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REACHABILITY MATRICES AND CYCLIC MATRICES*

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Abstract. We study reachability matrices $R(A,b) = [b, Ab, \dots, A^{n-1}b]$, where A is an $n \times n$ matrix over a field K and b is in K^n. We characterize those matrices that are reachability matrices for some pair (A,b). In the case of a cyclic matrix A and an n-vector of indeterminates x, we derive a factorization of the polynomial $\det\big(R(A,x)\big)$.

Key words. Reachability matrix, Krylow matrix, cyclic matrix, nonderogatory matrix, companion matrix, Vandermonde matrix, Hautus test.

AMS subject classifications. 15A03, 15A15, 93B05.

1. Introduction.

Let K be a field, and $A \in K^{n\times n}$, $b \in K^n$. The matrix

$$R(A,b) = [b, Ab, \dots, A^{n-1}b] \in K^{n\times n}$$

is the *reachability matrix* of the pair (A,b). A matrix A is called *cyclic* (e.g. in [3], [4]) or *nonderogatory* (e.g. in [2], [9]), if there exists a vector $b \in K^n$ such that

$$\operatorname{span}\{b, Ab, A^2b, \dots, A^{n-1}b\} = K^n. \tag{1.1}$$

In that case the pair (A,b) is said to be *reachable*. Let

$$a(z) = z^n - (a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) \tag{1.2}$$

be the characteristic polynomial of A. The matrix

$$F_a = \begin{bmatrix} 0 & . & & & a_0 \\ 1 & 0 & . & & a_1 \\ & . & . & . & \\ & & . & . & . \\ & & & . & . & . \\ 0 & 0 & & 1 & a_{n-1} \end{bmatrix} \tag{1.3}$$

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is the *companion matrix of the second type* [1] associated with (1.2). It is well known (see e.g. [4, p. 299]) that A is cyclic if and only if A is similar to the companion matrix F_a. Or equivalently, if $x_0, \ldots, x_{n-1}$ are indeterminates over K and $x := [x_0, \ldots, x_{n-1}]^\top$, then A is cyclic if and only if the polynomial $\det R(A, x)$ is not the zero polynomial.

In this note we are concerned with the following questions. When is a given matrix $M \in K^{n\times n}$ a reachability matrix? How can one factorize the polynomial $\det R(A, x)$?

2. Companion and reachability matrices. In this section we characterize those matrices that are reachability matrices for some pair (A, b). We first show that each nonsingular matrix M is a reachability matrix. Let

$$e_0 = [1, 0, \ldots, 0]^\top, \ \ldots, \ e_{n-1} = [0, \ldots, 0, 1]^\top,$$

be the unit vectors of K^n.

THEOREM 2.1. *Let $M = [v_0, v_1, \ldots, v_{n-1}] \in K^{n\times n}$ be nonsingular. Then $M = R(A, b)$ if and only if $b = v_0$ and $A = MF_cM^{-1}$ for some nonsingular companion matrix F_c. In particular, $M = R(A, v_0)$ with*

$$A = [v_1, \ldots, v_{n-1}, v_0]\, M^{-1}. \tag{2.1}$$

Proof. We have $e_0 = M^{-1}v_0$. Hence, if $b = v_0$ and $A = MF_cM^{-1}$ then $A^ib = MF_c^ie_0 = Me_i = v_i$, and thus $M = R(A, b)$. We obtain (2.1) if we choose $F_c = (e_1, e_2, \ldots, e_{n-1}, e_0)$. Conversely, if $M = R(A, b)$, then $b = v_0$, and $AM = MF_a$ for some companion matrix F_a. Hence if M is nonsingular then $A = MF_aM^{-1}$. ◻

THEOREM 2.2. *Let $M = [v_0, v_1, \ldots, v_{n-1}] \in K^{n\times n}$ and* $\operatorname{rank} M = r$. *The following statements are equivalent.*
(i) *M is a reachability matrix.*
(ii) *Either* $\operatorname{rank} M = n$, *i.e. the matrix M is nonsingular, or*

$$\operatorname{rank} M = \operatorname{rank}\,[v_0, v_1, \ldots, v_{r-1}] = r < n \tag{2.2}$$

and

$$\operatorname{Ker}[v_{k-1}, v_k, \ldots, v_{k-1+r}] \subseteq \operatorname{Ker}[v_k, v_{k+1}, \ldots, v_{k+r}], \quad k = 1, 2, \ldots, n-1-r. \tag{2.3}$$

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Proof. If (2.2) holds then (2.3) means that there exist $c_i \in K$, $i = 0, \dots, r-1$, such that

$$v_{r+k} = \sum_{i=0}^{r-1} c_i v_{i+k}, \;\; k = 0, 1 \dots, n-r-1. \tag{2.4}$$

(i) $\Rightarrow$ (ii) If M is a reachability matrix and $\operatorname{rank} M = r < n$ then it is obvious that the conditions (2.2) and (2.4) are satisfied.
(ii) $\Rightarrow$(i) If $\operatorname{rank} M = n$ then it follows from Theorem 2.1 that M is a reachability matrix. Now assume (2.2) and (2.4). Let $Q \in K^{n\times n}$ be nonsingular such that

$$Q\,[v_0, \dots, v_{r-1}] = \begin{bmatrix} I_r \\ 0 \end{bmatrix}.$$

Let $\hat{e}_0, \dots, \hat{e}_{r-1}$ be the canonical unit vectors of K^r. Then (2.4) implies

$$QM = \begin{bmatrix} w_0 & \dots & w_{r-1} & w_r & \dots & w_{n-1} \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \hat{e}_0 & \dots & \hat{e}_{r-1} & w_r & \dots & w_{n-1} \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{2.5}$$

and the vectors w_i satisfy

$$w_{r+k} = \sum_{i=0}^{r-1} c_i w_{i+k}, \;\; k = 0, 1 \dots, n-r-1.$$

Set

$$\hat{A} = \begin{bmatrix} 0 & 0 & & & c_0 \\ 1 & 0 & & & c_1 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & 0 & & 1 & c_{r-1} \end{bmatrix}. \tag{2.6}$$

Then

$$w_r = [c_0, c_1, \dots, c_{r-1}]^\top = \hat{A}\hat{e}_{r-1} = \hat{A}\,\hat{A}^{r-1}\hat{e}_0 = \hat{A}^r\hat{e}_0,$$

and

$$[\hat{e}_0,\, \hat{A}\hat{e}_0,\, \dots,\, \hat{A}^{r-1}\hat{e}_0,\, \hat{A}^r\hat{e}_0,\, \dots,\, \hat{A}^{n-1}\hat{e}_0] = [\hat{e}_0, \dots, \hat{e}_{r-1}, w_r, \dots, w_{n-1}].$$

Hence the matrix QM in (2.5) can be written as

$$QM = R(A, b) \quad \text{with} \quad A = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \in K^{n\times n},\; b = \begin{bmatrix} \hat{e}_0 \\ 0 \end{bmatrix} \in K^n,$$

and we obtain $M = R(Q^{-1}AQ, Q^{-1}b)$. ∎

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ELA
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3. A factorization theorem. We first characterize companion matrices in terms of reachability matrices. Let $b = (b_0, b_1, \dots, b_{n-1})^\top \in K^n$. We call

$$b(z) = (1, z, \dots, z^{n-1})\, b = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} \tag{3.1}$$

the polynomial associated to b.

Proposition 3.1. *Let $a(z) = z^n - \sum a_i z^i$ be the characteristic polynomial of $A \in K^{n\times n}$, and let F_a be the companion matrix in* (1.3)*. Then $A = F_a$ if and only if*

$$R(A, b) = b(A) \quad for\ all \quad b \in K^n. \tag{3.2}$$

Proof. It is obvious that A is a companion matrix of the form (1.3) if and only if

$$A[e_0, e_1, \dots, e_{n-2}] = [e_1, e_2, \dots, e_{n-1}]. \tag{3.3}$$

Assume now that (3.2) is satisfied. Choose $b = e_0$. Then $b(z) = 1$ and $b(A) = I$. Therefore

$$R(A, e_0) = [e_0, Ae_0, \dots, A^{n-1}e_0] = I = [e_0, e_1, \dots, e_{n-1}].$$

Hence we obtain (3.3), and we conclude that $A = F_a$. To prove the converse we have to show that

$$R(F_a, b) = b(F_a) \tag{3.4}$$

holds for all $b = \sum_{i=0}^{n-1} b_i e_i$. We have $e_i = F_a^i e_0,\ \ i = 0, \dots, n-1$. From $R(F_a, e_0) = I$ and $R(F_a, e_i) = F_a^i\, R(F_a, e_0)$ follows $R(F_a, e_i) = F_a^i$. Therefore

$$R(F_a, b) = \sum\nolimits_{i=0}^{n-1} b_i\, R(F_a, e_i) = \sum\nolimits_{i=0}^{n-1} b_i F_a^i = b(F_a). \square$$

Suppose A is cyclic. Let S be nonsingular such that $SAS^{-1} = F_a$, and let the polynomial $(Sb)(z)$ be defined in analogy to (3.1). Then $S\, R(A, b) = R(F_a, Sb)$. From (3.2) we obtain

$$R(A, b) = S^{-1}\ (Sb)(F_a). \tag{3.5}$$

Note that for all $b \in K^n$ we have $A\, R(A, b) = R(A, b)\, F_a$. Hence, if the pair (A, b) is reachable then the matrix $S = R(A, b)^{-1}$ satisfies

$$SAS^{-1} = F_a. \tag{3.6}$$

The identity (3.6) can be found in [6, Section 6.1].

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ELA
http://math.technion.ac.il/iic/ela

For the following observation we are indebted to a referee. Suppose A is a matrix with distinct eigenvalues, and $X^{-1}AX = D$ is a Jordan form. Then the corresponding companion matrix F_a is similar to the diagonal matrix D. The similarity transformation $VF_aV^{-1} = D$ is accomplished by a Vandermonde matrix V whose nodes are the eigenvalues of A (see e.g. [12, Section 1.11]). One can write V as a reachability matrix, that is, $V = R(D, e)$, where $e = (1, 1, \ldots, 1)^\top$.

Let $a_j(z)$, $j = 1, \ldots, r$, be the monic irreducible factors of the polynomial $a(z)$ in (1.2). Suppose $\deg a_j(z) = \ell_j$ and

$$a(z) = a_1(z)^{m_1} \cdots a_r(z)^{m_r}, \tag{3.7}$$

such that $\sum_{j=1}^{r} m_j\ell_j = n$. Let $F_{a_j} \in K^{\ell_j\times\ell_j}$ be the corresponding companion matrices. The main result of this section is the following.

THEOREM 3.2. *Let A be cyclic with characteristic polynomial $a(z)$, and let* (3.7) *be the prime factorization of $a(z)$. Suppose*

$$SAS^{-1} = F_a \quad \textit{and} \quad \det S = 1. \tag{3.8}$$

Set $g_j(x) = \det(Sx)(F_{a_j})$, $j = 1, \ldots, r$. Then

$$\det R(A, x) = \big(g_1(x)\big)^{m_1} \cdots \big(g_r(x)\big)^{m_r}. \tag{3.9}$$

The polynomials $g_1(x), \ldots, g_r(x)$ are irreducible, and homogeneous of degree $\ell_1, \ldots, \ell_r$, respectively.

Proof. Define

$$C(a_j, m_j) = \begin{pmatrix} F_{a_j} & I_{\ell_j} & 0 & . & 0 \\ 0 & F_{a_j} & I_{\ell_j} & . & 0 \\ . & . & . & . & . \\ . & . & . & F_{a_j} & I_{\ell_j} \\ . & . & . & . & F_{a_j} \end{pmatrix}_{m_j\ell_j\times m_j\ell_j}.$$

Then

$$TF_aT^{-1} = \operatorname{diag}\Big(C(a_1, m_1), \ldots, C(a_r, m_r)\Big) \tag{3.10}$$

for some $T \in K^{n\times n}$. The right hand side of (3.10) is the rational canonical form (the Frobenius canonical form) of F_a. Let $\hat{b} \in K^n$. From (3.4) be obtain

$$R(F_a, \hat{b}) = \hat{b}(F_a) = T^{-1}\,\hat{b}\Big[\operatorname{diag}\Big(C(a_1, m_1), \ldots, C(a_r, m_r)\Big)\Big]\,T.$$

Hence

$$\det R(F_a, \hat{b}) = \det \hat{b}(C(a_1, m_1)) \cdots \det \hat{b}(C(a_r, m_r)) = \big(\det \hat{b}(F_{a_1})\big)^{m_1} \cdots \big(\det \hat{b}(F_{a_r})\big)^{m_r}. \tag{3.11}$$

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ELA
http://math.technion.ac.il/iic/ela

Suppose $SAS^{-1} = F_a$ and $\det S = 1$. Then (3.5) implies $\det R(A, x) = \det R(F_a, Sx)$. Taking $\hat{b} = Sx$ in (3.11) we obtain (3.9).

Suppose one of the polynomials $g_j(x)$ is reducible. E.g. let $g_1(x) = p(x)q(x)$ and $\deg p \geq 1$, $\deg q \geq 1$. Then $x = S^{-1}(-1, z, 0, \ldots, 0)^\top$ yields $(Sx)(F_{a_1}) = -F_{a_1} + zI$. Hence $g_1(x) = \det(-F_{a_1} + zI) = a_1(z)$. On the other hand $g_1(x) = \tilde{p}(z)\tilde{q}(z)$, and $\deg \tilde{p} \geq 1$, $\deg \tilde{q} \geq 1$. This is a contradiction to the irreducibility of $a_1(z)$.

Since F_{a_j} is of size $\ell_j \times \ell_j$ we obtain $g_j(\lambda x) = \lambda^{\ell_j} g_j(x)$. Thus, $g_j(x)$ is homogeneous of degree ℓ_j. □

We now assume that the characteristic polynomial $a(z)$ of A splits over K. If $\lambda_1, \ldots, \lambda_r$ are the different eigenvalues of A then

$$a(z) = (z - \lambda_1)^{m_1} \ldots (z - \lambda_r)^{m_r}. \tag{3.12}$$

In that case $a_j(z) = (z - \lambda_j)$ and $F_{a_j} = (\lambda_j)$, and

$$g_j(x) = [1, \lambda_j, \lambda_j^2, \ldots, \lambda_j^{n-1}] S \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, \quad j = 1, \ldots, r. \tag{3.13}$$

The factors $g_j(x)$ in (3.13) are related to the Popov-Belevitch-Hautus controllability test (see e.g. [11, p. 93]). It is known that - up multiplicative constants - the row vectors

$$w_j^\top = [1, \lambda_j, \ldots, \lambda_j^{n-1}],\ j = 1, \ldots, r,$$

are the left eigenvectors of F_a. Then $SAS^{-1} = F_a$ implies that $v_j^\top = w_j^\top S$ are the left eigenvectors of A. Hence

$$v_j^\top b = g_j(b),\ j = 1, \ldots, r. \tag{3.14}$$

Therefore we obtain the PBH criterion in the special case of cyclic matrices.

Corollary 3.3. *Let A be cyclic. The following statements are equivalent.* (i) *The pair (A, b) is reachable.* (ii) *If*

$$v^\top (A - \lambda I) = 0,\ v \in K^n,\ v \neq 0,$$

then $v^\top b \neq 0$.

Proof. Because of (3.14) we can rewrite (ii) in the form

$$g_j(b) \neq 0,\ j = 1, \ldots, r. \tag{3.15}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

From (3.9) follows that (3.15) is equivalent to $\det R(A,b) \neq 0$. □

We illustrate Theorem 3.2 with an example. Consider

$$A = \begin{bmatrix} -6 & -38 & 6 & -4 & 281 \\ -11 & -131 & 10 & -5 & 928 \\ 11 & -155 & -6 & -16 & 1191 \\ 1 & -170 & 1 & -11 & 1253 \\ -1 & -21 & 1 & -1 & 151 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$R(A,b) = \begin{bmatrix} 3 & 6 & 5 & 2 & 6 \\ 0 & 7 & 0 & 0 & 8 \\ 4 & 9 & 6 & 3 & 4 \\ 0 & 7 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \ \det R(A,b) = 1,$$

and the pair (A,b) is reachable. Set $S = R(A,b)^{-1}$. Then

$$S = \begin{bmatrix} 3 & -16 & -2 & -3 & 133 \\ 0 & -1 & 0 & 0 & 8 \\ 0 & 2 & 0 & 1 & -21 \\ -4 & 19 & 3 & 2 & -150 \\ 0 & 1 & 0 & 0 & -7 \end{bmatrix},$$

and $SAS^{-1} = F_a$. We have

$$\det(zI - A) = a(z) = z^5 + 3z^4 - 6z^3 - 10z^2 - 21z - 9 = \\ (z-1)^3(z+3)^2 = \big(a_1(z)\big)^3\big(a_2(z)\big)^2.$$

Hence $\det R(A,x) \ = \ g_1(x)^3\, g_2(x)^2$ with

$$g_1(x) = [1,1,1,1,1]Sx = -x_0 + 5x_1 + x_2 - 37x_4$$

and

$$g_2(x) = [1,-3,9,-27,81]Sx = 111x_0 - 427x_1 - 83x_2 - 48x_3 + 3403x_4.$$

We conclude with some remarks which place our study into a larger context. Matrices of the form $\mathcal{K}_r(A,b) = [b, Ab, \ldots, A^{r-1}b]$, $1 \le r \le n$, are known as *Krylov matrices* (see e.g. [9, p. 646]). Thus $R(A,b) = \mathcal{K}_n(A,b)$. We refer to [8] for an investigation of numerical aspects of Krylov and reachability matrices. The concept of *Faddeev reachability matrix* was introduced in [5] and further elaborated in [10]. A "spectral factorization" of $R(A,b)$ is due to [7] (see also [13]).

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ELA
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