On the Fragility Index

Diana Tichy

Dissertation

Institut für Mathematik
Universität Würzburg

Lehrstuhl für Statistik
Institut für Mathematik
Universität Würzburg

On the Fragility Index

Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades der Bayerischen Julius Maximilians Universität Würzburg

vorgelegt von

DIANA TICHY

aus

Bad Brückenau

Dezember 2011

Contents

1. Introduction

Coast dwellers especially know that extreme wind speeds are accompanied by rough and heavy seas. The experience of an inshore storm can be an obvious example of the joint occurrence of extreme events that Multivariate Extreme Value Analysis deals with. The well-known and fascinating publication *Sea and Wind: Multivariate Extremes at Work* (1998) (cf. de Haan and Ronde [33]) analyzes this just mentioned extreme dependence. The article investigates the dependence structure between the co-players of storm events and engages in modeling and estimating the probability of failure of a dutch sea dike by order of the European Union project "Neptune" 1995-1997. But not only coast dwellers are threatened by extreme events like storm surges. Just the preceding years told us about flooding caused by heavy or long-lasting rainfall. Thus, we are not only interested in the maximum of a river's water level but also in how long the flood lasts and the question of which neighboring rivers of the water system are also affected by the flood.

This example of a storm upsurge may surely not be able to compare with the whole complexity of the current situation at the global financial market system, but it may serve as a first descriptive image. Geluk et al. [22] published a measure called the *fragility index*, which aims to describe the stability of a financial system under the influence of extreme events. Now, given a random system of dimension $d \in \mathbb{N}$ denoted by $\{Q_1, \dots, Q_d\}$. If one or several exceedances above a high threshold occur in a system, i.e., we observe $\{Q_j > s\}, s \nearrow$ for at least one $j \in \{1, \dots, d\}$, we are faced with the failure of at least one component, up to the total collapse of the whole system, under strong dependencies within the system. Considering a financial system of d banks, within which at least one bank crashes due to high liabilities, the *fragility index* (FI), if it exists, is defined as the limit of the expected number of exceedances above a high threshold among d random variables $Q_1, \dots, Q_d$ as the threshold increases, given that there is at least one exceedance. The larger the value of the FI, the stronger the dependence between extreme events among the system.

As we have already mentioned, the stability of a random system significantly depends on the probability of exceedance above a high threshold. According to Balkema and de Haan [3] and Pickands [53] the distribution of a random exceedance above a high threshold can be approximated by a generalized Pareto distribution. This fact is well-known as the *peaks-over-threshold approach* (POT), which goes back to Pickands [53], cf. for example, the summary in Chapter 5 in Reiss and Thomas [55] or Beirlant [5], as well as Chapter 7 in Embrechts et al. [13]. Since the fragility index is defined by means of a conditional expectation, its calculation is based on the random number of exceedances within the random system. However, in the current and past literature, less attention is paid to the distribution of the *number* of exceedances. We shall deal extensively with this topic within the work at hand. In doing so, we assume that the joint distribution F of the random vector representing the system $\{Q_1, \dots, Q_d\}$ belongs to the domain of attraction of a multivariate extreme value distribution (EVD). This is referred to as the *domain-of-attraction assumption*. It follows that the corresponding copula C_F can be approximated in its tail by a multivariate generalized Pareto distribution (GPD), which turns out to be a crucial result within this work. Further, conditions are provided under which the survival function of a GPD vanishes, i.e., there are no further possible exceedances in the random system. Another crucial result is the representation of the asymptotic conditional distribution of exceedance counts (ACDEC). The ACDEC serves as a main tool within the closed representation of the fragility index and its extension.

A multivariate extreme value distribution (EVD) is defined by the specification of its univariate margins and its so-called *dependence function* D, which describes the dependence structure among the univariate margins. Historically, several equivalent representations of an EVD evolved from the Balkema and Resnick representation, cf. Balkema and Resnick [4]. Within the work at hand, we prefer the representation of an EVD G by norms, i.e., $G(x_1, \dots, x_d) = \exp(\|(x_1, \dots, x_d)\|_D)$, where the so-called *D-norm* $\|\cdot\|_D$ is one possible representation of the dependence function D of an EVD among several others, cf. Falk et al. [19] or Hofmann [36]. Hence, we succeed in the representation of a compact form of the original representation of the fragility index in Geluk et al. [22]. Further, our representation of the fragility index coincides with the so-called *dependence coefficient* κ, defined by de Haan und Ferreira [29], cf. Section 7.4

therein. This is because the so-called *stable tail dependence function* l, cf. Huang [39], coincides with the D-norm, i.e. $l(x_1, \dots, x_d) = \|(x_1, \dots, x_d)\|_D$ for $x_j \leq 0, j \leq d \in \mathbb{N}$. Further analysis shows that the *extremal coefficient* ε, which goes back to Tiago de Oliveira [70] and was named by Smith [63], also coincides with the D-norm at point $(1, \dots, 1)$. The coefficient ε reflects the amount of dependence between the identically distributed univariate margins of an EVD. If we consider the exceedance of a component Q_j above a threshold that depends on the particular univariate margin F_j, a so-called *individual threshold*, we obtain the simple representation $FI = d/\varepsilon$ for the fragility index of the random system $\{Q_1, \dots, Q_d\}$. This is an important enhancement of the fragility index provided by Geluk et al. [22], who only consider exceedances above a threshold, which is common for every component of the system, a so called *common threshold*.

Another main result of the work at hand is the extension of the fragility index, the so-called *extended fragility index* $FI(m)$, which is the limit of the expected number of exceedances within the system, given there are at least m exceedances. Thus, we are able to capture the *development* of risk within the random system by an increasing number of collapsing components. We show that the existence of the $FI(m)$ for certain $m \leq d$ depends on the convergence behavior of the survival copulas corresponding to certain margins of the system‘s distribution F. If the $FI(m)$ exists for certain $m \leq d$, it can be represented as a function of the extremal coefficients corresponding to those margins G_K of the EVD G, for which we have $m \leq |K| \leq d$. Thereby we assume that the copula C_F of the random system $\{Q_1, \dots, Q_d\}$ belongs to the domain of attraction of an EVD. We have $FI(m) \in [m; d]$. Therefore, we call $\{Q_1, \dots, Q_d\}$ *m-stable* if $FI(m) = m$, and *fragile* if $FI(m) > m$. Hence, the $FI(m)$ turns out to be a suitable measure for the tail dependence (structure) of a multivariate distribution F, whenever F belongs to the domain of attraction of an EVD. Such a measure for tail dependence is not present within the literature at the time of writing. Those tail dependence measures that have been published so far mostly deal with the bivariate case, like the *upper and lower tail dependence coefficient* (cf. Geoffrey [25] and Sibuya [60]), or the multivariate *(stable) tail dependence function* (cf. Beirlant et al. [5], especially Section 9.4 as well as Section 8.2 and 8.3, or Heffernan [35] for an appealing summary for measures of tail dependence). Both the extremal coefficient and the (stable) tail dependence function have disadvantages, which the (extended) fragility index overcomes.

So far, we have considered the number of exceedances within a finite random system and captured its asymptotic stability by the (extended) fragility index. Within the work at hand we also want to deal with a stochastic process $(X_d)_{d\in\mathbb{N}}$, where we are interested in the number of *sequential* exceedances above a high threshold. This number is referred to as the *exceedance cluster length* and thus covers the (discrete) sojourn time of the stochastic process $(X_d)_{d\in\mathbb{N}}$. One may think of the earlier mentioned image of a river flooding its banks and of neighboring rivers, which are also affected by an extreme weather pattern. Beside our interest in the expected total number of threatened rivers within a certain area – captured by the $FI(m)$ – we are also interested in the *duration* of an extreme water level. With the existing tools taken from the work on the fragility index, we are able to provide the mean cluster size, i.e., the sojourn time of exceedance. This measure can be regarded as the fragility index for sojourn times and captures the amount of asymptotic dependence within a finite sequence of a stochastic system.
Further, it is well-known that the reciprocal of the mean cluster size coincides with the so-called *extremal index* (cf. Hsing et al. [38]). The extremal index captures the amount of dependence within a strictly stationary process, for an appealing summary cf. Section 8.1 in Embrechts et al. [13]. We show that the reciprocal of the fragility index of a finite sequence $(X_1, \dots, X_d)$ taken from $(X_d)_{d\in\mathbb{N}}$ coincides with the extremal index letting $d \to \infty$.

The last part of the work at hand provides a first approach towards an estimation of the (extended) fragility index. The main idea is based on the estimation of the extremal coefficients corresponding to the margins of the EVD G, to whose domain of attraction the system‘s distribution F belongs. Existing literature contains frequent procedures of estimating the stable tail dependence function, i.e. the dependence structure of an EVD. These estimators are predominantly based on the estimation of the Pickands dependence function (cf. for example Zhang et al. [71], Genest and Segers [24] or Gudendorf and Segers [27]). Further, they are restricted to the assumption that the observations follow exactly an EVD (EVD assumption). However, the fragility index is based on the weaker domain of attraction assumption. We therefore use an estimator for the extremal coefficient that is not restricted to the EVD assumption. Further, we want to use a nonparametric estimation approach, since we do not want to focus on a special parametric model for the dependence structure. The nonparametric estimator of

the stable tail dependence function provided in Section 7.2 in de Haan and Ferreira [29] turns out to be a suitable one for our purpose. The resulting nonparametric estimator of the fragility index is consistent and asymptotically normal distributed.
By means of a simulation study we investigate the behavior of the obtained estimator in dependence on the sample and chosen tail size as well as the amount of dependence between. We simulate from a 3-dimensional logistic EVD. Further, we investigate the mean squared error of the estimator and engage in the determination of an optimal tail size. Finally, we apply the obtained nonparametric estimator of the extended fragility index to a 3-dimensional German financial system and to an arbitrarily mixed system of companies, which are listed in the DAX. Thereby, the obvious assumption is confirmed that the financial system is of lower asymptotic stability than the mixed system.

Chapter 2 provides a short introduction to extreme value analysis and establishes necessary tools for the representation and extension of the fragility index with a special focus on the topics of tail dependence.

Chapter 3 covers the topic on events of exceedances, which serves as a crucial component in the framework of establishing the (extended) fragility index. Section 3.2 analyzes the conditions under which no further exceedances within a random system are possible. Section 3.3 is almost as important, since it provides the asymptotic conditional distribution of exceedance counts (ACDEC), which will be one of the two main tools for the representation of the (extended) fragility index.

Chapter 4 provides the representation of the fragility index by norms and its extension to a measure of the asymptotic system‘s stability where several exceedances within the system have already occurred, the extended fragility index. Thereby we distinguish between the approach of exceedances above an individual and a common threshold (cf. Section 4.2.1 and 4.2.2 and the discussion on the relevance of each approach in Section 4.2.3). Further, we shed light on the (extended) fragility index as a measure for tail dependence, which is new to the literature with respect to this approach. Section 4.4 provides the fragility index for sojourn times and links the fragility index with the well-known extremal index.

Chapter 5 shows a first approach in the estimation of the (extended) fragility index, which works under the domain-of-attraction approach. Thereby Section 5.1 engages in the theoretical background of the considered nonparametric estimator, while Section 5.2 provides a simulation study to investigate the behavior of the estimator with respect to sample size, tail size and tail dependence of the simulated data. At last we apply the estimator of the extended fragility index to financial data in Section 5.3.

Nomenclature, general information on this work at hand and basic auxiliary results, as well as outlined figures and tables, can be found in the appendix (cf. Chapter 7).

2. Setting the stage

This chapter provides the framework of multivariate extreme value analysis and other tools connected to multivariate analysis, which are used within this work at hand. Results shown here will be necessary for the theoretical results concerning events of exceedances and the construction of the fragility index and its applications.
The author wants the reader to know that this chapter is to be read as a "short story about things one should know to begin the story about the fragility index". The sections are not meant to divide this chapter into independent readable parts of this story and are only provided for sake of clarity. Since many of the following provided results can be found in literature, we often skimp the proofs or even refer to the literature.

2.1. A Short Introduction to Extreme Value Distributions

We want to start with a short preface to the family of extreme value distributions (EVD). Let $\boldsymbol{Y}_i = \left(Y_i^{(1)}, \dots, Y_i^{(d)}\right)$, $i \le n$, be iid d-variate random vectors with common df F. The d-variate maximum is defined by

$$\boldsymbol{M}_n := \max_{i \le n} \boldsymbol{Y}_i := \left(\max_{i \le n} Y_i^{(1)}, \dots, \max_{i \le n} Y_i^{(d)} \right),$$

that means the maximum of the random vectors is taken component-wise.
We call G a (multivariate) *extreme value distribution* (abbr. EVD), if there exist normalizing constants $a_n^{(j)} > 0$ and $b_n^{(j)} \in \mathbb{R}$, $j \le d$ such that

$$\begin{aligned} P(\boldsymbol{M}_n \le \boldsymbol{a}_n \boldsymbol{x} + \boldsymbol{b}_n) &= P\left(\frac{M_n^{(j)} - b_n^{(j)}}{a_n^{(j)}} \le x^{(j)}, j \le d \right) \\ &= F^n\left(a_n^{(1)} x^{(1)} + b_n^{(1)}, \dots, a_n^{(d)} x^{(d)} + b_n^{(d)} \right) \to G(\boldsymbol{x}) \qquad (2.1) \end{aligned}$$

holds for $n \to \infty$, where G is a non-degenerate df. G is uniquely defined by its dependence structure and its one-dimensional marginal distributions. The one-dimensional marginal dfs of G are univariate EVDs and turn out to be dfs of three possible types, characterized by the families of univariate extreme value distributions, namely the family of the Weibull, Fréchet or the Gumbel distributions. Further the components of the normalizing vectors $\boldsymbol{a}_n$, $\boldsymbol{b}_n$ are the normalizing constants in the univariate case.

The following results and their proofs as well (of this section) are taken from Resnick [56], Chapter 5.

A df F is called *max-stable* if for each $n \in \mathbb{N}$,

$$F^n(\boldsymbol{d}_n + \boldsymbol{c}_n \boldsymbol{x}) = F(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^d,$$

holds for some vectors $\boldsymbol{c}_n > 0$ and $\boldsymbol{d}_n \in \mathbb{R}^d$. Max-stable dfs are the limiting dfs of linearly normalized maxima. Hence, one is able to give a characterization of EVDs by the max-stable dfs.

Theorem 2.1.1 *The class of multivariate extreme value distributions is precisely the class of max-stable distribution functions with non degenerate margins.*

Proof: cf. Prop. 5.9 in Resnick [56]. □

Since we need the benefits of different representations of an EVD within our work on the fragility index, we want to introduce those in the following. In order to follow the historically development we will start with the so called max-id dfs, which are an extension of max-stable dfs.

A df F on $\mathbb{R}^d$ is called *max-infinitely divisible (max-id)* if for every n there exists a df F_n on $\mathbb{R}^d$ such that

$$F = F_n^n$$

holds that means $F^{1/n}$ is a df as well.

Lemma 2.1.2 *F is max-id if and only if F^t is a df for all $t > 0$.*

Proof: cf. Proposition 5.1 in Resnick [56]. □

It turns out that any max-id df F can be represented in terms of an exponent measure ν, i.e.

$$F(\boldsymbol{x}) = \exp\left(-\nu([-\boldsymbol{\infty}, \boldsymbol{x}]^{\complement})\right)$$

for every $\boldsymbol{x} \in [l, \infty]^d \backslash \{\boldsymbol{l}\}$ with $\boldsymbol{l} \in [-\infty, \infty)^d$ (cf. Section 5.3. in Resnick [56]).
Thereby, a σ-finite measure ν on the set $E := [l, \infty]^d \backslash \{\boldsymbol{l}\}$ for $\boldsymbol{l} \in [-\infty, \infty)^d$ is called an *exponent measure*, if it satisfies

$$\nu\left(\mathbb{R}^{j-1} \times [-\infty, \infty) \times \mathbb{R}^{d-j}\right) = \infty$$

for $j \leq d$ and

$$\nu\left([-\boldsymbol{\infty}, \boldsymbol{x}_0]^{\complement}\right) < \infty$$

for some $\boldsymbol{x}_0 \in \mathbb{R}^d$.

Proposition 2.1.3 (Balkema and Resnick) *A df F is max-id if and only if there exists an exponent measure ν on $E := [l, \infty]^d \backslash \{\boldsymbol{l}\}$ for some $\boldsymbol{l} \in [-\infty, \infty)^d$ such that*

$$F(\boldsymbol{x}) = \begin{cases} \exp\left(-\nu([-\boldsymbol{\infty}, \boldsymbol{x}]^{\complement})\right), & \boldsymbol{x} \geq \boldsymbol{l} \\ 0, & \textit{otherwise} \end{cases} \tag{2.2}$$

holds.

Proof: cf. Proposition 5.8 in Resnick [56] or Balkema and Resnick [4]. □

Since a max-stable df F is max-id, we get by Theorem 2.1.1 that an extreme value distribution G can be represented by

$$G(\boldsymbol{x}) = \exp\left(-\nu([-\boldsymbol{\infty}, \boldsymbol{x}]^{\complement})\right)$$

for $\boldsymbol{x} \geq \boldsymbol{l} \in [-\infty, \infty)^d$. In the following we want to mark the development of the de-Haan-Resnick-representation of an EVD coming from the representation via the exponent measure. For detailed information see Section 5.4. in Resnick [56].
Suppose, the convergence $P\left(\frac{M_n^{(j)} - b_n^{(j)}}{a_n^{(j)}} \leq x^{(j)}, j \leq d\right) \to G(\boldsymbol{x})$ in (2.1) holds.
The max-stability of a df $\tilde{G}$ having standard Fréchet margins $\tilde{G}_j(x) = \exp(-1/x)$, $x > 0, j \leq d$, implies $\tilde{G}^t(t\boldsymbol{x}) = \tilde{G}(\boldsymbol{x})$ for $\boldsymbol{x} > \boldsymbol{0}$. Since the margins of $\tilde{G}$ concentrate on $(0, \infty)$, the exponent measure ν_* regarding to $\tilde{G}$ concentrates on $E : [0, \infty]^d \backslash \{\boldsymbol{0}\}$. Further $\tilde{G}^t(t\boldsymbol{x}) = \tilde{G}(\boldsymbol{x})$ can be translated in a homogeneity property for ν_* that means it is equivalent to $\nu_*([\boldsymbol{0}, \boldsymbol{x}]^{\complement}) = t\nu_*(t[\boldsymbol{0}, \boldsymbol{x}]^{\complement})$ for $t > 0, \boldsymbol{x} \geq 0, \boldsymbol{x} \neq \boldsymbol{0}$ leading to $\nu_*(B) = t\nu_*(tB)$ for all Borel subsets of E. Suppose $\|\cdot\|$ to be an arbitrary norm in $\mathbb{R}^d$ and denote by $S_E := \{\boldsymbol{z} \in E : \|\boldsymbol{z}\| = 1\}$ the unit sphere in E. The transformation $\boldsymbol{x} \to (\|\boldsymbol{x}\|, \|\boldsymbol{x}\|^{-1}\boldsymbol{x})$ leads to

$$\nu_*\left(\boldsymbol{y} \in E : \|\boldsymbol{y}\| > r, \|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in A\right) = r^{-1}\mu(A)$$

with $r > 0$ for any Borel subset $A \in S_E$, where μ defined by

$$\mu(A) = \nu_* \left(\boldsymbol{x} : \|\boldsymbol{x}\| > 1, \|\boldsymbol{x}\|^{-1} \boldsymbol{x} \in A\right) \tag{2.3}$$

is called the *angular measure*. Now denote by $T : E \to ((0,\infty) \times S_E)$ via $T\boldsymbol{y} = \left(\|\boldsymbol{y}\|, \|\boldsymbol{y}\|^{-1} \boldsymbol{y}\right)$ the transformation of a vector onto its polar coordinates. Thus, with the well-known transformation for integrals we get $\nu_* \circ T^{-1}(dr, d\boldsymbol{a}) = r^{-2} dr \mu(d\boldsymbol{a})$. Finally note that $T([\boldsymbol{0}, \boldsymbol{x}]^{\complement}) = \left((r, \boldsymbol{a}) : r > \max_{j\leq d}\left(\frac{x_j}{a_j}\right)\right)$ and we have

$$\nu_* \left([\boldsymbol{0}, \boldsymbol{x}]^{\complement}\right) = \int_{S_E} \max_{i\leq d} \left(\frac{a_j}{x_j}\right) \mu(d\boldsymbol{a}) \, .$$

Since this is a very short abstract of the approach to the de Haan-Resnick representation of an EVD we refer to Section 5.4. in Resnick [56] where the sketch above is taken from. Now we are able to present the de Haan-Resnick representation of an EVD with standard Fréchet margins $G(x) = \exp\left(-x^{-1}\right), x > 0$, which goes back to de Haan and Resnick [31].

Theorem 2.1.4 (De Haan-Resnick Representation of an EVD) *Any max-stable df with standard Fréchet margins can be represented by*

$$\tilde{G}(\boldsymbol{x}) = \exp\left(-\int_{S_E} \max_{j\leq d}\left(\frac{a_j}{x_j}\right) d\mu(\boldsymbol{a})\right), \quad \boldsymbol{x} \in (\boldsymbol{0}, \boldsymbol{\infty}), \tag{2.4}$$

where $E := (\boldsymbol{0}, \boldsymbol{\infty})$, $S_E := \{\boldsymbol{z} \in E : \|\boldsymbol{z}\| = 1\}$ *is the unit sphere and* μ *the angular measure (cf. (2.3)) on* S_E *which satisfies*

$$\int_{S_E} a_j d\mu(\boldsymbol{a}) = 1\,, \quad j \leq d\,.$$

Proof: cf. Proposition 5.11. in Resnick [56]. □

We obtain the *Pickands representation* of a max-stable df G by transforming the margins of $\tilde{G}$ to standard reverse exponential margins. Note that the property of max-stability is preserved by the transformation of the univariate margins (cf. Lemma 5.6.8 in Falk et al. [19]). Suppose a max-stable df $\tilde{G}$ with standard Fréchet margins $G(x) = \exp\left(-x^{-1}\right)$, $x > 0$, then $G(\boldsymbol{x}) = \tilde{G}\left(-x_1^{-1}, \ldots, -x_d^{-1}\right)$ is a max-stable df with standard reverse exponential margins $G(x) = \exp(x), x \leq 0$, cf. Remark 2.1.12 for details on transformation. From Theorem 2.1.4 we attain

$$G(\boldsymbol{x}) \;=\; \tilde{G}\left(-\frac{1}{x_1}, \ldots, -\frac{1}{x_d}\right) = \exp\left(\int_{S_E} \min_{j\leq d}(a_j x_j) d\mu(\boldsymbol{a})\right), \; \boldsymbol{x} \leq \boldsymbol{0}. \tag{2.5}$$

Now choose the L_1-norm and S_E switches to the unit simplex $S := \Big\{\boldsymbol{u} : \sum_{j\le d} u_j = 1, u_j \ge 0\Big\}$.

The following theorem provides the representation of max-stable dfs with standard reverse exponential margins.

Theorem 2.1.5 (Pickands Representation of an EVD) *A function G is a max-stable, d-variate df with standard reverse exponential margins if and only if there exists an angular measure μ on the d-variate unit simplex $S = \Big\{\boldsymbol{u} : \sum_{j\le d} u_j = 1,\ u_j \ge 0\Big\}$ having the property $\int_S u_j d\mu(\boldsymbol{u}) = 1$ for $j \le d$, such that*

$$G(\boldsymbol{x}) = \exp\left(-\int_S \max_{j\le d}(-u_j x_j) d\mu(\boldsymbol{u})\right) \tag{2.6}$$

holds for $\boldsymbol{x} \le \mathbf{0}$.

Proof: For the if-part, cf. Theorem 4.3.1 in Falk et al. [19]. For the converse implication see Theorem 5.4.5 in Galambos [20]. As an alternative, Theorem 2.1.5 can be seen as a corollary of Theorem 2.1.4 by just switching the margins. □

Within the following part of this section we demonstrate how to provide a representation of EVDs by norms. The norm-representation will be crucial for applications on the fragility index.

Definition 2.1.6 *Let μ be an angular measure on the unit sphere S as defined in (2.3). Then the norm $\|\cdot\|_D : \mathbb{R}^d \to \mathbb{R}^d_+$ defined by*

$$\|\boldsymbol{x}\|_D := \int_S \max_{j\le d}(|x_j|\, u_j)\, \mu(d\boldsymbol{u})$$

is called D-norm on $\mathbb{R}^d$.

Since the angular measure uniquely determines the D-norm, we say the angular measure *induces* the D-norm. The following corollary arises from Theorem 2.1.5 and provides a characterization of a D-norm.

Corollary 2.1.7 *A norm $\|\cdot\|$ on $\mathbb{R}^d$ is a D-norm if and only if*

$$G(\boldsymbol{x}) := \exp\left(-\|\boldsymbol{x}\|\right), \qquad \mathbf{0} \ge \boldsymbol{x} \in \mathbb{R}^d,$$

defines a df with standard negative exponential margins.

Indeed, the function $\|\cdot\|_D$ defines a *norm* on $\mathbb{R}^d$. This can be shown by means of the *Pickands dependence function* $D : R \to [0, \infty)$, which is defined by

$$D(t_1, \ldots, t_{d-1}) := \int_S \max\left(u_1 t_1, \ldots, u_{d-1} t_{d-1}, u_d \left(1 - \sum_{j \le d-1} t_j\right)\right) d\mu(\boldsymbol{u}),$$

where the domain of D is given by $R := \{(t_1, \ldots, t_{d-1}) \in [0,1]^{d-1} \mid \sum_{j \le d-1} t_j \le 1\}$, cf. Section 4.4 in Falk et al. [19]. The D-norm, as defined in Definition 2.1.6, can be also defined by means of the Pickands dependence function, i.e.

$$\|\boldsymbol{x}\|_D := \left(\sum_{j \le d} |x_j|\right) D\left(\frac{|x_1|}{\sum_{j \le d} |x_j|}, \ldots, \frac{|x_{d-1}|}{\sum_{j \le d} |x_j|}\right) \tag{2.7}$$

with the convention $\|\mathbf{0}\|_D = 0$, cf. Equation (4.36) in Falk et al. [19]. Properties of the Pickands dependence function can be found in Section 4.3 in Falk et al. [19]. They especially imply that

$$\|\boldsymbol{x}\|_\infty := \max(|x_1|, \ldots, |x_d|) \le \|\boldsymbol{x}\|_D \le \|\boldsymbol{x}\|_1 := \sum_{j \le d} |x_j| \tag{2.8}$$

holds for every $\boldsymbol{x} \in \mathbb{R}^d$. Hence, any EVD G (recall that we can transform an arbitrary EVD to an EVD with standard negative exponential margins) satisfies the inequalities

$$\prod_{j \le d} \exp(x_j) \le G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D) \le \exp(\min(x_1, \ldots, x_d))$$

for $\boldsymbol{x} \le \mathbf{0}$. Furthermore, the D-norm is standardized, i.e. $\|\boldsymbol{e}_j\|_D = 1$, $j \le d$. The monotonicity of G implies that the D-norm is monotone, i.e. for arbitrary $\mathbf{0} \le \boldsymbol{x} \le \boldsymbol{y}$ we have $\|\boldsymbol{x}\|_D \le \|\boldsymbol{y}\|_D$.

The Pickands representation of a multivariate EVD G on $\mathbb{R}^d$ with standard negative exponential margins, cf. Theorem 2.1.5, further justifies the following characterization of a D-norm.

Lemma 2.1.8 *A norm $\|\cdot\|$ on $\mathbb{R}^d$ is a D-norm if, and only if, there exists a random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d) \in [0, c]^d$ for some $c \ge 1$, with $E(Z_j) = 1$, $1 \le j \le d$, such that*

$$\|\boldsymbol{x}\| = E\left(\max_{j \le d}(|x_j| Z_j)\right), \qquad \boldsymbol{x} \in \mathbb{R}^d. \tag{2.9}$$

In this case we call $\boldsymbol{Z}$ a generator of the D-norm $\|\cdot\|$.

Proof: The assertion follows with Definition 2.1.6 and the following transformation: note that μ/d defines a probability measure, hence there exists a random variable $\tilde{Z}$ which distribution is determined by μ/d. Now set $Z := d \cdot \tilde{Z}$. Then the assertion follows for $\boldsymbol{Z} \geq \mathbf{0}$ with $\|\boldsymbol{Z}\| = d$.
We want to remark that the assertion can be regarded as an implication of Proposition 2.3 in Aulbach et al. [2], too, which provides the representation of a max-stable process $\eta \in \bar{C}^-[0,1] := \{f \in C[0,1] : f \leq 0\}$. (2.9) is the assertion for the finite dimensional marginals of η. □

Of course it is possible to represent an EVD with *arbitrary* margins by means of the D-norm, cf. Proposition 2.1.10.

Furthermore it suggests itself to ask which conditions on a norm $\|\cdot\|$ have to be required such that this norm is a D-norm that means, $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|)$ defines an EVD with standard reverse exponential margins.
Indeed, this is a complex question and has been solved by Hofmann [36]. Since it is a basic result concerning the representation of the fragility index by norms, we want to provide it in the following theorem.

Theorem 2.1.9 *For any norm $\|\cdot\|$ on $\mathbb{R}^d$ the following assertions are equivalent:*

(i) the function $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|)$, $\boldsymbol{x} \leq \mathbf{0}$, defines a d-variate extreme value distribution (with standard reverse exponential margins)

(ii) there exists a measure ν on $[-\infty, \infty)\backslash\{-\infty\}$ with

$$\nu\left([-\infty, \boldsymbol{x}]^{\complement}\right) = \begin{cases} \|\boldsymbol{x}\|, & \boldsymbol{x} \leq \mathbf{0} \\ 0, & \textit{otherwise} \end{cases}$$

(iii) the norm satisfies

$$\sum_{\mathbf{m} \in \{0,1\}^d:\, m_i = 1,\, i \in K} (-1)^{d+1-\sum_{j \leq d} m_j} \left\| \left(b_1^{m_1} a_1^{1-m_1}, \dots, b_d^{m_d} a_d^{1-m_d}\right) \right\| \geq 0 \tag{2.10}$$

for every nonempty $K \subseteq \{1, \dots, d\}$, $K \neq \{1, \dots, d\}$, and $-\infty < a_j \leq b_j \leq 0$, $1 \leq j \leq d$.

A norm which fulfills one condition from above will be called a D-norm, now denoted by $\|\cdot\|_D$.

Proof: cf. Theorem 3.1 in Hofmann [36]. □

Hence, condition (2.10) is sufficient and necessary for $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|)$ defining an EVD with standard reverse exponential margins solely by means of elementary tools.

Since we do not want to restrict the representation of an EVD by norms on standard reverse exponential margins as done in Corollary 2.1.7, we provide the following proposition. It shows in detail how to work with an EVD with arbitrary margins using the representation by norms. Within our following work we will focus on the representation of EVDs by norms.

Proposition 2.1.10 *An arbitrary d-dim. EVD G can be represented by*

$$G_{\boldsymbol{\alpha}}(x_1, \ldots, x_d) = \exp(-\|\psi_{\alpha_1}(x_1), \ldots, \psi_{\alpha_d}(x_d)\|_D) \tag{2.11}$$

with $\psi_{\alpha_j}(x) := \log(G_{\alpha_j}(x))$, $j \leq d$, *where* G_{α_j} *is the j-th margin of the EVD G and therefore belongs to the family of non-degenerate univariate EVDs. In their standard version these are given by*

$$G_\alpha(x) := \begin{cases} \exp(-(-x)^\alpha), & x \leq 0 \\ 1, & x > 0 \end{cases} \quad \textit{for } \alpha > 0,$$

$$G_\alpha(x) := \begin{cases} 0, & x \leq 0 \\ \exp(-x^\alpha), & x > 0 \end{cases} \quad \textit{for } \alpha < 0$$

and

$$G_0(x) := \exp(-\exp(-x)), \quad x \in \mathbb{R}.$$

with the parameter $\alpha \in \mathbb{R}$. *Hence,* ψ *is precisely defined by with*

$$\psi_{\alpha_j}(x) = \log(G_{\alpha_j}(x)) = \begin{cases} -(-x)^{\alpha_j}, & x < 0, \quad \alpha_j > 0, \\ -x^{\alpha_j}, & x > 0, \quad \alpha_j < 0, \\ -\exp(-x), & x \in \mathbb{R}, \quad \alpha_j = 0, \end{cases} \tag{2.12}$$

The first family is the Weibull, the second the Fréchet and the third is the Gumbel distribution.

Proof: We have to show that $G_{\boldsymbol{\alpha}}$ defines a max-stable df with univariate margins $G_{\alpha_j}(x) = \exp(\psi_{\alpha_j}(x))$. This follows by Lemma 5.6.8 in Falk et al. [19]. □

The univariate margins of a multivariate EVD can be provided in a closed form as well. The following representation of an univariate EVD is especially important within estimation of the parameters defining the family of univariate EVD (cf. Section 5.1 in Beirlant et al. [5]).

Definition 2.1.11 (Jenkinson-von Mises representation) *The families of univariate extreme value distributions are given in the closed form*

$$H_\xi(x) := \begin{cases} \exp\left(-(1+\xi x)^{-1/\xi}\right), & \xi \neq 0, \\ \exp\left(-\exp(-x)\right), & \xi = 0, \end{cases}$$

where $1+\xi x > 0$. H_ξ *is called the standard generalized extreme value distribution (GEV). The corresponding location-scale family* $H_{\xi;\mu,\beta}$ *is defined by replacing* x *by* $(x-\mu)/\beta$ *for* $\mu \in \mathbb{R}$ *and* $\beta > 0$. *The support has to be adjusted accordingly. Note that* $\xi = -\alpha^{-1} < 0$ *corresponds to the Weibull,* $\xi = \alpha^{-1} > 0$ *corresponds to the Fréchet and* $\xi = 0$ *corresponds to the Gumbel distribution with* α *given in Proposition 2.1.10.*

The following remark shows in detail how to switch between the families of EVDs. Recall that the transformation of the univariate margins of an EVD G preserves the max-stability of G (cf. Lemma 5.6.8 in Falk et al. [19]. Further, Equation (2.13) is taken from (5.47) in Falk et al. [19]).

Remark 2.1.12 *Recall that if* $G_{\boldsymbol{\alpha}}$ *is max-stable with margins* $G_{\alpha_j}(x) = \exp(\psi_{\alpha_j}(x)), j \leq d$, *and* ψ_{α_i} *as defined in Proposition 2.1.10, then*

$$G_\alpha(\psi_{\alpha_1}^{-1}(x_1), \ldots, \psi_{\alpha_d}^{-1}(x_d)) = G_{(1,\ldots,1)}(x_1, \ldots, x_d) \tag{2.13}$$

is max stable with standard Weibull margins $G(x) = \exp(x)$, $x \leq 0$, *i.e. Weibull margins with parameter* $\alpha = 1$, *which are also denoted by standard reverse exponential margins. Hence, with the transformation in (2.13) one is able to transform an EVD* $G_{\boldsymbol{\alpha}}$ *with arbitrary margins to an EVD with standard reverse exponential margins.*
Hence, the assertion in Proposition 2.1.10 can be also formulated as

$$G_{\boldsymbol{\alpha}}(x_1, \ldots, x_d) = G_{(1,\ldots,1)}(\psi_{\alpha_1}(x_1), \ldots, \psi_{\alpha_d}(x_d)). \tag{2.14}$$

Furthermore, if one likes to transform the EVD $G_{\boldsymbol{\alpha}}$ to an EVD with standard Fréchet margins $G(x) = \exp(-\frac{1}{x})$, $x > 0$, one has to apply the transformation $\tilde{\psi}_{\alpha_j}$ with

$$\tilde{\psi}_{\alpha_j}(x) := -\frac{1}{\psi_{\alpha_j}(x)} = \frac{1}{-\log(G_{\alpha_j}(x))}, \quad j \leq d,$$

and

$$G_\alpha(\tilde{\psi}^{-1}_{\alpha_1}(x_1), \ldots, \tilde{\psi}^{-1}_{\alpha_d}(x_d)) \quad = \quad G_{(-1,\ldots,-1)}(x_1, \ldots, x_d)$$

is max-stable with standard Fréchet margins.
And if one likes to transform the EVD $G_{\boldsymbol{\alpha}}$ to an EVD with Gumbel margins $G_0(x) = \exp(-\exp(-x))$, $x \in \mathbb{R}$, one has to apply the transformation $\breve{\psi}_{\alpha_j}$ with

$$\breve{\psi}_{\alpha_j}(x) := -\log\left(-\psi_{\alpha_j}(x)\right) = -\log(-\log(G_{\alpha_j}(x))), \quad j \leq d,$$

and

$$G_\alpha(\breve{\psi}^{-1}_{\alpha_1}(x_1), \ldots, \breve{\psi}^{-1}_{\alpha_d}(x_d)) \quad = \quad G_{(0,\ldots,0)}(x_1, \ldots, x_d)$$

is max-stable with Gumbel margins.
On the other hand, i.e., if one wants to transform an EVD with standard reverse exponential margins to an EVD with standard Fréchet or Gumbel margins respectively, Equation (2.14) should be applied.
Since the parameters $\alpha = 1$ or $\alpha = -1$ lead to an EVD with standard Weibull, standard Fréchet margins respectively, we denote by $G_{(1,\ldots,1)}$ an EVD with standard Weibull margins and by $G_{(-1,\ldots,-1)}$ an EVD with standard Fréchet margins. As well we denote by $G_{(0,\ldots,0)}$ an EVD with Gumbel margins.

Note that the D-norm is invariant under the transformation of the margins, e.g. the dependence structure within the EVD remains maintained.

The type of transformation used is somehow according to taste. We will use the transformation to standard reverse exponential margins. For the sake of simplicity, we denote from now on by G a multivariate EVD with arbitrary margins and by G_* a multivariate EVD with standard reverse exponential margins $G_j(x) = \exp(x)$ for $x \leq 0$ and $j \leq d$ respectively. G_* can therefore be represented by

$$G_*(x_1, \ldots, x_d) = \exp\left(-\left\|(x_1, \ldots, x_d)\right\|_D\right), \boldsymbol{x} \leq \mathbf{0}.$$

The above results show that an arbitrary EVD G is uniquely defined by specifying the margins and the D-norm. If we focus on the d-variate unit sphere $S = \left\{\boldsymbol{u} : \sum_{j\leq d} u_j = 1,\ u_j \geq 0\right\}$ that means the unit sphere defined by the L_1-norm, the determination of the D-norm solely depends on the choice of the angular measure μ, i.e. there exists a direct link between the D-norm and μ. The two cases, of complete independence or respectively of dependence, between the margins of G, are of particular interest. We exemplarily provide these two cases.

Example 2.1.13 *Assume the angular measure μ on S_d which puts equal weight on the d points $\boldsymbol{e}_j$ for $j \leq d$, i.e. $\mu(\boldsymbol{e}_j) = 1$, $i \leq d$. Then we get*

$$\begin{aligned}\|\boldsymbol{x}\|_D &= \int_{S_d} \max_{j\leq d}(|x_j|\, u_j)\mu(d\boldsymbol{u}) \\ &= \sum_{\boldsymbol{e}_j, j\leq d} \max_{i\leq d}((|x_j|\, \boldsymbol{e}_j)^{(i)}) \\ &= \sum_{j\leq d} |x_j| = \|\boldsymbol{x}\|_1 .\end{aligned}$$

Hence, this is the case of complete independence of the margins, since $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_1) = \exp(-\sum_{j\leq d} |x_j|) = \prod_{j\leq d} G_j(x_j)$.
Now assume the angular measure μ on S_d which puts its whole weight on the point $\left(\frac{1}{d}, \ldots, \frac{1}{d}\right)$. Then we get

$$\begin{aligned}\|\boldsymbol{x}\|_D &= \int_{S_d} \max_{j\leq d}(|x_j|\, u_j)\mu(d\boldsymbol{u}) \\ &= \max_{j\leq d}\left(\frac{1}{d}|x_j|\right) \cdot d = \|\boldsymbol{x}\|_\infty .\end{aligned}$$

Hence, this is the case of complete dependence of the margins.

A very popular example of an extreme value distribution, especially within applications of extreme value theory, is the family of logistic EVD.

Example 2.1.14 (EVD of logistic type) *Consider the arbitrary L_λ-norm defined by $\|(x_1, \ldots, x_d)\|_\lambda := \left(\sum_{j=1}^d |x_j|^\lambda\right)^{1/\lambda}$ for $1 \leq \lambda < \infty$ and $\|(x_1, \ldots, x_d)\|_\infty := \max_{j\leq d} |x_j|$. Then*

$$G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_\lambda), \quad \boldsymbol{x} \leq \boldsymbol{0},$$

is a max-stable df which we call the EVD of logistic type (with standard reverse exponential margins).
The max-stability of the function $\exp(-\|\boldsymbol{x}\|)$ *can easily be shown, due to the homogeneity property of an arbitrary norm* $\|\cdot\|$, *i.e.*

$$\left(\exp\left(-\left\|\frac{1}{n}\boldsymbol{x}\right\|\right)\right)^n = \left(\exp\left(-\frac{1}{n}\|\boldsymbol{x}\|\right)\right)^n = \exp(-\|\boldsymbol{x}\|).$$

Further, $\exp(-\|\boldsymbol{x}\|_\lambda)$ *is a distribution function. This is shown by Kotz and Nadarajah [42], cf. the considerations following (3.27) in Section 3.5.1 therein.*
Note that since $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_\lambda)$ *is an EVD, this implies by means of Corollary 2.1.7 that the* L_λ*-norm is a D-norm. We have total independence between the margins of* G *if* $\lambda = 1$ *and total dependence between the margins for the maximum norm. Further, note that some authors, e.g. Stephenson [67], define the logistic EVD by means of the parameter* $\vartheta := 1/\lambda \in [0, 1]$. *The cases of complete dependence and independence has to be adjusted of course.*

2.2. The extremal coefficient

As already noted, the cases of complete independence and complete dependence are usually of special interest. The following theorem provides a characterization of independence and total dependence of the univariate margins of a multivariate EVD and is due to Takahashi [68].

Theorem 2.2.1 (Takahashi) *Let* G *be an arbitrary* d*-dimensional EVD with margins* G_j, $j \le d$. *We have*

(i) $G(\boldsymbol{x}) = \prod_{j\le d} G_j(x_j)$ *for each* $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ *if and only if there exists one* $\boldsymbol{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$ *with* $0 < G_j(y_j) < 1$, $j \le d$, *such that* $G(\boldsymbol{y}) = \prod_{j\le d} G_j(y_j)$.

(ii) $G(\boldsymbol{x}) = \min_{j\le d} G_j(x_j)$ *for each* $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ *if and only if there exists* $\boldsymbol{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$ *with* $0 < G_j(y_j) < 1$, $j \le d$, *such that* $G(\boldsymbol{y}) = G_1(y_1) = \dots = G_d(y_d)$.

Proof: cf. Theorem 2.2 and 3.1 in Takahashi [68]. □

The following result entails in particular that bivariate independence of the margins of a multivariate EVD is equivalent to complete independence of the margins. This is a specific characteristic for multivariate EVDs.

Theorem 2.2.2 *Let G be an arbitrary d-dimensional EVD with one dimensional margins G_j, $j \le d$. Suppose that for each bivariate margin $G_{(i,j)}$ of G there exists $\boldsymbol{y}_{(i,j)} = \left(y^{(1)}_{(i,j)}, y^{(2)}_{(i,j)}\right) \in \mathbb{R}^2$ with $0 < G_i\left(y^{(1)}_{(i,j)}\right), G_j\left(y^{(2)}_{(i,j)}\right) < 1$ such that $G_{(i,j)}(\boldsymbol{y}_{(i,j)}) = G_i\left(y^{(1)}_{(i,j)}\right) G_j\left(y^{(2)}_{(i,j)}\right)$. Then the margins of G are independent, i.e., $G(\boldsymbol{y}) = \prod_{j\le d} G_j(y_j)$ for all $\boldsymbol{y} \in \mathbb{R}^d$.*

Proof: cf. Theorem 4.3.3 in Falk et al. [19]. □

Furthermore, Takahashi's Theorem 2.2.1, for instance, can be provided by means of D-norms as follows. The proof solely contains the use of properties of the Pickands dependence function (cf. Section 4.3 in Falk et al. [19]), and the relation between the Pickands dependence function and the D-norm, cf. (2.7).

Corollary 2.2.3 (Takahashi in terms of the D-norm) *We have the following equivalences:*

(i) $\|\cdot\|_D = \|\cdot\|_1 \iff \|\boldsymbol{y}\|_D = \|\boldsymbol{y}\|_1$ *for at least one* $\boldsymbol{y} \in \mathbb{R}^d$*, whose components are all different from* 0*.*

(ii) $\|\cdot\|_D = \|\cdot\|_\infty \iff \|(1,\dots,1)\|_D = \left\|\sum_{j\le d} \boldsymbol{e}_j\right\|_D = 1.$

Hence, Corollary 2.2.3 provides necessary and sufficient condition for a D-norm characterizing complete independence or dependence.

Since we have $1 = \|(1,\dots,1)\|_\infty \le \|(1,\dots,1)\|_D \le \|(1,\dots,1)\|_1 = d$ with (2.8), the number $\|(1,\dots,1)\|_D$ can be used to quantify the amount of dependence between the margins of an EVD G with identical margins. The number $\|(1,\dots,1)\|_D$ will play a crucial role within our work on the fragility index. First of all, its significance as a measure for dependence has been mentioned in Tiago de Oliveira [70] as an index for extremal dependence between two variables. The naming goes back to Smith [63].

Definition 2.2.4 (Extremal Coefficient) *Let G be an EVD with identical margins $G_1(x) = \ldots = G_d(x) = \exp(x)$ for $x \le 0$ and D-norm $\|\cdot\|_D$. The number*

$$\varepsilon := \left\|\sum_{j\le d} \boldsymbol{e}_j\right\|_D = \|(1,\dots,1)\|_D \in [1,d] \tag{2.15}$$

satisfies $G(x,\dots,x) = G_1^{\varepsilon}(x)$, $x \in \mathbb{R}$. *We call it the extremal coefficient.*

Note that ε is invariant under monotone transformations of the identical univariate margins of an EVD G, since we have

$$G_{\boldsymbol{\alpha}}(x,\dots,x) = \exp\left(-\left\|\sum_{j\le d}\psi_{\alpha_1}(x)\boldsymbol{e}_j\right\|_D\right) = \exp(\varepsilon\psi_{\alpha_1}(x)) = \exp(\psi_{\alpha_1}(x))^{\varepsilon}.$$

Hence, it is obvious that above results (Takahashi) can be summarized as

$$\text{(2.16)} \qquad \varepsilon = 1 \;\Leftrightarrow\; \|\cdot\|_D = \|\cdot\|_\infty \;\Leftrightarrow\; \text{complete dependence of the margins}$$

and

$$\varepsilon = d \;\Leftrightarrow\; \|\cdot\|_D = \|\cdot\|_1 \;\Leftrightarrow\; \text{independence of the margins}\,.$$

Note that we have independence between the margins if we have $\|\boldsymbol{e}_i + \boldsymbol{e}_j\|_D = 2$ for $i \ne j \in \{1,\dots,d\}$. This goes back to Tiago de Oliveira [70].

Hence, the extremal coefficient measures the amount of dependence between the identical margins of an EVD and the amount of dependence increases with decreasing value of ε. The name *extremal coefficient* goes back to Smith [63]. The extremal coefficient can be regarded as a link between the correlation coefficient and the *extremal index* (cf. page 13 in Smith [63]). It is well-known that the correlation coefficient fails in measuring dependence between *extremal* events, not to mention that it is a measure for bivariate dependence (cf. Chapter 7 in Dempster [11] for an appealing summary regarding this issue). The extremal index (cf. Section 8.1 in Embrechts et al. [13]) measures the amount of dependence within a stochastic process, more precisely in the case of $\{X_k\}_{k\in\mathbb{N}}$ being a strictly stationary sequence of random numbers, cf. Section 4.4.1. Within this work, we will see that there exists a direct link between the fragility index, the extremal coefficient and the extremal index, which we consider extensively in Section 4.3 and 4.4.1 respectively.

From now on, denote by G_K the $|K|$-dimensional margin

$$\text{(2.17)} \qquad G_K := G \star (\pi_k)_{k\in K}\,,\ \text{with } \pi_k(x) := x_k \text{ for } \emptyset \ne K \subseteq \{1,\dots,d\}$$

of an EVD G.

If G_K is the $|K|$-variate margin of an EVD G with identical margins, we will see that the extremal coefficient of G_K is given by

$$\varepsilon_K := \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D , \tag{2.18}$$

cf. Lemma 3.1.1 and Section 4.2 for a detailed discussion on this topic. Hence the ε_K are the extremal coefficients of the $|K|$-dimensional margins of the EVD G having identical margins for an arbitrary subset $K \subseteq \{1, \dots, d\}$. If not stated otherwise, the simplification $\varepsilon := \varepsilon_d$ shall refer to the extremal coefficient of G. We will show in Section 4.2 that the ε_K will be crucial for the definition of the extended fragility index. More precisely, we will show that under certain conditions (e.g. exceedances above individual thresholds), the extended fragility index can be regarded as an extension of the extremal coefficient with dropping the requirements of identical margins. We provide this result in Section 4.3 and point out its relevance as a measure for tail dependence in Section 2.4. A similar consideration on those "marginal" extremal coefficients, of which there exist $2^d - 1$ for a d-variate distribution, can be found in Schlather et al. [58].

2.3. Copulas and the stable tail dependence function

Within the next subsection the reader will be introduced to the link between the popular use of copulas and the stable tail dependence function as a measure for tail dependence. It is crucial with respect to any part of the work at hand. This link is also crucially influenced by the generalized Pareto distributions.

Definition 2.3.1 (Copula) *Let $\boldsymbol{X}$ be a random variable in $\mathbb{R}^d$ with df F and denote by F_j its j-th marginal df. Suppose that F is continuous. The function*

$$C_F(\boldsymbol{u}) = F(F_j^{-1}(u_j),\ j \le d)\ , \quad \boldsymbol{u} \in [0,1]^d,$$

is the copula of F satisfying $F(\boldsymbol{x}) = C_F(F_j(x_j),\ j \le d)\ ,\ \boldsymbol{x} \in \mathbb{R}^d$. Further, C_F is a df on $[0,1]^d$ with uniform margins.

Obviously, the copula "couples" the joint df F to its margins in a specific, unique way (in case of continuity of F) - the name *copula* suggests itself. In view of the dependence structure, the copula describes the dependence structure of the df F without paying attention to the margins, since they are transformed to uniform distributed ones. Indeed it is not trivial that for every (continuous) multivariate df F, there exists a decomposition of margins and dependence structure as given in Definition 2.3.1 as a characteristic of the copula C_F. The following theorem provides this result. It firstly appeared in Sklar [61] for the bivariate case.

Theorem 2.3.2 (Sklar) *Let F be a joint distribution function on $\mathbb{R}^d$ with margins $F_j, j \le d$. Then there exists a copula denoted by C_F such that*

$$F(x_1, \dots, x_d) = C_F(F_1(x_1), \dots, F_d(x_d)) \tag{2.19}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^d$. If the margins $F_j, j \le d$, are continuous, then C_F is uniquely determined. Otherwise, C_F is uniquely determined on $Ran(F_1) \times \cdots \times Ran(F_d)$, where $Ran(F_j)$ defines the range of F_j for $j \le d$. Conversely, if C is a copula and F_j is a df for every $j \le d$, then the function F defined by (2.19) is a d-dimensional distribution function with margins $F_j, j \le d$.

Proof: cf. Theorem 2.10.9 in Nelsen [52]. □

The copula of an EVD is of special interest. A popular assumption within extreme value analysis is that a multivariate df F belongs to the so-called *domain of attraction of an EVD G*, which can be defined by the convergence in (2.1), if the limit G exists. In order to provide a more convenient characterization of the domain of attraction condition, we need the following definition.

Definition 2.3.3 *Any copula C which satisfies*

$$C^t(u_1, \dots, u_d) = C(u_1^t, \dots, u_d^t) \tag{2.20}$$

for $(u_1, \dots, u_d) \in [0,1]^d$, $t > 0$, is called an extreme value copula (EVC). It is the copula distribution function corresponding to an extreme value distribution.

The following result is due to the application of Sklar‘s Theorem to an arbitrary EVD and provides the representation of an EVC by norms.

Corollary 2.3.4 (Extreme Value Copula) *Let $G(\boldsymbol{x}) = \exp(-\|(\psi_1(x_1), \dots, \psi_d(x_d))\|_D)$ be an arbitrary EVD. The corresponding copula C_G is given by*

$$C_G(\boldsymbol{u}) = \exp(-\|(\log(u_1), \dots, \log(u_d))\|_D), \quad \boldsymbol{u} \in (0,1]^d. \tag{2.21}$$

The naming of this family is self-explanatory.

Proof: Note that the univariate margins of G are continuous functions, cf. Proposition 2.1.10. Check, that (2.21) fulfills $G(\boldsymbol{x}) = C_G(G_j(x), j \leq d)$ and C_G has uniform margins by means of $\psi_j(x) = \log(G_j(x))$ for $j \leq d$. □

Beside the well-known normal- and t-copula, a very popular example for a copula is the parametric family of the Gumbel copula.

Example 2.3.5 *Consider the logistic EVD $G_\lambda(\boldsymbol{x}) := \exp(-\|\boldsymbol{x}\|_\lambda)$ of example 2.1.14. Its corresponding copula is given by*

$$C_{Gu}(\boldsymbol{u}) = \exp\left(-\left(\sum_{i\leq d}(-\log(u_i))^\lambda\right)^{\frac{1}{\lambda}}\right) = \exp\left(-\|\log(\boldsymbol{u})\|_\lambda\right), \tag{2.22}$$

which is known as the Gumbel copula, since it goes back to Gumbel [28]. Note that one often refers the Gumbel copula to the so-called logistic copula, which is mostly defined by the parameter $\vartheta := 1/\lambda \in (0,1]$ and $\vartheta := 0$ for the maximum-norm in (7.1). We call the family of EVD defined by the arbitrary L_λ-norm (cf. Section A of the appendix) the logistic EVD with corresponding Gumbel copula.

Assume that the d-variate random vector $\boldsymbol{X}$ follows a df F. Then, with the well-known additivity formula, cf. Theorem A.6, the corresponding survival distribution function $\bar{F}$ is defined by

$$\begin{aligned}\bar{F}(\boldsymbol{x}) := P(\boldsymbol{X} \geq \boldsymbol{x}) &= 1 - \sum_{j\leq d}(-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=j}} P(X_j \leq x_j, j \in T)\\ &= 1 - \sum_{j\leq d}(-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=j}} F_T(x_j, j \in T),\end{aligned}$$

where F_T denotes the $|T|$-variate marginal df of F corresponding to $\emptyset \neq T \subseteq \{1,\dots,d\}$. The corresponding survival copula distribution function of $\bar{F}$ is defined as follows, e.g. cf. Nelsen [52], Section 2.6.

Definition 2.3.6 (Survival Copula) *Let $\boldsymbol{X}$ be a random vector in $\mathbb{R}^d$ with df F. Further denote by $\bar{F}$ the joint survival distribution function which is given by $\bar{F}(\boldsymbol{x}) =$*

$P(X_j > x_j,\, j \le d)$ and denote by $\bar{F}_j$ the j-th marginal survival function, i.e. $\bar{F}_j(x) = P(X_j > x)$. Suppose that F is continuous. The function

$$\tilde{C}_F(\boldsymbol{u}) := \bar{F}(\bar{F}_j^{-1}(u_j),\ j \le d) = P(1 - F_j(X_j) \le u_j, j \le d)\ , \quad \boldsymbol{u} \in [0,1]^d,$$

defines the survival copula of F in accordance to the definition of the copula C_F of F, cf. Definition 2.3.1. $\tilde{C}_F$ satisfies $\bar{F}(\boldsymbol{x}) = \tilde{C}_F(\bar{F}_j(x_j),\ j \le d)\,,\ \boldsymbol{x} \in \mathbb{R}^d$. Further, $\tilde{C}_F$ is a df on $[0,1]^d$ with uniform margins. Due to its construction the survival copula is also called tail copula.

Of course there is a connection between the copula C_F and the survival copula $\tilde{C}_F$ of the distribution F:

$$\begin{aligned}
\tilde{C}_F(u_j,\, j \le d) =& P\left(1 - F_j(X_j) \le u_j, j \le d\right) \\
=& P\left(F_j(X_j) \ge 1 - u_j, j \le d\right) \\
=& 1 - P\left(\bigcup_{j \le d} \{F_j(X_j) \le 1 - u_j\}\right) \\
=& 1 - \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|+1} P\left(\bigcap_{i \in T} \{F_i(X_i) \le 1 - u_i\}\right) \\
=& 1 - \sum_{j \le d} (-1)^{j+1} \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} C_{F_T}(1 - u_i,\, i \in T)\,,
\end{aligned}$$

where C_{F_T} denotes the copula corresponding to the $|T|$-variate margin F_T of F for $\emptyset \ne T \subseteq \{1,\dots,d\}$. For the bivariate case the above connection between the survival copula and the copula of the df F simplifies to

$$\tilde{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)\,.$$

Now we want to give a short introduction to multivariate generalized Pareto distributions. For detailed information cf. Chapter 5 in Falk et al. [19] or Michel [50].
The upper tail of an EVD G can be approximated by a class of multivariate functions, which are of the form $W(\boldsymbol{x}) = 1 + \log(G(\boldsymbol{x}))$, $\log(G(\boldsymbol{x})) \ge -1$, where G denotes a multivariate EVD. This class is called the Generalized Pareto functions (GP). See Section 5.1 in Falk et al. [19] for a motivation and Section 5.2, especially Theorem 5.2.3 therein for detailed information.

In the univariate and bivariate case, one can show that $W(x) = 1 + \log(G(x))$ defines a *distribution* function for any max-stable df G, if only $\log(G(x)) \geq -1$ holds, cf. Lemma 5.1.1 in Falk et al. [19]. This is not true for dimensions $d \geq 3$. For a discussion and proof see Section 5.1 in Falk et al. [19] and detailed information in Michel [50], Chapter 2.

Regarding Proposition 2.1.10 we provide the multivariate generalized Pareto *distribution* functions, which have the form $1+\log(G)$ in the upper tail of G, where G is a max-stable df in $\mathbb{R}^d$. Let us start with the following definition.

Definition 2.3.7 *Let $G_{\boldsymbol{\alpha}}$ be an arbitrary EVD. The function*

$$W_{\boldsymbol{\alpha}}(\boldsymbol{x}) \;\; := \;\; 1 + \log(G_{\boldsymbol{\alpha}}(\boldsymbol{x})), \quad \log(G_{\boldsymbol{\alpha}}(\boldsymbol{x})) \geq -1,$$

is called a generalized Pareto function (GP). For case of simplicity this is often shortened by $W_{\boldsymbol{\alpha}} = 1 + \log(G_{\boldsymbol{\alpha}})$. Further, we call $W_{\boldsymbol{\alpha}} = 1 + \log(G_{\boldsymbol{\alpha}})$ a generalized Pareto distribution (GPD), if there exists $\boldsymbol{x}_0 \in \mathbb{R}^d$, such that $W_{\boldsymbol{\alpha}}(\boldsymbol{x}) = 1 + \log(G_{\boldsymbol{\alpha}}(\boldsymbol{x}))$ is a distribution function for $\boldsymbol{x} \geq \boldsymbol{x}_0$. By means of Proposition 2.1.10, $W_{\boldsymbol{\alpha}}$ can be represented by

$$W_{\boldsymbol{\alpha}}(\boldsymbol{x}) \;\; = \;\; 1 - \|\psi_{\alpha_1}(x_1), \ldots, \psi_{\alpha_d}(x_d)\|_D \; ,$$

with

$$\psi_{\alpha_j}(x) \;\; = \;\; \log(G_{\alpha_j}(x)) = \begin{cases} -(-x)^{\alpha_j}, & x < 0, \quad \alpha_j > 0, \\ -x^{\alpha_j}, & x > 0, \quad \alpha_j < 0, \\ -\exp(-x), & x \in \mathbb{R}, \quad \alpha_j = 0, \end{cases}$$

for $j \leq d$ and the univariate margins G_{α_j} as defined in Proposition 2.1.10.

Hence, we have uniform margins W_{α_j} of the Pareto distribution W_α in the first case, Pareto margins in the second case and exponential margins in the third case.

Since any GPD can be transformed to uniform margins, we want to focus on the GPD corresponding to $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$. We call a d-dimensional df $W_{1,\ldots,1}$ a (multivariate) *generalized Pareto df* with *uniform margins*, if there exists an EVD with standard reverse exponential margins such that

$$W_{1,\ldots,1}(\boldsymbol{x}) = 1 + \log(G_{1,\ldots,1}(\boldsymbol{x})) = 1 - \|\boldsymbol{x}\|_D$$

for $\boldsymbol{x}$ in a left neighborhood of $\mathbf{0} \in \mathbb{R}^d$.
For a more precise definition of the "neighborhood of $\mathbf{0}$" in this specific case, one is able to choose the cube $[-1/d, 0]^d$ as a suitable area for this neighborhood.
That means that for every D-norm $\|\cdot\|_D$ there exists a df W on $(-\infty, 0]^d$, such that

$$W(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D$$

holds on $\boldsymbol{x} \in [-1/d, 0]^d$, cf. Theorem 6.2.1 in Hofmann [36]. In the case of independence between the margins, i.e. $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1$, the function $W(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_1$ is not a df for any dimension $d \geq 3$. The GPD $W_\lambda(\boldsymbol{x}) := 1 - \|\boldsymbol{x}\|_\lambda$ is called the GPD of logistic type and is the corresponding GPD to the EVD of logistic type, i.e. $G_\lambda(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_\lambda)$, cf. Example 2.1.14. Note that $W_\lambda(\boldsymbol{x})$ is not a df for *any* $\lambda \geq 1$ and $d \geq 3$. This is Proposition 5.1.3 in Falk et al. [19].

Definition 2.3.8 *The copula of a GPD W with uniform margins is given by*

$$C_W(\boldsymbol{u}) = W\left(W_j^{-1}(u_j),\, j \leq d\right)$$

for $u \in [0,1]^d$ and is itself a shifted GPD with uniform margins. Therefore, we call C_W a GPD-copula. For $\boldsymbol{u}$ close to $\mathbf{1}$, C_W can be represented by

$$C_W(\boldsymbol{u}) = 1 - \|\boldsymbol{u} - \mathbf{1}\|_D \,.$$

The following result characterizes a GPD with uniform margins in terms of rv. It provides in particular an easy way to generate a multivariate GPD, thus extending the bivariate approach proposed by Buishand et al. [7] to an arbitrary dimension. For a recent account on simulation techniques of multivariate GPDs we refer to Michel [50]. The following proposition and corollary is due to Aulbach et al. [1].

Proposition 2.3.9 *(i) Let W be a multivariate GPD with standard uniform margins in a left neighborhood of $\mathbf{0} \in \mathbb{R}^d$. Then there exists an rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $Z_j \in [0, d]$ and $E(Z_j) = 1, j \leq d$, and a vector $(-\frac{1}{d}, \dots, -\frac{1}{d})$
$\leq \boldsymbol{x}_0 < \mathbf{0}$ such that*

$$W(x) = P\left(-U\left(\frac{1}{Z_1}, \dots, \frac{1}{Z_d}\right) \leq \boldsymbol{x}\right), \quad \boldsymbol{x}_0 \leq \boldsymbol{x} \leq \mathbf{0}, \tag{2.23}$$

where the rv U is uniformly distributed on $(0,1)$ and independent of $\boldsymbol{Z}$.

(ii) *The rv* $-U(1/Z_1, \dots, 1/Z_d)$ *follows a GPD with standard uniform margins in a left neighborhood of* $\mathbf{0} \in \mathbb{R}^d$ *if* U *is independent of* $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ *and* $0 \le Z_j \le c_j$ *a.s. with* $E(Z_j) = 1, j \le d$*, for some* $c_1, \dots, c_d \ge 1$.

Proof: cf. Aulbach et al. [1], Proposition 2.4 □

We want to note that Equation (2.23) is well defined, since we may consider the rv $\boldsymbol{Y} := \max\left(-U\left(\frac{1}{Z_1}, \dots, \frac{1}{Z_d}\right), M\right)$ for $M > -\infty$ (this ensures the existence of $\frac{1}{Z_j}$), as Equation (2.23) is said to hold for $\boldsymbol{x}_0 \le \boldsymbol{x} \le \mathbf{0}$.

Further note that the case of a GPD W with arbitrary uniform margins $W_j(x) = 1 - a_j x$ in a left neighborhood of 0 with arbitrary scaling factors $a_j > 0, j \le d$, immediately follows from the preceding result by substituting Z_j by $a_j Z_j$.

Since we will need the above results due to Fréchet margins we provide the following corollary.

Corollary 2.3.10 *Let* $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ *satisfy* $Z_j \in [0, c_j]$ *for* $c_j \ge 1$ *and* $E(Z_j) = 1$ *for* $j \le d$*. Then we get*

(i) *Let* W *be a multivariate GPD with standard Pareto margins in the upper tail. Then there exists a rv* $\boldsymbol{Z}$ *as mentioned above and a vector* $(d, \dots, d) \le \boldsymbol{x}_0$ *such that*

$$W(x) = P\left(\frac{1}{U}\boldsymbol{Z} \le \boldsymbol{x}\right), \quad \boldsymbol{x} \ge \boldsymbol{x}_0,$$

where the rv U *is uniformly on distributed* $(0,1)$ *and independent of* $\boldsymbol{Z}$.

(ii) *The rv* $\frac{1}{U}(Z_1, \dots, Z_d)$ *follows a GPD with standard Pareto margins in a left neighborhood of* ∞ *if* U *is independent of* $\boldsymbol{Z} = (Z_1, \dots, Z_d)$*, hence we have*

$$P\left(\frac{1}{U}\boldsymbol{Z} \le \boldsymbol{x}\right) = 1 - \left\|\left(\frac{1}{x_1}, \dots, \frac{1}{x_d}\right)\right\|_D.$$

Proof: Assume that there exists a rv $\tilde{\boldsymbol{Q}} := -U(1/Z_1, \dots, 1/Z_d)$ as given in Proposition 2.3.9. Hence $\tilde{\boldsymbol{Q}}$ follows a GPD with uniform margins. With the transformation $x \mapsto -\frac{1}{x}$, cf. Remark 2.1.12, we transform the margins of $\tilde{\boldsymbol{Q}}$ to pareto margins and obtain for $\boldsymbol{Q} = (1/U)(Z_1, \dots, Z_d)$ that

$$P\left(\frac{1}{U}\boldsymbol{Z} \le \boldsymbol{x}\right) = 1 - \left\|\left(\frac{1}{x_1}, \dots, \frac{1}{x_d}\right)\right\|_D$$

holds for $\boldsymbol{x} > (d,\dots,d) \geq (0,\dots,0) \in \mathbb{R}^d$. This implies the assertion. □

In the very beginning of this section we introduced an EVD G as the limit distribution of normalized maxima of random variables following a df F. The following definition characterizes the property, if this limit exists.

Definition 2.3.11 (Domain of attraction) *Let F be a d-dimensional df. Then we say F belongs to the (maximum) domain of attraction of an EVD G, denoted by $F \in \mathcal{D}(G)$, if there exist vectors $\boldsymbol{a}_n > \boldsymbol{0}$, $\boldsymbol{b}_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that*

$$F^n(\boldsymbol{a}_n\boldsymbol{x} + \boldsymbol{b}_n) \to_{n\to\infty} G(\boldsymbol{x})\ , \quad \boldsymbol{x} \in \mathbb{R}^d\ .$$

From now on we refer to the assumption $F \in \mathcal{D}(G)$ as the domain-of-attraction assumption.

The following Theorem gives a necessary and sufficient condition for $F \in \mathcal{D}(G)$.

Theorem 2.3.12 (Domain of attraction) *Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots$ be i.i.d. d-dimensional random vectors with common df F and suppose constants $\boldsymbol{a}_n > \boldsymbol{0}$, $\boldsymbol{b}_n \in \mathbb{R}^d$, $n \in \mathbb{N}$. Then $F \in \mathcal{D}(G)$ if and only if this is true for each univariate margin of F, i.e. $F_j \in \mathcal{D}(G_j)$ holds for $j \leq d$, together with the convergence of the copula*

$$C_F^n(\boldsymbol{u}^{1/n}) \ \to_{n\to\infty} \ C_G(\boldsymbol{u}),\ \boldsymbol{u} \in (0,1)^d\ . \tag{2.24}$$

Proof: cf. Deheuvels [10] or Theorem 5.2.3 in Galambos [20]. □

Note that the copula C_G on the right side of (2.24) is an extreme value copula, cf. Definition 2.3.3. Loosely speaking, the convergence of (2.24) tells us that the df F is in the domain of attraction of an EVD G if and only if the same holds for the univariate margins and the convergence result for the copula of F in (2.24). In the following we will show that (2.24) offers a very useful equivalent convergence for our purpose.

Definition 2.3.13 (Stable tail dependence function) *Let F be an arbitrary d-dimensional df. If the limit*

$$\lim_{t\downarrow 0} t^{-1}(1 - C_F(\boldsymbol{1} + t\boldsymbol{x})) =: l_F(\boldsymbol{x}) \tag{2.25}$$

exists for $\boldsymbol{x} \leq \boldsymbol{0}$, then l_F is called the stable tail dependence function of the function F (Huang [39]).

If the df F belongs to the domain of attraction of a multivariate EVD G, then the limit in (2.25) exists and the stable tail dependence function l_F coincides with the stable tail dependence function l_G of the EVD G. Huang [39] defines l_G by

$$l_G(x_1,\dots,x_d) := -\log(G(\boldsymbol{x})),\ \boldsymbol{x} > \mathbf{0}, \tag{2.26}$$

where G is an EVD with Frèchet margins in this case. The following corollary contains this assertion and is itself a consequence of Theorem 2.3.12.

Corollary 2.3.14 *Let F be a multivariate continuous df. Then $F \in \mathcal{D}(G)$ if and only if this is true for each univariate margin of F, i.e. $F_j \in \mathcal{D}(G_j)$ holds for $j \le d$, together with the convergence of the copula*

$$t^{-1}(1 - C_F(\mathbf{1} + t\boldsymbol{x})) \to_{t\downarrow 0} -\log(C_G(\exp(\boldsymbol{x}))) = l_G(\boldsymbol{x}) \tag{2.27}$$

for $\boldsymbol{x} \le \mathbf{0}$.

Proof: The assertion follows from Theorem 2.3.12 with respect to the following equivalences. A definition of the so-called *Landau symbols o* and O is given in Definition A.1. By taking logarithm, we get

$$C_F^n\left(\boldsymbol{u}^{1/n}\right) \to_{n\to\infty} C_G(\boldsymbol{u}) \ \Leftrightarrow\ -n\ln\left(C_F\left(\boldsymbol{u}^{1/n}\right)\right) \to_{n\to\infty} -\ln\left(C_G(\boldsymbol{u})\right)\ . \tag{2.28}$$

With the Taylor expansion of the natural logarithm at point $\mathbf{1}$, i.e. $\ln\left(C_F\left(1+\frac{1}{n}\boldsymbol{x}\right)\right) = C_F\left(1+\frac{1}{n}\boldsymbol{x}\right) - 1 + O\left(\frac{1}{n^2}\right)$, entails that (2.28) is equivalent to

$$n\left(1 - C_F\left(\boldsymbol{u}^{1/n}\right)\right) \to_{n\to\infty} -\ln\left(C_G(\boldsymbol{u})\right)\ . \tag{2.29}$$

Now choose $\boldsymbol{u} = \exp(\boldsymbol{x})$ for $\boldsymbol{x} \le \mathbf{0}$. Then (2.29) is equivalent to

$$n\left(1 - C_F\left(\exp\left(\frac{1}{n}\boldsymbol{x}\right)\right)\right) \to_{n\to\infty} -\ln\left(C_G\left(\exp(\boldsymbol{x})\right)\right)\ . \tag{2.30}$$

By means of Corollary 2.3.4 and Equation (2.26), the right hand side of (2.30) turns into $l_G(\boldsymbol{x})$. Further Taylor expansion of the exponential function at point $\mathbf{0}$, i.e. $\exp\left(\frac{1}{n}\boldsymbol{x}\right) = 1 + \frac{1}{n}\boldsymbol{x} + o\left(\frac{1}{n^2}\right)$, together with the continuity of C_F close to 1 entails that (2.30) is equivalent to

$$n\left(1 - C_F\left(\mathbf{1} + \frac{1}{n}\boldsymbol{x}\right)\right) \to_{n\to\infty} l_G(\boldsymbol{x})\,, \tag{2.31}$$

where a continuous version of (2.31) is given by

$$t^{-1}\left(1 - C_F\left(\mathbf{1} + t\boldsymbol{x}\right)\right) \to_{t\downarrow 0} l_G(\boldsymbol{x})\,,$$

cf. e.g., Section 4.2 in de Haan and Ronde [33]. □

Remark 2.3.15 *Note that with the representation of an EVD by norms, cf. Corollary 2.3.4, we get*

$$C_G(\exp(\boldsymbol{x})) = \exp(-\left\|\log(\exp(\boldsymbol{x}))\right\|_D) = \exp(-\left\|\boldsymbol{x}\right\|_D) \tag{2.32}$$

for the expression $C_G(\exp(\boldsymbol{x}))$ *of the above proof. Hence, if* $F \in \mathcal{D}(G)$*, the copula* C_F *of a df* F *belongs to the domain of attraction of an EVD* G *with standard reverse exponential margins. Furthermore, the equation in (2.27) together with (2.32) implies*

$$l_G(\boldsymbol{x}) = \left\|\boldsymbol{x}\right\|_D \tag{2.33}$$

for $\boldsymbol{x} \leq \mathbf{0}$*. Hence, the stable tail dependence function is a norm and therefore fulfills the homogeneity condition. Furthermore, it is a continuous and convex function. Especially, listed properties of the stable tail dependence function, which can be found in de Haan and Ferreira [29], Proposition 6.1.21, are obvious.*

The following assertion follows directly from above considerations and will be crucial for theoretical results as well applications on the fragility index through the whole work at hand.

Corollary 2.3.16 *Assume the copula* C_F *of a multivariate df* F *belongs to the domain of attraction of an EVD* G*. Then the limit in Definition 2.3.13 exists, coincides with the D-norm and can be computed by*

$$\lim_{t\downarrow 0} \frac{1 - C_F(\mathbf{1} + t\boldsymbol{x})}{t} = \left\|\boldsymbol{x}\right\|_D \tag{2.34}$$

for $\boldsymbol{x} \leq \mathbf{0}$*.*

The above results – starting from Theorem 2.3.12 – concerning the characterization of the domain of attraction of a copula can be summarized in the following corollary.

Corollary 2.3.17 *Let C_F be the copula corresponding to a multivariate df F. Further denote by G an arbitrary EVD with dependence function $\|\boldsymbol{x}\|_D$ and by G_* an EVD with standard reverse exponential margins. Then*

$$C_F \in \mathcal{D}(G_*) \quad \text{if and only if} \quad \lim_{t\downarrow 0} \frac{1 - C_F(\mathbf{1} + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D \ , \ \boldsymbol{x} \leq \mathbf{0} \, .$$

Furthermore it holds that

$$F \in \mathcal{D}(G) \ \Leftrightarrow \ F_j \in \mathcal{D}(G_j),\, j \leq d \quad \text{and} \quad \lim_{t\downarrow 0} \frac{1 - C_F(\mathbf{1} + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D \ .$$

Due to the fact that if the df F is in the domain of attraction of an arbitrary EVD G, the corresponding copula C_F is in the domain of attraction of an EVD with standard Weibull margins, the following approximation of C_F is obvious. Both the theorem and the corollary are taken from Aulbach et al. [1]. For a definition of the Landau symbols o and O, cf. Definition A.1.

Theorem 2.3.18 *An arbitrary df F is in the domain of attraction of a multivariate EVD G if and only if this is true for the univariate margins together with the existence of a GPD W with uniform margins such that*

$$C_F(\boldsymbol{y}) = W(\boldsymbol{y} - \mathbf{1}) + o(\|\boldsymbol{y} - \mathbf{1}\|)$$

uniformly for $\boldsymbol{y} \in [0,1]^m$.

Proof: From Corollary 2.3.14 we get

$$\frac{1 - C_F(\mathbf{1} + t\boldsymbol{x})}{t} \to_{t\downarrow 0} l_G(\boldsymbol{x}) \, .$$

With Definition 2.3.7 and the equality $l_G(\boldsymbol{x}) = \|\boldsymbol{x}\|_D$, cf. (2.33) in Remark 2.3.15, we get $l_G(\boldsymbol{x}) = 1 - W(\boldsymbol{x})$ for $\boldsymbol{x}$ close to $\mathbf{0}$, where W is a multivariate GPD with uniform margins. This implies

$$\frac{W(\boldsymbol{y} - \mathbf{1}) - C_F(\boldsymbol{y})}{\|\boldsymbol{y} - \mathbf{1}\|} = \frac{1 - l_G(\boldsymbol{y} - \mathbf{1}) - C_F(\boldsymbol{y})}{\|\boldsymbol{y} - \mathbf{1}\|} \to 0$$

for $\boldsymbol{y} \uparrow \mathbf{1}$. Since the limit function W is continuous, above convergence is uniformly in $\boldsymbol{y}$ (cf. Section 3.1 in Gänsler and Stute [21]). Note that above convergence coincides with the notation $C_F(\boldsymbol{y}) = W(\boldsymbol{y} - \mathbf{1}) + o(\|\boldsymbol{y} - \mathbf{1}\|)$ by using the Landau-symbol o, cf.

Definition A.1. □

The next result is again provided by Aulbach et al. [1] and is in accordance with Rootzén and Tajvidi [57]. It shows, that the upper tail of the copula C_F of a df F can be approximated by a multivariate GPD with uniform margins, if $F \in \mathcal{D}(G)$.

Corollary 2.3.19 *A copula C is in the domain of attraction of an EVD G.*

$\iff$ *There exists a GPD function W with uniform margins such that*

$$C(\boldsymbol{y}) = W(\boldsymbol{y} - \mathbf{1}) + o\left(\|\boldsymbol{y} - \mathbf{1}\|\right),$$

uniformly for $\boldsymbol{y} \in [0,1]^m$. In this case $W = 1 + \log(G)$.

$\iff$ *There exists a norm $\|\cdot\|_D$ on $\mathbb{R}^m$ such that*

$$C(\boldsymbol{y}) = 1 - \|\boldsymbol{y} - \mathbf{1}\|_D + o\left(\|\boldsymbol{y} - \mathbf{1}\|_D\right),$$

uniformly for $\boldsymbol{y} \in [0,1]^m$. In this case $G(\boldsymbol{x}) = \exp\left(- \|\boldsymbol{x}\|_D\right)$, $\boldsymbol{x} \le \mathbf{0}$.

Proof: cf. Aulbach et al. [1], Corollary 2.2. □

Another characterization for F lying in the domain of attraction of a multivariate EVD goes back to Resnick [56]. It shows an alternative approach to provide sufficient and necessary conditions for $F \in \mathcal{D}(G)$. The difference to the approach provided in Theorem 2.3.12 lies in the kind of transformation of the margins of F, i.e. therein, the transformation to uniform margins is applied, which entails the copula C_F corresponding to F.

Proposition 2.3.20 *Let G be an arbitrary multivariate EVD and define by*

$$\tilde{G}(\boldsymbol{x}) = G(\tilde{\psi}_1^{-1}(x_1), \dots, \tilde{\psi}_d^{-1}(x_d))$$

with $\tilde{\psi}(x) := 1/(-\log(G_i))$, $x \ge 0$, $i \le d$ an EVD with standard Fréchet margins, cf. Remark 2.1.12. Further, consider $U_j := 1/(1 - F_j)$, $j \le d$, for a distribution F_j and define by $U_j^{\leftarrow}$ its inverse function. Now, set

$$\tilde{F}(\boldsymbol{x}) = F(U_1^{\leftarrow}(x_1), \dots, U_d^{\leftarrow}(x_d)), \quad \boldsymbol{x} \ge \mathbf{1}.$$

Then we have

(i)

$$\tilde{F} \in \mathcal{D}(\tilde{G}) \quad \text{if and only if} \quad \lim_{t\to\infty} \frac{1-\tilde{F}(t\boldsymbol{x})}{1-\tilde{F}(t\mathbf{1})} = \frac{\log \tilde{G}(\boldsymbol{x})}{\log \tilde{G}(\mathbf{1})}$$

holds for $\boldsymbol{x} > \mathbf{0}$.

(ii) $F \in \mathcal{D}(G)$ *if and only if* $F_j \in \mathcal{D}(G_j)$ *holds for every* $j \le d$ *and* $\tilde{F} \in \mathcal{D}(\tilde{G})$.

Proof: cf. Resnick [56], Proposition 5.15. □

Remark 2.3.21 *The limit in (i) of Proposition 2.3.20 turns to*

$$\lim_{t\to\infty} \frac{1-\tilde{F}(t\boldsymbol{x})}{1-\tilde{F}(t\mathbf{1})} = \frac{\log \tilde{G}(\boldsymbol{x})}{\log \tilde{G}(\mathbf{1})} = \varepsilon^{-1} \left\| \left(\frac{1}{x_1}, \dots, \frac{1}{x_d} \right) \right\|_D,$$

where ε *is the extremal coefficient, cf. Definition 2.2.4. Hence, (i) in Proposition 2.3.20 is equivalent to*

$$\tilde{F} \in \mathcal{D}(\tilde{G}) \quad \text{if and only if} \quad \lim_{t\to\infty} \frac{1-\tilde{F}(t\boldsymbol{x})}{1-\tilde{F}(t\mathbf{1})} = \varepsilon^{-1} \left\| \left(\frac{1}{x_1}, \dots, \frac{1}{x_d} \right) \right\|_D.$$

2.4. Measures for tail dependence

This section gives a short review of measures of tail dependence within the existing literature. In doing so, it makes no claim to be complete, since the literature concerning this topic is extensive. It provides instead a useful selection with respect to the progress of the work at hand. For example, an overview to the literature can be found in Chapter 9 of Beirlant et al. [5], especially Section 9.4 as well Section 8.2 and 8.3, or Heffernan [35], who concentrates on the bivariate case but therefore gives a detailed summary of the tail dependence coefficient. Those estimators presented therein provide answers to the question, "How to estimate dependence between the univariate tails of a multivariate distribution F" under specific assumptions on F, like the domain-of-attraction assumption or the assumption of a slowly varying tail.

Let us start with the bivariate case.

Definition 2.4.1 *Let* X_1 *and* X_2 *be two random variables having joint distribution function* F *with continuous margins* F_1 *and* F_2*. Then the upper and the lower tail dependence* *coefficients are defined by*

$$\lambda^{up} := \lim_{t\downarrow 0} P(X_2 > F_2^{-1}(1-t) \,|\, X_1 > F_1^{-1}(1-t))$$

and

$$\lambda^{lo} := \lim_{t\downarrow 0} P(X_2 \le F_2^{-1}(t) \,|\, X_1 \le F_1^{-1}(t)) \,.$$

If λ^{up} (λ^{lo}) exists and is positive, then we say that X_1 and X_2 are upper-tail (lower-tail) dependent. If the limit is zero, then X_1 and X_2 are tail-independent. The tail dependent coefficient λ goes back to Geoffrey [25] and Sibuya [60]. Further, note that the above definition yields that $\lambda^{up}, \lambda^{lo} \in [0,1]$. Note that with the definition of the survival copula, cf. Definition 2.3.6, and the assumption of continuous univariate tails, we have

$$\lambda^{up} = \lim_{t\downarrow 0} \frac{\tilde{C}(t,t)}{t} \quad \text{and} \quad \lambda^{lo} = \lim_{t\downarrow 0} \frac{C(t,t)}{t}.$$

Hence the upper and lower tail dependence coefficients refer to the limit behavior of the survival copula, the copula corresponding to F respectively .

Now consider a subset $\emptyset \neq S \subset \{1, \dots, d\}$. Then a possible extension of the upper tail dependence coefficient is given by

$$\lim_{t\downarrow 0} P(X_i > F_i^{-1}(1-t),\, i \in S \,|\, X_j > F_j^{-1}(1-t),\, j \in S^{\complement}), \tag{2.35}$$

where this limit crucially depends on the chosen subset S. Further, such an extension of the bivariate tail dependence coefficients, as provided in Definition 2.4.1, is not used in literature as a measure for tail dependence as far as is known to the author due to obvious necessary restrictions on S. But we want to mention that the limit distributions in (2.35) are the basic elements of a limit distribution, which are called the *asymptotic conditional distribution of exceedance counts* (ACDEC) and will be provided in Section 3.3 and used as a basic tool for the fragility index.

A second measure for bivariate tail dependence is provided by Ledford and Tawn [44], [45]. They assume the model

$$P(X_1 > s, X_2 > s) \sim \mathcal{L}(s) s^{-1/\eta}$$

for the joint survival function, where $\eta \in (0,1]$ is called the *coefficient of tail dependence* and $\mathcal{L}$ is a slowly varying function, i.e. $\mathcal{L}(ts)/\mathcal{L}(s) \to 1$ as $s \to \infty$. The cases of total independence and dependence can not be determined as easy. For example we have asymptotic independence if $\eta = 1$ and $\lim_{s\to\infty} \mathcal{L}(s) = c$ for some $0 < c \le 1$. The case of asymptotic dependence occurs if $0 < \eta < 1$ or if $\eta = 1$ together with $\lim_{s\to\infty} \mathcal{L}(s) = 0$, i.e. the coefficient of tail dependence does not determine the amount of dependence between the margins solely. Hence, its absolute value cannot be interpreted. For a detailed discussion see Section 9.5.1 in Beirlant et al. [5].

A well known measure for tail dependence is the stable tail dependence function as already defined in Definition 2.3.13. From the domain of attraction condition, cf. Theorem 2.3.12, we have for $\boldsymbol{x} \le \boldsymbol{0}$

$$
\begin{aligned}
l(x_1,\dots,x_d) &:= \lim_{t\downarrow 0} \frac{P(\{F_1(X_1) > 1+tx_1\} \cup \ldots \cup \{F_d(X_d) > 1+tx_d\})}{t} \\
&= \lim_{t\downarrow 0} \frac{1 - C_F(1+t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D
\end{aligned}
$$

if $F \in \mathcal{D}(G)$ with $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$. In close connection one can consider the *tail dependence function.*

Lemma 2.4.2 (Tail dependence function) *Suppose $F \in \mathcal{D}(G)$. Further denote by $\tilde{C}_F$ the tail copula corresponding to F, cf. Definition 2.3.6. Then, the limit*

$$
\lambda(x_1,\dots,x_d) := \lim_{t\downarrow 0} \frac{\tilde{C}_F(-t\boldsymbol{x})}{t} = \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{j\in T} x_j \boldsymbol{e}_j \right\|_D
$$

exists for $\boldsymbol{x} \le \boldsymbol{0}$ and is called the tail dependence function corresponding to the df F.

Proof: By means of Theorem A.6 and Corollary 2.3.17, we get

$$
\begin{aligned}
\lambda(x_1,\dots,x_d) &= \lim_{t\downarrow 0} \frac{\tilde{C}_F(-t\boldsymbol{x})}{t} = \lim_{t\downarrow 0} \frac{P(F_1(X_1) > 1+tx_1,\dots,F_d(X_d) > 1+tx_d)}{t} \\
&= \lim_{t\downarrow 0} \frac{1 - P(F_1(X_1) \le 1+tx_1 \cup \ldots \cup F_d(X_d) \le 1+tx_d)}{t} \\
&= \lim_{t\downarrow 0} \frac{1 - \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} C_{F_T}(1+t\boldsymbol{x})}{t} \\
&= \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \lim_{t\downarrow 0} \frac{(1 - C_{F_T}(1+t\boldsymbol{x}))}{t} \\
&= \sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{j\in T} x_j \boldsymbol{e}_j \right\|_D . \qquad (2.36)
\end{aligned}
$$

□

Note that (2.36) can be represented in terms of the stable tail dependence functions of the $|T|$-variate marginals G_T of G, cf. (2.17). Hence we get

$$\sum_{\emptyset \neq T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{j \in T} x_j \boldsymbol{e}_j \right\|_D = \sum_{\emptyset \neq T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} l_T(x_{i_1}, \dots, x_{i_T}), \tag{2.37}$$

where l_T denotes the stable tail dependence function of G_T. This can be seen by using the equality $l_G(\boldsymbol{x}) = \|\boldsymbol{x}\|_D$ in Remark 2.3.15 and the assertions of Lemma 3.1.1. Hence, (2.37) shows that there exists a direct link between the stable tail dependence function and the tail dependence function.

We will see in Section 4.2.1 that the sum in (2.36) will play a crucial role with respect to the extended fragility index.

In the bivariate case, (2.37) simplifies to

$$\lim_{t \downarrow 0} \frac{P(F_1(X_1) > 1 + tx_1, F_2(X_2) > 1 + tx_2)}{t} = x_1 + x_2 - l(x_1, x_2) = \lambda(x_1, x_2)$$

for $x_1, x_2 \leq 0$. Further, this implies that if $F \in \mathcal{D}(G)$, we have

$$\lambda^{up} = \lambda(1,1) = 2 - l(1,1) = 2 - \|(1,1)\|_D = 2 - \varepsilon, \tag{2.38}$$

where ε is the extremal coefficient, cf. Definition 2.2.4. As shown in Section 2.3, we know that the D-norm coincides with the stable tail dependence function, i.e. $\|\boldsymbol{x}\|_D = l(\boldsymbol{x})$. This implies $\varepsilon = l(1, \dots, 1)$, hence the extremal coefficient is the stable tail dependence function at point $(1, \dots, 1)$.

There exists frequent literature on estimating the stable tail dependence function (for example see Huang [39], Einmahl et al. [16] or de Haan et al. [32]).

Literature about estimating the tail dependence function are for example de Haan et al. [30], who provide a parametric estimation procedure as well a goodness-of-fit test. Further, a nonparametric estimation procedure for the tail dependence function is given by Hsing et al. [37] and a semi-parametric estimation procedure for the tail-copula is provided by Klüppelberg et al. [41]. Klüppelberg et al. [40] also provided an estimator for the tail dependence function in the special but popular case of an elliptical distribution.

As a general statement it is worth mentioning that a lot of estimators for the dependence structure between rare events are based on the Pickands dependence function or even the exponent or angular measure itself, since the representation of an EVD is mainly determined by them; see the representations in (2.2), (2.4) and (2.6). Most of these estimators are restricted to the knowledge about the univariate margins. Therefore, Genest and Segers [24] provide rank-based (bivariate) versions of Pickands-estimators (and CFG-estimators, see mentioned below) to overcome the restriction of the knowledge of the univariate margins. Their estimator, as most of the established estimators for the Pickands dependence function, is based on the assumption that the underlying distribution function is actually an EVD.

The domain-of-attraction assumption suggests to estimate the tail of a copula C_F by an extreme value copula, since the approximation

$$C_F(\boldsymbol{w}) \approx C_G^{1/n}(\boldsymbol{w}^n) = C_G(\boldsymbol{w}) \tag{2.39}$$

is the better the closer $\boldsymbol{w}$ to $\mathbf{1}$. If one therefore assumes a specific model for the extremal copula C_G, such as the popular logistic model (Gumbel-copula), the estimation procedure can be reduced to the estimation of the copula parameter, e.g. the parameter ϑ, if one considers the dependence function of the logistic model, i.e. the stable tail dependence function $l(\boldsymbol{x}) = \left(\sum_{j\le d} |x_j|^\vartheta\right)^{1/\vartheta}$. For example this is done by de Haan et al. [30] in the bivariate case with a possible extension to higher dimension. Doing so, one has to be aware of the fact that the *structure* of tail dependence is already determined by the choice of the parametric model. However, the *amount* of tail dependence is also controlled by the parameters of the parametric model.
Modeling tail dependence via extreme value copulas based on the approximation in (2.39) has been widely investigated by Tawn and others (see for example Ledford and Tawn [44], [45]). They suggest a (semi-) parametric maximum-likelihood estimation for so-called *censored data* in the bivariate case. The approach of censored data coupled with an estimation procedure for the dependence function of an EVD refers to the setting, where the considered df F belongs to the domain of attraction of an EVD instead of being an EVD itself, cf. Section 5.4 for a short outlook on this topic.
In close connection to (2.39) one may also consider a parametric estimation approach for the dependence structure in the tail using the fact that the tail of C_F can be approximated by a GPD, if C_F belongs to the domain of attraction of an EVD, cf. Corollary 2.3.19. A popular parametric model for the GPD is the *logistic GPD* defined by

$W(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_\lambda$, where $\|\cdot\|_\lambda$ denotes the L_λ-norm, cf. (7.1). Estimation of tail dependence via the logistic GPD is done by Michel [51] (see especially Chapter 6 therein, for example).

A semiparametric estimation of copulas has been provided by Genest et al. [23] and a nonparametric estimation in the bivariate case by Capéraà et al. [8], (so-called CFG-estimators), and extended to the multivariate case by Zhang et al. [71], to mention but a few. Recent work on modeling the dependence structure between rare events are established by Gudendorf, Segers and Genest, among others. Gudendorf and Segers [27] provide a nonparametric estimator for an EVD-copula, extending the idea of Capéraà et al. [8], who provided a purely nonparametric estimator of the Pickands dependence function by means of the empirical distribution function. The estimation procedures therein are restricted to the assumption that data comes from a multivariate distribution with an extreme value copula, which we call the *EVD-assumption*. We also want to mention the recent work of Einmahl et al. [17], who provided a parametric estimation procedure for the dependence structure of an EVD by a minimum distance estimator. Their approach works in arbitrary dimension under the assumption, that the underlying distribution belongs to the domain of attraction of an EVD, which we call the *domain-of-attraction-assumption*.

An appealing overview to the world of extreme value copulas, including parametric and nonparametric models as well as different estimation procedures, are provided by Gudendorf and Segers [26].

The extremal coefficient $\varepsilon := \|(1,\dots,1)\|_D$ in the d-variate case can be regarded as a measure for tail dependence between the margins of a multivariate distribution F, which belongs to the domain of attraction of an EVD G with identical margins, cf. Definition 2.2.4. Due to the inequalities (2.8) for the D-norm, we have $\varepsilon \in [1, d]$, hence the extremal coefficient can be transformed to the interval $[0, 1]$ by the transformation

$$T(\varepsilon) := \frac{\|(1,\dots,1)\|_1 - \varepsilon}{\|(1,\dots,1)\|_1 - \|(1,\dots,1)\|_\infty},$$

which enables us to interpret the *amount* of tail dependence between the identical margins of G. The tail dependence function as well as the stable tail dependence function are not restricted to measure the tail dependence between the margins of F, which belongs to an EVD with *identical* margins. Hence this is an advantage of these two measures

over the extremal coefficient. But their disadvantage is due to the fact that they are not bounded within a finite interval, as this holds for the extremal coefficient.
The fragility index combines the two mentioned advantages of compactness and validity for dependence between arbitrary margins. Furthermore, the *extended* fragility index measures the *persisting* dependence within a random system that already exhibits asymptotic dependent components. The (extended) fragility index can therefore be considered as an extension of the extremal coefficient and can be used as a powerful measure for tail dependence between the components of a random system that belongs to the domain of attraction of an extreme value distribution with arbitrary margins.

We will work out theoretical results on the representation and the extension of the fragility index in Chapters 3 and 4. All necessary previous knowledge is presented in Chapter 2. A nonparametric estimation procedure for the fragility index and applications on it are provided in Chapter 5.

3. Theoretical results for events of exceedance

This chapter contains preliminaries and results about the asymptotic conditional distribution of the number of exceedances. This distribution plays a crucial role within the representation and the extension of the fragility index as a measure for tail dependence. The chapter starts with some technical tools in Section 3.1. An insight to the problems that come up when looking at the asymptotic distribution of exceedances is given in Section 3.2. The main results from the events of exceedances are provided in Section 3.3. Therein we distinguish between exceedances above an *individual* and a *common* threshold for each component of the concerned random system. The latter approach is in line with Geluk et al. [22]. Section 3.3.3 provides examples for the asymptotic conditional distribution of exceedances counts from two different points of view.

3.1. Technical tools

At first, to provide the following condition, which is needed in the framework of events of exceedances above a common threshold, denote by F a multivariate df with univariate margins $F_j, j \leq d$, where $\omega(F_j) := \sup\{t \in \mathbb{R} : F_j(t) < 1\} \in (-\infty, \infty]$ is the upper endpoint of the univariate df F_j for $j \in \{1, \dots, d\}$.

Condition C: There exists an index $\kappa \in \{1, \dots, d\}$ with $\omega(F_\kappa) =: \omega^*$, such that

$$\lim_{s \uparrow \omega^*} \frac{1 - F_j(s)}{1 - F_\kappa(s)} \quad =: \quad \gamma_j \in [0, \infty), \quad 1 \leq j \leq d \tag{3.1}$$

holds. Note that $\omega(F_j) \leq \omega^*$ holds for $j \leq d$, since otherwise we would get $\gamma_j = \infty$, which is excluded. If $\omega(F_j) < \omega^*$ we get $\gamma_j = 0$. We have $\gamma_\kappa = 1$. If F has identical margins F_j, we get $\gamma_j = 1, i \leq d$.

For further considerations the case of $\gamma_j = 0$ for $j \in I \subset \{1, \dots, d\}$ will be of substantial interest. In Section 4.2, i.e. within the framework of exceedances above a common threshold, we will observe that the value of the extended fragility index depends on the number of such coefficients γ_j being zero. Note that γ_j equals zero if $\omega(F_j) < \omega^*$, which means the distribution function F_j has a finite upper endpoint. γ_j equals zero as well if $\omega(F_j) = \omega^*$ and $1 - F_j = o(1 - F_\kappa)$, which means the convergence rate of F_j is higher than that of F_κ, see Definition A.1.
The following lemma is necessary for further considerations.

Lemma 3.1.1 *Let G be an EVD with pertaining D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ and angular measure μ on the unit sphere S_d. Further consider $K \subseteq \{1, \dots, d\}$ and denote by G_K the corresponding $|K| =: m$-variate marginal df of G, cf. (2.17). Then the following assertions hold.*

(i) There exists an angular measure $\tilde{\mu}$ on S_m, such that the marginal df G_K of G can be represented by

$$G_K(\boldsymbol{x}) = \exp\left(\int_{S_m} \min_{i \le m}(u_i x_i) d\tilde{\mu}(\boldsymbol{u})\right)$$

for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^m$.

(ii) The angular measure $\tilde{\mu}$ induces a D-norm $\|\cdot\|_{\tilde{D}}$ on $\mathbb{R}^m$ such that

$$\left\|\sum_{j \le d} x_j \boldsymbol{e}_j\right\|_D = \left\|\sum_{j \in K} x_{i_j} \tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}$$

holds with $x_{i_j} = 0$ for $j \in K^{\complement}$, where $\boldsymbol{e}_j$ is the j-th unit vector in $\mathbb{R}^d$ and $\tilde{\boldsymbol{e}}_j$ is the j-th unit vector in $\mathbb{R}^m$.

(iii) Denote by G_K^{id} that $|K|$-variate margin of the EVD G one obtains, if we transform the margins of G_K to identical ones, cf. Remark 2.1.12. Then the extremal coefficient ε_K of G_K^{id} is given by

$$\varepsilon_K = \left\|\sum_{j \in K} \boldsymbol{e}_j\right\|_D = \left\|\sum_{j \le m} \tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}, \tag{3.2}$$

where $\boldsymbol{e}_j$ and $\tilde{\boldsymbol{e}}_j$ are the j-th unit vectors in $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively.

Proof: The first assertion is due to the fact that the marginals of max-stable distributions are max-stable again, hence, there exists an angular measure $\tilde{\mu}$ on S_m, such that G_K can be represented as given in (i). Now suppose $x_i = 0$ for $i \in K^{\complement}$ and $x_j < 0$ for $j \in K$. Then we have

$$G(x_1, \ldots, x_d) = \exp\left(-\left\|\sum_{j\in K} x_j \boldsymbol{e}_j\right\|_D\right), \tag{3.3}$$

where $\boldsymbol{e}_j$ denotes the j-th unit vector in $\mathbb{R}^d$. Thereby, the df in (3.3) is a $m := |K|$-variate margin of the EVD G. Since any margin of the EVD G is max-stable again, there exists a D-norm $\|\cdot\|_{\tilde{D}}$ defined by

$$\|(x_{i_1}, \ldots, x_{i_m})\|_{\tilde{D}} := \int_{S_m} \max_{j\le m}(\left|x_{i_j}\right| u_j)\tilde{\mu}(d\boldsymbol{u}),$$

cf. Definition 2.1.6. A m-variate margin G_K of G can therefore by provided by

$$G_K(x_{i_1}, \ldots, x_{i_m}) = \exp\left(-\left\|\sum_{j\le m} x_{i_j} \tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}\right), \tag{3.4}$$

where $\tilde{\boldsymbol{e}}_j$ is the j-th unit vector in $\mathbb{R}^m$.
Hence, the quality

$$\left\|\sum_{j\in K} x_j \boldsymbol{e}_j\right\|_D = \left\|\sum_{j\le m} x_{i_j} \tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}$$

holds under the condition $x_{i_j} = 0$ for $j \in K^{\complement}$. This shows the assertion in (ii).
Now, assume that we have $x_{i_j} = 0$ for $j \in K^{\complement}$ and $x_{i_j} = x < 0$ for $j \in K$. Then we get

$$\left\|\sum_{j\in K} \boldsymbol{e}_j\right\|_D = \left\|\sum_{j\le m} \tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}.$$

Since $G^{id}\left(\sum_{j\in K} x\boldsymbol{e}_j\right) = \exp\left(-\left\|\sum_{j\in K} x\boldsymbol{e}_j\right\|_D\right) = (\exp(-|x|))^{\left\|\sum_{j\in K}\boldsymbol{e}_j\right\|_D}$ is a $|K|$-variate margin with identical margins of the EVD G, the extremal coefficient of G^{id}, denoted by ε_K, can be represented by $\varepsilon_K = \left\|\sum_{j\in K} \boldsymbol{e}_j\right\|_D$, cf. Definition 2.2.4. This is assertion (iii). □

Noting that assertion (iii) of Lemma 3.1.1 will play a crucial role within the framework of the extended fragility index, cf. Section 4.3. Therefore we advise the reader not to confuse the terms "extremal coefficient *of* an EVD G", cf. Definition 2.2.4, and "extremal coefficient *corresponding* to an EVD G". The former requires that the EVD G has identical margins, whereas the latter shall be considered to be the extremal coefficient of the EVD G^{id} that one obtains by transforming the margins of G to identical ones. Therefore we provide the following definition.

Definition 3.1.2 (Corresponding extremal coefficient) *Let $G_{\boldsymbol{\alpha}}$ be an EVD with arbitrary, not necessarily identical margins $G_{\alpha_j}(x) = \exp(\psi_{\alpha_j}(x))$ for $j \leq d$ with ψ defined in (2.12), i.e. $G_{\boldsymbol{\alpha}}$ can be represented by*

$$G_{\boldsymbol{\alpha}}(x_1, \dots, x_d) = \exp(-\|\psi_{\alpha_1}(x_1), \dots, \psi_{\alpha_d}(x_d)\|_D),$$

cf. Proposition 2.1.10. Further denote by G^{id} the EVD one obtains, if we transform the margins of G_K to identical ones, e.g. $G^{id}_{\boldsymbol{\alpha}}(x_1, \dots, x_d) = G_{\alpha}(\psi^{-1}_{\alpha_1}(x_1), \dots,$ $\psi^{-1}_{\alpha_d}(x_d))$ is an EVD with standard exponential margins with identical D-norm to $G_{\boldsymbol{\alpha}}$, cf. Remark 2.1.12. The extremal coefficient of $G^{id}_{\boldsymbol{\alpha}}$ is defined by $\varepsilon = \left\|\sum_{j\leq d} \boldsymbol{e}_j\right\|_D$. This is called the extremal coefficient "corresponding" to the EVD $G_{\boldsymbol{\alpha}}$.

3.2. No Exceedances

Consider a random system $\{Q_1, \dots, Q_d)\}$, which shall be represented by the random vector $\boldsymbol{Q} := (Q_1, \dots, Q_d)$. Suppose that $\boldsymbol{Q}$ follows the multivariate df F, which is in the domain of attraction of an EVD G. Now, we may be interested in the event of exceedance $\{Q_j > s_j\}$ for any $j \leq d$, given that m exceedances have already occurred within the system. Under certain conditions we may be faced with the situation that no further exceedances are "possible". Within this section we will provide the tools that are necessary to investigate the situation of no exceedances.

We know from Theorem 2.3.18 that if $F \in \mathcal{D}(G)$, the copula corresponding to F can be approximated by a GPD in the upper tail. Now, suppose further that $\tilde{\boldsymbol{Q}}$ is any margin of $\boldsymbol{Q}$ of size $m \leq d$ denoted by $\tilde{\boldsymbol{Q}} := (Q_1, \dots, Q_m)$ for simplicity. Hence, with

Corollary 2.3.19 and Theorem A.6, we get that

$$\begin{aligned}
P(\tilde{\boldsymbol{Q}} > \boldsymbol{x}) &= P(F_1(Q_1) > F_1(x_1), \dots, F_m(Q_m) > F_m(x_m)) \\
&= 1 - \sum_{\emptyset \neq T \subseteq \{1,\dots,m\}} (-1)^{|T|-1} P(F_i(Q_i) \leq F_i(x_i), i \in T) \\
&= 1 - \sum_{\emptyset \neq T \subseteq \{1,\dots,m\}} (-1)^{|T|-1} C_{F_T}(F_i(x_i), i \in T) \\
&= \sum_{\emptyset \neq T \subseteq \{1,\dots,m\}} (-1)^{|T|-1} \left\| \sum_{i \in T} (F_i(x_i) - 1)\, \boldsymbol{e}_i \right\|_D \\
&\quad + o\left(\sup_{T \subseteq \{1,\dots,m\}} \{ \| F_i(x_i) - 1, i \in T \|_D \} \right)
\end{aligned}$$

holds in the upper tail of F, where C_{F_T} denotes the copula corresponding to the margin F_T of F and $\sum_{\emptyset \neq T \subseteq \{1,\dots,m\}} (-1)^{|T|-1} = 1$, see Lemma A.3.

Now suppose that the rv $\boldsymbol{U} \in \mathbb{R}^d$ follows a GPD-copula, cf. Definition 2.3.8, i.e. we have $P(\boldsymbol{U} \leq \boldsymbol{u}) = 1 - \|\boldsymbol{u} - 1\|_D$ for $\boldsymbol{u}$ close to $\mathbf{1}$. Then, the survival function of $\boldsymbol{U}$ is given by

$$\begin{aligned}
P(\boldsymbol{U} > \boldsymbol{u}) =& 1 - \left(\bigcup_{j \leq d} \{ U_j \leq u_j \} \right) \\
=& 1 - \sum_{\emptyset \neq \{1,\dots,d\}} (-1)^{|T|-1} \left(1 - \left\| \sum_{i \in T} (u_i - 1) \boldsymbol{e}_i \right\|_D \right) \\
=& \sum_{\emptyset \neq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{i \in T} (u_i - 1) \boldsymbol{e}_i \right\|_D
\end{aligned}$$

for $\boldsymbol{u}$ close to $\mathbf{1}$ where we use Theorem A.6 and Lemma A.3.

This implies that we may approximate the survival function of $\tilde{\boldsymbol{Q}}$ by means of the survival function of a GPD-copula.

Without loss of generality, we are able to consider the special case that F is in the domain of attraction of an EVD G_* with standard Weibull margins. Now, the following interesting question appears:

"Is there a GPD W, such that

$$P(\tilde{\boldsymbol{Q}} > \boldsymbol{c}) = 0 \tag{3.5}$$

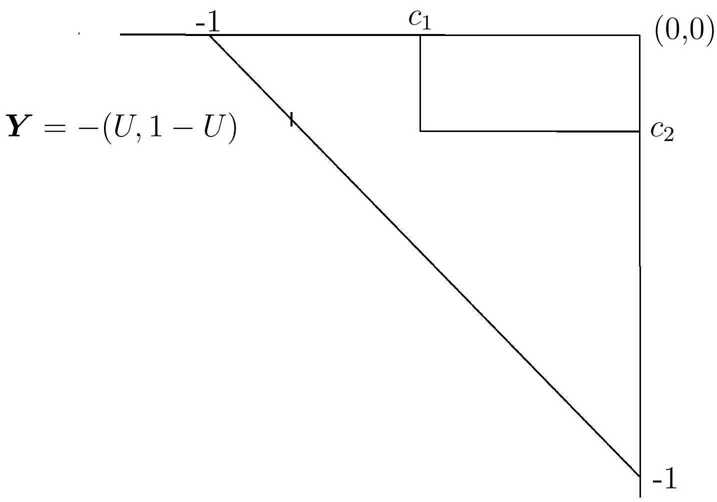


Figure 3.1.: Support line of $\boldsymbol{Y} = -(U, 1-U)$.

holds for some $\boldsymbol{c} := (c_1, \dots, c_m)$ in a left neighborhood of $\boldsymbol{0}$ with $P(Q_j > c_j) > 0$, $j \le m$, where the rv $\tilde{\boldsymbol{Q}}$ has df W?"

At first, let us consider the following bivariate example. Let the rv U be uniformly distributed on $(0, 1)$ and put

$$\boldsymbol{Y} = -(U, 1-U).$$

Then, obviously, we have

$$P(\boldsymbol{Y} > \boldsymbol{c}) = 0$$

if $\boldsymbol{c} = (c_1, c_2) < \boldsymbol{0}$ satisfies $c_1 + c_2 > -1$, see Figure 3.1.

Note that for $\boldsymbol{y} = (y_1, y_2) \le \boldsymbol{0}$, $y_1 + y_2 \ge -1$, we get

$$\begin{aligned} P(\boldsymbol{Y} \le \boldsymbol{y}) &= 1 + y_1 + y_2 \\ &= 1 - \|\boldsymbol{y}\|_1 \\ &= 1 + \log(G(\boldsymbol{y})), \end{aligned}$$

where

$$G(\boldsymbol{y}) = \exp\left(-\|\boldsymbol{y}\|_1\right), \qquad \boldsymbol{y} \le \boldsymbol{0},$$

is a bivariate EVD with standard negative exponential margins and, consequently,

$$W(\boldsymbol{y}) = P(\boldsymbol{Y} \le \boldsymbol{y}) = 1 + \log(G(\boldsymbol{y}))$$

is a GPD with uniform margins in a left neighborhood of $\mathbf{0}$. Hence, this shows that the rv $\boldsymbol{Y}$ follows a GPD with dependence structure $\|\cdot\|_D = \|\cdot\|_1$ in a left neighborhood of $\mathbf{0}$. It turns out that in the bivariate case, *any* rv $\boldsymbol{Y}$ satisfying condition (3.5) follows a GPD $W(\boldsymbol{y}) = 1 - \|y\|_1$.

Lemma 3.2.1 *Let $\boldsymbol{Y} \in [-\infty, 0]^2$ be an rv following a GPD with uniform margins. Further suppose there exist $c_1, c_2 < 0$ such that $P(\boldsymbol{Y} > \boldsymbol{c}) = 0$ and $P(Y_j > c_j) > 0$, $j \le 2$, holds. Then we have*

$$P(\boldsymbol{Y} > \boldsymbol{x}) = 0 \quad \textit{for all} \quad \boldsymbol{x} \ge \boldsymbol{c} \quad \Leftrightarrow \quad \|\cdot\|_D = \|\cdot\|_1 .$$

Proof: Assume that $P(\boldsymbol{Y} > \boldsymbol{x}) = 0$ holds for all $\boldsymbol{x} \ge \boldsymbol{c}$. Then

$$\begin{aligned}
0 &= P(\boldsymbol{Y} > \boldsymbol{x}) \\
&= 1 - P\left(\bigcup_{j\le 2} \{Y_j \le x_j\}\right) \\
&= 1 - \left(P(Y_1 \le x_1) + P(Y_2 \le x_2) - P(\boldsymbol{Y} \le \boldsymbol{x})\right) \\
&= 1 - \left(1 - \|(x_1, 0)\|_D + 1 - \|(0, x_2)\|_D - (1 - \|\boldsymbol{x}\|_D)\right) \\
&= \|(x_1, 0)\|_D + \|(0, x_2)\|_D - \|\boldsymbol{x}\|_D \\
&= |x_1| + |x_2| - \|\boldsymbol{x}\|_D , \qquad \boldsymbol{x}_0 \le \boldsymbol{x} \le \mathbf{0},
\end{aligned}$$

yields

$$\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1 , \qquad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d.$$

The other implication is obvious due to the above calculation. □

For higher dimensions $d \ge 3$ the assertions of Lemma 3.2.1 is not true. Indeed one can show that if the threshold $\boldsymbol{x}$ is a common one, say $\boldsymbol{x} := (x, \dots, x)$, we have $P(\boldsymbol{Y} > \boldsymbol{x}) = 0$ for $\boldsymbol{x} \ge \boldsymbol{c}$, if $\|\cdot\|_D = \|\cdot\|_1$.

Lemma 3.2.2 *Let $\boldsymbol{Y}$ be an rv in $\mathbb{R}^m, m \ge 2$, following a GPD with uniform margins. Further suppose that there exists $\boldsymbol{c} := \sum_{j\le m} c\boldsymbol{e}_j < \mathbf{0}$ such that $P(\boldsymbol{Y} > \boldsymbol{c}) = 0$ and $P(Y_j > c) = |c|$ for $j \le m$ and consider the common threshold $\boldsymbol{x} := \sum_{j\le m} x\boldsymbol{e}_j = (x, \dots, x)$. Then we have*

$$P(\boldsymbol{Y} > \boldsymbol{x}) = 0 \quad \textit{for all} \quad \boldsymbol{x} \ge \boldsymbol{c}$$

if $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1$.

Proof: At first, note that the survival function of a GPD W with uniform margins can be represented by

$$(3.6)\qquad \begin{aligned} P(\boldsymbol{Y} > \boldsymbol{x}) =& 1 - P\left(\bigcup_{j\le m}\{Y_j \le x_j\}\right) \\ =& 1 - \sum_{\emptyset\neq T\subseteq\{1,\dots,m\}} (-1)^{|T|-1} P(Y_i \le x_i, i\in T) \\ =& \sum_{\emptyset\neq T\subseteq\{1,\dots,m\}} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D \end{aligned}$$

for $\boldsymbol{x}$ in a left neighborhood of $\mathbf{0}$. Together with Lemma A.4 and the condition $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1$, we get

$$\begin{aligned} P(\boldsymbol{Y} > \boldsymbol{x}) &= \sum_{\emptyset\neq T\subseteq\{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{i\in T} x \boldsymbol{e}_i \right\|_1 \\ &= |x| \sum_{\emptyset\neq T\subseteq\{1,\dots,d\}} (-1)^{|T|-1} |T| \\ &= |x| \sum_{1\le j\le d} \sum_{T\subseteq\{1,\dots,d\}} (-1)^{j-1} j \\ &= |x| \sum_{1\le j\le d} \binom{d}{j} (-1)^{j-1} j \\ &= (-|x|) \underbrace{\sum_{j=0}^{d} \binom{d}{j} (-1)^{j-1} j}_{=0} = 0 \end{aligned}$$

□

As already noted, the assertion of Lemma 3.2.1 is not true for higher dimension $d \ge 3$, see Example 3.2.4 as a counter example. That means, $\|\cdot\|_D = \|\cdot\|_1$ is in general neither a necessary nor sufficient condition for (3.5) in higher dimensions.

The following lemma gives necessary and sufficient conditions such that condition (3.5) is fulfilled.

Lemma 3.2.3 *Let $\|\cdot\|_D$ be a D-norm on $\mathbb{R}^m$ with pertaining angular measure μ on S_m. Then we have*

$$(3.7)\qquad \sum_{\emptyset\neq T\subseteq\{1,\dots,m\}} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D = 0 \quad \textit{for some} \quad \boldsymbol{x} < \mathbf{0}$$

if and only if

$$\mu\left(\left\{\boldsymbol{u}\in S_m:\ \min_{j\le m}u_j=0\right\}\right)=\mu\left(\left\{\boldsymbol{u}\in S_m:\ \prod_{j\le m}u_j=0\right\}\right)=m. \tag{3.8}$$

Proof: Without loss of generality we can show the assertion for any $(\tilde{x}_1,\dots,\tilde{x}_m) := \frac{1}{a}(x_1,\dots,x_m) < (0,\dots,0)$, since (3.7) remains true if we divide it by any positive number $a>0$, i.e.

$$(3.7)\ \Leftrightarrow \sum_{\emptyset\neq T\subseteq\{1,\dots,m\}}(-1)^{|T|-1}\left\|\sum_{i\in T}x_i\boldsymbol{e}_i\right\|_D=0\quad \textit{for all}\quad \boldsymbol{x}\le\boldsymbol{0}. \tag{3.9}$$

Hence assume a vector $\boldsymbol{x}_0$ as close to $\boldsymbol{0}$, such that there exists a random vector $\boldsymbol{Y}$ which follows a GPD with uniform margins on $(\boldsymbol{x}_0,\boldsymbol{0}]$. Hence we have $P(\boldsymbol{Y}>\boldsymbol{x}) = \sum_{\emptyset\neq T\subseteq\{1,\dots,m\}}(-1)^{|T|-1}\left\|\sum_{i\in T}x_i\boldsymbol{e}_i\right\|_D$ for $\boldsymbol{x}>\boldsymbol{x}_0$. Since $\boldsymbol{Y}$ follows a GPD, Proposition 2.3.9 yields that there exists an rv $\boldsymbol{Z}$ with $Z_j\in[0,m]$ and $E(Z_j)=1, j\le m$, and a vector $(-\frac{1}{m},\dots,-\frac{1}{m})\le\boldsymbol{x}_0<\boldsymbol{0}$ such that $P(\boldsymbol{Y}>\boldsymbol{x}) = P\left(-U\left(\frac{1}{Z_1},\dots,\frac{1}{Z_m}\right)>\boldsymbol{x}\right)$ for $\boldsymbol{x}>\boldsymbol{x_0}$. Further note that the generator $\boldsymbol{Z}$ has probability measure μ/m. Hence we get

$$\begin{aligned}
P(\boldsymbol{Y}>\boldsymbol{x}) &= P\left(-U\left(\frac{1}{Z_1},\dots,\frac{1}{Z_m}\right)>\boldsymbol{x}\right)\\
&= P\left(U\left(\frac{1}{Z_1},\dots,\frac{1}{Z_m}\right)<-\boldsymbol{x}\right)\\
&= P\left(U<\min_{j\le m}(-x_jZ_j)\right)\\
&= \int_0^1 P\left(u<-\min_{j\le m}(x_jZ_j)\,|\,U=u\right)(P*U)(du)\quad \text{for all}\quad \boldsymbol{x}>\boldsymbol{x_0}\\
&= \int_0^1 P\left(u<-\min_{j\le m}(x_jZ_j)\right)du\quad \text{for all}\quad \boldsymbol{x}>\boldsymbol{x_0}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
\int_0^1 P\left(u < -\min_{j\le m}(x_j Z_j)\right) du &= 0 \Leftrightarrow \\
P\left(-\min_{j\le m}(x_j Z_j) > u\right) &= 0 \quad a.s. \text{ for } 0 < u < 1 \Leftrightarrow \\
P\left(-\min_{j\le m}(x_j Z_j) = 0\right) &= 1 \Leftrightarrow \\
P(\min_{j\le m} Z_j = 0) &= 1 \Leftrightarrow \\
\mu\left(\left\{u \in S_m \,:\, \min_{j\le m} u_j = 0\right\}\right) &= m
\end{aligned}$$

which shows the assertion. □

We want to note that the right hand side of (3.9) is equivalent to

$$\sum_{\emptyset\ne T\subseteq\{1,\dots,m\}} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D = 0 \quad \text{for all} \quad \boldsymbol{x} \ge \boldsymbol{0}$$

since any norm fulfills $\|\boldsymbol{x}\| = \|-\boldsymbol{x}\|$, $\boldsymbol{x} \in \mathbb{R}^d$.

The condition concerning the angular measure in (3.8) states that μ puts its whole mass on the boundary of the m-variate simplex $S_m := \{\boldsymbol{u} \ge \boldsymbol{0} \,:\, \|\boldsymbol{u}\|_1 = 1\}$. Note that we have independence between the margins, if the measure μ concentrates on the edges of S_m, more precisely on $\{\boldsymbol{e}_j, j \le d\}$, see Corollary 5.25 in Resnick [56]. Geometric interpretation of condition (3.8) is visualized in Figure 3.2.

Further note that Condition (3.8) is equivalent to $\|\cdot\|_D = \|\cdot\|_1$ in dimension $m = 2$ while this is not true in higher dimensions, hence we provide the sufficient and necessary condition on the angular measure corresponding to the D-norm, see Condition (3.7). The following example provides a choice for an angular measure which fulfills Condition (3.8) but does not induce the L_1-norm. Hence it is a counterexample for Lemma 3.2.1 in dimension $d \ge 3$.

Example 3.2.4 *Consider that angular measure μ, which puts equal weight* 1 *on each of the m points of the set*

$$\left\{\left(0, \frac{1}{m-1}, \dots, \frac{1}{m-1}\right), \dots, \left(\frac{1}{m-1}, \dots, \frac{1}{m-1}, 0\right)\right\}$$
$$= \left\{\frac{1}{m-1} \sum_{j\le m,\, j\ne i} \boldsymbol{e}_j,\ i \le m\right\}.$$

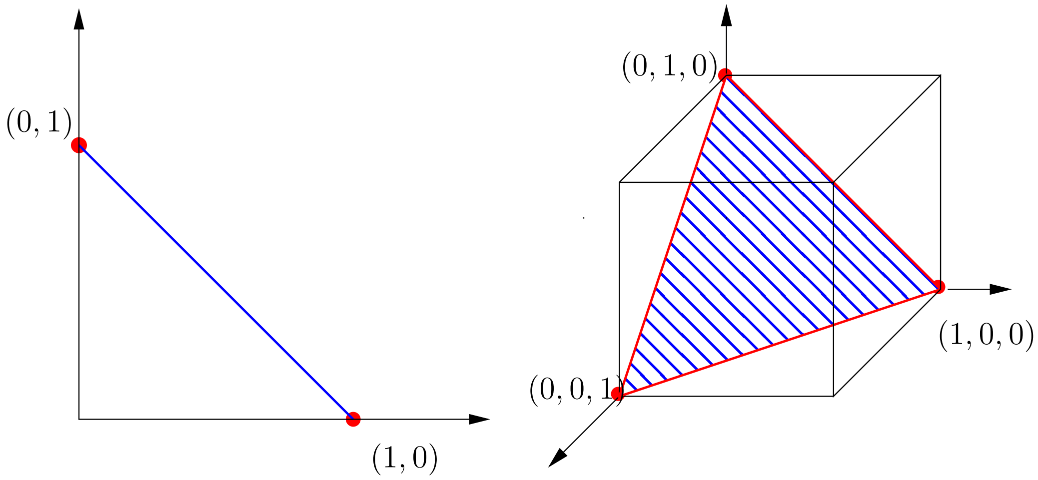


Figure 3.2.: Above figure show the valid set defined by the angular measure under Condition (3.8). The blue areas shows the simplex and the red area the set under (3.8). Shown is the two- and three-dimensional case.

Obviously, this angular measure fulfills Condition (3.8) and therefore implies that Equation (3.7) holds, where the induced D-norm is

$$\begin{aligned} \|\boldsymbol{x}\|_D &= \int_{S_m} \max_{k\le m}(x_k u_k)\,\mu(d\boldsymbol{u}) \\ &= \sum_{i\le m} \int_{\left\{\frac{1}{m-1}\sum_{j\le m,\, j\ne i} \boldsymbol{e}_j\right\}} \max_{k\le m}(x_k u_k)\,\mu(d\boldsymbol{u}) \\ &= \sum_{i\le m} \frac{1}{m-1} \max_{j\le m, j\ne i} |x_j| \\ &= \frac{1}{m-1} \sum_{i\le m} \left(\max_{j\le m, j\ne i} |x_j| \right), \qquad \boldsymbol{x} \in \mathbb{R}^m. \end{aligned}$$

This D-norm does not coincide with the L_1*-norm for* $m \ge 3$.

Condition (3.8) quantifies a certain kind of independence, which can also be characterized by means of the exponent measure.

Let G be an EVD with standard negative exponential margins, i.e. $G(\boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right)$ for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$ and some D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$, or, equivalently, there exists a σ-finite measure ν on $[-\boldsymbol{\infty}, \boldsymbol{0}] \setminus \{-\boldsymbol{\infty}\}$ with $G(\boldsymbol{x}) = \exp\left(-\nu\left([-\boldsymbol{\infty}, \boldsymbol{x}]^{\complement}\right)\right)$ for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, where ν is the exponent measure, see Proposition 2.1.3 and preceding notes.

Lemma 3.2.5 *Let G be an EVD on $\mathbb{R}^d$ with standard negative exponential margins, corresponding D-norm $\|\cdot\|_D$ and exponent measure ν. Then we have for $\boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$*

$$\nu(\boldsymbol{x}, \mathbf{0}] = \sum_{\emptyset \neq T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{i \in T} x_i \boldsymbol{e}_i \right\|_D .$$

Proof: Since ν is σ-finite, there exists a sequence of measurable subsets $B_1 \subset B_2 \subset \dots$ of $\Omega := [-\boldsymbol{\infty}, \mathbf{0}] \setminus \{-\boldsymbol{\infty}\}$ with $\bigcup_{n\in\mathbb{N}} B_n = \Omega$ and $\nu(B_n) =: b_n < \infty$, $n \in \mathbb{N}$.

Put

$$\nu_n(\cdot) := \nu(\cdot \cap B_n), \qquad n \in \mathbb{N}.$$

Then ν_n, $n \in \mathbb{N}$, defines a sequence of finite measures on Ω, $\nu_n(\Omega) = b_n$, $n \in \mathbb{N}$, with

$$\lim_{n\to\infty} \nu_n(B) = \nu(B)$$

for any measurable subset B of Ω.

The Δ-monotonicity of an arbitrary finite measure implies

$$\nu_n(\boldsymbol{x}, \boldsymbol{y}] = \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right] \right) \ge 0$$

for any $-\boldsymbol{\infty} < \boldsymbol{x} \le \boldsymbol{y} \le \mathbf{0}$ and, thus, switching to complements,

$$\begin{aligned} \nu_n(\boldsymbol{x}, \boldsymbol{y}] &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} \left(b_n - \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right]^{\complement} \right) \right) \\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right]^{\complement} \right) \end{aligned}$$

for any $n \in \mathbb{N}$; note that $\sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} = \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{\sum_{j\le d} m_j}$ $= \sum_{k=0}^{d} (-1)^k \binom{d}{k} = 0$.

We thus obtain,

$$\begin{aligned}
\nu(\boldsymbol{x},\boldsymbol{y}] &= \lim_{n\to\infty} \nu_n(\boldsymbol{x},\boldsymbol{y}] \\
&= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \lim_{n\to\infty} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i}\boldsymbol{e}_i\right]^{\complement}\right) \\
&= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \nu\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i}\boldsymbol{e}_i\right]^{\complement}\right) \\
&= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \left\|\sum_{i\le d} y_i^{m_i} x_i^{1-m_i}\boldsymbol{e}_i\right\|_D .
\end{aligned}$$

Putting $\boldsymbol{y}=\boldsymbol{0}$ and substituting m_i by $1-m_i$ we obtain

$$\begin{aligned}
\nu(\boldsymbol{x},\boldsymbol{0}] &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \left\|\sum_{i\le d} 0^{m_i} x_i^{1-m_i}\boldsymbol{e}_i\right\|_D \\
&= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{1+\sum_{j\le d} m_j} \left\|\sum_{i\le d} 0^{1-m_i} x_i^{m_i}\boldsymbol{e}_i\right\|_D \\
&= \sum_{\emptyset\ne T\subseteq\{1,\dots,d\}} (-1)^{|T|-1} \left\|\sum_{i\in T} x_i\boldsymbol{e}_i\right\|_D .
\end{aligned}$$

□

Recall that any random vector $\boldsymbol{Z}=(Z_1,\dots,Z_d)\in[0,c]^d$ for some $c\ge 1$, which satisfies $E(Z_j)=1$ for $j\le d$ generates a D-norm by $\|\boldsymbol{x}\|_D = E(\max_{j\le d}|x_j|\, Z_j)$, see Lemma 2.1.8. Further note that $\sum_{T\subseteq K}(-1)^{|T|-1}\max_{i\in T} a_i = \min_{k\in K} a_k$ for any set $\{a_k : k\in K\}$ of real numbers, which can be seen by induction.
The following further equivalences will be useful for providing conditions under which the extended fragility index exists.

Corollary 3.2.6 *Let $\|\cdot\|_D$ be a D-norm on $\mathbb{R}^m$ with pertaining angular measure μ on S_m and exponent measure ν respectively. Further denote by $Z_j, j\le m$, the generator of the D-norm. Then we have the following list of equivalences:*

(i) $E(\min_{j\le m}|x_j|\, Z_j)=0$ *for some* $\boldsymbol{x}:=(x_1,\dots,x_m)<\boldsymbol{0}$.

(ii) $\sum_{\emptyset\ne T\subseteq\{1,\dots,m\}}(-1)^{|T|-1}\left\|\sum_{i\in T} x_i\boldsymbol{e}_i\right\|_D = 0$ *for all* $\boldsymbol{x}\le\boldsymbol{0}$.

(iii) $\mu\left(\{\boldsymbol{u} \in S_m : \min_{j\le m} u_j = 0\}\right) = m$.

(iv) ν *is the null measure on* $(-\boldsymbol{\infty}, \mathbf{0}]$*, i.e.,* $\nu(-\boldsymbol{\infty}, \mathbf{0}] = 0$.

(v) ν *has all its mass (which is infinite) on the set* $\{\boldsymbol{x} \in [-\boldsymbol{\infty}, \mathbf{0}] \setminus \{-\boldsymbol{\infty}\} : \min_{j\le m} x_j = -\infty\}$.

Proof: We have for any $K \subseteq \{1, \dots, m\}$

$$\sum_{T\subseteq K} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D = \sum_{T\subseteq K} (-1)^{|T|-1} E\left(\max_{i\in T} |x_i|\, Z_i\right) = E\left(\min_{i\in K} |x_i|\, Z_i\right)$$

with Definition 2.1.8 and thus

$$\sum_{T\subseteq K} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D = 0 \iff E\left(\min_{i\in K} |x_i|\, Z_i\right) = 0\,.$$

The equivalence of (ii) and (iv) follows from Lemma 3.2.5, (ii) ⇔ (iii) follows from Lemma 3.2.3 and (iv) ⇔ (v) is due to the definition of the exponent measure. □

3.3. Asymptotic distribution of exceedance counts

This chapter presents necessary tools to provide the representation and extension of the fragility index. We will mainly focus on the asymptotic behavior of the conditional distribution function of the number of exceedances among the random system $\{Q_1, \dots, Q_d\}$.

This section is divided into two parts, which mainly cover two different approaches to the asymptotic conditional distribution of exceedance counts. This will lead to two possible representations of the extended fragility index in Chapter 4, which depend on the type of event of exceedance one considers. Therefore, Section 3.3.1 will focus on exceedances above a common high threshold within the system. This is in accordance with the approach of Geluk et al. [22] who established the fragility index as a measure for the stability of a random system. Section 3.3.2 provides the asymptotic distribution of exceedance counts with respect to exceedances above a threshold, which is an individual one for every component of the system.

We will start with the approach of exceedances above a common threshold for every component of the random system $\{Q_1, \dots, Q_d\}$.

3.3.1. A common threshold for events of exceedance

Consider a random system $\{Q_1, \dots, Q_d\}$ of size d. Let s be a high threshold and define by

$$\{Q_j > s\}$$

for any $j \le d$ the exceedance of the component Q_j above a high threshold s. Thereby the threshold s should be chosen high enough such that $\{Q_j > s\}$ is an *extreme* event. Further define by

$$N_s := \sum_{j=1}^{d} \mathbb{1}_{(s,\infty)}(Q_j) \tag{3.10}$$

the number of exceedances within the system. The aim of this section is to provide the asymptotic distribution of N_s given at least $m \le d$ exceedances for any $1 \le m \le d$ have occurred. We start with some technical results.

Lemma 3.3.1 *Suppose $(Q_1, \dots, Q_d) \sim F$ is continuous for $\boldsymbol{x}$ close to $\omega(F) := (\omega(F_1), \dots, \omega(F_d))$ and the pertaining copula satisfies $C_F \in \mathcal{D}(G)$. Furthermore suppose that condition C, see (3.1), holds. Then we have for arbitrary $K \subseteq \{1, \dots, d\}$*

$$P(Q_i \le s,\ i \in K) = 1 - c(s) \left\| \sum_{i \in K} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \tag{3.11}$$

for $s \uparrow \omega^*$ with $c(s) := 1 - F_\kappa(s) \downarrow_{s \uparrow \omega^*} 0$.

Proof: With Definition 2.3.1 and Theorem 2.3.18, Corollary 2.3.19 respectively, we get

$$
\begin{aligned}
P(Q_i \le s,\ i \in K) &= P(F_i(Q_i) \le F_i(s),\ i \in K) = C_F\left(\sum_{j\in K} F_j(s)\boldsymbol{e}_j + \sum_{i\notin K}\boldsymbol{e}_i\right)\\
&= 1 - \left\|\sum_{i\in K}(F_i(s)-1)\boldsymbol{e}_i\right\|_D + o\left(\left\|\sum_{i\in K}(F_i(s)-1)\boldsymbol{e}_i\right\|_D\right)\\
&= 1 - \left\|\sum_{i\in K}\frac{1-F_i(s)}{1-F_\kappa(s)}(1-F_\kappa(s))\boldsymbol{e}_i\right\|_D + o\left(\left\|\sum_{i\in K}\frac{1-F_i(s)}{1-F_\kappa(s)}(1-F_\kappa(s))\boldsymbol{e}_i\right\|_D\right)\\
&= 1-(1-F_\kappa(s))\cdot\left\|\sum_{i\in K}\gamma_i\boldsymbol{e}_i\right\|_D + o\left((1-F_\kappa(s))\cdot\underbrace{\left\|\sum_{i\in K}\gamma_i\boldsymbol{e}_i\right\|_D}_{const.}\right)\\
&= 1-(1-F_\kappa(s))\cdot\left\|\sum_{i\in K}\gamma_i\boldsymbol{e}_i\right\|_D + o(1-F_\kappa(s)),
\end{aligned}
$$

hence the assertion follows with $1-F_\kappa(s)\downarrow 0$ for $s\uparrow\omega^*$ and $c(s):=1-F_\kappa(s)$. □

Corollary 3.3.2 *Assume the same assumptions as in Lemma 3.3.1. Denote by* $\emptyset \ne I \subset \{1,\dots,d\}$ *the set of indices with* $\gamma_i = 0$ *and choose an arbitrary* $K \subseteq \{1,\dots,d\}$*. Then the distribution in (3.11) simplifies to*

(i) $P(Q_i \le s,\, i\in K) = 1 - o(c(s))$ *for* $s\uparrow\omega^*$ *and* $K\subseteq I$,

(ii) $P(Q_i \le s,\, i\in K) = 1 - \left\|\sum_{i\in K\cap I^\complement}\gamma_i\boldsymbol{e}_i\right\|_D + o(c(s))$ *for* $s\uparrow\omega^*$.

Now we are able to provide the asymptotic distribution of the number of exceedances above a common threshold, denoted by N_s.

Lemma 3.3.3 *Assume the same assumptions as in Lemma 3.3.1.*
Let $N_s := \sum_{j\le d}\mathbb{1}_{(s,\infty)}(Q_j)$ *be the number of exceedances above a common threshold* s

within the random system $\{Q_1, \dots, Q_d\}$. *Then we have*

$$\begin{aligned}
P(N_s = 0) &= 1 - c(s) \left\| \sum_{1 \le j \le d} \gamma_j \boldsymbol{e}_j \right\|_D + o(c(s)) =: 1 - c(s) a_0(\boldsymbol{\gamma}) + o(c(s))\,, \\
P(N_s = k) &= c(s) \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \\
&=: c(s)\, a_k(\boldsymbol{\gamma}) + o(c(s)) \quad \textit{for} \quad 1 \le k \le d-1 \quad \textit{and} \\
P(N_s = d) &= c(s) \cdot \sum_{j=1}^{d} (-1)^{j+1} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \\
&=: c(s)\, a_d(\boldsymbol{\gamma}) + o(c(s))
\end{aligned}$$

for $s \uparrow \omega^*$ *with* $c(s) = 1 - F_\kappa(s)$.

Proof: Lemma 3.3.1 immediately implies

$$P(N_s = 0) = P(Q_j \le s,\ 1 \le j \le d) = 1 - c(s) \left\| \sum_{j=1}^{d} \gamma_j \boldsymbol{e}_j \right\|_D + o(c(s))$$

for $s \uparrow \omega^*$. For $1 \le k \le d-1$ we get due to disjoint events and Theorem A.6

$$\begin{aligned}
P(N_s = k) &= \sum_{\substack{S \subseteq \{1,\dots,d\} \\ |S| = k}} P(Q_i > s,\ i \in S,\ Q_j \le s,\ j \in S^\complement) \\
&= \sum_{\substack{S \subseteq \{1,\dots,d\} \\ |S| = k}} P(Q_i > s,\ i \in S \,|\, Q_j \le s,\ j \in S^\complement) \cdot P(Q_j \le s, j \in S^\complement) \\
&= \sum_{\substack{S \subseteq \{1,\dots,d\} \\ |S| = k}} \Bigg[P(Q_j \le s, j \in S^\complement) \cdot \\
&\qquad \left(1 - \sum_{r \le |S|} (-1)^{r+1} \sum_{\substack{K \subseteq S \\ |K| = r}} P(Q_i \le s,\ i \in K \,|\, Q_j \le s,\ j \in S^\complement) \right) \Bigg] \\
&= \sum_{\substack{S \subseteq \{1,\dots,d\} \\ |S| = k}} \Bigg[P(Q_j \le s, j \in S^\complement) \cdot \\
&\qquad \left(1 - \sum_{r \le |S|} (-1)^{r+1} \sum_{\substack{K \subseteq S \\ |K| = r}} \frac{P(Q_i \le s,\ i \in K \cup S^\complement)}{P(Q_j \le s,\ j \in S^\complement)} \right) \Bigg]
\end{aligned}$$

$$= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \Bigg[P(Q_j \le s, j \in S^{\complement})\cdot \left(1 - \frac{1}{P(Q_j \le s,\ j \in S^{\complement})} \sum_{r\le|S|} (-1)^{r+1} P(Q_i \le s,\ i \in K \cup S^{\complement})\right)\Bigg]$$

$$= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \left[P(Q_j \le s,\ j \in S^{\complement}) - \sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} P(Q_i \le s,\ i \in K \cup S^{\complement})\right].$$

Now let $s \uparrow \omega^*$ with $c(s) := 1 - F_\kappa(s)$. Then it follows from Lemma 3.3.1

$$P(N_s = k) = \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \Bigg[1 - c(s)\left\|\sum_{j\in S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s)) - \sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left(1 - c(s)\left\|\sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s))\right)\Bigg]$$

$$= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \Bigg[1 - c(s)\left\|\sum_{j\in S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s)) - \underbrace{\sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} 1}_{=1} + \sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left(c(s)\left\|\sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s))\right)\Bigg]$$

$$= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \Bigg[1 - c(s)\left\|\sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s)) - 1 + c(s)\cdot\sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left\|\sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D + o(c(s)) \underbrace{\sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} 1}_{=1}\Bigg]$$

$$= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \left[c(s)\cdot\left(\sum_{r\le|S|}(-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left\|\sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D - \left\|\sum_{j\in S^{\complement}} \gamma_j \boldsymbol{e}_j\right\|_D\right)\right.$$

$$
\begin{aligned}
&+o(c(s))] \\
&= \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \left[o(c(s)) + c(s)\cdot \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left\| \sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D \right] \\
&= c(s)\cdot \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left\| \sum_{j\in K\cup S^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D + o(c(s))\,.
\end{aligned}
$$

With an index transformation we get

$$
\begin{aligned}
P(N_s = k) &= c(s)\cdot \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subseteq S\\|K|=r}} \left\| \sum_{j\in \underbrace{K\cup S^{\complement}}_{\substack{:=T,\\|T|=r+d-k}}} \gamma_j e_j \right\|_D \\
&\quad +o(c(s)) \\
&= c(s)\cdot \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{S\subseteq\{1,\dots,d\}\\|S|=k}} \sum_{\substack{K\subseteq S\\|K|=r}} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s)) \\
&= c(s)\cdot \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{K\subseteq\{1,\dots,d\}\\|K|=r}} \sum_{\substack{K\subseteq T\\|T|=r+d-k}} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s)) \\
&= c(s)\cdot \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=r+d-k}} \sum_{\substack{K\subseteq T\\|K|=r}} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s)) \\
&= c(s)\cdot \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=r+d-k}} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D \underbrace{\sum_{\substack{K\subseteq T\\|K|=r}} 1}_{=\binom{|T|}{r}} + o(c(s)) \\
&= c(s)\cdot \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=r+d-k}} \binom{r+d-k}{r} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s)) \\
&\overset{j=k-r}{=} c(s)\cdot \sum_{0\le j\le k} (-1)^{k-j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=d-j}} \binom{d-j}{k-j} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s))
\end{aligned}
$$

$$= c(s) \cdot \sum_{0\le j\le k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \left\| \sum_{i\in T} \gamma_i e_i \right\|_D + o(c(s)) \ ,$$

and

$$\begin{aligned} P(N_s = d) &= P(Q_j > s,\ j \le d) = 1 - P(\cup_{j\le d}\{Q_j \le s\}) \\ &= 1 - \sum_{j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}} P(Q_i \le s,\ i \in T) \\ &= 1 - \sum_{j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}} \left(1 - c(s) \cdot \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \right) \\ &= c(s) \cdot \sum_{j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \end{aligned}$$

for $s \uparrow \omega^*$. Note that one also gets the assertion for $P(N_s = d)$ by putting $k = d$ in the above step of calculation. □

The following corollary is an immediate consequence of Lemma 3.3.3.

Corollary 3.3.4 *We have $a_0(\boldsymbol{\gamma}) > 0$, since $\gamma_\kappa = 1$, $a_k(\boldsymbol{\gamma}) \ge 0\,,\ 1 \le k \le d$ and $a_0(\boldsymbol{\gamma}) = \sum_{k=1}^d a_k(\boldsymbol{\gamma})$. Furthermore it holds*

$$a_k(\boldsymbol{\gamma}) = \lim_{s\uparrow\omega^*} \frac{P(N_s = k)}{c(s)} = \sum_{0\le j\le k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D$$

for $1 \le k \le d-1$,

$$a_d(\gamma) = \lim_{s\uparrow\omega^*} \frac{P(N_s = d)}{c(s)} = \sum_{1\le j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\},\\ |T|=j}} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D$$

and

$$a_0(\boldsymbol{\gamma}) = \lim_{s\uparrow\omega^*} \frac{P(N_s > 0)}{c(s)} = \left\| \sum_{1\le j\le d} \gamma_j \boldsymbol{e}_j \right\|_D \ .$$

In the following we simplify the above result for the case $\gamma_i = 0$ for $i \in I \subset \{1, \dots, d\}$. Therefore assume that the set I of indices with $\gamma_i = 0$ is not empty. The case $I = \{1, \dots, d\} \backslash \{\kappa\}$ is of special interest. We also include the results for arbitrary $\emptyset \ne I \subset \{1, \dots, d\}$ for the sake of completeness even if they are obvious.

Corollary 3.3.5 *Denote by $\emptyset \neq I \subset \{1,\dots,d\}$ the set of indices with $\gamma_i = 0$. Then we get*

$$\begin{aligned} a_0(\boldsymbol{\gamma}) &= \lim_{s\uparrow\omega^*} \frac{P(N_s>0)}{c(s)} = \left\| \sum_{j\in I^\complement} \gamma_j \boldsymbol{e}_j \right\|_D , \\ a_k(\boldsymbol{\gamma}) &= \lim_{s\uparrow\omega^*} \frac{P(N_s=k)}{c(s)} \\ &= \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \left\| \sum_{i\in T\cap I^\complement} \gamma_i \boldsymbol{e}_i \right\|_D \end{aligned}$$

for $1\le k\le d-1$ and

$$a_d(\gamma) = \sum_{1\le j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\},\\ |T|=j}} \left\| \sum_{i\in T\cap I^\complement} \gamma_i \boldsymbol{e}_i \right\|_D .$$

For the special case of $I^\complement = \{\kappa\}$ the limits above simplify to

$$a_0(\boldsymbol{\gamma}) = a_1(\boldsymbol{\gamma}) = 1$$

and

$$a_k(\boldsymbol{\gamma}) = 0 \quad \textit{for} \quad 2\le k\le d\,.$$

Proof: The first part of the corollary is obvious, we provided it for sake of completeness. Hence, we only prove the assertion for the special case of $I^\complement = \{\kappa\}$, which is the second part of the corollary. Note that $\gamma_\kappa = 1$, hence we get $a_0(\boldsymbol{\gamma}) = 1$. For $k=1$ the assertion is a straightforward calculation. Now, we show $a_k(\boldsymbol{\gamma}) = 0$ for $2\le k\le d$. Note that there are $\binom{d-1}{d-j-1}$ subsets of $\{1,\dots,d\}$ of length $d-j$ for $j\le k$ and $k\le d$, which contain the index κ. For $I^\complement = \{\kappa\}$ we get $\left\|\sum_{i\in T\cap I^\complement} \gamma_i \boldsymbol{e}_i\right\|_D = \|\boldsymbol{e}_\kappa\|_D = 1$, if $\kappa\in T$ (else we sum up over the empty set) and therefore the sum $\sum_{0\le j\le k}(-1)^{k-j+1}\binom{d-j}{k-j}\sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \left\|\sum_{i\in T\cap I^\complement} \gamma_i \boldsymbol{e}_i\right\|_D$ simplifies to

$$\sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \binom{d-1}{d-j-1} . \tag{3.12}$$

Note that (3.12) equals 1 for $k = 1$. For $2 \le k \le d$ we get

$$
\begin{aligned}
&\sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \binom{d-1}{d-j-1} \\
&= \binom{d}{k} \frac{1}{d} \sum_{0\le j\le k} (-1)^{k-j+1} \binom{k}{j} (d-j) \\
&= \binom{d}{k} \sum_{0\le j\le k} (-1)^{k-j+1} \binom{k}{j} - \binom{d}{k} \frac{1}{d} \sum_{0\le j\le k} (-1)^{k-j+1} \binom{k}{j} j \\
&= 0 - \frac{1}{d} \binom{d}{k} \sum_{1\le j\le k} (-1)^{k-j+1} k \binom{k-1}{j-1} \\
&= (-1) \frac{k}{d} \binom{d}{k} \sum_{0\le j\le k-1} (-1)^{k-j} \binom{k-1}{j} \\
&= (-1) \frac{k}{d} \binom{d}{k} \sum_{0\le j\le k-1} (-1)^{k-1-j} \binom{k-1}{j} = 0\,,
\end{aligned}
$$

which proves the assertion. □

The following corollary is a consequence of Lemma 3.3.3 and directly follows by the preceding Corollary 3.3.5.

Corollary 3.3.6 *Assume the same setting as in Lemma 3.3.3. Denote by $I \neq \emptyset \subset \{1, \ldots, d\}$ the set of indices with $\gamma_i = 0$. Then the distribution of N_c in (3.11) can be written as*

$$
\begin{aligned}
P(N_s = 0) &= 1 - c(s) \left\| \sum_{j\in I^{\complement}} \gamma_j \boldsymbol{e}_j \right\|_D + o(c(s)) =: 1 - c(s) a_0(\boldsymbol{\gamma}) + o(c(s))\,, \\
P(N_s = k) &= c(s) \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subseteq\{1,\ldots,d\}\\ |T|=d-j}} \left\| \sum_{i\in T\cap I^{\complement}} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \\
&=: c(s)\, a_k(\boldsymbol{\gamma}) + o(c(s)) \quad \textit{for} \quad 1 \le k \le d-1 \quad \textit{and} \\
P(N_s = d) &= c(s) \cdot \sum_{j=1}^{d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\ldots,d\}\\ |T|=j}} \left\| \sum_{i\in T\cap I^{\complement}} \gamma_i \boldsymbol{e}_i \right\|_D + o(c(s)) \\
&=: c(s)\, a_d(\boldsymbol{\gamma}) + o(c(s))
\end{aligned}
$$

for $s \uparrow \omega^*$ *with* $c(s) = 1 - F_\kappa(s)$. *In the special case of* $I^{\complement} = \{\kappa\}$ *we get the simplification*

$$\begin{aligned} P(N_s = 0) &= 1 - c(s) + o(c(s)), \\ P(N_s = 1) &= c(s) + o(c(s)) \quad \text{and} \\ P(N_s = k) &= o(c(s)) \quad \text{for} \quad 2 \le k \le d. \end{aligned}$$

In the following we want to provide the asymptotic conditional distribution of exceedance counts among $\{Q_1, \dots, Q_d\}$ (ACDEC). The limit of $p_k := P(N_s = k \mid N_s > 0)$, as the threshold increases, turns out to be the ratio of a_k and a_0 as given in Corollary 3.3.4. The following result is an implication of the above considerations and will be the main result within this section. Since it will be a very important tool for the extension of the fragility index, we will provide the ACDEC as a theorem.

Theorem 3.3.7 (ACDEC) *Set* $p_k(\boldsymbol{\gamma}) := a_k(\boldsymbol{\gamma})/a_0(\boldsymbol{\gamma})$, $1 \le k \le d,$. *Then* $p_k(\boldsymbol{\gamma})$ *defines a probability distribution on* $\{1, \dots, d\}$ *and is the asymptotic conditional distribution function of exceedance counts, abbreviated by ACDEC.*

$$\lim_{s \uparrow \omega^*} P(N_s = k \mid N_s > 0) =$$
$$\frac{\sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \|_D}{\| \sum_{j=1}^d \gamma_j \boldsymbol{e}_j \|_D} =: p_k(\boldsymbol{\gamma})$$

for $1 \le k \le d-1$ *and*

$$\lim_{s \uparrow \omega^*} P(N_s = d \mid N_s > 0) =$$
$$\frac{\sum_{1 \le j \le d} (-1)^{j+1} \sum_{\substack{T \subseteq \{1,\dots,d\}, \\ |T| = j}} \| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \|_D}{\| \sum_{j=1}^d \gamma_j \boldsymbol{e}_j \|_D} =: p_d(\boldsymbol{\gamma}).$$

Proof: The assertion follows from the equation

$$\lim_{s \uparrow \omega^*} P(N_s = k \mid N_s > 0) = \lim_{s \uparrow \omega^*} \frac{P(N_s = k)/c(s)}{P(N_s > 0)/c(s)}$$

for $k \le d$ together with the definition of $a_k(\boldsymbol{\gamma})$, see Corollary 3.3.4, for $1 \le k \le d$, which are finite numbers, and $a_0(\boldsymbol{\gamma}) > 0$. Hence the limits $p_k(\boldsymbol{\gamma})$, $k \le d$, are well defined and exist. □

For sake of completeness we establish the representation of the ACDEC under the special case of $|I| \ge 1$.

Corollary 3.3.8 *Denote by $\emptyset \neq I \subset \{1,\dots,d\}$ the set of indices with $\gamma_i = 0$ and choose an arbitrary $K \subseteq \{1,\dots,d\}$. Then the ACDEC can be represented by*

$$p_k(\gamma) \;=\; \frac{\sum_{0\le j\le k}(-1)^{k-j+1}\begin{pmatrix} d-j \\ k-j \end{pmatrix}\sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \|\sum_{i\in T\cap I^{\complement}}\gamma_i \boldsymbol{e}_i\|_D}{\|\sum_{j\in I^{\complement}}\gamma_j \boldsymbol{e}_j\|_D}$$

for $1 \le k \le d-1$ and

$$p_d(\boldsymbol{\gamma}) \;=\; \frac{\sum_{1\le j\le d}(-1)^{j+1}\sum_{\substack{T\subseteq\{1,\dots,d\},\\ |T|=j}} \|\sum_{i\in T\cap I^{\complement}}\gamma_i \boldsymbol{e}_i\|_D}{\|\sum_{j\in I^{\complement}}\gamma_j \boldsymbol{e}_j\|_D}.$$

For the special case of $I^{\complement} = \{\kappa\}$ the ACDEC simplifies to

$$p_1(\boldsymbol{\gamma}) = \frac{a_1(\boldsymbol{\gamma})}{a_0(\boldsymbol{\gamma})} = 1$$

and

$$p_k(\gamma) = \frac{a_k(\boldsymbol{\gamma})}{a_0(\boldsymbol{\gamma})} = 0 \quad \textit{for} \quad 2 \le k \le d\,.$$

3.3.2. An individual threshold for events of exceedance

Hitherto we have worked with events of exceedance $\{Q_j > s\}$, $j \in \{1,\dots,d\}$, for s close to ω^*. Hence we considered a *common threshold* for components of the random system $\{Q_1,\dots,Q_d\}$.

We assume in the following that the joint df F is continuous in its upper tail. Now, as an alternative approach to Section 3.3.1, we want to consider

$$\{Q_j > F_j^{-1}(1-c)\} \quad \text{for} \quad c \downarrow 0 \tag{3.13}$$

as an event of exceedance among $\{Q_1,\dots,Q_d\}$. Since the threshold depends on the margin F_j this is a so-called *individual threshold* for every component of the system, more precisely $\tilde{s}_j := F_j^{-1}(1-c)$ for $j \le d$. This is once again an extreme event if c is chosen close enough to 0. Hence, the individual thresholds $\tilde{s}_j$ may differ, but the probability of exceedance remains the same for every component $Q_j, j \le d$.

Due to the equality $F^{-1}(q) > t \Leftrightarrow q > F(t), q \in (0,1), t \in \mathbb{R}$, the event (3.13) is equivalent to

$$\{F_j(Q_j) > 1-c\} \quad \text{for} \quad c \downarrow 0$$

as an event of exceedance in the random system $\{F_1(Q_1), \dots, F_d(Q_d)\}$. Hence we consider the system of the copula, i.e. $(U_1, \dots, U_d) := (F_1(Q_1), \dots, F_d(Q_d)) \sim C_F$, if F is continuous.

Further we note that with respect to the fragility index the approach of exceedances above an *individual* threshold is new to the literature. For example, Geluk et al. [22] solely consider exceedances above a common threshold in order to provide a fragility index for the stability of a random system. Of course, there exists broad literature concerning asymptotic distribution results for multivariate exceedances above a high individual threshold; see e.g. Buishand [7], Section 2.2 or Rootzén and Tajvidi [57], who define a multivariate curve $\{\boldsymbol{u}(t) \,|\, t \in [1, \infty)\}$ to which certain normalized excesses are considered, see page 921ff therein.
In the following we want to give a short summary of the results obtained by working with an individual threshold as defined in (3.13). The procedure is the same as in Section 3.3.1.

Lemma 3.3.9 *Suppose* $(Q_1, \dots, Q_d) \sim F$ *and* F *is continuous in the neighborhood of* $\omega(F) := (\omega(F_1), \dots, \omega(F_d)) \in (-\infty, \infty]^d$ *and the pertaining copula satisfies* $C_F \in \mathcal{D}(G)$. *Then it holds for any* $K \subseteq \{1, \dots, d\}$

$$P(F_j(Q_j) \le 1 - c,\ j \in K) = 1 - c \cdot \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D + o(c)$$

for $c \downarrow 0$.

Proof: First, note that if the copula C_F belongs to the domain of attraction of an EVD G and the $|K|$-variate margin of the copula C_F, denoted by C_{F_K}, belongs to the domain of attraction of the $|K|$-variate margin G_K of the EVD G. Then the assertion immediately follows with Corollary 2.3.19, i.e.

$$\begin{aligned} P(F_j(Q_j) \le 1 - c,\, j \in K) &= C_F\left(\sum_{j \in K} (1 - c)\boldsymbol{e}_j + \sum_{j \notin K} \boldsymbol{e}_j \right) \\ &= 1 - c \cdot \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D + o(c)\,. \end{aligned}$$

□

Define by

$$N_c := \sum_{j \le d} \mathbb{1}_{(1-c,1]}(F_j(Q_j)) \tag{3.14}$$

the number of exceedances of $F_j(Q_j)$ above $1-c$ among the system $\{Q_1, \ldots, Q_d\}$ for any $j \le d$. The following lemma plays the crucial role with regard to the representation of the fragility index based on individual thresholds.

Lemma 3.3.10 *Assume the same assumptions as in Lemma 3.3.9.*
Let $N_c := \sum_{j \le d} \mathbb{1}_{(1-c,1]}(F_j(Q_j))$ *be the number of exceedances above* $1-c$. *Then we have*

(i) $P(N_c = 0) = 1 - c \cdot \left\| \sum_{1 \le j \le d} \boldsymbol{e}_j \right\|_D + o(c) =: 1 - c \cdot a_0 + o(c)\,,\ c \downarrow 0,$

(ii) and for $k \le d-1$ *we have*

$$\begin{aligned} P(N_c = k) &= c \cdot \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\ldots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D + o(c) \\ &=: c \cdot a_k + o(c) \end{aligned}$$

for $c \downarrow 0$ *and*

(iii)

$$\begin{aligned} P(N_c = d) &= c \cdot \sum_{j=1}^{d} (-1)^{j+1} \sum_{\substack{T \subseteq \{1\ldots,d\} \\ |T| = j}} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D + o(c) \\ &=: c \cdot a_d + o(c)\,,\ c \downarrow 0\,. \end{aligned}$$

Proof: The proof is analogue to the proof of Lemma 3.3.3 by means of Lemma 3.3.9.
□

Corollary 3.3.11 *Lemma 3.3.10 implies*

$$\begin{aligned} a_k &= \lim_{c \downarrow 0} \frac{P(N_c = k)}{c} \\ &= \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\ldots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D \end{aligned}$$

for $1 \le k \le d$ *and* $a_0 = \lim_{c \downarrow 0} \frac{P(N_c > 0)}{c} = \left\| \sum_{1 \le j \le d} \boldsymbol{e}_j \right\|_D$.

Using Lemma 3.3.10 we get the following result concerning the asymptotic distribution of N_c given that we have already observed at least one exceedance. As we do in Section 3.3.1, we call this distribution the asymptotic conditional distribution of exceedance counts, ACDEC. Although it is an implication of preceding results, we provide it as a theorem due to its importance for future considerations.

Theorem 3.3.12 (ACDEC) *Under the same assumptions of Lemma 3.3.9 we get*

$$\lim_{c\downarrow 0} P(N_c = k \mid N_c > 0) = $$
$$= \frac{\sum_{0\le j\le k}(-1)^{k-j+1}\begin{pmatrix} d-j \\ k-j \end{pmatrix}\sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}}\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_D}{\left\|\sum_{j=1}^d \boldsymbol{e}_j\right\|_D}$$
$$=: p_k$$

for $1 \le k \le d-1$ *and*

$$\lim_{c\downarrow 0} P(N_c = d \mid N_c > 0) = \frac{\sum_{1\le j\le d}(-1)^{j+1}\sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}}\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_D}{\left\|\sum_{1\le j\le d}\boldsymbol{e}_j\right\|_D} =: p_d\ .$$

Since $p_j \ge 0,\ 1 \le j \le d,\ \sum_{j=1}^d p_j = 1$ *holds,* $p_1, \dots, p_d$ *is a probability-distribution on* $\{1, \dots, d\}$.

Proof: The assertion immediately follows from Corollary 3.3.11 and the notes in the proof of Theorem 3.3.7. □

Recall the definition of the extremal coefficient $\varepsilon := \|(1,\dots,1)\|_D \in [1,d]$ corresponding to an EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, cf. Definition 2.2.4, and the term "corresponding extremal coefficient", cf. Remark 3.1.2. Further, recall (2.18). Denote by G_K the $|K|$-variate margin of the EVD G corresponding to $K \subset \{1,\dots,d\}$. Then the extremal coefficient corresponding to G_K is defined by $\varepsilon_K := \left\|\sum_{j\in K}\boldsymbol{e}_j\right\|_D = \left\|\sum_{j\le m}\tilde{\boldsymbol{e}}_j\right\|_{\tilde{D}}$, where $\|\cdot\|_{\tilde{D}}$ is the D-norm corresponding to G_K, see Lemma 3.1.1. Hence the ACDEC $p_k, k \le d$, is a composition of the extremal coefficients of the margins G_K of G (included G itself) where $|K| \ge d-k$, i.e. p_k, depends on the extremal coefficients corresponding to the margins G_K of dimension larger or equal $d-k$.

Remark 3.3.13 *Consider the EVD* $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$ *and denote by* $\varepsilon_K := \left\|\sum_{j\in K}\boldsymbol{e}_j\right\|_D$ *the extremal coefficient corresponding to the* $|K|$*-variate margin* G_K *of* G*,* $\emptyset \neq K \subset \{1,\dots,d\}$*, see Definition 3.1.2. Then the ACDEC (*p_k *respectively) as provided in Theorem 3.3.12 can be represented in terms of the extremal coefficients corresponding to the margins of* G *with dimension* $|K| \geq d-k$ *for any* $k \leq d$*. Precisely we have*

$$p_k := \frac{1}{\varepsilon}\sum_{0\leq j\leq k}(-1)^{k-j+1}\binom{d-j}{k-j}\sum_{\substack{T\subseteq\{1,\dots,d\}\\|T|=d-j}}\varepsilon_T \tag{3.15}$$

for any $k \leq d$*. Thereby recall the terming "corresponding extremal coefficient", cf. Remark 3.1.2.*

3.3.3. Examples for the ACDEC

Within this section we want to provide some examples for the ACDEC, which may give a first insight into applications on it. This section will be divided into two parts. The first part sets the focus on the D-norm. The second part focuses on the condition $C_F \in \mathcal{D}(G)$, which leads to the "corresponding" D-norm via the limit in (2.34) within Corollary 2.3.16.

3.3.3.1. Certain examples for the D-norm

Let us start with an easy example, which turns out to be crucial within applications on the fragility index. With respect to the popular logistic EVD, cf. Example 2.1.14, we want to consider the arbitrary L_λ-norm, which includes the two extreme cases of maximum dependence and full independence in the tail of $(Q_1,\dots,Q_d)$.

Lemma 3.3.14 *Let* $\left\|\sum_{j\in A}\boldsymbol{e}_j\right\|_D = \left\|\sum_{j\in B}\boldsymbol{e}_j\right\|_D$ *for arbitrary* $A,B \subseteq \{1,\dots,d\}$ *with* $|A| = |B|$*. Then we get*

$$p_k = \frac{(-1)^{k+1}\binom{d}{k}\sum_{0\leq j\leq k}(-1)^j\binom{k}{j}\left\|\sum_{i=1}^{d-j}\boldsymbol{e}_i\right\|_D}{\left\|\sum_{1\leq j\leq d}\boldsymbol{e}_j\right\|_D},\ 1\leq k\leq d-1$$

and

$$p_d = \frac{\sum_{1\le j\le d}(-1)^{j+1}\begin{pmatrix} d \\ j \end{pmatrix}\left\|\sum_{i=1}^{j}\boldsymbol{e}_i\right\|_D}{\left\|\sum_{1\le j\le d}\boldsymbol{e}_j\right\|_D}.$$

Proof: This is elementary:

$$\begin{aligned}
&\sum_{0\le j\le k}(-1)^{k-j+1}\begin{pmatrix} d-j \\ k-j \end{pmatrix}\sum_{\substack{T\subseteq\{1,\dots,d\},\\ |T|=d-j}}\left\|\sum_{j\in T}\boldsymbol{e}_j\right\|_D \\
&= \sum_{0\le j\le k}(-1)^{k-j+1}\underbrace{\begin{pmatrix} d-j \\ k-j \end{pmatrix}\begin{pmatrix} d \\ d-j \end{pmatrix}}_{=\begin{pmatrix} d \\ k \end{pmatrix}\begin{pmatrix} k \\ j \end{pmatrix}}\left\|\sum_{i=1}^{d-j}\boldsymbol{e}_i\right\|_D \\
&= (-1)^{k+1}\begin{pmatrix} d \\ k \end{pmatrix}\sum_{0\le j\le k}(-1)^{j}\begin{pmatrix} k \\ j \end{pmatrix}\left\|\sum_{i=1}^{d-j}\boldsymbol{e}_i\right\|_D .
\end{aligned}$$

□

Note that the L_λ-norm fulfills the required symmetry condition in Lemma 3.3.14.

Corollary 3.3.15 *(i) The choice of the L_λ-norm, cf. (7.1) for $1<\lambda<\infty$ implies $\|\sum_{j=1}^{k}\boldsymbol{e}_j\|_\lambda = k^{1/\lambda}$ and therefore we get*

$$p_k = (-1)^{k+1}\begin{pmatrix} d \\ k \end{pmatrix}\sum_{0\le j\le k}(-1)^{j}\begin{pmatrix} k \\ j \end{pmatrix}\left(1-\frac{j}{d}\right)^{1/\lambda},\ 1\le k\le d-1$$

and

$$p_d = \sum_{1\le j\le d}(-1)^{j+1}\begin{pmatrix} d \\ j \end{pmatrix}\left(\frac{j}{d}\right)^{1/\lambda}.$$

(ii) For the maximum-norm, cf. (7.1), we get $\left\|\sum_{j=1}^{k}\boldsymbol{e}_j\right\|_\infty = 1$ and therefore

$$p_k = (-1)^{k+1}\begin{pmatrix} d \\ k \end{pmatrix}\sum_{0\le j\le k}(-1)^{j}\begin{pmatrix} k \\ j \end{pmatrix} = 0 \quad \textit{for} \quad 1\le k\le d-1$$

and

$$p_d = \sum_{1\le j\le d} (-1)^{j+1} \binom{d}{j} = 1 \, .$$

Hence in the case of the maximum-norm the ACDEC vanishes for $k \le d-1$, i.e. the whole mass of the distribution is on $k = d$.

(iii) For the L_1-norm, cf. (7.1) for $\lambda = 1$, we get $\left\|\sum_{j=1}^k \boldsymbol{e}_j\right\|_1 = k$ and hence

$$\begin{aligned} p_k &= \frac{(-1)^{k+1}\binom{d}{k}\sum_{0\le j\le k}(-1)^j\binom{k}{j}(d-j)}{d} \\ &= 0 + \frac{1}{d}(-1)^k k \sum_{0\le j\le k-1} (-1)^j \binom{k-1}{j} = 0\, , \; 2 \le k \le d \end{aligned}$$

and $p_1 = 1$.

Hence, in the case of the L_1-norm, the ACDEC vanishes for $k \ge 2$, i.e. the whole mass of the distribution is on $k = 1$.

We want to use the two extreme cases $\|\cdot\|_1$ and $\|\cdot\|_\infty$ to provide another example.

Example 3.3.16 (Marshall-Olkin) *The convex combination of the maximum- and the L_1-norm*

$$\|\boldsymbol{x}\|_{MO} := \vartheta \|\boldsymbol{x}\|_1 + (1-\vartheta)\|\boldsymbol{x}\|_\infty \, , \qquad \boldsymbol{x} \in \mathbb{R}^d, \, \vartheta \in [0,1]$$

is called the Marshall-Olkin norm (see Falk et al. [19], Example 4.3.2, for the so called Marshall-Olkin df). Note that $\|\cdot\|_{MO}$ defines a D-norm, since the convex combination of two Pickands dependence functions is a Pickands dependence function (see Section 4.3 in Falk et al. [19]) and the D-norm can be represented by means of the Pickands dependence function, see (2.7).

By means of Theorem 3.3.12, Lemma A.4 and the binomial formula $\sum_{j=0}^m (-1)^j \binom{m}{j} = (1+(-1))^m = 0$ we have

$$p_k := \frac{\sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \left(\vartheta \left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_1 + (1-\vartheta)\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_\infty\right)}{\vartheta\left\|\sum_{j=1}^d \boldsymbol{e}_j\right\|_1 + (1-\vartheta)\left\|\sum_{j=1}^d \boldsymbol{e}_j\right\|_\infty}$$

$$
\begin{aligned}
&= \frac{\sum_{0\le j\le k}(-1)^{k-j+1}\begin{pmatrix} d-j \\ k-j \end{pmatrix}\begin{pmatrix} d \\ d-j \end{pmatrix}(\vartheta(d-j)+(1-\vartheta))}{\vartheta d+(1-\vartheta)} \\
&= \frac{\begin{pmatrix} d \\ k \end{pmatrix}\sum_{0\le j\le k}(-1)^{k-j+1}\begin{pmatrix} k \\ j \end{pmatrix}(\vartheta(d-1)+1-\vartheta j))}{\vartheta d+(1-\vartheta)} \\
&= \frac{\begin{pmatrix} d \\ k \end{pmatrix}(-1)^k\vartheta\sum_{0\le j\le k}(-1)^j\begin{pmatrix} k \\ j \end{pmatrix} j}{\vartheta d+(1-\vartheta)} \\
&= \begin{cases} \frac{\binom{d}{k}(-1)^k\vartheta(-1)^k k!}{\vartheta d+1-\vartheta}, & k=1 \\ 0, & 2\le k\le d-1\,. \end{cases}
\end{aligned}
$$

for $k \le d-1$ and

$$
\begin{aligned}
p_d &= \frac{\sum_{1\le j\le k}(-1)^{j+1}\sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}}\left(\vartheta\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_1+(1-\vartheta)\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_\infty\right)}{\vartheta\left\|\sum_{j=1}^d\boldsymbol{e}_j\right\|_1+(1-\vartheta)\left\|\sum_{j=1}^d\boldsymbol{e}_j\right\|_\infty} \\
&= \frac{\sum_{1\le j\le k}(-1)^{j+1}\begin{pmatrix} d \\ j \end{pmatrix}(\vartheta j+(1-\vartheta))}{\vartheta d+(1-\vartheta)} \\
&= \frac{1-\vartheta}{\vartheta d+1-\vartheta}
\end{aligned}
$$

for $k = d$. Hence we obtain

$$
p_1 = \frac{\vartheta d}{\vartheta d+1-\vartheta}, \quad p_d = \frac{1-\vartheta}{\vartheta d+1-\vartheta}
$$

and $p_k = 0$ for $2 \le k \le d-1$, i.e. the Marshall-Olkin norm implies that the whole mass of the distribution is on $k = 1$ and $k = d$. For $\vartheta = 1$ we get the L_1-norm and for $\vartheta = 0$ we get the maximum-norm. Hence the parameter ϑ determines the degree of dependence between the margins of G (the asymptotic dependence between the margins of F respectively) and covers the cases of total dependence as well independence. For further results ascribed to the Marshall-Olkin norm, see Example 4.2.19 and Remark 4.4.3.

In the particular case, where the D-norm is the usual L_λ-norm with $\lambda \in [0,\infty]$, we can derive the limit

$$\lim_{d\to\infty} p_k = \lim_{d\to\infty} p_k(d) \tag{3.16}$$

of the ACDEC as the dimension d increases.
With the results of Corollary 3.3.15, (ii) and (iii), the limit behavior of p_k in (3.16) is clear for $\lambda \in \{1,\infty\}$. We, therefore, restrict ourselves in the following to $\lambda \in (0,\infty)$.

The next proposition provides the asymptotic ACDEC for the L_λ-norm.

Proposition 3.3.17 (Asymptotic ACDEC) *Suppose that the underlying D-norm is the L_λ-norm with $1 < \lambda < \infty$. Then we have for $k \in \mathbb{N}$*

$$p_k^*(\lambda) := \lim_{d\to\infty} p_k = \frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right).$$

Proof: First, recall the assertion of Lemma A.4. Further, recall that

$$p_k = p_k(d) = \binom{d}{k}\sum_{0\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda}, \qquad 1\le k\le d.$$

Put $f(x) := x^{1/\lambda}$, $x \ge 0$. The Taylor expansion of length k implies for $\varepsilon \in (0,1)$

$$f(1-\varepsilon) = f(1) + \sum_{1\le i\le k-1}\frac{f^{(i)}(1)}{i!}(-\varepsilon)^i + \frac{f^{(k)}(\xi)}{k!}(-\varepsilon)^k,$$

where $\xi \in (1-\varepsilon, 1)$ and

$$f^{(i)}(x) = x^{\frac{1}{\lambda}-i}\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right).$$

We, thus, obtain for $1 \le j \le k < d$ with $\varepsilon = j/d$

$$\begin{aligned}\left(1-\frac{j}{d}\right)^{1/\lambda} &= 1 + \sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^i\frac{\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right)}{i!}\\ &\qquad + \xi_j^{\frac{1}{\lambda}-k}\left(-\frac{j}{d}\right)^k\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{k!},\end{aligned}$$

where $\xi_j \in (1-j/d, 1)$. This implies for fixed $1 \le k < d$

$$p_k = \binom{d}{k}\sum_{0\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda}$$

$$
\begin{aligned}
&= \binom{d}{k}\left((-1)^{k+1} + \sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda}\right)\\
&= \binom{d}{k}\left((-1)^{k+1} + \sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left\{1+\sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^{i}\frac{\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right)}{i!}\right.\right.\\
&\qquad\qquad\left.\left.+\ \xi_j^{\frac{1}{\lambda}-k}\left(-\frac{j}{d}\right)^{k}\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{k!}\right\}\right)\\
&= \binom{d}{k}\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left\{\sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^{i}\frac{\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right)}{i!}\right.\\
&\qquad\qquad\left.+\ \xi_j^{\frac{1}{\lambda}-k}\left(-\frac{j}{d}\right)^{k}\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{k!}\right\}\\
&= \binom{d}{k}\sum_{1\le i\le k-1}\frac{\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right)}{i!}\left(\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(-\frac{j}{d}\right)^{i}\right)\\
&\qquad + \binom{d}{k}\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{k!}\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(-\frac{j}{d}\right)^{k}\xi_j^{\frac{1}{\lambda}-k}.
\end{aligned}
$$

The first term in the final equation on the right hand side above vanishes by Lemma A.4. For fixed k and $d\to\infty$, the second term converges to

$$
\begin{aligned}
\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{(k!)^2}\sum_{1\le j\le k}(-1)^{-j+1}\binom{k}{j}j^k &= (-1)^{k-1}\frac{\prod_{0\le r\le k-1}\left(\frac{1}{\lambda}-r\right)}{k!}\\
&= \frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right)
\end{aligned}
$$

by Lemma A.4. □

Note that $p_k^*(\lambda) = 1/(\lambda k)\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right)$, $k\in\mathbb{N}$, is the distribution of a stopping time: Let $X_1, X_2\dots$ be an independent rv with values in $\{0,1\}$ and

$$P(X_j = 0) = 1-\frac{1}{j\lambda} = 1-P(X_j=1), \qquad j\in\mathbb{N}.$$

Put

$$\tau(\lambda) := \min\{j\in\mathbb{N} : X_j = 1\}.$$

Then, obviously,

$$P(\tau(\lambda)=k) = \frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right) = p_k^*(\lambda), \qquad k\in\mathbb{N}.$$

Note that $P(\tau(\lambda) < \infty) = 1$, $1 \leq \lambda < \infty$, whereas $P(\tau(\infty) = \infty) = 1$, if we include $\lambda \in \{1, \infty\}$ in our considerations.

Denote by P_λ the ACDEC as given in 3.3.15, i.e., $P_\lambda(k) = p_k(d)$, $k \in \mathbb{N}$. Then Proposition 3.3.17 can be formulated as follows, where $\to_D$ denotes weak convergence.

Proposition 3.3.18 *We have for $\lambda \in [1, \infty)$ as $d \to \infty$*

$$P_\lambda \to_D \tau(\lambda).$$

The following Example is motivated by a similar one taken from Geluk et al. [22], Example 6 together with an extension of Theorem 3 therein.

Example 3.3.19 (Weighted Pareto) *Let $X_1, \dots, X_m$ be an independent and identically Pareto distributed rv with parameter $\alpha > 0$. Put*

$$Q_i := \sum_{j=1}^{m} \lambda_{ij} X_j, \qquad 1 \leq i \leq d,$$

where the weights λ_{ij} are nonnegative and satisfy $\sum_{j=1}^{m} \lambda_{ij}^{\alpha} = 1$, $1 \leq i \leq d$.

The df of the rv $\boldsymbol{Q} = (Q_1, \dots, Q_d)$ is in the domain of attraction of the EVD

$$G^*(\boldsymbol{s}) = \exp\left(-\sum_{j=1}^{m} \max_{i \leq d} \left(\frac{\lambda_{ij}}{s_i}\right)^{\alpha}\right), \qquad \boldsymbol{s} = (s_1, \dots, s_d) > \boldsymbol{0}, \tag{3.17}$$

with standard Fréchet margins $G_k(\boldsymbol{s}) = \exp\left(-s^{-\alpha}\right)$, $s > 0$, $1 \leq k \leq d$. This can be seen by proving that

$$\begin{aligned} &P\left(\sum_{j=1}^{m} \lambda_{ij} X_j \leq n^{1/\alpha} s_i,\, 1 \leq i \leq d\right) \\ &= 1 - \frac{1}{n}\left(\sum_{j=1}^{m} \max_{i \leq d} \left(\frac{\lambda_{ij}}{s_i}\right)^{\alpha} + o(1)\right), \qquad \boldsymbol{s} > \boldsymbol{0} \in \mathbb{R}^d, \end{aligned}$$

which follows from tedious but elementary computations, using conditioning on $X_j = x_j$, $j = 2, \dots, m$.

As a consequence, we obtain that the copula pertaining to $\boldsymbol{X}$ is in the domain of attraction of $G(\boldsymbol{x}) = \exp\left(-\left\|\boldsymbol{x}\right\|_D\right)$, $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, where the D-norm is

$$\left\|\boldsymbol{x}\right\|_D := \sum_{j=1}^{m} \left(\max_{i \leq d} \left(\lambda_{ij}^{\alpha} \left|x_i\right|\right)\right), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

see (3.17).

3.3.3.2. Certain examples for the copula

In the following we show a certain choice of parametric models for the copula C_F, on which we require that it is in the domain of attraction of an EVD G with corresponding D-norm $\|\cdot\|_D$.

Even if we are in the situation that we know the df F, it will be hard work to determine the domain of attraction F belongs to (in case it exists) in the majority of cases. Concerning this matter, one can find characterizations of multivariate domains of attraction in Falk et al. [19], Theorem 5.3.1 or in Resnick [56], Proposition 5.15. One might check the sufficient conditions for $F \in \mathcal{D}(G)$ given there.
We want to follow another approach within this Section. It is common to model the dependence structure of a df F with an arbitrary family of multivariate parametric copula function C. This involves elliptical copulas, such as the Normal-copula and the t-copula, as well as the family of Archimedean copulas, with popular members as the Frank and the Clayton copulas, for example. For more detailed information, see Nelsen [52].

Recall that we are able to compute the D-norm by means of the copula. The necessary tools have already been provided in Chapter 2.
Suppose that $(Q_1, \dots, Q_d) \sim F \in \mathcal{D}(G)$. Then the corresponding copula C_F converges to the copula C_G of G, see Theorem 2.3.12. Furthermore, we have shown that the convergence of the copulas is equivalent to

$$\lim_{t \downarrow 0} \frac{1 - C_F(1 + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D \,, \tag{3.18}$$

see Corollary 2.3.16.

Now we want to provide some examples for the ACDEC, using the limit result in (3.18).
A popular parametric copula model in financial mathematics and insurances is the family of Archimedean copulas; see Chapter 4 in Nelsen [52] for an appealing overview. The following definition is taken from McNeil and Něslehová [48], Definition 2.2.

Definition 3.3.20 *A nonincreasing and continuous function* $\varphi : [0, 1] \to [0, \infty)$*, which satisfies*

(i) φ *is strictly decreasing on* $(0, 1]$,

(ii) $\varphi(1) = 0$,

(iii) $\lim_{x \downarrow 0} \varphi(x) = \infty$

is called an Archimedean generator. A d-dimensional copula C_φ is called Archimedean if it permits the representation

$$C_\varphi(\boldsymbol{u}) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_m)), \quad \boldsymbol{u} \in [0,1]$$

for some Archimedean generator φ and its inverse $\varphi^{-1}(t) := \inf\{x > 0 \,:\, \varphi(x) \leq t\}$.

Note that McNeil and Nĕslehová [48] define Archimedean copulas in terms of φ instead of φ^{-1}, as done in Definition 3.3.20. We use the above definition, since it is in line with the majority of authors defining Archimedean copulas. The following theorem is Theorem 2.2 in McNeil and Nĕslehová [48] and provides necessary and sufficient conditions on φ such that C_φ is a d-dimensional copula on $[0,1]^d$.

Theorem 3.3.21 *Let φ be an Archimedean generator as defined in Definition 3.3.20. Then $C : [0,1]^d \to [0,1]$ given by*

$$c(\boldsymbol{u}) = \varphi\left(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d)\right), \quad \boldsymbol{u} \in [0,1]^d$$

is a d-dimensional copula if and only if φ is d-monotone on $[0, \infty)$ (see Definition A.2 for the definition of d-monotonicity).

Proof: See McNeil and Nĕslehová [48].

The following proposition shows that under certain additional conditions on the generator φ, the limit in (2.34) equals the L_1-norm.

Proposition 3.3.22 *Let C_φ be an Archimedean Copula and assume that the generator φ is differentiable to the left in $x = 1$ and that the left derivative of φ in $x = 1$ is not equal to 0. Then we get*

$$\lim_{t \downarrow 0} \frac{1 - C_\varphi(\mathbf{1} + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_1 \,, \quad \boldsymbol{x} \leq \mathbf{0} \,.$$

Proof: At first note that $C_\varphi(\mathbf{1} + t\boldsymbol{x}) = \varphi^{-1}\left(\sum_{j \leq d} \varphi(1 + tx_j)\right)$, $x_j \leq 0$, $j \leq d$ follows from definition. Therefore we get

$$\begin{aligned}
\lim_{t\downarrow 0}\frac{1-C_\varphi(\mathbf{1}+t\boldsymbol{x})}{t} &= \lim_{t\downarrow 0}\frac{\varphi^{-1}\left(\sum_{j\le d}\varphi(1)\right)-\varphi^{-1}\left(\sum_{j\le d}\varphi(1+tx_j)\right)}{t}\\
&= \lim_{t\downarrow 0}\frac{\varphi^{-1}\left(\sum_{j\le d}\varphi(1)\right)-\varphi^{-1}\left(\sum_{j\le d}\varphi(1)+\sum_{j\le d}\varphi(1+tx_j)\right)}{\sum_{j\le d}\varphi(1+tx_j)}\cdot\\
&\qquad\sum_{j\le d}\frac{\varphi(1+tx_j)}{tx_j}x_j\\
&= \lim_{t\downarrow 0}(-1)\frac{\varphi^{-1}\left(\sum_{j\le d}\varphi(1)+\sum_{j\le d}\varphi(1+tx_j)\right)-\varphi^{-1}\left(\sum_{j\le d}\varphi(1)\right)}{\sum_{j\le d}\varphi(1+tx_j)}\cdot\\
&\qquad\sum_{j\le d}\frac{\varphi(1+tx_j)-\varphi(1)}{tx_j}x_j\\
&= \lim_{t\downarrow 0}\frac{\varphi^{-1}\left(\sum_{j\le d}\varphi(1)+\sum_{j\le d}\varphi(1+tx_j)\right)-\varphi^{-1}\left(\sum_{j\le d}\varphi(1)\right)}{\sum_{j\le d}\varphi(1+tx_j)}\\
&\qquad\lim_{t\downarrow 0}\sum_{j\le d}\frac{\varphi(1+tx_j)-\varphi(1)}{tx_j}(-x_j)\\
&= \left(\varphi^{-1}(0)\right)'\cdot\sum_{j\le d}\left(\varphi'(1)(-x_j)\right)=\frac{1}{\varphi'(\varphi^{-1}(0))}\cdot\sum_{j\le d}\left(\varphi'(1)(-x_j)\right)\\
&= \sum_{j\le d}(-x_j)=\sum_{j\le d}|x_j|=\|\boldsymbol{x}\|_1
\end{aligned}$$

for $x_j\le 0,\ j\le d$. □

Note that the cases $\|\boldsymbol{x}\|_D=\|\boldsymbol{x}\|_\infty$ and $\|\boldsymbol{x}\|_D=\|\boldsymbol{x}\|_1$ represent the two extreme scenarios of asymptotic dependence, independence in the system $\{Q_1,\dots,Q_d\}$ respectively. Therefore, the above result shows that the application of Archimedean copulas, which fulfills the conditions in Proposition 3.3.22, leads to a model that inhibits asymptotic independence among the margins of $(Q_1,\dots,Q_d)$.

Example 3.3.23 (Archimedean copula) *Take for example the Clayton copula with generator*

$$\varphi_C(x)=\frac{1}{\vartheta}\left((x)^{-\vartheta}-1\right),\ \vartheta\in[-1,\infty)\backslash 0$$

or the Frank copula with generator

$$\varphi_F(x)=-\log\left(\frac{\exp(-\vartheta x)-1}{\exp(-\vartheta)-1}\right),\ \vartheta>0\,,$$

which are popular copulas within the application of finance and economics (see Nelsen [52], Chapter 4). Both generator functions fulfill the conditions in Proposition 3.3.22. Hence we get $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_1$*, i.e. the margins of* $(Q_1, \ldots, Q_d)$ *are asymptotically independent. Hence the ACDEC is given by* $p_1 = 1$ *and* $p_k = 0$ *for* $2 \le k \le d$*, see Corollary 3.3.15, (iii).*

Take now the Gumbel copula with generator

$$\varphi_G(x) = (-\log(x))^\lambda,\ \lambda \in [1, \infty)\,.$$

Hence the expression for the Gumbel copula is

$$C_{Gu}(\boldsymbol{u}) = \exp\left(-\left(\sum_{j\le d}(-\log(u_j))^\lambda\right)^{\frac{1}{\lambda}}\right) = \exp\left(-\|\log(\boldsymbol{u})\|_\lambda\right)\ , \tag{3.19}$$

see also Example 2.3.5.

The derivative of the generator φ_G *equals* 0 *in* $x = 1$*, hence the condition in Proposition 3.3.22 is not fulfilled.*
With the Taylor approximation $\exp(x) = 1 + x + o(x^2)$, $x \in \mathbb{R}$ *and* $\ln(1+x) = x + O(x^2)$ *for* $x \le 0$ *close to* 0 *we get*

$$\begin{aligned}
1 - C_{Gu}(1 + t\boldsymbol{x}) &= 1 - \exp(-\|-\log(1 + tx_j),\, j \le d\|_\lambda)\\
&= 1 - \exp(-\left\|-tx_j + O(t^2),\, j \le d\right\|_\lambda)\\
&= 1 - (1 - \left\|-tx_j + O(t^2),\, j \le d\right\|_\lambda + o(\left\|-tx_j + O(t^2),\, j \le d\right\|_\lambda^2))\\
&= \left\|-tx_j + O(t^2),\, j \le d\right\|_\lambda + o(\left\|-tx_j + O(t^2),\, j \le d\right\|_\lambda^2)\,.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1 - C_{Gu}(1 + t\boldsymbol{x})}{t}\\
&= \lim_{t\downarrow 0}\frac{t\left(\|-x_j + O(t),\, j \le d\|_\lambda + o(\|-tx_j + O(t),\, j \le d\|_\lambda^2\right)}{t} = \|\boldsymbol{x}\|_\lambda\,.
\end{aligned} \tag{3.20}$$

Hence, the Gumbel copula corresponds to the arbitrary L_λ*-norm and therefore covers as well the case of asymptotic full dependence and independence between the margins of* $(Q_1, \ldots, Q_d)$*. The ACDEC is provided by Corollary 3.3.15.*

The Ali-Mikhail-Haq family of copulas with generator function

$$\varphi_{AMH}(x) = -\ln\left(\frac{1 - \theta(1 - x)}{x}\right),\ \theta \in (-1, 1)$$

fulfills the condition in Proposition 3.3.22 and hence the ACDEC is given by $p_1 = 1$ *and* $p_k = 0$ *for* $2 \leq k \leq d$*, see Corollary 3.3.15, (iii).*

Also, the Fairlie-Gumbel-Morgenstern copula (see Example 3.12 in Nelsen [52]), which is popular within financial applications, leads to the L_1-norm using (3.18) and the rule of L'Hospital.

We want to note that the majority of generators of Archimedean copulas (see for example the listing in Table 4.1 in Nelsen [52]) fulfill the conditions in Proposition 3.3.22 and this therefore implies asymptotic independence between the margins of an rv $(Q_1, \ldots, Q_d)$ if its copula belongs to the domain of attraction of an EVD.
We summarize to inform the interested reader that, unfortunately, a wide range of copulas, mainly Archimedean copulas, provide a parametric model that inhibits asymptotic independence. This is an undesirable property if one wishes to model the obvious present tail dependence in random systems like the financial one. With respect to above results, the Gumbel copula for $\lambda > 1$ provides a useful parametric model for tail dependence.

Applying Corollary 2.3.19, the tail of an arbitrary copula, which belongs to the domain of attraction of an EVD can be approximated by

$$C(\boldsymbol{y}) \sim 1 - \|\boldsymbol{y} - \boldsymbol{1}\|_D, \quad \boldsymbol{y} \in [0,1]^d,$$

Hence, the following approximation for the resulting D-norm concerning the Gumbel copula

$$\|\boldsymbol{x}\|_D \sim 1 - \exp\left(- \|\log(\boldsymbol{x} + \boldsymbol{1})\|_\lambda\right)$$

holds for $\boldsymbol{x} < \boldsymbol{0}$ close to $\boldsymbol{0}$. In particular, we conclude that in the tail of the Gumbel copula, the D-norm is a function that can be approximated by a transformed exponential function.

4. Representation and Extension of the Fragility Index

This chapter aims to provide and extend a measure for the stability of a random system, namely the fragility index. This work is based on the results of Geluk et al. [22], who established the fragility index in the framework of financial systems. We want to generalize their approach to an arbitrary setting where we consider a stochastic system with special interest to its tail dependence structure.

With the focus on the system‘s stability, we provide an extension of the fragility index that is even able to capture the amount of risk when some components of the system have already exceeded their critical threshold. It turns out that the fragility index is a suitable measure for tail dependence, even if the margins of the system‘s distribution do not belong to the same domain of attraction. More precisely, we regard the fragility index to be a generalization of the extremal coefficient (see Smith [64]) by dropping the assumption that the distribution of the system has identical margins. We focus on this topic in Section 4.3.

A stochastic system might also be considered as the finite sequence taken from a stochastic process. Doing so we provide in Section 4.4 the so-called *sojourn index*, which measures the excursion time of a stochastic process exceeding a high threshold within a finite sequence at a certain time point. Therefore, the amount of asymptotic dependence within a finite sequence of a stochastic process can be captured by the number of sequential exceedances in comparison to the fragility index, which provides the expected total number of exceedances given at least one (or several) exceedance(s).

We start with the representation of the fragility index by norms within Section 4.1. Then we continue with the extension of the fragility index in Section 4.2. As already mentioned, Section 4.3 aims to point out the meaning of the fragility index in comparison to the extremal coefficient and Section 4.4 provides a short excursion of the fragility index to the world of stochastic processes.

4.1. The Fragility Index

Based on the results for the ACDEC - see Section 3.3 - we are able to present the fragility index by norms. We start with the representation of the fragility index by means of exceedances above a common threshold and continue by means of exceedances above an individual threshold.

Definition 4.1.1 *Denote by N_s the number of exceedances within the random system $\{Q_1, \dots, Q_d\}$ as defined in (3.10).*
The limit

$$FI := \lim_{s \nearrow} E(N_s \mid N_s > 0)$$

is called the fragility index, abbr. FI, if it exists. The FI defines a measure for asymptotic stability at the system's level.

Hence, the fragility index is the limit of the expected number of exceedances above a high threshold within the random system as the threshold increases, given there is at least one exceedance. The following proposition provides the representation of the FI by norms.

Proposition 4.1.2 *Under the assumptions of Lemma 3.3.1 we get*

$$FI = \frac{\sum_{j=1}^d \gamma_j}{\left\| \sum_{j=1}^d \gamma_j \boldsymbol{e}_j \right\|_D}. \tag{4.1}$$

Proof: Recall Condition (3.1), ω^* and γ_j defined therein and the assertion of Lemma 3.3.3. Then we get

$$\begin{aligned}\lim_{s\uparrow\omega^*} E(N_s \mid N_s > 0) &= \sum_{j=1}^d \lim_{s\uparrow\omega^*} E\left(\mathbb{1}_{(s,\infty)}(Q_j) \mid N_s > 0\right) \\ &= \sum_{j=1}^d \lim_{s\uparrow\omega^*} \frac{P(Q_j > s)}{1 - P(N_s = 0)}\end{aligned}$$

$$= \sum_{j=1}^{d} \lim_{s \uparrow \omega^*} \frac{1 - F_j(s)}{1 - F_\kappa(s)} \frac{1 - F_\kappa(s)}{1 - P(N_s = 0)}$$
$$= \frac{\sum_{j=1}^{d} \gamma_j}{\left\| \sum_{j=1}^{d} \gamma_j \boldsymbol{e}_j \right\|_D}.$$

□

Remark 4.1.3 *Note that $FI \in [1, d]$ and we call the financial system $\{Q_1, \ldots, Q_d\}$ weak fragile if $FI = 1$ and strongly fragile if $FI > 1$. We also refer to these cases as asymptotic stability, asymptotic instability of a random system respectively. This coincides with the notation in Geluk et al. [22].*

Now we want to consider the case $\gamma_j = 1$, $j \leq d$, which follows from identical or tail-equivalent margins in the framework of a common threshold. In this case, the resulting fragility index coincides with the approach of exceedances above an individual threshold, see Section 3.3.2. We will interpret this case and its implications extensively in Section 4.2.2 and for this moment request to suppose the case $\gamma_j = 1$, $j \leq d$.

Corollary 4.1.4 *Assume the conditions of Lemma 3.3.1 and further suppose that $\gamma_j = 1$ holds for $j \leq d$. Then we get*

$$FI = \frac{d}{\left\| \sum_{j \leq d} \boldsymbol{e}_j \right\|_D} = \frac{d}{\varepsilon},$$

where ε is the extremal coefficient corresponding to the EVD G, see Definition 2.2.4. Further note that in the bivariate case, we have

$$FI = \frac{2}{\varepsilon} = \frac{2}{2 - \lambda^{up}},$$

where λ^{up} is the well known upper tail dependence coefficient (Geoffrey [25] and Sibuya [60]), see Section 2.4. Recall that λ_{up} only measures upper tail dependence in the bivariate case, where the FI serves as a measure for tail dependence in arbitrary dimensions.

Proof: The first assertion is a direct implication of Proposition 4.1.2 and the definition of the extremal coefficient, cf. Definition 2.2.4. The second assertion follows by Equation (2.38). □

Hence we obtain the representation $FI = d/\varepsilon$ in case of identical margins $F_j, j \le d$. However, if one considers the approach of an individual threshold, we obtain the same representation of the FI. This shows the following corollary.

Corollary 4.1.5 *Consider the approach of exceedances above an individual threshold, cf. Section 3.3.2, i.e. exceedances defined by* $\{Q_j > F_j^{-1}(1-c)\}$ *for* $c \downarrow 0$ *and the number of exceedances defined by* $N_c := \sum_{j\le d} \mathbb{1}_{(1-c,1]}(F_j(Q_j))$. *Then the fragility index is defined by* $FI := \lim_{c\downarrow 0} E(N_c \mid N_c > 0)$ *and we get*

$$FI = \frac{d}{\varepsilon}. \tag{4.2}$$

Proof: Recall Lemmata 3.3.9 and 3.3.10. Hence we get

$$\begin{aligned} FI &= \lim_{c\downarrow 0} E(N_c \mid N_c > 0) = \lim_{c\downarrow 0} \sum_{j\le d} E\left(\mathbb{1}_{(1-c,1]}(F_j(Q_j)) \mid N_c > 0\right) \\ &= \sum_{j\le d} \lim_{c\downarrow 0} \frac{P(F_j(Q_j) > 1-c)}{1-P(N_c = 0)} = \frac{d}{\varepsilon}. \end{aligned}$$

□

We want to mention that the resulting representation of the fragility index in (4.2) is in line with results provided in de Haan and Ferreira [29], Section 7.4, who consider a dependence coefficient called κ defined by

$$\kappa := \lim_{t\to\infty} E(K(t) \mid K(t) \ge 1),$$

where $K(t)$ is the number of exceedances within the system $\{X_1, \dots, X_d\}$. Thereby they define the exceedance $\{X_j \ge U_j(t)\}$ with the threshold $U_j := \left(\frac{1}{1-F_j}\right)^{\leftarrow}$, i.e. they focus on the domain of attraction of an EVD with Fréchet margins. Their dependence coefficient κ coincides with the fragility index as given in Corollary 4.1.4, since we know that the stable tail dependence function as named by L in de Haan and Ferreira [29], Section 6.1.5, is equal to the D-norm together with the equality $\|(1,\dots,1)\|_D = \varepsilon$. This shows that the fragility index based on the approach of Section 3.3.2 and under the assumption $\gamma_j = 0$ for $j \le d$ respectively, leads to the same results as provided by de Haan and Ferreira [29].

Finally we want to present an example of the fragility index. It shows the FI under the popular parametric family of Gumbel copula.

Corollary 4.1.6 *Consider the setting of Corollary 4.1.4, Corollary 4.1.5 respectively and assume $\|\cdot\|_D = \|\cdot\|_\lambda$, i.e. the arbitrary L_λ-norm as defined in (7.1). This corresponds to the assumption that the df F belongs to the domain of attraction of the logistic EVD. Then we get*

$$FI = \begin{cases} 1, & \|\cdot\|_D = \|\cdot\|_1 \\ d^{1-1/\lambda}, & \|\cdot\|_D = \|\cdot\|_\lambda, \quad 1 < \lambda < \infty \\ d, & \|\cdot\|_D = \|\cdot\|_\infty \end{cases} .$$

Hence, the L_λ-norm covers the whole range of asymptotic dependence within a random system covering the case of asymptotic dependence as well asymptotic independence, see Remark 4.1.3. This means the larger λ the higher the amount of tail dependence within the regarded random system.

Remark 4.1.7 *Note that under the setting of Corollary 4.1.4, Corollary 4.1.5 respectively, we get*

$$FI = 1 \;\Leftrightarrow\; \|\cdot\|_D = \|\cdot\|_1 \quad \textit{and} \quad FI = d \;\Leftrightarrow\; \|\cdot\|_D = \|\cdot\|_\infty ,$$

i.e. the L_1-norm corresponds to asymptotic stability of the random system where the maximum-norm corresponds to asymptotic instability.

4.2. Extension of the Fragility Index

Within the preceding section we provided the fragility index as a measure for tail dependence, which can also be used to characterize the asymptotic stability of a random system.

Now we want to extend this approach. By means of the *extended* fragility index we are additionally able to capture the amount of risk as well as the development of the risk-structure in a random system $\{Q_1, \dots, Q_d\}$ if the system already indicates instability, i.e. there have already occurred at least $m \geq 2$ exceedances.

The fragility index depends on whether one considers exceedances above a common or an individual threshold, which may lead to different values for the fragility index independently from the univariate margins and the tail dependence structure of the df F of random system $\{Q_1, \dots, Q_d\}$. We want to start with the former one. The latter approach leads to a fragility index whose *representation*, i.e. its value in case of existence, can be embedded in the approach of common threshold exceedances. The difference between the two approaches is crucial with respect to the existence as well as the representation of the extended fragility index. Hence we provide both approaches separately from each other.

4.2.1. Approach of a common threshold

Consider the event of exceedance $\{Q_j > s\}$ for $j \leq d$ above a common threshold s required to be high enough. Further, denote by $N_s := \sum_{j=1}^{d} \mathbb{1}_{(s,\infty)}(Q_j)$ the number of exceedances as already defined in (3.10) with respect to considerations leading to the asymptotic distribution of exceedance counts.

The extended FI is the asymptotic expected number of exceedances above a high threshold, conditional on the assumption that there are at least $m \geq 1$ exceedances. The following definition is based on the definition of the conditional expectation of a discrete random variable. Note that N_s is a discrete random variable with values in $[m; d]$.

Definition 4.2.1 *Denote by* $N_s := \sum_{j=1}^{d} \mathbb{1}_{(s,\infty)}(Q_j)$ *the number of exceedances among* $\{Q_1, \dots, Q_d\}$*. The limit*

$$FI(m) \; := \; \lim_{s \uparrow \omega^*} E(N_s \,|\, N_s \geq m), \quad m \leq d,$$

is called the extended fragility index whenever it exists for certain $m \leq d$*, where* ω^* *is defined in (3.1). It defines a measure for asymptotic stability of a random system* $\{Q_1, \dots, Q_d\}$ *in the situation of at least* m *exceedances within the system. We call the system* $\{Q_1, \dots, Q_d\}$ *m-stable if* $FI(m) = m$ *and fragile if* $FI(m) > m$*. According to Geluk et al. [22] and Remark 4.1.3 we also refer to the asymptotic behaviour of* $\{Q_1, \dots, Q_d\}$ *by weakly fragile, resp. strongly fragile.*

Especially for $m = 1$ we get the FI as defined in Definition 4.1.1.

The conditional expectation of N_s given the event $\{N_s \geq m\}$ is defined by

$$E(N_s \,|\, N_s \geq m) \; = \; \sum_{k=1}^{d} k \cdot P(N_s = k \,|\, N_s \geq m)$$

$$= \frac{1}{P(N_s \geq m)} \sum_{k=m}^{d} k \cdot P(N_s = k)$$
$$= \frac{\sum_{k=m}^{d} k \cdot P(N_s = k)}{\sum_{k=m}^{d} P(N_s = k)}$$

for $m \leq k \leq d$, given that the denominator is larger than 0. By means of the ACDEC (see Theorem 3.3.7), the extended fragility index is therefore provided by

$$FI(m) := \frac{\sum_{k=m}^{d} k p_k(\boldsymbol{\gamma})}{\sum_{k=m}^{d} p_k(\boldsymbol{\gamma})} \tag{4.3}$$

for those $m \leq d$ for which $\sum_{k=m}^{d} p_k(\boldsymbol{\gamma}) > 0$ holds.

Hence the following considerations aim to give necessary and sufficient conditions under which the extended fragility index is well-defined for certain $m \leq d$, i.e. the denominator satisfies $\sum_{k=m}^{d} p_k(\boldsymbol{\gamma}) > 0$.
The content of the following lemma will be crucial for the main result concerning sufficient and necessary conditions under which the $FI(m)$ exists. Hence we explicitly provide it, although the statements are rather obvious.

Lemma 4.2.2 *Assume $d \in \mathbb{N}$ and an arbitrary subset $M \subseteq \{1, \ldots, d\}$ with $|M| =: m \leq d$. Then it holds*

(i)

$$P\left(\bigcup_{k=m}^{d} \bigcup_{\substack{S \subseteq \{1,\ldots,d\} \\ |S|=k}} \{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\} \right) = \sum_{k=m}^{d} \sum_{\substack{S \subseteq \{1,\ldots,d\} \\ |S|=k}} P(Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}),$$

(ii) $\{Q_i > s, i \in M\} \subseteq \left\{ \bigcup_{k=m}^{d} \bigcup_{\substack{S \subseteq \{1,\ldots,d\} \\ |S|=k}} \{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\} \right\}.$

Proof: For any $S \neq \tilde{S} \subseteq \{1, \ldots, d\}$ the sets $\{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\}$ and $\{Q_i > s, i \in \tilde{S}, Q_j \leq s, j \in \tilde{S}^{\complement}\}$ are disjoint. This shows the assertion in (i).
Now we have to show (ii).The following considerations are based on the repeated application of separation of an event into disjoint ones. In order to give an insight in

the following argumentation, we start with an example: the event $\{Q_i > s, i \in M\}$ can be separated into the disjoint events $\{Q_i > s, i \in M, Q_{j_1} \leq s, j_1 \in M^{\complement}\}$ and $\{Q_i > s, i \in M, Q_{j_1} > s, j_1 \in M^{\complement}\}$. If we want to provide a decomposition concerning to $j_1, j_2, \ldots$, the considerations will be more complicated. Hence, let $A := \{Q_i, i \in M\}$ and $B_K := \{Q_{j_1} \leq s, \ldots, Q_{i_L} \leq s, Q_{j_{L+1}} > s, \ldots, Q_{j_{|K|}} > s\}$ with $0 \leq |K| \leq |M^{\complement}|$, $K = \{j_1, \ldots, j_k\} \subseteq I$ and $0 \leq |L| \leq |K| =: k$. There exist a number of $|K|!$ disjoint events B_K. We want to separate $\{Q_i > s, i \in M\}$ into the disjoint events $\{Q_i > s, i \in M\} \cap B_K$. We obtain with the above considerations

$$\begin{aligned}
\{Q_i > s, i \in M\} &= \bigcup_{K \subseteq M^{\complement}} \{Q_i > s, i \in M, Q_j \leq s, j \in K, Q_l > s, l \in M^{\complement} \backslash K\} \\
&= \bigcup_{K \subseteq M^{\complement}} \{Q_i > s, i \in \underbrace{M \cup (i \backslash K)}_{=:S^{\complement}}, Q_j \leq s, j \in \underbrace{K}_{=S}\} \\
&= \bigcup_{\substack{S \subseteq \{1,\ldots,d\} \\ M \subseteq S,\, |S| \geq |M|}} \{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\} \\
&= \bigcup_{k=m}^{d} \bigcup_{\substack{S \subseteq \{1,\ldots,d\} \\ |S| = k,\, M \subseteq S}} \{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\} \\
&\subseteq \bigcup_{k=m}^{d} \bigcup_{\substack{S \subseteq \{1,\ldots,d\} \\ |S| = k}} \{Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement}\}
\end{aligned}$$

since within the last step the condition $M \subseteq S$ is omitted. □

Remember, we assume $\{Q_1, \ldots, Q_d\} \sim F$ where F is in the domain of attraction of an EVD G with arbitrary margins. We shall see that the dependence structure within the EVD G – represented by the D-norm, which is induced by the angular measure μ (see Corollary 2.1.7) – plays the crucial role with respect to the question, for which $m \leq d$ the extended fragility index is well-defined. In the framework of exceedances above a common threshold, this further depends on the margins of F. The following proposition excludes the cases under which the $FI(m)$ is not defined with respect to the number m. Together with Proposition 4.2.4 we provide a sufficient and necessary condition under which the $FI(m)$ is defined for certain $m \leq d$ depending on the asymptotic behavior of the margins of F, cf. condition C in (3.1) and the dependence structure of the EVD to whose domain of attraction the df F belongs to. The main result concerning the extension of the fragility index is obtained by Theorem 4.2.5.

Proposition 4.2.3 *Suppose* $\{Q_1, \ldots, Q_d\} \sim F$ *with corresponding copula* C_F*, which belongs to the domain of attraction of an EVD G with pertaining D-norm. Furthermore, suppose*

$$\gamma_i := \lim_{s \uparrow \omega^*} \frac{1 - F_i(s)}{1 - F_\kappa(s)} = 0$$

for $i \in I$*, where* $I \subset \{1, \ldots, d\}$ *or* $I = \emptyset$*,* $0 \leq |I| \leq d-1$ *and* $\gamma_j > 0$ *for* $j \in I^{\complement}$*. Let* ω^* *be defined as in condition C, see (3.1). Recall the limits*

$$a_k(\boldsymbol{\gamma}) = \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{1 - F_\kappa(s)}$$

as defined in Corollary 3.3.5 for $0 \leq k \leq d$*. Further set* $m^* := d - |I|$*. Then we get* $\sum_{k=m}^d a_k(\boldsymbol{\gamma}) = 0$ *for* $m > m^*$*.*

Proof: Consider $S \subseteq \{1, \ldots, d\}$ with $|S| \geq m^* + 1 = d - |I| + 1$. Hence we have $|S^{\complement}| \leq |I| - 1$, i.e. $S \cap I \neq \emptyset$, namely there exists an index $\tau_k \in S \cap I$ depending on $k \geq m^*$ and therefore $\lim_{s \uparrow \omega^*} \frac{1 - F_{\tau_k}(s)}{1 - F_\kappa(s)} = 0$. We get for any $k \geq m^* + 1$

$$\begin{aligned}
a_k(\boldsymbol{\gamma}) &= \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{1 - F_\kappa(s)} = \lim_{s \uparrow \omega^*} \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \frac{P(Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement})}{1 - F_\kappa(s)} \\
&= \limsup_{s \uparrow \omega^*} \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \frac{P(Q_i > s, i \in S \backslash \{\tau_k\}, Q_{\tau_k} > s, Q_j \leq s, j \in S^{\complement})}{1 - F_\kappa(s)} \\
&\leq \limsup_{s \uparrow \omega^*} \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \frac{P(Q_{\tau_k} > s)}{1 - F_\kappa(s)} \\
&= \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \underbrace{\limsup_{s \uparrow \omega^*} \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \frac{1 - F_{\tau_k}(s)}{1 - F_\kappa(s)}}_{=0} = 0,
\end{aligned}$$

which implies $a_k(\boldsymbol{\gamma}) = 0$ for $k \geq m^*$, since $a_k \geq 0$ for any $k \geq 0$. □

Now we are ready to give a sufficient and necessary condition for a well-defined extended fragility index.

Proposition 4.2.4 *Assume the same assumptions as in Proposition 4.2.3 and put* $I = \{i \in \{1, \ldots, d\} : \gamma_i = 0\}$*. Consider an arbitrary but fixed* $m \leq m^* = |I^{\complement}|$*. Then we have*

$\sum_{k=m}^{d} a_k(\boldsymbol{\gamma}) > 0$ *if and only if there exists* $K \subseteq I^{\complement}$ *with* $m \leq |K| \leq m^*$ *such that*

$$\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D > 0 . \tag{4.4}$$

Proof: First assume $\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D > 0$ holds for $K \subseteq I^{\complement}$. By means of Lemma A.3, Theorem A.6 and Corollary 2.3.19 we further have

$$\begin{aligned}
\lim_{s \uparrow \omega^*} \frac{P(Q_i > s, i \in K)}{1 - F_\kappa(s)} &= \lim_{s \uparrow \omega^*} \frac{P(F_i(Q_i) > F_i(s), i \in K)}{1 - F_\kappa(s)} \\
&= \lim_{s \uparrow \omega^*} \frac{1 - \sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} P(F_i(Q_i) \leq F_i(s), i \in T)}{1 - F_\kappa(s)} \\
&= \lim_{s \uparrow \omega^*} \frac{1 - \sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} C_T(F_i(s), i \in T)}{1 - F_\kappa(s)} \\
&= \lim_{s \uparrow \omega^*} \frac{\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} (1 - C_T(F_i(s), i \in T))}{1 - F_\kappa(s)} \\
&= \lim_{s \uparrow \omega^*} \frac{\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} (1 - C_T(1 - (1 - F_i(s)), i \in T))}{1 - F_\kappa(s)} \\
&= \lim_{s \uparrow \omega^*} \sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \frac{1}{1 - F_\kappa(s)} \cdot \\
&\qquad \left(1 - C_T\left(1 - (1 - F_\kappa(s)) \frac{1 - F_i(s)}{1 - F_\kappa(s)}, i \in T\right)\right) \\
&= \sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D .
\end{aligned} \tag{4.5}$$

Hence (4.4) is equivalent to $\lim_{s \uparrow \omega^*} c(s)^{-1} P(Q_i > s, i \in K) > 0$. Since this limit exists we get with Lemma 4.2.2

$$\begin{aligned}
&\lim_{s \uparrow \omega^*} \frac{P(Q_i > s, i \in K)}{c(s)} \\
&\qquad \leq \lim_{s \uparrow \omega^*} \sum_{k=|K|}^{d} \sum_{\substack{S \subseteq \{1, \ldots, d\} \\ |S| = k}} \frac{P(Q_i > s, i \in S, Q_j \leq s, j \in S^{\complement})}{c(s)} = a_{|K|}(\boldsymbol{\gamma}) .
\end{aligned}$$

Hence, we have $a_{|K|}(\boldsymbol{\gamma}) > 0$, which implies $\sum_{k=m}^{d} a_k(\boldsymbol{\gamma}) > 0$ for every $m \leq |K|$. This is the sufficient condition.

Now assume $\sum_{k=m}^{d} a_k(\boldsymbol{\gamma}) > 0$ for every $m \le |K|$. Recall that

$$\sum_{k=m}^{d} a_k(\boldsymbol{\gamma}) = \lim_{s\uparrow\omega^*} \sum_{k=m}^{d} \frac{1}{c(s)} \sum_{\substack{S\subseteq\{1,\dots,d\}\\ |S|=k}} P(Q_i > s, i \in S, Q_j \le s, j \in S^{\complement}) .$$

Hence this implies that there exists an index set $K \subseteq \{1,\dots,d\}$ with $m \le |K| \le m^*$, such that

$$\lim_{s\uparrow\omega^*} \frac{P(Q_i > s, i \in K, Q_j \le s, j \in K^{\complement})}{c(s)} > 0$$

holds. Now the inequalities

$$\begin{aligned}
0 &< \lim_{s\uparrow\omega^*} \frac{P(Q_i > s, i \in K, Q_j \le s, j \in K^{\complement})}{c(s)} \\
&= \limsup_{s\uparrow\omega^*} \frac{P(Q_i > s, i \in K, Q_j \le s, j \in K^{\complement})}{c(s)} \\
&\le \limsup_{s\uparrow\omega^*} \frac{P(Q_i > s, i \in K)}{c(s)} = \lim_{s\uparrow\omega^*} \frac{P(Q_i > s, i \in K)}{c(s)} \\
&= \sum_{\emptyset\ne T\subseteq K} (-1)^{|T|-1} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D
\end{aligned}$$

yield the assertion for the necessary condition where the last identity is due to the considerations following (4.5) downward. □

The above considerations allow us to give a necessary and sufficient condition under which the extended fragility index is well-defined. It is the main result within this section.

Theorem 4.2.5 (Extended Fragility Index) *Consider a random system $\{Q_1,\dots,Q_d\}$, which can be represented by the random vector $(Q_1,\dots,Q_d)$. Assume $(Q_1,\dots,Q_d) \sim F$, where F is continuous in its upper tail and its corresponding copula C_F belongs to the domain of attraction of an EVD G with pertaining D-norm. Further put $I := \{i \in \{1,\dots,d\} : \gamma_i = 0\}$. We have $|I| \le d-1$ and $I = \emptyset$ is allowed, too. Furthermore, define $m^* := |I^{\complement}|$. There exists an index set $K \subseteq I^{\complement}$ (hence $|K| \le m^*$) such that*

$$\sum_{\emptyset\ne T\subseteq K} (-1)^{|T|-1} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D > 0 \tag{4.6}$$

holds, if and only if the $FI(m)$ is well-defined for $m \le |K|$. In this case it is given by

$$
\begin{aligned}
FI(m) &= \frac{\sum_{k=m}^{d} k \cdot p_k(\boldsymbol{\gamma})}{\sum_{k=m}^{d} p_k(\boldsymbol{\gamma})} \\
&= \frac{\sum_{k=m}^{d} k \cdot \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \|\sum_{i\in T}\gamma_i \boldsymbol{e}_i\|_D}{\sum_{k=m}^{d} \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \|\sum_{i\in T}\gamma_i \boldsymbol{e}_i\|_D},
\end{aligned}
$$

where the ACDEC $p_k(\boldsymbol{\gamma})$ is defined in Theorem 3.3.7.

Proof: Note that we have $\sum_{k=m}^{d} p_k(\boldsymbol{\gamma}) > 0$ if and only if $\sum_{k=m}^{d} a_k(\boldsymbol{\gamma}) > 0$, where $p_k := a_k(\boldsymbol{\gamma})/a_0(\boldsymbol{\gamma})$ and $a_0(\boldsymbol{\gamma}) > 0$ holds, see Corollary 3.3.4. Then the assertion follows together with Proposition 4.2.4. The representation of the $FI(m)$ follows by the considerations leading to (4.3) and the representation of the ACDEC $p_k(\boldsymbol{\gamma})$, see Theorem 3.3.7. □

Remark 4.2.6 *Theorem 4.2.5 implies that in the situation of a common threshold s the value of FI(m) depends on the univariate margins of the df F of $(Q_1, \dots, Q_d)$, since γ_i is the limit of $\frac{1-F_i(s)}{1-F_\kappa(s)}$ for $s \uparrow \omega^*$. This connection between the value of the FI(m) and the type of margins F_j plays the central and crucial role in Geluk et al. [22] – therein, the results are restricted to the FI.*
Further, we remark that the domain of the $FI(m)$ depends on the univariate margins of F as well as on the dependence structure, captured by the D-norm, of the $|K|$-variate margin G_K of the EVD G to which domain of attraction the df F belongs to.

Of course one expects the extension of the fragility index for $m = 1$ to coincide with the representation of the fragility index as given in Proposition 4.1.2. Indeed this is true, which will be shown next.
The representation of the extended fragility index for $m = 1$ can be derived as follows from the representation of the $FI(m)$ in Theorem 4.2.5. At first, recall that we have $\sum_{k=1}^{d} p_k(\boldsymbol{\gamma}) = 1$ by Theorem 3.3.7. Hence we get

$$FI = \sum_{k=1}^{d} k p_k(\boldsymbol{\gamma})$$

$$(4.7) \qquad = \sum_{k=1}^{d} k \cdot \frac{\sum_{k=1}^{d} k \cdot \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=d-j}} \| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \|_D}{\| \sum_{j \le d} \gamma_j \boldsymbol{e}_j \|_D}.$$

We show that (4.7) equals (4.1).

Recall the definition of the ACDEC as given in Theorem 3.3.7. We get

$$\begin{aligned}
&\sum_{1 \le k \le d} k \sum_{0 \le j \le k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D \\
&= \sum_{0 \le j \le d} \sum_{1 \le k \le d} k(-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D \\
&= \sum_{0 \le j \le d} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D \sum_{1 \le k \le d} k(-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \\
&= \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=1}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D = \sum_{j \le d} \| \gamma_j \boldsymbol{e}_j \|_D = \sum_{j \le d} \gamma_j
\end{aligned}$$

where the last but one equation follows by Lemma A.5.

In Section 3.2 we provided an extensive discussion on equivalent conditions to the condition in (4.6). Recall that the D-norm is induced by an angular measure μ on the d-variate unit sphere S_d and the angular measure corresponds to an exponent measure ν on $[-\infty, 0]^d \backslash \{-\boldsymbol{\infty}\}$. Further note that under the assumption $C_F \in \mathcal{D}(G)$ the tail of the df F can be approximated by that of a GPD, cf. Corollary 2.3.19. With respect to these considerations, Condition (4.6) can be interpreted as follows. There exists a GPD $W(\boldsymbol{x}) = 1 - \left\| \left(\frac{1}{x_1}, \dots, \frac{1}{x_d} \right) \right\|_D$ with $|K|$-variate marginal df W_K having a survival function for which Condition (4.6) is fulfilled with $x_i := 1/\gamma_i$ for $i \in K$. Based on the results of Section 3.2 we want to consider conditions equivalent to (4.6) in order to characterize sufficient and necessary conditions under which the $FI(m)$ is defined for certain $1 \in \{m \dots, d\}$.

First note that Condition (4.6) is equivalent to

$$\lim_{s \uparrow \omega^*} \frac{P(Q_i > s, i \in K)}{c(s)} > 0\,,$$

cf. the first part of the proof of Proposition 4.2.4, i.e. the convergence rate of the survival function of the $|K|$-variate margin F_K of F is not larger than the convergence rate at

which the univariate tail $1 - F_\kappa =: c(s)$ converges to 0.
Note that further equivalences of Condition (4.6) can be taken from Corollary 3.2.6.

However, Theorem 4.2.5 only provides the *existence* of such an index set $|K|$. Indeed, in any case, there exists such an index set, take $K = \{\kappa\}$, which obviously fulfills Condition (4.4). Of course K is not uniquely defined, and there may exist several index sets which fulfill (4.4). Needless to say we are also interested in that index set K, which is the largest one among those fulfilling Condition (4.4). Indicating such a largest $K_{max} \subseteq I^{\complement}$, we define by

$$m_{max} := \max\{1 \le m \le m^* \,:\, FI(m) \text{ is well defined}\}$$

the largest number for which the extended fragility index $FI(m)$ exists.
The largest possible index set is $I^{\complement}$. It can be regarded to be the most desired index set, since this implies that the $FI(m)$ is well-defined on the maximum range $\{1, \dots, m^*\}$. Although the following assertion is a special case of Theorem 4.2.5, i.e. put $K = I^{\complement}$, we provide it as a corollary.

Corollary 4.2.7 *Assume the same assumptions as in Theorem 4.2.5. Put $I := \{i \in \{1, \dots, d\} \,:\, \gamma_i = 0\}$ and $m^* = \left|I^{\complement}\right|$. Then if and only if*

$$\sum_{\emptyset \neq T \subseteq I^{\complement}} (-1)^{|T|-1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D > 0 \tag{4.8}$$

holds, the maximum range, on which the extended fragility index is well-defined, equals $\{1, \dots, m^*\}$, i.e. $FI(m)$ is well-defined for any $m \le m^*$.

We further want to remark that $FI(m)$ is an increasing function in m, which obviously follows by the definition of a conditional expectation.

Remark 4.2.8 *We are faced with the case of $\gamma_j = 1$ for all $j \le d$ under several settings. The first one is the case that the df F has identical margins. The second one occurs if the margins of F are "tail equivalent", i.e. $\lim_{s \uparrow \omega^*} \frac{1-F_j(s)}{1-F_\kappa(s)} = 1$ holds for every $j \le d$, see Condition (3.1).*
We remark that in the case of $\gamma_j = 1, j \le d$, the behavior of the marginal tails is negligible, which implies that the $FI(m)$ is independent of the marginals. In this case,

the domain of the $FI(m)$ only depends on the dependence structure, captured by the D-norm, of the $|K|$-variate margin G_K of the EVD G to which domain of attraction the df F belongs to (see (4.6) in Theorem 4.2.5).

The *representation* of the fragility index in the case of $\gamma_j = 1, j \le d$, is equivalent to the representation of the $FI(m)$ within the approach of exceedances above an individual threshold. Since the definition of the events of exceedances within the approach of a common, individual threshold respectively, is a different one, but crucial for applications on the extended fragility index, we provide the following section separately from the preceding one.

4.2.2. Approach of an individual threshold

The preceding section provided the representation of the extended fragility index and conditions under which it is well-defined on a certain subset of $\{1, \dots, d\}$. It turns out that the $FI(m)$ depends crucially on the tail behavior of the univariate margins of F. This is not a surprising result if one keeps in mind that we consider events of exceedances above a common threshold for *all* components Q_j for $j \le d$. Thereby, the exceedance probability of different components might differ from each other, if the univariate margins of F are not identical. In order to pay attention to this fact we consider exceedances above an individual threshold separately for each component of the system.

Now suppose we want to consider extreme events concerning each component of $\{Q_1, \dots, Q_d\}$, which have equal exceedance probabilities. Therefore, with respect to the application of the extended fragility index, it might be reasonable to consider events of exceedance above an *individual* threshold since we cannot assume that the univariate margins exhibit equal tail properties that would allow us to define events of exceedance with respect to a common threshold.

Now consider the event of exceedance $\{Q_j > F_j^{-1}(1-c)\}$ for $j \le d$ above an individual threshold $s_j := F_j^{-1}(1-c)$ and denote by $N_c := \sum_{j \le d} \mathbb{1}_{(1-c,1]}(F_j(Q_j))$ the number of exceedances above $1-c$, c close to 0, as already defined in (3.14).

In principle, the representation of the extended fragility index, as well as the condition on it to be well-defined within the approach of an individual threshold, is obtained by the

approach of a common threshold considering the case $I = \{i \in \{1, \ldots, d\} : \gamma_i = 1\} = \{1, \ldots, d\}$. But, to state once again, only the *resulting representation* of the $FI(m)$ can be derived from the approach of a common threshold, while the construction of events of exceedance among the system distinguishes between the two approaches. We will see that the definition of events of exceedance crucially affects the domain and the value of the $FI(m)$.
Therefore we dedicate the extended fragility index under the approach of an individual threshold to an extra section and want to provide a shortcut of the results.

The extended fragility index under the approach of an individual threshold is analogue to Definition 4.2.1 defined as

$$FI(m) := \lim_{c \downarrow 0} E(N_c \mid N_c \geq m), \tag{4.9}$$

whenever it exists for certain $m \leq d$.

By means of the ACDEC, see Theorem 3.3.12, the $FI(m)$ is provided by

$$FI(m) = \frac{\sum_{k=m}^{d} k p_k}{\sum_{k=m}^{d} p_k}$$

for those $m \leq d$ for which $\sum_{k=m}^{d} p_k > 0$ holds. Hence we once again provide sufficient and necessary conditions under which the extended fragility is well-defined on a certain subset of $\{1, \ldots, d\}$.

Proposition 4.2.9 *Consider a random system $\{F_1(Q_1), \ldots, F_d(Q_d)\}$, which belongs to the domain of attraction of an EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$ with pertaining D-norm, angular measure μ respectively. Further denote by p_k the ACDEC as given in Theorem 3.3.12 for $k \leq d$. Choose $m \in \{1, \ldots, d\}$. Then we have the following list of equivalences.*

(i) *There exists an index set $K \subseteq \{1, \ldots, d\}$ with $m \leq |K| \leq d$, such that* $\lim_{c \downarrow 0} \frac{P(F_j(Q_j) > 1-c, j \in K)}{c} > 0$

(ii) $\sum_{k=m}^{d} p_k > 0$

(iii) $\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D > 0$

(iv) $\mu\left(\{\boldsymbol{u} \in S_d : \min_{i \in K} u_i = 0\}\right) < |K|$

Proof: Equivalence (i) $\Leftrightarrow$ (ii) is true by the first part of the proof of Proposition 4.2.4 with $\gamma_j = 1, j \leq d$. Equivalence (ii) $\Leftrightarrow$ (iii) follows by Proposition 4.2.4 with $\gamma_j = 1$ for $j \leq d$, i.e. we have $I^{\complement} = \{1, \dots, d\}$ and $\sum_{k=m}^{d} a_k > 0 \Leftrightarrow \sum_{k=m}^{d} p_k > 0$. Equivalence (iii) $\Leftrightarrow$ (iv) follows by Lemma 3.2.3 for $x_j = 1, j \leq d$, with the remark of (3.9), since (3.9) remains true by dividing each side of the equation by any negative number, i.e. $\sum_{\emptyset \neq T \subseteq \{1,\dots,m\}} (-1)^{|T|-1} \left\| \sum_{i \in T} x_i \boldsymbol{e}_i \right\|_D = 0$ for all $\boldsymbol{x} \geq \boldsymbol{0}$. □

A geometrical interpretation of Condition (iv) in Proposition 4.2.9 can be found in Section 3.2, cf. Figure 3.2.

By means of Proposition 4.2.9 we are able to provide the representation of the $FI(m)$ within the approach of an individual threshold. It is equivalent to Theorem 4.2.5 for $\gamma_j = 1, j \leq d$. Due to its importance as the main result within this section – and in case one is only interested in exceedances above an individual threshold – we provide it separately.

Theorem 4.2.10 (Extended Fragility Index) *Consider a random system $\{Q_1, \dots, Q_d\}$, which can be represented by the vector $(Q_1, \dots, Q_d)$. Assume $(Q_1, \dots, Q_d) \sim F$, where F is continuous in its upper tail and its corresponding copula C_F belongs to the domain of attraction of an EVD G with pertaining D-norm. There exists an index set $K \subseteq \{1, \dots, d\}$ with $|K| =: \tilde{m}$, such that*

$$\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D > 0 \tag{4.10}$$

holds, if and only if the $FI(m)$ is well-defined for $m \leq \tilde{m}$. In this case it is given by

$$\begin{aligned} FI(m) &= \frac{\sum_{k=m}^{d} k \cdot p_k}{\sum_{k=m}^{d} p_k} \\ &= \frac{\sum_{k=m}^{d} k \cdot \sum_{j=0}^{k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \| \sum_{i \in T} \boldsymbol{e}_i \|_D}{\sum_{k=m}^{d} \sum_{j=0}^{k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \| \sum_{i \in T} \boldsymbol{e}_i \|_D}, \end{aligned}$$

where p_k is defined in Theorem 3.3.12.

Proof: The assertion follows by Proposition 4.2.9 ($(ii) \Leftrightarrow (iii)$) and Theorem 3.3.12. □

Once again note that Theorem 4.2.10 only states that there exists an index set $K \subseteq \{1,\dots,d\}$, such that the $FI(m)$ is well-defined for any $m \le |K|$. Obviously, any $K \subseteq \{1,\dots,d\}$ that fulfills one of the conditions in Proposition 4.2.9 provides a sufficient condition for the $FI(m)$ being well-defined for $m \le |K|$. Of course those situations are desirable for which the $FI(m)$ is defined on the maximum range $\{1,\dots,d\}$. This is the index set $K := \{1,\dots,d\}$.

Corollary 4.2.11 *Assume the same assumptions as in Theorem 4.2.10. Then the $FI(m)$ is well-defined for any $m \le d$ if and only if*

$$\sum_{\emptyset \ne T \subseteq \{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D > 0 \tag{4.11}$$

holds.

Remember that with Proposition 4.2.9, Condition (4.11) is equivalent to

$$\lim_{c \downarrow 0} \frac{P(F_j(Q_j) > 1-c, j \le d)}{c} = \lim_{c \downarrow 0} \frac{\tilde{C}_F\left(\sum_{j \le d} c\boldsymbol{e}_j\right)}{c} > 0\,,$$

where $\tilde{C}_F$ denotes the survival copula corresponding to the df F (see Definition 2.3.6). Further note that the above limit simplifies to

$$\lim_{c \downarrow 0} \frac{\tilde{C}_F\left(\sum_{j \le d} c\boldsymbol{e}_j\right)}{c} = \lim_{c \downarrow 0} \frac{\tilde{C}_F(c, j \le d)}{c} = \lambda_F(1,\dots,1) > 0,$$

where $\lambda_F(\boldsymbol{x})$ is called the tail dependence function for $\boldsymbol{x} > \boldsymbol{0}$ (see Section 2.4).
Hence, in order to get the maximum domain for the $FI(m)$, i.e. $FI(m)$ exists for any $m \in \{1,\dots,d\}$, we require that the convergence rate of the survival copula $\tilde{C}_F$ is at most that of a linear function.
In the case of $K \ne \{1,\dots,d\}$, we obviously have

$$\text{Condition (4.11) holds} \quad \Leftrightarrow \quad \lim_{c \downarrow 0} \frac{\tilde{C}_{F_K}(c, j \le d)}{c} =: \lambda_{F_K}(1,\dots,1) > 0, \tag{4.12}$$

which is a condition on the survival copula, the tail dependence function at point $(1,\dots,1)$ respectively, corresponding to the $|K|$-variate marginal df of F; see Section 2.4.

With the focus on the application of the fragility index one may be interested in checking Condition (4.11) via statistical tools. Since the D-norm coincides with the stable tail dependence function, one may use estimators for the stable tail dependence function, of which there exist several in the literature; see Section 2.4 for an incomplete overview. With the equivalence in (4.12) one is also able to use estimators of the tail dependence function; see once again Section 2.4. We suggest to use semiparametric or nonparametric estimators, since the usage of a parametric estimator may already inhibit an assumption about the *structure* of the model. For example, de Haan et al. [30] pay attention to this restriction, as they suggest a parametric estimation procedure of the tail dependence function *together* with a goodness-of-fit test, which tests as well the validity of the parametric model assumption.

4.2.3. Trade-off between common and individual threshold

The definition of the (extended) fragility index is based on the *type of extreme* event one considers. We established two alternatives. I.e., the event $\{Q_j > s\}$ for $j \in \{1, \dots, d\}$, and a threshold s high enough that corresponds to a common threshold for the components $Q_j, j \le d$, of a random system $\{Q_1, \dots, Q_d\}$. On the other hand, we may consider the event $\{Q_j > F_j^{-1}(1-c)\}$ for $j \le d$ and c in a right neighborhood of 0, which corresponds to an individual threshold for every component Q_j of a random system $\{Q_1, \dots, Q_d\}$. Hence, the deciding difference between these approaches is the question whether we want to consider exceedances above a common or an individual threshold. This question plays a crucial role within the application of the (extended) fragility index as a measure for tail dependence. Thereby, the goodness of an estimator of the fragility index also depends on the question whether the considered *finite* thresholds really correspond to univariate *tails* and nothing else of the random system.

Further, we want to mention that the approach of an individual threshold can be imbedded in the approach of a common threshold with respect to the *representation* of the (extended) fragility index. More precisely, if $\gamma_j = 1$ holds for every $j \le d$, the representation of the (extended) fragility index is independent of the type of event of exceedance one considers, which means the resulting (extended) fragility index is the same. For example, this occurs if the univariate margins $F_j, j \le d$, are identical or at least tail equivalent. If this assumption is supportable, we do not have to know the univariate margins in order to provide the fragility index. If the univariate margins are different, they have to be known in order to establish the (extended) fragility index, both in the framework of exceedances above a *common* threshold and in the framework of exceedances above an *individual* threshold. With respect to applications on the (extended) fragility index, we do not think that the required knowledge of the univariate margins is a big problem, since there exist frequent approaches for estimating those, take for example the empirical df of F_j or the well-known Peaks-over-threshold approach; see Chapter 5 in Reiss and Thomas [55] for a summary. In the original literature (see Geluk et al. [22]), the fragility index is based on the exceedances above a common threshold. We extended this approach for the mentioned reasoning.

4.2.4. The Fragility Index in case of a GPD

Let us consider once again a random system $\{Q_1, \dots, Q_d\}$ with corresponding joint distribution function F. Within this section we want to strengthen the domain of attraction condition $F \in \mathcal{D}(G)$ to the assumption that the df F follows a multivariate GPD in its upper tail, i.e. we assume that there exists a vector $\boldsymbol{x}_0 < \mathbf{0}$ close enough to $\mathbf{0}$ and a norm $\|\cdot\|_D$, as defined in Section 2.1, such that

$$F(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D \tag{4.13}$$

for $\boldsymbol{x} \geq \boldsymbol{x}_0$. That means the tail of $\{Q_1, \dots, Q_d\}$ follows a generalized Pareto distribution with uniform margins on $[-1, 0]^d$, which we call W (see Section 2.3). As we did in the preceding sections, we want to calculate the extended fragility index, but now under assumption (4.13), which we call the GPD-assumption. Therefore consider the event of exceedance $\{Q_j > W_j^{-1}(1-c)\}$ for $j \leq d$ above an individual threshold $s_j := W_j^{-1}(1-c)$ and denote by $N_c := \sum_{j \leq d} \mathbb{1}_{(1-c,1]}(W_j(Q_j))$ the number of exceedances above $1 - c$. This is analogue to Section 4.2.2. Note that the above definition of an exceedance event implies that we consider the random copula system $\{W_1(Q_1), \dots, W_d(Q_d)\}$ following a GPD-copula on $[-1, 0]^d$; see Definition 2.3.8.

We will see that the deciding difference to the domain of attraction assumption is the fact that under the GPD-assumption, the (extended) fragility index attains its "limit" for a finite threshold.

The following results follow easily from the considerations in Section 3.3.

Lemma 4.2.12 *Denote by* $\{Q_1, \dots, Q_d\}$ *a random system that follows a GPD* W *in its upper tail. Hence, suppose that there exists a vector* $\boldsymbol{u}_0 \in [0,1]^d$ *close enough to* $\mathbf{1}$ *and a norm* $\|\cdot\|_D$*, as defined in Section 2.1, such that*

$$(W_1(Q_1), \dots, W_d(Q_d)) \sim C_W(\boldsymbol{u}) = 1 - \|\boldsymbol{u} - \mathbf{1}\|_D$$

for $\boldsymbol{u} \geq \boldsymbol{u}_0$*. Then*

$$P(W_j(Q_j) \leq 1 - c,\ j \in K) = 1 - c \cdot \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D$$

holds for any $K \subseteq \{1, \dots, d\}$ *and all* $c \leq 1 - \max_{j \leq d}(u_{0,j})$.

Proof: The assertion follows by the definition of the GPD-copula, see Definition 2.3.8, i.e. there exists $\boldsymbol{u}_0 \in [0,1]^d$ close to $\mathbf{1}$, such that for all $(c, \dots, c) \leq \mathbf{1} - \boldsymbol{u}_0$ we have

$$\begin{aligned} P(W_j(Q_j) \leq 1 - c, j \in K) =& 1 - \left\| \sum_{j \in K} (1-c)\boldsymbol{e}_j + \sum_{j \notin K} \boldsymbol{e}_j - \sum_{j \leq d} \boldsymbol{e}_j \right\|_D \\ =& 1 - c \cdot \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D . \end{aligned}$$

□

Now we provide the distribution of the number of exceedances under the GPD-assumption.

Lemma 4.2.13 *Assume the same assumptions as in Lemma 4.2.12. Let* $N_c = \sum_{j \leq d} \mathbb{1}_{(1-c,1]}(W_j(Q_j))$ *be the number of exceedances above* $1 - c$*. Then we have*

(i) $P(N_c = 0) = 1 - c \cdot \left\| \sum_{1 \leq j \leq d} \boldsymbol{e}_j \right\|_D =: 1 - c \cdot a_0$

(ii) and for $k \leq d - 1$ *we have*

$$\begin{aligned} P(N_c = k) &= c \cdot \sum_{0 \leq j \leq k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D \\ &=: c \cdot a_k \end{aligned}$$

(iii)

$$\begin{aligned} P(N_c = d) &= c \cdot \sum_{j=1}^{d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1\dots,d\}\\ |T|=j}} \left\| \sum_{i\in T} \boldsymbol{e}_i \right\|_D \\ &=: c \cdot a_d \end{aligned}$$

Proof: The proof is analogue to the proof of Lemma 3.3.3 with obvious respect to the GPD assumption instead of the use of Corollary 2.3.19 as referenced in Lemma 3.3.3. □

In order to provide the extended fragility index within the framework of the GPD-assumption, we need the conditional distribution of exceedance counts. It is based on the assumptions and the results of preceding considerations.

Corollary 4.2.14 (ACDEC under the GPD-assumption) *Set $p_k := a_k/a_0$ for $1 \le k \le d$. Then p_k defines a probability distribution on $\{1, \dots, d\}$ and is the conditional distribution function of exceedance counts. Further assume the same assumptions as in Lemma 4.2.12. Then we have*

$$P(N_c = k \mid N_c > 0) = \frac{P(N_c = k)}{P(N_c > 0)} = \\ \frac{\sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=d-j}} \| \sum_{i\in T} \boldsymbol{e}_i \|_D}{\| \sum_{j=1}^{d} \boldsymbol{e}_j \|_D} =: p_k$$

for $1 \le k \le d-1$ and

$$P(N_c = d \mid N_c > 0) = \frac{P(N_c = k)}{P(N_c > 0)} = \\ \frac{\sum_{1\le j\le d} (-1)^{j+1} \sum_{\substack{T\subseteq\{1,\dots,d\},\\ |T|=j}} \| \sum_{i\in T} \boldsymbol{e}_i \|_D}{\| \sum_{j=1}^{d} \boldsymbol{e}_j \|_D} =: p_d \, .$$

Proof: The assertion follows immediately from Lemma 4.2.12 and 4.2.13. □

Note that the fact that the ACDEC under the GPD-assumption attains its limit at a finite threshold is related to the multivariate POT-stability of a GPD, see Falk and Guillou [18]. They show that the excess distribution of an rv $\boldsymbol{X}$ is invariant under the chosen thresholds, if $\boldsymbol{X}$ follows a multivariate GPD.

Remark 4.2.15 *Denote by N_c the number of exceedances among $\{Q_1, \dots, Q_d\}$ as defined in Lemma 4.2.13. In the case of $(Q_1, \dots, Q_d)$ following a GPD in the upper tail, the conditional expectation*

$$E(N_c \mid N_c \geq m), \quad m \leq d$$

for c close but finite to 0 serves as a measure for system stability and is called the extended fragility index ($FI(m)$) under the GPD-assumption whenever it exists for certain $m \leq d$. This is analogue to the definition of the extended fragility index under the approach of an individual threshold; see the definition in (4.9), with the crucial difference that, under the GPD-assumption, the $FI(m)$ attains its limit at a finite threshold.

Now we are ready to represent the (extended) fragility index in case of the GPD-assumption.

In analogy to the maximum of attraction condition we have to ensure that the extended fragility index is well-defined. Remember that under the domain of attraction assumption $(Q_1, \dots, Q_d) \sim F \in \mathcal{D}(G)$ with $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$ we set conditions on the survival function of $W(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D$. Now we assume $F = W$ in the upper tail.

Theorem 4.2.16 (Extended Fragility Index) *Consider a random system $\{Q_1, \dots, Q_d\}$ which can be represented by the vector $(Q_1, \dots, Q_d)$. Assume that the df of $(Q_1, \dots, Q_d)$ coincides with a GPD W in the upper tail, i.e. there exists $\boldsymbol{x}_0 < \boldsymbol{0}$ and a norm $\|\cdot\|_D$, such that $W(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_D$ holds for $\boldsymbol{x} \geq \boldsymbol{x}_0$. There exists an index set $K \subseteq \{1, \dots, d\}$ with $|K| =: \tilde{m}$, such that*

$$\sum_{\emptyset \neq T \subseteq K} (-1)^{|T|-1} \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D > 0 \tag{4.14}$$

holds, if and only if the $FI(m)$ is well-defined for $m \leq \tilde{m}$. In this case the representation of the $FI(m)$ coincides with the representation of the $FI(m)$ under the domain of attraction assumption in case of an individual threshold (see Theorem 4.2.10).

Proof: Since a GPD belongs to the maximum domain of attraction of an EVD, the assertion follows by Theorem 4.2.10. □

Note that condition (4.14) is equivalent to those in Proposition 4.2.9.

The considerations within this section can be used as follows. Under the domain of attraction condition, the stability of a system can be *theoretically* captured by the limit of the expected number of collapses, given several have already occurred. Within applications on the extended fragility index we will use a finite approximation to the (extended) fragility Index as defined hitherto. The problem at hand is obvious: How to choose a threshold *high enough*, such that the obtained tail is indeed a tail, which entails that the approximation of the fragility index is already quite good?
Under the GPD-assumption, we do not have to care about the goodness of approximation, since we can find a threshold $\boldsymbol{u}_0 < \mathbf{0}$, such that the fragility index exists as a finite number for all $c \leq 1 - \max_{j \leq d}(u_{0,j})$. As one may guess, we have to pay for that benefit. The GPD-assumption is much stronger than the domain of attraction condition.

4.2.5. Examples of the Fragility Index

To get an impression of the (extended) fragility index *at work* we want to establish a few examples. The representation of the $FI(m)$ is mainly based on the D-norm. In order to compute the $FI(m)$ for a certain setting $(Q_1, \dots, Q_d) \sim F \in \mathcal{D}(G)$ we need to know the D-norm corresponding to the EVD G. If we know the df F we are *in principle* aware of the domain of attraction to which F belongs to (in case of existence) and therefore we are able to determine the D-norm even if this might be difficult (cf. introduction of Section 3.3.3.2).

We want to provide examples for the $FI(m)$ under several settings. Some of the represented examples have already been prepared within the examples of the ACDEC (see Section 3.3.3).
Let us start with a very simple bivariate example, which shows in detail how to compute a D-norm.

Example 4.2.17 (Extension of Example 5.16 in Resnick [56]) *Suppose* (Q_1, Q_2) *follows the bivariate distribution function*

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \frac{1}{\exp(x) + \exp(y) - 1}, \; x, y \geq 0\,.$$

Hence, we have margins $F_1(x) := \lim_{y\to\infty} F(x, y) = 1 - \exp(-x)$ *and* $F_2(y) := \lim_{x\to\infty} F(x, y) = 1 - \exp(-y)$*, i.e. exponential margins.* F *is in the domain of attraction of a* *bivariate EVD*

$$G(x,y) = \exp\left(-\left(\exp(-x)+\exp(-y)-\frac{1}{\exp(x)+\exp(y)}\right)\right)$$

with margins $G_1(x) = G_2(x) = \exp(-\exp(-x))$, *i.e. Gumbel margins. (See Example 5.16 in Resnick [56].)*
Since we have identical margins, we may apply Theorem 4.2.10 or Theorem 4.2.5 with $\gamma_j = 1, j \le d$, *i.e. we have*

$$FI(1) = \frac{\|(0,1)\|_D + \|(1,0)\|_D}{\|(1,1)\|_D} = \frac{2}{\|(1,1)\|_D}\,. \tag{4.15}$$

Transformation of G *to an EVD with standard Frèchet margins (see Remark 2.1.12) leads to*

$$\begin{aligned} &G_{(0,0)}\left(-\log\left(\frac{1}{x}\right), -\log\left(\frac{1}{x}\right)\right)\\ &= \exp\left(-\left(\frac{1}{x}+\frac{1}{y}-\left(\exp\left(-\log\left(\frac{1}{x}\right)\right)+\exp\left(-\log\left(\frac{1}{x}\right)\right)\right)^{-1}\right)\right)\\ &= \exp\left(-\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{x+y}\right)\right) = G_{(1,1)}(x,y)\,,\ x,y>0. \end{aligned}$$

Hence we get

$$\|(x,y)\|_D = \frac{1}{|x|}+\frac{1}{|y|}-\frac{1}{|x|+|y|}.$$

This implies $\|(1,1)\|_D = \frac{3}{2}$ *and with (4.15) we have* $FI(1) = \frac{4}{3}$. *It follows that in this situation, the system* $\{Q_1, Q_2\}$ *is strongly fragile. Note that* $FI(2) = 2$.

We want to continue with an example on the popular logistic EVD. We already know from Corollary 3.3.15 and Example 3.3.23 that the arbitrary L_λ-norm is the limit of $1 - C_\lambda(1+t\boldsymbol{x})/t$ where C_λ denotes the Gumbel copula. Hence, if the df F belongs to the domain of attraction of a logistic EVD, the corresponding D-norm is the L_λ-norm.

Example 4.2.18 (Fragility Index under the logistic EVD) *Assume that* $(Q_1,\dots,Q_d)$ *follows a df* F *which is in the domain of attraction of a logistic EVD, i.e,* $F \in \mathcal{D}(G)$ *with* $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_\lambda)$. *Further consider the approach of exceedances above an individual threshold. Under this setting, Corollary 3.3.15 provides the ACDEC and the extended fragility index is provided as follows.*

(i) For $1 < \lambda < \infty$ *we get*

$$FI(m) = \frac{\sum_{m \le k \le d} k(-1)^{k+1} \begin{pmatrix} d \\ k \end{pmatrix} \sum_{0 \le j \le k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \left(1 - \frac{j}{d}\right)^{1/\lambda}}{\sum_{m \le k \le d} (-1)^{k+1} \begin{pmatrix} d \\ k \end{pmatrix} \sum_{0 \le j \le k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \left(1 - \frac{j}{d}\right)^{1/\lambda}}$$

for $m \le d$*. In the special case of* $m = 1$ *a less complex representation of the FI under the* L_λ*-norm is provided by Corollary 4.1.6. It coincides with the representation given here, see therefore the general considerations which follow after Theorem 4.2.5.*

(ii) For the maximum-norm we have $p_k = 0$ *for* $1 \le k \le d-1$ *and* $p_d = 1$ *. Hence, the* $FI(m)$ *is defined for every* $m \le d$ *and we get* $FI(m) = \frac{dp_d}{p_d} = d$ *for every* $m \le d$*, which is the case of total dependence in the tail of* $(Q_1, \ldots, Q_d)$*.*

(iii) For the L_1*-norm we have* $p_1 = 1$ *and* $p_k = 0$ *for* $2 \le k \le d$*. Hence the* $FI(m)$ *is only defined for* $m = 1$ *and we get* $FI(1) = FI = \frac{1p_1}{p_1} = 1$*, which is the case of total independence. Note that the* $FI(m)$ *is not defined for* $m > 1$*, since all mass of the distribution* p_k *is concentrated on* $k = 1$ *(see Corollary 3.3.15).*

Note that we obtain the same results for the $FI(m)$*, if we consider the approach of a common threshold together with* $\gamma_j = 1$ *for every* $j \le d$ *(see Remark 4.2.8).*

Hence, this example shows that we have the maximum of dependence in the tail of $(Q_1, \ldots, Q_d)$, e.g. total dependency, if we choose the L_∞-norm. But the maximum norm represents only a sufficient condition for asymptotic instability of the system (see the next example).
Figure 4.1 visualizes Example 4.2.18. It shows the extended fragility index under certain choices for the λ. We observe that the $FI(m)$ is an increasing function in λ. Furthermore, the figure indicates that under the L_λ-norm the $FI(m)$ is a concave function in m.

The following example is also crucial with respect to the *need* for the extended fragility index. It is the answer to the question "Why do we need an *extension* of the fragility index, since the fragility index already describes the stability of a random system?"

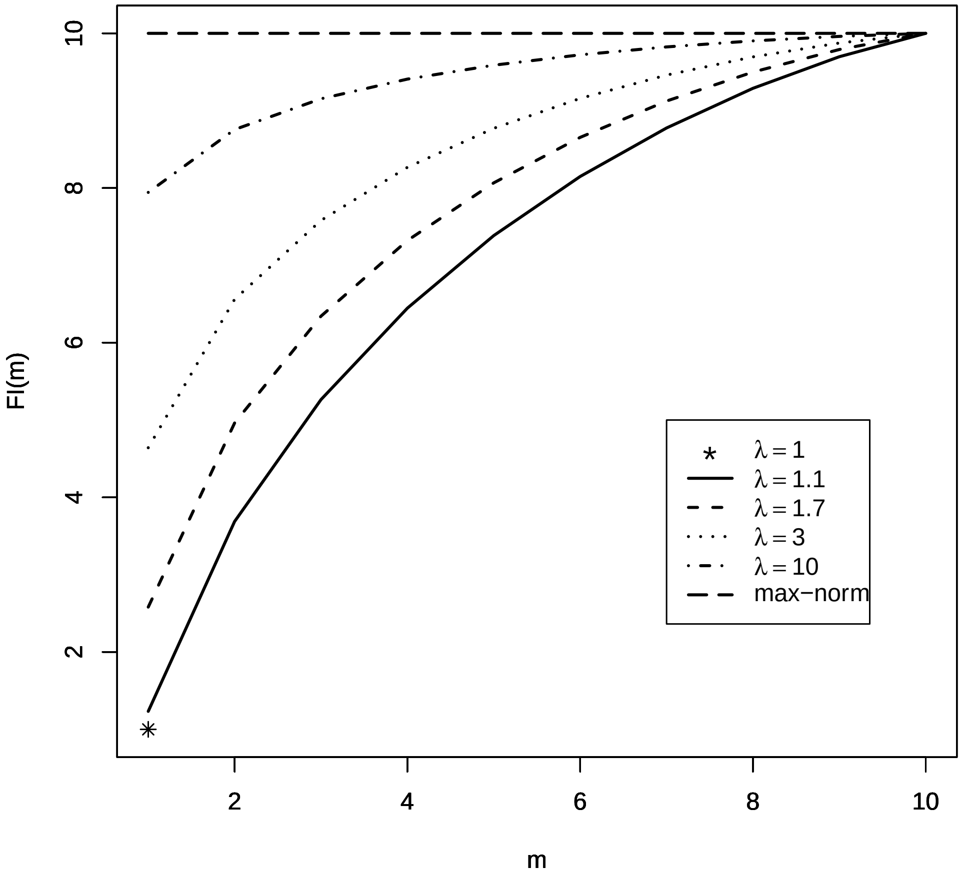


Figure 4.1.: The $FI(m)$ for a random system of dimension $d = 10$ with respect to certain choices of the L_λ-norm, especially including $\lambda = 1$ and the maximum-norm. The figure visualizes the results of Example 4.2.18.

Example 4.2.19 (Fragility Index under the Marshall-Olkin norm) *Assume that the copula corresponding to $(Q_1, \dots, Q_d)$ belongs to the domain of attraction of the Marshall-Olkin copula, i.e. the copula corresponding to $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_{MO})$, where we denote by*

$$\|\boldsymbol{x}\|_{MO} := \vartheta \|\boldsymbol{x}\|_1 + (1 - \vartheta) \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^d,\ \vartheta \in [0, 1]$$

the Marshall-Olkin norm. By Example 3.3.16 we have

$$p_1 = \frac{\vartheta d}{\vartheta d + 1 - \vartheta}, \quad p_d = \frac{1-\vartheta}{\vartheta d + 1 - \vartheta}, \quad p_k = 0, \qquad 2 \le k \le d-1,$$

and this yields

$$FI(1) = FI = \frac{d}{\vartheta(d-1)+1} \quad \text{and} \quad FI(m) = d \quad \text{for every} \quad 2 \le m \le d$$

for $\vartheta \in [0,1)$ *and*

$$FI(1) = 1$$

for $\vartheta = 1$. *In the latter case, the Marshall-Olkin norm coincides with the* L_1*-norm, i.e. we have asymptotic independence and the* $FI(m)$ *is not defined for* $m \ge 2$*, see Example 4.2.18. The former case, i.e.* $\vartheta \in [0,1)$*, leads to full asymptotic dependence when looking at the extended* $FI(m)$*, although the* $FI(1)$ *does not necessarily show full asymptotic dependence.*

Recall that for the fragility index we have $FI = d$ if and only if $\|\cdot\|_D = \|\cdot\|_\infty$, see Corollary 4.1.6. This is not true for $m \ge 2$ (see the previous Example 4.2.19). That means, the choice of the maximum norm is only a sufficient condition for full asymptotic instability of the system with respect to the *extended* fragility index, i.e. in situations where we have already observed at least more than one exceedance among the random system $\{Q_1, \dots, Q_d\}$.

Further note that the Marshall-Olkin norm represents an important example with respect to the *extension* of the fragility index, since the fragility index (i.e. $FI(1)$) may take any value in $[1, d]$, depending on the value of $\vartheta \in [0,1]$, whereas the extended fragility index $FI(m)$ takes the value d for every $m \ge 2$ in case of $\vartheta < 1$. This can be seen as follows. Suppose that $\vartheta \in [0,1)$, then we get $FI(m) = d$ independently of the number m and the value of ϑ. That means, if we observe more than one exceedance among the random system, i.e. $FI(1) > 1$, then we have to expect the total collapse of the whole system, hence the asymptotic expected number of exceedances "jumps" from $FI(1) = a > 1$ to $FI(m) = d$ for $m = 1$ to $m \ge 2$ even if a may be very close to 1 (this is ϑ very close to 1). This highlights the necessity of the *extended* fragility index, since one may get fooled by the assumptions that there is no strong tail dependence within a random system, if the value of the fragility index $FI(1)$ is close to 1 and one does not look at the extended fragility index!

Example 4.2.20 (Archimedean copulas with asymptotic independence) *Suppose that the corresponding copula to* $(Q_1, \dots, Q_d)$ *is an Archimedean copula defined by* $C_\varphi(\boldsymbol{u}) = \varphi^{-1}(\varphi(u_1 + \dots + \varphi(u_m))$ *with the generator* φ *as defined in Definition 3.3.20. Further suppose that the generator* φ *is differentiable to the left in* $x = 1$ *and its left derivation in* $x = 1$ *is not equal to* 0*. Then we get with Proposition 3.3.22 that the corresponding D-norm to* C_φ *is the* L_1*-norm. Hence, the extended fragility index is only defined for* $m = 1$ *and we get* $FI = 1$*, hence, the random system inhibits asymptotic stability. For example, this is true for the Frank and the Clayton copulas and the Ali-Mikail-Haq family, which are popular copulas within finance and insurance (see Example 3.3.23).*

Example 4.2.21 (Weighted Pareto) *Assume the setting as in Example 3.3.19, i.e. assume* $X_1, \dots, X_m$ *to be independent and identically Pareto distributed rv with parameter* $\alpha > 0$*. Put*

$$Q_i := \sum_{j=1}^{m} \lambda_{ij} X_j, \qquad 1 \le i \le d,$$

where the weights λ_{ij} *are nonnegative and satisfy* $\sum_{j=1}^{m} \lambda_{ij}^{\alpha} = 1$, $1 \le i \le d$*. Then we obtain the D-norm*

$$\|\boldsymbol{x}\|_D := \sum_{j=1}^{m} \left(\max_{i \le d} \left(\lambda_{ij}^{\alpha} |x_i| \right) \right), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

see Example 3.3.19.

Let us assume that we consider events of exceedance above a common threshold. From Lemma A 3.26 in [13] we obtain that the df F_i *of* X_i *satisfies*

$$1 - F_i(s) \sim s^{-\alpha} \sum_{j=1}^{m} \lambda_{ij}^{\alpha} = s^{-\alpha}, \qquad 1 \le i \le d,$$

as $s \to \infty$ *and, thus,*

$$\gamma_i = \lim_{s \to \infty} \frac{1 - F_i(s)}{1 - F_\kappa(s)} = 1, \qquad 1 \le i \le d,$$

where $\kappa \in \{1, \dots, d\}$ *can be chosen arbitrarily. As a consequence, using Proposition 4.1.2, we obtain for the fragility index*

$$FI = \frac{\sum_{i=1}^{d} \gamma_i}{\left\| \sum_{i=1}^{d} \gamma_i \boldsymbol{e}_i \right\|_D} = \frac{d}{\sum_{j=1}^{m} \max_{i \le d} \lambda_{ij}^{\alpha}}.$$

The following example shows that the assertion of Remark 4.1.7 is only true under $\gamma_j = 1, j \leq d$, the approach of an individual threshold respectively. If one considers events of exceedances above a common threshold, i.i.d or tail equivalent margins $F_j, j \leq d$, the convergence behavior of the marginal df may influence seriously the value of the fragility index.

Example 4.2.22 *Assume the bivariate case $(Q_1, Q_2) \sim F$ with corresponding copula belonging to the domain of attraction of an EVD. Further assume that the margins F_1 and F_2 of F belong to the family of subexponential distributions, e.g. assume $F_1(x) = 1 - \exp(-\lambda x)$, $\lambda > 0$ (exponential) and $F_2(x) = 1 - \left(\frac{\vartheta}{\vartheta + x}\right)^\alpha$, ϑ, $\alpha > 0$, $x \geq 0$ (Pareto). The margins therefore belong to the domain of attraction of the Gumbel and the Frèchet distribution respectively (see, e.g., Section 3.4 in Embrechts et al [13]). We want to consider the approach of a common threshold. Therefore, we have to check Condition C. Since*

$$\lim_{x\to\infty} \frac{\vartheta^\alpha(\vartheta + x)^{-\alpha}}{\exp(-\lambda x)} = \lim_{x\to\infty} \vartheta^\alpha \frac{\exp(\lambda x)}{(\vartheta + x)^\alpha} = \infty\,,$$

we set $\omega^* := \omega(F_2) = \infty$. Then we get

$$\gamma_1 = \lim_{x\to\infty} \frac{\exp(-\lambda x)}{\vartheta^\alpha(\vartheta + x)^{-\alpha}} = \lim_{x\to\infty} \frac{(\vartheta + x)^\alpha}{\vartheta^\alpha \exp(\lambda x)} = 0$$

for $\alpha > 0$ and $\gamma_2 = 1$.

Condition C is fulfilled. The fragility index takes the values (see Proposition 4.1.2)

$$FI(1) = \frac{\gamma_1 + \gamma_2}{\|(\gamma_1, \gamma_2)\|_D} = \frac{0 + 1}{\|\boldsymbol{e}_2\|_D} = 1$$

and $FI(2) = 2$ for an arbitrary D-norm. This example provides a scenario which leads to asymptotic stability of the system $\{Q_1, Q_2\}$ independently of the chosen D-norm!
As a generalization to an arbitrary dimension $d > 2$, we get $FI(1) = 1$ and the $FI(m)$ is not defined for $m \geq 2$ if the convergence behavior of a single margin F_κ of F is predominated by all the others, i.e. $\gamma_i = 0, i \neq \kappa$. This follows by Corollary 3.3.8, which states that if $I^\complement = \{\kappa\}$, we get $p_1 = 1$ and $p_k = 0$ for $k \geq 2$.

Further note that if we consider exceedances above an individual threshold within the same provided setting, we get $FI(1) = \frac{2}{\|(1,1)\|_D}$. Hence, in this case, we get $FI(1) = 1$ if and only if $\|(1,1)\|_D = 2$. By Remark 4.1.7 this is equivalent to $\|\cdot\|_D = \|\cdot\|_1$, hence the approach of an individual threshold within the provided setting above only leads to asymptotic stability of the system in case of the L_1-norm.

4.3. The Fragility Index as an extension of the extremal coefficient

This section emphasizes on the fragility index as an extension of the extremal coefficient. Therefore, the fragility index can be interpreted as a measure for tail dependence, which is new to the literature and overcomes the deficiency of the extremal coefficient and the (stable) tail dependence function. As already discussed in Section 2.4 the extremal coefficient ε can be regarded to be a measure for tail dependence, since it is that number which satisfies $G(x, \dots, x) = G_1^{\varepsilon}(x)$ where G is an EVD with identical margins G_1. Hence, it is a measure for tail dependence between the margins of a multivariate distribution F that belongs to the domain of attraction of an EVD G with identical margins, cf. Definition 2.2.4. It is obvious that this is a restriction, which one desires to overcome.

It is well-known that the stable tail dependence function determines the dependence structure within an EVD. Furthermore, in case of its existence, the stable tail dependence function arises as the crucial limit under the domain of attraction condition on a copula. Furthermore, it coincides with the D-norm. However, the stable tail dependence function is not bounded and, therefore, fails as a coefficient of tail dependence, since its value cannot be interpreted with the focus of identifying the amount of dependence nor the special cases of full independence and dependence.

The (extended) fragility index is both bounded and not restricted to identical margins. It serves as a measure for tail dependence within a random system $(Q_1, \dots, Q_d)$, which follows an arbitrary multivariate df F. We only require that the corresponding copula of F belongs to the domain of attraction of an EVD G. If the limit exists, the extended fragility index is defined as the asymptotic expected number of exceedances among $(Q_1, \dots, Q_d)$ given there have already occurred $m \leq d$ exceedances, i.e. $FI(m) := \lim_{s \nearrow} E(N_s \mid N_s \geq m)$ for $m \leq d$ (cf. Definition 4.2.1). Hence its range is $FI(m) \in [m; d]$. Therefore we apply the following transformation.

Definition 4.3.1 *Consider the random system $\{Q_1, \dots, Q_d\}$ of dimension $d \in \mathbb{N}$. The transformation*

$$TFI(m) := \frac{FI(m) - m}{d - m} \in [0, 1], \quad 1 \leq m \leq d - 1,$$

shifts the $FI(m)$ onto the interval $[0,1]$ and is denoted by the transformed fragility index together with $TFI(d) := 1$ (since $FI(d) = d$ by construction), given $FI(m)$ exists for certain $m \in \{1, \dots, d\}$. We call the system $\{Q_1, \dots, Q_d\}$ weakly fragile, resp. m-stable (cf. Definition 4.2.1), if $TFI(m) = 0$ for every $m \leq d$. Otherwise (that means $TFI(m) > m$ for at least one $m \leq d-1$), we call the system strongly fragile.

The above transformation provides a measure for tail dependence that is independent of the dimension of the random system. Further, in connection to dependence measures like the coefficient of correlation, it can be easily interpreted, since the closer $TFI(m)$ is to 1, the higher the amount of dependence within the random system, and the closer $TFI(m)$ is to 0, the less votes against total asymptotic independence within the system. As already mentioned in Section 2.2, the extremal coefficient corresponding to an EVD G with identical margins is given by $\varepsilon = \|(1, \dots, 1)\|_D$ (cf. Definition 2.2.4). Further denote by G_K a K-dimensional margin of G, hence its corresponding extremal coefficient (cf. Definition 3.1.2 for the meaning of "corresponding" within this framework) is defined by

$$\varepsilon_K := \left\| \sum_{j \in K} \boldsymbol{e}_j \right\|_D = \left\| \sum_{j \leq m} \tilde{\boldsymbol{e}}_j \right\|_{\tilde{D}} = \|(1, \dots, 1)\|_{\tilde{D}}, \tag{4.16}$$

where $\|\cdot\|_{\tilde{D}}$ corresponds to G_K of dimension $|K| = m$ (cf. Section 2.2 and Lemma 3.1.1). The existence of $\|\cdot\|_{\tilde{D}}$ is clear, since the margins of a max-stable df are again max-stable, which enables the definition in (4.16).
Recall the representation of the extended fragility index within the approach of an individual threshold (cf. Theorem 4.2.10). Define $\varepsilon_T := \left\| \sum_{i \in T} \boldsymbol{e}_i \right\|_D$ for $T \subseteq \{1, \dots, d\}$. Then the $FI(m)$ is given by

$$FI(m) = \frac{\sum_{k=m}^{d} k \cdot \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \varepsilon_T}{\sum_{k=m}^{d} \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = d-j}} \varepsilon_T}. \tag{4.17}$$

Consider an arbitrary but fixed $m \leq d-1$. The extended fragility index $FI(m)$ enables us to calculate the tail dependence among the random "subsystem" $\{Q_{i_1}, \dots, Q_{i_t}\}$ consisting of those components of the full system $\{Q_1, \dots, Q_d\}$, which still fall below

their threshold. Thereby we account for the past exceedances among the system and provide a tail dependence measure that captures the amount of risk that is still inherent in the subsystem based on the tail information of the full system. This statement is based on the fact that the $FI(m)$ consists of the extremal coefficients corresponding to those margins G_T whose dimension is at least $d - m$ (this coincides with the number of components still falling below the threshold), cf. the range of the set T in (4.17).
This shows that the extended fragility index is a measure for tail dependence, which factors those events having already occurred into the calculation of the $FI(m)$. This enables dynamic and variable conclusions about the tail *behavior*. Thereby past events serve as a type of predictor for the *risk* of further exceedances, which may lead to a total breakdown of the system as the worst scenario. We want to advise the reader that a random system, which shows to be low tail dependence based on at least one exceedance, may exhibit high risk when there have occurred several exceedances nonetheless, cf. Example 4.2.19 and the following discussion there.

4.4. Sojourn times

Hitherto we considered exceedances above a high threshold among the components of a stochastic system $\{Q_1, \dots, Q_d\}$. Within Section 3.3 we provided the asymptotic conditional distribution of exceedance counts, which is the basic tool of the expected number of exceedances among $\{Q_1, \dots, Q_d\}$, given there is at least one exceedance. Now, we investigate the aspect of tail dependence from another point of view. Consider a stationary process $(X_d)_{d \in \mathbb{N}}$ on the natural numbers, hence we denote it a stationary sequence. A finite sequence taken from $(X_d)_{d \in \mathbb{N}}$ is a random vector and therefore its fragility index can be derived as done in Section 4.2.1 and 4.2.2. We provide the link between the fragility index and the extremal index in Section 4.4.1.

Further, we compute the expected excursion time as a measure for asymptotic dependence within a stochastic process in Section 4.4.2. It can be regarded to be a measure for tail dependence as well, with the focus on sequential dependence, since we consider the amount of tail dependence by the *sojourn time* of an exceedance. Remember that in the framework of the fragility index, we captured the amount of risk by the *number* of exceedances among a random system.

4.4.1. Link between the Fragility Index and the Extremal Index

In what follows we show that the reciprocal of the fragility index, as a function of the dimension d, converges to the extremal index of a strictly stationary sequence. Hence, within this section we use the notation $FI^{(d)}$. Therefore consider a strictly stationary sequence $(X_d)_{d\in\mathbb{N}}$ of rv, which means $(X_{d+k})_{d\in\mathbb{N}}$ has the same distribution as $(X_d)_{d\in\mathbb{N}}$ for any integer $k > 0$.

Definition 4.4.1 (Extremal index) *Let $(X_d)_{d\in\mathbb{N}}$ be a strictly stationary sequence of rv and θ a number in $[0, 1]$. Assume that for every $\tau > 0$ there exists a sequence $(u_d)_{d\in\mathbb{N}}$ of numbers such that*

$$\lim_{d\to\infty} d(1 - F(u_d)) = \tau, \tag{4.18}$$

where F is the df of X_1, and

$$\lim_{d\to\infty} P\left(\max_{1\le k\le d} X_k \le u_d\right) = \exp(-\theta\tau). \tag{4.19}$$

Then θ is called the extremal index of the sequence $(X_d)_{d\in\mathbb{N}}$.

We refer to Section 8.1 in Embrechts et al. [13] for an appealing summary and discussion of the extremal index.

Denote by $N_n(\cdot) := \sum_{i=1}^{d} \varepsilon_{n^{-1}i}(\cdot)\mathbb{1}_{\{X_i>u_d\}}$ the point process of the exceedances of u_d by $X_1, \ldots, X_d$. Then one can show that N_n converges weakly to a compound Poisson process $N_\infty := \sum_{i=1}^{\infty} \xi_i\varepsilon_{\Gamma_i}$ (cf. Section 5.5.1 in Embrechts et al. [13]), where ξ_i are i.i.d cluster sizes and Γ_i are the points of a homogeneous Poisson process.

It is in particular well-known (cf. Hsing et al. [38]) that the extremal index is the reciprocal of the mean cluster size of the limiting compound process associated with the point process of the exceedances by $X_1, \ldots, X_d$ for $d \to \infty$.

For the sake of simplicity denote by $(X_1, \ldots, X_d)$ an arbitrary finite dimensional rv taken from the stochastic process $(X_d)_{d\in\mathbb{N}}$. Hence, its joint distribution $F^{(d)}$ is a d-variate marginal distribution of the process $(X_d)_{d\in\mathbb{N}}$, which has identical margins F by requirements. Further denote by $C_F^{(d)}$ the corresponding copula to $F^{(d)}$. The number of exceedances above

s by $X_1, \dots, X_d$ are defined by $N_s := \sum_{k \le d} \mathbb{1}_{(s,\infty)}(X_k)$. Within this framework the fragility index can be defined by

$$FI^{(d)} := \lim_{s \nearrow} E\left(N_s \mid N_s > 0\right)$$

analogue to Definition 4.1.1.

The following result links the fragility index with the extremal index.

Theorem 4.4.2 *Let $(X_d)_{d\in\mathbb{N}}$ be a strictly stationary sequence with extremal index θ. Suppose that the copula $C^{(d)}$ associated with the vector $\boldsymbol{X}^{(d)} = (X_1, \dots, X_d)$ satisfies the expansion*

$$C^{(d)}(\boldsymbol{y}) = 1 - \|\mathbf{1} - \boldsymbol{y}\|_{D^{(d)}} + o(d\,|1-y|) \tag{4.20}$$

with $\boldsymbol{y} = (y, \dots, y)$ uniformly for $y \in [0,1]$ and $d \in \mathbb{N}$, where $\|\cdot\|_{D^{(d)}}$ is a D-norm on $\mathbb{R}^d$, cf. Definition 2.1.6. Then the fragility index $FI = FI^{(d)}$ exists for $\boldsymbol{X}^{(d)}$ for each $d \in \mathbb{N}$, i.e.

$$FI^{(d)} = \frac{d}{\|\mathbf{1}\|_{D^{(d)}}}$$

and we have

$$\lim_{d\to\infty} \frac{1}{FI^{(d)}} = \theta.$$

Note that Condition (4.20) is derived from Corollary 2.3.19 in a natural way using the fact that every D-norm is bounded above by the L_1-norm.

Proof: We have

$$\begin{aligned} FI^{(d)} &= \lim_{s\nearrow} \sum_{k=1}^{d} E\left(1_{(s,\infty)}(X_k) \mid \max_{1\le k\le d} X_k > s\right) \\ &= \lim_{s\nearrow} \frac{d(1-F(s))}{1 - P(X_k \le s,\, 1 \le k \le d)} \\ &= \lim_{s\nearrow} \frac{d(1-F(s))}{1 - C^{(d)}(F(s), \dots, F(s))} \\ &= \frac{d}{\|\mathbf{1}\|_{D^{(d)}}} \end{aligned}$$

by Condition (4.20). We have, moreover, by the same condition

$$P\left(\max_{1\le k\le d} X_k \le u_d\right)$$

$$
\begin{aligned}
&= C^{(d)}(F(u_d), \dots, F(u_d)) \\
&= 1 - (1 - F(u_d)) \|\mathbf{1}\|_{D^{(d)}} + o(d(1 - F(u_d)) \\
&= 1 - \frac{d(1 - F(u_d))}{FI^{(d)}} + o(d(1 - F(u_d))
\end{aligned}
$$

and, thus, by Condition (4.18) and (4.19)

$$\exp(-\theta\tau) + o(1) = 1 - \frac{\tau + o(1)}{FI^{(d)}} + o(\tau)$$

as $d \to \infty$. This implies

$$\frac{1}{FI^{(d)}} = \frac{1 - \exp(-\theta\tau) + o(\tau)}{\tau}$$

as $d \to \infty$. Letting now τ converge to 0 yields the assertion. □

The preceding result enables a further interpretation of the extremal index. Take again the Marshall-Olkin D-norm, i.e., the convex combination of the L_1- and the maximum norm, which are the two extremal D-norms representing independence and complete dependence of the margins of the associated EVD:

$$\|\cdot\|_{MO} = \vartheta \|\cdot\|_1 + (1 - \vartheta) \|\cdot\|_\infty ,$$

where $\vartheta \in [0, 1]$ (cf. Section A and Example 3.3.16). Take now an arbitrary D-norm $\|\cdot\|_{D^{(d)}}$ on $\mathbb{R}^d$. Since every D-norm is bounded above by the L_1-norm and bounded below by the maximum-norm, there exists a unique $\vartheta_d \in [0, 1]$ such that $\|\mathbf{1}\|_{D^{(d)}}$ coincides with the pertaining Marshall-Olkin norm of $\mathbf{1}$, i.e.,

$$\|\mathbf{1}\|_{D^{(d)}} = \vartheta_d \|\mathbf{1}\|_1 + (1 - \vartheta_d) \|\mathbf{1}\|_\infty = 1 + (d - 1)\vartheta_d.$$

We thus obtain that the sequence of reciprocals $\|\mathbf{1}\|_{D^{(d)}} / d$ of the fragility index $FI^{(d)}$ converges as $d \to \infty$ if, and only if, $\lim_{d\to\infty} \vartheta_d \in [0, 1]$ exists. Theorem 4.4.2 now yields that $\lim_{d\to\infty} \vartheta_d = \theta$, the extremal index.

Remark 4.4.3 *With respect to the previous considerations, the extremal index can be considered as the "proportion of tail independence" contained in the vector $\boldsymbol{X}^{(d)}$ for large d, as the L_1-norm represents the case of independence of the margins of the limiting extreme value distribution $G^{(d)}(\boldsymbol{x}) = \exp\left(- \|\boldsymbol{x}\|_{D^{(d)}}\right)$, $\boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$ associated with $\boldsymbol{X}^{(d)}$.*

The following example shows that the extremal index of a GPD-process is 0.

Example 4.4.4 (GPD-Process) Let $(Z_k)_{k\in\mathbb{N}}$ be a strictly stationary process with $0 < Z_1 \le c$ almost surely for some $c > 1$ and $E(Z_1) = 1$. Let U be a uniformly on $(0,1)$ distributed rv, which is independent of the process $(Z_k)_{k\in\mathbb{N}}$ and put

$$X_k := 1 - \frac{U}{Z_k}, \qquad k \in \mathbb{N}.$$

Then the process $(X_k)_{k\in\mathbb{N}}$ is a *GPD-process* (cf. Buishand et al. [7]). It is obviously strictly stationary and the copula $C^{(d)}$ corresponding to $(X_1,\dots,X_d)$ is a GPD-copula.

We show in the following that $C^{(d)}$ satisfies Condition (4.20) and that the extremal index corresponding to $(X_k)_{k\in\mathbb{N}}$ is 0.

Note that we have for $1 - 1/c \le x_k \le 1$, $k \le d$,

$$\begin{aligned} P(X_k \le x_k,\, 1 \le k \le d) &= 1 - \int \max_{1\le k\le d}((1-x_k)z_k)\,(P*(Z_1,\dots,Z_d))(d\boldsymbol{z}) \\ &= 1 - E\left(\max_{1\le k\le d}((1-x_k)Z_k)\right) \\ &= 1 - \|(1-x_1,\dots,1-x_d)\|_{D^{(d)}}\,, \end{aligned}$$

with the definition of the D-norm (cf. Lemma 2.1.8), i.e.

$$\|\boldsymbol{y}\|_{D^{(d)}} := E\left(\max_{1\le k\le d}(|y_k|\, Z_k)\right), \qquad \boldsymbol{y} \in \mathbb{R}^d,$$

defines a D-norm on $\mathbb{R}^d$ for each $d \in \mathbb{N}$. Condition (4.20) is, therefore, obviously satisfied.

Next we show that the extremal index of $(X_k)_{k\in\mathbb{N}}$ exists and that it is equal to 0. With $d = 1$ we obtain for $1 - 1/c \le x \le 1$

$$P(X_1 \le x) = 1 - (1-x)E(Z_1) = x$$

and, thus, with $u_d := 1 - \tau/d$, $\tau > 0$, we have

$$d(1 - P(X_1 \le u_d)) = \tau$$

for d large.

Finally, we obtain

$$\begin{aligned} P\left(\max_{1\le k\le d} X_k \le u_d\right) &= C^{(d)}(F(u_d),\dots,F(u_d)) \\ &= 1 - \|(1-u_d,\dots,1-u_d)\|_{D^{(d)}} \end{aligned}$$

$$= 1 - \frac{\tau}{d} \|(1, \dots, 1)\|_{D^{(d)}}$$
$$\to_{d\to\infty} 1,$$

as $\|(1, \dots, 1)\|_{D^{(d)}} = E\left(\max_{1\le k\le d} Z_k\right) \le c$ and thus, by (4.19), the extremal index of $(X_k)_{k\in\mathbb{N}}$ is $\theta = 0$.

4.4.2. Fragility Index in terms of a stochastic system

Within this section we consider a stochastic process $(X_d)_{d\in\mathbb{N}}$, which is not required to be strictly stationary as in Section 4.4.1. The total number of sequential time points at which $(X_d)_{d\in\mathbb{N}}$ exceeds a high threshold is called an *excursion time*. The mathematical tools developed in the preceding sections enable the computation of its distribution as well. Precisely, denote by $L_\kappa(s)$ the number of sequential exceedances above the threshold s, if we have an exceedance at $\kappa \in \{1, \dots, d\}$, i.e.

$$L_\kappa(s) := \sum_{k=0}^{d-\kappa} k \cdot \mathbb{1}_{(X_\kappa > s, \dots, X_{\kappa+k} > s, X_{\kappa+k+1} \le s)}.$$

We have, in particular, $L_d(s) := 0 = L_\kappa(s)$, if $X_{\kappa+1} \le s$.

We suppose throughout this section that Condition (3.1) holds for the index $\kappa \in \{1, \dots, d\}$.

Within this section the definition of the D-norm by the so-called generator $\boldsymbol{Z}$ (cf. Lemma 2.1.8), i.e.

$$\|\boldsymbol{x}\|_D = E\left(\max_{1\le j\le d}(|x_j|\, Z_j)\right), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{4.21}$$

will be quite useful.

Note that the distribution of the generator of a D-norm is in general not uniquely determined. Put, for example, $\boldsymbol{Z} = 2\boldsymbol{U}$, where $\boldsymbol{U}$ follows an arbitrary copula on $[0,1]^d$ with $P(U_1 + \dots + U_d = d/2) < 1$, i.e. $P(Z_1 + \dots + Z_d = d) < 1$. The rv $\boldsymbol{Z}$ is the generator of a D-norm $\|\cdot\|_D$ as in Lemma 2.1.8, i.e. $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, defines an EVD with standard negative exponential margins. But the Pickands-de Haan-Resnick representation of G implies the existence of another generator $\widetilde{\boldsymbol{Z}}$ of $\|\cdot\|_D$ with the additional property $\widetilde{Z}_1 + \dots + \widetilde{Z}_d = d$.

The following auxiliary result will be crucial.

Lemma 4.4.5 *Assume the conditions of Lemma 3.3.1. Then we obtain for* $\kappa \in \{1, \dots, d\}$ *as* $s \nearrow \omega^*$

$$\begin{aligned} P\left(L_\kappa(s) \ge k \mid X_\kappa > s\right) &= \sum_{\emptyset \ne T \subseteq \{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1) \\ &=: s_\kappa(k) + o(1), \qquad 0 \le k \le d - \kappa. \end{aligned}$$

Proof: By means of Theorem A.6 we obtain

$$\begin{aligned} &P\left(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s\right) \\ &= \frac{1 - P\left(\bigcup_{0 \le i \le k} \{X_{\kappa+i} \le s\}\right)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \ne T \subseteq \{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} P\left(X_i \le s,\, i \in T\right)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \ne T \subseteq \{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \left(1 - c(s) \left\|\sum_{i \in T} \gamma_i \boldsymbol{e}_i\right\|_D\right) + o(1 - F_\kappa(s))}{1 - F_\kappa(s)} \\ &= \sum_{\emptyset \ne T \subseteq \{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1). \end{aligned}$$

□

If we represent the D-norm by means of an expectation, c.f. Lemma 2.1.8, we get an equivalent formulation of Lemma 4.4.5. At first, we need the following Lemma.

Lemma 4.4.6 *For arbitrary numbers* $a_1, \dots, a_d \in \mathbb{R}$ *and* $d \in \mathbb{N}$ *we have*

$$\min_{1 \le i \le d} a_i = \sum_{\emptyset \ne L \subseteq \{1,\dots,d\}} (-1)^{|L|-1} \max_{k \in L} a_k.$$

Proof: We show the assertion by induction over d. The assertion is obvious for $d = 1$. Suppose that it is true for $d \in \mathbb{N}$. Without loss of generality suppose $a_1 \le a_2 \le \dots \le a_{d+1}$. Now we have

$$\begin{aligned} &\sum_{1 \le j \le d+1} (-1)^{j-1} \sum_{\substack{T \subseteq \{1,\dots,d+1\} \\ |T| = j}} \max_{k \in T}(a_k) \\ &= \sum_{1 \le j \le d} (-1)^{j-1} \left(\sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T| = j}} \max_{k \in T}(a_k) + \sum_{\substack{T \subseteq \{1,\dots,d+1\} \\ |T| = j, \{d+1\} \in T}} \max_{k \in T}(a_k) \right) + (-1)^d \max_{k \in \{1,\dots,d+1\}}(a_k) \end{aligned}$$

$$
\begin{aligned}
&= \min_{1\le i\le d}(a_i) + \sum_{1\le j\le d} (-1)^{j-1} \sum_{\substack{T\subseteq\{1,\dots,d+1\}\\ |T|=j,\{d+1\}\in T}} \max_{k\in T}(a_k) + (-1)^d \max_{k\in\{1,\dots,d+1\}}(a_k)\\
&= \min_{1\le i\le d}(a_i) + \sum_{0\le j\le d} (-1)^{j} \sum_{\substack{T\subseteq\{1,\dots,d\}\\ |T|=j}} \max_{k\in\{T\cup\{d+1\}\}}(a_k)\\
&= \min_{1\le i\le d}(a_i) + \underbrace{\sum_{0\le j\le d} (-1)^{j}\binom{d}{j}}_{=0} a_{d+1} = \min_{1\le i\le d+1}(a_i)
\end{aligned}
$$

since we supposed $a_1 \le a_2 \le \ldots \le a_{d+1}$. □

Corollary 4.4.7 *Suppose in addition to the assumptions in Lemma 3.3.1 that $\boldsymbol{Z}$ is a generator of the D-norm $\|\cdot\|_D$. Then we obtain for $\kappa \in \{1,\dots,d\}$ as $s \nearrow \omega^*$*

$$
P\left(L_\kappa(s) \ge k \mid X_\kappa > s\right) = E\left(\min_{\kappa\le i\le\kappa+k}(\gamma_i Z_i)\right) + o(1),
$$

for $0 \le k \le d-\kappa$.

Proof: By means of Lemma 4.4.6 we get

$$
\begin{aligned}
&\sum_{\emptyset\ne T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \left\|\sum_{i\in T}\gamma_i \boldsymbol{e}_i\right\|_D + o(1)\\
&= \sum_{\emptyset\ne T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} E\left(\max_{i\in\{\kappa,\dots,\kappa+k\}}(\gamma_i Z_i)\right) + o(1)\\
&= E\left(\sum_{\emptyset\ne T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \max_{i\in\{\kappa,\dots,\kappa+k\}}(\gamma_i Z_i) + o(1)\right)\\
&= E\left(\min_{i\in\{\kappa,\dots,\kappa+k\}}(\gamma_i Z_i)\right) + o(1)\,.
\end{aligned}
$$

□

The asymptotic distribution of the excursion time, conditional on the assumption that there is an exceedance at time point $\kappa \in \{1,\dots,d\}$, is an immediate consequence of Lemma 4.4.5.

Proposition 4.4.8 *Assume the conditions of Lemma 3.3.1. Then we have for $\kappa < d$ as $s \nearrow \omega^*$*

$$
P(L_\kappa(s) = k \mid X_\kappa > s)
$$

$$= \begin{cases} \sum_{\emptyset\neq T\subseteq\{\kappa,\dots,d\}}(-1)^{|T|+1}\left\|\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D + o(1), \\ \qquad\qquad\qquad\qquad\qquad k = d-\kappa, \\ \sum_{T\subseteq\{\kappa,\dots,\kappa+k\}}(-1)^{|T|+1}\left\|\gamma_{\kappa+k+1}\boldsymbol{e}_{\kappa+k+1}+\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D + o(1), \\ \qquad\qquad\qquad\qquad\qquad 0\le k< d-\kappa. \end{cases}$$

Proof: First note that $P(L_\kappa(s)=0 \mid X_\kappa > s)=1$ for $\kappa=d$. For $k=d-\kappa$, the assertion is clear by $P(L_\kappa(s)=d-\kappa \mid X_\kappa>s)=P(L_\kappa(s)\ge d-\kappa \mid X_\kappa>s)$ and the use of Lemma 4.4.5. For $0\le k<d-\kappa$, we get with Lemma 4.4.5

$$\begin{aligned} P(L_\kappa(s)=k \mid X_\kappa>s) =& P(L_\kappa(s)\ge k \mid X_\kappa>s) - P(L_\kappa(s)\ge k+1 \mid X_\kappa>s) \\ =& \sum_{\emptyset\neq T\subseteq\{\kappa,\dots,\kappa+k\}}(-1)^{|T|+1}\left\|\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D \\ & - \sum_{\emptyset\neq T\subseteq\{\kappa,\dots,\kappa+k+1\}}(-1)^{|T|+1}\left\|\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D + o(1) \\ =& -\sum_{\substack{T\cup\{\kappa+k+1\},\\ T\subseteq\{\kappa,\dots,\kappa+k\}}}(-1)^{|T|}\left\|\sum_{i\in\{T\cup\{\kappa+k+1\}\}}\gamma_i\boldsymbol{e}_i\right\|_D + o(1) \\ =& \sum_{T\subseteq\{\kappa,\dots,\kappa+k+1\}}(-1)^{|T|+1}\left\|\gamma_{\kappa+k+1}\boldsymbol{e}_{\kappa+k+1}+\sum_{i\in T}\gamma_i\boldsymbol{e}_i\right\|_D \\ & + o(1), \end{aligned}$$

which completes the proof. □

In terms of a generator $\boldsymbol{Z}$ of a D-norm, Proposition 4.4.8 becomes the following result.

Corollary 4.4.9 *Assume in addition to the conditions of Lemma 3.3.1 that* $\boldsymbol{Z}$ *is a generator of the D-norm* $\|\cdot\|_D$. *Then we have for* $\kappa<d$ *as* $s\nearrow\omega^*$

(i) $P(L_\kappa(s)=k \mid X_\kappa>s)$

$$= \begin{cases} E\left(\min_{\kappa\le i\le d}(\gamma_i Z_i)\right)+o(1), & k=d-\kappa \\ E\left(\min_{\kappa\le i\le \kappa+k}(\gamma_i Z_i)-\min_{\kappa\le i\le\kappa+k+1}(\gamma_i Z_i)\right)+o(1), & 0\le k<d-\kappa. \end{cases}$$

(ii) $P(L_\kappa(s)\le k \mid X_\kappa>s)$

$$= \begin{cases} 1, & k=d-\kappa \\ 1-E\left(\min_{\kappa\le i\le\kappa+k+1}(\gamma_i Z_i)\right)+o(1), & 0\le k<d-\kappa. \end{cases}$$

Proof: By means of Lemma 4.4.6 we have

$$\begin{aligned}
P(L_\kappa(s)=k \mid X_\kappa > s) =& \sum_{\emptyset\neq T\subseteq\{\kappa,\dots,d\}} (-1)^{|T|+1} \left\| \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1) \\
=& E\left(\sum_{\emptyset\neq T\subseteq\{\kappa,\dots,d\}} (-1)^{|T|+1} \max_{i\in T}(\gamma_i Z_i) + o(1)\right) \\
=& E\left(\min_{\kappa\leq i\leq d}(\gamma_i Z_i)\right) + o(1)
\end{aligned}$$

for $k=d-\kappa$ and

$$\begin{aligned}
P(L_\kappa(s)=k \mid X_\kappa > s) =& \sum_{T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \left\| \gamma_{\kappa+k+1}\boldsymbol{e}_{\kappa+k+1} + \sum_{i\in T} \gamma_i \boldsymbol{e}_i \right\|_D + o(1) \\
=& \sum_{T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} E\left(\max_{i\in\{T\cup\{\kappa+k+1\}\}}(\gamma_i Z_i)\right) + o(1) \\
=& E\left(\sum_{\substack{T\cup\{\kappa+k+1\},\\ T\subseteq\{\kappa,\dots,\kappa+k\}}} (-1)^{|T|+1} \max_{i\in\{T\cup\{\kappa+k+1\}\}}(\gamma_i Z_i)\right) + o(1) \\
=& E\left(\sum_{\emptyset\neq T\subseteq\{\kappa,\dots,\kappa+k\}} (-1)^{|T|+1} \max_{i\in T}(\gamma_i Z_i)\right. \\
& \left. - \sum_{\emptyset\neq T\subseteq\{\kappa,\dots,\kappa+k+1\}} (-1)^{|T|+1} \max_{i\in T}(\gamma_i Z_i)\right) + o(1) \\
=& E\left(\min_{k\leq i\leq\kappa+k}(\gamma_i Z_i) - \min_{k\leq i\leq\kappa+k+1}(\gamma_i Z_i)\right) + o(1)
\end{aligned}$$

for $0\leq k< d-\kappa$, which shows (i). Further we have

$$P(L_\kappa(s)\leq d-\kappa \mid X_\kappa > s) = 1 - \underbrace{P(L_\kappa(s) > d-\kappa \mid X_\kappa > s)}_{=0} = 1$$

and by means of Corollary 4.4.7 we get

$$\begin{aligned}
P(L_\kappa(s)\leq k \mid X_\kappa > s) =& 1 - P(L_\kappa(s)\geq k+1 \mid X_\kappa > s) \\
=& 1 - E\left(\min_{\kappa\leq i\leq\kappa+k+1}(\gamma_i Z_i)\right) + o(1)
\end{aligned}$$

for $0 \leq k < d - \kappa$, which completes the proof of (ii). $\square$

We thus obtain the limit distribution of the excursion time:

$$Q_\kappa([0,k]) := \lim_{s \nearrow \omega^*} P(L_\kappa(s) \leq k \mid X_\kappa > s)$$
$$= \begin{cases} 1, & k = d - \kappa \\ 1 - E\left(\min_{\kappa \leq i \leq \kappa+k+1}(\gamma_i Z_i)\right), & 0 \leq k < d - \kappa. \end{cases}$$

Take, for example, the generator $\boldsymbol{Z} = 2(U_1, \dots, U_d)$, where the U_i are independent and uniformly on $(0,1)$ distributed rv. If, in addition, $\gamma_i = 1$, $\kappa \leq i \leq d$, then we obtain

$$Q_\kappa([0,k]) = \begin{cases} 1, & k = d - \kappa \\ 1 - \frac{2}{k+3}, & 0 \leq k < d - \kappa. \end{cases}$$

Note that $\boldsymbol{Z}$ is not a generator of the L_1-norm $\|\boldsymbol{x}\|_1 = \sum_{i=1}^d |x_i|$, which yields the EVD $G(\boldsymbol{x}) = \exp\left(-\sum_{i=1}^d |x_i|\right)$, $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, with independent margins (cf. Example 4.4.13).

Next we compute the asymptotic mean excursion time. It can be interpreted as the fragility index for sojourn times, hence we call it the *sojourn index* (SI).

Definition 4.4.10 (Sojourn Index) *The sojourn index is defined by the limit*

$$\lim_{s \nearrow \omega^*} E\left(L_\kappa(s) \mid X_\kappa > s\right) = \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} s_\kappa(k) & \text{else} \end{cases}$$
$$= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} E\left(\min_{\kappa \leq i \leq \kappa+k}(\gamma_i Z_i)\right) & \text{else} \end{cases}$$

and defines a measure for asymptotic dependence within a finite sequence of a stochastic process.

Note that we have

$$E\left(L_\kappa(s) \mid X_\kappa > s\right) \in [1, d - \kappa + 1]$$

by construction. Hence, in a period of $d+1$ sequential time points, we may expect the duration of exceedance by at least 1 and at most $d+1$ time units if we have an exceedance at the starting time point. In comparison to the fragility index, we capture the amount of asymptotic dependence within a finite sequence of a stochastic process by the limit of the expected excursion time within a finite sequence of the stochastic process $(X_d)_{d\in\mathbb{N}}$. By the way, considering a strictly stationary stochastic process, the results due to the finite sequence can be carried over to the stochastic process if one applies a transformation of the sojourn index to the interval $[0,1]$ as done for the extended fragility index in Definition 4.3.1 in a similar way.

Proposition 4.4.11 (Sojourn Index) *Assume the conditions of Lemma 3.3.1 and let* $\boldsymbol{Z}$ *be a generator of the D-norm* $\|\cdot\|_D$. *Then we have for* $1 \le \kappa \le d$

$$E\left(L_\kappa(s) \mid X_\kappa > s\right) = \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1) & \text{else} \end{cases}$$
$$= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} E\left(\min_{\kappa\le i\le \kappa+k}(\gamma_i Z_i)\right) + o(1) & \text{else.} \end{cases}$$

Proof: Since $L_\kappa(s)$ attains only nonnegative values, we have for $\kappa < d$

$$\begin{aligned} E\left(L_\kappa(s) \mid X_\kappa > s\right) &= \int_0^\infty P\left(L_\kappa(s) \ge t \mid X_\kappa > s\right)\, dt \\ &= \sum_{k=1}^{d-\kappa} P\left(L_\kappa(s) \ge k \mid X_\kappa > s\right) \\ &= \sum_{k=1}^{d-\kappa} P\left(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s\right) \\ &= \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1), \end{aligned}$$

where s_κ is defined in Lemma 4.4.5. □

Corollary 4.4.12 *Under the conditions of Proposition 4.4.11 we have for* $\kappa < d$, *if* $\gamma_k > 0$, $1 \le k \le d$,

$$\lim_{s\uparrow\omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0$$

if and only if

$$\|x\boldsymbol{e}_\kappa + y\boldsymbol{e}_{\kappa+1}\|_D = \|x\boldsymbol{e}_\kappa + y\boldsymbol{e}_{\kappa+1}\|_1 = x + y, \qquad x, y \ge 0.$$

Proof: Note that $s_\kappa(1) \geq \dots \geq s_\kappa(d-\kappa)$. We, thus, obtain from Proposition 4.4.11

$$\lim_{s\uparrow\omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0 \iff s_\kappa(1) = 0.$$

And further we have $s_\kappa(1) = 0$ if and only if $\|\gamma_\kappa \boldsymbol{e}_\kappa + \gamma_{\kappa+1}\boldsymbol{e}_{\kappa+1}\|_D = \gamma_\kappa + \gamma_{\kappa+1}$ for arbitrary $\gamma_\kappa, \gamma_{\kappa+1} > 0$. Hence the assertion follows in analogy to the proof of Lemma 3.2.1. $\square$

The following considerations will be used in Example 4.4.13. Suppose in addition to the assumptions of Lemma 3.3.1 that the components $X_1, \dots, X_d$ of the rv $\boldsymbol{X}$ are exchangeable. Then we have $\gamma_1 = \dots = \gamma_d = 1$, as well as

$$\left\| \sum_{i\in T} \boldsymbol{e}_i \right\|_D = \left\| \sum_{i=1}^{|T|} \boldsymbol{e}_i \right\|_D$$

for any nonempty subset $T \subseteq \{1, \dots, d\}$. As a consequence we obtain

$$s_\kappa(k) = \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D, \qquad 0 \leq k \leq d-\kappa,$$

and, thus, by rearranging sums,

$$\begin{aligned}
\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) &= \sum_{k=1}^{d-\kappa} s_\kappa(k)\\
&= \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D \sum_{k=\max(1,j-1)}^{d-\kappa} \binom{k+1}{j}\\
&= -1 + \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \binom{d-\kappa+2}{j+1} \left\| \sum_{i=1}^{j} \boldsymbol{e}_i \right\|_D, \qquad (4.22)
\end{aligned}$$

where the final equality follows from the general equation $\sum_{r=n}^{N} \binom{r}{n} = \binom{N+1}{n+1}$.

We want to finish this section with an example for the sojourn index.

Example 4.4.13 (Marshall-Olkin D-norm) *Consider the Marshall-Olkin D-norm*

$$\|\boldsymbol{x}\|_{\mathrm{MO}} = \vartheta \|\boldsymbol{x}\|_1 + (1-\vartheta)\|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^d,\ \vartheta \in [0,1],$$

cf. Section A and Example 3.3.16. In this case, i.e. $\|\cdot\|_D = \|\cdot\|_{MO}$, we obtain from Equation (4.22)

$$\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) = (1-\vartheta)(d-\kappa),$$

which can be seen as follows. By means of Lemma A.4 and the binomial formula $\sum_{j=0}^{m}(-1)^j\binom{m}{j} = (1+(-1))^m = 0$, *we get*

$$
\begin{aligned}
&\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) \\
&= -1 + \sum_{j=1}^{d-\kappa+1} (-1)^{j+1}\binom{d-\kappa+2}{j+1}\left\|\sum_{i=1}^{j}\boldsymbol{e}_i\right\|_{MO} \\
&= -1 + \sum_{j=1}^{d-\kappa+1} (-1)^{j+1}\binom{d-\kappa+2}{j+1} j\vartheta + (1-\vartheta)\sum_{j=1}^{d-\kappa+1}(-1)^{j+1}\binom{d-\kappa+2}{j+1} \\
&= -1 + \vartheta\sum_{j=2}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j} j - \vartheta\sum_{j=2}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j} \\
&\qquad + \sum_{j=2}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j} - \vartheta\sum_{j=2}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j} \\
&= -1 + \vartheta\left(\underbrace{\sum_{j=0}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j} j}_{=0} + (d-\kappa+2)\right) \\
&\qquad - \vartheta\left(\underbrace{\sum_{j=0}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j}}_{=0} -1 + d - \kappa + 2\right) \\
&\qquad + \underbrace{\sum_{j=0}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j}}_{=0} -1 + d - \kappa + 2 \\
&\qquad - \vartheta\left(\underbrace{\sum_{j=0}^{d-\kappa+2}(-1)^j\binom{d-\kappa+2}{j}}_{=0} -1 + d - \kappa + 2\right) \\
&= -1 + \vartheta(d-\kappa+2) - \vartheta(d-\kappa+1) + d - \kappa + 1 - \vartheta(d-\kappa+1) \\
&= (d-\kappa)(1-\vartheta)\,.
\end{aligned}
$$

In the case $\vartheta = 0$ of complete tail dependence of the margins we therefore obtain

$$\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) = d - \kappa,$$

which is the full possible length, whereas in the tail independence case $\vartheta = 1$ we obtain the shortest length

$$\lim_{s\nearrow} E\left(L_\kappa(s) \mid X_\kappa > s\right) = 0,$$

which is in complete accordance with Corollary 4.4.12.

Hence this example covers the whole range of what is possible with respect to the *amount* of asymptotic dependence within a finite sequence of the stochastic process $(X_d)_{d\in\mathbb{N}}$. To investigate the dependence structure, one may consider different choices for the D-norm as done in Section 4.2.5.

5. Applications on the Fragility Index

This chapter engages in the estimation of the (extended) fragility index and its application on random systems in order to shed light on their asymptotic stability. Thereby we focus on a nonparametric estimation procedure for the dependence structure of the EVD to which domain of attraction the random system‘s representing distribution function belongs to. A nonparametric approach is reasonable since we do not want to make assumptions on a parametric model for the EVD. The estimation of the (extended) fragility index is of great importance with respect to the fact that it can be considered as a tail dependence measure under the domain-of-attraction condition for arbitrary dimensions (cf. Section 4.3).

This chapter is structured as follows. Section 5.1 summarizes a nonparametric estimation procedure of the stable tail dependence function and carries over to the nonparametric estimation approach of the (extended) fragility index. Section 5.2 provides a simulation study of the nonparametric estimator for the extremal coefficient and the thereof derived (extended) fragility index. Therein we simulate the distribution and the mean squared error of the estimators and comment on an optimal tail fraction. Section 5.3 presents the application of the (extended) fragility index on two random systems represented by stock prices taken from the DAX over the last ten years.

5.1. Estimation of the Fragility Index

This section aims to provide an estimation procedure for the fragility index. The representation of the fragility index, as well as the extended fragility index, is based on the extremal coefficients corresponding to the margins of the EVD G to which domain of attraction the df F of the random system $\{Q_1 \dots, Q_d\}$ belongs to (cf. (4.17) in Section 4.3). The extremal coefficient *corresponding* to an EVD G with arbitrary margins is defined by means of the D-norm (cf. Definition 2.2.4). Further we know that the D-norm coincides with the stable tail dependence function (cf. Section 2.3). There exists a multitude of possibilities to estimate the stable tail dependence function (cf. Section 2.4 for an overview and corresponding literature). Most of those estimators focus on the fact that the stable tail dependence function represents the dependence structure of an EVD, hence most estimators of the stable tail dependence function are provided under the assumption that the underlying multivariate distribution function is actually an EVD (we call it the EVD-assumption). For example, the CFG-estimators for the Pickands-dependence function (cf. Genest et al. [23] and Capéraà et al. [8] for the bivariate case, extended to the multivariate case by Zhang et al. [71]) are restricted to the EVD-assumption. This is also true for recent work of Genest and Seghers [24] or Gudendorf and Seghers [27], who extensively work on the estimation of the dependence structure of an EVD.

Our (extended) fragility index is based on the assumption that the copula of the underlying distribution belongs to the domain of attraction of an EVD (we call it the domain-of-attraction-assumption). In this case we have

$$C_F \in \mathcal{D}(G) \;\Leftrightarrow\; \lim_{t \downarrow 0} \frac{1 - C_F(1 + t\boldsymbol{x})}{t} = \|\boldsymbol{x}\|_D \,, \tag{5.1}$$

cf. Corollary 2.3.17. Hence we use a weaker assumption than the EVD-assumption. In the following we focus on the nonparametric estimation of the D-norm. At this point we want to note that (5.1) also suggests a parametric estimation approach based on the copula C_F (cf. Section 5.4 for an outlook on this topic).
The nonparametric estimator of the stable tail dependence function, provided in de Haan and Ferreira [29], Section 7.2, emerged to be a suitable estimator for our purpose. In the following we want to present this estimator and the results concerning consistency and the asymptotic behavior of its distribution function. We start with the derivation of the estimator (cf. de Haan and Ferreira [29], Section 7.2).

Due to Definition 2.3.1, the right hand side of (5.1) is equivalent to

$$\lim_{t \to \infty} t \cdot \left(1 - F\left(U_j\left(\frac{t}{x_j}\right), j \le d\right)\right) = l(\boldsymbol{x}) \,, \tag{5.2}$$

with $U_j(x) := F_j^{-1}\left(1-\frac{1}{x}\right)$, where F_j is continuous for $j \le d$, where l denotes the stable tail dependence function as defined in Definition 2.3.13.

With regard to certain regularity conditions, which we will need later, we require $k = k(n) \to \infty$, $k/n \to 0$, $n \to \infty$. Then (5.2) is equivalent to

$$\lim_{n\to\infty} \frac{n}{k}\left(1 - F\left(U_j\left(\frac{n}{kx_j}\right), j \le d\right)\right) = l(x_1, \ldots, x_d)\,.$$

Now, use empirical counterparts for the function U_j and F to derive an estimator, i.e. use the order statistic $Y^{(j)}_{n-[kx_j]+1,n}$ instead of $U_j\left(\frac{n}{kx_j}\right) = F_j^{-1}\left(1-\frac{k}{n}x_j\right)$ and take the empirical df $\hat{F}$ of F. Hence, we get for $x_1, \ldots, x_d > 0$

$$(5.3) \qquad \|(x_1,\ldots,x_d)\|_{\hat{D}} := \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i^{(1)} \ge Y^{(1)}_{n-[kx_1]+1,n} \text{ or } \ldots \text{ or } Y_i^{(d)} \ge Y^{(d)}_{n-[kx_d]+1,n}\}}$$

as a nonparametric estimator for the D-norm. Note that (5.3) is invariant under monotone transformations of the margins. The following results concerning consistency and asymptotic normality of $\|(x_1,\ldots,x_d)\|_{\hat{D}}$ are provided by Theorem 7.2.1 and 7.2.2. of de Haan and Ferreira [29] for the bivariate case. The proofs are rather extensive, but obvious for extension to higher dimensions (cf. the notes in the first paragraph of Section 6.1.2 of de Haan and Ferreira [29]). We want to start with the consistency of $\|\cdot\|_{\hat{D}}$.

Theorem 5.1.1 *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ to be i.i.d. random vectors with joint df $F \in \mathcal{D}(G)$ and continuous margins. Denote by $\|\cdot\|_D$ the D-norm corresponding to G. Further, suppose that we have $k = k(n) \to \infty, k/n \to 0$ for $k < n$ and $n \to \infty$. Then we get for $\boldsymbol{T} > \boldsymbol{0}$*

$$\sup_{0 \le \boldsymbol{x} \le T} |\, \|\boldsymbol{x}\|_D - \|\boldsymbol{x}\|_{\hat{D}}\,| \to_P 0$$

as $n \to \infty$, where $\to_P$ denotes convergence in probability.

Proof: Cf. Theorem 7.2.1 in de Haan and Ferreira [29]. □

The next theorem states the asymptotic normality of $\|\cdot\|_{\hat{D}}$.

Theorem 5.1.2 *Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ to be i.i.d. random vectors with joint df $F \in \mathcal{D}(G)$ and continuous margins. Denote by $L_i(\boldsymbol{x}) := \frac{\partial \|\boldsymbol{x}\|_D}{\partial x_j}$ the continuous first-order j-th partial derivative of the function $\|\boldsymbol{x}\|_D$, for which it holds that $\|x\boldsymbol{e}_i\|_D = \|x\boldsymbol{e}_j\|_D = x$ for $i, j \le d$*

and $x \geq 0$. *Furthermore, suppose that for some* $\alpha > 0$ *and for all* $\boldsymbol{x} > \mathbf{0}$, *we have*

$$t\left(1 - F\left(U_j\left(\frac{t}{x_j}\right), j \leq d\right)\right) = \|\boldsymbol{x}\|_D + O(t^{-\alpha})$$

for $t \to \infty$ *uniformly on the unit simplex* $S = \left\{\boldsymbol{u} : \sum_{j \leq d} u_j = 1,\ u_j \geq 0\right\}$, *where* U_j *is defined in (5.2) for* $j \leq d$. *Further, suppose that we have* $k = k(n) \to \infty$ *with* $k(n) = o\left(n^{2\alpha/(1+2\alpha)}\right)$, $\alpha > 0$ *and* $k/n \to 0$ *for* $k < n$ *and* $n \to \infty$. *Then we have for* $n \to \infty$

$$\sqrt{k}\left(\|(x_1, \dots, x_d)\|_D - \|(x_1, \dots, x_d)\|_{\hat{D}}\right) \to_{\mathcal{D}} B(x_1, \dots, x_d)$$

in $D([0,T]^d)$ *for every* $T > 0$ *and* $\boldsymbol{x} \in \mathbb{R}^d_+$, *where*

$$B(x_1, \dots, x_d) = W(x_1, \dots, x_d) - \sum_{j \leq d} L_j(x_1, \dots, x_d) W(x_j \boldsymbol{e}_j)$$

and W *is a continuous Gaussian process with mean zero and covariance structure* $E(W(x_1, \dots, x_d)W(y_1, \dots, y_d)) = \mu\left(R(x_1, \dots, x_d) \cap R(y_1, \dots, y_d)\right)$ *with* $R(x_1, \dots, x_d) := \left\{(u_1, \dots, u_d) \in \mathbb{R}^d_+ : \bigcup_{j \leq d}\{0 \leq u_j \leq x_j\}\right\}$. *Thereby,* μ *is defined by means of the exponent measure* ν *(cf. Proposition 2.1.3), i.e* $\mu := \nu * T$ *defined by*

$$\mu(A) = \nu(T^{-1}(A)) \tag{5.4}$$

with $T : x \mapsto -x$ *for* $x \leq 0$ *and* $A \subseteq [0, \infty]^d \backslash \{\boldsymbol{\infty}\}$.

Proof: Cf. Theorem 7.2.2 in de Haan and Ferreira [29]. □

Further information about results of Theorems 5.1.1 and 5.1.2 can be found in de Haan and de Ronde [33], Section 5.1. Of special interest regarding Theorem 5.1.2 is the recent work of Einmahl et al. [17]. Theorem 4.6 therein states, that asymptotic normality of the nonparametric estimator $\|\cdot\|_{\hat{D}}$ for the stable tail dependence function holds under a weaker smoothness condition on $\|\cdot\|_D$, namely $L_j(\boldsymbol{x})$ is continuous on a certain set of points instead of the whole interval $[0, \infty]^d$. Furthermore, they succeed to provide a parametric estimator for the stable tail dependence function in arbitrary dimensions, which exhibits asymptotic normality without any differentiability conditions on $\|\cdot\|_D$.

Note that the basic element of the extended fragility index is

$$\varepsilon_K := \left\|\sum_{j \in K} \boldsymbol{e}_j\right\|_D, \quad \text{for} \quad K \subseteq \{1, \dots, d\}, \tag{5.5}$$

cf. Theorem 4.2.10. Thereby ε_K is the extremal coefficient of the $|K|$-dimensional margin $G_K(\boldsymbol{x}) = G(\sum_{j\in K} x_j \boldsymbol{e}_j)$ of an EVD G. Recall the considerations on the extremal coefficient; cf. Definition 2.2.4 and the following discussion there. Using the nonparametric estimator for the stable tail dependence function, as given in (5.3), we obtain

$$\hat{\varepsilon}_K := \left\| \sum_{j\in K} \boldsymbol{e}_j \right\|_{\hat{D}} := \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}_{\{\bigcup_{j\in K}\{Y_i^{(j)} \geq Y_{n-k+1,n}^{(j)}\}\}} \tag{5.6}$$

as a nonparametric estimator for the extremal coefficient corresponding to the EVD G_K. Applying Theorem 5.1.1 and 5.1.2 we obtain consistency and asymptotic normality for the extremal coefficient estimator $\hat{\varepsilon}_K$ by considering the special case $x_j = 1$ for $j \in K$. For the sake of simplicity, we restrict ourselves to the case $K = \{1, \ldots, m\}$. Therefore, denote by $\boldsymbol{e}_j$ the j-th unit vector in $\mathbb{R}^m$.

Corollary 5.1.3 *Denote by $W(\mathbf{1})$ the Gaussian process and $L_j(\mathbf{1}), j \leq d$, as defined in Theorem 5.1.2 at point $\mathbf{1} := \sum_{j\leq m} \boldsymbol{e}_j$. Under the conditions of Theorem 5.1.2, we get for $n \to \infty$*

$$\sqrt{k}\,(\varepsilon - \hat{\varepsilon}) \to_{\mathcal{D}} B(\mathbf{1}),$$

where

$$B(\mathbf{1}) = W(\mathbf{1}) - \sum_{j\leq m} L_j(\mathbf{1}) W(\boldsymbol{e}_j). \tag{5.7}$$

In the specific situation of Corollary 5.1.3 we are able to specify the normal distribution in (5.7).

Recall that we have $\|\boldsymbol{x}\|_D = \nu\{[-\boldsymbol{\infty}, \boldsymbol{x}]^c\}$ for $\boldsymbol{x} \leq \mathbf{0} \in \mathbb{R}^m$, cf. Theorem 2.1.9, where ν denotes the exponent measure, cf. proposition 2.1.3. More precisely, we have

$$\|\boldsymbol{x}\|_D = \nu\left(\left\{\boldsymbol{s} \in [-\infty, 0]^m \backslash \{-\boldsymbol{\infty}\} \,:\, \bigcup_{j\leq m}\{s_j > x_j\}\right\}\right)$$

for $\boldsymbol{x} \leq \mathbf{0}$. With the definition of the measure μ in (5.4) we have

$$\mu\left(\left\{(s_1, \ldots, s_m) \in [0, \infty]^m \backslash \{\boldsymbol{\infty}\} \,:\, \bigcup_{j\leq m}\{s_j < x_j\}\right\}\right)$$

$$:= \nu\left(T^{-1}\left(\left\{(s_1,\dots,s_m)\in[0,\infty]^m\backslash\{\boldsymbol{\infty}\}\,:\,\bigcup_{j\le m}\{s_j<x_j\}\right\}\right)\right)$$
$$= \nu\left(\left\{(s_1,\dots,s_m)\in[-\infty,0]^m\backslash\{-\boldsymbol{\infty}\}\,:\,\bigcup_{j\le m}\{s_j>x_j\}\right\}\right).$$

Hence this implies

$$\|\boldsymbol{x}\|_D = \mu\left(\left\{(s_1,\dots,s_m)\in[0,\infty]^m\backslash\{\boldsymbol{\infty}\}\,:\,\bigcup_{j\le m}\{s_j<x_j\}\right\}\right)$$

for $\boldsymbol{x}\ge\boldsymbol{0}$.

With the definition of the covariance structure of the Gaussian process W as given in Theorem 5.1.2, we get

$$\begin{aligned}Var(W(\mathbf{1})) =& E(W(\mathbf{1})W(\mathbf{1})) = \mu(R(\mathbf{1}))\\ =&\mu\left(\left\{\boldsymbol{u}\in\mathbb{R}^m_+\,:\,\bigcup_{j\le m}\{0\le u_j\le 1\}\right\}\right) = \|\mathbf{1}\|_D=\varepsilon.\end{aligned}$$

Further we have

$$\begin{aligned}Var(W(\boldsymbol{e}_j)) =& E(W(\boldsymbol{e}_j)W(\boldsymbol{e}_j)) = \mu(R(\boldsymbol{e}_j))\\ =&\mu\left(\left\{\boldsymbol{u}\in\mathbb{R}^m_+\,:\,\{0\le u_j\le 1\}\cup\bigcup_{i\ne j}\{u_i=0\}\right\}\right) = \|\boldsymbol{e}_j\|_D=1,\\ E(W(\mathbf{1})\cdot W(\boldsymbol{e}_j)) =& \mu(R(\boldsymbol{e}_j))=1,\end{aligned}$$

and

$$\begin{aligned}&E(W(\boldsymbol{e}_i)\cdot W(\boldsymbol{e}_j)) = \mu(R(\boldsymbol{e}_i)\cap R(\boldsymbol{e}_j))\\ &= \mu(R(\boldsymbol{e}_i))+\mu(R(\boldsymbol{e}_j))-\mu(R(\boldsymbol{e}_i)\cup R(\boldsymbol{e}_j))\\ &= 1+1-\mu\left(\left\{\boldsymbol{u}\in\mathbb{R}^m_+\,:\,\{0\le u_i\le 1\}\cup\{0\le u_j\le 1\}\cup\bigcup_{k\ne i,j}\{u_k=0\}\right\}\right)\\ &= 2-\|\boldsymbol{e}_i+\boldsymbol{e}_j\|_D = 2-\varepsilon_{\{i,j\}}\,.\end{aligned}$$

Hence we get

$$Var\left(W(\mathbf{1})-\sum_{j\le m}L_j(\mathbf{1})W(\boldsymbol{e}_j)\right)$$

$$
\begin{aligned}
&= E\left(\left(W(\mathbf{1}) - \sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j) - E\left(W(\mathbf{1}) - \sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j)\right)\right)^2\right)\\
&= E\left(\left(W(\mathbf{1}) - \sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j)\right)^2\right)\\
&= E\left(W^2(\mathbf{1}) - 2\cdot W(\mathbf{1})\sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j) + \left(\sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j)\right)^2\right)\\
&= Var(W(\mathbf{1})) - 2\cdot\sum_{j\le m} L_j(\mathbf{1})E(W(\mathbf{1})W(\boldsymbol{e}_j))\\
&\qquad + \sum_{i\le m}\sum_{j\le m} L_i(\mathbf{1})L_j(\mathbf{1})E(W(\boldsymbol{e}_i)W(\boldsymbol{e}_j))\\
&= \varepsilon - 2\cdot\sum_{j\le m} L_j(\mathbf{1}) + \sum_{i\le m}\sum_{\substack{j\le m\\ j\ne i}} L_i(\mathbf{1})L_j(\mathbf{1})\left(2-\varepsilon_{\{i,j\}}\right)\\
&\qquad + \sum_{j\le m} L_j^2(\mathbf{1})Var(W(\boldsymbol{e}_j))\\
&= \varepsilon + \sum_{j\le m}\left(L_i^2(\mathbf{1}) - 2L_i(\mathbf{1})\right) + \sum_{i\le m}\sum_{\substack{j\le m\\ j\ne i}} L_i(\mathbf{1})L_j(\mathbf{1})\left(2-\varepsilon_{\{i,j\}}\right)\,.
\end{aligned}
$$

We finally obtain

$$
W(\mathbf{1}) - \sum_{j\le m} L_j(\mathbf{1})W(\boldsymbol{e}_j) =_{\mathcal{D}} N(0,\sigma), \tag{5.8}
$$

with

$$
\sigma^2 := \varepsilon + \sum_{j\le m}(L_j^2(\mathbf{1}) - 2L_j(\mathbf{1})) + \sum_{j\le m}\sum_{\substack{i\le m\\ i\ne j}} L_j(\mathbf{1})L_i(\mathbf{1})(2-\varepsilon_{i,j})\,, \tag{5.9}
$$

cf. Section 7.4. in de Haan and Ferreira [29].

The fragility index can be regarded as a tail dependence coefficient based on the extremal coefficient, see Section 4.3. With the nonparametric estimator in (5.6), we obtain an estimator for the extended fragility index $FI(m), m \le d$, of the random system $\{Q_1,\dots,Q_d\}$ via

$$
\hat{FI}(m) := \frac{\sum_{k=m}^{d} k\hat{p_k}}{\sum_{k=m}^{d}\hat{p_k}} \tag{5.10}
$$

with

$$\hat{p_k} := 1/\hat{\varepsilon} \cdot \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subseteq \{1,\dots,d\} \\ |T|=d-j}} \hat{\varepsilon}_T \,,$$

cf. Theorem 3.3.12 for the definition of p_k and Remark 3.3.13 for the representation of p_k by the extremal coefficients ε_T. The fragility index as provided in Corollary 4.1.4 is given by

$$\hat{FI} := \hat{FI}(1) = \frac{d}{\hat{\varepsilon}} \tag{5.11}$$

Note that the above estimator for the (extended) fragility index works under the approach of an individual threshold for every component Q_j of the random system $(Q_1, \dots, Q_d\}$ or within the situation of identically distributed and tail equivalent components respectively (cf. the discussion in Section 4.2.3). However, in order to obtain the individual thresholds, one is able to estimate the univariate margins by a parametric POT-approach (cf. Chapter 7 in Embrechts et al. [13] for an appealing summary), or a nonparametric approach using the empirical distribution function, if the tail size is chosen not too small.

For the specific case $m = 1$, i.e. the fragility index FI, we can derive the following asymptotic distribution result for FI. Under the assumption that the estimator $\hat{FI}$ is consistent, i.e. the convergence $\hat{FI} \to FI$ holds in probability, we obtain

$$\sqrt{k}\left(\hat{FI} - FI\right) \to_{\mathcal{D}} N\left(0, \frac{d\sigma}{\varepsilon^2}\right) . \tag{5.12}$$

This is due to Cramer‘s Delta method, cf. Theorem A.7, which can be seen as follows. Consider the function $h : [1, d] \to [1, d]$ defined by $x \mapsto \frac{d}{x}$ for $d \in \mathbb{N}$. Note that h is continuous in x and

$$\frac{\partial h}{\partial x}(\varepsilon) = \frac{(-1)d}{\varepsilon^2} \neq 0 \,.$$

Hence, the assertion follows by means of Theorem A.7.

Above provided properties of the estimator for the extremal coefficient and the derived fragility index will be investigated by means of a simulation study, cf. Section 5.2. Therein we also provide simulations of the estimator‘s distribution for the extended fragility index $FI(m)$, $m \le d$. However, we are not able to provide theoretical results on its asymptotic behavior except in the case of $m = 1$.

5.2. Simulation Study

This section aims to show simulation results for the estimator of the extremal coefficient and the fragility index provided in Section 5.1. Further we show simulation results for the extended fragility index, although we do not provide theoretical results concerning consistency and normality. This shall serve as a first insight into the properties of the nonparametric estimator for the extended fragility index established in (5.10) with respect to its mean squared error and its distribution according to sample and tail size. Section 5.2.1 refers to the simulation technique for the extremal coefficient and the (extended) fragility index respectively. Results of the simulation study are provided in Section 5.2.2 and Section 5.2.3 aims to suggest a choice for an optimal tail fraction to which the estimation of the extremal coefficient is applied to.

5.2.1. Simulation of the estimators distribution

We simulate samples of size n of three-dimensional random vectors coming from an EVD with logistic dependence structure and Fréchet margins with parameters $\xi_j = 0.5$, $\mu_j = 0$ and $\beta_j = 1$ for $j \leq d$ according to the Jenkinson-von Mises representation of an univariate EVD (cf. Definition 2.1.11). Although the extremal coefficient is invariant under transformation of the univariate margins of an EVD, we have to choose the type of simulated margins. With respect to further application to financial data, which mostly exhibit heavy tails, we therefore simulate Fréchet margins. Hence we simulate from the multivariate EVD

$$G(\boldsymbol{x}) = \exp\left(-\left(\sum_{j\leq d}(x_j^{-2})^{\lambda}\right)^{1/\lambda}\right) \tag{5.13}$$

with $\lambda \in [1;\infty]$, hence the logistic EVD, cf. Example 2.1.14. The copula belonging to an EVD of logistic type is just the Gumbel copula (cf. Example 2.3.5). Thereby note that some authors, e.g. Stephenson [67], refer to the logistic EVD and corresponding copula by means of (5.13) with dependence parameter $\vartheta := 1/\lambda \in (0,1]$. The multivariate logistic model with standard Fréchet margins goes back to Gumbel [28]. For the generation of EVD of the logistic type we use an algorithm provided by Stephenson [67]. It is based on the so-called *Shi transformation*, which is given by $(X_i)^{-1/\vartheta} \mapsto (X_iV)^{-1/\vartheta}$

with $V := \left(\sum_{j\le d} X_j^{-1/\vartheta}\right)^{\vartheta}$ for $j \le d$, i.e. the transformation of standard Fréchet margins to the $d-1$ dimensional unit simplex (cf. Shi [59]). In the framework of logistic dependence structure, the Shi transformation is also used by Michel [51], who provided an algorithm in order to simulate from a multivariate generalized Pareto distribution of logistic type.

For each simulated sample from the logistic EVD we apply the nonparametric estimator for the extremal coefficient ε_K as defined in Section 5.1, i.e.

$$(5.14) \qquad \hat{\varepsilon}_K := \left\| \sum_{j\in K} \boldsymbol{e}_j \right\|_{\hat{D}} := \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}_{\{\bigcup_{j\in K}\{Y_i^{(j)} \ge Y_{n-k+1,n}^{(j)}\}\}},$$

which is the corresponding extremal coefficient (cf. Definition 3.1.2) to each marginal distribution G_K of G. We repeat this simulation and estimation procedure m times. Hence, we obtain an empirical distribution of the estimator $\hat{\varepsilon}_K$ of size m. Then we compute the estimator for the fragility index and the extended fragility index via (5.11) and (5.10) based on the estimation results of $\hat{\varepsilon}_K$. Hence we also obtain an empirical distribution of each of the estimators for $FI(m)$.
To investigate the influence on the size of the tail (denoted by γ), the amount of dependence (denoted by λ) and the sample size of simulated EVDs (denoted by n) on the estimation results, we took simulations of various combinations of the parameters γ, λ and n.

Since the nonparametric estimator $\hat{\varepsilon}_K$ depends on the number of observations in the tail, denoted by k in (5.14), we chose a set of increasing tail fractions $\gamma_s := k_s/n, s \in S$. The choice of the tail was determined by k/n heavily influences consistency and asymptotic normality of the estimator (cf. Theorem 5.1.1 and 5.1.2). The smaller γ_s, the more variability will be inherent in the estimator due to less observations, but on the other hand, it should show higher accuracy the smaller the tail, due to the construction of the estimator serving for estimating *tail*-dependence. This is the well-known trade-off in the framework of tail estimation. With regard to the question, "Which tail fraction is an *optimal* choice in the face of above mentioned trade-off?" we refer to Section 5.2.3, where we investigate the mean squared error of the estimator.

5.2.2. Simulation Results

This section represents the simulation results of the nonparametric estimator of the extremal coefficient provided in (5.3) and the derived estimator for the (extended) fragility index (cf. 5.10 in Section 5.1).
In doing so, our simulation study aims to provide the empirical distribution of the nonparametric estimator for the extremal coefficient $\varepsilon := \|(1,1,1)\|_D$ (cf. Definition 2.2.4 and Section 2.4). The simulated data, to which the nonparametric estimation procedure is applied to, come from a logistic EVD with dependence parameter λ, cf. (5.13). Hence we wish to estimate the true underlying extremal coefficient $\|(1,1,1)\|_\lambda$ by means of the nonparametric estimator $\hat{\varepsilon} := \|(1,1,1)\|_{\hat{D}}$ provided in (5.3). In the following we refer to this estimator by $\hat{\varepsilon}$.

Within this section we present figures of boxplots of the nonparametric estimator for the extremal coefficient and the (extended) fragility index. We decided to represent and discuss the simulation results by means of boxplots, since they easily summarize the empirical distribution of the estimator in contrast to the use of tables representing basic statistics like sample mean and standard deviation, which may lie above our capacity for the huge amount of presented information. The boxplots correspond to the combination of simulation parameters like simulated sample size n, tail fraction γ and dependence parameter λ, $\varepsilon = \|(1,1,1)\|_\lambda$ respectively, of the logistic model. For sake of simplicity, we shorten "simulated sample size" by "sample size" and ask the reader not to confuse the sample size of the datasets (i.e. the number of simulated rv coming from a logistic EVD), which is denoted by n, with the sample size underlying each boxplot, which is fixed to $m = 10000$, i.e. the number of datasets generated from the mentioned logistic EVD.

First have a look at Figure 5.1, which represents the estimation results corresponding to $\lambda = 1.7$, which reflects a medium amount of dependence, since $\|(1,1,1)\|_{1.7} \approx 1.9$ (cf. 2.15). Each *floor* represents the boxplot's corresponding level to a specific choice of sample size n, here we choose $n = 500, 1000, 2500, 5000, 10000$ and 20000, i.e. we see six floors. Within one floor each step in the boxplot corresponds to a specific choice of tail fraction, here we choose $\gamma = 0.002, 0.004, 0.006, 0.008, 0.01, 0.02, 0.03, 0.04, 0.05, 0.075, 0.1, 0.2, 0.3$, i.e. each
floor consists of 13 steps. Each boxplot shows the dataset of $m = 10000$ replicates of the

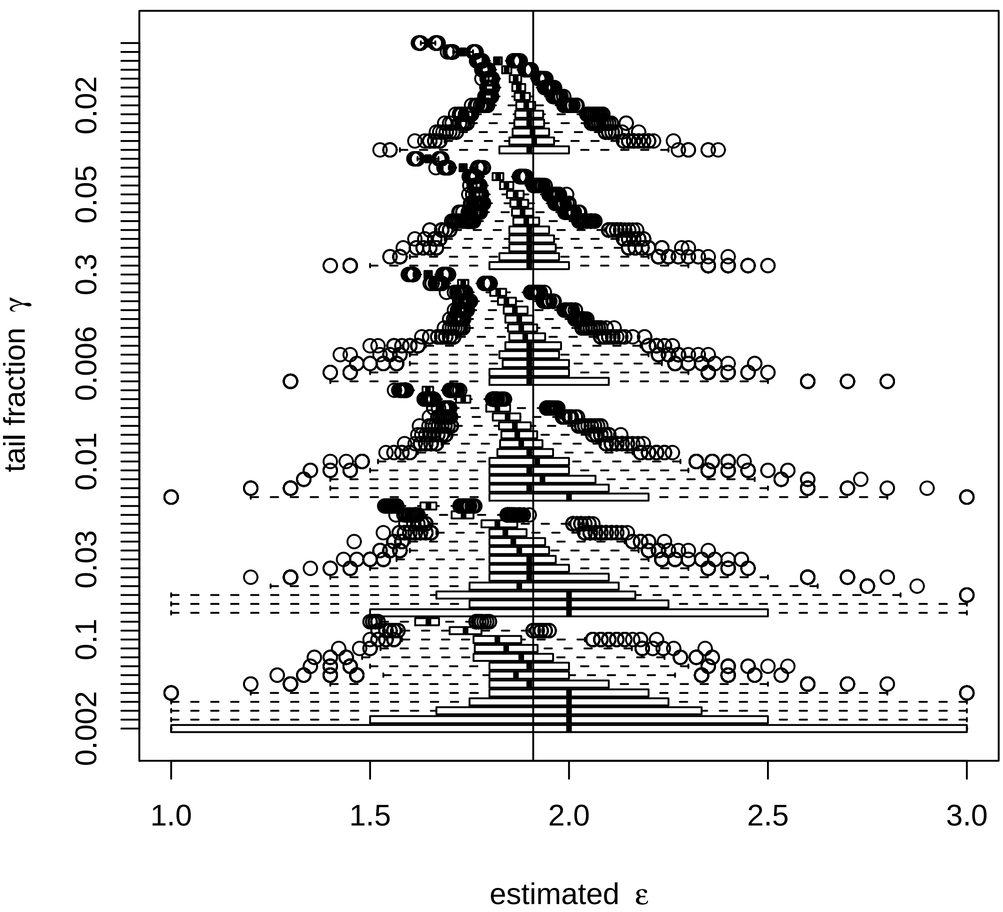


Figure 5.1.: Shown is the wave plot corresponding to the estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$ (represented by the vertical line). The floors from the bottom to the top show the increasing sample sizes $n = 500, 1000, 2500, 5000, 10000, 20000$. Boxplots are also grouped by tail fractions γ. The data shown in boxplots represent the simulation results of the nonparametric estimation approach for the extremal coefficient by means of the estimator in (5.3), cf. Section 5.1.

estimator $\hat{\varepsilon}$ corresponding to varying combinations of λ, n and γ. Thereby, the median is represented by the line within the box, the sample mean (if shown) is visualized by an asterisk and observations beyond the whiskers are represented by circles (whiskers are defined by $[x_{25\%} - 1.5 \cdot \mathrm{IQR}]$ and $[x_{75\%} + 1.5 \cdot \mathrm{IQR}]$ respectively, where $x_{25\%}$ and $x_{75\%}$

define the lower and upper quartile and IQR denotes the inter quartile range).

We observe that for increasing sample sizes n the variance of $\hat{\varepsilon}$ decreases, this is also true for the increasing tail fractions γ within each floor of the figure. With respect to the standard deviation, the same behavior is shown by the boxplots corresponding to $\lambda = 1.1$ and $\lambda = 3$ (cf. Figure C.1 and C.2 in Chapter 7, Section C).
The vertical line through Figure 5.1 represents the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$. The larger the tail fraction, the more bias arises by the use of the nonparametric estimator $\hat{\varepsilon}$. We observe that for too large tail fractions, this bias leads to complete underestimating of the true underlying value of the extremal coefficient. Thereby the tail fraction γ, which leads to bias of the estimator, depends on the sample size n of the underlying dataset. For example, for $n = 500$ we observe bias to the left for tail fractions of about $\gamma \geq 0.05$, whereas for $n = 5000$ bias already occurs for about $\gamma \geq 0.02$. In the following we explain this fact.
However, we know from Theorem 5.1.1 that $\hat{\varepsilon}$ is asymptotically unbiased, even consistent. The crucial requirement under which this is true regards to the convergence rate of the sequences n and corresponding tail size $k(n)$. Indeed we have the requirement $k = k(n) \to \infty, k/n \to 0$ for $k < n$ and $n \to \infty$. This means that k crucially depends on n with the restriction that the convergence rate of k is of a lower rate than that of a linear function. In view of Theorem 5.1.1 one is advised to choose $\gamma := k/n$ the smaller the higher the sample size n is chosen, for example choose $k := \sqrt{n}$. We will extensively discuss the choice of k later on, especially in Section 5.2.3.
With respect to the described behavior of $\hat{\varepsilon}$, we call such a plot a *wave plot*.
We also want to remark that the extreme bias, which we observe for large tail fractions, seems to be reduced by increasing values for the dependence parameter λ (cf. Figure C.3 in Section C for a representative choice of $n = 2500$).
Hitherto we discussed the behavior of $\hat{\varepsilon}$ with respect to its variance and bias; more precisely we investigated the consistency of $\hat{\varepsilon}$ by means of the so-called *mean squared error* (MSE). Since the MSE will play a crucial role within the question of which tail size k should be chosen in order to reduce variance and bias, i.e. the MSE, we will discuss the MSE of $\hat{\varepsilon}$ separately and extensively in Section 5.2.3.

Now we want to continue with discussing the skewness of the estimator. Therefore we representatively choose the wave corresponding to $n = 2500$ of Figure 5.1.

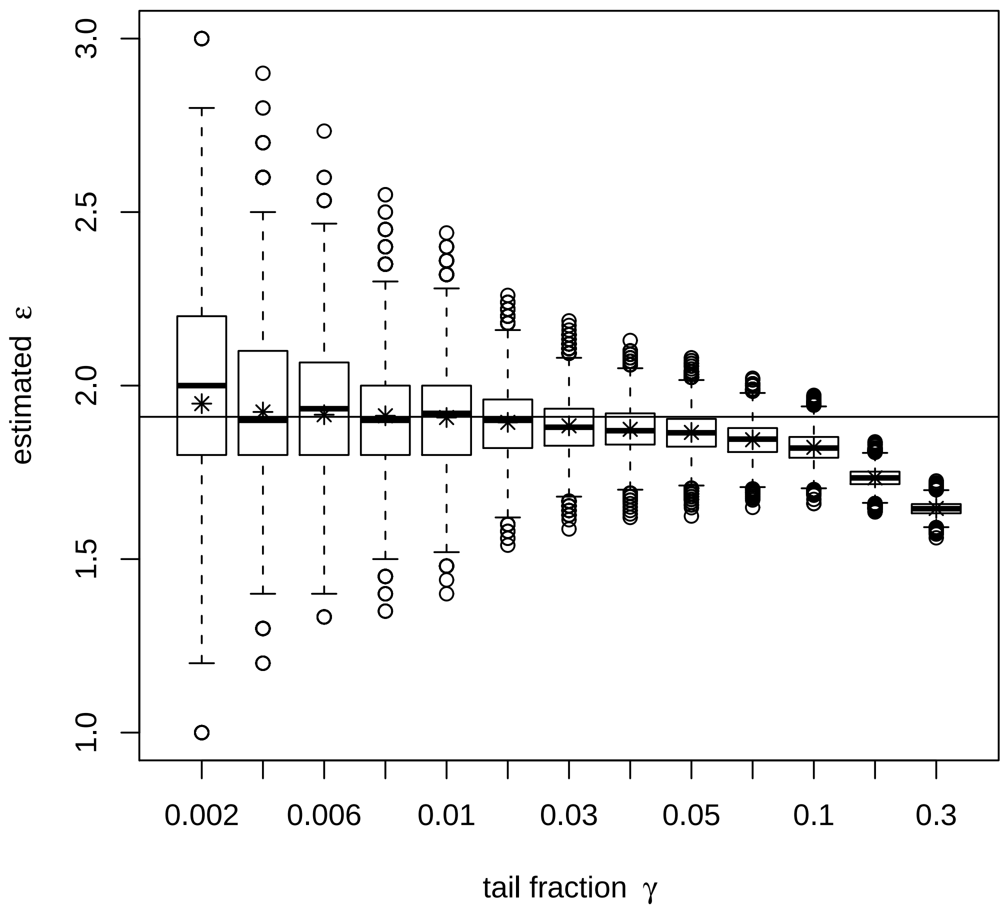


Figure 5.2.: Shown are the boxplots of the estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$ and $n = 2500$. This is the wave corresponding to $n = 2500$ of Figure 5.1.

The boxplots of Figure 5.2 show an increasing goodness of approximation of the sample mean (represented by an asterisk) to the median with increasing tail fraction γ. Thereby the minimal value of γ at which the mean fits the median decreases with increasing sample size n (cf. the represented selection of n in Figure C.4 and C.5 in Section C). Further, Figure 5.2, C.4 and C.5 show that in general neither a tendency to skewness to the left nor a skewness to the right can be observed. But in general, skewness vanishes for increasing tail size.

Furthermore, we want to mention that the number of outliers, defined as those observations that lie beyond the whiskers of the boxplot, amounts at most to about 1 % of the data for arbitrary combination of λ, n and γ.

Lastly, we want to check the property of (asymptotic) normality of the estimator $\hat{\varepsilon}$, since this result is provided in Theorem 5.1.2 and Corollary 5.1.3 respectively. For this we use normal probability plots (NP plots) of the simulated samples of $\hat{\varepsilon}$ additionally to the above considerations regarding skewness and the number of outliers.

Figure 5.3 shows NP plots of $\hat{\varepsilon}$ for sample size $n = 2500$ and a representative selection of tail fractions γ. Thereby the solid line connects the first and third quartile of the dataset, where the dashed line represents the line, which shall be approximated by the NP plot under the validity of normality of $\hat{\varepsilon}$ (see later). First note that the NP plots indicate a very good fit to a linear function, which shall not let us doubt in the assumption of normality. Of course, the smaller the tail fraction γ the more ties appear in the dataset of $m = 10000$ replicates of $\hat{\varepsilon}$, because $\hat{\varepsilon}$ is then based on the very small number $k := \gamma \cdot n$ of tail observations. This deficiency is displayed by the occurrence of steps within the NP-plot. Furthermore, the interpretation of the NP plots is done based on a finite number of observations, hence we have to neglect the violation of the continuity property of the normal distribution. Anyway, we are only able to check the normality assumption at finiteness in order to investigate the influence of sample size n and tail fraction γ on the goodness of fit of the corresponding empirical distribution of $\hat{\varepsilon}$ to the normal distribution. Based on the theoretical asymptotic results of Theorem 5.1.1 and 5.1.2, the estimator $\hat{\varepsilon}$ in (5.3) is approximately normal distributed with mean value $\mu_\varepsilon = \varepsilon$ and standard deviation $\sigma_\varepsilon = \sigma/\sqrt{k}$ where σ is given in (5.9) and $k := \gamma \cdot n$ (cf. Corollary 5.1.3). Of course, the approximation is as better as larger n and k with respect to the sufficient condition $k/n \to 0$. The first two NP plots of Figure 5.3 show an intercept μ_ε of the line $y = \mu_\varepsilon + \sigma_\varepsilon \cdot x$ which is satisfyingly close to $\varepsilon = 1.9$ under the validity of $\hat{\varepsilon} \sim N(\mu_\varepsilon, \sigma_\varepsilon)$. This is not true for the last two NP plots of Figure 5.3, which correspond to the larger tail fractions $\gamma = 0.075, 0.2$, which exhibit a noteworthy bias of the estimator. The same holds for NP plots corresponding to $\varepsilon = 1.4, 2.7$ (figures are not shown).

Table 5.1 shows a listing of the absolute relative deviation $s_{sd,\sigma_\varepsilon}$ between the empirical standard deviation of $\hat{\varepsilon}$, denoted by sd, and the approximative standard deviation $\sigma_\varepsilon =$

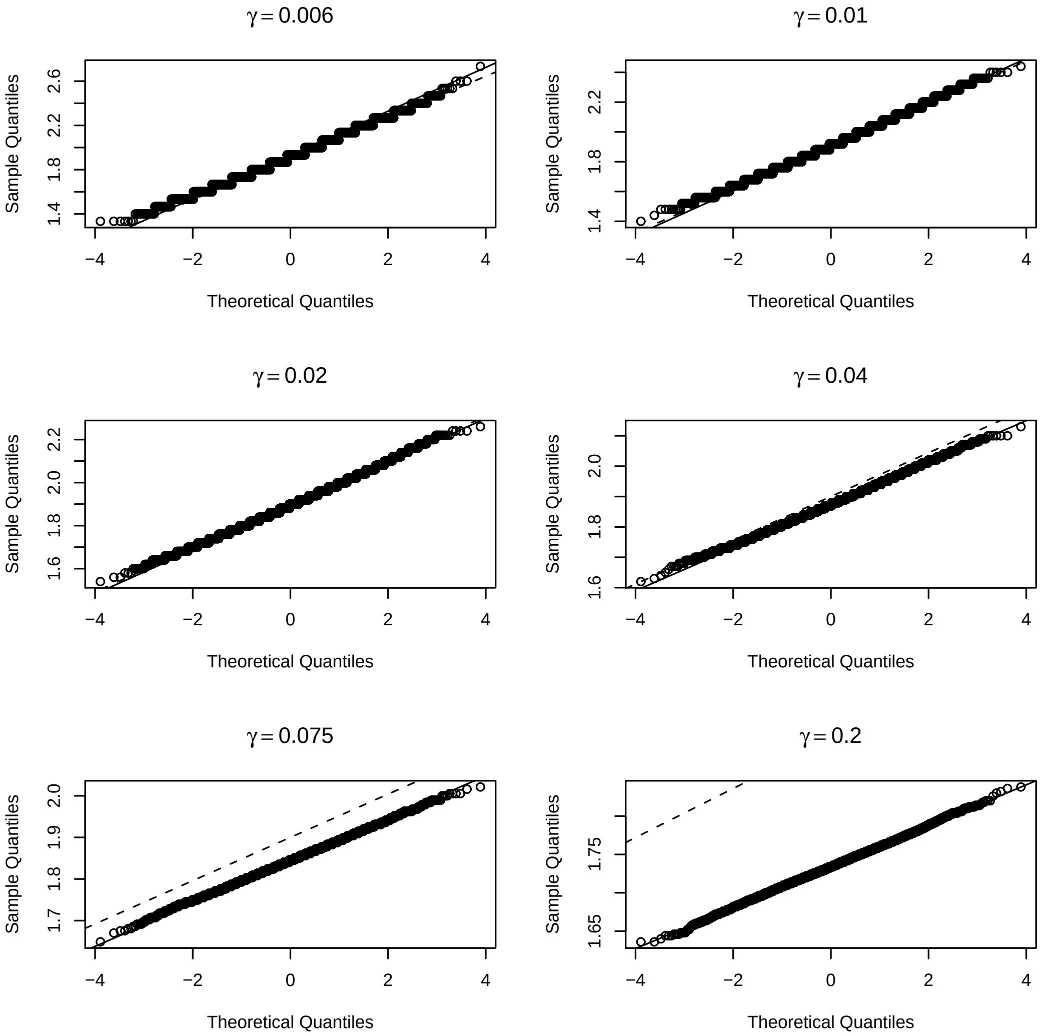


Figure 5.3.: The figure shows the NP plots of the estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$ for sample size $n = 2500$ and a representative selection of γ. The solid line connects the first and third quartile of the dataset. The dashed line represents the line $y = \mu_\varepsilon + \sigma_\varepsilon \cdot x$, where $\mu_\varepsilon = \varepsilon$ and $\sigma_\varepsilon = \sigma/\sqrt{k}$ with σ is given in (5.9) and $k := \gamma \cdot n$. We expect $\hat{\varepsilon} \sim N(\mu_\varepsilon, \sigma_\varepsilon)$.

$\sigma/\sqrt{k}$ of $\hat{\varepsilon}$ obtained by Corollary 5.1.3, i.e. $d_{sd,\sigma_\varepsilon} := |sd - \sigma_\varepsilon| / \sigma_\varepsilon$. These values are based on the simulation data of the extremal coefficient $\varepsilon = \|(1,1,1)\|_\lambda$ for $\lambda = 1.7$.
We summarize that the tail fraction at which the minimum of absolute relative deviation in $d_{sd,\sigma_\varepsilon}$ is obtained decreases with increasing sample size. If one wants to generalize this assertion with respect to the sample size $n \in [500; 20000]$ one may choose tail fractions

$\gamma \in [0.006; 0.1]$ in order to obtain as less values for $d_{sd,\sigma_\varepsilon}$. This is also true for the results corresponding to $\lambda = 1.1$ and $\lambda = 3$ (table is not shown). The NP plots of Figure 5.3 show a good fit to the required normal distribution (provided by Corollary 5.1.3) for tail fractions $\gamma \leq 0.02$.

We close this section with the representation of the boxplots corresponding to the fragility index FI and the extended fragility index $FI(2)$. Analogue to Figure 5.1, we provide the wave plots for the estimator $\widehat{FI}$ in (5.11) and $\widehat{FI}(2)$ in (5.10) corresponding to $\lambda = 1.7$ (cf. Figure C.6 and C.7 in Section C). The boxplots of Figure C.6 show a quite similar behavior as the boxplots of $\hat{\varepsilon}$ in Figure 5.1, which is not surprising, since we have $\widehat{FI} = d/\hat{\varepsilon}$. However, the boxplots of the estimator $\widehat{FI}(2)$ in Figure C.7 indicate that there is almost no bias of $\widehat{FI}(2)$, which is surprising in view of the observed bias of $\hat{\varepsilon}$ and $\widehat{FI}$.

Figure 5.4 represents the boxplots of $\widehat{FI}$ and $\widehat{FI}(2)$ corresponding to $\lambda = 1.7$ and the representative sample size $n = 2500$.

The boxplots show a very good approximation of the arithmetic mean to the true underlying values $FI \approx 1.57$ and $FI(2) \approx 2.57$ for tail fractions $\gamma \leq 0.02$. This is in line with the behavior of $\hat{\varepsilon}$. Significantly, for the estimator $\widehat{FI}(2)$, this approximation is still excellent for larger tail fractions up to $\gamma = 0.2$; hence there is almost no bias, which neither can be observed for $\hat{\varepsilon}$ nor $\widehat{FI}$.

Lastly we want to check the normal behavior of $\widehat{FI}$. We know from (5.12) that the estimator $\widehat{FI}$ is approximately normal distributed with mean value FI and standard deviation $(d\sigma)/(\varepsilon^2\sqrt{k})$, where σ is given in (5.9) and $k = n \cdot \gamma$. Figure 5.5 shows the NP plots of $\widehat{FI}$ corresponding to $\lambda = 1.7$ and $n = 2500$ for a representative selection of tail fractions γ. For $\gamma \leq 0.01$ the NP plots show a convex behavior, which votes against the assumption of normality. Skewness seems to vanish for larger tail fractions, say $\gamma \geq 0.02$, but unfortunately we then observe an increasing bias, since the NP plots show a worsening goodness of fit to the dashed line for increasing tail fractions γ.

	γ												
$\lambda = 1.7$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
500	0.112	0.034	0.008	0.001	0.001	0.005	0.014	0.018	0.049	0.075	0.08	0.161	0.246
1000	0.028	0.001	0.011	0.011	0.009	0.024	0.019	0.045	0.051	0.067	0.077	0.169	0.249
2500	0.004	0.000	0.009	0.006	0.000	0.011	0.031	0.035	0.046	0.062	0.071	0.156	0.272
5000	0.000	0.012	0.004	0.009	0.011	0.021	0.041	0.031	0.049	0.079	0.093	0.16	0.242
10000	0.006	0.009	0.014	0.012	0.007	0.011	0.031	0.021	0.031	0.081	0.115	0.187	0.234
20000	0.000	0.012	0.004	0.009	0.009	0.021	0.007	0.051	0.027	0.079	0.062	0.204	0.242

Table 5.1.: Shown is the absolute relative deviation between the empirical standard deviation of $\hat{\varepsilon}$ and the approximative standard deviation $\sigma_\varepsilon = \sigma/\sqrt{k}$ of $\hat{\varepsilon}$ corresponding to $\lambda = 1.7$, i.e. the entries of the table represent $d_{sd,\sigma_\varepsilon} := |sd - \sigma_\varepsilon| / \sigma_\varepsilon$. Tabled values of 0.000 correspond to those ones for which $d_{sd,\sigma_\varepsilon} < 5 \cdot 10^{-4}$ holds.

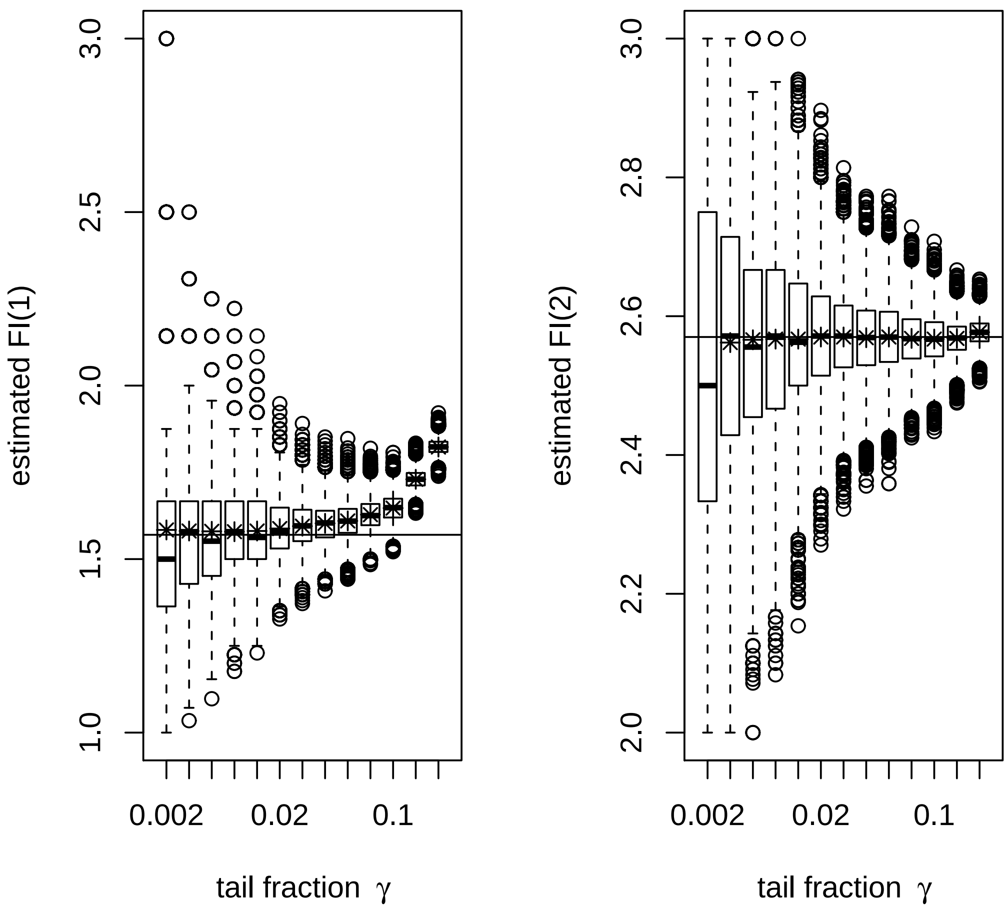


Figure 5.4.: Boxplots of the estimator of FI and $FI(2)$, provided in (5.10) and (5.11) respectively, corresponding to $\lambda = 1.7$ and $n = 2500$. The horizontal line represents the value $FI \approx 1.57$ respectively $FI(2) \approx 2.57$.

Hence, for $n = 2500$ and $\lambda = 1.7$, the choice $\gamma = 0.01$ can be a trade-off between the validity of the normal assumption and a negligible bias.
Figure C.8 shows the NP plots of $\widehat{FI}$ corresponding to $\lambda = 1.7$ and $n = 10000$. Here, for $n = 10000$, the skewness of $\widehat{FI}$ is already negligible for $\gamma \geq 0.008$ in contrast to $n = 2500$. For $\gamma \leq 0.01$ we observe no bias.

For the estimator of the extended fragility index $FI(2)$ we do not have any theoretical results concerning normality. In order to get an impression of a possible normal behavior

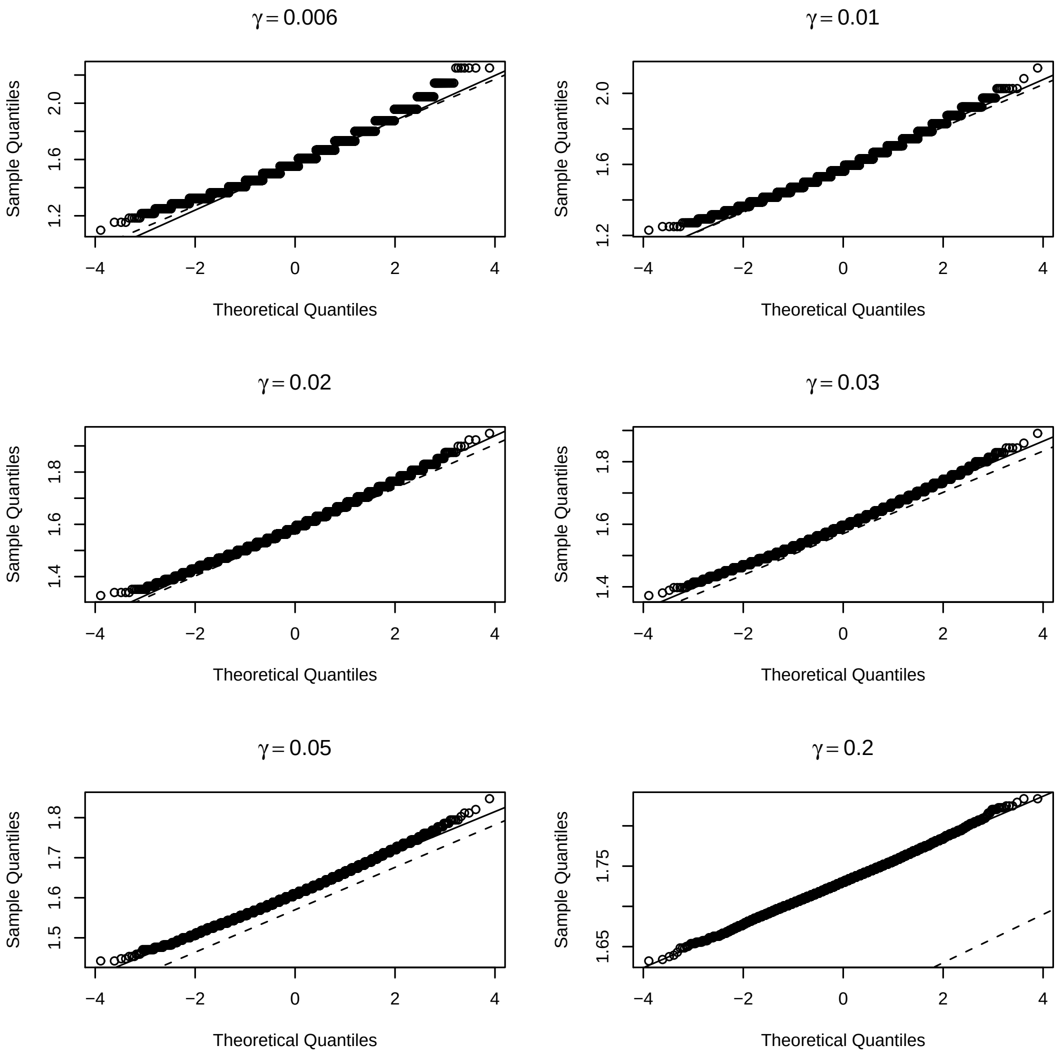


Figure 5.5.: NP plots of the estimator $\widehat{FI}$ corresponding to $\lambda = 1.7$ for sample size $n = 2500$ and a representative selection of γ. The solid line connects the first and third quartile of the dataset. The dashed line represents the line $y = \mu_\varepsilon + \sigma_\varepsilon \cdot x$, where $\mu_{FI} = FI \approx 1.57$ and $\sigma_{FI} = (d\sigma)/(\varepsilon^2\sqrt(k))$ with σ is given in (5.9) and $k := \gamma \cdot n$. We expect $\widehat{FI} \sim N(\mu_\varepsilon, \sigma_\varepsilon)$.

we once again look at NP plots. Figure 5.6 presents the NP plots of the estimator $\widehat{FI}(2)$ of the extended fragility index. The combination of $\lambda = 1.7$ and $n = 2500$ and a representative selection of γ is shown. We observe that the NP plots show a very good fit to the solid line even for very small tail fractions of $\gamma = 0.006, 0.008$.
The boxplots and NP plots of the (extended) fragility index corresponding to $\lambda = 1.1$

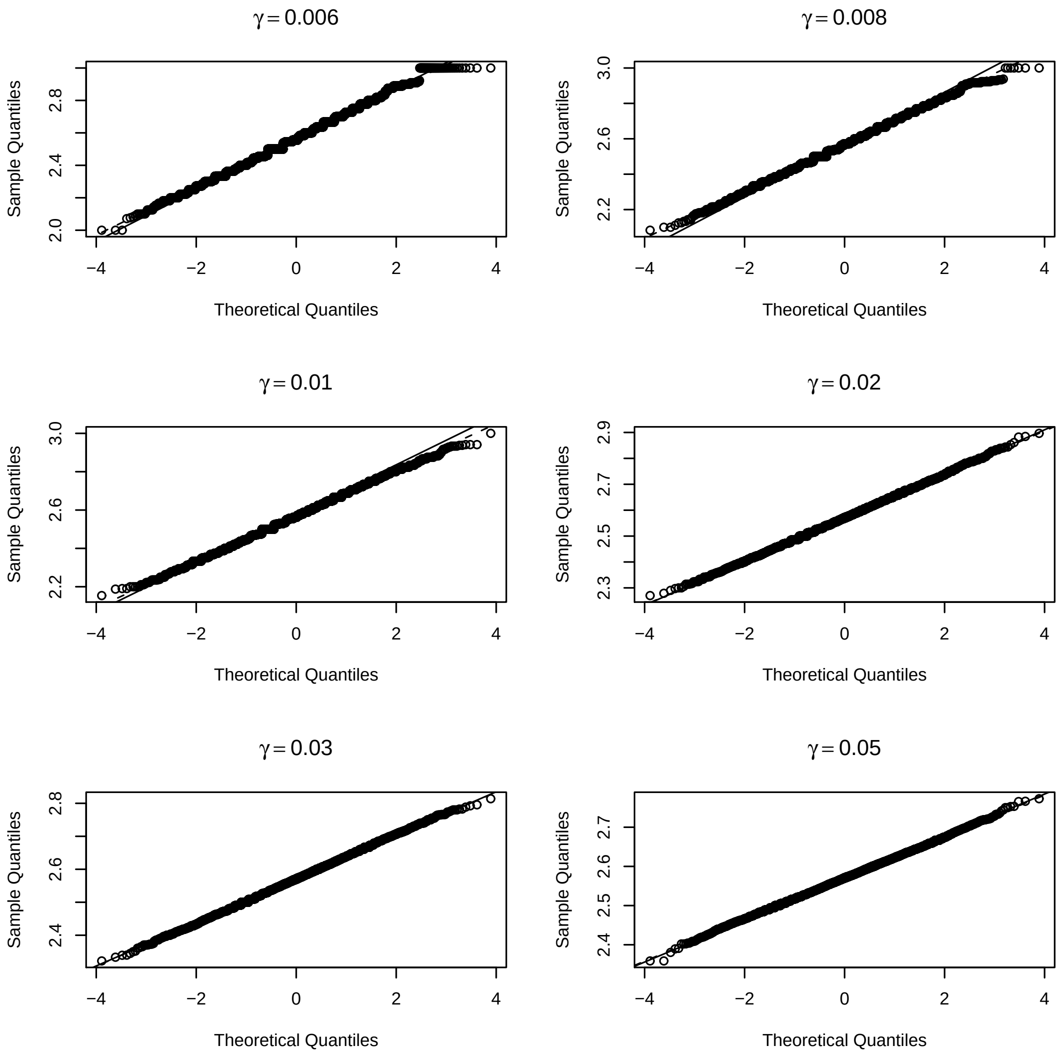


Figure 5.6.: NP plots of the estimator $\widehat{FI}(2)$ for sample size $n = 2500$ and a representative selection of γ. The solid line connects the first and third quartile of the dataset.

and $\lambda = 3$ behave similarly and are therefore not shown.

5.2.3. An optimal tail fraction

This section aims to provide an *optimal* choice for the tail fraction within the nonparametric estimation procedure for the extremal coefficient provided in Section 5.1. The accurate estimation namely depends on a precise choice of number of observations in the tail, to sufficiently allow for the trade-off between them, but only *tail* observations are taken into account.

One of the most important properties of a point estimator is its consistency. A sequence of estimators δ_n of $g(\delta)$ is called *consistent* if

$$\lim_{n\to\infty} P(|\delta_n - g(\delta)| < \varepsilon) = 1$$

holds for every $g(\delta)$. (More precisely, this is weak consistency, whereas strong consistency is almost surely convergence of δ_n to $g(\delta)$.)
The estimator $\hat{\varepsilon}$ in (5.3) for the extremal coefficient is consistent, cf. Theorem 5.1.1. But consistency in this case crucially depends on the condition that there exists a sequence $k(n)$ representing the size of the tail, for which we have $k = k(n) \to \infty, k/n \to 0$ for $k < n$ and $n \to \infty$. To investigate the influence of this condition on the consistency of $\hat{\varepsilon}$ we study the behavior of the so-called *mean squared error*

$$MSE(\delta_n) := E\left[(\delta_n - g(\delta))^2\right]$$

of the estimator δ_n. If $E\left[(\delta_n - g(\delta))^2\right] \to 0$ for all δ, then δ_n is a consistent estimator of $g(\delta)$ (cf. Theorem 8.2 in Section 1 of Lehmann and Casella [46] for example). Note that the MSE decomposes into the sum of the variance and the squared bias of an estimator, i.e. we have $MSE(\delta_n) = Var(\delta_n) + bias^2(\delta_n)$.

Within the simulation procedure established in Section 5.2.1 we also computed the MSE of the estimator $\hat{\varepsilon}$ based on the sample of estimators $\hat{\varepsilon}(n,\gamma)_1, \dots, \hat{\varepsilon}(n,\gamma)_m$ by means of

$$\widehat{MSE}\left(\hat{\varepsilon}(n,\gamma)\right) := \frac{1}{m}\sum_{i\le m}\left(\hat{\varepsilon}(n,\gamma)_i - \varepsilon\right)^2 \tag{5.15}$$

separately for each combination of the size of the data sets n and the tail fraction γ. Table 5.2 provides an overview of the (estimated) mean squared error of $\hat{\varepsilon}$. We observe that $\widehat{MSE}$ decreases with increasing sample sizes n for every tail fraction γ. Thereby the strength of the decrease of $\widehat{MSE}$ regarding n depends on γ, e.g. the decrease almost vanishes for $\gamma \geq 0.1$. This is not surprising, since we observe a huge bias of $\hat{\varepsilon}$ for larger tail fractions for which the bias is then balanced by small variance.

With respect to the MSE, the estimator $\hat{\varepsilon}$ in (5.3) mostly behaves as well as other nonparametric estimators of the dependence structure of an EVD. Based on a simulation study taken from Zhang et al. [71], the Pickands estimator, provided first in Pickands [54], shows a maximum MSE, which is in the same order of magnitude than the MSE of the estimator in (5.3) taken from Table 5.2. The new estimator provided in Zhang et al. [71], which is an extension of the bivariate CFG estimator (cf. Capéraà et al. [8] and Section 2.4) to the multivariate setting of an EVD, shows a maximum MSE, which is of order 10^{-1} regarding to the MSE of $\hat{\varepsilon}$. The same holds for the estimator of the Pickands dependence function provided by Gudendorf and Segers [27]. But note that the estimators provided in [27] and [71] are based on the EVD-assumption. The estimator in (5.3), which we used, serves as an estimator for tail dependence under the domain-of-attraction-assumption. Hence, the worse MSE of $\hat{\varepsilon}$ may be because estimation is done under the weaker domain-of-attraction-assumption. However, beside the disadvantage of $\hat{\varepsilon}$ in (5.3) with respect to a larger MSE, $\hat{\varepsilon}$ can also be applied in the framework of the domain-of-attraction-assumption, which is our existing framework regarding to the estimation of the (extended) fragility index.

In order to pay attention to the increasing bias regarding γ, Table 5.3 represents the estimator

$$\widehat{bias}^2\left(\hat{\varepsilon}(n,\gamma)\right) := \left(\frac{1}{m}\sum_{i\leq m}\hat{\varepsilon}(n,\gamma)_i \; - \; \varepsilon\right)^2 \tag{5.16}$$

for the squared bias of $\hat{\varepsilon}$ based on our simulation data. Table 5.3 clearly visualizes the dependence of the bias on the combination of sample size n and tail fraction γ. We observe that the minimum of bias is obtained at those tail fractions γ, which can be taken as smaller the larger the sample size n is considered. This fact is visualized by the belt of zeros (representing a bias, for which $bias^2 < 5 \cdot 10^{-5}$ holds), whose position also depends on the logistic dependence parameter λ of the simulated EVD.
To provide an *optimal* tail fraction γ for the nonparametric estimator $\hat{\varepsilon}$ in (5.3), one has to clarify the *definition* of optimality. We require:

An optimal γ should minimize the MSE of $\hat{\varepsilon}$, where the minimization of the bias should be of greater importance than the minimization of the variance.

Hence we have to take into account that the tail fraction γ at which the bias is minimized tends to be smaller than that γ at which the MSE is minimized in general. Unfortunately, the location of the belt of minimized MSE (typed in bold numbers, cf. Figure 5.2), as well as bias depends on the simulated logistic dependence parameter λ, which is unknown and its knowledge is precisely the aim of estimation given a real dataset.

Hence, first we assume that one is aware of an estimate for λ (hypothesized λ) given a certain dataset based on the assumption of goodness of fit of the logistic model. Then we suggest to take that tail fraction γ, which corresponds to the left boundary of the belt of minimized MSE of Table 5.2 with respect to a hypothesized λ and certain (known) n. For example, for a hypothesized $\lambda \approx 1.7$ and $n = 2500$ we suggest to take the optimal tail fraction of $\gamma = 0.03$. This shall be an acceptable guideline reflecting our forced criterion on optimality. Note that for this example we should have taken $\gamma = 0.05$ if we only focused on the minimization of the MSE with no special weight on the minimization of the bias.

Usually, when faced with a real dataset, one does not know the parameter λ reflecting the amount of tail dependence within the dataset. In this case, once again based on minimization of the MSE and with the main focus on the minimization of bias, we suggest to take that γ, which lies in the interval corresponding to the sample size n of the dataset at hand (cf. Table 5.4). Thereby, the lower the amount of assumed tail dependence in the dataset (e.g. based on analysis of dependence from previous historical data), the closer should be the choice of γ to the left boundary of the interval. Of course this is only a heuristic procedure and therefore can only serve as a rough guideline.

		γ												
	$\lambda = 1.1$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.2689	0.1488	0.1018	0.0815	0.0654	**0.0357**	**0.0276**	**0.0256**	**0.0272**	**0.0385**	0.0588	0.2003	0.4214
	1000	0.149	0.0797	0.0541	0.0425	0.0342	**0.019**	**0.0164**	**0.0172**	**0.0206**	0.0342	0.0557	0.1981	0.4199
n	2500	0.0645	0.0337	0.023	0.0173	0.0144	**0.0091**	**0.0095**	**0.012**	0.0158	0.0313	0.0526	0.197	0.4197
	5000	0.0334	0.0169	0.0117	**0.0088**	**0.0074**	**0.0056**	**0.007**	0.0104	0.0146	0.0303	0.0525	0.1968	0.4191
	10000	0.0167	0.0086	0.006	**0.0047**	**0.004**	**0.0039**	0.006	0.0094	0.0139	0.0299	0.052	0.1966	0.4194
	20000	0.0085	0.0044	**0.0031**	**0.0025**	**0.0023**	**0.0031**	0.0054	0.009	0.0136	0.0297	0.0518	0.1966	0.4192

	$\lambda = 1.7$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.6532	0.2803	0.1767	0.1297	0.1032	0.0506	**0.0332**	**0.0251**	**0.0198**	**0.0158**	**0.0157**	**0.0339**	0.0719
	1000	0.2791	0.1296	0.0878	0.063	0.0503	0.0245	**0.0166**	**0.0128**	**0.0108**	**0.01**	**0.0118**	0.0327	0.0703
n	2500	0.1047	0.0514	0.0335	0.0253	0.0205	0.0103	**0.0071**	**0.006**	**0.0058**	**0.0067**	0.0095	0.0317	0.0702
	5000	0.0511	0.0249	0.017	0.0125	0.0101	**0.0052**	**0.0038**	**0.0036**	**0.0038**	**0.0055**	0.0086	0.0314	0.0699
	10000	0.0252	0.0125	0.0083	0.0062	0.0052	**0.0028**	**0.0024**	**0.0025**	**0.0029**	0.0051	0.0082	0.0311	0.07
	20000	0.0128	0.0063	0.0042	0.0032	**0.0027**	**0.0016**	**0.0016**	**0.0019**	**0.0024**	0.0047	0.0081	0.0311	0.0699

	$\lambda = 3$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.4839	0.1905	0.1157	0.0804	0.0637	0.0286	0.0182	0.0133	0.0104	**0.0069**	**0.0057**	**0.0062**	0.0113
	1000	0.1985	0.082	0.0506	0.0368	0.0288	0.0138	0.0092	0.0067	**0.0054**	**0.0039**	**0.0033**	**0.0053**	0.0108
n	2500	0.0637	0.0293	0.0189	0.014	0.0113	0.0054	0.0037	**0.0028**	**0.0023**	**0.0019**	**0.0019**	0.0048	0.0105
	5000	0.0299	0.0142	0.0095	0.007	0.0055	0.0027	**0.0019**	**0.0014**	**0.0013**	**0.0012**	**0.0015**	0.0045	0.0104
	10000	0.0146	0.0072	0.0047	0.0035	0.0027	0.0014	**$9e-04$**	**$8e-04$**	**$8e-04$**	**$9e-04$**	0.0013	0.0045	0.0103
	20000	0.007	0.0034	0.0023	0.0017	0.0014	**$7e-04$**	**$5e-04$**	**$5e-04$**	**$5e-04$**	**$7e-04$**	0.0012	0.0044	0.0103

Table 5.2.: Shown is the estimated mean squared error $\widehat{MSE}(\hat{\varepsilon}(n,\gamma))$ of the nonparametric estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_\lambda$ for simulated logistic data with dependence parameter λ for increasing sample sizes n and tail fractions γ. Thereby the estimator in (5.15) is used.

	$\lambda = 1.1$	γ 0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.0046	9e-04	3e-04	0	1e-04	0.0015	0.0044	0.0079	0.0127	0.029	0.0515	0.1969	0.4195
	1000	9e-04	2e-04	0	1e-04	2e-04	0.0019	0.0044	0.0085	0.0133	0.0293	0.0521	0.1964	0.4189
n	2500	1e-04	0	1e-04	2e-04	4e-04	0.0021	0.0048	0.0084	0.0129	0.0294	0.0511	0.1963	0.4193
	5000	0	0	2e-04	3e-04	5e-04	0.0021	0.0047	0.0086	0.0131	0.0294	0.0517	0.1965	0.4189
	10000	0	1e-04	2e-04	3e-04	5e-04	0.0021	0.0048	0.0085	0.0132	0.0294	0.0516	0.1964	0.4193
	20000	0	1e-04	2e-04	3e-04	5e-04	0.0022	0.0048	0.0086	0.0132	0.0294	0.0516	0.1965	0.4192

	$\lambda = 1.7$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.0216	0.0072	0.0035	0.0017	8e-04	0	2e-04	6e-04	0.0014	0.0041	0.0071	0.0303	0.07
	1000	0.0085	0.0019	7e-04	4e-04	1e-04	1e-04	4e-04	0.001	0.0016	0.0041	0.0075	0.031	0.0694
n	2500	0.0014	2e-04	0	0	0	3e-04	7e-04	0.0013	0.002	0.0043	0.0078	0.031	0.0698
	5000	3e-04	0	0	0	1e-04	3e-04	7e-04	0.0012	0.002	0.0044	0.0077	0.031	0.0697
	10000	0	0	0	0	1e-04	3e-04	7e-04	0.0013	0.002	0.0045	0.0078	0.0309	0.0699
	20000	0	0	0	1e-04	1e-04	3e-04	8e-04	0.0013	0.002	0.0044	0.0078	0.031	0.0698

	$\lambda = 3$	0.002	0.004	0.006	0.008	0.01	0.02	0.03	0.04	0.05	0.075	0.1	0.2	0.3
	500	0.0135	0.0055	0.0043	0.0024	0.0018	2e-04	0	0	0	3e-04	8e-04	0.004	0.0101
	1000	0.0068	0.0022	0.0012	9e-04	4e-04	0	0	0	1e-04	5e-04	9e-04	0.0042	0.0102
n	2500	0.0017	4e-04	2e-04	1e-04	0	0	0	1e-04	2e-04	5e-04	0.001	0.0044	0.0103
	5000	5e-04	1e-04	1e-04	0	0	0	1e-04	1e-04	2e-04	6e-04	0.001	0.0043	0.0103
	10000	1e-04	0	0	0	0	0	1e-04	1e-04	2e-04	6e-04	0.001	0.0044	0.0102
	20000	0	0	0	0	0	0	1e-04	2e-04	2e-04	6e-04	0.001	0.0044	0.0103

Table 5.3.: Shown is the estimated squared bias $\widehat{bias}^2(\varepsilon(n,\gamma))$ of the nonparametric estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_\lambda$ for simulated logistic data with dependence parameter λ for increasing sample sizes n and tail fractions γ. Thereby the estimator in (5.16) is used. Displayed values of 0 represent a bias, for which $bias^2 < 5 \cdot 10^{-5}$ holds.

n	interval of optimal γ	n	interval of optimal γ
500	$[0.02; 0.075]$	1000	$[0.02; 0.05]$
2500	$[0.02; 0.04]$	5000	$[0.008; 0.03]$
10000	$[0.008; 0.03]$	20000	$[0.006; 0.02]$

Table 5.4.: The presented table shows the interval for an optimal choice of tail fraction γ corresponding to certain sample sizes n of present dataset. Optimality is based on minimized MSE and bias of $\hat{\lambda}$ based on Table 5.2 and Table 5.3.

Given a real dataset, Danielsson et al. [9] suggested a bootstrap based method to choose an optimal tail fraction within the tail index estimation of a df F having a regularly varying tail. This procedure aims for an optimal tail fraction, which minimizes the MSE of Hill‘s estimator of the tail index. Beside the fact that Danielsson et al. [9] focus on the estimation of the tail index, we do not adopt, but comply with, their optimality criterion: Based on the fact that the estimator of the extremal coefficient in (5.3) exhibits a huge bias if the tail fraction is chosen too large, we use the minimization of the MSE with special focus on the minimization of the bias as an optimality criterion for the tail size k in (5.3).
Further, we remark that the preassigned optimal choice for the tail fraction also shows an acceptable fit of the simulated empirical df of $\hat{\varepsilon}$ to the approximated normal distribution (cf. the discussion in Section 5.2.2 based on Corollary 5.1.3).

5.3. Application to Financial Data

In the following we provide an application of the (extended) fragility index on financial data. The estimation procedure is based on the nonparametric estimator (5.3) for the extremal coefficient, established in the former sections. Thereby we apply a Monte Carlo simulation (cf. Section B) in order to obtain a confidence interval of the estimated (extended) fragility index of a system of stock prices taken from the DAX over the last ten years. Hence we shed light on the amount of tail dependence within a German financial system and provide a statement about its asymptotic stability and behavior under the stepwise breakdown of the system.

Within finance and economy it is common to analyze stock prices by use of so-called *log returns* defined by the daily logarithmic difference

$$R_t := \ln\left(\frac{X_t}{X_{t-1}}\right),$$

where $\{X_t\}_{t\in\mathbb{N}}$ is a real-valued process (cf. Embrechts et al. [13] for an appealing introduction). Since we have $R_t = \ln\left(\frac{X_t}{X_{t-1}}\right) \approx \frac{X_t - X_{t-1}}{X_{t-1}}, t \in \mathbb{N}$, we consider daily relative returns, i.e. only the relative change over time is of interest. Within this framework a large value of relative change presents an extreme event. Now assume a 3-dimensional financial system $\left(R_i^{(1)}, R_i^{(2)}, R_i^{(3)}\right), i \leq n$, on n consecutive days. With the focus on the stability of the system we are interested in the dependence structure between extreme losses of stock prices, i.e. we consider the highest negative values of log returns. Hence we work on the dataset

$$\left(Q_i^{(1)}, Q_i^{(2)}, Q_i^{(3)}\right) := \left(-R_i^{(1)}, -R_i^{(2)}, -R_i^{(3)}\right), \quad i \leq n,$$

whose upper tail plays the crucial role within the nonparametric estimation approach of Section 5.1. If the considered time series $(\boldsymbol{Q}_i)_{i\in\mathbb{N}} := \left(Q_i^{(1)}, Q_i^{(2)}, Q_i^{(3)}\right)_{i\in\mathbb{N}}$ can be assumed to be stationary, the theory of multivariate EVD also holds within this setting, instead of considering i.i.d. observations (cf. Section 7.1.3 in McNeil et al. [47]). It is an accepted working hypothesis that time series of daily log returns can be considered to be stationary.

In the following we select the stock prices from the three DAX indices, "Deutsche Bank", "Commerzbank" and "Allianz", denoted by the financial system $\{D, C, A\}$. At the end of this section, we also look at the mixed DAX system $\{D, A, B\}$, containing the indices of the companies "Deutsche Bank", "Adidas" and "Bayer AG", in order to compare its asymptotic stability with that of the financial system $\{D, C, A\}$.

The system $\{D, C, A\}$ shall be considered to represent a system of German financial institutions. Thereby we analyze the 3-dimensional dataset containing the log returns of their closing prices during the ten-year period of 6th September 2001 to 5th September 2011. This results in 2577 daily returns. Figure 5.7 shows the two-dimensional scatterplots of the daily negative log returns of $\{D, C, A\}$. The three-dimensional plot is not shown with respect to the restriction of the two-dimensional output on this work. By means of a bootstrap procedure we obtain the empirical distribution of bootstrap estimators for the extremal coefficient. We use the Monte Carlo simulation as described in

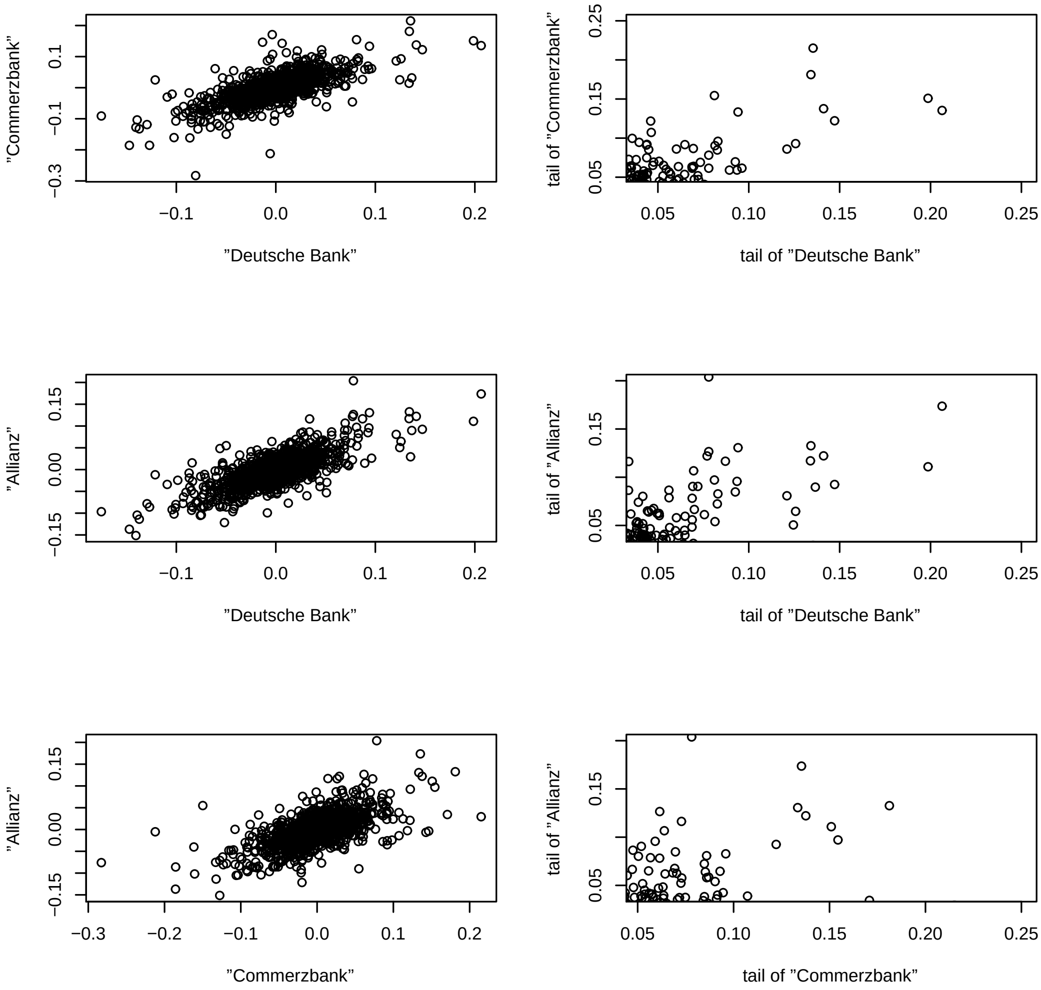


Figure 5.7.: Two-dimensional scatterplots of the financial DAX system $\{D, C, A\}$ during the ten-year period Sept. 2001 - Sept. 2011. The sample size is $n = 2577$. The complete two-dimensional datasets are shown on the left, where the corresponding upper tails are shown on the right. The tail fraction of the univariate tails is $\gamma = 0.04$.

Section B. Based on this bootstrap data we derive the empirical distribution of the non-parametric estimator of the (extended) fragility index as provided in (5.11) and (5.10). The number of bootstrap resamples is 10000, which coincides with the number m of simulated datasets coming from a logistic EVD (cf. Section 5.2.2).

Figure 5.8 shows the bootstrap samples of the estimator (5.3) for the extremal coefficient corresponding to increasing tail fractions γ. We observe decreasing variance of

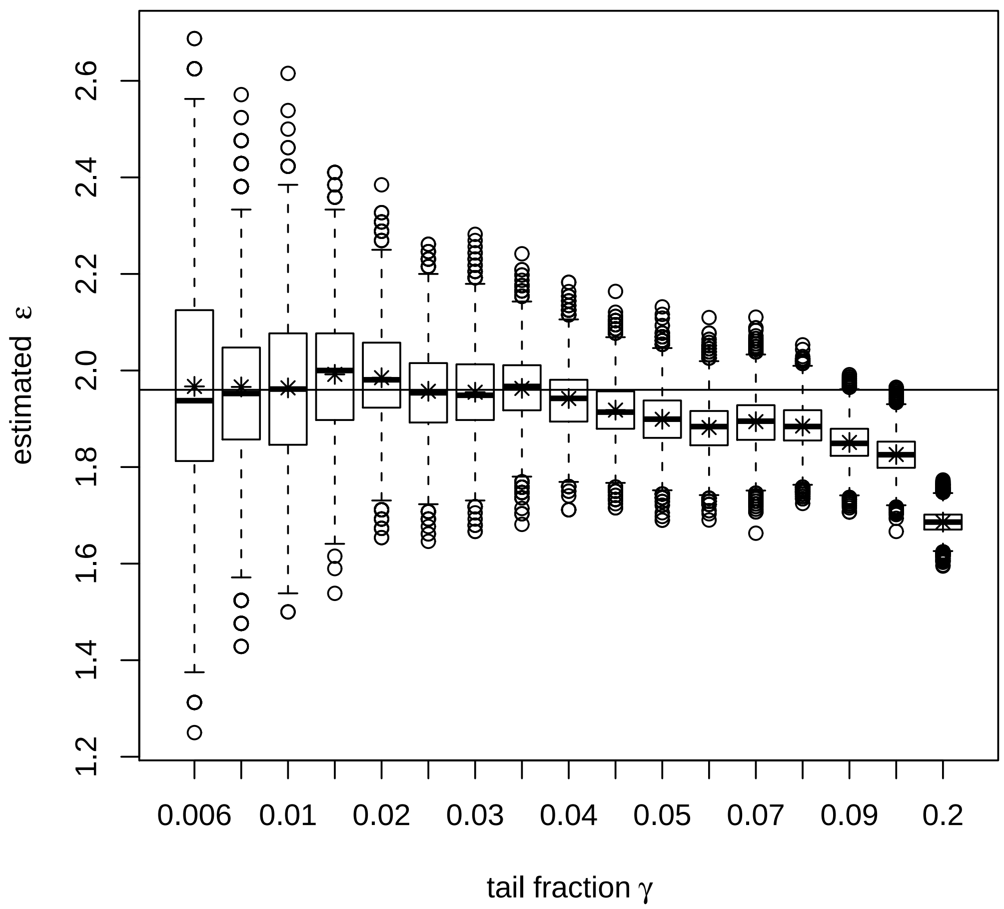


Figure 5.8.: Boxplots of the bootstrap estimates of the extremal coefficient ε corresponding to the financial system $\{D, C, A\}$ during the ten-year period Sept. 2001 - Sept. 2011. Shown are the tail fractions $\gamma = 0.006, 0.008, 0.01, 0.015,$ $0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2$. The sample size is $n = 2577$. The sample size is $n = 2577$. The horizontal line represents the value $\varepsilon = 1.96$.

the bootstrap estimator and increasing influence of bias for increasing γ. This behavior of the bootstrap estimator of the extremal coefficient has already been observed within the simulation study in Section 5.2.2. The horizontal line $\varepsilon = 1.96$ in Figure 5.8 represents an approximation on the median of the bootstrap samples based on the interval $[0.02; 0.04]$ for γ, cf. Table 5.4 and the justification of the chosen interval. Due to the symmetric behavior of the bootstrap estimator, the value $\varepsilon = 1.96$ also represents a good approximation on the mean of the bootstrap samples for $\gamma \in [0.02; 0.04]$. Further, the NP plots in Figure C.9 do not vote against the assumption of normality of the estimator of the extremal coefficient.

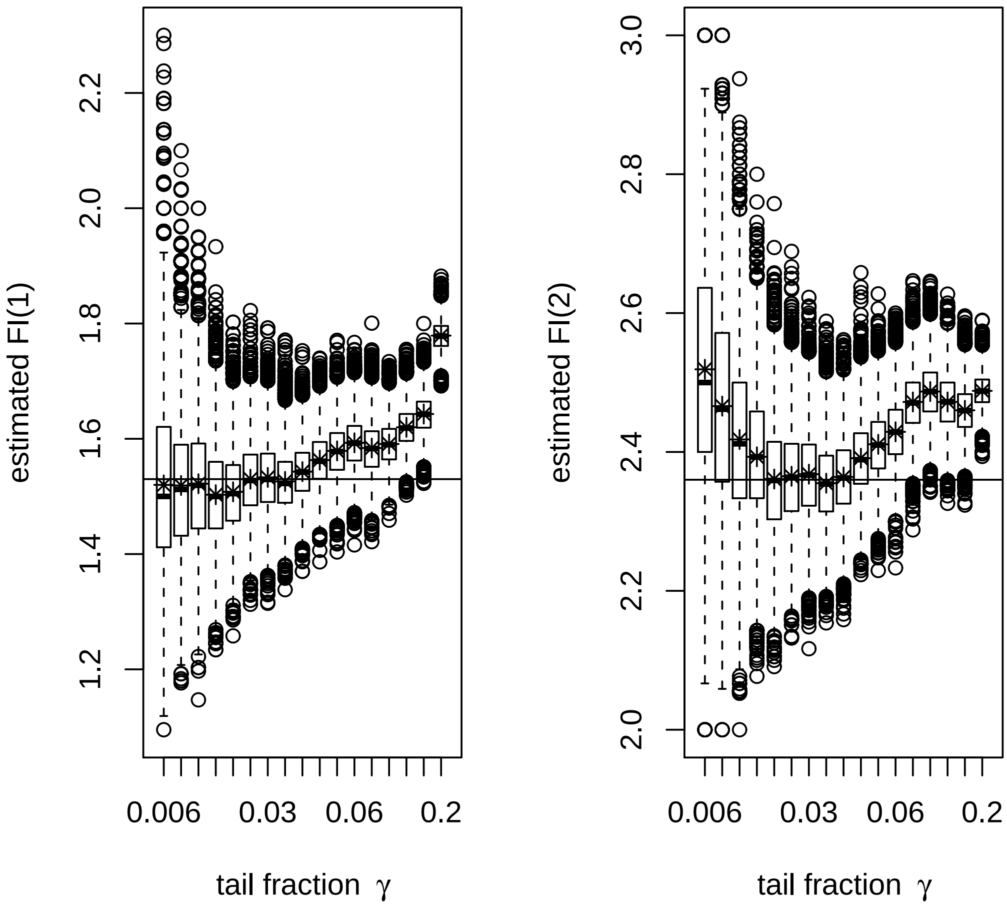


Figure 5.9.: Boxplots of the bootstrap estimates of the fragility index and the extended fragility index $FI(2)$ corresponding to the financial system $\{D, C, A\}$ during the ten-year period Sept. 2001 - Sept. 2011. Shown are the same tail fractions as in Figure 5.8. The sample size is $n = 2577$. The horizontal line represents the value $FI = 1.53$, resp. $FI(2) = 2.36$.

Figure 5.9 shows the bootstrap samples of the estimator for the fragility index, (5.11), the extended fragility index (5.10) respectively. Once again we observe the influence of bias on the estimator $\widehat{FI}$ for increasing tail fractions. The boxplots of bootstrap samples for $\widehat{FI}(2)$ (right figure of Figure 5.9) represent a so-called *bathtub curve*, i.e. the median of $\widehat{FI}(2)$ starts with high values for small tail fractions, stays at a constant level for central tail fractions of about $\gamma = 0.02$ to $\gamma = 0.04$ and increases once again by increasing tail fractions $\gamma > 0.04$. Thereby the horizontal lines in Figure 5.9 represent the value $FI = 1.53$ (left), resp. $FI(2) = 2.36$ (right). These values represent the average of means corresponding to $\gamma \in [0.02; 0.04]$ in analogy to Figure 5.8. The shown behavior of the bootstrap estimators in Figure 5.9 confirm the suggestion for an optimal tail fraction, based on our simulation results, represented in Table 5.4. To specify the value for γ, we look at the behavior of the MSE of the bootstrap estimates for the extremal coefficient. The simulation of the MSE has been done via (5.15) based on a Monte Carlo simulation. For an increasing sequence of tail fractions γ, we computed the MSE based on the m bootstrap estimators for the extremal coefficient, where the true but unknown value ε in (5.15) is replaced by the mean of bootstrap estimates for ε corresponding to $\gamma \in \{0.01, 0.02, 0.03\}$. This procedure is not completely in line with the one suggested in Danielsson et al. [9], since we have to account for an increasing bias of $\hat{\varepsilon}$ shown by our simulation results.

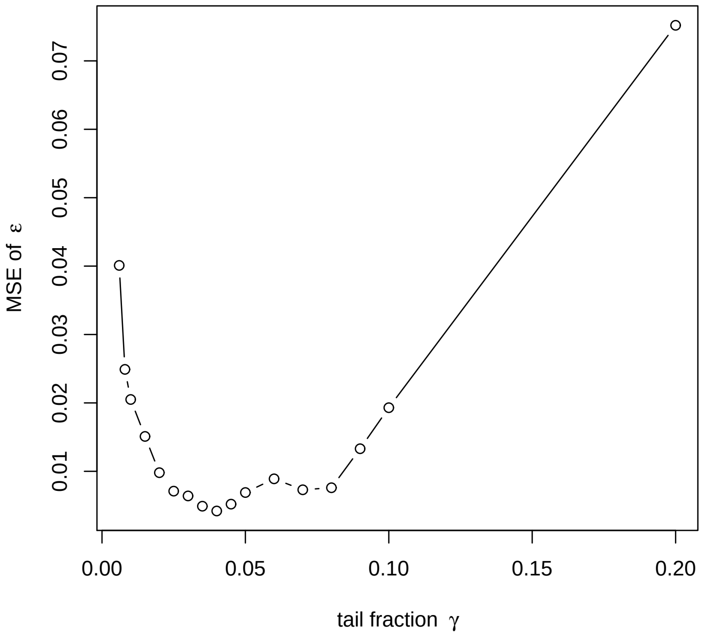
Table 5.5 shows the bootstrap estimates for the MSE of ε for increasing tail fractions γ. There exists an unique value at which the MSE takes on its minimum. Hence, we suggest the optimal tail fraction of $\gamma = 0.04$ for our DAX-example of the financial system $\{D, C, A\}$.

Based on the tail fraction $\gamma = 0.04$, a 95%- bootstrap confidence interval for the extremal coefficient ε and the (extended) fragility index of the financial system $\{D, C, A\}$ is given by

$$BC(\varepsilon) = [1.82; 2.06], \tag{5.17}$$

$$BC(FI) = [1.45; 1.65], \quad BC(FI(2)) = [2.22; 2.44], \tag{5.18}$$

respectively, cf. (7.11) in Section B for computation and references. For a listing of mean and standard deviation of the bootstrap estimator of the extremal coefficient, FI and $FI(2)$, cf. Table 7.1 in Section C. Furthermore, Table 7.2 represents the 95%-bootstrap confidence interval of ε, FI and $FI(2)$ based on further tail fractions. The

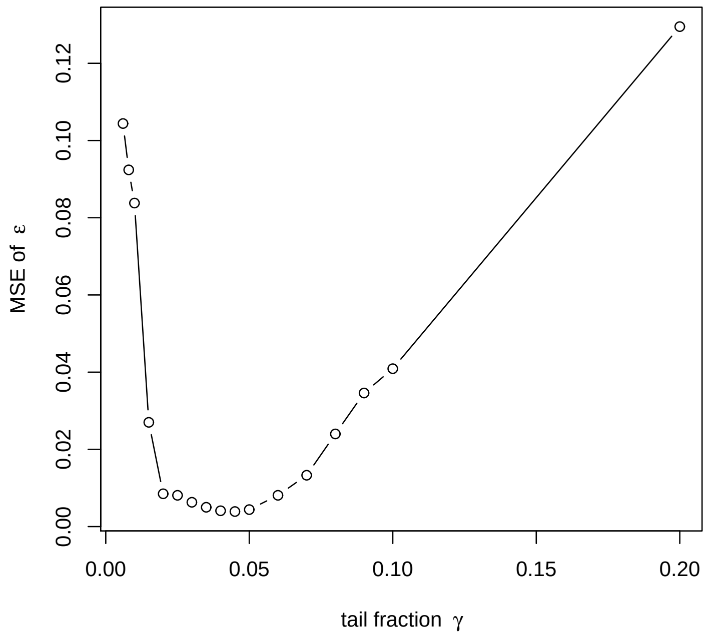


γ	0.006	0.008	0.01	0.015	0.02	0.025	0.03	0.035	0.04
MSE	0.0401	0.0249	0.0205	0.0151	0.0098	0.0071	0.0064	0.0049	0.0042
γ	0.045	0.05	0.06	0.07	0.08	0.09	0.1	0.2	
MSE	0.0052	0.0069	0.0089	0.0073	0.0076	0.0133	0.0193	0.0752	

Table 5.5.: Shown is the simulated mean squared error (MSE) of the bootstrap estimator of the extremal coefficient corresponding to the financial system $\{D, C, A\}$. The sample size is $n = 2577$.

numbers of the bootstrap confidence interval for the (extended) fragility index in (5.18) represent an estimator for the stability of the financial system $\{D, C, A\}$. Thereby we focus on the approach of exceedances of the components of $\{D, C, A\}$ above an individual threshold, cf. Section 4.2.2. The individual thresholds of the components D, C and A are given by the empirical 96%-quantile $\hat{s}_D := \hat{x}^D_{96\%} = 0.041$, $\hat{s}_C := \hat{x}^C_{96\%} = 0.052$ and $\hat{s}_A := \hat{x}^A_{96\%} = 0.040$ respectively.
Hence, with respect to Remark 4.2.1 and Definition 4.1.3, the system $\{D, C, A\}$ is (strongly) fragile.

At last, we want to present another example taken from the DAX, namely the mixed system of the companies "Deutsche Bank", "Adidas" and "Bayer AG", $\{D, A, B\}$. We analyzed the log return of stock prices over the same ten-year period Sept. 2001 - Sept. 2011 as done for the financial system $\{D, C, A\}$. Two-dimensional scatterplots of the system $\{D, A, B\}$ are shown in Section C, Figure C.10. Figure C.11 and Figure C.12 show the boxplots of the simulation results for the extremal coefficient and the (extended) fragility index of the mixed Dax system $\{D, A, B\}$. In order to suggest an optimal tail fraction for the system $\{D, A, B\}$, we look at the MSE of the estimated extremal coefficient, cf. Table 5.6.



γ	0.006	0.008	0.01	0.015	0.02	0.025	0.03	0.035	0.04
MSE	0.1044	0.0924	0.0838	0.027	0.0085	0.0081	0.0063	0.005	0.0041
γ	0.045	0.05	0.06	0.07	0.08	0.09	0.1	0.2	
MSE	0.0039	0.0044	0.0081	0.0133	0.024	0.0346	0.0409	0.1295	

Table 5.6.: Shown is the simulated mean squared error (MSE) of the bootstrap estimator of the extremal coefficient corresponding to the mixed DAX system $\{D, A, B\}$. The sample size is $n = 2577$.

The MSE plot suggests to choose the optimal tail fraction of $\gamma = 0.045$. Based on

$\gamma = 0.045$, the 95%- bootstrap confidence interval for the extremal coefficient ε and the (extended) fragility index of the mixed DAX system $\{D, A, B\}$ is given by

$$BC(\varepsilon) = [2.14; 2.37]$$

and

$$BC(FI) = [1.27; 1.40], \quad BC(FI(2)) = [2.12; 2.32],$$

respectively, cf. preceding procedure corresponding to the financial system $\{D, C, A\}$. Thereby the individual thresholds of the system $\{D, A, B\}$ are given by the empirical 95.5%-quantile $\hat{s}_D := \hat{x}^D_{95.5\%} = 0.039$, $\hat{s}_A := \hat{x}^A_{95.5\%} = 0.034$ and $\hat{s}_B := \hat{x}^B_{95.5\%} = 0.032$

The resulting BC-intervals corresponding to the mixed DAX system $\{D, A, B\}$ are lying beyond the BC-interval corresponding to the financial system $\{D, C, A\}$, cf. (5.17) and (5.18). Hence, the estimated FI (tendency for $FI(2)$) of the mixed DAX system is lower than that of the financial system, i.e. we observe less tail dependence within the mixed DAX system. However, the system $\{D, A, B\}$ is (strongly) fragile too.

The simulations and the figures have been done with the software `R, Version 2.11.1`, cf. http://www.r-project.org/.

5.4. Outlook on Parametric Estimation Approach for the (Extended) Fragility Index

As far as known to the author, the estimation approach provided in Section 5.1 is the first attempt within research about the fragility index. Since we do not want to restrict ourselves to a parametric EVD model for the first time, we have used a nonparametric estimation approach for the D-norm in the domain-of-attraction condition, cf. (5.1). As already noted, (5.1) also allows for a parametric estimation approach, e.g. the parametric estimation of the copula C_G, to which the domain of attraction C_F belongs. One therefore assumes a certain parametric model for the extreme value copula C_G. Since the approximation in (2.39) is only good in the tail of the copula, one might precede as follows. Choose a parametric model for C_G which can be assumed to fit well. Additionally, the univariate tails can be estimated by a peaks-over-threshold method (POT

method), cf. Leadbetter [43] or Section 6.5.1 in Embrechts et al. [13] for an appealing introduction to the POT method. (If one does, not only the copula C_F but also the whole df F is estimated in its tail.) Then couple a parametric estimation approach, like the maximum-likelihood (ML) method, with the so-called *censored data* approach. The approach of censored data takes into account that, under the domain-of-attraction assumption, only tail data should be used within the estimation procedure. This is called the *multivariate threshold exceedances for censored data* and has been extensively applied by Ledford and Tawn [44] and Smith et al. [66]. They especially use the parametric EVD model of the Gumbel copula and a POT approach for the univariate tails. The former article provides a nice introduction to the above described ML method for censored data. Unfortunately, at this time, there does not exist any result – beside some results under certain restrictions – concerning the asymptotic behavior of the thus obtained ML estimator. It remains an open question whether the ML estimator also exhibits asymptotic normality and consistency in combination with the censored data approach, as one is used to expect from the ML estimators under efficient likelihood estimation (cf. page 462 and Theorem 5.1 of Section 6 in Lehmann and Casella [46]).
We have extended the bivariate approach for censored data as published in Ledford and Tawn [44] and Smith et al. [66] to arbitrary dimension for the logistic model. Simulation studies, within which we apply the ML method for censored data to simulated three-dimensional data coming from a logistic EVD with different dependence parameters, show promising results. For example, the ML estimator of the logistic dependence parameter is unbiased for tail fractions $\gamma \leq 0.1$ and its corresponding NP plots do not vote against the assumption of normality if the tail size is chosen not too small. This might give reason for theoretical research, including consistency and asymptotic normality and an optimal tail size, on the topic of ML estimators for censored data in the framework of a parametric estimation procedure for the (extended) fragility index. However, recall that a parametric copula model will restrict the dependence structure; hence, the addressed parametric estimation approach can only serve as a suitable estimation procedure if the assumption about the parametric copula model can be assumed to fit the data of interest. De Haan et al. [30] therefore provide a goodness of fit test within extreme-value dependence. They also establish a parametric estimator for the bivariate tail dependence function, which can serve as an alternative to the approach of

Ledford and Tawn [44] in the bivariate case. Another recent approach in estimating the dependence parameter in a parametric EVD copula model is provided by Einmahl et al. [17]. They provided a minimum distance estimator for the dependence structure of an EVD under the domain-of-attraction condition in arbitrary dimension.

At last, we remark that the presented nonparametric estimation procedure for the (extended) fragility index of Sections 5.1 to 5.3 can be easily applied to arbitrary dimensions. In order to shed light on the asymptotic stability of a random system, it is an interesting question whether the total breakdown of this system announces itself early or only after the situation has become serious. This can be answered by the investigation of the curvilinear behavior of the extended fragility index $FI(m)$ as a function of the number m.

6. Final Remarks

The main results of the work at hand include the compact representation of the fragility index by norms and its extension to a measure for system stability, which also captures the persisting amount of risk within a step-by-step collapsing system, namely the extended fragility index. Thereby we require that the copula C_F, which captures the dependence structure of the random system $\{Q_1, \dots, Q_d\}$, belongs to the domain of attraction of an EVD. Further we provide the fragility index based on the approach of exceedances above an individual threshold, i.e. the threshold $s_j := F_j^{-1}(1-c), j \leq d$, c close to zero, depends on the univariate margins of the distribution F. The representation by norms and the use of an individual threshold improves the fragility index as defined in the original literature Geluk et al. [22]. We consider the domain-of-attraction assumption to be a suitable condition, which facilitates the handling of the FI, in contrast to Geluk et al. [22] who work out the FI by means of the univariate margins and the derived joint distribution F (cf., for example, Theorem 3 in [22]). Further, we think that the approach of an individual threshold supersedes the originally by Geluk et al. [22] defined approach of a common threshold. To apply the (extended) fragility index as a measure for tail dependence we have to ensure that the *tail* and nothing else is estimated. If one considers exceedances above a common threshold s, this fixed threshold s may not be high enough to define a tail event for *every* component of the random system. To overcome this problem we prefer exceedances above an individual threshold with a certain fixed exceedance probability being small enough. In the framework of exceedances above an common threshold, we only have equal exceedance probabilities if the univariate margins are identical on which we do not want to restrict ourselves. Further quantiles of exceedance can be estimated quite easily, see the well known theorem of Glivenko-Cantelli or the POT-approach, cf. Chapter 5 in Beirlant et al. [5]. See Section 4.2.3 for a discussion on the use of an individual or a common threshold.

Within the framework of the extended fragility index we provided equivalent conditions under which the $FI(m)$ is well-defined for certain $m \in \{1, \ldots, d\}$, see Lemma 3.2.3 and Corollary 3.2.6. These conditions are quite technical, since they deal with the abstract dependence function of an EVD represented by the angular measure μ or the D-norm. Within the approach of an individual threshold we may simplify this condition (cf. Corollary 4.2.11 and the following considerations there). Hence, if the tail dependence function λ_F is larger than 0, the $FI(m)$ is well-defined for any $m \leq d$. By use of estimators for the tail dependence function (see de Haan et al. [30] or Hsing et al. [37]) one is able to provide a statistical condition under which the extended fragility index is well-defined. This might be useful with respect to applications on the (extended) fragility index and has not been considered within the work at hand.

Section 5.1 provides a nonparametric estimation approach for the (extended) fragility index. The estimation procedure is based on a quite simple nonparametric estimator for the stable tail dependence function (cf. Section 7.2 in de Haan and Ferreira [29]), more precisely the extremal coefficients corresponding to the margins G_K, $m \leq |K| \leq d$, of that EVD G for which we have $F \in \mathcal{D}(G)$. The obtained estimator for the FI is consistent and asymptotic normal. This cannot be derived as easily for the extended fragility index and is still an open question within the work at hand.
Further, one may be interested in a parametric estimation approach for the extended fragility index. Based on simulation studies, we think that the maximum likelihood approach for censored data (cf. Ledford and Tawn [44] as well as Smith et al. [66]) may serve as a promising approach for estimation the $FI(m)$ under a parametric point of view. Unfortunately, there do not exist theoretical results like consistency and asymptotic normality for the obtained parametric estimator of the dependence parameter of an EVD as efficient estimators like ML estimators usually exhibit. We did not engage in this topic since the focus on the work at hand is on the representation and extension of the fragility index. Tawn [69] states that checking sufficient conditions for asymptotic normality of the ML estimator (cf. Lehman and Casella [46], Chapter 6, Theorem 2.6 and 3.10) fails in the framework of *censored data*. However, the approach of censored data is a quite simple approach for using parametric EVD models, although data are obtained under the domain-of-attraction assumption instead of an EVD assumption, cf. Section 5.4 on this topic. Hence we think that further research on the asymptotic

behavior of ML estimators for censored data is worthwhile, not only in the framework of a parametric estimation approach for the (extended) fragility index.

7. Appendix

A. Abbreviations, auxiliary definitions and propositions

This section provides a summary of abbreviations, standard definitions and basic propositions used within the work at hand.

Throughout this thesis we denote by $\boldsymbol{e}_j$ the j-th unit vector in $\mathbb{R}^d$. If not stated otherwise, we have $\boldsymbol{x} \in \mathbb{R}^d$, i.e. $\boldsymbol{x} := (x_1, \dots, x_d)$ is d-variate. All operations on vectors are meant component wise, i.e. $\boldsymbol{x} \leq \boldsymbol{0}$ is defined by $x_j \leq 0$ for every $j \leq d$. Thereby, we set $\mathbb{R}^d_+ := \{\boldsymbol{x} \in \mathbb{R} : \boldsymbol{x} > \boldsymbol{0}\}$ and $\mathbb{R}^d_0 := \{\boldsymbol{x} \in \mathbb{R} : \boldsymbol{x} \geq \boldsymbol{0}\}$. Further we denote by the symbol $\subset$ a real subset, where $\subseteq$ also contains equality of sets and the symbol $^{\complement}$ denotes the complement of a set.

We denote by $\|\cdot\|_\lambda$ the arbitrary L_λ-norm defined by

$$\|(x_1, \dots, x_d)\|_\lambda := \left(\sum_{j=1}^{d} |x_j|^\lambda \right)^{1/\lambda} \tag{7.1}$$

for $1 \leq \lambda < \infty$ and $\|(x_1, \dots, x_d)\|_\infty := \max_{j \leq d} |x_j|$ for any $\boldsymbol{x} \in \mathbb{R}^d$. The convex-combination of the L_1-norm and the maximum-norm, i.e.

$$\|\boldsymbol{x}\|_{MO} := \vartheta \|\boldsymbol{x}\|_1 + (1 - \vartheta) \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^d,\ \vartheta \in [0, 1]$$

is called the *Marshall-Olkin* norm and defines a D-norm (cf. Section4.3 in Falk et al. [19]).

Definition A.1 (Landau symbols) *Let $a \in \mathbb{R} \cup \{-\infty, \infty\}$ and assume $f, g : \mathbb{R} \to \mathbb{R}$ to be continuous in a neighborhood U of a, denoted by $U(a)$. Then we define the so-called*

Landau symbols by

$$f(x) = o(g(x))$$

if and only if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ *and*

$$f(x) = O(g(x))$$

if and only if there exists $C \geq 0$ *such that there exists* $U(a)$ *and* $|f(x)| \leq C\,|g(x)|$ *holds for every* $x \in U$.
Further we write

$$f(x) \sim g(x)$$

if and only if there exists $C \in \mathbb{R}\backslash 0$, *such that* $\lim_{x\to a} \frac{f(x)}{g(x)} = A \neq 0$ *holds. If* $a \in \{-\infty, \infty\}$ *we call* f *and* g *tail-equivalent.*

Definition A.2 *A real function* f *is called d-monotone in* (a,b), *where* $a, b \in [-\infty, \infty]$ *and* $d \geq 2$, *if it is differentiable there up to the order* $d-2$ *and the derivatives satisfy*

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \ldots, d-2$$

for any $x \in (a,b)$ *and further if* $(-1)^{d-2} f^{(d-2)}$ *is nonincreasing and convex in* (a,b). *For* $d = 1$, f *is called 1-monotone in* (a,b) *if it is nonnegative and nonincreasing there. If* f *has derivatives of all orders in* (a,b) *and if* $(-1)^k f^{(k)}(x) \geq 0$ *for any* x *in* (a,b), *then* f *is called completely monotone.*
Further a real function f *on an interval* $I \subseteq [-\infty, \infty]$ *is d-monotone (completely monotone) on* I, $d \in \mathbb{N}$, *if it is continuous there and if* f *restricted to the interior int*(I) *of* I *is d-monotone (completely monotone) on int*(I).

Lemma A.3 *Consider any finite set* M *with* $|M| = m \in \mathbb{N}$. *Then we have*

$$\sum_{\emptyset \neq T \subseteq M} (-1)^{|T|-1} = 1\,.$$

Proof: By the binomial formula we get

$$\sum_{\emptyset \neq T \subseteq M} (-1)^{|T|-1} \;=\; (-1)\left(\sum_{\emptyset \subseteq T \subseteq M} (-1)^{|T|} - 1\right)$$

$$= \quad (-1)\left(\left(\sum_{k=0}^{|M|}\begin{pmatrix}|M|\\ k\end{pmatrix}(-1)^k 1^{|M|-k}\right)-1\right)=1\,.$$

□

Lemma A.4 *We have for $k \in \mathbb{N}$*

$$\sum_{0\le j\le k}(-1)^j\binom{k}{j}j^i=\begin{cases}0, & 0\le i\le k-1,\\ (-1)^k k!, & i=k.\end{cases}$$

Proof: We establish the first case, where $0 \le i \le k-1$, by induction over k. The assertion is obvious for $k = 1$. Suppose that it is true for $k \in \mathbb{N}$; note that $\sum_{0\le j\le k}(-1)^j\binom{k}{j} = (1+(-1))^k = 0$ for any $k \in \mathbb{N}$. Then, for $1 \le i \le k$,

$$\begin{aligned}
&\sum_{0\le j\le k+1}(-1)^j\binom{k+1}{j}j^i\\
&=\sum_{1\le j\le k+1}(-1)^j\binom{k+1}{j}j^i\\
&=(k+1)(-1)\sum_{1\le j\le k+1}(-1)^{j-1}\binom{k}{j-1}(1+(j-1))^{i-1}\\
&=-(k+1)\sum_{0\le j\le k}(-1)^j\binom{k}{j}(1+j)^{i-1}\\
&=-(k+1)\sum_{0\le j\le k}(-1)^j\binom{k}{j}\sum_{r=0}^{i-1}\binom{i-1}{r}1^{i-1-r}j^r\\
&=-(k+1)\sum_{0\le j\le k}(-1)^j\binom{k}{j}\underbrace{\sum_{r=0}^{i-1}\binom{i-1}{r}j^r}_{(*)} \qquad (7.2)\\
&=0
\end{aligned}$$

by applying the induction hypothesis to (*) for every $r \le i-1$. This establishes the first equation. Next we establish the assertion in case $i = k$, again by induction.

The assertion is obviously true for $k = 1$. Suppose it is true for k. Then we obtain

$$\sum_{0\le j\le k+1}(-1)^j\binom{k+1}{j}j^{k+1}$$

$$= (k+1) \sum_{1 \le j \le k+1} (-1)^j \binom{k}{j-1} (1 + (j-1))^k$$

$$= -(k+1) \sum_{0 \le j \le k} (-1)^j \binom{k}{j} (1+j)^k \tag{7.3}$$

$$= -(k+1) \sum_{0 \le j \le k} (-1)^j \binom{k}{j} (j)^k \tag{7.4}$$

$$= -(k+1)(-1)^k k! = (-1)^{k+1}(k+1)!, \tag{7.5}$$

where step (7.3) to step (7.4) follows in analogy to the argumentation in (7.2) and step (7.4) to step (7.5) follows by the application of the induction hypothesis. □

Lemma A.5 *For arbitrary $d \in \mathbb{N}$ and $j \le d$ we get*

$$\sum_{k=1}^{d} k(-1)^{k-j+1} \binom{d-j}{k-j} = \begin{cases} 1 & \text{for } j = d-1 \\ -d & \text{for } j = d \\ 0 & \text{else} \end{cases}$$

Proof: The assertion is clear for $j = d$. Note, that $\binom{d}{i} = 0$ for $i < 0$. For $j \le d-1$ we get

$$\sum_{k=1}^{d} k(-1)^{k-j+1} \binom{d-j}{k-j}$$

$$= \sum_{k=j}^{d} k(-1)^{k-j+1} \binom{d-j}{k-j}, \quad k \mapsto k-j,\ j \ne d$$

$$= \sum_{k=0}^{d-j} (k+j)(-1)^{k+1} \binom{d-j}{k}$$

$$= \sum_{k=0}^{d-j} k(-1)^{k+1} \binom{d-j}{k} + j \underbrace{\sum_{k=0}^{d-j} (-1)^{k+1} \binom{d-j}{k}}_{=(-1)(-1+1)^{d-j}=0}, \ j \ne d$$

$$= (d-j) \sum_{k=1}^{d-j} (-1)^{k+1} \binom{d-j-1}{k-1} \quad k \mapsto k-1$$

$$= (d-j) \sum_{k=0}^{d-j-1} (-1)^{k} \binom{d-j-1}{k}$$

$$= (d-j)(-1+1)^{d-j-1} = \begin{cases} 1 & \text{for } j = d-1 \\ 0 & \text{for } j < d-1 \end{cases}$$

□

Theorem A.6 (Additive law of probability) *Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then we have*

$$P(A_1 \cup \ldots \cup A_m) = \sum_{\emptyset \neq T \subseteq \{1,\ldots,m\}} (-1)^{|T|+1} P\left(\bigcap_{i \in T} A_i\right)$$

for $A_1, \ldots, A_m \in \mathcal{A}$.

Proof: The proof is done by induction over $m \in \mathbb{N}$. For $m = 2$ the assertion is obvious. Now assume, the assertion holds for arbitrary but fixed $m \in \mathbb{N}$. We get

$$\begin{aligned} P((A_1 \cup \ldots \cup A_m) \cup A_{m+1}) &= P(A_1 \cup \ldots \cup A_m) + P(A_{m+1}) \\ &-P((A_1 \cap A_{m+1}) \cup (A_2 \cap A_{m+1}) \cup \ldots \cup (A_m \cap A_{m+1})) \\ &= \sum_{\emptyset \neq T \subseteq \{1,\ldots,m\}} (-1)^{|T|+1} P\left(\bigcap_{i \in T} A_i\right) + P(A_{m+1}) \\ &- \sum_{\emptyset \neq T \subseteq \{1,\ldots,m\}} (-1)^{|T|+1} P\left(\bigcap_{i \in T} (A_i \cap A_{m+1})\right) \\ &= \sum_{\emptyset \neq T \subseteq \{1,\ldots,m+1\}, m+1 \notin T} (-1)^{|T|+1} P\left(\bigcap_{i \in T} A_i\right) + P(A_{m+1}) \\ &+ \sum_{T \subseteq \{1,\ldots,m+1\}, m+1 \in T, T \cap \{1,\ldots,m\} \neq \emptyset} (-1)^{|T|+1} P\left(\bigcap_{i \in T} A_i\right) \\ &= \sum_{\emptyset \neq T \subseteq \{1,\ldots,m+1\}} (-1)^{|T|+1} P\left(\bigcap_{i \in T} A_i\right), \end{aligned}$$

which shows the assertion. □

Theorem A.7 (Cramer‘s Delta method) *Denote by T_n a sequence of estimators for θ which satisfies*

$$\sqrt{n}(T_n - \theta) \to_D N(0, \sigma^2), \tag{7.6}$$

where $N(0, \sigma^2)$ *is the normal distribution with mean value* 0 *and variance* σ^2*. Now, denote by* h *a continuous and differentiable function. Then we get*

$$\sqrt{n}\,(h(T_n) - h(\theta)) \to_D N(0, \sigma^2(h'(\theta)^2)),$$

provided the derivative $h'(\theta)$ *exists and is not zero.*

Proof: By Taylor expansion of the function h at point θ we get

$$h(T_n) = h(\theta) + h'(\theta)(T_n - \theta) + R_n\,, \tag{7.7}$$

where the remainder term R_n is continuous in θ with $h(\theta) = 0$ defined by $R_n := \frac{h''(\xi)}{2}(T_n - \theta)^2$ for the applied Taylor approximation of first degree. Thereby, ξ lies between T_n and θ and $R_n \to 0$ if $T_n \to \theta$. Further we have

$$\sqrt{n}(h(T_n) - h(\theta)) = \sqrt{n}h'(\theta)(T_n - \theta) + \sqrt{n}R_n \tag{7.8}$$

is equivalent to (7.7). Since the convergence in (7.6) holds, we get $T_n \to \theta$ in probability and hence $R_n \to 0$ in probability. Now due to the assumption $\sqrt{n}(T_n - \theta) \to_D N(0, \sigma^2)$ and the equality in (7.8), we immediately obtain $\sqrt{n}\,(h(T_n) - h(\theta)) \to_D N(0, \sigma^2(h'(\theta)^2))$ for the left hand side of (7.8), since the normal distributed rv $Y := T_n - \theta$ is multiplied by the constant $h'(\theta)$, which implies that $h'(\theta)Y$ is normal distributed with variance $h'(\theta)^2\sigma^2$. □

B. Bootstrap Procedures

The Bootstrap method is a "resampling" method, which provides the approximation of the theoretical distribution of a random variable on the basis of a single given sample. It goes back to Bradley Efron, who made a considerable contribution to the work on non-parametric procedures with his work on *Bootstrap Methods: Another Look at the Jacknife* (see Efron [14]). With the soaring capacity of present-day computers, it poses as a successful method. Furthermore, bootstrap methods are accepted within scientific research if theoretical problems cannot be solved otherwise.

Suppose, we have observed a sample $S = \{x_1, x_2, \ldots, x_n\}$ of size n from an unknown population $\mathcal{G}$ with population parameter μ. The estimator $\hat{\mu}$ based on S is a sample estimator of μ. Obviously, we are faced with the questions "How accurate is this sample estimator? And which distribution does it follow?". Hence we are interested in the distribution F of the random variable

$$Z := \frac{\hat{\mu} - \mu}{s(\hat{\mu})},$$

with $s(\hat{\mu})$ being the standard deviation of $\hat{\mu}$, and an (approximative) confidence interval for μ with coverage $1 - \alpha$ respectively. Unfortunately, there do not exist in any case theoretical results concerning these questions. Resampling techniques, like the bootstrap method, turn out to be an appropriate and satisfying alternative.

Note that the following summary is based on Efron and Tibshirani [15].
The procedure is quite simple. Draw a random sample S^* from the sample S with replacement, i.e. we get $S^* = \{x_1^*, \ldots, x_n^*\}$. Then compute an estimator μ^* for μ based on the sample S^*. Repeat this procedure B times. This procedure is called *resampling* and the samples and obtained estimators are therefore called *resamples*, *bootstrap estimators* respectively. Hence we get a sample of bootstrap estimators

$$\{\mu_1^*, \ldots, \mu_B^*\},$$

which can be regarded as a kind of empirical distribution estimator of the unknown distribution of μ. Based on the bootstrap estimators we compute their mean $\mu_\cdot^* := \frac{1}{B}\sum_{i \leq B} \mu_i^*$ and standard deviation $s^* := (\frac{1}{B-1}\sum_{i \leq B}(\mu_i^* - \mu_\cdot^*)^2)^{1/2}$. Hence, s^* is a bootstrap estimate of the standard error of $\hat{\mu}$. The numbers $z_i^* = (\mu_i^* - \hat{\mu})/s^*$ can be regarded as realizations of the random variable

$$Z^* = \frac{\hat{\mu}^* - \hat{\mu}}{s^*},$$

which distribution function shall be denoted by F^*, the *bootstrap distribution* of the bootstrap parameter μ^*. The empirical distribution function of Z^* - given by the realizations $z_i^*, i \leq B$ - is obviously an approximation on the bootstrap distribution F^* and as well as an approximation on the true but unknown distribution F of the rv Z. Theoretically, it is possible to specify the bootstrap distribution F^* of the random variable Z^* exactly by generating the finite number of possible bootstrap populations. Since this is too time-consuming, one simplifies this procedure by just generating by random a number of B bootstrap replicates. Of course, B shall be therefore large enough to get a good replicate of the true underlying population parameter μ. This simulation based procedure is referred to as *Monte Carlo simulation* and will be carried out by computer. By means on the above mentioned procedure we are able to provide a confidence interval (CI) for θ. The easiest method is to construct a quantile-based CI based on the empirical df $\hat{F}^*$, i.e.

$$\left[\hat{\mu}^*_{\alpha/2}\,;\,\hat{\mu}^*_{1-\alpha/2}\right]\,, \tag{7.9}$$

where $\hat{\mu}^*_{\alpha/2}$ is denoted to be the empirical $\alpha/2$-quantile, analogue for $1-\alpha/2$, i.e. the $[B \times \alpha/2]$-th value in the ordered list of bootstrap replicates ($[\cdot]$ shall denote the nearest integer). This (approximate) bootstrap CI is called *quantile confidence interval* (for detailed information see Chapter 13 in Efron and Tibshirani [15]).
An alternative to (7.9) is to construct a confidence interval based on the student‘s t distribution and the standard error. Therefore compute the bootstrap standard error s^* as defined above and denote by $t^{\alpha/2}$ the $\alpha/2$-quantile of the t-distribution. Hence, a bootstrap estimate of the $1-\alpha$-confidence interval $[\hat{\mu} - t^{1-\alpha/2}\hat{se}; \hat{\mu} + t^{\alpha/2}\hat{se}]$ of μ is given by

$$[\hat{\mu} - t^{1-\alpha/2}s^*; \hat{\mu} + t^{\alpha/2}s^*] \tag{7.10}$$

based on the original sample S, cf. Section 12 in Efron and Tibshirani [15]. This kind of CI is called the *bootstrap-t* confidence interval.
Unfortunately neither the quantile- nor the bootstrap-t confidence interval exhibit two important properties. A confidence interval is called *first order accurate* if

$$P(\mu \le \hat{\mu}[\alpha/2]) = \alpha + O(n^{-1/2})$$

and *second order accurate* if

$$P(\mu \le \hat{\mu}[\alpha/2]) = \alpha + O(n^{-1})\,.$$

The quantile CI is first order accurate, the bootstrap-t CI is second order accurate (see Section 14.3 in Efron and Tibshirani [15]). Hence one may suggest to choose the latter because of "better" coverage, i.e. we expect the true but unknown parameter μ to be found in the $(1-\alpha)$ CI with probability $(1-\alpha)$. Unfortunately, the bootstrap-t CI does not carry another nice property, that of *transformation invariance.* If we want to provide a CI of a function g of the parameter μ we just have to apply the function g

to the endpoints of the CI. This procedure results in a CI for the parameter $g(\mu)$ with the same properties like the CI of μ carries, if g is a monotone increasing function (see Section 14.3 in Efron and Tibshirani [15]).

At last we want to introduce a confidence interval based on bootstrap methods, one which fulfills the property of second order accuracy as well as the transformation invariance. This CI is based on the bootstrap quantiles corresponding to an $(1-2\alpha)$-CI with an adjustment due to bias and acceleration and is therefore called *BC_a-confidence interval* (**b**ias-**c**orrected and **a**ccelerated). For intended coverage of $1-2\alpha$, it is given by

$$BC_a := [\hat{\mu}^*_{\alpha_1}\,;\,\hat{\mu}^*_{\alpha_2}], \tag{7.11}$$

where $\hat{\mu}^*_{\alpha_1}$ and $\hat{\mu}^*_{\alpha_2}$ are the empirical α_1-, respectively α_2-quantiles of the sample of bootstrap estimators $\{\mu^*_1,\ldots,\mu^*_B\}$. α_1 and α_2 are computed by

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(\alpha)})}\right) \text{ and } \alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(1-\alpha)})}\right).$$

Thereby Φ denotes the standard normal df, and $z^{(\alpha)}$ is the $100*\alpha$-th quantile of Φ. The numbers of the bias-correction $\hat{z}_0$ and the acceleration $\hat{a}$ are also easy to compute (see page 186 in Efron and Tibshirani [15]).
One can show that the BC_α confidence interval is second order accurate and invariant under monotone increasing transformations. The proof can be found in Hall [34].

C. Further figures and tables

This section contains further figures and tables regarding the estimation results of Section 5.2.2 and Section 5.3 and which are not represented there for reasons of clarity.

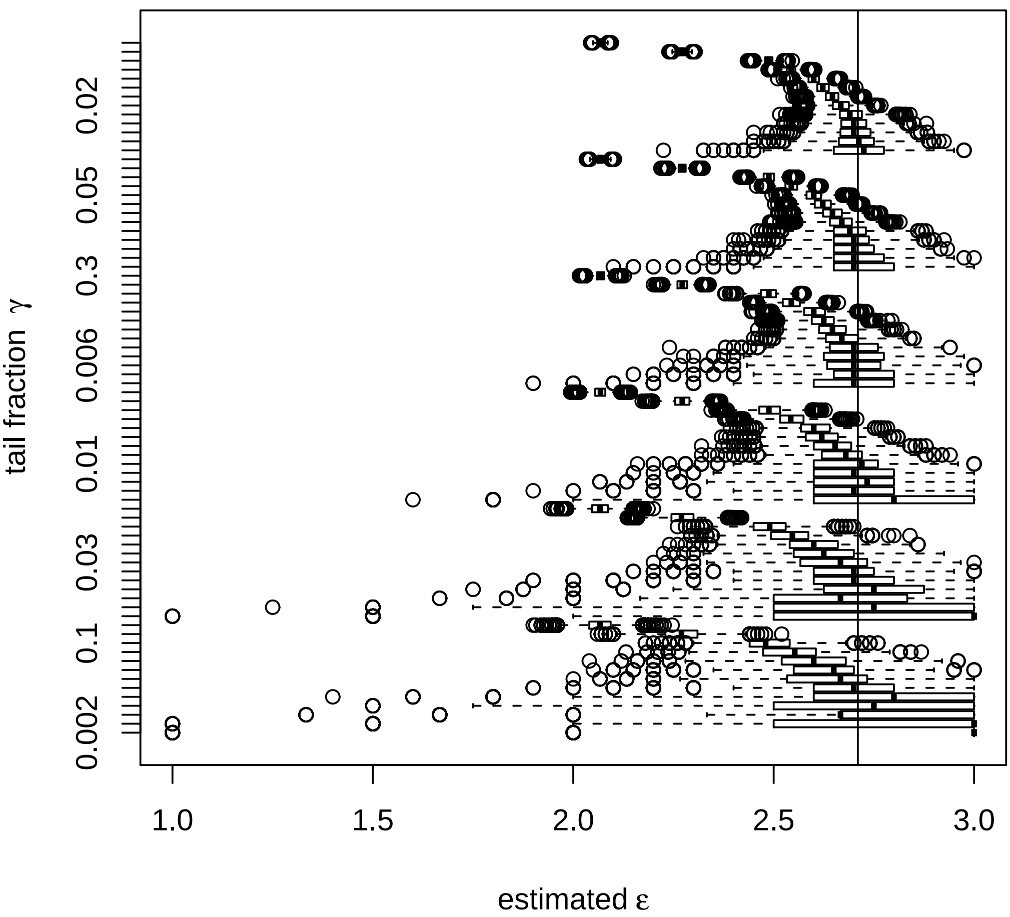


Figure C.1.: Shown is the wave plot corresponding to the estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.1} \approx 2.7$ (represented by the vertical line). The floors from bottom to the top show the increasing sample sizes $n = 500, 1000, 2500, 5000, 10000, 20000$. Boxplots are also grouped by tail fractions γ. Shown data in boxplots represent the simulation results of the nonparametric estimation approach for the extremal coefficient by means of the estimator in (5.3), see Section 5.1.

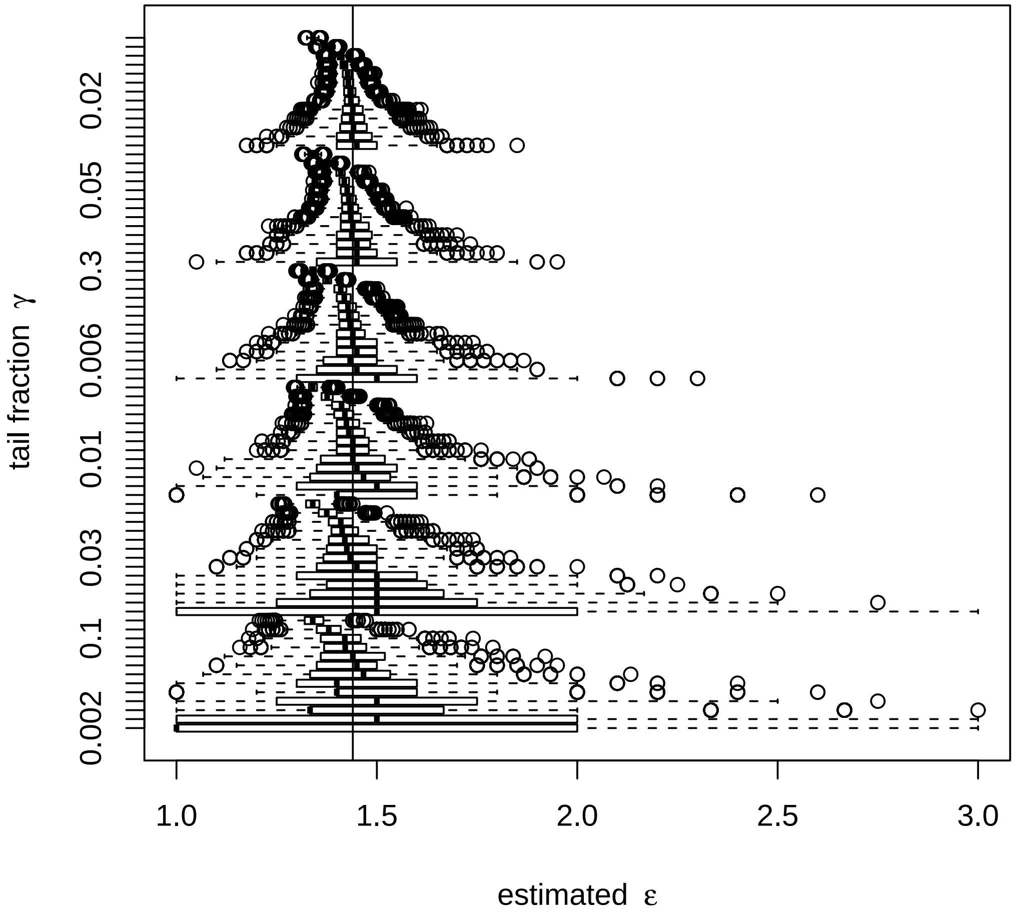


Figure C.2.: Shown is the wave plot corresponding to the estimator of the extremal coefficient $\varepsilon = \|(1,1,1)\|_3 \approx 1.4$ (represented by the vertical line). The floors from bottom to the top show the increasing sample sizes $n = 500, 1000, 2500, 5000, 10000, 20000$. Boxplots are also grouped by tail fractions γ. Shown data in boxplots represent the simulation results of the nonparametric estimation approach for the extremal coefficient by means of the estimator in (5.3), see Section 5.1.

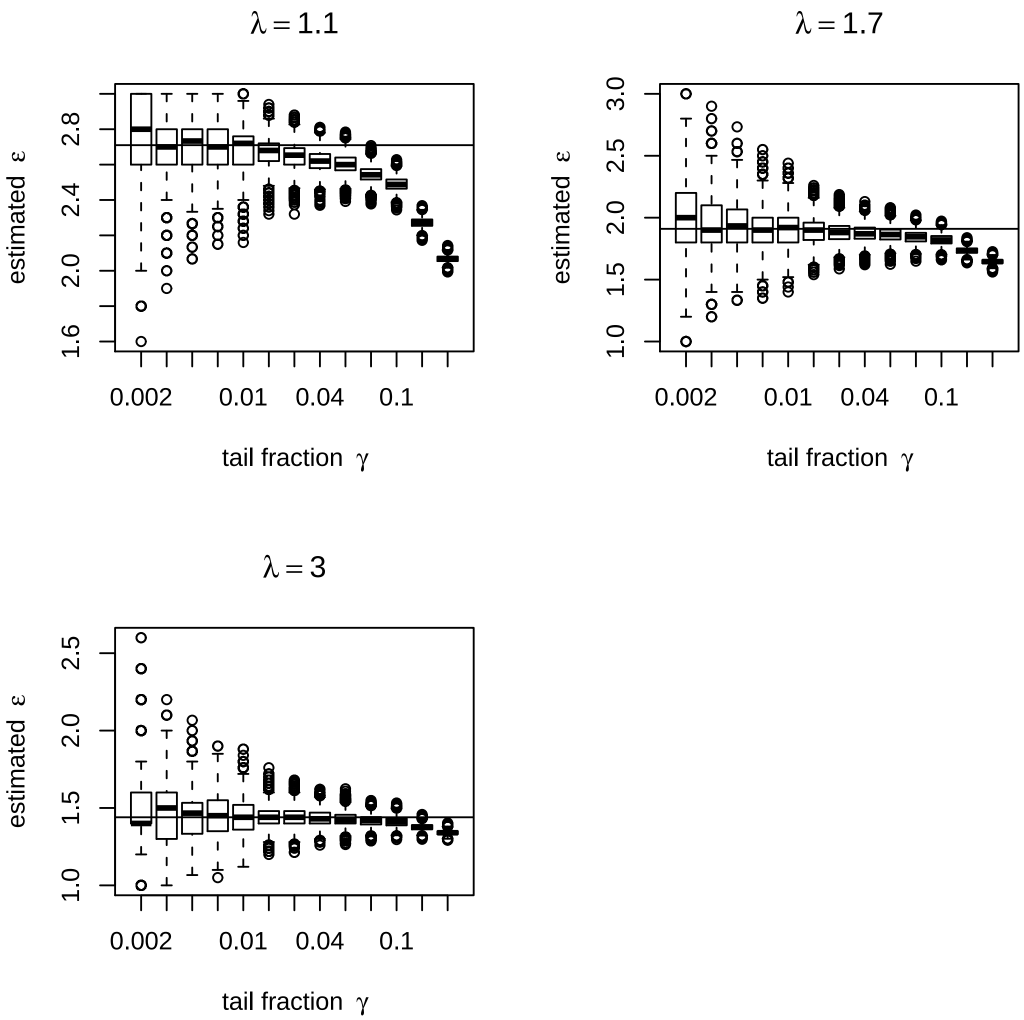


Figure C.3.: The three figures show the boxplots of the estimator (5.3) of the extremal coefficient $\varepsilon = \|(1,1,1)\|_\lambda \in \{2.7, 1.9, 1.4\}$ (represented by the horizontal line) which corresponds to the logistic dependence parameter $\lambda \in \{1.1, 1.7, 3\}$ for simulated sample size $n = 2500$. Shown data represent the simulation results of the nonparametric estimator of the extremal coefficient, see (5.3) in Section 5.1.

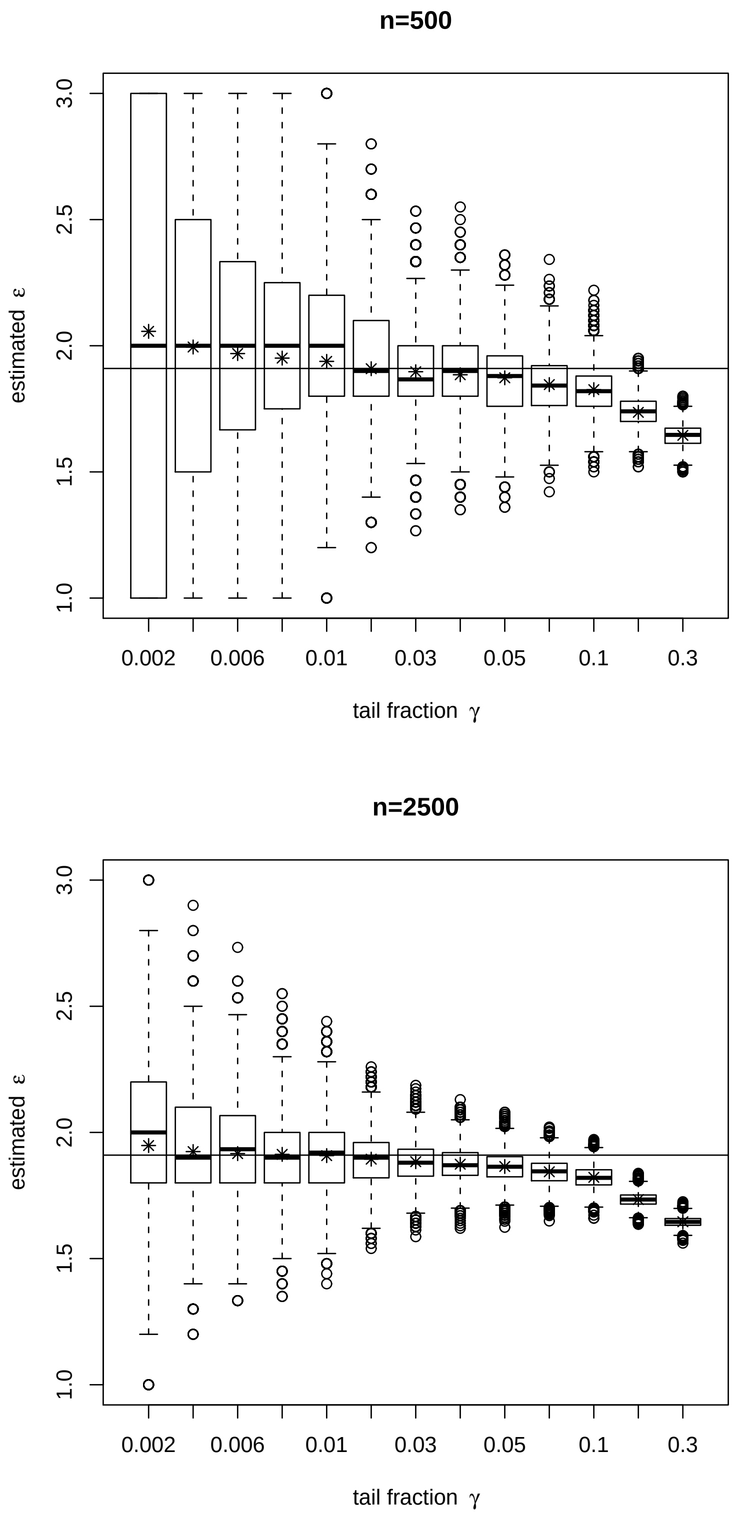
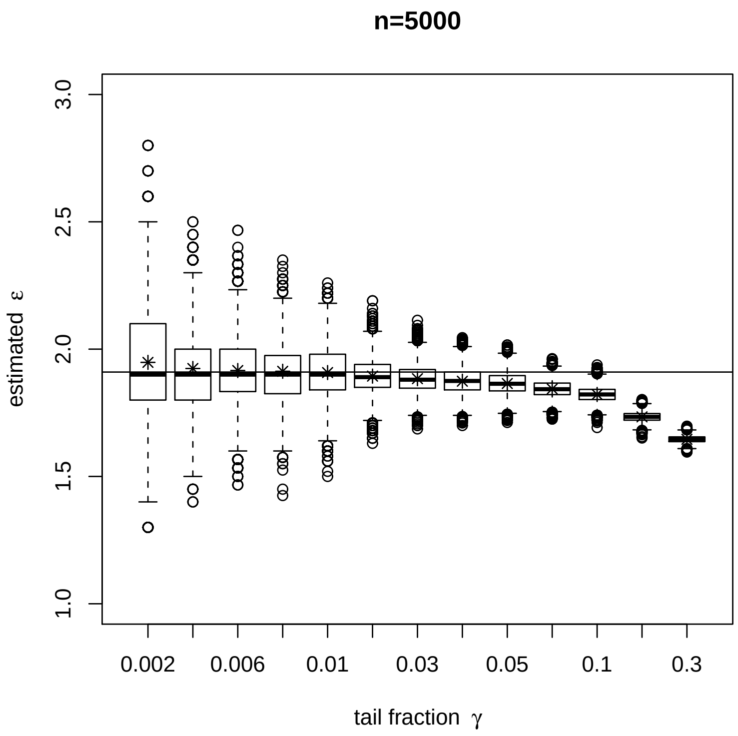


Figure C.4.: The figures show the boxplots of the estimator (5.3) of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$ for simulated sample sizes $n \in \{500, 2500\}$. Shown data represent the simulation results of the nonparametric estimator of the extremal coefficient, see (5.3) in Section 5.1.



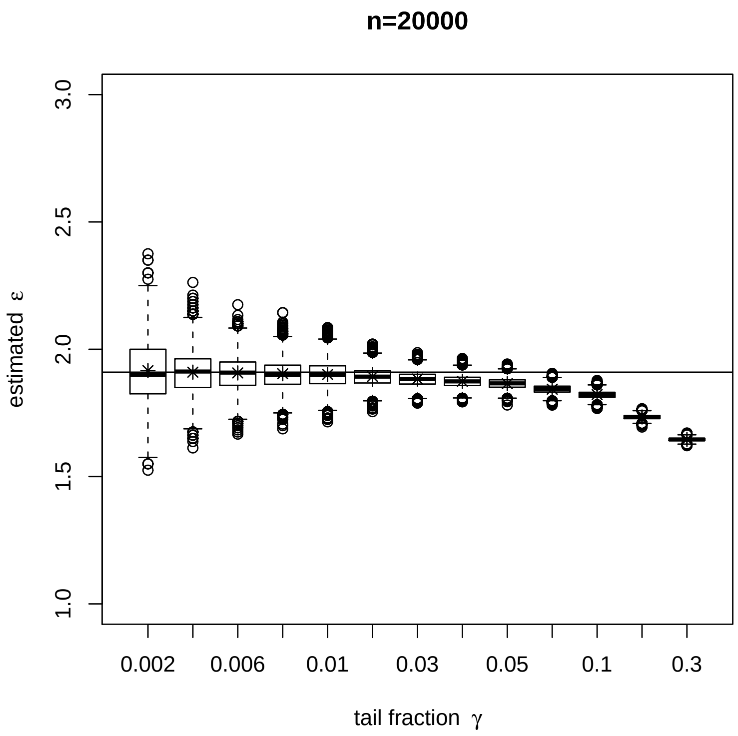


Figure C.5.: The figures show the boxplots of the estimator (5.3) of the extremal coefficient $\varepsilon = \|(1,1,1)\|_{1.7} \approx 1.9$ for simulated sample sizes $n \in \{5000, 20000\}$. Shown data represent the simulation results of the nonparametric estimator of the extremal coefficient, see (5.3) in Section 5.1.

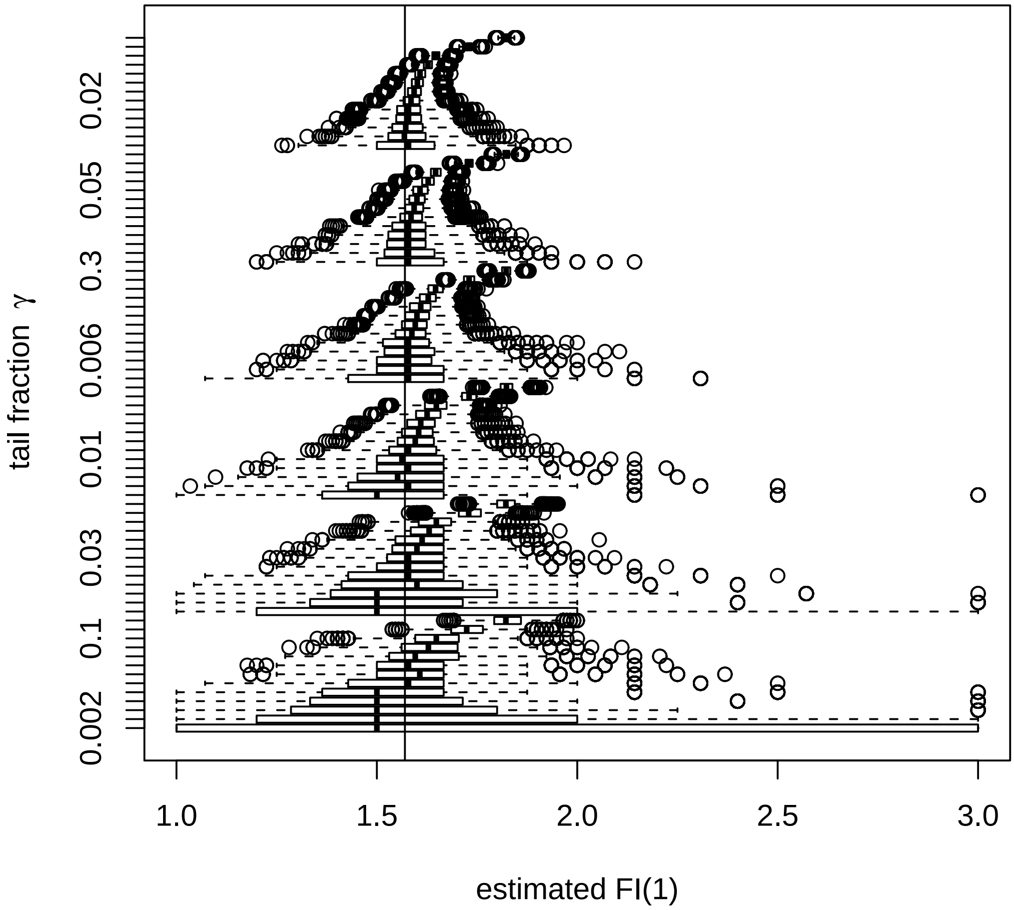


Figure C.6.: Shown is the wave plot of the estimator of the fragility index FI corresponding to the logistic dependence parameter $\lambda = 1.7$. The vertical line represents the true value $FI(1) \approx 1.57$. The floors from bottom to the top show the increasing sample sizes $n = 500, 1000, 2500, 5000, 10000, 20000$. Boxplots are also grouped by tail fractions γ. Shown data in boxplots represent the simulation results of the nonparametric estimation approach for the fragility index, see (5.11) in Section 5.1.

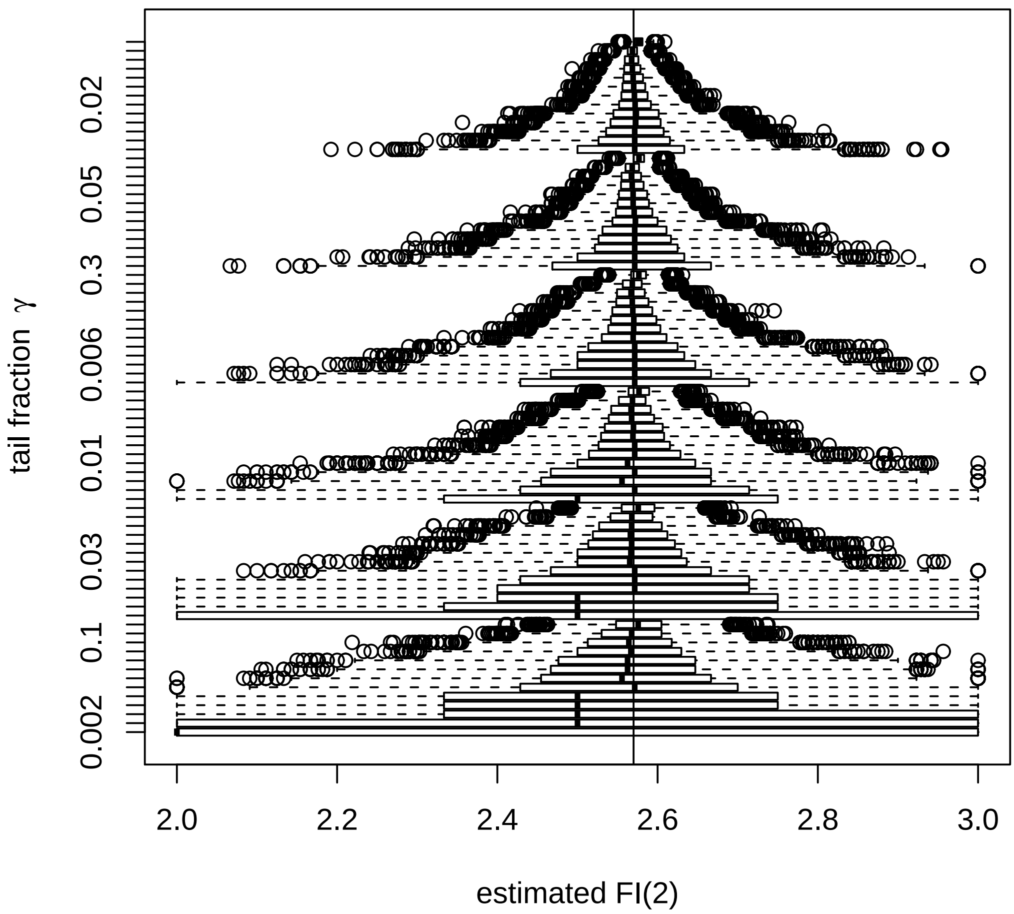


Figure C.7.: Shown is the wave plot of the estimator of the extended fragility index $FI(2)$ corresponding to the logistic dependence parameter $\lambda = 1.7$. The vertical line represents the true value $FI(2) \approx 2.57$. The floors from bottom to the top show the increasing sample sizes $n = 500, 1000, 2500, 5000, 10000, 20000$. Boxplots are also grouped by tail fractions γ. Shown data in boxplots represent the simulation results of the nonparametric estimation approach for extended fragility index by means of the estimator in (5.10), see Section 5.1.

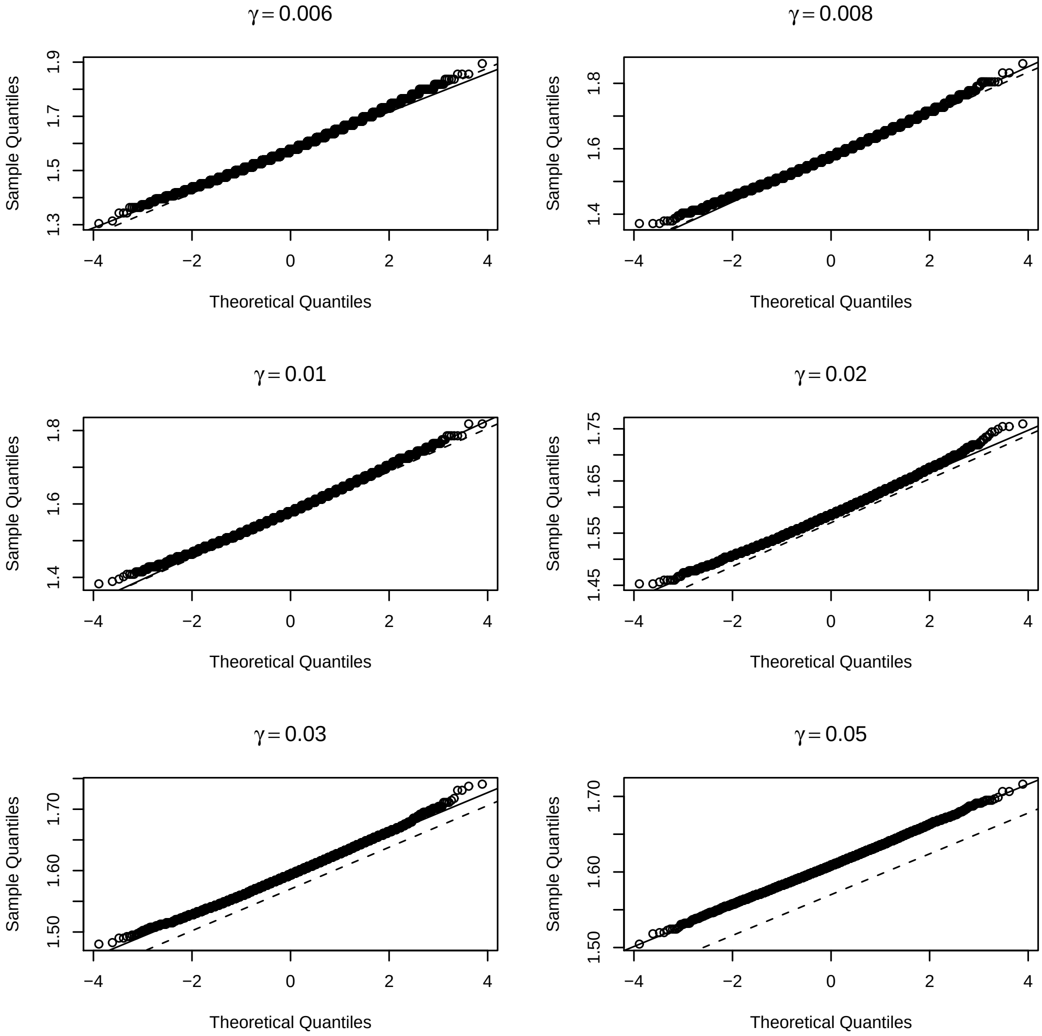


Figure C.8.: The figure shows the NP plots of the estimator $\hat{FI}$ corresponding to the logistic dependence parameter $\lambda = 1.7$ for simulated sample size $n = 10000$ and a representative selection of γ. The solid line connects the first and third quartile of the dataset. The dashed line represents the line $y = \mu_{FI} + \sigma_{FI} * x$, where $\mu_{FI} = FI \approx 1.57$ and $\sigma_{FI} = (d\sigma)/(\varepsilon^2\sqrt{(}k))$ with σ given in (5.9) and $k := \gamma * n$. We expect $\hat{FI} \sim N(\mu_{FI}, \sigma_{FI})$.

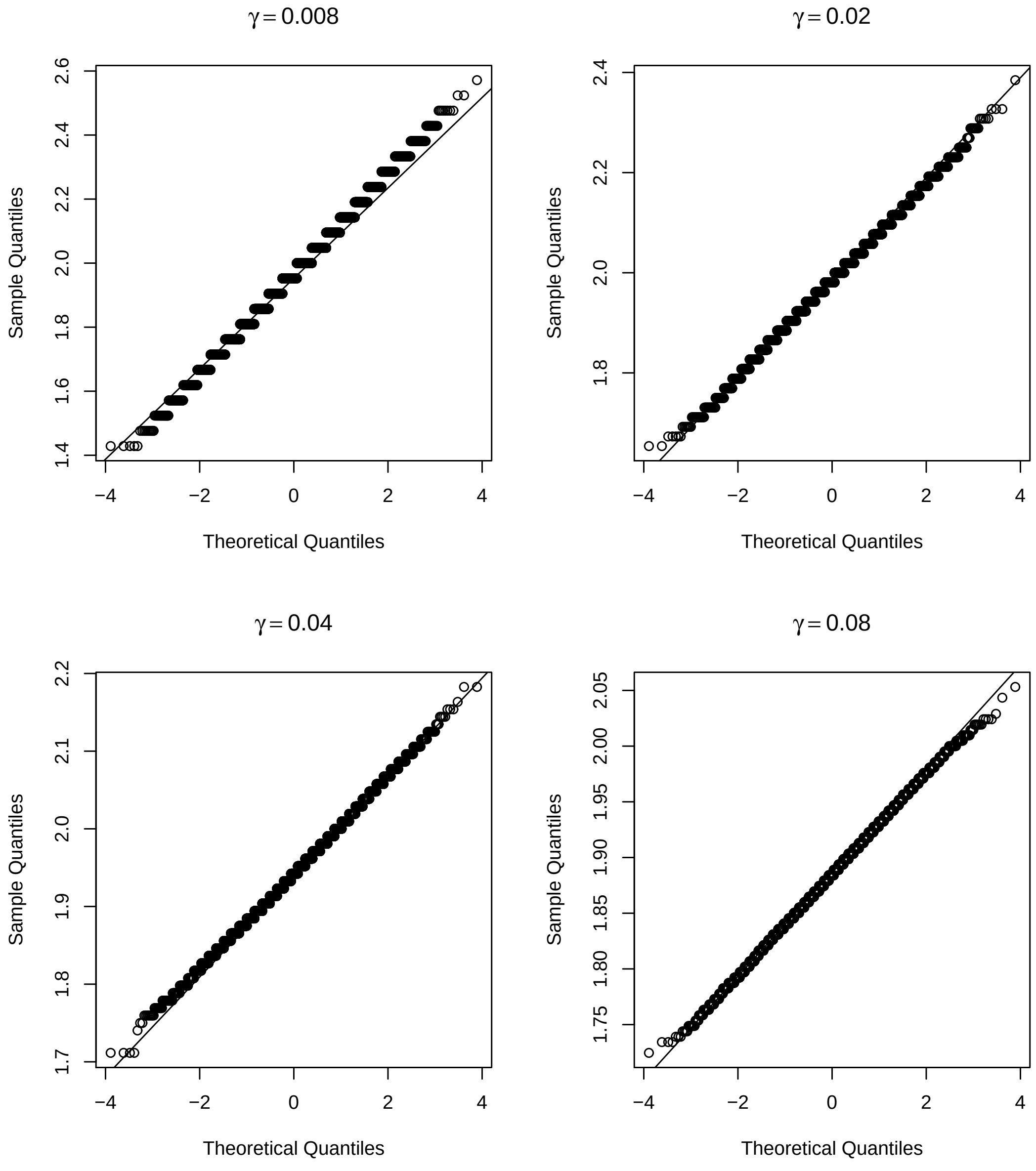


Figure C.9.: NP plots of the bootstrap estimates of the extremal coefficient ε corresponding to the boxplots for $\gamma = 0.008, 0.02, 0.04, 0.08$ in Figure 5.8 of the financial system $\{D, C, A\}$. The sample size is $n = 2577$. The solid line connects the first and third empirical quantile.

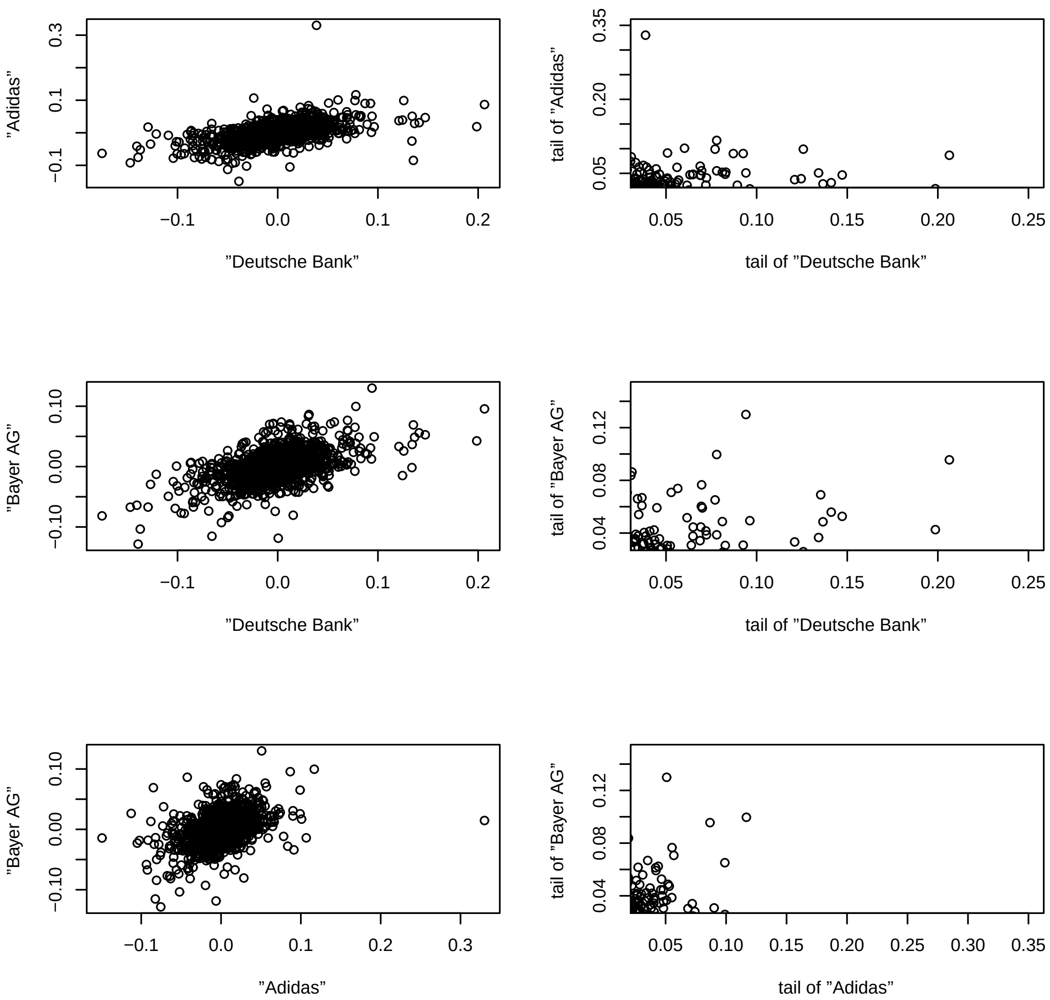


Figure C.10.: Two-dimensional scatterplots of the mixed DAX system $\{D, A, B\}$ during the ten-year period Sept. 2001 - Sept. 2011, cf. Section 5.3. The sample size is $n = 2577$. The complete two-dimensional datasets are shown on the left, where the corresponding upper tails are shown on the right. The tail fraction of the univariate tails is $\gamma = 0.045$.

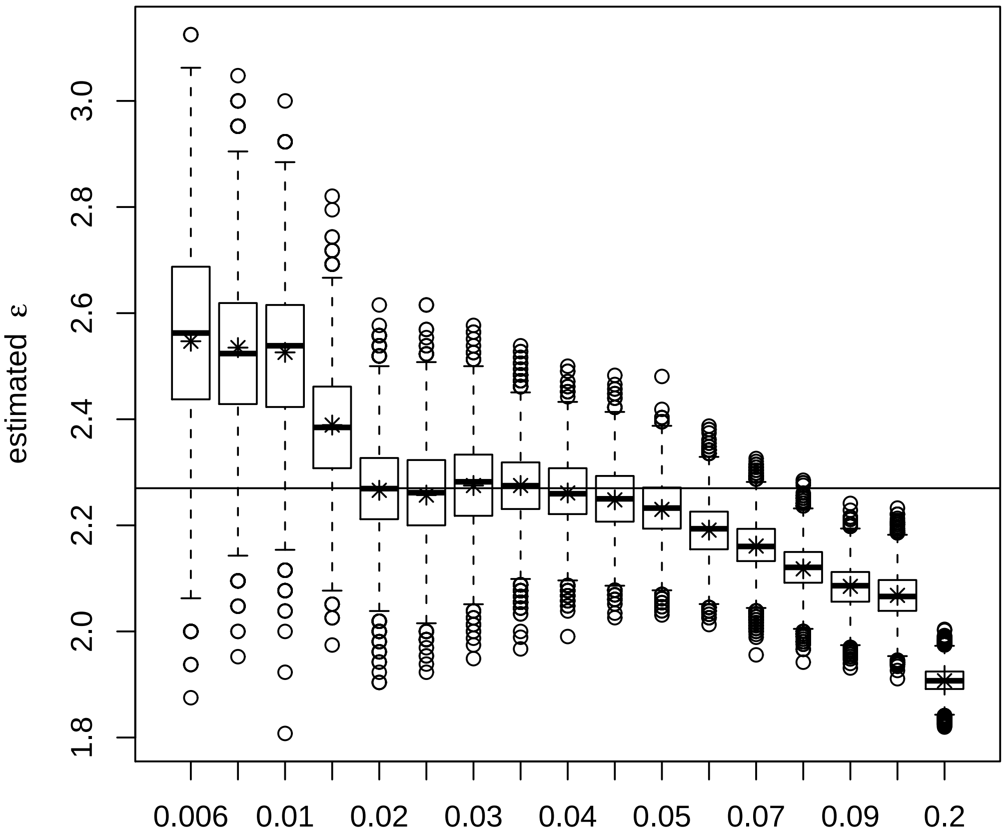


Figure C.11.: Boxplots of the bootstrap estimates of the extremal coefficient ε corresponding to the mixed DAX system $\{D, A, B\}$ during the ten-year period Sept. 2001 - Sept. 2011. Shown are the tail fractions $\gamma = 0.006, 0.008, 0.01, 0.015,$ $0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2$. The sample size is $n = 2577$. The horizontal line represents the value $\varepsilon = 2.27$.

γ	0.008	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
mean(ε)	1.966	1.964	1.985	1.955	1.942	1.899	1.882	1.894	1.885	1.851	1.826
sd(ε)	0.158	0.143	0.096	0.080	0.062	0.057	0.054	0.055	0.045	0.040	0.038
mean(FI)	1.519	1.522	1.508	1.533	1.543	1.579	1.593	1.584	1.591	1.620	1.643
sd(FI)	0.122	0.110	0.073	0.062	0.049	0.047	0.045	0.046	0.038	0.035	0.034
mean(FI(2))	2.466	2.418	2.361	2.368	2.364	2.411	2.429	2.472	2.487	2.472	2.460
sd(FI(2))	0.155	0.126	0.083	0.066	0.055	0.050	0.047	0.044	0.042	0.041	0.035

Table 7.1.: Shown are the sample mean and standard deviation of the bootstrap estimator of the extremal coefficient ε and the (extended) fragility index FI, $FI(2)$ for the financial system $\{D, C, A\}$ respectively, which is investigated in Section 5.3. The table shows a representative selection of tail fractions γ.

γ	0.008	0.01	0.02	0.03	0.04
BC(ε)	[1.71; 2.33]	[1.73; 2.31]	[1.81; 2.19]	[1.78; 2.09]	[1.82; 2.06]
BC(FI)	[1.27; 1.72]	[1.34; 1.77]	[1.39; 1.68]	[1.43; 1.68]	[1.45; 1.65]
BC(FI(2)	[2.28; 2.92]	[2.21; 2.71]	[2.22; 2.55]	[2.26; 2.52]	[2.22; 2.44]
γ	0.05	0.06	0.07	0.08	0.1
BC(ε)	[1.77; 1.99]	[1.75; 1.96]	[1.78; 1.99]	[1.78; 1.96]	[1.78; 1.92]
BC(FI)	[1.51; 1.69]	[1.53; 1.72]	[1.50; 1.68]	[1.53; 1.68]	[1.56; 1.70]
BC(FI(2)	[2.33; 2.53]	[2.30; 2.48]	[2.41; 2.59]	[2.41; 2.58]	[2.38; 2.52]

Table 7.2.: The table shows the bootstrap confidence interval (BC) of the estimators of the extremal coefficient ε, FI and $FI(2)$ for the financial system $\{D, C, A\}$, which is analyzed in Section 5.3. Thereby a representative selection of the tail fraction γ is provided. See (7.11) in Section B for the computation of BC.

γ	0.008	0.01	0.02	0.03	0.04	0.045	0.05	0.06	0.07	0.08	0.1
mean(ε)	2.535	2.526	2.266	2.275	2.261	2.248	2.230	2.191	2.161	2.118	2.068
sd(ε)	0.145	0.130	0.093	0.079	0.063	0.060	0.056	0.051	0.046	0.043	0.041
mean(FI)	1.175	1.180	1.321	1.317	1.325	1.333	1.344	1.368	1.387	1.415	1.450
sd(FI)	0.065	0.060	0.054	0.046	0.037	0.036	0.033	0.032	0.030	0.029	0.029
mean(FI(2))	2.216	2.271	2.241	2.247	2.234	2.220	2.228	2.248	2.247	2.247	2.280
sd(FI(2))	0.185	0.177	0.082	0.066	0.057	0.052	0.049	0.044	0.041	0.037	0.034

Table 7.3.: Shown are the sample mean and standard deviation of the bootstrap estimator of the extremal coefficient ε, FI and $FI(2)$ for the mixed DAX system $\{D, A, B\}$ in Section 5.3.

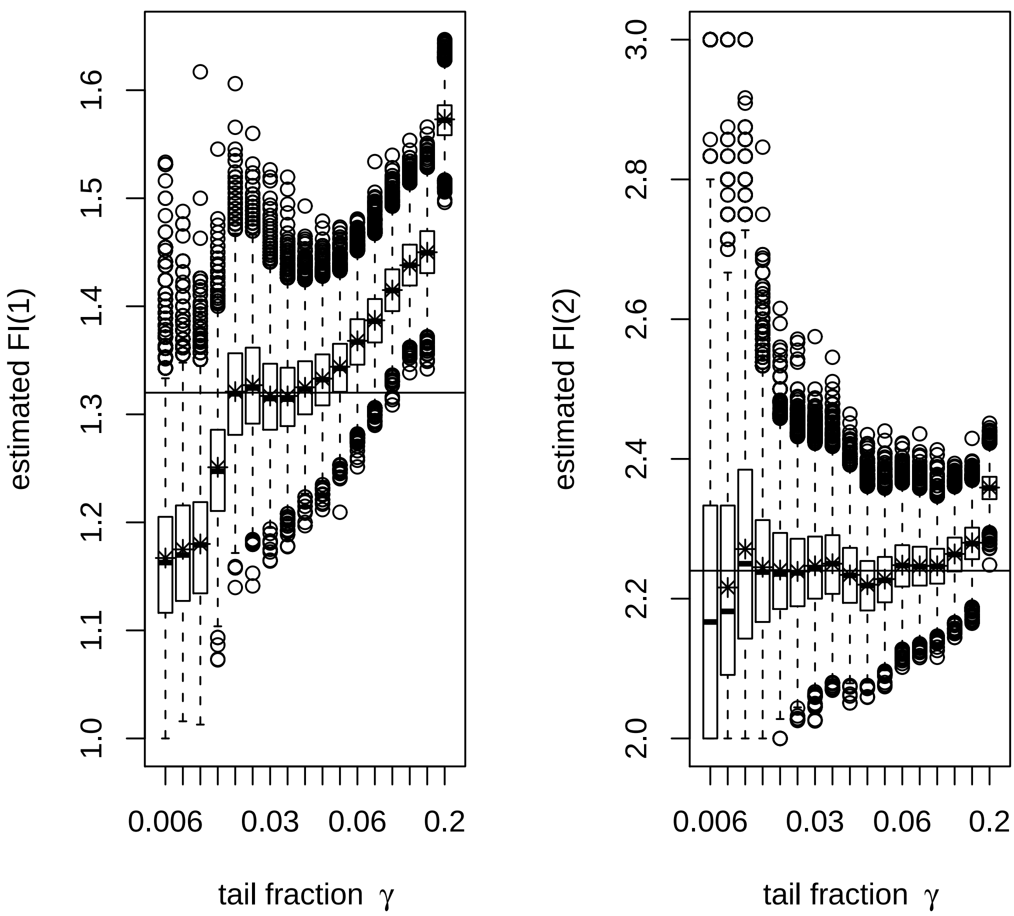


Figure C.12.: Boxplots of the bootstrap estimates of the fragility index FI and the extended fragility index $FI(2)$ corresponding to the mixed DAX system $\{D, A, B\}$ during the ten-year period Sept. 2001 - Sept. 2011, see Section 5.3. Shown are the same tail fractions as in Figure 5.8. The sample size is $n = 2577$. The horizontal line represents the value $FI = 1.32$ (left), resp. $FI(2) = 2.24$ (right).

γ	0.008	0.01	0.02	0.03	0.04	0.045
BC(ε)	[2.24; 2.81]	[2.42; 2.92]	[2.04; 2.40]	[2.14; 2.45]	[2.14; 2.39]	[2.14; 2.37]
BC(FI)	[1.07; 1.31]	[1.06; 1.28]	[1.25; 1.47]	[1.23; 1.41]	[1.26; 1.41]	[1.27; 1.40]
BC(FI(2)	[2.00; 2.50]	[2.13; 3.00]	[2.14; 2.48]	[2.14; 2.41]	[2.11; 2.33]	[2.12; 2.32]
γ	0.05	0.06	0.07	0.08	0.1	
BC(ε)	[2.13; 2.35]	[2.08; 2.27]	[2.12; 2.30]	[2.01; 2.18]	[1.97; 2.13]	
BC(FI)	[1.28; 1.41]	[1.32; 1.44]	[1.30; 1.42]	[1.38; 1.49]	[1.41; 1.52]	
BC(FI(2)	[2.12; 2.32]	[2.15; 2.32]	[2.16; 2.31]	[2.17; 2.32]	[2.22; 2.35]	

Table 7.4.: The table shows the bootstrap confidence interval (BC) of the estimators of the extremal coefficient ε, FI and $FI(2)$ for the mixed DAX system $\{D, A, B\}$, which is analyzed in Section 5.3, cf. Table 7.2. See (7.11) in Section B for the computation of BC.

Danksagung

Meinem Doktorvater, Herrn Prof. Falk, möchte ich an dieser Stelle für sieben lehrreiche Jahre danken.

Ebenso möchte ich mich bei meiner Familie und meinen Freunden bedanken, die mich in den letzten Jahren auf unterschiedlichste Weise verstanden und unterstützt haben.

Dieser Dank ist nicht weiter in Worte zu fassen.

Bibliography

[1] Aulbach, S., Bayer, V. and Falk, M. (2011). A multivariate piecing together approach with an application to operational loss data. To appear in: Bernoulli.

[2] Aulbach, S., Falk, M. and Hofmann, M. (2011). On Max-Stable Processes and the Functional D-Norm. Submitted, preprint available under http://arxiv.org/abs/1107.5136.

[3] Balkema, A.A. and de Haan, L. (1974). Residual life time at great age. Annals of Probability **2**, 792:804.

[4] Balkema, A.A. and Resnick, S.I. (1977). Max-infinite divisibility. Journal of Applied Probability **14**, 309-319.

[5] Beirlant, J., Goegebeur. Y., Segers, J. and Teugels, J. (2004). Statistics of Extremes: Theory and Applications. John Wiley & Sons.

[6] Berman, SM (1961). Convergence to bivariate limiting extreme value distribution. Annals of the Institute for Statistical Mathematics **13**, 217-223.

[7] Buishand, T.A., de Haan, L. and Zhou, C. (2008). On spatial extremes: With application to the rainfall problem. Annals of Applied Statistics, **2**, 624-642.

[8] Capéraà, P., Fougéres and Genest, C. (1997). A nonparametric estimation procedure for bivariate extreme value copulas. Biometrika **84**(3), 567-577.

[9] Danielsson, J., de Haan, L., Peng, L. and de Vries, C.G. (2001). Using a Bootstrap Method to Choose the Sample Fraction in Tail Index Estimation. Journal of Multivariate Analysis **76**, 226-248.

[10] Deheuvels, P. (1984). Probabilistic aspects of multivariate extremes. In *Statistical Extremes and Applications* (Tiago de Oliveira ed.), 117-130. D. Reidel Publishing Company.

[11] Dempster, M. A. H. (2001). Risk Management: Value at Risk and Beyond. Cambridge University Press.

[12] Draisma, G., Drees, H., Ferreira, A. and de Haan, L. (2004). Bivariate tail estimation: Dependence in asymptotic independence. Bernoulli **10**, 251-280.

[13] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). Modeling extreme events for Insurance and Finance. Springer. Heidelberg.

[14] Efron, B. (1979). Bootstrap Methods: Another Look at the Jacknife. Annals of Statistics **7**, 1-26.

[15] Efron, B and Tibshirani, R.J. (1993). An introduction on bootstrap. Chapman and Hall.

[16] Einmahl, J., de Haan, L. and Huang, X. (1993). Estimating a multidimensional extreme value distribution. Journal of Multivariate Analysis, **47**(1), 35-47.

[17] Einmahl, J., Krajina, A. and Segers, J. (2011). An M-estimator for tail dependence in arbitrary dimensions. Technical report.

[18] Falk, M. and Guillou, A. (2007). Peaks-over-threshold stability of multivariate generalized Pareto distributions. Journal of Multivariate Analysis **99**, 715-734.

[19] Falk, M., Hüsler, J. and Reiss, R.-D (2010). Laws of small numbers: Extremes and Rare Events. Third edition. Birkhäuser, Basel.

[20] Galambos, J. (1987). The Asymptotic Theory of Extreme Order Statistics. 2nd edition, Krieger, Malabar.

[21] Gänsler, P. und Stute, W. (1977). Wahrscheinlichkeitstheorie. Springer, Berlin.

[22] Geluk, J.L., De Haan, L. and De Vries, C.G. (2007). Weak and strong financial fragility. Tinbergen Institute Discussion Paper. TI 2007-023/2.

[23] Genest, C., Ghoudi, K. and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika **82**(3), 543-552.

[24] Genest, C. and Segers, J. (2009). Rank-based Inference for Bivariate Extreme-Value Copulas. Annals of Statistics, **37**(5B), 2990-3022.

[25] Geoffroy, J. Contributions à la theorie des valeurs extrêmes (1958). Publ. Inst. Statist. Univ. Paris **8**, 37-185.

[26] Gudendorf, G. and Segers, J. (2009). Extreme-Value Copulas. Technical report. Institut de Statistique, Universitè catholique de Louvain.

[27] Gudendorf, G. and Segers, J. (2011). Nonparametric estimation of an extreme-value copula in arbitrary dimensions. Journal of Multivariate Analysis **102**, 37-47.

[28] Gumbel, E.J. (1960). Distributions des valeurs extremes en plusiers dimensions. Publ. Inst. Statist. Univ. Paris **9**, 171-173.

[29] de Haan, L. and Ferreira, A. (2006). Extreme Value Theory. An Introduction. Springer.

[30] de Haan, L., Neves, C. and Peng, L. (2008). Parametric tail copula estimation and model testing. Journal of Multivariate Analysis, **99**, 1260-1275.

[31] de Haan, L. and Resnick, S. (1977). Limit theory of multivariate sample extremes. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **40**, 317-337.

[32] de Haan, L. and Resnick, S. (1993). Estimating the limit distribution of multivariate extremes. Comm. Statist. Stochastic Models, **9**(2), 275-309.

[33] de Haan, L. and de Ronde, J. (1998). Sea and Wind: Multivariate Extremes at Work. Extremes **1**(1), 7-45.

[34] Hall, P. (1988). Theoretical comparison of bootstrap confidence intervals (with discussion). The Annals of Statistics, **16**, 927-953.

[35] Heffernan, J. E. (2000). A Directory of Coefficients of Tail Dependence. Extremes, **3**(3), 279-290.

[36] Hofmann, D. (2009). Characterization of the D-Norm Corresponding to a Multivariate Extreme Value Distribution. PhD-thesis, University of Würzburg.

[37] Hsing, T., Klüppelberg, C. and Kuhn, G. (2004). Dependence Estimation and Visualization in Multivariate Extremes with Applications to Financial Data. Extremes **7**(2), 99-121.

[38] Hsing, T., Hüsler, J. and Leadbetter, M.R. (1988). On the Exceedance Point Process for a Stationary Sequence. Probability Theory and Related Fields **78**, 97-112.

[39] Huang,X. (1992). Statistics of Bivariate Extreme Values. PhD-thesis, Tinbergen Institute Research Series.

[40] Klüppelberg, C. Kuhn, G. and Liang, P. (2007). Estimating the tail dependence function of an elliptical distribution. Bernoulli **13**(1), 229-251.

[41] Klüppelberg, Kuhn, Peng (2008). Semi-Parametric Models for the Multivariate Tail Dependence Function - the Asymptotically Dependent Case. Scandinavian Journal of Statistics. **35** (4), 701-718.

[42] Kotz, S. and Nadarajah, S. (2000). Extreme Value Distributions. Theory and Applications. Imperial College Press, London.

[43] Leadbetter, M.R. (1991). On a basis for 'peaks over threshold' modeling. Statistical Probability Letters **12**, 357-362.

[44] Ledford, A.W. and Tawn, J.A. (1996). Statistics for near independence in multivariate extreme values. Biometrika, **83**, 169-187.

[45] Ledford, A.W. and Tawn, J.A. (1997). Modeling dependence within joint tail regions. Journal of the Royal Statistic Society, **B 59**, 475-499.

[46] Lehmann, E.L. and Casella, G. (1998). Theory of Point Estimation. Springer, New York.

[47] McNeil, A., Frey, R., and Embrechts, P. (2005). Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press, Princeton.

[48] McNeil, A.J. and Něslehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and L_1-norm symmetric distributions. Annals of Statistics, **37**(5B), 3059-3097.

[49] Michel, R. (2007). Simulation and Estimation in Multivariate Generalized Pareto Models. Dissertation, Bayerische Julius−Maximilians−Universität Würzburg.

[50] Michel, R. (2008). Some notes on multivariate generalized Pareto distributions. Journal of Multivariate Analysis **99**(6), 1288-1301.

[51] Michel, R. (2007). Simulation of certain multivariate generalized Pareto distributions. Extremes **10**, 83-107.

[52] Nelsen, R.B. (2006). An Introduction to Copulas. Springer. New York.

[53] Pickands, J. (1975). Statistical inference using extreme order statistics. Annals of Statistics **3**, 119-131.

[54] Pickands, J. (1981). Multivariate extreme value distributions. Bull. Internat. Statist. Inst. 49, 859-878.

[55] Reiss, R.-D. and Thomas, M. (2007). Statistical Analysis of Extreme Values. Birkhäuser, Basel.

[56] Resnick, S.I. (1987). Extreme Values, Regular Variation, and Point Processes. Springer, New York.

[57] Rootzén, H. and Tajvidi, N. (2006). Multivariate generalized Pareto distributions. Bernoulli **12**, 917-930.

[58] Schlather, M. and Tawn, J. Inequalities for the Extremal Coefficients of Multivariate Extreme Value Distributions (2002). Extremes, **5**(1), 87-102.

[59] Shi, D. (1995). Fisher information for the multivariate extreme value distribution. Biometrika **82**, 644-699.

[60] Sibuya, M. (1960). Bivariate extreme statistics. Ann. Inst. Statist. Math. Tokyo. **11**, 195-210.

[61] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. Publ Inst Statist Univ Paris 8:229-231.

[62] Smith, R.L.(1987). Estimating tails of probability distributions. Annals of Statistics **15**, 1174-1207.

[63] Smith, R.L. (1990a). Max stable processes and spatial extremes. Technical report.

[64] Smith, R.L. (1990b). Extreme Value Theory. in Handbook of applicable Mathematics (W. Ledermann, ed), vol.7. Wiley, New York.

[65] Smith, R.L. (1994). Multivariate Threshold Methods. In: Extreme Value Theory an Application. Ed. J. Galambos, J. Lechner and E. Simiu, 225-248. Dordrecht: Kluwer.

[66] Smith, R.L., Tawn, J.A. and Coles, S.G. (1997). Markov chain models for threshold exceedances. Biometrika, **84**(2), 249-268.

[67] Stephenson, A. (2003). Simulating Multivariate Extreme Value Distributions of Logistic Type. Extremes **6**(1), 49-59.

[68] Takahashi, R. (1994). Asymptotic independence and perfect dependence of vector components of multivariate extreme statistics. Statistics and Probability Letters **19**, 19-26.

[69] Tawn, J.A. (1988). Bivariate extreme value theory: Models and estimation. Biometrika **75**, pp 397-415.

[70] Tiago de Oliveira, J. (1962/63). Structure theory of bivariate extremes, extensions. Est. Mat., Estat. e Econ. 7, 165-195.

[71] Zhang, D., Wells, M.Z. and Peng, L. (2008). Nonparametric estimation of the dependence function for a multivariate extreme value distribution. Journal of Multivariate Analysis **99**(4), 577-588.