B}) := (T(\bar{A}_0 + \bar{B}_0\bar{F})T^{-1}, T\bar{B}_0 S^{-1})$$

is a matrix representation of a restriction of (A, B) to $\mathcal{V}$.

Let $Z_0 \in \mathcal{F}^{n\times k}$ with $\mathcal{V} = \operatorname{Im} Z_0$ and $W_0 \in \mathcal{F}^{m\times l}$ with $B^{-1}(\mathcal{V}) = \operatorname{Im} W_0$ be the bases corresponding to $(\bar{A}_0, \bar{B}_0)$. Then the defining diagram yields

$$(A + BF_0)Z_0 = Z_0\bar{A}_0 \quad \text{and} \quad BW_0 = Z_0\bar{B}_0$$

and $Z = Z_0T^{-1}$ and $W = W_0S^{-1}$ are the bases corresponding to $(\bar{A}, \bar{B})$. It follows $Z\bar{B} = Z_0T^{-1}T\bar{B}_0S^{-1} = BW_0S^{-1} = BW$. Furthermore

$$\begin{aligned} Z\bar{A} &= Z_0T^{-1}T(\bar{A}_0 + \bar{B}_0\bar{F})T^{-1} \\ &= Z_0\bar{A}_0T^{-1} + Z_0\bar{B}_0\bar{F}T^{-1} \\ &= (A + BF_0)Z_0T^{-1} + BW_0\bar{F}T^{-1} \\ &= (A + BF_0)Z + BW_0\bar{F}T^{-1}\ . \end{aligned}$$

Take $F \in \mathcal{F}^{m\times n}$ with $FZ_0 = W_0\bar{F}$. Then $Z\bar{A} = (A + BF_0)Z + BFZ_0T^{-1} = (A + B(F_0 + F))Z$.

Again the result on corestrictions follows along the same lines. □

Now Proposition 2.18 immediately yields the following corollary.

Corollary 2.27. *Given an (A, B)-invariant subspace $\mathcal{V}$ with controllable (co)restriction, there exists a feedback matrix F and a matrix representation of the resulting (co)restriction, which is in Brunovsky canonical form.*

Sometimes this special matrix representation is referred to as *the* (co)restriction of the pair (A, B) to $\mathcal{V}$ or *the* (co)restricted system on $\mathcal{V}$. Propositions 2.23 and 2.26 together yield the following result on the degree of uniqueness of matrix representations of (co)restrictions.

Proposition 2.28. *Let $(\bar{A}_0, \bar{B}_0)$ be a matrix representation of a restriction of (A, B) to $\mathcal{V}$. Then the set*

$$\{(T(\bar{A}_0 + \bar{B}_0\bar{F})T^{-1}, T\bar{B}_0S^{-1}) \,|\, T \in \mathcal{F}^{k\times k}, S \in \mathcal{F}^{l\times l}, \bar{F} \in \mathcal{F}^{l\times k},$$
$$T, S \text{ invertible}\}$$

is the set of all matrix representations of all restrictions of (A, B) to $\mathcal{V}$. For corestrictions the analogous result holds.

In the dual setting of Section 6.1 a similar but slightly different notion of restrictions (which play the role of corestrictions then) involving unipotent transformations will be used to obtain the cell decomposition of Section 6.3. The following proposition characterizes the admissible corestriction indices of a given pair (A, B).

Proposition 2.29. *Let (A, B) have controllability indices $(\kappa_1, \ldots, \kappa_m)$ and let $(\lambda_1, \ldots, \lambda_{m-l})$ be the corestriction indices of (A, B) with respect to the (A, B)-invariant subspace $\mathcal{V}$. Then*

$$\lambda_i \leq \kappa_i\ , \qquad i = 1, \ldots, m-l\ . \tag{2.3}$$

Conversely, for all integers $0 \leq l < m$ and all lists of integers $(\lambda_1, \ldots, \lambda_{m-l})$ with $\lambda_1 \geq \cdots \geq \lambda_{m-l} \geq 1$ and $\lambda_1 + \cdots + \lambda_{m-l} = n - k \leq n$ satisfying equation (2.3) there exists an (A, B)-invariant subspace $\mathcal{V}$ of dimension k, such that (A, B) has corestriction indices $(\lambda_1, \ldots, \lambda_{m-l})$ with respect to $\mathcal{V}$.

The first statement of (the first part of) Proposition 2.29 the author could find is contained in a paper by Heymann [Hey76, Theorem 7.3]. Another reference (for the full statement) is Baragaña and Zaballa [BZ95], who discuss also the general case of corestrictions, where the pair (A, B) is not controllable.

Heymann also derived a result on the restriction indices of a controllable pair (A, B) with respect to a controllability subspace [Hey76, Theorem 7.5]. Baragaña and Zaballa [BZ97] obtained similar results for restrictions of a controllable pair (A, B) to an (A, B)-invariant subspace. In a recent paper [BZ99] they address restrictions in the general case.

2.2 Duality

Duality is a fundamental tool in linear systems theory. The basic idea is the following: associate to any system another system, the dual system, and to any property of a system another property, the dual property, such that a system exhibits a special property if and only if the dual system exhibits the dual property. Instead of proving that a given system has a special property, it is possible to prove that its dual system has the dual property, then. If both properties happen to be of interest, characterizations for both of them can be derived in one step.

Of course, the notion of duality comes from linear algebra. In this section the connection between system theoretic duality and duality of linear maps is explained. The presentation follows the book by Knobloch and Kappel [KK74].

Consider system (2.1). The way it acts on a finite time interval $[0, T]$ can be characterized in terms of three linear maps:

1. The *input-state function*

$$\begin{array}{rcl} \sigma_T : \mathrm{C}^m_{\mathrm{pw}} & \longrightarrow & \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} \ , \\ u(.) & \mapsto & (0, Bu(.)) = (0, g(.)) \ , \end{array}$$

   which associates to an input function $u(.)$ the starting point 0 together with the inhomogeneous part $g(.) = Bu(.)$ of the differential equation.

2. The *state-state function*

$$\begin{array}{rcl} \omega_T : \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} & \longrightarrow & \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} \ , \\ (x_0, g(.)) & \mapsto & (\mathrm{e}^{AT} x_0 + \int\limits_0^T \mathrm{e}^{A(T-\tau)} g(\tau)\mathrm{d}\tau, \\ & & \qquad \mathrm{e}^{A(.)} x_0 + \int\limits_0^{.} \mathrm{e}^{A(.-\tau)} g(\tau)\mathrm{d}\tau) \\ & & = (x_T, x(.)) \ , \end{array}$$

   which associates to a starting point x_0 together with an inhomogeneous part $g(.)$ of the differential equation an end point x_T together with the solution trajectory $x(.)$.

3. The *state-output function*

$$\begin{array}{rcl} \tau_T : \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} & \longrightarrow & \mathrm{C}^p_{\mathrm{pw}} \ , \\ (x_T, x(.)) & \mapsto & Cx(.) = y(.) \ , \end{array}$$

which associates to an end point x_T together with a state function $x(.)$ an output function $y(.) = Cx(.)$.

Here $\mathrm{C}^k_{\mathrm{pw}}$ denotes the space of all piecewise continuous functions $f : [0, T] \longrightarrow \mathcal{F}^k$. This function space is chosen for simplicity. The analogous discussion in $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathcal{F}^k)$, the space of locally integrable functions $f : \mathbb{R} \longrightarrow \mathcal{F}^k$ would require considerably more effort, while the result is the same. Consider the standard inner products on $\mathcal{F}^k$ and $\mathrm{C}^k_{\mathrm{pw}}$:

$$\begin{array}{rcl} (x_1, x_2) & = & x_1^* x_2 \quad \text{and} \\ (f_1(.), f_2(.)) & = & \int\limits_0^T f_1^*(\tau) f_2(\tau) \mathrm{d}\tau \ . \end{array}$$

With respect to these inner products the adjoint linear maps to σ_T, ω_T and τ_T are the following.

1. The state-output function

$$\begin{array}{rcl} \sigma_T^* : \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} & \longrightarrow & \mathrm{C}^m_{\mathrm{pw}} \ , \\ (x_0, x(.)) & \mapsto & B^* x(.) = u(.) \ . \end{array}$$

2. The state-state function

$$\begin{array}{rcl} \omega_T^* : \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} & \longrightarrow & \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} \ , \\ (x_T, g(.)) & \mapsto & (\mathrm{e}^{-A^*(-T)} \, x_T - \int\limits_T^0 \mathrm{e}^{-A^*(-\tau)} \, g(\tau) \mathrm{d}\tau, \\ & & \qquad \mathrm{e}^{-A^*(.-T)} \, x_T - \int\limits_T^{.} \mathrm{e}^{-A^*(.-\tau)} \, g(\tau) \mathrm{d}\tau) \\ & & = (x_0, x(.)) \ . \end{array}$$

3. The input-state function

$$\begin{array}{rcl} \tau_T^* : \mathrm{C}^p_{\mathrm{pw}} & \longrightarrow & \mathcal{F}^n \times \mathrm{C}^n_{\mathrm{pw}} \ , \\ y(.) & \mapsto & (0, C^* y(.)) = (0, g(.)) \ . \end{array}$$

In the same way as above these three maps characterize the system

$$\begin{aligned} \dot{x} &= -A^*x - C^*y \\ u &= B^*x \ , \end{aligned}$$

on the finite interval $[0,T]$, where the time is reversed, i.e. goes from T to 0. The minus signs in the first equation are due to the requirement that $x(.)$ has to be a solution trajectory of the system (note that $g(.) = +C^*y(.)$). This system is usually replaced by the following *dual system* to system (2.1), which is equivalent by time reversal:

$$\begin{aligned} \dot{x} &= A^*x + C^*y \\ u &= B^*x \ . \end{aligned} \tag{2.4}$$

Note, that in the dual system the roles of input and output are exchanged. For example, system (2.1) is controllable if and only if the dual system (2.4) is observable.

2.3 Almost (C, A)-invariant subspaces

In this section the dual notions to those of Section 2.1 are defined. The concept of almost (C, A)-invariant subspaces was introduced by Willems [Wil82].

As stated in Section 2.1.2, all types of invariant subspaces of system (2.1) that have been introduced in Section 2.1 do only depend on the pair of matrices (A, B). In view of Section 2.2, especially system (2.4), the dual notions should only depend on the pair of matrices (C, A). Indeed, this is the case. The following definitions make use of the feedback characterizations of the dual case (cf. Section 2.1). They translate to *output injection* characterizations here. Dynamical characterizations in terms of observers are derived in Chapter 3.

Definition 2.30. A linear subspace $\mathcal{V}$ of the state space $\mathcal{F}^n$ of system (2.1) is called (C, A)-*invariant*, if there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ such that $A^J\mathcal{V} \subset \mathcal{V}$, where $A^J = A - JC$. Such a J is called a *friend of* $\mathcal{V}$.

A linear subspace $\mathcal{V}_a$ of the state space $\mathcal{F}^n$ of system (2.1) is called *almost* (C, A)-*invariant*, if for every $\varepsilon > 0$ there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ such that $\operatorname{dist}(\mathrm{e}^{A^J t}x_0, \mathcal{V}_a) < \varepsilon$ for all $t \geq 0$ and $x_0 \in \mathcal{V}_a$.

In view of Propositions 2.3 and 2.7 there are the following geometric characterizations.

Proposition 2.31. *A subspace $\mathcal{V}$ is (C, A)-invariant if and only if $\mathcal{V}^\perp$ is (A^*, C^*)-invariant or, equivalently, if and only if $A(\mathcal{V} \cap \operatorname{Ker} C) \subset \mathcal{V}$.*

A subspace $\mathcal{V}_a$ is almost (C, A)-invariant if and only if ${\mathcal{V}_a}^\perp$ is almost (A^*, C^*)-invariant.

Next the duals of (almost) controllability subspaces are introduced. Recall the notion of $\mathbb{C}_g$, a 'good' part of the complex plane (cf. Section 2.1). Let

$$\mathcal{N}(C, A) := \operatorname{Ker} O_n(C, A) := \operatorname{Ker} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

denote the *unobservable subspace* of system (2.1).

Definition 2.32. A linear subspace $\mathcal{O}$ of the state space $\mathcal{F}^n$ of system (2.1) is called an *observability subspace*, if for every monic polynomial of degree $\operatorname{codim} \mathcal{O}$ there exists an output injection matrix $J \in \mathcal{F}^{n \times p}$ such that $A^J \mathcal{O} \subset \mathcal{O}$ and the characteristic polynomial of $A^J|_{\mathcal{F}^n/\mathcal{O}}$ is equal to p.

A linear subspace $\mathcal{O}_a$ of the state space $\mathcal{F}^n$ of system (2.1) is called an *almost observability subspace*, if for every set $\mathbb{C}_g$ and every $\varepsilon > 0$ there exists an output injection matrix $J \in \mathcal{F}^{n \times p}$ such that $\operatorname{dist}(\mathrm{e}^{A^J t} x_0, \mathcal{O}_a) < \varepsilon$ for all $t \geq 0$ and $x_0 \in \mathcal{O}_a$, while the spectrum of $A^J|_{\mathcal{F}^n/\mathcal{N}(C,A)}$ lies in $\mathbb{C}_g$.

The duals of controllability subspaces have already been studied by Morse [Mor73], they were named (complementary) observability subspaces by Willems and Commault [WC81]. Note that $\mathbb{C}_g$ can be the same for every ε in the definition of almost observability subspaces. This corresponds to T not depending on ε in the definition of almost controllability subspaces (cf. Definition 2.4). There are the following characterizations, dual to Proposition 2.5.

Proposition 2.33. *A subspace $\mathcal{O}$ is an observability subspace with respect to the pair (C, A) if and only if $\mathcal{O}^\perp$ is a controllability subspace with respect to the pair (A^*, C^*). This is the case if and only if there exists an output injection matrix $J \in \mathcal{F}^{n \times p}$ and a linear space $\operatorname{Ker} C \subset \mathcal{K}_1 \subset \mathcal{F}^n$ such that*

$$\mathcal{O} = \mathcal{K}_1 \cap (A^J)^{-1} \mathcal{K}_1 \cap \cdots \cap (A^J)^{-n+1} \mathcal{K}_1 \ .$$

A subspace $\mathcal{O}_a$ is an almost observability subspace with respect to the pair (C, A) if and only if ${\mathcal{O}_a}^{\perp}$ is an almost controllability subspace with respect to the pair (A^*, C^*). This is the case if and only if there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ and a chain of linear spaces $\operatorname{Ker} C \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_r \subset \mathcal{F}^n$ such that

$$\mathcal{O}_a = \mathcal{K}_1 \cap (A^J)^{-1}\mathcal{K}_2 \cap \cdots \cap (A^J)^{-r+1}\mathcal{K}_r \ .$$

Since $\mathcal{N}(C, A - JC) = \mathcal{N}(C, A)$ for every output injection matrix $J \in \mathcal{F}^{n\times p}$, any (almost) observability subpace contains the unobservable subpace.

Now it is possible to formulate a geometric characterization of almost (C, A)-invariant subspaces, dual to Proposition 2.7.

Proposition 2.34. *A subspace $\mathcal{V}_a$ is almost (C, A)-invariant if and only if there exist a (C, A)-invariant subspace $\mathcal{V}$ and an almost observability subspace $\mathcal{O}_a$ such that*

$$\mathcal{V}_a = \mathcal{V} \cap \mathcal{O}_a \ .$$

To achieve further insight into the structure of almost (C, A)-invariant subspaces, the duals to coasting and sliding subspaces are introduced. The definition uses the lower *semilattice structure* of the set of (C, A)-invariant subspaces containing a given one (dual result to Proposition 2.9).

Proposition 2.35. *The intersection of any two (C, A)-invariant subspaces is a (C, A)-invariant subspace. Let any linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$ of system* (2.1) *be given. The set of all (C, A)-invariant subspaces containing $\mathcal{U}$ forms a lower semilattice with respect to $\subset, \cap$, hence it admits an infimum, the minimal (C, A)-invariant subspace containing $\mathcal{U}$, which will be denoted by $\mathcal{V}_*(\mathcal{U})$.*

The analogous result holds for almost (C, A)-invariant subspaces, observability subspaces and almost observability subspaces. The corresponding infima will be denoted by $\mathcal{V}_{a*}(\mathcal{U})$, $\mathcal{O}_*(\mathcal{U})$ and $\mathcal{O}_{a*}(\mathcal{U})$, respectively.

Definition 2.36. A linear subspace $\mathcal{T}$ of the state space $\mathcal{F}^n$ of system (2.1) is called *tight*, if $\mathcal{T}$ is (C, A)-invariant and $\mathcal{O}_*(\mathcal{T}) = \mathcal{F}^n$ holds.

A linear subspace $\mathcal{I}$ of the state space $\mathcal{F}^n$ of system (2.1) is called *instantaneous*, if $\mathcal{I}$ is an almost observability subspace and $\mathcal{O}_*(\mathcal{I}) = \mathcal{F}^n$ holds.

Tight and instantaneous subspaces are (C, A)-invariant subspaces or almost observability subspaces, respectively, which are not contained in any observability subspace other than $\mathcal{F}^n$. The name tight subspace was first used by Fuhrmann and Helmke [FH97]. For the duals of sliding subspaces there seems to be no well accepted terminology in the literature. Willems called the observers appearing in the dynamic characterization of $\mathcal{I}$ with $\mathcal{I}^\perp$ sliding *instantaneously acting observers* (cf. Chapter 3). Hence the name instantaneous subspace is chosen here.

The next result follows immediately by dualizing Proposition 2.10.

Proposition 2.37. *A subspace $\mathcal{T}$ is tight if and only if $\mathcal{T}^\perp$ is coasting. A subspace $\mathcal{I}$ is instantaneous if and only if $\mathcal{I}^\perp$ is sliding.*

Using these concepts, the dual version of Proposition 2.11 can be formulated. It refines Proposition 2.34.

Proposition 2.38. *For every almost (C, A)-invariant subspace $\mathcal{V}_a$ there exist a tight subspace $\mathcal{T}$ and an instantaneous subspace $\mathcal{I}$ such that*

$$\begin{aligned} \mathcal{V}_a &= \mathcal{T} \cap \mathcal{O} \cap \mathcal{I} \ , \\ \mathcal{O} &= \mathcal{O}_*(\mathcal{V}_a) \ , \\ \mathcal{T} \cap \mathcal{O} &= \mathcal{V}_*(\mathcal{V}_a) \ \textit{and} \\ \mathcal{O} \cap \mathcal{I} &= \mathcal{O}_{a*}(\mathcal{V}_a) \ . \end{aligned}$$

Here all intersections are transversal*, i.e.* $\mathcal{T} + \mathcal{O} = \mathcal{O} + \mathcal{I} = \mathcal{T} + \mathcal{I} = \mathcal{V}_*(\mathcal{V}_a) + \mathcal{I} = \mathcal{T} + \mathcal{O}_{a*}(\mathcal{V}_a) = \mathcal{F}^n$*. If $\mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$ then $\mathcal{T}$ and $\mathcal{I}$ are uniquely determined.*

Note the following consequence of this result: If an almost (C, A)-invariant subspace is both (C, A)-invariant and an almost observability subspace, then it is an observability subspace.

2.3.1 Subspace algorithms II

In this section the algorithms for computing minimal invariant subspaces are presented. They are dual to the algorithms of Section 2.1.1.

Proposition 2.39. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then $\mathcal{V}_*(\mathcal{U})$ coincides with the limit $\mathcal{V}_\infty(\mathcal{U})$ of the* conditioned invariant subspace algorithm*:*

$$\begin{aligned} \mathcal{V}_1(\mathcal{U}) &= \mathcal{U} \ , \\ \mathcal{V}_{i+1}(\mathcal{U}) &= \mathcal{U} + A(\mathcal{V}_i(\mathcal{U}) \cap \operatorname{Ker} C) \ . \end{aligned} \tag{CISA}$$

Proposition 2.40. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then $\mathcal{O}_{a*}(\mathcal{U})$ coincides with the limit $\mathcal{O}_\infty(\mathcal{U})$ of the* almost observability subspace algorithm*:*

$$\begin{aligned} \mathcal{O}_0(\mathcal{U}) &= \mathcal{F}^n \ , \\ \mathcal{O}_{i+1}(\mathcal{U}) &= \mathcal{U} + (A^{-1}\mathcal{O}_i(\mathcal{U}) \cap \operatorname{Ker} C) \ . \end{aligned} \tag{AOSA}$$

Using Proposition 2.38, similar results for observability subspaces and almost (C, A)-invariant subspaces are achieved.

Proposition 2.41. *Consider an arbitrary linear subspace $\mathcal{U}$ of the state space $\mathcal{F}^n$. Then*

(1) $\mathcal{O}_*(\mathcal{U}) = \mathcal{V}_\infty(\mathcal{U}) + \mathcal{O}_\infty(\mathcal{U}) = \mathcal{O}_\infty(\mathcal{V}_\infty(\mathcal{U})) = \mathcal{V}_\infty(\mathcal{O}_\infty(\mathcal{U}))$ *and*

(2) $\mathcal{V}_{a*}(\mathcal{U}) = \mathcal{V}_\infty(\mathcal{U}) \cap \mathcal{O}_\infty(\mathcal{U})$.

Again these results can be used to obtain geometric characterizations of tight and instantaneous subspaces, respectively. These characterizations are dual to those of Proposition 2.16.

Proposition 2.42. *A subspace $\mathcal{T}$ is tight if and only if $\mathcal{T}$ is (C, A)-invariant and $\mathcal{O}_\infty(\mathcal{T}) = \mathcal{F}^n$ or, equivalently, $\mathcal{T} + \operatorname{Ker} C = \mathcal{F}^n$ holds.*

A subspace $\mathcal{I}$ is instantaneous if and only if $\mathcal{I}$ is an almost observability subspace and $\mathcal{V}_\infty(\mathcal{I}) = \mathcal{F}^n$ holds.

2.3.2 Output injections

In this section the results dual to those of Section 2.1.2 are presented.

All types of invariant subspaces introduced in Section 2.3 depend only on the linear maps represented by the pair of matrices (C, A) and have output injection characterizations. That is to say, these concepts are invariant under *output injection transformations*

$$((T, J, S), (C, A)) \mapsto (SCT^{-1}, T(A - JC)T^{-1}) \ ,$$

where $J \in \mathcal{F}^{n\times p}$ is an arbitrary matrix and $T \in \mathcal{F}^{n\times n}$, $S \in \mathcal{F}^{p\times p}$ are invertible. In fact, T describes the choice of a basis in the state space $\mathcal{F}^n$, S describes the choice of a basis in the output space $\mathcal{F}^p$, J describes constant output injection. All such transformations form a group, the *output injection transformation group*, which acts on the set of observable pairs (C, A). Let Ω be the orbit space of this action. Two pairs belonging to the same orbit are called *output injection equivalent*.

Proposition 2.43. *There is a bijection from Ω onto the set of all lists of integers $\mu = (\mu_1, \dots, \mu_p)$ with $\mu_1 \geq \dots \geq \mu_p \geq 0$ and $\mu_1 + \dots + \mu_p = n$.*

Given an observable pair (C, A), the corresponding numbers $(\mu_1, \dots, \mu_p)$ are called the *observability indices* of the pair.

Proposition 2.44. *An observable pair (C, A) with observability indices $(\mu_1, \dots, \mu_p)$ is output injection equivalent to the pair*

$$\mathsf{C} = \left(\begin{array}{c|c|c} \underbrace{0 \ \dots \ 0 \ 1}_{\mu_1} & & \\ \hline & \ddots & \\ \hline & & \underbrace{0 \ \dots \ 0 \ 1}_{\mu_p} \end{array}\right) \quad \textit{and} \quad \mathsf{A} = \left(\begin{array}{c|c|c} \underbrace{\begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix}}_{\mu_1} & & \\ \hline & \ddots & \\ \hline & & \underbrace{\begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix}}_{\mu_p} \end{array}\right) .$$

This pair is said to be in dual Brunovsky canonical form*. Actually, the dual Brunovsky canonical form is a normal form for the output injection equivalence relation.*

For the purpose of parametrization or classification of invariant subspaces (of any of the above types) of a given observable system (2.1), it is sometimes convenient to assume that the system, i.e. the matrix pair (C, A), is in dual Brunovsky canonical form. This can be done without loss of generality. Note that the previous proposition implies that the number of nonzero observability indices equals $\operatorname{rk} C$.

As in the dual case consider the following subgroup of the output injection transformation group. A *restricted output injection transformation* is defined by

$$((T, J, U), (C, A)) \mapsto (UCT^{-1}, T(A - JC)T^{-1}) ,$$

where $J \in \mathcal{F}^{n\times p}$ is an arbitrary matrix, $T \in \mathcal{F}^{n\times n}$ is invertible and $U \in \mathcal{F}^{p\times p}$ is *unipotent* lower triangular (i.e. U is invertible and lower triangular with all diagonal entries equal to 1). The choice of a basis in the output space $\mathcal{F}^p$ is thus restricted to unipotent transformations of the standard basis. All such transformations form a group, the *restricted output injection transformation group*, which acts on the set of all observable pairs (C, A). Let $\hat{\Omega}$ be the orbit space of this action. Two pairs belonging to the same orbit are called *restricted output injection equivalent*.

Proposition 2.45. *There is a bijection from $\hat{\Omega}$ onto the set of all unordered lists of nonnegative integers $K = (K_1, \ldots, K_p)$ with $K_1 + \cdots + K_p = n$.*

Given an observable pair (C, A), the corresponding numbers are called the *Kronecker indices* of the pair.

Proposition 2.46. *An observable pair (C, A) with Kronecker indices $(K_1, \ldots, K_p)$ is restricted output injection equivalent to a pair of the form shown in Proposition* 2.44 *but with block sizes $K_1, \ldots, K_p$. This pair is said to be in* dual Kronecker canonical form. *Actually, the dual Kronecker canonical form is a normal form for the restricted output injection equivalence relation.*

Again the only difference between observability indices and Kronecker indices is that the former are ordered while the later need not to be ordered. As in the dual case the precise connection is as follows.

Proposition 2.47. *Let the observable pair (C, A) have observability indices $(\mu_1, \ldots, \mu_p)$ and Kronecker indices $(K_1, \ldots, K_p)$. Then there exists a permutation $\pi \in S(p)$ such that*

$$(K_1, \ldots, K_p) = (\mu_{\pi(1)}, \ldots, \mu_{\pi(p)}) .$$

As a consequence of this statement the orbits of the output injection transformation group action (*Brunovsky strata*) decompose into orbits of the restricted output injection transformation group action (*Kronecker strata*) and this decomposition is parametrized by the permutations $\pi \in S(p)$. This fact will be used in Section 6.3 to identify a certain cell decomposition as a Bruhat decomposition.

Remark 2.48. The (restricted) output injection transformation group also acts on the set of all matrix pairs $(C, A) \in \mathcal{F}^{p\times n} \times \mathcal{F}^{n\times n}$. The notion of (restricted) output injection equivalence will also be used in the general case. A complete set of output injection invariants is provided by the observability indices together with the *invariant factors* (*hidden modes*) of (C, A). For restricted output injection equivalence a corresponding result seems to be unknown.

2.3.3 Restricted and corestricted systems II

In this section the meaning of "restricting" and "corestricting" system (2.1) to a given (C, A)-invariant subspace $\mathcal{V}$ is clarified. Again system (2.1) is assumed to be observable.

As in the dual case (Section 2.1.3) the notion of (co)restriction can be easily understood if one considers the linear maps represented by the matrices C, A and so on. Given a (C, A)-invariant subspace $\mathcal{V}$, there exists an output injection matrix J such that $(A - JC)\mathcal{V} \subset \mathcal{V}$, i.e. $\mathcal{V}$ is an invariant subspace of the linear map represented by $A^J = A - JC$. But then the commutative diagram of linear maps

$$\begin{tikzcd}
\mathcal{F}^n/\mathcal{V} \arrow[r, "\tilde{A}"] & \mathcal{F}^n/\mathcal{V} \arrow[r, "\tilde{C}"] & \mathcal{F}^p/C(\mathcal{V}) \\
\mathcal{F}^n \arrow[u, two heads] \arrow[r, "A^J"] & \mathcal{F}^n \arrow[u, two heads] \arrow[r, "C"] & \mathcal{F}^p \arrow[u, two heads] \\
\mathcal{V} \arrow[u, hook] \arrow[r, "\bar{A}"] & \mathcal{V} \arrow[u, hook] \arrow[r, "\bar{C}"] & C(\mathcal{V}) \arrow[u, hook]
\end{tikzcd}$$

defines a *restriction* $(\bar{C}, \bar{A})$ and a *corestriction* $(\tilde{C}, \tilde{A})$ of the pair (C, A) to $\mathcal{V}$ in terms of linear maps. Note, that the roles of restriction and corestriction are interchanged when compared to the dual case. Of course, both of them depend on the choice of J. Let $k = \operatorname{codim} \mathcal{V}$ and $q = \operatorname{codim} C(\mathcal{V})$. The matrix representations $(\bar{C}, \bar{A}) \in \mathcal{F}^{(p-q)\times(n-k)} \times \mathcal{F}^{(n-k)\times(n-k)}$ and $(\tilde{C}, \tilde{A}) \in \mathcal{F}^{q\times k} \times \mathcal{F}^{k\times k}$ of the (co)restriction depend on the choice of a basis in $\mathcal{V}$ and $C(\mathcal{V})$, also. More precisely, the following holds.

Proposition 2.49. *Let $\mathcal{V}$ be a (C, A)-invariant subspace. Any two matrix representations of* (*co*)*restrictions of the pair (C, A) to $\mathcal{V}$ are output injection equivalent.*

Remark 2.50. Note that the corestriction $(\tilde{C}, \tilde{A})$ of an observable pair (C, A) to $\mathcal{V}$ is observable if and only if $\mathcal{V}$ is an observability subspace, while the restriction $(\bar{C}, \bar{A})$ is always observable. Note further that $\tilde{C}$ and $\bar{C}$ are both of full row rank if C has full row rank.

Corollary 2.51. *The observability indices of all matrix representations of observable restrictions* (*resp. corestrictions*) *coincide. They are called the* (co)restriction indices *of (C, A) with respect to $\mathcal{V}$.*

Again, there is a canonical choice for a matrix representation of an observable (co)restriction of the pair (C, A) to $\mathcal{V}$.

Proposition 2.52. *Given a (C, A)-invariant subspace $\mathcal{V}$ and a matrix representation of a* (*co*)*restriction of (C, A) to $\mathcal{V}$, every output injection equivalent matrix pair is also a matrix representation of a* (*co*)*restriction of (C, A) to $\mathcal{V}$.*

Corollary 2.53. *Given a (C, A)-invariant subspace $\mathcal{V}$ with observable* (*co*)*-restriction, there exists an output injection matrix J and a matrix representation of the resulting* (*co*)*restriction, which is in dual Brunovsky canonical form.*

Sometimes this special matrix representation is referred to as *the* (co)restriction of the pair (C, A) to $\mathcal{V}$ or *the* (co)restricted system on $\mathcal{V}$. Propositions 2.49 and 2.52 together yield the following result on the degree of uniqueness of matrix representations of (co)restrictions.

Proposition 2.54. *Let $(\bar{C}_0, \bar{A}_0)$ be a matrix representation of a restriction of (C, A) to $\mathcal{V}$. Then the set*

$$\begin{aligned}\{(S\bar{C}_0T^{-1}, T(\bar{A}_0 - \bar{J}\bar{C}_0)T^{-1}) \,|\, T \in \mathcal{F}^{(n-k)\times(n-k)}, S \in \mathcal{F}^{(p-q)\times(p-q)},\\ \bar{J} \in \mathcal{F}^{(n-k)\times(p-q)},\ T, S \ \textit{invertible}\}\end{aligned}$$

is the set of all matrix representations of all restrictions of (C, A) to $\mathcal{V}$. For corestrictions the analogous result holds.

In Section 6.1 a similar but slightly different notion of restrictions involving unipotent transformations will be used to obtain the cell decomposition of Section 6.3. The following proposition (dual to Proposition 2.29) characterizes the admissible restriction indices of a given pair (C, A).

Proposition 2.55. *Let (C, A) have observability indices $(\mu_1, \ldots, \mu_p)$ and let $(\lambda_1, \ldots, \lambda_{p-q})$ be the restriction indices of (C, A) with respect to the (C, A)-invariant subspace $\mathcal{V}$. Then*

$$\lambda_i \leq \mu_i \ , \qquad i = 1, \ldots, p - q \ . \tag{2.5}$$

Conversely, for all integers $0 \leq q < p$ and all lists of integers $(\lambda_1, \ldots, \lambda_{p-q})$ with $\lambda_1 \geq \cdots \geq \lambda_{p-q} \geq 1$ and $\lambda_1 + \cdots + \lambda_{p-q} = n - k \leq n$ satisfying equation (2.5) there exists a (C, A)-invariant subspace $\mathcal{V}$ of codimension k, such that (C, A) has restriction indices $(\lambda_1, \ldots, \lambda_{p-q})$ with respect to $\mathcal{V}$.

Chapter 3

Observers

In this chapter dynamic characterizations of various kinds of almost (C, A)-invariant subspaces (cf. Section 2.3) are derived. The precise relation between observers and these subspaces is explored. Again linear finite-dimensional time-invariant control systems of the following form are considered.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \ , \end{aligned} \tag{3.1}$$

where $A \in \mathcal{F}^{n\times n}$, $B \in \mathcal{F}^{n\times m}$ and $C \in \mathcal{F}^{p\times n}$, $\mathcal{F} = \mathbb{R}, \mathbb{C}$. Here $x \in \mathcal{F}^n$ is referred to as the *state*, $u \in \mathcal{F}^m$ as the *control* or *input*, $y \in \mathcal{F}^p$ as the *output* of system (3.1), respectively. Various kinds of observers for system (3.1) are considered throughout the literature, identity observers, tracking observers and asymptotic observers being the most popular. Unfortunately the literature in this field is a kind of fragmentary and contains a number of flaws and misunderstandings. Hence the whole theory is redeveloped from scratch, here.

3.1 Identity observers

Definition 3.1. An *identity observer* for system (3.1) is a dynamical system

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \ . \end{aligned} \tag{3.2}$$

$L \in \mathcal{F}^{n\times p}$ is called the *observer gain matrix*.

Apparently system (3.2) is driven by the input u of system (3.1) and by the difference $y-\hat{y}$ of the output of system (3.1) and the output of the observer. $\hat{x}$ is considered to be an *estimate* for the state x of system (3.1). The dynamics of the *estimation error* $e(t) := x(t) - \hat{x}(t)$ is given by

$$\dot{e} = (A - LC)e \ .$$

If (C, A) is detectable, the matrix L can be chosen such that $A - LC$ is stable, which implies that the estimation error goes to zero for every initial value $e(0) = x(0) - \hat{x}(0)$. The state x of system (3.1) is then asymptotically identified by the observer. Any identity observer (not only those with $A-LC$ stable) has the *tracking property*: If the estimation error is initially zero ($e(0) = 0$ or equivalently $x(0) = \hat{x}(0)$) it stays zero all the time ($e(t) = 0$ or equivalently $x(t) = \hat{x}(t)$ for all $t \in \mathbb{R}$). The combined dynamics of system (3.1) and the identity observer (3.2) is visualized in Figure 3.1.

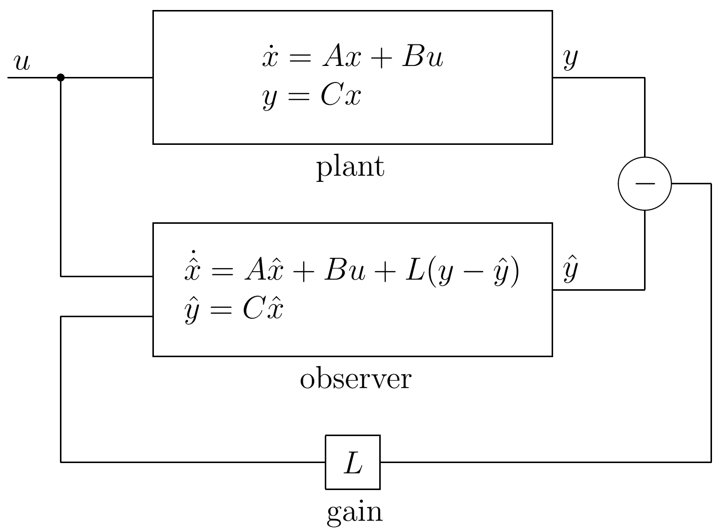

Figure 3.1: Combined dynamics of a system and an identity observer

The idea of estimating the state of system (3.1) using an identical copy of the system driven by the *innovations* $y - \hat{y}$ goes back to the original work of Luenberger [Lue64, Lue66, Lue71]. The identity observers are also called *full order observers*, since the observer state $\hat{x}$ has as many components as the system state x has. In applications it is often not necessary to estimate the whole state x of system (3.1), but rather a part Vx of it, where $V \in \mathcal{F}^{k\times n}$.

Of course this could still be done using an identity observer, but in this case it is more natural to search for an observer of lower order (think especially of the case where $k = 1$, i.e. where V is nothing but a single linear functional of the state). However it is not immediately clear how to construct such *partial observers*. It will be shown below that in principle it suffices to consider corestrictions (cf. Section 2.3.3) of identity observers to (C, A)-invariant subspaces.

Remark 3.2.
1. Using $\hat{y} = C\hat{x}$ the observer equation can be rewritten as
$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu \ .$$
2. It is well known that the full state of system (3.1) can also be estimated by an observer of order $n - \operatorname{rk} C$, the so called *reduced order observer*, which uses *direct feedthrough* of the output $y = Cx$ (cf. Section 3.2.4 and Section 3.3.1).

3.2 Tracking observers

The simplest kind of partial observers are tracking observers. They can be used to characterize (C, A)-invariant subspaces dynamically. In a first step observers without an output equation are considered. The observer state is used as an estimate for a linear function of the state of the observed system. Observers with output will be discussed below (Sections 3.2.3 and 3.2.4).

Definition 3.3. A *tracking observer* for the linear function Vx of the state of system (3.1), $V \in \mathcal{F}^{k\times n}$, is a dynamical system
$$\dot{v} = Kv + Ly + Mu \ , \tag{3.3}$$
$K \in \mathcal{F}^{k\times k}$, $L \in \mathcal{F}^{k\times p}$ and $M \in \mathcal{F}^{k\times m}$, which is driven by the input u and by the output y of system (3.1) and has the *tracking property*: For every $x(0) \in \mathcal{F}^n$, every $v(0) \in \mathcal{F}^k$ and every input function $u(.)$
$$v(0) = Vx(0) \ \Rightarrow \ v(t) = Vx(t) \text{ for all } t \in \mathbb{R}.$$
k is called the *order* of the observer.

Note that the tracking property makes a statement about all trajectories of system (3.1): whatever starting point $x(0)$ and whatever input $u(t)$ is chosen, setting $v(0) := Vx(0)$ must make the observer track the given function. Note further that an identity observer is a tracking observer for Ix. The following characterization of tracking observers has been proposed by Luenberger [Lue64].

Theorem 3.4. *System* (3.3) *is a tracking observer for Vx if and only if*

$$\begin{aligned} VA - KV &= LC \\ M &= VB \ . \end{aligned} \tag{3.4}$$

In this case the tracking error $e(t) = v(t) - Vx(t)$ *is governed by the differential equation $\dot{e} = Ke$.*

Proof. Let the system (3.3) satisfy equations (3.4). Set $e(t) = v(t) - Vx(t)$. Then

$$\begin{aligned} \dot{e} &= \dot{v} - V\dot{x} \\ &= (Kv + Ly + Mu) - V(Ax + Bu) \\ &= Kv + LCx + Mu - VAx - VBu \\ &= Kv - KVx + KVx + LCx - VAx + Mu - VBu \\ &= K(v - Vx) - (VA - KV - LC)x + (M - VB)u \\ &= Ke \ , \end{aligned}$$

where the last equation follows from (3.4). Now $e(0) = 0$, i.e. $v(0) = Vx(0)$, implies $e(t) = 0$, i.e. $v(t) = Vx(t)$, for all $t \in \mathbb{R}$.

Conversely let (3.3) be a tracking observer for Vx. Again set $e(t) = v(t) - Vx(t)$. Then

$$\dot{e} = Ke - (VA - KV - LC)x + (M - VB)u \ .$$

Let $x(0)$ and $u(0)$ be given and set $v(0) = Vx(0)$, i.e. $e(0) = 0$. Then $e(t) = 0$ for all $t \in \mathbb{R}$ implies

$$\begin{aligned} \dot{e}(0) &= Ke(0) - (VA - KV - LC)x(0) + (M - VB)u(0) \\ &= (VA - KV - LC)x(0) + (M - VB)u(0) \\ &= 0 \ . \end{aligned}$$

Since $x(0)$ and $u(0)$ were arbitrary it follows $VA - KV - LC = M - VB = 0$, i.e. equations (3.4). □

Note that the characterization by equations (3.4) implies that a tracking observer for Ix, i.e. for the full state, is necessarily an identity observer.

Now consider the composite system

$$\begin{aligned} \begin{pmatrix} x \\ v \end{pmatrix}^{\!\cdot} &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ M \end{pmatrix} u \ , \\ e &= \begin{pmatrix} -V & I \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \ . \end{aligned} \tag{3.5}$$

Denote

$$A_c := \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \ , \quad B_c := \begin{pmatrix} B \\ M \end{pmatrix} \ \text{ and } \ C_c := \begin{pmatrix} -V & I \end{pmatrix} \ .$$

In order to derive a characterization of tracking observers in terms of the composite system matrices the following result is needed.

Proposition 3.5. *The output $y(.)$ of system* (3.1) *has the property that for every $x(0) \in \mathcal{F}^n$ and every input function $u(.)$*

$$y(0) = 0 \ \Rightarrow \ y(t) = 0 \ \text{ for all } t \in \mathbb{R}$$

if and only if $\operatorname{Ker} C$ *is A-invariant and contains* $\operatorname{Im} B$*.*

Proof. Let $\operatorname{Ker} C$ be A-invariant and contain $\operatorname{Im} B$. Since the *reachable subspace*

$$\mathcal{R}(A,B) := \operatorname{Im} R_n(A,B) := \operatorname{Im} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

is the smallest A-invariant subspace containing $\operatorname{Im} B$ it follows $\mathcal{R}(A,B) \subset \operatorname{Ker} C$. Let $y(0) = 0$, i.e. $x(0) \in \operatorname{Ker} C$. Let $t \in \mathbb{R}$ and $u(.)$ be arbitrary. Since $\operatorname{Ker} C$ is A-invariant it follows $\mathrm{e}^{At}\, x(0) \in \operatorname{Ker} C$. But then the variation of constant formula implies

$$y(t) = C\,\mathrm{e}^{At}\, x(0) + C \int_0^t \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\mathrm{d}\tau = 0$$

since the integral lies in $\mathcal{R}(A,B) \subset \operatorname{Ker} C$.

Conversely let $y(0) = 0$ imply $y(t) = 0$ for all $t \in \mathbb{R}$. Assume $\operatorname{Ker} C$ is not A-invariant. Then there exist $x_0 \in \operatorname{Ker} C$ and $t \in \mathbb{R}$ such that $\mathrm{e}^{At}\, x_0 \notin \operatorname{Ker} C$. But then the trajectory corresponding to $x(0) = x_0$ and $u(.) \equiv 0$ satisfies $y(0) = 0$ but $y(t) \neq 0$, a contradiction. Hence $\operatorname{Ker} C$ is A-invariant. Since every point $x \in \mathcal{R}(A,B)$ is reachable from $x(0) = 0$ (which implies $y(0) = 0$) in finite time $T > 0$, the hypothesis $y(T) = 0$ implies $x = x(T) \in \operatorname{Ker} C$, i.e. $\operatorname{Im} B \subset \mathcal{R}(A,B) \subset \operatorname{Ker} C$. □

Corollary 3.6. *System* (3.3) *is a tracking observer for* Vx *if and only if* $\operatorname{Ker} C_c$ *is* A_c*-invariant and contains* $\operatorname{Im} B_c$*.*

Equations (3.4) allow a geometric characterization of the existence of tracking observers or, depending on the point of view, a dynamic characterization of (C, A)-invariant subspaces. The concept of (C, A)-invariant (or *conditioned invariant*) subspaces has been introduced by Basile and Marro [BM69] as the notion dual to (A, B)*-invariant* (or *controlled invariant*) subspaces. Recall the following definition from Section 2.3.

Definition 3.7. A linear subspace $\mathcal{V}$ of the state space $\mathcal{F}^n$ of system (3.1) is called (C, A)*-invariant*, if there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ such that $A^J\mathcal{V} \subset \mathcal{V}$, where $A^J = A - JC$. Such a J is called a *friend of* $\mathcal{V}$. An equivalent condition is $A(\mathcal{V} \cap \operatorname{Ker} C) \subset \mathcal{V}$.

The next theorem provides the link to observer theory.

Theorem 3.8. *There exists a tracking observer for the linear function* Vx *of the state of system* (3.1) *if and only if* $\mathcal{V} = \operatorname{Ker} V$ *is* (C, A)*-invariant.*

Proof. Let the system (3.3) be a tracking observer for Vx. According to Theorem 3.4 it follows $VA - KV = LC$. Let $x \in \operatorname{Ker} V \cap \operatorname{Ker} C$. Then $VAx = VAx - KVx = LCx = 0$ and $Ax \in \operatorname{Ker} V$. With $\mathcal{V} = \operatorname{Ker} V$ it follows $A(\mathcal{V} \cap \operatorname{Ker} C) \subset \mathcal{V}$ and $\mathcal{V}$ is (C, A)-invariant.

Conversely let $V \in \mathcal{F}^{k\times n}$ and let $\mathcal{V} = \operatorname{Ker} V$ be (C, A)-invariant. There exists $J \in \mathcal{F}^{n\times p}$ such that $(A-JC)\mathcal{V} \subset \mathcal{V}$. But then there exists a matrix $K \in \mathcal{F}^{k\times k}$ such that $V(A-JC) = KV$. Setting $L := VJ$ yields $VA-KV = LC$. Define $M := VB$. According to Theorem 3.4 the system $\dot{v} = Kv + Ly + Mu$ is a tracking observer for Vx. □

Remark 3.9.

1. The idea of using the existence of observers as a definition of (C, A)-invariant subspaces originates from the work of Willems and Commault [WC81], which contains also a proof of Theorem 3.8.

2. In the nonlinear setting a generalization of Theorem 3.8 has been proved by van der Schaft [vdS85] and Krener [Kre86].

3.2.1 Singular (partially) tracking observers

In this section the notion of tracking observers is generalized in two ways. First, they may be singular systems themselves to allow a dynamic characterization of almost (C, A)-invariant subspaces. Second, they may be required to track a given function only when started with zero (hence the name *partially* tracking observers).

Recall the following facts about solutions of singular systems (see e.g. Lewis [Lew86] for a detailed discussion). The system

$$E\dot{v} = Kv \ , \tag{3.6}$$

where $E, K \in \mathcal{F}^{k\times k}$, is called *admissible* if the *matrix pencil* $\lambda E + \mu K$ is *regular*, i.e. $\det(\lambda E + \mu K) \neq 0$ for some $\lambda, \mu \in \mathbb{C}$. The so called *initial manifold* of such an admissible system is the set of starting points $v(0) \in \mathcal{F}^k$ admitting a solution $t \mapsto v(t)$ which is continuous at $t = 0$. Such solutions are uniquely determined by $v(0)$. The initial manifold is known to coincide with the supremal subspace $\mathcal{S} \subset \mathcal{F}^k$ such that

$$K\mathcal{S} \subset E\mathcal{S}$$

holds.

Remark 3.10. If the admissible system is given in *Weierstraß form*, i.e. if $K = \mathrm{diag}(K_1, I)$ and $E = \mathrm{diag}(I, N)$, where I denotes the identity matrix and N is nilpotent, then the initial manifold is formed by all those elements of $\mathcal{F}^k$ whose second block coordinate is zero.

A singular system with input

$$E\dot{v} = Kv + Rz \ , \tag{3.7}$$

where $R \in \mathcal{F}^{k\times q}$, is called *admissible* if the system (3.6) without input is admissible. Apparently, not every input function $z(.)$ will lead to a continuous solution, even if $v(0)$ lies in the initial manifold. On the other hand, there exists a certain set of starting points $v(0)$ for which there exist input functions $z(.)$ that lead to continuous solutions. Again, such solutions are uniquely determined by $v(0)$ and $z(.)$. That set of admissible initial conditions is known to coincide with the supremal $(K, E, \mathrm{Im}\, R)$-invariant subspace, i.e. the supremal subspace $\mathcal{S} \subset \mathcal{F}^k$ such that

$$K\mathcal{S} \subset E\mathcal{S} + \mathrm{Im}\, R$$

holds.

Definition 3.11. A *singular tracking observer* for the linear function Vx of the state of system (3.1), $V \in \mathcal{F}^{k\times n}$, is a (*possibly* singular but admissible) dynamical system

$$E\dot{v} = Kv + Ly + Mu, \tag{3.8}$$

with $E, K \in \mathcal{F}^{k\times k}$, $L \in \mathcal{F}^{k\times p}$ and $M \in \mathcal{F}^{k\times m}$, which is driven by the input u and by the output y of system (3.1) and has the *tracking property*: For every $x(0) \in \mathcal{F}^n$ and every input function $u(.)$ the corresponding output function $y(.)$ and $v(0) := Vx(0)$ lead to a continuous solution $v(.)$ and, furthermore,

$$v(0) := Vx(0) \ \Rightarrow\ v(t) = Vx(t) \text{ for all } t > 0.$$

Again k is called the *order* of the observer. The observer (3.8) is called a *singular partially tracking observer* if it has the *partial tracking property*: For every $x(0) \in \operatorname{Ker} V$ and every input function $u(.)$ the corresponding output function $y(.)$ and $v(0) := Vx(0) = 0$ lead to a continuous solution $v(.)$ and, furthermore,

$$v(0) := Vx(0) = 0 \ \Rightarrow\ v(t) = Vx(t) \text{ for all } t > 0.$$

Note that the partial tracking property makes a statement about special trajectories of system (3.1): whatever starting point $x(0) \in \operatorname{Ker} V$ and whatever input $u(.)$ is chosen, setting $v(0) := Vx(0) = 0$ must make the observer track the given function. Apparently, any singular tracking observer is also a singular partially tracking observer.

Remark 3.12. Since the singular observer is used to track a solution of a non singular system, it suffices to consider continuous trajectories of the observer. The situation becomes more complicated, when the observed system is singular itself. However, that case is not considered here.

There is the following characterization of singular (partially) tracking observers.

Theorem 3.13. *System* (3.8) *is a singular tracking observer for Vx if and only if*

$$\begin{aligned} EVA - KV &= LC \\ M &= EVB\ . \end{aligned} \tag{3.9}$$

System (3.8) *is a singular partially tracking observer for Vx if and only if*

$$\begin{aligned} (EVA - KV - LC)|_{\mathcal{R}(\operatorname{Ker} V)} &= 0 \\ M &= EVB\ . \end{aligned} \tag{3.10}$$

Here $\mathcal{R}(\operatorname{Ker} V) \subset \mathcal{F}^n$ denotes the subspace of the state space $\mathcal{F}^n$ of system (3.1) consisting of all points $x(T) \in \mathcal{F}^n$, which are reachable from a point $x(0) \in \operatorname{Ker} V$ in time $T \geq 0$.

Proof. Let the system (3.8) be a singular partially tracking observer for Vx. Then $x(0) \in \operatorname{Ker} V$ and $v(0) := Vx(0) = 0$ lead to a continuous solution $v(.)$ for every choice of $u(.)$. Furthermore, it follows $v(t) = Vx(t)$ for all $t > 0$ and hence for all $t \geq 0$. But this implies $\dot{v}(t) = V\dot{x}(t)$ for all $t \geq 0$. Now calculate

$$\begin{aligned} 0 &= E(\dot{v}(t) - V\dot{x}(t)) \\ &= Kv(t) + Ly(t) + Mu(t) - EV(Ax(t) - Bu(t)) \\ &= (KV + LC - EVA)x(t) + (M - EVB)u(t) \end{aligned}$$

for all $t \geq 0$. In particular, setting $x(0) := 0 \in \operatorname{Ker} V$, it follows $0 = (M - EVB)u(0)$ and hence $M = EVB$, since $u(0) \in \mathcal{F}^m$ was arbitrary. But then $0 = (KV + LC - EVA)x(t)$ for all $t \geq 0$ and all $x(0) \in \operatorname{Ker} V$ implies $(EVA - KV - LC)|_{\mathcal{R}(\operatorname{Ker} V)} = 0$, i.e. the equations (3.10) are satisfied.

If the system (3.8) is a singular tracking observer for Vx then the same argument holds for every $x(0) \in \mathcal{F}^n$, hence this implies equations (3.9).

Conversely, let the system (3.8) satisfy equations (3.9). Let $x(0) \in \mathcal{F}^n$ and $u(.)$ be arbitrary and consider the corresponding solution $x(.)$ of system (3.1). Then the continuous function $t \mapsto Vx(t) =: v(t)$, $t \geq 0$, is a solution of the observer equation (3.8), since

$$\begin{aligned} E\dot{v}(t) &= EV\dot{x}(t) \\ &= EV(Ax(t) + Bu(t)) \\ &= (KV + LC)x(t) + Mu(t) \\ &= Kv(t) + Ly(t) + Mu(t) \end{aligned}$$

for all $t \geq 0$. Since continuous solutions are uniquely determined, this implies that $v(0) := Vx(0)$ leads to a continuous solution satisfying $v(t) = Vx(t)$ for all $t > 0$. Hence the system (3.8) is a singular tracking observer for Vx.

If the system (3.8) satisfies equations (3.10) then the same argument holds for every $x(0) \in \operatorname{Ker} V$, since this implies $x(t) \in \mathcal{R}(\operatorname{Ker} V)$ for all $t \geq 0$. Hence $v(0) := Vx(0) = 0$ leads to a continuous solution satisfying $v(t) = Vx(t)$ for all $t > 0$ and the system (3.8) is a singular partially tracking observer for Vx. □

Since $\mathcal{R}(\operatorname{Ker} V) \supset \mathcal{R}(A, B) = \mathcal{R}(\{0\})$, for controllable systems Theorem 3.13 reduces to the following.

Corollary 3.14. *Let the system* (3.1) *be controllable. System* (3.8) *is a singular partially tracking observer for Vx if and only if it is a singular tracking observer for Vx. This is the case if and only if equations* (3.9) *are satisfied.*

As the following example shows, for non controllable systems there might exist singular partially tracking observers which are *not* singular tracking observers.

Example 3.15. Let

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B := 0 \quad \text{and} \quad C := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let further

$$V := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$E := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L := 0 \quad \text{and} \quad M := 0 .$$

Since $A(\operatorname{Ker} V) = \{0\}$ it follows $\mathcal{R}(\operatorname{Ker} V) = \operatorname{Ker} V$, and hence $M = EVB$ and $EVA - KV - LC = -V$ implies that $\dot{v} = Kv + Ly + Mu$ is a singular partially tracking observer for Vx, while it is not a singular tracking observer for Vx.

However, as the next theorem shows, the (full) tracking property can be enforced by a slight modification of the singular observer even in the non controllable case. Hence the generalization to *partially* tracking observers yields no new class of observable functions.

Theorem 3.16. *If there exists a singular partially tracking observer for Vx, then there exists a singular tracking observer for Vx. If there exists a non-singular partially tracking observer for Vx, then there exists a tracking observer for Vx.*

Proof. Let (3.8) be a singular partially tracking observer for Vx. Using Theorem 3.13 it follows $(EVA - KV - LC)|_{\mathrm{Ker}\,V} = 0$ and $M = EVB$. Since $V|_{\mathrm{Ker}\,V}$ is injective, there exists a matrix $Q \in \mathcal{F}^{k\times k}$ such that $EVA - KV - LC = QV$. Set $K_1 := K + Q$, $L_1 := L$ and $M_1 := M$, then $EVA - K_1V = L_1C$ and $M_1 = EVB$, i.e. equations (3.9) hold for the system $E\dot{v} = K_1v + L_1y + M_1u$. According to Theorem 3.13 this is a singular tracking observer for Vx. Now let E be invertible. Set $K_2 := E^{-1}K_1$, $L_2 := E^{-1}L_1$ and $M_2 := E^{-1}M_1$, then $VA - K_2V = L_2C$ and $M_2 = VB$. According to Theorem 3.4 the system $\dot{v} = K_2v + L_2y + M_2u$ is a tracking observer for Vx, then. $\square$

Since any tracking observer is also a partially tracking observer, Theorem 3.8 immediately yields the following Corollary.

Corollary 3.17. *There exists a singular partially tracking observer for the linear function Vx of the state of system* (3.1) *if and only if there exists a singular tracking observer for Vx. There exists a non-singular partially tracking observer for Vx if and only if there exists a tracking observer for Vx. The latter is equivalent to $\mathcal{V} = \mathrm{Ker}\,V$ being (C, A)-invariant.*

Note that a tracking observer can be interpreted as a "singular" (partially) tracking observer with $E = I$.

Now recall the following definitions from Section 2.3. $\mathbb{C}_g$ denotes any subset of $\mathbb{C}$ which is symmetric (i.e. $\lambda \in \mathbb{C}_g \Leftrightarrow \bar{\lambda} \in \mathbb{C}_g$, where $\bar{\lambda}$ denotes the complex conjugate of λ) and contains a left semi infinite real interval (i.e. there exists a number $c \in \mathbb{R}$ such that $]-\infty, c] \subset \mathbb{C}_g$). The prototype of such a 'good' part of $\mathbb{C}$ is $\mathbb{C}^- = \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\}$.

Definition 3.18. A linear subspace $\mathcal{V}$ of the state space $\mathcal{F}^n$ of system (3.1) is called *almost* (C, A)*-invariant*, if for every $\varepsilon > 0$ there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ such that $\mathrm{dist}(\mathrm{e}^{A^J t}x_0, \mathcal{V}) < \varepsilon$ for all $t \geq 0$ and $x_0 \in \mathcal{V}$.

A linear subspace $\mathcal{V}$ of the state space $\mathcal{F}^n$ of system (3.1) is called an *almost observability subspace*, if for every set $\mathbb{C}_g$ and every $\varepsilon > 0$ there exists an output injection matrix $J \in \mathcal{F}^{n\times p}$ such that $\mathrm{dist}(\mathrm{e}^{A^J t}x_0, \mathcal{V}) < \varepsilon$ for all $t \geq 0$ and $x_0 \in \mathcal{V}$, while the spectrum of $A^J|_{\mathcal{F}^n/\mathcal{N}(C,A)}$ lies in $\mathbb{C}_g$. Here $\mathcal{N}(C, A)$ denotes the *unobservable subspace* of system (3.1).

To link these concepts to observer theory, characterizations of almost (C, A)-invariant subspaces and almost observability subspaces in terms of rational and polynomial matrices are needed. Since the (dual) characterizations of almost (A, B)-invariant subspaces and almost controllability subspaces which are contained in Willems' paper [Wil82, Proposition A.3] and in Trentelman's thesis [Tre85, Lemma 5.7] are partly incorrect, a correct version of these results (Proposition 3.20) is derived, first. Let $\mathcal{F}^k[s]$ (respectively $\mathcal{F}^k(s)$) denote the space of all k-vectors whose components are polynomials (respectively rational functions) with coefficients in $\mathcal{F}$. Similarly, for a subspace $\mathcal{U} \subset \mathcal{F}^k$ let $\mathcal{U}[s]$ (respectively $\mathcal{U}(s)$) denote the space of all elements $\xi(s) \in \mathcal{F}^k[s]$ (respectively $\xi(s) \in \mathcal{F}^k(s)$) with the property that $\xi(s) \in \mathcal{U}$ for all $s \in \operatorname{dom} \xi(s)$. The following result can be found in Trentelman's thesis [Tre85, Theorem 2.15 and Theorem 2.16].

Proposition 3.19. *Consider system* (3.1).

(1) $\mathcal{V} \subset \mathcal{F}^n$ *is almost* (A, B)*-invariant if and only if for every* $x_0 \in \mathcal{V}$ *there exist* $\xi(s) \in \mathcal{V}(s)$ *and* $\omega(s) \in \mathcal{F}^m(s)$ *such that*

$$x_0 = (sI - A)\xi(s) + B\omega(s) \ . \tag{3.11}$$

(2) $\mathcal{V} \subset \mathcal{F}^n$ *is an almost controllability subpace if and only if for every* $x_0 \in \mathcal{V}$ *there exist* $\xi(s) \in \mathcal{V}[s]$ *and* $\omega(s) \in \mathcal{F}^m[s]$ *such that equation* (3.11) *holds.*

From this result the following characterization can be derived.

Proposition 3.20. *Consider system* (3.1), *and let* $G \in \mathcal{F}^{n \times k}$ *and* $H \in \mathcal{F}^{k \times n}$ *be such that* $\mathcal{V} := \operatorname{Im} G = \operatorname{Ker} H$.

(1) $\mathcal{V}$ *is almost* (A, B)*-invariant if and only if there exists a rational matrix* $W(s)$ *such that*

$$H(sI - A)^{-1}BW(s) = H(sI - A)^{-1}G \ . \tag{3.12}$$

(2) $\mathcal{V}$ *is an almost controllability subspace if and only if there exists a polynomial matrix* $W(s)$ *such that equation* (3.12) *holds and* $(sI - A)^{-1}(G - BW(s))$ *is polynomial.*

Proof. Ad (1): Let the equation (3.12) have the rational solution $W(s)$. Let $x_0 \in \mathcal{V} = \operatorname{Im} G$. Then there exists $v_0 \in \mathcal{F}^k$ such that $x_0 = Gv_0$. Define $\xi(s) := (sI - A)^{-1}(G - BW(s))v_0 \in \mathcal{F}^n(s)$ and $\omega(s) := W(s)v_0 \in \mathcal{F}^m(s)$, then it follows $(sI-A)\xi(s)+B\omega(s) = (G-BW(s))v_0+BW(s)v_0 = Gv_0 = x_0$. Furthermore, equation (3.12) implies $H\xi(s) = 0$ for all $s \in \operatorname{dom}\xi(s)$. Since $\operatorname{Ker} H = \mathcal{V}$ this yields $\xi(s) \in \mathcal{V}(s)$. But then it follows from Proposition 3.19 that $\mathcal{V}$ is almost (A, B)-invariant.

Conversely let $\mathcal{V} = \operatorname{Im} G$ be almost (A, B)-invariant. According to Proposition 3.19 there exist $\xi_i(s) \in \mathcal{V}(s)$ and $\omega_i(s) \in \mathcal{F}^m(s)$, $i = 1, \ldots, k$, such that $G\mathrm{e}_i = (sI - A)\xi_i(s) - B\omega_i(s)$, where $\mathrm{e}_i \in \mathcal{F}^k$ denotes the i-th standard basis vector. Define $W(s) := \begin{pmatrix}\omega_1(s) & \ldots & \omega_k(s)\end{pmatrix}$, then $H(sI - A)^{-1}(BW(s) - G)\mathrm{e}_i = H(sI - A)^{-1}B\omega_i(s) - H\xi_i(s) - H(sI - A)^{-1}B\omega_i(s) = 0$ for all $i = 1, \ldots, k$, since $\xi_i(s) \in \mathcal{V}(s)$ and $\mathcal{V} = \operatorname{Ker} H$. But this implies equation (3.12).

The proof of (2) follows along the same lines. The extra condition on $W(s)$ comes from the requirement that $\xi(s)$ has to be polynomial, here. □

As the following example shows, the additional condition on $W(s)$ in (2) is indispensible.

Example 3.21. Let

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Let further

$$G := \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Then $\operatorname{Im} G = \operatorname{Ker} H$ and $H(sI - A)^{-1}G = 0$. Hence the equation (3.12) has the polynomial solution $W(s) = 0$, although $\mathcal{V} := \operatorname{Im} G = \operatorname{Ker} H$ is *not* an almost controllability subpace with respect to the pair (A, B). In fact, $\mathcal{V}$ is a tight (A, B)-invariant subspace. Indeed, one calculates

$$(sI - A)^{-1}(G - BW(s)) = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 0 \\ 0 & \frac{1}{s} \\ 0 & 0 \end{pmatrix} .$$

Dualizing Proposition 3.20 yields the following corollary.

Corollary 3.22. *Consider system* (3.1), *let* $V \in \mathcal{F}^{k\times n}$.

(1) $\mathcal{V} = \operatorname{Ker} V$ *is almost* (C, A)*-invariant if and only if there exists a rational matrix* $W(s)$ *such that*

$$W(s)C(sI - A)^{-1}|_{\mathcal{V}} = V(sI - A)^{-1}|_{\mathcal{V}} \ . \tag{3.13}$$

(2) $\mathcal{V} = \operatorname{Ker} V$ *is an almost observability subspace if and only if there exists a polynomial matrix* $W(s)$ *such that equation* (3.13) *holds and* $(V - W(s)C)(sI - A)^{-1}$ *is polynomial.*

Now the following characterization of almost (C, A)-invariant subspaces and almost observability subspaces in terms of the existence of singular tracking observers can be derived. The proof uses the following Lemma.

Lemma 3.23. *Let* $E \in \mathcal{F}^{k\times k}$ *be nilpotent and let* $EVA = V + LC$. *Then*

$$(V - (sE - I)^{-1}LC)(sI - A)^{-1} = -\sum_{i=0}^{k-2} s^i E^{i+1} V \ . \tag{3.14}$$

Proof. Multiplying with $(sE - I)$ from the left and with $sI - A$ from the right, equation (3.14) is equivalent to

$$sEV - V - LC = -(sE - I)\left(\sum_{i=0}^{k-2} s^i E^{i+1} V\right)(sI - A) \ .$$

The right hand side is equal to

$$\begin{aligned}
&(I - sE)\left(\sum_{i=0}^{k-2} s^{i+1}E^{i+1}V - \sum_{i=0}^{k-2} s^i E^{i+1} VA\right) = \\
&\sum_{i=0}^{k-2} s^{i+1}E^{i+1}V - \sum_{i=0}^{k-2} s^i E^{i+1}VA - \sum_{i=0}^{k-2} s^{i+2}E^{i+2}V + \sum_{i=0}^{k-2} s^{i+1}E^{i+2}VA = \\
&\sum_{i=0}^{k-2} s^{i+1}E^{i+1}V - \sum_{i=0}^{k-2} s^i E^{i+1}VA - \sum_{i=1}^{k-1} s^{i+1}E^{i+1}V + \sum_{i=1}^{k-1} s^{i}E^{i+1}VA = \\
&sEV - EVA - s^k E^k V + s^{k-1}E^k VA = \\
&sEV - EVA \ ,
\end{aligned}$$

where the last equality follows from $E^k = 0$. Now $EVA = V + LC$ yields the desired result. ∎

Theorem 3.24. (1) *There exists a singular tracking observer for the linear function Vx of the state of system* (3.1) *if and only if $\mathcal{V} = \operatorname{Ker} V$ is almost (C, A)-invariant.*

(2) *There exists a singular tracking observer for the linear function Vx of the state of system* (3.1) *with E nilpotent and $K = I$ if and only if $\mathcal{V} = \operatorname{Ker} V$ is an almost observability subspace.*

Proof. Ad (1): Let the system (3.8) be a singular tracking observer for Vx, let $x(0) \in \operatorname{Ker} V$ and let $u(t) = 0$ for all $t \in \mathbb{R}$. Set $v(0) := Vx(0)$. Taking Laplace transforms in (3.1) and (3.8) yields

$$\begin{aligned} sX(s) - x(0) &= AX(s) \quad \text{and} \\ sEV(s) - Ev(0) &= KV(s) + LCX(s) \; . \end{aligned}$$

Note that $v(.)$ is continuous, hence the Laplace transform can be applied as usual. It follows

$$X(s) = (sI - A)^{-1}x(0)$$

and

$$\begin{aligned} V(s) =& (sE - K)^{-1}EVx(0) + (sE - K)^{-1}LCX(s) \\ =& (sE - K)^{-1}LC(sI - A)^{-1}x(0) \; . \end{aligned}$$

Now $v(t) = Vx(t)$ for all $t \geq 0$ implies $V(s) = VX(s)$ hence

$$(sE - K)^{-1}LC(sI - A)^{-1}x(0) = V(sI - A)^{-1}x(0) \; .$$

Since $x(0) \in \operatorname{Ker} V =: \mathcal{V}$ was arbitrary it follows

$$(sE - K)^{-1}LC(sI - A)^{-1}|_{\mathcal{V}} = V(sI - A)^{-1}|_{\mathcal{V}} \; .$$

Applying Corollary 3.22 with $W(s) := (sE - K)^{-1}L$ shows that $\operatorname{Ker} V$ is almost (C, A)-invariant.

Conversely, let $\mathcal{V} = \operatorname{Ker} V$ be almost (C, A)-invariant. According to Corollary 3.22 there exists a rational matrix $W(s)$ such that equation (3.13) holds. Realize $W(s)$ as singular system of the form (3.8). It follows

$$(sE - K)^{-1}LC(sI - A)^{-1}x(0) = V(sI - A)^{-1}x(0)$$

for all $x(0) \in \operatorname{Ker} V$. Since solutions of (3.8) are not necessarily continuous, the following Laplace transform argument is carried out in the space of distributions. See e.g. Kailath [Kai80] for an explanation of the concept of left and right initial values ($v(0-)$ and $v(0+)$) and the corresponding unilateral Laplace transforms. Choose $x(0) \in \operatorname{Ker} V$ and set $v(0-) := Vx(0) = 0$. For the moment assume that $B = 0$ and $M = 0$. Taking Laplace transforms in (3.1) and in (3.8) yields

$$\begin{aligned} VX(s) &= V(sI - A)^{-1}x(0) \\ &= (sE - K)^{-1}LC(sI - A)^{-1}x(0) \end{aligned}$$

and

$$\begin{aligned} V(s) =&(sE - K)^{-1}Ev(0-) + (sE - K)^{-1}LCX(s) \\ =&(sE - K)^{-1}EVx(0) + (sE - K)^{-1}LC(sI - A)^{-1}x(0) \\ =&(sE - K)^{-1}LC(sI - A)^{-1}x(0) \ , \end{aligned}$$

then. It follows $V(s) = VX(s)$ and $v(t) = Vx(t)$ for all $t > 0$. But then $v(0+) = Vx(0) = 0$ and hence $v(0) := Vx(0) = 0$ leads to a continuous solution satisfying $v(t) = Vx(t)$ for all $t > 0$. Hence (3.8) with $M = 0$ is a singular partially tracking observer for the function Vx of the state of system (3.1) with $B = 0$. According to Theorem 3.16 this observer can be modified such that $EVA - KV = LC$ holds. Now let B be arbitrary and set $M := EVB$. Applying Theorem 3.13 completes the proof.

Ad (2): The first part of the 'only if' direction follows along the same lines as the 'only if' part of (1) using the fact that $W(s) := (sE-I)^{-1}L$ is polynomial if E is nilpotent. Furthermore, Theorem 3.13 yields $EVA - V = LC$, and hence by Lemma 3.23 $(V-W(s)C)(sI-A)^{-1} = (V-(sE-I)^{-1}LC)(sI-A)^{-1}$ is polynomial, too. Apply Corollary 3.22.

The proof of the 'if' direction uses kernel representations of almost observability subspaces and is hence postponed until later (cf. Theorem 5.36). It seems that the method used in the proof of the 'if' part of (1) does not apply here: In that proof the matrix K was modified using Theorem 3.16. □

Although rather counter to intuition, the use of a singular system to observe a non-singular system is of theoretical interest. The observer matrices are directly related to the matrices appearing in the kernel representations of (almost) (C, A)-invariant subspaces and almost observability subspaces derived in Chapter 5 (cf. especially Section 5.7).

Remark 3.25. A similar characterization of almost (C, A)-invariant subspaces and almost observability subspaces in terms of *PID-observers* and *PD-observers*, repectively, has been derived by Willems [Wil82] (see also Trentelman [Tre85, Theorem 5.6]). Although based on a partly incorrect characterization of almost controllability subspaces (cf. the above comments to

Proposition 3.20), the observer type characterizations seem to be correct themselves.

3.2.2 Uniqueness vs. pole assignment

In this section equations (3.4) are used to obtain (non)uniqueness results for tracking observers. It is shown that the source of nonuniqueness is either a lack of rank (in V or in C) or a freedom of pole placement (in K).

Theorem 3.26. *Let $V \in \mathcal{F}^{k\times n}$. Let*

$$\mathrm{Obs}_k(V) = \{(K, L, M) \in \mathcal{F}^{k\times(k+p+m)} \mid VA - KV = LC, M = VB\}$$

be the set of all order k tracking observers for Vx. If $\mathrm{Obs}_k(V)$ is nonempty it is an affine space of dimension

$$\dim \mathrm{Obs}_k(V) = k(\operatorname{def} C + \operatorname{def} V + [n - \dim(\operatorname{Ker} V + \operatorname{Ker} C)]) \ .$$

Here def *denotes the defect, i.e. the dimension of the codomain minus the rank.*

Proof. Being the solution set of the linear matrix equation

$$X \cdot Q := \begin{pmatrix} K & L & M \end{pmatrix} \cdot \begin{pmatrix} V & 0 \\ C & 0 \\ 0 & I \end{pmatrix} = V \begin{pmatrix} A & B \end{pmatrix} \ ,$$

$\mathrm{Obs}_k(V)$ is either empty or an affine space. Since the number of rows in X is k, its dimension is $k \cdot \dim \operatorname{Ker} Q^*$. But

$$\begin{aligned}
\dim \operatorname{Ker} Q^* &= \dim(\operatorname{Im} Q)^\perp \\
&= k + p + m - \operatorname{rk} Q \\
&= k + p + m - (n + m - \dim \operatorname{Ker} Q) \\
&= k + p - n + \dim(\operatorname{Ker} V \cap \operatorname{Ker} C) \\
&= k + p - n + \dim \operatorname{Ker} V + \dim \operatorname{Ker} C - \dim(\operatorname{Ker} V + \operatorname{Ker} C) \\
&= k + p - n + (n - \operatorname{rk} V) + (n - \operatorname{rk} C) - \dim(\operatorname{Ker} V + \operatorname{Ker} C) \\
&= (k - \operatorname{rk} V) + (p - \operatorname{rk} C) + n - \dim(\operatorname{Ker} V + \operatorname{Ker} C) \ .
\end{aligned}$$

□

In applications it causes no loss of generality to assume that C and V are both of full row rank. In this case the dimension formula is especially nice.

Corollary 3.27. *Let C be of full row rank p and let $\mathcal{V} := \operatorname{Ker} V$ be a codimension k subspace (which is equivalent to V being of full row rank k). If $\mathcal{V}$ is (C, A)-invariant then the set $\operatorname{Obs}_k(V)$ of all tracking observers for Vx is an affine space of dimension $k[n - \dim(\mathcal{V} + \operatorname{Ker} C)]$.*

Corollary 3.27 suggests the following definition (see also Section 2.3 and Proposition 2.42).

Definition 3.28. A (C, A)-invariant subspace $\mathcal{V}$ is called *ρ-tight* if

$$\dim(\mathcal{V} + \operatorname{Ker} C) = \rho \ .$$

A n-tight subspace is also called *tight*.

Since $\dim(\mathcal{V}+\operatorname{Ker} C) = n$ implies $\mathcal{V}+\operatorname{Ker} C = \mathcal{F}^n$, this definition of tightness coincides with that of Section 2.3. The name tight subspace was first used by Fuhrmann and Helmke [FH97]. The *degree of tightness* ρ measures the (non)uniqueness of tracking observers.

Corollary 3.29. *Let C in system (3.1) be of full row rank p. If $\operatorname{Ker} V$ is a codimension k ρ-tight (C, A)-invariant subspace then the set $\operatorname{Obs}_k(V)$ of all tracking observers for Vx is an affine space of dimension $k(n - \rho)$. There exists a unique tracking observer for Vx if and only if $\operatorname{Ker} V$ is a codimension k tight (C, A)-invariant subspace.*

If C has full row rank it is possible to characterize tightness of $\mathcal{V}$ in terms of the map $A - JC$, where J is a friend of $\mathcal{V}$.

Proposition 3.30. *Let C have full row rank. A (C, A)-invariant subspace $\mathcal{V}$ is tight if and only if the map $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is the same for every friend J of $\mathcal{V}$, i.e. if and only if $(A - J_1C)\mathcal{V} \subset \mathcal{V}$ and $(A - J_2C)\mathcal{V} \subset \mathcal{V}$ imply $(A - J_1C)|_{\mathcal{F}^n/\mathcal{V}} = (A - J_2C)|_{\mathcal{F}^n/\mathcal{V}}$ for every choice of J_1 and J_2.*

Proof. Let $(A - J_1C)\mathcal{V} \subset \mathcal{V}$ and $(A - J_2C)\mathcal{V} \subset \mathcal{V}$. Then $(A - J_1C)|_{\mathcal{F}^n/\mathcal{V}} = (A - J_2C)|_{\mathcal{F}^n/\mathcal{V}}$ if and only if

$$\begin{aligned}\left[(A - J_1C)|_{\mathcal{F}^n/\mathcal{V}} - (A - J_2C)|_{\mathcal{F}^n/\mathcal{V}}\right](x + \mathcal{V}) &= (J_2 - J_1)C|_{\mathcal{F}^n/\mathcal{V}}(x + \mathcal{V}) \\ &= (J_2 - J_1)Cx + \mathcal{V} \\ &= \mathcal{V}\end{aligned}$$

for all $x \in \mathcal{F}^n$. Since C is surjective this is equivalent to $\operatorname{Im}(J_2 - J_1) \subset \mathcal{V}$. Recall that $\mathcal{V}$ is tight if and only if $\mathcal{V} + \operatorname{Ker} C = \mathcal{F}^n$, i.e. if and only if $C|_{\mathcal{V}}$ is surjective.

Let $\mathcal{V}$ be tight and let $(A - J_1C)\mathcal{V} \subset \mathcal{V}$ and $(A - J_2C)\mathcal{V} \subset \mathcal{V}$. Then $(J_2 - J_1)C\mathcal{V} \subset \mathcal{V}$ and $C|_{\mathcal{V}}$ being surjective implies $\operatorname{Im}(J_2 - J_1) \subset \mathcal{V}$. But then $(A - J_1C)|_{\mathcal{F}^n/\mathcal{V}} = (A - J_2C)|_{\mathcal{F}^n/\mathcal{V}}$.

Conversely let $(A-J_1C)\mathcal{V} \subset \mathcal{V}$ and $(A-J_2C)\mathcal{V} \subset \mathcal{V}$ imply $(A-J_1C)|_{\mathcal{F}^n/\mathcal{V}} = (A-J_2C)|_{\mathcal{F}^n/\mathcal{V}}$ for every choice of J_1 and J_2. Then $\operatorname{Im}(J_2-J_1) \subset \mathcal{V}$. Define a full row rank matrix V by $\mathcal{V} =: \operatorname{Ker} V$. Then the diagram (3.15) yields $K_1V = K_2V$ for every choice of tracking observers $\dot{v} = K_1v + L_1y + M_1u$ and $\dot{v} = K_2v + L_2y + M_2u$ for Vx. Since V has full row rank it follows $K_1 = K_2$. Furthermore $M_1 = VB = M_2$ and $L_1 = VJ_1 = VJ_2 = L_2$ since $\operatorname{Im}(J_2 - J_1) \subset \mathcal{V} = \operatorname{Ker} V$. But then Corollary 3.29 implies that $\mathcal{V}$ is tight. □

Now recall the following definition from Section 2.3.

Definition 3.31. A (C, A)-invariant subspace $\mathcal{V}$ is called *observability subspace* if for every monic polynomial p of degree $\operatorname{codim} \mathcal{V}$ there exists a friend J of $\mathcal{V}$ such that the characteristic polynomial of $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is equal to p.

The dual concept of *controllability subspaces* has been introduced by Wonham and Morse [WM70]. Morse [Mor73] first studied observability subspaces (talking only about duals of controllability subspaces), which were named later by Willems and Commault [WC81].

Comparing the last definition with Proposition 3.30, observabilty subspaces and tight subspaces play a kind of complementary roles. This idea is supported by the following result due to Willems [Wil82]. An extensive proof (of the dual result) can be found in Trentelman's thesis [Tre85]. See also Proposition 2.38.

Proposition 3.32. *For every (C, A)-invariant subspace $\mathcal{V}$ there exists a tight (C, A)-invariant subspace $\mathcal{T}$ such that*

$$\mathcal{V} = \mathcal{T} \cap \mathcal{O}_*(\mathcal{V}) \quad \text{and} \quad \mathcal{T} + \mathcal{O}_*(\mathcal{V}) = \mathcal{F}^n .$$

Here $\mathcal{O}_*(\mathcal{V})$ denotes the smallest observability subspace containing $\mathcal{V}$ (cf. Section 2.3).

If V is of full row rank k then the spectrum of a corestriction (cf. Section 2.3.3) of A to $\operatorname{Ker} V$, i.e. of the map $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ where J is a friend of $\mathcal{V}$, is reflected in the matrix K of an appropriate tracking observer for Vx.

Theorem 3.33. *Let $V \in \mathcal{F}^{k\times n}$ be of full row rank k. For every friend $J \in \mathcal{F}^{n\times p}$ of $\mathcal{V} := \operatorname{Ker} V$ there exists a unique tracking observer for Vx such that K is similar to $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$. Conversely, for every tracking observer $\dot{v} = Kv+Ly+Mu$ for Vx there exists a friend J of $\mathcal{V}$ such that $(A-JC)|_{\mathcal{F}^n/\mathcal{V}}$ is similar to K.*

Proof. Let $(A - JC)\mathcal{V} \subset \mathcal{V}$ then there exists a matrix $K \in \mathcal{F}^{k\times k}$ such that $V(A - JC) = KV$, i.e. such that the following diagram commutes. Since V has full row rank, K is uniquely determined.

$$\begin{tikzcd}
\mathcal{F}^n \arrow[r, "A-JC"] \arrow[d, "V"'] & \mathcal{F}^n \arrow[d, "V"] \\
\mathcal{F}^k \arrow[r, "K"] & \mathcal{F}^k
\end{tikzcd}$$

This induces a quotient diagram with the induced map $\bar{V}$ an isomorphism.

$$\begin{tikzcd}
\mathcal{F}^n/\mathcal{V} \arrow[r, "(A-JC)_{\mathcal{F}^n/\mathcal{V}}"] \arrow[d, "\bar{V}"'] & \mathcal{F}^n/\mathcal{V} \arrow[d, "\bar{V}"] \\
\mathcal{F}^k \arrow[r, "K"] & \mathcal{F}^k
\end{tikzcd} \tag{3.15}$$

But then K is similar to $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$. Define $L := VJ$ then the first diagram yields $VA-LC = KV$. Define $M := VB$. It follows by Theorem 3.4 that $\dot{v} = Kv + Ly + Mu$ is a tracking observer for Vx.

Conversely let $\dot{v} = Kv + Ly + Mu$ be a tracking observer for Vx. It follows by Theorem 3.4 that $VA - KV = LC$. Since V is surjective there exists $J \in \mathcal{F}^{n\times p}$ such that $L = VJ$. But then $V(A - JC) = KV$ and hence $(A - JC)\mathcal{V} \subset \mathcal{V}$, i.e. J is a friend of $\mathcal{V}$. Furthermore, Diagram (3.15) yields that $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is similar to K. $\square$

In this sense a tracking observer is nothing else but a corestriction of an identity observer to a (C, A)-invariant subspace.

Theorem 3.33 allows to relate (non)uniqueness of observers to the following classical pole placement result. The dual version can e.g. be found in Wonham's book [Won74, Corollary 5.2]. It has been extended by Schumacher [Sch80a, Sch81].

Proposition 3.34. *Let $\mathcal{V}$ be (C, A)-invariant. If J is a friend of $\mathcal{V}$ then*

$$\sigma((A - JC)|_{\mathcal{F}^n/\mathcal{V}}) = \sigma_J \cup \sigma_{\text{fix}} \ ,$$

where

$$\sigma_J := \sigma((A - JC)|_{\mathcal{F}^n/\mathcal{O}_*(\mathcal{V})})$$

is freely assignable by a suitable choice of J, and

$$\sigma_{\text{fix}} := \sigma((A - JC)|_{\mathcal{O}_*(\mathcal{V})/\mathcal{V}})$$

is fixed for all J.

Note that Proposition 3.34 implies that any friend J of a (C, A)-invariant subspace $\mathcal{V}$ is automatically a friend of $\mathcal{O}_*(\mathcal{V})$. Let $\mathcal{T}$ be a tight complement of $\mathcal{O}_*(\mathcal{V})$ (cf. Proposition 3.32) then $\mathcal{O}_*(\mathcal{V})/\mathcal{V}$ is naturally isomorphic to $\mathcal{F}^n/\mathcal{T}$. In view of Theorem 3.33, Proposition 3.34 indeed makes the following statement on *observer poles*.

Theorem 3.35. *Any tracking observer for Vx, V of full row rank, has some completely arbitrary poles (corresponding to $\mathcal{F}^n/\mathcal{O}_*(\operatorname{Ker} V)$) and some fixed poles (corresponding to $\mathcal{F}^n/\mathcal{T}$ for any tight complement $\mathcal{T}$ of $\mathcal{O}_*(\operatorname{Ker} V)$). The spectrum of the observer is fixed if and only if $\operatorname{Ker} V$ is tight. It is completely variable if and only if $\operatorname{Ker} V$ is an observability subspace.*

Remark 3.36. The statement of the last theorem was already contained in a paper by Willems [Wil82]. The paper does not contain a proof, though.

The following example shows that there is no direct relation between the number of free parameters in K (i.e. the degree of tightness ρ) and the number of free eigenvalues (i.e. the codimension of $\mathcal{O}_*(\mathcal{V})$), even if C is of full row rank p.

Example 3.37. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

1. Consider
$$V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$
Then $\mathcal{V}_1 = \operatorname{Ker} V_1$ is an observability subspace of codimension 3, i.e. there are 3 free eigenvalues in any tracking observer for Vx. Solving $V_1 A - K_1 V_1 = L_1 C$ for K_1 yields
$$K_1 = \begin{pmatrix} k_1 & 1 & 0 \\ k_2 & 0 & 1 \\ k_3 & 0 & 0 \end{pmatrix} ,$$
where $k_i \in \mathcal{F}$, $i = 1, 2, 3$ are arbitrary ($\rho_1 = 4$). Note that K_1 is in (permuted) companion form.

2. Consider
$$V_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$
Then $\mathcal{V}_2 = \operatorname{Ker} V_2$ is (C, A)-invariant but not an observability subspace. It is $\operatorname{codim} \mathcal{O}_*(\mathcal{V}_2) = 2$, i.e. there are 2 free eigenvalues in any tracking observer for Vx. But solving $V_2 A - K_2 V_2 = L_2 C$ for K_2 yields
$$K_2 = \begin{pmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ 0 & 1 & k_3 \end{pmatrix} ,$$
where $k_i \in \mathcal{F}$, $i = 1, 2, 3$ are arbitrary ($\rho_2 = 4$). Though having the same number of free parameters as K_1, K_2 has a fixed zero eigenvalue.

With a little more effort the rank condition on V can be dropped in Theorem 3.33, at least in the case where the spectrum is completely variable.

Theorem 3.38. *For every monic polynomial p of degree k there exists a tracking observer for the linear function Vx of the state of system* (3.1) *such that the characteristic polynomial of K is equal to p if and only if $\mathcal{V} := \operatorname{Ker} V$ is an observability subspace.*

Proof. Assume that for every monic polynomial p of degree k there exists a tracking observer for Vx such that the characteristic polynomial of K is equal to p. According to Theorem 3.8 $\mathcal{V} := \operatorname{Ker} V$ is (C, A)-invariant. Let J_0 be a friend of $\mathcal{V}$ then $(A - J_0 C)\mathcal{V} \subset \mathcal{V}$ and the quotient map

$$\begin{aligned} A_0 := (A - J_0 C)|_{\mathcal{F}^n/\mathcal{V}} : \mathcal{F}^n/\mathcal{V} &\longrightarrow \mathcal{F}^n/\mathcal{V} \ , \\ x + \mathcal{V} &\mapsto (A - J_0 C)x + \mathcal{V} \end{aligned}$$

is well defined. Let

$$\begin{aligned} \pi : \mathcal{F}^n &\longrightarrow \mathcal{F}^n/\mathcal{V} \ , \\ x &\mapsto x + \mathcal{V} \end{aligned}$$

be the canonic projection. Choose a map $S : \mathcal{F}^p \longrightarrow \mathcal{F}^p$ such that $\operatorname{Ker} SC = \operatorname{Ker} C + \mathcal{V}$. Choose a map

$$C_0 : \ \mathcal{F}^n/\mathcal{V} \longrightarrow \mathcal{F}^p$$

such that $C_0\pi = SC$. Such a map exists since $\mathcal{V} \subset \operatorname{Ker} SC$.

Let $\lambda \in \mathbb{C}$ and choose a tracking observer $\dot{v} = Kv + Ly + Mu$ for Vx such that $\lambda \notin \sigma(K)$. According to Theorem 3.4 it follows $VA - KV - LC = 0$. Let $x + \mathcal{V} \in \operatorname{Ker}(A_0 - \lambda \cdot \mathrm{id}_{\mathcal{F}_n/\mathcal{V}}) \cap \operatorname{Ker} C_0$. Then $SCx = C_0\pi x = C_0(x + \mathcal{V}) = 0$ hence $x \in \operatorname{Ker} C + \mathcal{V}$ and there exists $x' \in \operatorname{Ker} C$ such that $x + \mathcal{V} = x' + \mathcal{V}$. But then $Ax' + \mathcal{V} = (A - JC)x' + \mathcal{V} = A_0(x' + \mathcal{V}) = A_0(x + \mathcal{V}) = \lambda(x + \mathcal{V}) = \lambda x' + \mathcal{V}$ and hence $KVx' = (KV + LC)x' = VAx' = \lambda Vx'$. Since $\lambda \notin \sigma(K)$ this implies $Vx' = 0$ hence $x + \mathcal{V} = x' + \mathcal{V} = \mathcal{V}$. Since $\lambda \in \mathbb{C}$ was arbitrary it follows by the Hautus test that the pair (C_0, A_0) is observable.

Let p be a monic polynomial of degree $\operatorname{codim} \mathcal{V}$. Then there exists an output injection $J_1 : \mathcal{F}^p \longrightarrow \mathcal{F}^n/\mathcal{V}$ such that the characteristic polynomial of $(A_0 - J_1 C_0)$ is equal to p. Choose $J_2 : \mathcal{F}^p \longrightarrow \mathcal{F}^n$ such that $\pi J_2 = J_1$. Define $J := J_0 + J_2 S$. Then $\mathcal{V} \subset \operatorname{Ker} SC$ implies $(A - JC)\mathcal{V} = (A - J_0 C)\mathcal{V} \subset \mathcal{V}$ hence J is a friend of $\mathcal{V}$. Furthermore

$$\begin{aligned} \pi(A - JC) &= \pi(A - J_0 C) - \pi J_2 SC \\ &= A_0\pi - J_1 C_0 \pi \\ &= (A_0 - J_1 C_0)\pi \ . \end{aligned}$$

But then $(A - JC)|_{\mathcal{F}^n/\mathcal{V}} = A_0 - J_1 C_0$ has the characteristic polynomial p. It follows that $\mathcal{V}$ is an observability subspace.

Conversely let $V \in \mathcal{F}^{k\times n}$ and let $\mathcal{V} := \operatorname{Ker} V$ be an observability subspace. There exists $J_0 \in \mathcal{F}^{n\times p}$ such that $(A - J_0 C)\mathcal{V} \subset \mathcal{V}$ and hence there exists a matrix $K_0 \in \mathcal{F}^{k\times k}$ such that $V(A - J_0 C) = K_0 V$. Setting $L_0 := V J_0$ yields $VA - K_0 V = L_0 C$. Let $N \in \mathcal{F}^{k\times k}$ be such that $\operatorname{Ker} N = V(\operatorname{Ker} C)$.

Let $\lambda \in \mathbb{C}$ and choose a friend J of $\mathcal{V}$ such that $\lambda \notin \sigma(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$. Let $z \in \operatorname{Ker}(K_0 - \lambda I) \cap \operatorname{Ker} N$ then $z \in V(\operatorname{Ker} C)$, i.e. there exists $x \in \operatorname{Ker} C$ such that $z = Vx$. Furthermore $x \in \operatorname{Ker} C$ implies $V(A - JC)x = V(A - J_0 C)x = K_0 V x = \lambda V x$. It follows $(A - JC)_{\mathcal{F}^n/\mathcal{V}}(x + \mathcal{V}) = \lambda(x + \mathcal{V})$. Since λ is not an eigenvalue of $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ this implies $x + \mathcal{V} = \mathcal{V}$ hence $z = Vx = 0$. Since $\lambda \in \mathbb{C}$ was arbitrary it follows by the Hautus test that the pair (N, K_0) is observable.

Let p be a monic polynomial of degree k. Then there exists an output injection $Q \in \mathcal{F}^{k\times k}$ such that the characteristic polynomial of $K := K_0 - QN$ is equal to p. For $x \in \operatorname{Ker} C$ it follows $(VA - KV)x = (VA - K_0 V)x = L_0 C x = 0$, i.e. $\operatorname{Ker} C \subset \operatorname{Ker} VA - KV$. But then there exists $L \in \mathcal{F}^{k\times p}$ such that $LC = VA - KV$. Set $M := VB$. According to Theorem 3.4 the system $\dot{v} = Kv + Ly + Mu$ is a tracking observer for Vx. □

Remark 3.39. A statement of Theorem 3.38 as an exercise can be found in the textbook by Trentelman, Stoorvogel and Hautus [TSH01]. They use the same techniques as in the above proof to prove a geometric characterization of the existence of tracking observers with stable K. Since the latter are related to asymptotic observers (cf. Section 3.3), the statement and proof of this result is postponed until later (Theorem 3.66).

3.2.3 Tracking output observers

Until now the observer *state* has been required to track the given function Vx of the system state. Therefore the order of the observer has always been equal to the number of rows in V. In this section an output equation is attached to the observer and this observer *output* is required to track the function Vx. It is shown that the only difference to tracking observers without output results from the possibility to apply *direct feedthrough* of the system output y.

Definition 3.40. A *tracking output observer* for the linear function Vx of the state of system (3.1), $V \in \mathcal{F}^{k\times n}$, is a dynamical system

$$\begin{aligned} \dot{v} &= Kv + Ly + Mu \ , \\ w &= Pv + Qy \ , \end{aligned} \tag{3.16}$$

$K \in \mathcal{F}^{q\times q}$, $L \in \mathcal{F}^{q\times p}$, $M \in \mathcal{F}^{q\times m}$, $P \in \mathcal{F}^{k\times q}$ and $Q \in \mathcal{F}^{k\times p}$ which is driven by the input u and by the output y of system (3.1) and has the *tracking output property*: For every $x(0) \in \mathcal{F}^n$, every $v(0) \in \mathcal{F}^k$ and every input function $u(.)$

$$w(0) = Vx(0) \ \Rightarrow \ w(t) = Vx(t) \text{ for all } t \in \mathbb{R}.$$

Here q is called the *order* of the observer.

Note that the tracking output property leaves some freedom in the choice of the starting state $v(0)$ of the observer. To obtain tracking, it is only required to be chosen such that $w(0) = Pv(0) + QCx(0)$ has the appropriate value. There is the following characterization of tracking output observers.

Theorem 3.41. *System* (3.16) *is a tracking output observer for* Vx *if and only if there exists a matrix* $\hat{K} \in \mathcal{F}^{k\times k}$ *such that*

$$\begin{aligned} (V-QC)A - \hat{K}(V-QC) &= PLC \ , \\ PM &= (V-QC)B \ , \\ PK &= \hat{K}P \ . \end{aligned} \tag{3.17}$$

Then $\dot{\hat{v}} = \hat{K}\hat{v} + PLy + PMu$ *is a tracking observer for* $(V-QC)x$.

Proof. Apply Proposition 3.5 to the composite system

$$\begin{aligned} \dot{\begin{pmatrix} x \\ v \end{pmatrix}} = \dot{x}_c = A_c x_c + B_c u &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ M \end{pmatrix} u \ , \\ e = w - Vx = C_c x_c &= \begin{pmatrix} QC - V & P \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \ . \end{aligned}$$

Then the system (3.16) is a tracking output observer for Vx if and only if $\operatorname{Ker} C_c$ is A_c-invariant and contains $\operatorname{Im} B_c$. The latter is equivalent to $C_c B_c = 0$, i.e. to $(QC-V)B + PM = 0$, which is the second equation in (3.17). $\operatorname{Ker} C_c$ is A_c-invariant if and only if there exists a matrix $\hat{K} \in \mathcal{F}^{k\times k}$ such that $C_c A_c = \hat{K} C_c$, i.e.

$$\begin{pmatrix} (QC-V)A + PLC & PK \end{pmatrix} = \begin{pmatrix} \hat{K}(QC-V) & \hat{K}P \end{pmatrix} \ .$$

This yields the other two equations in (3.17). The last statement follows from Theorem 3.4. □

On the other hand, assume that $\dot{v} = Kv + Ly + Mu$ is a tracking observer for $(V - QC)x$. Then by definition the system

$$\begin{aligned} \dot{v} &= Kv + Ly + Mu \ , \\ w &= Iv + Qy \end{aligned}$$

is a tracking output observer for Vx. Since $P = I$ is invertible, there is no freedom in the choice of the starting state $v(0)$, here. Using Theorem 3.41 and Theorem 3.8 this immediately yields the following corollary.

Corollary 3.42. *There exists a tracking output observer for Vx with direct feedthrough matrix Q if and only if there exists a tracking observer for $(V - QC)x$. The latter is equivalent to* $\operatorname{Ker}(V - QC)$ *being (C, A)-invariant.*

In particular, there exists a tracking output observer for Vx without direct feedthrough, i.e. with $Q = 0$, if and only if there exists a tracking observer for Vx. The latter is equivalent to $\operatorname{Ker} V$ *being (C, A)-invariant.*

3.2.4 Tracking observers with output

In this section tracking (state) observers with an extra output equation are discussed. Theoretically, they allow the tracking of functions which are not related to (C, A)-invariant subspaces using low dimensional observers. But on the other hand tracking observers with output require more knowledge about the starting state of the system than tracking observers or tracking output observers would, which sometimes might reduce their practical usefullness. The observer dimension can be reduced further by direct feedthrough of the system output (Luenberger observer).

Definition 3.43. A linear function Ux, $U \in \mathcal{F}^{q \times n}$, of the state of system (3.1) is said to *contain* the linear function Vx, $V \in \mathcal{F}^{k \times n}$, if there exists a matrix $P \in \mathcal{F}^{k \times q}$ such that $V = PU$ or, equivalently, if $\operatorname{Ker} U \subset \operatorname{Ker} V$.

Using this concept it is easy to construct an observer which tracks an arbitrary linear function Vx of the state of system (3.1). Simply choose a function Ux which contains Vx, and for which $\operatorname{Ker} U$ is (C, A)-invariant. Since $\operatorname{Ker} I = \{0\}$ is (C, A)-invariant and is trivially contained in $\operatorname{Ker} V$, such a function exists. Therefore it is even possible to choose such a U of minimal dimension q (i.e. maximal dimension of $\operatorname{Ker} U$). Next, construct a tracking observer

$$\dot{v} = Kv + Ly + Mu$$

for Ux and attach to it the output equation

$$w = Pv \ ,$$

where $V = PU$. Now, setting $v(0) = Ux(0)$ yields $v(t) = Ux(t)$ and hence $w(t) = Pv(t) = PUx(t) = Vx(t)$ for all $t \in \mathbb{R}$. Since tracking observers always relate to (C, A)-invariant subspaces (Theorem 3.8), designing such an observer of minimal order results in finding a maximal dimensional (C, A)-invariant subspace contained in $\operatorname{Ker} V$ (cf. Section 5.6, Theorem 5.25). Since the sum of two (C, A)-invariant subspaces is not necessarily (C, A)-invariant, there might exist many such subspaces.

As has been mentioned before, setting $v(0) = Ux(0)$ requires more knowledge about the starting state $x(0)$ of the system (3.1) than just $Vx(0)$, which would be needed for a tracking observer or a tracking output observer for Vx.

Remark 3.44. Observers of the above type have been named *preobservers* by Fuhrmann and Helmke [FH01]. They appear in the characterization of asymptotic output observers (cf. Section 3.3.1).

As an application of the above design procedure, the following result due to Schumacher (see e.g. [Sch81, Lemma 2.9]) is used to construct the *reduced order observer* or *Luenberger observer* for the full state x of system (3.1), which is required to be detectable here. The reduced order observer uses *direct feedthrough* of the system output y. In the (more special) case of an observable system the corresponding result has been derived by Wonham [Won70]. The proposition refers to the following concept introduced by Schumacher [Sch81] and by Willems and Commault [WC81] as the notion dual to *stabilizability subspaces*. The latter were introduced by Wonham [Won74] and named by Hautus [Hau80].

Definition 3.45. A (C, A)-invariant subspace $\mathcal{V}$ is called *outer detectable* if there exists a friend J of $\mathcal{V}$ such that $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is stable.

By some authors an outer detectable subspace is also called *detectability subspace*.

Proposition 3.46. *Let $(C, A) \in \mathcal{F}^{p\times n} \times \mathcal{F}^{n\times n}$ be detectable. Then there exists an outer detectable subspace $\mathcal{V}$ such that $\mathcal{V} \oplus \operatorname{Ker} C = \mathcal{F}^n$. (Note that such a $\mathcal{V}$ is tight.)*

Using this result a reduced order observer for the full state x of system (3.1) is constructed as follows. Let $\mathcal{V} \subset \mathcal{F}^n$ be outer detectable such that $\mathcal{V} \oplus$

$\operatorname{Ker} C = \mathcal{F}^n$. Let $U \in \mathcal{F}^{(n-\operatorname{rk} C)\times n}$ such that $\operatorname{Ker} U = \mathcal{V}$. Then U has full row rank $n - \operatorname{rk} C$. According to Theorem 3.33 there exists a tracking observer $\dot{v} = Kv + Ly + Mu$ for Ux such that K is stable. According to Theorem 3.4 then the *tracking error* $e = v - Ux$ is governed by $\dot{e} = Ke$. Since

$$\operatorname{Ker} \begin{pmatrix} U \\ C \end{pmatrix} = \{0\} = \operatorname{Ker} I$$

there exist matrices $P \in \mathcal{F}^{n\times(n-\operatorname{rk} C)}$ and $Q \in \mathcal{F}^{n\times p}$ such that $PU + QC = I$. Since Ux by definition contains PUx, adding the output equation

$$w = Pv$$

to the observer and setting $v(0) = Ux(0)$ yields $w(t) = PUx(t) = (I - QC)x(t)$ for all $t \in \mathbb{R}$. Hence adding the output equation

$$\tilde{w} = Pv + Qy \ ,$$

with the direct feedthrough term $Qy = QCx$, yields an observer of reduced order $n - \operatorname{rk} C$ with

$$v(0) = Ux(0) \ \Rightarrow \ \tilde{w}(t) = x(t) \text{ for all } t \in \mathbb{R} \ .$$

Furthermore, since K is stable and $\dot{e} = Ke$, for arbitrary $v(0) \in \mathcal{F}^{n-\operatorname{rk} C}$ it follows $\lim_{t\to\infty}(\tilde{w}(t) - x(t)) = \lim_{t\to\infty} Pe(t) = 0$, i.e. the system state is asymptotically identified by the observer.

Remark 3.47. A generalization of Proposition 3.46 using an almost observability subspace instead of $\operatorname{Ker} C$ has been derived by Trentelman [Tre84]. He uses direct feedthrough of derivatives of the output y for a further reduction of the observer order (*PID-observer*).

3.3 Asymptotic observers

Asymptotic observers in general do not track (part of) the system state but identify it asymptotically, i.e. for the time going to infinity, while the starting state of the observer can be chosen arbitrarily. In this section the precise relation between asymptotic observers and tracking observers is discussed. In a first step observers without output are considered. Adding an output to an asymptotic observer yields a practically useful new type of observer which can be used to observe arbitrary (linear) functions of the system state. It is then possible to talk about minimal observers, i.e. observers of minimal order.

Definition 3.48. An *asymptotic observer* for the linear function Vx of the state of system (3.1), $V \in \mathcal{F}^{k\times n}$, is a dynamical system

$$\dot{v} = Kv + Ly + Mu \ , \tag{3.18}$$

$K \in \mathcal{F}^{k\times k}$, $L \in \mathcal{F}^{k\times p}$ and $M \in \mathcal{F}^{k\times m}$, which is driven by the input u and the output y of system (3.1) and has the property that

$$\lim_{t\to\infty}(v(t) - Vx(t)) = 0$$

for every choice of $x(0)$, $v(0)$ and the input function $u(.)$, i.e. the observer state converges to the to be estimated function of the system state. As before k is called the *order* of the observer.

Note that any tracking observer with stable K is an asymptotic observer (Theorem 3.4), especially an identity observer with $A - LC$ stable is an asymptotic observer for Ix. It will be shown below (Corollary 3.58) that the converse is also true if the system (3.1) is controllable.

Example 3.49. Let $\dot{x} = Ax$ be a free (totally uncontrollable) and stable system. Let $V = I$. Then $\lim_{t\to\infty} Vx(t) = 0$ for every starting point $x(0) \in \mathcal{F}^n$. Any other free and stable system $\dot{v} = Kv$, $K \in \mathcal{F}^{n\times n}$, is an asymptotic observer for Vx, then. But apparently it is not a tracking observer for Vx unless $K = A$.

If the system (3.1) is merely partially controllable then the Kalman decomposition into a controllable and a free subsystem allows a similar construction. Hence, in general, asymptotic observers do not have the tracking property.

To deduce a characterization of asymptotic observers consider the following result on linear systems. Let

$$\mathcal{R}(A,B) := \operatorname{Im} R_n(A,B) := \operatorname{Im}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

be the *reachable subspace* and let

$$\mathcal{N}(C,A) := \operatorname{Ker} O_n(C,A) := \operatorname{Ker}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

be the *unobservable subspace* of system (3.1).

Proposition 3.50. *The output $y(.)$ of system* (3.1) *has the property that $\lim_{t\to\infty} y(t) = 0$ for every choice of $x(0)$ and $u(.)$ if and only if*

(1) $\mathcal{R}(A, B) \subset \operatorname{Ker} C$ *and*

(2) *A is stable on $\mathcal{F}^n/\mathcal{N}(C, A)$.*

Proof. Let $\lim_{t\to\infty} y(t) = 0$ for every choice of $x(0)$ and $u(.)$. Assume there exists $x_0 \in \mathcal{R}(A, B)$ with $Cx_0 \neq 0$. Since $x_0 \in \mathcal{R}(A, B)$ there exists a control $u(.)$ and a corresponding trajetory $x(.)$ that oscillates between 0 and x_0, contradicting $y(t) = Cx(t) \to 0$ for $t \to \infty$. Hence (1) holds. If $\mathcal{N}(C, A) = \mathcal{F}^n$ there is nothing to prove in (2). Assume $\mathcal{N}(C, A) \neq \mathcal{F}^n$ and assume that A is not stable on $\mathcal{F}^n/\mathcal{N}(C, A)$. Then there exists $0 \neq x_0 \in \mathcal{F}^n$ with $x_0 \notin \mathcal{N}(C, A)$ and $Ax_0 = \lambda x_0$ for a $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. It is $x_0 \notin \operatorname{Ker} C$ since the span of x_0 is A-invariant but $\mathcal{N}(C, A)$ is the largest A-invariant subspace of $\operatorname{Ker} C$. Choosing $x(0) = x_0$ and $u(.) \equiv 0$ yields a trajectory with $y(t) = Cx(t) \not\to 0$ for $t \to \infty$. Hence (2) holds.

Conversely let (1) and (2) hold. Let $x(0) = x_0 \in \mathcal{F}^n$ and let $u(.)$ be arbitrary. Then

$$y(t) = Cx(t) = C\,\mathrm{e}^{At}\,x_0 + C\int_0^t \mathrm{e}^{A(t-\tau)}\,Bu(\tau)\mathrm{d}\tau\ ,$$

where the integral is an element of $\mathcal{R}(A, B)$ and hence of $\operatorname{Ker} C$. Let $\mathcal{F}^n = \mathcal{N}(C, A) \oplus \mathcal{W}$ and decompose $x_0 = n_0 + w_0$ with $n_0 \in \mathcal{N}(C, A)$ and $w_0 \in \mathcal{W}$. Since $\mathcal{N}(C, A)$ is A-invariant and contained in $\operatorname{Ker} C$ it follows $y(t) = C\,\mathrm{e}^{At}\,w_0$. But then (2) implies $\lim_{t\to\infty} y(t) = 0$. □

Apparently, applying Proposition 3.50 to the composite system

$$\begin{aligned} \dot{\begin{pmatrix} x \\ v \end{pmatrix}} &= \dot{x}_c = A_c x_c + B_c u = \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ M \end{pmatrix} u\ , \\ e &= v - Vx = C_c x_c = \begin{pmatrix} -V & I \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} \end{aligned} \tag{3.19}$$

results in the following characterization of asymptotic observers.

Corollary 3.51. *System* (3.18) *is an asymptotic observer for Vx if and only if $\mathcal{R}(A_c, B_c) \subset \operatorname{Ker} C_c$ and A_c is stable on $\mathcal{F}^{n+k}/\mathcal{N}(C_c, A_c)$.*

In a more general setting (cf. Section 3.3.1) this characterization has been used by Schumacher [Sch80b] as a definition of what he called a *stable observer*. In the following a more elaborate version of this characterization is deduced. The following two lemmas are technical.

Lemma 3.52. *Let $x \in \mathcal{F}^n$ and $v \in \mathcal{F}^k$. Then*

$$\begin{pmatrix} -V & I \end{pmatrix} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{j-1} \begin{pmatrix} x \\ v \end{pmatrix} = 0 \tag{3.20}$$

for all $j \in \mathbb{N}$ if and only if $v = Vx$ and $(VA - KV - LC)A^{i-1}x = 0$ for all $i \in \mathbb{N}$.

Proof. Let (3.20) hold for all $j \in \mathbb{N}$. Setting $j = 1$ yields $v = Vx$. It will be proved by induction that $(VA - KV - LC)A^{i-1}x = 0$ for all $i \in \mathbb{N}$. It is

$$\begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} A \\ LC + KV \end{pmatrix} x \tag{3.21}$$

and

$$\begin{pmatrix} -V & I \end{pmatrix} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = -(VA - LC - KV)x \tag{3.22}$$

hence setting $j = 2$ yields $(VA - KV - LC)x = 0$. Let

$$\begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{i} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} A \\ LC + KV \end{pmatrix} A^{i-1}x \tag{3.23}$$

and $(VA - KV - LC)A^{i-1}x = 0$ for a fixed $i \in \mathbb{N}$. Then

$$\begin{aligned} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{i+1} \begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} A \\ LC + KV \end{pmatrix} A^{i-1}x \\ &= \begin{pmatrix} A^{i+1} \\ LCA^{i} + K(LC + KV)A^{i-1} \end{pmatrix} x \\ &= \begin{pmatrix} A^{i+1} \\ LCA^{i} + K(VA)A^{i-1} \end{pmatrix} x \\ &= \begin{pmatrix} A \\ LC + KV \end{pmatrix} A^{i}x \end{aligned} \tag{3.24}$$

and

$$\begin{pmatrix} -V & I \end{pmatrix} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{i+1} \begin{pmatrix} x \\ v \end{pmatrix} = -(VA - LC - KV)A^{i}x \tag{3.25}$$

hence $(VA - KV - LC)A^i x = 0$. This completes the induction.

Conversely let $v = Vx$ and let $(VA - KV - LC)A^{i-1}x = 0$ for all $i \in \mathbb{N}$. Then $v = Vx$ immediately implies (3.20) for $j = 1$. It will be proved by induction that (3.20) holds for all $j \in \mathbb{N}$, $j \geq 2$. (3.21) and (3.22) imply (3.20) for $j = 2$. Let $j \in \mathbb{N}$, $j \geq 2$ be fixed, let (3.23) be true for $i = j - 1$ and let (3.20) be true for j. Then (3.24) holds and (3.25) implies (3.20) for $j + 1$. This completes the induction. □

Lemma 3.53. *The following statements are equivalent.*

(1) $(VA - KV - LC)R_n(A, B) = 0$,

(2) $\mathcal{R}(A, B) \subset \operatorname{Ker}(VA - KV - LC)$,

(3) $\mathcal{R}(A, B) \subset \mathcal{N}(VA - KV - LC, A)$ *and*

(4) $\operatorname{Im} B \subset \mathcal{N}(VA - KV - LC, A)$.

Proof. (1)⇔(2) follows from $\mathcal{R}(A, B) = \operatorname{Im} R_n(A, B)$. (2) implies (3) since $\mathcal{R}(A, B)$ is A-invariant but $\mathcal{N}(VA - KV - LC, A)$ is the largest A-invariant subspace of $\operatorname{Ker} VA - KV - LC$. (3) implies (2) since $\mathcal{N}(VA - KV - LC, A) = \operatorname{Ker} O_n(VA - KV - LC, A)$. (3) implies (4) since $\operatorname{Im} B \subset \mathcal{R}(A, B)$ and (4) implies (3) since $\mathcal{N}(VA - KV - LC, A)$ is A-invariant. □

The first condition of Corollary 3.51 states that the system input (Schumacher [Sch80b] talks of the *disturbance*) is decoupled from the *estimation error*, i.e. the input and the output of the composite system are decoupled. The following Lemma implies that this is indeed a tracking property on the controllable subspace (cf. Theorem 3.4).

Lemma 3.54. *The following statements are equivalent.*

(1) $\mathcal{R}(A_c, B_c) \subset \operatorname{Ker} C_c$,

(2) $C_c R_{n+k}(A_c, B_c) = 0$ *and*

(3) $(VA - KV - LC)R_n(A, B) = 0$ *and* $M = VB$.

Proof. (1)⇔(2) follows from $\mathcal{R}(A_c, B_c) = \operatorname{Im} R_{n+k}(A_c, B_c)$. To get (2)⇔(3) apply Lemma 3.52 with $u \in \mathcal{F}^m$ arbitrary, $x = Bu$ and $v = Mu$. □

The next lemma is an easy consequence of Lemma 3.52.

Lemma 3.55.

$$\mathcal{N}(C_c, A_c) = \begin{pmatrix} I \\ V \end{pmatrix} \mathcal{N}(VA - KV - LC, A) \ . \tag{3.26}$$

Proof. By definition $\mathcal{N}(C_c, A_c) = \operatorname{Ker} O_{n+k}(C_c, A_c)$. Use Lemma 3.52. $\square$

The following lemma is due to Schumacher [Sch80b].

Lemma 3.56.

$$\sigma(A_c|_{\mathcal{F}^{n+k}/\mathcal{N}(C_c,A_c)}) = \sigma(A|_{\mathcal{F}^n/\mathcal{N}(VA-KV-LC,A)}) \cup \sigma(K) \ .$$

Proof. By Lemma 3.55 it follows

$$\begin{aligned} \mathcal{N}_p :&= \left\{ x \in \mathcal{F}^n \mid \exists_{v \in \mathcal{F}^k} \begin{pmatrix} x \\ v \end{pmatrix} \in \mathcal{N}(C_c, A_c) \right\} \\ &= \mathcal{N}(VA - KV - LC, A) \ . \end{aligned} \tag{3.27}$$

Furthermore, $\begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathcal{N}(C_c, A_c) \subset \operatorname{Ker} C_c = \operatorname{Ker} \begin{pmatrix} -V & I \end{pmatrix}$ implies $v = 0$. Hence

$$\dim \mathcal{N}_0 := \dim \left\{ v \in \mathcal{F}^k \mid \begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathcal{N}(C_c, A_c) \right\} = 0 \ .$$

On the other hand clearly $\dim \mathcal{N}(C_c, A_c) = \dim \mathcal{N}_p + \dim \mathcal{N}_0$ hence (3.27) implies $\dim \mathcal{N}(C_c, A_c) = \dim \mathcal{N}(VA - KV - LC, A)$. Let

$$P : \mathcal{N}(C_c, A_c) \longrightarrow \mathcal{N}_p = \mathcal{N}(VA - KV - LC, A)$$

be the natural projection, then P is an isomorphism. Moreover, the following diagram commutes.

$$\begin{tikzcd} \mathcal{N}(C_c, A_c) \arrow[r, "A_c"] \arrow[d, "P"'] & \mathcal{N}(C_c, A_c) \arrow[d, "P"] \\ \mathcal{N}(VA - KV - LC, A) \arrow[r, "A"] & \mathcal{N}(VA - KV - LC, A) \end{tikzcd}$$

Hence
$$\sigma(A_c|_{\mathcal{N}(C_c,A_c)}) = \sigma(A|_{\mathcal{N}(VA-KV-LC,A)}) \ .$$
On the other hand clearly
$$\sigma(A_c) = \sigma(A) \cup \sigma(K) \ .$$
Combining the two equations yields the desired result. □

Now applying Proposition 3.50 to the composite system (3.19) and using Lemmas 3.54 and 3.56 yields the following characterization of asymptotic observers.

Theorem 3.57. *System* (3.18) *is an asymptotic observer for Vx if and only if*

(1) $(VA - KV - LC)R_n(A,B) = 0$ *and* $M = VB$,

(2) K *is stable and*

(3) A *is stable on* $\mathcal{F}^n/\mathcal{N}(VA - KV - LC, A)$.

For curiosity note that by Proposition 3.50 the first part of (1) and (3) amount to $\lim_{t\to\infty} z(t) = 0$ for the system $\dot{x} = Ax + Bu$, $z = (VA - KV - LC)x$. In the controllable case Theorem 3.57 specializes to the following.

Corollary 3.58. *Let the system* (3.1) *be controllable. System* (3.18) *is an asymptotic observer for Vx if and only if it is a tracking observer for Vx with stable K, i.e. if and only if K is stable and equations* (3.4) *are satisfied.*

Proof. If the system (3.1) is controllable then $\mathcal{R}(A,B) = \mathcal{F}^n$ and $R_n(A,B)$ has full rank n. Hence (1) of Theorem 3.57 is equivalent to equations (3.4). Using Lemma 3.53 the first equation of (1) is equivalent to $\mathcal{R}(A,B) \subset \mathcal{N}(VA - KV - LC, A)$ hence $\mathcal{N}(VA - KV - LC, A) = \mathcal{F}^n$ and (3) is guaranteed. □

Note again that the characterization by equations (3.4) implies that an asymptotic observer for Ix, i.e. for the full state, of a controllable system is necessarily an identity observer with $K = A - LC$ stable.

Remark 3.59. Although Corollary 3.58 and its generalization to asymptotic observers with output (cf. Theorem 3.70) is considered to be well known throughout the literature, there seems to be only a very recent paper by Fuhrmann and Helmke [FH01], in which a rigorous proof is carried out. Most authors cite the original work of Luenberger [Lue64, Lue66, Lue71], which does only contain the 'if'-part. The general characterization of Theorem 3.57 has not been in the literature in this form.

Remark 3.60. Some authors, see e.g. Fortmann and Williamson [FW72], Moore and Ledwich [ML75] or Sirisena [Sir79], use the stronger condition

$$\lim_{t\to\infty} \frac{\mathrm{d}^j}{\mathrm{d}t^j}(v(t) - Vx(t)) = 0 \quad \text{for all } j = 0, 1, 2, \ldots$$

as the definition of an asymptotic observer for Vx. However, by Corollary 3.58 and Theorem 3.4 the equality for $j = 0$ implies $\dot{e} = Ke$, where $e(t) := v(t) - Vx(t)$ and K is stable, and hence implies the equality for $j = 1, 2, \ldots$ So, in the case of a controllable system, the above condition is equivalent to the definition of an asymptotic observer as it has been given here.

Theorem 3.57 allows a geometric characterization of the existence of asymptotic observers. It uses the concept of *outer detectable subspace* (cf. Section 3.2.4).

Theorem 3.61. *There exists an asymptotic observer for Vx if and only if there exists an A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ such that $\mathcal{V} = \operatorname{Ker} V \cap \mathcal{W}$ is outer detectable with a friend J that satisfies $\operatorname{Im} J \subset \mathcal{W}$.*

The proof of Theorem 3.61 uses the following two lemmas.

Lemma 3.62. *Let $\mathcal{W}$ be an A-invariant subspace, let $(VA - KV - LC)|_{\mathcal{W}} = 0$ and let K be stable. Set $\mathcal{V} := \operatorname{Ker} V \cap \mathcal{W}$. Then there exists an output injection J such that $\operatorname{Im} J \subset \mathcal{W}$, $(A - JC)\mathcal{V} \subset \mathcal{V}$ and $(A - JC)|_{\mathcal{W}/\mathcal{V}}$ is stable.*

Proof. Consider $V|_{\mathcal{W}}$ as surjective linear map

$$V|_{\mathcal{W}} : \mathcal{W} \longrightarrow V(\mathcal{W}) \ .$$

The key point in the proof is to modify K in such a way that it maps $V(\mathcal{W})$ into itself and is still stable. Let $x \in \mathcal{W} \cap \operatorname{Ker} C$ then $KVx = VAx - LCx = VAx$ and since $\mathcal{W}$ is A-invariant this implies $KVx \in V(\mathcal{W})$. Hence

$$K|_{V(\mathcal{W}\cap\operatorname{Ker} C)} : V(\mathcal{W} \cap \operatorname{Ker} C) \subset V(\mathcal{W}) \longrightarrow V(\mathcal{W}) \ . \tag{3.28}$$

Since K is stable, the restriction of (3.28) to the largest K-invariant subspace of $V(\mathcal{W} \cap \operatorname{Ker} C)$ is stable. The same is true for every linear extension $K^{(1)} : V(\mathcal{W}) \longrightarrow V(\mathcal{W})$ of (3.28). Let $K^{(1)}$ be such an extension. Let $N : V(\mathcal{W}) \longrightarrow V(\mathcal{W})$ be any linear map with $\operatorname{Ker} N = V(\mathcal{W} \cap \operatorname{Ker} C)$ then the pair $(N, K^{(1)})$ is detectable (note that the choice of the codomain of N is not vital). Hence there exists an output injection $Q : V(\mathcal{W}) \longrightarrow V(\mathcal{W})$ such that $K^{(2)} := K^{(1)} - QN$ is stable. The definition of N implies that $K^{(2)}|_{V(\mathcal{W}\cap \operatorname{Ker} C)} = K|_{V(\mathcal{W}\cap \operatorname{Ker} C)}$.

Having modified K it is now possible to modify L in such a way that it maps $C(\mathcal{W})$ into $V(\mathcal{W})$. For $x \in \mathcal{W} \cap \operatorname{Ker} C$ it follows $K^{(2)}Vx - VAx = KVx - VAx = 0$, i.e. $\mathcal{W} \cap \operatorname{Ker} C = \operatorname{Ker} C|_{\mathcal{W}} \subset \operatorname{Ker}(K^{(2)}V - VA)|_{\mathcal{W}}$. But then there exists a linear map $L^{(1)} : C(\mathcal{W}) \longrightarrow V(\mathcal{W})$ such that $L^{(1)}C|_{\mathcal{W}} = (K^{(2)}V - VA)|_{\mathcal{W}}$.

In a last step the modified L is used to define J. Let $S : V(\mathcal{W}) \longrightarrow \mathcal{W}$ be a right inverse of V, i.e. let $VS = \mathrm{id}_{V(\mathcal{W})}$. Define $J := -SL^{(1)} : C(\mathcal{W}) \longrightarrow \mathcal{W}$. Extend J by 0 to a map $J^{(1)} : \mathcal{F}^p \longrightarrow \mathcal{F}^n$. Then $\operatorname{Im} J^{(1)} \subset \mathcal{W}$ and it follows $(A - J^{(1)}C)\mathcal{W} = (A - JC)|_{\mathcal{W}}\mathcal{W} \subset \mathcal{W}$. Furthermore $V(A - JC)|_{\mathcal{W}} = (VA + VSL^{(1)}C)|_{\mathcal{W}} = (VA + L^{(1)}C)|_{\mathcal{W}} = K^{(2)}V|_{\mathcal{W}}$. But then $\mathcal{V} = \mathcal{W} \cap \operatorname{Ker} C$ implies $(A - J^{(1)}C)\mathcal{V} = (A - JC)|_{\mathcal{W}}\mathcal{V} \subset \mathcal{V}$. Let $\lambda \in \mathbb{C}$ and let $x \in \mathcal{W}$ such that $Vx \neq 0$. Then $(A - J^{(1)}C)|_{\mathcal{W}/\mathcal{V}}(x + \mathcal{V}) = \lambda(x + \mathcal{V})$ implies $V(A - J^{(1)}C)x = \lambda Vx$. Hence $K^{(2)}(Vx) = V(A - JC)|_{\mathcal{W}}x = V(A - J^{(1)}C)|_{\mathcal{W}}x = \lambda(Vx)$. Since $K^{(2)}$ is stable this implies $\operatorname{Re} \lambda < 0$. This means $(A - J^{(1)}C)|_{\mathcal{W}/\mathcal{V}}$ is stable. $\square$

Lemma 3.63. *Let $\mathcal{W}$ and $\mathcal{V} = \operatorname{Ker} V \cap \mathcal{W}$ both be $(A - JC)$-invariant subspaces. Let $(A - JC)|_{\mathcal{W}/\mathcal{V}}$ be stable. Then there exist a stable matrix $K \in \mathcal{F}^{k\times k}$ and a matrix $L \in \mathcal{F}^{k\times p}$ such that $(VA - KV - LC)|_{\mathcal{W}} = 0$.*

Proof. Since $\mathcal{V}$ is $(A - JC)$-invariant there exists a linear map $K_0 : V(\mathcal{W}) \longrightarrow V(\mathcal{W})$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "(A-JC)_{\mathcal{W}}"] \arrow[d, "V"'] & \mathcal{W} \arrow[d, "V"] \\
V(\mathcal{W}) \arrow[r, "K_0"] & V(\mathcal{W})
\end{tikzcd}$$

This induces a quotient diagram with the induced map $\bar{V}$ an isomorphism.

$$\begin{tikzcd}
\mathcal{W}/\mathcal{V} \arrow[r, "(A-JC)_{\mathcal{W}/\mathcal{V}}"] \arrow[d, "\bar{V}"'] & \mathcal{W}/\mathcal{V} \arrow[d, "\bar{V}"] \\
V(\mathcal{W}) \arrow[r, "K_0"] & V(\mathcal{W})
\end{tikzcd}$$

But then K_0 is similar to $(A - JC)|_{\mathcal{W}/\mathcal{V}}$ and hence stable. Extend K to a stable map $K : \mathcal{F}^k \longrightarrow \mathcal{F}^k$. Define $L := VJ$ then the first diagram yields $(VA - KV - LC)|_{\mathcal{W}} = (VA - K_0V - LC)|_{\mathcal{W}} = 0$. □

Proof of Theorem 3.61. Let the system (3.18) be an asymptotic observer for Vx. According to Theorem 3.57 it follows $(VA - KV - LC)R_n(A, B) = 0$ which by Lemma 3.53 implies $\operatorname{Im} B \subset \mathcal{N}(VA - KV - LC, A) =: \mathcal{W}$. Then $\mathcal{W}$ is A-invariant and $\mathcal{W} \subset \operatorname{Ker} VA - KV - LC$. Set $\mathcal{V} := \operatorname{Ker} V \cap \mathcal{W}$. Since K is stable (Theorem 3.57) Lemma 3.62 yields the existence of an output injection J such that $\operatorname{Im} J \subset \mathcal{W}$, $(A - JC)\mathcal{V} \subset \mathcal{V}$ and $(A - JC)|_{\mathcal{W}/\mathcal{V}}$ is stable. But then $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is stable since $A|_{\mathcal{F}^n/W} = (A - JC)|_{\mathcal{F}^n/W}$ is stable (Theorem 3.57). Hence $\mathcal{V}$ is outer detectable.

Conversely let $V \in \mathcal{F}^{k\times n}$ and let $\mathcal{W} \supset \operatorname{Im} B$ be an A-invariant subspace such that $\mathcal{V} := \operatorname{Ker} V \cap \mathcal{W}$ is outer detectable with friend J and $\operatorname{Im} J \subset \mathcal{W}$. Then $\mathcal{V}$ and $\mathcal{W}$ are both $(A - JC)$-invariant. Since $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is stable so is $(A - JC)|_{\mathcal{W}/\mathcal{V}}$. Applying Lemma 3.63 yields the existence of a stable matrix K and of a matrix L such that $(VA - KV - LC)|_{\mathcal{W}} = 0$. Since $\mathcal{R}(A, B)$ is the smallest A-invariant subspace containing $\operatorname{Im} B$, $\operatorname{Im} B \subset \mathcal{W}$ implies $\mathcal{R}(A, B) \subset \mathcal{W}$. It follows $(VA - KV - LC)R_n(A, B) = 0$. Define $M := VB$. Since $\operatorname{Im} J \subset \mathcal{W}$ it is $A|_{\mathcal{F}^n/\mathcal{W}} = (A - JC)|_{\mathcal{F}^n/\mathcal{W}}$ and the later is stable since $\mathcal{V} \subset \mathcal{W}$. Furthermore $\mathcal{W} \subset \operatorname{Ker} VA - KV - LC$ implies $\mathcal{W} \subset \mathcal{N}(VA - KV - LC, A)$ (the later being the largest A-invariant subspace in $\operatorname{Ker} VA - KV - LC$). It follows that $A|_{\mathcal{F}^n/\mathcal{N}(VA-KV-LC,A)}$ is stable. According to Theorem 3.57 the system $\dot{v} = Kv + Ly + Mu$ is an asymptotic observer for Vx. □

If the system (3.1) is controllable then Theorem 3.61 reduces to the following.

Corollary 3.64. *Let the system* (3.1) *be controllable. There exists an asymptotic observer for* Vx *if and only if* $\mathcal{V} := \operatorname{Ker} V$ *is outer detectable.*

Proof. If the system (3.1) is controllable then $\mathcal{R}(A, B) = \mathcal{F}^n$. Since $\mathcal{R}(A, B)$ is the smallest A-invariant subspace containing $\operatorname{Im} B$, any A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ is equal to $\mathcal{F}^n$, then. □

Remark 3.65.

1. The proof of Theorem 3.61 has been adopted from Schumacher [Sch80b]. As has been pointed out before he used the composite system (3.19) to define asymptotic observers (and called them stable observers).

2. The controllable case of Corollary 3.64 has also been treated by Kawaji [Kaw80] who implicitely assumes that V is of full row rank. Furthermore he relies on Corollary 3.58 (cf. Remark 3.59).

3. Both of them discussed asymptotic observers with output, cf. Section 3.3.1.

Another consequence of Lemma 3.62 and Lemma 3.63 is the following geometric charcterization of the existence of tracking observers with stable K. It also provides a dynamic characterization of outer detectable subspaces. Note that there is no hypothesis on the pair (C, A) nor on the matrix V. As has been pointed out before (Remark 3.39), it could also be proved in the same way as Theorem 3.38.

Theorem 3.66. *There exists a tracking observer for Vx with stable K if and only if $\mathcal{V} := \operatorname{Ker} V$ is outer detectable.*

Proof. Let the system (3.3) be a tracking observer for Vx with stable K. According to Theorem 3.4 it follows $VA - KV = LC$. Set $\mathcal{V} = \operatorname{Ker} V$. Applying Lemma 3.62 with $\mathcal{W} = \mathcal{F}^n$ yields the existence of an output injection J such that $(A - JC)\mathcal{V} \subset \mathcal{V}$ and $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is stable. Hence $\mathcal{V}$ is outer detectable.

Conversely let $V \in \mathcal{F}^{k \times n}$ and let $\mathcal{V} = \operatorname{Ker} V$ be outer detectable. There exists $J \in \mathcal{F}^{n \times p}$ such that $(A - JC)\mathcal{V} \subset \mathcal{V}$ and $(A - JC)|_{\mathcal{F}^n/\mathcal{V}}$ is stable. Applying Lemma 3.63 with $\mathcal{W} = \mathcal{F}^n$ yields the existence of a stable matrix K and of a matrix L such that $VA - KV - LC = 0$. Define $M := VB$. According to Theorem 3.4 the system $\dot{v} = Kv + Ly + Mu$ is a tracking observer for Vx. □

3.3.1 Asymptotic output observers

In this section an output is attached to the observer, and this output is required to identify the given (linear) function of the system state asymptotically. Since the starting state of an asymptotic observer can be chosen arbitrarily anyway, there is no need to make a difference analogously to Section 3.2.3 vs. Section 3.2.4. Again, *direct feedthrough* of the system output is used to reduce the observer order.

Definition 3.67. An *asymptotic output observer* for the linear function Vx of the state of system (3.1), $V \in \mathcal{F}^{k\times n}$, is a dynamical system

$$\begin{aligned} \dot{v} &= Kv + Ly + Mu \ , \\ w &= Pv + Qy \ , \end{aligned} \tag{3.29}$$

$K \in \mathcal{F}^{q\times q}$, $L \in \mathcal{F}^{q\times p}$, $M \in \mathcal{F}^{q\times m}$, $P \in \mathcal{F}^{k\times q}$ and $Q \in \mathcal{F}^{k\times p}$, which is driven by the input u and by the output y of system (3.1) and has the property that

$$\lim_{t\to\infty}(w(t) - Vx(t)) = 0$$

for every choice of $x(0)$, $v(0)$ and the input function $u(.)$, i.e. the observer output converges to the to be estimated function of the system state. Here q is called the *order* of the observer. The observer (3.29) is called *observable* if the pair (P, K) is observable.

Applying Proposition 3.50 to the composite system

$$\begin{aligned} \dot{\begin{pmatrix} x \\ v \end{pmatrix}} = \dot{x}_c = A_c x_c + B_c u &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ M \end{pmatrix} u \ , \\ e = w - Vx = C_c x_c &= \begin{pmatrix} QC - V & P \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{aligned} \tag{3.30}$$

results in the following characterization of asymptotic output observers.

Theorem 3.68. *System* (3.29) *is an asymptotic output observer for Vx if and only if $\mathcal{R}(A_c, B_c) \subset \operatorname{Ker} C_c$ and A_c is stable on $\mathcal{F}^{n+q}/\mathcal{N}(C_c, A_c)$.*

In a slightly more general setting, allowing a function $r = Hu$ instead of u in the observer equation, this characterization has been used by Schumacher [Sch80b] as a definition of what he called a *stable observer*.

To require the observer (3.29) to be observable does not really impose a limitation, since non observable observers would anyway be of unnecessarily large order.

Proposition 3.69. *If there exists an asymptotic output observer for Vx then there exists an observable asymptotic output observer for Vx of less or equal order. Both observers contain the same direct feedthrough term, i.e. they have the same Q.*

Proof. Consider the dual Kalman decomposition for the pair (P, K): There exists an invertible $S \in \mathcal{F}^{q\times q}$ such that

$$SKS^{-1} = \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \quad \text{and} \quad PS^{-1} = \begin{pmatrix} P_1 & 0 \end{pmatrix} ,$$

while the pair (P_1, K_{11}) is observable. Now split

$$SL = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} , \quad SM = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \quad \text{and} \quad Sv = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

analogously. Then starting the system

$$\begin{aligned} \dot{v}_1 &= K_{11} v_1 + L_1 y + M_1 u , \\ w &= P_1 v_1 + Qy \end{aligned}$$

with $v_1(0)$ generates the same output as starting the original observer with $v(0)$. ☐

As has been pointed out before (cf. Remark 3.59), the following characterization of observable asymptotic output observers for controllable systems has a long history in the literature. The proof given here uses ideas developed by Fuhrmann and Helmke [FH01], who treated the case $Q = 0$ (no direct feedthrough) and V of full row rank.

Theorem 3.70. *Let the system* (3.1) *be controllable. System* (3.29) *is an observable asymptotic output observer for Vx if and only if there exists a matrix $U \in \mathcal{F}^{q\times n}$ such that*

$$\begin{aligned} UA - KU &= LC , \\ M &= UB , \\ V &= PU + QC , \end{aligned} \tag{3.31}$$

(P, K) is observable, and K is stable.

Proof. Let there be a matrix $U \in \mathcal{F}^{q\times n}$ such that the system (3.29) satisfies equations (3.31). Let (P, K) be observable, and let K be stable. According to Theorem 3.4 then $e = v - Ux$ is governed by $\dot{e} = Ke$, which implies $\lim_{t\to\infty} e(t) = 0$. It follows $\lim_{t\to\infty}(w(t) - Vx(t)) = \lim_{t\to\infty} Pe(t) = 0$, i.e. the system (3.29) is an observable asymptotic output observer for Vx.

Conversely let the system (3.29) be an observable asymptotic output observer for Vx. By Theorem 3.68 it follows $\mathcal{R}(A_c, B_c) \subset \operatorname{Ker} C_c$, i.e.

$$\begin{pmatrix} QC - V & P \end{pmatrix} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{j-1} \begin{pmatrix} B \\ M \end{pmatrix} = 0 \quad \text{for all } j \in \mathbb{N} . \tag{3.32}$$

It will be proved by induction that

$$\begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{j-1} \begin{pmatrix} B \\ M \end{pmatrix} = \begin{pmatrix} A^{j-1}B \\ K^{j-1}M + \sum\limits_{i=0}^{j-2} K^i LCA^{j-i-2}B \end{pmatrix} .$$

Obviously, this is true for $j = 1$. Let it be true for a fixed $j \in \mathbb{N}$ now. It follows

$$\begin{aligned} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{j} \begin{pmatrix} B \\ M \end{pmatrix} &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix}^{j-1} \begin{pmatrix} B \\ M \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ LC & K \end{pmatrix} \begin{pmatrix} A^{j-1}B \\ K^{j-1}M + \sum\limits_{i=0}^{j-2} K^i LCA^{j-i-2}B \end{pmatrix} \\ &= \begin{pmatrix} A^{j}B \\ K^{j}M + LCA^{j-1}B + \sum\limits_{i=1}^{j-1} K^i LCA^{j-i-1}B \end{pmatrix} \\ &= \begin{pmatrix} A^{j}B \\ K^{j}M + \sum\limits_{i=0}^{j-1} K^i LCA^{(j+1)-i-2}B \end{pmatrix} , \end{aligned}$$

which completes the induction. But now equation (3.32) implies

$$(V - QC)A^{j-1}B = P \left(K^{j-1}M + \sum_{i=0}^{j-2} K^i LCA^{j-i-2}B \right) \quad \text{for all } j \in \mathbb{N} .$$

Since (A, B) is controllable, the reachability matrix

$$R_n(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

has full rank and hence it follows

$$V - QC = P\tilde{U} \ ,$$

where

$$\begin{aligned}\tilde{U} = &\left[\sum_{j=1}^{n}\left(K^{j-1}M + \sum_{i=0}^{j-2} K^i LCA^{j-i-2}B\right)B^*(A^*)^{j-1}\right]\cdot \\ &\cdot (R_n(A,B)R_n(A,B)^*)^{-1} \ .\end{aligned} \tag{3.33}$$

Now set $e(t) = w(t) - Vx(t)$ and consider the composite system (3.30) with Laplace transform

$$\begin{aligned}\begin{pmatrix} X(s) \\ V(s) \end{pmatrix} = &\begin{pmatrix} (sI-A)^{-1} & 0 \\ (sI-K)^{-1}LC(sI-A)^{-1} & (sI-K)^{-1} \end{pmatrix}\cdot \\ &\cdot \left(\begin{pmatrix} B \\ M \end{pmatrix} U(s) + \begin{pmatrix} x(0) \\ v(0) \end{pmatrix}\right) \ .\end{aligned}$$

Then the Laplace transform of $e(t)$ is

$$\begin{aligned}E(s) &= W(s) - VX(s) = PV(s) + (QC - V)X(s) = PV(s) - P\tilde{U}X(s) \\ &= \Big\{P(sI-K)^{-1}M + \Big[P(sI-K)^{-1}LC - P\tilde{U}\Big](sI-A)^{-1}B\Big\}U(s) + \\ &\quad \Big[P(sI-K)^{-1}LC - P\tilde{U}\Big](sI-A)^{-1}x(0) + P(sI-K)^{-1}v(0) \ .\end{aligned}$$

Let $x(0) = 0$ and $u(t) = 0$ for all $t \in \mathbb{R}$. Then $\lim_{t\to\infty} e(t) = 0$ for every choice of $v(0) \in \mathcal{F}^q$ implies $P(sI-K)^{-1}$ being stable. Since the observer and hence the pair (P,K) is observable, this implies the stability of K.

Now let $x(0) = v(0) = 0$. Then $\lim_{t\to\infty} e(t) = 0$ for every choice of $u(t)$ implies that the term in braces is zero: Assume an entry in the i-th row is nonzero then $U(s)$ can be chosen such that the i-th component of $E(s)$ is $\frac{1}{s}$ contradicting $\lim_{t\to\infty} e(t) = 0$. It follows

$$P(sI-K)^{-1}M = -\Big[P(sI-K)^{-1}LC - P\tilde{U}\Big](sI-A)^{-1}B \ , \tag{3.34}$$

and since (P,K) is observable this implies $\operatorname{Ker} B \subset \operatorname{Ker} M$. Hence there exists $W \in \mathcal{F}^{q\times n}$ such that $M = WB$. But then (3.34) implies

$$P(sI-K)^{-1}\Big[W(sI-A) + LC - (sI-K)\tilde{U}\Big](sI-A)^{-1}B = 0 \ .$$

For every $J \in \mathcal{F}^{q\times k}$ the equality

$$P(sI-K+JP)^{-1} = (I + P(sI-K)^{-1}J)^{-1}P(sI-K)^{-1}$$

holds, and hence it follows

$$P(sI-(K-JP))^{-1}\Big[W(sI-A)+LC-(sI-K)\tilde{U}\Big](sI-A)^{-1}B=0\ . \quad (3.35)$$

Since (P,K) is observable, the matrix J can be chosen such that $(K-JP)$ and A have disjoint spectra.

According to Gantmacher [Gan59, Chapter 8, §3] then the Sylvester equation

$$(K-JP)X-XA=-\tilde{U}A+K\tilde{U}+LC \quad (3.36)$$

has a unique solution $X\in\mathcal{F}^{q\times n}$. Set $Y=W+X-\tilde{U}$. Then

$$Y(sI-A)-(sI-(K-JP))X=W(sI-A)+LC-(sI-K)\tilde{U} \quad (3.37)$$

and hence by (3.35) it follows

$$\begin{aligned}P(sI-(K-JP))^{-1}\Big[Y(sI-A)-(sI-(K-JP))X\Big](sI-A)^{-1}B=&\\ P(sI-(K-JP))^{-1}YB-PX(sI-A)^{-1}B=0\ .&\end{aligned}$$

Since $(K-JP)$ and A have disjoint spectra this yields

$$P(sI-(K-JP))^{-1}YB=PX(sI-A)^{-1}B=0\ .$$

Since (P,K) is observable, so is $(P,K-JP)$, and it follows $YB=0$. Furthermore, from the controllability of the pair (A,B) it follows $PX=0$.

Now comparing the constant terms in (3.37) yields

$$-YA+KX+WA-LC-K\tilde{U}=0\ .$$

Define

$$U:=W-Y=\tilde{U}-X\ . \quad (3.38)$$

Then it follows $UA-KU-LC=0$, i.e. the first equality in (3.31). Furthermore, it is $UB=WB-YB=M$ and $PU=P\tilde{U}-PX=V-QC$, which are the second and third equalities in (3.31). This completes the proof. □

Remark 3.71. The 'if'-direction of Theorem 3.70 could as well be formulated without the observability hypothesis: If the equations (3.31) are satisfied, and if K is stable, then the system (3.29) is an asymptotic output observer for Vx. Observability of (P,K) has not been used for this conclusion in the above proof.

Note that for a controllable system the characterization by equations (3.31) implies (Theorem 3.4) that an observable asymptotic output observer for Vx with direct feedthrough matrix Q tracks a function Ux containing $(V-QC)x$ (cf. Definition 3.43). A formula for such a U is given by equations (3.33), (3.36) and (3.38). This means, a reduced order asymptotic output observer (Luenberger observer) for the full state of a controllable system is necessarily of the form shown in Section 3.2.4.

Unfortunately, it seems to be quite hard to derive a generalization of Theorem 3.70 to non controllable systems, as has been done for observers without output (Theorem 3.57).

Remark 3.72. It has been pointed out before (Remark 3.60) that some authors use the stronger condition

$$\lim_{t\to\infty} \frac{\mathrm{d}^j}{\mathrm{d}t^j}(w(t) - Vx(t)) = 0 \quad \text{for all } j = 0, 1, 2, \dots$$

as the definition of an asymptotic output observer for Vx. However, the equality for $j = 0$ implies $\dot{e} = Ke$, where $e(t) := v(t) - Ux(t)$ and K is stable, as has just been shown in the proof of Theorem 3.70. This in turn implies

$$\lim_{t\to\infty} \frac{\mathrm{d}^j}{\mathrm{d}t^j}(w(t) - Vx(t)) = \lim_{t\to\infty} \frac{\mathrm{d}^j}{\mathrm{d}t^j} Pe(t) = 0 \quad \text{for } j = 1, 2, \dots$$

So, in the case of a controllable system and an observable observer, the above condition is equivalent to the definition of asymptotic output observers as it has been given here. A proof deducing equations (3.31) from the above condition (for $j = 0, 1, 2, \dots$) can be found in the paper by Fortmann and Williamson [FW72]. The alternative proof given by Moore and Ledwich [ML75] is incomplete, they use a misshaped reachability matrix.

Equations (3.31) allow the following geometric characterization of the existence of observable asymptotic output observers for a controllable system.

Theorem 3.73. *Let the system* (3.1) *be controllable. There exists an observable asymptotic output observer for Vx if and only if there exists an outer detectable subspace $\mathcal{U}$ with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$.*

There exists an observable asymptotic output observer for Vx without direct feedthrough, i.e. with $Q = 0$, if and only if there exists an outer detectable subspace $\mathcal{U} \subset \operatorname{Ker} V$.

Proof. Let the system (3.29) be an observable asymptotic output observer for Vx. According to Theorem 3.70 and Theorem 3.4 then the system $\dot{v} = Kv + Ly + Mu$ is a tracking observer for a function Ux, $U \in \mathcal{F}^{q\times n}$, and K is stable. Furthermore it is

$$V = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} U \\ C \end{pmatrix} , \tag{3.39}$$

which implies $\operatorname{Ker} U \cap \operatorname{Ker} C \subset \operatorname{Ker} V$. According to Theorem 3.66, $\mathcal{U} = \operatorname{Ker} U$ is outer detectable.

Conversely let $\mathcal{U}$ be outer detectable with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$. Set $q = \operatorname{codim} \mathcal{U}$ and choose $U \in \mathcal{F}^{q\times n}$ such that $\operatorname{Ker} U = \mathcal{U}$. Then there exist matrices $P \in \mathcal{F}^{k\times q}$ and $Q \in \mathcal{F}^{k\times p}$ such that equation (3.39), i.e. $V = PU + QC$, holds. According to Theorem 3.66 there exists a tracking observer $\dot{v} = Kv+Ly+Mu$ for Ux with stable K. By Theorem 3.4 this implies $UA-KU = LC$ and $M = UB$. According to Theorem 3.70 and Remark 3.71 then the system (3.29) is an asymptotic output observer for Vx. By Proposition 3.69 there exists an observable asymptotic output observer for Vx.

The proof of the second statement follows along the same lines. □

Careful tracking of the various sizes of matrices appearing in the above proof yields the following corollary.

Corollary 3.74. *Let the system* (3.1) *be controllable. If there exists an order q observable asymptotic output observer for Vx* (*without direct feedthrough*) *then there exists an outer detectable subspace $\mathcal{U}$ of codimension less or equal than q with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$* ($\mathcal{U} \subset \operatorname{Ker} V$).

If there exists a codimension q outer detectable subspace $\mathcal{U}$ with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$ ($\mathcal{U} \subset \operatorname{Ker} V$) *then there exists an observable asymptotic output observer for Vx* (*without direct feedthrough*) *of order less or equal than q.*

An immediate consequence of Corollary 3.74 and Proposition 3.69 is the following characterization of the *minimal order* of asymptotic output observers.

Theorem 3.75. *Let the system* (3.1) *be controllable. The minimal order of an asymptotic output observer for Vx is equal to the minimal codimension of an outer detectable subspace $\mathcal{U}$ with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$. Such an observer is necessarily observable.*

The minimal order of an asymptotic output observer for Vx without direct feedthrough, i.e. with $Q = 0$, is equal to the minimal codimension of an outer detectable subspace $\mathcal{U} \subset \operatorname{Ker} V$. Such an observer is necessarily observable.

As a consequence of Theorem 3.75, for controllable systems, a lower bound for the minimal order of an (observable) asymptotic output observer for Vx without direct feedthrough is given by the minimal codimension of a (C, A)-invariant subspace contained in $\operatorname{Ker} V$ (cf. Section 5.6, Theorem 5.25).

Another consequence of Theorem 3.75 is the following result on the minimal order of an (observable) asymptotic output observer for the full state of a controllable and detectable system (Luenberger observer).

Theorem 3.76. *Let the system* (3.1) *be controllable and detectable. Then the minimal order of an* (*observable*) *asymptotic output observer for the full state is $n - \operatorname{rk} C$. The minimal order of an* (*observable*) *asymptotic output observer for the full state without direct feedthrough, i.e. with $Q = 0$, is n.*

Proof. An observer of order $n - \operatorname{rk} C$ has been constructed in Section 3.2.4. Now let the system (3.29) be an asymptotic output observer for Ix of minimal order q. Then $q \leq n - \operatorname{rk} C$ holds. According to Theorem 3.75 the observer is observable, and there exists a codimension q outer detectable subspace $\mathcal{U}$ with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} I = \{0\}$, i.e. $\mathcal{U} \cap \operatorname{Ker} C = \{0\}$. Since $\mathcal{U} \subset \mathcal{F}^n$ it follows

$$
\begin{aligned}
q &= \dim \mathcal{F}^n - \dim \mathcal{U} \\
&\geq \dim(\mathcal{F}^n \cap \operatorname{Ker} C) - \dim(\mathcal{U} \cap \operatorname{Ker} C) \\
&= \dim \operatorname{Ker} C = n - \operatorname{rk} C \ ,
\end{aligned}
$$

i.e. $q = n - \operatorname{rk} C$.

To construct an observer of order n without direct feedthrough take an identity observer with $K = A - LC$ stable (cf. Section 3.1) and add to it the output equation $w = I\hat{x}$. Now let the system (3.29) be an asymptotic output observer for Ix without direct feedthrough, of minimal order q. According to Theorem 3.75 then the observer is observable, and there exists a codimension q outer detectable subspace $\mathcal{U} \subset \operatorname{Ker} I = \{0\}$, i.e. $\mathcal{U} = \{0\}$. It follows $q = n$. $\square$

In the remaining part of this section analogous results for general (not necessarily controllable) systems are stated. They have been derived by Schumacher [Sch80b].

Theorem 3.77. *There exists an observable asymptotic output observer for Vx if and only if there exist an A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ and an outer detectable subspace $\mathcal{U} \subset \mathcal{W}$ with a friend J satisfying $\operatorname{Im} J \subset \mathcal{W}$ such that $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} V$.*

There exists an observable asymptotic output observer for Vx without direct feedthrough, i.e. with $Q = 0$, if and only if there exist an A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ and an outer detectable subspace $\mathcal{U} \subset \mathcal{W} \cap \operatorname{Ker} V$ with a friend J satisfying $\operatorname{Im} J \subset \mathcal{W}$.

In both cases the observer order can be chosen to be less or equal than $\dim \mathcal{W} - \dim \mathcal{U}$.

Proof. This is the statement of Theorem 1 and Theorem 2 of Schumacher [Sch80b] in the case $H = I$, i.e. $\mathcal{B}_0 = \{0\}$. □

Theorem 3.78. *The minimal order of an asymptotic output observer for Vx (with or without direct feedthrough) is equal to the minimal dimension difference $\dim \mathcal{W} - \dim \mathcal{U}$ of a pair $(\mathcal{U}, \mathcal{W})$ satisfying the (respective) requirements of Theorem 3.77. Such an observer is necessarily observable.*

Proof. This is the statement of Theorem 3 of Schumacher [Sch80b] combined with Proposition 3.69. □

If the system (3.1) is controllable, it follows $\mathcal{R}(A, B) = \mathcal{F}^n$. Since $\mathcal{R}(A, B)$ is the smallest A-invariant subspace containing $\operatorname{Im} B$, any A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ is equal to $\mathcal{F}^n$, then. Hence in this case Theorem 3.77 and Theorem 3.78 reduce to Theorem 3.73 and Theorem 3.75, respectively.

As before, these results can be used to determine the minimal order of an (observable) asymptotic output observer for the full state of system (3.1). It turns out that the system has to be detectable in order to allow such an observer. The proof given here is due to Schumacher [Sch80b, Theorem 4]. For a matrix $A \in \mathcal{F}^{n \times n}$ let $\mathcal{E}_-(A)$ and $\mathcal{E}_+(A)$ denote the sums of the generalized eigenspaces associated with eigenvalues with negative and nonnegative real part, respectively. Apparently, it is $\mathcal{F}^n = \mathcal{E}_-(A) \oplus \mathcal{E}_+(A)$.

Theorem 3.79. *There exists an observable asymptotic output observer for the full state of system (3.1) if and only if the pair (C, A) is detectable. The minimal order of such an observer is $\dim((\mathcal{E}_+(A) + \mathcal{R}(A, B)) \cap \operatorname{Ker} C)$. The minimal order of an (observable) asymptotic output observer for the full state without direct feedthrough, i.e. with $Q = 0$, is $\dim(\mathcal{E}_+(A) + \mathcal{R}(A, B))$.*

Proof. Let there exist an asymptotic output observer for Ix. By Theorem 3.77 there exists an outer detectable subspace $\mathcal{U}$ with $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} I = \{0\}$, i.e. $\mathcal{U} \cap \operatorname{Ker} C = \{0\}$. But then there exists a matrix $J \in \mathcal{F}^{n \times p}$

such that $(A-JC)\mathcal{U} \subset \mathcal{U}$ and $(A-JC)|_{\mathcal{F}^n/\mathcal{U}}$ is stable. Since $\mathcal{U} \cap \operatorname{Ker} C = \{0\}$, the map $C|_{\mathcal{U}} : \mathcal{U} \longrightarrow C(\mathcal{U})$ is injective and hence the restriction $(\bar{C}, \bar{A})$ of (C, A) to $\mathcal{U}$ with corresponding output injection J (cf. Section 2.3.3) is observable. Hence there exists a map $\bar{J} : C(\mathcal{U}) \longrightarrow \mathcal{U}$ such that $\bar{A} - \bar{J}\bar{C}$ is stable. Extend $\bar{J}$ by zero to a map $J_1 : \mathcal{F}^p \longrightarrow \mathcal{F}^n$ then $\operatorname{Im} J_1 \subset \mathcal{U}$ and it follows $(A-(J+J_1)C)\mathcal{U} \subset \mathcal{U}$ and $(A-(J+J_1)C)|_{\mathcal{F}^n/\mathcal{U}} = (A-JC)|_{\mathcal{F}^n/\mathcal{U}}$. Furthermore $(A-(J+J_1)C)|_{\mathcal{U}} = \bar{A} - \bar{J}\bar{C}$ implies

$$\sigma(A-(J+J_1)C) = \sigma((A-JC)|_{\mathcal{F}^n/\mathcal{U}}) \cup \sigma(\bar{A}-\bar{J}\bar{C}) \subset \mathbb{C}_- \ ,$$

But then (C, A) is necessarily detectable. For the remainder of this proof let the pair (C, A) be detectable.

To construct an observer of order less or equal than $\dim((\mathcal{E}_+(A)+\mathcal{R}(A,B)) \cap \operatorname{Ker} C)$ set $\mathcal{W} := \mathcal{E}_+(A) + \mathcal{R}(A,B)$. Then $\mathcal{W} \supset \operatorname{Im} B$ is A-invariant. Apply Proposition 3.46 to the restriction $(\bar{C}, \bar{A})$ of (C, A) to $\mathcal{W}$ (cf. Section 2.3.3), which is detectable too, and get a subspace $\mathcal{U} \subset \mathcal{W}$ with $\mathcal{W} = \mathcal{U} \oplus (\mathcal{W} \cap \operatorname{Ker} C)$ and a map $\bar{J} : C(\mathcal{W}) \longrightarrow \mathcal{W}$ such that $(\bar{A}-\bar{J}\bar{C})\mathcal{U} \subset \mathcal{U}$ and $(\bar{A}-\bar{J}\bar{C})|_{\mathcal{W}/\mathcal{U}}$ is stable. Extend $\bar{J}$ by zero to a map $J : \mathcal{F}^p \longrightarrow \mathcal{F}^n$ then $\operatorname{Im} J \subset \mathcal{W}$ and it follows $(A-JC)\mathcal{W} \subset \mathcal{W}$ and $(A-JC)|_{\mathcal{F}^n/\mathcal{W}} = A|_{\mathcal{F}^n/\mathcal{W}}$. Furthermore, $(A-JC)|_{\mathcal{W}} = \bar{A} - \bar{J}\bar{C}$ by $\mathcal{U} \subset \mathcal{W}$ implies $(A-JC)\mathcal{U} \subset \mathcal{U}$ and

$$\sigma((A-JC)|_{\mathcal{F}^n/\mathcal{U}}) = \sigma(A|_{\mathcal{F}^n/\mathcal{W}}) \cup \sigma((\bar{A}-\bar{J}\bar{C})|_{\mathcal{W}/\mathcal{U}}) \subset \mathbb{C}_- \ ,$$

where $A|_{\mathcal{F}^n/\mathcal{W}}$ is stable since $\mathcal{E}_+(A) \subset \mathcal{W}$. From $\mathcal{U} \subset \mathcal{W}$ and $\mathcal{U} \cap (\mathcal{W} \cap \operatorname{Ker} C) = \{0\}$ it follows $\mathcal{U} \cap \operatorname{Ker} C = \{0\}$, i.e. $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} I = \{0\}$. According to Theorem 3.77 there exists an (observable) asymptotic output observer for Ix of order less or equal than $\dim \mathcal{W} - \dim \mathcal{U} = \dim(\mathcal{W} \cap \operatorname{Ker} C) = \dim((\mathcal{E}_+(A)+\mathcal{R}(A,B)) \cap \operatorname{Ker} C)$.

Now let the system (3.29) be an asymptotic output observer for Ix of minimal order q. Then $q \le \dim((\mathcal{E}_+(A)+\mathcal{R}(A,B)) \cap \operatorname{Ker} C)$. According to Theorem 3.78 the observer is observable, and there exist an A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ and an outer detectable subspace $\mathcal{U} \subset \mathcal{W}$ with a friend J satisfying $\operatorname{Im} J \subset \mathcal{W}$ such that $\mathcal{U} \cap \operatorname{Ker} C \subset \operatorname{Ker} I = \{0\}$, i.e. $\mathcal{U} \cap \operatorname{Ker} C = \{0\}$, and $\dim \mathcal{W} - \dim \mathcal{U} = q$. Since $\operatorname{Im} J \subset \mathcal{W}$ and since $\mathcal{W}$ is A-invariant, $\mathcal{W}$ is also $(A-JC)$-invariant, as is $\mathcal{U}$. Furthermore, it is $(A-JC)|_{\mathcal{F}^n/\mathcal{W}} = A|_{\mathcal{F}^n/\mathcal{W}}$. It follows

$$\mathbb{C}_- \supset \sigma((A-JC)|_{\mathcal{F}^n/\mathcal{U}}) = \sigma(A|_{\mathcal{F}^n/\mathcal{W}}) \cup \sigma((A-JC)|_{\mathcal{W}/\mathcal{U}}) \ ,$$

and hence $\mathcal{E}_+(A) \subset \mathcal{W}$. Since $\mathcal{W}$ is A-invariant and contains $\operatorname{Im} B$, it also contains $\mathcal{R}(A,B)$ being the smallest A-invariant subspace containing $\operatorname{Im} B$.

It follows $\mathcal{E}_+(A) + \mathcal{R}(A,B) \subset \mathcal{W}$. But then $\mathcal{U} \subset \mathcal{W}$ implies

$$\begin{aligned} q &= \dim \mathcal{W} - \dim \mathcal{U} \\ &\geq \dim(\mathcal{W} \cap \operatorname{Ker} C) - \dim(\mathcal{U} \cap \operatorname{Ker} C) \\ &\geq \dim((\mathcal{E}_+(A) + \mathcal{R}(A,B)) \cap \operatorname{Ker} C) \ , \end{aligned}$$

and hence $q = \dim((\mathcal{E}_+(A) + \mathcal{R}(A,B)) \cap \operatorname{Ker} C)$.

To construct an observer of order less or equal than $\dim(\mathcal{E}_+(A) + \mathcal{R}(A,B))$ without direct feedthrough set $\mathcal{W} := \mathcal{E}_+(A) + \mathcal{R}(A,B)$. Then $\mathcal{W} \supset \operatorname{Im} B$ is A-invariant and the restriction $(\bar{C}, \bar{A})$ of (C, A) to $\mathcal{W}$ (cf. Section 2.3.3) is detectable, too. Hence there exists a map $\bar{J} : C(\mathcal{W}) \longrightarrow \mathcal{W}$ such that $(\bar{A} - \bar{J}\bar{C})$ is stable. Extend $\bar{J}$ by zero to a map $J : \mathcal{F}^p \longrightarrow \mathcal{F}^n$ then $\operatorname{Im} J \subset \mathcal{W}$ and it follows $(A - JC)\mathcal{W} \subset \mathcal{W}$ and $(A - JC)|_{\mathcal{F}^n/\mathcal{W}} = A|_{\mathcal{F}^n/\mathcal{W}}$. Furthermore, $(A - JC)|_{\mathcal{W}} = \bar{A} - \bar{J}\bar{C}$ implies

$$\sigma(A - JC) = \sigma(A|_{\mathcal{F}^n/\mathcal{W}}) \cup \sigma(\bar{A} - \bar{J}\bar{C}) \subset \mathbb{C}_- \ ,$$

where $A|_{\mathcal{F}^n/\mathcal{W}}$ is stable since $\mathcal{E}_+(A) \subset \mathcal{W}$. Set $\mathcal{U} := \{0\} \subset \mathcal{W}$ then $\mathcal{U} \subset \mathcal{W} \cap \operatorname{Ker} I = \{0\}$ is outer detectable since $A - JC = (A - JC)|_{\mathcal{F}^n/\mathcal{U}}$ is stable. According to Theorem 3.77 there exists an (observable) asymptotic output observer for Ix of order less or equal than $\dim \mathcal{W} - \dim \mathcal{U} = \dim \mathcal{W} = \dim(\mathcal{E}_+(A) + \mathcal{R}(A,B))$.

Now let the system (3.29) be an asymptotic output observer for Ix without direct feedthrough, of minimal order q. Then $q \leq \dim(\mathcal{E}_+(A) + \mathcal{R}(A,B))$. According to Theorem 3.78 the observer is observable, and there exist an A-invariant subspace $\mathcal{W} \supset \operatorname{Im} B$ and an outer detectable subspace $\mathcal{U} \subset \mathcal{W} \cap \operatorname{Ker} I = \{0\}$, i.e. $\mathcal{U} = \{0\}$, with a friend J satisfying $\operatorname{Im} J \subset \mathcal{W}$ such that $q = \dim \mathcal{W} - \dim \mathcal{U} = \dim \mathcal{W}$. As in the first half of this proof it follows $\mathcal{E}_+(A) + \mathcal{R}(A,B) \subset \mathcal{W}$, hence $q \geq \dim(\mathcal{E}_+(A) + \mathcal{R}(A,B))$ and finally $q = \dim(\mathcal{E}_+(A) + \mathcal{R}(A,B))$. $\square$

Remark 3.80. As far as is known to the author, it is an unsolved problem to find a closed formula (not an iterative computation algorithm) for the minimal order of an asymptotic output observer for a general function Vx. Nevertheless, much work on lower bounds for this order has been done, at least in the case of a controllable system. Roman and Bullock [RB75] exploit the connection between partial observers and partial realizations (cf. Section 5.6) to obtain such a lower bound and to outline a design algorithm for minimal observers which uses this bound. The same program is carried out by Moore and Ledwich [ML75] but using decision methods instead of partial realizations. Sirisena [Sir79] proposed a rather algebraic algorithm, while Kimura [Kim77] and Kawaji [Kaw80] relie on Wonham's geometric algorithms. A comparison between the various algorithms can be found in Siefert [Sie97].

Chapter 4

Moduli spaces of linear systems

In this chapter the moduli spaces of linear systems used in Chapter 5 are introduced. Some of their topological properties are stated. The moduli space of tracking observer parameters (cf. Section 3.2) is shown to be a smooth manifold, which allows to make the connection between (C, A)-invariant subspaces and tracking observers established in Theorem 3.33 more precise using a vector bundle structure over this manifold.

4.1 Controllable pairs

The moduli space $\Sigma_{k,p}(\mathcal{F})$ consisting of similarity classes of controllable pairs has been examined since the 1970s now. An extensive treatment can e.g. be found in the monograph by Helmke [Hel92].

Consider the *similarity action*

$$\begin{aligned} \sigma : \mathrm{GL}(\mathcal{F}^k) \times (\mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}) &\longrightarrow \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} , \\ (S, (A, B)) &\mapsto (SAS^{-1}, SB) \end{aligned}$$

on the set of controllable pairs (A, B). The orbit space (cf. Appendix A) of this action, i.e. the manifold of all *similarity classes*

$$[A, B]_\sigma = \{(SAS^{-1}, SB) \mid S \in \mathrm{GL}(\mathcal{F}^k)\}$$

of controllable pairs (endowed with the quotient topology) is denoted by $\Sigma_{k,p}(\mathcal{F})$.

This moduli space will be used in Chapter 5 to describe the set of all codimension k (C, A)-invariant subspaces of an observable pair (C, A) in dual Brunovsky form with observability indices $(\mu_1, \dots, \mu_p)$ (cf. Section 2.3.2). Following Antoulas [Ant83] and Fuhrmann and Helmke [FH00] the description uses the following object. The terminology used here has been introduced by Fuhrmann and Helmke.

Definition 4.1. Let $1 \leq k \leq n$ and $\mu = (\mu_1, \dots, \mu_p)$ with $\mu_1 \geq \cdots \geq \mu_p > 0$ and $\mu_1 + \cdots + \mu_p = n$. The *$\mu$-partial reachability matrix* $R_\mu(A, B)$ of a controllable pair $(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$ is the $k \times n$-matrix

$$R_\mu(A, B) = (b_1, Ab_1, \dots, A^{\mu_1 - 1} b_1, \dots, b_p, Ab_p, \dots, A^{\mu_p - 1} b_p) \ ,$$

where b_i denotes the i-th column of B, $i = 1, \dots, p$.

Using this matrix, some special kinds of controllable pairs are defined.

Definition 4.2. A pair (A, B) is called *μ-regular*, if the μ-partial reachability matrix has full row rank, i.e.

$$\operatorname{rk} R_\mu(A, B) = k \ .$$

The pair is called *μ-tight*, if it is $\mu - 1 = (\mu_1 - 1, \dots, \mu_p - 1)$-regular, i.e. if

$$\operatorname{rk} R_{\mu-1}(A, B) = \operatorname{rk}(b_1, \dots, A^{\mu_1 - 2} b_1, \dots, b_p, \dots, A^{\mu_p - 2} b_p) = k \ .$$

Obviously, any μ-tight pair is μ-regular and μ-regularity implies controllability. Both notions are invariant under the similarity action. Therefore it is possible to define the set of all similarity classes of μ-regular and μ-tight pairs, respectively.

$$\begin{aligned} \Sigma_{k,p}(\mu) &:= \{[A, B]_\sigma \in \Sigma_{k,p}(\mathcal{F}) \mid \operatorname{rk} R_\mu(A, B) = k\} \subset \Sigma_{k,p}(\mathcal{F}) \ , \\ \Sigma^{\mathrm{t}}_{k,p}(\mu) &:= \{[A, B]_\sigma \in \Sigma_{k,p}(\mathcal{F}) \mid \operatorname{rk} R_{\mu-1}(A, B) = k\} \subset \Sigma_{k,p}(\mu) \ . \end{aligned}$$

In Section 4.2 another condition on (A, B) will show up: A being nilpotent. Since nilpotency is another similarity invariant, the following subsets of $\Sigma_{k,p}(\mathcal{F})$ are well defined.

$$\begin{aligned} \mathcal{N}_{k,p}(\mathcal{F}) &:= \{[A, B]_\sigma \in \Sigma_{k,p}(\mathcal{F}) \mid A \text{ nilpotent}\} \ , \\ \mathcal{N}_{k,p}(\mu) &:= \{[A, B]_\sigma \in \Sigma_{k,p}(\mu) \mid A \text{ nilpotent}\} \ , \\ \mathcal{N}^{\mathrm{t}}_{k,p}(\mu) &:= \{[A, B]_\sigma \in \Sigma^{\mathrm{t}}_{k,p}(\mu) \mid A \text{ nilpotent}\} \ , \end{aligned}$$

In this context a different partial reachability matrix, the *reverse μ-partial reachability matrix*

$$\overleftarrow{R}_\mu(A, B) = (A^{\mu_1 - 1} b_1, \dots, Ab_1, b_1, \dots, A^{\mu_p - 1} b_p, \dots, Ab_p, b_p)$$

is used. Apparently $\operatorname{rk} R_\mu(A, B) = \operatorname{rk} \overleftarrow{R}_\mu(A, B)$, so μ-regularity and μ-tightness stays the same.

4.1.1 Topology of μ-regular pairs

It is well known that the moduli space $\Sigma_{k,p}(\mathcal{F})$ of controllable pairs is a smooth connected manifold of dimension kp over $\mathcal{F}$ (see e.g. [Hel92]). Being defined by a 'full rank' condition the subset $\Sigma_{k,p}(\mu)$ of similarity classes of μ-regular pairs is open and dense in $\Sigma_{k,p}(\mathcal{F})$, hence a smooth manifold of dimension kp over $\mathcal{F}$. Note that $\Sigma_{k,p}(\mu)$ is only defined for $1 \le k \le n$ and is nonempty as is its subset $\mathcal{N}_{k,p}(\mu)$, an element of which can be constructed by the method shown in the proof of Lemma 4.3 below.

Since the subset $\Sigma^{\mathrm{t}}_{k,p}(\mu)$ of similarity classes of μ-tight pairs is defined by a 'full rank' condition, it is either empty or open and dense in $\Sigma_{k,p}(\mu)$, hence either empty or a smooth manifold of dimension kp over $\mathcal{F}$.

Lemma 4.3. *$\Sigma^{\mathrm{t}}_{k,p}(\mu)$ is nonempty if and only if $k \le n-p$. The same is true for $\mathcal{N}^{\mathrm{t}}_{k,p}(\mu)$.*

Proof. Let $(A,B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$. Then $R_{\mu-1}(A,B)$ is a $k \times (n-p)$-matrix. For $k > n-p$ it follows $\operatorname{rk} R_{\mu-1}(A,B) < k$ and therefore $\Sigma^{\mathrm{t}}_{k,p}(\mu)$ and $\mathcal{N}^{\mathrm{t}}_{k,p}(\mu)$ are empty.

Let $k \le n-p$. Let

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathcal{F}^{k\times k} \ .$$

Note that A is nilpotent. Let e_i, $i = 1,\dots,k$, be the standard basis vectors of $\mathcal{F}^k$. Let $j \in \mathbb{N}$ be the maximal j with $k = (\sum_{i=1}^{j-1}(\mu_i - 1)) + l$ and $l \in \mathbb{N}_0$. Let $B \in \mathcal{F}^{k\times p}$ be defined by the following equation:

$$\begin{array}{l} (\; b_1 \ \dots \ A^{\mu_1-2}b_1 \ \ \dots \ \ b_j \ \dots \ A^{l-1}b_j \ A^l b_j \ \dots \ A^{\mu_j-2}b_j \ \ \dots \ \ b_p \ \dots \ A^{\mu_p-2}b_p \;) = \\ (\; e_k \ \dots \ e_{k-\mu_1+1} \ \ \dots \ \ e_l \ \dots \ \ e_1 \quad 0 \ \dots \quad 0 \quad \dots \ 0 \ \dots \quad 0 \quad) \end{array}$$

Then $\operatorname{rk} R_{\mu-1}(A,B) = k$ and $[A,B]_\sigma \in \mathcal{N}^{\mathrm{t}}_{k,p}(\mu) \subset \Sigma^{\mathrm{t}}_{k,p}(\mu)$. □

$\Sigma_{k,p}(\mathcal{F})$ can be embedded into the *Grassmann manifold* $\mathrm{G}_{n+p-k}(\mathcal{F}^{n+p})$, i.e. the manifold of all codimension k linear subspaces of $\mathcal{F}^{n+p}$, where $n = kp$. The embedding map

$$\begin{aligned} R_{k+1} : \Sigma_{k,p}(\mathcal{F}) &\longrightarrow \mathrm{G}_{n+p-k}(\mathcal{F}^{n+p}) \ , \\ [A,B]_\sigma &\mapsto \operatorname{Ker} R_{k+1}(A,B) := \operatorname{Ker} \begin{pmatrix} B & AB & \dots & A^k B \end{pmatrix} \end{aligned}$$

is called the *Kalman embedding* of $\Sigma_{k,p}(\mathcal{F})$ (see [Haz77], [BH78]). In fact this embedding can be obtained as a special case ($\mu_1 = \cdots = \mu_p = k$) of the following new result.

Proposition 4.4. *Let $1 \leq k \leq n$ and let $\mu = (\mu_1, \ldots, \mu_p)$ with $\mu_1 \geq \cdots \geq \mu_p \geq 1$ and $\mu_1 + \cdots + \mu_p = n$. Then the map*

$$\begin{aligned} R_{\mu+1} : \Sigma_{k,p}(\mu) &\longrightarrow \mathrm{G}_{n+p-k}(\mathcal{F}^{n+p}) \ , \\ [A, B]_\sigma &\mapsto \operatorname{Ker} R_{\mu+1}(A, B) \end{aligned}$$

*is an embedding, the μ-*partial Kalman embedding.

Proof. Since $\operatorname{Ker} R_{\mu+1}(A, B)$ is invariant under the similarity action and $\operatorname{rk} R_\mu(A, B) = k$ hence $\operatorname{rk} R_{\mu+1}(A, B) = k$ for $(A, B) \in \Sigma_{k,p}(\mu)$ the map $R_{\mu+1}$ is well defined. Clearly it is continuous.

To show that the map $R_{\mu+1}$ is injective let $[A_1, B_1]_\sigma, [A_2, B_2]_\sigma \in \Sigma_{k,p}(\mu)$ with $\operatorname{Ker} R_{\mu+1}(A_1, B_1) = \operatorname{Ker} R_{\mu+1}(A_2, B_2)$. Then there exists $S \in \mathrm{GL}(\mathcal{F}^k)$ such that $R_{\mu+1}(A_1, B_1) = SR_{\mu+1}(A_2, B_2) = R_{\mu+1}(SA_2S^{-1}, SB_2)$. But then it might be assumed w.l.o.g. (i.e. up to similarity) that $R_{\mu+1}(A_1, B_1) = R_{\mu+1}(A_2, B_2)$, i.e. $B_1 = B_2$ and $A_1 R_\mu(A_1, B_1) = A_2 R_\mu(A_2, B_2)$. Since $R_\mu(A_1, B_1) = R_\mu(A_2, B_2)$ it follows $(A_1 - A_2) R_\mu(A_1, B_1) = 0$. But then $\operatorname{rk} R_\mu(A_1, B_1) = k$ yields $A_1 - A_2 = 0$, i.e. $(A_1, B_1) = (A_2, B_2)$.

To show that the map $R_{\mu+1}$ is open onto its image $R_{\mu+1}(\Sigma_{k,p}(\mu))$, it suffices to show that its inverse $R_{\mu+1}^{-1} : R_{\mu+1}(\Sigma_{k,p}(\mu)) \longrightarrow \Sigma_{k,p}(\mu)$ is sequentially continuous (the sets $R_{\mu+1}(\Sigma_{k,p}(\mu))$ and $\Sigma_{k,p}(\mu)$ are aubspaces of manifolds, hence metrizable). Let $([A_j, B_j]_\sigma)_{j \in \mathbb{N}} \subset \Sigma_{k,p}(\mu)$ be a sequence with $\operatorname{Ker} R_{\mu+1}(A_j, B_j) \to \operatorname{Ker} R_{\mu+1}(A_\infty, B_\infty)$ for $j \to \infty$ and a $[A_\infty, B_\infty]_\sigma \in \Sigma_{k,p}(\mu)$. Then w.l.o.g. $R_{\mu+1}(A_j, B_j) \to R_{\mu+1}(A_\infty, B_\infty)$ holds. It follows $B_j \to B_\infty$ and $(A_j - A_\infty) R_\mu(A_j, B_j) \to 0$. Since $R_\mu(A_j, B_j) \to R_\mu(A_\infty, B_\infty)$ and $\operatorname{rk} R_\mu(A_\infty, B_\infty) = k$ it follows $A_j - A_\infty \to 0$, i.e. $(A_j, B_j) \to (A_\infty, B_\infty)$. □

Helmke [Hel92] used the following modification of the Kalman embedding

$$\begin{aligned} \tilde{R}_k : \Sigma_{k,p}(\mu) &\longrightarrow \mathcal{F}^k \times \mathrm{G}_{n-k}(\mathcal{F}^n) \ , \\ [A, B]_\sigma &\mapsto (c(A), \operatorname{Ker} R_k(A, B)) \ , \end{aligned}$$

where $n = kp$ and $c(A) = (c_0, \ldots, c_{k-1}) \in \mathcal{F}^k$ denotes the vector of coefficients of the characteristic polynomial of A

$$\det(sI - A) = \sum_{j=0}^{k} c_j s^j \ , \quad c_k = 1$$

to show that $\mathcal{N}_{k,p}(\mathcal{F})$, which consists of all $[A,B]_\sigma \in \Sigma_{k,p}(\mathcal{F})$ with $c(A)=0$, is compact. The proof uses the Cayley-Hamilton Theorem and can thus not easily be generalized to $\mathcal{N}_{k,p}(\mu)$. In fact, being defined by a 'full rank' condition, $\mathcal{N}_{k,p}(\mu)$ is open in $\mathcal{N}_{k,p}(\mathcal{F})$. It can only be compact if it is also closed in $\mathcal{N}_{k,p}(\mathcal{F})$. To demonstrate the topological locus of the 'gap' between $\mathcal{N}_{k,p}(\mu)$ and $\mathcal{N}_{k,p}(\mathcal{F})$, in the following proof an element of the difference set and a sequence in $\mathcal{N}_{k,p}(\mu)$ converging to this element is constructed.

Proposition 4.5. *Let $1 \le k \le n$ and let $\mu = (\mu_1, \dots, \mu_p)$ with $\mu_1 \ge \dots \ge \mu_p \ge 1$ and $\mu_1 + \dots + \mu_p = n$. For $k \le \mu_p$ the set $\mathcal{N}_{k,p}(\mu)$ is equal to $\mathcal{N}_{k,p}(\mathcal{F})$, hence it is compact. For $k > \mu_p$ the set $\mathcal{N}_{k,p}(\mu)$ is not closed in $\mathcal{N}_{k,p}(\mathcal{F})$, hence it is not compact.*

Proof. Let $k \le \mu_p$ and let $[A,B]_\sigma \in \mathcal{N}_{k,p}(\mathcal{F})$. Then $\operatorname{rk} R_k(A,B) = k$. Since $\mu_1 \ge \dots \ge \mu_p \ge k$ this implies $\operatorname{rk} R_\mu(A,B) = k$ and $[A,B]_\sigma \in \mathcal{N}_{k,p}(\mu)$. Using $\mathcal{N}_{k,p}(\mu) \subset \mathcal{N}_{k,p}(\mathcal{F})$ it follows $\mathcal{N}_{k,p}(\mu) = \mathcal{N}_{k,p}(\mathcal{F})$ and hence $\mathcal{N}_{k,p}(\mu)$ is compact.

Let $k > \mu_1$. Let

$$A_\alpha = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathcal{F}^{k \times k} .$$

Note that A_α is nilpotent. Let e_i, $i = 1, \dots, k$, be the standard basis vectors of $\mathcal{F}^k$. Let $j \in \mathbb{N}$ be the maximal j with $k = (\sum_{i=1}^{j-1}(\mu_i - 1)) + l$ and $l \in \mathbb{N}_0$. Let $B_\alpha \in \mathcal{F}^{k \times p}$ be defined by

$$(B_\alpha)_\beta = \begin{cases} e_k & , \ \beta = 1 \\ \frac{1}{\alpha} e_{(\sum_{i=1}^{j-\beta}(\mu_i - 1)) + l} & , \ \beta = 2, \dots, j \\ 0 & , \ \beta = j+1, \dots, p \end{cases} .$$

Note that (A_α, B_α) is the same as the pair (A,B) considered in the proof of Lemma 4.3 but with $b_2, \dots, b_j$ premultiplied by $\frac{1}{\alpha}$. Then

$$A_\infty = \lim_{\alpha \to \infty} A_\alpha = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathcal{F}^{k \times k}$$

and

$$B_\infty = \lim_{\alpha\to\infty} B_\alpha = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{F}^{k\times p} \ .$$

It follows $\operatorname{rk} R_k(A_\alpha, B_\alpha) = \operatorname{rk} R_k(A_\infty, B_\infty) = k$. Hence $[A_\alpha, B_\alpha]_\sigma \in \mathcal{N}_{k,p}(\mathcal{F})$ and $[A_\infty, B_\infty]_\sigma \in \mathcal{N}_{k,p}(\mathcal{F})$. Furthermore $\operatorname{rk} R_\mu(A_\alpha, B_\alpha) = k$, i.e. $[A_\alpha, B_\alpha]_\sigma \in \mathcal{N}_{k,p}(\mu)$. But $\operatorname{rk} R_\mu(A_\infty, B_\infty) = \mu_1 < k$, i.e. $[A_\infty, B_\infty]_\sigma \notin \mathcal{N}_{k,p}(\mu)$ and $\mathcal{N}_{k,p}(\mu)$ is not closed in $\mathcal{N}_{k,p}(\mathcal{F})$.

Let finally $\mu_1 \geq k > \mu_p$. Let

$$A_\alpha = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathcal{F}^{k\times k} \ .$$

Let $B_\alpha \in \mathcal{F}^{k\times p}$ be defined by

$$(B_\alpha)_\beta = \begin{cases} \frac{1}{\alpha} e_k & , \ \beta = 1 \\ 0 & , \ \beta = 2, \dots, p-1 \\ e_k & , \ \beta = p \end{cases} \ .$$

Then

$$A_\infty = \lim_{\alpha\to\infty} A_\alpha = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathcal{F}^{k\times k}$$

and

$$B_\infty = \lim_{\alpha\to\infty} B_\alpha = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathcal{F}^{k\times p} \ .$$

It follows $\operatorname{rk} R_k(A_\alpha, B_\alpha) = \operatorname{rk} R_k(A_\infty, B_\infty) = k$. Hence $[A_\alpha, B_\alpha]_\sigma \in \mathcal{N}_{k,p}(\mathcal{F})$ and $[A_\infty, B_\infty]_\sigma \in \mathcal{N}_{k,p}(\mathcal{F})$. Furthermore $\operatorname{rk} R_\mu(A_\alpha, B_\alpha) = k$ since $\mu_1 \geq k$, i.e. $[A_\alpha, B_\alpha]_\sigma \in \mathcal{N}_{k,p}(\mu)$. But $\operatorname{rk} R_\mu(A_\infty, B_\infty) = \mu_p < k$, i.e. $[A_\infty, B_\infty]_\sigma \notin \mathcal{N}_{k,p}(\mu)$ and $\mathcal{N}_{k,p}(\mu)$ is not closed in $\mathcal{N}_{k,p}(\mathcal{F})$. $\square$

Further results on the topology of $\mathcal{N}_{k,p}(\mu)$ can be found in Helmke [Hel92]. In Theorem 2.24 of that paper Helmke proposed that the inclusion map of $\mathcal{N}_{k,p}(\mathcal{F})$ into $\Sigma_{k,p}(\mathcal{F})$ is a homotopy equivalence, a rigorous proof of which has yet to be given.

4.2 Controllable triples

The moduli space $C_{k,p}(\mathcal{F})$ of restricted system equivalence classes of controllable triples is well understood. It is a compactification of the moduli space $\Sigma_{k,p}(\mathcal{F})$ of similarity classes of controllable pairs [Hel92]. Basic facts about matrix triples (in the sense of descriptor systems, singular control systems, linear DAEs) can e.g. be found in the lecture notes by Dai [Dai89].

Consider the *restricted system equivalence action*

$$\begin{aligned}\eta : (\mathrm{GL}(\mathcal{F}^k) \times \mathrm{GL}(\mathcal{F}^k)) \times (\mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}) &\longrightarrow \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p},\\ ((S,T),(E,A,B)) &\mapsto (SET^{-1}, SAT^{-1}, SB)\end{aligned}$$

on the set of *admissible triples* (E, A, B), i.e. triples for which the *matrix pencil* $\lambda E+\mu A$ is *regular*, i.e. $\det(\lambda E+\mu A) \neq 0$ for some $\lambda, \mu \in \mathbb{C}$. It is well known (see e.g. the textbook by Gantmacher [Gan59]) that for an admissible triple (E, A, B) there exists a transformation (S, T) such that

$$\begin{aligned} SET^{-1} &= \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} , \\ SAT^{-1} &= \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix} \text{ and} \\ SB &= \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} , \end{aligned}$$

where I is the identity matrix and N is nilpotent (*Weierstraß decomposition*). This decomposition is not unique but the following holds ([Dai89, Theorem 1-3.1]):

Lemma 4.6. *Let (S, T) and (S', T') be two transformations, which decompose the given triple (E, A, B) as above. Then there exist invertible matrices U_1, U_2 with*

$$\begin{aligned} S' &= \mathrm{diag}(U_1, U_2)S , \quad T' = \mathrm{diag}(U_1, U_2)T , \\ A_1' &= U_1 A_1 U_1^{-1} , \\ N' &= U_2 N U_2^{-1} , \\ B_1' &= U_1 B_1 , \quad B_2' = U_2 B_2 , \end{aligned}$$

i.e. the pairs (A_1, B_1) and (A_1', B_1') and the pairs (N, B_2), (N', B_2') are similar, respectively.

As a consequence of this, any similarity invariants of the pairs (A_1, B_1) and (N, B_2) of any Weierstraß decomposition of the admissible triple (E, A, B) together form a restricted system equivalence invariant of the triple. As a first example controllability of a triple is defined.

Definition 4.7. An admissible triple (E, A, B) is called *controllable*, if both of the pairs (A_1, B_1) and (N, B_2) of any Weierstraß decomposition are controllable.

A triple is controllable if and only if the associated descriptor system (singular control system, linear DAE)

$$E\dot{x} = Ax + Bu$$

is controllable (see e.g. Dai [Dai89]). Apparently the restricted system equivalence action η restricts to the set of controllable triples. The orbit space (cf. Appendix A) of this action, i.e. the manifold of all *restricted system equivalence classes*

$$[E, A, B]_\eta = \{(SET^{-1}, SAT^{-1}, SB) \,|\, (S, T) \in \mathrm{GL}(\mathcal{F}^k) \times \mathrm{GL}(\mathcal{F}^k)\}$$

of controllable triples (endowed with the quotient topology) is denoted by $C_{k,p}(\mathcal{F})$.

This moduli space will be used in Chapter 5 to describe the set of all almost (C, A)-invariant subspaces of an observable pair (C, A) in dual Brunovsky form with observability indices $(\mu_1, \ldots, \mu_p)$ (cf. Section 2.3.2). The description uses the following object.

Definition 4.8. Let $1 \leq r \leq k \leq n$ and $\mu = (\mu_1, \ldots, \mu_p)$ with $\mu_1 \geq \cdots \geq \mu_p > 0$ and $\mu_1 + \cdots + \mu_p = n$. The *combined μ-partial reachability matrix* $R_\mu(A_1, B_1, N, B_2)$ of a pair of matrix pairs $((A_1, B_1), (N, B_2)) \in (\mathcal{F}^{r\times r} \times \mathcal{F}^{r\times p}) \times (\mathcal{F}^{(k-r)\times(k-r)} \times \mathcal{F}^{(k-r)\times p})$ is the $k \times n$-matrix

$$R_\mu(A_1, B_1, N, B_2) = \begin{pmatrix} R_\mu(A_1,B_1) \\ \overleftarrow{R}_\mu(N,B_2) \end{pmatrix} =$$
$$\begin{pmatrix} b_{11} & A_1 b_{11} & \ldots & A_1^{\mu_1-1} b_{11} & \ldots & b_{1p} & A_1 b_{1p} & \ldots & A_1^{\mu_p-1} b_{1p} \\ N^{\mu_1-1} b_{21} & N^{\mu_1-2} b_{21} & \ldots & b_{21} & \ldots & N^{\mu_p-1} b_{2p} & N^{\mu_p-2} b_{2p} & \ldots & b_{2p} \end{pmatrix},$$

where b_{ij} denotes the j-th column of B_i, $i = 1, 2$, $j = 1, \ldots, p$.

Using this matrix, some special kinds of admissible triples are defined.

Definition 4.9. An admissible triple (E, A, B) is called *μ-regular*, if for any (and therefore for all) Weierstraß decomposition $((A_1, B_1), (N, B_2))$ of (E, A, B) the combined μ-partial reachability matrix has full row rank, i.e.

$$\operatorname{rk} R_\mu(A_1, B_1, N, B_2) = k \ .$$

The triple is called *μ-tight*, if it is $\mu - 1 = (\mu_1 - 1, \dots, \mu_p - 1)$-regular, i.e. if

$$\begin{aligned} \operatorname{rk} R_{\mu-1}(A_1, B_1, N, B_2) &= \operatorname{rk} \begin{pmatrix} b_{11} & \dots & A_1^{\mu_1-2} b_{11} & \dots & b_{1p} & \dots & A_1^{\mu_p-2} b_{1p} \\ N^{\mu_1-2} b_{21} & \dots & b_{21} & \dots & N^{\mu_p-2} b_{2p} & \dots & b_{2p} \end{pmatrix} \\ &= \operatorname{rk} \begin{pmatrix} R_{\mu-1}(A_1, B_1) \\ \overleftarrow{R}_{\mu-1}(N, B_2) \end{pmatrix} = k \ . \end{aligned}$$

Apparently any μ-regular triple and any μ-tight triple is controllable. Both notions are invariant under the restricted system equivalence action. Therefore it is possible to define the set of all restricted system equivalence classes of μ-regular and μ-tight triples, respectively.

$$\begin{aligned} C_{k,p}(\mu) &:= \{[E, A, B]_\eta \in C_{k,p}(\mathcal{F}) \mid \operatorname{rk} R_\mu(A_1, B_1, N, B_2) = k\} \subset C_{k,p}(\mathcal{F}) \ , \\ C^{\mathrm{t}}_{k,p}(\mu) &:= \{[E, A, B]_\eta \in C_{k,p}(\mathcal{F}) \mid \operatorname{rk} R_{\mu-1}(A_1, B_1, N, B_2) = k\} \subset C_{k,p}(\mathcal{F}) \ . \end{aligned}$$

It will follow from Proposition 5.13 that μ-tightness of (E, A, B) implies μ-regularity (cf. Corollary 5.16), i.e. $C^{\mathrm{t}}_{k,p}(\mu) \subset C_{k,p}(\mu)$ holds.

As has already been indicated before, the moduli space $C_{k,p}(\mathcal{F})$ of controllable triples is a smooth compact manifold of dimension kp over $\mathcal{F}$ ([Hel92]).

If the triple

$$(E, A, B) = \left(\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right)$$

is given in Weierstraß form then

$$\begin{aligned} R_\mu(A_1, B_1, N, B_2) = \big(& E^{\mu_1-1} b_1 \ E^{\mu_1-2} A b_1 \ \dots \ E A^{\mu_1-2} b_1 \ A^{\mu_1-1} b_1 \\ & \dots \ E^{\mu_1-1} b_p \ E^{\mu_1-2} A b_p \ \dots \ E A^{\mu_1-2} b_p \ A^{\mu_1-1} b_p \big) \ , \end{aligned}$$

where b_i, $i = 1, \dots, p$, denotes the i-th column of B. Unfortunately, the object on the right hand side does not behave well under the restricted system equivalence action unless $S = T$, which does not leave enough freedom to transform an arbitrary (admissible or even controllable) triple into Weierstraß form. Hence its rank, in general, is no restricted system equivalence invariant as the following example shows.

Example 4.10. Let $p = 1$, $\mu_1 = 2$,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ .$$

Then

$$\mathrm{rk} \begin{pmatrix} EB & AB \end{pmatrix} = \mathrm{rk} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \ ,$$

but with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

it follows

$$\mathrm{rk} \begin{pmatrix} SET^{-1}SB & SAT^{-1}SB \end{pmatrix} = \mathrm{rk} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 1 \ .$$

In view of this example the object seems to be no suitable tool to derive topological properties of $C_{k,p}(\mu)$ and $C^{\mathrm{t}}_{k,p}(\mu)$.

Furthermore, there exists no *continuous* transformation of the set of controllable triples into Weierstraß form ([Hel92]). Consequently, results on the topology of $C_{k,p}(\mu)$ and $C^{\mathrm{t}}_{k,p}(\mu)$ would be much harder to derive than those for $\Sigma_{k,p}(\mu)$ and $\Sigma^{\mathrm{t}}_{k,p}(\mu)$ in the case of controllable pairs.

4.3 The manifold of tracking observers

If the system (3.1) is observable then the connection between (C, A)-invariant subspaces and tracking observers established in Theorem 3.33 can be made more precise using the following manifold structure.

Theorem 4.11. *Let the system* (3.1) *be observable and let*

$$\mathrm{Obs}_k = \{(K, L, M, V) \in \mathcal{F}^{k\times(k+p+m+n)} \mid VA - KV = LC, \ M = VB\}$$

be the set of all order k tracking observer parameters for system (3.1)*.* Obs_k *is a smooth submanifold of $\mathcal{F}^{k\times(k+p+m+n)}$ of dimension* $\dim \mathrm{Obs}_k = k^2 + kp$*. Its tangent space at the point $(K, L, M, V) \in \mathrm{Obs}_k$ is*

$$T_{(K,L,M,V)}\mathrm{Obs}_k = \{(\dot{K}, \dot{L}, \dot{M}, \dot{V}) \mid -\dot{K}V - \dot{L}C + \dot{V}A - K\dot{V} = \dot{M} - \dot{V}B = 0\} \ .$$

Proof. Consider the map

$$\begin{aligned} f : \mathcal{F}^{k\times(k+p+m+n)} &\longrightarrow \mathcal{F}^{k\times(n+m)}, \\ (K, L, M, V) &\mapsto (VA - KV - LC, M - VB) \ . \end{aligned}$$

It will be shown that $(0, 0)$ is a regular value of f, hence $\mathrm{Obs}_k = f^{-1}(0, 0)$ is a smooth submanifold of $\mathcal{F}^{k\times(k+p+m+n)}$. The derivative of f at a point (K, L, M, V) is given by

$$Df : \ (\dot{K}, \dot{L}, \dot{M}, \dot{V}) \mapsto (-\dot{K}V - \dot{L}C + \dot{V}A - K\dot{V}, \dot{M} - \dot{V}B) \ ,$$

where $(\dot{K}, \dot{L}, \dot{M}, \dot{V}) \in T_{(K,L,M,V)}(\mathcal{F}^{k\times(k+p+m+n)})$.

An element $(\xi, \eta) \in T_{f(K,L,M,V)}(\mathcal{F}^{k\times(n+m)})$ is orthogonal to the image of Df if and only if

$$\operatorname{tr} \xi^*(-\dot{K}V - \dot{L}C + \dot{V}A - K\dot{V}) + \operatorname{tr} \eta^*(\dot{M} - \dot{V}B) = 0$$

for all $(\dot{K}, \dot{L}, \dot{M}, \dot{V}) \in T_{(K,L,M,V)}(\mathcal{F}^{k\times(k+p+m+n)})$. This is equivalent to

$$\begin{aligned} V\xi^* &= 0 & (4.1) \\ C\xi^* &= 0 & (4.2) \\ \eta^* &= 0 & (4.3) \\ A\xi^* - \xi^* K &= 0 \ . & (4.4) \end{aligned}$$

From (4.4) it follows by induction $A^i\xi^* - \xi^* K^i = 0$ for all $i \in \mathbb{N}$. Together with (4.2) this yields

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \xi^* = 0 \ .$$

Since (C, A) is observable this yields $\xi^* = 0$. It follows that Df is surjective and $(0, 0)$ is a regular value of f (in fact f is a submersion).

The dimension of $\mathrm{Obs}_k = f^{-1}(0, 0)$ is $k(k+p+m+n) - k(n+m) = k^2 + kp$. From the fibre theorem it follows $T_{(K,L,M,V)}\mathrm{Obs}_k = (Df)^{-1}(0, 0)$. $\square$

Corollary 4.12. *Being an open subset of* Obs_k *the set*

$$\mathrm{Obs}_{k,k} = \{(K, L, M, V) \in \mathrm{Obs}_k \mid \operatorname{rk} V = k\}$$

is a smooth submanifold of $\mathcal{F}^{k\times(k+p+m+n)}$ *of dimension* $k^2 + kp$.

Now consider the *similarity action* on $\mathrm{Obs}_{k,k}$

$$\begin{aligned}\sigma : \mathrm{GL}(\mathcal{F}^k) \times \mathrm{Obs}_{k,k} &\longrightarrow \mathrm{Obs}_{k,k}\ ,\\ (S,(K,L,M,V)) &\mapsto (SKS^{-1},SL,SM,SV)\end{aligned}$$

and the induced *similarity classes*

$$[K,L,M,V]_\sigma = \{(SKS^{-1},SL,SM,SV)\,|\,S \in \mathrm{GL}(\mathcal{F}^k)\}\ .$$

Note that σ is well defined since $VA - KV = LC$ and $M = VB$ imply $SVA - SKS^{-1}SV = S(VA - KV) = SLC$ and $SM = SVB$.

Theorem 4.13. *The orbit space*

$$\mathrm{Obs}^\sigma_{k,k} = \{[K,L,M,V]_\sigma\,|\,(K,L,M,V) \in \mathrm{Obs}_{k,k}\}$$

of similarity classes of order k tracking observer parameters for system (3.1) *is a smooth manifold of dimension* $\dim \mathrm{Obs}^\sigma_{k,k} = kp$.

Proof. Since V has full row rank k for $(K,L,M,V) \in \mathrm{Obs}_{k,k}$, the similarity action is free and has a closed graph mapping (cf. Appendix A): $SV = V$ implies $S = I$, furthermore $V_j \to V$ and $S_jV_j \to W$ imply $S_j \to S$ and $W = SV$. Hence the orbit space of σ is a smooth manifold of dimension $\dim \mathrm{Obs}^\sigma_{k,k} = \dim \mathrm{Obs}_{k,k} - \dim \mathrm{GL}(\mathcal{F}^k) = k^2 + kp - k^2 = kp$. □

Now the following theorem refines Theorem 3.33. Let $G_{n-k}(\mathcal{F}^n)$ denote the *Grassmann manifold* of all codimension k linear subspaces of $\mathcal{F}^n$.

Theorem 4.14. *Let the system* (3.1) *be observable. For each k the set*

$$\mathrm{InvJ}_{n-k} = \{(\mathcal{V},J) \in \mathrm{G}_{n-k}(\mathcal{F}^n) \times \mathcal{F}^{n\times p}\,|\,(A-JC)\mathcal{V} \subset \mathcal{V}\}$$

is a smooth manifold of dimension $\dim \mathrm{InvJ}_{n-k} = np$. *The map*

$$\begin{aligned}\bar{f} : \mathrm{InvJ}_{n-k} &\longrightarrow \mathrm{Obs}^\sigma_{k,k}\ ,\\ (\mathcal{V},J) &\mapsto [K,L,M,V]_\sigma\ ,\end{aligned}$$

defined by $\mathrm{Ker}\,V = \mathcal{V}$, $M = VB$, $L = VJ$ *and* $KV = VA - LC = V(A-JC)$ *is a smooth vector bundle with fiber* $\mathcal{F}^{(n-k)\times p}$.

Proof. Consider the set

$$\mathcal{M}_{n-k} = \{(V, J) \in \mathrm{St}(k, n) \times \mathcal{F}^{n\times p} \,|\, (A - JC)\,\mathrm{Ker}\, V \subset \mathrm{Ker}\, V\}\ ,$$

where $\mathrm{St}(k, n)$ denotes the set of full row rank $k \times n$ matrices (*Stiefel manifold*). Apparently, if $(V, J) \in \mathcal{M}_{n-k}$ then $\mathrm{Ker}\, V$ is a codimension k (C, A)-invariant subspace with friend J. Consider the map

$$\begin{aligned} f : \mathcal{M}_{n-k} &\longrightarrow \mathrm{Obs}_{k,k}\ , \\ (V, J) &\mapsto (K, L, M, V)\ , \end{aligned}$$

where $L = VJ$, $M = VB$ and K is defined as the unique solution of the equation $KV = VA - LC = V(A - JC)$ (cf. Theorem 3.33, Part 1). By Theorem 3.33, Part 2, the map f is surjective. Since $K = V(A - JC)V^*(VV^*)^{-1}$, the map f is continuous. Moreover, it is the restriction of a smooth map defined on $\mathrm{St}(k, n) \times \mathcal{F}^{n\times p}$, which is an open subset of $\mathcal{F}^{k\times n} \times \mathcal{F}^{n\times p}$. According to Corollary 4.12 the set $\mathrm{Obs}_{k,k}$ is a smooth submanifold of $\mathcal{F}^{k\times(k+p+m+n)}$.

Given V and $L = VJ$, the solution set of $VX = VJ$ is the affine space $V^*(VV^*)^{-1}(VJ) + \prod_{i=1}^{p} \mathrm{Ker}\, V$. Furthermore, $\dim \mathrm{Ker}\, V = n - k$. Therefore, for every $(K, L, M, V) \in \mathrm{Obs}_{k,k}$ the fiber $f^{-1}(K, L, M, V)$ is an affine space of dimension $(k - n)p$.

Let $V_0 \in \mathrm{St}(k, n)$. Since V_0 has full row rank there exists a permutation matrix P_0 such that $V_0P_0 = \begin{pmatrix} X_0 & Y_0 \end{pmatrix}$ with $X_0 \in \mathcal{F}^{k\times k}$ invertible. Then $W = \{\begin{pmatrix} X & Y \end{pmatrix} P_0^{-1} \,|\, X \text{ invertible}\}$ is an open neighborhood of V_0 in $\mathrm{St}(k, n)$ and $\mathrm{Ker}\, V = \{P_0(-X^{-1}Yy, y)^\top \,|\, y \in \mathcal{F}^{n-k}\}$ for every $V = \begin{pmatrix} X & Y \end{pmatrix} P_0^{-1} \in W$. But then

$$\begin{aligned} \varphi_W : W \times \mathcal{F}^{n-k} &\longrightarrow W \times \mathcal{F}^n\ , \\ (V, y) &\mapsto (V, P_0[I_n - \begin{pmatrix} I_k \\ 0 \end{pmatrix} (VP_0 \begin{pmatrix} I_k \\ 0 \end{pmatrix})^{-1} VP_0] \begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix} y) \\ & \quad (V, P_0(-X^{-1}Yy, y)^\top) \end{aligned}$$

is a smooth injective map mapping $(V, \mathcal{F}^{n-k})$ onto $(V, \mathrm{Ker}\, V)$ for every $V \in W$. Hence φ_W is a homeomorphism onto its image. The inverse map

$$\begin{aligned} \varphi_W^{-1} : \varphi_W(W \times \mathcal{F}^{n-k}) &\longrightarrow W \times \mathcal{F}^{n-k}\ , \\ (V, z) &\mapsto (V, \begin{pmatrix} 0 & I_{n-k} \end{pmatrix} P_0^{-1} z) \end{aligned}$$

is the restriction of a smooth map defined on all of $\mathcal{F}^{k\times n} \times \mathcal{F}^n$. If $V = \begin{pmatrix} X_1 & Y_1 \end{pmatrix} P_1^{-1} = \begin{pmatrix} X_2 & Y_2 \end{pmatrix} P_2^{-1} \in W_1 \cap W_2$ then the change of coordinate function $\varphi_{W_2}^{-1} \circ \varphi_{W_1}$ induces the invertible linear transformation

$$\vartheta_{W_2,W_1,V} : \; y \mapsto \begin{pmatrix} 0 & I_{n-k} \end{pmatrix} P_2^{-1} P_1 \begin{pmatrix} -X_1^{-1} Y_1 \\ I_{n-k} \end{pmatrix} y$$

on $\mathcal{F}^{n-k}$.

Using p-fold products of φ_W it is now easy to construct local trivializations of f. Given $(K_0, L_0, M_0, V_0) \in \mathrm{Obs}_{k,k}$ choose the neighborhood

$$U := (\mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} \times \mathcal{F}^{k\times m} \times W) \cap \mathrm{Obs}_{k,k} \; ,$$

which is open in $\mathrm{Obs}_{k,k}$. Let pr_2 denote the projection $(V, y) \mapsto y$ and consider the map

$$\begin{aligned} \phi_U : U \times \mathcal{F}^{(n-k)\times p} &\longrightarrow f^{-1}(U) \; , \\ (K, L, M, V, \begin{pmatrix} y_1 & \cdots & y_p \end{pmatrix}) &\mapsto \\ (V, V^*(VV^*)^{-1}L + \begin{pmatrix} \mathrm{pr}_2(\varphi_W(V,y_1)) & \cdots & \mathrm{pr}_2(\varphi_W(V,y_p)) \end{pmatrix}) \; , \end{aligned}$$

where y_i, $i = 1, \ldots, p$, denotes the i-th column of the matrix $Y \in \mathcal{F}^{(n-k)\times p}$. Apparently, $f(\phi_U(K, L, M, V, Y)) = (K, L, M, V)$ for all $(K, L, M, V) \in U$ and all $Y \in \mathcal{F}^{(n-k)\times p}$. Furthermore, ϕ_U is bijective by construction. Since φ_W is smooth, so is ϕ_U. Let e_i, $i = 1, \ldots, p$, denote the i-th standard basis vector of $\mathcal{F}^p$. The inverse map

$$\begin{aligned} \phi_U^{-1} : f^{-1}(U) &\longrightarrow U \times \mathcal{F}^{(n-k)\times p} \; , \\ (V, J) &\mapsto (f(V, J), g(V, J)) \; , \end{aligned}$$

where

$$g(V, J) = \begin{pmatrix} \mathrm{pr}_2(\varphi_W^{-1}(V, [J - V^*(VV^*)^{-1}(VJ)]\mathrm{e}_1)) & \cdots & \mathrm{pr}_2(\varphi_W^{-1}(V, [J - V^*(VV^*)^{-1}(VJ)]\mathrm{e}_\mathrm{p})) \end{pmatrix} \; ,$$

is the restriction of a smooth map defined on $\mathrm{St}(k, n) \times \mathcal{F}^{n\times p}$, which is an open subset of $\mathcal{F}^{k\times n} \times \mathcal{F}^{n\times p}$. It follows that ϕ_U is a homeomorphism. Finally, if $(K, L, M, V) \in U_1 \cap U_2 = (\mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} \times \mathcal{F}^{k\times m} \times (W_1 \cap W_2)) \cap \mathrm{Obs}_{k,k}$ then the change of coordinate function $\phi_{U_2}^{-1} \circ \phi_{U_1}$ induces the invertible linear transformation

$$\theta_{U_2,U_1,(K,L,M,V)} : \begin{pmatrix} y_1 & \cdots & y_p \end{pmatrix} \mapsto \begin{pmatrix} \vartheta_{W_2,W_1,V}(y_1) & \cdots & \vartheta_{W_2,W_1,V}(y_p) \end{pmatrix}$$

on $\mathcal{F}^{(n-k)\times p}$.

According to Theorem B.2 the set $\mathcal{M}_{n-k}$ is a smooth submanifold of $\mathcal{F}^{k\times n} \times \mathcal{F}^{n\times p}$ of dimension $\dim \mathrm{Obs}_{k,k} + (n-k)p = k^2 + kp + (n-k)p = k^2 + np$. Furthermore, the map f is a differentiable vector bundle with fiber $\mathcal{F}^{(n-k)\times p}$.

As has been shown in the proof of Theorem 4.13, the similarity action σ on $\mathrm{Obs}_{k,k}$ is free and proper (cf. Theorem A.5). By the same arguments this is also true for the similarity action on $\mathcal{M}_{n-k}$:

$$\begin{aligned}\sigma : \mathrm{GL}(\mathcal{F}^k) \times \mathcal{M}_{n-k} &\longrightarrow \mathcal{M}_{n-k} \ , \\ (S,(V,J)) &\mapsto (SV,J) \ .\end{aligned}$$

As is well known, the quotient space $\mathrm{St}(k,n)/\mathrm{GL}(\mathcal{F}^k)$ is diffeomorphic to $\mathrm{G}_{n-k}(\mathcal{F}^n)$ via $[V]_\sigma \mapsto \operatorname{Ker} V$, hence the quotient $\mathcal{M}_{n-k}/\mathrm{GL}(\mathcal{F}^k)$ is diffeomorphic to InvJ_{n-k} and the latter is a smooth manifold of dimension $\dim \mathcal{M}_{n-k} - \dim \mathrm{GL}(\mathcal{F}^k) = k^2 + np - k^2 = np$.

Apparently, $(K,L,M,V) \in U$ implies $(SKS^{-1}, SL, SM, SV) \in U$ since S being invertible and $V \in W$ implies $SV \in W$. Furthermore,

$$\begin{aligned}\phi_U(\sigma(S,(K,L,M,V)),Y) &= \phi_U((SKS^{-1}, SL, SM, SV), Y) \\ &= \big(SV, (SV)^*(SV(SV)^*)^{-1}SL + \big(\mathrm{pr}_2(\varphi_W(SV,y_1)) \dots\big)\big) \\ &= \big(SV, V^*(VV^*)^{-1}L + \big(\mathrm{pr}_2(\varphi_W(V,y_1)) \dots\big)\big) \\ &= \sigma\big(S, \big(V, V^*(VV^*)^{-1}L + \big(\mathrm{pr}_2(\varphi_W(V,y_1)) \dots\big)\big)\big) \\ &= \sigma(S, \phi_U((K,L,M,V),Y)) \ .\end{aligned}$$

But then Theorem B.3 implies that $\bar{f}$ is a smooth vector bundle with fiber $\mathcal{F}^{(n-k)\times p}$, which completes the proof. $\square$

Chapter 5

Kernel representations

In this chapter the moduli spaces presented in Chapter 4 are used to gain some insight into the topology and geometry of the set of all codimension k almost (C,A)-invariant subspaces of a given observable pair (C,A) in dual Brunovsky form. The idea of using controllable pairs to parametrize (C,A)-invariant subspaces goes back to the pioneering work of Hinrichsen, Münzner and Prätzel-Wolters [HMP81]. Permuted and truncated observability matrices were first used by Antoulas [Ant83] to characterize (A,B)-invariant subspaces. The results presented in this chapter are generalizations of the results achieved by Fuhrmann and Helmke [FH97, FH00, FH01] in the tight (not almost) (C,A)-invariant case (Section 5.1). In Section 5.5 the restriction pencil used by Jaffe and Karcanias [JK81] and Schumacher [Sch83] to characterize almost invariant subspaces is shown to be directly related to the parametrizing triples. In Section 5.6 the connection of kernel representations to partial realization theory is explored. The relevance of these results for observer design is explained in Section 5.7.

5.1 The tight case

In this section some of the results achieved by Antoulas [Ant83] and Fuhrmann and Helmke [FH97, FH00, FH01] are presented. The notation used follows Fuhrmann and Helmke. They use controllable pairs (cf. Section 4.1) to describe the set of all codimension k (C,A)-invariant subspaces of a given observable pair (C,A) in dual Brunovsky form with observability indices $\mu = (\mu_1,\dots,\mu_p)$ (cf. Section 2.3.2). The methods used in the proofs can also be applied in the instantaneous case (Section 5.2).

Proposition 5.1. *Let $1 \leq k \leq n$ and let $(A,B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$.*

(1) $\mathcal{V} = \operatorname{Ker} R_\mu(A,B)$ *is* (C,A)*-invariant,*

(2) $\operatorname{codim}\mathcal{V} = k$ *if and only if* (A,B) *is* μ*-regular and*

(3) $\mathcal{V}$ *is tight of codimension* k *if and only if* (A,B) *is* μ*-tight.*

Proof. Ad (1): According to Proposition 2.31 it is sufficient to show that $\mathsf{A}(\mathcal{V} \cap \operatorname{Ker}\mathsf{C}) \subset \mathcal{V}$. Let $x \in \mathcal{V} \cap \operatorname{Ker}\mathsf{C}$. Then

$$x = \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^{\mu_1-2} \\ 0 \\ \vdots \\ x_p^0 \\ \vdots \\ x_p^{\mu_p-2} \\ 0 \end{pmatrix} \in \operatorname{Ker}\mathsf{C}$$

with

$$R_\mu(A,B)x = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j x_j^l = 0 \ .$$

It follows

$$\begin{aligned} R_\mu(A,B)\mathsf{A}x &= \sum_{j=1}^{p} \sum_{l=1}^{\mu_j-1} A^l b_j x_j^{l-1} \\ &= A \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j x_j^l = 0 \ , \end{aligned}$$

i.e. $\mathsf{A}x \in \mathcal{V}$.

Ad (2): By definition (A,B) is μ-regular if and only if $\operatorname{codim}\operatorname{Ker} R_\mu(A,B) = \operatorname{rk} R_\mu(A,B) = k$.

Ad (3): Let (A,B) be μ-tight, then (A,B) is μ-regular and $\mathcal{V}$ is a codimension k (C,A)-invariant subspace. According to Proposition 2.42 $\mathcal{V}$ is tight if and only if $\mathcal{V} + \operatorname{Ker}\mathsf{C} = \mathcal{F}^n$.

Let

$$x = \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^{\mu_1-1} \\ \vdots \\ x_p^0 \\ \vdots \\ x_p^{\mu_p-1} \end{pmatrix} \in \mathcal{F}^n \ .$$

Since $\operatorname{rk} R_{\mu-1}(A,B) = k$, any linear combination of the vectors $A^{\mu_j-1}b_j$, $j = 1,\ldots,p$, can be written as a linear combination of the columns of $R_{\mu-1}(A,B)$, i.e. there exist numbers $y_j^l \in \mathcal{F}$, $j = 1,\ldots,p$ and $l = 0,\ldots,\mu_j-2$, with

$$\sum_{j=1}^{p} A^{\mu_j-1} b_j x_j^{\mu_{j-1}} = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j y_j^l \ . \tag{5.1}$$

Consider the decomposition

$$x = \begin{pmatrix} -y_1^0 \\ \vdots \\ -y_1^{\mu_1-2} \\ x_1^{\mu_1-1} \\ \vdots \\ -y_p^0 \\ \vdots \\ -y_p^{\mu_p-2} \\ x_p^{\mu_p-1} \end{pmatrix} + \begin{pmatrix} x_1^0 + y_1^0 \\ \vdots \\ x_1^{\mu_1-2} + y_1^{\mu_1-2} \\ 0 \\ \vdots \\ x_p^0 + y_p^0 \\ \vdots \\ x_p^{\mu_p-2} + y_p^{\mu_p-2} \\ 0 \end{pmatrix} =: y + z \ .$$

Then (5.1) yields $y \in \operatorname{Ker} R_\mu(A,B) = \mathcal{V}$ and obviously $z \in \operatorname{Ker} \mathsf{C}$, i.e. $x \in \mathcal{V} + \operatorname{Ker} \mathsf{C}$. Since $x \in \mathcal{F}^n$ was arbitrary, this yields $\mathcal{V} + \operatorname{Ker} \mathsf{C} = \mathcal{F}^n$ and $\mathcal{V}$ is tight.

Conversely let $\mathcal{V}$ be tight of codimension k. Then (A,B) is μ-regular. Assume, that (A,B) is *not* μ-tight, i.e. $\operatorname{rk} R_{\mu-1}(A,B) < k$. Since $\operatorname{rk} R_\mu(A,B) = k$, there exists a linear combination of the vectors $A^{\mu_j-1}b_j$, $j = 1,\ldots,p$, which is not contained in $\operatorname{columnspan} R_{\mu-1}(A,B)$. This means that there exist numbers $x_j^{\mu_j-1} \in \mathcal{F}$, $j = 1,\ldots,p$, such that for every choice of numbers $x_j^l \in \mathcal{F}$, $j = 1,\ldots,p$ and $l = 0,\ldots,\mu_j-2$,

$$\sum_{j=1}^{p} \sum_{l=0}^{\mu_j-1} A^l b_j x_j^l \neq 0 \ . \tag{5.2}$$

It will be shown that

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1^{\mu_1 - 1} \\ \vdots \\ 0 \\ \vdots \\ 0 \\ x_p^{\mu_p - 1} \end{pmatrix} \notin \mathcal{V} + \operatorname{Ker} \mathsf{C} \ .$$

$x \in \mathcal{V} + \operatorname{Ker} \mathsf{C}$ would imply $x = y + z$ with $y \in \mathcal{V}$ and $z \in \operatorname{Ker} \mathsf{C}$, i.e. there would exist numbers $z_j^l \in \mathcal{F}$, $j = 1, \ldots, p$ and $l = 0, \ldots, \mu_j - 2$, such that

$$y = x - z = \begin{pmatrix} -z_1^0 \\ \vdots \\ -z_1^{\mu_1 - 2} \\ x_1^{\mu_1 - 1} \\ \vdots \\ -z_p^0 \\ \vdots \\ -z_p^{\mu_p - 2} \\ x_p^{\mu_p - 1} \end{pmatrix} \in \mathcal{V} = \operatorname{Ker} R_\mu(A, B) \ .$$

But then the choice $x_j^l = -z_j^l$, $j = 1, \ldots, p$ and $l = 0, \ldots, \mu_j - 2$, contradicts (5.2). Hence $x \notin \mathcal{V} + \operatorname{Ker} \mathsf{C}$.

It follows $\mathcal{V} + \operatorname{Ker} \mathsf{C} \neq \mathcal{F}^n$, a contradiction to $\mathcal{V}$ being tight. Hence (A, B) was μ-tight. $\square$

Proposition 5.2. *Let $\mathcal{V} \subset \mathcal{F}^n$ be (C, A)-invariant with* $\operatorname{codim} \mathcal{V} = k$. *Then there exists a μ-regular pair $(A, B) \in \mathcal{F}^{k \times k} \times \mathcal{F}^{k \times p}$ with $\mathcal{V} = \operatorname{Ker} R_\mu(A, B)$.*

Since the existence part of the proof turns out to be more instructive in the dual case, this result is obtained by dualizing the following proposition.

Proposition 5.3. *Let (A, B) be a controllable pair in Brunovsky form with controllability indices $\kappa = (\kappa_1, \ldots, \kappa_m)$ (cf. Section 2.1.2). For any k-dimensional (A, B)-invariant subspace $\mathcal{V} \subset \mathcal{F}^n$ there exists a pair $(C, A) \in$*

$\mathcal{F}^{m\times k} \times \mathcal{F}^{k\times k}$ with

$$\mathcal{V} = \mathrm{Im}\ O_\kappa(C,A) = \mathrm{Im}\begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{\kappa_1 - 1} \\ \vdots \\ c_m \\ \vdots \\ c_m A^{\kappa_m - 1} \end{pmatrix} ,$$

where c_i, $i = 1, \dots, m$, denotes the i-th row of C.

Proof. Let $\mathcal{V} \subset \mathcal{F}^n$ be a k-dimensional (A,B)-invariant subspace and let

$$X = \begin{pmatrix} x_1^0 \\ \vdots \\ x_1^{\kappa_1 - 1} \\ \vdots \\ x_m^0 \\ \vdots \\ x_m^{\kappa_m - 1} \end{pmatrix} \in \mathcal{F}^{n\times k}$$

be a rank k matrix with $\mathrm{Im}\, X = \mathcal{V}$. Proposition 2.3 yields $\mathrm{Im}\, \mathsf{A}X \subset \mathrm{Im}\, X + \mathrm{Im}\, \mathsf{B}$. Equivalently, there exist matrices $A \in \mathcal{F}^{k\times k}$ and $Y \in \mathcal{F}^{m\times k}$ with

$$\mathsf{A}X = XA + \mathsf{B}Y \ .$$

Written out row by row this is equivalent to

$$\begin{aligned} x_j^{l+1} &= x_j^l A \ , \quad j = 1, \dots, m \text{ and } l = 0, \dots, \kappa_j - 2 \\ y_j &= -x_j^{\kappa_j - 1} A \ , \quad j = 1, \dots, m \ , \end{aligned}$$

where y_j, $j = 1, \dots, m$, denotes the j-th row of Y. Setting $c_j = x_j^0$ for $j = 1, \dots, m$ yields the desired result. □

Proof of Proposition 5.2. The existence of such a pair (A, B) follows by dualizing Proposition 5.3. The μ-regularity of (A, B) follows from $\mathrm{codim}\, \mathcal{V} = \mathrm{codim}\, \mathrm{Ker}\, R_\mu(A,B) = \mathrm{rk}\, R_\mu(A,B) = k$. □

Since the linear space $\operatorname{Ker} R_\mu(A,B)$ is invariant under the similarity action, i.e. $\operatorname{Ker} R_\mu(A,B) = \operatorname{Ker} R_\mu(A',B')$ for $(A',B') \in [A,B]_\sigma$, and in view of Proposition 5.1 the following map is well defined:

$$\begin{aligned} \rho_\mu : \Sigma_{k,p}(\mathcal{F}) &\longrightarrow \mathcal{V}(\mathsf{C},\mathsf{A}) \ , \\ [A,B]_\sigma &\mapsto \operatorname{Ker} R_\mu(A,B) \ . \end{aligned}$$

Here $\mathcal{V}(\mathsf{C},\mathsf{A})$ denotes the set of all (C,A)-invariant subspaces. As a consequence of Proposition 5.1 the following two restrictions of this map are well defined. $\mathcal{V}_k(\mathsf{C},\mathsf{A})$ and $\mathcal{T}_k(\mathsf{C},\mathsf{A})$ denote the sets of all codimension k (C,A)-invariant subspaces or tight subspaces, respectively.

Theorem 5.4. *The map*

$$\begin{aligned} \rho_\mu : \Sigma_{k,p}(\mu) &\longrightarrow \mathcal{V}_k(\mathsf{C},\mathsf{A}) \ , \\ [A,B]_\sigma &\mapsto \operatorname{Ker} R_\mu(A,B) \end{aligned}$$

is a surjective algebraic map. It restricts to a bijection

$$\rho_\mu : \ \Sigma^{\mathrm{t}}_{k,p}(\mu) \longrightarrow \mathcal{T}_k(\mathsf{C},\mathsf{A}) \ .$$

Proof. Surjectivity of the first map follows from Proposition 5.2. Surjectivity of the second map follows from surjectivity of the first map and Proposition 5.1. Injectivity of the second map follows from the same arguments as injectivity of the μ-partial Kalman embedding (cf. Proposition 4.4) with $\mu+1$ replaced by μ. □

To bring in the topology, the spaces $\mathcal{T}_k(\mathsf{C},\mathsf{A})$ and $\mathcal{V}_k(\mathsf{C},\mathsf{A})$ are viewed as subsets of the *Grassmann manifold* $\mathrm{G}_{n-k}(\mathcal{F}^n)$.

Theorem 5.5. *The map*

$$\begin{aligned} \rho_\mu : \Sigma^{\mathrm{t}}_{k,p}(\mu) &\longrightarrow \mathcal{T}_k(\mathsf{C},\mathsf{A}) \ , \\ [A,B]_\sigma &\mapsto \operatorname{Ker} R_\mu(A,B) \end{aligned}$$

is a homeomorphism.

Proof. According to Theorem 5.4 ρ_μ is bijective. Furthermore it is clearly continuous. It remains to show that the map ρ_μ is open. This follows from the same arguments as openness of the μ-partial Kalman embedding (cf. Proposition 4.4) with $\mu+1$ replaced by μ. □

Corollary 5.6. *For $k \leq n-p$, $\mathcal{T}_k(\mathsf{C},\mathsf{A})$ is a smooth manifold of dimension kp over $\mathcal{F}$, which is embedded into $\mathrm{G}_{n-k}(\mathcal{F}^n)$ via ρ_μ. For $k > n-p$, $\mathcal{T}_k(\mathsf{C},\mathsf{A})$ is empty.*

5.2 The instantaneous case

In this section a result analogous to the result of Section 5.1 is achieved in the instantaneous case. Here controllable pairs (A, B) with A nilpotent (cf. Section 4.1) are used to describe the set of all codimension k almost observability subspaces with respect to a given observable pair (C, A) in dual Brunovsky form with observability indices $\mu = (\mu_1, \dots, \mu_p)$ (cf. Section 2.3.2).

Proposition 5.7. *Let $1 \leq k \leq n$ and let $(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$, where A is nilpotent.*

(1) $\mathcal{O}_a = \operatorname{Ker} \overleftarrow{R}_\mu(A, B)$ *is an almost observability subspace with respect to the pair* (C, A)*,*

(2) $\operatorname{codim} \mathcal{O}_a = k$ *if and only if* (A, B) *is μ-regular and*

(3) $\mathcal{O}_a$ *is instantaneous of codimension k if and only if* (A, B) *is μ-tight.*

Proof. Ad (1): According to Proposition 2.40 $\mathcal{O}_a$ is an almost observability subspace if and only if $\mathcal{O}_\infty(\mathcal{O}_a) = \mathcal{O}_a$, where $\mathcal{O}_\infty$ refers to the limit of AOSA. Note that $\mathcal{O}_a \subset \mathcal{O}_\infty$ is trivial.

Let $n_1(A) = \min\{l \mid A^l = 0\}$ be the *nilpotency index* of A. It will be shown by induction that

$$\mathcal{O}_i \subset \operatorname{Ker} A^{n_1(A)-i} \overleftarrow{R}_\mu(A, B)$$

for $i = 0, \dots, n_1(A)$. For $i = 0$ this is obvious. Let it be true for a fixed $0 \leq i < n_1(A)$, now. Then

$$\mathcal{O}_{i+1} \subset \operatorname{Ker} \overleftarrow{R}_\mu(A, B) + (\mathsf{A}^{-1} \operatorname{Ker} A^{n_1(A)-i} \overleftarrow{R}_\mu(A, B) \cap \operatorname{Ker} \mathsf{C}) \ .$$

Clearly,

$$\operatorname{Ker} \overleftarrow{R}_\mu(A, B) \subset \operatorname{Ker} A^{n_1(A)-(i+1)} \overleftarrow{R}_\mu(A, B) \ ,$$

so it remains to show that

$$\mathsf{A}^{-1} \operatorname{Ker} A^{n_1(A)-i} \overleftarrow{R}_\mu(A, B) \cap \operatorname{Ker} \mathsf{C} \subset \operatorname{Ker} A^{n_1(A)-(i+1)} \overleftarrow{R}_\mu(A, B) \ .$$

Let $x \in \mathsf{A}^{-1} \operatorname{Ker} A^{n_1(A)-i} \overleftarrow{R}_\mu(A,B) \cap \operatorname{Ker} \mathsf{C}$. Then

$$x = \begin{pmatrix} x_1^{\mu_1-1} \\ \vdots \\ x_1^1 \\ 0 \\ \vdots \\ x_p^{\mu_p-1} \\ \vdots \\ x_p^1 \\ 0 \end{pmatrix} \in \operatorname{Ker} \mathsf{C}$$

and $\mathsf{A}x \in \operatorname{Ker} A^{n_1(A)-i} \overleftarrow{R}_\mu(A,B)$, i.e.

$$\begin{aligned} A^{n_1(A)-i} \overleftarrow{R}_\mu(A,B)\mathsf{A}x &= \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^{n_1(A)-i} A^l b_j x_j^{l+1} \\ &= \sum_{j=1}^{p} \sum_{l=1}^{\mu_j-1} A^{n_1(A)-(i+1)} A^l b_j x_j^{l} \\ &= A^{n_1(A)-(i+1)} \overleftarrow{R}_\mu(A,B) x = 0 \ . \end{aligned}$$

But that means $x \in \operatorname{Ker} A^{n_1(A)-(i+1)} \overleftarrow{R}_\mu(A,B)$ and hence the induction is complete.

Now $\mathcal{O}_{n_1(A)} \subset \operatorname{Ker} \overleftarrow{R}_\mu(A,B) = \mathcal{O}_a$ and therefore $\mathcal{O}_\infty = \mathcal{O}_a$.

Ad (2): By definition (A,B) is μ-regular if and only if $\operatorname{codim} \operatorname{Ker} \overleftarrow{R}_\mu(A,B) = \operatorname{rk} \overleftarrow{R}_\mu(A,B) = k$.

Ad (3): Let (A,B) be μ-tight, then (A,B) is μ-regular and $\mathcal{O}_a$ is a codimension k almost observability subspace. According to Proposition 2.42 $\mathcal{O}_a$ is instantaneous if and only if $\mathcal{V}_\infty(\mathcal{O}_a) = \mathcal{F}^n$, where $\mathcal{V}_\infty$ refers to the limit of CISA.

It will be shown by induction that

$$\mathcal{V}_i \supset \operatorname{Ker} A^{i-1} \overleftarrow{R}_\mu(A,B)$$

for all $i \in \mathbb{N}$. Obviously, this is true for $i = 1$. Let it be true for a fixed $i \in \mathbb{N}$, now. Then

$$\mathcal{V}_{i+1} \supset \operatorname{Ker} \overleftarrow{R}_\mu(A,B) + \mathsf{A}(\operatorname{Ker} A^{i-1} \overleftarrow{R}_\mu(A,B) \cap \operatorname{Ker} \mathsf{C}) \ .$$

Let

$$x = \begin{pmatrix} x_1^{\mu_1-1} \\ \vdots \\ x_1^0 \\ \vdots \\ x_p^{\mu_p-1} \\ \vdots \\ x_p^0 \end{pmatrix} \in \operatorname{Ker} A^i \overleftarrow{R}_\mu(A,B) \ ,$$

i.e.

$$A^i \overleftarrow{R}_\mu(A,B)x = A^i \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-1} A^l b_j x_j^l = 0 \ . \tag{5.3}$$

Since $\operatorname{rk} \overleftarrow{R}_{\mu-1}(A,B) = k$, any linear combination of the vectors $A^{\mu_j-1}b_j$, $j = 1,\ldots,p$, can be written as a linear combination of the columns of $\overleftarrow{R}_{\mu-1}(A,B)$, i.e. there exist numbers $y_j^l \in \mathcal{F}$, $j = 1,\ldots,p$ and $l = 0,\ldots,\mu_j-2$, with

$$\sum_{j=1}^{p} A^{\mu_j-1} b_j x_j^{\mu_j-1} = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j y_j^l \ . \tag{5.4}$$

Then (5.3) yields

$$A^{i-1} \sum_{j=1}^{p} \sum_{l=1}^{\mu_j-1} A^l b_j (x_j^{l-1} + y_j^{l-1}) = A^i \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j (x_j^l + y_j^l) = 0 \ . \tag{5.5}$$

Consider the decomposition

$$x = \begin{pmatrix} x_1^{\mu_1-1} \\ -y_1^{\mu_1-2} \\ \vdots \\ -y_1^0 \\ \vdots \\ x_p^{\mu_p-1} \\ -y_p^{\mu_p-2} \\ \vdots \\ -y_p^0 \end{pmatrix} + \mathsf{A} \begin{pmatrix} x_1^{\mu_1-2} + y_1^{\mu_1-2} \\ \vdots \\ x_1^0 + y_1^0 \\ 0 \\ \vdots \\ x_p^{\mu_p-2} + y_p^{\mu_p-2} \\ \vdots \\ x_p^0 + y_p^0 \\ 0 \end{pmatrix} =: y + \mathsf{A}z \ .$$

Then (5.4) yields $y \in \operatorname{Ker} \overleftarrow{R}_\mu(A,B)$ and (5.5) yields $z \in \operatorname{Ker} A^{i-1}\overleftarrow{R}_\mu(A,B)$. Obviously, $z \in \operatorname{Ker} \mathsf{C}$. Hence

$$x \in \operatorname{Ker} \overleftarrow{R}_\mu(A,B) + \mathsf{A}(\operatorname{Ker} A^{i-1}\overleftarrow{R}_\mu(A,B) \cap \operatorname{Ker} \mathsf{C}) \subset \mathcal{V}_{i+1}$$

and $\mathcal{V}_{i+1} \supset \operatorname{Ker} A^i \overleftarrow{R}_\mu(A,B)$. This completes the induction.

Since A is nilpotent, $A^{i-1} = 0$ and hence $\mathcal{V}_i \supset \operatorname{Ker} 0 \cdot \overleftarrow{R}_\mu(A,B) = \mathcal{F}^n$ for i big enough, i.e. $\mathcal{V}_\infty = \mathcal{F}^n$ and $\mathcal{O}_a$ is instantaneous.

Conversely let $\mathcal{O}_a$ be instantaneous of codimension k. Then (A,B) is μ-regular. Assume, that (A,B) is *not* μ-tight, i.e. $\operatorname{rk} \overleftarrow{R}_{\mu-1}(A,B) < k$. Since $\operatorname{rk} \overleftarrow{R}_\mu(A,B) = k$, there exists a linear combination of the vectors $A^{\mu_j-1}b_j$, $j = 1,\ldots,p$, which is not contained in columnspan $\overleftarrow{R}_{\mu-1}(A,B)$. This means that there exist numbers $x_j^{\mu_j-1} \in \mathcal{F}$, $j = 1,\ldots,p$, such that for every choice of numbers $x_j^l \in \mathcal{F}$, $j = 1,\ldots,p$ and $l = 0,\ldots,\mu_j - 2$,

$$\sum_{j=1}^{p} \sum_{l=0}^{\mu_j-1} A^l b_j x_j^l \neq 0 \ . \tag{5.6}$$

It will be shown by induction that

$$x := \begin{pmatrix} x_1^{\mu_1-1} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ x_p^{\mu_p-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \notin \mathcal{V}_i$$

for all $i \in \mathbb{N}$. For $i = 1$ choose $x_j^l = 0$, $j = 1,\ldots,p$ and $l = 0,\ldots,\mu_j - 2$, then (5.6) yields $x \notin \operatorname{Ker} \overleftarrow{R}_\mu(A,B) = \mathcal{O}_a = \mathcal{V}_1$. Let it be true for a fixed $i \in \mathbb{N}$, now. Then $x \in \mathcal{V}_{i+1} = \mathcal{O}_a + \mathsf{A}(\mathcal{V}_i + \operatorname{Ker} \mathsf{C})$ would imply $x = y + \mathsf{A}z$ with $y \in \mathcal{O}_a$, i.e. there would exist numbers $z_j^l \in \mathcal{F}$, $j = 1,\ldots,p$ and $l = 1,\ldots,\mu_j - 1$, such that

$$y = x - \mathsf{A}z = \begin{pmatrix} x_1^{\mu_1-1} \\ -z_1^{\mu_1-1} \\ \vdots \\ -z_1^1 \\ \vdots \\ x_p^{\mu_p-1} \\ -z_p^{\mu_p-1} \\ \vdots \\ -z_p^1 \end{pmatrix} \in \mathcal{O}_a = \operatorname{Ker} \overleftarrow{R}_\mu(A,B) \ .$$

But then the choice $x_j^l = -z_j^{l+1}$, $j = 1, \ldots, p$ and $l = 0, \ldots, \mu_j - 2$, contradicts (5.6). Hence $x \notin \mathcal{V}_{i+1}$ and the induction is complete.

It follows $x \notin \mathcal{V}_\infty$, i.e. $\mathcal{V}_\infty \neq \mathcal{F}^n$, a contradiction to $\mathcal{O}_a$ being instantaneous. Hence (A, B) was μ-tight. $\square$

Proposition 5.8. *Let $\mathcal{O}_a \subset \mathcal{F}^n$ be an almost observability subspace with respect to the pair (C, A) with $\operatorname{codim} \mathcal{O}_a = k$. Then there exists a μ-regular pair $(A, B) \in \mathcal{F}^{k \times k} \times \mathcal{F}^{k \times p}$ with A nilpotent such that $\mathcal{O}_a = \operatorname{Ker} \overleftarrow{R}_\mu(A, B)$.*

Since the existence part of the proof turns out to be more instructive in the dual case, this result is obtained by dualizing the following proposition.

Proposition 5.9. *Let (A, B) be a controllable pair in Brunovsky form with controllability indices $\kappa = (\kappa_1, \ldots, \kappa_m)$ (cf. Section 2.1.2). For any k-dimensional almost controllability subspace $\mathcal{R}_a \subset \mathcal{F}^n$ with respect to the pair (A, B) there exists a pair $(C, N) \in \mathcal{F}^{m \times k} \times \mathcal{F}^{k \times k}$ with N nilpotent and*

$$\mathcal{R}_a = \operatorname{Im} \overleftarrow{O}_\kappa(C, N) = \operatorname{Im} \begin{pmatrix} c_1 N^{\kappa_1 - 1} \\ \vdots \\ c_1 \\ \vdots \\ c_m N^{\kappa_m - 1} \\ \vdots \\ c_m \end{pmatrix},$$

where c_i, $i = 1, \ldots, m$, denotes the i-th row of C.

Proof. According to Proposition 2.5 there exists a feedback matrix $F \in \mathcal{F}^{m \times n}$ and a chain of subspaces $\operatorname{Im} \mathsf{B} \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_r$ such that

$$\mathcal{R}_a = \mathcal{B}_1 \oplus \mathsf{A}_F \mathcal{B}_2 \oplus \cdots \oplus \mathsf{A}_F^{r-1} \mathcal{B}_r$$

and $n_i := \dim \mathcal{B}_i = \dim \mathsf{A}_F^{i-1} \mathcal{B}_i$, $i = 1, \ldots, r$, where

$$\mathsf{A}_F = \mathsf{A} + \mathsf{B}F = \begin{array}{c} \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} & & \\ \hline \multicolumn{3}{c}{f_1} \\ \hline & \ddots & \\ \hline & & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \multicolumn{3}{c}{f_m} \end{array}\right) \\ \underbrace{\qquad\qquad}_{\kappa_1} \quad \cdots \quad \underbrace{\qquad\qquad}_{\kappa_m} \end{array}$$

and f_i, $i = 1, \ldots, m$, denotes the i-th row of F. It follows $n_1 + \cdots + n_r = k$. It is possible to choose a basis $b_1, \ldots, b_{n_1}$ of $\mathcal{B}_1$ such that $b_1, \ldots, b_{n_i}$ is a basis of $\mathcal{B}_i$, $i = 1, \ldots, r$. The set of vectors

$$
\begin{array}{lllllll}
\mathsf{A}_F^{r-1}b_1 \ldots \mathsf{A}_F^{r-1}b_{n_r} & & & & \\
\mathsf{A}_F^{r-2}b_1 \ldots \mathsf{A}_F^{r-2}b_{n_r} & \mathsf{A}_F^{r-2}b_{n_r+1} \ldots \mathsf{A}_F^{r-2}b_{n_{r-1}} & & & \\
\quad\vdots \qquad\quad \vdots & \quad\vdots \qquad\quad \vdots & & & \\
\mathsf{A}_F b_1 \ \ldots \ \mathsf{A}_F b_{n_r} & \mathsf{A}_F b_{n_r+1} \ \ldots \ \mathsf{A}_F b_{n_{r-1}} & \ldots & \mathsf{A}_F b_{n_3+1} \ldots \mathsf{A}_F b_{n_2} & \\
\ \ b_1 \quad \ldots \quad b_{n_r} & \ \ b_{n_r+1} \quad \ldots \quad b_{n_{r-1}} & \ldots & \ \ b_{n_3+1} \ \ldots \ b_{n_2} & b_{n_2+1} \ldots b_{n_1}
\end{array}
$$

forms a basis of $\mathcal{R}_a$, then. Read this tabular from bottom to top and from left to right and form subspaces

$$
\mathcal{U}_i = \mathsf{b}_i \oplus \mathsf{A}_F \mathsf{b}_i \oplus \cdots \oplus \mathsf{A}_F^{r_i - 1} \mathsf{b}_i \quad , i = 1, \ldots, n_1 \ ,
$$

where $r_i = \max\{\rho \,|\, i \leq n_\rho\}$ and $\mathsf{b}_i = \operatorname{columnspan} b_i$. Then $r_1 + \cdots + r_{n_1} = k$ and

$$
\mathcal{R}_a = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_{n_1} \ .
$$

Fix an $i \in \{1, \ldots, n_1\}$. Since $b_i \in \operatorname{Im} \mathsf{B}$, there exist numbers $c_{ij}^1 \in \mathcal{F}$, $j = 1, \ldots, m$, with

$$
b_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{i1}^1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ c_{im}^1 \end{pmatrix} \ .
$$

Assume that for $l < r_i$

$$
\mathsf{A}_F^{l-1} b_i = \begin{pmatrix} c_{i1}^{l-\kappa_1+1} \\ \vdots \\ c_{i1}^{l} \\ \vdots \\ c_{im}^{l-\kappa_m+1} \\ \vdots \\ c_{im}^{l} \end{pmatrix} \ ,
$$

where $c_{ij}^{s} = 0$ for $s \le 0$, then

$$\mathsf{A}_F^l b_i = \begin{pmatrix} c_{i1}^{l-\kappa_1+2} \\ \vdots \\ c_{i1}^{l+1} \\ \vdots \\ c_{im}^{l-\kappa_m+2} \\ \vdots \\ c_{im}^{l+1} \end{pmatrix} ,$$

where $c_{ij}^{l+1} \in \mathcal{F}$ are suitable numbers. By induction

$$\mathcal{U}_i = \operatorname{Im} \begin{pmatrix} U_{i1} \\ \vdots \\ U_{im} \end{pmatrix} ,$$

where

$$U_{ij} = \operatorname{Im} \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & c_{ij}^{1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & c_{ij}^{r_i-2} \\ 0 & c_{ij}^{1} & \dots & c_{ij}^{r_i-1} \\ c_{ij}^{1} & c_{ij}^{2} & \dots & c_{ij}^{r_i} \end{pmatrix}$$

if $r_i \le \kappa_j$ and

$$U_{ij} = \operatorname{Im} \begin{pmatrix} 0 & 0 & \dots & c_{ij}^{1} & \dots & c_{ij}^{r_i-\kappa_j+1} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & & c_{ij}^{\kappa_j-2} & \dots & c_{ij}^{r_i-2} \\ 0 & c_{ij}^{1} & \dots & c_{ij}^{\kappa_j-1} & \dots & c_{ij}^{r_i-1} \\ c_{ij}^{1} & c_{ij}^{2} & \dots & c_{ij}^{\kappa_j} & \dots & c_{ij}^{r_i} \end{pmatrix}$$

if $r_i > \kappa_j$, respectively. Choosing

$$C_i = \begin{pmatrix} c_{i1}^{1} & \dots & c_{i1}^{r_i} \\ \vdots & & \vdots \\ c_{im}^{1} & \dots & c_{im}^{r_i} \end{pmatrix} \text{ and } N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

it follows

$$\mathcal{U}_i = \operatorname{Im} \begin{pmatrix} c_{i1} N_i^{\kappa_1 - 1} \\ \vdots \\ c_{i1} \\ \vdots \\ c_{im} N_i^{\kappa_m - 1} \\ \vdots \\ c_{im} \end{pmatrix} .$$

Now the choice

$$C = (C_1 \ \ldots \ C_{n_1}) \text{ and } N = \operatorname{diag}(N_1, \ldots, N_{n_1})$$

yields the desired result. □

Proof of Proposition 5.8. The existence of such a pair (A, B) follows by dualizing Proposition 5.9. The μ-regularity of (A, B) follows from $\operatorname{codim} \mathcal{O}_a = \operatorname{codim} \operatorname{Ker} \overleftarrow{R}_\mu(A, B) = \operatorname{rk} \overleftarrow{R}_\mu(A, B) = k$. □

Since the linear space $\operatorname{Ker} \overleftarrow{R}_\mu(A, B)$ is invariant under the similarity action, i.e. $\operatorname{Ker} \overleftarrow{R}_\mu(A, B) = \operatorname{Ker} \overleftarrow{R}_\mu(A', B')$ for $(A', B') \in [A, B]_\sigma$, and in view of Proposition 5.7 the following map is well defined:

$$\begin{aligned} \overleftarrow{\rho}_\mu : \mathcal{N}_{k,p}(\mathcal{F}) &\longrightarrow \mathcal{O}_a(\mathsf{C}, \mathsf{A}) \ , \\ [A, B]_\sigma &\mapsto \operatorname{Ker} \overleftarrow{R}_\mu(A, B) \ . \end{aligned}$$

Here $\mathcal{O}_a(\mathsf{C}, \mathsf{A})$ denotes the set of all almost observability subspaces with respect to the pair (C, A). As a consequence of Proposition 5.7 the following two restrictions of this map are well defined. ${\mathcal{O}_a}^k(\mathsf{C}, \mathsf{A})$ and $\mathcal{I}_k(\mathsf{C}, \mathsf{A})$ denote the sets of all codimension k almost observability subspaces or instantaneous subspaces with respect to the pair (C, A), respectively.

Theorem 5.10. *The map*

$$\begin{aligned} \overleftarrow{\rho}_\mu : \mathcal{N}_{k,p}(\mu) &\longrightarrow {\mathcal{O}_a}^k(\mathsf{C}, \mathsf{A}) \ , \\ [A, B]_\sigma &\mapsto \operatorname{Ker} \overleftarrow{R}_\mu(A, B) \end{aligned}$$

is a surjective algebraic map. It restricts to a bijection

$$\overleftarrow{\rho}_\mu : \ \mathcal{N}^{\mathrm{t}}_{k,p}(\mu) \longrightarrow \mathcal{I}_k(\mathsf{C}, \mathsf{A}) \ .$$

Proof. Surjectivity of the first map follows from Proposition 5.8. Surjectivity of the second map follows from surjectivity of the first map and Proposition 5.7. Injectivity of the second map follows as in the proof of Theorem 5.4. □

To bring in the topology, the spaces $\mathcal{I}_k(\mathsf{C},\mathsf{A})$ and ${\mathcal{O}_a}^k(\mathsf{C},\mathsf{A})$ are viewed as subsets of the *Grassmann manifold* $\mathrm{G}_{n-k}(\mathcal{F}^n)$.

Theorem 5.11. *The map*

$$\overleftarrow{\rho}_\mu : \mathcal{N}^{\mathrm{t}}_{k,p}(\mu) \longrightarrow \mathcal{I}_k(\mathsf{C},\mathsf{A}) \ , \qquad [A,B]_\sigma \mapsto \operatorname{Ker} \overleftarrow{R}_\mu(A,B)$$

is a homeomorphism.

Proof. According to Theorem 5.10 $\overleftarrow{\rho}_\mu$ is bijective. Furthermore it is clearly continuous. Openness of $\overleftarrow{\rho}_\mu$ follows as in the proof of Theorem 5.5. □

Corollary 5.12. *For $k \le n-p$, $\mathcal{I}_k(\mathsf{C},\mathsf{A})$ is locally compact. If in addition $k \le \mu_p - 1$ then $\mathcal{I}_k(\mathsf{C},\mathsf{A})$ is compact. For $k > n-p$, $\mathcal{I}_k(\mathsf{C},\mathsf{A})$ is empty.*

5.3 The $\mathcal{O}_* = \mathcal{F}^n$ case

In this section the results of Sections 5.1 and 5.2 are combined to describe the set of all codimension k almost (C,A)-invariant subspaces of a given observable pair (C,A) in dual Brunovsky form with observability indices $\mu = (\mu_1,\dots,\mu_p)$ (cf. Section 2.3.3). This description uses controllable triples (cf. Section 4.2).

Proposition 5.13. *Let $1 \le r \le k \le n$ and let $((A_1,B_1),(N,B_2)) \in (\mathcal{F}^{r\times r} \times \mathcal{F}^{r\times p}) \times (\mathcal{F}^{(k-r)\times(k-r)} \times \mathcal{F}^{(k-r)\times p})$ be a pair of matrix pairs with N nilpotent.*

(1) $\mathcal{V}_a = \operatorname{Ker} R_\mu(A_1,B_1,N,B_2)$ *is an almost* (C,A)*-invariant subspace,*

(2) $\operatorname{codim} \mathcal{V}_a = k$ *if and only if the triple*

$$(E,A,B) = \left(\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right)$$

is μ-regular and

(3) $\mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$ *and* $\operatorname{codim} \mathcal{V}_a = k$ *if and only if* (E, A, B) *is* μ*-tight.*

Some parts of the proof turn out to be easier in the dual case and will be obtained by dualizing the following Lemmas.

Lemma 5.14. *Let* (A, B) *be a controllable pair in Brunovsky form with controllability indices* $\kappa = (\kappa_1, \dots, \kappa_m)$ *(cf. Section 2.1.2). Let* $1 \le r \le k \le n$ *and let* $((C_1, A_1), (C_2, N)) \in (\mathcal{F}^{p\times r} \times \mathcal{F}^{r\times r}) \times (\mathcal{F}^{p\times(k-r)} \times \mathcal{F}^{(k-r)\times(k-r)})$ *be a pair of matrix pairs with* N *nilpotent. Let*

$$\mathcal{V}_a = \operatorname{Im} O_\kappa(C_1, A_1, C_2, N) = \operatorname{Im} \left(O_\kappa(C_1,A_1) \; \overleftarrow{O}_\kappa(C_2,N) \right)$$

$$= \operatorname{Im} \begin{pmatrix} c_{11} & c_{21}N^{\kappa_1-1} \\ \vdots & \vdots \\ c_{11}A_1^{\kappa_1-1} & c_{21} \\ \vdots & \vdots \\ c_{1m} & c_{2m}N^{\kappa_m-1} \\ \vdots & \vdots \\ c_{1m}A_1^{\kappa_m-1} & c_{2m} \end{pmatrix} \in \mathcal{F}^{n\times k} ,$$

where c_{ij} *denotes the* j*-th row of* C_i, $i = 1, 2$, $j = 1, \dots, m$. *If the matrix*

$$O_{\kappa-1}(C_1, A_1, C_2, N) = \left(O_{\kappa-1}(C_1,A_1) \; \overleftarrow{O}_{\kappa-1}(C_2,N) \right) \in \mathcal{F}^{(n-m)\times k}$$

has full (column) rank k *then for all* $i \in \mathbb{N}$

$$\mathcal{V}^i(\mathcal{V}_a) \subset \operatorname{Im} \left(O_\kappa(C_1,A_1) \; \overleftarrow{O}_\kappa(C_2,N)N^{i-1} \right) ,$$

where $\mathcal{V}^i$ *refers to the steps of ISA.*

Proof. The statement will be proved by induction. Obviously it is true for $i = 1$. Let it be true for a fixed $i \in \mathbb{N}$ now. Then

$$\mathcal{V}^{i+1} = \mathcal{V}_a \cap \mathsf{A}^{-1}(\mathcal{V}^i + \operatorname{Im} \mathsf{B})$$

$$\subset \operatorname{Im} \begin{pmatrix} c_{11} & c_{21}N^{\kappa_1-1} \\ \vdots & \vdots \\ c_{11}A_1^{\kappa_1-1} & c_{21} \\ \vdots & \vdots \\ c_{1m} & c_{2m}N^{\kappa_m-1} \\ \vdots & \vdots \\ c_{1m}A_1^{\kappa_m-1} & c_{2m} \end{pmatrix} \cap \operatorname{Im} \begin{pmatrix} 1 & & 0 & 0 \\ 0 & & c_{11} & c_{21}N^{\kappa_1-1}N^{i-1} \\ \vdots & & \vdots & \vdots \\ 0 & & c_{11}A_1^{\kappa_1-2} & c_{21}NN^{i-1} \\ & \ddots & \vdots & \vdots \\ & 1 & 0 & 0 \\ & 0 & c_{1m} & c_{2m}N^{\kappa_m-1}N^{i-1} \\ & \vdots & \vdots & \vdots \\ & 0 & c_{1m}A_1^{\kappa_m-2} & c_{2m}NN^{i-1} \end{pmatrix} .$$

For $y \in \mathcal{V}^{i+1}$ it follows the existence of $x_1 \in \mathcal{F}^r$, $x_2 \in \mathcal{F}^{k-r}$, $x_3 \in \mathcal{F}^m$, $x_4 \in \mathcal{F}^r$ and $x_5 \in \mathcal{F}^{k-r}$ such that

$$
\begin{aligned}
y &= \begin{pmatrix} c_{11} \\ \vdots \\ c_{11}A_1^{\kappa_1-1} \\ \vdots \\ c_{1m} \\ \vdots \\ c_{1m}A_1^{\kappa_m-1} \end{pmatrix} x_1 + \begin{pmatrix} c_{21}N^{\kappa_1-1} \\ \vdots \\ c_{21} \\ \vdots \\ c_{2m}N^{\kappa_m-1} \\ \vdots \\ c_{2m} \end{pmatrix} x_2 \\
&= \begin{pmatrix} \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ c_{11} \\ \vdots \\ c_{11}A_1^{\kappa_1-2} \\ \vdots \\ 0 \\ c_{1m} \\ \vdots \\ c_{1m}A_1^{\kappa_m-2} \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ c_{21}N^{\kappa_1-1}N^{i-1} \\ \vdots \\ c_{21}NN^{i-1} \\ \vdots \\ 0 \\ c_{2m}N^{\kappa_m-1}N^{i-1} \\ \vdots \\ c_{2m}NN^{i-1} \end{pmatrix} x_5 \ .
\end{aligned}
$$

This implies

$$
O_{\kappa-1}(C_1, A_1)A_1x_1 + \overleftarrow{O}_{\kappa-1}(C_2, N)x_2 = O_{\kappa-1}(C_1, A_1)x_4 + \overleftarrow{O}_{\kappa-1}(C_2, N)N^ix_5 \ .
$$

But then $O_{\kappa-1}(C_1, A_1, C_2, N)$ having full rank k yields

$$
O_{\kappa-1}(C_1, A_1)(A_1x_1 - x_4) = \overleftarrow{O}_{\kappa-1}(C_2, N)(N^ix_5 - x_2) = 0 \ .
$$

Since $O_{\kappa-1}(C_1, A_1)$ has full rank r and $O_{\kappa-1}(C_2, N)$ has full rank $k - r$ it follows $A_1x_1 = x_4$ and $N^ix_5 = x_2$. But then

$$
y = \begin{pmatrix} c_{11} \\ \vdots \\ c_{11}A_1^{\kappa_1-1} \\ \vdots \\ c_{1m} \\ \vdots \\ c_{1m}A_1^{\kappa_m-1} \end{pmatrix} x_1 + \begin{pmatrix} c_{21}N^{\kappa_1-1} \\ \vdots \\ c_{21} \\ \vdots \\ c_{2m}N^{\kappa_m-1} \\ \vdots \\ c_{2m} \end{pmatrix} N^ix_5 \in \operatorname{Im}\left(O_\kappa(C_1,A_1) \ \overleftarrow{O}_\kappa(C_2,N)N^i \right) \ ,
$$

which completes the induction. □

Lemma 5.15. *Let* (A, B) *be a controllable pair in Brunovsky form with controllability indices* $\kappa = (\kappa_1, \ldots, \kappa_m)$ *(cf. Section 2.1.2). Let* $1 \leq r \leq k \leq n$ *and let* $((C_1, A_1), (C_2, N)) \in (\mathcal{F}^{p\times r} \times \mathcal{F}^{r\times r}) \times (\mathcal{F}^{p\times(k-r)} \times \mathcal{F}^{(k-r)\times(k-r)})$ *be a pair of matrix pairs with* N *nilpotent. Let* $O_\kappa(C_1, A_1, C_2, N)$ *have full rank* k *and let* $\mathcal{V}_a = \operatorname{Im} O_\kappa(C_1, A_1, C_2, N)$*. Let* $O_{\kappa-1}(C_1, A_1)$ *have full rank* r *and let* $\overleftarrow{O}_{\kappa-1}(C_2, N)$ *have full rank* $k - r$*. If* $\operatorname{rk} O_{\kappa-1}(C_1, A_1, C_2, N) < k$ *then* $\mathcal{V}^\infty(\mathcal{V}_a) \neq \mathcal{V} := \operatorname{Im} O_\kappa(C_1, A_1)$*, where* $\mathcal{V}^\infty$ *refers to the limit of ISA.*

Proof. According to Proposition 5.1 $\mathcal{V}$ is (A,B)-invariant. By definition $\mathcal{V} \subset \mathcal{V}_a$ holds. According to Proposition 2.13 $\mathcal{V}^\infty(\mathcal{V}_a)$ is the largest (A,B)-invariant subspace contained in $\mathcal{V}_a$. It follows $\mathcal{V} \subset \mathcal{V}^\infty$ hence $\mathcal{V} \subset \mathcal{V}^i$ for all $i \in \mathbb{N}$. $O_{\kappa-1}(C_1, A_1)$ and $\overleftarrow{O}_{\kappa-1}(C_2, N)$ having full rank, respectively, while $\operatorname{rk} O_{\kappa-1}(C_1, A_1, C_2, N) < k$, yields the existence of $x_1 \in \mathcal{F}^r$ and $x_2 \in \mathcal{F}^{k-r}$ with

$$O_{\kappa-1}(C_1, A_1)x_1 = \overleftarrow{O}_{\kappa-1}(C_2, N)x_2 \neq 0 \; .$$

But then $y = \overleftarrow{O}_\kappa(C_2, N)x_2 \neq 0$ and $y \notin \mathcal{V}$ since $O_\kappa(C_1, A_1, C_2, N)$ has full rank. It will be shown by induction that $y \in \mathcal{V}^i$ for all $i \in \mathbb{N}$. Obviously, this is true for $i = 1$. Let it be true for a fixed $i \in \mathbb{N}$ now. Then

$$\mathsf{A}y = \begin{pmatrix} c_{21}N^{\kappa_1-2} \\ \vdots \\ c_{21} \\ 0 \\ \vdots \\ c_{2m}N^{\kappa_m-2} \\ \vdots \\ c_{2m} \\ 0 \end{pmatrix} x_2 = \begin{pmatrix} c_{11} \\ \vdots \\ c_{11}A_1^{\kappa_1-1} \\ 0 \\ \vdots \\ c_{1m} \\ \vdots \\ c_{1m}A_1^{\kappa_m-1} \\ 0 \end{pmatrix} x_1 \in \mathcal{V} + \operatorname{Im}\mathsf{B} \subset \mathcal{V}^i + \operatorname{Im}\mathsf{B} \; .$$

But then $y \in \mathsf{A}^{-1}(\mathcal{V}^i + \operatorname{Im}\mathsf{B})$ and therefore $y \in \mathcal{V}^{i+1}$ which completes the induction. It follows $y \in \mathcal{V}^\infty$ and the proof is complete. □

Proof of Proposition 5.13. By Proposition 5.1 and Proposition 5.7 the subspace $\mathcal{V} := \operatorname{Ker} R_\mu(A_1, B_1)$ is (C,A)-invariant while the subspace $\mathcal{O}_a := \operatorname{Ker} \overleftarrow{R}_\mu(N, B_2)$ is an almost observability subspace with respect to the pair (C,A). By definition it is $\mathcal{V}_a = \mathcal{V} \cap \mathcal{O}_a$.

Ad (1): According to Proposition 2.34 $\mathcal{V}_a$ is an almost (C,A)-invariant subspace.

Ad (2): (E, A, B) is μ-regular if and only if $\operatorname{codim}\operatorname{Ker} R_\mu(A_1, B_1, N, B_2) = \operatorname{rk} R_\mu(A_1, B_1, N, B_2) = k$.

Ad (3): Let (E, A, B) be μ-tight. According to (1) $\mathcal{V}_a$ is an almost (C,A)-invariant subspace. Dualizing Lemma 5.14 yields

$$\mathcal{V}_i(\mathcal{V}_a) \supset \operatorname{Ker}\begin{pmatrix} R_\mu(A_1,B_1) \\ N^{i-1}\overleftarrow{R}_\mu(N,B_2) \end{pmatrix}$$

for all $i \in \mathbb{N}$, where $\mathcal{V}_i$ refers to the steps of CISA. Since N is nilpotent, there exists $i \in \mathbb{N}$ such that $N^{i-1} = 0$ and hence $\mathcal{V}_i \supset \operatorname{Ker} R_\mu(A_1, B_1) = \mathcal{V}$, i.e. $\mathcal{V}_a \subset \mathcal{V} \subset \mathcal{V}_\infty(\mathcal{V}_a) = \mathcal{V}_*(\mathcal{V}_a)$ (for the last equality cf. Proposition 2.39). Since

$\mathcal{V}_*(\mathcal{V}_a)$ is the minimal (C,A)-invariant subspace containing $\mathcal{V}_a$ and since $\mathcal{V}$ is (C,A)-invariant (it is even tight since (A_1,B_1) is μ-tight, cf. Proposition 5.1), it follows $\mathcal{V}_*(\mathcal{V}_a) = \mathcal{V}_\infty(\mathcal{V}_a) = \mathcal{V}$. But then according to Proposition 2.41 $\mathcal{O}_*(\mathcal{V}_a) = \mathcal{O}_\infty(\mathcal{V}_\infty(\mathcal{V}_a)) = \mathcal{O}_\infty(\mathcal{V}) = \mathcal{F}^n$, since $\mathcal{V}$ is tight (Proposition 2.42). It follows from Proposition 2.38, that $\mathcal{V} + \mathcal{O}_a = \mathcal{F}^n$. But then $\operatorname{codim} \mathcal{V}_a = \operatorname{codim} \mathcal{V} + \operatorname{codim} \mathcal{O}_a$. Since (A_1,B_1) and (N,B_2) are both μ-tight and hence are both μ-regular, $\operatorname{codim} \mathcal{V} = r$ (Proposition 5.1) and $\operatorname{codim} \mathcal{O}_a = k - r$ (Proposition 5.7). It follows $\operatorname{codim} \mathcal{V}_a = k$.

Conversely let $\mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$ and let $\operatorname{codim} \mathcal{V}_a = k$. Then (E,A,B) is μ-regular, i.e. $\operatorname{rk} R_\mu(A_1,B_1,N,B_2) = k$. Since $\mathcal{V} \supset \mathcal{V}_a$ it is $\mathcal{O}_*(\mathcal{V}) \supset \mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$. Hence $\mathcal{O}_*(\mathcal{V}) = \mathcal{F}^n$ and $\mathcal{V}$ is tight. According to Proposition 5.1 this means $\operatorname{rk} R_{\mu-1}(A_1,B_1) = r$. On the other hand $\mathcal{R}_a \supset \mathcal{V}_a$ and therefore $\mathcal{O}_*(\mathcal{R}_a) \supset \mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$. Hence $\mathcal{O}_*(\mathcal{R}_a) = \mathcal{F}^n$ and $\mathcal{R}_a$ is instantaneous. According to Proposition 5.7 this means $\operatorname{rk} R_{\mu-1}(N,B_2) = k - r$. Assume, that (E,A,B) is *not* μ-tight, i.e. $\operatorname{rk} R_{\mu-1}(A_1,B_1,N,B_2) < k$. Dualizing Lemma 5.15 yields $\mathcal{V}_\infty(\mathcal{V}_a) = \mathcal{V}_*(\mathcal{V}_a) \neq \mathcal{V}$. According to Proposition 2.38 the decomposition $\mathcal{V}_a = \mathcal{V} \cap \mathcal{O}_a$ is unique, since $\mathcal{O}_*(\mathcal{V}_a) = \mathcal{F}^n$. It follows $\mathcal{V}_*(\mathcal{V}_a) = \mathcal{V}$, a contradiction. Hence (E,A,B) was μ-tight. $\square$

Corollary 5.16. *Any μ-tight triple (E,A,B) is μ-regular.*

Proof. Let (E,A,B) be μ-tight, and let $((A_1,B_1),(N,B_2))$ be any Weierstraß decomposition of (E,A,B). Then the triple in number (2) of Proposition 5.13 is restricted system equivalent to (E,A,B). Hence it is also μ-tight and $\mathcal{V}_a = \operatorname{Ker} R_\mu(A_1,B_1,N,B_2)$ is a codimension k almost (C,A)-invariant subspace. But then this triple is μ-regular and therefore (E,A,B) is μ-regular. $\square$

Proposition 5.17. *Let $\mathcal{V}_a \subset \mathcal{F}^n$ be an almost (C,A)-invariant subspace with $\operatorname{codim} \mathcal{V}_a = k$. Then there exists a pair of matrix pairs $((A_1,B_1),(N,B_2)) \in (\mathcal{F}^{r\times r} \times \mathcal{F}^{r\times p}) \times (\mathcal{F}^{(k-r)\times(k-r)} \times \mathcal{F}^{(k-r)\times p})$ with N nilpotent such that $\mathcal{V}_a = \operatorname{Ker} R_\mu(A_1,B_1,N,B_2)$ and $\operatorname{rk} R_\mu(A_1,B_1,N,B_2) = k$.*

Proof. According to Proposition 2.38 there exists an instantaneous subspace $\mathcal{I}$ such that $\mathcal{V}_a = \mathcal{V} \cap \mathcal{I}$, where $\mathcal{V} = \mathcal{V}_*(\mathcal{V}_a)$ is (C,A)-invariant and $\mathcal{V}+\mathcal{I} = \mathcal{F}^n$. Let $r = \operatorname{codim} \mathcal{V}$, then $\operatorname{codim} \mathcal{I} = \operatorname{codim} \mathcal{V}_a - \operatorname{codim} \mathcal{V} + \operatorname{codim}(\mathcal{V}+\mathcal{I}) = k - r + 0 = k - r$. According to Proposition 5.2 there exists a matrix pair $(A_1,B_1) \in \mathcal{F}^{r\times r} \times \mathcal{F}^{r\times p}$ with $\mathcal{V} = \operatorname{Ker} R_\mu(A_1,B_1)$. According to Proposition 5.8 there exists a matrix pair $(N,B_2) \in \mathcal{F}^{(k-r)\times(k-r)} \times \mathcal{F}^{(k-r)\times p}$ with N nilpotent and $\mathcal{I} = \operatorname{Ker} \overleftarrow{R}_\mu(N,B_2)$. It follows $\mathcal{V}_a = \operatorname{Ker} R_\mu(A_1,B_1,N,B_2)$ and $\operatorname{codim} \mathcal{V}_a = \operatorname{codim} \operatorname{Ker} R_\mu(A_1,B_1,N,B_2) = \operatorname{rk} R_\mu(A_1,B_1,N,B_2) = k$. $\square$

Apparently the previous two Propositions set up a relation between controllable triples and almost (C,A)-invariant subspaces via the Weierstraß decomposition. Since the linear space $\operatorname{Ker} R_\mu(A_1,B_1,N,B_2)$ is invariant under the action induced on the Weierstraß pairs by the restricted system equivalence action (cf. Lemma 4.6) and in view of Proposition 5.13 the following map is well defined:

$$\begin{aligned}\rho_\mu : C_{k,p}(\mathcal{F}) &\longrightarrow \mathcal{V}_a(\mathsf{C},\mathsf{A}) \ ,\\ [E,A,B]_\eta &\mapsto \operatorname{Ker} R_\mu(A_1,B_1,N,B_2) \ ,\end{aligned}$$

where A_1, B_1, N and B_2 are taken from any Weierstraß decomposition of (E,A,B). Here $\mathcal{V}_a(\mathsf{C},\mathsf{A})$ denotes the set of all almost (C,A)-invariant subspaces. As a consequence of Proposition 5.13 the following two restrictions of this map are well defined. $\mathcal{V}_a^k(\mathsf{C},\mathsf{A})$ and $\mathcal{F}_k(\mathsf{C},\mathsf{A})$ denote the sets of all codimension k almost (C,A)-invariant subspaces or almost (C,A)-invariant subspaces $\mathcal{V}_a$ with $\mathcal{O}_*(\mathcal{V}_a)=\mathcal{F}^n$, respectively.

Theorem 5.18. *The map*

$$\begin{aligned}\rho_\mu : C_{k,p}(\mu) &\longrightarrow \mathcal{V}_a^k(\mathsf{C},\mathsf{A}) \ ,\\ [E,A,B]_\eta &\mapsto \operatorname{Ker} R_\mu(A_1,B_1,N,B_2)\end{aligned}$$

is a surjective algebraic map. It restricts to a bijection

$$\rho_\mu : \ C^{\mathrm{t}}_{k,p}(\mu) \longrightarrow \mathcal{F}_k(\mathsf{C},\mathsf{A}) \ .$$

Proof. Surjectivity of the first map follows from Proposition 5.17. Surjectivity of the second map follows from surjectivity of the first map and Proposition 5.13. Injectivity of the second map follows from the fact that (A_1,B_1) and (N,B_2) are both μ-tight if (E,A,B) is μ-tight, from the uniqueness of the decomposition $\mathcal{V}_a=\mathcal{V}\cap\mathcal{O}_a$ in the $\mathcal{O}_*=\mathcal{F}^n$ case (Proposition 2.38) and from Theorem 5.4 and Theorem 5.10. □

5.4 The observability case

In view of the decomposition result of Proposition 2.38, in order to parametrize the set of all codimension k almost (C,A)-invariant subspaces, i.e. to cover also the $\mathcal{O}_*\neq\mathcal{F}^n$ case, a description of the set of all observability subspaces with respect to (C,A) is needed. Unfortunately the description obtained in the following Proposition does not lead to a bijective correspondence.

Proposition 5.19. *Let* $1 \le k \le n$ *and let* $(A,B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$*, where* A *is nilpotent.* $\mathcal{O} = \operatorname{Ker} \overleftarrow{R}_\mu(A,B)$ *is an observability subspace with respect to the pair* (C,A) *if and only if* (A,B) *satisfies the* complementary condition

$$\operatorname{Ker} A \cap \operatorname{Im} \overleftarrow{R}_{\mu-1}(A,B) = \{0\}\ .$$

Proof. According to Proposition 5.7 $\mathcal{O}$ is an almost observability subspace with respect to the pair (C,A). Hence it is an observability subspace if and only if it is also (C,A)-invariant, i.e. if and only if $\mathsf{A}(\mathcal{O} \cap \operatorname{Ker} \mathsf{C}) \subset \mathcal{O}$. Equivalently, for any $x \in \operatorname{Ker} \mathsf{C}$, i.e.

$$x = \begin{pmatrix} x_1^{\mu_1-1} \\ \vdots \\ x_1^1 \\ 0 \\ \vdots \\ x_p^{\mu_p-1} \\ \vdots \\ x_p^1 \\ 0 \end{pmatrix} ,$$

with $x \in \mathcal{O} = \operatorname{Ker} \overleftarrow{R}_\mu(A,B)$, i.e.

$$\begin{aligned} \overleftarrow{R}_\mu(A,B)x &= \sum_{j=1}^{p} \sum_{l=1}^{\mu_j-1} A^l b_j x_j^l = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^{l+1} b_j x_j^{l+1} \\ &= A \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j x_j^{l+1} = 0\ , \end{aligned}$$

it follows $\mathsf{A}x \in \mathcal{O} = \operatorname{Ker} \overleftarrow{R}_\mu(A,B)$, i.e.

$$\overleftarrow{R}_\mu(A,B)\mathsf{A}x = \overleftarrow{R}_\mu(A,B) \begin{pmatrix} 0 \\ x_1^{\mu_1-1} \\ \vdots \\ x_1^1 \\ \vdots \\ 0 \\ x_p^{\mu_p-1} \\ \vdots \\ x_p^1 \end{pmatrix} = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j-2} A^l b_j x_j^{l+1} = 0\ .$$

But this is equivalent to the following: for any $y \in \operatorname{Im} \overleftarrow{R}_{\mu-1}(A, B)$, i.e.

$$y = \sum_{j=1}^{p} \sum_{l=0}^{\mu_j - 2} A^l b_j x_j^{l+1} ,$$

with $Ay = 0$ it follows $y = 0$. This in turn is equivalent to $\operatorname{Ker} A \cap \operatorname{Im} \overleftarrow{R}_{\mu-1}(A, B) = \{0\}$. $\square$

5.5 The restriction pencil

The aim of this section is to relate kernel representations of almost invariant subspaces to the matrix pencil characterizations of almost invariant subspaces obtained by Jaffe and Karcanias [JK81] and used by Schumacher [Sch83].

Definition 5.20. Let $(A, B) \in \mathcal{F}^{n\times n} \times \mathcal{F}^{n\times m}$ with $\operatorname{rk} B = m$. Let $N \in \mathcal{F}^{n\times n}$ be the orthogonal projector onto $(\operatorname{Im} B)^\perp$ with respect to the standard inner product on $\mathcal{F}^n$. Let $V \in \mathcal{F}^{n\times k}$ with $\operatorname{rk} V = k$. The *restriction pencil of* V is the pencil $sNV - NAV$, $s \in \mathbb{C}$.

Remark 5.21. Jaffe and Karcanias [JK81] speak of the restriction pencil of the *subspace* $\mathcal{V} := \operatorname{Im} V$. Since $\mathcal{V} = \operatorname{Im} VS^{-1}$ for every $S \in \mathrm{GL}(\mathcal{F}^k)$, this notion is not well defined. Furthermore, they do not require N to be an orthogonal projector onto $(\operatorname{Im} B)^\perp$ but rather a left annihilator of B. Hence any TN for $T \in \mathrm{GL}(\mathcal{F}^n)$ can be used instead. But then at least the restricted system equivalence class

$$[sNV - NAV]_\eta = \{sTNVS^{-1} - TNAVS^{-1} \,|\, T \in \mathrm{GL}(\mathcal{F}^n), S \in \mathrm{GL}(\mathcal{F}^m)\}$$

(cf. Section 4.2) of the restriction pencil is uniquely determined by the subspace $\mathcal{V}$. Since they use only restricted system equivalence invariants in their characterizations of almost invariant subspaces, this is good enough.

The following Theorem relates the notion of restriction pencil to the image representations of almost invariant subspaces, the duals of which have been derived in Section 5.3 (being then kernel representations, of course).

Theorem 5.22. *Let* (A, B) *be a controllable pair in Brunovsky form with controllability indices* $\kappa = (\kappa_1, \ldots, \kappa_m)$ *(cf. Section* 2.1.2*). Let* $1 \le r \le k \le n$

and let $((C_1, A_1), (C_2, N)) \in (\mathcal{F}^{p\times r} \times \mathcal{F}^{r\times r}) \times (\mathcal{F}^{p\times(k-r)} \times \mathcal{F}^{(k-r)\times(k-r)})$ *be a pair of matrix pairs with* N *nilpotent. Let*

$$V = O_\kappa(C_1, A_1, C_2, N) = \begin{pmatrix} O_\kappa(C_1,A_1) & \overleftarrow{O}_\kappa(C_2,N) \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{21}N^{\kappa_1-1} \\ \vdots & \vdots \\ c_{11}A_1^{\kappa_1-1} & c_{21} \\ \vdots & \vdots \\ c_{1m} & c_{2m}N^{\kappa_m-1} \\ \vdots & \vdots \\ c_{1m}A_1^{\kappa_m-1} & c_{2m} \end{pmatrix} \in \mathcal{F}^{n\times k} \ ,$$

where c_{ij} *denotes the* j*-th row of* C_i*,* $i = 1, 2$*,* $j = 1, \dots, m$*. If the matrix*

$$O_{\kappa-1}(C_1, A_1, C_2, N) = \begin{pmatrix} O_{\kappa-1}(C_1,A_1) & \overleftarrow{O}_{\kappa-1}(C_2,N) \end{pmatrix} \in \mathcal{F}^{(n-m)\times k}$$

has full (column) rank k *then the restriction pencil* $s\mathsf{N}V - \mathsf{NA}V$ *is restricted system equivalent to the pencil*

$$s\begin{pmatrix} I & 0 \\ 0 & N \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A_1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \ .$$

Proof. Since

$$\mathsf{N} = \left(\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & & \\ \hline & & & \ddots & & & \\ \hline & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ & & & & & & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\kappa_1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\kappa_m \end{array} \ ,$$

it follows

$$s\mathsf{N}V - \mathsf{NA}V = \begin{pmatrix} sc_{11} - c_{11}A_1 & sc_{21}N^{\kappa_1-1} - c_{21}N^{\kappa_1-2} \\ \vdots & \vdots \\ sc_{11}A_1^{\kappa_1-2} - c_{11}A_1^{\kappa_1-1} & sc_{21}N - c_{21} \\ 0 & 0 \\ \vdots & \vdots \\ sc_{1m} - c_{1m}A_1 & sc_{2m}N^{\kappa_m-1} - c_{2m}N^{\kappa_m-2} \\ \vdots & \vdots \\ sc_{1m}A_1^{\kappa_m-2} - c_{1m}A_1^{\kappa_m-1} & sc_{2m}N - c_{2m} \\ 0 & 0 \end{pmatrix}$$

and hence

$$
s\mathsf{N}V - \mathsf{N}\mathsf{A}V = \begin{pmatrix} c_{11} & c_{21}N^{\kappa_1-2} & * \\ \vdots & \vdots & \vdots \\ c_{11}A_1^{\kappa_1-2} & c_{21} & * \\ 0 & 0 & * \\ \vdots & \vdots & \vdots \\ c_{1m} & c_{2m}N^{\kappa_m-2} & * \\ \vdots & \vdots & \vdots \\ c_{1m}A_1^{\kappa_m-2} & c_{2m} & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} sI - A_1 & 0 \\ 0 & sN - I \\ 0 & 0 \end{pmatrix} .
$$

Since $O_{\kappa-1}(C_1, A_1, C_2, N)$ has full column rank, the same is true for the submatrix formed of the two leftmost block columns of the square matrix on the left hand side of this expression. Hence the $*$-entries can be chosen such that this matrix has full rank. □

Under the hypotheses of Theorem 5.22 the subspace $\mathcal{V} := \operatorname{Im} V$ is a k-dimensional almost (A, B)-invariant subspace with $\mathcal{R}_*(\mathcal{V}) = \{0\}$ (dualize Theorem 5.13). Hence in this special case the characterizations achieved by Jaffe and Karcanias [JK81] and by Schumacher [Sch83] can easily be verified using the results of this Chapter.

5.6 Partial realizations

It has been pointed out recently by Fuhrmann and Helmke [FH01] that there is a close relation between partial realization theory and kernel representations of (C, A)-invariant subspaces which has been discovered by Antoulas [Ant83]. The presentation here follows Fuhrmann and Helmke [FH01].

As is well known the *transfer function* $H(s) = C(sI - A)^{-1}B$, $s \in \mathbb{C} \setminus \sigma(A)$, describes the steady state frequency response of system (2.1). Its Laurent expansion $H(s) = \sum_{i=1}^{\infty} H^{(i)} s^{-i}$ uniquely defines the *Markov parameters* $H^{(i)} = CA^{i-1}B$, $i \in \mathbb{N}$. The *partial realization problem* introduced by Kalman (see e.g. [KFA69]) is the following: Given fixed Markov parameters $H^{(i)} \in \mathcal{F}^{p \times m}$, $i = 1, \ldots, N$, find a system of form (2.1), i.e. find matrices $A \in \mathcal{F}^{n \times n}$, $B \in \mathcal{F}^{n \times m}$, $C \in \mathcal{F}^{p \times n}$, such that $H^{(i)} = CA^{i-1}B$ for $i = 1, \ldots, N$ holds. A generalization of this problem proposed by Antoulas is the following *nice partial realization problem*:

Let $\mu = (\mu_1, \ldots, \mu_p)$ be a list of integers (indices) with $\mu_1 \geq \cdots \geq \mu_p \geq 1$ and $\mu_1 + \cdots + \mu_p = n$. Let $r_\nu = \#\{\mu_j \mid \mu_j \geq \nu\}$, $\nu = 1, \ldots, \mu_1$, be the *conjugate indices* (# means cardinality). Note that $p = r_1 \geq \cdots \geq r_{\mu_1} \geq 1$ and $r_1 + \cdots + r_{\mu_1} = n$. Let $V \in \mathcal{F}^{k \times n}$ be partitioned as

$$V = \begin{pmatrix} V_1 & \dots & V_p \end{pmatrix} ,$$

where $V_j \in \mathcal{F}^{k \times \mu_j}$, $j = 1, \dots, p$. Let the *nice sequence* $(H^{(\nu)})$, $H^{(\nu)} \in \mathcal{F}^{k \times r_\nu}$, $\nu = 1, \dots, \mu_1$, be defined by

$$H_j^{(\nu)} = V_{j\nu} , \quad j = 1, \dots, r_\nu, \ \nu = 1, \dots, \mu_1 ,$$

where $H_j^{(\nu)}$ denotes the j-th column of $H^{(\nu)}$ and $V_{j\nu}$ denotes the ν-th column of V_j. Then $H^{(\nu)}$ consists of the ν-th columns of the blocks $V_1, \dots, V_{r_\nu}$. The blocks V_j, $j = r_\nu + 1, \dots, p$ have less than ν columns. Find matrices $A \in \mathcal{F}^{q \times q}$, $B \in \mathcal{F}^{q \times p}$ and $C \in \mathcal{F}^{k \times q}$, such that

$$H_j^{(\nu)} = CA^{\nu-1} b_j , \quad j = 1, \dots, r_\nu, \ \nu = 1, \dots, \mu_1 , \tag{5.7}$$

where b_j denotes the j-th column of B. Note that for $p = r_1 = \cdots = r_{\mu_1}$ (equivalently $\mu_1 = \cdots = \mu_p$) this is the usual partial realization problem. A triple (A, B, C) of matrices satisfying (5.7) is called a *partial realization* of the nice sequence $(H^{(\nu)})$ of *McMillan degree* q.

Proposition 5.23. *Let* (C, A) *be an observable pair in dual Brunovsky form with observability indices* $\mu = (\mu_1, \dots, \mu_p)$ *(cf. Section* 2.3.2*). Let* $V \in \mathcal{F}^{k \times n}$ *define the nice sequence* $(H^{(\nu)})$ *as above and let* $1 \leq q \leq n$*. If there exists a partial realization of* $(H^{(\nu)})$ *of* minimal *McMillan degree* q *then there exists a codimension* q (C, A)*-invariant subspace* $\mathcal{U} \subset \operatorname{Ker} V$*. Conversely, if there exists a codimension* q (C, A)*-invariant subspace* $\mathcal{U} \subset \operatorname{Ker} V$ *then there exists a partial realization of* $(H^{(\nu)})$ *of* (not *necessarily minimal*) *McMillan degree* q.

Proof. Assume that there exists a partial realization (A, B, C) of $(H^{(\nu)})$ of minimal McMillan degree q. Then (5.7) yields

$$V = CR_\mu(A, B) . \tag{5.8}$$

According to Proposition 5.1 $\mathcal{U} := \operatorname{Ker} R_\mu(A, B)$ is (C, A)-invariant. Furthermore (5.8) yields $\mathcal{U} \subset \operatorname{Ker} V$. Assume that $\operatorname{rk} R_\mu(A, B) < q$. Then there exists a state space transformation $S \in \mathrm{GL}(\mathcal{F}^k)$ such that

$$SR_\mu(A, B) = R_\mu(SAS^{-1}, SB) = R_\mu(\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}) = \begin{pmatrix} R_\mu(A_1, B_1) \\ 0 \end{pmatrix}$$

with $(A_1, B_1) \in \mathcal{F}^{q_1 \times q_1} \times \mathcal{F}^{q_1 \times p}$ and $q_1 < q$. Partitioning $CS^{-1} = (C_1 \ C_2)$ with $C_1 \in \mathcal{F}^{k \times q_1}$ then yields $V = C_1 R_\mu(A_1, B_1)$ and (A_1, B_1, C_1) is a partial realization of $(H^{(\nu)})$ of McMillan degree $q_1 < q$ contradicting the assumption of q being minimal. Hence $\operatorname{rk} R_\mu(A,B) = q$ holds and (A,B) is μ-regular. But then, according to Proposition 5.1, $\operatorname{codim} \mathcal{U} = q$.

Conversely, let $\mathcal{U} \subset \operatorname{Ker} V$ be (C, A)-invariant of codimension q. According to Proposition 5.2 there exists a μ-regular pair $(A,B) \in \mathcal{F}^{q\times q} \times \mathcal{F}^{q\times p}$ such that $\mathcal{U} = \operatorname{Ker} R_\mu(A,B)$. Since $\operatorname{Ker} R_\mu(A,B) \subset \operatorname{Ker} V$ there exists a matrix $C \in \mathcal{F}^{k\times q}$ such that (5.8) and hence (5.7) holds. Then (A,B,C) is a partial realization of $(H^{(\nu)})$ of McMillan degree q. □

Corollary 5.24. *Let* (C, A) *be an observable pair in dual Brunovsky form with observability indices* $\mu = (\mu_1, \dots, \mu_p)$ *(cf. Section 2.3.2). Let* $V \in \mathcal{F}^{k\times n}$ *define the nice sequence* $(H^{(\nu)})$ *as above. The minimal McMillan degree* q *of a partial realization of* $(H^{(\nu)})$ *is equal to the minimal codimension of a* (C, A)*-invariant subspace* $\mathcal{U} \subset \operatorname{Ker} V$.

Proof. According to Proposition 5.23 there exists a (C, A)-invariant subspace $\mathcal{U} \subset \operatorname{Ker} V$ of codimension q. Assume that there exists a (C, A)-invariant subspace $\tilde{\mathcal{U}} \subset \operatorname{Ker} V$ with $\operatorname{codim} \tilde{\mathcal{U}} < q$. Then Proposition 5.23 yields the existence of a partial realization of $(H^{(\nu)})$ of McMillan degree $\operatorname{codim} \tilde{\mathcal{U}} < q$, a contradiction to q being minimal. □

Gohberg, Kaashoek and Lerer [GKL91, Theorem 2.2] have derived a formula for the minimal McMillan degree of a partial realization of the nice sequence $(H^{(\nu)})$. Applying Corollary 5.24 then immediately yields the following formula for the minimal codimension of a (C, A)-invariant subspace $\mathcal{U} \subset \operatorname{Ker} V$. Let

$$\mathsf{H}_i := \begin{pmatrix} H^{(1)}_{1\dots r_i} & H^{(2)}_{1\dots r_{i+1}} & H^{(3)}_{1\dots r_{i+2}} & \dots & H^{(\mu_1-i+1)}_{1\dots r_{\mu_1}} \\ H^{(2)}_{1\dots r_i} & H^{(3)}_{1\dots r_{i+1}} & \dots & & \vdots \\ H^{(3)}_{1\dots r_i} & \dots & & & \vdots \\ \vdots & & & & \vdots \\ H^{(i)}_{1\dots r_i} & \dots & & & H^{(\mu_1)}_{1\dots r_{\mu_1}} \end{pmatrix}$$

and

$$\tilde{\mathsf{H}}_i := \begin{pmatrix} H^{(1)}_{1\dots r_{i+1}} & H^{(2)}_{1\dots r_{i+2}} & H^{(3)}_{1\dots r_{i+3}} & \dots & H^{(\mu_1-i)}_{1\dots r_{\mu_1}} \\ H^{(2)}_{1\dots r_{i+1}} & H^{(3)}_{1\dots r_{i+2}} & \dots & & \vdots \\ H^{(3)}_{1\dots r_{i+1}} & \dots & & & \vdots \\ \vdots & & & & \vdots \\ H^{(i)}_{1\dots r_{i+1}} & \dots & & & H^{(\mu_1-1)}_{1\dots r_{\mu_1}} \end{pmatrix},$$

where $H^{(\nu)}_{1\dots j}$ denotes the submatrix of $H^{(\nu)}$ consisting of the first j columns.

Theorem 5.25. *The minimal codimension of a* (C,A)*-invariant subspace* $\mathcal{U} \subset \operatorname{Ker} V$ *is equal to*

$$\sum_{i=1}^{\mu_1} \operatorname{rk} \mathsf{H}_i - \sum_{i=1}^{\mu_1 - 1} \operatorname{rk} \tilde{\mathsf{H}}_i \ .$$

As has been pointed out in Section 3.3.1, this formula provides a lower bound for the minimal order of an asymptotic output observer for the function Vx (at least if the observed system is controllable).

The following theorem extends and corrects a result of Fuhrmann and Helmke [FH01, Theorem 5.2]. If (A, B, C) is a partial realization of the nice sequence $(H^{(\nu)})$ of McMillan degree q then so is (SAS^{-1}, SB, CS) for every $S \in \mathrm{GL}(\mathcal{F}^q)$. For parametrization purposes it is therefore convenient to introduce *similarity classes*

$$[(A, B, C)]_\sigma = \{(SAS^{-1}, SB, CS) \mid S \in \mathrm{GL}(\mathcal{F}^q)\}$$

of partial realizations. Let $\mathcal{P}_q(H^{(\nu)})$ denote the set of all similarity classes of partial realizations of $(H^{(\nu)})$ of McMillan degree q. Let $\mathcal{V}_q^{\operatorname{Ker} V}(\mathsf{C},\mathsf{A})$ denote the set of all codimesion q (C,A)-invariant subspaces contained in $\operatorname{Ker} V$.

Theorem 5.26. *Let* (C,A) *be an observable pair in dual Brunovsky form with observability indices* $\mu = (\mu_1, \ldots, \mu_p)$ *(cf. Section* 2.3.2*). Let* $V \in \mathcal{F}^{k \times n}$ *define the nice sequence* $(H^{(\nu)})$ *as above. Let* q *be the McMillan degree of a minimal partial realization of* $(H^{(\nu)})$*. The map*

$$\begin{aligned} \rho_\mu : \mathcal{P}_q(H^{(\nu)}) &\longrightarrow \mathcal{V}_q^{\operatorname{Ker} V}(\mathsf{C},\mathsf{A}) \ , \\ [A, B, C]_\sigma &\mapsto \operatorname{Ker} R_\mu(A, B) \end{aligned}$$

is a surjection. If $\mathcal{V}_q^{\operatorname{Ker} V}(\mathsf{C},\mathsf{A})$ *happens to consist only of tight subspaces then* ρ_μ *is bijective.*

Proof. Since $\operatorname{Ker} R_\mu(A, B)$ is invariant under the similarity action and according to Proposition 5.23 the map ρ_μ is well defined and surjective. In the tight case the pair (A, B) is (up to similarity) uniquely determined by $\operatorname{Ker} R_\mu(A, B)$ (Theorem 5.4). Since $\operatorname{rk} R_\mu(A, B) = q$, equation (5.8) determines C uniquely. □

In general partial realizations of nice sequences are not unique – even in the tight and minimal case – as the following example shows.

Example 5.27. Let (C, A) be observable in dual Brunovsky form with observability indices $\mu_1 = 3$ and $\mu_2 = 2$, i.e. $p = 2$ and $n = \mu_1 + \mu_2 = 5$. Consider

$$V = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} .$$

Let $q = \operatorname{rk} V = 2$. Observe that there exists no pair $(A, B) \in \mathcal{F}^{q\times q} \times \mathcal{F}^{q\times p}$ with $V = R_\mu(A, B)$. Hence Proposition 5.1 yields that $\operatorname{Ker} V$ is *not* (C, A)-invariant (but of codimesion 2). But then according to Proposition 5.23 there exists no partial realization of the nice sequence

$$H^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad H^{(2)} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} , \quad H^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

of McMillan degree $q = 2$. The McMillan degree of a minimal partial realization of $H^{(\nu)}$ is hence ≥ 3. Now consider the family

$$A(x, y, z, v, w) = \begin{pmatrix} -\frac{x}{y-v} & -\frac{v}{y-v} & \frac{1}{y-v} \\ 1 + \frac{x}{y-v} & 1 + \frac{v}{y-v} & -\frac{1}{y-v} \\ y - x\frac{z-w}{y-v} & w - v\frac{z-w}{y-v} & \frac{z-w}{y-v} \end{pmatrix} ,$$

$$B(x, y, z, v, w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x & v \end{pmatrix} , \qquad x, y, z, v, w \in \mathcal{F} , \quad y \neq v$$

of μ-tight pairs $(A(.), B(.))$. According to Proposition 5.1 the subspaces

$$\mathcal{T}(.) = \operatorname{Ker} R_\mu(A(.), B(.)) = \operatorname{Ker} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ x & y & z & v & w \end{pmatrix}$$

are tight of codimension 3 (note that $y \neq v$). Apparently $\mathcal{T}(.) \subset \operatorname{Ker} V$ holds for all values of the parameters. But then the family $(A(.), B(.), C(.))$, where

$$C(.) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

consists of (obviously different) partial realizations of $(H^{(\nu)})$ of minimal McMillan degree 3. In fact, every partial realization of $(H^{(\nu)})$ of McMillan degree 3 comes from a kernel representation of the above type (Theorem 5.26), where $y \neq v$ is necessary since

$$A \cdot \begin{pmatrix} 0 \\ 1 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix} \quad \text{and} \quad A \cdot \begin{pmatrix} 0 \\ 1 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ w \end{pmatrix}$$

implies $a_{12} + ya_{13} = 1$ and $a_{12} + va_{13} = 0$ hence $(y - v)a_{13} = 1$ and $y - v \neq 0$. Consequently all minimal partial realizations of $(H^{(\nu)})$ are tight.

5.7 Observers II

In this section the connection between kernel representations of (almost) (C, A)-invariant subspaces and observer theory as well as the link of the latter to partial realization theory is explained.

Consider the system

$$\begin{aligned} \dot{x} &= \mathsf{A}x + \mathsf{B}u \\ y &= \mathsf{C}x \ , \end{aligned} \tag{5.9}$$

where the pair (C, A) is observable and in dual Brunovsky form with observability indices $\mu = (\mu_1, \ldots, \mu_p)$ (cf. Section 2.3.2). In this set-up the following three propositions provide an alternative proof for Theorem 3.8. The new proof clarifies the dynamical meaning of the matrix pair (A, B) appearing in the kernel representation of a (C, A)-invariant subpace $\mathcal{V} = \operatorname{Ker} R_\mu(A, B)$.

Proposition 5.28. *Let $V \in \mathcal{F}^{k \times n}$ and let $\mathcal{V} := \operatorname{Ker} V$ be (C, A)-invariant. Then there exists a (not necessarily controllable) pair $(A, B) \in \mathcal{F}^{k \times k} \times \mathcal{F}^{k \times p}$ such that $V = R_\mu(A, B)$.*

Proof. Let $q := \operatorname{codim} \mathcal{V}$, then $q \leq k$. According to Proposition 5.2 there exists a (μ-regular) pair $(\tilde{A}, \tilde{B}) \in \mathcal{F}^{q \times q} \times \mathcal{F}^{q \times p}$ such that $\mathcal{V} = \operatorname{Ker} R_\mu(\tilde{A}, \tilde{B}) = \operatorname{Ker} V$. But then there exists a rank q matrix $P \in \mathcal{F}^{k \times q}$ such that $V = PR_\mu(\tilde{A}, \tilde{B})$. Since $\operatorname{Ker} P = \{0\}$, there exists an invertible matrix $S \in \mathcal{F}^{k \times k}$ such that

$$SP = \begin{pmatrix} I \\ 0 \end{pmatrix} \ .$$

Now define

$$A := S^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} S \quad \text{and} \quad B := S^{-1} \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} \ .$$

It follows

$$\begin{aligned} R_\mu(A, B) &= S^{-1} \begin{pmatrix} R_\mu(\tilde{A}, \tilde{B}) \\ 0 \end{pmatrix} \\ &= S^{-1} SPR_\mu(\tilde{A}, \tilde{B}) \\ &= V \ . \end{aligned}$$

□

Proposition 5.29. *Let $(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$, and set $V := R_\mu(A, B)$, $K := A$, $L := -\begin{pmatrix} A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p \end{pmatrix}$ and $M := V\mathsf{B}$. Then $\dot{v} = Kv + Ly + Mu$ is a tracking observer for Vx.*

Proof. Calculate

$$\begin{aligned} V\mathsf{A} - KV &= R_\mu(A, B)\mathsf{A} - AR_\mu(A, B) \\ &= \begin{pmatrix} 0 & \dots & 0 & -A^{\mu_1}b_1 & \dots & 0 & \dots & 0 & -A^{\mu_p}b_p \end{pmatrix} \\ &= L\mathsf{C} \end{aligned}$$

and apply Theorem 3.4. □

Proposition 5.30. *Let $V \in \mathcal{F}^{k\times n}$ and let $\dot{v} = Kv + Ly + Mu$ be a tracking observer for Vx. Set*

$$A := K \quad \textit{and} \quad B := V\left(\begin{array}{c|c|c} 1 & & \\ 0 & & \\ \vdots & & \\ 0 & & \\ \hline & \ddots & \\ \hline & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right\}\mu_1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right\}\mu_p \end{array}.$$

Then $R_\mu(A, B) = V$, $A = K$, $-\begin{pmatrix} A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p \end{pmatrix} = L$ and $V\mathsf{B} = M$. It follows that $\mathcal{V} := \operatorname{Ker} V$ is (C, A)-invariant.

Proof. According to Theorem 3.4 it is $M = V\mathsf{B}$ and $V\mathsf{A} - KV = L\mathsf{C}$, i.e. $Kv_i = v_{i+1}$, $i = \mu_1 + \cdots + \mu_{j-1} + 1, \dots, \mu_1 + \cdots + \mu_j - 1$, and $Kv_{\mu_1+\cdots+\mu_j} = -l_j$, where $j = 1, \dots, p$ and v_i, l_i denote the i-th column of V and L, respectively. By induction this yields $A^i b_j = K^i v_{\mu_1+\cdots+\mu_{j-1}+1} = v_{\mu_1+\cdots+\mu_{j-1}+1+i}$, $i = 0, \dots, \mu_j - 1$ and $A^{\mu_j} b_j = K^{\mu_j} v_{\mu_1+\cdots+\mu_{j-1}+1} = -l_j$, where $j = 1, \dots, p$ and b_j denotes the j-th column of B. $\mathcal{V}$ being (C, A)-invariant follows from Proposition 5.1. □

Let $V \in \mathcal{F}^{k\times n}$ be such that $\mathcal{V} := \operatorname{Ker} V$ is (C, A)-invariant. Let

$$P_k^{\mathrm{id}}(V) := \{(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} \mid R_\mu(A, B) = V\}$$

be the set of μ-representations of V and let

$$\mathrm{Obs}_k(V) = \{(K, L, M) \in \mathcal{F}^{k\times(k+p+m)} \mid V\mathsf{A} - KV = L\mathsf{C}, M = V\mathsf{B}\}$$

be the set of all order k tracking observers for Vx. By the previous propositions the map

$$\begin{aligned}\rho_V : P_k^{\mathrm{id}}(V) &\longrightarrow \mathrm{Obs}_k(V)\ ,\\ (A,B) &\mapsto (K,L,M) = \left(A, -\begin{pmatrix}A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p\end{pmatrix}, R_\mu(A,B)\mathsf{B}\right)\end{aligned}$$

is a surjection. Since $V = R_\mu(A,B)$ defines B uniquely, ρ_V is also injective, hence a bijection. An already well known (cf. Theorem 5.4) consequence of this fact is that V has a unique representation $V = R_\mu(A,B)$ if and only if V has full row rank k and $\mathcal{V} = \operatorname{Ker} V$ is tight (Corollary 3.29). If V has full row rank k then the matrices A appearing in $P_k^{\mathrm{id}}(V)$ are nothing else but the corestrictions (cf. Section 2.3.3) of A to $\mathcal{V}$ (Theorem 3.33). Hence the spectral properties of $\mathcal{V}$ (e.g. being outer detectable or an observability subspace) are reflected in $P_k^{\mathrm{id}}(V)$ in this case. The formulation of the corresponding statements is left as an exercise to the reader.

The following theorem provides the link between the previous results and partial realizations (cf. Section 5.6). It extends a result achieved by Fuhrmann and Helmke [FH01, Theorem 5.5]. The proof uses the following lemma on uniqueness of tracked functions.

Lemma 5.31. *Let the system* (5.9) *be controllable. Consider a tracking observer* $\dot v = Kv + Ly + Mu$ *for the function* Ux*, where* $U \in \mathcal{F}^{q\times n}$*. Then* $U\mathsf{A} - KU = L\mathsf{C}$ *and* $M = U\mathsf{B}$*. If the observer tracks a second function* $\tilde U x$*, where* $\tilde U \in \mathcal{F}^{q\times n}$*, i.e. if* $\tilde U\mathsf{A} - K\tilde U = L\mathsf{C}$ *and* $M = \tilde U\mathsf{B}$*, then* $U = \tilde U$*.*

Proof. The equalities $U\mathsf{A} - KU = L\mathsf{C}$ and $M = U\mathsf{B}$ follow from Theorem 3.4. If also $\tilde U\mathsf{A} - K\tilde U = L\mathsf{C}$ and $M = \tilde U\mathsf{B}$ then $U\mathsf{B} = \tilde U\mathsf{B}$ and $U\mathsf{AB} = KU\mathsf{B} + L\mathsf{CB} = K\tilde U\mathsf{B} + L\mathsf{CB} = \tilde U\mathsf{AB}$ and by induction it follows $UR_n(\mathsf{A},\mathsf{B}) = \tilde U R_n(\mathsf{A},\mathsf{B})$. By controllability of (A,B) this implies $U = \tilde U$. $\square$

If the system (5.9) is not controllable then tracked functions in general are not unique as the following example shows.

Example 5.32. Let $p = 2$, $\mu_1 = \mu_2 = 2$,

$$\mathsf{A} = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}, \quad \mathsf{B} = 0 \quad \text{and} \quad \mathsf{C} = \begin{pmatrix}0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Let $K = 0$, $L = 0$ and $M = 0$, then $U = \begin{pmatrix}u_1 & u_2 & u_3 & u_4\end{pmatrix}$ satisfies $U\mathsf{A} - KU = L\mathsf{C}$ and $M = U\mathsf{B}$ if and only if $u_2 = u_4 = 0$, while u_1 and u_3 are arbitrary.

Theorem 5.33. *Let the system* (5.9) *be controllable. Let* $V \in \mathcal{F}^{k\times n}$, *and let*

$$\begin{aligned}\mathrm{Obs}_q^{\mathrm{out}}(V) := \{(K,L,M,P) \in \mathcal{F}^{q\times q} \times \mathcal{F}^{q\times p} \times \mathcal{F}^{q\times m} \times \mathcal{F}^{k\times q} \mid \\ \exists U \in \mathcal{F}^{q\times n}\ U\mathsf{A} - KU = L\mathsf{C}, M = U\mathsf{B}, V = PU\}\end{aligned}$$

be the set of order q *tracking observers for* Vx *with output* (*cf. Section* 3.2.4). *Let*

$$P_q(V) = \{(A,B,C) \in \mathcal{F}^{q\times q} \times \mathcal{F}^{q\times p} \times \mathcal{F}^{k\times q} \mid CR_\mu(A,B) = V\}$$

be the set of partial realizations of V (*respectively of the nice sequence* $(H^{(\nu)})$ *defined by* V, *cf. Section* 5.6) *of McMillan degree* q. *The map*

$$\begin{aligned}&\rho_V : P_q(V) \longrightarrow \mathrm{Obs}_q^{\mathrm{out}}(V)\ ,\\ &\quad (A,B,C) \mapsto (K,L,M,P) = (A, -\begin{pmatrix} A^{\mu_1}b_1 & \ldots & A^{\mu_p}b_p \end{pmatrix}, R_\mu(A,B)\mathsf{B}, C)\end{aligned}$$

is a bijection. It restricts to a bijection from the set of McMillan degree q *observable and stable partial realizations of* V *onto the set of order* q *observable asymptotic output observers for* Vx *without direct feedthrough.*

Proof. Let $(A,B,C) \in P_q(V)$ then, according to the above considerations, the (K,L,M)-part of $\rho_V(A,B,C)$ forms a tracking observer for the function $Ux := R_\mu(A,B)x$. Hence $\rho_V(A,B,C) \in \mathrm{Obs}_q^{\mathrm{out}}(V)$ and ρ_V is well defined. Since $\rho_U : P_q^{\mathrm{id}}(U) \longrightarrow \mathrm{Obs}_q(U)$ is surjective for every $U \in \mathcal{F}^{q\times n}$ for which $\mathrm{Ker}\, U$ is (C,A)-invariant (or, equivalently, for which there exists a tracking observer for Ux), so is ρ_V. By Lemma 5.31 the (K,L,M)-part of $\rho_V(A,B,C)$ defines U uniquely, which is therefore equal to $R_\mu(A,B)$. The latter defines B uniquely, and since $K = A$ and $P = C$ it follows that (A,B,C) is uniquely defined by $\rho_V(A,B,C)$, i.e. ρ_V is injective. The second statement follows from Theorem 3.70. □

Since minimal asymptotic output observers for Vx are necessarily observable (Proposition 3.69), the minimal order of an (observable) asymptotic output observer without direct feedthrough for the function Vx of the state of the controllable system (5.9) is consequently equal to the minimal McMillan degree of an observable stable partial realization of V. As has been pointed out before, a formula for this minimal order (in the manner of Theorem 5.25) is not yet available.

The next results are the counterparts of Proposition 5.28 and Proposition 5.29 for almost observability subspaces.

Proposition 5.34. *Let $V \in \mathcal{F}^{k\times n}$ and let $\mathcal{V} := \operatorname{Ker} V$ be an almost observability subspace with respect to the pair (C, A). Then there exists a (not necessarily controllable) pair $(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$ such that A is nilpotent and $V = \overleftarrow{R}_\mu(A, B)$.*

Proof. Let $q := \operatorname{codim} \mathcal{V}$, then $q \leq k$. According to Proposition 5.8 there exists a (μ-regular) pair $(\tilde{A}, \tilde{B}) \in \mathcal{F}^{q\times q} \times \mathcal{F}^{q\times p}$ such that $\tilde{A}$ is nilpotent and $\mathcal{V} = \operatorname{Ker} \overleftarrow{R}_\mu(\tilde{A}, \tilde{B}) = \operatorname{Ker} V$. But then there exists a rank q matrix $P \in \mathcal{F}^{k\times q}$ such that $V = P\overleftarrow{R}_\mu(\tilde{A}, \tilde{B})$. Since $\operatorname{Ker} P = \{0\}$, there exists an invertible matrix $S \in \mathcal{F}^{k\times k}$ such that

$$SP = \begin{pmatrix} I \\ 0 \end{pmatrix} .$$

Now define

$$A := S^{-1} \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} S \quad \text{and} \quad B := S^{-1} \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} .$$

It follows that A is nilpotent and

$$\begin{aligned} \overleftarrow{R}_\mu(A, B) &= S^{-1} \begin{pmatrix} \overleftarrow{R}_\mu(\tilde{A}, \tilde{B}) \\ 0 \end{pmatrix} \\ &= S^{-1} S P \overleftarrow{R}_\mu(\tilde{A}, \tilde{B}) \\ &= V . \end{aligned}$$

□

Proposition 5.35. *Let $(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$ with A nilpotent, and set $V := \overleftarrow{R}_\mu(A, B)$, $E := A$, $K := I$, $L := -B$ and $M := EV\mathsf{B}$. Then $EV\mathsf{A} - KV = L\mathsf{C}$ and $E\dot{v} = Kv + Ly + Mu$ is a singular tracking observer for Vx (with E nilpotent and $K = I$).*

Proof. Calculate

$$\begin{aligned} EV\mathsf{A} - KV &= A\overleftarrow{R}_\mu(A, B)\mathsf{A} - \overleftarrow{R}_\mu(A, B) \\ &= \begin{pmatrix} 0 & \dots & 0 & -b_1 & \dots & 0 & \dots & 0 & -b_p \end{pmatrix} \\ &= L\mathsf{C} \end{aligned}$$

and apply Theorem 3.13. □

The following theorem now fills the gap left in the proof of Theorem 3.24. Recall that there was made no assumption on the pair (C, A) in the formulation of that theorem.

Theorem 5.36. *Consider the system* (3.1). *Let* $V \in \mathcal{F}^{k\times n}$ *and let* $\mathcal{V} := \operatorname{Ker} V$ *be an almost observability subspace. Then there exists a singular tracking observer for* Vx *with* E *nilpotent and* $K = I$ *(cf. Section* 3.2.1*).*

Proof. Consider the dual Kalman decomposition of the pair (C, A): There exist invertible matrices $T \in \mathcal{F}^{n\times n}$ and $S \in \mathcal{F}^{p\times p}$ such that

$$C^{(1)} := SCT^{-1} = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A^{(1)} := TAT^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathcal{F}^{r\times r}$, $C_{11} \in \mathcal{F}^{q\times r}$, (C_{11}, A_{11}) is observable and C_{11} has full row rank q.

Let $i \in \mathbb{N}$, $J \in \mathcal{F}^{n\times p}$ and $\mathcal{K}_i \supset \operatorname{Ker} C$. Then there exists a matrix $K_i \in \mathcal{F}^{p\times p}$ such that $\mathcal{K}_i = \operatorname{Ker} K_i C$. But then

$$\begin{aligned} x \in (A - JC)^{-i+1}\mathcal{K}_i &\iff \\ K_i C (A - JC)^{i-1} x = 0 &\iff \\ K_i S^{-1} SCT^{-1} T (A - JS^{-1}SC)^{i-1} T^{-1} T x = 0 &\iff \\ (K_i S^{-1}) SCT^{-1} (TAT^{-1} - (TJS^{-1}) SCT^{-1})^{i-1} T x = 0 &\iff \\ Tx \in (A^{(1)} - (TJS^{-1}) C^{(1)})^{-i+1} \operatorname{Ker}(K_i S^{-1}) C^{(1)} & \end{aligned}$$

by Proposition 2.33 implies that $\operatorname{Ker} VT^{-1}$ is an almost observability subspace with respect to the pair $(C^{(1)}, A^{(1)})$. Hence

$$\operatorname{Ker} VT^{-1} \supset \mathcal{N}(C^{(1)}, A^{(1)}) = \operatorname{Ker} \begin{pmatrix} I & 0 \end{pmatrix}$$

and $V^{(1)} := VT^{-1} = \begin{pmatrix} V_1 & 0 \end{pmatrix}$, where $V_1 \in \mathcal{F}^{k\times r}$.

Again let $i \in \mathbb{N}$, $J^{(1)} \in \mathcal{F}^{n\times p}$ and $\mathcal{K}_i^{(1)} = \operatorname{Ker} K_i^{(1)} C^{(1)} \supset \operatorname{Ker} C^{(1)}$. Then

$$\begin{aligned} x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in (A^{(1)} - J^{(1)} C^{(1)})^{-i+1} \mathcal{K}_i^{(1)} &\iff \\ \left(\begin{smallmatrix} K_{i,1} & K_{i,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} C_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right) \left(\left(\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right) - \left(\begin{smallmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{smallmatrix}\right) \left(\begin{smallmatrix} C_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right) \right)^{i-1} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = 0 &\iff \\ \left(\begin{smallmatrix} K_{i,1} C_{11} & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} A_{11} - J_{11} C_{11} & 0 \\ A_{21} - J_{21} C_{11} & A_{22} \end{smallmatrix}\right)^{i-1} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = 0 &\iff \\ K_{i,1} C_{11} (A_{11} - J_{11} C_{11})^{i-1} x_1 = 0 &\iff \\ x_1 \in (A_{11} - J_{11} C_{11})^{-i+1} \operatorname{Ker} K_{i,1} C_{11} & \end{aligned}$$

implies that $\operatorname{Ker} V_1$ is an almost observability subspace with respect to the pair (C_{11}, A_{11}).

According to Proposition 2.44 there exist invertible matrices $T_1 \in \mathcal{F}^{r\times r}$ and $S_1 \in \mathcal{F}^{q\times q}$ and a matrix $J_1 \in \mathcal{F}^{r\times q}$ such that the pair $(C^{(2)}, A^{(2)}) := (S_1 C_{11} T_1^{-1}, T_1(A_{11} - J_1 C_{11})T_1^{-1})$ is in dual Brunovsky form with observability indices $\mu = (\mu_1, \dots, \mu_q)$.

Once again let $i \in \mathbb{N}$, $J \in \mathcal{F}^{r\times q}$ and $\mathcal{K}_i = \operatorname{Ker} K_i C_{11} \supset \operatorname{Ker} C_{11}$. As before

$$x \in (A_{11} - JC_{11})^{-i+1}\mathcal{K}_i \Longleftrightarrow$$
$$T_1 x \in (A^{(2)} - (T_1(J - J_1)S_1^{-1})C^{(2)})^{-i+1} \operatorname{Ker}(K_i S_1^{-1})C^{(2)}$$

implies that $\operatorname{Ker} V^{(2)} := \operatorname{Ker} V_1 T_1^{-1}$ is an almost observability subspace with respect to the pair $(C^{(2)}, A^{(2)})$.

By Proposition 5.34 and Proposition 5.35 there exist a nilpotent matrix $E \in \mathcal{F}^{k\times k}$ and a matrix $L_1 \in \mathcal{F}^{k\times q}$ such that $EV^{(2)}A^{(2)} - V^{(2)} = L_1 C^{(2)}$. Setting

$$L := \begin{pmatrix} L_1 S_1 + EV_1 J_1 & 0 \end{pmatrix} S \in \mathcal{F}^{k\times p}$$

yields

$$\begin{aligned}
(EVA - V)T^{-1} &= E(VT^{-1})(TAT^{-1}) - VT^{-1} \\
&= E\begin{pmatrix} V_1 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} - \begin{pmatrix} V_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} EV_1 A_{11} - V_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} E(V_1 T_1^{-1})(T_1(A_{11} - J_1 C_{11})T_1^{-1})T_1 - (V_1 T_1^{-1})T_1 & 0 \end{pmatrix} + \\
&\quad\ \begin{pmatrix} EV_1 J_1 C_{11} & 0 \end{pmatrix} \\
&= \begin{pmatrix} (EV^{(2)}A^{(2)} - V^{(2)})T_1 + EV_1 J_1 C_{11} & 0 \end{pmatrix} \\
&= \begin{pmatrix} L_1 C^{(2)} T_1 + EV_1 J_1 C_{11} & 0 \end{pmatrix} \\
&= \begin{pmatrix} (L_1 S_1 + EV_1 J_1) C_{11} & 0 \end{pmatrix} \\
&= \begin{pmatrix} (L_1 S_1 + EV_1 J_1) & 0 \end{pmatrix}\begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} \\
&= (LS^{-1})(SCT^{-1}) \\
&= LCT^{-1}\ .
\end{aligned}$$

It follows that $EVA - V = LC$. Set $M := EVB$. According to Theorem 3.13 the system $E\dot{v} = Iv + Ly + Mu$ is a singular tracking observer for Vx, then. $\square$

Remark 5.37. The above proof uses the fact that any almost observability subspace contains the unobservable subspace (cf. Section 2.3). Since there is no analogous result for (C, A)-invariant subspaces, the techniques used above can not be applied to generalize Proposition 5.29 to general pairs (C, A). That generalization has been proved in Section 3.2 by different methods, though (Theorem 3.8).

The following result is the counterpart to Proposition 5.30 for almost observability subspaces.

Proposition 5.38. *Let $V \in \mathcal{F}^{k\times n}$ and let $E\dot{v} = v + Ly + Mu$ be a singular tracking observer for Vx with E nilpotent. Set*

$$A := E \quad \text{and} \quad B := V \left(\begin{array}{c|c|c} \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{array} \right) \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}}\right\} \mu_1 \\ \vdots \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}}\right\} \mu_p \end{array} .$$

Then $\overleftarrow{R}_\mu(A, B) = V$, $A = E$, $-B = L$ and $EV\mathsf{B} = M$. It follows that $\mathcal{V} := \operatorname{Ker} V$ is an almost observability subspace with respect to the pair (C, A).

Proof. According to Theorem 3.13 it is $M = EV\mathsf{B}$ and $EV\mathsf{A} - V = L\mathsf{C}$, i.e. $v_i = Ev_{i+1}$, $i = \mu_1 + \cdots + \mu_{j-1} + 1, \ldots, \mu_1 + \cdots + \mu_j - 1$, and $v_{\mu_1+\cdots+\mu_j} = -l_j$, where $j = 1, \ldots, p$ and v_i, l_i denote the i-th column of V and L, respectively. By induction this yields $b_j = v_{\mu_1+\cdots+\mu_j} = -l_j$ and $A^i b_j = E^i v_{\mu_1+\cdots+\mu_j} = v_{\mu_1+\cdots+\mu_j - i}$, where $i = 0, \ldots, \mu_j - 1$, $j = 1, \ldots, p$ and b_j denotes the j-th column of B. $\mathcal{V}$ being an almost observability subspace with respect to the pair (C, A) follows from Proposition 5.7. $\square$

Let $V \in \mathcal{F}^{k\times n}$ be such that $\mathcal{V} := \operatorname{Ker} V$ is an almost observability subspace with respect to the pair (C, A). Let

$$\overleftarrow{P}_k^{\mathrm{id}}(V) := \{(A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} \mid A^k = 0, \overleftarrow{R}_\mu(A, B) = V\}$$

be the set of reverse μ-representations of V and let

$$\begin{aligned} \operatorname{Obs}_k^{\mathrm{nil}}(V) = \{&(E, K, L, M) \in \mathcal{F}^{k\times(k+k+p+m)} \mid \\ &E^k = 0, K = I, EV\mathsf{A} - KV = L\mathsf{C}, M = EV\mathsf{B}\} \end{aligned}$$

be the set of all order k singular tracking observers for Vx with E nilpotent and $K = I$. By the previous propositions the map

$$\begin{aligned} \overleftarrow{\rho}_V : \overleftarrow{P}_k^{\mathrm{id}}(V) &\longrightarrow \mathrm{Obs}_k^{\mathrm{nil}}(V) \ , \\ (A, B) &\mapsto (E, K, L, M) = (A, I, -B, AR_\mu(A, B)\mathsf{B}) \end{aligned}$$

is a surjection. Obviously it is also injective, hence a bijection.

The next results are the counterparts to Propositions 5.28 and 5.29 (respectively Propositions 5.34 and 5.35) for almost (C, A)-invariant subspaces.

Proposition 5.39. *Let $V \in \mathcal{F}^{k\times n}$ and let $\mathcal{V} := \operatorname{Ker} V$ be almost (C, A)-invariant. Then there exists a not necessarily controllable but admissible (cf. Section 4.2) triple $(E, A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$ such that $EA = AE$ and*

$$V = R_\mu(E, A, B) := \left(\, {\scriptstyle E^{\mu_1-1}b_1 \ E^{\mu_1-2}Ab_1 \ \ldots \ EA^{\mu_1-2}b_1 \ A^{\mu_1-1}b_1 \ \ldots \ E^{\mu_p-1}b_p \ \ldots \ A^{\mu_p-1}b_p} \, \right) ,$$

where b_j, $j = 1, \ldots, p$, denotes the j-th column of B.

Proof. Let $q := \operatorname{codim} \mathcal{V}$, then $q \le k$. According to Proposition 5.17 there exists a pair of matrix pairs $(\tilde{A}_1, \tilde{B}_1) \in \mathcal{F}^{r\times r} \times \mathcal{F}^{r\times p}$ and $(\tilde{N}, \tilde{B}_2) \in \mathcal{F}^{(q-r)\times(q-r)} \times \mathcal{F}^{(q-r)\times p}$ with $\tilde{N}$ nilpotent such that $\mathcal{V} = \operatorname{Ker} R_\mu(\tilde{A}_1, \tilde{B}_1, \tilde{N}, \tilde{B}_2) = \operatorname{Ker} V$ and $\operatorname{rk} R_\mu(\tilde{A}_1, \tilde{B}_1, \tilde{N}, \tilde{B}_2) = q$. But then there exists a rank q matrix $P \in \mathcal{F}^{k\times q}$ such that $V = PR_\mu(\tilde{A}_1, \tilde{B}_1, \tilde{N}, \tilde{B}_2)$. Since $\operatorname{Ker} P = \{0\}$, there exists an invertible matrix $S \in \mathcal{F}^{k\times k}$ such that

$$SP = \begin{pmatrix} I \\ 0 \end{pmatrix} \ .$$

Now define

$$E := S^{-1} \begin{pmatrix} \tilde{I} & 0 & 0 \\ 0 & \tilde{N} & 0 \\ 0 & 0 & 0 \end{pmatrix} S, \ \ A := S^{-1} \begin{pmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} S \text{ and } B := S^{-1} \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \end{pmatrix} .$$

It follows

$$\begin{aligned} R_\mu(E, A, B) &= S^{-1} \begin{pmatrix} R_\mu(\tilde{A}_1, \tilde{B}_1) \\ \overleftarrow{R}_\mu(\tilde{N}, \tilde{B}_2) \\ 0 \end{pmatrix} \\ &= S^{-1}SPR_\mu(\tilde{A}_1, \tilde{B}_1, \tilde{N}, \tilde{B}_2) \\ &= V \ . \end{aligned}$$

Apparently, $EA = AE$ holds and (E, A, B) is admissible. $\square$

Proposition 5.40. *Let $(E, A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p}$ be admissible with $EA = AE$. Set $V := R_\mu(E, A, B)$, $K := A$, $L := -\begin{pmatrix} A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p \end{pmatrix}$ and $M := EV\mathsf{B}$. Then $E\dot{v} = Kv + Ly + Mu$ is a singular tracking observer for Vx.*

Proof. Calculate

$$\begin{aligned} EV\mathsf{A} - KV &= ER_\mu(E, A, B)\mathsf{A} - AR_\mu(E, A, B) \\ &= \begin{pmatrix} 0 & \dots & 0 & -A^{\mu_1}b_1 & \dots & 0 & \dots & 0 & -A^{\mu_p}b_p \end{pmatrix} \\ &= L\mathsf{C} \end{aligned}$$

and apply Theorem 3.13. □

The next proposition is the counterpart to Proposition 5.30 (respectively Proposition 5.38) for almost (C, A)-invariant subpaces. The proof uses the following result on the quasi Weierstraß form of a matrix pencil, which is due to Helmke (private communication).

Theorem 5.41. *Let $(E, A) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k}$ be admissible with $EA = AE$. Then there exists an invertible matrix $S \in \mathcal{F}^{k\times k}$ such that*

$$SES^{-1} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad \text{and} \quad SAS^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} ,$$

where $E_1 \in \mathcal{F}^{r\times r}$ and $A_2 \in \mathcal{F}^{(k-r)\times(k-r)}$ are invertible, E_2 is nilpotent, $E_1A_1 = A_1E_1$ and $E_2A_2 = A_2E_2$. The transformed pencil is said to be in quasi Weierstraß form*, then. If both (E, A) and (SES^{-1}, SAS^{-1}) are in quasi Weierstraß form then*

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} .$$

Proof. Let $S \in \mathcal{F}^{k\times k}$ be invertible and such that

$$SES^{-1} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

with $E_1 \in \mathcal{F}^{r\times r}$ invertible and E_2 nilpotent (such an S exists by the spectral decomposition theorem). Partition

$$SAS^{-1} = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$$

such that $A_1 \in \mathcal{F}^{r\times r}$. Then the equality $AE = EA$ implies $SES^{-1}SAS^{-1} = SAS^{-1}SES^{-1}$ and hence $E_1A_1 = A_1E_1$, $E_2A_2 = A_2E_2$, $E_1X = XE_2$ and $E_2Y = YE_1$.

Since E_2 is nilpotent, there exists $q \geq 2$ such that $E_2^q = 0$. Let $v \in \mathcal{F}^{k-r}$, then $E_1XE_2^{q-1}v = XE_2E_2^{q-1}v = XE_2^qv = 0$. Since E_1 is invertible it follows $XE_2^{q-1}v = 0$ and v being arbitrary yields $XE_2^{q-1} = 0$. But then $E_1XE_2^{q-2}v = XE_2E_2^{q-2}v = XE_2^{q-1}v = 0$ yields $XE_2^{q-2}v = 0$ and hence $XE_2^{q-2} = 0$. Iterating this argument finally yields $X = 0$, and $Y = 0$ follows along the same lines.

Since (E, A) is admissible, there exist $\lambda, \mu \in \mathbb{C}$ such that $\det(\lambda E + \mu A) = \det(\lambda E_1 + \mu A_1)\det(\lambda E_2 + \mu A_2) \neq 0$ and hence $\det(\lambda E_2 + \mu A_2) \neq 0$. Assume that A_2 is singular, i.e. $\operatorname{Ker} A_2 \neq \{0\}$. Since $E_2A_2 = A_2E_2$ it follows that $\operatorname{Ker} A_2$ is E_2-invariant. Since E_2 is nilpotent, so is $E_2|_{\operatorname{Ker} A_2}$. But then there exists $v \in \operatorname{Ker} A_2 \setminus \{0\}$ such that $E_2v = 0$. It follows that $(\lambda E_2 + \mu A_2)v = 0$ and hence $\det(\lambda E_2 + \mu A_2) = 0$, a contradiction. Consequently, A_2 is invertible.

Let finally $S \in \mathcal{F}^{k\times k}$ be invertible, and let

$$E = \begin{pmatrix} E_1' & 0 \\ 0 & E_2' \end{pmatrix} \quad \text{and} \quad SES^{-1} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

be such that E_1, E_1' are invertible and E_2, E_2' are nilpotent. Partition

$$S = \begin{pmatrix} S_1 & X \\ Y & S_2 \end{pmatrix} ,$$

then $SE = (SES^{-1})S$ implies $YE_1' = E_2Y$ and $XE_2' = E_1X$. As before it follows $X = 0$ and $Y = 0$. $\square$

Proposition 5.42. *Let $V \in \mathcal{F}^{k\times n}$ and let $E\dot{v} = Kv + Ly + Mu$ be a singular tracking observer for Vx with $EK = KE$. Set $A := K$ and $B := \begin{pmatrix} b_1 & \dots & b_p \end{pmatrix}$, where*

$$b_j := S^{-1}\begin{pmatrix} E_1^{-(\mu_j-1)} & 0 \\ 0 & 0 \end{pmatrix} Sv_{\mu_1+\dots+\mu_{j-1}+1} + S^{-1}\begin{pmatrix} 0 & 0 \\ 0 & A_2^{-(\mu_j-1)} \end{pmatrix} Sv_{\mu_1+\dots+\mu_j}$$

for $j = 1, \dots, p$. Here E_1 and A_2 are taken from any quasi Weierstraß form of (E, A), and S is the corresponding transformation. v_i denotes the i-th column of V. Then $R_\mu(E, A, B) = V$, $A = K$, $-\begin{pmatrix} A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p \end{pmatrix} = L$ and $EV\mathsf{B} = M$. It follows that $\mathcal{V} := \operatorname{Ker} V$ is almost (C, A)-invariant.

Proof. According to Theorem 3.13 it is $M = EV\mathsf{B}$ and $EV\mathsf{A} - KV = L\mathsf{C}$, i.e. $Av_i = Kv_i = Ev_{i+1}$, $i = \mu_1 + \dots + \mu_{j-1} + 1, \dots, \mu_1 + \dots + \mu_j - 1$, and $Av_{\mu_1+\dots+\mu_j} = -l_j$, where $j = 1, \dots, p$ and l_i denotes the i-th column of L. By induction this yields

$$
\begin{aligned}
E^{\mu_j-i-1}A^i b_j &= S^{-1}\begin{pmatrix} E_1^{\mu_j-i-1}A_1^i & 0 \\ 0 & E_2^{\mu_j-i-1}A_2^i \end{pmatrix} Sb_j \\
&= S^{-1}\begin{pmatrix} E_1^{-i}A_1^i & 0 \\ 0 & 0 \end{pmatrix} Sv_{\mu_1+\dots+\mu_{j-1}+1} + \\
&\quad S^{-1}\begin{pmatrix} 0 & 0 \\ 0 & E_2^{\mu_j-i-1}A_2^{-(\mu_j-i-1)} \end{pmatrix} Sv_{\mu_1+\dots+\mu_j} \\
&= S^{-1}\begin{pmatrix} E_1^{-i}E_1^i & 0 \\ 0 & 0 \end{pmatrix} Sv_{\mu_1+\dots+\mu_{j-1}+i+1} + \\
&\quad S^{-1}\begin{pmatrix} 0 & 0 \\ 0 & A_2^{-(\mu_j-i-1)}A_2^{\mu_j-i-1} \end{pmatrix} Sv_{\mu_j+\dots+\mu_{j-1}+i+1} \\
&= v_{\mu_j+\dots+\mu_{j-1}+i+1}
\end{aligned}
$$

for $i = 0, \dots, \mu_j - 1$ and $A^{\mu_j}b_j = AE^0A^{\mu_j-1}b_j = Av_{\mu_1+\dots+\mu_j} = -l_j$, where $j = 1, \dots, p$.

Define $\tilde{B}_1$ and $\tilde{B}_2$ by

$$
\begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} := SB \ .
$$

Set $B_1 := \begin{pmatrix} E_1^{\mu_1-1}\tilde{b}_{11} & \dots & E_1^{\mu_p-1}\tilde{b}_{1p} \end{pmatrix}$ and $B_2 := \begin{pmatrix} A_2^{\mu_1-1}\tilde{b}_{21} & \dots & A_2^{\mu_p-1}\tilde{b}_{2p} \end{pmatrix}$, where $\tilde{b}_{ij}$, $i = 1, 2$ and $j = 1, \dots, p$, denotes the j-th column of $\tilde{B}_i$. Then $\mathcal{V} = \operatorname{Ker} V$ being almost (C, A)-invariant follows from

$$
\begin{aligned}
V &= R_\mu(E, A, B) \\
&= S^{-1}R_\mu(SES^{-1}, SAS^{-1}, SB) \\
&= S^{-1}\begin{pmatrix} R_\mu(E_1^{-1}A_1, B_1) \\ \overleftarrow{R}_\mu(E_2A_2^{-1}, B_2) \end{pmatrix}
\end{aligned}
\tag{5.10}
$$

and Proposition 5.13 (note that $E_2A_2^{-1}$ is nilpotent). $\square$

Let $V \in \mathcal{F}^{k\times n}$ be such that $\mathcal{V} := \operatorname{Ker} V$ is almost (C, A)-invariant. Let

$$
\begin{aligned}
\hat{P}_k^{\mathrm{id}}(V) := \{&(E, A, B) \in \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times k} \times \mathcal{F}^{k\times p} \mid \\
&(E, A, B) \text{ admissible}, EA = AE, R_\mu(E, A, B) = V\}
\end{aligned}
$$

be the set of μ-representations of V and let

$$\begin{aligned}\mathrm{Obs}_k^{\mathrm{sing.}}(V) = \{(E,K,L,M) \in \mathcal{F}^{k\times(k+k+p+m)} \mid (E,K) \text{ admissible},\\ EK = KE, EV\mathsf{A} - KV = L\mathsf{C}, M = EV\mathsf{B}\}\end{aligned}$$

be the set of all order k singular tracking observers for Vx. By the previous propositions the map

$$\begin{aligned}\rho_V : \hat{P}_k^{\mathrm{id}}(V) &\longrightarrow \mathrm{Obs}_k^{\mathrm{sing.}}(V)\ ,\\ (E,A,B) &\mapsto (E,K,L,M) =\\ &\qquad (E, A, -\begin{pmatrix} A^{\mu_1}b_1 & \dots & A^{\mu_p}b_p\end{pmatrix}, ER_\mu(E,A,B)\mathsf{B})\end{aligned}$$

is a surjection. Assume that $\rho_V(E,A,B) = \rho_V(E',A',B')$. Then $E = E'$, $A = A'$ and $V = R_\mu(E,A,B) = R_\mu(E,A,B')$. Choose an S which transforms (E,A) to quasi Weierstraß form. Look at equation (5.10). Apparently, V and S define B_1 and B_2 uniquely, i.e. $B_1 = B_1'$ and $B_2 = B_2'$. Since E_1 and A_2 are invertible, it follows $\tilde{B}_1 = \tilde{B}_1'$ and $\tilde{B}_2 = \tilde{B}_2'$ and $B = B'$. But then ρ_V is also injective, hence a bijection.

Remark 5.43. In the special case of system (5.9) the last three propositions provide an alternative proof for Theorem 3.24, (1). The new proof clarifies the dynamical meaning of the matrices appearing in the kernel representation of an almost (C,A)-invariant subspace.

Chapter 6

Image representations

In this chapter image representations (i.e. explicit bases arranged in a matrix) for tight (C,A)-invariant subspaces are constructed, where (C,A) is a given observable pair in dual Brunovsky form. The set of all codimension k tight (C,A)-invariant subspaces is shown to be the union of smooth manifolds (the Brunovsky strata) consisting of the subspaces with fixed restriction indices. These strata are shown to be diffeomorphic to a quotient of a certain set of block Toeplitz type matrices, which provide bases for the subspaces under consideration. This result is due to Ferrer, F. Puerta and X. Puerta [FPP98]. Then a slight variation of the method is applied to obtain another block Toeplitz type description corresponding to Kronecker strata instead of Brunovsky strata. The two results are combined to obtain a Bruhat type cell decomposition of the Brunovsky strata.

6.1 Brunovsky and Kronecker strata

The results presented in this Section extend the work of Ferrer, F. Puerta and X. Puerta [FPP98] on Brunovsky strata of (C,A)-invariant subspaces.

Let (C,A) be an observable pair in dual Brunovsky form with observability indices $\mu = (\mu_1,\dots,\mu_p)$ (cf. Section 2.3.2). The following lemma is due to Fuhrmann and Helmke [FH00].

Lemma 6.1. *A codimension (C,A)-invariant subspace $\mathcal{V}$ is tight if and only if $\operatorname{rk}\bar{C} = p$ for any (and thus for all) restriction $(\bar{C},\bar{A})$ of (C,A) to $\mathcal{V}$, i.e. if and only if the smallest restriction index λ_p satisfies $\lambda_p \geq 1$.*

Proof. Let $\mathcal{V}$ be tight. According to Proposition 2.42 $\mathcal{V} + \operatorname{Ker} \mathsf{C} = \mathcal{F}^n$, i.e. for all $x \in \mathcal{F}^n$ there exist $u \in \mathcal{V}$ and $z \in \operatorname{Ker} \mathsf{C}$ with $x = u + z$. But then for all $x \in \mathcal{F}^n$ there exists $u \in \mathcal{V}$ with $\mathsf{C}x = \mathsf{C}u$. Since C is surjective this implies $\mathsf{C}|_{\mathcal{V}}$ being surjective. Hence $\bar{C}$ has full rank p.

Conversely, if $\bar{C}$ has full rank p then $\mathsf{C}|_{\mathcal{V}}$ is surjective and for all $y \in \mathcal{F}^p$ there exists $v \in \mathcal{V}$ with $y = \mathsf{C}v$. This implies that for all $x \in \mathcal{F}^n$ there exists $v \in \mathcal{V}$ with $\mathsf{C}x = \mathsf{C}v$, but then $x = v + (x - v) \in \mathcal{V} + \operatorname{Ker} \mathsf{C}$. It follows $\mathcal{F}^n \subset \mathcal{V} + \operatorname{Ker} \mathsf{C}$ and hence $\mathcal{F}^n = \mathcal{V} + \operatorname{Ker} \mathsf{C}$. According to Proposition 2.42 $\mathcal{V}$ is tight, then. $\square$

The following results are specializations of the results of Ferrer, F. Puerta and X. Puerta [FPP98] on (C, A)-invariant subspaces to the tight case. The presentation follows mainly Fuhrmann and Helmke [FH00].

Theorem 6.2. *$\mathcal{V} \subset \mathcal{F}^n$ is tight (C, A)-invariant of codimension k if and only if there exists an observable pair $(\bar{C}, \bar{A}) \in \mathcal{F}^{p\times(n-k)} \times \mathcal{F}^{(n-k)\times(n-k)}$ of matrices in dual Brunovsky form with $\operatorname{rk} \bar{C} = p$ and observability indices $\lambda = (\lambda_1, \ldots, \lambda_p)$ and a full rank matrix $Z \in \mathcal{F}^{n\times(n-k)}$ with $\mathcal{V} = \operatorname{Im} Z$, such that*

(1) $\mathsf{A}Z = Z\bar{A} + \mathsf{A}Z\bar{C}^\top\bar{C}$

(2) $\mathsf{C}Z = \mathsf{C}Z\bar{C}^\top\bar{C}$

(3) $\mathsf{C}Z\bar{C}^\top \in \mathcal{F}^{p\times p}$ *is invertible.*

Then $(\bar{C}, \bar{A})$ is a matrix representation of a restriction of (C, A) to $\mathcal{V}$ with restriction indices *$\lambda = (\lambda_1, \ldots, \lambda_p)$. It is $\lambda_i \leq \mu_i$ for all $i = 1, \ldots, p$.*

Proof. Let $\mathcal{V}$ be tight of codimension k. According to Proposition 2.52 there exists an output injection matrix J and an observable matrix representation $(\bar{C}, \bar{A}) \in \mathcal{F}^{p\times(n-k)} \times \mathcal{F}^{(n-k)\times(n-k)}$ of the resulting restriction of (C, A) to $\mathcal{V}$ which is in dual Brunovsky form. The matrix $\bar{C}$ consisting of p rows follows from Lemma 6.1. Let Z, S be the matrix representations of the inclusions $i : \mathcal{V} \longrightarrow \mathcal{F}^n$ and $i : \mathsf{C}(\mathcal{V}) \longrightarrow \mathcal{F}^p$ with respect to the associated bases, respectively. According to Lemma 6.1 it is $\operatorname{rk} \bar{C} = \operatorname{rk} S = p$. From the defining diagram (see Section 2.3.3) it follows $(\mathsf{A} - J\mathsf{C})Z = Z\bar{A}$ and

$\mathsf{C}Z = S\bar{C}$. Note that $\bar{C}\bar{C}^\top = I$ since $\bar{C}$ is in dual Brunovsky form. But then $\mathsf{C}Z\bar{C}^\top = S$ has rank p and (3) follows. Furthermore it is

$$\begin{aligned}
\mathsf{A}Z &= Z\bar{A} + J\mathsf{C}Z \\
&= Z\bar{A} + JS\bar{C} \\
&= Z\bar{A} + J\mathsf{C}Z\bar{C}^\top\bar{C} \\
&= Z\bar{A} + J\mathsf{C}Z\bar{C}^\top\bar{C} - \mathsf{A}Z\bar{C}^\top\bar{C} + \mathsf{A}Z\bar{C}^\top\bar{C} \\
&= Z\bar{A} - (\mathsf{A} - J\mathsf{C})Z\bar{C}^\top\bar{C} + \mathsf{A}Z\bar{C}^\top\bar{C} \\
&= Z\bar{A} - Z\bar{A}\bar{C}^\top\bar{C} + \mathsf{A}Z\bar{C}^\top\bar{C} \\
&= Z\bar{A} + \mathsf{A}Z\bar{C}^\top\bar{C}\ ,
\end{aligned}$$

where the last equation follows from $\bar{A}\bar{C}^\top = 0$ (dual Brunovsky form). This yields (1). Equation (2) follows from $\mathsf{C}Z = S\bar{C} = \mathsf{C}Z\bar{C}^\top\bar{C}$. Now let $\lambda = (\lambda_1, \dots, \lambda_p)$ be the observability indices of $(\bar{C}, \bar{A})$. By definition these are the restriction indices of (C, A) with respect to $\mathcal{V}$. From Proposition 2.55 it follows $\lambda_i \le \mu_i$ for all $i = 1, \dots, p$.

Conversely let $(\bar{C}, \bar{A})$ and Z be given as stated. Let $\mathcal{V} = \operatorname{Im} Z \subset \mathcal{F}^n$. Using (1)-(3) it follows

$$\begin{aligned}
\mathsf{A}Z &= Z\bar{A} + \mathsf{A}Z\bar{C}^\top\bar{C} \\
&= Z\bar{A} + \mathsf{A}Z\bar{C}^\top(\mathsf{C}Z\bar{C}^\top)^{-1}\mathsf{C}Z \\
&= Z\bar{A} + J\mathsf{C}Z\ ,
\end{aligned}$$

where $J := \mathsf{A}Z\bar{C}^\top(\mathsf{C}Z\bar{C}^\top)^{-1}$. It follows $(\mathsf{A} - J\mathsf{C})Z = Z\bar{A}$, i.e. $(\mathsf{A} - J\mathsf{C})\mathcal{V} \subset \mathcal{V}$ and $\mathcal{V}$ is (C, A)-invariant. Setting $S := \mathsf{C}Z\bar{C}^\top$, equation (2) yields $\mathsf{C}Z = S\bar{C}$. Together this means that $(\bar{C}, \bar{A})$ is a matrix representation of a restriction of (C, A) to the (C, A)-invariant subspace $\mathcal{V}$ with restriction indices $\lambda = (\lambda_1, \dots, \lambda_p)$ (cf. Section 2.3.3). It follows from Lemma 6.1 that $\mathcal{V}$ is tight. Again, $\lambda_i \le \mu_i$ for all $i = 1, \dots, p$ follows from Proposition 2.55. $\square$

Theorem 6.2 characterizes the image representations (i.e. the bases) of tight (C, A)-invariant subspaces. The next task is to derive a uniqueness result for this kind of representation. Given a fixed tight subspace $\mathcal{V} \subset \mathcal{F}^n$, the pair $(\bar{C}, \bar{A})$ appearing in the characterizing equations is uniquely determined by $\mathcal{V}$. In fact, it is *the* restriction of (C, A) to $\mathcal{V}$, which is in dual Brunovsky canonical form (cf. Section 2.3.3). Concerning the matrix Z there is the following result.

Definition 6.3. Given (C, A) and $(\bar{C}, \bar{A})$ with $\operatorname{rk} \bar{C} = p$ in dual Brunovsky form, i.e. given μ and λ, the set of all full rank matrices $Z \in \mathcal{F}^{n \times (n-k)}$ satisfying equations (1)-(3) of Theorem 6.2 is denoted by $M(\lambda, \mu)$. Set $\Gamma(\lambda) := M(\lambda, \lambda)$.

Theorem 6.4. *Let $Z_1, Z_2 \in M(\lambda, \mu)$. Then $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ if and only if there exists $S \in \Gamma(\lambda)$ such that $Z_2 = Z_1 S^{-1}$. S is uniquely determined. It is*

$$\begin{aligned}\Gamma(\lambda) = \{S \in \operatorname{GL}(\mathcal{F}^{n-k}) \,|\, (W\bar{C}S^{-1}, S(\bar{A} - L\bar{C})S^{-1}) = (\bar{C}, \bar{A})& \\ \textit{for suitable } L \in \mathcal{F}^{(n-k)\times p}, W \in \mathcal{F}^{p \times p},& \\ W \textit{ invertible}\}&\end{aligned}$$

and $\Gamma(\lambda)$ is a subgroup of $\operatorname{GL}(\mathcal{F}^{n-k})$ that acts freely on $M(\lambda, \mu)$ via $Z \mapsto ZS^{-1}$.

Proof. Note that any $S \in \Gamma(\lambda) \subset \mathcal{F}^{(n-k)\times(n-k)}$ has full rank and is hence invertible. Calculating

$$(\bar{C}S^{-1}\bar{C}^\top)(\bar{C}S\bar{C}^\top) = (\bar{C}S^{-1}\bar{C}^\top\bar{C})(S\bar{C}^\top) = \bar{C}S^{-1}S\bar{C}^\top = \bar{C}\bar{C}^\top = I$$

shows that the inverse of $\bar{C}S\bar{C}^\top$ is $\bar{C}S^{-1}\bar{C}^\top$. It is easily checked that $\Gamma(\lambda)$ is indeed a subgroup of $\operatorname{GL}(\mathcal{F}^{n-k})$: Let $S_1, S_2 \in \Gamma(\lambda)$. Then

$$\begin{aligned}\bar{C}S_1S_2 &= \bar{C}S_1\bar{C}^\top\bar{C}S_2 \\ &= \bar{C}S_1\bar{C}^\top\bar{C}S_2\bar{C}^\top\bar{C} \\ &= \bar{C}S_1S_2\bar{C}^\top\bar{C}\end{aligned}$$

and

$$\begin{aligned}\bar{A}S_1S_2 &= S_1\bar{A}S_2 + \bar{A}S_1\bar{C}^\top\bar{C}S_2 \\ &= S_1S_2\bar{A} + S_1\bar{A}S_2\bar{C}^\top\bar{C} + \bar{A}S_1\bar{C}^\top\bar{C}S_2 \\ &= S_1S_2\bar{A} + \bar{A}S_1S_2\bar{C}^\top\bar{C} - \bar{A}S_1\bar{C}^\top\bar{C}S_2\bar{C}^\top\bar{C} + \bar{A}S_1\bar{C}^\top\bar{C}S_2 \\ &= S_1S_2\bar{A} + \bar{A}S_1S_2\bar{C}^\top\bar{C}\ ,\end{aligned}$$

where the last equality follows from $\bar{C}S_2\bar{C}^\top\bar{C} = \bar{C}S_2$. Furthermore it is $\bar{C}S_1S_2\bar{C}^\top = \bar{C}S_1\bar{C}^\top\bar{C}S_2\bar{C}^\top$ and hence is invertible as a product of invertible matrices. It follows $S_1S_2 \in \Gamma(\lambda)$. Next, let $S \in \Gamma(\lambda)$. Then $\bar{C}S = \bar{C}S\bar{C}^\top\bar{C}$

implies $\bar{C} = (\bar{C}S\bar{C}^\top)^{-1}\bar{C}S = \bar{C}S^{-1}\bar{C}^\top\bar{C}S$ and $\bar{C}S^{-1} = \bar{C}S^{-1}\bar{C}^\top\bar{C}$. Multiplying $S\bar{A} = \bar{A}S - \bar{A}S\bar{C}^\top\bar{C}$ by S^{-1} on both sides yields

$$\begin{aligned}\bar{A}S^{-1} &= S^{-1}\bar{A} - S^{-1}\bar{A}S\bar{C}^\top\bar{C}S^{-1}\\ &= S^{-1}\bar{A} - S^{-1}\bar{A}S\bar{C}^\top\bar{C}S^{-1}\bar{C}^\top\bar{C}\\ &= S^{-1}\bar{A} - S^{-1}(\bar{A}S - S\bar{A})S^{-1}\bar{C}^\top\bar{C}\\ &= S^{-1}\bar{A} + \bar{A}S^{-1}\bar{C}^\top\bar{C} - S^{-1}\bar{A}\bar{C}^\top\bar{C}\\ &= S^{-1}\bar{A} + \bar{A}S^{-1}\bar{C}^\top\bar{C}\ ,\end{aligned}$$

where the last equality follows from $\bar{A}\bar{C}^\top = 0$. Furthermore $\bar{C}S^{-1}\bar{C}^\top$ is the inverse of $\bar{C}S\bar{C}^\top$ hence itself invertible. It follows $S^{-1} \in \Gamma(\lambda)$ and $\Gamma(\lambda)$ is a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$.

Now let $Z_1, Z_2 \in M(\lambda,\mu)$. If $Z_2 = Z_1S^{-1}$ for any $S \in \Gamma(\lambda)$ then $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ is clear. Conversely, from $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ it follows $Z_2 = Z_1S^{-1}$ for a suitable $S \in \mathrm{GL}(\mathcal{F}^{n-k})$. Since Z_1 and Z_2 have full rank, respectively, S is uniquely determined. Since $Z_1, Z_2 \in M(\lambda,\mu)$ it follows $\mathsf{C}Z_1S^{-1} = \mathsf{C}Z_2 = \mathsf{C}Z_2\bar{C}^\top\bar{C} = \mathsf{C}Z_1S^{-1}\bar{C}^\top\bar{C} = \mathsf{C}Z_1\bar{C}^\top\bar{C}S^{-1}\bar{C}^\top\bar{C}$ and $\mathsf{C}Z_1S^{-1} = \mathsf{C}Z_1\bar{C}^\top\bar{C}S^{-1}$. Since $\mathsf{C}Z_1\bar{C}^\top$ has full rank these two equations imply $\bar{C}S^{-1} = \bar{C}S^{-1}\bar{C}^\top\bar{C}$. Furthermore

$$\begin{aligned}\mathsf{A}Z_1S^{-1} &= \mathsf{A}Z_2\\ &= Z_2\bar{A} + \mathsf{A}Z_2\bar{C}^t\bar{C}\\ &= Z_1S^{-1}\bar{A} + \mathsf{A}Z_1S^{-1}\bar{C}^\top\bar{C}\\ &= Z_1S^{-1}\bar{A} + Z_1\bar{A}S^{-1}\bar{C}^\top\bar{C} + \mathsf{A}Z_1\bar{C}^\top\bar{C}S^{-1}\bar{C}^\top\bar{C}\\ &= Z_1S^{-1}\bar{A} + Z_1\bar{A}S^{-1}\bar{C}^\top\bar{C} + \mathsf{A}Z_1\bar{C}^\top\bar{C}S^{-1}\end{aligned}$$

and $\mathsf{A}Z_1S^{-1} = Z_1\bar{A}S^{-1} + \mathsf{A}Z_1\bar{C}^\top\bar{C}S^{-1}$. Using that Z_1 has full rank these two equations imply $\bar{A}S^{-1} = S^{-1}\bar{A} + \bar{A}S^{-1}\bar{C}^\top\bar{C}$. Finally, $(\bar{C}S^{-1}\bar{C}^\top)(\bar{C}S\bar{C}^\top) = (\bar{C}S^{-1}\bar{C}^\top\bar{C})(S\bar{C}^\top) = \bar{C}S^{-1}S\bar{C}^\top = \bar{C}\bar{C}^\top = I$ yields that $\bar{C}S^{-1}\bar{C}^\top$ is invertible. It follows $S^{-1} \in \Gamma(\lambda)$ and since $\Gamma(\lambda)$ is a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$ this yields $S \in \Gamma(\lambda)$.

Since any $Z \in M(\lambda,\mu)$ has full rank, $ZS^{-1} = Z$ for $S \in \Gamma(\lambda)$ implies $S = I$ and hence $\Gamma(\lambda)$ acts freely on $M(\lambda,\mu)$.

If $S \in \Gamma(\lambda)$ then Theorem 6.2 yields that $(\bar{C},\bar{A})$ is a matrix representation of a restriction of $(\bar{C},\bar{A})$ to $\mathcal{V} = \operatorname{Im} S$. By the defining diagram (cf. Section 2.3.3) this means that there exist a matrix $L \in \mathcal{F}^{(n-k)\times p}$ and a matrix $W \in \mathcal{F}^{p\times p}$ of full rank such that $S^{-1}(\bar{A} - L\bar{C})S = \bar{A}$ and $W\bar{C}S = \bar{C}$. But then $W^{-1}\bar{C}S^{-1} = \bar{C}$ and $S(\bar{A} - (-S^{-1}LW^{-1})\bar{C})S^{-1} = \bar{A}$ and S is contained in the set stated in the theorem.

Conversely, let $S \in \mathrm{GL}(\mathcal{F}^{n-k})$ and let here exist matrices $L \in \mathcal{F}^{(n-k)\times p}$ and $W \in \mathcal{F}^{p\times p}$, W of full rank, such that $S(\bar{A} - L\bar{C})S^{-1} = \bar{A}$ and $W\bar{C}S^{-1} = \bar{C}$. Then $\bar{C}S^{-1}\bar{C}^\top = W^{-1}\bar{C}\bar{C}^\top = W^{-1}$ implies that $\bar{C}S^{-1}\bar{C}^\top$ is invertible and that $\bar{C}S^{-1}\bar{C}^\top\bar{C} = W^{-1}\bar{C} = \bar{C}S^{-1}$. Furthermore,

$$\begin{aligned}
\bar{A}S^{-1} &= S^{-1}\bar{A} + L\bar{C}S^{-1} \\
&= S^{-1}\bar{A} + L\bar{C}S^{-1}\bar{C}^\top\bar{C} \\
&= S^{-1}\bar{A} + L\bar{C}S^{-1}\bar{C}^\top\bar{C} - \bar{A}S^{-1}\bar{C}^\top\bar{C} + \bar{A}S^{-1}\bar{C}^\top\bar{C} \\
&= S^{-1}\bar{A} - (\bar{A} - L\bar{C})S^{-1}\bar{C}^\top\bar{C} + \bar{A}S^{-1}\bar{C}^\top\bar{C} \\
&= S^{-1}\bar{A} - S^{-1}\bar{A}\bar{C}^\top\bar{C} + \bar{A}S^{-1}\bar{C}^\top\bar{C} \\
&= S^{-1}\bar{A} + \bar{A}S^{-1}\bar{C}^\top\bar{C}\ ,
\end{aligned}$$

and hence $S^{-1} \in \Gamma(\lambda)$. Since $\Gamma(\lambda)$ is a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$ this yields $S \in \Gamma(\lambda)$. $\square$

Theorem 6.4 shows that $\Gamma(\lambda)$ measures the degree of uniqueness of image representations of tight (C, A)-invariant subspaces. The structure of $\Gamma(\lambda)$ (being the state space similarity transformations occuring in the *stabilizer subgroup* of output injection transformations, cf. Section 2.3.2) is well known and has been determined e.g. by Münzner and Prätzel-Wolters [MPW78] or by Fuhrmann and Willems [FW79]. The description of $\Gamma(\lambda)$ in terms of equations (1)-(3) in Theorem 6.2 is due to Ferrer, F. Puerta and X. Puerta [FPP98].

The characterization of image representations (i.e. of the bases) of tight (C, A)-invariant subspaces in terms of equations (1)-(3) of Theorem 6.2 allows to specify the structure of these bases explicitely. This is done by solving the equations (1) and (2) for Z. The details are carried out in the paper by Ferrer, F. Puerta and X. Puerta [FPP98]. The result is the following description of *block Toeplitz* type. Of course, the same technique applies to $\Gamma(\lambda) = M(\lambda, \lambda)$, resulting in a description of the state space similarity transformations occuring in the stabilizer subgroup of output injection transformations. In a recent preprint F. Puerta, X. Puerta and I. Zaballa [PPZ00] use this description to construct overlapping coordinate charts for the manifold $M(\lambda, \mu)/\Gamma(\lambda)$ via parameter elimination.

Theorem 6.5. *$\mathcal{V}$ is tight (C,A)-invariant of codimension k with restriction indices λ if and only if $\mathcal{V} = \operatorname{Im} Z$ with*

$$Z = \begin{pmatrix} Z_{11} & \dots & Z_{1p} \\ \vdots & & \vdots \\ Z_{p1} & \dots & Z_{pp} \end{pmatrix} ,$$

where each block $Z_{ij} \in \mathcal{F}^{\mu_i \times \lambda_j}$, $i, j = 1, \dots, p$, is of the form

$$Z_{ij} = \begin{cases} \begin{pmatrix} z_1^{ij} & 0 & \dots & 0 \\ z_2^{ij} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{\mu_i-\lambda_j}^{ij} & & \ddots & z_1^{ij} \\ z_{\mu_i-\lambda_j+1}^{ij} & \ddots & & z_2^{ij} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{\mu_i-\lambda_j}^{ij} \\ 0 & \dots & 0 & z_{\mu_i-\lambda_j+1}^{ij} \end{pmatrix} & , \ \lambda_j \leq \mu_i \\ \\ \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} & , \ \lambda_j > \mu_i \end{cases}$$

and $\mathsf{C} Z \bar{C}^\top$ (i.e. the matrix consisting of the lower right corner entries of each block) is invertible. $M(\lambda, \mu)$ consists exactly of all matrices of this structure.

Note that $\mathsf{C} Z \bar{C}^\top$ being invertible implies $\lambda_i \leq \mu_i$, $i = 1, \dots, p$. The following theorem exploits the results reached so far.

Theorem 6.6. *Let $\mathcal{T}_k(\lambda, \mu) \subset \mathcal{T}_k(\mathsf{C}, \mathsf{A})$ denote the set of codimension k tight subspaces with restriction indices λ* (Brunovsky stratum). *$\mathcal{T}_k(\lambda, \mu)$ is nonempty if and only if $\lambda_i \leq \mu_i$, $i = 1, \dots, p$. Then $\mathcal{T}_k(\lambda, \mu)$ is a smooth submanifold of $\mathrm{G}_{n-k}(\mathcal{F}^n)$ which is diffeomorphic to the orbit space $M(\lambda, \mu)/\Gamma(\lambda)$ via $Z \cdot \Gamma(\lambda) \mapsto \operatorname{Im} Z$.*

Proof. $\mathcal{T}_k(\lambda, \mu)$ being nonempty if and only if $\lambda_i \leq \mu_i$, $i = 1, \dots, p$, follows from Theorem 6.5. The set $\mathcal{T}_k(\lambda, \mu)$ being a smooth manifold diffeomorphic to $M(\lambda, \mu)/\Gamma(\lambda)$ via $Z \cdot \Gamma(\lambda) \mapsto \operatorname{Im} Z$ is an immediate consequence of Theorem 6.2 and Theorem 6.4 (cf. Appendix A for the notion of orbit space). $\mathcal{T}_k(\lambda, \mu)$ being a submanifold of $\mathrm{G}_{n-k}(\mathcal{F}^n)$ follows from a local section argument which is carried out in Ferrer, F. Puerta and X. Puerta [FPP00]. $\square$

A slight modification of the above arguments leads to the following new results. See Section 2.3.2 for an explanation of the dual Kronecker canonical form.

Theorem 6.7. *$\mathcal{V} \subset \mathcal{F}^n$ is tight (C, A)-invariant of codimension k if and only if there exists an observable pair $(\bar{C}, \bar{A}) \in \mathcal{F}^{p\times(n-k)} \times \mathcal{F}^{(n-k)\times(n-k)}$ of matrices in dual Kronecker form with $\operatorname{rk} \bar{C} = p$ and Kronecker indices $K = (K_1, \ldots, K_p)$ and a full rank matrix $Z \in \mathcal{F}^{n\times(n-k)}$ with $\mathcal{V} = \operatorname{Im} Z$, such that*

(1) $\mathsf{A}Z = Z\bar{A} + \mathsf{A}Z\bar{C}^\top\bar{C}$

(2) $\mathsf{C}Z = \mathsf{C}Z\bar{C}^\top\bar{C}$

(3) $\mathsf{C}Z\bar{C}^\top \in \mathcal{F}^{p\times p}$ *is unipotent lower triangular.*

Then $(\bar{C}, \bar{A})$ is a restriction of (C, A) to $\mathcal{V}$ with restriction indices $\lambda = (\lambda_1, \ldots, \lambda_p) = (K_{\pi^{-1}(1)}, \ldots, K_{\pi^{-1}(p)})$, where $\pi^{-1} \in S(p)$ is a permutation that orders the K_i, $i = 1, \ldots, p$, decreasingly. It is $\lambda_i \leq \mu_i$ for all $i = 1, \ldots, p$.

Proof. Let $\mathcal{V}$ be tight of codimension k. Let the observable pair $(\bar{C}_0, \bar{A}_0) \in \mathcal{F}^{p\times(n-k)} \times \mathcal{F}^{(n-k)\times(n-k)}$ be any matrix representation of any restriction of (C, A) to $\mathcal{V}$. Let Z_0, S_0 be the matrix representations of the inclusions $i : \mathcal{V} \longrightarrow \mathcal{F}^n$ and $i : \mathsf{C}(\mathcal{V}) \longrightarrow \mathcal{F}^p$ with respect to the associated bases, respectively. According to Lemma 6.1 it is $\operatorname{rk} \bar{C}_0 = \operatorname{rk} S_0 = p$. According to Proposition 2.54 the pair $(\bar{C}_1, \bar{A}_1) = (S_0\bar{C}_0, \bar{A}_0)$ is another matrix representation of a restriction of (C, A) to $\mathcal{V}$. $Z_1 := Z_0$ and $S_1 := I$ are the matrix representations of the inclusions $i : \mathcal{V} \longrightarrow \mathcal{F}^n$ and $i : \mathsf{C}(\mathcal{V}) \longrightarrow \mathcal{F}^p$ with respect to the (new) associated bases, respectively. According to Propositions 2.46 and 2.54 there are matrices $\bar{J} \in \mathcal{F}^{(n-k)\times p}$, $T \in \mathrm{GL}(\mathcal{F}^{n-k})$ and $U \in \mathcal{F}^{p\times p}$ unipotent lower triangular such that $(\bar{C}, \bar{A}) = (U\bar{C}_1T^{-1}, T(\bar{A}_1 - \bar{J}\bar{C}_1)T^{-1})$ is a matrix representation of a restriction of (C, A) to $\mathcal{V}$ which is in dual Kronecker canonical form. $Z := Z_1T^{-1}$ and $S := U^{-1}$ are the matrix representations of the inclusions $i : \mathcal{V} \longrightarrow \mathcal{F}^n$ and $i : \mathsf{C}(\mathcal{V}) \longrightarrow \mathcal{F}^p$ with respect to the (new) associated bases, respectively. Note that S is unipotent lower triangular. Note further that $\bar{C}\bar{C}^\top = I$ and $\bar{A}\bar{C}^\top = 0$ are also valid for a pair in dual Kronecker form. Then (1)-(3) follow as in the proof of Theorem 6.2. Let $K = (K_1, \ldots, K_p)$ be the Kronecker indices of $(\bar{C}, \bar{A})$. According to Proposition 2.47 $\lambda = (\lambda_1, \ldots, \lambda_p) = (K_{\pi^{-1}(1)}, \ldots, K_{\pi^{-1}(p)})$, where $\pi^{-1} \in S(p)$ is a permutation that orders the K_i, $i = 1, \ldots, p$, decreasingly, are the observability indices of $(\bar{C}, \bar{A})$. By definition these are the restriction indices of

(C,A) with respect to $\mathcal{V}$. $\lambda_i \le \mu_i$ for all $i = 1,\dots,p$ follows from Proposition 2.55.

Conversely let $(\bar{C},\bar{A})$ and Z be given as stated. Let $\mathcal{V} = \operatorname{Im} Z \subset \mathcal{F}^n$. It follows as in the proof of Theorem 6.2 that $(\bar{C},\bar{A})$ is a matrix representation of a restriction of (C,A) to the (C,A)-invariant subspace $\mathcal{V}$ with restriction indices $\lambda = (\lambda_1,\dots,\lambda_p) = (K_{\pi^{-1}(1)},\dots,K_{\pi^{-1}(p)})$, where $\pi^{-1} \in S(p)$ is a permutation that orders the K_i, $i = 1,\dots,p$, decreasingly (cf. Proposition 2.47 and Section 2.3.3). Again, $\lambda_i \le \mu_i$ for all $i = 1,\dots,p$ follows from Proposition 2.55. $\square$

Again the pair $(\bar{C},\bar{A})$ appearing in the characterizing equations is uniquely determined by $\mathcal{V} \subset \mathcal{F}^n$ as the following proposition shows.

Definition 6.8. Given (C,A) and $(\bar{C},\bar{A})$ with $\operatorname{rk}\bar{C} = p$, i.e. given μ and K, the set of all full rank matrices $Z \in \mathcal{F}^{n\times(n-k)}$ satisfying equations (1)-(3) of Theorem 6.7 is denoted by $M(K,\mu)$. Set $\Gamma(K) := M(K,K)$.

Proposition 6.9. *Let $Z \in M(K,\mu)$ and $Z' \in M(K',\mu)$ with $\mathcal{V} := \operatorname{Im} Z = \operatorname{Im} Z'$. Then $K = K'$.*

Proof. Since $\operatorname{Im} Z = \operatorname{Im} Z'$ there exists $S \in \mathrm{GL}(\mathcal{F}^{n-k})$ such that $Z' = ZS^{-1}$. From $\mathsf{C}Z'\bar{C}'^\top\bar{C}' = \mathsf{C}Z'$ and $\mathsf{C}Z\bar{C}^\top\bar{C} = \mathsf{C}Z$ it follows $(\mathsf{C}Z'\bar{C}'^\top)\bar{C}' = \mathsf{C}Z' = \mathsf{C}ZS^{-1} = (\mathsf{C}Z\bar{C}^\top)\bar{C}S^{-1}$ and hence $\bar{C} = U^{-1}\bar{C}'S$, where $U^{-1} := (\mathsf{C}Z\bar{C}^\top)^{-1}(\mathsf{C}Z'\bar{C}'^\top)$ is unipotent lower triangular.

Now $\mathsf{A}Z = Z\bar{A} + \mathsf{A}Z\bar{C}^\top\bar{C}$ implies $\mathsf{A}Z' = \mathsf{A}ZS^{-1} = Z\bar{A}S^{-1} + \mathsf{A}Z\bar{C}^\top\bar{C}S^{-1}$. On the other hand

$$\begin{aligned}
\mathsf{A}Z' &= Z'\bar{A}' + \mathsf{A}Z'\bar{C}'^\top\bar{C}' \\
&= ZS^{-1}\bar{A}' + \mathsf{A}ZS^{-1}\bar{C}'^\top\bar{C}' \\
&= ZS^{-1}\bar{A}' + Z\bar{A}S^{-1}\bar{C}'^\top\bar{C}' + \mathsf{A}Z\bar{C}^\top\bar{C}S^{-1}\bar{C}'^\top\bar{C}' \; .
\end{aligned}$$

But then

$$\begin{aligned}
\mathsf{A}Z\bar{C}^\top\bar{C}S^{-1}\bar{C}'^\top\bar{C}' &= \mathsf{A}Z\bar{C}^\top U^{-1}\bar{C}'\bar{C}'^\top\bar{C}' \\
&= \mathsf{A}Z\bar{C}^\top U^{-1}\bar{C}' \\
&= \mathsf{A}Z\bar{C}^\top\bar{C}S^{-1}
\end{aligned}$$

implies $Z\bar{A}S^{-1} = ZS^{-1}\bar{A}' + Z\bar{A}S^{-1}\bar{C}'^\top\bar{C}'$ and hence $Z\bar{A} = ZS^{-1}(\bar{A}' - J\bar{C}')S$, where $J := -S\bar{A}S^{-1}\bar{C}'^\top$. Since Z has full rank this implies $\bar{A} = S^{-1}(\bar{A}' - J\bar{C}')S$.

Together it follows that $(\bar{C}, \bar{A})$ and $(\bar{C}', \bar{A}')$ are restricted output injection equivalent. Since both pairs are in dual Kronecker canonical form this implies $(\bar{C}, \bar{A}) = (\bar{C}', \bar{A}')$ and hence $K = K'$. □

As a consequence of Proposition 6.9 it is possible to define the *Kronecker indices* of a tight subspace $\mathcal{V} \subset \mathcal{F}^n$ to be the Kronecker indices of *the* pair $(\bar{C}, \bar{A})$ appearing in equations (1)-(3) of Theorem 6.7. Concerning the matrix Z there is the following result.

Theorem 6.10. *Let $Z_1, Z_2 \in M(K, \mu)$. Then $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ if and only if there exists $S \in \Gamma(K)$ such that $Z_2 = Z_1 S^{-1}$. S is uniquely determined. It is*

$$\begin{aligned}\Gamma(K) = \{S \in \mathrm{GL}(\mathcal{F}^{n-k}) \mid (U\bar{C}S^{-1}, S(\bar{A} - L\bar{C})S^{-1}) = (\bar{C}, \bar{A})\\ \textit{for suitable } L \in \mathcal{F}^{(n-k)\times p}, U \in \mathcal{F}^{p\times p},\\ U \textit{ unipotent lower triangular}\}\end{aligned}$$

and $\Gamma(K)$ is a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$ that acts freely on $M(K, \mu)$ via $Z \mapsto ZS^{-1}$.

Proof. $\Gamma(K)$ being a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$ follows as in the proof of Theorem 6.4 (use the fact that the unipotent lower triangular matrices form a group).

Now let $Z_1, Z_2 \in M(K, \mu)$. If $Z_2 = Z_1 S^{-1}$ for any $S \in \Gamma(K)$ then $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ is clear. Conversely, from $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ it follows as in the proof of Theorem 6.4 that $Z_2 = Z_1 S^{-1}$ for a uniquely defined $S^{-1} \in \mathrm{GL}(\mathcal{F}^{n-k})$ which satisfies equations (1) and (2) of Theorem 6.7. Since $\mathsf{C} Z_1 \bar{C}^\top$ and $\mathsf{C} Z_2 \bar{C}^\top = \mathsf{C} Z_1 S^{-1} \bar{C}^\top$ are both unipotent lower triangular, from $\mathsf{C} Z_1 = \mathsf{C} Z_1 \bar{C}^\top \bar{C}$ and hence $\mathsf{C} Z_1 S^{-1} \bar{C}^\top = (\mathsf{C} Z_1 \bar{C}^\top)(\bar{C} S^{-1} \bar{C}^\top)$ it follows that $\bar{C} S^{-1} \bar{C}^\top$ is unipotent lower triangular. Hence $S^{-1} \in \Gamma(K)$ and since $\Gamma(K)$ is a subgroup of $\mathrm{GL}(\mathcal{F}^{n-k})$ this yields $S \in \Gamma(K)$.

Since any $Z \in M(K, \mu)$ has full rank, $ZS^{-1} = Z$ for $S \in \Gamma(K)$ implies $S = I$ and hence $\Gamma(K)$ acts freely on $M(K, \mu)$. The characterization of $\Gamma(K)$ stated in the Theorem follows as in the proof of Theorem 6.4 (use the identity $\bar{C} S \bar{C}^\top = U$). □

Since the *block Toeplitz* type structure of the matrices in $M(\lambda, \mu)$ already follows from equations (1) and (2) of Theorem 6.2 (which are the same as equations (1) and (2) of Theorem 6.7), there is the following analogous description of matrices in $M(K, \mu)$. The condition $K_i \leq \mu_i$ for all $i = 1, \dots, p$

follows directly from the structure of the matrices and from the requirement that $\mathsf{C}Z\bar{C}^\top$ (i.e. the matrix consisting of the lower right corner entries of each block) has to be unipotent. Note again that this description covers also $\Gamma(K) = M(K, K)$.

Theorem 6.11. *$\mathcal{V}$ is tight (C, A)-invariant of codimension k with Kronecker indices K if and only if $\mathcal{V} = \operatorname{Im} Z$ with*

$$Z = \begin{pmatrix} Z_{11} & \dots & Z_{1p} \\ \vdots & & \vdots \\ Z_{p1} & \dots & Z_{pp} \end{pmatrix},$$

where each block $Z_{ij} \in \mathcal{F}^{\mu_i \times K_j}$, $i, j = 1, \dots, p$, is of the form

$$Z_{ij} = \begin{cases} \begin{pmatrix} z_1^{ij} & 0 & \dots & 0 \\ z_2^{ij} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{\mu_i - K_j}^{ij} & & \ddots & z_1^{ij} \\ z_{\mu_i - K_j + 1}^{ij} & \ddots & & z_2^{ij} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{\mu_i - K_j}^{ij} \\ 0 & \dots & 0 & z_{\mu_i - K_j + 1}^{ij} \end{pmatrix} & , \ K_j \leq \mu_i \\ \\ \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} & , \ K_j > \mu_i \end{cases}$$

and $\mathsf{C}Z\bar{C}^\top$ (i.e. the matrix consisting of the lower right corner entries of each block) is unipotent lower triangular. Furthermore it is $K_i \leq \mu_i$ for all $i = 1, \dots, p$. $M(K, \mu)$ consists exactly of all matrices of this structure.

As before, the results reached so far yield the following manifold structure.

Theorem 6.12. *Let $\mathcal{T}_k(K, \mu) \subset \mathcal{T}_k(\mathsf{C}, \mathsf{A})$ denote the set of codimension k tight subspaces with Kronecker indices K* (Kronecker stratum). *$\mathcal{T}_k(K, \mu)$ is nonempty if and only if $K_i \leq \mu_i$ and $\lambda_i \leq \mu_i$, $i = 1, \dots, p$. Here $\lambda_i = K_{\pi^{-1}(i)}$, $i = 1, \dots, p$, as in Theorem 6.7. Then $\mathcal{T}_k(K, \mu)$ is a smooth submanifold of $\mathrm{G}_{n-k}(\mathcal{F}^n)$ which is diffeomorphic to the orbit space $M(K, \mu)/\Gamma(K)$ via $Z \cdot \Gamma(K) \mapsto \operatorname{Im} Z$.*

The preceding results are used in Section 6.3 to obtain a Bruhat type cell decomposition of $\mathcal{T}_k(\lambda, \mu)$. The order presented in the following section will be the adherence order of this decomposition.

6.2 Orders on combinations

In this section the relation between the Kronecker order, the Bruhat order and the dominance order on a subset of the set of combinations $K_{n,m}$ is explored.

Let

$$K_{n,m} = \{(K_1, \ldots, K_m) \in \mathbb{N}^m \mid K_1 + \cdots + K_m = n\}$$

be the set of all *combinations* of m natural numbers with sum n. To each element $K = (K_1, \ldots, K_m) \in K_{n,m}$ one associates the set

$$Y_K = \{(i, j) \in \mathbb{N}^2 \mid 1 \le j \le m \text{ and } 1 \le i \le K_j\} \ ,$$

which can be visualized by a *Young diagram* as shown in Figure 6.1

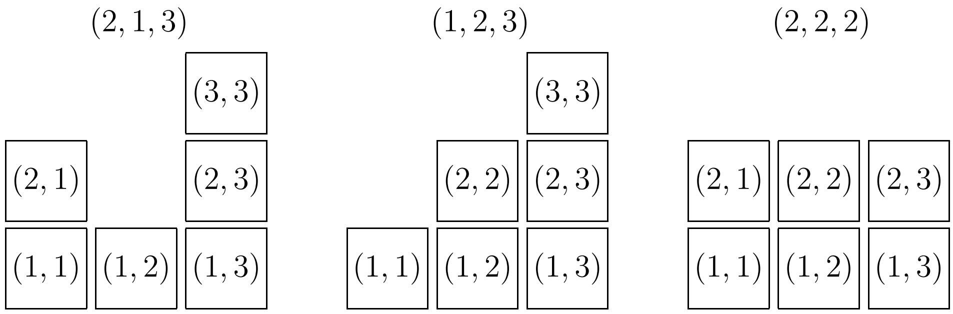

Figure 6.1: Young diagrams visualizing combinations

Since $K_1 + \cdots + K_m = n$ holds, Y_K consists of exactly n pairs, say $(i_1, j_1) < \cdots < (i_n, j_n)$, where $<$ denotes the *lexicographic order* on $\mathbb{N}^2$, i.e.

$$(i_1, j_1) \le (i_2, j_2) \iff \begin{cases} i_1 < i_2 \text{ or} \\ i_1 = i_2 \text{ and } j_1 \le j_2 \end{cases} .$$

Let

$$p(K) = ((i_1, j_1), \ldots, (i_n, j_n))$$

be the *position* of K, e.g. $p(K) = ((1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 3))$ for $K = (2, 1, 3)$ (see Figure 6.1). The lexicographic order on $\mathbb{N}^2$ induces the following product order on the set of all positions of combinations:

$$p(L) \le_{\text{Lex}} p(K) \iff (i'_\alpha, j'_\alpha) \le (i_\alpha, j_\alpha) \ , \quad \alpha = 1, \ldots, n \ ,$$

where $p(L) = ((i'_1, j'_1), \ldots, (i'_n, j'_n))$.

Definition 6.13. The *Kronecker order* on $K_{n,m}$ is the partial order defined by

$$K \leq_{\mathrm{Kro}} L \iff p(L) \leq_{\mathrm{Lex}} p(K)$$

for all $K, L \in K_{n,m}$.

In [Hel82] the *covers* of this partial order are characterized (see also [Hel85]). See e.g. the textbook by Aigner [Aig79] for an explanation of the concept of covers.

Definition 6.14. Let $1 \leq i < j \leq m$. The *transposition* $\bar{K} = T_{ij}K \in K_{n,m}$ of the combination $K \in K_{n,m}$ is obtained in the following way:

(1) If $K_i < K_j$ then

$$\begin{aligned} \bar{K}_i &= K_j \ , \\ \bar{K}_j &= K_i \ , \\ \bar{K}_\alpha &= K_\alpha \ , \quad \alpha \notin \{i,j\} \ . \end{aligned}$$

(2) If $K_i \in \{K_j, K_j + 1\}$ then $\bar{K} = K$.

(3) If $K_i > K_j + 1$ then

$$\begin{aligned} \bar{K}_i &= K_j + 1 \ , \\ \bar{K}_j &= K_i - 1 \ , \\ \bar{K}_\alpha &= K_\alpha \ , \quad \alpha \notin \{i,j\} \ . \end{aligned}$$

It can be shown, that the covers of a combination K relative to the Kronecker order are the transpositions $T_{ij}K$ which are minimal in a certain sense (see the above references for details). The exact characterization of covers is not needed here, but all covers being transpositions yields the following important property of the Kronecker order on $K_{n,m}$.

Proposition 6.15. *Let $K, L \in K_{n,m}$. If $K \leq_{\mathrm{Kro}} L$ then there exists a sequence $(T_{i_\alpha j_\alpha})_{\alpha=1}^{l}$ of transpositions such that $L = T_{i_l j_l} \dots T_{i_1 j_1} K$.*

Now fix an element $\kappa = (\kappa_1, \dots, \kappa_m) \in K_{n,m}$ with $\kappa_1 \geq \cdots \geq \kappa_m$. The subset of $K_{n,m}$ under consideration is the $S(m)$ orbit

$$S(m) \cdot \kappa = \{(\kappa_{\pi^{-1}(1)}, \dots, \kappa_{\pi^{-1}(m)}) \mid \pi \in S(m)\}$$

of κ in $K_{n,m}$. To get a non redundant description of $S(m)\cdot\kappa$, permutations π and π' of κ leading to the same K must be identified. Usually this is done in the following way (cf. Hiller [Hil82]). Let $\kappa_i^* := \#\{j \in \{1,\dots,m\} \mid \kappa_j \geq i\}$, $i = 1,\dots,\kappa_1$, be the *conjugate indices* of κ. Then $m = \kappa_1^* \geq \cdots \geq \kappa_{\kappa_1}^* \geq 1$. Consider the subgroup $S_\kappa(m)$ of $S(m)$ generated by the set $\{(i,i+1) \mid 1 \leq i < m,\ i \notin \{\kappa_1^*,\dots,\kappa_{\kappa_1}^*\}\}$ consisting of elementary transpositions. Then $\pi\cdot\kappa = \pi'\cdot\kappa$ if and only if there exists $\tau \in S_\kappa(m)$ such that $\pi' = \pi\circ\tau$. Hence the coset space $S(m)/S_\kappa(m)$ is the right space to work with.

Now consider the *length function* $l(\pi) = \sum_{j=1}^{m-1} \#\{i > j \mid \pi(i) < \pi(j)\}$ for $\pi \in S(m)$. It is well known, that each coset $\pi\circ S_\kappa(m)$, $\pi \in S(m)$, contains a unique element of minimal length [Hil82, Corollary I.5.4]. Let $S^\kappa(m)$ denote the set of all these minimal length coset representatives. Any $\pi \in S(m)$ has a unique decomposition $\pi = \sigma\circ\tau$ with $\sigma \in S^\kappa(m)$ and $\tau \in S_\kappa(m)$. The set $S^\kappa(m)$ is clearly isomorphic to the coset space $S(m)/S_\kappa(m)$, hence $S(m)\cdot\kappa = S^\kappa(m)\cdot\kappa$ and the latter is the required non redundant description.

The *Bruhat order* on $S(m)$ (see e.g. Deodhar [Deo77] for various characterizations) induces an order on $S(m)\cdot\kappa = S^\kappa(m)\cdot\kappa$ via

$$\pi\cdot\kappa \leq_{\mathrm{Bru}} \pi'\cdot\kappa \iff \exists_{\tau,\tau'\in S_\kappa(m)}\ \pi\circ\tau \leq \pi'\circ\tau' .$$

According to the previous considerations this is well defined. There is the following strong relation between the Kronecker order and the Bruhat order on $S(m)\cdot\kappa$.

Theorem 6.16. *The Kronecker order on $K_{n,m}$ restricts to the reverse Bruhat order on $S(m)\cdot\kappa$, i.e.*

$$K \leq_{\mathrm{Kro}} L \iff L \leq_{\mathrm{Bru}} K$$

for all $K, L \in S(m)\cdot\kappa$.

The first step in the proof of Theorem 6.16 is to refine Proposition 6.15. This is done in Proposition 6.19 using the following lemmas.

Lemma 6.17. *A sequence of transpositions containing a transposition of type* (3) *must leave $S(m)\cdot\kappa$.*

Proof. For a combination $K \in K_{n,m}$ let

$$r_\nu(K) = \#\{\alpha \in \{1,\dots,m\} \mid K_\alpha \geq \nu\}\ , \quad \nu \in \mathbb{N}$$

be the conjugate partition (# means cardinality). Observe that $K \in S(m)\cdot\kappa$ if and only if $r_\nu(K) = r_\nu(\kappa)$ for all $\nu \in \mathbb{N}$ and $r_\nu(K) = 0$ for $\nu > \kappa_1$. Transpositions of type (1) and (2) leave the conjugate indices invariant hence stay on the same $S(m)$ orbit. A transposition T_{ij} of type (3) has the following effect on the conjugate indices: In the case $K_i > K_j + 2$ the conjugate indices of $\bar{K} = T_{ij}K$ are

$$\begin{aligned}
r_{K_j}(\bar{K}) &= r_{K_j}(K) - 1 \\
r_{K_j+1}(\bar{K}) &= r_{K_j+1}(K) + 1 \\
r_{K_i-1}(\bar{K}) &= r_{K_i-1}(K) + 1 \\
r_{K_i}(\bar{K}) &= r_{K_i}(K) - 1 \\
r_\nu(\bar{K}) &= r_\nu(K)\ , \quad \nu \notin \{K_j, K_j+1, K_i-1, K_i\}\ ,
\end{aligned}$$

in the case $K_i = K_j + 2$ they are

$$\begin{aligned}
r_{K_j}(\bar{K}) &= r_{K_j}(K) - 1 \\
r_{K_j+1}(\bar{K}) &= r_{K_j+1}(K) + 2 \\
r_{K_i}(\bar{K}) &= r_{K_i}(K) - 1 \\
r_\nu(\bar{K}) &= r_\nu(K)\ , \quad \nu \notin \{K_j, K_j+1, K_i\}\ .
\end{aligned}$$

Nevertheless, the conjugate indices are decreased in the lexicographic order hence $\bar{K}$ and K do not lie on the same $S(m)$ orbit. Furthermore a sequence of transpositions containing any positive number of transpositions of type (3) decreases the conjugate indices in the lexicographic order and must hence leave $S(m)\cdot\kappa$. $\square$

Lemma 6.18. *Let $K \in S(m)\cdot\kappa$ and let T_{ij} be a transposition of type* (1). *Then $K \leq_{\mathrm{Kro}} T_{ij}K$.*

Proof. If the position of K is

$$\begin{aligned}
p(K) = (&(1,1), \dots, (K_i, i), \dots, (K_i, j), \dots, \\
&(K_i+1, l), (K_i+1, m), \dots, (K_i+1, j), \dots)
\end{aligned}$$

with $l < i < m < j$, then the position of $\bar{K} := T_{ij}K$ is

$$\begin{aligned}
p(\bar{K}) = (&(1,1), \dots, (K_i, i), \dots, (K_i, j), \dots, \\
&(K_i+1, l), (K_i+1, i), (K_i+1, m), \dots)\ .
\end{aligned}$$

It follows $p(\bar{K}) <_{\mathrm{Lex}} p(K)$ and hence $K \leq_{\mathrm{Kro}} \bar{K}$. $\square$

Proposition 6.19. *Let $K, L \in S(m) \cdot \kappa$. Then $K \leq_{\mathrm{Kro}} L$ if and only if there exists a sequence $(T_{i_\alpha j_\alpha})_{\alpha=1}^{l}$ of transpositions of type (1) such that $L = T_{i_l j_l} \ldots T_{i_1 j_1} K$.*

Proof. Let $K \leq_{\mathrm{Kro}} L$. According to Proposition 6.15 there exists a sequence $(T_{i_\alpha j_\alpha})_{\alpha=1}^{l}$ of transpositions such that $L = T_{i_l j_l} \ldots T_{i_1 j_1} K$. Since K and L lie on the same $S(m)$ orbit this series can't contain transpositions of type (3) (Lemma 6.17). Transpositions of type (2) are the identity hence can be left out.

Conversely let $L = T_{i_l j_l} \ldots T_{i_1 j_1} K$ with transpositions $T_{i_\alpha j_\alpha}$, $\alpha = 1, \ldots, l$, of type (1). Using Lemma 6.18 repeatedly yields $K \leq_{\mathrm{Kro}} T_{i_1 j_1} K \leq_{\mathrm{Kro}} \ldots \leq_{\mathrm{Kro}} T_{i_l j_l} \ldots T_{i_1 j_1} K = L$. □

The following nice characterization of the Bruhat order on $S(m)$ can be found in a textbook by Fulton on Young diagrams ([Ful97, §10.5, Corollary 1]).

Proposition 6.20. *Let $\pi, \tau \in S(m)$. Then $\pi \leq \tau$ if and only if there exists a sequence $((i_\alpha, j_\alpha))_{\alpha=1}^{l}$ of transpositions with $i_\alpha < j_\alpha$ for all $\alpha = 1, \ldots, l$, such that with $\tau_0 := \tau$ and $\tau_\alpha := \tau \cdot (i_1, j_1) \cdot \cdots \cdot (i_\alpha, j_\alpha)$, $\alpha = 1, \ldots, l$, it is $\tau_{\alpha-1}(i_\alpha) > \tau_{\alpha-1}(j_\alpha)$ for all $\alpha = 1, \ldots, l$ and $\tau_l = \pi$.*

Now the proof of Theorem 6.16 is straight forward using the following lemmas.

Lemma 6.21. *Let $K \in S(m) \cdot \kappa$ and let T_{ij} be a transposition of type (1). Then $T_{ij} K \leq_{\mathrm{Bru}} K$.*

Proof. Since T_{ij} is a transposition of type (1) it is $i < j$ and $K_i < K_j$. Let $K = \pi \cdot \kappa$. Then $l := \pi^{-1}(i) > \pi^{-1}(j) =: m$ since κ is ordered with $\kappa_1 \geq \cdots \geq \kappa_m$. So it is $m < l$ and $j = \pi(m) > \pi(l) = i$. According to Proposition 6.20 it is $\pi \cdot (m, l) \leq \pi$. It follows $(i, j)\pi = \pi(m, l) \leq \pi$ and $T_{ij} K \leq_{\mathrm{Bru}} K$. □

Lemma 6.22. *Let $i < j$ and let $\pi \in S(m)$ such that $\pi(i) > \pi(j)$. Then $\pi \cdot \kappa \leq_{\mathrm{Kro}} \pi \cdot (i, j) \cdot \kappa$.*

Proof. Let $K := \pi \cdot \kappa$ and let $l := \pi(i)$ and $m := \pi(j)$. Then $m < l$ and $(m, l) \cdot \pi = \pi \cdot (i, j)$. Since $K_l = \kappa_i$ and $K_m = \kappa_j$ it follows $K_m < K_l$ ($i < j$ and κ is ordered with $\kappa_1 \geq \cdots \geq \kappa_m$). Hence $\pi \cdot (i, j) \cdot \kappa = (m, l) \cdot \pi \cdot \kappa = T_{ml} K$ and T_{ml} is a transposition of type (1). But then Proposition 6.19 yields $\pi \cdot \kappa = K \leq_{\mathrm{Kro}} T_{ml} K = \pi \cdot (i, j) \cdot \kappa$. □

Proof of Theorem 6.16. Let $K \leq_{\mathrm{Kro}} L$. According to Proposition 6.19 there exists a sequence $(T_{i_\alpha j_\alpha})_{\alpha=1}^{l}$ of transpositions of type (1) such that $L = T_{i_l j_l} \dots T_{i_1 j_1} K$. Using Lemma 6.21 repeatedly yields $L = T_{i_l j_l} \dots T_{i_1 j_1} K \leq_{\mathrm{Bru}} \dots \leq_{\mathrm{Bru}} T_{i_1 j_1} K \leq_{\mathrm{Bru}} K$.

Let $\pi \cdot \kappa = L \leq_{\mathrm{Bru}} K = \tau \cdot \kappa$. According to Proposition 6.20 there exists a sequence $((i_\alpha, j_\alpha))_{\alpha=1}^{l}$ of transpositions with $i_\alpha < j_\alpha$ for all $\alpha = 1, \dots, l$, such that with $\tau_0 := \tau$ and $\tau_\alpha := \tau \cdot (i_1, j_1) \cdot \dots \cdot (i_\alpha, j_\alpha)$, $\alpha = 1, \dots, l$, it is $\tau_{\alpha-1}(i_\alpha) > \tau_{\alpha-1}(j_\alpha)$ for all $\alpha = 1, \dots, l$ and $\tau_l = \pi$. Using Lemma 6.22 repeatedly yields $K = \tau_0 \cdot \kappa \leq_{\mathrm{Kro}} \tau_1 \cdot \kappa \leq_{\mathrm{Kro}} \dots \leq_{\mathrm{Kro}} \tau_l \cdot \kappa = L$. □

There is another interesting order on $K_{n,m}$, the *dominance order* defined by

$$K \leq_{\mathrm{Dom}} L \iff \sum_{\alpha=1}^{l} K_\alpha \leq \sum_{\alpha=1}^{l} L_\alpha \text{ for all } l = 1, \dots, m \; .$$

There is the following strong relation between the Kronecker order and the dominance order on $S(m) \cdot \kappa$.

Theorem 6.23.

$$K \leq_{\mathrm{Kro}} L \Longrightarrow K \leq_{\mathrm{Dom}} L$$

for all $K, L \in S(m) \cdot \kappa$.

The proof of Theorem 6.23 uses the following lemma.

Lemma 6.24. *Let* $K \in S(m) \cdot \kappa$ *and let* T_{ij} *be a transposition of type* (1). *Then* $K \leq_{\mathrm{Dom}} T_{ij} K$.

Proof. Since T_{ij} is a transposition of type (1) it is $i < j$ and $K_i < K_j$. T_{ij} swaps K_i and K_j hence with $\bar{K} := T_{ij} K$ it follows

$$\sum_{\alpha=1}^{l} K_\alpha = \sum_{\alpha=1}^{l} \bar{K}_\alpha \text{ for all } l = 1, \dots, i-1 \text{ and } l = j, \dots, m \; ,$$

while

$$\sum_{\alpha=1}^{l} K_\alpha < \sum_{\alpha=1}^{l} \bar{K}_\alpha \text{ for all } l = i, \dots, j-1 \; .$$

It follows $K \leq_{\mathrm{Dom}} \bar{K}$. □

Proof of Theorem 6.23. Let $K \leq_{\mathrm{Kro}} L$. According to Proposition 6.19 there exists a sequence $(T_{i_\alpha j_\alpha})_{\alpha=1}^{l}$ of transpositions of type (1) such that $L = T_{i_l j_l} \dots T_{i_1 j_1} K$. Using Lemma 6.24 repeatedly yields $K \leq_{\mathrm{Dom}} T_{i_1 j_1} K \leq_{\mathrm{Dom}} \dots \leq_{\mathrm{Dom}} T_{i_l j_l} \dots T_{i_1 j_1} K = L$. □

Now Theorem 6.16 yields that the reverse order of the dominance order, the so called *specialization order*, is finer than the Bruhat order on $S(m) \cdot \kappa$. Various computations suggest that these two orders are in fact equivalent on $S(m) \cdot \kappa$, i.e. the converse of Theorem 6.23 is also true, but there seems to be no easy proof of this statement.

Remark 6.25. In a nice survey Hazewinkel and Martin [HM83] pointed out that there are many relations between the various orders that have been discussed in this Section. Nevertheless, they focus on the subset of $K_{n,m}$ formed by the *partitions*, i.e. the ordered combinations. So their results do not apply here.

6.3 A cell decomposition

In this section a cell decomposition of a Brunovsky stratum $\mathcal{T}_k(\lambda, \mu)$ of codimension k tight subspaces is constructed. It is shown to be induced by a Bruhat decomposition of a generalized flag manifold which is a retract of $\mathcal{T}_k(\lambda, \mu)$ as has been shown in a recent article by X. Puerta and Helmke [PH00]. After finishing this work the author got knowledge of another paper by F. Puerta, X. Puerta and Zaballa [PPZ01] in which the same cell decomposition is deduced by totally different methods (direct calculations). Nevertheless the author believes that the proofs presented here (besides being more elegant) allow much deeper insight into the structure of this cell decomposition.

Theorem 6.26. *The manifold $\mathcal{T}_k(\lambda, \mu)$ of all codimension k tight subspaces with restriction indices $\lambda = (\lambda_1, \dots, \lambda_p)$ decomposes into a finite number of cells, the* Kronecker cells *$\mathcal{T}_k(K, \mu)$, each consisting of the subspaces with Kronecker indices $K = (K_1, \dots, K_p) = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(p)})$, where $\pi \in S(p)$. (Recall that $\mathcal{T}_k(K, \mu)$ is nonempty if and only if $K_i \leq \mu_i$ and $\lambda_i \leq \mu_i$, $i = 1, \dots, p$, and note that different πs might yield the same K.)*

Proof. Since any tight subspace has some Kronecker indices $(K_1, \dots, K_p)$, the manifold $\mathcal{T}_k(\lambda, \mu)$ decomposes into a finite number of $\mathcal{T}_k(K, \mu)$ sets. K

running through the permutations of λ follows from Theorem 6.7. It remains to show that $\mathcal{T}_k(K,\mu)$ is a cell, i.e. homeomorphic to $\mathcal{F}^N$, $N \in \mathbb{N}$.

According to Theorem 6.12 it suffices to show that the equivalence relation on $M(K,\mu)$ defined by $Z' \equiv Z$ if and only if $Z' = ZS^{-1}$ with $S \in \Gamma(K)$ has a normal form homeomorphic to $\mathcal{F}^N$. This is done by showing that the free parameters in S^{-1} can be used to eliminate parameters in Z. This can be done block columnwise. In fact, since $\Gamma(K)$ operates on $M(K,\mu)$, i.e. S^{-1} preserves the structure of Z, it suffices to consider the first column of each block.

To make it easier to understand the following construction, it is visualized by an example with $\mu = (4,2,2)$ and $K = (3,1,2)$. Recall the Toeplitz structure of each block (Theorem 6.11). For example the free parameters (visualized by asterisks) in the upper left block of Z are all equal.

$$Z = \begin{pmatrix} * & 0 & 0 & * & * & 0 \\ 1 & * & 0 & * & * & * \\ 0 & 1 & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & * & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & \boldsymbol{*} & * & 0 \\ 0 & 1 & 0 & \boldsymbol{*} & 0 & * \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix}.$$

Choose a block column, say number i, in S^{-1} and consider the first column of this block (bold face in the example). In block row number j of Z, $j = 1,\dots,p$, delete $\mu_j - K_j \geq 0$ rows starting with the first row of the block. According to Theorem 6.11 (applied with $\mu = K$) the resulting square matrix Z_1 has full column rank, hence also full row rank for every choice of the free parameters.

$$Z_1 = \begin{pmatrix} 1 & * & 0 & \boldsymbol{*} & * & * \\ 0 & 1 & * & \boldsymbol{*} & 0 & * \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & \boldsymbol{*} & * & 0 \\ 0 & 1 & 0 & \boldsymbol{*} & 0 & * \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix}.$$

As a consequence of Theorem 6.11 the 1s in Z_1 all lie on the main diagonal, and the first column of block column number i of Z (bold face in the example) coincides with the corresponding column in S^{-1}. Let $j_1, \ldots, j_q$ be the row numbers of the zero entries of this column. Let A_i be the row number of the 1 in this column. Delete the rows $j_1, \ldots, j_q$ and the row A_i from Z_1. Then the resulting matrix Z_2 still has full row rank for each choice of the free parameters.

$$Z_2 = \begin{pmatrix} 1 & * & 0 & \boldsymbol{*} & * & * \\ 0 & 1 & * & \boldsymbol{*} & 0 & * \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & \boldsymbol{*} & * & 0 \\ 0 & 1 & 0 & \boldsymbol{*} & 0 & * \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix}.$$

By construction the column A_i of Z_2 only consists of free parameters. Since the 1s in Z_1 are on the main diagonal, the columns $j_1, \ldots, j_q$ and the column A_i of Z_2 contain no 1. Since Z_2 has full row rank for every choice of the free parameters, especially if the free parameters in the columns $j_1, \ldots, j_p$ and in column A_i are set to zero, deleting these columns from Z_2 results in a full rank square matrix Z_3.

$$Z_3 = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following picture visualizes the locus of Z_3 (bold face) in Z.

$$Z = \begin{pmatrix} * & 0 & 0 & * & * & 0 \\ \mathbf{1} & \boldsymbol{*} & 0 & \boxed{*} & \boldsymbol{*} & \boldsymbol{*} \\ \mathbf{0} & \mathbf{1} & * & \boxed{*} & \mathbf{0} & \boldsymbol{*} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \boxed{*} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & \boxed{*} & \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & \boldsymbol{*} & * & 0 \\ 0 & 1 & 0 & \boldsymbol{*} & 0 & * \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 1 & 0 \\ 0 & 0 & 0 & \boldsymbol{*} & 0 & 1 \end{pmatrix}.$$

From the construction it is now clear that there is a unique choice of the free parameters in column A_i of S^{-1} (bold face in the example, they meet Z_3 when multiplying Z by S^{-1}) which zeros out column A_i in Z_2 (framed entries in the example) when multiplying Z by S^{-1}. As has been said before, these entries are all free parameters. Note that the position of these entries in Z does only depend on i, K and μ, but not on the values of the free parameters in Z.

Iterating this construction over all block columns of S^{-1} results in a normal form Z_{norm} of Z which is homeomorphic to $\mathcal{F}^N$, where N is the difference of the number of free parameters in Z and the number of free parameters in S^{-1}.

$$Z_{\text{norm}} = \begin{pmatrix} * & 0 & 0 & * & * & 0 \\ 1 & * & 0 & \mathbf{0} & \mathbf{0} & * \\ 0 & 1 & * & \mathbf{0} & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 1 \end{pmatrix} .$$

□

Corollary 6.27. *If $K_i \leq \mu_i$, $i = 1, \ldots, p$, then*

$$\dim \mathcal{T}_k(K, \mu) = \sum_{i,j=1}^{p} \max\{\mu_i - K_j + 1, 0\} - \sum_{i=1}^{p} i - \sum_{i,j=1}^{p} \max\{K_i - K_j + 1, 0\} + \#\{K_i \geq K_j \mid 1 \leq i \leq j \leq p\} .$$

Proof. Count the free parameters in $M(K, \mu)$ and $\Gamma(K)$. □

To get an index set for the decomposition of $\mathcal{T}_k(\lambda, \mu)$ in Kronecker cells, permutations π and π' of λ leading to the same K must be identified. This is done using the set $S^\lambda(p)$ of minimal length coset representatives of $S(p)/S_\lambda(p)$, cf. page 160. The required index set is the set consisting of all $\pi \in S^\lambda(p)$ satisfying $\lambda_{\pi(j)} \leq \mu_j$ for all $j = 1, \ldots, p$.

In order to see that the decomposition of $\mathcal{T}_k(\lambda, \mu)$ in Kronecker cells is indeed a cell decomposition in the topological sense it is shown to be induced by a Bruhat decomposition of a generalized flag manifold. Recall the following definition of a cell decomposition.

Definition 6.28. Let X be a Hausdorff space. A decomposition $(X_i)_{i \in I}$ of X into disjoint subsets is called a *cell decomposition* if

(1) $(X_i)_{i\in I}$ is locally finite and each X_i is homeomorphic to some $\mathbb{R}^{n_i}$,

(2) $\bar{X}_i \backslash X_i$ is contained in the union of *cells* X_j of strictly smaller dimension, where $\bar{X}_i$ denotes the topological closure of X_i.

The cell decomposition is called *finite*, if I is a finite set. The cell decomposition is said to satisfy the *frontier condition*, if additionally

(3) $X_j \cap \bar{X}_i \neq \emptyset$ if and only if $X_j \subset \bar{X}_i$ for all $i, j \in I$.

Let $b = (b_1, \ldots, b_{\mu_1})$ be the *conjugate indices* of μ read from right to left, i.e. let $b_i = \#\{j \in \{1, \ldots, p\} \,|\, \mu_j \geq \mu_1 - i + 1\}$ for $i = 1, \ldots, \mu_1$. Then $1 \leq b_1 \leq \cdots \leq b_{\mu_1} = p$. Let $a = (a_1, \ldots, a_{\mu_1})$ be the *conjugate indices* of λ read from right to left and brought into line with b, i.e. let $a_i = \#\{j \in \{1, \ldots, p\} \,|\, \lambda_j \geq \mu_1 - i + 1\}$ for $i = 1, \ldots, \mu_1$. Then $0 \leq a_1 \leq \cdots \leq a_{\mu_1} = p$, and $\lambda_i \leq \mu_i$ for all $i = 1, \ldots, p$ implies $a_i \leq b_i$ for all $i = 1, \ldots, \mu_1$.

Consider the compact analytic manifold of *partial flags* of *type* a

$$\mathrm{Flag}(a, \mathcal{F}^p) = \{(\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) \in \prod_{i=1}^{\mu_1} G_{a_i}(\mathcal{F}^p) \,|\, \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{\mu_1}\}$$

(see e.g. Hiller [Hil82]). It is well known that each (complete) *reference flag*

$$\mathsf{V}^* = (\mathcal{V}_1^*, \ldots, \mathcal{V}_p^*) \ , \quad \mathcal{V}_1^* \subset \cdots \subset \mathcal{V}_p^* \quad \text{and} \quad \dim \mathcal{V}_j^* = j \text{ for } j = 1, \ldots, p$$

induces a finite cell decomposition on $\mathrm{Flag}(a, \mathcal{F}^p)$ via the following construction. For a k-dimensional subspace $\mathcal{V} \subset \mathcal{F}^p$ define the *signature* $\mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}) = \{s_1, \ldots, s_k\}$, where $s_1 < \cdots < s_k$ are the "jump points" of $\mathcal{V}$ with respect to the reference flag V^*, i.e. ($\mathcal{V}_0^* := \{0\}$)

$$\mathcal{V} \cap \mathcal{V}_{s_i - 1}^* \neq \mathcal{V} \cap \mathcal{V}_{s_i}^* \ , \quad i = 1, \ldots, k \ .$$

Then $\mathcal{V} \subset \mathcal{V}'$ implies $\mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}) \subset \mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}')$. In particular, for any flag $\mathsf{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) \in \mathrm{Flag}(a, \mathcal{F}^p)$ there is an increasing sequence of signatures $\mathrm{sig}_{\mathsf{V}^*}(\mathsf{V}) := (\mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_1), \ldots, \mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_{\mu_1}))$ with

(1) $\mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_1) \subset \cdots \subset \mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_{\mu_1}) \subset \{1, \ldots, p\}$,

(2) $\# \,\mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_i) = a_i$ for $i = 1, \ldots, \mu_1$.

Any such sequence $S = (S_1, \ldots, S_{\mu_1})$ of subsets of $\{1, \ldots, p\}$ satisfying (1) and (2) is called a *flag symbol* of type a. Let $F(a)$ denote the set of all such flag symbols. Now for any flag symbol $S \in F(a)$ the set

$$B^S_{\mathsf{V}^*} := \{(\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) \in \mathrm{Flag}(a, \mathcal{F}^p) \,|\, \mathrm{sig}_{\mathsf{V}^*}(\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) = S\}$$

is a cell, a so called *Bruhat cell* of $\mathrm{Flag}(a, \mathcal{F}^p)$ with respect to the given reference flag V^*. All these cells form a (obviously finite) cell decomposition $(B^S_{\mathsf{V}^*})_{S \in F(a)}$ of $\mathrm{Flag}(a, \mathcal{F}^p)$ which satisfies the frontier condition.

If in particular the reference flag V^* is chosen to be the *standard flag*

$$\mathsf{V}^0 = (\mathcal{V}^0_1, \ldots, \mathcal{V}^0_p) \ , \quad \mathcal{V}^0_j = \mathrm{span}\{\mathrm{e}_1, \ldots, \mathrm{e}_j\} \text{ for } j = 1, \ldots, p \ ,$$

where e_j, $j = 1, \ldots, p$, denotes the j-th standard basis vector of $\mathcal{F}^p$, then the corresponding (classical) *Bruhat decomposition* $(B^S_{\mathsf{V}^0})_{S \in F(a)}$ restricts to the compact analytic *generalized flag manifold* (cf. Helmke and Shayman [HS87])

$$\mathrm{Flag}(a, b, \mathcal{F}^p) = \{(\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) \in \mathrm{Flag}(a, \mathcal{F}^p) \,|\, \mathcal{V}_i \subset \mathcal{V}^0_{b_i}, i = 1, \ldots, \mu_1\} \ .$$

Apparently, for any flag symbol $S \in F(a)$ the intersection $B^S_{\mathsf{V}^0} \cap \mathrm{Flag}(a, b, \mathcal{F}^p)$ is nonempty if and only if the flag symbol $S = (S_1, \ldots, S_{\mu_1})$ is (a, b)-*admissible*, i.e. $S_i \subset \{1, \ldots, b_i\}$ for $i = 1, \ldots, \mu_1$. In this case $B^S_{\mathsf{V}^0} \subset \mathrm{Flag}(a, b, \mathcal{F}^p)$ holds. Let $F(a, b)$ denote the set of all (a, b)-admissible flag symbols of type a, then $(B^S_{\mathsf{V}^0})_{S \in F(a,b)}$ is a cell decomposition of $\mathrm{Flag}(a, b, \mathcal{F}^p)$ which satisfies the frontier condition.

A convenient way to describe the boundary relation of a cell decomposition satisfying the frontier condition is the so called *adherence order*, a partial order $\leq$ defined on the index set I which has the property that $X_j \subset \bar{X}_i$ if and only if $j \leq i$. For the Bruhat decomposition $(B^S_{\mathsf{V}^0})_{S \in F(a)}$ of $\mathrm{Flag}(a, \mathcal{F}^p)$ this adherence order is induced by the well known *Bruhat order* on the symmetric group $S(p)$: To any flag symbol

$$S = (S_1, \ldots, S_{\mu_1}) = (\{s_{11}, \ldots, s_{1a_1}\}, \ldots, \{s_{\mu_1 1}, \ldots, s_{\mu_1 a_{\mu_1}}\}) \in F(a)$$

associate a permutation $\pi_{\mathsf{V}^0}(S)$ in the following way. Consider the sequence $r_{\mathsf{V}^0}(S) := (r^1_{\mathsf{V}^0}(S), \ldots, r^p_{\mathsf{V}^0}(S))$ with $\{r^{a_{i-1}+1}_{\mathsf{V}^0}(S), \ldots, r^{a_i}_{\mathsf{V}^0}(S)\} = S_i \setminus S_{i-1}$ and $(r^{a_{i-1}+1}_{\mathsf{V}^0}(S), \ldots, r^{a_i}_{\mathsf{V}^0}(S))$ ordered increasingly, $i = 1, \ldots, \mu_1$ (set $a_0 := 0$ and $S_0 := \emptyset$). Since $S_{i-1} \subset S_i$, $i = 1, \ldots, \mu_1$, it follows $\{r^1_{\mathsf{V}^0}(S), \ldots, r^p_{\mathsf{V}^0}(S)\} = \{1, \ldots, p\}$. Hence there exists a unique permutation $\pi_{\mathsf{V}^0}(S) \in S(p)$ with $\pi_{\mathsf{V}^0}(S)(j) = r^j_{\mathsf{V}^0}(S)$, $j = 1, \ldots, p$.

Example 6.29. Let $a := (0, 0, 1, 3, 3, 5, 6)$ and

$$S := (\emptyset, \emptyset, \{2\}, \{1, 2, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 6\}) \in F(a) \ .$$

Then $r_{\mathsf{V}^0}(S) = (2, 1, 4, 3, 6, 5)$ and $\pi_{\mathsf{V}^0}(S) = (1, 2)(3, 4)(5, 6)$.

From the construction it follows that $\pi_{\mathsf{V}^0}(F(a)) = S^\lambda(p) \subset S(p)$ (for the meaning of $S^\lambda(p)$ see page 160). Now the Bruhat order on $S(p)$ yields a partial order on $S^\lambda(p) = \pi_{\mathsf{V}^0}(F(a))$ via

$$\pi \le \pi' \iff \exists_{\tau,\tau' \in S_\lambda(p)}\ \pi \circ \tau \le \pi' \circ \tau' \ .$$

This order in turn induces a partial order on $F(a)$ via $\pi_{\mathsf{V}^0}(.)$. This order, denoted by $\le_{\text{Bru}}$ and also called Bruhat order, turns out to be the adherence order of $(B^S_{\mathsf{V}^0})_{S \in F(a)}$. Apparently, the restriction of this order to $F(a, b)$ then is the adherence order of the Bruhat decomposition $(B^S_{\mathsf{V}^0})_{S \in F(a,b)}$ of $\mathrm{Flag}(a, b, \mathcal{F}^p)$.

Now consider the *reverse standard flag*

$$\mathsf{V}^p = (\mathcal{V}^p_1, \dots, \mathcal{V}^p_p) \ , \quad \mathcal{V}^p_j = \mathrm{span}\{\mathrm{e}_{p-j+1}, \dots, \mathrm{e}_p\} \text{ for } j = 1, \dots, p \ ,$$

and the corresponding Bruhat decomposition $(B^S_{\mathsf{V}^p})_{S \in F(a)}$ of $\mathrm{Flag}(a, \mathcal{F}^p)$. This decomposition also satisfies the frontier condition. Associate to any flag symbol

$$S = (S_1, \dots, S_{\mu_1}) = (\{s_{11}, \dots, s_{1a_1}\}, \dots, \{s_{\mu_1 1}, \dots, s_{\mu_1 a_{\mu_1}}\}) \in F(a)$$

a permutation $\pi_{\mathsf{V}^p}(S)$ by the following construction. Consider the sequence of numbers $r_{\mathsf{V}^p}(S) := (r^1_{\mathsf{V}^p}(S), \dots, r^p_{\mathsf{V}^p}(S))$ where $\{r^{a_{i-1}+1}_{\mathsf{V}^p}(S), \dots, r^{a_i}_{\mathsf{V}^p}(S)\} = \{p - r^{a_{i-1}+1}_{\mathsf{V}^0}(S) + 1, \dots, p - r^{a_i}_{\mathsf{V}^0}(S) + 1\}$ and $(r^{a_{i-1}+1}_{\mathsf{V}^p}(S), \dots, r^{a_i}_{\mathsf{V}^p}(S))$ is ordered increasingly, $i = 1, \dots, \mu_1$ (set $a_0 := 0$ and $S_0 := \emptyset$). As before it follows $\{r^1_{\mathsf{V}^p}(S), \dots, r^p_{\mathsf{V}^p}(S)\} = \{1, \dots, p\}$ and there exists a unique permutation $\pi_{\mathsf{V}^p}(S) \in S(p)$ with $\pi_{\mathsf{V}^p}(S)(j) = r^j_{\mathsf{V}^p}(S)$, $j = 1, \dots, p$.

Example 6.30. Consider Example 6.29. Here $r_{\mathsf{V}^p}(S) = (5, 3, 6, 1, 4, 2)$ and $\pi_{\mathsf{V}^p}(S) = (1, 4, 5)(6, 3, 2)$.

It is $\pi_{\mathsf{V}^p}(F(a)) = \pi_{\mathsf{V}^0}(F(a)) = S^\lambda(p) \subset S(p)$ and the Bruhat order on $S(p)$ induces a partial order on $F(a)$ via $\pi_{\mathsf{V}^p}(.)$. By symmetry considerations it follows that this order is the reverse order of the Bruhat order $\le_{\text{Bru}}$. It turns out that the adherence order of $(B^S_{\mathsf{V}^p})_{S \in F(a)}$ is the Bruhat order $\le_{\text{Bru}}$, i.e. the same order as the adherence order of $(B^S_{\mathsf{V}^0})_{S \in F(a)}$.

A flag symbol $(S_1, \ldots, S_{\mu_1}) \in F(a)$ is called *reversely* (a,b)*-admissible* if $S_i \subset \{p - b_i + 1, \ldots, p\}$, $i = 1, \ldots, \mu_1$, holds. Let $F_p(a,b) \subset F(a)$ denote the set of all reversely (a,b)-admissible flag symbols. The next proposition shows that the Bruhat decomposition of $\mathrm{Flag}(a, \mathcal{F}^p)$ with respect to the reverse standard flag V^p induces a cell decomposition with index set $F_p(a,b)$ on $\mathrm{Flag}(a, b, \mathcal{F}^p)$ by restriction. The proof uses the following well known homogeneous space descriptions of $\mathrm{Flag}(a, \mathcal{F}^p)$ and $\mathrm{Flag}(a, b, \mathcal{F}^b)$.

Let $V(a,b)$ denote the set of those matrices in $\mathrm{GL}(\mathcal{F}^p)$ for which the last $p - b_i$ entries in columns $a_{i-1} + 1, \ldots, a_i$ (set $a_0 := 0$) are zero, $i = 1, \ldots, \mu_1$. Then $P(a) := V(a,a)$ is a *parabolic subgroup* of $\mathrm{GL}(\mathcal{F}^p)$. Furthermore $\mathrm{Flag}(a, \mathcal{F}^p)$ is diffeomorphic to $\mathrm{GL}(\mathcal{F}^p)/P(a)$ and $\mathrm{Flag}(a, b, \mathcal{F}^p)$ is diffeomorphic to $V(a,b)/P(a)$: Any matrix $Z \in \mathrm{GL}(\mathcal{F}^p)$ (resp. $Z \in V(a,b)$) represents a flag $\mathsf{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_{\mu_1}) \in \mathrm{Flag}(a, \mathcal{F}^p)$ (resp. $\mathsf{V} \in \mathrm{Flag}(a, b, \mathcal{F}^p)$) via $\mathcal{V}_i := \mathrm{columnspan}\begin{pmatrix} z_1 & \ldots & z_{a_i} \end{pmatrix}$, $i = 1, \ldots, \mu_1$, where z_j, $j = 1, \ldots, p$, denotes the j-th column of Z. Z and Z' represent the same flag if and only if $Z' = ZS^{-1}$ with $S \in P(a)$.

Proposition 6.31. $(B^S_{\mathsf{V}^p} \cap \mathrm{Flag}(a, b, \mathcal{F}^p))_{S \in F_p(a,b)}$ *is a cell decomposition of* $\mathrm{Flag}(a, b, \mathcal{F}^p)$ *which satisfies the frontier condition. Its adherence order is the restriction of the Bruhat order* $\leq_{\mathrm{Bru}}$ *to* $F_p(a,b) \subset F(a)$.

Proof. Let $S \in F(a)$. A flag $\mathsf{V} \in B^S_{\mathsf{V}^p} \subset \mathrm{Flag}(a, \mathcal{F}^p)$ has a unique representative $Z = \begin{pmatrix} z_1 & \ldots & z_p \end{pmatrix} \in \mathrm{GL}(\mathcal{F}^p)$ in *generalized Echelon normal form*:

$$z_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ ? \\ \vdots \\ ? \end{pmatrix}, \quad j = 1, \ldots, p \ ,$$

where the 1-entry is in row number $r^j_{\mathsf{V}^p}(S)$, and the ?-entry in row number $k \in \{r^j_{\mathsf{V}^p}(S) + 1, \ldots, p\}$ is zero if $k \in \{r^1_{\mathsf{V}^p}(S), \ldots, r^{a(j)}_{\mathsf{V}^p}(S)\}$, where $a(j) := \min\{a_i \geq j \,|\, i = 1, \ldots, \mu_1\}$, otherwise it is a free parameter. To avoid misunderstandings, the normal form for Example 6.29/6.30 is given here.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

From this normal form it is clear that $B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p) \neq \emptyset$ if and only if the row positions of the 1-entries in columns $1, \ldots, a_i$ are smaller or equal than b_i, $i = 1, \ldots, \mu_1$, i.e. if and only if $p - s + 1 \leq b_i$ for all $s \in S_i$, $i = 1, \ldots, \mu_1$, i.e. if and only if $S_i \subset \{p - b_i + 1, \ldots, p\}$, $i = 1, \ldots, \mu_1$, i.e. if and only if $S \in F_p(a, b)$. In this case the last $p - b_i$ entries in columns $a_{i-1} + 1, \ldots, a_i$, $i = 1, \ldots, \mu_1$, are all free parameters. Setting them to zero hence yields a normal form for elements of $B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)$, which is therefore a cell, i.e. homeomorphic to some $\mathbb{R}^{N_S}$. These cells clearly decompose $\operatorname{Flag}(a, b, \mathcal{F}^p)$. Furthermore $\dim B^S_{\mathsf{V}^p} < \dim B^{S'}_{\mathsf{V}^p}$ if and only if $\dim(B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)) < \dim(B^{S'}_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p))$, whenever $S, S' \in F_p(a, b)$.

Consider the natural projections $p_i : G_{a_i}(\mathcal{F}^p) \longrightarrow G_{a_i}(\mathcal{F}^{b_i})$ induced by the linear projections $\mathcal{F}^p \longrightarrow \mathcal{F}^{b_i}$ with kernel $\operatorname{span}\{\mathrm{e}_{b_i+1}, \ldots, \mathrm{e}_p\}$, $i = 1, \ldots, \mu_1$, where e_j, $j = 1, \ldots, p$, denotes the j-th standard basis vector of $\mathcal{F}^p$. Then the projection $p := (p_1 \times \cdots \times p_{\mu_1})|_{\operatorname{Flag}(a, \mathcal{F}^p)}$ restricts to a homeomorphism $p|_{\operatorname{Flag}(a,b,\mathcal{F}^p)} : \operatorname{Flag}(a, b, \mathcal{F}^p) \longrightarrow p(\operatorname{Flag}(a, b, \mathcal{F}^p))$. It follows from the above considerations that $p(B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)) = p(B^S_{\mathsf{V}^p})$ for all $S \in F_p(a, b)$.

Now let $S, S' \in F_p(a, b)$ and let $S \leq_{\text{Bru}} S'$, then $B^S_{\mathsf{V}^p} \subset \overline{B^{S'}_{\mathsf{V}^p}}$. Since p is continuous this implies

$$\begin{aligned} p(B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)) &= p(B^S_{\mathsf{V}^p}) \\ &\subset p(\overline{B^{S'}_{\mathsf{V}^p}}) \\ &\subset \overline{p(B^{S'}_{\mathsf{V}^p})} \\ &= \overline{p(B^{S'}_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p))} \ . \end{aligned}$$

But then $(p|_{\operatorname{Flag}(a,b,\mathcal{F}^p)})^{-1}$ being continuous yields $B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p) \subset \overline{B^{S'}_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)}$.

Conversely, let $(B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)) \cap \overline{(B^{S'}_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p))} \neq \emptyset$. Then clearly $S, S' \in F_p(a, b)$ and $B^S_{\mathsf{V}^p} \cap \overline{B^{S'}_{\mathsf{V}^p}} \neq \emptyset$, i.e. $B^S_{\mathsf{V}^p} \subset \overline{B^{S'}_{\mathsf{V}^p}}$ (frontier condition). But this implies $S \leq_{\text{Bru}} S'$ and $B^S_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p) \subset \overline{B^{S'}_{\mathsf{V}^p} \cap \operatorname{Flag}(a, b, \mathcal{F}^p)}$, as has just been shown. This completes the proof. $\square$

The following result due to X. Puerta and Helmke [PH00] relates $\mathcal{T}_k(\lambda,\mu)$ and $\mathrm{Flag}(a,b,\mathcal{F}^p)$.

Theorem 6.32. *The surjective smooth and closed maps*

$$\begin{aligned}\gamma :& M(\lambda,\mu) \longrightarrow V(a,b)\ ,\\ & Z \mapsto \mathsf{C}Z\bar{C}^\top\end{aligned}$$

and

$$\begin{aligned}\gamma :& \Gamma(\lambda) \longrightarrow P(a)\ ,\\ & S \mapsto \bar{C}S\bar{C}^\top\end{aligned}$$

induce a surjective smooth and closed map

$$\tilde{\gamma} : \mathcal{T}_k(\lambda,\mu) \equiv M(\lambda,\mu)/\Gamma(\lambda) \longrightarrow V(a,b)/P(a) \equiv \mathrm{Flag}(a,b,\mathcal{F}^p)$$

on quotients. Furthermore, $\mathrm{Flag}(a,b,\mathcal{F}^p)$ *is a* strong deformation retract *of* $\mathcal{T}_k(\lambda,\mu)$*, hence* $\mathcal{T}_k(\lambda,\mu)$ *and* $\mathrm{Flag}(a,b,\mathcal{F}^p)$ *are* homotopy equivalent.

Note that $\bar{C}$ denotes a matrix in dual Brunovsky canonical form with indices λ, here. As the next proposition shows, the map $\tilde{\gamma}$ behaves well with respect to the decompositions in cells constructed above.

Proposition 6.33. *$\tilde{\gamma}$ maps Kronecker cells onto Bruhat cells with respect to the reverse standard flag.*

Proof. Let $\pi \in S^\lambda(p)$ and let $K = (K_1,\dots,K_p) = (\lambda_{\pi(1)},\dots,\lambda_{\pi(p)})$ be such that $K_j \le \mu_j$ for all $j = 1,\dots,p$. Let $\mathcal{V} \in \mathcal{T}_k(K,\mu) \subset \mathcal{T}_k(\lambda,\mu)$. According to Theorem 6.11 there exists $Z \in M(K,\mu)$ such that $\mathcal{V} = \operatorname{Im} Z$. Let $P_{\pi^{-1}}$ be the standard permutation matrix that permutes the block columns of Z according to π^{-1} when multiplied to Z from the right. Then $\mathcal{V} = \operatorname{Im} Z'$ where $Z' := ZP_{\pi^{-1}}$. From Theorem 6.5 it follows that $Z' \in M(\lambda,\mu)$. Furthermore, since the matrix formed of the lower right corner entries of each block of Z is unipotent lower triangular, the matrix $\gamma(Z') = \mathsf{C}Z'\bar{C}^\top = \begin{pmatrix} g_1 & \dots & g_p\end{pmatrix}$ formed of the lower right corner entries of each block of Z' has the following structure:

$$g_j = \begin{pmatrix} 0\\ \vdots\\ 0\\ 1\\ ?\\ \vdots\\ ?\end{pmatrix}\ ,$$

where the 1-entry is in row number $\pi^{-1}(j)$, $j = 1, \ldots, p$. Since $\gamma(Z') \in V(a,b)$, it represents a flag $\mathsf{V} \in \mathrm{Flag}(a,b,\mathcal{F}^p)$. From the structure of $\gamma(Z')$ it follows that $\mathrm{sig}_{\mathsf{V}^p}(\mathsf{V})$ is such that $r^j_{\mathsf{V}^p}(\mathrm{sig}_{\mathsf{V}^p}(\mathsf{V})) = \pi^{-1}(j)$, $j = 1, \ldots, p$, i.e. such that $\pi_{\mathsf{V}^p}(\mathrm{sig}_{\mathsf{V}^p}(\mathsf{V})) = \pi^{-1}$. Hence $\mathrm{sig}_{\mathsf{V}^p}(\mathsf{V})$ does only depend on the choice of π and not on Z' or Z.

It follows that $\tilde{\gamma}$ maps any Kronecker cell into a corresponding Bruhat cell with respect to the reverse standard flag (cf. Proposition 6.31). Since $\tilde{\gamma}$ is surjective, the statement follows. □

Now the promised topological result follows immediately.

Theorem 6.34. *The decomposition of $\mathcal{T}_k(\lambda,\mu)$ into Kronecker cells is a finite cell decomposition. Furthermore, $\mathcal{T}_k(K,\mu) \cap \overline{\mathcal{T}_k(K',\mu)} \neq \emptyset$ if and only if $K \leq_{\mathrm{Kro}} K'$ (cf. Section 6.2).*

Proof. For $K = (K_1, \ldots, K_p) = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(p)})$, where $\pi \in S^\lambda(p)$ and $K_j \leq \mu_j$ for all $j = 1, \ldots, p$, let $S(K)$ denote the flag symbol of the Bruhat cell $(B^S_{\mathsf{V}^p} \cap \mathrm{Flag}(a,b,\mathcal{F}^p))$ on which $\mathcal{T}_k(K,\mu)$ is mapped by $\tilde{\gamma}$ (cf. Proposition 6.33). Let $K = (K_1, \ldots, K_p) = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(p)})$ and $K' = (K'_1, \ldots, K'_p) = (\lambda_{\pi'(1)}, \ldots, \lambda_{\pi'(p)})$ be as stated. According to Theorem 6.16 it is $K \leq_{\mathrm{Kro}} K'$ if and only if $\pi' \leq_{\mathrm{Bru}} \pi$. The latter is equivalent to $(\pi')^{-1} \leq_{\mathrm{Bru}} \pi^{-1}$, i.e. to $\pi_{\mathsf{V}^p}(S(K')) \leq_{\mathrm{Bru}} \pi_{\mathsf{V}^p}(S(K))$ as has been shown in the proof of Proposition 6.33. Since $\pi_{\mathsf{V}^p}(.)$ induces the reverse order of the Bruhat order on $F(a)$, it follows that $K \leq_{\mathrm{Kro}} K'$ if and only if $S(K) \leq_{\mathrm{Bru}} S(K')$.

Let $\mathcal{T}_k(K,\mu) \cap \overline{\mathcal{T}_k(K',\mu)} \neq \emptyset$ then $\tilde{\gamma}(\mathcal{T}_k(K,\mu)) \cap \tilde{\gamma}(\overline{\mathcal{T}_k(K',\mu)}) \neq \emptyset$. Since $\tilde{\gamma}$ is continuous, it is $\tilde{\gamma}(\overline{\mathcal{T}_k(K',\mu)}) \subset \overline{\tilde{\gamma}(\mathcal{T}_k(K',\mu))}$. But then it follows $\tilde{\gamma}(\mathcal{T}_k(K,\mu)) \cap \overline{\tilde{\gamma}(\mathcal{T}_k(K',\mu))} \neq \emptyset$, in other words

$$(B^{S(K)}_{\mathsf{V}^p} \cap \mathrm{Flag}(a,b,\mathcal{F}^p)) \cap \overline{(B^{S(K')}_{\mathsf{V}^p} \cap \mathrm{Flag}(a,b,\mathcal{F}^p))} \neq \emptyset \ .$$

By the frontier condition this implies

$$B^{S(K)}_{\mathsf{V}^p} \cap \mathrm{Flag}(a,b,\mathcal{F}^p) \subset \overline{B^{S(K')}_{\mathsf{V}^p} \cap \mathrm{Flag}(a,b,\mathcal{F}^p)} \tag{6.1}$$

and hence $S(K) \leq_{\mathrm{Bru}} S(K')$, i.e. $K \leq_{\mathrm{Kro}} K'$.

Conversely let $K \leq_{\mathrm{Kro}} K'$, then $S(K) \leq_{\mathrm{Bru}} S(K')$ implies (6.1), i.e.

$$\tilde{\gamma}(\mathcal{T}_k(K,\mu)) \subset \overline{\tilde{\gamma}(\mathcal{T}_k(K',\mu))} = \tilde{\gamma}(\overline{\mathcal{T}_k(K',\mu)}) \ ,$$

where the last equality follows from $\tilde{\gamma}$ being continuous and closed. Since $\tilde{\gamma}^{-1}(\tilde{\gamma}(\mathcal{T}_k(K,\mu))) = \mathcal{T}_k(K,\mu)$, this implies $\mathcal{T}_k(K,\mu) \cap \overline{\mathcal{T}_k(K',\mu)} \neq \emptyset$.

Now let $K \leq_{\mathrm{Kro}} K'$ and $K \neq K'$. It remains to show that $\dim \mathcal{T}_k(K,\mu) < \dim \mathcal{T}_k(K',\mu)$. In view of Corollary 6.27 this is equivalent to

$$\#\{K_i \geq K_j \,|\, 1 \leq i \leq j \leq p\} < \#\{K'_i \geq K'_j \,|\, 1 \leq i \leq j \leq p\}$$

since the other terms in the dimension formula for $\mathcal{T}_k(K,\mu)$ do not depend on the order of the elements of K. But then the claim follows from Proposition 6.19. □

Remark 6.35. (1) It is not at all clear if the decomposition of $\mathcal{T}_k(\lambda,\mu)$ in Kronecker cells satisfies the frontier condition or not. A proof or disproof of this should make heavy use of the structure of matrices in $M(K,\mu)$.

(2) A generalization of the results presented in this chapter to the non tight case, which is also covered by the work of Ferrer, F. Puerta and X. Puerta [FPP98], X. Puerta and Helmke [PH00] and F. Puerta, X. Puerta and Zaballa [PPZ01], is subject to future research.

Appendix A

Topology of orbit spaces

In this appendix some facts about the topology of orbit spaces of Lie group actions on manifolds are presented.

Definition A.1. A *Lie group action* of a Lie group G on a manifold M is a differentiable map

$$\Phi : G \times M \longrightarrow M,\ (g,p) \mapsto \Phi(g,p) =: g \cdot p$$

with $e \cdot p = p$ and $(gh) \cdot p = g \cdot (h \cdot p)$ for all $g, h \in G$, $p \in M$. Sometimes G is called *Lie transformation group*, the pair (M, Φ) is often referred to as a *G-manifold*.

The *isotropy subgroup* or *stabilizer subgroup* of a point $p \in M$ is the closed subgroup $G_p = \{g \in G \,|\, g \cdot p = p\}$ of G. The action Φ is called *free*, if every isotropy subroup is trivial, i.e. $G_p = \{e\}$ for every $p \in M$.

The *orbit* of a point $p \in M$ is the subset $G \cdot p = \{g \cdot p \,|\, g \in G\}$ of M. The action Φ is called *transitive* if for one (and therefore for every) point $p \in M$ the orbit $G \cdot p$ is all of M. One says, *G acts transitively* on M, then.

For non-transitive actions Φ one consideres the *orbit space* $M/G = M/\sim_\Phi$, where $\sim_\Phi$ denotes the equivalence relation on M set up by Φ: $m \sim_\Phi m'$ if there exists a $g \in G$ with $m' = g \cdot m$. The orbit space M/G is equipped with the quotient topology, i.e. the finest topology for which the *natural projection*

$$\pi : M \longrightarrow M/G,\ m \mapsto [m]_{\sim_\Phi}$$

is continous.

The *graph map* $\hat{\Phi}$ associated to Φ is the map

$$\hat{\Phi} : G \times M \longrightarrow M \times M,\ (g, p) \mapsto (p, g \cdot p)\ .$$

The image of $\hat{\Phi}$ is nothing else but the relation $\sim_\Phi$ seen as a subspace of $M \times M$.

Under certain circumstances the orbit space M/G is a manifold again. The following necessary and sufficient condition can be found in [Die82, Theorem 16.10.3].

Theorem A.2. *There is a unique manifold structure on M/G such that the natural projection π is a submersion if and only if the image of the graph map $\operatorname{Im}\hat{\Phi}$ is a closed submanifold of $M \times M$.*

If the action Φ is free, $\operatorname{Im}\hat{\Phi}$ being a closed submanifold can be checked easily.

Proposition A.3. *Let Φ be a free Lie group action of G on M. Then $\operatorname{Im}\hat{\Phi}$ is a closed submanifold of $M \times M$ if and only if $\hat{\Phi}$ is a closed map, i.e. maps closed sets to closed sets.*

Proof. Since Φ is free, $\hat{\Phi}$ is injective. The map

$$\Phi_p : G \longrightarrow M,\ g \mapsto g \cdot p$$

is a subimmersion ([Die82, Proposition 16.10.2]) and hence Φ being free implies $\operatorname{Ker} T_g\Phi_p = T_g\Phi_p^{-1}(g \cdot p) = T_g\{g\} = \{0\}$. Therefore $T_g\Phi_p$ is injective and Φ_p is an immersion. Since the projections

$$pr_1 : G \times M \longrightarrow G,\ (g, p) \mapsto g \quad \text{and} \quad pr_2 : G \times M \longrightarrow M,\ (g, p) \mapsto p$$

are immersions and $\hat{\Phi} = pr_2 \times (\Phi_p \circ pr_1)$, it follows that $\hat{\Phi}$ is an injective immersion. But then $\hat{\Phi}$ is an embedding with $\operatorname{Im}\hat{\Phi}$ closed if and only if $\hat{\Phi}$ is a closed map. The statement follows. $\square$

In the case of a free action Φ there is a sequence criterion for $\hat{\Phi}$ being a closed map.

Proposition A.4. *Let Φ be a free Lie group action of G on M. Then $\hat{\Phi}$ is a closed map if and only if for every sequence $((g_i, p_i))_{i\in\mathbb{N}}$ in $G \times M$ for which the sequence $(\hat{\Phi}(g_i, p_i))_{i\in\mathbb{N}}$ in $M \times M$ converges, the sequence $(g_i)_{i\in\mathbb{N}}$ in G has an accumulation point.*

Proof. Let $\hat{\Phi}$ be a closed map. Let $((g_i, p_i))_{i\in\mathbb{N}}$ be a sequence for which the sequence $(\hat{\Phi}(g_i, p_i) = (p_i, g_i \cdot p_i))_{i\in\mathbb{N}}$ converges to (p, p'). Since $\operatorname{Im}\hat{\Phi}$ is closed, there exists a $g \in G$ with $p' = g \cdot p$. Assume that the sequence $(g_i)_{i\in\mathbb{N}} \subset G$ has no accumulation point. Then $((g_i, p_i))_{i\in\mathbb{N}}$ has no accumulation point, either. But that means the set $\{(g_i, p_i) \,|\, i \in \mathbb{N}\}$ is closed and therefore the set $\{\hat{\Phi}(g_i, p_i) = (p_i, g_i \cdot p_i) \,|\, i \in \mathbb{N}\}$ is closed, too, and contains the point $(p, g \cdot p)$. Since Φ is free it follows $(g, p) \in \{(g_i, p_i) \,|\, i \in \mathbb{N}\}$. Assume the set $I = \{i \in \mathbb{N} \,|\, (g_i, p_i) = (g, p)\}$ is finite. Then repeat the same argument for the still closed set $\{(g_i, p_i) \,|\, i \in \mathbb{N} \setminus I\}$ which then contains the point (g, p), a contradiction. So, I must be infinite. But then (g, p) is an accumulation point of $((g_i, p_i))_{i\in\mathbb{N}}$, a contradiction.

Conversely let $A \subset G \times M$ be closed and let $(p_i, g_i \cdot p_i)$ be a convergent sequence in $\hat{\Phi}(A)$ with limit (p, p'). Then the sequence $(g_i)_{i\in\mathbb{N}}$ has an accumulation point g, i.e. there exists a subsequence $(h_j)_{j\in\mathbb{N}}$ of $(g_i)_{i\in\mathbb{N}}$ with $h_j \to g$. Then $(h_j, p_j) \to (g, p)$ and $(g, p) \in A$ since A is closed. Furthermore $(p_j, h_j \cdot p_j) \to (p, g \cdot p) = (p, p')$ since $\hat{\Phi}$ is continous. But then $(p, p') \in \hat{\Phi}(A)$ and $\hat{\Phi}(A)$ is closed. Note that Φ was not required to be free in this direction of the proof. ☐

In the literature another condition for M/G being a manifold can be found.

Theorem A.5. *Let Φ be a free Lie group action of G on M. Then M/G is a manifold if and only if Φ is proper. Especially, if Φ is a free Lie group action of a compact Lie group G on M then M/G is a manifold.*

Proof. Φ is a proper action if and only if $\hat{\Phi}$ is a proper map. If Φ is a free action, $\hat{\Phi}$ is injective and hence a proper map if and only if it is a closed map ([Bou66, Chapter I, §10.1, Proposition 2]). If G is compact, Φ is necessarily proper ([Bou66, Chapter III, §4.1, Proposition 2]). ☐

In the proper context, the sequence criterion of Proposition A.4 can be found in [Bou66, Chapter III, §4.1].

The closedness of $\operatorname{Im}\hat{\Phi}$ plays an important role in the whole theory. M/G is a Hausdorff space if and only if $\operatorname{Im}\hat{\Phi}$ is closed ([Bou66, Chapter I, §8.3, Proposition 8 and Chapter III, §2.4, Lemma 2]). Nevertheless, M/G being a manifold requires more than that. In the following example of a free Lie group action $\operatorname{Im}\hat{\Phi}$ is a closed set but $\hat{\Phi}$ is not a closed map.

Example A.6. Note first, that a global analytic injective flow on $\mathbb{R}^2$

$$\Phi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (t,(x,y)) \mapsto \Phi_t(x,y)$$

is a free analytic Lie group action of the group $G := (\mathbb{R},+)$ on the manifold $\mathbb{R}^2$. If Φ is invariant under the transformation

$$\tau : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x,y) \mapsto (x+2,-y)\ ,$$

i.e. $\Phi \circ (\mathrm{id}_{\mathbb{R}} \times \tau) = \tau \circ \Phi$, then Φ can be seen as a global analytic flow on the *Möbiusband* $M := \mathbb{R}^2/ <\tau>$, where $<\tau>$ denotes the subgroup of $\mathrm{End}(\mathbb{R}^2)$ generated by τ. Then $\Phi : G \times M \longrightarrow M$ is a free analytic Lie group action. Consider now the injective flow on $\mathbb{R}^2$ induced by the vector field

$$\begin{aligned} x' &= (1-\sin^2(\frac{\pi}{2}x))^2 \\ y' &= -\sin(\frac{\pi}{2}x)\ . \end{aligned}$$

It is obviously τ-invariant. Seen on M, $\mathrm{Im}\,\hat{\Phi}$ is closed since M/G is Hausdorff, but $\hat{\Phi}$ is not closed, since the sequence criterion of Proposition A.4 is violated for a sequence $(x_i,y_i) \to (-1,-y)$ and $\Phi_{t_i}(x_i,y_i) \to (1,y)$ ($\equiv (-1,-y)$ in M) monotonically, because then $t_i \to \infty$ monotonically and hence $(t_i)_{i\in\mathbb{N}}$ has no accumulation point.

The sequence criterion of Proposition A.4 is used by Helmke [Hel92] to show that the orbit space $\Sigma_{k,p}(\mathcal{F})$ of the similarity action σ (cf. Section 4.1) is a manifold. In particular, he shows that σ is a free action when restricted to the space of controllable pairs.

Appendix B

On the differentiability of vector bundles

In this appendix a sufficient condition for a (topological) vector bundle being a differentiable vector bundle is presented. The result also provides a sufficient condition for the preimage of a differentiable submanifold being a differentiable submanifold again. Second, quotients of differentiable vector bundles with respect to free and proper Lie group actions are shown to be differentiable vector bundles.

Definition B.1. Let X and B be Hausdorff spaces and let

$$f: \ X \longrightarrow B$$

be a continuous surjection. Let $p \in \mathbb{N}$ be fixed. For each point $x \in B$ let there exist an open neighborhood U such that there is a homeomorphism

$$\phi_U: \ U \times \mathcal{F}^p \longrightarrow f^{-1}(U)$$

such that

$$f(\phi_U(x,y)) = x$$

for all $x \in U$ and all $y \in \mathcal{F}^p$. Such a homeomorphism is called a *local trivialization* of f. For each pair ϕ_U and ϕ_V of local trivializations and each point $x \in U \cap V$ let there exist a map $\theta_{V,U,x} \in \mathrm{GL}(\mathcal{F}^p)$ such that

$$\phi_V^{-1} \circ \phi_U(x,y) = (x, \theta_{V,U,x}(y))$$

for all $y \in \mathcal{F}^p$, i.e. the induced change of coordinates function on $\mathcal{F}^p$ is linear. If all these hypotheses hold then f is called a *vector bundle* with *fiber* $\mathcal{F}^p$. If X and B are differentiable manifolds, f is a differentiable map and each ϕ_U is a diffeomorphism then the bundle f is called *differentiable*.

Theorem B.2. *Let $X \subset \mathcal{F}^n$ and $B \subset \mathcal{F}^m$ be topological subspaces and let $f : X \longrightarrow B$ be a vector bundle such that f is the restriction of a differentiable map*

$$F : \ U_X \longrightarrow \mathcal{F}^m \ ,$$

where U_X is an open neighborhood of X in $\mathcal{F}^n$. Let B be a q-dimensional differentiable submanifold of $\mathcal{F}^m$ and let each local trivialization $\phi_U : U \times \mathcal{F}^p \longrightarrow f^{-1}(U) \subset \mathcal{F}^n$ of f be differentiable and such that $\phi_U^{-1} : f^{-1}(U) \longrightarrow U \times \mathcal{F}^p$ is the restriction of a differentiable map

$$\Phi_{U,\mathrm{inv}} : \ U_{f^{-1}(U)} \longrightarrow \mathcal{F}^m \times \mathcal{F}^p \ ,$$

where $U_{f^{-1}(U)}$ is an open neighborhood of $f^{-1}(U)$ in $\mathcal{F}^n$. Then X is a $(q+p)$-dimensional submanifold of $\mathcal{F}^n$ and f is a differentiable vector bundle.

Proof. Let $x_0 \in X$, then there exists an open neighborhood U_1 of $f(x_0)$ in B and a local trivialization $\phi_{U_1} : U_1 \times \mathcal{F}^p \longrightarrow f^{-1}(U_1)$ of f. Furthermore, there exists an open neighborhood U_2 of $f(x_0)$ in B and a local coordinate chart $\varphi_{U_2} : U_2 \longrightarrow \varphi_{U_2}(U_2) \subset \mathcal{F}^q$ of B around $f(x_0)$. Set $U := U_1 \cap U_2$, then U is open in B and $\phi_U := \phi_{U_1}|_{U \times \mathcal{F}^p} : U \times \mathcal{F}^p \longrightarrow f^{-1}(U)$ is a local trivialization of f. Furthermore, $\varphi_U := \varphi_{U_2}|_U : U \longrightarrow \varphi_U(U) = \varphi_{U_2}(U)$ is a local coordinate chart of B around $f(x_0)$. Define a local coordinate chart $\psi_{f^{-1}(U)} : f^{-1}(U) \longrightarrow \varphi_U(U) \times \mathcal{F}^p$ of X around x_0 by

$$x \mapsto (\varphi_U \circ \mathrm{pr}_1 \circ \phi_U^{-1}(x), \mathrm{pr}_2 \circ \phi_U^{-1}(x)) \ .$$

Here pr_1 and pr_2 denote the projections on the first and second factor of $U \times \mathcal{F}^p$, respectively. Note that $f^{-1}(U)$ is open in X since f is continuous. Note further that $\varphi_U(U) \times \mathcal{F}^p$ is open in $\mathcal{F}^q \times \mathcal{F}^p$ since φ_U is a local coordinate chart of B. Since ϕ_U^{-1} and φ_U are both bijective, so is $\psi_{f^{-1}(U)}$. Furthermore, $\psi_{f^{-1}(U)}$ is continuous as concatenation of continuous maps, hence it is a homeomorphism.

Now let $\psi_{f^{-1}(U)}$ and $\psi_{f^{-1}(V)}$ be two such local coordinate charts of X and let $f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$. Then

$$\psi_{f^{-1}(V)} \circ \psi_{f^{-1}(U)}^{-1} : \ \varphi_U(U \cap V) \times \mathcal{F}^p \longrightarrow \varphi_V(U \cap V) \times \mathcal{F}^p \ ,$$

which is given by

$$\begin{aligned}(z, y) \ \mapsto & \ (\varphi_V \circ \mathrm{pr}_1 \circ \phi_V^{-1} \circ \phi_U(\varphi_U^{-1}(z), y), \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_U(\varphi_U^{-1}(z), y)) \ = \\ & \ (\varphi_V \circ \varphi_U^{-1}(z), \theta_{V,U,\varphi_U^{-1}(z)}(y)) \ ,\end{aligned}$$

is a diffeomorphism since $\varphi_V \circ \varphi_U^{-1}$ is a diffeomorphism (because B is a differentiable manifold) and $\theta_{V,U,\varphi_U^{-1}(z)} \in \mathrm{GL}(\mathcal{F}^p)$. It follows that X is a $(p+q)$-dimensional differentiable manifold. Since the local coordinate charts of X are continuous, the preimage of any open set in $\mathcal{F}^q \times \mathcal{F}^p$ under any chart is open in X. Since these preimages form a basis of the topology τ induced on X by its differentiable structure, τ coincides with the given topology on X, which is the subspace topology induced by the topology on $\mathcal{F}^n$. Hence X is a submanifold of $\mathcal{F}^n$.

Since f and the inverse maps ϕ_U^{-1} of all local trivializations ϕ_U of f are restrictions of differentiable maps defined on open subsets of $\mathcal{F}^n$, they are differentiable themselves. Since each ϕ_U is also differentiable, it follows that f is a differentiable vector bundle. □

Theorem B.3. *Let $f : X \longrightarrow B$ be a differentiable vector bundle with fiber $\mathcal{F}^p$ and let*

$$\phi_X : \ G \times X \longrightarrow X$$

and

$$\phi_B : \ G \times B \longrightarrow B$$

be free and proper actions of the Lie group G on X and B, respectively. For every local trivialization ϕ_U of f let U consist of full G-orbits (i.e. $x \in U$ implies $\phi_B(g,x) \in U$ for all $g \in G$) and let

$$\phi_U(\phi_B(g,x),y) = \phi_X(g,\phi_U(x,y)) \tag{B.1}$$

for all $g \in G$, $x \in U$ and $y \in \mathcal{F}^p$. Then

$$\begin{aligned} \bar{f} : X/G &\longrightarrow B/G \ , \\ [x]_{\sim_{\phi_X}} &\mapsto [f(x)]_{\sim_{\phi_B}} \end{aligned}$$

is a differentiable vector bundle with fiber $\mathcal{F}^p$.

Proof. Let $x \in X$ and $g \in G$ be arbitrary. Then $f(x) \in B$ and hence there exists a neighborhood U of $f(x)$ and a local trivialization ϕ_U such that $x = \phi_U(z,y)$ for appropriate $z \in U$ and $y \in \mathcal{F}^p$. But then (B.1) implies $f(\phi_X(g,x)) = f(\phi_X(g,\phi_U(z,y))) = f(\phi_U(\phi_B(g,z),y)) = \phi_B(g,z)$. Taking $g = e$ yields $f(x) = z$. But this means

$$f \circ \phi_X(g,x) = \phi_B(g,f(x)) \tag{B.2}$$

for all $g \in G$ and $x \in X$.

From (B.2) it follows that $\bar{f}$ is well defined. By Theorem A.5 the spaces X/G and B/G are both differentiable manifolds. Now consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] \arrow[dr] & B \arrow[d, "\pi_B"] \\ X/G \arrow[r, "\bar{f}"] & B/G \end{tikzcd}$$

Apparently the map $\pi_B \circ f$ is differentiable and hence $\bar{f}$ is differentiable by the universal property of quotients ([Die82, Proposition 16.10.4]).

For every local trivialization ϕ_U of f define a local trivialization of $\bar{f}$ by

$$\bar{\phi}_U : \pi_B(U) \times \mathcal{F}^p \longrightarrow \pi_X(f^{-1}(U)) \ ,$$
$$([x]_{\sim_{\phi_B}}, y) \mapsto [\phi_U(x,y)]_{\sim_{\phi_X}} \ .$$

From (B.1) it follows that $\bar{\phi}_U$ is well defined. Since π_B is an open map, it follows that $\pi_B(U)$ is open in B/G. Since π_B is surjective, the sets $\pi_B(U)$ cover B/G. As before, the commutative diagram

$$\begin{tikzcd} U \times \mathcal{F}^p \arrow[r, "\phi_U"] \arrow[d, "\pi_B \times \mathrm{id}"'] \arrow[dr] & f^{-1}(U) \arrow[d, "\pi_X"] \\ \pi_B(U) \times \mathcal{F}^p \arrow[r, "\bar{\phi}_U"] & \pi_X(f^{-1}(U)) \end{tikzcd}$$

implies that $\bar{\phi}_U$ is differentiable. Since π_X and ϕ_U are both surjective so is $\pi_X \circ \phi_U$, and hence $\bar{\phi}_U$ is surjective. To see that $\bar{\phi}_U$ is also injective consider $x, x' \in U$ and $y, y' \in \mathcal{F}^p$ with $[\phi_U(x,y)]_{\sim_{\phi_X}} = [\phi_U(x',y')]_{\sim_{\phi_X}}$. Then there exists $g \in G$ with $\phi_U(x,y) = \phi_X(g, \phi_U(x',y')) = \phi_U(\phi_B(g,x'),y')$, i.e. $x = \phi_B(g,x')$ and $y = y'$, since ϕ_U is injective. It follows that $([x]_{\sim_{\phi_B}}, y) = ([x']_{\sim_{\phi_B}}, y')$ and $\bar{\phi}_U$ is injective. Now the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\phi_U^{-1}"] \arrow[d, "\pi_X"'] \arrow[dr] & U \times \mathcal{F}^p \arrow[d, "\pi_B \times \mathrm{id}"] \\ \pi_X(f^{-1}(U)) \arrow[r, "\bar{\phi}_U^{-1}"] & \pi_B(U) \times \mathcal{F}^p \end{tikzcd}$$

implies that $\bar{\phi}_U^{-1}$ is differentiable, hence $\bar{\phi}_U$ is a diffeomorphism.

Now let $x \in U$ and $y \in \mathcal{F}^p$. Then

$$\begin{aligned}\bar{f}(\bar{\phi}_U([x]_{\sim_{\phi_B}}, y)) &= \bar{f}([\phi_U(x,y)]_{\sim_{\phi_X}}) \\ &= [f(\phi_U(x,y))]_{\sim_{\phi_B}} \\ &= [x]_{\sim_{\phi_B}} \ .\end{aligned}$$

If ϕ_V is another local trivialization of f, $x \in U \cap V$ and $y \in \mathcal{F}^p$ then

$$\begin{aligned}&\bar{\phi}_V([\mathrm{pr}_1 \circ \phi_V^{-1} \circ \phi_U(x,y)]_{\sim_{\phi_B}}, \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_U(x,y)) = \\ &[\phi_V(\mathrm{pr}_1 \circ \phi_V^{-1} \circ \phi_U(x,y), \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_U(x,y))]_{\sim_{\phi_X}} = \\ &\qquad\qquad\qquad\qquad [\phi_U(x,y)]_{\sim_{\phi_X}}\end{aligned}$$

implies

$$\begin{aligned}\bar{\phi}_V^{-1} \circ \bar{\phi}_U([x]_{\sim_{\phi_B}}, y) &= \bar{\phi}_V^{-1}([\phi_U(x,y)]_{\sim_{\phi_X}}) \\ &= ([\mathrm{pr}_1 \circ \phi_V^{-1} \circ \phi_U(x,y)]_{\sim_{\phi_B}}, \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_U(x,y)) \\ &= ([x]_{\sim_{\phi_B}}, \theta_{V,U,x}(y)) \ .\end{aligned}$$

Let furthermore $g \in G$ be arbitrary then successive use of (B.1) implies

$$\begin{aligned}\theta_{V,U,\phi_B(g,x)}(y) &= \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_U(\phi_B(g,x), y) \\ &= \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_X(g, \phi_U(x,y)) \\ &= \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_X(g, \phi_V(\phi_V^{-1} \circ \phi_U(x,y))) \\ &= \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_X(g, \phi_V(x, \theta_{V,U,x}(y))) \\ &= \mathrm{pr}_2 \circ \phi_V^{-1} \circ \phi_V(\phi_B(g,x), \theta_{V,U,x}(y)) \\ &= \theta_{V,U,x}(y) \ .\end{aligned}$$

Hence

$$\theta_{V,U,[x]_{\sim_{\phi_B}}} := \theta_{V,U,x}$$

is well defined and

$$\bar{\phi}_V^{-1} \circ \bar{\phi}_U([x]_{\sim_{\phi_B}}, y) = ([x]_{\sim_{\phi_B}}, \theta_{V,U,[x]_{\sim_{\phi_B}}}(y)) \ .$$

It follows that $\bar{f}$ is a differentiable vector bundle with fiber $\mathcal{F}^p$. $\square$

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Notation

Structures

$\mathbb{C}^-$	stable part of the complex plane $\mathbb{C}$	16
$\mathbb{C}_g$	symmetric subset of $\mathbb{C}$ containing $]-\infty, c]$	16
$\mathcal{F}$	the underlying field, here $\mathbb{R}$ or $\mathbb{C}$	13
$F(a)$	set of flag symbols of type a	169
$F(a,b)$	set of (a,b)-admissible flag symbols	169
$F_p(a,b)$	set of reversely (a,b)-admissible flag symbols	171
$\mathcal{F}^k[s]$	set of k-vectors with polynomial entries	50
$\mathcal{F}^k(s)$	set of k-vectors with rational entries	50
$K_{n,m}$	set of combinations of m numbers with sum n	158
$S(m), S(p)$	the symmetric group	21,34,159,169
$S^\kappa(m), S^\lambda(p)$	group of min. length coset representatives	160,167
$\mathcal{U}[s]$	set of vectors in $\mathcal{U}$ with polynomial entries	50
$\mathcal{U}(s)$	set of vectors in $\mathcal{U}$ with rational entries	50

Matrices

A_c, B_c, C_c	composite system matrices	43,63,68,77
A^J	short for $A - JC$	28
A_F	short for $A + BF$	15
$\mathsf{A}, \mathsf{B}, \mathsf{C}$	system matrices in (dual) Brunovsky form	20,33
$\tilde{A}, \tilde{B}, \tilde{C}$	corestricted system matrices	23,35
$\bar{A}, \bar{B}, \bar{C}$	restricted system matrices	23,35
$H^{(i)}$	Markov parameters	128
$H(s)$	transfer matrix	128
$O_n(C,A)$	observability matrix of (C,A)	29
$O_\kappa(C,A)$	κ-partial observability matrix of (C,A)	109
$\overleftarrow{O}_\kappa(C,A)$	reverse κ-partial observability matrix of (C,A)	115
$O_\kappa(C_1,A_1,C_2,N)$	combined κ-partial observability matrix	120

$R_n(A,B)$	reachability matrix of (A,B)	16
$R_\mu(A,B)$	μ-partial reachability matrix of (A,B)	90
$\overleftarrow{R}_\mu(A,B)$	reverse μ-partial reachability matrix of (A,B)	90
$R_\mu(A_1,B_1,N,B_2)$	combined μ-partial reachability matrix	96
$R_\mu(E,A,B)$	μ-partial reachability matrix of (E,A,B)	141

Subspaces

$\mathcal{E}_-(A), \mathcal{E}_+(A)$	sums of certain generalized eigenspaces	85
$\mathcal{N}(C,A)$	unobservable subspace of $\dot{x} = Ax$, $y = Cx$	29
$\mathcal{O}_*(\mathcal{U})$	min. observability subspace containing $\mathcal{U}$	30
$\mathcal{O}_{a*}(\mathcal{U})$	min. almost observab. subspace containing $\mathcal{U}$	30
$\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_\infty$	subspace sequence of AOSA and its limit	32
$\mathcal{R}(A,B)$	reachable subspace of $\dot{x} = Ax + Bu$	16
$\mathcal{R}(\mathcal{U})$	subspace of points reachable from $x(0) \in \mathcal{U}$	47
$\mathcal{R}^*(\mathcal{U})$	max. controllability subsp. contained in $\mathcal{U}$	17
$\mathcal{R}_a^*(\mathcal{U})$	max. almost controllab. subsp. contained in $\mathcal{U}$	17
$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^\infty$	subspace sequence of ACSA and its limit	19
$\mathcal{V}^*(\mathcal{U})$	max. (A,B)-invariant subsp. contained in $\mathcal{U}$	17
${\mathcal{V}_a}^*(\mathcal{U})$	max. almost (A,B)-inv. subsp. contained in $\mathcal{U}$	17
$\mathcal{V}^1, \mathcal{V}^2, \ldots, \mathcal{V}^\infty$	subspace sequence of ISA and its limit	19
$\mathcal{V}_*(\mathcal{U})$	min. (C,A)-invariant subsp. containing $\mathcal{U}$	30
$\mathcal{V}_{a*}(\mathcal{U})$	min. almost (C,A)-inv. subsp. containing $\mathcal{U}$	30
$\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_\infty$	subspace sequence of CISA and its limit	32

Maps

η	r.s.e. action on admissible triples	95
R_{k+1}	Kalman embedding	91
$R_{\mu+1}$	μ-partial Kalman embedding	92
σ	similarity action on controllable pairs	89

Topological spaces

$B^S_{\mathsf{V}^*}$	Bruhat cell	169
$\mathrm{C}_{\mathrm{abs}}(\mathbb{R}, \mathcal{F}^i)$	absolutely continuous funct. $f : [0,T] \longrightarrow \mathcal{F}^i$	14
$C_{k,p}(\mathcal{F})$	r.s.e. classes of controllable triples	96
$C_{k,p}(\mu)$	r.s.e. classes of μ-regular triples	97
$C^{\mathrm{t}}_{k,p}(\mu)$	r.s.e. classes of μ-tight triples	97
$\mathrm{C}^k_{\mathrm{pw}}$	piecewise continuous functions $f : [0,T] \longrightarrow \mathcal{F}^k$	26

Notation

System and subspace invariants

Miscellaneus

Index

Eingereicht am:

bei der Fakultät für Mathematik und Informatik.

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