Holographic Description of Curved-Space Quantum Field Theory and Gravity

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There is something fascinating about science.
One gets such wholesale returns of conjecture
out of such a trifling investment of fact.

- *Mark Twain*

Abstract

The celebrated AdS/CFT dualities provide a window to strongly-coupled quantum field theories (QFTs), which are realized in nature at the most fundamental level on the one hand, but are hardly accessible for the standard mathematical tools on the other hand. The prototype examples of AdS/CFT relate classical supergravity theories on $(d+1)$-dimensional anti-de Sitter space (AdS) to strongly-coupled d-dimensional conformal field theories (CFTs). The AdS spacetimes are characterized by a constant negative curvature and admit a timelike conformal boundary, on which the dual CFT is defined. In that sense the AdS/CFT dualities are holographic, and this new approach has led to remarkable progress in understanding strongly-coupled QFTs defined on Minkowski space and on the Einstein cylinder. On the other hand, the study of QFT on more generic curved spacetimes is of fundamental interest and non-trivial already for free theories. Moreover, understanding the properties of gravity – the actual description of spacetime dynamics – as a quantum theory remains among the hardest problems to solve in physics. Both of these issues can be studied holographically and we are interested here in generalizations of AdS/CFT involving on the lower-dimensional side QFTs on curved backgrounds and as a further generalization gravity.

In the first part we take the natural first step from flat-space QFT towards gravity and expand on the holographic description of QFT on fixed curved backgrounds. The description of a CFT on a specific background involves gravity on an asymptotically-AdS space with that prescribed boundary structure. We discuss geometries with de Sitter and AdS as conformal boundary to holographically describe CFTs on these maximally symmetric spacetimes. After setting up the procedure of holographic renormalization we study the reflection of CFT unitarity properties in the dual bulk description. The geometry with AdS on the boundary exhibits a number of interesting features, mainly due to the fact that the boundary itself has a boundary. We study both cases and resolve potential tensions between the unitarity properties of the bulk and boundary theories, which would be incompatible with a duality. The origin of these tensions is partly in the structure of the geometry with AdS conformal boundary, while another one arises for a particular limiting case where the bulk and boundary descriptions naïvely disagree. Besides technical challenges, the interesting hierarchy of boundaries for the geometry with AdS conformal boundary offers the possibility of multi-layered AdS/CFT dualities. Namely, having the dual theory on the conformal boundary itself defined on an AdS space of codimension 1 offers the logical possibility of implementing a second instance of AdS/CFT. We discuss an appropriate geometric setting allowing for the notion of the boundary of a boundary and draw conclusions on limitations for such multi-layered dualities from our previous investigation of holographic renormalization.

In the second part we consider five-dimensional supergravities resulting from string theory as low-energy limits and whose solutions can be lifted to actual string-theory backgrounds. We work out the asymptotic structure of the theories on asymptotically-AdS spaces, carry out

the holographic renormalization and calculate the Weyl anomaly of the dual CFTs. These holographic calculations confirm the expectations from the field-theory side and provide a non-trivial test of the AdS/CFT conjecture. Moreover, the holographic renormalization also alters the symplectic structure of the bulk theory, and building on the previous results we show that in addition to Dirichlet also Neumann or mixed boundary conditions can be imposed. That deformation of AdS/CFT by changing the boundary conditions in particular promotes the boundary metric to a dynamical quantity and is expected to yield a holographic relation between a conformal supergravity on the boundary and the bulk theory. The boundary theory obtained this way exhibits pathologies such as perturbative ghosts, which is in fact expected for a conformal gravity. The fate of these ghosts beyond perturbation theory actually is a longstanding question and our setting provides a starting point to study it from the string-theory perspective. That discussion leads to a regime where the holographic description of the boundary theory requires quantization of the bulk supergravity.

A necessary ingredient of any supergravity is a number of gravitinos as superpartners of the graviton, for which we thus need an effective-QFT description – to make sense of AdS/CFT beyond the limit where bulk theory becomes classical, but also more generally e.g. for cosmological applications. In particular, quantization should be possible not only on rigid AdS, but also on generic asymptotically-AdS spacetimes which may not be Einstein. In the third part we study the quantization and causality properties of the gravitino on Friedmann-Robertson-Walker spacetimes to explicitly show that a consistent quantization can be carried out also on non-Einstein spaces, in contrast to claims in the recent literature. Furthermore, this reveals interesting non-standard effects for the gravitino propagation, which in certain cases is restricted to regions more narrow than the expected light cones.

Zusammenfassung

Die bemerkenswerten AdS/CFT-Dualitäten ermöglichen einen Zugang zu stark gekoppelten Quantenfeldtheorien (QFT), welche einerseits für die Beschreibung der Natur auf fundamentaler Ebene eine große Rolle spielen, andererseits aber mittels der üblichen mathematischen Methoden schwer zu behandeln sind. Die etablierten Beispiele solcher AdS/CFT-Dualitäten liefern eine Identifikation von klassischen supersymmetrischen Gravitationstheorien auf $(d+1)$-dimensionalen anti-de Sitter Räumen (AdS) mit d-dimensionalen stark gekoppelten konformen Feldtheorien (CFT). Die AdS Raumzeiten zeichnen sich durch konstante negative Krümmung aus und besitzen einen zeitartigen konformen Rand, auf dem die duale CFT definiert ist. In diesem Sinn sind die Dualitäten also holographisch, und dieser neue Zugang hat zu beachtlichen Fortschritten im Verständnis von CFT auf der Minkowski Raumzeit und dem Einstein-Zylinder geführt. Auf der anderen Seite ist das Verständnis von QFT auf allgemeineren gekrümmten Mannigfaltigkeiten von besonderem Interesse und nicht-trivial bereits für freie Theorien. Darüber hinaus bleibt das Verständnis von Gravitation, der eigentlichen Beschreibung von dynamischer Raumzeit, als Quantentheorie eines der schwierigsten Probleme in der Physik. Beide Fragestellungen können holographisch untersucht werden, und wir sind hier an Verallgemeinerungen der üblichen AdS/CFT-Dualitäten interessiert, welche auf der niederdimensionalen Seite QFT auf gekrümmten Räumen und als weitere Verallgemeinerung auch Gravitation beschreiben.

Im ersten Teil beschäftigen wir uns mit dem natürlichen ersten Schritt von QFT auf flachen Räumen in Richtung Gravitation und erweitern die Beschreibung von QFT auf gekrümmten Raumzeiten. Die Beschreibung einer CFT auf einem bestimmten Hintergrund bedient sich einer Gravitationstheorie auf einer asymptotisch-AdS Raumzeit mit dieser gegebenen Randstruktur. Wir diskutieren Geometrien, deren konformer Rand mit de Sitter oder AdS Raumzeiten identifiziert werden kann, um CFTs auf diesen maximal symmetrischen Räumen holographisch zu beschreiben. Nachdem wir die holographische Renormierung auf diesen Raumzeiten etabliert haben, studieren wir die Widerspiegelung von Unitaritätseigenschaften der CFT in der dualen *bulk*-Beschreibung. Die Geometrie mit AdS als Rand zeigt eine Reihe von interessanten Eigenschaften, was hauptsächlich darauf zurückzuführen ist, dass der Rand dieser Geometrie selbst einen Rand hat. Wir untersuchen beide Geometrien und lösen potenzielle Differenzen in den Eigenschaften der Rand- und *bulk*-Theorien, welche mit einer Dualität inkompatibel wären. Der Ursprung dieser Differenzen liegt zum einen in der Struktur der Geometrie mit AdS als Rand und rührt zum anderen von einem speziellen Grenzfall, in dem sich die beiden Beschreibungen auf den ersten Blick unterscheiden. Neben technischen Herausforderungen bietet die Hierarchie von Rändern bei der Geometrie mit AdS als Rand die interessante Möglichkeit von mehrstufigen AdS/CFT-Dualitäten. Mit der dualen CFT wiederum definiert auf einem AdS Raum von Kodimension 1 besteht zumindest prinzipiell die Möglichkeit, eine weitere Instanz von AdS/CFT zu implementieren. Wir

diskutieren den passenden geometrischen Rahmen, in dem der Begriff des Randes eines Randes ein wohldefiniertes Konzept ist, und leiten aus unseren vorherigen Untersuchungen erste Schlussfolgerungen über die Möglichkeiten solcher mehrstufigen Dualitäten ab.

Im zweiten Teil behandeln wir fünfdimensionale supersymmetrische Gravitationstheorien, welche als Niederenergie-Grenzfälle aus Stringtheorie resultieren und deren Lösungen als Stringtheorie-Konfigurationen interpretiert werden können. Wir arbeiten die asymptotische Struktur dieser Theorien auf asymptotisch-AdS Räumen heraus, führen die holographische Renormierung durch und berechnen die Weyl-Anomalie der dualen CFTs. Diese holographischen Rechnungen bestätigen die Erwartungen von der Feldtheorieseite und liefern damit einen nicht-trivialen Test der AdS/CFT-Vermutung. Zudem beeinflusst die holographische Renormierung auch die symplektische Struktur der Gravitationstheorien, und aufbauend auf den vorherigen Resultaten zeigen wir, dass zusätzlich zu den üblichen Dirichlet- auch Neumann- oder gemischte Randbedingungen gestellt werden können. Mit dieser Deformation der ursprünglichen AdS/CFT-Dualität durch eine Veränderung der Randbedingungen wird insbesondere die Randmetrik zu einer dynamischen Größe, und man erwartet eine holographische Dualität zwischen der *bulk*-Theorie und einer konformen supersymmetrischen Gravitationstheorie auf dem Rand. Die so erhaltene Randtheorie weist pathologische Eigenschaften wie perturbative Geister auf, was für konforme Gravitationstheorien tatsächlich zu erwarten ist. Die Rolle dieser Geister über die Störungstheorie hinaus ist eine seit langem offene Frage und unsere Konstruktion bietet einen Startpunkt, sie von der Stringtheorie-Perspektive zu untersuchen. Dies führt uns in einen Bereich, in dem die holographische Beschreibung der Randtheorie eine Betrachtung der Gravitationstheorie als effektive QFT erfordert.

Ein notwendiger Bestandteil einer supersymmetrischen Gravitationstheorie ist eine Anzahl von Gravitinos als Superpartner des Gravitons, für welche wir daher eine Beschreibung in Form von effektiver QFT benötigen. Dies gilt speziell um AdS/CFT-Dualitäten über den Grenzfall mit klassischer *bulk*-Theorie hinaus verstehen zu können, aber auch allgemeiner für z.B. kosmologische Anwendungen. Insbesondere sollte die Quantisierung nicht nur auf AdS selbst, sondern auch allgemeiner auf z.B. asymptotisch-AdS Raumzeiten möglich sein, die nicht notwendig die Einstein-Bedingung erfüllen. Im dritten Teil studieren wir die Quantisierung und Kausalitätseigenschaften des Gravitinos auf Friedmann-Robertson-Walker Raumzeiten. Dabei zeigen wir explizit, dass eine konsistente Quantisierung auch auf Raumzeiten möglich ist, die nicht der Einstein-Bedingung genügen, im Gegensatz zu anderslautenden Schlussfolgerungen in der aktuellen Literatur. Darüber hinaus finden wir interessante Effekte für die Propagation der Gravitinos, welche in bestimmten Fällen auf echte Teilmengen der zu erwartenden Lichtkegel eingeschränkt ist.

Contents

1 Introduction and Outline

Our current understanding of nature on shortest length scales is based on the mathematical description in terms of quantum field theory (QFT). The predictions obtained from the so-called Standard Model of particle physics are mostly based on perturbative expansions in the coupling constants and as of now agree astonishingly well with the results of collider experiments. On the other hand, coping with the theory in situations where perturbative expansions are not available, e.g. understanding the spectra of hadrons and mesons, remains a technically very challenging issue. Besides lattice simulations and descriptions in terms of effective field theories, which both have their limitations, the holographic dualities among seemingly very different theories as obtained from string theory are a promising alternative. In addition to the prospects for their practical applications these dualities are of interest also from a more conceptual perspective. Namely, they reduce the number of truly different QFTs and may in turn also be used to study gravitational puzzles like the black hole information paradox via conformal field theory (CFT).

Historically, a first indication that holography could play a role at a fundamental level was the Bekenstein bound, which gives an upper bound on the entropy in a spacetime region in terms of the surface area of its boundary [1]. This led to the proposal of the holographic principle by 't Hooft and Susskind [2]. On the other hand, the relation of U(N) Yang-Mills theory at large N and string theory has been known for long [3] and a duality of open and closed strings has also been investigated early, see e.g. the review [4]. However, only with the AdS/CFT conjecture formulated by Maldacena [5] and subsequently worked out in more detail by Witten [6] and Gubser, Klebanov and Polyakov [7], contact was made with concrete applications to describe strongly-coupled QFTs in terms of string theory. This is accomplished by identifying classical supergravity theories on anti-de Sitter space (AdS) with strongly-coupled CFTs on the conformal boundary of AdS. The duality has since been applied to describe a variety of phenomena ranging from particle physics right up to solid state physics [8]. Nevertheless, although being well tested these dualities are still conjectural and the range of their validity is not clear, such that non-trivial tests are still called for. We review this in more detail in Sec. 2 where some preliminary material is covered. The main part of this thesis will be concerned with generalizations of the AdS/CFT correspondence to cover on the boundary also QFTs on curved spacetimes and gravity.

QFT on curved spacetimes, the tool to study fundamental phenomena like Hawking radiation, is frequently complicated already for free theories, such that alternative descriptions are desirable. Beyond the prime AdS/CFT examples involving gravity on global/Poincaré AdS, which is dual to CFTs on the Einstein cylinder/Minkowski space, the holographic study of CFT on more general curved spacetimes is therefore of particular interest. This is true also from a more conceptual perspective, to see how far the dualities actually extend. The holographic description of a CFT on a specific background involves gravity on an asymptotically-AdS space with that prescribed boundary structure. We study in Sec. 3 geometries with de Sitter (dS) and AdS as conformal boundary, to holographically describe CFTs on these maximally symmetric spacetimes. We focus on unitarity properties and the possibility of multi-layered holographic dualities, which is offered by the setting with AdS on the boundary. The former entails a proper setup of holographic renormalization for these geometries and we study the reflection of the CFT unitarity properties in the spectra of bulk scalar fields in Sec 3.1. For a specific limiting case the 'remarkable' singleton representation of the AdS isometry group, discovered by Dirac [9], turns out to play a crucial role as we discuss in Sec. 3.2. These results were published in [10] and [11]. In Sec. 3.3 we discuss $\langle n \rangle$-manifolds as an appropriate geometric setting for multi-layered holography and construct an example which at least in principle allows for hierarchies of AdS/CFT dualities. From our previous investigations we can draw first conclusions on the prospects for such hierarchies of dualities.

In the second part we study the natural further generalization including gravity on the boundary. Understanding the properties of gravity as a quantum theory remains among the hardest problems to solve in physics, and a holographic description is a promising route to gain insights. A prominent example of qualitative questions where AdS/CFT provides insight is the black-hole information paradox [12]. However, the study of gravitational theories via their CFT dual in that setting is limited to the AdS-type bulk supergravities. That limitation can be avoided with a gravitational theory on the boundary, to study e.g. strongly-coupled Minkowski-space CFTs coupled to gravity. The boundary theory is again expected to possess conformal invariance, and in fact conformal gravity has long been of interest as a possible UV completion of general relativity, although its perturbative treatments are plagued by ghosts. In Sec. 4 we study complete supergravities on asymptotically-AdS spaces. The boundary values of the bulk fields constitute gravity supermultiplets providing the background on which the dual CFT is defined. In generic gravitational backgrounds the scale invariance of a CFT may be broken by the so-called Weyl anomaly [13] (for physical consequences see also [14]), and we calculate holographically the Weyl anomaly of the dual CFTs in generic conformal supergravity backgrounds. This provides a test of the standard AdS/CFT conjecture and these results were published in [15]. The boundary values of the bulk fields are fixed in the standard AdS/CFT setting by Dirichlet boundary conditions, and we proceed by establishing boundary conditions promoting these boundary conformal supergravity fields to dynamical quantities. This deformation of the original AdS/CFT correspondence is expected to yield a dual bulk description of gravity on the boundary, which should in particular provide insights regarding the fate of the ghosts. That discussion leads to the issue of quantizing the bulk theory.

The AdS/CFT prescription relates CFTs in the limit of infinite rank N of the gauge group to classical bulk supergravities, and corrections to that limit are related to quantum corrections in the bulk theory. Supergravities inevitably contain as superpartner for the graviton a gravitino, for which a consistent quantization prescription is therefore crucial. In particular, a quantization prescription is needed not only on rigid AdS, but also on more generic non-Einstein spaces. In Sec. 5 we study the quantization and causality properties of the gravitino on Friedmann-Robertson-Walker spacetimes, to explicitly show that a consistent quantization can be carried out also on non-Einstein spaces, in contrast to claims in the literature. These results can also be found in [16].

2 Prelude: AdS/CFT 101

The AdS/CFT dualities relate supergravities arising as low-energy limits of string theory to certain CFTs emerging in a similar way. In the following we provide a short introduction emphasizing the aspects which will be relevant for the main part. A crucial ingredient are the specific geometric properties of AdS, which we describe in Sec. 2.1, along with some implications for field theory on that spacetime. We briefly introduce string theory and supergravity in Sec. 2.2 and discuss some facts about CFTs in Sec. 2.3, before turning to AdS/CFT itself in Sec. 2.4.

2.1 Geometry of Anti-de Sitter Space

AdS is one of the maximally symmetric spaces[1] which have a Riemann tensor proportional to the metric, i.e. $\mathcal{R}_{\mu\nu\rho\sigma} = \alpha(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. There is a unique space for each α, which is de Sitter/Minkowski/anti-de Sitter space for positive/vanishing/negative curvature. The $(d+1)$-dimensional Euclidean and Lorentzian AdS spaces may be defined as the hyperboloid

$$X^M X^N \eta_{MN} = -R^2 \tag{2.1}$$

in $\mathbb{R}^{1,d+1}$ and $\mathbb{R}^{2,d}$, respectively. The isometries are given by the transformations in the ambient spaces preserving (2.1), which yields $\mathrm{SO}(1, d+1)$ and $\mathrm{SO}(2, d)$, respectively. Possible extensions of the corresponding Lie algebras to superalgebras have been classified in [17]. The example of Lorentzian AdS_2 as hyperboloid in $\mathbb{R}^{2,1}$ with metric $\eta = \mathrm{diag}(-1, 1, -1)$ is shown in Fig. 2.1, together with the cone to which the hyperboloid asymptotes. The timelike direction is along constant X^1, such that there are closed timelike curves making the lack of global hyperbolicity obvious. Moreover, any two points on AdS can be connected by causal curves, such that the notion of causal commutativity is somewhat obscure. While these problems can be solved easily by passing to the universal cover, the resulting space again fails to be globally hyperbolic as we discuss in more detail below. Global coordinates may be introduced on (the covering of) AdS_{d+1} by solving (2.1) with

$$X^0 = R\cos\tau\sec z \ , \qquad X^{d+1} = R\sin\tau\sec z \ , \qquad X^i = R\,\Omega^i\tan z \ , \tag{2.2}$$

where $z \in [0, \frac{\pi}{2})$, $i = 1, \ldots, d$ and $\sum_{i=1}^d (\Omega^i)^2 = 1$. For AdS we have $\tau \in [0, 2\pi)$ and $\tau \in \mathbb{R}$ for its universal cover. In the following we will only make the explicit distinction between AdS and its covering where necessary. The resulting line element reads

$$ds^2 = \frac{R^2}{\cos^2 z}\Big(-d\tau^2 + dz^2 + \sin^2 z\, d\Omega_{d-1}^2\Big) \ . \tag{2.3}$$

[1] The Killing equation $\mathcal{L}_X g = 0$ has the maximal number of $d(d+1)/2$ independent solutions, which are the Killing vector fields generating symmetries of the spacetime.

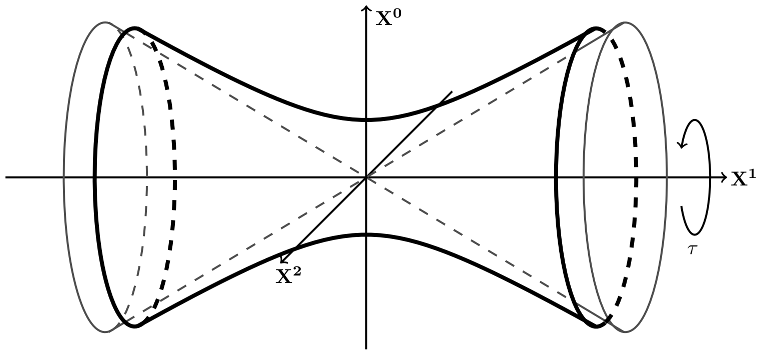


Figure 2.1: Lorentzian AdS_2 as hyperboloid in $\mathbb{R}^{2,1}$ together with its asymptotic cone. The timelike direction is along constant X^1.

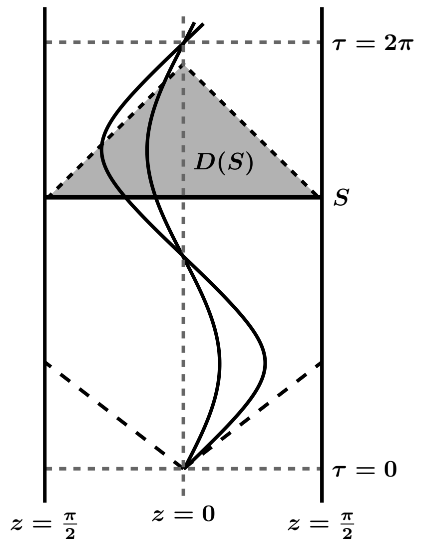
AdS may now be visualized as a cylinder, a timelike section of which is shown in Fig. 2.2(a). The geometry of a spatial section is illustrated in Fig. 2.2(b). The solid and dashed curves starting at $t = 0$ in 2.2(a) are timelike and null geodesics, respectively. As remnant of the time-periodicity of AdS the geodesics are periodic on the cover of AdS. This in particular implies that not all points which are in causal contact can be connected by causal geodesics. The null geodesics go off to infinity at $z = \frac{\pi}{2}$ in finite coordinate time. Thus, for any surface in AdS one easily constructs maximal causal curves which do not intersect it, such that there are no Cauchy surfaces. By the classical result of [19] AdS thus fails to be globally hyperbolic. The grey shaded region in 2.2(a) shows the domain of dependence $D(S)$ for a spatial section S of the AdS cylinder, which certainly is not the complete spacetime. The situation is similar to considering a strip of Minkowski space and one can indeed formulate well-defined dynamics if boundary conditions are specified. This analogy can be made more precise by noting that the causal structure does not depend on the geometry but only on the conformal structure. AdS_{d+1} can be conformally embedded into the globally hyperbolic Einstein static universe $\mathbb{R} \times S^d$, see Fig. 2.3. The metric is that of (2.3) without the overall factor $R^2/\cos^2 z$, and z extended to the range $[0, \pi)$. This embedding of AdS may be used to formulate well-defined dynamics for conformally invariant theories by pulling back a theory on the Einstein cylinder to AdS. Another coordinate system which is frequently employed for AdS are so-called Poincaré coordinates, obtained by setting

$$X^0 = \frac{y}{2} + \frac{R^2 - t^2 + \vec{x}^2}{2y} \ , \quad X^i = \frac{Rx^i}{y} \ , \quad X^d = \frac{y}{2} - \frac{R^2 + t^2 - \vec{x}^2}{2y} \ , \quad X^{d+1} = \frac{Rt}{y} \ , \tag{2.4}$$

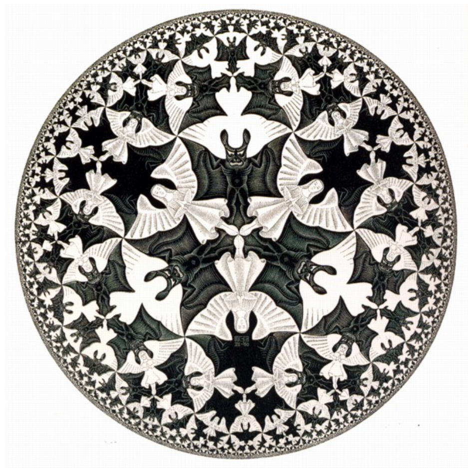
where $y > 0$ and $i = 1 \dots d-1$. This solves (2.1) and the resulting line element reads

$$ds^2 = \frac{R^2}{y^2}\left(dy^2 - dt^2 + d\vec{x}^2\right) \ . \tag{2.5}$$

These coordinates cover a patch of global AdS which is bounded by a horizon at $y \to \infty$. We have $X^0 - X^d = R^2/y$, so the patch is given by the part of the AdS hyperboloid for which $X^0 > X^d$, compare Fig. 2.1 where it corresponds to the part with $X^0 > X^1$. A more general class of spacetimes which is relevant for AdS/CFT are asymptotically-AdS spaces, which at spacelike infinity asymptote to an AdS geometry. They will be discussed in Sec. 4.1.



(a)



(b)

Figure 2.2: A timelike section of the cover of AdS realized as the interior of a cylinder is shown in 2.2(a). For AdS itself the surfaces $\tau = 0$ and $\tau = 2\pi$ are to be identified. The boundary of the cylinder corresponds to spacelike infinity. The geometry of a spatial slice of AdS is illustrated by Escher's 'Circle Limit IV' [18] in 2.2(b).

Conformal compactification

We now discuss the conformal compactification of rigid AdS. A more general discussion of conformally compact manifolds is given in Sec. 4.1. Consider AdS in global coordinates (2.2), i.e. as the interior of a cylinder as shown in Fig. 2.2(a). The metric can not be extended directly to the boundary of the cylinder as the overall factor $1/\cos^2 z$ diverges at $z = \pi/2$. However, given a function f which vanishes on the boundary with $df \neq 0$ and is positive in the interior, we can define an unphysical metric $\overline{ds}^2 = f^2 ds^2$, which does extend to the boundary. The simplest option is to just remove the overall factor in (2.3) with $f = R^{-1}\cos z$, but of course this choice is not unique and the metric induced on the boundary is therefore not well defined. The procedure does, however, yield a well-defined conformal structure on the boundary, i.e. a class of metrics which are related by conformal rescalings. For the case of AdS itself the boundary is $S^1 \times S^{d-1}$, and $\mathbb{R} \times S^{d-1}$ for the covering. These are a compactification of Minkowski space and an ∞-sheeted cover thereof, see Sec. 2.3. Performing the same construction for the Poincaré patch of AdS with metric (2.5) yields a more direct identification of the conformal boundary at $y = 0$ with Minkowski space.

Field Theory on AdS

We close with some remarks on (scalar) QFT on AdS, which will be relevant in particular for the holographic discussion of unitarity in Sec. 3. A group-theoretical discussion of certain aspects including masslessness and conformal invariance for AdS field theories can be found

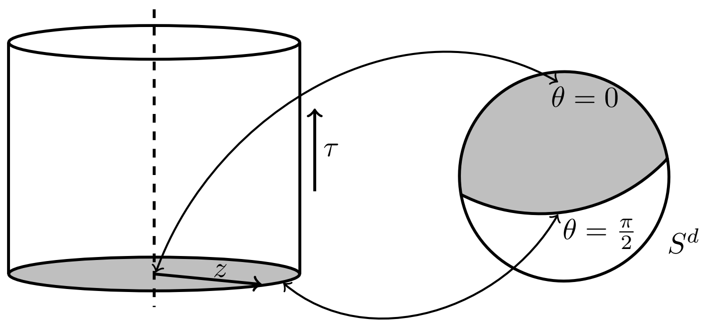


Figure 2.3: The conformal embedding of global AdS_{d+1} into half of a $(d+1)$-dimensional Einstein universe $\mathbb{R} \times S^d$.

in [20]. In [21] a nice treatment for conformally coupled scalar fields on AdS is given, based on the conformal embedding into the Einstein cylinder described above. The procedure is to define a QFT on the globally hyperbolic Einstein cylinder using the Cauchy surfaces there. Employing the periodicity properties of the solutions, this can be reinterpreted as specifying initial data on two complete spatial hypersurfaces inside AdS. Demanding then energy conservation on the subspace of the Einstein cylinder corresponding to AdS yields the boundary conditions suitable for the specific case of a conformally coupled scalar on AdS.

Scalar fields with generic mass were investigated in [22]. The derivation of suitable boundary conditions is based on demanding finiteness and conservation of the Killing energy for solutions of the Klein-Gordon equation. The definition of the Killing energy employs the fact that there is a global timelike Killing vector field ξ on AdS (simply ∂_τ in the coordinates (2.3)), such that one can define the – up to boundary terms – conserved energy

$$E = \int_\Sigma n^\mu T_{\mu\nu} \xi^\nu \ , \tag{2.6}$$

where Σ is a spatial hypersurface with timelike unit normal vector field n and $T_{\mu\nu}$ the energy-momentum tensor. Conservation of the energy functional is achieved by imposing generalizations of the familiar Dirichlet and Neumann boundary conditions as follows. The solutions to the field equation have a definite asymptotic behaviour as the conformal boundary of AdS is approached: They split into two groups according to the leading term in their asymptotic expansion around the conformal boundary. Selecting the set with dominant/subdominant behaviour corresponds to generalized Neumann/Dirichlet boundary conditions, respectively. These boundary conditions also yield a well-defined Cauchy problem on AdS. As it turns out, even for tachyonic scalars with negative squared mass m^2, such that the potential is not bounded from below, the energy of the fluctuations around the maximum of the potential at $\phi \equiv 0$ is positive provided that $m^2 \geq -d^2/4$ on AdS_{d+1}. This bound is referred to as Breitenlohner-Freedman bound and the theory is stable if it is satisfied. Normalizability of the solutions allows for both choices of boundary conditions only in an interval just above the Breitenlohner-Freedman bound, while in the generic case only the Dirichlet boundary condition is allowed. For a more explicit discussion see Sec. 3.1.

2.2 String Theory and Supergravity in a Nutshell

The basic concept underlying string theory is that the elementary building blocks of matter and the mediators of interactions are one-dimensional extended objects rather than pointlike particles. The theory a priori depends on only a single parameter l_s setting the fundamental length scale. On sufficiently low energy scales where the string length can not be resolved, these objects appear as pointlike particles and different spins correspond to different vibrational modes of the strings. The interactions are due to joining and splitting of the strings and their strength is controlled by the dilaton, which itself arises as a scalar excitation of closed strings. The spectrum of excitations is obtained by quantizing the Polyakov action for the embedding functions of the string worldsheet into the target spacetime, which is classically equivalent to the area of the worldsheet swept out by the propagating string. Depending on the amount of supersymmetry and its realization there are five consistent theories, all defined in ten-dimensional spacetime: type I, IIA/B and two heterotic. Common to all of them is a graviton describing fluctuations of the background geometry, a two-form field B and a dilaton ϕ in the massless spectrum of closed strings, and a Yang-Mills vector field in that of open strings.

In addition to the fundamental strings there are solitonic extended objects which were first found in terms of the corresponding supergravity solutions [23]. These branes are higher-dimensional analogs of black holes and carry charges corresponding to the various fields in the massless closed-string spectrum. Besides the NS-branes corresponding to the Kalb-Ramond field B and the fundamental string corresponding to the dilaton there are Dp-branes of various dimension in the type II theories. They correspond in perturbative string theory to $(p+1)$-dimensional hypersurfaces on which open strings end. These open-string excitations describe the fluctuations of the brane. The different descriptions of D-branes are at the heart of the AdS/CFT dualities, as we review for the case of D3-branes in Sec. 2.4. More complicated brane configurations will play a role in Sec. 4.

To define for the worldsheet theory describing excitations of the string a consistent quantum theory, where negative-norm states decouple, its conformal invariance is a crucial requirement. Considering the background fields like the target-space metric as (an infinite series of) coupling constants for the embedding functions, worldsheet conformal invariance requires their β-functions to vanish. This requirement yields at leading order in $\alpha' \sim l_s^2$ the supergravity field equations as constraint on the allowed background geometries [24], and thus the intimate relation of string theory and supergravity. The higher-order contributions to the β-functions provide the specific string-theory corrections to the supergravity description.

Supergravity theories (see e.g. [25] for reviews) are of interest in their own right, as e.g. the four-dimensional $\mathcal{N}$=8 supergravity could possibly provide a finite theory of quantum gravity [26]. Here, however, our interest is for their role in holography, which arises due to the intimate relation to string theory. From the type IIA and IIB string theories one obtains the corresponding ten-dimensional IIA/B supergravities. These are $\mathcal{N}$=2 supersymmetric extensions of the pure Einstein-Hilbert action with the field content matching the massless spectrum of the corresponding string theories. An interesting supergravity which can not be obtained directly from string theory arises as follows. Demanding the theories to contain no fields of spin greater than two yields upper bounds on the spacetime dimension and the amount of supersymmetry. The maximal theory in that sense is eleven-dimensional $\mathcal{N}=1$ supergravity, also called M-theory, which yields IIA supergravity upon dimensional reduction. The presence of p-form gauge fields for various p in these theories immediately allows for Freund-Rubin type solutions of the form $\text{AdS}_n\times\text{S}^m$ [27]. Truncating the theory to a finite number of Kaluza-Klein modes on S^m is in many cases consistent in the sense that solutions of the truncated lower-dimensional theory can be lifted to solutions of the original theory, see e.g. [28]. These truncations yield so-called gauged supergravities, which were first constructed independently by promoting part of the global R-symmetry to an actual gauge symmetry. The corresponding gauge fields are employed from the Abelian vector fields of the original supergravity. This procedure introduces new interactions and breaks supersymmetry in the first place, and invariance under supersymmetry transformations is then restored by adding terms proportional to the gauge coupling. This in particular includes cosmological constants such that the theories have AdS vacua. In Sec. 4 we study specific five-dimensional gauged supergravities.

The supergravity description of the string-theory branes involves a kind of solution which interpolates between the maximally symmetric $\text{AdS}_n\times\text{S}^m$ and Minkowski solutions. The metric for these extremal p-branes which are BPS objects and break half the supersymmetry generically takes the form

$$ds^2 = H(r)^{-1/2}\eta_{\mu\nu}\,dx^\mu\otimes dx^\nu + H(r)^{1/2}\sum_\alpha dr^\alpha\otimes dr^\alpha \ , \tag{2.7}$$

where η is the $(p+1)$-dimensional Minkowski metric, $\mu,\nu=0\dots p$, $\alpha=1\dots 9-p$ and $H(r)$ is a harmonic function. The dilaton is given by $e^\phi = g_s H(r)^{(3-p)/4}$ and depending on the choice of the harmonic function H the solution describes different brane configurations. For a single stack of N p-branes we have $H(r)=1+d_p g_s N(l_s/r)^{7-p}$ with the numerical factor $d_p=(4\pi)^{(5-p)/2}\Gamma(\frac{7-p}{2})$. We come back to these geometries in Sec. 2.4.

Supergravity theories inevitably contain fermions as superpartner for the metric. In the absence of a global Lorentz symmetry with respect to which fermions can be defined as spinor representations, one introduces local orthonormal frames in the tangent spaces. One may then freely pass between the coordinate basis and this new non-coordinate basis, and the transformation is described by the vielbein field e^a_μ. The introduction of orthonormal frames is not unique and different choices are related by local Lorentz transformations. However, as geometric quantities should not depend on the specific choice of frames we have thus introduced a local Lorentz symmetry, and fermions can be defined as spacetime scalars or vector fields which transform as spinors under local Lorentz transformations. The corresponding connection, the spin connection ${\omega_\mu}^{ab}$, turns out to be non-dynamical in supersymmetrizations of Einstein-Hilbert gravity. In the so-called 1.5^{th}-order formulation, employed e.g. for the theories in Sec. 4, it is fixed to its on-shell value but not inserted explicitly. In variations of the action it is then not necessary to vary the vielbein and possible other fields 'inside' the spin connection: Considering for the sake of argument the spin connection and vielbein alone, such that the variation of the action reads

$$\delta S = \frac{\delta S}{\delta e^a_\mu}\delta e^a_\mu + \frac{\delta S}{\delta\omega^{ab}_\mu}\frac{\delta\omega^{ab}_\mu}{\delta e^c_\nu}\delta e^c_\nu \ , \tag{2.8}$$

we note that the second term vanishes since the spin connection extremizes the action. A technical issue which we have to deal with in Sec. 4 concerns the variational problem for (super)gravity on a manifold with boundary. In the metric formulation, the variation with respect to the metric yields a boundary term including the derivative of the metric. To have a well-defined variational problem for $\delta\hat{g}$ subject only to the condition $\delta\hat{g}|_{\partial M} \equiv 0$, one has to add the Gibbons-Hawking-York term [29]. As explained above it is usually not necessary to vary the vielbein and possible other fields inside the spin connection. However, in the presence of a boundary the variation of the action with respect to the spin connection vanishes only up to a boundary term, which includes the spin connection and hence derivatives of the vielbein. Thus, the same problem occurs and one has to add the GHY boundary action

$$S = \int_{\mathcal{M}} \mathcal{L} + \int_{\partial\mathcal{M}} \frac{1}{2} g^{\mu\nu} K_{\mu\nu} \ , \tag{2.9}$$

where the extrinsic curvature $K_{\mu\nu} := P_\mu{}^\rho P_\nu{}^\sigma \nabla_\rho n_\sigma$ is defined from the outward-pointing (unit) normal vector field n and the projector $P_\mu{}^\rho = g_\mu{}^\rho - n_\mu n^\rho / g(n,n)$.

2.3 Aspects of Conformal Field Theory

In the best-understood examples of holographic dualities, the theories appearing on the lower-dimensional side are invariant under conformal transformations. Intuitively, this means that there is no fundamental scale in the theory, like in four-dimensional massless ϕ^4 or Yang-Mills theory which as classical theories are scale invariant. Under certain assumptions one can show that this scale invariance for theories on d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ actually implies invariance under the full conformal group $\mathrm{SO}(2,d)$, see e.g. [30]. These transformations are easily derived in infinitesimal form from the conformal Killing equation $\left(\mathcal{L}_X \eta\right)_{\mu\nu} = \Lambda(x)\eta_{\mu\nu}$, and are given for $d > 2$ by the Lorentz transformations $P_\mu = \partial_\mu$, $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ supplemented by the generators of dilatation $D = x^\mu\partial_\mu$ and special conformal transformations $K_\mu = 2x_\mu x^\nu \partial_\nu - x^2\partial_\mu$. The exponentiated form of the special conformal transformations reads $x^\mu \to x'^\mu = (x^\mu - b^\mu x^2)/(1 - 2b\cdot x + b^2x^2)$ and it maps points of Minkowski space to infinity. To have a well-defined action of the conformal group it is thus necessary to consider a compactification of Minkowski space.

A construction going back to [31] is to consider in the real projective space $\mathbb{RP}^{d+1} := (\mathbb{R}^{d+2} \setminus \{0\})/\!\sim$, where the equivalence relation is given by $X \sim \lambda X$ for $\lambda \in \mathbb{R}\setminus\{0\}$, the quadric defined by

$$Q := \{X \in \mathbb{RP}^{d+1} \,|\, \eta(X,X) = 0\} \ , \qquad \eta = \mathrm{diag}(-1,1,\dots,1,-1) \ . \tag{2.10}$$

The condition $\eta(X,X) = 0$ is invariant under rescalings of X such that Q is well defined and it is also invariant under $\mathrm{SO}(2,d)$ transformations acting linearly on $\mathbb{R}^{d+2}$. Choosing specific representatives of the equivalence classes this can be written as

$$Q = \{X \mid (X^0)^2 + (X^{d+1})^2 = 1 = (X^1)^2 + \dots + (X^d)^2\}/(X \sim -X) \ , \tag{2.11}$$

so we have $Q \cong S^1 \times S^{d-1}/\mathbb{Z}_2$. To identify d-dimensional Minkowski space as a subspace one introduces coordinates (κ, x^μ, λ) on $\mathbb{R}^{d+2}$ with

$$\kappa = X^d + X^{d+1}\ , \qquad x^\mu = X^\mu/\kappa\ , \qquad \lambda = X^M X_M\ , \tag{2.12}$$

where $M = 0, .., d+1$ and $\mu = 0, .., d-1$. That coordinate transformation is non-singular except for $\kappa = 0$. The quadric (2.10) corresponds to $\lambda = 0$ and using the freedom to rescale X^M we fix $\kappa = 1$, such that we find $\mathbb{R}^{1,d-1} \cong \{X \in \mathbb{R}^{d+2} \,|\, \kappa = 1, \lambda = 0\} \subset Q$. Theories with manifest conformal invariance may now be constructed by defining them directly on Q.

The introduction of compactified Minkowski space, on which the action of the conformal group is well defined, brings about another problem. Namely, with time being S^1 there are closed timelike curves and the notion of a global causal ordering is lost. This is avoided by the concept of weak conformal invariance as follows. For a Minkowski-space QFT satisfying the Wightman axioms the Wightman functions can be extended to a complex domain with an ∞-sheeted cover of Minkowski space as real boundary. The QFT possesses weak conformal invariance if the restriction of these extended Wightman functions to the Euclidean are invariant under the Euclidean conformal group. This implies invariance of the Minkowski-space Wightman functions under infinitesimal conformal transformations and the real boundary has a global conformally invariant causal ordering. For more details see [32] and references therein.

With that concept of conformal invariance we are ready to study Minkowski-space CFTs which are invariant under the infinitesimal conformal group. The fields and composite operators form representations of the conformal group. Restricting attention to operators which are polynomials in the fundamental fields, they have a definite behaviour under dilatations, i.e. $\mathcal{O}(x) \to \mathcal{O}'(x) = \lambda^\Delta \mathcal{O}(\lambda X)$ where Δ is the scaling dimension. While P_μ raises the scaling dimension, K_μ lowers it and there is a lower bound on the scaling dimension for unitary representations. Thus, in each unitary representation there is a so-called primary operator which is annihilated by K_μ. The corresponding representations containing the primary and all descendants obtained by acting with the generators on the primary are classified by the respective Lorentz representation and the scaling dimension. A complete classification of the unitary representations in four dimensions has been given in [33] and for the supersymmetric extensions of the conformal group in [34]. In [35] restrictions on the scaling dimensions have been discussed in more general spacetime dimensions. We close the discussion of theories on a fixed background by pointing out that, while the construction of theories which are conformally invariant at the classical level is rather straightforward, relatively few examples are known where this extends to the quantized theories. The prime example of the latter category in four dimensions is $\mathcal{N}$=4 supersymmetric Yang-Mills theory (SYM), which exhibits conformal invariance at least to all orders in perturbation theory, see [36] for a review and references.

Conformal gravity and the Weyl anomaly

The concept of scale invariance, which we have discussed so far for theories on a fixed geometry, can be extended to gravitational theories and this will play a role in Sec. 4 for the discussion of gravity on the boundary. The relevant symmetry in addition to the diffeomorphisms is the Weyl symmetry. It amounts to rescaling the metric locally by a non-vanishing function, $g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = e^{-2\lambda(x)} g_{\mu\nu}(x)$, and similarly for possible matter fields which transform as $\phi(x) \to \phi'(x) = e^{\alpha\lambda(x)}\phi(x)$ according to their Weyl weight α. The linearization of such a theory around a fixed background is conformally invariant, the conformal transformations arising as combinations of conformal isometries and compensating Weyl rescalings. In $d=4$ the action for a pure gravity theory with Weyl invariance is constructed from the Weyl tensor $C^{\lambda\mu}{}_{\nu\rho} = \mathcal{R}^{\lambda\mu}{}_{\nu\rho} - 2\delta^{[\lambda}_{[\nu}\mathcal{R}^{\mu]}_{\rho]} + \frac{1}{3}\delta^{\lambda}_{[\mu}\delta^{\nu}_{\rho]}\mathcal{R}$ which is a traceless version of the Riemann tensor. The action is given by

$$S = \frac{1}{2}\int d^4x\sqrt{g}\, C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = \int d^4x\sqrt{g}\left(\mathcal{R}^{\mu\nu}R_{\mu\nu} - \frac{1}{3}\mathcal{R}^2 + E_4\right)\,, \tag{2.13}$$

where the four-dimensional Euler density $E_4 = \frac{1}{4}\epsilon_{\mu\nu\lambda\rho}\epsilon^{\alpha\beta\gamma\delta}\mathcal{R}^{\mu\nu}{}_{\alpha\beta}\mathcal{R}^{\lambda\rho}{}_{\gamma\delta}$ yields a boundary term. To construct free Weyl-invariant matter actions one compensates for the non-covariant parts of the transformation of the kinetic term by adding suitable curvature couplings[2], e.g. the quadratic Weyl-invariant action for a scalar with Weyl weight $d/2-1$ reads

$$S = \int d^dx\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}\frac{d-2}{d-1}\mathcal{R}\phi^2\right)\,. \tag{2.14}$$

The resulting Weyl-covariant Laplacian will play a role later and is given by $\Box^W = \Box - \frac{1}{4}\frac{d-2}{d-1}\mathcal{R}$. For a more general discussion of conformally covariant differential operators see [37]. We also note that Yang-Mills theory in four dimensions is already Weyl invariant with Weyl weight zero for the connection. Furthermore, the Weyl-invariant action for a four-dimensional antisymmetric, anti-selfdual rank 2 tensor field $C_{\mu\nu}$ reads

$$S = \int d^4x\sqrt{g}\big(2D_\mu\overline{C^{\mu\nu}}D_\rho C^\rho{}_\nu - \mathcal{R}_{\mu\nu}\overline{C^{\mu\rho}}C_{\nu\rho}\big)\,, \tag{2.15}$$

where the bar denotes complex conjugation. Supersymmetric gravity theories with Weyl invariance are constructed by combining (2.13) with suitable matter field actions and Weyl-invariant couplings. A review and constructions of theories with up to $\mathcal{N}=4$ supersymmetry can be found in [38].

An interesting effect which we will study holographically in Sec. 4 is the Weyl anomaly. Coupling a theory which is conformally invariant in flat space to gravity and considering it in general backgrounds does not necessarily preserve the conformal invariance. The coupled system of CFT and gravity is Weyl invariant and classically this implies tracelessness of the energy-momentum tensor $g^{\mu\nu}T_{\mu\nu} = 0$. The failure of this tracelessness for the quantized CFT in the classical gravitational background, $g^{\mu\nu}\langle T_{\mu\nu}\rangle \neq 0$, is referred to as Weyl anomaly and signals the breakdown of conformal invariance. For reviews see [13, 39]. The form of the anomalous trace of the energy-momentum tensor is rather restricted and given by curvature invariants, see [40]. In four dimensions it is a combination of the Euler density and the squared Weyl tensor

$$\mathcal{A} := \langle T_\mu{}^\mu\rangle = a\,E_4 + b\,C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\,, \tag{2.16}$$

[2] The transformed d-dimensional Christoffel symbols and Ricci tensor are $\Gamma'^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \delta^{\rho}_{\mu}\xi_\nu - \delta^{\rho}_{\nu}\xi_\mu + g_{\mu\nu}\xi^\rho$ and $\mathcal{R}'_{\mu\nu} = \mathcal{R}_{\mu\nu} + (d-2)\,(D_\mu\xi_\nu + \xi_\mu\xi_\nu) + g_{\mu\nu}\,(D_\rho\xi^\rho - (d-2)\xi_\rho\xi^\rho)$ with $\xi_\nu = \partial_\nu\lambda$.

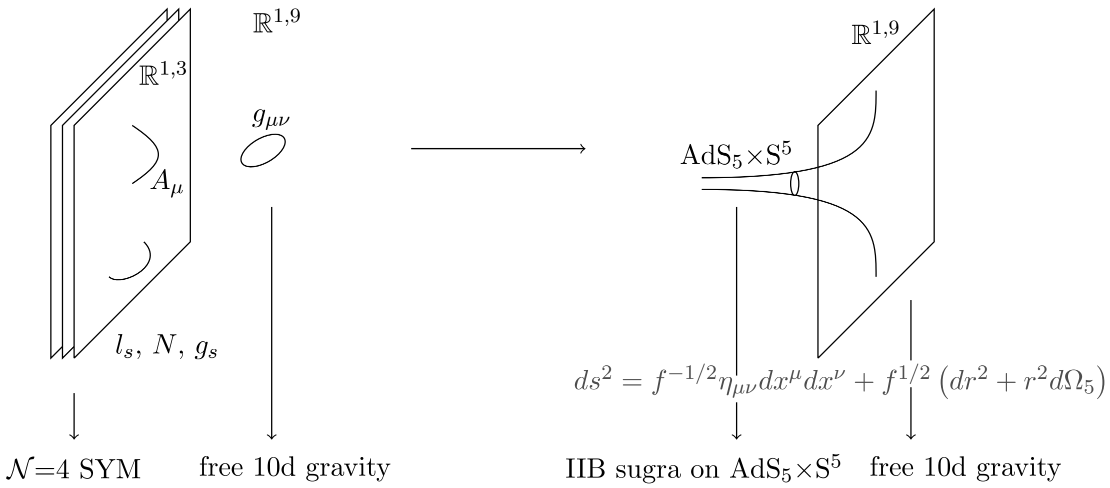


Figure 2.4: The low-energy limits of the configuration with N D3-branes yielding the duality of four-dimensional $\mathcal{N}$=4 SYM theory and IIB supergravity on $\text{AdS}_5 \times S^5$.

with the constants a, b depending on the specific CFT. The anomaly vanishes for a Minkowski background but is non-zero in general. In particular, this anomaly is non-vanishing also for the $\mathcal{N}$=4 SYM theory discussed above. The anomaly can be avoided by tuning the matter content such that the contributions to $\mathcal{A}$ from the various fields exactly cancel. This approach was followed in [41] where $\mathcal{N}$=4 SYM theory with a gauge group with four-dimensional Lie algebra (e.g. U(2) or $\text{U}(1)^4$) coupled to $\mathcal{N}$=4 conformal supergravity was found as a suitable combination.

2.4 AdS/CFT – Low-Energy Limits of String Theory

The AdS/CFT dualities are obtained by considering particular low-energy limits of brane configurations in string theory [5, 6, 7], which yields two descriptions of the setup, both consisting of two decoupled sectors. While the sector asymptotically far from the branes agrees in the two descriptions, the near-brane theories are different and identifying them yields the conjectured correspondence of a gravitational and a Yang-Mills theory. Reviews can be found in [42, 43]. Following [42] we describe the limits and the line of thought for the specific example of a stack of D3-branes, resulting in the celebrated duality of four-dimensional $\mathcal{N}$=4 SYM theory with gauge group SU(N) and type IIB supergravity on $\text{AdS}_5\times\text{S}^5$.

Perturbative string theory on flat ten-dimensional spacetime with a stack of N D3-branes as illustrated in Fig. 2.4 contains two kinds of excitations: closed strings propagating in the bulk and open strings ending on the branes. Interactions in particular include the joining of open strings on the brane to form a closed string which is radiated off and the inverse process. In a somewhat schematic form the theory is described by a bulk action for the closed-string sector, a brane action for the open-string sector and an interaction part. The spectrum of massless excitations of the closed strings is given by the fields of type IIB supergravity, while the open-string massless spectrum is given by an $\mathcal{N}$=4 vector multiplet. An effective action describing the massless excitations can be obtained by integrating out the massive excitations. This yields the total action

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \ , \tag{2.17}$$

where the bulk part is given by the type IIB supergravity action with an additional series of higher-derivative terms and likewise the brane part is given by the action of $\mathcal{N}$=4 SYM theory with a series of higher-derivative terms. One now performs the specific low-energy limit $l_s \to 0$ while keeping the dimensionless parameters g_s and N fixed. In that limit the gravitational coupling $\kappa \propto g_s \alpha'^2$ vanishes such that the interaction of the bulk and brane sectors becomes negligible and the bulk gravity becomes free. At the same time, the higher-derivative terms in the effective bulk and brane actions vanish, such that we have two decoupled sectors comprising free supergravity in the bulk and pure $\mathcal{N}$=4 SYM theory on the branes.

Another way of looking at the system is from the supergravity perspective. Here the D3-branes are understood as massive charged objects which source the supergravity fields. The corresponding type IIB supergravity solution is given by $e^{\phi} = g_s$ and

$$ds^2 = f^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + f^{1/2}\big(dr^2 + r^2 d\Omega_5^2\big) \ , \quad F_5 = (\mathbb{1} + \star) dx_0 dx_1 dx_2 dx_3 df^{-1} \ , \tag{2.18}$$

where $f = 1 + R^4/r^4$ and $R^4 = 4\pi g_s \alpha'^2 N$ (compare (2.7)). The energy E_r of an object as measured by an observer at constant r and the energy E measured by another one at asymptotically large r are related by the redshift factor $E = f^{-1/4} E_r$. Thus, to an observer at infinity the energy of an object seems to vanish as it moves towards $r = 0$. In the corresponding low-energy limit there are therefore two kinds of excitations: massless particles with large wavelengths and any kind of excitation in the region close to $r = 0$. These two sectors decouple, as a calculation of the absorption cross section of the branes shows [44], and we again have a description of the brane setup in terms of two decoupled sectors. One of them is free flat-space supergravity at large r where $f \approx 1$ and the other one supergravity on the near-horizon geometry at $r \ll R$. In that near-horizon limit $f \approx R^4/r^4$ and the geometry becomes $\text{AdS}_5 \times \text{S}^5$. Comparing the two descriptions of the brane setup we have in both cases two decoupled sectors one of which is free supergravity. Identifying the remaining sectors yields the conjectured duality of the four-dimensional $\mathcal{N}$=4 SYM theory and type IIB string theory on $\text{AdS}_5 \times \text{S}^5$. The respective coupling constants are related by $g_s \propto g_{\text{YM}}^2$, see Fig. 2.5.

Figure 2.5: The closed-string vertex $\propto g_s$ regarded as two glued open-string vertices $\propto g_{\text{YM}}^2$ yields the identification of open and closed string coupling strengths.

The weakest form of the correspondence is obtained by considering the limit of large N and large $\lambda := g_{\rm YM}^2 N$. In this limit we have $l_s/R \ll 1$ and $\kappa \ll 1$ such that the gravitational description becomes classical type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$. In the stronger forms of the conjecture the expansions in α' and the string loop expansion in g_s are assumed to be dual to the expansions in $\lambda^{-1/2}$ and $1/N$ on the Yang-Mills side, respectively.

Geometrically, the conformal boundary of AdS can be identified with Minkowski space as discussed in Sec. 2.1. The CFT on the lower-dimensional side of the AdS/CFT duality, which is $\mathcal{N}$=4 SYM theory here, can therefore be regarded as defined on the conformal boundary of AdS, such that the duality indeed is holographic. In fact, the quadric (2.10) discussed in Sec. 2.3 can be understood as the lightcone $\{X \in \mathbb{R}^{2,d}\,|\,\eta(X,X)=0\}$ modulo the identification $X \sim \lambda X$ for $\lambda \in \mathbb{R}\setminus\{0\}$. It thus corresponds to the asymptotic cone shown in Fig. 2.1. Identifying its 'celestial sphere' at infinity with the boundary of AdS, we manifestly obtain the projective model of conformal Minkowski space discussed in Sec. 2.3 at the conformal boundary of AdS, with the $\mathrm{SO}(2,d)$ AdS isometries acting as the conformal transformations on the Minkowski space boundary. Likewise, the boundary of the cover of AdS is the cover of conformal Minkowski space discussed in Sec. 2.3.

The theories involved on both sides of the duality are very different concerning the field content and the structure of interactions. However, the symmetry groups on both sides agree, the AdS_5 isometries corresponding to the conformal symmetry of the $\mathcal{N}$=4 SYM theory, the S^5 isometries corresponding to its R-symmetry and similarly for the fermionic symmetries. To make the duality explicit it is crucial that the AdS theory depends on the choice of boundary conditions on the timelike conformal boundary. The boundary values of the gravitational fields are interpreted as sources for gauge-invariant operators of the gauge theory, and the bulk partition function as functional of the boundary conditions is identified with the generating functional of connected correlation functions for the boundary theory. For example, the metric/vielbein field which is present in any of the supergravity theories is related to the energy-momentum tensor of the dual CFT. The explicit formula in the Euclidean case reads

$$\mathcal{Z}_{\rm string}\big|_{\phi|_{\partial\mathcal{M}}=\phi_0} = \langle e^{\int_{\partial\mathcal{M}}\phi_0\mathcal{O}}\rangle_{\rm CFT} \ , \tag{2.19}$$

where ϕ denotes the collection of bulk fields, $\phi|_{\partial\mathcal{M}}$ their residue at the conformal boundary and $\mathcal{O}$ represents the dual gauge-invariant operators of the boundary theory. For the Lorentzian case see (4.41) in Sec. 4.1.3. In the limit where the bulk theory reduces to classical supergravity the left hand side reduces to the exponentiated bulk action evaluated on shell. This on-shell action in general is divergent and calls for a 'holographic renormalization', which we discuss in more detail in the main part. We thus have an explicit relation between – in the weakest form of the correspondence – classical supergravity in the bulk and a strongly coupled gauge theory on the boundary. This in fact makes the duality not only hard to test or disprove, but also extremely useful as the technically very accessible weakly-coupled regime of the bulk theory is mapped to the much more involved strong-coupling regime of the boundary theory. Of course, this reasoning may also be reversed to study the bulk gravity by means of the dual gauge-theory description. Tests of the correspondence are possible by comparing gravity calculations to quantities of the dual gauge theory which are known to agree in the strong and weak coupling regimes. Such tests have been performed in great number and so far only provided support for the conjecture, see [42, 43] for references.

An alleged puzzle arises for the gauge group of the $\mathcal{N}$=4 SYM theory. Namely, the gauge group of the worldvolume theory of N D3-branes naturally is U(N), which is locally isomorphic to SU(N)⊗U(1). But the decoupled U(1) sector is not found in the dual string-theory description, where all fields interact at least gravitationally, suggesting that the dual description covers the SU(N) sector only. This is resolved by noting that the bulk supergravity can be formulated with a topological sector employing the singleton representation of the AdS isometry group, to which we come back in the holographic discussion of unitarity in Sec. 3.2. The AdS_5 supersingleton is equivalent to a four-dimensional Maxwell supermultiplet on the boundary of AdS_5 [45]. This topological sector thus corresponds to the U(1) factor and describes the center of mass motion of the brane stack [46]. For explicit constructions of singleton theories on the boundary of AdS_4 and the connection to M-theory branes see [47].

The dualities derived from string theory are still unproven, although in special situations like topological string theory the arguments can be made more precise [48]. On the other hand, holographic identities can be proven in the axiomatic framework of algebraic QFT [49]. The proof provides for a given QFT on AdS the construction of an equivalent CFT on Minkowski space of one dimension less, and vice versa. This is technically astonishingly simple and not restricted to specific limits of the involved theories. However, the physical interpretation is less accessible and identifying the dual theory constructed this way with known theories is not straightforward. Moreover, as this algebraic holography is not restricted to gravitational theories but actually relates 'ordinary' QFTs, it is somewhat complementary to AdS/CFT (for a critical discussion see also [50]). Nevertheless, the construction shows that the specific features of AdS allow for holographic relations also in a broader sense than suggested by the specific string-theory setups employed in AdS/CFT.

Normalizability vs. Boundary Conditions on AdS

The canonical quantization of a classical theory employs a complete set of solutions equipped with a symplectic structure and a notion of positive frequency, see e.g. [51]. On AdS the existence of a well-defined symplectic structure plays a crucial role for admissible boundary conditions, while the split into positive and negative frequency is straightforward thanks to the timelike Killing vector field. Crucial e.g. for our holographic investigation of unitarity in Sec. 3 is the observation [52] that a serious treatment of boundary terms yields a well-defined symplectic structure for much more general boundary conditions than naïvely expected. The argument rests upon a universal construction of a conserved current for a given Lagrangian field theory [53, 54], which we review below. This construction will serve us well also for the quantization of the gravitino in Sec. 5.

The construction is based on a geometrization of the calculus of variations, which can be made precise using ∞-jet bundles [55, 56]. Consider an n-dimensional spacetime manifold $\mathcal{M}$ and a fibre bundle $E \xrightarrow{\pi} \mathcal{M}$ over it such that a field configuration corresponds to a section of E over $\mathcal{M}$. For a real scalar field as example, the fibre would be $\mathbb{R}$. We denote by $\mathcal{S}$ the space of sections of E over $\mathcal{M}$, i.e. the space of field configurations. On the product space the de Rham complex is bigraded, $\Omega(\mathcal{M} \times \mathcal{S}) = \oplus_{p,q}\Omega^{p,q}(\mathcal{M} \times \mathcal{S})$. Accordingly the exterior differential d on $\mathcal{M} \times \mathcal{S}$ splits into $d = D + \delta$ with D of degree $(1,0)$ corresponding to the usual spacetime exterior derivative along $\mathcal{M}$ and δ of degree $(0,1)$ corresponding to the notion of variation, satisfying $d^2 = D^2 = \delta^2 = 0$. A classical field theory defined by a least action principle is then given by a Lagrangian form L of degree $(n,0)$ on $\mathcal{M}\times\mathcal{S}$. Note that L is defined on $\mathcal{M}\times\mathcal{S}$, not just on $\mathcal{M}$. To recover from a form J of degree (p,q) on $\mathcal{M}\times\mathcal{S}$ a form on $\mathcal{M}$ one evaluates it for a specific field configuration and specific variations. That is, one fixes a section ψ, tangent vectors $\delta_1\psi,\ldots,\delta_q\psi \in T_\psi\mathcal{S}$ and defines $J(\psi,\delta_1\psi,\ldots,\delta_q\psi)(x) := i_{\delta_1\psi}\ldots i_{\delta_q\psi}J(x,\psi) \in \Omega^p(\mathcal{M})$ where $i_{\delta\psi}$ is the contraction of a tangent vector with a form.

Using that geometric variational calculus one can make precise the following properties [53]. The variation of the Lagrangian splits into $\delta L = E + D\Theta$ with a unique source form E of degree $(n,1)$ yielding the equations of motion ($\psi\in\mathcal{S}$ extremizes the action iff $E(\psi,\delta\psi) = 0$ for all variations $\delta\psi$) and a local form Θ of degree $(n-1,1)$, unique up to D-exact forms. Furthermore, $\mathsf{u} := \delta\Theta$ satisfies $\delta\mathsf{u} = 0$ and $D\mathsf{u} = \delta E$. This u is also uniquely determined by L up to D-exact forms and its restriction to $\mathcal{M}$ yields a conserved current. More precisely, let $\mathcal{S}_\mathrm{L}$ be the variety of extremals for L and $T_\psi\mathcal{S}_\mathrm{L}$ be the corresponding subspace of the tangent bundle $T_\psi\mathcal{S}$. By Theorem 10 of [53] the restriction of u from $\mathcal{M}\times\mathcal{S}_\mathrm{L}$ to $\mathcal{M}$ defines a closed $(n-1)$-form, the universal conserved current associated to L. Note that u is thus closed (only) when evaluated on solutions of the linearized field equations.

A pairing of solutions to the linearized field equations is defined by fixing a Cauchy surface Σ (a complete spatial section for AdS) and defining $\langle\delta_1\psi,\delta_2\psi\rangle = \int_\Sigma \iota_{\delta_1\psi}\iota_{\delta_2\psi}\mathsf{u}$. With $\iota_{\delta_1\psi}\iota_{\delta_2\psi}\mathsf{u}$ being a closed $(n-1)$-form on $\mathcal{M}$, this pairing is conserved up to possible boundary terms by virtue of Stokes' theorem. For a Klein-Gordon field, this yields the standard symplectic structure. The fact that u is unique only up to D-exact forms was exploited in [52] to cancel divergences in the symplectic structure on AdS for Neumann or mixed boundary conditions by taking into account the natural contributions from boundary terms arising in the holographic renormalization.

3 Holographic Description of Curved-Space QFT and Multi-Layered AdS/CFT

In this part we study holographically CFTs defined on the maximally symmetric de Sitter (dS) and AdS spacetimes. This is not only a natural first step from flat to generic curved spacetimes, but also provides a link to CFTs on manifolds with boundary (BCFT) [57], since global AdS is conformally related to half of the Einstein static universe. BCFTs have received attention recently, e.g. in the context of brane configurations with branes ending on branes [58, 59, 60]. Furthermore, the case with AdS on the boundary offers an interesting possibility for multi-layered AdS/CFT dualities.

The holographic description of a CFT on a specific background involves gravity on an asymptotically-AdS space with that prescribed boundary structure. The geometries for a dual description of CFTs on dS and AdS have been discussed recently in [61] and [62], respectively, and earlier related works can be found in [63]. It is sufficient in these cases to choose specific coordinates on global AdS_{d+1} such that it is sliced by $(\text{A})\text{dS}_d$ hypersurfaces and perform the conformal compactification adapted to these coordinates. For the AdS slicing this results in two copies of AdS_d on the boundary and a single AdS_d boundary is obtained by taking a $\mathbb{Z}_2$ quotient of AdS_{d+1}. The bulk theory then depends on boundary conditions on the hypersurface which is fixed under the $\mathbb{Z}_2$ action, and the resulting geometry resembles the general construction for BCFT duals outlined in [64, 65].

In Sec. 3.1 we examine how violations of the unitarity bound in CFTs defined on dS and AdS are recovered in the dual bulk description. We consider a Klein-Gordon field on the geometries with (A)dS conformal boundary and choose masses and boundary conditions such that the corresponding boundary operator violates the CFT unitarity bound. The setup with AdS_d boundary has a particularly interesting structure since the boundary itself has a boundary. Indeed, the bulk theory turns out to crucially depend on the choice of boundary conditions on the boundary of the AdS_d slices. We find that violations of the unitarity bound in CFTs on dS_d and AdS_d are reflected in the bulk through the presence of ghost excitations. In Sec. 3.2 we turn to a puzzle already present on global AdS but also for the geometries with (A)dS conformal boundary. Namely, the standard Klein-Gordon field corresponding to an operator saturating the unitarity bound contains ghosts, although a unitary CFT exists. We investigate the holographic description of CFTs on the cylinder and on AdS, which include an operator saturating the unitarity bound, and identify a limit in which the singleton field theory is obtained from the bulk Klein-Gordon theory with renormalized inner product. This provides the unitary bulk theory corresponding to an operator which saturates the unitarity bound. In Sec. 3.3 we discuss multi-layered AdS/CFT and single out particular $\langle n\rangle$-manifolds as an appropriate geometric setting. Employing this notion we first construct geometries which at least in principle allow for an extreme case, where a chain of dualities could relate a theory on AdS_{d+1} for generic d eventually to a theory on the boundary of AdS_2. Building on the previous results we identify obstructions to such multiply nested dualities. We then turn to double-layered holography and give an outlook on a concrete realization involving the worldvolume theory of M2-branes.

3.1 Beyond the Unitarity Bound in AdS/CFT$_{\mathrm{(A)dS}}$

Facilitated by the matching of bulk isometries and boundary conformal symmetries, the AdS/CFT correspondence provides a concrete map between the bulk and boundary Hilbert spaces. For a free scalar field ϕ with mass m on AdS with unit curvature radius there are in principle two dual operators with conformal dimensions $\Delta_\pm = d/2 \pm \sqrt{d^2/4 + m^2}$, up to $1/N$ corrections. This is related to the fact that solutions to the second-order Klein-Gordon equation are characterized by two asymptotic scalings near the conformal boundary. Imposing boundary conditions such that the slower/faster fall-off is fixed, which we shall refer to as Dirichlet and Neumann boundary conditions below, yields a bulk field dual to an operator of dimension Δ_+/Δ_- [66, 67]. Note that the conformal dimensions are real so long as the Breitenlohner-Freedman (BF) stability bound $m^2 > -d^2/4 =: m^2_{\mathrm{BF}}$ [22] is respected. For $m^2_{\mathrm{BF}} < m^2 < m^2_{\mathrm{BF}} + 1$ Dirichlet and Neumann boundary conditions yield well-defined theories [22], and in fact even more general boundary conditions can be imposed [68]. On the other hand, as noted in [66, 67], Neumann boundary conditions for $m^2 > m^2_{\mathrm{BF}} + 1$ lead to $\Delta_- < d/2 - 1$, in conflict with unitarity bounds in the CFT [33, 34, 35]. Consequently, the freedom in the choice of boundary conditions was expected to break down for $m^2 > m^2_{\mathrm{BF}} + 1$. This expectation was recently confirmed for global and Poincaré AdS in [69][1]. A crucial point is that normalizability of the Neumann modes requires a modification of the symplectic structure [52], sacrificing manifest positivity of the associated inner product. Interestingly, the pathologies in the bulk theory show up in different ways for the two cases. While on global AdS the Neumann theories contain ghosts for $m^2 > m^2_{\mathrm{BF}} + 1$, such that unitarity in the bulk is explicitly violated, on Poincaré AdS there is no manifest violation of bulk unitarity. Instead, the 2-point function for the Neumann theories is found to be ill-defined even at large separations.

In this section we take a further step towards a holographic understanding of (A)dS CFTs. We consider a scalar field with $m^2 \geq m^2_{\mathrm{BF}} + 1$ on AdS_{d+1} and choose coordinates and compactification such that the boundary is $\mathrm{(A)dS}_d$. Imposing Neumann boundary conditions in this mass range is dual to a CFT on $\mathrm{(A)dS}_d$ with an operator of scaling dimension $\Delta \leq d/2 - 1$. We will investigate the precise way in which this violation of the CFT unitarity bound is reproduced by the dual bulk theory. In the setup with the boundary CFT defined on AdS_d, the bulk theory depends not only on the boundary conditions on the AdS_{d+1} conformal boundary, which we refer to as $\mathrm{Neumann}_{d+1}/\mathrm{Dirichlet}_{d+1}$, but also on the orbifold boundary conditions and on the boundary conditions on the boundary of the AdS_d slices, referred to as $\mathrm{Neumann}_d/\mathrm{Dirichlet}_d$ in the following. Furthermore, due to the fact that the AdS_d boundary itself has a conformal boundary, the structure of divergences is more involved

[1] The ghosts found there may be eliminated by imposing a radial cut-off [70], which corresponds to breaking conformal invariance in the boundary theory and therefore is not in conflict with the unitarity bound.

than for global or Poincaré AdS. Thus, in order to properly deal with this configuration we have to adapt the well-established procedure of holographic renormalization [71, 72, 73]. The choice of Neumann$_d$/Dirichlet$_d$ turns out to be quite crucial. For Dirichlet$_d$ the adaption of regularization and renormalization is straightforward, and we find the complete sets of Dirichlet$_{d+1}$ and Neumann$_{d+1}$ modes normalizable with respect to the renormalized inner product. On the other hand, our construction of the theory with Neumann$_d$ boundary condition leads to a drastically reduced spectrum of normalizable modes, making the AdS$_{d+1}$ theory equivalent to an AdS$_d$ theory in a trivial way. This will allow us to draw some conclusions on the possibility of multi-layered holographic dualities. The setup with dS on the boundary, on the other hand, is obtained from global AdS by a coordinate transformation which merely results in a rescaling of the boundary metric, such that this setting is more closely related to global AdS. However, the dS$_d$ slicing covers only a patch of AdS$_{d+1}$ bounded by a horizon, analogous to the Lorentzian Poincaré AdS. We will investigate whether there is a similarly tricky manifestation of the pathologies as found for Poincaré AdS in [69].

In Sec. 3.1.1 we introduce the setups for a holographic description of CFTs on (A)dS and give the relevant properties of the Klein-Gordon field in these settings. Unitarity of the bulk theories for AdS$_d$ and dS$_d$ on the boundary is studied in Sec. 3.1.2 and 3.1.3, respectively. In Sec. 3.1.4 we discuss a scalar field with tachyonic mass below the BF bound on global AdS. This research was carried out in collaboration with Tomás Andrade and published in [10].

3.1.1 (A)dS$_\text{d}$ slicings of AdS$_\text{d+1}$

In this section we introduce the foliations of AdS that will be relevant for the subsequent analysis and discuss some generic features of the Klein-Gordon field in these coordinates. We consider AdS$_{d+1}$ with curvature radius L in global coordinates $(\rho, \zeta, t) \in [0,\infty)\times[0,\pi]\times\mathbb{R}$ such that the line element takes the form

$$ds^2 = -\big(1+\rho^2/L^2\big)dt^2 + \frac{1}{1+\rho^2/L^2}d\rho^2 + \rho^2 d\Omega_{d-1}^2 \,, \qquad d\Omega_{d-1}^2 = d\zeta^2 + \sin^2\zeta d\Omega_{d-2}^2 \,. \tag{3.1}$$

In the following we discuss coordinate transformations resulting in a metric of the form

$$ds^2 = dR^2 + \lambda(R)^2\, \gamma_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.2}$$

with $R \in [0,\infty)$ and the conformal boundary of AdS at $R=\infty$. The slicing by dS$_d$ hypersurfaces with Hubble constant H is obtained by the coordinate transformation $(\rho, t, \Omega_{d-1}) \to (R, \tau, \Omega_{d-1})$ with $\tau \in \mathbb{R}$ and

$$\rho = L\cosh(H\tau)\sinh\frac{R}{L} \,, \qquad \tan(t) = L\sinh(H\tau)\tanh\frac{R}{L} \,. \tag{3.3}$$

The resulting metric is of the form (3.2) with

$$\gamma^{\text{dS}}_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + H^{-2}\cosh^2(H\tau)\, d\Omega^2_{\text{d-1}} \,, \qquad \lambda_{\text{dS}}(R) = LH\sinh\frac{R}{L} \,. \tag{3.4}$$

Note that (3.3) implies $|\tan(t)/\rho| < 1$, which restricts the range of t to $|t| < \arctan(\rho) < \pi/2$. The coordinates (R,τ) therefore cover a patch as shown in Fig. 3.1(b). The patch is bounded by a causal horizon at $|\tan(t)/\rho| \to 1$, which is an infinite-redshift surface as $\lambda_{\rm dS}^2$ vanishes there. The conformal boundary of the patch at $R \to \infty$ is part of the AdS_{d+1} conformal boundary, and from (3.4) we see that the boundary metric at $R = \infty$ is that of global dS_d, as desired.

The foliation of AdS_{d+1} by AdS_d hypersurfaces with curvature radius l is obtained from the transformation $(\rho, \zeta, t, \Omega_{d-2}) \to (R, z, \tau, \Omega_{d-2})$ with $z \in (0, \pi/2]$, $\tau \in \mathbb{R}$ and

$$\frac{\rho^2}{L^2} = \csc^2 z \cosh^2 \frac{R}{L} - 1 \ , \qquad \rho^2 \sin^2 \zeta = L^2 \cot^2 z \cosh^2 \frac{R}{L} \ , \qquad t = L\tau \ . \tag{3.5}$$

The resulting metric again is of the form (3.2) but with

$$\gamma_{\mu\nu}^{\rm AdS} dx^\mu dx^\nu = \frac{l^2}{\sin^2 z}\big(- d\tau^2 + dz^2 + \cos^2 z \, d\Omega_{d-2}^2 \big) \ , \qquad \lambda_{\rm AdS}(R) = \frac{L}{l} \cosh \frac{R}{L} \ . \tag{3.6}$$

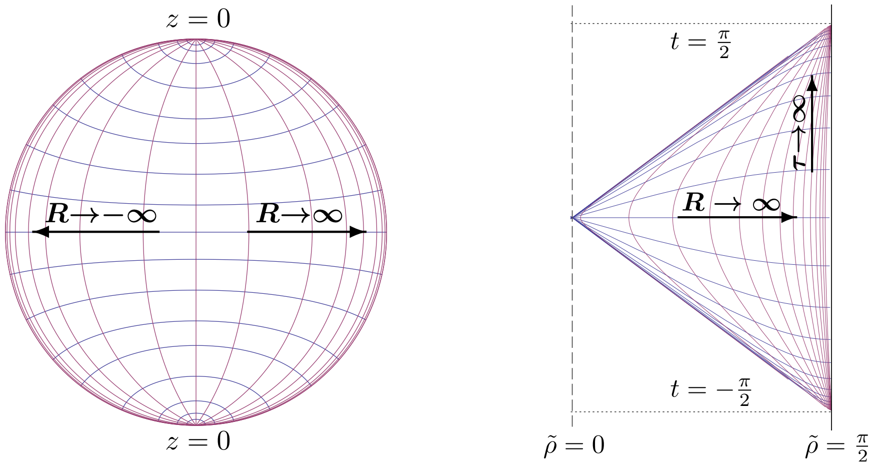
As we have to choose the domain for the sine in the 2nd equation in (3.5) to be either $\zeta \in [0, \pi/2)$ or $\zeta \in (\pi/2, \pi]$ we need two patches to cover the full AdS_{d+1}. The patches are 'joined' at $\zeta = \pi/2$, the equator of S^{d-1}. This is realized in [62] by letting R run on $(-\infty, \infty)$ and choosing the appropriate domains for ζ on the two half lines. To obtain a holographic description of a CFT on a single copy of AdS_d we consider the $\mathbb{Z}_2$ quotient of global AdS identifying the two patches, as discussed in [62]. This quotient is covered by the coordinates discussed above for any of the two choices for the domain of ζ. In turn, this implies that the fields under consideration should have definite $\mathbb{Z}_2$ parity, which imposes boundary conditions at $R = 0$, as will be discussed in Sec. 3.1.2.1. Furthermore, note that the resulting single copy of AdS_d at the conformal boundary of AdS_{d+1} has itself a conformal boundary, which, in the coordinate system (3.6), corresponds to the locus $z = 0$.

The setup for a holographic description of CFTs on AdS_d discussed above resembles the holographic description for generic BCFT proposed in [64, 65]. For a CFT on a d-dimensional manifold M with boundary it was proposed there to consider as dual a gravitational theory on a d+1-dimensional asymptotically-AdS manifold with conformal boundary M and an additional boundary Q, such that $\partial Q = \partial M$. The AdS_d slice at $R \to \infty$ in our setup corresponds to M, the $\mathbb{Z}_2$-fixed hypersurface at $R = 0$ to Q, and imposing even/odd $\mathbb{Z}_2$ parity translates to Neumann/Dirichlet boundary conditions on Q. This similarity of the setups can be understood as a consequence of the relation of CFTs on AdS to BCFTs discussed in the introduction.

3.1.1.1 Klein-Gordon field

We consider a free, massive Klein-Gordon field on AdS_{d+1} foliated by $\mathrm{(A)dS}_d$ and discuss the features that apply to both slicings in parallel. Our starting point is the 'bare' bulk action for a free scalar field,

$$S = -\frac{1}{2} \int d^{d+1}x \, \sqrt{g} \Big(g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2 \Big) \ , \tag{3.7}$$



(a) (b)

Figure 3.1: The slicing of global AdS_{d+1} by (A)dS_d hypersurfaces. 3.1(a) shows the Poincaré disk representation of AdS_{d+1} sliced by AdS_d in the (R, z) coordinates used in (3.2), (3.6). Horizontal/vertical curves have constant z/R. The boundary consists of two copies of AdS_d joined at their boundaries at $z \to 0$. 3.1(b) shows the dS_d slicing (3.2), (3.4) of AdS_{d+1} as cylinder with radial coordinate $\tilde{\rho} = \arctan \rho$ and the Ω_{d-1} part suppressed. Horizontal/vertical curves have constant τ/R.

which will later be augmented by boundary terms. For a metric of the form (3.2) the resulting Klein-Gordon equation reads

$$\partial_R^2 \phi + d\, \frac{\lambda'(R)}{\lambda(R)}\, \partial_R \phi + \lambda(R)^{-2}\, \Box_\gamma \phi = m^2 \phi \ . \tag{3.8}$$

We separate the radial and transverse parts by choosing the ansatz $\phi(x, R) = \varphi(x) f(R)$, such that φ are the modes on the (A)dS_d slices and f are the radial modes. Introducing M as separation constant, (3.8) separates into the radial equation

$$f'' + d\, \frac{\lambda'}{\lambda}\, f' = \left(m^2 - M^2 \lambda^{-2}\right) f \ , \tag{3.9}$$

and the (A)dS_d hypersurface part $\Box_\gamma \varphi = M^2 \varphi$. The latter is a Klein-Gordon equation for the transverse part with 'boundary mass' M. Note that (3.9) can be written in Sturm-Liouville form,

$$Lf = \alpha f \ , \qquad \text{where} \ \ L = \frac{1}{w(R)} \left[-\frac{d}{dR} \left(p(R) \frac{d}{dR} \right) + q(R) \right] \ . \tag{3.10}$$

Fixing $p(R) = \lambda(R)^d$, $w(R) = \lambda(R)^{d-2}$ and $q(R) = m^2 \lambda(R)^d$ reproduces (3.9) with $\alpha = M^2$. The inner product defined from the 'bare' symplectic current associated to (3.7) is the standard Klein-Gordon product

$$\langle \delta_1 \phi, \delta_2 \phi \rangle_{\mathcal{M}} = -i \int_\Sigma d^d x_\Sigma \sqrt{g_\Sigma} n^M\, \delta_1 \phi^* \overset{\leftrightarrow}{\partial}_M \delta_2 \phi \ . \tag{3.11}$$

The spacelike hypersurface Σ can be chosen constant along the radial coordinate labeling the (A)dS$_d$ slices, such that the unit normal vector field n has no radial component $n^M = (0, n^\mu)$. The inner product (3.11) can then also be factorized. In fact, with $n^\mu =: \lambda(R)^{-1} n^\mu_\gamma$ such that n^μ_γ is normalized with respect to $\gamma_{\mu\nu}$, (3.11) becomes

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{M}} = \langle \varphi_1, \varphi_2 \rangle_{\text{slice}} \langle f_1, f_2 \rangle_{\text{SL}} , \tag{3.12}$$

where $\langle \varphi_1, \varphi_2 \rangle_{\text{slice}}$ is the Klein-Gordon inner product on the (A)dS$_d$ slice and $\langle f_1, f_2 \rangle_{\text{SL}}$ is the Sturm-Liouville inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\text{slice}} = -i \int_{\partial\Sigma} d^{d-1} x_{\partial\Sigma} \sqrt{\gamma_{\partial\Sigma}} n^\mu_\gamma \big(\varphi_1^* \overset{\leftrightarrow}{\partial}_\mu \varphi_2 \big) , \quad \langle f_1, f_2 \rangle_{\text{SL}} = \int_0^\infty dR\, \lambda^{d-2} f_1^* f_2 . \tag{3.13}$$

Using integration by parts and (3.9) yields[2]

$$\langle f_1, f_2 \rangle_{\text{SL}} = \frac{1}{M_1^{*2} - M_2^2} \lim_{a \to 0, b \to \infty} \Big[\lambda^d \left(f_1^* f_2' - f_1'^* f_2 \right) \Big]_a^b . \tag{3.14}$$

The inner product (3.11) is finite and conserved for Dirichlet and Neumann boundary conditions if $m^2 < m^2_{\text{BF}} + 1$. However, for larger masses the holographic renormalization of the bulk action introduces derivative terms on the boundary, which in turn induce the necessary renormalization of the inner product [52]. We shall discuss this issue in detail in Sec. 3.1.2.

3.1.1.2 Asymptotic solutions

The covariant boundary terms introduced by the holographic renormalization of the bulk theory are crucial for the construction of the renormalized inner product. The construction of these terms involves the asymptotic expansion of the on-shell bulk field, which we shall now discuss. The relevant computations are most conveniently carried out with the metric in Fefferman-Graham form. For the dS$_d$ slicing (3.2), (3.4) this form is obtained by the coordinate transformation $y := 2H^{-1} e^{-R/L} \in (0, 2H^{-1}]$, resulting in the metric

$$ds^2 = \frac{L^2}{y^2} \Big(dy^2 + \Big(1 - \frac{H^2 y^2}{4} \Big)^2 \gamma^{\text{dS}}_{\mu\nu} dx^\mu dx^\nu \Big) . \tag{3.15}$$

Likewise, for the AdS$_d$ slicing (3.2), (3.6) the transformation $y := 2l e^{-R/L} \in (0, 2l]$ yields

$$ds^2 = \frac{L^2}{y^2} \Big(dy^2 + \Big(1 + \frac{y^2}{4l^2} \Big)^2 \gamma^{\text{AdS}}_{\mu\nu} dx^\mu dx^\nu \Big) . \tag{3.16}$$

The conformal boundary of AdS$_{d+1}$ is at $y = 0$ in both cases. The asymptotic expansion of ϕ in these coordinates is obtained by solving the Klein-Gorden equation expanded around the conformal boundary. With $m^2 L^2 =: -\frac{d^2}{4} + \nu^2$ we obtain

$$\phi(x^\mu, y) = y^{\frac{d}{2} - \nu} \phi_{\text{D}}(x^\mu, y) + y^{\frac{d}{2} + \nu} \phi_{\text{N}}(x^\mu, y) , \tag{3.17}$$

[2] Although the derivation of (3.14) is only valid for $M_1^* \neq M_2$, (3.14) can be continued to $M_1^* = M_2$ by taking the appropriate limits, as we discuss later. We also note that for continuous boundary mass (3.14) has to be understood in the distributional sense. This procedure is justified by the fact that the obtained results exhibit conservation and finiteness of the symplectic structure.

where $\phi_{N/D}$ have regular power-series expansions around $y = 0$, and in particular

$$\phi_D = \phi_D^{(0)} + y^2 \phi_D^{(2)} + \dots \,, \qquad \phi_D^{(2)} = \frac{1}{4(\nu-1)} \Box_\gamma^W \phi_D^{(0)} \,, \qquad \nu \in (1,2) \,, \tag{3.18a}$$

$$\phi_D = \phi_D^{(0)} + y^2 \log(y) \phi_D^{(2)} + \dots \,, \qquad \phi_D^{(2)} = -\frac{1}{2} \Box_\gamma^W \phi_D^{(0)} \,, \qquad \nu = 1 \,. \tag{3.18b}$$

Here we have defined $\Box_\gamma^W := \Box_\gamma - \frac{d-2\nu}{4(d-1)} R[\gamma]$, with $R[\gamma]$ denoting the curvature of the hypersurface metric. For $\nu = 1$ this is the conformal Laplacian discussed in Sec. 2.3. The curvature convention is such that $\mathcal{R}[\gamma^{\mathrm{AdS}}] = -l^{-2} d(d-1)$ and $\mathcal{R}[\gamma^{\mathrm{dS}}] = H^2 d(d-1)$.

3.1.2 AdS on the boundary

In this section we study the case of AdS on the boundary. After setting up the regularization and renormalization procedure we discuss the Dirichlet$_d$ theory in the mass range dual to a CFT beyond the unitarity bound and discuss the special properties of the Neumann$_d$ theories.

3.1.2.1 Renormalization and boundary conditions

We consider the AdS$_d$ slicing of AdS$_{d+1}$ using the coordinates (y, z, τ, Ω) such that the metric is of the Fefferman-Graham form (3.16). The action (3.7) evaluated on-shell is divergent as a power series in a vicinity of the boundary at $y = 0$, and we also expect divergences from $z = 0$. To renormalize the divergences we introduce cut-offs at $y = \epsilon_1$, $z = \epsilon_2$ and boundary counterterms to render the asymptotic expansions in y, z finite as the cut-offs are removed by $\epsilon_{1/2} \to 0$. The form of the cut-offs is the standard prescription adapted to the current slicing, and is illustrated in Fig. 3.2(a). We use the notation $\mathcal{M} = \mathrm{AdS}_{d+1}$ and parametrize the boundary $\partial\mathcal{M}$ of the regularized $\mathcal{M}$ as follows: $\partial_0\mathcal{M} := \{\mathcal{M} \,|\, y = 2l, z > \epsilon_2\}$ is the hypersurface which is invariant under the orbifold action, $\partial_1\mathcal{M} := \{\mathcal{M} \,|\, y = \epsilon_1, z > \epsilon_2\}$ denotes the (regularized) AdS$_d$ part at large R, $\partial_2\mathcal{M} := \{\mathcal{M} \,|\, \epsilon_1 < y < 2l, z = \epsilon_2\}$ consists of the boundaries of the AdS$_d$ slices and $\partial\partial\mathcal{M} := \{\mathcal{M} \,|\, y = \epsilon_1, z = \epsilon_2\}$ is the boundary of $\partial_1\mathcal{M}$, see Fig. 3.2(a).

We briefly discuss the boundary conditions to be imposed on the various parts of the boundary. On the $\mathbb{Z}_2$-fixed part at $R = 0/y = 2l$, definite orbifold parity demands either vanishing function value $\phi = 0$ or vanishing normal derivative $\partial_R \phi = 0$. In view of the decomposition $\phi = \varphi f$ discussed in Sec. 3.1.1.1, this places restrictions on f. Further restrictions are imposed on f by Dirichlet$_{d+1}$/Neumann$_{d+1}$ or more general mixed boundary conditions at $R \to \infty/y = 0$. For non-Dirichlet boundary conditions and $\nu \geq 1$, the inner product needs to be properly renormalized, as usual. On the remaining part, which is the boundary of the AdS$_d$ slices at $z = 0$, Dirichlet$_d$/Neumann$_d$ or mixed boundary conditions can be imposed. We focus on Dirichlet$_d$ first and discuss non-Dirichlet boundary conditions in Sec. 3.1.2.4. Finally, regularity and normalizability at the origin of the AdS$_d$ slices at $z = \pi/2$ places restrictions on the AdS$_d$ modes φ. This condition is satisfied by choosing for φ the

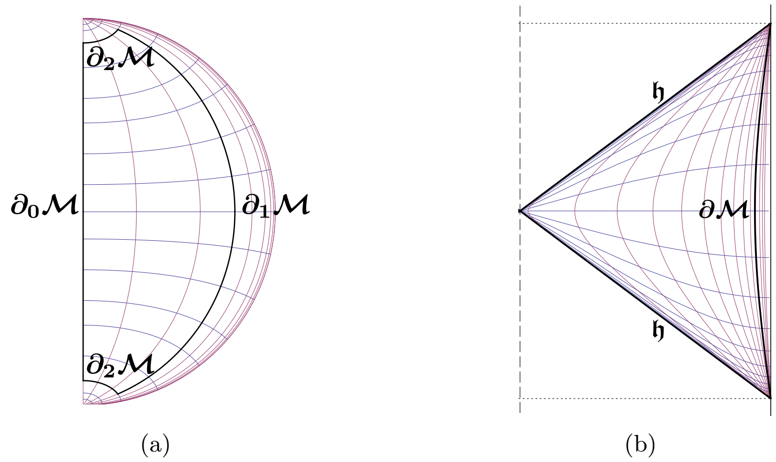


Figure 3.2: The boundaries of the regularized geometries. 3.2(a) shows the regularization of the AdS slicing discussed in Sec. 3.1.2.1. The intersection of $\partial_1\mathcal{M}$ and $\partial_2\mathcal{M}$ is $\partial\partial\mathcal{M}$. 3.2(b) shows the regularization of the dS slicing discussed in Sec. 3.1.3.

modes discussed in [67], which decay as a power-law at the origin. In the following we set $L = l = 1$.

Focusing on Dirichlet$_d$ boundary conditions we now construct the counterterms that render the action finite and stationary when both the equations of motion and the boundary conditions hold. Using integration by parts and dropping terms which vanish by orbifold parity and normalizability at $z = \pi/2$ the action (3.7) reads

$$S = \frac{1}{2}\int_{\partial_1\mathcal{M}} \phi\sqrt{g^{yy}}\partial_y\phi + \frac{1}{2}\int_{\partial_2\mathcal{M}} \phi\sqrt{g^{zz}}\partial_z\phi + \text{EOM}\ . \tag{3.19}$$

The volume forms are suppressed throughout, they are the standard forms constructed from the (induced) metric on the respective (sub)manifold. Note that both terms are divergent for $\epsilon_1 \to 0$. The familiar divergence of the first term has to be cancelled by counterterms on $\partial_1\mathcal{M}$. The second term is divergent due to the integral over $y \in (\epsilon_1, 2l)$. Expanding the integrand in y and performing the integral order by order we isolate the divergent part, which is to be cancelled by a counterterm on $\partial\partial\mathcal{M}$. For $\nu \in (1,2)$ we find the counterterms

$$S_{\partial\mathcal{M}} = -\frac{1}{2}\int_{\partial_1\mathcal{M}} \left[\left(\frac{d}{2}-\nu\right)\phi^2 + \frac{1}{2(\nu-1)}\phi\,\Box^W_{g_{\text{ind}}}\phi\right] + \frac{1}{4(\nu-1)}\int_{\partial\partial\mathcal{M}} \phi\mathcal{L}_n\phi\ , \tag{3.20a}$$

where $\mathcal{L}_n$ is the Lie derivative along $n = -\sqrt{g^{zz}}\partial_z$, which is the outward-pointing normalized vector field in $T\partial_1\mathcal{M}$ normal to $\partial\partial\mathcal{M}$. $\Box^W_{g_{\text{ind}}}$ is defined below (3.18) and g_{ind} is the induced metric. For $\nu \in (0,1)$ the second term in the first integral in (3.20a) is absent, such that the boundary terms do not contain derivatives. For $\nu = 1$ we find

$$S_{\partial\mathcal{M}} = -\frac{1}{2}\int_{\partial_1\mathcal{M}} \left[\left(\frac{d}{2}-1\right)\phi^2 - \big(\log y + \kappa\big)\phi\,\Box^W_{g_{\text{ind}}}\phi\right] - \frac{1}{2}\int_{\partial\partial\mathcal{M}} (\log y + \kappa)\phi\mathcal{L}_n\phi\ . \tag{3.20b}$$

Here we have included, with an arbitrary coefficient κ, a combination of boundary terms which is compatible with all bulk symmetries and finite for $\nu = 1$. Note that invariance under radial isometries, corresponding to conformal transformations on the boundary, is broken by the $\log y$ terms. We also emphasize that integration by parts in the counterterms has to be carried out carefully, since e.g. $\partial_1\mathcal{M}$ itself has a boundary. The counterterms also enter the symplectic structure and the associated inner product. Following [52], we find

$$\langle\phi_1,\phi_2\rangle_{\text{ren}} = \langle\phi_1,\phi_2\rangle_{\mathcal{M}} - \frac{1}{2(\nu-1)}\langle\phi_1,\phi_2\rangle_{\partial_1\mathcal{M}} \;, \qquad \nu \in (1,2) \;, \tag{3.21a}$$

$$\langle\phi_1,\phi_2\rangle_{\text{ren}} = \langle\phi_1,\phi_2\rangle_{\mathcal{M}} + \big(\log\epsilon_1 + \kappa\big)\langle\phi_1,\phi_2\rangle_{\partial_1\mathcal{M}} \;, \qquad \nu = 1 \;. \tag{3.21b}$$

For calculating CFT correlation functions we need the variations of the action to be finite when evaluated on-shell. The variation of $S_{\text{ren}} := S + S_{\partial\mathcal{M}}$ reads

$$\delta S_{\text{ren}} = \text{EOM} - \int_{\partial_0\mathcal{M}} \delta\phi\sqrt{g^{yy}}\partial_y\phi + \int_{\partial_2\mathcal{M}} \delta\phi\sqrt{g^{zz}}\partial_z\phi + \delta S^{\nu}_{\text{ren}} \;. \tag{3.22}$$

The first boundary term vanishes for solutions with definite $\mathbb{Z}_2$ parity. The $\partial_2\mathcal{M}$ integral is divergent for $\epsilon_1 \to 0$ and the remaining part is

$$\delta S^{\nu}_{\text{ren}} = \int_{\partial_1\mathcal{M}} 2\nu\phi^{(0)}_{\text{N}}\delta\phi^{(0)}_{\text{D}} + \frac{1}{2(1-\nu)}\int_{\partial\partial\mathcal{M}} \delta\phi\sqrt{g^{zz}}\partial_z\phi \;, \tag{3.23a}$$

$$\delta S^{\nu}_{\text{ren}} = \int_{\partial_1\mathcal{M}} \delta\phi^{(0)}_{\text{D}}\big(2\phi^{(0)}_{\text{N}} + (1-2\kappa)\phi^{(2)}_{\text{D}}\big) + \int_{\partial\partial\mathcal{M}} (\log y + \kappa)\delta\phi\sqrt{g^{zz}}\partial_z\phi \;, \tag{3.23b}$$

for $\nu \in (1,2)$ and $\nu = 1$, respectively. The $\partial\partial\mathcal{M}$ terms are divergent for $\epsilon_1 \to 0$ and combine with the divergent $\partial_2\mathcal{M}$ term in (3.22) to render the variation finite as we remove the cut-off on y. For fixed Dirichlet$_d$ boundary conditions there are no divergences for $\epsilon_2 \to 0$, such that the limit $\epsilon_{1/2} \to 0^+$ is finite and independent of the order in which the limits are performed. Thus, we have renormalized the theory such that we have finite variations with respect to the boundary data at $y = 0$, while keeping fixed Dirichlet boundary conditions at $z = 0$. This allows to compute correlators for the dual CFT on AdS$_d$ with fixed boundary conditions.

In summary, the renormalized action is stationary for solutions of the Klein-Gordon equation with Dirichlet$_d$ boundary conditions, provided they have definite $\mathbb{Z}_2$ parity such that the $\partial_0\mathcal{M}$ integral in (3.22) vanishes and satisfy either the Dirichlet$_{d+1}$ condition $\delta\phi^{(0)}_{\text{D}} = 0$ or the Neumann$_{d+1}$ condition

$$\phi^{(0)}_{\text{N}} = 0 \;, \quad \nu \in (1,2) \;, \qquad 2\phi^{(0)}_{\text{N}} + (1-2\kappa)\phi^{(2)}_{\text{D}} = 0 \;, \quad \nu = 1 \;, \tag{3.24}$$

such that the $\partial_1\mathcal{M}$ integral in (3.23) vanishes. The remaining finite combination of the $\partial_2\mathcal{M}$ and $\partial\partial\mathcal{M}$ integrals vanishes for Dirichlet$_d$ boundary conditions. This can be seen as follows, expanding

$$\varphi = z^{\frac{d-1}{2}-\mu}\big(\varphi^{(0)}_{\text{D}} + \dots\big) + z^{\frac{d-1}{2}+\mu}\big(\varphi^{(0)}_{\text{N}} + \dots\big) \;, \tag{3.25}$$

where μ is defined in (3.26), and using the fixed Dirichlet$_d$ boundary condition $\varphi^{(0)}_{\text{D}} = \delta\varphi^{(0)}_{\text{D}} = 0$, bilinears in ϕ, $\delta\phi$ scale at least as $z^{d-1+2\mu}$. $\sqrt{g^{zz}}\partial_z$ does not decrease the order in z and the volume forms on $\partial_2\mathcal{M}$, $\partial\partial\mathcal{M}$ are $\propto z^{-(d-1)}$. Thus, the overall scaling is with a positive power of z and as the integrations are performed for fixed $z = \epsilon_2$ the integrands vanish for $\epsilon_2 \to 0$.

3.1.2.2 Dirichlet$_d$ beyond the unitarity bound

With the renormalization set up in the previous section, we now study the bulk theory in the mass range corresponding to a CFT with an operator violating the unitarity bound. We use the decomposition $\phi = \varphi f$ discussed in Sec. 3.1.1.1 and determine the spectrum from the boundary conditions at $y = 0$ and $\mathbb{Z}_2$ parity, which impose restrictions on the radial profiles f. This yields a quantization condition on the 'AdS$_d$ mass' M introduced in Sec. 3.1.1.1, which we parametrize by an in general complex complex parameter μ as

$$M^2 =: -\frac{(d-1)^2}{4} + \mu^2 \; . \tag{3.26}$$

Note that modes with $\mu \in \mathbb{R}$ respect the AdS$_d$ BF bound. We start with non-integer ν and discuss the case $\nu = 1$ separately. For completeness we discuss both Neumann$_{d+1}$ and Dirichlet$_{d+1}$ boundary conditions, but of course expect unitarity violations only for the former.

The two independent solutions to the radial equation (3.9) for non-integer ν are given by

$$f_{\mathrm{N/D}} = (\cosh R)^{-\frac{d}{2}}\, P^{a_{\mathrm{N/D}}\nu}_{\mu-\frac{1}{2}}\big(\tanh R\big) \; , \qquad a_{\mathrm{N}} = 1, \quad a_{\mathrm{D}} = -1 \; , \tag{3.27}$$

where P^{β}_{α} are the generalized Legendre functions. For the discussion of Dirichlet$_{d+1}$ and Neumann$_{d+1}$ boundary conditions we use the radial variable $y = 2e^{-R}$, see Sec. 3.1.1.2. The asymptotic expansions of the radial modes (3.27) around the conformal boundary at $y = 0$ are given by $f_{\mathrm{N/D}} = y^{\frac{d}{2}-a_{\mathrm{N/D}}\nu}\big(2^{a_{\mathrm{N/D}}\nu}/\Gamma(1-a_{\mathrm{N/D}}\nu) + \dots\big)$, where the ellipsis denotes subleading terms of integer order. Hence, we conclude that modes with radial profile $f_{\mathrm{N}}/f_{\mathrm{D}}$ satisfy Neumann$_{d+1}$/Dirichlet$_{d+1}$ boundary conditions. Imposing definite $\mathbb{Z}_2$ parity translates to the conditions $f|_{R=0} = 0$ for odd and $f'|_{R=0} = 0$ for even parity. For the modes (3.9) we have

$$f_{\mathrm{D/N}}(0) = \frac{\sqrt{\pi}\, 2^{a_{\mathrm{D/N}}\nu}}{\Gamma\left(\frac{3}{4} - \frac{\mu}{2} - a_{\mathrm{D/N}}\frac{\nu}{2}\right)\Gamma\left(\frac{3}{4} + \frac{\mu}{2} - a_{\mathrm{D/N}}\frac{\nu}{2}\right)} \; , \tag{3.28a}$$

$$f'_{\mathrm{D/N}}(0) = \frac{-\sqrt{\pi}\, 2^{1+a_{\mathrm{D/N}}\nu}}{\Gamma\left(\frac{1}{4} - \frac{\mu}{2} - a_{\mathrm{D/N}}\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4} + \frac{\mu}{2} - a_{\mathrm{D/N}}\frac{\nu}{2}\right)} \; . \tag{3.28b}$$

The expressions on the right hand sides vanish when the appropriate Γ-functions in the denominator have a pole, which is for non-positive integer arguments. The spectrum can therefore be read off from (3.28a) for odd and (3.28b) for even $\mathbb{Z}_2$ parity, which yields

$$\mu^2_{\mathrm{D/N,even/odd}} = \left(2n + \frac{1}{2} - a_{\mathrm{D/N}}\nu + b_{\mathrm{even/odd}}\right)^2 \; , \qquad n \in \mathbb{N} \; , \tag{3.29}$$

where $b_{\mathrm{even}} = 0$ and $b_{\mathrm{odd}} = 1$ for even and odd parity, respectively. Note that these μ are real, such that the transverse modes φ of the bulk field with Dirichlet$_{d+1}$/Neumann$_{d+1}$ boundary conditions do not violate the AdS$_d$ BF bound. However, there can be modes with $\mu = 0$ which saturate the BF bound for half-integer ν and Neumann$_{d+1}$ boundary condition.

For a concrete realization of the transverse modes φ we use the AdS modes discussed in [67]. Imposing normalizability at the origin and boundary conditions on the conformal boundary of the AdS_d slices yields a quantization of their frequencies depending on μ. For the Dirichlet_d case, all the modes are normalizable with respect to the usual symplectic structure and the frequencies ω are given by

$$\omega_{\text{D/N,even/odd}} = \pm \left[\ell + 2p + \frac{d-1}{2} + \mu_{\text{D/N,even/odd}}\right] , \qquad p \in \mathbb{N} , \tag{3.30}$$

where ℓ denotes the principal angular momentum. Note that the subscripts D/N in (3.30) refer to $\text{Dirichlet}_{d+1}/\text{Neumann}_{d+1}$ boundary conditions on the conformal boundary of AdS_{d+1}. For the case of Neumann_d to be discussed in Sec. 3.1.2.4, the frequencies are given by (3.30) with $\mu_{\text{D/N,even/odd}} \to -\mu_{\text{D/N,even/odd}}$.

For Dirichlet_d boundary conditions the AdS_d norms are positive [69], so the existence of ghosts depends only on the norms of the radial modes, which we now calculate. With the decomposition $\phi = \varphi(x) f(R)$ the renormalized inner product (3.21a) reads $\langle \phi_1, \phi_2 \rangle_{\mathcal{M}} = \langle \varphi_1, \varphi_2 \rangle_{\text{slice}} \langle f_1, f_2 \rangle_{\text{SL,ren}}$, where the renormalized SL product is given by

$$\langle f_1, f_2 \rangle_{\text{SL,ren}} = \langle f_1, f_2 \rangle_{\text{SL}} - \frac{1}{2(\nu-1)} (\cosh R)^{d-2} f_1^* f_2 \big|_{R\to\infty} . \tag{3.31}$$

We evaluate (3.31) using (3.14) for the modes f_N and f_D given in (3.27), which satisfy Neumann_{d+1} and Dirichlet_{d+1} boundary conditions for all μ, respectively. The divergence of the bare SL product $\langle f_1, f_2 \rangle_{\text{SL}}$ for Neumann_{d+1} and $\nu \in (1,2)$ is cancelled by the boundary term, such that the inner product is finite. Furthermore, the finite contribution from $R = \infty$ vanishes for all μ if $\text{Neumann}_{d+1}/\text{Dirichlet}_{d+1}$ boundary conditions are satisfied. The inner product thus evaluates to

$$\langle f_1, f_2 \rangle_{\text{SL,ren}} = -\frac{1}{M_1^2 - M_2^2} (\cosh R)^d (f_1^* f_2' - f_1'^* f_2) \big|_{R=0} . \tag{3.32}$$

Note that the term in parenthesis on the right hand side vanishes if orbifold boundary conditions are satisfied, and therefore modes with different boundary mass are orthogonal. The expression (3.32) as it stands is not defined for $M_1 = M_2$. However, it can be extended continuously to coinciding masses given by (3.29), since in that case the term in parenthesis vanishes as well. Defining $||f||^2 := \langle f, f \rangle_{\text{SL,ren}}$, the inner product for AdS_{d+1} fields $\phi_{\text{D/N}}$ with $\text{Dirichlet}_{d+1}/\text{Neumann}_{d+1}$ boundary conditions reads

$$\begin{aligned} \langle \phi_{\text{D},1}, \phi_{\text{D},2} \rangle_{\mathcal{M}} &= \delta_{M_1 M_2} \langle \varphi_1, \varphi_2 \rangle_{\text{slice}} \, ||f_{\text{D}}||^2 , \\ \langle \phi_{\text{N},1}, \phi_{\text{N},2} \rangle_{\mathcal{M}} &= \delta_{M_1 M_2} \langle \varphi_1, \varphi_2 \rangle_{\text{slice}} \, ||f_{\text{N}}||^2 . \end{aligned} \tag{3.33}$$

For the Dirichlet_{d+1} modes with even/odd $\mathbb{Z}_2$ parity we find

$$||f_{\text{D,even/odd}}||^2 = \frac{(2n + b_{\text{even/odd}})!}{(1 + 2\nu + 4n + 2b_{\text{even/odd}})\Gamma(1 + 2\nu + 2n + b_{\text{even/odd}})} . \tag{3.34}$$

As expected, these are positive for all $n \in \mathbb{N}$ and $\nu \geq 0$. Thus, since $< \cdot, \cdot >_{\text{slice}}$ is non-negative for Dirichlet$_d$ boundary conditions, the spectrum is ghost-free. Similarly, for Neumann$_{d+1}$ boundary conditions we find the norms

$$||f_{\text{N,even/odd}}||^2 = \frac{(2n + b_{\text{even/odd}})!}{(1 - 2\nu + 4n + 2b_{\text{even/odd}})\Gamma(1 - 2\nu + 2n + b_{\text{even/odd}})} \ , \tag{3.35}$$

which are positive for $\nu \in [0, 1)$ as expected. For $\nu > 1$ we first consider $\nu \notin \mathbb{Z} + \frac{1}{2}$. If $m := \lfloor 2\nu - b_{\text{even/odd}} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part, is even, the $n = 0$ mode has negative norm since the coefficient of the Γ-function in the denominator is negative while the Γ-function itself is positive. If m is odd the $n = (m - 1)/2$ mode is of negative norm since the coefficient is positive while the Γ-function is negative. As $< \cdot, \cdot >_{\text{slice}}$ is non-negative we thus have ghosts in the spectrum for non-half-integer ν in $(1, 2)$, such that the non-unitarity of the dual boundary theory is nicely reflected in the bulk. For $\nu = k + \frac{1}{2}$, $1 \leq k \in \mathbb{Z}$ the AdS$_d$ modes have integer μ and by continuity of (3.35) there are modes with vanishing or negative norm. The $n = 0$ and $n = k - b_{\text{even/odd}}$ modes are of norm zero and yield the same μ^2 unless $k = 1$ with odd $\mathbb{Z}_2$ parity[3]. This degeneracy in the spectrum indicates that the basis of solutions we are using is incomplete, so we expect 'logarithmic modes', analogous to those in [74]. Once the log-modes are incorporated, continuity of the spectrum indicates that ghosts must be present [75]. For $k = 1$ with odd parity the $n = 0$ mode is of negative norm and the others are positive, such that the non-unitarity of the dual theory is reproduced in the bulk.

We close this section noting that the results established explicitly here for $\nu < 2$ extend to higher $\nu \notin \mathbb{Z}$ even without knowledge of the exact counterterms. We consider the expansions of the Dirichlet and Neumann bulk fields near the conformal boundary $\partial_1 \mathcal{M}$

$$\phi_{\text{D}} = y^{\frac{d}{2} - \nu} \sum_k \phi_{\text{D}}^{(2k)} y^{2k} \ , \qquad \phi_{\text{N}} = y^{\frac{d}{2} + \nu} \sum_k \phi_{\text{N}}^{(2k)} y^{2k} \ , \tag{3.36}$$

where k is a non-negative integer. For $2\nu \notin \mathbb{Z}$ the only way to get boundary terms which are quadratic in the field and have an integer scaling – finite terms, in particular – is through the combination $\phi_{\text{D}} \phi_{\text{N}}$. However, since $\phi_{\text{D}} \phi_{\text{N}} = \mathcal{O}(y^d)$ and derivative/curvature-terms are subleading with even powers of y, while the volume form is $\mathcal{O}(y^{-d})$, only the boundary term without derivatives can yield a finite part. This implies that there are no extra finite contributions to the norm from the additional boundary terms. For half-integer ν the combination of two ϕ_{D} fields with the volume form scales as an odd power of y, so such terms can again not yield finite contributions. The results (3.34), (3.35) are therefore also valid for generic $\nu \notin \mathbb{Z}$.

3.1.2.3 Dirichlet$_d$ at the unitarity bound

We now consider the special case $\nu = 1$, corresponding on the boundary to an operator which saturates the unitarity bound[4]. The modes (3.27) are not linearly independent for integer ν,

[3] For k large enough there are further pairs of modes with norm zero and the same μ^2, which are of the form $(n_1, n_2) = (1, n - b_{\text{even/odd}} - 1)$, $(2, n - b_{\text{even/odd}} - 2)$ etc.

[4] Here we slightly abuse notation – boundary conformal symmetry is broken for integer ν due to the logarithmic counterterms, such that the unitarity bound does not strictly apply.

so we instead use the basis of radial profiles

$$f_i(R) = u^{2c_i-\frac{3}{2}}\left(1-u^2\right)^{\frac{d+2}{4}}\,{}_2F_1\Big(c_i-\frac{\mu}{2}, c_i+\frac{\mu}{2}; 2c_i-1; u^2\Big)\ , \qquad i=1,2\ , \tag{3.37}$$

where $u = \tanh(R)$ and $c_1 = 3/4$, $c_2 = 5/4$. Since $f_1(0) = 1$, $f_1'(0) = 0$ and $f_2(0) = 0$, $f_2'(0) = 1$, the modes are independent and f_1/f_2 has even/odd $\mathbb{Z}_2$ parity. The expansions around $y = 0$ are

$$f_i = \sqrt{\pi}\,2^{1-i}y^{\frac{d}{2}-1}\big(f_i^{(0)} + y^2\log(y)f_i^{(1)} + y^2 f_i^{(2)} + \dots\big)\ , \tag{3.38}$$

where $f_i^{(0)} = 1/\big(\Gamma\left(c_i-\frac{\mu}{2}\right)\Gamma\left(c_i+\frac{\mu}{2}\right)\big)$, $f_i^{(1)} = \frac{1}{8}(1-4\mu^2)f_i^{(0)}$ and

$$f_i^{(2)} = \frac{\left(1-4\mu^2\right)\left(\psi(c_i-\frac{\mu}{2})+\psi(c_i+\frac{\mu}{2})+2\gamma-1\right)-2d-(-1)^i 4}{16\Gamma\left(c_i-\frac{\mu}{2}\right)\Gamma\left(c_i+\frac{\mu}{2}\right)}\ . \tag{3.39}$$

Here, γ is the Euler-Mascheroni constant and ψ the digamma function.

We now discuss the spectrum, which for even/odd $\mathbb{Z}_2$ parity is found by imposing Dirichlet$_{d+1}$ or Neumann$_{d+1}$ boundary conditions on f_1/f_2. We first consider Dirichlet boundary conditions, which amount to setting to zero the leading coefficient in (3.38), i.e. $f_i^{(0)} = 0$. This yields $\mu_{i,\mathrm{D}} = \pm 2(n + c_i)$ with $n \in \mathbb{N}$. Note that for these choices of μ the coefficients $f_i^{(2)}$ are finite despite the pole in the denominator, namely $f_i^{(2)}\big|_{\mu_{i,\mathrm{D}}} = (-1)^n(n+2)!/\Gamma(n-1+2c_i)$. This ensures that the modes are non-trivial. The norms for the Dirichlet case are positive $\forall n \in \mathbb{N}$, as expected:

$$||f_{i,\mathrm{D}}||^2 = \pi n!(n+1)!\frac{n+2c_i-1}{4^i(n+c_i)\Gamma(n+2c_i)^2}\ . \tag{3.40}$$

We now come to the Neumann$_{d+1}$ boundary condition which, as seen in (3.24), amounts to

$$2f_i^{(2)} + (1-2\kappa)f_i^{(1)} = 0\ . \tag{3.41}$$

The specific solution $\mu^2 = \frac{1}{4}$ only exists for $d = 2$ and even parity. For the remaining solutions we note that, as $f_i^{(2)}$ is finite for the choices of μ which make its denominator diverge while $f_i^{(1)}$ vanishes, those μ do not yield solutions. Therefore, (3.41) is equivalent to

$$\psi(c_i-\frac{\mu}{2})+\psi(c_i+\frac{\mu}{2})+2(\gamma-\kappa) = \frac{2d+(-1)^i 4}{1-4\mu^2}\ . \tag{3.42}$$

We first argue that there are only real or purely imaginary solutions. Assume that we have $\mu = a + ib$ with $a, b \neq 0$ satisfying (3.42). Due to the non-vanishing imaginary part of μ the arguments of the digamma functions in (3.42) are non-integer, such that we can expand $\psi(1+z)+\gamma = \sum_{n=1}^{\infty}\frac{z}{n(n+z)}$ [76]. Taking the imaginary part of (3.42) then yields

$$-\sum_{n=0}^{\infty}\frac{n+c_i}{\left|n+c_i+\mu/2\right|^2\left|n+c_i-\mu/2\right|^2} = 16\,\frac{d+(-1)^i 2}{|1-4\mu^2|^2}\ . \tag{3.43}$$

Since each term in the sum on the left hand side is positive, the overall left hand side is negative. On the other hand, the right hand side is non-negative, which yields a contradiction. The assumption that there are solutions with $a, b \neq 0$ therefore has to be dropped and we only have real or purely imaginary solutions. For such μ's the modes (3.37) are real.

We now focus on the lowest-M^2 solutions. To see whether we have tachyons, i.e. states with negative mass squared below the BF bound, we consider purely imaginary $\mu = i\lambda$. Equation (3.42) then becomes

$$\mathrm{Re}\,\psi(c_i + \frac{i\lambda}{2}) + \gamma - \kappa = \frac{d + (-1)^i 2}{1 + 4\lambda^2} \ . \tag{3.44}$$

Both sides of the equation are monotonic functions of $|\lambda|$. While the right hand side decreases with $|\lambda|$ and tends to zero, the left hand side increases and tends to infinity. Thus, we have one tachyonic state if at $\lambda = 0$ the right hand side is greater than the left hand side, and none otherwise. This translates to $\kappa > 2 - d - \log 8 - (-1)^i \frac{\pi}{2}$ as condition for the existence of a tachyon in the spectrum. A discussion of such a tachyonic scalar field with negative mass squared below the BF bound in global AdS is given in Sec. 3.1.4. If κ is such that there is no tachyon, the left hand side of (3.42) is greater than or equal to the right hand side at $\mu = 0$. Since the left hand side is bounded for real $\mu \in [0, \frac{1}{2})$, while the right hand side tends to $+\infty$ for $\mu \to \frac{1}{2}$, the lowest-M^2 solution in that case has $\mu \in [0, \frac{1}{2})$.

With the properties of the spectrum discussed above we can now examine the norms. Calculating the renormalized inner product for the modes (3.37) and using (3.42) to simplify it, we find

$$||f_{i,\mathrm{N}}||^2 = \pi \frac{-8\frac{d+(-1)^i 2}{1-4\mu^2} + \frac{1-4\mu^2}{4\mu}\left(\psi^{(1)}(c_i + \frac{\mu}{2}) - \psi^{(1)}(c_i - \frac{\mu}{2})\right)}{2^{4c_i}\Gamma(c_i - \frac{\mu}{2})^2\,\Gamma(c_i + \frac{\mu}{2})^2} \ , \tag{3.45}$$

where $\psi^{(1)}$ is the trigamma function. The denominator is positive for μ real or purely imaginary, so the sign of the norm only depends on the numerator. For any choice of κ, $d \geq 2$ and $\mu \in [0, \frac{1}{2})$ or μ purely imaginary, the first term in the numerator is non-positive and the second one negative, such that the norm is negative and we find ghosts in all these cases.

3.1.2.4 Neumann$_d$ and dimensional reduction

In this section we study the case of Neumann$_d$ boundary conditions at the boundary of the AdS$_d$ slices for non-integer ν. It turns out that normalizability in that case is quite delicate, as we will see shortly. We expect similar features for all boundary conditions which allow the Neumann$_d$ modes to fluctuate. The values of μ^2 corresponding to Neumann$_{d+1}$/Dirichlet$_{d+1}$ and even/odd orbifold parity were given in (3.29) and read

$$\mu^2_{\mathrm{D/N,even/odd}} = \left(2n + \frac{1}{2} - a_{\mathrm{D/N}}\nu + b_{\mathrm{even/odd}}\right)^2 \ , \qquad n \in \mathbb{N} \ . \tag{3.46}$$

In the previous section we have discussed Dirichlet$_d$ boundary conditions. In that case, normalizability of the inner product (3.12) was equivalent to normalizability of the radial part $\langle f_1, f_2 \rangle_{\mathrm{SL}}$ since the transverse part $\langle \varphi_1, \varphi_2 \rangle_{\mathrm{slice}}$ was finite. On the other hand, for Neumann$_d$ normalizability of $\langle \varphi_1, \varphi_2 \rangle_{\mathrm{slice}}$ is not given a priori. Instead, counterterms on $\partial_2\mathcal{M}$ with their contribution to the symplectic structure are needed to render the associated inner product $\langle\varphi_1,\varphi_2\rangle_{\text{slice,ren}}$ finite for $\mu^2>1$. Additional terms on $\partial\partial\mathcal{M}$ may also be required to cancel combined divergences in the radial and slice parts of the inner product. The (standard) geometric counterterm action on $\partial_2\mathcal{M}$ can be arranged as a series of terms with decreasing degree of divergence as $z\to 0$:

$$S_{\partial_2\mathcal{M}}=\int_{\partial_2\mathcal{M}}\alpha\,\phi^2+\beta\,\phi\Box_{g_{\text{ind}}}\phi+\gamma\,\mathcal{R}[g_{\text{ind}}]\phi^2+\dots\,, \tag{3.47}$$

with fixed coefficients α, β etc. Note that, to render modes with $\mu^2>1$ normalizable, the precise relation between the coefficients and μ would be required, as seen for the corresponding relation to ν in (3.20a). Thus, with fixed coefficients we could at most render the modes for one $\mu^2>1$ normalizable[5]. Moreover, calculating explicitly the contribution to the inner product and factorizing off the radial part similarly to (3.12), we find $\langle\phi_1,\phi_2\rangle_{\partial_2\mathcal{M}}=\langle\varphi_1,\varphi_2\rangle\int_0^\infty dR\,\lambda^{d-3}f_1^*f_2$, where $\langle\varphi_1,\varphi_2\rangle$ denotes the R-independent part at fixed z. Note that the radial part is not the Sturm-Liouville inner product and therefore – in contrast to the counterterm contribution from $\partial_1\mathcal{M}$ – the renormalization can not be absorbed by renormalizing only one of the factors in (3.12). Thus, the counterterms can not cancel the divergences coming solely from the $\langle.,.\rangle_{\text{slice}}$ part of the inner product (3.12) and only the modes with $\mu^2<1$ are normalizable. Finally, we note that since the boundary geometry is global AdS$_d$ the results of [69] apply and we conclude that even if the AdS$_d$ part $\langle\varphi_1,\varphi_2\rangle_{\text{slice}}$ of (3.12) was properly renormalized, it would be indefinite for $\mu^2>1$. The bulk theory would then contain ghosts immediately and we therefore choose to add no counterterms on $\partial_2\mathcal{M}$.

In summary, the Neumann$_d$ theory is specified by the AdS$_{d+1}$ action (3.7) with mass parametrized by ν, and the ν-dependent counterterms discussed in Sec. 3.1.2.1 to render the inner product finite as $\epsilon_1\to 0$. The spectrum of normalizable excitations is given by the modes with μ^2 as in (3.46) subject to the condition that the AdS$_d$ part of the inner product $\langle\varphi_1,\varphi_2\rangle_{\text{slice,ren}}$ is finite. This is only the case for $\mu^2<1$, i.e.

$$-\frac{3}{2}<2n-a_{\text{D/N}}\nu+b_{\text{even/odd}}<\frac{1}{2}\,. \tag{3.48}$$

This immediately implies that there is at most one normalizable radial mode. For Dirichlet$_{d+1}$ boundary conditions there are only normalizable modes for even parity and $\nu\in[0,\frac{1}{2})$. They have $n=0$ and, as seen in (3.34), their norm is positive, such that there are no ghosts for Dirichlet$_{d+1}$ as expected. The spectrum is slightly richer for Neumann$_{d+1}$. For even $\mathbb{Z}_2$ parity there is a normalizable radial mode for all $\nu\geq 0$, namely $n=\lfloor(2\nu+1)/4\rfloor$. For odd $\mathbb{Z}_2$ parity we find a normalizable mode only for $\nu>1/2$, and it has $n=\lfloor(2\nu-1)/4\rfloor$. In both of the Neumann$_{d+1}$ cases the normalizable modes – if any – have positive norm for $\nu<1$ and negative norm for $\nu\in(1,2)$, see (3.35). The spectrum, although somewhat trivial, is therefore ghost-free for $\nu\in(0,1)$ and contains ghosts for $\nu\in(1,2)$, which matches our expectations based on boundary unitarity. The discussion extends to $\nu>2$ for the same reason as discussed in Sec. 3.1.2.2. Interestingly, for even $\mathbb{Z}_2$ parity and $\nu\in(2,\frac{5}{2})$, we find

[5] We have not attempted to generate coefficients like $1/(1-\mu)$ by acting on ϕ with non-local operators containing e.g. inverse z-derivatives.

that the only normalizable modes have $n = 1$ and positive norm, see (3.35). Thus, the spectrum, although very simple, is free of ghosts in that case. A similar mechanism applies to odd $\mathbb{Z}_2$ parity and $\nu \in (\frac{5}{2}, 3)$.

What we see here is a kind of dimensional reduction – the radial dependence of the AdS$_{d+1}$ field ϕ is completely fixed, such that it has only the degrees of freedom of φ, i.e. of the AdS$_d$ field. The space of normalizable solutions to the Klein-Gordon equation on AdS$_{d+1}$ with mass $m^2 = -d^2/4 + \nu^2$ and Neumann$_d$ boundary condition is thus isomorphic to the space of solutions to the Klein-Gordon equation on AdS$_d$ with Neumann$_d$ boundary condition and mass $M^2 = -(d-1)^2/4 + \mu^2$, where μ^2 is given by the only one normalizable $\mu^2 < 1$ of (3.46). In that sense, the bulk theory is equivalent to a boundary theory in a trivial way.

Some further comments are in order. We notice that the bulk theory is non-local in the sense that with only the solutions for one μ^2 available it is impossible to localize initial data along the radial direction. Furthermore, we note that the boundary theory lacks conformal invariance, as it is simply a scalar field with a fixed mass, which fails to be conformally coupled[6]. Hence, we do not expect the unitarity bound to hold, which explains the cases with only positive-norm modes found above.

3.1.3 dS on the boundary

In the present section we study the Klein-Gordon theory defined on AdS$_{d+1}$ foliated by dS$_d$ slices which, as noted in the introduction, corresponds to a boundary dual theory defined on dS$_d$. More precisely, we take the metric to be (3.4) and set $L = H = 1$ henceforth. As mentioned in the introduction, this bulk set up is closely related to the global AdS$_{d+1}$ case discussed in [69] and one could in principle argue that the results should translate from those in the global case, at least in conformal invariant scenarios. However, as the dS$_d$ slicing covers only a patch of AdS$_{d+1}$ and a horizon is present, one could expect the bulk manifestations of the unitarity violations in the boundary theory to resemble those in Poincaré AdS. Furthermore, there are cases of interest in which conformal invariance is broken, and this motivates our study of the dS$_d$ case in detail. We shall also see that the spectrum possesses an interesting structure, making this discussion worthwhile.

Our main interest is to find possible violations of unitarity in the bulk when the dual theory contains an operator whose dimension violates the unitarity bound. Thus, we will focus on Neumann boundary conditions with mass $m^2 = -d^2/4 + \nu^2$ and $0 < \nu < 2$. For comparison, we shall also include the Dirichlet results. The boundary of the patch covered by the dS$_d$ foliation consists of the causal horizon located in the interior, where R goes to zero in the coordinate system (3.1), (3.3), and a piece of the conformal boundary where R goes to infinity. Below, we impose normalizability on the causal horizon and shall not add counterterms in this region, in analogy to the usual treatment of the Poincaré horizon in the Poincaré patch of AdS, see e.g. [52], [69], [77]. On the other hand, on the conformal boundary we will

[6] The field equation for a conformally coupled scalar in d dimensions is $\Box\phi - \frac{1}{4}\frac{d-2}{d-1}\mathcal{R}\phi = 0$, compare (2.14), where in our case $\mathcal{R} = -d(d-1)$. This corresponds to $\mu^2 = \frac{1}{4}$, which is only possible for integer ν as can be seen in (3.46). Although we have not discussed the integer-ν cases in detail for Neumann$_d$, we expect tachyons/ghosts similar to the situation in Sec. 3.1.2.3.

require the usual Dirichlet or Neumann boundary conditions, which can be implemented by adding the familiar counterterms.

In order to solve the wave equation we use the mode decomposition discussed in Sec. 3.1.1.1 with the dS_d harmonics $\varphi = Y_{\sigma,\vec{j}}$. Since these will play an important role in our analysis, we now review their main properties following [78]. We refer the reader to [79] for a more extensive discussion. By definition, the dS_d harmonics satisfy eigenvalue equation

$$\Box_\gamma Y_{\sigma,\vec{j}} = -\sigma(\sigma + d - 1) Y_{\sigma,\vec{j}} \,, \tag{3.49}$$

where σ is an arbitrary complex parameter. The collection $\vec{j}$ corresponds to $(d-1)$ angular momentum quantum numbers, i.e. the components of $\vec{j}$ are non-negative integers such that $j_{d-1} > j_{d-2} > \ldots |j_1|$. Note that, as a consequence of the definition (3.49), the dS harmonics are unchanged under the replacement $\sigma \to -(\sigma + d - 1)$. Thus, without loss of generality, we can restrict ourselves to $\mathrm{Re}\,\sigma > -(d-1)/2$. The space spanned by the dS harmonics is endowed with the inner product

$$\langle Y_{\sigma,\vec{j}}, Y_{\sigma,\vec{k}} \rangle_{\text{slice}} = -i \int_{\partial\Sigma} d\Omega \sqrt{g_{\partial\Sigma}}\, n^i Y^*_{\sigma,\vec{j}} \overleftrightarrow{\partial}_i \, Y_{\sigma,\vec{k}} \,. \tag{3.50}$$

With the convention $\mathrm{Re}\,\sigma > -(d-1)/2$, it can be shown that the dS harmonics furnish unitary representations, i.e. that (3.50) is positive definite, if σ belongs to one of the following

- Principal series: $\sigma = -\frac{d-1}{2} + i\rho$, with $\rho \in \mathbb{R}$,
- Complementary series: $-\frac{d-1}{2} < \sigma < 0$, with $\sigma \in \mathbb{R}$.

Some comments are in order here. First, it is important to keep in mind that, since we are interested in searching for ghosts/violations of unitarity, we must consider all σ's allowed by normalizability, and not restrict ourselves to modes in the principal or complementary series. Second, we have defined ghosts as solutions with positive frequency and negative norm. The notion of positive frequency we shall adopt here is closely analogous to that depicted in [80] in the context of asymptotically flat spaces foliated by dS slices. That is, we shall choose $Y_{\sigma,\vec{j}}$ such that $\phi = Y_{\sigma,\vec{j}} f_\sigma$ is positive frequency in the usual sense in AdS_{d+1}. Third, we note that σ_1, σ_2 in the principal series are indistinguishable if $\sigma_1 = \sigma_2^*$. We shall remove this ambiguity by taking $\rho > 0$ below.

3.1.3.1 Renormalization

The goal of this section is to find a properly renormalized action and symplectic product for the case of the dS_d slicing. As mentioned above, we will only add to the action counterterms on the conformal boundary of the patch of AdS_{d+1}. Having found a satisfactory (i.e. finite and stationary) action, we shall follow the prescription of [52] to determine the renormalized symplectic structure. In analogy to the AdS_d case, the action we consider is $S^{\mathrm{dS}}_{\mathrm{ren}} = S + S_{\partial\mathcal{M}}$, where S is given by (3.7) and

$$S_{\partial\mathcal{M}} = -\frac{1}{2}\int_{\partial\mathcal{M}}\left[\left(\frac{d}{2}-\nu\right)\phi^2 + \frac{1}{2(\nu-1)}\,\phi\Box^{\mathrm{W}}_{g_{\mathrm{ind}}}\phi\right] \qquad \text{for } \nu\neq 1\ , \tag{3.51a}$$

$$S_{\partial\mathcal{M}} = -\frac{1}{2}\int_{\partial M}\left[\left(\frac{d}{2}-1\right)\phi^2 - (\log y+\kappa)\,\phi\Box^{\mathrm{W}}_{g_{\mathrm{ind}}}\phi\right] \qquad \text{for } \nu = 1\ . \tag{3.51b}$$

Here $\Box^{\mathrm{W}}_{g_{\mathrm{ind}}}$ is the differential operator defined in Sec. 3.1.1.2, the radial variable y is defined via $y = 2e^{-R}$ and $\partial\mathcal{M}$ denotes the part of the boundary at the radial cut-off $y=\epsilon$, see Fig. 3.2(b). Note that in (3.51b) we have introduced an extra finite counterterm with an arbitrary coefficient κ. Using the results of Sec. 3.1.1.2, it is not hard to verify that $S^{\mathrm{dS}}_{\mathrm{ren}}$ provides a well-defined variational principle for the relevant boundary conditions. In fact, taking an arbitrary on-shell variation we obtain

$$\delta S^{\mathrm{dS}}_{\mathrm{ren}} = -2\nu\int_{\partial\mathcal{M}}\phi^{(0)}_{\mathrm{N}}\delta\phi^{(0)}_{\mathrm{D}} \qquad \text{for } \nu\neq 1\ , \tag{3.52a}$$

$$\delta S^{\mathrm{dS}}_{\mathrm{ren}} = -\int_{\partial\mathcal{M}}\left(2\phi^{(0)}_{\mathrm{N}} + (1-2\kappa)\phi^{(2)}_{\mathrm{D}}\right)\delta\phi^{(0)}_{\mathrm{D}} \qquad \text{for } \nu = 1\ , \tag{3.52b}$$

where the coefficients of the asymptotic expansion are those given in Sec. 3.1.1.2. For $0<\nu<2$, with $\nu\neq 1$, we observe from (3.52a) that $\delta S^{\mathrm{dS}}_{\mathrm{ren}}$ is indeed finite and stationary for either Dirichlet boundary conditions, $\phi^{(0)}_{\mathrm{D}}=0$, for all ν or Neumann boundary conditions, $\phi^{(0)}_{\mathrm{N}}=0$. In the $\nu=1$ case, (3.52b) reveals that $\delta S^{\mathrm{dS}}_{\mathrm{ren}}$ is finite and stationary for the Dirichlet boundary condition $\phi^{(0)}_{\mathrm{D}}=0$ and for

$$2\phi^{(0)}_{\mathrm{N}} + (1-2\kappa)\phi^{(2)}_{\mathrm{D}} = 0\ , \tag{3.53}$$

which we shall refer to as Neumann. The renormalized inner products constructed along the lines of [52] read

$$\langle\phi_1,\phi_2\rangle_{\mathrm{ren}} = \langle\phi_1,\phi_2\rangle_{\mathcal{M}} - \frac{1}{2(\nu-1)}\langle\phi_1,\phi_2\rangle_{\partial\mathcal{M}} \qquad \text{for } \nu\neq 1\ , \tag{3.54a}$$

$$\langle\phi_1,\phi_2\rangle_{\mathrm{ren}} = \langle\phi_1,\phi_2\rangle_{\mathcal{M}} + (\log\epsilon+\kappa)\langle\phi_1,\phi_2\rangle_{\partial\mathcal{M}} \qquad \text{for } \nu = 1\ , \tag{3.54b}$$

where the subscripts $\mathcal{M}$, $\partial\mathcal{M}$ indicate the slices in which the usual KG products are to be evaluated. As outlined in Sec. 3.1.1.1, after inserting the mode decomposition $\phi = Yf$ in (3.54a) we find

$$\langle\phi_1,\phi_2\rangle_{\mathrm{ren}} = \langle Y_{\sigma_1,\vec{j}_1}, Y_{\sigma_2,\vec{j}_2}\rangle_{\mathrm{slice}}\langle f_1,f_2\rangle_{\mathrm{SL,ren}}\ , \tag{3.55}$$

where

$$\langle f_1,f_2\rangle_{\mathrm{SL,ren}} = \langle f_1,f_2\rangle_{\mathrm{SL}} - \frac{(\sinh R)^{d-2}}{2(\nu-1)}f_1^*f_2\big|_{R=\infty} \qquad \text{for } \nu\neq 1\ , \tag{3.56a}$$

$$\langle f_1,f_2\rangle_{\mathrm{SL,ren}} = \langle f_1,f_2\rangle_{\mathrm{SL}} + (\kappa+\log 2-R)\frac{(\sinh R)^{d-2}}{2(\nu-1)}f_1^*f_2\big|_{R=\infty} \qquad \text{for } \nu = 1\ . \tag{3.56b}$$

Note that the first factor in the right hand side of (3.55) corresponds to the inner product of two dS harmonics with different values of σ. At first sight, this might seem problematic since the inner product (3.50) was only defined for two harmonics with the same boundary mass. However, as we shall see shortly, the renormalized SL factor defined in (3.56a) vanishes for $\sigma_1 \neq \sigma_2$, so no inconsistency arises. Finally, as mentioned in Sec. 3.1.1.1, we note that the unrenormalized SL product can be evaluated by means of (3.14) with $\lambda = (\sinh R)^{d-2}$.

We shall explicitly verify below that the inner product (3.55) is finite and conserved in the cases of interest, namely, Dirichlet boundary conditions for all ν and Neumann boundary conditions for non-integer ν in the range $0 < \nu < 2$.

3.1.3.2 Beyond the unitarity bound

Let us now study the ghost content of the theories defined by the boundary conditions of interest. To this end we first determine the spectrum of normalizable solutions and then compute the norms of the various modes. We focus on the requirement of normalizability in the interior, i.e. $R = 0$ in the coordinate system (3.4), since normalizability at the boundary is either automatic (for Dirichlet boundary conditions and Neumann boundary conditions for $0 < \nu < 1$) or guaranteed by the presence of the boundary terms (for Neumann boundary conditions and $\nu \geq 1$). In the present section we restrict ourselves to non-integer ν and postpone the analysis of the special case $\nu = 1$ until Sec. 3.1.3.3.

As stated in Sec. 3.1.1.1, the equation of motion is given by (3.8) with $\lambda = \sinh R$. Using the mode decomposition and the property (3.49), this reduces to (3.9) with $M^2 = -\sigma(\sigma + d - 1)$, which, according to the general discussion in Sec. 3.1.1.1, can be written as a SL problem with eigenvalue $\alpha = -\sigma(\sigma + d - 1)$. Studying the radial equation near $R = 0$, we find that the two characteristic behaviors are $f \approx R^\sigma$ and $f \approx R^{1-d-\sigma}$. Inspecting (3.54a), we conclude that for $\operatorname{Re}\sigma > -(d-1)/2$ only $f \approx R^\sigma$ is normalizable near the origin, while for σ in the principal series, both fall-offs are δ-function normalizable at the horizon[7].

In order to write down the full solution, we introduce $x = (\cosh R)^{-1}$, so the boundary is located at $x = 0$ while the deep interior corresponds to $x = 1$. In terms of this variable, the two independent solutions can be expressed as

$$f_{\mathrm{D/N}} = x^{d/2 + a_{\mathrm{D/N}}\nu}(1 - x^2)^{\sigma/2}\ {}_2F_1\Big(c_{\mathrm{D/N}}\,,\, c_{\mathrm{D/N}} + \frac{1}{2}\,;\, 1 + a_{\mathrm{D/N}}\nu\,,\, x^2\Big)\ , \tag{3.57}$$

where $c_{\mathrm{D/N}} = (d + 2\sigma + 2a_{\mathrm{D/N}}\nu)/4$ and $a_{\mathrm{D}} = 1$, $a_{\mathrm{N}} = -1$. Near the boundary, the radial profiles (3.57) behave as

$$f_{\mathrm{D}} = x^{d/2+\nu}(1 + O(x^2))\ , \qquad f_{\mathrm{N}} = x^{d/2-\nu}(1 + O(x^2))\ , \tag{3.58}$$

where the sub-leading terms consist solely of integer powers of x. Noting that near the boundary we have $x = y + O(y^3)$ and comparing (3.58) with (3.17), we conclude that the profile f_{D} satisfies Dirichlet boundary conditions while f_{N} satisfies Neumann boundary conditions.

[7] They oscillate near $R = 0$ in such a way that we can construct wave packets that decay faster than any power law. Instead of constructing these wave packets, one can treat the norms of the modes in the principal series as distributions. We shall do so below and obtain well-defined results. As anticipated above, this resembles the behavior of timelike modes near the Poincaré horizon of Poincaré AdS.

It is convenient to organize the following discussion according to the value of σ that characterizes the radial profiles. For σ in the principal series, we have seen that both characteristic behaviors are allowed near the origin. This implies that the spectrum is continuous. In fact, Dirichlet/Neumann modes are simply given by the profiles (3.57) with $\sigma = -(d-1)/2 + i\rho$. We now proceed to compute the respective norms, following [69]. We observe that since the spectrum is continuous, the norms are understood in the distributional sense. Taking $\rho > 0$ by convention, we find for the symplectic product of the modes in the principal series

$$\langle \phi_1, \phi_2 \rangle_{\text{ren}} = \delta_{\vec{j}_1, \vec{j}_2} \delta(\rho_1 - \rho_2) \left[\frac{\pi \Gamma(1 + a_{D/N}\nu)}{2^\nu \rho \sinh(\pi\rho)} \right]^2 \left| \frac{1}{\Gamma(1 - a_{D/N}\nu + i\rho)\Gamma(i\rho)} \right|^2 . \tag{3.59}$$

It should be noted that the renormalized SL product yields the factor of $\delta(\rho_1 - \rho_2)$ in (3.59). Thanks to this property, we only need to compute the dS inner product for modes of the same boundary mass. Moreover, consistently with the fact that σ belongs to the principal series, we have assumed the dS harmonics to be normalized as $\langle Y_{\sigma,\vec{j}}, Y_{\sigma,\vec{k}} \rangle_{\text{slice}} = \delta_{\vec{j},\vec{k}}$. Note that (3.59) is positive definite for both Dirichlet and Neumann boundary conditions. Also, we emphasize that, for $1 < \nu < 2$ and Neumann boundary conditions, the explicit boundary contribution in (3.54a) exactly cancels a divergence coming from the bulk term so that (3.59) is finite. Finally, we note that (3.59) does not mix modes of different quantum numbers.

Let us consider now the case $\operatorname{Re}\sigma > -(d-1)/2$. With this restriction, only the solution that behaves as $f \approx R^\sigma$ near $R = 0$ is normalizable, which implies that the allowed values of σ form a discrete set. In fact, expanding (3.57) near $R = 0$, we find

$$f_{\text{D/N}} = 2^{a_{\text{D/N}}\nu} \pi^{-\frac{1}{2}} \Gamma(1 + a_{\text{D/N}}\nu) \Big(C^{(1)}_{\text{D/N}} (R^\sigma + \dots) + C^{(2)}_{\text{D/N}} (R^{1-d-\sigma} + \dots) \Big) , \tag{3.60}$$

where the ellipses denote subleading terms and

$$C^{(1)}_{\text{D/N}} = \frac{2^{-\frac{1}{2}(d+\sigma)} \Gamma\left(\frac{1}{2} - \frac{d}{2} - \sigma\right)}{\Gamma\left(1 - \frac{d}{2} + a_{\text{D/N}}\nu - \sigma\right)} , \qquad C^{(2)}_{\text{D/N}} = \frac{2^{\frac{1}{2}(\sigma-1)} \Gamma\left(\frac{d-1}{2} + \sigma\right)}{\Gamma\left(\frac{d}{2} + a_{\text{D/N}}\nu + \sigma\right)} . \tag{3.61}$$

As stated above, normalizability requires $C^{(2)}_{\text{D/N}} = 0$, which translates into the quantization condition

$$\sigma = \sigma_{\text{D/N}} := -\frac{(d-1)}{2} - n - \left(a_{\text{D/N}}\nu + \frac{1}{2} \right) \qquad \text{for } n \in \mathbb{N} \cup \{0\} . \tag{3.62}$$

Note that σ_{D} violates the assumption $\operatorname{Re}\sigma > -(d-1)/2$ for all n, so there are no Dirichlet modes in this class. On the other hand, depending on the value of ν, some discrete modes are allowed for Neumann boundary conditions. In particular, restricting ourselves to $0 < \nu < 2$, we note that the mode $n = 0$ is allowed for $\nu > 1/2$, while $n = 1$ is allowed for $\nu > 3/2$.

Now, the norm of the Neumann modes that satisfy (3.62) is given by

$$\langle \phi_1, \phi_2 \rangle_{\text{ren}} = \langle Y_{\sigma_1,\vec{j}_1}, Y_{\sigma_2,\vec{j}_2} \rangle_{\text{slice}} \delta_{\sigma_1,\sigma_2} (-1)^n \frac{n! \csc(\pi\nu)}{2^{3+2n-2\nu}} \frac{(2\nu - 2n - 1)\Gamma\left(\nu - n - \frac{1}{2}\right)^2}{\Gamma(2\nu - n)} . \tag{3.63}$$

Note that for $1/2 < \nu < 1$ the $n = 0$ mode has positive SL norm and $\sigma < 0$, such that the overall norm is positive. On the other hand, this mode has negative SL norm for $1 < \nu < 2$, and also the $n = 1$ mode which belongs to the spectrum for $\nu > 3/2$ has negative SL norm. Since the slice part $\langle Y_{\sigma_1,\vec{j}_1}, Y_{\sigma_2,\vec{j}_2}\rangle_{\rm slice}$ is positive for $\sigma < 0$ and indefinite for $\sigma \geq 0$, we find ghosts in any case for $1 < \nu < 2$.

Summing up, we conclude that the Neumann spectrum is free of ghosts for $\nu < 1$, while for $\nu > 1$ the norm becomes indefinite. In addition, the Dirichlet spectrum is ghost-free for all ν, in complete agreement with the CFT unitarity bound. It is worthwhile noting that it is possible to have a unitary theory that contains one discrete – yet degenerate – mode if the dimension of the operator is above but sufficiently close to the unitarity bound, i.e. for $d/2 - 1 < \Delta_- < d/2 - 1/2$. Assuming that $\Delta_- = d/2 - \nu > 0$, so that the boundary operator is relevant, it follows from (3.62) that this discrete mode always occurs in the complementary series, which is in principle a continuous series. Interestingly enough, the authors of [81] encountered an analogous structure in the corrected 2-point function in the weakly interacting scalar theory in dS_d. We can partly understand this qualitative agreement between the strongly and weakly coupled regimes from the argument that operators close to the unitarity bound should interact weakly, since operators saturating the unitarity bound must be free fields in a unitary theory.

3.1.3.3 Saturating the unitarity bound

So far we have assumed that ν is not an integer. In this section we tackle the case $\nu = 1$ paying special attention to the Neumann-like boundary condition (3.53), although we shall also include the Dirichlet results. The main feature of the integer ν cases is the presence of logarithms of the radial coordinate in the asymptotic expansion. As a result, the counterterms required to renormalize the action contain the radial variable explicitly so conformal invariance is broken, see (3.51b). The intuition about the existence of ghosts developed in the conformally invariant setups does therefore not transfer straightforwardly to this case.

We first proceed to find the spectrum of normalizable solutions. Again, we use the mode decomposition $\phi = Yf$, where Y is a dS harmonic and f a radial profile. Introducing the variable $x = (\cosh R)^{-1}$, so that the boundary is at $x = 0$, the two independent solutions to the radial equation read

$$f_1 = x^{d/2-1}(1-x^2)^{\sigma/2}\,{}_2F_1\Big(\frac{\sigma-1}{2}+\frac{d}{4}\,,\,\frac{\sigma}{2}-\frac{d}{4}\,;\,\frac{d+1}{2}+\sigma\,,\,1-x^2\Big)\,, \tag{3.64a}$$

$$f_2 = x^{d/2-1}(1-x^2)^{(1-d-\sigma)/2}\,{}_2F_1\Big(-\frac{d}{4}-\frac{\sigma}{2}\,,\,\frac{1-\sigma}{2}-\frac{d}{4}\,;\,\frac{3-d}{2}-\sigma\,,\,1-x^2\Big)\,. \tag{3.64b}$$

As in the non-integer ν case, both characteristic behaviors are allowed near the origin for modes in the principal series. Thus, for both Dirichlet and Neumann boundary conditions there are continuous families of modes in the princhipal series and one can readily verify that the norms are positive definite in this subspace.

Let us now study the discrete part of the spectrum. Defining $\sigma = -(d-1)/2 + \lambda$, the candidates for discrete modes are those with $\mathrm{Re}\,\lambda > 0$. This is because in that case only (3.64a) is regular at the origin, which implies that the boundary conditions at the conformal boundary will fix the value of λ. Letting $y = 2e^{-R}$, we find that the near-boundary expansion of (3.64a) is of the form (3.17) with

$$f_{1\mathrm{D}}^{(0)} = \frac{2^{\lambda+\frac{1}{2}}\Gamma(1+\lambda)}{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{3}{2}\right)} , \qquad f_{1\mathrm{D}}^{(2)} = \frac{2^{\lambda-\frac{1}{2}}\Gamma(1+\lambda)}{\sqrt{\pi}\,\Gamma\left(\lambda-\frac{1}{2}\right)} , \tag{3.65a}$$

$$f_{1\mathrm{N}}^{(0)} = \frac{f_{1\mathrm{D}}^{(0)}}{16}\left[2d-4+\left(4\lambda^2-1\right)\left(2\psi(\lambda+1/2)+2\gamma-1-\log 4\right)\right] , \tag{3.65b}$$

where ψ is the digamma function and γ is the Euler-Mascheroni constant. Dirichlet boundary conditions require $f_{1\mathrm{D}}^{(0)} = 0$, which according to (3.65a) implies $\lambda = -(n+3/2)$, where n is a non-negative integer. Since this violates our assumption $\mathrm{Re}\,\lambda > 0$ for all n, we conclude that there are no Dirichlet modes in this class. We now study the discrete Neumann modes. It follows from (3.53) that these must satisfy $2f_{1\mathrm{N}}^{(0)} + (1-2\kappa)f_{1\mathrm{D}}^{(2)} = 0$. We note that $\lambda = 1/2$ is a solution only for $d = 2$. Now, assuming $\lambda \neq 1/2$ and given (3.65), the Neumann condition translates into

$$b(\lambda) := \frac{d-2}{4\lambda^2-1} - \tilde{\kappa} + \psi\left(\frac{1}{2}+\lambda\right) = 0 , \tag{3.66}$$

where $\tilde{\kappa} = \kappa - \gamma + \log 2$. Though we have not found the spectrum in closed form, it is still possible to extract the relevant physics. In order to do so, we first recall that complex solutions constitute a pair of ghost/antighosts, so we only need to examine the norms of the real λ solutions. Assuming that such solutions exist, the norm of the corresponding modes can be written as

$$\langle \phi_1, \phi_2 \rangle_{\mathrm{ren}} = \delta_{\vec{j}_1,\vec{j}_2}\delta_{\sigma_1,\sigma_2}\langle Y_{\sigma_1,\vec{j}_1}, Y_{\sigma_2,\vec{j}_2}\rangle_{\mathrm{slice}}\langle f, f\rangle_{\mathrm{SL,ren}} , \tag{3.67}$$

where

$$\langle f, f\rangle_{\mathrm{SL,ren}} = A(\lambda)(1-4\lambda^2)\frac{d}{d\lambda}b(\lambda) \tag{3.68}$$

with $A(\lambda) = 4^{\lambda-1}\Gamma(\lambda)\Gamma(1+\lambda)/[\pi\Gamma\left(\frac{3}{2}+\lambda\right)^2] > 0$. In (3.68), λ is given implicitly by the real solutions of (3.66) that satisfy $\lambda > 0$. Note that in writing (3.67) we have not assumed that the dS harmonics belong to a unitary representation.

Let us now study the existence of solutions to (3.66). We first note that $b(0) = 2-d-\log 8-\kappa$ and that $b \to -\infty$ as $\lambda \to 1/2^-$. Furthermore, $b \to \infty$ when $\lambda \to 1/2^+$ and $b \to \infty$ when $\lambda \to \infty$. If $\kappa < \kappa_{c,1} := 2-d-\log 8$, we have $b(0) > 0$ and thus there is a real solution λ_0 in the range $(0, 1/2)$, which in fact is the only one. Moreover, this solution is such that $b'(\lambda_0) < 0$, so it follows from (3.68) that the associated mode is a ghost. If we increase κ above $\kappa_{c,1}$, we find two real solutions in $(0, 1/2)$ as long as $\kappa < \kappa_{c,2}$, where $\kappa_{c_1} < \kappa_{c,2} < 0$. In this regime, the solution with higher value of λ is a ghost. Further increasing κ, the solutions move to the complex plane. Finally, there is another threshold $\kappa_{c,3} > 0$ such that for $\kappa > \kappa_{c,3}$ there are real solutions in $(1/2, \infty)$. To see this we note that $b(\lambda) \to \infty$ for $\lambda \to 1/2^+$ and for $\lambda \to \infty$. It therefore has a minimum in $(1/2, \infty)$ with a minimum value $b_{\mathrm{min}} = b_{\mathrm{min}}|_{\tilde{\kappa}=0} - \tilde{\kappa}$. For sufficiently large $\tilde{\kappa}$ the minimum value is negative and we thus find real solutions. For at least one of them we have $\frac{d}{d\lambda}b > 0$, such that it has negative SL norm. Since $\langle Y_{\sigma,\vec{j}}, Y_{\sigma,\vec{j}}\rangle_{\mathrm{slice}}$ is either positive for $\sigma < 0$ or indefinite for $\sigma \geq 0$ this means we have ghosts in any case.

In summary, we have established analytically that theories with Dirichlet boundary conditions have a ghost-free spectrum for $\nu = 1$. On the other hand, for the family of Neumann-like boundary conditions we have found that there are ghosts for all values of κ.

3.1.4 Below the BF bound on global AdS

In Sec. 3.1.2.3 we found tachyonic AdS modes below the BF bound, which we discuss in more detail now. We consider a scalar field ϕ with squared mass $m^2 = -d^2/4 + (i\lambda)^2$, $\lambda \in \mathbb{R}$, below the BF bound on global AdS$_{d+1}$ with metric

$$ds^2 = \sec^2\rho \left(- dt^2 + d\rho^2\right) + \tan^2\rho \, d\Omega_{d-1} \; . \tag{3.69}$$

Changing the radial coordinate to $r = \cos\rho$ such that the boundary is located at $r = 0$, the asymptotic expansion of the field reads

$$\phi = r^{d/2+i\lambda}\phi^{(+)} + r^{d/2-i\lambda}\phi^{(-)} \; . \tag{3.70}$$

The symplectic structure constructed from the symplectic current $\omega_\mu(\phi_1, \phi_2) = i(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1)$ is finite without adding counterterms, and we shall impose boundary conditions at the conformal boundary such that it is conserved. The flux through the boundary is given by

$$\mathcal{F} = 2i\lambda \int_{\partial M} \left(\phi_1^{(-)}\phi_2^{(+)} - \phi_2^{(-)}\phi_1^{(+)}\right) \; . \tag{3.71}$$

We choose a boundary condition which makes $\mathcal{F}$ vanish and is compatible with reality of ϕ as a formal power series[8]

$$\phi^{(+)}\big|_{r=0} \; = \; \phi^{(-)}\big|_{r=0} \; . \tag{3.72}$$

It should be noted that the boundary condition (3.72) breaks invariance under radial isometries, as it relates the coefficients of different powers of r. Alternatively, from the boundary perspective conformal invariance is broken since the operators associated to $\phi^{(+)}$ and $\phi^{(-)}$ have different conformal dimensions.

In order to solve the Klein-Gordon equation we employ the mode decomposition $\phi = e^{-i\omega t}Y_L(\Omega)\psi(r)$ where Y_L is a spherical harmonic on S^{d-1} satisfying $\Delta_{\Omega_{d-1}}Y_L = -L(L + d-2)Y_L$. For notational convenience we introduce $a_\pm := c \pm \frac{\omega}{2}$ and $b_\pm := c^* \pm \frac{\omega}{2}$, where $c = (d + 2L - 2i\lambda)/4$. For $\lambda \in \mathbb{R}$ the solution which is regular at the origin is (see e.g. [69])

$$\psi(r) = r^{\frac{d}{2}-i\lambda}\left(1-r^2\right)^{\frac{L}{2}} \, {}_2F_1\Big(a_- \, , \, a_+ \, , \, \frac{d}{2} + L \, , \, 1-r^2\Big) \; . \tag{3.73}$$

Note that, using ${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, {}_2F_1(c-a, c-b; c; z)$, one can show that the radial profile (3.73) is real for $\omega^* = \pm\omega$. From (3.73) we find $\phi^{(\pm)}\big|_{r=0} = e^{-i\omega t}Y_L(\Omega)\psi_\pm$ where

$$\psi_+ = \frac{\pi \text{csch}(\pi\lambda)\Gamma\left(\frac{d}{2}+L\right)}{\lambda\,\Gamma(i\lambda)\Gamma\left(a_-\right)\Gamma\left(a_+\right)} \; , \qquad \psi_- = \frac{\pi \text{csch}(\pi\lambda)\Gamma\left(\frac{d}{2}+L\right)}{\lambda\,\Gamma(-i\lambda)\Gamma\left(b_-\right)\Gamma\left(b_+\right)} \; . \tag{3.74}$$

The boundary condition (3.72) therefore amounts to $\psi_+ = \psi_-$. This is equivalent to[9]

$$\frac{\Gamma(i\lambda)}{\Gamma(-i\lambda)} = \frac{\Gamma\left(b_-\right)\Gamma\left(b_+\right)}{\Gamma(a_-)\Gamma(a_+)} \; . \tag{3.75}$$

[8] The boundary condition (3.72) can be generalized to include a phase as $\phi^{(+)}\big|_{r=0} = e^{i2\alpha}\phi^{(-)}\big|_{r=0}$, $\alpha \in \mathbb{R}$. This corresponds to rescaling the coordinate r as can be seen from (3.70), and we therefore set $\alpha = 0$ without loss of generality.

[9] The Γ-functions in the denominator only have poles or zeros if $\text{Im}(\omega) = \pm\lambda$. This, however, does not yield solutions since due to the structure of the arguments the pole/zero always appears in one of $\psi_\pm$ only.

We first show that there are only real or purely imaginary solutions. Using the Weierstraß form $\Gamma(z)^{-1} = ze^{\gamma z}\prod_{k=1}^{\infty}(1+z/k)e^{-z/k}$, the modulus of (3.75) yields

$$\begin{aligned} 1 = \left|\frac{\Gamma(b_+)\Gamma(b_-)}{\Gamma(a_+)\Gamma(a_-)}\right|^2 &= \prod_{k=0}^{\infty}\left|\frac{(k+a_+)(k+a_-)}{(k+b_+)(k+b_-)}\right|^2 \\ &= \prod_{k=0}^{\infty}\left(1+\frac{(d/2+L+2k)\lambda\,\mathrm{Re}(\omega)\,\mathrm{Im}(\omega)}{|k+b_+|^2\,|k+b_-|^2}\right)\ . \end{aligned} \tag{3.76}$$

The first equality follows from (3.75), the second one by using the Weierstraß form and the third one by evaluating each factor. Depending on the sign of $\lambda\,\mathrm{Re}(\omega)\,\mathrm{Im}(\omega)$, either each factor in the product is greater than one, or each factor is less than one. As that makes the entire product on the right different from 1, we conclude that there are no solutions with $\mathrm{Re}(\omega) \neq 0$ and $\mathrm{Im}(\omega) \neq 0$.

For ω real or purely imaginary the modulus of both sides of (3.75) is identically 1. We first analyze purely imaginary ω. In this case we can use the asymptotic expansion $\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-1/2}\big(1+\mathcal{O}(|z|^{-2})\big)$ which holds if z is bounded away from the negative real axis ($\exists\,\delta>0:\ |\arg z| < \pi-\delta$). Parametrizing $\omega = i(\lambda+2e^{\tau})$ we find

$$\frac{\Gamma(b_-)\,\Gamma(b_+)}{\Gamma(a_-)\Gamma(a_+)} = e^{-2i\lambda}\sqrt{\frac{a_+a_-}{b_+b_-}}\,\frac{b_-^{b_-}\,b_+^{b_+}}{a_-^{a_-}\,a_+^{a_+}} + \mathcal{O}(|\omega|^{-2}) = e^{2i\lambda\tau} + \mathcal{O}(e^{-2\tau})\ . \tag{3.77}$$

The second equality follows from the asymptotic expansion for generic large ω and the third one by expanding the result for large imaginary part, i.e. large e^{τ}. Therefore, in the regime of large τ, solving (3.75) becomes equivalent to solving $2\lambda\tau = 2\arg\Gamma(i\lambda) \mod 2\pi$. This yields a discrete series of solutions which for large $|\omega|$ is well approximated by $\omega = \pm i(\lambda+2e^{\tau})$ with $\tau = \lambda^{-1}(\arg\Gamma(i\lambda)+\pi k)$, $k\in\mathbb{Z}$. Note that for $\lambda\to 0$ the imaginary frequency solutions go off to $\pm i\infty$, consistent with the fact that there are no complex solutions for $\lambda = 0$. We stress that the presence of these imaginary frequency solutions indicates the expected instabilities that are known to occur for masses below the BF bound. Moreover, as argued in [69], the imaginary frequency solutions constitute a pair 'ghost/anti-ghost'.

For the real solutions we assume without loss of generality $\omega > 0$. Using $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to rewrite (3.75) such that all the arguments of the Γ function have positive real part and then using the asymptotic expansion discussed above yields

$$\frac{\Gamma(b_-)\,\Gamma(b_+)}{\Gamma(a_-)\Gamma(a_+)} = \frac{\sin(\pi a_-)}{\sin(\pi b_-)}\frac{\Gamma(b_+)\Gamma(1-a_-)}{\Gamma(a_+)\Gamma(1-b_-)} = \left(\frac{\omega}{2}\right)^{2i\lambda}\frac{\sin\left(\frac{\pi}{2}(\omega+i\lambda)\right)}{\sin\left(\frac{\pi}{2}(\omega-i\lambda)\right)} + \mathcal{O}(\omega^{-1})\ . \tag{3.78}$$

Equation (3.75) for large ω then becomes

$$\frac{\Gamma(i\lambda)}{\Gamma(-i\lambda)} = \left(\frac{\omega}{2}\right)^{2i\lambda}\frac{\sin\left(\frac{\pi}{2}(\omega+i\lambda)\right)}{\sin\left(\frac{\pi}{2}(\omega-i\lambda)\right)} = e^{2i\left(\lambda\log\frac{\omega}{2}+\arctan(\tanh\frac{\pi\lambda}{2}\cot\frac{\pi\omega}{2})\right)} =: e^{i\vartheta(\omega)}\ . \tag{3.79}$$

While $e^{i\vartheta(\omega)}$ is of course single-valued, the arctan is single-valued only up to addition of integer multiples of π. We choose the values within these classes such that $\arctan(a\cot\frac{\pi\omega}{2})$ becomes a continuous function on $\mathbb{R}$, e.g. $\arctan(a\cot\frac{\pi\omega}{2}) = \arctan_0(a\cot\frac{\pi\omega}{2}) - \mathrm{sign}(a)\pi\left\lfloor\frac{\omega}{2}\right\rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer smaller than x and $\arctan_0$ is the principal value in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This makes ϑ a continuous function which tends to $-\infty$ for $\omega \to \infty$. Due to the periodicity of $e^{i\vartheta(\omega)}$ this shows that there is a series of solutions to (3.79). Thus, we also have a series of real solutions to (3.75). Finally, we note that the approximations derived above in (3.77), (3.78) provide an accurate description already for moderately large arguments of the Γ-functions.

3.2 Saturating the Unitarity Bound in AdS/CFT$_{\text{(AdS)}}$

In our investigation of how the CFT unitarity properties are reflected in the dual bulk description in the previous section we have found ghosts for the case where the dual operator violates the unitarity bound, consistent with the expectations from the CFT side. On the other hand, the case where the dual operator actually saturates the unitarity bound is of particular interest, as fields with the corresponding mass frequently appear in supergravity spectra on geometries relevant for AdS/CFT, see e.g. [82]. As found in Sec. 3.1 the standard Klein-Gordon field contains ghosts in that case, although a unitary representation of the conformal group exists. The same applies to the Klein-Gordon field on global AdS with the dual CFT defined on the cylinder [69]. In this section we are interested in the particular case where the unitarity bound in the dual CFT is saturated, for which the singleton field theory turns out to play a crucial role.

The singleton [9, 83, 84, 85] is a particular representation of the isometry group SO(2,d) of AdS$_{d+1}$. The maximal compact subgroup is SO(2)⊗SO(d) and representations $D(E, j)$ are characterized by an energy E (the lowest weight of SO(2)) and a set of SO(d) quantum numbers j. The scalar singleton is realized as an indecomposable representation $D(d/2-1, 0) \to D(d/2+1, 0)$ for $d > 2$ (see [86] for AdS$_3$). The structure can be extended to a Gupta-Bleuler triple of scalar → physical → gauge modes as [83, 87]

$$D(d/2+1, 0) \to D(d/2-1, 0) \to D(d/2+1, 0) \ , \tag{3.80}$$

and it can be formulated using a fourth-order action [88]. Among the remarkable properties of this representation is that it allows for the construction of a gauge theory for a scalar field with mass $m^2 = -d^2/4 + \nu^2$ if $\nu^2 = 1$. Its role in AdS/CFT has been emphasized and discussed in [89]. See also [90] where the singleton appears as a special case in the discussion of gravity duals for logarithmic CFTs.

The particular inner product used for the singleton field theory was obtained in [83] as the limit $\nu \to 1$ of the non-renormalized inner product of solutions to the Klein-Gordon equation with generic $\nu < 1$. Taking into account the contribution of the holographic counterterms to the inner product [52] we identify an alternative limit yielding the singleton theory for fixed $\nu = 1$. This allows for a direct application of the standard AdS/CFT dictionary, showing that the unitary singleton describes a free field on the boundary – as expected for a field saturating the CFT unitarity bound.

We then turn to the holographic description of CFTs which are itself defined on AdS$_d$, employing the geometry discussed in Sec. 3.1.1. As discussed in Sec. 3.1.2.4, there is an additional subtlety if Neumann boundary conditions are chosen on the boundary of the AdS_d hypersurfaces, resulting in a breaking of the bulk isometries or unitarity. We discuss the singleton on the AdS_d slicing of AdS_{d+1}, yielding also for that case a unitary bulk theory for $\nu = 1$. Furthermore, we find that the normalizability issues for Neumann boundary conditions on the boundary of AdS_d are avoided.

In Sec. 3.2.1 we derive the singleton theory on global AdS_{d+1} from the Klein-Gordon field with renormalized inner product and discuss its role for the unitarity bound. In Sec. 3.2.2 we perform the same construction on the geometry with AdS_d conformal boundary and discuss the normalizability issues found previously for the standard Klein-Gordon field. This work was published in collaboration with Thorsten Ohl in [11].

3.2.1 The singleton on AdS_{d+1} in global coordinates

To fix notation we recapitulate in this section the standard construction of the singleton on AdS in global coordinates [83, 84]. We also offer a new perspective on the choice of the inner product in the light of [52]. We choose global coordinates (z, τ, Ω_{d-1}) on AdS_{d+1} such that the line element reads

$$ds^2 = \frac{l^2}{\sin^2 z}\big(- d\tau^2 + dz^2 + \cos^2 z \, d\Omega_{d-1}^2 \big) \ , \tag{3.81}$$

and consider a Klein-Gordon field with mass $m^2 l^2 = -d^2/4 + \nu^2$ and action

$$S = -\frac{1}{2} \int d^{d+1}x \, \sqrt{g} \Big(g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2 \Big) \ . \tag{3.82}$$

Our focus here is on the case $\nu = 1$. The standard inner product associated to (3.82) reads

$$\langle \phi_1, \phi_2 \rangle = \int_\Sigma \sqrt{g_{\mathrm{ind}}} \, n^\mu \Big(\phi_1^\star \overleftrightarrow{\partial}_\mu \, \phi_2 \Big) \ , \tag{3.83}$$

where Σ is a spacelike hypersurface with unit normal vector field $n^\mu \partial_\mu$. To solve the field equations we employ the ansatz

$$\phi(z, \tau, \Omega_{d-1}) = e^{-i\omega\tau} \, Y_{\vec{L}}(\Omega_{d-1}) \, f(z) \ , \tag{3.84}$$

where $Y_{\vec{L}}$ are the spherical harmonics on S^{d-1} satisfying $\triangle_{S^{d-1}} Y_{\vec{L}} = -L(L+d-2) Y_{\vec{L}}$. The resulting equation for the radial modes $f(z)$ can be written as Sturm-Liouville problem

$$Kf = \omega^2 f \ , \qquad K = \frac{1}{w(z)} \left[-\partial_z p(z) \partial_z + q(z) \right] \ , \tag{3.85}$$

where $w(z) = p(z) = \cot^{d-1} z$ and $q(z) = \tan^{d-1} z \left[L(L+d-2)\cos^{-2} z + m^2 l^2 \sin^{-2} z \right]$. Choosing for Σ a surface of constant τ we find

$$\langle \phi_1, \phi_2 \rangle = \delta_{\vec{L}_1, \vec{L}_2} (\omega_1 + \omega_2) e^{i(\omega_1 - \omega_2)\tau} l^{d-1} \langle f_1, f_2 \rangle_{\mathrm{SL}} \ , \tag{3.86}$$

where the Sturm-Liouville inner product is given by

$$\langle f_1, f_2\rangle_{\rm SL} = \int dz \cot^{d-1} z\, f_1^\star f_2 \ . \tag{3.87}$$

Using integration by parts and (3.85) yields

$$\langle f_1, f_2\rangle_{\rm SL} = \frac{1}{\omega_1^{*2} - \omega_2^2} \left[\cot^{d-1} z \left(f_1^* f_2' - f_1'^* f_2\right)\right]_0^{\pi/2} \ , \tag{3.88}$$

which is to be understood in the distributional sense. The two independent sets of solutions to (3.85) are

$$f(z) = (\sin z)^{\frac{d}{2}-\nu} (\cos z)^L \, {}_2F_1\Big(\frac{a-\omega-\nu}{2}, \frac{a+\omega-\nu}{2}; a; \cos^2 z\Big) \ , \tag{3.89}$$

where $a = d/2 + L$, and a second set which is $f\big|_{L\to 2-d-L}$ for odd d and a combination involving explicit logarithms for even d [76]. Demanding the solutions to be regular at the origin $z = \pi/2$ selects the modes (3.89). For the singleton theory, instead of deriving the frequency spectrum from a vanishing-flux boundary condition, one imposes [84]

$$\pm\omega = a - 1 + 2n \ , \qquad n \in \mathbb{N} \cup \{0\} \ . \tag{3.90}$$

The $n \geq 1$ modes are the standard Dirichlet solutions, i.e. they are $\mathcal{O}(z^{d/2+1})$ in the boundary limit. They form the representation $D(d/2 + 1)$ with lowest-weight state given by the $n = 1$ mode with $\omega = d/2 + 1$. Adding the $n = 0$ solutions yields the representation $D(d/2 - 1)$. The $n = 0$ mode is $\mathcal{O}(z^{d/2-1})$ and is not normalizable with respect to (3.83). Following [83, 84] we replace the radial part of the inner product by

$$\langle f_1, f_2\rangle_{\rm sing} = \lim_{\nu\to 1^-} (1-\nu)\, \langle f_1, f_2\rangle_{\rm SL} \ , \tag{3.91}$$

where the f_i on the right hand side are the modes f of (3.89) for generic $\nu < 1$. Evaluating this inner product yields

$$\langle f_1, f_2\rangle_{\rm sing} = \frac{1}{2}\delta_{n_1,0}\delta_{n_2,0} \ . \tag{3.92}$$

Thus, except for the $n = 0$ mode which has positive norm all other modes are of norm zero, i.e. pure gauge. The singleton representation is induced on the quotient space obtained by identifying in the space spanned by the $n \geq 0$ solutions those which differ only by $n \geq 1$ modes. It has only a single (E, j) trajectory, hence the name.

3.2.1.1 Relation to the renormalized inner product

As discussed in detail in [52] the contribution of the holographic counterterms is crucial for dealing with the divergences in the symplectic structure and inner product. We now discuss the singleton representation from that perspective. The action (3.82) reduces on shell to a boundary term $S_{\rm on-shell} = \frac{1}{2}\int_{z=0} \phi\sqrt{g^{zz}}\partial_z\phi$ which is divergent for $\nu \geq 1$. We suppress the standard volume form constructed from the (induced) metric from here on. For

$\nu = 1$ $S_{\text{on-shell}}$ contains a logarithmic divergence and is rendered finite by regularizing the geometry with a cut-off $z \geq \epsilon$ and adding boundary terms at $z = \epsilon$. The renormalized action is $S_{\text{ren}} := S + S_{\text{ct}}$ with

$$S_{\text{ct}} = -\frac{1}{2}\int_{z=\epsilon}\left[\left(\frac{d}{2}-1\right)\phi^2 - \big(\log z + \kappa\big)\phi\,\Box_{g_{\text{ind}}}\phi\right] . \tag{3.93}$$

Note that we have included, with an arbitrary coefficient κ, a boundary term which is compatible with all symmetries and finite for $\nu = 1$. With the asymptotic expansion of ϕ given by

$$\phi = \phi^{(0)} z^{\frac{d}{2}-1} + \phi^{(1)} z^{\frac{d}{2}+1}\log z + \phi^{(2)} z^{\frac{d}{2}+1} + \dots \tag{3.94}$$

the variation of the renormalized action reads

$$\delta S_{\text{ren}} = \text{EOM} + \int_{z=\epsilon}\delta\phi^{(0)}\big(2\phi^{(2)} + (1-2\kappa)\phi^{(1)}\big) . \tag{3.95}$$

The boundary conditions for a stationary action are therefore either the Dirichlet condition $\delta\phi^{(0)} = 0$ or the Neumann condition

$$2\phi^{(2)} + (1-2\kappa)\phi^{(1)} = 0 . \tag{3.96}$$

The inner product associated to the renormalized action takes a form similar to (3.86). The contribution of the counterterms to the inner product can be absorbed into a renormalized Sturm-Liouville product, which then reads

$$\langle f_1, f_2\rangle_{\text{ren}} = \langle f_1, f_2\rangle_{\text{SL}} + (\log z + \kappa)\cot^{d-1} z \sin z\, f_1^* f_2\big|_{z\to 0} . \tag{3.97}$$

We now consider a particular limit which yields the frequency quantization (3.90) and the inner product (3.92) such that we obtain the singleton. To this end we rescale the field as $\phi \to \phi' = \kappa^{-1/2}\phi$ and perform a limit $\kappa \to \infty$. We consider the family of theories for $\kappa \in \mathbb{R}^+$. The variation of the action reads

$$\delta S = -\frac{1}{2}\int \kappa^{-1}\delta\phi'\big(-\Box + m^2\big)\phi' + \int_{z=\epsilon}\delta\phi'^{(0)}\big(2\kappa^{-1}\phi'^{(2)} + (\kappa^{-1}-2)\phi'^{(1)}\big) . \tag{3.98}$$

The bulk part has to vanish for any finite κ and so the bulk field equation also applies as we consider the limit $\kappa \to \infty$. However, had we included interaction terms in (3.82) they would become negligible with respect to the quadratic part. The field rescaling ensures that we get a finite on-shell action. In the boundary part of the variation the κ^{-1}-terms become negligible with respect to the remaining term, so the variation reduces to

$$\delta S_{\text{ren}} = \text{EOM} - 2\int_{z=\epsilon}\delta\phi'^{(0)}\phi'^{(1)} . \tag{3.99}$$

The Neumann boundary condition (3.96) thus becomes $\phi'^{(1)} = 0$. With the expansion $f = z^{-d/2-1}(f^{(0)} + f^{(1)}z^2\log z + f^{(2)}z^2 + \dots)$ we then have to solve $f^{(1)} = 0$. For the modes (3.89) with $\nu = 1$ we have

$$f^{(1)} = \frac{2\Gamma(a)}{\Gamma\left(\frac{a-\omega-1}{2}\right)\Gamma\left(\frac{a+\omega-1}{2}\right)} . \tag{3.100}$$

Solving $f^{(1)} = 0$ amounts to demanding the Γ-functions in the denominator to have a pole, which yields the frequency quantization (3.90). Note that (3.90) can thus be understood as solving vanishing-flux boundary conditions for the renormalized symplectic structure, see Sec. 2 of [52]. For the inner product we find

$$\langle \phi_1', \phi_2' \rangle = \lim_{\kappa \to \infty} \delta_{\vec{L}_1, \vec{L}_2} (\omega_1 + \omega_2) e^{i(\omega_1 - \omega_2)\tau} l^{d-1} \langle f_1, f_2 \rangle_\kappa \ , \tag{3.101}$$

where $\langle f_1, f_2 \rangle_\kappa = \kappa^{-1} \langle f_1, f_2 \rangle_{\text{ren}}$. With the notation $\tilde{f}_n := f\big|_{\omega=(d-2)/2+L+2n}$ we find for the radial part

$$\begin{aligned} \langle \tilde{f}_0, \tilde{f}_0 \rangle_\kappa &= \frac{1}{2\kappa} \Big(2\kappa - \psi^{(0)}(a) - \gamma \Big) \ , \qquad \langle \tilde{f}_0, \tilde{f}_n \rangle_\kappa = -\frac{1}{2\kappa} (-1)^n B(a,n) \ , \\ \langle \tilde{f}_n, \tilde{f}_m \rangle_\kappa &= \delta_{nm} \frac{n(a+n-1)}{2\kappa(a+2n-1)} B(n,a)^2 \ , \end{aligned} \tag{3.102}$$

where $n, m > 0$ and $B(x,y)$ is the Euler beta function. Clearly, in the limit $\kappa \to \infty$ only $\langle \tilde{f}_0, \tilde{f}_0 \rangle_\kappa$ is non-vanishing and in fact positive, such that we recover (3.92) up to an overall factor.

This can also be understood from a scaling argument as follows. We argued above that the action does not simply reduce to the boundary terms for $\kappa \to \infty$, as the bulk field equation applies for any finite κ while the boundary terms merely affect the boundary conditions. However, the inner product associated to the renormalized action is just the sum of the bulk part (3.83) and the boundary contributions derived from (3.93). Thus, it indeed reduces to the boundary part arising from the term proportional to κ as we take the limit $\kappa \to \infty$ with the corresponding field rescaling. This remaining part now vanishes for the $n > 0$ modes as they satisfy the standard Dirichlet boundary condition.

3.2.1.2 AdS/CFT at the unitarity bound

Realizing the singleton as discussed in the previous section allows for a direct interpretation in the AdS/CFT context. Fluctuations of a scalar with Neumann boundary condition correspond to a deformation of the dual CFT by an operator $\mathcal{O}$ with scaling dimension $d/2 - \nu$ [68]. Performing the Legendre transform

$$S_{\text{ren}} \to S_{\text{ren}}^{\text{N}} := S_{\text{ren}} - \int_{z=\epsilon} \phi^{(0)} \big(2\phi^{(2)} + (1 - 2\kappa)\phi^{(1)} \big) \tag{3.103}$$

we find

$$\delta S_{\text{ren}}^{\text{N}} = \text{EOM} - \int_{z=\epsilon} \phi^{(0)} \, \delta \big(2\phi^{(2)} + (1 - 2\kappa)\phi^{(1)} \big) \ , \tag{3.104}$$

and the on-shell action becomes a functional of the Neumann boundary data $2\phi^{(2)} + (1-2\kappa)\phi^{(1)}$. For $\kappa \to \infty$ with the field rescaling $\phi \to \phi' = \kappa^{-1/2}\phi$ discussed above, we find $\delta S_{\text{ren}}^{\text{N}} = \text{EOM} + \int_{z=\epsilon} 2\phi'^{(0)} \delta\phi'^{(1)}$. Following the familiar AdS/CFT identification of bulk partition function and the generating functional for boundary correlation functions, functional differentiation of $S^{\mathrm{N}}_{\mathrm{ren}}$ with respect to $\phi'^{(1)}$ yields the connected correlation functions of the dual operator $\mathcal{O}$ of the CFT. We find

$$\langle\mathcal{O}\rangle = \frac{1}{\sqrt{g}}\frac{\delta S^{\mathrm{N}}_{\mathrm{ren}}}{\delta\phi'^{(1)}} = 2\phi'^{(0)} \ , \qquad \langle\mathcal{O}\mathcal{O}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\langle\mathcal{O}\rangle}{\delta\phi'^{(1)}} \ , \tag{3.105}$$

where $\phi'^{(1)} = -\frac{1}{2}\big(\Box_{g^{(0)}} - \frac{1}{4}\frac{d-2}{d-1}R[g^{(0)}]\big)\phi'^{(0)}$ for a generic asymptotically-AdS metric of the form $r^{-2}(dr\otimes dr - g)$. The n-point functions with $n\geq 3$ vanish unless interactions of the bulk scalar are included. However, for the singleton there are no gauge-invariant bulk interactions as the field is gauge-equivalent to zero in any compact region, so the higher correlation functions vanish. This is characteristic of (generalized) free fields and the singleton therefore yields the dual description of a free field on the boundary, consistent with the fact that this is the only way of realizing a unitary representation of the conformal group for $\Delta = d/2-1$.

3.2.2 The singleton on the $\mathbf{AdS_d}$ slicing of $\mathbf{AdS_{d+1}}$

We now turn to the holographic description of CFTs defined on AdS_d. A geometry for the dual description has been proposed in [62] and was discussed in the context of unitarity from the holographic perspective in Sec. 3.1.2. As shown there, the standard Klein-Gordon theory yields ghosts for $\nu\geq 1$ and the renormalization turns out to be nontrivial if Neumann boundary conditions are chosen at the boundary of AdS_d. We come back to that issue at the end of the section.

3.2.2.1 The geometry

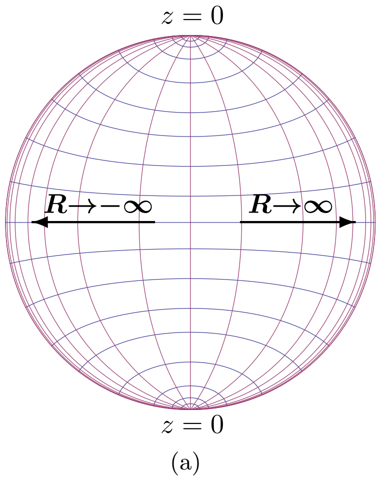
As explained in detail in Sec. 3.1.1 the slicing of AdS_{d+1} with curvature radius L by AdS_d hypersurfaces with curvature radius l is obtained by transforming the global coordinates with line element (3.1) according to (3.5). The resulting line element reads

$$ds^2 = dR^2 + \frac{L^2}{l^2}\cosh^2\frac{R}{L}\,ds^2_{\mathrm{AdS}_d} \ , \tag{3.106}$$

where $ds^2_{\mathrm{AdS}_d} = l^2\csc^2 z\,(-d\tau^2 + dz^2 + \cos^2 z\, d\Omega^2_{d-2})$. Two patches are needed to cover the full AdS_{d+1} and the conformal boundary of the resulting geometry comprises two copies of AdS_d, see Fig. 3.3(a). A geometry with a single AdS_d conformal boundary is obtained by taking a $\mathbb{Z}_2$ quotient identifying the two patches. This implies that we have to choose a definite $\mathbb{Z}_2$ parity for the Klein-Gordon field, which imposes boundary conditions at $R=0$.

As usual this geometry needs to be regularized to account for divergences in the on-shell action and inner products. This was done in the previous section by imposing cut-offs on $y := 2le^{-R/L}$ and z, i.e. $y\geq\epsilon_1$, $z\geq\epsilon_2$. The resulting geometry with its boundary is illustrated in Fig. 3.3(b). The renormalized action $S_{\mathrm{ren}} := S + S_{\mathrm{ct}}$ was constructed in Sec. 3.1 for $L=l=1$, which we fix henceforth, with the counterterms

$$S_{\mathrm{ct}} = -\frac{1}{2}\int_{\partial_1\mathcal{M}}\left[\left(\frac{d}{2}-1\right)\phi^2 - \big(\log y+\kappa\big)\phi\,\Box^W_{g_{\mathrm{ind}}}\phi\right] - \frac{1}{2}\int_{\partial\partial\mathcal{M}}(\log y+\kappa)\phi\mathcal{L}_n\phi \ . \tag{3.107}$$



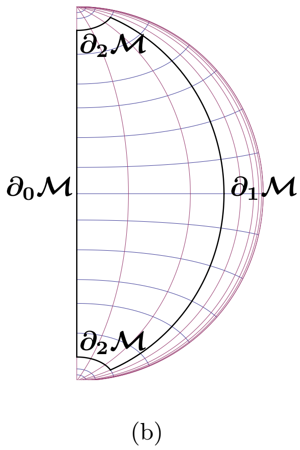


Figure 3.3: Poincaré disk representation of AdS_{d+1} sliced by AdS_d hypersurfaces and the boundary of the regularized $\mathbb{Z}_2$-quotient, for more details see Sec. 3.1.1.

The associated inner product is given by (3.21b) where $\langle\cdot,\cdot\rangle_{\partial_1\mathcal{M}}$ denotes the AdS_d inner product evaluated at fixed $y=\epsilon_1$.

3.2.2.2 AdS/CFT$_{\mathrm{AdS}}$ at the unitarity bound

We now construct the singleton theory on this geometry analogously to the construction in Sec. 3.2.1.1. Dropping terms which vanish upon imposing the field equations or $\mathbb{Z}_2$ parity, the variation of the renormalized action reads

$$\delta S_{\mathrm{ren}} = \int\limits_{\partial_2\mathcal{M}} \delta\phi\sqrt{g^{zz}}\partial_z\phi + \int\limits_{\partial_1\mathcal{M}} \delta\phi^{(0)}\big(2\phi^{(2)}+(1-2\kappa)\phi^{(1)}\big) + \int\limits_{\partial\partial\mathcal{M}} (\log y+\kappa)\delta\phi\sqrt{g^{zz}}\partial_z\phi\ , \quad (3.108)$$

where $\partial\partial\mathcal{M}=\partial_1\mathcal{M}\cap\partial_2\mathcal{M}$ and similar to (3.94)

$$\phi = \phi^{(0)}y^{\frac{d}{2}-1} + \phi^{(1)}y^{\frac{d}{2}+1}\log y + \phi^{(2)}y^{\frac{d}{2}+1} + \dots\ . \quad (3.109)$$

Demanding the $\partial_1\mathcal{M}$ boundary term to vanish imposes boundary conditions on f and we choose the Neumann condition

$$2\phi^{(2)} + (1-2\kappa)\phi^{(1)} = 0\ . \quad (3.110)$$

As discussed in detail in Sec. 3.1.1.2 the Klein-Gordon equation is conveniently solved by a separation ansatz $\phi=\varphi f$ where φ satisfies an AdS_d Klein-Gordon equation with mass $M^2=-(d-1)^2/4+\mu^2$. The independent solutions to the radial part are

$$f_i = u^{2c_i-\frac{3}{2}}\left(1-u^2\right)^{\frac{d+2}{4}}\,{}_2F_1\big(c_i-\frac{\mu}{2},c_i+\frac{\mu}{2};2c_i-1;u^2\big)\ , \quad (3.111)$$

where $i = 1, 2$, $u = \tanh(R)$ and $c_1 = 3/4$, $c_2 = 5/4$. f_1 and f_2 have even and odd $\mathbb{Z}_2$ parity, respectively. Rescaling $\phi \to \phi' = \kappa^{-1/2}\phi$ and considering the limit $\kappa \to \infty$ the $\partial_1\mathcal{M}$ term of the variation (3.108) becomes $\delta S_{\text{ren}}|_{\partial_1\mathcal{M}} = -2\int_{\partial_1\mathcal{M}} \delta\phi'^{(0)}\phi'^{(1)}$ and the boundary condition (3.110) becomes $\phi'^{(1)} = 0$, demanding the log-term in the expansion of $f_i(y)$ around $y = 0$ to vanish. This yields the spectrum of μ for which we find $\mu = 1/2$ and $\mu = 2(c_i + n)$ with $n \in \mathbb{N} \cup \{0\}$. Solutions corresponding to the latter choice of μ are subdominant in the boundary limit.

For the solutions constructed by means of our separation ansatz we find

$$\langle \phi'_1, \phi'_2 \rangle_{\text{ren}} = \langle \varphi_1, \varphi_2 \rangle \, \langle f_1, f_2 \rangle_\kappa \ , \tag{3.112}$$

where $\langle \varphi_1, \varphi_2 \rangle$ is the standard AdS$_d$ inner product and $\langle f_1, f_2 \rangle_\kappa = \kappa^{-1}\langle f_1, f_2 \rangle_{\text{ren}}$. With $\tilde{\kappa} = \kappa + \log 2$ the renormalized Sturm-Liouville inner product with the counterterm contributions according to (3.107) is given by

$$\langle f_1, f_2 \rangle_{\text{ren}} = \langle f_1, f_2 \rangle_{\text{SL}} - (R - \tilde{\kappa}) \cosh^{d-2} R \; f_1^* f_2 \big|_{R\to\infty} \ . \tag{3.113}$$

Denoting $\tilde{f}_i := f_i|_{\mu=1/2}$ and $\tilde{f}_i^n := f_i|_{\mu=2(c_i+n)}$ we find

$$\begin{aligned} ||\tilde{f}_i||_\kappa^2 &= \frac{1}{\kappa}\Big(\tilde{\kappa} + \frac{3}{2} - 2c_i\Big) \ , \qquad\qquad \langle \tilde{f}_i, \tilde{f}_i^n \rangle_\kappa = \frac{\sqrt{2\pi}(-1)^n n!}{2^{2c_i}\kappa\Gamma(2c_i + n)} \ , \\ \langle \tilde{f}_i^n, \tilde{f}_i^m \rangle_\kappa &= \delta_{nm} \frac{2\pi (n!)^2 \left(2c_i n + 2c_i + n^2 - 1\right)}{2^{4c_i}\kappa(c_i + n)\Gamma(2c_i + n)^2} \ . \end{aligned} \tag{3.114}$$

Clearly, for $\kappa \to \infty$ only $||\tilde{f}_i||_\kappa^2$ is non-vanishing and in fact positive. Thus, in that limit all the subdominant modes $\tilde{f}_i^n$ become pure gauge while the dominant $\tilde{f}_i$ remains physical, and we obtain the singleton field on the geometry with AdS on the boundary. The choice of $\mathbb{Z}_2$ parity has little effect – it only alters the spectrum of gauge modes and the form of the radial profile of the physical $\mu = 1/2$ mode close to the $\mathbb{Z}_2$-fixed hypersurface at $R = 0$.

In Sec. 3.1.2 it was shown that pushing the bulk scalar on the geometry considered here beyond the unitarity bound yields ghosts in the spectrum. Likewise, ghosts were also found for the standard Klein-Gordon field with mass such that the dual operator saturates the unitarity bound, although in that case a unitary representation is expected to exist. The discussion of the correlation functions of the dual CFT obtained from the singleton theory in Sec. 3.2.1.2 immediately applies to the singleton theory on the geometry considered in this section. Thus, the singleton yields the unitary bulk dual of a boundary free field also for the dual CFT defined on AdS$_d$, completing the discussion in Sec. 3.1.2. Furthermore, it offers a way to avoid the issues with normalizability found there for Neumann boundary conditions along z, as we discuss in more detail now.

3.2.2.3 Renormalization and Neumann$_\text{d}$ boundary conditions

The normalizability issues found in Sec. 3.1.2 for Neumann boundary conditions at $z = 0$ (referred to as 'Neumann$_d$' there) for the standard Klein-Gordon field arise for any choice of the bulk mass and are rooted in the AdS$_d$ factor of the inner product $\langle \phi_1, \phi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle \, \langle f_1, f_2 \rangle$.

Depending on the renormalization it either fails to be finite on the full solution space or becomes indefinite for Neumann boundary conditions along z. The result is either a drastic truncation of the spectrum of AdS_d modes such that the bulk field fails to carry a representation of the AdS isometries, or the appearance of ghosts such that it fails to carry a unitary one. More precisely, the solutions we constructed by means of the separation ansatz $\phi = \varphi f$ comprise an infinite series of AdS_d modes corresponding to $\mu = 1/2$ and $\mu = 2(c_i + n)$ with the associated radial modes. The AdS_d factor $\langle \varphi_1, \varphi_2 \rangle$ of the inner product is divergent for the $\mu^2 \geq 1$ solutions, leaving only a drastically reduced set of normalizable modes. On the other hand, rendering that part of the inner product finite by adding counterterms on $\partial_2 \mathcal{M}$ – if possible – would spoil positive definiteness of the inner product. The special structure of the singleton field theory automatically avoids these issues. In fact, since the radial part of the norm vanishes for all $\mu^2 > 1$ modes, finiteness of the inner products as the cut-offs on y and z are removed does not require any additional counterterm contributions to the AdS_d factor. For the physical $\mu = \frac{1}{2}$ mode the AdS_d factor of the norm is positive and so is the radial part (3.114). We thus have a well-defined semidefinite inner product on the set of all modes also for Neumann boundary conditions along z and the drastic reduction of the spectrum of AdS_d modes found in Sec. 3.1.2 is avoided. Although promoting the $\mu^2 > 1$ modes to pure gauge in the $\kappa \to \infty$ limit is in fact a similar reduction of the physical spectrum, this way of realizing the Neumann boundary condition is compatible with the symmetries and with unitarity.

3.3 Multi-Layered AdS/CFT

In Sec. 3.1.1 we have discussed an AdS_{d+1} geometry where the conformal boundary is AdS_d, and we studied the holographic description of CFTs defined on that boundary in Sec. 3.1.2 and 3.2.2. Fig. 3.4 highlights the particular structure of the boundary, offering the possibility to implement a second instance of AdS/CFT-type duality – either by realizing a gravitational theory on the boundary and using AdS/CFT again, or by using the boundary CFT as starting point for algebraic holography (see Sec. 2.4). This would relate the AdS_d boundary theory to a theory on the 'boundary of the boundary'. In this section we discuss such multi-layered dualities in more detail[10]. However, this in the first place needs the notion of the boundary of a boundary, which does not exist in the context of smooth manifolds with boundary. We therefore first discuss in more detail the geometric nature of the $(d{+}1)$-dimensional setup with AdS_d boundary. The setting of $\langle n \rangle$-manifolds, where the boundary of a boundary indeed is a well-defined concept, can in fact be pushed even further. In Sec. 3.3.2 we construct a geometry with three instances of AdS spaces, i.e. where the boundary of the boundary is again AdS, thus allowing for hierarchies of AdS/CFT dualities. While the construction of the geometry is rather straightforward, one expects problems in trying to realize such hierarchies of dualities. From the results of the previous sections we can conclude that iterating AdS/CFT more than twice does not yield non-trivial relations, at least with the renormalization prescription used here. We then turn to double-layered holography and comment on prospects for a concrete realization within M-theory in Sec. 3.3.3. The bulk

[10] An option which we do not discuss here is to combine AdS/CFT with a more speculative dS/CFT correspondence [91] by using the dS_d slicing of AdS_{d+1}.

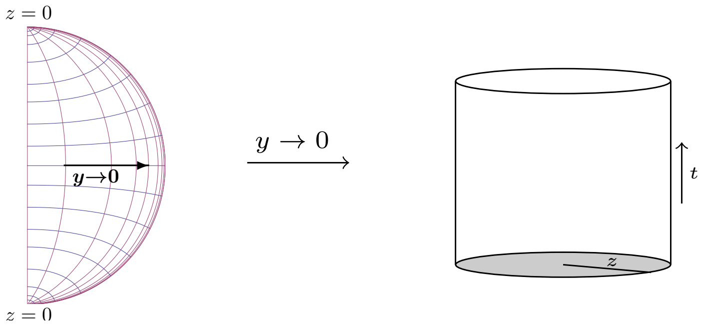


Figure 3.4: The boundary of $AdS_{d+1}/\mathbb{Z}_2$ is AdS_d which again has a conformal boundary. A second instance of holographic duality can thus be implemented if the boundary theory on AdS_d admits a dual description in terms of a CFT_{d-1} on the boundary of the boundary.

theory in AdS/CFT necessarily is a gravitational one, since the boundary theory contains an energy-momentum tensor as observable. Implementing nested AdS/CFT type dualities therefore also needs gravitational theories on the boundary, an issue which we will discuss in more detail in Sec. 4.

3.3.1 Conformally compact $\langle n \rangle$-manifolds

The geometry with AdS on the boundary discussed in Sec. 3.1.1 and shown in Fig. 3.3(a), 3.3(b) can not be described in the usual setting of a conformally compact metric on a smooth manifold with boundary[11]. The full AdS_{d+1} geometry shown in Fig. 3.3(a) is a smooth manifold with boundary, but the metric (3.16) on the interior fails to be conformally compact – rescaling it to cancel the divergence in y as the boundary is approached still leaves the $\gamma^{\text{AdS}}_{\mu\nu}$-part divergent as $z = 0$ is approached, which can not be cured by extracting a global factor. On the other hand, the $\mathbb{Z}_2$-quotient of the geometry shown in Fig. 3.3(b) – although being a topological manifold with boundary – does not fit into the class of smooth manifolds with boundary. Thus, conformally compact metrics on smooth manifolds with boundary are not the appropriate framework to be used here. In particular, the boundary of a smooth manifold with boundary does not have a boundary again. However, the setup can be understood in the more general context of manifolds with corners. We briefly introduce that concept and some of the relevant properties following [93] and [94]. The prototype of a smooth manifold with corners is the closed positive quadrant

$$\overline{\mathbb{R}^n_+} := \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 \,\forall i = 1 \,..\, n\} \,, \tag{3.115}$$

which is a topological manifold with boundary but not a smooth manifold with boundary. A general manifold with corners locally looks like $\overline{\mathbb{R}^n_+}$ and the differential structure is defined as follows.

[11] A metric $\hat{g}$ on the interior of a manifold with boundary is called conformally compact if, for a defining function of the boundary f, the rescaled metric $f^2\hat{g}$ extends to the full manifold as metric [92].

Definition 3.3.1 A smooth manifold with corners $\mathcal{M}$ is covered by (relatively) open sets $U_i \subset \mathcal{M}$ together with homeomorphisms $\phi_i : U_i \to V_i \subset \overline{\mathbb{R}^n_+}$ which are smoothly compatible in the sense that $\phi_i \circ \phi_j^{-1} : \phi_j^{-1}(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ admits a smooth extension to $\mathbb{R}^n$.

The boundary is the set of points where at least one of the coordinates vanishes and the points where more than one coordinate vanishes are called corner points. More precisely, we denote for a point $p \in \mathcal{M}$ by $c(p)$ the number of vanishing coordinates of $\phi(p)$ in a chart (U, ϕ). This quantity is independent of the choice of chart, and $\partial\mathcal{M} := \{p \in \mathcal{M} \,|\, c(p) \geq 1\}$ constitutes the boundary. Points with $c(p) > 1$ are called corner points. With AdS/CFT applications in mind we also need some structure on the boundary. The boundary of a general smooth manifold with corners is not necessarily a smooth manifold with corners again. However, for a manifold with faces which we introduce now, it is a finite union of such. More precisely, each point $p \in \mathcal{M}$ belongs to the closure of at most $c(p)$ connectedness components of $\{q \in \mathcal{M} \,|\, c(q) = 1\}$. $\mathcal{M}$ is called a manifold with faces if this upper bound is saturated for all its points. The closure of a connectedness component of $\{q \in \mathcal{M} \,|\, c(q) = 1\}$ is called a connected face of $\mathcal{M}$ and then has itself the structure of a manifold with corners. This leads to the structure of $\langle n \rangle$-manifolds [95]:

Definition 3.3.2 A manifold with faces $\mathcal{M}$ with a set $\partial_0\mathcal{M}, \ldots, \partial_{n-1}\mathcal{M}$ of faces of $\mathcal{M}$ is called an $\langle n \rangle$-manifold if $\partial\mathcal{M} = \cup_{i=0}^{n-1} \partial_i\mathcal{M}$ and $\partial_i\mathcal{M} \cap \partial_j\mathcal{M}$ is a face of $\partial_i\mathcal{M}$ and $\partial_j\mathcal{M}$ for $i \neq j$. The intersections $\partial_{i_1}\mathcal{M} \cap .. \cap \partial_{i_k}\mathcal{M}$ are called $\langle n-k \rangle$-faces of $\mathcal{M}$.

Smooth manifolds are recovered as $\langle 0 \rangle$-manifolds and smooth manifolds with boundary as $\langle 1 \rangle$-manifolds. The geometry discussed in Sec. 3.1 before taking the $\mathbb{Z}_2$ quotient can be understood as a $\langle 2 \rangle$-manifold. To illustrate this we start from AdS_3 in global coordinates (τ, ρ, ζ) with line element $ds^2 = \sec^2\!\rho\,(d\rho^2 - d\tau^2 + \cos^2\!\rho d\zeta^2)$. In the usual discussion global AdS_3 is realized as the interior of a cylinder in $\mathbb{R}^3$ equipped with that metric. The compactification is then performed as appropriate for that embedding, i.e. by adding the boundary of the cylinder. Note that no reference is made to a metric on the ambient space. The crucial point for us is that the compactification of a topological space is by no means unique[12]. We exploit the freedom to choose a compactification as follows to obtain a $\langle 2 \rangle$-manifold. Instead of the embedding $(\tau, \rho, \zeta) \mapsto (\tau, \rho\cos\zeta, \rho\sin\zeta)$ into $\mathbb{R}^3$ which results in the realization as a cylinder shown in Fig. 3.5(a) we use

$$(\tau,\, \rho,\, \zeta) \mapsto (\tau,\, \rho\cos\zeta\sqrt{1 - \frac{4\rho^2}{\pi^2}\sin^2\zeta},\, \rho\sin\zeta)\ . \tag{3.116}$$

Although that embedding is not differentiable for $\rho = \pi/2$ and $\zeta \in \{0, 2\pi\}$, it is smooth for $\rho < \pi/2$ and we thus have a smooth embedding of AdS_3 itself, which corresponds to $\rho < \pi/2$. This yields the realization shown in Fig. 3.5(b). Switching now to the slicing by AdS_2 hypersurfaces in coordinates (τ, y, z) we end up with the geometry illustrated in Fig. 3.5(c). The closure now clearly displays the structure of a $\langle 2 \rangle$-manifold with the $\langle 1 \rangle$-faces corresponding to $y = 0$ and the $\langle 0 \rangle$-faces given by the corners at $z = 0$. (τ, y, z) are valid coordinates only in the interior of that disc with two corners and the metric still is well-defined only in the interior.

[12] While the one-point or Alexandroff compactification existing under rather mild assumptions is unique, this does not hold for the type of compactifications discussed here.

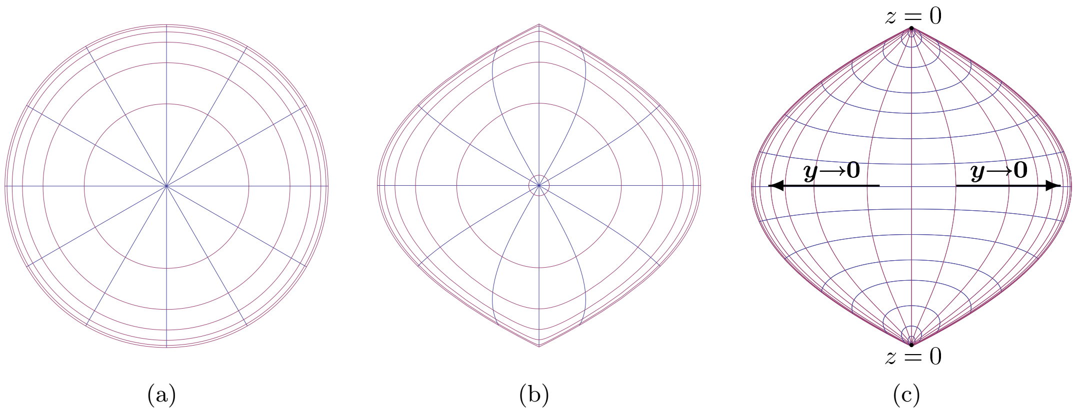


Figure 3.5: The Poincaré disk representation 3.5(a) of AdS_3 with lines of constant ρ and ζ in red and blue, respectively. Note that AdS_3 itself is only the interior of the disc. 3.5(b) shows the same geometry embedded according to (3.116), and the closure clearly displays the structure of a $\langle 2\rangle$-manifold. In 3.5(c) the slicing by AdS_2 surfaces is illustrated and horizontal/vertical curves correspond to constant z/y.

For AdS/CFT applications we need the notion of a conformal structure on the boundary, where the dual CFT shall be defined. This entails an extension of the usual notion of conformal compactness to $\langle n\rangle$-manifolds. As straightforward generalization we start with

Definition 3.3.3 A metric $\hat{g}$ on the interior of an $\langle n\rangle$-manifold $\mathcal{M}$ with $n \geq 1$ is said to be conformally compact if, for a defining function of the boundary f, the rescaled metric $\overline{g} := f^2\hat{g}$ extends as a metric to the interior of the $\langle n-1\rangle$-faces of $\mathcal{M}$. In this context we call a function f, which is differentiable on the interior of $\mathcal{M}$ and on the interior of the $\langle n-1\rangle$-faces $\partial_i\mathcal{M}$, a defining function of the boundary if $f|_{\text{int}\mathcal{M}} > 0$ while on the interior of the $\langle n-1\rangle$-faces we have $f = 0$ with $df \neq 0$.

Having in mind the geometry with AdS conformal boundary we can not generally expect the rescaled metric to extend directly to the corner points. The usual definition of conformal compactness is recovered for $\langle 1\rangle$-manifolds. On the other hand, the geometry displayed in Fig. 3.5(c) is now also conformally compact in this sense with a defining function $f = \sin(\pi y/4)$, which behaves like y as $y = 0$ is approached. Note that simply choosing $f = y$ is not differentiable. For the discussion of multi-layered holography a more restrictive class of conformally compact $\langle n\rangle$-manifolds with even more structure will be of particular interest:

Definition 3.3.4 A metric $\hat{g}$ on the interior of an $\langle n\rangle$-manifold $\mathcal{M}$ $(n \geq 2)$ is said to be conformally $\langle 2\rangle$-compact if it is conformally compact in the sense of Def. 3.3.3 and $\overline{g}$ yields conformally compact metrics on the $\langle n-1\rangle$-faces. Conformally $\langle m\rangle$-compact $\langle n\rangle$-manifolds for $n \geq m$ are defined by the straightforward generalization.

A conformally $\langle m\rangle$-compact $\langle n\rangle$-manifold $(m \leq n)$ yields conformally compact metrics on the $\langle p\rangle$-faces down to $p > n-m$ and a metric for $p = n-m$. Of course, all these metrics depend on the choice of defining function and we are in fact dealing only with the corresponding conformal structures which are well defined. The example of the AdS_3 geometry sliced by AdS_2 hypersurfaces discussed above is in fact conformally $\langle 2\rangle$-compact. The metrics induced on the $\langle 1 \rangle$-faces are AdS_2 and conformally compact. They do not contain a dy^2-part anymore and induce a conformal structure on the boundaries of the faces at $z = 0$. Note that this is different from directly inducing a conformal structure on the corner points, which is not possible. The situation is similar for generic AdS_{d+1} sliced by AdS_d, as illustrated in Fig. 3.6 for AdS_4 sliced by AdS_3 to which we will come in more detail later. It again is a conformally $\langle 2 \rangle$-compact $\langle 2 \rangle$-manifold with the upper and lower halfs of the surface as $\langle 1 \rangle$-faces and the equator as $\langle 0 \rangle$-face.

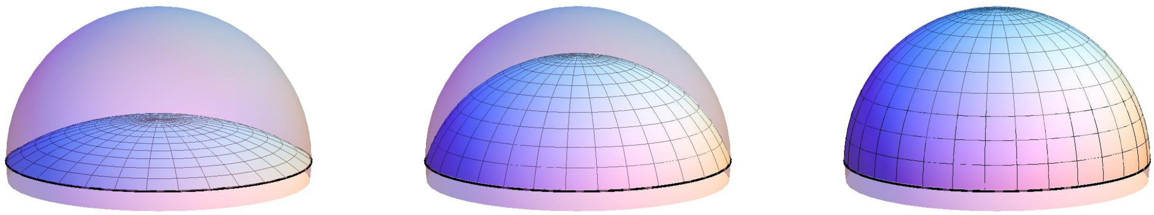


Figure 3.6: The constant-time section of AdS_4 is the three-dimensional ball B^3 and the figure illustrates its slicing by AdS_3, compare Fig. 3.5(a). The AdS_3 slices interpolate between the equatorial plane and half of the boundary in each of the two patches. Regarded as $\langle 2 \rangle$-manifold the boundary splits into the upper and lower hemispheres, which are $\langle 1 \rangle$-faces, and the equator as the $\langle 0 \rangle$-face.

In the previous sections we were interested in having a single copy of AdS_d as conformal boundary, which is achieved by taking a $\mathbb{Z}_2$-quotient of that geometry. The resulting setup has a boundary in the usual sense as well as a conformal boundary and is therefore not conformally compact in the sense of Def. 3.3.3. A class of geometries with similar properties which can also be understood better in the context of $\langle n \rangle$-manifolds are the duals to CFTs on manifolds with boundary, as discussed in the context of AdS/BCFT [64, 65]. The structure of $\langle n \rangle$-manifolds allows for the following generalization which nicely captures those geometries.

Definition 3.3.5 A metric $\hat{g}$ on the interior of an $\langle n \rangle$-manifold $\mathcal{M}$ with $n \geq 1$ is called conformally compact with boundary if there is a function f such that

(i) $f|_{\mathrm{int}\mathcal{M}} > 0$

(ii) on the interior of each $\langle n-1 \rangle$-face we either have $f > 0$, or $f = 0$ with $df \neq 0$

(iii) $\overline{g} := f^2 \hat{g}$ extends as a metric to the interior of the $\langle n-1 \rangle$-faces of $\mathcal{M}$.

The generalization to $\langle n \rangle$-manifolds which are conformally $\langle m \rangle$-compact with boundary follows the definition of conformally $\langle m \rangle$-compact $\langle n \rangle$-manifolds.

If f is positive on a face $\partial_i \mathcal{M}$, this face is part of a boundary in the usual sense, otherwise it is part of the conformal boundary and we call $\partial_i \mathcal{M}$ a conformal face of $\mathcal{M}$. The usual notions of conformally compact manifolds and of manifolds with boundary are recovered for $\langle 1 \rangle$-manifolds, and this class of $\langle n \rangle$-manifolds also includes those of Def. 3.3.3. The geometries discussed in [64, 65] are conformally compact with boundary in this sense. The $\mathbb{Z}_2$-quotient of the AdS_d slicing of AdS_{d+1} discussed in the previous section is a $\langle 2 \rangle$-manifold and is conformally $\langle 2 \rangle$-compact with boundary. The $\langle 1 \rangle$-faces are given by the closures of $\partial_0 \mathcal{M} = \{p \in \mathcal{M} \,|\, y_p = 2l\}$, which is the $\mathbb{Z}_2$-fixed surface and a boundary in the usual sense, and $\partial_1\mathcal{M} = \{p \in \mathcal{M} \,|\, y_p = 0\}$ as conformal boundary. The corner points are the $\langle 0\rangle$-face $\{p \in \mathcal{M} \,|\, y_p = z_p = 0\}$. The regularized geometry $\tilde{\mathcal{M}}$ shown in Fig. 3.2(a) is a $\langle 3\rangle$-manifold with $\langle 2\rangle$-faces $\partial_0\tilde{\mathcal{M}} = \{p \in \tilde{\mathcal{M}} \,|\, y_p = 2l\}$, $\partial_1\tilde{\mathcal{M}} = \{p \in \tilde{\mathcal{M}} \,|\, y_p = \epsilon_1\}$, $\partial_2\tilde{\mathcal{M}} = \{p \in \tilde{\mathcal{M}} \,|\, z_p = \epsilon_2\}$. The corner points are the $\langle 1\rangle$-faces $\{p \in \tilde{\mathcal{M}} \,|\, y_p = 2l, z_p = \epsilon_2\}$ and $\{p \in \tilde{\mathcal{M}} \,|\, y_p = \epsilon_1, z_p = \epsilon_2\}$. As discussed in [93] the definition of the usual objects of differential geometry, such as differential forms, on manifolds with corners is straightforward. In particular, the integration of differential forms can be defined and Stokes' theorem can be extended to manifolds with corners. This gives a more general justification for our treatment of the Klein-Gordon field in the previous sections where our discussion of the corners was adapted to the specific situation at hand. An interesting question left for future research is whether the total boundary blow-up [96] could be employed to give an improved renormalization prescription and possibly avoid the issues with Neumann boundary conditions found in Sec. 3.1.2.

3.3.2 Multi-Layered AdS/CFT to the extreme

Having found an appropriate differential-geometric structure for our geometry, which in particular allows for the notion of the boundary of a boundary, we can now discuss prospects for multi-layered AdS/CFT. More precisely, using the notation of Sec. 3.3.1, we have discussed in Sec. 3.1.2 a bulk theory on the $\langle 2\rangle$-manifold $\mathcal{M}$ which is dual to a CFT on the $\langle 1\rangle$-face $\partial_1\mathcal{M}$. As the $\langle 1\rangle$-face $\partial_1\mathcal{M}$ is AdS_d and itself has a boundary given by the $\langle 0\rangle$-face $\{p \in \mathcal{M} \,|\, y_p = z_p = 0\}$, we may ask under which circumstances a second instance of AdS/CFT can be applied. We come back to that specific question later and discuss a more extreme case first. To push the line of thought leading to multi-layered AdS/CFT to the extreme, we discuss a particular example of an $\langle n\rangle$-manifold which is conformally $\langle n\rangle$-compact with boundary. The interiors of the faces which are conformal boundaries are given by AdS spaces of appropriate dimension, such that this setting allows – at least in principle – for hierarchies of AdS/CFT dualities.

In the following we outline in detail the construction of the case $n = 3$ and comment on the generalization afterwards. We start with AdS_4 represented as $\mathbb{R} \times B^3$, where B^n denotes the open ball of radius $\pi/2$ in $\mathbb{R}^n$. We choose global coordinates (τ, ρ, Ω_2), $\Omega_2 = (\zeta, \phi)$ with line element (3.1), such that the boundary of the cylinder corresponds to $\rho \to \infty$. We then transform coordinates to $(\tau, u, \tilde{z}, \phi)$ by

$$\rho^2 = \csc^2\tilde{z}\,\Big(\frac{u}{4} + \frac{1}{u}\Big)^2 - 1\ , \qquad \rho^2 \sin^2\zeta = \cot^2\tilde{z}\Big(\frac{u}{4} + \frac{1}{u}\Big)^2\ , \tag{3.117}$$

compare (3.5) with $u = 2e^{-R}$ (we fix unit curvature radii). This yields the slicing by AdS_3 surfaces coordinatized by $(\tau, \tilde{z}, \phi)$ shown in Fig. 3.6. The boundary of B^3 is S^2 and splits into the upper and lower hemispheres which are each a copy of AdS_3, joined at the equator which is their conformal boundary. The AdS_3 surfaces smoothly interpolate between the equatorial plane and half of the surface of B^3. On the AdS_3 slices we then transform coordinates to (τ, y, z) by

$$\cot^2\tilde{z} = \csc^2 z\,\Big(\frac{y}{4} + \frac{1}{y}\Big)^2 - 1\ , \qquad \cot^2\tilde{z}\,\sin^2\phi = \cot^2 z\,\Big(\frac{y}{4} + \frac{1}{y}\Big)^2\ , \tag{3.118}$$

such that it is itself sliced by AdS_2. Fig. 3.3(a) shows a constant-time section of that slicing of the AdS_3 cylinder $\mathbb{R} \times B^2$ by AdS_2 surfaces, and the structure as $\langle 2 \rangle$-manifold is emphasized in Fig. 3.5(c). Altogether we have a number of patches with coordinates (τ, u, y, z), which coordinatize AdS_4 sliced by AdS_3 in coordinates (τ, y, z) such that it is itself sliced by AdS_2 in coordinates (τ, z). The result is illustrated in Fig. 3.7, where the surfaces of constant u are shown for various u. These slices of constant u are coordinatized by (y, z) and the lines shown on the images of B^2 in B^3 are those of Fig. 3.3(a). The resulting line element for AdS_4 reads

$$ds^2_{\mathrm{AdS}_4} = \frac{1}{u^2}\left[du^2 + \left(1 + \frac{u^2}{4}\right)^2 ds^2_{\mathrm{AdS}_3}\right] , \tag{3.119}$$

where the line element of AdS_3 sliced by AdS_2 is given by

$$ds^2_{\mathrm{AdS}_3} = \frac{1}{y^2}\left[dy^2 + \left(1 + \frac{y^2}{4}\right)^2 ds^2_{\mathrm{AdS}_2}\right] , \qquad ds^2_{\mathrm{AdS}_2} = \csc^2 z (dz^2 - d\tau^2) . \tag{3.120}$$

The boundary of B^3 now splits into the upper and lower hemispheres which are AdS_3, the

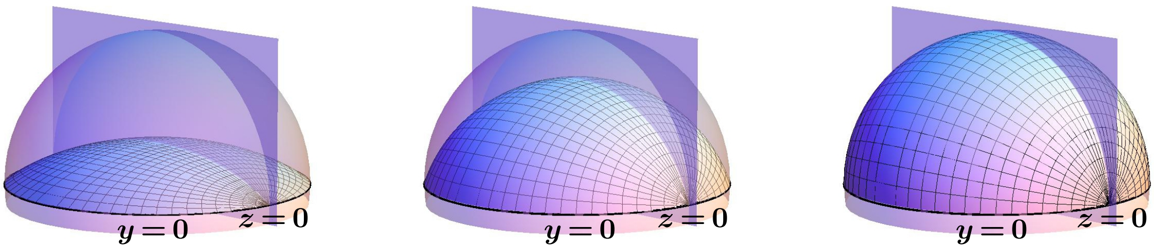


Figure 3.7: The constant-time section of AdS_4 with the embedding of the AdS_3 surfaces shown for $u \in \{1.3, 0.5, 0.01\}$ from left to right. The AdS_3 surfaces are themselves sliced by AdS_2 as shown in Fig. 3.5(c).

two parts of the equator with the two points at $z = 0$ removed, which are each a copy of AdS_2, and the two points corresponding to $z = 0$. Note that this geometry does not have the structure of a manifold with faces, as the points $z = 0$ correspond to 3 vanishing coordinates but are only part of the closure of the two connectedness components of $\{p \in \mathcal{M} \,|\, c(p) = 1\}$. However, to obtain at each level a geometry with a single copy of AdS as conformal boundary we have to take the quotient with respect to $\mathbb{Z}_2 \times \mathbb{Z}_2$ anyway. The first $\mathbb{Z}_2$ identifies the upper and lower hemispheres of B^3, such that the conformal boundary of AdS_4 comprises only a single copy of AdS_3. The second factor identifies the two patches needed to cover AdS_3 by AdS_2 slicings, compare again Fig. 3.3. It identifies the two hemispheres obtained by cutting along the plane shown in Fig. 3.7. This yields one fourth of B^3, which is one of the two parts of the upper half shown in Fig. 3.7. That resulting geometry then is indeed a $\langle 3 \rangle$-manifold with $\langle 2 \rangle$-faces given by a quarter of S^2 and half of each of the two $\mathbb{Z}_2$-fixed surfaces, see Fig. 3.8. The $\langle 1 \rangle$-faces are half of the equator and the intersection of the $\mathbb{Z}_2$-fixed surfaces. Finally, the $\langle 0 \rangle$-faces are the two points corresponding to $z = 0$. On the interior of the conformal boundary of this $\mathrm{AdS}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ geometry, given by the conformal $\langle 2 \rangle$-face which is part of S^2, the metric of AdS_3 as given in (3.120) is induced. That induced metric on the

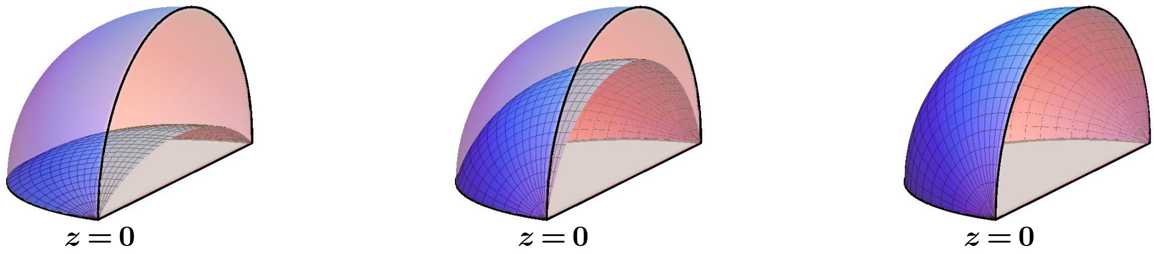


Figure 3.8: Embedding of the $\mathrm{AdS}_3/\mathbb{Z}_2$ surfaces into $\mathrm{AdS}_4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ for $u \in \{1.3, 0.5, 0.01\}$ from left to right. The $\mathbb{Z}_2$ parities identify the upper with the lower half of B^3 and the two halfs obtained by cutting along the hyperplane shown in Fig. 3.7.

conformal $\langle 2 \rangle$-face induces on its conformal $\langle 1 \rangle$-face at the equator of B^3, where $y = 0$, the metric of AdS_2. With the chosen coordinates this metric again induces on the $\langle 0 \rangle$-faces a metric, such that the manifold is conformally $\langle 3 \rangle$-compact with boundary in the sense of Def. 3.3.5. It thus allows – at least in principle – for three instances of AdS/CFT dualities, each relating the theory on AdS_d to a theory on the conformal boundary for $d = 2, 3, 4$.

The construction outlined above can be carried out starting with AdS_{d+1} for arbitrary $d > 2$ and nesting the slicings by codimension-1 AdS hypersurfaces down to AdS_2. The result will then be $\mathbb{R} \times B^d/(\mathbb{Z}_2)^{d-1}$ as $\langle d \rangle$-manifold which is conformally $\langle d \rangle$-compact with boundary for AdS_{d+1}. One may therefore ask, whether – by a suitable choice of boundary conditions – one can obtain hierarchies of theories on the $\langle n \rangle$-faces corresponding to conformal boundaries, which are related by hierarchies of AdS/CFT dualities. Ultimately, one could then relate a gravitational theory on AdS_{d+1} to a theory on the boundary of AdS_2 via nested AdS/CFT dualities. In the usual setting of AdS/CFT this requires gravitational theories on AdS, since sensible boundary theories supposedly have an energy-momentum tensor as observable, to which AdS/CFT associates the bulk metric as dual field. Thus, one would need nested Neumann or mixed boundary conditions on the various instances of AdS spaces to allow for a dynamical boundary metric. However, extrapolating the results for a scalar field obtained in Sec. 3.1 to the graviton[13], this trivializes the theories in the sense that the normalizable modes are so sparse that the theories effectively reduce to boundary theories in a trivial way. This clearly is an obstruction to realizing non-trivial relations via such multiple iterations of AdS/CFT. A possible cure for the normalizability issues may be the total boundary blow-up [96], and employing a singleton formulation for gravity may be another possible route to follow. Realizing gravity on the boundary is an interesting task in the first place, which will be discussed in more detail in Sec. 4.

3.3.3 Outlook on a concrete realization

In this part we come back to the issue of double-layered dualities which is not affected by the obstructions found in the previous section, and discuss prospects for a specific realization

[13] The obstructions found there to renormalizing the bulk theory such that the full set of Neumann modes is rendered normalizable did not rely on the field under consideration being scalar.

within string theory. To keep the technical level assessable it would be desirable to have an example where the involved theories are defined on spacetimes of rather low dimension. A distinguished case is where the CFT on the lower-dimensional end of the dualities is a two-dimensional one. Interestingly enough, multi-layered holography has been speculated to play a role in such a setting, and our detailed discussion of the four-dimensional geometry shown in Fig. 3.6 in the previous section turns out to be quite useful in that respect. The concrete setting involves ABJM theory [97], an $\mathcal{N}$=6 supersymmetric Chern-Simons theory with gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ which is understood as the worldvolume theory of stacks of M2-branes. In appropriate regimes it admits a dual description in terms of M-theory on $\mathrm{AdS}_4\times \mathrm{S}^7/\mathbb{Z}_k$ (for $N\gg k^5$) or type IIA string theory on $\mathrm{AdS}_4\times\mathbb{CP}^3$ (for larger Chern-Simons level k). The theory has a 't Hooft limit $N,k\to\infty$ with fixed 't Hooft coupling $\lambda=N/k\gg 1$, and in that limit the geometry in the dual description becomes weakly curved such that the supergravity approximation is valid. The dual gravity theories are understood as usual with Dirichlet boundary conditions and the spectra have been obtained in [98]. Omitting the compact factor of the M-theory background we thus have an AdS/CFT duality of a four-dimensional gravity theory and a three-dimensional CFT

$$\text{M-theory on AdS}_4 \quad\longleftrightarrow\quad \text{ABJM} = \text{CFT}_3\ .$$

The ABJM theory may now be coupled to conformal supergravity, see [99] where the result was called topologically gauged ABJM theory. This theory in turn admits a Higgsing which corresponds to a theory on D2-branes and has an AdS_3 vacuum solution [99, 100]. As a gravitational theory on AdS_3 it may therefore be dual to a two-dimensional CFT on the boundary, yielding a second duality

$$\text{Higgsed top. gauged ABJM on AdS}_3 \quad\longleftrightarrow\quad \text{CFT}_2\ .$$

Thus, there is a chance to realize nested AdS/CFT by combining these dualities, relating a version of M-theory on AdS_4 to the Higgsed topologically gauged ABJM, which itself is a gravitational theory on AdS_3 with a CFT_2 dual, as speculated in [101]:

$$\text{M-theory} \quad\longleftrightarrow\quad \text{Higgsed top. gauged ABJM on AdS}_3 \quad\longleftrightarrow\quad \text{CFT}_2\ .$$

This, however, requires a precise understanding of the coupling of ABJM theory to gravity and the subsequent Higgsing in the holographic picture in M-theory/IIA supergravity.

The two steps which have to be understood from the M-theory perspective are the topological gauging of ABJM in the first place, i.e. its coupling to conformal supergravity, and then the subsequent Higgsing, resulting in a theory with AdS_3 solution. The coupling to gravity is naturally implemented by twisting the boundary conditions [52]. Restricting to the pure metric part, the variation of the renormalized bulk action reads

$$\delta S = \text{EOM} + \int_{\partial\mathcal{M}} \delta g^{(0)}_{\mu\nu} T^{\mu\nu}, \tag{3.121}$$

with some finite $T_{\mu\nu}$ which yields the expectation value of the CFT energy-momentum tensor and $g^{(0)}_{\mu\nu}$ the leading coefficient of the on-shell asymptotic expansion of the metric

$$ds^2 = \frac{1}{r^2}\big(dr^2 + g_{\mu\nu}(x,r)dx^\mu dx^\nu\big)\ , \qquad g_{\mu\nu}(x,r) = g^{(0)}_{\mu\nu}(x) + \mathcal{O}(r^2)\ . \tag{3.122}$$

Switching from the usual Dirichlet boundary condition fixing $g^{(0)}_{\mu\nu}$ to Neumann amounts to fixing $T_{\mu\nu} = 0$. Explicitly adding to the bulk theory dynamics on the boundary described by an action S_{bndy} then further modifies the boundary conditions resulting in

$$T_{\mu\nu} = \frac{\delta S_{\text{bndy}}}{\delta g^{\mu\nu}} \ , \tag{3.123}$$

coupling the dual CFT to the explicit dynamics added on the boundary. The coupling of ABJM theory to conformal supergravity can therefore be understood holographically by coupling the M-theory/IIA reductions on global AdS_4 to conformal supergravity on the three-dimensional boundary. One can now speculate that the Higgsing procedure, resulting in a theory with an AdS_3 vacuum, should be reproduced by the change of coordinates to the AdS_3 slicing of AdS_4 with the subsequent $\mathbb{Z}_2$-orbifolding. While the change of foliation does not affect the amount of preserved symmetry, the orbifolding breaks the radial isometries of AdS as they do not preserve the $\mathbb{Z}_2$-fixed surface, corresponding to $R = 0$ in Fig. 3.1(a). These should be precisely the bulk transformations acting as conformal symmetries on the boundary. The failure of the linearized boundary theory to be conformally invariant may then be understood as the breaking of radial isometries by the orbifolding in the bulk theory, yielding the desired reflection of the Higgsing on the boundary in the dual bulk description. Likewise, only a subset of the AdS Killing spinors are invariant under the orbifolding and it therefore also affects the amount of preserved supersymmetry. We expect that those of the fermionic symmetries which correspond to the special conformal transformations are broken.

Having discussed how the topological gauging and Higgsing procedures can be understood from the M-theory perspective it still remains to sort out what a possible CFT_2 dual of the Higgsed topologically gauged ABJM theory could be. From this perspective the discussion of the singleton theory in the previous chapter turns out to be of interest. Noting that Chern-Simons theory can be formulated as a singleton theory [86], and an explicit CFT_2 description has been discussed in [102], see also [103], it seems natural to expect a similar possibility for the supersymmetrization. The same may even be possible for gravity [104] and one can then attempt to formulate the whole topologically gauged ABJM theory as a singleton theory to get a handle on a possible equivalent boundary theory. Whether a singleton formulation provides the possibility for further nestings of dualities, since – as we found for the scalar in Sec. 3.2.2 – it allows to formulate a unitary theory with Neumann-like boundary conditions on AdS_3 without the normalizability issues discussed in Sec. 3.1, may be an interesting question way beyond the current discussion.

3.4 Discussion

In the first part of this section we have studied unitarity violations in CFTs defined on the maximally symmetric dS and AdS spacetimes from a holographic perspective. For this purpose we have considered a scalar field on AdS_{d+1} conformally compactified such that the conformal boundary is (A)dS_d. The mass and boundary conditions on the AdS_{d+1} conformal boundary were chosen such that the bulk theories provide a dual description of a

CFT that contains an operator violating known unitarity bounds, i.e. $m^2 \geq m^2_{\rm BF} + 1$ and Neumann$_{(d+1)}$.

Starting with the case of AdS$_d$ on the boundary, we have adapted the well-known procedure of holographic renormalization to this setting and found that the qualitative features of the bulk theory strongly depend on the choice of boundary conditions on the AdS$_d$ boundary. While the Dirichlet$_d$ boundary condition yields a full set of normalizable modes, choosing Neumann$_d$ drastically reduces the spectrum. For Dirichlet$_d$ we have found that for even and odd $\mathbb{Z}_2$ parity the spectrum of the bulk theory contains ghosts for $m^2_{\rm BF} + 1 < m^2 < m^2_{\rm BF} + 2$ combined with Neumann$_{d+1}$. Thus, we have found that the non-unitarity of the dual CFT is well reflected in the bulk theory for the Dirichlet$_d$ cases. As argued in the main text, it is also possible to extrapolate our results to higher values of the bulk mass even in the absence of an explicit expression for the renormalized action and inner products. This has shown that also for higher values of ν the boundary non-unitarity is recovered in the bulk theory. For Neumann$_d$ on the other hand, we have also found ghosts in the spectrum for Neumann$_{d+1}$ and $m^2_{\rm BF} + 1 < m^2 < m^2_{\rm BF} + 2$, but extrapolating our results to higher values of the bulk mass we have found that in certain cases Neumann$_d$ boundary conditions yield – contrary to expectations based on the unitarity bound – a ghost-free spectrum. This can be traced back to the special structure of the boundary theory, which lacks conformal invariance. Summing up, we find that the boundary unitarity bound is well reflected in the bulk theories in the cases where it is expected to hold.

It is interesting to compare these results in more detail to the expectations based on the field theory reasoning. On the one hand, the presence of the boundary breaks the symmetry group from $\mathrm{SO}(2,d)$ to $\mathrm{SO}(2,d-1)$, such that one might expect the relevant unitarity bound to be that of $d-1$ dimensions. On the other hand, for observables localized away from the boundary the relevant unitarity bound should still be the d-dimensional one. Thus, for degrees of freedom that are not confined to the boundary we still expect the d-dimensional unitarity bound to be relevant. Our results, which state that the relevant bound is the d-dimensional one, are in good agreement with this picture, as we have not included degrees of freedom that solely reside on the boundary of AdS$_d$, which could however be done along the lines of [52, 64].

For the case of dS$_d$ on the boundary the involved geometry is an open patch of global AdS$_{d+1}$, bounded by a causal horizon. Although the setup is similar to Poincaré AdS in that respect, we found – in contrast to Poincaré AdS – a straightforward reflection of the boundary non-unitarity since the spectrum of the bulk theory contains ghosts. The difference in the two settings is that in our setup the dS$_d$ slices have compact spatial sections, which is different from Poincaré AdS where the d-dimensional slices are Minkowski. This suggests that the tricky manifestation of the boundary non-unitarity in the bulk found for Poincaré AdS is related to the non-compactness of the boundary, rather than to the appearance of a horizon in the bulk. To further investigate this point one could study the case with dS on the boundary using an open slicing instead of global dS$_d$ coordinates.

We have also included the cases with Dirichlet boundary conditions on the conformal boundary of AdS_{d+1} for generic[14] ν, and the fact that we found ghost-free spectra in that case shows that the condition $\Delta \geq d/2 - 1$, derived in [35] as necessary condition for unitarity, is indeed also sufficient for CFTs which have a holographic description in terms of the setups we have considered. Likewise, we have included the cases with Neumann boundary conditions on the conformal boundary of AdS_{d+1} and $m^2 = m^2_{\mathrm{BF}} + 1$. This corresponds to a CFT with an operator saturating the unitarity bound and we have shown that for the standard Klein-Gordon field ghosts are present in the bulk.

We have then discussed the specific case where the unitarity bound in the dual CFT is saturated in more detail in Sec. 3.2. As found in Sec. 3.1.2 and for global AdS in [69], the standard Klein-Gordon field yields ghosts for mass and boundary condition such that the dual operator saturates the unitarity bound, although a unitary representation of the conformal group exists. We have derived the singleton field theory as a particular limit of the Klein-Gordon field with standard renormalized inner product, which allows for a direct AdS/CFT interpretation. It provides the dual description of a free field on the boundary, as expected for an operator saturating the unitarity bound. This extends the thorough discussion of unitarity from the holographic perspective for global AdS in [69] to the case where the unitarity bound is saturated and resolves the tension between bulk and boundary unitarity for that case. We have also formulated the singleton field theory on the geometry with AdS_d on the conformal boundary of AdS_{d+1}, extending the discussion of unitarity in Sec. 3.1.2 accordingly. Remarkably, the singleton field on the AdS_d slicing of AdS_{d+1} does not suffer from the normalizability issues found for $\mathrm{Neumann}_d$ boundary conditions.

In Sec. 3.3 we have introduced an appropriate geometric framework for the setup with AdS on the boundary and extended the usual definitions of conformal compactness to that setting of $\langle n \rangle$-manifolds. This in particular allows to make sense of the notion of the boundary of a boundary, as needed to implement nested AdS/CFT. The more general perspective opened the discussion of an extreme geometry where subsequent slicings of AdS spacetimes by codimension-1 hypersurfaces which are again AdS are used. This at least in principle offers the possibility to impose hierarchies of Neumann boundary conditions, thus leading to hierarchies of gravitational theories on AdS spaces possibly connected by nested AdS/CFT-type dualities. From the result of the previous investigations, that choosing $\mathrm{Neumann}_d$ boundary conditions reduces the bulk theory to a boundary theory in a trivial way by leaving only a very sparse set of normalizable solutions, we could draw first conclusions on the prospects for such a construction. Namely, at least in the renormalization framework we have set up before, nesting more than two instances of AdS/CFT dualities does not yield non-trivial relations. However, the discussion of $\langle n \rangle$-manifolds also revealed a possible route to curing that problem using the total boundary blow-up [96]. Also the singleton theory offers interesting prospects as it was found to avoid the normalizability issues for nested Neumann conditions. In the last part of Sec. 3.3 we have discussed prospects for a concrete realization of a double-layered AdS/CFT duality within M-theory. In the language developed in Sec. 3.3.1, this involves topologically gauged ABJM theory on the conformal $\langle 1 \rangle$-face of a four-dimensional $\langle 2 \rangle$-manifold which is conformally $\langle 2 \rangle$-compact with boundary. The $\langle 1 \rangle$-face is AdS_3 with a $\langle 0 \rangle$-face as boundary, where the expected dual two-dimensional CFT

[14] The results for $\mathrm{Dirichlet}_{d+1}$ are insensitive to the ν-dependent explicit form of the counterterms due to the fast fall-off of the field.

is defined. On the other hand, the specific version of M-theory in the four-dimensional bulk of the $\langle 2 \rangle$-manifold, as discussed in Sec. 3.3.3, is expected to yield another dual description of topologically gauged ABJM theory. With a more detailed understanding of the deformations of the usual M-theory description of ABJM theory to arrive at the anticipated dual for the Higgsed topologically gauged version, one could then establish a double-layered duality. The discussions so far involved gravitational theories on the boundary already at several places and we consider this issue in more detail in the next section.

4 The Boundary unleashed – CFT Weyl Anomaly and Gravity on the Boundary

We now want to extend the discussion of boundary CFTs defined on fixed curved spacetimes towards dynamical gravity on the boundary, which is likewise expected to possess conformal invariance. Perturbative treatments of conformal gravity are plagued by ghosts and unitarity may be restored only by genuine strong-coupling effects [105], an issue on which a holographic description – e.g. in terms of a ghost-free dual string theory – may provide useful insights. An argument for the existence of a dual description of gravitational theories on the boundary is that, beyond the strict decoupling limit discussed in Sec. 2.4, the worldvolume theory of a stack of D-branes couples to the gravitational closed-string sector. Thus, if AdS/CFT holds beyond the strict limits of large N and λ, an appropriate limit of string theory should provide a dual description of e.g. $\mathcal{N}$=4 SYM theory coupled to conformal supergravity, as argued already in [106].

Coupling the boundary CFT to gravity can be achieved holographically by promoting the otherwise fixed residue of the bulk metric on the boundary to a dynamical quantity. This entails a generalization of the usual Dirichlet boundary conditions to Neumann or mixed boundary conditions, which was discussed for metric perturbations around rigid AdS in [52]. As first step towards such a duality involving concrete string theories, we extend these results to complete five-dimensional supergravities arising as low-energy limits of string theory. More concretely, the near-horizon geometry of the p-brane solutions relevant for AdS/CFT is typically given by a product of AdS space and a compact manifold, on which one can perform a Kaluza-Klein expansion of the ten-dimensional supergravities arising from string theory. Lower-dimensional gauged supergravities on the AdS spaces then describe the Kaluza-Klein expanded ten-dimensional theory truncated to a finite number of Kaluza-Klein modes, and consequently also the corresponding sector of the dual superconformal field theory (SCFT) [107].

We consider five-dimensional $\mathcal{N}$=4 and $\mathcal{N}$=2 gauged supergravities whose solutions can be lifted to specific brane configurations in string and M-theory in Sec. 4.1 and 4.2, respectively. We first study the dual CFTs in generic backgrounds, for which we have to generalize the setting of a fixed AdS background and instead only restrict the configuration space to generic asymptotically-AdS geometries. We determine the asymptotic structure of the bulk theories and carry out the holographic renormalization. In the usual AdS/CFT context the boundary values of the bulk fields are fixed and provide the background in which the dual CFT is defined. With the results on the asymptotic structure of the supergravities we calculate the Weyl anomaly of the dual CFTs in generic backgrounds and compare to the expectation from the field-theory side, providing a non-trivial test of the AdS/CFT conjecture. Building on these results we then establish the availability of Neumann and mixed boundary conditions for the full supergravities in Sec. 4.3. This promotes the conformal supergravity multiplets arising as boundary values of the bulk fields to dynamical quantities and we discuss the features of the resulting boundary theories.

4.1 Boundary Multiplet of $\mathcal{N}$=4 SU(2)⊗U(1) Gauged Supergravity on Asymptotically-AdS$_5$ and the Weyl Anomaly

In this section we consider five-dimensional half-maximally supersymmetric gauged supergravity. We determine the asymptotic structure and calculate the Weyl anomaly of the dual $\mathcal{N}$=2 SCFTs. The general gauged matter-coupled $\mathcal{N}$=4 supergravities in five dimensions were constructed in [108, 109], and it was noted in [108] that AdS ground states are only possible if the gauge group is a product of a one-dimensional Abelian factor and a semi-simple group. We focus on the $\mathcal{N}$=4 SU(2)⊗U(1) gauged supergravity constructed by Romans [110], the only gauging of the pure supergravity without additional matter multiplets which admits an AdS vacuum. Solutions of this theory can be lifted to solutions of the IIB supergravity [111] where they correspond to product geometries involving S^5, and also to warped-product solutions of IIA supergravity and the maximal d=11 supergravity [112, 113]. We restrict the configuration space to generic asymptotically-AdS$_5$ geometries with an arbitrary four-dimensional boundary metric. By a near-boundary analysis we determine the boundary-dominant components of the bulk fields from their partially gauge-fixed field equations. Subdominant components are projected out in the boundary limit and we find a reduced set of boundary fields constituting an $\mathcal{N}$=2 Weyl multiplet. The residual bulk symmetries act on the boundary fields as four-dimensional diffeomorphisms, $\mathcal{N}$=2 supersymmetry and (super-)Weyl transformations. Thus, the on-shell $\mathcal{N}$=4 supergravity multiplet yields the $\mathcal{N}$=2 Weyl multiplet on the boundary with the appropriate local $\mathcal{N}$=2 superconformal transformations. This limiting procedure does not rely on the choice of boundary conditions, and similar calculations have previously been carried out for bulk theories in d=3, 6, 7 dimensions and for $\mathcal{N}$=2 supergravity in d=5 [114]. For the bosonic sector of the bulk supergravity we then carry out the holographic renormalization [71, 72] and calculate the Weyl anomaly of the dual four-dimensional SCFTs in a generic bosonic $\mathcal{N}$=2 conformal supergravity background. This extends the existing results for nontrivial metric and dilaton backgrounds [71, 115, 116][1].

We review the $\mathcal{N}$=4 SU(2)⊗U(1) gauged supergravity in Sec. 4.1.1 and construct the multiplet of fields along with the symmetry transformations induced on the conformal boundary of asymptotically-AdS spaces in Sec. 4.1.2. In Sec. 4.1.3 we carry out the holographic renormalization and calculate the Weyl anomaly of the dual SCFTs in an external bosonic $\mathcal{N}$=2 conformal supergravity background. This work was published in collaboration with Thorsten Ohl in [15].

[1] For the maximally supersymmetric case a discussion of the SCFT effective action, the conformal anomaly and the role of conformal supergravity in AdS/CFT can be found in [106]. Explicit constructions for the boundary of AdS are given there for the metric-dilaton sector.

4.1.1 Romans' $\mathcal{N}$=4 SU(2)⊗U(1) gauged supergravity

We briefly discuss the five-dimensional gauged supergravity [110] in order to fix notation. The theory has $\mathcal{N}$=4 supersymmetry (counted in terms of symplectic Majorana spinors) with R-symmetry group USp(4), of which an SU(2)⊗U(1) subgroup is gauged. The symplectic metric is denoted by Ω, and exploiting the isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$ the Lie algebra generators are given by $\Gamma_{mn} := \frac{1}{2}[\Gamma_m, \Gamma_n]$ with $\mathfrak{so}(5)$ vector indices m, n, and Γ_m satisfying the five-dimensional Euclidean Clifford algebra relation[2] $\{\Gamma_m, \Gamma_n\} = 2\delta_{mn}\mathbb{1}$. With the obvious embedding of $\mathfrak{su}(2)\oplus\mathfrak{u}(1) \cong \mathfrak{so}(3)\oplus\mathfrak{so}(2)$ into $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$, the vector index m decomposes into $m = (I, \alpha)$ with $I = 1, 2, 3$ and $\alpha = 4, 5$. We consider the theory referred to as $\mathcal{N}$=4$^+$ in [110], for which the SU(2) gauge coupling g_2 is fixed in terms of the U(1) coupling g_1 by $g_2 = +\sqrt{2}g_1 =: g$. For this choice of couplings the theory admits an AdS solution. The bosonic field content is given by the vielbein e_μ^a, two antisymmetric tensor fields $B_{\mu\nu}^\alpha$, the SU(2) and U(1) gauge fields A_μ^I and a_μ, respectively, and a scalar φ. The four gravitinos ψ_μ^i and four spin-$\frac{1}{2}$ fermions χ^i comprising the fermionic field content are in the spinor $\mathbf{4}$ of $\mathfrak{usp}(4)$, which decomposes as $\mathbf{4} \to \mathbf{2}_{1/2} + \mathbf{2}_{-1/2}$. The vector and tensor fields originate from the vector representation, decomposing as $\mathbf{5} \to \mathbf{3}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$. With the charge conjugation matrix C satisfying $C\gamma_\mu C^{-1} = \gamma_\mu^T$, $C^T = C^{-1} = -C$ and $C^\star = C$ the supercharges and hence all the spinors satisfy the symplectic Majorana condition $\bar{\chi}^i = \left(\chi^i\right)^T C$ with the conjugate $\bar{\chi}^i := (\chi_i)^\dagger \gamma_0$. The metric is of signature $(+,-,-,-,-)$ and the γ-matrices are chosen such that $\gamma_{abcde} = \epsilon_{abcde}$ with $\epsilon_{01234} = 1$. From this point on we denote five-dimensional objects with hat and four-dimensional ones without, e.g. five-dimensional spacetime indices $\hat{\mu} = (\mu, r)$ with $\mu = 0, 1, 2, 3$. The Lagrangian as given up to four-fermion terms in [110] is

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}\hat{e}\hat{\mathcal{R}}(\hat{\omega}) - \frac{1}{2}i\hat{e}\hat{\bar{\psi}}_{\hat{\mu}}^i\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{D}_{\hat{\nu}}\hat{\psi}_{\hat{\rho}i} + \frac{3}{2}i\hat{e}T_{ij}\hat{\bar{\psi}}_{\hat{\mu}}^i\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\psi}_{\hat{\nu}}^j - i\hat{e}A_{ij}\hat{\bar{\psi}}_{\hat{\mu}}^i\hat{\gamma}^{\hat{\mu}}\hat{\chi}^j + \frac{1}{2}i\hat{e}\hat{\bar{\chi}}^i\hat{\gamma}^{\hat{\mu}}\hat{D}_{\hat{\mu}}\hat{\chi}_i \\
&+ i\hat{e}\Big(\frac{1}{2}T_{ij} - \frac{1}{\sqrt{3}}A_{ij}\Big)\hat{\bar{\chi}}^i\hat{\chi}^j + \frac{1}{2}\hat{e}\hat{D}^{\hat{\mu}}\hat{\varphi}\hat{D}_{\hat{\mu}}\hat{\varphi} + \hat{e}P(\hat{\varphi}) - \frac{1}{4}\hat{e}\,\xi^2\hat{B}^{\hat{\mu}\hat{\nu}\alpha}\hat{B}_{\hat{\mu}\hat{\nu}}^{\ \alpha} \\
&+ \frac{1}{4g_1}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\epsilon_{\alpha\beta}\hat{B}_{\hat{\mu}\hat{\nu}}^{\ \alpha}\hat{D}_{\hat{\rho}}\hat{B}_{\hat{\sigma}\hat{\tau}}^{\ \beta} - \frac{1}{4}\hat{e}\,\xi^{-4}\hat{f}^{\hat{\mu}\hat{\nu}}\hat{f}_{\hat{\mu}\hat{\nu}} - \frac{1}{4}\hat{e}\,\xi^2\hat{F}^{\hat{\mu}\hat{\nu}I}\hat{F}_{\hat{\mu}\hat{\nu}}^{\ I} - \frac{1}{4}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\hat{F}_{\hat{\mu}\hat{\nu}}^{\ I}\hat{F}_{\hat{\rho}\hat{\sigma}}^{\ I}\hat{a}_{\hat{\tau}} \\
&+ \frac{1}{4\sqrt{2}}i\hat{e}\Big(H_{\hat{\mu}\hat{\nu}}^{ij} + \frac{1}{\sqrt{2}}h_{\hat{\mu}\hat{\nu}}^{ij}\Big)\hat{\bar{\psi}}_i^{\hat{\rho}}\hat{\gamma}_{[\hat{\rho}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\gamma}_{\hat{\sigma}]}\hat{\psi}_j^{\hat{\sigma}} + \frac{1}{2\sqrt{6}}i\hat{e}\Big(H_{\hat{\mu}\hat{\nu}}^{ij} - \sqrt{2}h_{\hat{\mu}\hat{\nu}}^{ij}\Big)\hat{\bar{\psi}}_i^{\hat{\rho}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\gamma}_{\hat{\rho}}\hat{\chi}_j \\
&- \frac{1}{12\sqrt{2}}i\hat{e}\Big(H_{\hat{\mu}\hat{\nu}}^{ij} - \frac{5}{\sqrt{2}}h_{\hat{\mu}\hat{\nu}}^{ij}\Big)\hat{\bar{\chi}}_i\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\chi}_j + \frac{1}{\sqrt{2}}i\hat{e}\,(\partial_{\hat{\nu}}\hat{\varphi})\,\hat{\bar{\psi}}_{\hat{\mu}}^i\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\mu}}\hat{\chi}_i\ ,
\end{aligned} \tag{4.1}$$

with $\xi := \exp\sqrt{\frac{2}{3}}\hat{\varphi}$ and the scalar potential $P(\hat{\varphi}) := \frac{1}{8}g^2\left(\xi^{-2} + 2\xi\right)$. Antisymmetrization of indices is defined as $X_{[\mu}Y_{\nu]} := \frac{1}{2}(X_\mu Y_\nu - X_\nu Y_\mu)$. Furthermore,

$$\begin{aligned}
T^{ij} &:= \frac{g}{12\sqrt{2}}\left(2\xi^{-1} + \xi^2\right)(\Gamma_{45})^{ij}\,, \qquad & A^{ij} &:= \frac{g}{2\sqrt{6}}\left(\xi^{-1} - \xi^2\right)(\Gamma_{45})^{ij}\ , \\
H_{\hat{\mu}\hat{\nu}}^{ij} &:= \xi\left(\hat{F}_{\hat{\mu}\hat{\nu}}^I(\Gamma_I)^{ij} + \hat{B}_{\hat{\mu}\hat{\nu}}^\alpha(\Gamma_\alpha)^{ij}\right)\ , \qquad & h_{\hat{\mu}\hat{\nu}}^{ij} &:= \xi^{-2}\Omega^{ij}\hat{f}_{\hat{\mu}\hat{\nu}}\ .
\end{aligned} \tag{4.2}$$

[2] The Γ_m can all be chosen hermitian, such that $\Gamma_{mn}^\dagger + \Gamma_{mn} = 0$. With the charge conjugation matrix C_E satisfying $C_E\Gamma_m C_E^{-1} = \Gamma_m^T$, we can identify $\Omega := C_E$ and have $\Omega\Gamma_{mn} + \Gamma_{mn}^T\Omega = 0$, providing the isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$.

The covariant derivative on the spinor **4** of $\mathfrak{usp}(4)$ is given by

$$\hat{D}_{\hat{\mu}} v_i = \hat{\nabla}_{\hat{\mu}} v_i + \frac{1}{2} g_1 \hat{a}_{\hat{\mu}} \left(\Gamma_{45}\right)_i{}^j v_j + \frac{1}{2} g_2 \hat{A}^I_{\hat{\mu}} \left(\Gamma_{I45}\right)_i{}^j v_j \ , \tag{4.3}$$

with the spacetime-covariant derivative $\hat{\nabla}_{\hat{\mu}}$ and $\Gamma_{IJ} = -\epsilon^{IJK}\Gamma_{K45}$. Acting on a spinor $\hat{\nabla}_{\hat{\mu}} = \partial_{\hat{\mu}} + \frac{1}{4}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\gamma}_{\hat{a}\hat{b}}$, and the curvatures are defined by

$$\left[\hat{D}_{\hat{\mu}}, \hat{D}_{\hat{\nu}}\right]\hat{\epsilon}_i =: \frac{1}{4}\hat{\mathcal{R}}^{\hat{a}\hat{b}}_{\hat{\mu}\hat{\nu}}(\hat{\omega})\,\hat{\gamma}_{\hat{a}\hat{b}}\,\hat{\epsilon}_i + \frac{1}{2} g_1 \hat{f}_{\hat{\mu}\hat{\nu}} \left(\Gamma_{45}\right)_i{}^j \hat{\epsilon}_j + \frac{1}{2} g_2 \hat{F}^I_{\hat{\mu}\hat{\nu}} \left(\Gamma_{I45}\right)_i{}^j \hat{\epsilon}_j \ . \tag{4.4}$$

On the vector **5** of $\mathfrak{usp}(4)$ the covariant derivative is given by

$$\hat{D}_{\hat{\mu}} v^{I\alpha} = \hat{\nabla}_{\hat{\mu}} v^{I\alpha} + g_1 \hat{a}_{\hat{\mu}} \epsilon^{\alpha\beta} v^{I\beta} + g_2 \epsilon^{IJK} \hat{A}^J_{\hat{\mu}} v^{K\alpha} \ . \tag{4.5}$$

The supersymmetry transformations to leading order in the fermionic terms are

$$\begin{aligned}
\delta_{\hat{\epsilon}} \hat{e}^{\hat{a}}_{\hat{\mu}} &= i\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{a}}\hat{\varepsilon}_i \ , \qquad \delta_{\hat{\epsilon}}\hat{A}^I_{\hat{\mu}} = \Theta^{ij}_{\hat{\mu}}\left(\Gamma^I\right)_{ij} , \qquad \delta_{\hat{\epsilon}}\hat{\varphi} = \frac{1}{\sqrt{2}} i\hat{\bar{\chi}}^i\hat{\varepsilon}_i \ , \\
\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}i} &= \hat{D}_{\hat{\mu}}\hat{\varepsilon}_i + \hat{\gamma}_{\hat{\mu}} T_{ij}\hat{\varepsilon}^j - \frac{1}{6\sqrt{2}}\left(\hat{\gamma}_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}} - 4\delta_{\hat{\mu}}{}^{\hat{\nu}}\hat{\gamma}^{\hat{\rho}}\right)\left(H_{\hat{\nu}\hat{\rho}ij} + \frac{1}{\sqrt{2}} h_{\hat{\nu}\hat{\rho}ij}\right)\hat{\varepsilon}^j \ , \\
\delta_{\hat{\epsilon}}\hat{a}_{\hat{\mu}} &= \frac{1}{2} i\xi^2\left(\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\varepsilon}_i + \frac{2}{\sqrt{3}}\hat{\bar{\chi}}^i\hat{\gamma}_{\hat{\mu}}\hat{\varepsilon}_i\right) \ , \\
\delta_{\hat{\epsilon}}\hat{\chi}_i &= \frac{1}{\sqrt{2}}\hat{\gamma}^{\hat{\mu}}\left(\partial_{\hat{\mu}}\hat{\varphi}\right)\hat{\varepsilon}_i + A_{ij}\hat{\varepsilon}^j - \frac{1}{2\sqrt{6}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\left(H_{\hat{\mu}\hat{\nu}ij} - \sqrt{2} h_{\hat{\mu}\hat{\nu}ij}\right)\hat{\varepsilon}^j \ , \\
\delta_{\hat{\epsilon}}\hat{B}^\alpha_{\hat{\mu}\hat{\nu}} &= 2\hat{D}_{[\hat{\mu}}\Theta^{ij}_{\hat{\nu}]}\left(\Gamma^\alpha\right)_{ij} - \frac{i g_1}{\sqrt{2}}\epsilon^{\alpha\beta}\left(\Gamma_\beta\right)_{ij}\xi\left(\hat{\bar{\psi}}^i_{[\hat{\mu}}\hat{\gamma}_{\hat{\nu}]}\hat{\epsilon}^j + \frac{1}{2\sqrt{3}}\hat{\bar{\chi}}^i\hat{\gamma}_{\hat{\mu}\hat{\nu}}\hat{\epsilon}^j\right) \ ,
\end{aligned} \tag{4.6}$$

where $\Theta^{ij}_{\hat{\mu}} = \sqrt{\frac{1}{2}} i\xi^{-1}\left(-\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\epsilon}^j + \sqrt{\frac{1}{3}}\hat{\bar{\chi}}^i\hat{\gamma}_{\hat{\mu}}\hat{\epsilon}^j\right)$. The commutator of two supersymmetries is – to leading order in the fermionic fields – given by

$$\left[\delta_{\hat{\epsilon}_2}, \delta_{\hat{\epsilon}_1}\right] = \delta_{\hat{X}} + \delta_{\hat{\Sigma}} + \delta_{\hat{\sigma}} + \delta_{\hat{\tau}^I} \ , \tag{4.7}$$

where $\delta_{\hat{X}}$ denotes a diffeomorphism with $\hat{X}^{\hat{\mu}} = -i\hat{\bar{\epsilon}}^i_1\hat{\gamma}^{\hat{\mu}}\hat{\epsilon}_{2i}$, $\delta_{\hat{\Sigma}}$ is a local Lorentz transformation with

$$\hat{\Sigma}^{\hat{a}\hat{b}} = \hat{X}^{\hat{\mu}}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}} + 2i\hat{\bar{\epsilon}}^i_1\left(-\hat{\gamma}^{\hat{a}\hat{b}} T_{ij} + \frac{1}{6\sqrt{2}}\left(\hat{\gamma}^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} + 4\delta^{\hat{a}}_{\hat{c}}\delta^{\hat{b}}_{\hat{d}}\right)\left(H^{\hat{c}\hat{d}}_{ij} + \frac{1}{\sqrt{2}} h^{\hat{c}\hat{d}}_{ij}\right)\right)\hat{\epsilon}^j_2 \ , \tag{4.8}$$

and $\delta_{\hat{\sigma}}$ and $\delta_{\hat{\tau}^I}$ denote U(1) and SU(2) gauge transformations, respectively, with

$$\hat{\sigma} = \hat{X}^{\hat{\mu}}\hat{a}_{\hat{\mu}} + \frac{1}{2} i\xi^2\hat{\bar{\epsilon}}^i_1\hat{\epsilon}_{2i} \ , \qquad \hat{\tau}^I = \hat{X}^{\hat{\mu}}\hat{A}^I_{\hat{\mu}} - \frac{1}{\sqrt{2}} i\xi^{-1}\left(\Gamma^I\right)_{ij}\hat{\bar{\epsilon}}^i_1\hat{\epsilon}^j_2 \ . \tag{4.9}$$

Summary of conventions: For easier reference we summarize the conventions for the $\mathfrak{usp}(4)$ generators and the index notation, which agree with those of [110]. The $\mathfrak{usp}(4)$ symplectic metric Ω and its inverse satisfy $\Omega_{ij}\Omega^{jk} = \delta_i{}^k$, $\Omega^{ij} = (\Omega_{ji})^*$ and spinor indices are raised and lowered via $\epsilon^i = \Omega^{ij}\epsilon_j$ and $\epsilon_i = \Omega_{ij}\epsilon^j$. The $\mathfrak{so}(5)$ Clifford algebra generators Γ_m satisfy $\left(\Gamma_m\right)_i{}^k\left(\Gamma_n\right)_k{}^j + \left(\Gamma_n\right)_i{}^k\left(\Gamma_m\right)_k{}^j = 2\delta_{mn}\delta_i{}^j$, which yields canonical Clifford matrices only for these specific index positions. With the charge conjugation matrix Ω we have $\Omega^{ik}\left(\Gamma_m\right)_k{}^j =: \left(\Gamma_m\right)^{ij} = -\left(\Gamma_m\right)^{ji}$. The conjugate is denoted by $\left(\Gamma_m\right)_{ij} = \left(\left(\Gamma_m\right)^{ij}\right)^*$ and the $\mathfrak{so}(5)$ generators satisfy $\left(\Gamma_{mn}\right)^{ij} = \left(\Gamma_{mn}\right)^{ji}$. Note also that $\epsilon_{45} = \epsilon^{45} = 1$. Fermionic fields are by convention anticommuting and complex conjugation changes their order.

4.1.2 Local $\mathcal{N}=2$ superconformal symmetry on the boundary of asymptotically-AdS configurations

We now restrict the configuration space of the theory discussed in the previous section to geometries which are asymptotically AdS_5, and discuss the fields and symmetries induced on the conformal boundary. We give a brief discussion of asymptotically-AdS spaces in the following, and refer to [92, 117] for more details. The metric signature and curvature conventions are those of Sec. 4.1.1 and [110], i.e. AdS has positive curvature.

A metric $\hat{g}$ on the interior of a compact manifold X with boundary ∂X is called conformally compact if, for a defining function r of the boundary (meaning that $r|_{\partial X} = 0$, $dr|_{\partial X} \neq 0$ and $r|_{\mathrm{int}X} > 0$), the rescaled metric $\bar{g} := r^2 \hat{g}$ extends to all of X as a metric. For such a conformally compact metric $\hat{g}$ the conformal structure $[\bar{g}|_{T\partial X}]$ induced on ∂X and the boundary restriction of the function $|dr|^2_{\bar{g}} := \bar{g}^{\,-1}(dr, dr)$ are independent of the choice of defining function. The curvature of the metric $\hat{g}$ is given by[3]

$$\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = -|dr|^2_{\bar{g}} \left(\hat{g}_{\hat{\mu}\hat{\rho}}\hat{g}_{\hat{\nu}\hat{\sigma}} - \hat{g}_{\hat{\mu}\hat{\sigma}}\hat{g}_{\hat{\nu}\hat{\rho}}\right) + \mathcal{O}(r^{-3}) \ , \tag{4.10}$$

where we denote tangent-space indices on TX with hat, e.g. $\hat{\mu}$, $\hat{\nu}$, and tangent-space indices on $T\partial X$ are denoted without hat. Asymptotically, $\hat{g}$ thus has constant sectional curvature given by $-|dr|^2_{\bar{g}}$, and we call a conformally compact metric $\hat{g}$ an asymptotically-AdS metric if the value of the sectional curvature is positive and constant on the boundary, i.e. $|dr|^2_{\bar{g}} = -1/R^2$ on ∂X for some constant R. Note that we do not demand $\hat{g}$ to be Einstein.

A representative metric $g^{(0)}$ of the boundary conformal structure uniquely determines a defining function r such that $g^{(0)} = \frac{r^2}{R^2}\hat{g}\,|_{T\partial X}$ and $|dr|^2_{\bar{g}} = -1/R^2$ in a neighbourhood of ∂X. Choosing this defining function as radial coordinate, the metric $\hat{g}$ takes the Fefferman-Graham form

$$\hat{g} = \frac{R^2}{r^2}\left(g_{\mu\nu}dx^\mu \otimes dx^\nu - dr \otimes dr\right) , \qquad g_{\mu\nu}(x,r) = g^{(0)}_{\mu\nu}(x) + \frac{r^2}{R^2}g^{(2)}_{\mu\nu}(x) + \dots \tag{4.11}$$

with g of signature $(+,-,-,-)$ and the limit $r \to 0$ corresponding to the conformal boundary. The expansion of g in powers of r is justified when $\hat{g}$ satisfies vacuum Einstein equations, which, however, we do not assume here. For the time being we will still use that expansion and refer the discussion of its validity to Sec. 4.1.2.2. We note that the geometries with AdS on the boundary discussed in Sec. 3.1.1, 3.3.2 – although they are not conformally compact in the usual sense and are better understood as $\langle n\rangle$-manifolds – can also be described by a metric of the form (4.11). Thus, the results we shall obtain here apply also to these backgrounds.

According with the Fefferman-Graham form of the metric, we partially gauge-fix the local Lorentz symmetry such that the vielbein is of the form

$$\hat{e}^a_\mu(x,r) = \frac{R}{r}e^a_\mu(x,r) \ , \qquad \hat{e}^r_\mu = \hat{e}^a_r = 0 \ , \qquad \hat{e}^r_r = \frac{R}{r} \ , \tag{4.12}$$

[3] With the usual Landau notation $f \overset{x\to x_0}{=} \mathcal{O}(g) \Longleftrightarrow \limsup_{x\to x_0}|f/g| < \infty$ and $f \overset{x\to x_0}{=} o(g) \Longleftrightarrow \lim_{x\to x_0}|f/g| = 0$.

with $e_\mu^a(x,r) = e_\mu^{(0)a}(x) + r e_\mu^{(1)a}(x) + \dots$. We denote Lorentz indices by $\hat{a} = (a, \underline{r})$ with an underline below r to avoid confusion. For the gravitinos and the SU(2)⊗U(1) gauge fields we employ axial gauges $\hat{\psi}_{ri} \equiv \hat{A}_r^I \equiv \hat{a}_r \equiv 0$.

In this setting we construct the fields induced on the conformal boundary in Sec. 4.1.2.1. For the discussion of the induced symmetry transformations we will be interested in the residual bulk symmetries preserving the gauge-fixing conditions. These are determined as solutions to

$$\left(\delta_{\hat{X}} + \delta_{\hat{\Sigma}} + \delta_{\hat{\epsilon}_i} + \delta_{\mathrm{U}(1)} + \delta_{\mathrm{SU}(2)}\right) \{\hat{e}_r^{\underline{r}}, \hat{e}_r^a, \hat{e}_\mu^{\underline{r}}, \hat{a}_r, \hat{A}_r^I, \hat{\psi}_{ri}\} = 0 \,, \tag{4.13}$$

where $\delta_{\hat{X}}$, $\delta_{\hat{\Sigma}}$, $\delta_{\hat{\epsilon}}$ denote diffeomorphisms, local Lorentz and supersymmetry transformations, respectively. The solutions and their action on the boundary fields will be discussed in Sec. 4.1.2.3.

The spin connection is treated in 1.5^{th}-order formalism and fixed by its equation of motion as derived from (4.1). We split $\hat{\omega}_{\hat{\mu}\hat{a}\hat{b}} = \hat{\omega}_{\hat{\mu}\hat{a}\hat{b}}(\hat{e}) + \hat{\omega}_{\hat{\mu}\hat{a}\hat{b}}(\hat{e}, \hat{\psi}, \hat{\chi})$ where the torsion-free part $\hat{\omega}_{\hat{\mu}\hat{a}\hat{b}}(\hat{e})$ calculated from (4.12) has the non-vanishing components

$$\hat{\omega}_\mu{}^{ab}(\hat{e}) = \omega_\mu{}^{ab}(e) \,, \qquad \hat{\omega}_\mu{}^{a\underline{r}}(\hat{e}) = \frac{1}{r} e_\mu^a - \frac{1}{2} e^{\rho a} \partial_r g_{\mu\rho} \,, \qquad \hat{\omega}_r{}^{ab}(\hat{e}) = e^{\mu[a} \partial_r e_\mu^{b]} \,, \tag{4.14}$$

and for the remaining part involving fermions we find

$$\hat{\omega}_{\hat{\mu}\hat{a}\hat{b}}(\hat{e}, \hat{\psi}, \hat{\chi}) = -\frac{1}{2} i \left(\hat{\bar{\psi}}^i_{\hat{a}} \hat{\gamma}_{\hat{\mu}} \hat{\psi}_{\hat{b}i} + 2 \hat{\bar{\psi}}^i_{\hat{\mu}} \hat{\gamma}_{[\hat{a}} \hat{\psi}_{\hat{b}]i} \right) - \frac{1}{4} i \hat{\bar{\psi}}^i_{\hat{\lambda}} \hat{\gamma}_{\hat{\mu}\hat{a}\hat{b}}{}^{\hat{\lambda}\hat{\tau}} \hat{\psi}_{\hat{\tau}i} - \frac{1}{4} i \hat{\bar{\chi}}^i \hat{\gamma}_{\hat{\mu}\hat{a}\hat{b}} \hat{\chi}_i \,. \tag{4.15}$$

Thus, the Lorentz-covariant derivative on spinor fields reads

$$\begin{aligned} \hat{\nabla}_\mu &= \nabla_\mu^{(e)} + \frac{1}{2r} \gamma_\mu \gamma_{\underline{r}} - Z_\mu + \frac{1}{4} \hat{\omega}_\mu{}^{\hat{a}\hat{b}}(\hat{e}, \hat{\psi}, \hat{\chi}) \hat{\gamma}_{\hat{a}\hat{b}} \quad =: \nabla_\mu + \frac{1}{2r} \gamma_\mu \gamma_{\underline{r}} \,, \\ \hat{\nabla}_r &= \partial_r - Z_r + \frac{1}{4} \hat{\omega}_r{}^{\hat{a}\hat{b}}(\hat{e}, \hat{\psi}, \hat{\chi}) \hat{\gamma}_{\hat{a}\hat{b}} \,, \end{aligned} \tag{4.16}$$

where $\hat{\gamma}_{\hat{\mu}} = \hat{e}_{\hat{\mu}}^{\hat{a}} \gamma_{\hat{a}}$, $\gamma_\mu = e_\mu^a \gamma_a$. For notational convenience we defined $\nabla_\mu^{(e)} := \partial_\mu + \frac{1}{4} \omega_\mu{}^{ab}(e) \gamma_{ab}$ and $Z_\mu := \frac{1}{4} (\partial_r g_{\mu\rho}) \gamma^\rho \gamma_{\underline{r}}$, $Z_r := \frac{1}{4} (\partial_r e_\mu^a) \gamma_a{}^\mu$.

4.1.2.1 Boundary fields

In this part we construct the fields induced on the conformal boundary. Similar to the construction of the induced conformal structure on the boundary, we define the classical boundary field as follows. For a bulk field $\hat{\phi}$ with asymptotic r-dependence $\hat{\phi}(x,r) = \mathcal{O}(f(r))$, we define the rescaled field $\phi(x,r) := f(r)^{-1} \hat{\phi}(x,r)$. This rescaled field then admits a finite, non-vanishing boundary limit, which is interpreted as the boundary field[4].

Therefore, to determine the multiplet of boundary fields we have to fix the asymptotic scaling of the various bulk fields. To this end we consider their equations of motion linearized in all fields but the metric/vielbein and decomposed into boundary-irreducible components,

[4] This is also the classical analog to the construction for the Wightman field in [118].

e.g. into four-dimensional chiral components for a bulk spinor field. The leading order in the boundary limit turns out to be an ordinary differential equation in r, and is solved by fixing the scalings of the different boundary-irreducible bulk field components. The rescaled field is defined by extracting the asymptotic r-dependence of the dominant field component, thereby subdominant components are projected out in the definition of the boundary field. The results obtained in this way on the basis of the linearized field equations are extended to the nonlinear theory in Sec. 4.1.2.2.

We start with the vielbein, for which the asymptotic r-dependence is already fixed by (4.11), (4.12) and the induced boundary field is given by $e^a_\mu(x,0)$. As discussed in [110], Einstein's equations as derived from (4.1) in a pure metric-dilaton background read

$$\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{g}_{\hat{\mu}\hat{\nu}}\hat{\mathcal{R}} + 2\hat{g}_{\hat{\mu}\hat{\nu}}P(\hat{\varphi}) = 0 \ , \tag{4.17}$$

and the scalar potential $P(\hat{\varphi})$, having exactly one extremum $\big(\hat{\varphi}, P(\hat{\varphi})\big) \equiv \big(0, \frac{3}{8}g^2\big)$, provides a cosmological constant such that AdS_5 is a vacuum solution. Here we do not restrict the theory to the metric-dilaton sector and only demand (4.17) to be solved at leading order in the boundary limit. From (4.10) we find that $\hat{g}$ indeed solves the leading order provided that the asymptotic curvature radius R is fixed in terms of the gauge coupling as $R^2 = 8/g^2$. In Sec. 4.1.2.2 we show that – with the scalings obtained in this section – all other terms in the complete Einstein equations contribute to the subleading orders only. In the following we fix $g = 2\sqrt{2}$ such that $R = 1$.

For the gravitinos, which we consider next, the nonlinear equation of motion reads

$$\begin{aligned}\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{D}_{\hat{\nu}}\hat{\psi}_{\hat{\rho}i} - 3T_{ij}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\psi}^j_{\hat{\nu}} = &-\frac{1}{2\sqrt{2}}\Big(H^{\hat{\rho}\hat{\sigma}}{}_i{}^j + \frac{1}{\sqrt{2}}h^{\hat{\rho}\hat{\sigma}}{}_i{}^j\Big)\hat{\gamma}^{[\hat{\mu}}\hat{\gamma}_{\hat{\rho}\hat{\sigma}}\hat{\gamma}^{\hat{\nu}]}\hat{\psi}_{\hat{\nu}j} - A_{ij}\hat{\gamma}^{\hat{\mu}}\chi^j \\ &-\frac{1}{2\sqrt{6}}\Big(H_{\hat{\rho}\hat{\sigma}i}{}^j - \sqrt{2}h_{\hat{\rho}\hat{\sigma}i}{}^j\Big)\hat{\gamma}^{\hat{\rho}\hat{\sigma}}\hat{\gamma}^{\hat{\mu}}\chi_j + \frac{1}{\sqrt{2}}\left(\partial_{\hat{\nu}}\hat{\varphi}\right)\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\mu}}\chi_i \ .\end{aligned}\tag{4.18}$$

To fix T_{ij} (see (4.2)) we note that Γ_{45}, since it squares to $-\mathbb{1}$ and is traceless, has eigenvalues $\pm i$, each with multiplicity 2. We choose a $\mathfrak{usp}(4)$ basis where Γ_{45} is diagonal $(\Gamma_{45})_i{}^j = i\kappa_i\delta_i{}^j$ and split $i = (i_+, i_-)$ such that $\kappa_{i_\pm} = \pm 1$. Since $\{\Omega, \Gamma_{45}\} = 0$ we have $\Omega^{i_+j_+} = \Omega^{i_-j_-} = 0$. Defining four-dimensional chirality projectors $P_{\mathrm{L/R}} := \frac{1}{2}\left(1 \pm i\gamma^{\underline{r}}\right)$, the L/R projections of the linearized equation (4.18) for $\hat{\mu} = \mu$ read

$$\gamma^{\mu\nu\rho}\nabla^{(e)}_\nu\hat{\psi}^{\mathrm{R/L}}_{\rho i} - \left(\gamma^{\mu\nu\rho}Z_\nu \pm i\gamma^{\mu\rho}Z_r\right)\hat{\psi}^{\mathrm{L/R}}_{\rho i} + i\gamma^{\mu\rho}\left(\pm\partial_r \mp \frac{1}{r} + \frac{3\kappa_i}{2r}\right)\hat{\psi}^{\mathrm{L/R}}_{\rho i} = 0 \ . \tag{4.19}$$

Since the $\hat{\psi}^{\mathrm{L/R}}_{\mu i_-}$ are related to the conjugates of $\hat{\psi}^{\mathrm{R/L}}_{\mu i_+}$ by the symplectic Majorana condition, it is sufficient to consider the i_+-components. Solving (4.19) at leading order in r yields the two independent solutions $\hat{\psi}_{\mu i_+} = r^{-1/2}\psi^{\mathrm{L}}_{\mu i_+} + o(r^{-1/2})$ and $\hat{\psi}_{\mu i_+} = r^{5/2}\psi^{\mathrm{R}}_{\mu i_+} + o(r^{5/2})$ with $\lim_{r\to 0}\psi^{\mathrm{L/R}}_{\mu i_+}$ finite. Thus, the gravitinos lose half of their components in the boundary limit and the rescaled field $\psi_{\mu i_+} := r^{1/2}\hat{\psi}_{\mu i_+}$ yields the two chiral gravitinos $\psi^{\mathrm{L}}_{\mu i_+}|_{r=0}$ as boundary fields.

Proceeding with the fermionic fields we now discuss the spin-$\frac{1}{2}$ fermions $\hat{\chi}_i$. Their equation of motion is given by

$$\begin{aligned}\hat{\gamma}^{\hat{\mu}}\hat{D}_{\hat{\mu}}\hat{\chi}_i + T_{ij}\hat{\chi}^j = \frac{2}{\sqrt{3}}A_{ij}\hat{\chi}^j + A_{ij}\hat{\gamma}^{\hat{\mu}}\hat{\psi}^j_{\hat{\mu}} + \frac{1}{2\sqrt{6}}\Big(H_{\hat{\mu}\hat{\nu}i}{}^j - \sqrt{2}h_{\hat{\mu}\hat{\nu}i}{}^j\Big)\hat{\gamma}^{\hat{\rho}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\psi}_{\hat{\rho}j} \\ - \frac{1}{6\sqrt{2}}\Big(H_{\hat{\mu}\hat{\nu}i}{}^j - \frac{5}{\sqrt{2}}h_{\hat{\mu}\hat{\nu}i}{}^j\Big)\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\chi}_j + \frac{1}{\sqrt{2}}\left(\partial_{\hat{\nu}}\hat{\varphi}\right)\hat{\gamma}^{\hat{\mu}}\hat{\gamma}^{\hat{\nu}}\hat{\psi}_{\hat{\mu}i} \ .\end{aligned} \tag{4.20}$$

Solving the linearized L/R projections, given by

$$\gamma^\mu\nabla^{(e)}_\mu\hat{\chi}^{\mathrm{R/L}}_i - \left(\gamma^\mu Z_\mu \mp iZ_r\right)\hat{\chi}^{\mathrm{L/R}}_i - i\left(\pm\partial_r + \frac{\kappa_i \mp 4}{2r}\right)\hat{\chi}^{\mathrm{L/R}}_i = 0 \ , \tag{4.21}$$

at leading order for $i = i_+$ we find as dominant solution $\hat{\chi}_{i_+} = r^{3/2}\chi^{\mathrm{L}}_{i_+} + o(r^{3/2})$. Similarly to the gravitinos, the $\hat{\chi}_{i_+}$ become chiral in the boundary limit and we have the two left handed Weyl fermions $\chi^{\mathrm{L}}_{i_+}|_{r=0}$ as boundary fields.

Coming to the tensor fields $\hat{B}^\alpha_{\hat{\mu}\hat{\nu}}$ we define $\hat{C}_{\hat{\mu}\hat{\nu}} := \frac{1}{\sqrt{2}}(\hat{B}^4_{\hat{\mu}\hat{\nu}} - i\hat{B}^5_{\hat{\mu}\hat{\nu}})$ and, with the four-dimensional Hodge dual $\star\hat{C}_{\mu\nu} := \frac{1}{2}e^{-1}\epsilon_{\mu\nu}{}^{\rho\sigma}\hat{C}_{\rho\sigma}$, the (anti-)selfdual parts of $\hat{C}_{\mu\nu}$ are defined as $\hat{C}^\pm_{\mu\nu} := \frac{1}{2}(\hat{C}_{\mu\nu} \pm i\star\hat{C}_{\mu\nu})$. The equation of motion reads

$$\frac{i}{g_1}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\hat{D}_{\hat{\rho}}\hat{C}_{\hat{\sigma}\hat{\tau}} - \hat{e}\,\xi^2\hat{C}^{\hat{\mu}\hat{\nu}} = -\frac{1}{2}\hat{e}\,\xi\Big(\frac{1}{2}J_1{}^{\hat{\mu}\hat{\nu}}_{ij} + \frac{1}{\sqrt{3}}J_2{}^{\hat{\mu}\hat{\nu}}_{ij} - \frac{1}{6}J_3{}^{\hat{\mu}\hat{\nu}}_{ij}\Big)\left(\Gamma_4 - i\Gamma_5\right)^{ij} \ , \tag{4.22}$$

with $J_1{}^{\hat{\mu}\hat{\nu}}_{ij} = i\bar{\hat{\psi}}{}^{\hat{\rho}}_i\hat{\gamma}_{[\hat{\rho}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\gamma}_{\hat{\sigma}]}\hat{\psi}^{\hat{\sigma}}_j$, $J_2{}^{\hat{\mu}\hat{\nu}}_{ij} = i\bar{\hat{\psi}}{}^{\hat{\rho}}_i\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\gamma}_{\hat{\rho}}\hat{\chi}_j$ and $J_3{}^{\hat{\mu}\hat{\nu}}_{ij} = i\bar{\hat{\chi}}_i\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\chi}_j$. From the μr-components of the linearized equation $\hat{C}_{\mu r}$ is fixed in terms of $\hat{C}_{\mu\nu}$ by $\hat{C}_{\mu r} = \frac{1}{2}ire^{-1}\epsilon_\mu{}^{\rho\sigma\tau}\partial_\rho\hat{C}_{\hat{\sigma}\hat{\tau}}$, and is of higher order in r. The (anti-)selfdual parts of the linearized $\mu\nu$-components,

$$\frac{1}{2}e^{-1}\epsilon_{\mu\nu}{}^{\rho\sigma}\left(\partial_r\hat{C}_{\mu\nu} + 2\partial_\rho\hat{C}_{\sigma r}\right) = -\frac{i}{r}\hat{C}_{\mu\nu} \ , \tag{4.23}$$

then yield the solutions $\hat{C}_{\mu\nu} = r^{-1}C^-_{\mu\nu} + o(r^{-1})$ and $\hat{C}_{\mu\nu} = r\,C^+_{\mu\nu} + o(r)$. Thus, the anti-selfdual part $\hat{C}^-$ is dominant in the boundary limit and the selfdual part $\hat{C}^+$ is projected out in the definition of the boundary field.

For the U(1) and SU(2) gauge fields the equations of motion are

$$\begin{aligned}\partial_{\hat{\nu}}\left(\hat{e}\xi^{-4}\hat{f}^{\hat{\mu}\hat{\nu}}\right) = \frac{1}{4}\hat{e}g_1\left(\Gamma_{45}\right)_i{}^j J_4{}^{\hat{\mu}i}{}_j - \frac{1}{4}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\left(\hat{B}^\alpha_{\hat{\nu}\hat{\rho}}\hat{B}^\alpha_{\hat{\sigma}\hat{\tau}} + \hat{F}^I_{\hat{\nu}\hat{\rho}}\hat{F}^I_{\hat{\sigma}\hat{\tau}}\right) \\ + \Omega^{ij}\partial_{\hat{\nu}}\left(\hat{e}\xi^{-2}\left(\frac{1}{4}J_1{}^{\hat{\mu}\hat{\nu}}_{ij} - \frac{1}{\sqrt{3}}J_2{}^{\hat{\mu}\hat{\nu}}_{ij} + \frac{5}{12}J_3{}^{\hat{\mu}\hat{\nu}}_{ij}\right)\right) \ ,\end{aligned} \tag{4.24}$$

$$\hat{D}_{\hat{\nu}}\left(\hat{e}\xi^2\hat{F}^{I\hat{\mu}\hat{\nu}}\right) = \frac{1}{4}\hat{e}g_2\left(\Gamma_{I45}\right)_i{}^j J_4{}^{\hat{\mu}i}{}_j - \hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\hat{D}_{\hat{\nu}}\left(\hat{F}^I_{\hat{\rho}\hat{\sigma}}\hat{a}_{\hat{\tau}}\right) + \frac{1}{\sqrt{2}}\hat{D}_{\hat{\nu}}\left(\hat{e}\xi K_I^{\hat{\mu}\hat{\nu}}\right) \ , \tag{4.25}$$

with $J_4{}^{\hat{\mu}i}{}_j = i\bar{\hat{\chi}}{}^i\hat{\gamma}^{\hat{\mu}}\hat{\chi}_j - i\bar{\hat{\psi}}{}^i_{\hat{\nu}}\hat{\gamma}^{\hat{\nu}\hat{\mu}\hat{\rho}}\hat{\psi}_{\hat{\rho}j}$ and $K_I^{\hat{\mu}\hat{\nu}} = \left(\Gamma_I\right)^{ij}\left(\frac{1}{2}J_1{}^{\hat{\mu}\hat{\nu}}_{ij} + \frac{1}{\sqrt{3}}J_2{}^{\hat{\mu}\hat{\nu}}_{ij} - \frac{1}{6}J_3{}^{\hat{\mu}\hat{\nu}}_{ij}\right)$. For the ansatz $\hat{a}_\mu = r^\alpha a_\mu$ the leading order of the linearized equation yields $\alpha \in \{0, 2\}$, and similarly for $\hat{A}^I_\mu$. Thus, $\hat{a}_\mu$ and $\hat{A}_\mu$ are itself finite in the boundary limit and define boundary vector fields without rescaling.

It remains to analyze the scalar field $\hat{\varphi}$ with equation of motion

$$\begin{aligned}\hat{\Box}_{\hat{g}}\hat{\varphi} - P'(\hat{\varphi}) = &- \frac{i}{\sqrt{2}}A_{ij}\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\psi}^j_{\hat{\nu}} - iA'_{ij}\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}}\hat{\chi}^j - \frac{i}{\sqrt{3}}\Big(A'_{ij} + \frac{1}{\sqrt{6}}A_{ij}\Big)\hat{\bar{\chi}}^i\hat{\chi}^j \\ &- \sqrt{\frac{2}{3}}\xi^2\hat{\bar{C}}_{\hat{\mu}\hat{\nu}}\hat{C}^{\hat{\mu}\hat{\nu}} + \sqrt{\frac{2}{3}}\xi^{-4}\hat{f}_{\hat{\mu}\hat{\nu}}\hat{f}^{\hat{\mu}\hat{\nu}} - \frac{1}{\sqrt{6}}\xi^2\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^{I\hat{\mu}\hat{\nu}} \\ &+ \frac{1}{4\sqrt{3}}\Big(H^{ij}_{\hat{\mu}\hat{\nu}} - \sqrt{2}h^{ij}_{\hat{\mu}\hat{\nu}}\Big)J_1{}^{\hat{\mu}\hat{\nu}}_{ij} + \frac{1}{6}\Big(H^{ij}_{\hat{\mu}\hat{\nu}} + 2\sqrt{2}h^{ij}_{\hat{\mu}\hat{\nu}}\Big)J_2{}^{\hat{\mu}\hat{\nu}}_{ij} \\ &- \frac{1}{12\sqrt{3}}\Big(H^{ij}_{\hat{\mu}\hat{\nu}} + 5\sqrt{2}h^{ij}_{\hat{\mu}\hat{\nu}}\Big)J_3{}^{\hat{\mu}\hat{\nu}}_{ij} - \frac{1}{\sqrt{2}}\hat{e}^{-1}\partial_{\hat{\nu}}\left(i\hat{e}\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\mu}}\hat{\chi}_i\right) \ ,\end{aligned} \tag{4.26}$$

where $A'^{ij} := -\frac{1}{6}g\left(\xi^{-1} + 2\xi^2\right)(\Gamma_{45})^{ij}$. The linearized equation is given by

$$r^2\Box_g\hat{\varphi} - \frac{1}{2}r^2\left(g^{\mu\nu}\partial_r g_{\mu\nu}\right)\partial_r\hat{\varphi} - \mathcal{D}_r^2\hat{\varphi} = 0 \ , \tag{4.27}$$

with $\mathcal{D}_r = r\partial_r - 2$, and the leading-order part is solved by $r^2\varphi_1(x,r)$ and $r^2\log(r)\,\varphi_2(x,r)$ with $\varphi_{1/2}|_{r=0}$ finite. The boundary scalar field is thus defined by extracting the dominant scaling $\hat{\varphi} =: r^2\log(r)\varphi$ and restricting φ to the boundary. In summary, the multiplet of boundary fields is given by $(e^a_\mu, \psi^{\mathrm{L}}_{\mu\, i_+}, C^-_{\mu\nu}, A^I_\mu, a_\mu, \chi^{\mathrm{L}}_{i_+}, \varphi)|_{r=0}$.

4.1.2.2 Nonlinear theory and subdominant components

The splitting into dominant and subdominant components and the scaling of the dominant parts as obtained above from the linearized equations of motion fixes the definition of the boundary fields. It remains to be checked whether the obtained scaling behaviour is consistent in the nonlinear theory. Furthermore, the subdominant components of some of the fields are required for the symmetry transformations to be discussed in Sec. 4.1.2.3. These two points are addressed in the following. Note that this discussion does not include the four-fermion terms which are not spelled out in [110]. However, as we find quite some cancellations taking place to ensure consistency of the previously obtained results at the leading orders in the fermions, we expect that this consistency is not accidental and extends to the four-fermion terms as well.

Since the analysis of Sec. 4.1.2.1 crucially relies on the form of the metric (4.11) in a neighbourhood of the boundary, the first thing to be checked is the validity of the Fefferman-Graham form. Considering the terms in the Lagrangian (4.1) with the scaling of the fields as obtained in the previous section, $\hat{e}\hat{\mathcal{R}}(\hat{\omega})$ and the cosmological constant $\hat{e}P(0)$ are $\mathcal{O}(r^{-5})$ while the other terms are $\mathcal{O}(r^{-3})$. Thus, the leading order of Einstein's equations reduces to the form discussed in the previous section and the Fefferman-Graham form of the metric (4.11) is justified. In particular, since there are no $\mathcal{O}(r^{-4})$ terms in the Lagrangian, there is no $\mathcal{O}(r)$ contribution to $g_{\mu\nu}(x,r)$ and the expansion in (4.11) is justified. Next, we consider the spin connection (4.14), (4.15). With the scaling as obtained before, $\hat{\omega}_{\mu ab}(\hat{e},\hat{\psi},\hat{\chi}) = \mathcal{O}(r^0)$ and the other components of $\hat{\omega}_{\hat{\mu}\hat{a}\hat{b}}(\hat{e},\hat{\psi},\hat{\chi})$ are of $\mathcal{O}(r)$. Therefore, the fermionic terms do not alter the $\mathcal{O}(r^{-1})$ part of the covariant derivative (4.16), which was relevant for the previous section. For the four-dimensional Lorentz-covariant derivative ∇_μ defined in (4.16) we find $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}$ with

$$\omega_{\mu ab}\big|_{r=0} = \omega_{\mu ab}(e) - \frac{1}{2}\left(i\bar{\psi}_a^{\mathrm{L}i_+}\gamma_\mu\psi^{\mathrm{L}}_{bi_+} + 2i\bar{\psi}_\mu^{\mathrm{L}i_+}\gamma_{[a}\psi^{\mathrm{L}}_{b]i_+} + \text{c.c.}\right) . \tag{4.28}$$

From (4.3) the four-dimensional gauge and Lorentz covariant derivative acting on a boundary spinor is

$$D_\mu v_{i_+} = \nabla_\mu v_{i_+} + \frac{1}{2}ig_1 a_\mu v_{i_+} + \frac{1}{2}ig_2 A^I_\mu \left(\Gamma_I\right)_{i_+}{}^{j_+} v_{j_+} . \tag{4.29}$$

For the remaining fields we study the interaction terms of (4.1) directly in the field equations. They turn out to be subdominant in the equations for the boundary-dominant field components, such that their scaling is not affected. They do, however, alter the subdominant components, some of which are in fact not subdominant but play the role of auxiliary fields on the boundary. We start with the gravitinos, for which the scaling of $\hat{\psi}^{\mathrm{L}}_{\mu i_+}$ was determined from the P_{L}-projection of (4.18) at $\mathcal{O}(r^{3/2})$. One easily verifies that the interaction terms in (4.18) are of $\mathcal{O}(r^{5/2})$ and thus the analysis of the previous section is not affected. To determine the subdominant components we consider the P_{R} projection of the $\hat{\mu}{=}\mu$ components. Noting that $\left(\Gamma_\alpha\right)_{i_+}{}^{j_+} = \left(\Gamma_\alpha\right)_{i_-}{}^{j_-} = 0$ due to $\{\Gamma_\alpha, \Gamma_{45}\} = 0$, and $\hat{B}^\alpha_{\hat{\mu}\hat{\nu}}\left(\Gamma_\alpha\right)_{i_+}{}^{j_-} = \sqrt{2}\hat{C}_{\hat{\mu}\hat{\nu}}\left(\Gamma_4\right)_{i_+}{}^{j_-}$, we find $\hat{\psi}_{\mu i_+} = r^{-1/2}\psi^{\mathrm{L}}_{\mu i_+} + r^{1/2}\Phi^{\mathrm{R}}_{\mu i_+}$ with

$$\Phi^{\mathrm{R}}_{\mu i_+}\Big|_{r=0} = -\frac{1}{2}i\Big(\gamma_\mu{}^{\nu\rho} - \frac{2}{3}\gamma_\mu\gamma^{\nu\rho}\Big)\Big(D_\nu\psi^{\mathrm{L}}_{\rho i_+} - \frac{1}{4}\gamma\cdot C^-_{i_+j_+}\gamma_\nu\psi^{\mathrm{R}j_+}_\rho\Big) , \tag{4.30}$$

where $\gamma\cdot C := \gamma^{\mu\nu}C_{\mu\nu}$ and $C_{\hat{\mu}\hat{\nu}\,i_+j_+} := C_{\hat{\mu}\hat{\nu}}\left(\Gamma_4\right)_{i_+j_+}$. Note that $\psi^{\mathrm{R}i_+}_\mu = C\big(\bar{\psi}^{\mathrm{L}i_+}_\mu\big)^T$ by the symplectic Majorana condition, and a possible C^+-contribution drops out due to $\gamma\cdot C^\pm = \gamma\cdot C^\pm P_{\mathrm{R/L}}$. For later convenience we define the quantity

$$R_{\mu\nu\,i_+}(Q) := D_{[\mu}\psi^{\mathrm{L}}_{\nu]i_+} - i\gamma_{[\mu}\Phi^{\mathrm{R}}_{\nu]i_+} - \frac{1}{4}\gamma\cdot C^-_{i_+j_+}\gamma_{[\mu}\psi^{\mathrm{R}j_+}_{\nu]} , \tag{4.31}$$

and note that it is anti-selfdual $i\star R_{\mu\nu\,i_+}(Q) = -R_{\mu\nu\,i_+}(Q)$ and satisfies $\gamma^\mu R_{\mu\nu\,i_+}(Q) = 0$.

We continue with the tensor field $\hat{C}_{\hat{\mu}\hat{\nu}}$. Using $\frac{1}{2}\left(\gamma_{\mu\nu} \pm i\star\gamma_{\mu\nu}\right) = \gamma_{\mu\nu}P_{\mathrm{R/L}}$ we find the interaction terms subdominant in the anti-selfdual part of (4.22) with $\hat{\mu}\hat{\nu}{=}\mu\nu$, which was used to determine the scaling $\hat{C}^-_{\mu\nu} = r^{-1}C^-_{\mu\nu}$. In the selfdual part the interaction terms are not subdominant, but rather fix $\hat{C}^+_{\mu\nu} = r^{-1}C^+_{\mu\nu}$ and

$$C^+_{\mu\nu}\big|_{r=0} = \frac{1}{4}i\left(\Gamma_4\right)^{i_+j_+}\bar{\psi}^{\mathrm{R}}_{\rho i_+}\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]}\psi^{\mathrm{L}}_{\sigma j_+} . \tag{4.32}$$

Thus, $\hat{C}^+_{\mu\nu}$ is in fact not subdominant with respect to $\hat{C}^-_{\mu\nu}$. However, since its boundary value is completely fixed in terms of the other boundary fields, it plays the role of an auxiliary field on the boundary. From the $\hat{\mu}\hat{\nu}{=}\mu r$ components we find for the subdominant $\hat{C}_{\mu r} = C_{\mu r}$

$$C_{\mu r}\big|_{r=0} = \frac{1}{2}ire^{-1}\epsilon_\mu{}^{\nu\rho\sigma}D_\nu\hat{C}_{\rho\sigma} + \bar{\psi}^{\mathrm{R}}_{\rho i_+}\big(\gamma_\mu{}^{\rho\sigma}\Phi^{\mathrm{R}}_{\sigma j_+} + \frac{1}{\sqrt{3}}\gamma_\mu\gamma^\rho\chi^{\mathrm{L}}_{j_+}\big)\left(\Gamma_4\right)^{i_+j_+} . \tag{4.33}$$

For the spin-$\frac{1}{2}$ fermions $\hat{\chi}^{\mathrm{L}}_{i_+} = r^{3/2}\chi^{\mathrm{L}}_{i_+}$ was obtained from the P_{L} projection of (4.20) at $\mathcal{O}(r^{3/2})$. The only additional contribution at that order is $\propto \gamma^\rho\gamma\cdot C^+_{i_+j_+}\psi^{\mathrm{R}j_+}_\rho$ which is a three-fermion term by (4.32) and we expect it to be cancelled by contributions of four-fermion terms in (4.1). We conclude that – up to the four-fermion terms not considered here – the obtained scaling for $\hat{\chi}^{\mathrm{L}}_{i_+}$ is not affected by the interaction terms. The subdominant right handed part is fixed from the P_{R}-projection of (4.20) and we find $\hat{\chi}_{i_+} = r^{3/2}\chi^{\mathrm{L}}_{i_+} + r^{5/2}\log(r)\chi^{\mathrm{R}}_{i_+}$ with

$$\begin{aligned}\chi^{\mathrm{R}}_{i_+}\Big|_{r=0} = i\not{D}\chi^{\mathrm{L}}_{i_+} - \frac{1}{\sqrt{2}}\varphi\gamma^\mu\psi^{\mathrm{L}}_{\mu i_+} - \frac{i}{2\sqrt{6}}\gamma^\rho\gamma^{\mu\nu}\left(F^I_{\mu\nu}\left(\Gamma_I\right)_{i_+}{}^{j_+} - \sqrt{2}f_{\mu\nu}\delta_{i_+}{}^{j_+}\right)\psi^{\mathrm{L}}_{\rho j_+} \\ + \frac{i}{2\sqrt{3}}\gamma^\rho\,\gamma\cdot C^-_{i_+j_+}\Phi^{j_+\mathrm{L}}_\rho - \frac{1}{\sqrt{3}}\gamma^\rho\gamma^\mu C_{\mu r\, i_+j_+}\psi^{j_+\mathrm{R}}_\rho\ .\end{aligned} \tag{4.34}$$

In the equations for the gauge fields (4.24), (4.25) the leading-order terms are those involving ${J_4}^{\hat{\mu}i}{}_j$ (the gravitino part thereof) and $J_{1\,ij}^{\mu\nu}$, both of which are of $\mathcal{O}(r^{-3})$. However, since $(\Gamma_I)_{i_+}{}^{j_-} = (\Gamma_I)_{i_-}{}^{j_+} = 0$ due to $[\Gamma_I, \Gamma_{45}] = 0$, their leading-order parts cancel exactly in both equations, such that the previous analysis of the linearized equations is not altered. For the scalar field we have to check that the interaction terms are subdominant with respect to the $\mathcal{O}(r^2)$ and $\mathcal{O}(r^2\log(r))$ parts of (4.26). Similar to the case of the gauge fields, there are cancellations between different terms at leading order. From (4.32) the $J^{\mu\nu}_{1ij}$ term and the $\hat{\bar{C}}_{\mu\nu}\hat{C}^{\mu\nu}$ term add up to zero at leading order, and also $-iA'_{ij}\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{\mu}}\hat{\chi}^j$ and $-\frac{1}{\sqrt{2}}\hat{e}^{-1}\partial_{\hat{\nu}}\left(i\hat{e}\hat{\bar{\psi}}^i_{\hat{\mu}}\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\mu}}\hat{\chi}_i\right)$ cancel. The remaining terms are subleading and thus the cancellations justify the analysis of the linearized equations also for $\hat{\varphi}$. We conclude that the scaling behaviours obtained from the linearized equations of motion with the modifications for the subdominant components discussed here are consistent in the nonlinear theory as given by (4.1).

4.1.2.3 Induced boundary symmetries

Having obtained the multiplet of boundary fields in the previous section we now discuss the symmetries on the boundary. To this end we determine the residual bulk symmetries from the constraints (4.13) and examine their action on the boundary fields, which is defined straightforwardly, e.g. $\delta\phi := \lim_{r\to0} f(r)^{-1}\hat{\delta}\hat{\phi}$ for a boundary field $\phi = \lim_{r\to0} f(r)^{-1}\hat{\phi}$. Relevant to us are solutions to the constraints (4.13) which act non-trivially on the boundary fields, and in the following we discuss certain special solutions which generate the general symmetry transformation of the boundary fields.

The constraint that $\hat{e}^r_r$ and $\hat{e}^a_r$ be preserved yields that, for an arbitrary $\lambda(x)$,

$$\hat{X}^r = r\lambda(x)\ , \qquad \hat{\Sigma}^a{}_{\underline{r}} = -e^a_\mu\partial_r\hat{X}^\mu\ . \tag{4.35}$$

We parametrize the U(1) and SU(2) gauge transformations by $\hat{\sigma}(x,r)$ and $\hat{\tau}^I(x,r)$, respectively, and using (4.35) the remaining constraints are

$$\partial_r \hat{X}^\mu = g^{\mu\rho}\big(r\partial_\rho\lambda(x) + i\hat{\bar{\psi}}^i_\rho\hat{\gamma}^r\hat{\epsilon}_i\big) \ , \tag{4.36a}$$

$$\partial_r\hat{\sigma} = \hat{a}_\mu\partial_r\hat{X}^\mu + \frac{1}{\sqrt{3}}i\xi^2\hat{\bar{\chi}}^i\hat{\gamma}_r\hat{\epsilon}_i \ , \tag{4.36b}$$

$$\partial_r\hat{\tau}^I = \hat{A}^I_\mu\partial_r\hat{X}^\mu + \frac{1}{\sqrt{6}}i\xi^{-1}\hat{\bar{\chi}}^i\hat{\gamma}_r\hat{\epsilon}^j\left(\Gamma^I\right)_{ij} \ , \tag{4.36c}$$

$$\hat{\nabla}_r\hat{\epsilon}_i + \hat{\gamma}_r T_{ij}\hat{\epsilon}^j = -\big(\partial_r\hat{X}^\mu\big)\hat{\psi}_{\mu i} + \frac{1}{6\sqrt{2}}\Big(\hat{\gamma}_r{}^{\hat{\nu}\hat{\rho}} - 4\delta^{\hat{\nu}}_r\hat{\gamma}^{\hat{\rho}}\Big)\Big(H_{\hat{\nu}\hat{\rho}ij} + \frac{1}{\sqrt{2}}h_{\hat{\nu}\hat{\rho}ij}\Big)\hat{\epsilon}^j \ . \tag{4.36d}$$

Thus, (4.13) is solved for $\hat{\epsilon}\equiv 0$, $\lambda\equiv 0$ and $\hat{X}^{\hat{\mu}} = (X^\mu(x),0)$, $\hat{\Sigma}^{\hat{a}}{}_{\hat{c}} = \delta^{\hat{a}}{}_a\delta_{\hat{c}}{}^c\,\Sigma^a{}_c(x)$, $\hat{\tau}^I = \tau^I(x)$ and $\hat{\sigma} = \sigma(x)$. There are no fixed relations among the parameters such that this yields four-dimensional diffeomorphisms δ_X, local Lorentz transformations δ_Σ and SU(2)⊗U(1) gauge transformations δ_{τ^I}, δ_σ, respectively, on the boundary fields.

There is another bosonic symmetry transformation which acts non-trivially on the boundary fields. Namely, we consider $\hat{\delta}_{\mathrm{w}}{:=}\delta_{\hat{X}_{\mathrm{w}}}{+}\delta_{\hat{\epsilon}_{\mathrm{w}}}{+}\delta_{\hat{\Sigma}_{\mathrm{w}}}{+}\delta_{\hat{\sigma}_{\mathrm{w}}}{+}\delta_{\hat{\tau}^I_{\mathrm{w}}}$ with nonzero $\hat{X}^r = r\lambda$. The conditions (4.35), (4.36) can be solved by $\hat{\Sigma}^a_{\mathrm{w}\,b} = 0$ and $\hat{\Sigma}^a_{\mathrm{w}\,\underline{r}} = \mathcal{O}(r)$ along with $\hat{X}^\mu_{\mathrm{w}}$, $\hat{\sigma}_{\mathrm{w}}$, $\hat{\tau}^I_{\mathrm{w}}$ of $\mathcal{O}(r^2)$ and $\hat{\epsilon}_{\mathrm{w}i} = \mathcal{O}(r^{3/2})$. All transformations preserve the boundary fields, except for $\delta_{\hat{X}_{\mathrm{w}}}$ which acts as a Weyl rescaling. The Weyl weights of the boundary fields are fixed by the scaling of the bulk fields from which they are defined, e.g. for $\phi := \lim_{r\to 0} r^\alpha\hat{\phi}$ we have $\delta_{\mathrm{w}}\phi := \lim_{r\to 0} r^\alpha\hat{\delta}_{\mathrm{w}}\hat{\phi} = -\alpha\lambda(x)\phi$.

Finally, coming to the fermionic symmetries we set $\lambda\equiv 0$ and consider non-vanishing $\hat{\epsilon}_i$ solving (4.36d). Similarly to the mass terms in the spinor field equations, the T_{ij}-term in (4.36d) affects a splitting of the chiral components when solving the leading order in r. We find the two independent solutions $\hat{\epsilon}_{i_+} = r^{-1/2}\epsilon^{\mathrm{L}}_{i_+} + o(r^{1/2})$ and $\hat{\epsilon}_{i_+} = r^{1/2}\epsilon^{\mathrm{R}}_{i_+} + o(r^{1/2})$ with $\epsilon^{\mathrm{L/R}}_{i_+}|_{r=0}$ finite. $\hat{X}^\mu$, $\hat{\sigma}$ and $\hat{\tau}^I$ of $\mathcal{O}(r^2)$ and $\hat{\Sigma}^a{}_{\underline{r}} = \mathcal{O}(r)$ are fixed by solving the remaining constraints, such that $\delta_{\hat{X},\hat{\Sigma},\hat{\sigma},\hat{\tau}^I}$ transform the subleading modes of the bulk fields only. On the boundary fields we thus have a purely fermionic transformation $\hat{\delta}_{\hat{\epsilon}}$.

We define $\zeta_{i_+}{:=}\,\epsilon^{\mathrm{L}}_{i_+}(x,0)$, $\zeta^{i_+}{:=}\,\epsilon^{\mathrm{R}i_+}(x,0)$, such that ζ^{i_+} is related to ζ_{i_+} by the symplectic Majorana condition, and similarly $\eta_{i_+}{:=}\,\epsilon^{\mathrm{R}}_{i_+}(x,0)$, $\eta^{i_+}{:=}\,\epsilon^{\mathrm{L}i_+}(x,0)$. To leading order in the fermionic fields the ζ-transformations of the boundary fields are

$$\begin{aligned}
\delta_\zeta e^a_\mu &= i\bar{\psi}^{\mathrm{L}i_+}_\mu\gamma^a\zeta_{i_+} + \text{c.c.}\ , \qquad \delta_\zeta\psi^{\mathrm{L}}_{\mu i_+} = D_\mu\zeta_{i_+} - \frac{1}{4}\,\gamma\cdot C^-_{i_+j_+}\gamma_\mu\zeta^{j_+}\ ,\\
\delta_\zeta A^I_\mu &= \frac{1}{\sqrt{2}}i\Big(\bar{\Phi}^{\mathrm{R}i_+}_\mu\zeta_{j_+} - \frac{1}{\sqrt{3}}\bar{\chi}^{\mathrm{L}i_+}\gamma_\mu\zeta_{j_+}\Big)\left(\Gamma^I\right)_{i_+}{}^{j_+} + \text{c.c.}\ ,\\
\delta_\zeta a_\mu &= \frac{1}{2}i\Big(\bar{\Phi}^{\mathrm{R}i_+}_\mu\zeta_{i_+} + \frac{2}{\sqrt{3}}\bar{\chi}^{\mathrm{L}i_+}\gamma_\mu\zeta_{i_+}\Big) + \text{c.c.}\ , \qquad \delta_\zeta\varphi = \frac{1}{\sqrt{2}}i\bar{\chi}^{\mathrm{R}i_+}\zeta_{i_+} + \text{c.c.}\ ,\\
\delta_\zeta\chi^{\mathrm{L}}_{i_+} &= \frac{1}{2\sqrt{6}}\gamma^{\mu\nu}\left(F^I_{\mu\nu}\left(\Gamma_I\right)_{i_+}{}^{j_+} - \sqrt{2}f_{\mu\nu}\delta_{i_+}^{j_+}\right)\zeta_{j_+} - \frac{1}{\sqrt{3}}i\gamma^\mu C_{\mu r\,i_+j_+}\zeta^{j_+} - \frac{1}{\sqrt{2}}i\varphi\zeta_{i_+}\ ,\\
\delta_\zeta C^-_{ab} &= 2i\left(\Gamma_4\right)^{i_+j_+}\left(\bar{\zeta}_{i_+}\hat{R}_{ab\,j_+}(Q) + \frac{1}{4}\eta_{ac}\bar{\psi}^{\mu\mathrm{R}}_{i_+}\gamma^{[\nu}\gamma_{b\mu}\gamma^{c]}\delta_\zeta\psi^{\mathrm{L}}_{\nu j_+}\right)\ ,
\end{aligned} \tag{4.37}$$

	e^a_μ	$\psi^L_{\mu i_+}$	a_μ, A^I_μ	$\chi^L_{i_+}$	$C^-_{\mu\nu}$	φ
w	-1	$-\frac{1}{2}$	0	$\frac{3}{2}$	-1	2
s	2	$\frac{3}{2}$	1	$\frac{1}{2}$	1	0
n	5	-8	3	-4	6	1
c	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0

Table 4.1: Boundary fields with Weyl weights w, spin s and n off-shell degrees of freedom. The fermions are SU(2) doublets and c denotes the U(1) charges.

where $\hat{R}_{\mu\nu\, i_+}(Q) := R_{\mu\nu\, i_+}(Q) - \frac{1}{2\sqrt{3}} i\gamma_{\mu\nu}\chi^{\rm L}_{i_+}$. These correspond to $\mathcal{N}$=2 (Q-)supersymmetry transformations of the boundary fields. The η-transformations are given by

$$\begin{aligned}
&\delta_\eta e^a_\mu = 0\ , \qquad \delta_\eta \psi^{\rm L}_{\mu i_+} = -i\gamma_\mu \eta_{i_+}\ , \qquad \delta_\eta a_\mu = \frac{1}{2} i\bar{\psi}^{{\rm L}i_+}_\mu \eta_{i_+} + \text{c.c.}\ ,\\
&\delta_\eta C^-_{ab} = \frac{1}{2} i\, (\Gamma_4)^{i_+ j_+}\, \eta_{ac}\bar{\psi}^{\mu{\rm R}}_{i_+}\gamma^{[\nu}\gamma_{b\mu}\gamma^{c]}\delta_\eta\psi^{\rm L}_{\nu j_+}\ , \qquad \delta_\eta\varphi = 0\ ,\\
&\delta_\eta \chi^{\rm L}_{i_+} = -\frac{1}{2\sqrt{3}}\,\gamma\cdot C^-_{i_+ j_+}\,\eta^{j_+}\ , \qquad \delta_\eta A^I_\mu = \frac{1}{\sqrt{2}} i\bar{\psi}^{{\rm L}i_+}_\mu \eta_{j_+}\left(\Gamma^I\right)_{i_+}{}^{j_+} + \text{c.c.}\ ,
\end{aligned} \tag{4.38}$$

and correspond to special conformal (S-)supersymmetry or super-Weyl transformations. The constrained field components $\Phi^{\rm R}_{\mu i_+}$, $C^+_{\mu\nu}$ and $C_{\mu r}$ are given by (4.30), (4.32) and (4.33), respectively, and the covariant derivative by (4.29). With $\chi^{\rm R}_{i_+}$ as given in (4.34) the transformation of the scalar field may be rewritten as

$$\delta_\zeta\varphi = \frac{1}{\sqrt{2}}\bar{\zeta}^{i_+}\gamma^\mu\Big(D_\mu - \delta_\zeta(\psi_\mu) - \delta_\eta(\Phi_\mu)\Big)\chi^{\rm L}_{i_+} + \text{c.c.}\ , \tag{4.39}$$

where $\delta_\zeta(\psi_\mu)$ denotes a field-dependent ζ-supersymmetry transformation with parameter $\zeta_{i_+} = \psi^{\rm L}_{\mu i_+}$, and analogously for $\delta_\eta(\Phi_\mu)$ with $\eta_{i_+} = \Phi^{\rm R}_{\mu i_+}$.

The commutators of Q- and S-supersymmetries can be derived from (4.7) and we find

$$[\delta_{\zeta_2}, \delta_{\zeta_1}] = \delta_{X_\zeta} + \delta_\Sigma\big(X^\mu_\zeta\omega_\mu{}^{ab}\big) + \delta_\Sigma\big(2i\bar{\zeta}^{i_+}_1\zeta^{j_+}_2 C^{-\,ab}{}_{i_+j_+} + \text{c.c.}\big) + \delta_{\sigma_\zeta} + \delta_{\tau^I_\zeta}\ , \tag{4.40a}$$

$$[\delta_\eta, \delta_\zeta] = \delta_{\rm Weyl}\big(\bar{\zeta}^{i_+}\eta_{i_+} + \text{c.c.}\big) + \delta_\Sigma\big(-\bar{\zeta}^{i_+}\gamma^{ab}\eta_{i_+} + \text{c.c.}\big) + \delta_{\sigma_{\eta\zeta}} + \delta_{\tau^I_{\eta\zeta}}\ , \tag{4.40b}$$

$$[\delta_{\eta_2}, \delta_{\eta_1}] = 0\ , \tag{4.40c}$$

where in (4.40a) the diffeomorphism is $X^\mu_\zeta = -i\bar{\zeta}^{i_+}_1\gamma^\mu\zeta_{2i_+}$+c.c. and the gauge transformations are $\sigma_\zeta = X^\mu_\zeta a_\mu$, $\tau^I_\zeta = X^\mu_\zeta A^I_\mu$. The gauge transformations in (4.40b) are $\sigma_{\eta\zeta} = \frac{1}{2}i\bar{\zeta}^{i_+}\eta_{i_+}$+ c.c. and $\tau^I_{\eta\zeta} = \frac{1}{\sqrt{2}}i\,(\Gamma_I)_{i_+}{}^{j_+}\,\bar{\zeta}^{i_+}\eta_{j_+}$+ c.c..

Altogether, we find the boundary degrees of freedom with properties as given in Tab. 4.1 and with the fermionic symmetry transformations (4.37), (4.38). The off-shell degrees of freedom are given as the difference of field components and gauge degrees of freedom, e.g. for the chiral gravitino we count 16 components from which $2\cdot 4$ degrees of freedom are removed for the chiral ζ and η supersymmetry transformations. Likewise, of the 16 vielbein components 4 degrees of freedom are subtracted for diffeomorphisms, 6 for local Lorentz and 1 for Weyl transformations. As seen from Tab. 4.1, the total numbers of bosonic and fermionic degrees of freedom, both being 24, match nicely, and the boundary fields fill the $\mathcal{N}$=2 Weyl multiplet, see [119, 38]. The bulk SU(2)⊗U(1) gauge symmetry has become the chiral U(2) transformations contained in SU(2, 2|2) to close the commutator of Q- and S-supersymmetries.

4.1.3 Holographic Weyl anomaly

Building on the previous results we use in this section AdS/CFT to study holographically the dual CFT in external supergravity backgrounds. As noted in the introduction, solutions of Romans' theory can be lifted to the ten-dimensional IIA/B supergravities and to the maximal d=11 supergravity. In particular, the AdS_5 vacuum lifts to $AdS_5\times S^5$ in IIB supergravity [111] and to a solution describing the near-horizon limit of a semi-localized system of two sets of M5-branes in M-theory [112]. The latter solution can be understood as uplift of a solution in the IIA theory describing an elliptic brane system with D4 and NS5 branes [120]. Thus, the fluctuations around AdS are understood as a dual description of a subsector of $\mathcal{N}$=4 SYM theory via the lift to IIB supergravity, and as dual to the $\mathcal{N}$=2 SCFTs on the M5-brane intersection and on the D4-branes via the lifts to M-theory and IIA supergravity, respectively.

An important result in AdS/CFT is that the appearance of a Weyl anomaly in the SCFT in an external supergravity background can be understood holographically as follows [71, 72, 121, 73]. In the limit where string theory is appropriately described by supergravity, the generating functional of the SCFT correlation functions in the conformal supergravity background $g_{\mu\nu},\dots$ with sources $\delta g_{\mu\nu},\dots$ is related to the path integral of the dual supergravity as a functional of the boundary conditions by [122]

$$\int\limits_{[t_-,t_+]} \mathcal{D}\hat{g}\Big|_{r^2\hat{g}|_{\partial X}=g+\delta g} e^{iS_{\text{sugra}}[\hat{g}]}\,\langle\hat{\beta}|\hat{g},t_+\rangle\langle\hat{g},t_-|\hat{\alpha}\rangle \;=\; \langle\beta|Te^{i\int_{\partial X}\frac{1}{2}\delta g^{\mu\nu}T_{\mu\nu}}|\alpha\rangle_{\text{SCFT}}\;. \tag{4.41}$$

The remaining supergravity fields and boundary conditions on the left hand side and the remaining SCFT operators and sources on the right hand side are implicit. In the limit where the bulk supergravity becomes classical, the path integral reduces to the integrand evaluated on the solution of the classical field equations[5].

The on-shell supergravity action, however, is divergent and has to be regularized, e.g. by introducing an IR cutoff on the radial coordinate $r\geq\epsilon$. This reflects the need for regularization of UV divergences on the CFT side. The renormalized supergravity action

$$S^{\text{ren}}_{\text{sugra}}=\lim_{\epsilon\to0}\left(S_{\text{sugra},\epsilon}+S_{\text{GHY}}+S_{\text{ct}}+S^{\log}_{\text{ct}}\right)\,,\qquad S_{\text{sugra},\epsilon}=\int_{r\geq\epsilon}d^5x\,\mathcal{L}_{\text{sugra}}\,, \tag{4.42}$$

is then constructed from the regularized action $S_{\text{sugra},\epsilon}$, the Gibbons-Hawking-York term S_{GHY} for a well-defined variational principle and the counterterm action to render the limit

[5] Which involve also boundary conditions at $t_\pm$. However, for the calculation of the anomaly only the boundary conditions on ∂X are relevant, since it does not depend on the choice of SCFT state.

$\epsilon \to 0$ finite. It turns out that the $1/\epsilon^k$ divergences can be removed by adding (bulk-)covariant boundary terms S_{ct}, constructed e.g. from the induced metric. In odd dimensions, however, there is also a $\log\epsilon$ divergence, the counterterm for which explicitly depends on (the coordinate of) the cutoff ϵ. Due to this explicit cutoff dependence the renormalization breaks invariance under those bulk diffeomorphisms that act as Weyl rescalings on the boundary. Applying such a diffeomorphism inducing a Weyl transformation on the boundary, as discussed in Sec. 4.1.2.3, yields the anomalous boundary Ward identity corresponding to Weyl invariance.

The anomalous trace of the energy-momentum tensor in pure-metric backgrounds has been calculated in [71] from pure gravity in the bulk. The extension to dilaton gravity can be found in [115] and higher-order curvature terms in the bulk arising from higher orders in the effective string-theory action are discussed in [116]. We now study the Weyl anomaly of the SCFTs dual to Romans' theory in generic bosonic $\mathcal{N}$=2 conformal supergravity backgrounds. To this end we truncate the five-dimensional $\mathcal{N}$=4 supergravity to its bosonic sector, which is consistent because the fermionic field equations are solved trivially by $\hat{\psi}_{\hat{\mu}i} \equiv \hat{\chi}_i \equiv 0$.

The bosonic part of the $\mathcal{N}$=2 Weyl multiplet of boundary fields as determined in the previous section is given by $\left(e^a_\mu, A^I_\mu, a_\mu, C^-_{\mu\nu}, \varphi\right)$, and we label the bosonic part of the dual multiplet of SCFT currents by $\left(T^a_\mu, J^I_\mu, j_\mu, L_{\mu\nu}, \phi\right)$. T^a_μ, J^I_μ and j_μ are the classically conserved currents[6] and $L_{\mu\nu}$, ϕ complete the bosonic part of the supermultiplet. Equation (4.41) then yields

$$\delta S^{\text{ren}}_{\text{sugra}} = \int_{r=0} d^4x\, e\,\Big(\delta e^\mu_a\,\langle T^a_\mu\rangle + \delta a_\mu\,\langle j^\mu\rangle + \delta A^I_\mu\,\langle J^{\mu I}\rangle + \delta C^-_{\mu\nu}\,\langle L^{\mu\nu}\rangle + \delta\varphi\,\langle\phi\rangle\Big)\ . \tag{4.43}$$

We choose the variations of the boundary conditions such that they correspond to a Weyl transformation, $\delta e^a_\mu = -\lambda e^a_\mu$ and likewise for the remaining fields. Extending them into the bulk to a diffeomorphism as discussed in Sec. 4.1.2.3, generated by the vector field $(X^\mu, \lambda r)$ with $\partial_r X^\mu = r g^{\mu\rho}\partial_\rho\lambda$ and $X^\mu|_{r=0} = 0$, yields the anomalous Ward identity

$$\langle T_\mu{}^\mu\rangle - C^-_{\mu\nu}\langle L^{\mu\nu}\rangle + 2\varphi\langle\phi\rangle = \mathcal{A}\ , \qquad \mathcal{A} := \lim_{\epsilon\to 0}\frac{1}{e}\frac{\delta}{\delta\lambda}S^{\log}_{\text{ct}}\ . \tag{4.44}$$

In the remaining part of this section we will determine $\mathcal{A}$. To this end we have to solve the nonlinear field equations of the various fields as asymptotic series in a vicinity of the boundary, which then allows us to determine the divergences of the on-shell action and the required counterterms.

4.1.3.1 On-shell bulk fields as asymptotic series

We now determine the required subleading modes of the bulk fields from their field equations. For the matter fields the equations have been given in Sec. 4.1.2.1 and Einstein's equations for the bosonic sector read[7]

$$\begin{aligned}\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}(\hat{\omega}) = {}&\frac{4}{3}P(\hat{\varphi})\hat{g}_{\hat{\mu}\hat{\nu}} + 2\hat{D}_{\hat{\mu}}\hat{\varphi}\hat{D}_{\hat{\nu}}\hat{\varphi} - \xi^{-4}\big(2\hat{f}_{\hat{\mu}\hat{\rho}}\hat{f}_{\hat{\nu}}{}^{\hat{\rho}} - \frac{1}{3}\hat{g}_{\hat{\mu}\hat{\nu}}\hat{f}^{\hat{\rho}\hat{\sigma}}\hat{f}_{\hat{\rho}\hat{\sigma}}\big)\\ &- \xi^2\Big(2\hat{B}^{\ \alpha}_{\hat{\mu}\hat{\rho}}\hat{B}_{\hat{\nu}}{}^{\hat{\rho}\alpha} + 2\hat{F}^{\ I}_{\hat{\mu}\hat{\rho}}\hat{F}_{\hat{\nu}}{}^{\hat{\rho}I} - \frac{1}{3}\hat{g}_{\hat{\mu}\hat{\nu}}\big(\hat{B}^{\hat{\rho}\hat{\sigma}\alpha}\hat{B}^{\ \alpha}_{\hat{\rho}\hat{\sigma}} + \hat{F}^{\hat{\rho}\hat{\sigma}I}\hat{F}^{\ I}_{\hat{\rho}\hat{\sigma}}\big)\Big)\ .\end{aligned} \tag{4.45}$$

[6] The dual theory has SU(2)⊗U(1) R-symmetry.

[7] Our conventions are $\hat{\mathcal{R}}_{\hat{\mu}}{}^{\hat{a}}(\hat{\omega}) = \hat{e}^{\hat{\nu}}_{\hat{b}}\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}(\hat{\omega})$ and $\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}(\hat{\omega}) = \hat{e}_{\hat{\nu}\hat{b}}\hat{\mathcal{R}}_{\hat{\mu}}{}^{\hat{a}}(\hat{\omega})$.

The coupled system of equations can be solved order by order in an expansion around the asymptotic boundary. The leading order has been discussed in Sec. 4.1.2.1 and 4.1.2.2. To consistently solve the Dirichlet problem $\log r$ terms have to be included in the expansions, which yields the asymptotic forms

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + r^2 g^{(2)}_{\mu\nu} + r^3 g^{(3)}_{\mu\nu} + r^4 (\log r)^2 h^{(0)}_{\mu\nu} + r^4 \log r\, h^{(1)}_{\mu\nu} + r^4 \tilde{g}^{(4)}_{\mu\nu} + o(r^4) \ , \\ \hat{a}_\mu &= a^{(0)}_\mu + o(r) \ , \\ \hat{A}^I_\mu &= A^{I(0)}_\mu + o(r) \ , \\ \hat{C}_{\mu\nu} &= r^{-1} C^{-(0)}_{\mu\nu} + r \log r\, C^{+(1)}_{\mu\nu} + r\, \tilde{C}^{(2)}_{\mu\nu} + o(r) \ , \quad \hat{C}_{\mu r} = C^{(0)}_{\mu r} + \mathcal{O}(r^2 \log r) \ , \\ \hat{\varphi} &= r^2 \log r\, \varphi^{(0)} + r^2 \tilde{\varphi}^{(1)} + o(r^2) \ . \end{aligned} \tag{4.46}$$

The leading mode $C^{-(0)}_{\mu\nu}$ of the tensor field is anti-selfdual and the $r \log r$ term $C^{+(1)}_{\mu\nu}$ selfdual. Note the additional $r^4 (\log r)^2$ term in the metric expansion as compared to the pure-gravity case. This is necessary due to the $\hat{\varphi}^2$ term in (4.45). As result of the additional log-terms and the fact that $h^{(1)}_{\mu\nu}$ is not traceless, as will be seen below, the bulk-covariant counterterms canceling the $1/\epsilon^k$-divergences do contribute additional log-divergences, in contrast to the pure-gravity case, and we have to determine them first. The 2^{nd}-order field equations fix the bulk fields in terms of two sets of boundary data. Namely, the bulk metric $\hat{g}$ is fixed from the boundary metric $g^{(0)}$ and the traceless and divergence-free part of $\tilde{g}^{(4)}$, the two-form field $\hat{C}$ is determined by specifying the anti-selfdual boundary field $C^{-(0)}$ and the selfdual part of $\tilde{C}^{(2)}$, and the boundary data for $\hat{\varphi}$ is given by $\varphi^{(0)}$ and $\tilde{\varphi}^{(1)}$. Thus, only the leading modes of the on-shell bulk fields are fixed in terms of the boundary fields alone. The second set of boundary data is linked to the choice of SCFT states in (4.41).

To determine $g^{(2)}_{\mu\nu}$ we need the $\mu\nu$-components of the Ricci tensor for the metric (4.11). With a prime denoting differentiation with respect to r and $R_{\mu\nu}(\omega)$ being the curvature of the four-dimensional spin connection $\omega_{\mu ab}$ they read

$$\hat{R}_{\mu\nu}(\hat{\omega}) = R_{\mu\nu}(\omega) + \frac{4}{r^2} g_{\mu\nu} - \frac{3}{2r} g'_{\mu\nu} + \frac{\operatorname{tr} g^{-1} g'}{4} \Big(g'_{\mu\nu} - \frac{2}{r} g_{\mu\nu} \Big) + \frac{1}{2} g''_{\mu\nu} - \frac{1}{2} g'_{\mu\rho} g^{\rho\sigma} g'_{\sigma\nu} \ . \tag{4.47}$$

Solving the $\mu\nu$-components of (4.45) at $\mathcal{O}(r^{-1})$ shows that there is no contribution to $g_{\mu\nu}(x,r)$ linear in r. Solving at $\mathcal{O}(r^0)$ shows

$$g^{(2)}_{\mu\nu} = \frac{1}{2} \Big(\mathcal{R}^{(0)}_{\mu\nu}(\omega) - \frac{1}{6} \mathcal{R}^{(0)}(\omega) g^{(0)}_{\mu\nu} + 4\, \overline{C^{-(0)}_{\mu\rho}}\, C^{-(0)}{}_\nu{}^\rho \Big) \ . \tag{4.48}$$

Note that the last term is real due to the anti-selfduality of $C^{-(0)}_{\mu\nu}$. For the gauge fields we find from (4.24) and (4.25) that the first subleading modes are $o(r)$. Equation (4.22) yields

$$C^{(0)}_{\mu r} = \frac{1}{2} i e^{(0)-1} \epsilon_\mu{}^{\rho\sigma\tau} D_\rho C^{-(0)}_{\sigma\tau} \ , \qquad C^{+(1)}_{\mu\nu} = \big(\mathbb{1} + i \star^{(0)} \big) \big(g^{(2)\,\rho}_{[\mu} C^{-(0)}_{\nu]\rho} - D_{[\mu} C^{(0)}_{\nu]r} \big) \ . \tag{4.49}$$

For the on-shell action we also need the expansion of the vielbein determinant

$$e = e^{(0)} \Big[1 + \frac{r^2}{2} t^{(2)} + \frac{r^4}{2} (\log r)^2 u^{(0)} + \frac{r^4}{2} \log r\, u^{(1)} + \frac{r^4}{2} \big(t^{(4)} + \frac{1}{4} (t^{(2)})^2 - \frac{1}{2} t^{(2,2)} \big) \Big] + o(r^4) \ , \tag{4.50}$$

where $t^{(n)} := \operatorname{tr} g^{(0)\,-1} g^{(n)}$, $u^{(n)} := \operatorname{tr} g^{(0)-1} h^{(n)}$ and $t^{(2,2)} := \operatorname{tr} g^{(0)\,-1} g^{(2)} g^{(0)\,-1} g^{(2)}$. These traces can be determined from the rr-components of (4.45) with

$$\hat{\mathcal{R}}_{rr}(\hat{\omega}) = -\frac{4}{r^2} + \frac{1}{2r}\operatorname{tr} g^{-1}g' - \frac{1}{2}\operatorname{tr} g^{-1}g'' + \frac{1}{4}\operatorname{tr} g^{-1}g'g^{-1}g' \ . \tag{4.51}$$

For notational convenience we define $\hat{\overline{C}}_{\rho\sigma}\hat{C}^{\rho\sigma} =: r^4 \log r\, c^{(0)} + r^4 c^{(1)} + o(r^4)$. The leading term, which would be $\mathcal{O}(r^2)$, vanishes due to the anti-selfduality of $C^{-(0)}_{\rho\sigma}$. With $b^{(0)} := r^{-2}\hat{B}_{r\rho}{}^{\alpha}\hat{B}_r{}^{\rho\alpha}|_{r=0}$ we find $t^{(3)} = 0$ and

$$\begin{aligned} u^{(0)} &= -\frac{4}{3}\varphi^{(0)2} \ , \qquad u^{(1)} = -\frac{8}{3}\varphi^{(0)}\tilde{\varphi}^{(1)} + \frac{1}{6}c^{(0)} \ , \\ t^{(2,2)} - 4t^{(4)} &= \frac{16}{3}\tilde{\varphi}^{(1)2} - \frac{1}{2}u^{(0)} - \frac{1}{3}F^{(0)\rho\sigma I}F^{(0)I}_{\rho\sigma} - \frac{1}{3}f^{(0)\rho\sigma}f^{(0)}_{\rho\sigma} - \frac{4}{3}b^{(0)} - \frac{2}{3}c^{(1)} + \frac{1}{2}c^{(0)} \ . \end{aligned} \tag{4.52}$$

Note the dependences on $c^{(1)}$ and $\tilde{\varphi}^{(1)}$ which are not fixed by the near-boundary analysis.

4.1.3.2 Holographic renormalization

Having calculated the necessary terms in the asymptotic expansions of the bulk fields we now determine the divergences of the on-shell action and the necessary counterterms. Using (4.45) and (4.22) the Lagrangian (4.1) truncated to the bosonic sector reads

$$\mathcal{L}_{\text{on-shell}} = -2\hat{e} - \frac{4}{3}\hat{e}\hat{\varphi}^2 + \frac{1}{12}\hat{e}\xi^2\hat{B}^{\hat{\mu}\hat{\nu}\alpha}\hat{B}^{\ \alpha}_{\hat{\mu}\hat{\nu}} - \frac{1}{6}\hat{e}\big(\hat{F}^{\hat{\mu}\hat{\nu}I}\hat{F}^{\ I}_{\hat{\mu}\hat{\nu}} + \hat{f}^{\hat{\mu}\hat{\nu}}\hat{f}_{\hat{\mu}\hat{\nu}}\big) + \mathcal{O}(r^0) \ . \tag{4.53}$$

Naïvely, one may expect terms of order $r^{-1}(\log r)^2$ and $r^{-1}\log r$ in $\mathcal{L}_{\text{on-shell}}$, e.g. due to the scalar and tensor field terms. This potentially leads to $(\log\epsilon)^3$ and $(\log\epsilon)^2$ divergences in the on-shell action. However, it turns out that the contributions from $\hat{e}$ to these terms cancel the others, such that only terms proportional to r^{-n} with $n = 5, 3, 1$ and $\mathcal{O}(r^0)$ are non-vanishing in $\mathcal{L}_{\text{on-shell}}$. As may be verified with the expansions of the previous section, $S_{\text{sugra},\epsilon} + S_{\text{GHY}} + S_{\text{ct}}$ with

$$S_{\text{GHY}} = \frac{1}{2}\int\limits_{r=\epsilon} d^4x\,\hat{e}^*\hat{K} \ , \qquad S_{\text{ct}} = \int\limits_{r=\epsilon} d^4x\,\hat{e}^*\Big(-\frac{3}{2} + \frac{1}{8}\mathcal{R}^*(\omega) - \hat{\varphi}^2 + \alpha\,\hat{\overline{C}}^{*\mu\nu}\hat{C}^*_{\mu\nu}\Big) \ , \tag{4.54}$$

only has a logarithmic divergence in the limit $\epsilon \to 0$, i.e. all $1/\epsilon^k$ divergences are cancelled. The $*$ denotes induced quantities on the boundary, e.g. the pullback of the vielbein and the two-form field $\hat{C}$, and indices are contracted with the induced vielbein and metric. $\hat{K} := \hat{e}^{\hat{\mu}}_{\hat{a}}\hat{K}_{\hat{\mu}}{}^{\hat{a}}$ is the trace of the extrinsic curvature of the boundary[8], $\hat{K}_{\hat{\mu}}{}^{\hat{a}} = \hat{\omega}_{\hat{\mu}}{}^{\hat{a}\underline{r}}$. We note that also S_{GHY} and S_{ct} separately only have power-law and $\log\epsilon$ divergences, e.g. the $\hat{\varphi}^2$ term in S_{ct} cancels the $(\log\epsilon)^2$ divergence in the cosmological-constant term $-\frac{3}{2}\hat{e}^*$. Thus, the only remaining divergence is $\log\epsilon$, which is consistent with the expectation that the Weyl anomaly of the dual theory is exhausted at one-loop[9]. A slight subtlety arises for the

[8] The extrinsic curvature $\hat{K}_{\hat{\mu}\hat{\nu}}$ is defined below (2.9) in Sec. 2.2. With the outward-pointing unit normal vector field $\hat{n}^{\hat{\mu}} = \hat{e}^{\hat{\mu}\underline{r}}$ and the vielbein postulate this yields $\hat{K}_{\hat{\mu}}{}^{\hat{a}} = \hat{\omega}_{\hat{\mu}}{}^{\hat{a}\underline{r}}$.

[9] It shares a multiplet with the chiral anomaly which receives no contributions beyond one-loop order.

$\hat{e}^*\hat{\overline{C}}^{*\mu\nu}\hat{C}^*_{\mu\nu}$ term in $S_{\rm ct}$. Since the leading term vanishes on-shell thanks to the anti-selfduality of $C^{-(0)}_{\mu\nu}$, it only contributes a logarithmic divergence. However, as it does not explicitly depend on the cutoff and therefore does not contribute to the Weyl anomaly we include it in $S_{\rm ct}$ with a for now arbitrary numerical coefficient α.

The remaining counterterm required to cancel the $\log\epsilon$ divergence depends on α and is given by

$$\begin{aligned} S^{\log}_{\rm ct} = \int_{r=\epsilon} d^4x\, \hat{e}^* \bigg(& \frac{1}{16}\Big(\mathscr{R}_{\mu\nu}\mathscr{R}^{\mu\nu} - \frac{1}{3}\mathscr{R}^2\Big)\log\epsilon - \frac{1}{4}\big(\hat{F}^{*I}_{\mu\nu}\hat{F}^{*\mu\nu I} + \hat{f}^*_{\mu\nu}\hat{f}^{*\mu\nu}\big)\log\epsilon \\ & - \frac{1}{2}\hat{\varphi}^2(\log\epsilon)^{-1} - \big(D_a\hat{\overline{C}}{}^*{}_b{}^a\big)\big(D_c\hat{C}^{*bc}\big)\log\epsilon + (1-4\alpha)\mathcal{B}\log\epsilon\bigg)\ , \end{aligned} \tag{4.55}$$

where we defined the modified curvature $\mathscr{R}_{\mu\nu} := \mathcal{R}^*_{\mu\nu}(\omega) + 4\,\hat{\overline{C}}{}^*_{\mu\rho}\,\hat{C}^*{}_\nu{}^\rho$ and D_a is the covariant derivative with the four-dimensional spin connection $\omega_{\mu ab}$. The dependence on α is seen in the last term, where

$$\mathcal{B} = \big(D_a\hat{\overline{C}}{}^*{}_b{}^a\big)\big(D_c\hat{C}^{*bc}\big) - \frac{1}{2}\mathscr{R}^{\mu\nu}\,\hat{\overline{C}}{}^*_{\mu\rho}\,\hat{C}^*{}_\nu{}^\rho - \frac{1}{2}D_aD_b\big(\hat{\overline{C}}{}^{*ac}\hat{C}^{*b}{}_c\big)\ . \tag{4.56}$$

Clearly, the choice of α will affect the Weyl anomaly, so it has to be fixed. As the renormalized bulk action should yield finite correlation functions for the boundary theory, we calculate the one-point function of the energy-momentum tensor of the dual Yang-Mills theory. According to (4.43) it is given by

$$\langle T^a_\mu\rangle \;=\; \frac{1}{e^{(0)}}\frac{\delta S^{\rm ren}_{\rm sugra}}{\delta e^{\mu(0)}_a} \;=\; \lim_{\epsilon\to 0}\epsilon^{-3}\frac{1}{\hat{e}^*}\frac{\delta S^{\rm ren}_{{\rm sugra},\epsilon}}{\delta\hat{e}^{*\mu}_a} \;=:\; \lim_{\epsilon\to 0}\epsilon^{-3}\,\mathcal{T}^a_\mu, \tag{4.57}$$

where $S^{\rm ren}_{{\rm sugra},\epsilon}$ is the action defined in (4.42) before taking the limit $\epsilon\to 0$. $\mathcal{T}^a_\mu$ is the Brown-York quasilocal energy-momentum tensor [123] of the bulk supergravity with regularization $r\geq\epsilon$ and supplemented by the counterterms. We find

$$\begin{aligned} \mathcal{T}^a_\mu \;=\; & \frac{1}{2}\big(\hat{K}_\mu{}^a - \hat{e}^{*a}_\mu\hat{K}\big) + \frac{3}{2}\hat{e}^{*a}_\mu + \frac{1}{4}\big(\mathcal{R}^{*a}_\mu(\omega) - \frac{1}{2}\hat{e}^{*a}_\mu\mathcal{R}^*(\omega)\big) \\ & + 2\alpha\big(\hat{\overline{C}}{}^*_{\mu\nu}\hat{C}^{*a\nu} + {\rm c.c.}\big) - \alpha\hat{e}^{*a}_\mu\,\hat{\overline{C}}{}^*_{\nu\rho}\hat{C}^{*\nu\rho} + \frac{1}{e^*}\frac{\delta S^{\log}_{\rm ct}}{\delta\hat{e}^{*\mu}_a}\ . \end{aligned} \tag{4.58}$$

Inserting the on-shell expansion of the fields as obtained in Sec. 4.1.3.1, the leading part of $\hat{\overline{C}}{}^*_{\mu\nu}\hat{C}^{*a\nu}$ does not vanish and contributes at $\mathcal{O}(\epsilon)$. Demanding a finite limit in (4.57) then fixes $\alpha = \frac{1}{4}$. Similarly, finiteness of $\langle L^{\mu\nu}\rangle$ also requires this choice of α. The reason why finiteness of the one-point functions requires a fixed α while finiteness of the on-shell action does not can be seen as follows. The vanishing of the leading order of the counterterm $\hat{\overline{C}}{}^{*\mu\nu}\hat{C}^*_{\mu\nu}$ due to the anti-selfduality of $C^{-(0)}_{\mu\nu}$ relies on the contraction of the two-form fields with the metric. Therefore, finiteness of the action evaluated on solutions of the classical field equations does not guarantee finiteness of the variations with respect to the metric or the two-form field evaluated on classical solutions.

Finally, we obtain the anomalous contribution to the Ward identity (4.44) from the variation of (4.55) for $\alpha = \frac{1}{4}$ and find, with $\mathscr{R}^{(0)}_{\mu\nu} = \mathcal{R}^{(0)}_{\mu\nu}(\omega) + 4\,\overline{C^{-(0)}_{\mu\rho}}\,C^{-(0)}{}_{\nu}{}^{\rho}$,

$$\begin{aligned}\mathcal{A} = &-\frac{1}{16}\Big(\mathscr{R}^{(0)}_{\mu\nu}\mathscr{R}^{(0)\mu\nu} - \frac{1}{3}\mathscr{R}^{(0)2}\Big) + \overline{D_a C^{-(0)}{}_b{}^a}\,D_c C^{-(0)bc}\\ &-\frac{1}{2}\varphi^{(0)2} + \frac{1}{4}\Big(F^{(0)I}_{\mu\nu}F^{(0)\mu\nu I} + f^{(0)}_{\mu\nu}f^{(0)\mu\nu}\Big)\,.\end{aligned} \tag{4.59}$$

The curvature-squared part of the first term yields the difference of the squared Weyl tensor and the four-dimensional Euler density, and the mixed terms complete the kinetic term of the two-form field to its Weyl-invariant form, compare (2.15). Note that the anomaly depends on the boundary fields only, i.e. the dependences on $\tilde{\varphi}^{(1)}$ and $c^{(1)}$, which are not fixed by the near-boundary analysis, have dropped out. This is to be expected as the anomaly is a UV effect in the dual theory. From the dual Yang-Mills theory point of view, the Weyl anomaly of $\mathcal{N}$=4 SYM theory should be given by the Lagrangian of $\mathcal{N}$=4 conformal supergravity [106]. As noted before, the bulk theory discussed here provides a holographic description of a subsector of that theory and thus the Weyl anomaly should correspond to a subsector of the $\mathcal{N}$=4 conformal supergravity Lagrangian. Comparing the holographic Weyl anomaly (4.59) to the construction of four-dimensional extended conformal supergravity in [124], it indeed matches the bosonic part of the $\mathcal{N}$=2 conformal supergravity Lagrangian (5.18) of [124]. Thus, our result gives further support to the AdS/CFT conjecture.

4.2 $\mathcal{N}$=2 Gauged Supergravity and the Weyl Anomaly of $\mathcal{N}$=1 SCFT

In the previous section we have determined the asymptotic structure of $\mathcal{N}$=4 gauged supergravity and calculated the Weyl anomaly of the dual $\mathcal{N}$=2 SCFTs in bosonic conformal supergravity backgrounds. It was sufficient for that purpose to carry out the holographic renormalization for the bosonic sector only. Having in mind that the counterterms will be crucial to render the Neumann modes normalizable we now extend this result to also include the fermionic parts. To keep the calculations at a reasonable level we discuss in detail a subsector of the theory which is $\mathcal{N}$=2 U(1) gauged supergravity, and calculate the full Weyl anomaly of the dual $\mathcal{N}$=1 SCFTs in generic, possibly fermionic backgrounds. The truncation to that subsector is consistent in the sense that any solution to the field equations of the truncated theory can be lifted to a solution of the full theory. In particular, solutions lift to the previously discussed string-theory/M-theory backgrounds via the lift to the $\mathcal{N}$=4 theory. On the other hand, more general classes of compactifications yield pure $\mathcal{N}$=2 gauged supergravity as a consistent reduction, see e.g. [125].

In Sec. 4.2.1 we construct the $\mathcal{N}$=2 U(1) gauged supergravity as a consistent truncation. We discuss the asymptotic structure in Sec. 4.2.2 and calculate the Weyl anomaly of the dual SCFTs in generic, possibly fermionic conformal supergravity backgrounds.

4.2.1 Consistent truncation of $\mathcal{N}$=4 SU(2)⊗U(1) gauged supergravity

We derive the minimal gauged supergravity as consistent truncation of the $\mathcal{N}$=4 gauged supergravity discussed in Sec. 4.1.1. Choosing a $\mathfrak{usp}(4)$ basis where Γ_1 is diagonal[10], we split $i=(i^+,i^-)$ such that $(\Gamma_1)_i{}^j=\lambda_i\delta_i{}^j$ with $\lambda_{i^\pm}=\pm 1$. The symplectic metric Ω commutes with Γ_1 such that $\Omega^{i^+j^-}=0$. Thus, the symplectic Majorana condition relates the i^+ components of a spinor $\hat\epsilon_i$ to each other and likewise the i^- components. Of the $\mathcal{N}$=4 supergravity multiplet we keep only $\hat e_{\hat\mu}^{\hat a}$, $\hat\psi_{\hat\mu i^+}$ and $\hat{\mathcal{A}}_{\hat\mu}:=\sqrt{\tfrac{3}{8}}\big(\hat A^1_{\hat\mu}+\sqrt{2}\hat a_{\hat\mu}\big)$, i.e. we set

$$\hat\psi_{\hat\mu i^-}\equiv 0,\quad \hat\chi_i\equiv 0,\quad \hat A^1_{\hat\mu}-\sqrt{2}\hat a_{\hat\mu}\equiv \hat A^2_{\hat\mu}\equiv \hat A^3_{\hat\mu}\equiv \hat C_{\hat\mu\hat\nu}\equiv\hat\varphi\equiv 0\ . \tag{4.60}$$

This ansatz solves the gravitino field equation (4.18) for $i=i^-$, and similarly the equations (4.20) for the spin-$\frac{1}{2}$ fields $\hat\chi_i$, (4.22) for the tensor field $\hat C_{\hat\mu\hat\nu}$ and (4.26) for the scalar $\hat\varphi$. Of the equations for the gauge fields, (4.24) and (4.25), the ansatz (4.60) solves $(4.25)|_{I=2,3}$ and the combination $(4.25)|_{I=1}-\sqrt{2}(4.24)$. The remaining equations are the field equations for $\hat e_{\hat\mu}^{\hat a}$, $\hat\psi_{\hat\mu i^+}$ and $\hat{\mathcal{A}}_{\hat\mu}$. Thus, the reduction (4.60) is consistent and solutions of the reduced theory can be lifted to the $\mathcal{N}$=4 theory. Setting $\hat\epsilon_{i^-}\equiv 0$ in the supersymmetry transformations (4.6), the remaining transformations parametrized by $\hat\epsilon_{i^+}$ close on $\hat e_{\hat\mu}^{\hat a}$, $\hat\psi_{\hat\mu i^+}$ and $\hat{\mathcal{A}}_{\hat\mu}$, and they preserve (4.60). The reduced theory therefore has $\mathcal{N}$=2 supersymmetry.

We use the redefined gauge coupling $g':=\sqrt{\tfrac{3}{8}}g$ and derive the Lagrangian and transformation rules from (4.1) and (4.6). Dropping the superscript + on indices i^+ and the prime on g' we find the following $\mathcal{N}$=2 gauged supergravity. The covariant derivative reads

$$\hat D_{\hat\mu}\hat\psi_{\hat\nu i}=\hat\nabla_{\hat\mu}\hat\psi_{\hat\nu i}+g\hat{\mathcal{A}}_{\hat\mu}(\Gamma_{45})_i{}^j\hat\psi_{\hat\nu i}\ , \tag{4.61}$$

where $\hat\nabla_{\hat\mu}=\partial_{\hat\mu}+\frac{1}{4}\hat\omega_{\hat\mu}^{\hat a\hat b}\hat\gamma_{\hat a\hat b}$, and the curvatures are defined by

$$\big[\hat D_{\hat\mu},\hat D_{\hat\nu}\big]\hat\epsilon_i=:\frac{1}{4}\hat{\mathcal{R}}_{\hat\mu\hat\nu}{}^{\hat a\hat b}(\hat\omega)\,\hat\gamma_{\hat a\hat b}\,\hat\epsilon_i+g\,\hat{\mathcal{F}}_{\hat\mu\hat\nu}\,(\Gamma_{45})_i{}^j\,\hat\epsilon_j\ . \tag{4.62}$$

The Lagrangian obtained from the $\mathcal{N}$=4 theory reads

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}\hat e\hat{\mathcal{R}}(\hat\omega)+g^2\hat e-\frac{1}{2}i\hat e\,\bar{\hat\psi}{}^i_{\hat\mu}\hat\gamma^{\hat\mu\hat\nu\hat\rho}\hat D_{\hat\nu}\hat\psi_{\hat\rho i}+\frac{3}{2}i\hat e\,T_{ij}\bar{\hat\psi}{}^i_{\hat\mu}\hat\gamma^{\hat\mu\hat\nu}\hat\psi^j_{\hat\nu}-\frac{1}{4}\hat e\,\hat{\mathcal{F}}^{\hat\mu\hat\nu}\hat{\mathcal{F}}_{\hat\mu\hat\nu}\\&-\frac{1}{6\sqrt{3}}\hat\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\hat{\mathcal{F}}_{\hat\mu\hat\nu}\hat{\mathcal{F}}_{\hat\rho\hat\sigma}\hat{\mathcal{A}}_{\hat\tau}-\frac{\sqrt{3}}{8}i\hat e\hat{\mathcal{F}}^{\hat\mu\hat\nu}\bar{\hat\psi}{}^i_{\hat\rho}\hat\gamma^{[\hat\rho}\hat\gamma_{\hat\mu\hat\nu}\hat\gamma^{\hat\sigma]}\hat\psi_{\hat\sigma i}\ ,\end{aligned} \tag{4.63}$$

with $T^{ij}=\frac{g}{2\sqrt{3}}(\Gamma_{45})^{ij}$. The supersymmetry transformations are

$$\begin{aligned}&\delta_{\hat\epsilon}\hat e_{\hat\mu}^{\hat a}=i\bar{\hat\psi}{}^i_{\hat\mu}\hat\gamma^{\hat a}\hat\varepsilon_i\ ,\qquad \delta_{\hat\epsilon}\hat{\mathcal{A}}_{\hat\mu}=\frac{1}{2}\sqrt{3}\,i\bar{\hat\psi}{}^i_{\hat\mu}\hat\varepsilon_i\ ,\\&\delta_{\hat\epsilon}\hat\psi_{\hat\mu i}=\hat D_{\hat\mu}\hat\varepsilon_i+\hat\gamma_{\hat\mu}T_{ij}\hat\varepsilon^j+\frac{1}{4\sqrt{3}}\big(\hat\gamma_{\hat\mu}{}^{\hat\nu\hat\rho}-4\delta_{\hat\mu}{}^{\hat\nu}\hat\gamma^{\hat\rho}\big)\hat{\mathcal{F}}_{\hat\nu\hat\rho}\,\hat\varepsilon_i\ .\end{aligned} \tag{4.64}$$

[10] Since Γ_1 and Γ_{45} can be diagonalized simultaneously, $[\Gamma_I,\Gamma_{45}]=0$, this is compatible with the choice made in Sec. 4.1.2. Furthermore, since Γ_1 is hermitian, traceless and squares to the identity it has eigenvalues ± 1 each with multiplicity 2.

For notational convenience we introduce the supercovariant gravitino and U(1) curvatures

$$\begin{aligned}\hat{\mathcal{R}}_{\hat\mu\hat\nu i}(\hat\psi) &= 2\hat{D}_{[\hat\mu}\hat\psi_{\hat\nu]i} + 2T_{ij}\hat\gamma_{[\hat\mu}\hat\psi^j_{\hat\nu]} + \frac{1}{2\sqrt{3}}\mathcal{F}_{\hat\sigma\hat\tau}\big(\hat\gamma_{[\hat\mu}{}^{\hat\sigma\hat\tau} - 4\delta_{[\hat\mu}{}^{\hat\sigma}\hat\gamma^{\hat\tau}\big)\hat\psi_{\hat\nu]i} \ , \\ \hat{\mathcal{R}}_{\hat\mu\hat\nu}(\hat{\mathcal{A}}) &= \hat{\mathcal{F}}_{\hat\mu\hat\nu} + \frac{1}{2}\sqrt{3}i\bar{\hat\psi}^i_{\hat\mu}\hat\psi_{\hat\nu i} \ .\end{aligned} \tag{4.65}$$

The Lagrangian then reads

$$\mathcal{L} = -\frac{1}{4}\hat{e}\hat{\mathcal{R}}(\hat\omega) + g^2\hat{e} - \frac{1}{4}i\hat{e}\,\bar{\hat\psi}^i_{\hat\mu}\hat\gamma^{\hat\mu\hat\nu\hat\rho}\hat{\mathcal{R}}_{\hat\nu\hat\rho i}(\hat\psi) - \frac{1}{4}\hat{e}\,\hat{\mathcal{F}}^{\hat\mu\hat\nu}\hat{\mathcal{F}}_{\hat\mu\hat\nu} - \frac{1}{6\sqrt{3}}\hat\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\hat{\mathcal{F}}_{\hat\mu\hat\nu}\hat{\mathcal{F}}_{\hat\rho\hat\sigma}\hat{\mathcal{A}}_{\hat\tau} \ . \tag{4.66}$$

The field equations derived from (4.63) comprise Einstein's equations and the gravitino and U(1) gauge field equations and they read

$$\begin{aligned}\hat{\mathcal{R}}^{\hat a}_{\hat\mu}(\hat\omega) - \frac{1}{2}\hat{\mathcal{R}}(\hat\omega)\hat{e}^{\hat a}_{\hat\mu} &= i\hat{e}^{-1}\hat{e}^{\hat a}_{\hat\sigma}\hat\epsilon^{\hat\nu\hat\rho\hat\sigma\hat\lambda\hat\tau}\bar{\hat\psi}^i_{\hat\lambda}\big(\hat\gamma_{\hat\mu\hat\tau}\hat{D}_{\hat\nu}\hat\psi_{\hat\rho i} - \frac{3}{2}T_{ij}\hat\gamma_{\hat\rho\hat\mu\hat\tau}\hat\psi^j_{\hat\nu} + \frac{\sqrt{3}}{4}\hat{\mathcal{F}}_{\hat\rho\hat\tau}\hat\gamma_{\hat\mu}\hat\psi_{\hat\nu i}\big) \\ &\quad - 2g^2\hat{e}^{\hat a}_{\hat\mu} + \frac{1}{2}\hat{\mathcal{R}}^{\hat\nu\hat\rho}(\hat{\mathcal{A}})\hat{\mathcal{R}}_{\hat\nu\hat\rho}(\hat{\mathcal{A}})\hat{e}^{\hat a}_{\hat\mu} - 2\hat{\mathcal{R}}^{\hat a\hat\nu}(\hat{\mathcal{A}})\hat{\mathcal{R}}_{\hat\mu\hat\nu}(\hat{\mathcal{A}}) \ ,\end{aligned} \tag{4.67}$$

$$\hat\gamma^{\hat\mu\hat\nu\hat\rho}\hat{\mathcal{R}}_{\hat\nu\hat\rho i}(\hat\psi) = 0 \ , \tag{4.68}$$

$$\partial_{\hat\nu}\hat{e}\hat{\mathcal{R}}^{\hat\mu\hat\nu}(\hat{\mathcal{A}}) = -\sqrt{3}\,i\hat{e}\,T_i{}^j\bar{\hat\psi}^i_{\hat\nu}\hat\gamma^{\hat\nu\hat\mu\hat\rho}\hat\psi_{\hat\rho j} - \frac{\sqrt{3}}{6}\hat\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\big(\hat{\mathcal{F}}_{\hat\nu\hat\rho}\hat{\mathcal{F}}_{\hat\sigma\hat\tau} + \frac{3}{2}i\partial_{\hat\nu}\bar{\hat\psi}^i_{\hat\rho}\hat\gamma_{\hat\tau}\hat\psi_{\hat\sigma i}\big) \ . \tag{4.69}$$

For later convenience we give the traced Einstein equation (4.67), simplified using (4.68). With $\hat{L}_\psi := iT_{ij}\bar{\hat\psi}^i_\mu\hat\gamma^{\mu\nu}\hat\psi^j_\nu$, $\hat{L}_{\mathcal{F}\psi\psi} := i\hat{\mathcal{F}}_{\hat\mu\hat\nu}\bar{\hat\psi}^i_\rho\hat\gamma^{\rho\hat\mu\hat\nu\sigma}\hat\psi_{\sigma i}$ and $\hat{L}_{\mathcal{F}\mathcal{F}\psi,\alpha} := \hat{\mathcal{F}}^{\hat\nu\hat\rho}\big(\hat{\mathcal{F}}_{\hat\nu\hat\rho} - \alpha\sqrt{3}i\bar{\hat\psi}^i_{\hat\nu}\hat\psi_{\hat\rho i}\big)$ it reads

$$\hat{\mathcal{R}}(\hat\omega) = \frac{20}{3}g^2 + 2\hat{L}_\psi - \frac{1}{3}\hat{L}_{\mathcal{F}\mathcal{F}\psi,1} + \frac{1}{2\sqrt{3}}\hat{L}_{\mathcal{F}\psi\psi} \ . \tag{4.70}$$

The spin connection is fixed to its on-shell value as usual in 1.5^{th}-order formalism. Splitting off the torsion-free part which for asymptotically-AdS backgrounds as described by (4.12) has the non-vanishing components given in (4.14), we have $\hat\omega_{\hat\mu\hat a\hat b} = \hat\omega_{\hat\mu\hat a\hat b}(\hat e) + \hat\omega_{\hat\mu\hat a\hat b}(\hat\psi)$ with the remaining part

$$\hat\omega_{\hat\mu\hat a\hat b}(\hat\psi) = -\frac{1}{2}i\left(\bar{\hat\psi}^i_{\hat a}\hat\gamma_{\hat\mu}\hat\psi_{\hat b i} + 2\bar{\hat\psi}^i_{\hat\mu}\hat\gamma_{[\hat a}\hat\psi_{\hat b]i}\right) - \frac{1}{4}i\bar{\hat\psi}^i_{\hat\lambda}\hat\gamma_{\hat\mu\hat a\hat b}{}^{\hat\lambda\hat\tau}\hat\psi_{\hat\tau i} \ . \tag{4.71}$$

4.2.2 Holographic renormalization: Weyl anomaly of $\mathcal{N}$=1 SCFT

We restrict the configuration space of the theory to asymptotically-AdS geometries as discussed in Sec. 4.1.2 and use the local symmetries to fix (4.12) and

$$\hat\psi_r \equiv 0 \ , \qquad \mathcal{A}_r \equiv 0 \ . \tag{4.72}$$

As derived in Sec. 4.1.2, one of the symmetries remaining after the gauge fixings is generated by Lorentz transformations $\delta_{\hat\Sigma}$ with $\hat\Sigma^{\hat a\hat b}(x,r) = \delta^{\hat a}_a\delta^{\hat b}_b\Sigma^{ab}(x,r)$. To solve Einstein's equations for the first few orders of the vielbein in terms of its boundary value, we fix a gauge such that only the boundary Lorentz symmetry $\hat{\Sigma}^{\hat{a}\hat{b}}(x,r) = \delta^{\hat{a}}_{a}\delta^{\hat{b}}_{b}\Sigma^{ab}(x)$ remains. As we discuss below, there exists a Lorentz transformation to set

$$\hat{\omega}_r{}^{ab} \equiv 0 \ , \tag{4.73}$$

which we choose as the additional gauge fixing condition. In the discussion of the residual symmetries in Sec. 4.1.2 we now have the additional constraint $(\delta_{\hat{X}} + \delta_{\hat{\Sigma}} + \delta_{\hat{\epsilon}})\hat{\omega}_{rab} = 0$, which indeed leaves only the boundary Lorentz transformations as an independent symmetry.

We now discuss the existence of the finite Lorentz transformation yielding $\hat{\omega}_r{}^{ab} \equiv 0$ in the class of residual gauge transformations. More concretely, we want to find a finite Lorentz transformation $\hat{\Lambda} \in \mathrm{SO}(1,4)$, generated by a $\hat{\Sigma} \in \mathfrak{so}(1,4)$ of the form $\hat{\Sigma}^{\hat{a}\hat{b}}(x,r) = \delta^{\hat{a}}_{a}\delta^{\hat{b}}_{b}\Sigma^{ab}(x,r)$, such that $\hat{\omega}_{rab}$ is transformed to zero. Thus, we have to find a $\hat{\Lambda}^{\hat{a}\hat{b}} = \delta^{\hat{a}}_{a}\delta^{\hat{b}}_{b}\Lambda^{ab} + \delta^{\hat{a}}_{\underline{r}}\delta^{\hat{b}}_{\underline{r}}$ such that

$$\omega_r{}^{ab} \ \rightarrow \ \omega_r'{}^{ab} = \Lambda^a{}_c\Lambda^b{}_d\omega_r{}^{cd} + \Lambda^{[a}{}_d\partial_r\Lambda^{b]d} \ \stackrel{!}{=} \ 0 \ . \tag{4.74}$$

The transformation follows e.g. from (4.14), (4.71). Using that $\hat{\Lambda} \in \mathrm{SO}(1,4)$ this can be recast as

$$-\hat{\omega}_{ra}{}^c\Lambda^{\mathrm{T}}{}_c{}^b = \partial_r\Lambda^{\mathrm{T}}{}_a{}^b \ . \tag{4.75}$$

Following [126], a solution for this kind of differential equation of the form $A(r)Y(r) = Y'(r)$ with $Y(0) = Y_0$ can be given as exponential $Y(r) = e^{\Omega(r)}Y_0$, with Ω given as a series $\Omega(r) = \sum_n \Omega^{(n)}(r)$. Each term in this series is calculated from integrals of sums of nested commutators of $A(r)$, and there is an $r_0 > 0$ such that this series converges for $r \in [0, r_0)$. For the case that A is in some Lie algebra and Y_0 in the corresponding Lie group, one notes that – since the operations by which the $\Omega^{(n)}$ are obtained from A, i.e. sums, commutators and integrations, all close in the Lie algebra – Ω also is an element of the Lie algebra and $Y = e^{\Omega}$ thus is in the corresponding Lie group. In the case at hand, we have $\omega_r{}^{ab} \in \mathfrak{so}(1,4)$ and we set $\Lambda_a{}^b|_{r=0} = \delta_a{}^b \in \mathrm{SO}(1,4)$ such that from the results quoted above there indeed is a finite Lorentz transformation setting $\hat{\omega}_r{}^{ab}$ to zero.

On-shell fields as asymptotic series

To determine the divergent part of the on-shell action we now solve the field equations in an asymptotic expansion in a vicinity of the conformal boundary at $r = 0$. Henceforth, we set the curvature radius of the asymptotic AdS_5 geometry to $R = 1$. From Sec. 4.1.2 we already know that

$$\hat{\psi}^{\mathrm{L}}_{\mu i_+} = r^{-1/2}\psi^{\mathrm{L}}_{\mu i_+} \ , \qquad \hat{\psi}^{\mathrm{R}}_{\mu i_+} = r^{1/2}\psi^{\mathrm{R}}_{\mu i_+} \ , \qquad \hat{\mathcal{A}}_\mu = \mathcal{A}_\mu \ , \tag{4.76}$$

and we find the non-vanishing terms in the asymptotic series

$$\begin{aligned} e^a_\mu &= e^a_\mu + r^2 e^{(2)a}_\mu + r^4(\log r)^2\tilde{e}^{(2)a}_\mu + r^4\log r\,\tilde{e}^{(3)a}_\mu + r^4 e^{(4)a}_\mu + \dots \ , \\ \hat{\mathcal{A}}_\mu &= \mathcal{A}_\mu + r^2\log r\,\tilde{\mathcal{A}}^{(1)}_\mu + r^2\mathcal{A}^{(2)}_\mu + \dots \ , \qquad \psi^{\mathrm{L}}_{\mu i_+} = \Psi_{\mu i_+} + r^2\psi^{\mathrm{L}(2)}_{\mu i_+} + \dots \ , \\ \psi^{\mathrm{R}}_{\mu i_+} &= \Phi_{\mu i_+} + r^2(\log r)^2\tilde{\psi}^{\mathrm{R}(0)}_{\mu i_+} + r^2\log r\,\tilde{\psi}^{\mathrm{R}(1)}_{\mu i_+} + r^2\psi^{\mathrm{R}(2)}_{\mu i_+} + \dots \ . \end{aligned} \tag{4.77}$$

Note that $e_\mu^{(4)a}$, $\mathcal{A}_\mu^{(2)}$ and $\psi_{\mu i_+}^{\mathrm{R}(2)}$ can not be determined by a near-boundary analysis alone. However, the trace of $e_\mu^{(4)a}$ can be determined. Note also that for notational convenience the leading-order terms are indicated by a different font, rather than using a superscript. Likewise, we use $\gamma_\mu = \gamma_\mu^{(0)}$ and $\mathcal{F}_{\mu\nu} = F_{\mu\nu}^{(0)}$.

For the gauge field we need the expression for the subleading term only to leading order in the fermions and we find $\tilde{\mathcal{A}}^{(1)\mu} = -\frac{1}{2} D_\nu \mathcal{F}^{\mu\nu}$. The components of the gravitino are

$$\Phi_{\mu i_+} = -\frac{1}{2} i \big(\gamma_\mu{}^{\nu\rho} - \frac{2}{3}\gamma_\mu \gamma^{\nu\rho}\big) D_\nu \Psi_{\rho i_+} \ , \tag{4.78a}$$

$$\gamma^{\mu\rho}\psi_{\rho i_+}^{\mathrm{L}(2)} = \frac{1}{2} i \gamma^{\mu\nu\rho}\big(D_\nu \Phi_{\rho i_+} - \frac{i}{2} g_{\nu\sigma}^{(2)} \gamma^\sigma \Psi_{\rho i_+}\big) + \frac{\sqrt{3}}{8} i \mathcal{F}^{\rho\sigma}\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\Psi_{\nu i_+} \ , \tag{4.78b}$$

$$\tilde{\psi}_{\mu i_+}^{\mathrm{R}(0)} = 2\sqrt{3}\big(\gamma_\mu{}^{\nu\rho} - \frac{2}{3}\gamma_\mu \gamma^{\nu\rho}\big)\tilde{\mathcal{A}}_\nu^{(1)}\Psi_{\rho i_+} \ , \tag{4.78c}$$

$$\begin{aligned}\gamma^{\mu\rho}\tilde{\psi}_{\rho i_+}^{\mathrm{R}(1)} &= \frac{2}{3} i \gamma^{\mu\nu\rho} D_\nu \psi_{\rho i_+}^{\mathrm{L}(2)} + \frac{1}{6} i \gamma^{\mu\nu\rho}\gamma^{\lambda\tau}\big(D_\tau g_{\nu\lambda}^{(2)}\big)\Psi_{\rho i_+} + \frac{2}{3} i \gamma^{(2)\mu\nu\rho} D_\nu \Psi_{\rho i_+} \\ &+ \frac{1}{2\sqrt{3}} i \big(2\mathcal{F}^{\mu\rho} + \mathcal{F}_{\sigma\tau}\gamma^{\mu\rho\sigma\tau}\big)\Phi_{\rho i_+} \ .\end{aligned} \tag{4.78d}$$

The expansion of the vielbein determinant is given by the expression analogous to (4.50) and for $\hat{L}_\psi$, $\hat{L}_{\mathcal{F}\psi\psi}$, $\hat{L}_{\mathcal{F}\mathcal{F}\psi,\alpha}$ defined above (4.70) we find

$$\begin{aligned}\hat{L}_{\mathcal{F}\psi\psi} &= r^4(\log r \tilde{L}_{\mathcal{F}\psi\psi}^{(0)} + L_{\mathcal{F}\psi\psi}^{(0)} + \dots) \ , \qquad \hat{L}_{\mathcal{F}\mathcal{F}\psi,\alpha} = r^4 L_{\mathcal{F}\mathcal{F}\psi,\alpha}^{(0)} + \dots \ , \\ \hat{L}_\psi &= r^2(L_\psi^{(0)} + r^2(\log r)^2 \tilde{L}_\psi^{(0)} + r^2 \log r \tilde{L}_\psi^{(1)} + r^2 L_\psi^{(2)} + \dots) \ .\end{aligned} \tag{4.79}$$

Moreover, for $\hat{L}_{\mathcal{F}r} := i\hat{\mathcal{F}}_{\mu r}\hat{\bar{\psi}}_\tau^i \hat{\gamma}^{\mu r\tau\nu}\hat{\psi}_{\nu i}$ and $\hat{L}_{\psi D\psi} := i\hat{\bar{\psi}}_\mu^i \hat{\gamma}^{\mu r\rho}\hat{D}_r \hat{\psi}_{\rho i}$ we find

$$\begin{aligned}\hat{L}_{\mathcal{F}r} &= r^4(\log r \tilde{L}_{\mathcal{F}r}^{(0)} + L_{\mathcal{F}r}^{(0)} + \dots) \ , \\ \hat{L}_{\psi D\psi} &= r^2(L_{\psi D\psi}^{(0)} + r^2(\log r)^2 \tilde{L}_{\psi D\psi}^{(0)} + r^2 \log r \tilde{L}_{\psi D\psi}^{(1)} + r^2 L_{\psi D\psi}^{(2)} + \dots) \ .\end{aligned} \tag{4.80}$$

The leading terms are $\tilde{L}_{\psi D\psi}^{(0)} = 3\tilde{L}_\psi^{(0)}$, $\tilde{L}_{\psi D\psi}^{(1)} = 3\tilde{L}_\psi^{(1)} + 2\tilde{L}_\psi^{(0)}$ and

$$\begin{aligned}L_\psi^{(0)} &= L_{\psi D\psi}^{(0)} = \bar{\Psi}_\mu^{i_+}\gamma^{\mu\nu}\Phi_{\nu i_+} + \text{c.c.} \ , \qquad \tilde{L}_\psi^{(0)} = \bar{\Psi}_\mu^{i_+}\gamma^{\mu\nu}\tilde{\psi}_{\nu i_+}^{\mathrm{R}(0)} + \text{c.c.} \ , \\ \tilde{L}_\psi^{(1)} &= \bar{\Psi}_\mu^{i_+}\gamma^{\mu\nu}\tilde{\psi}_{\nu i_+}^{\mathrm{R}(1)} + \text{c.c.} \ , \qquad L_{\mathcal{F}\mathcal{F}\psi,\alpha}^{(0)} = \mathcal{F}^{\mu\nu}\Big(\mathcal{F}_{\mu\nu} - 2\alpha\sqrt{3}\big(i\bar{\Psi}_\mu^{i_+}\Phi_{\nu i_+} + \text{c.c.}\big)\Big) \ .\end{aligned} \tag{4.81}$$

Supercovariant quantities / group curvatures

The various pieces in the log-divergent part of the Lagrangian are most conveniently expressed in terms of quantities defined from the boundary fields which are covariant with respect to the boundary superconformal symmetry. The right handed part of the gravitinos corresponds to the gauge field of S-supersymmetry transformations in the group manifold construction of conformal supergravity. It is fixed in this approach by a constraint on the gravitino curvature (see (2.2b) of [127]). Here it is fixed by solving the gravitino field equation as asymptotic series, which yields (4.78a), see also (4.30). The group curvature corresponding to Q-supersymmetry $R_{\mu\nu\, i_+}(Q)$ is then defined as (note the factor 2 compared to (4.31))

$$\mathcal{R}_{\mu\nu\, i_+}(Q) := 2D_{[\mu}\Psi_{\nu]i_+} - 2i\gamma_{[\mu}\Phi_{\nu]i_+} \ . \tag{4.82}$$

It vanishes upon contraction with γ^μ, which is the constraint used in [127], and it is anti-selfdual,

$$\gamma^\mu R_{\mu\nu\, i_+}(Q) = 0 \ , \qquad i \star R_{\mu\nu\, i_+}(Q) = -R_{\mu\nu\, i_+}(Q) \ . \tag{4.83}$$

The supercovariant curvature of the U(1) gauge field is

$$\mathcal{R}_{\mu\nu}(\mathcal{A}) = 2\partial_{[\mu}\mathcal{A}_{\nu]} + \sqrt{3}\big(i\Psi^{i_+}_{[\mu}\Phi_{\nu]i_+} + \text{c.c.}\big) \ . \tag{4.84}$$

In addition we define the group curvature corresponding to Lorentz transformations[11]

$$\hat{\mathcal{R}}_{\mu\nu}{}^{ab}(M) = \mathcal{R}_{\mu\nu}{}^{ab}(\omega^{(0)}) - 2\big(\bar{\Psi}^{i_+}_{[\mu}\gamma^{ab}\Phi_{\nu]i_+} + \text{c.c.}\big) \ , \tag{4.85}$$

and the supercovariantized version (compare (2.8) of [127])

$$\mathcal{R}^{\text{cov}}_{\mu\nu ab}(M) = \hat{\mathcal{R}}_{\mu\nu ab}(M) - \big(2i\bar{\Psi}^{i_+}_{[\mu}\gamma_{\nu]}\mathcal{R}_{ab i_+}(Q) + \text{c.c.}\big) \ . \tag{4.86}$$

Ricci tensor and vielbein expansion

To solve Einstein's equations we need explicit expressions for the Ricci curvature of the metric in Fefferman-Graham form, which we provide in this paragraph. The curvature of the spin connection is $\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}(\hat{\omega}) = 2\partial_{[\hat{\mu}}\hat{\omega}_{\hat{\nu}]}{}^{\hat{a}\hat{b}} + 2\hat{\omega}_{[\hat{\mu}}{}^{\hat{a}\hat{c}}\hat{\omega}_{\hat{\nu}]\hat{c}}{}^{\hat{b}}$, and the Ricci tensor is defined as $\hat{\mathcal{R}}^{\hat{a}}_{\hat{\mu}}(\hat{\omega}) = \hat{e}^{\hat{\nu}}_{\hat{b}}\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}(\hat{\omega})$. With the splitting of the spin connection into the torsion-free and the gravitino part, the Ricci tensor also splits and (up to 4-fermion terms) we have

$$\hat{\mathcal{R}}^{\hat{a}}_{\hat{\mu}}(\hat{\omega}) = \hat{\mathcal{R}}^{\hat{a}}_{\hat{\mu}}(\hat{\omega}(\hat{e})) + 2\hat{e}^{\hat{\nu}}_{\hat{b}}\hat{D}^{(e)}_{[\hat{\mu}}\hat{\omega}_{\hat{\nu}]}{}^{\hat{a}\hat{b}}(\hat{\psi}) \ . \tag{4.87}$$

For the scaling of the torsion part of the spin connection we find that $\hat{\omega}_{rab}(\hat{\psi})$, $\hat{\omega}_{\mu a \underline{r}}(\hat{\psi})$ are $\mathcal{O}(r^1)$, $\hat{\omega}_{ra\underline{r}}(\hat{\psi}) = 0$ and $\hat{\omega}_{\mu ab}(\hat{\psi}) = \mathcal{O}(r^0)$ with

$$\hat{\omega}_{\mu ab}(\hat{\psi}) = -\frac{1}{2}i\left(\hat{\bar{\psi}}^i_a\hat{\gamma}_\mu\hat{\psi}_{bi} + 2\hat{\bar{\psi}}^i_\mu\hat{\gamma}_{[a}\hat{\psi}_{b]i}\right) \ . \tag{4.88}$$

Furthermore, we find $2\hat{e}^{\hat{\nu}}_{\hat{b}}\hat{D}^{(e)}_{[r}\hat{\omega}_{\hat{\nu}]}{}^{\underline{r}\hat{b}}(\hat{\psi}) = 0$ and

$$2\hat{e}^{\hat{\nu}}_{\hat{b}}\hat{D}^{(e)}_{[\mu}\hat{\omega}_{\hat{\nu}]}{}^{a\hat{b}}(\hat{\psi}) = r\Big(2e^{\nu}_{b}D^{(e)}_{[\mu}\omega_{\nu]}{}^{ab}(\psi) - 4e^{\nu}_{b}\hat{\omega}_{[\mu}{}^{[a}{}_{\underline{r}}(\hat{e})\hat{\omega}_{\nu]}{}^{b]\underline{r}}(\hat{\psi}) - 2\hat{D}_{[r}\hat{\omega}_{\mu]}{}^{a\underline{r}}(\hat{\psi})\Big) \ . \tag{4.89}$$

[11] More precisely, it is the group curvature with the gauge field of proper conformal transformations f^a_μ set to zero, see Tab. II of [128].

Thus, with $\mathcal{R}^a_\mu(\omega) = \mathcal{R}^a_\mu(\omega(e)) + 2e^\nu_b D^{(e)}_{[\mu}\omega_{\nu]}{}^{ab}(\psi)$, we get

$$\begin{aligned}\hat{\mathcal{R}}^a_\mu(\hat\omega) = r\Big(&\mathcal{R}^a_\mu(\omega) + \frac{d}{r^2}e^a_\mu - \frac{d-1}{2r}e^{\rho a}g'_{\rho\mu} - \frac{1}{2}\operatorname{tr} g^{-1}g'(\frac{1}{r}e^a_\mu - \frac{1}{2}e^{\rho a}g'_{\rho\mu}) \\ &- \frac{1}{2}e^{\rho a}g'_{\rho\nu}g^{\nu\sigma}g'_{\sigma\mu} + \frac{1}{2}e^{\rho a}g''_{\rho\mu} - 4e^\nu_b\hat\omega_{[\mu}{}^{[a}{}_{\underline{r}}(\hat e)\hat\omega_{\nu]}{}^{b]\underline{r}}(\hat\psi) - 2\hat D_{[r}\hat\omega_{\mu]}{}^{a\underline{r}}(\hat\psi)\Big) ,\end{aligned} \tag{4.90a}$$

$$\hat{\mathcal{R}}^{\underline{r}}_r(\hat\omega) = r\Big(\frac{d}{r^2} - \frac{1}{2r}\operatorname{tr} g^{-1}g' + \frac{1}{2}\operatorname{tr} g^{-1}g'' - \frac{1}{4}\operatorname{tr} g^{-1}g'g^{-1}g'\Big) , \tag{4.90b}$$

$$\begin{aligned}\hat{\mathcal{R}}(\hat\omega) = r^2\Big(&\mathcal{R}(\omega) + \frac{d(d+1)}{r^2} - \frac{d}{r}\operatorname{tr} g^{-1}g' + \frac{1}{4}(\operatorname{tr} g^{-1}g')^2 - \frac{3}{4}\operatorname{tr} g^{-1}g'g^{-1}g' \\ &+ \operatorname{tr} g^{-1}g'' - 4e^\mu_a e^\nu_b\hat\omega_{[\mu}{}^{[a}{}_{\underline{r}}(\hat e)\hat\omega_{\nu]}{}^{b]\underline{r}}(\hat\psi)\Big) .\end{aligned} \tag{4.90c}$$

With these expressions we can actually evaluate the asymptotic expansion of Einstein's equations. From the first two orders we find $g^2 = 3/R^2$ and $g^{(1)}_{\mu\nu} = 0$. From the traced Einstein equation (4.70) we find for the traces $t^{(n)} := \operatorname{tr} g^{(0)-1}g^{(n)}$, $\tilde t^{(n)} := \operatorname{tr} g^{(0)-1}\tilde g^{(n)}$ and $t^{(2,2)} := \operatorname{tr} g^{(0)-1}g^{(2)}g^{(0)-1}g^{(2)}$

$$\begin{aligned} t^{(2)} &= \frac{1}{2(d-1)}\mathcal{R}^{\text{cov}}(M) , \qquad \tilde t^{(2)} = \frac{1}{4}\big(\tilde L^{(0)}_\psi - \tilde L^{(0)}_{\psi D\psi}\big) , \\ \tilde t^{(3)} &= \frac{1}{4}\big(\tilde L^{(1)}_\psi - \tilde L^{(1)}_{\psi D\psi} + \frac{1}{4\sqrt3}\tilde L^{(0)}_{\mathcal{F}\psi\psi} - \frac{\sqrt3}{2}\tilde L^{(0)}_{\mathcal{F}r} - 6\tilde t^{(2)}\big) , \\ \big(4t^{(4)} - t^{(2,2)}\big) &= L^{(2)}_\psi - L^{(2)}_{\psi D\psi} + \frac{1}{4\sqrt3}L^{(0)}_{\mathcal{F}\psi\psi} - \frac{\sqrt3}{2}L^{(0)}_{\mathcal{F}r} + \frac{1}{3}L^{(0)}_{\mathcal{F}\mathcal{F}\psi,-\frac12} - \tilde t^{(2)} - 3t^{(3)} .\end{aligned} \tag{4.91}$$

The subleading term in the metric expansion as obtained from (4.67) reads

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\left(\mathcal{R}^{\text{cov}}_{\{\mu\nu\}}(M) - \frac{1}{6}\mathcal{R}^{\text{cov}}(M)g^{(0)}_{\mu\nu}\right) . \tag{4.92}$$

For the remaining combination of traces to be determined this yields

$$t^{(2,2)} - (t^{(2)})^2 = \frac{1}{4}\Big(\mathcal{R}^{\text{cov}\{\mu\nu\}}(M)\mathcal{R}^{\text{cov}}_{\{\mu\nu\}}(M) - \frac{1}{3}\mathcal{R}^{\text{cov}}(M)^2\Big) . \tag{4.93}$$

Holographic counterterms and the Weyl anomaly

With the asymptotic expansions of the fields as obtained in the previous discussion we can now carry out the holographic renormalization of the $\mathcal{N}$=2 supergravity on asymptotically-AdS spaces. Using the gravitino field equation (4.68) and the traced Einstein equation (4.70) we obtain from (4.63) the on-shell Lagrangian

$$\mathcal{L}_{\text{on-shell}} = -\frac{2}{3}g^2\hat e - \frac{1}{2}\hat e\hat L_\psi - \frac{1}{6}\hat L_{\mathcal{F}\mathcal{F}\psi,-\frac12} - \frac{1}{8\sqrt3}\hat e\hat L_{\mathcal{F}\psi\psi} - \frac{1}{6\sqrt3}\hat\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\hat{\mathcal{F}}_{\hat\mu\hat\nu}\hat{\mathcal{F}}_{\hat\rho\hat\sigma}\hat{\mathcal{A}}_{\hat\tau} . \tag{4.94}$$

As discussed already in Sec. 4.1.3, the on-shell action is divergent due to this non-vanishing on-shell Lagrangian and the infinite volume of asymptotically-AdS spaces, and this is a long-distance IR divergence in the bulk corresponding to a UV divergence in the dual CFT.

The divergence is regularized by cutting off spacetime at $r = \epsilon$ and adding the Gibbons-Hawking-York boundary term for a well-defined variational problem, along with appropriate counterterms. The pure-metric counterterms consisting of the boundary cosmological constant and Einstein-Hilbert term have been calculated already in Sec. 4.1.3 and are also added here

$$S_\epsilon = \int_{r\geq\epsilon} dr\, d^4x \mathcal{L} + \int_{r=\epsilon R} d^4x\, \hat{e}^\star \Big(\frac{1}{2}\hat{K} - \frac{3}{2} + \frac{1}{8}\mathcal{R}^*(\omega)\Big) \;, \tag{4.95}$$

with $\mathcal{L}$ as given in (4.63). In fact, this turns out to be sufficient to cancel all power-law divergences. This may be somewhat unexpected – while the gravitino part of the action vanishes on shell, it still contributes to the asymptotic expansion of the vielbein resulting in power divergences. However, these are also cancelled by the counterterms given in (4.95). The asymptotic expansion of this partly renormalized action then reads

$$\begin{aligned} S_\epsilon = \int\limits_{r=\epsilon} d^4x e \frac{1}{4} \Bigg[& \Big(\tilde{L}^{(1)}_\psi - 3\tilde{t}^{(2)} + 2\tilde{t}^{(3)} + \frac{\sqrt{3}}{4}\tilde{L}^{(0)}_{\mathcal{F}\psi\psi}\Big)\log\epsilon + 2L^{(2)}_\psi + t^{(2)}L^{(0)}_\psi - 2\tilde{t}^{(2)} - 3\tilde{t}^{(3)} \\ & + (t^{(2)})^2 + t^{(4)} - 2t^{(2,2)} + \frac{1}{2\sqrt{3}}L^{(0)}_{\mathcal{F}\psi\psi} + \frac{2}{3}L^{(0)}_{\mathcal{F}\mathcal{F}\psi,-\frac{1}{2}}\Bigg] \log\epsilon + \mathcal{O}(\epsilon^0) \;. \end{aligned} \tag{4.96}$$

To cancel the $(\log\epsilon)^2$ divergence without reintroducing power-law divergences we have to employ the combination of boundary terms

$$S_{\rm ct} = \int_{r=\epsilon} d^4x\, \hat{e}^* \frac{1}{4}\big(i\hat{\bar{\psi}}^{\mathrm{L}i_+}_\mu \hat{\gamma}^{\mu\nu\rho} D^\star_\nu \hat{\psi}^{\mathrm{L}}_{\rho i_+} - 2\hat{\bar{\psi}}^{\mathrm{L}i_+}_\mu \hat{\gamma}^{\mu\rho}\hat{\psi}^{\mathrm{R}}_{\rho i_+} + \text{c.c.}\big) \;. \tag{4.97}$$

Interestingly, taking into account the full non-linear structure of the theory therefore fixes the coefficients which had to be left arbitrary in studies of the linearized gravitino [129, 130]. Note also that the split into left and right handed components using the projectors $P_{\mathrm{L/R}} = \frac{1}{2}(1 \pm i(\hat{n}\cdot\hat{\gamma}))$ is perfectly covariant on the boundary. The counterterm (4.97) contributes another set of log-divergences (due to $\hat{\mathcal{A}}_\mu$ in the first term and $\hat{\Psi}^{\mathrm{R}}_{\mu i_+}$ in the second) and the total renormalized action is then given by

$$S_{\rm ren} = S_\epsilon + S_{\rm ct} - \int_{r=\epsilon} d^4x e^* \log\epsilon \mathcal{A} \;, \tag{4.98}$$

with the part yielding the Weyl anomaly given by

$$\begin{aligned} \mathcal{A} = & -\frac{1}{16}\Big(\mathcal{R}^{\mathrm{cov}\mu\nu}(M)\mathcal{R}^{\mathrm{cov}}_{\mu\nu}(M) - \frac{1}{3}\mathcal{R}^{\mathrm{cov}}(M)^2\Big) + \frac{1}{4}\mathcal{R}^{\mu\nu}(\mathcal{A})\mathcal{R}_{\mu\nu}(\mathcal{A}) \\ & + \frac{1}{2}i\bar{\Phi}^{i_+}_\mu \gamma^{\mu\nu\rho} D_\nu \Phi_{\rho i_+} + \frac{\sqrt{3}}{2}(\star\mathcal{F})^{\mu\nu}\bar{\Psi}^{i_+}_\mu \Phi_{\nu i_+} + \text{c.c.} \\ & - \frac{1}{16}\mathcal{R}^{\mathrm{cov}\mu\rho}(M)\big(\bar{\Psi}^{i_+}_\mu \gamma_\rho{}^\nu \Phi_{\nu i_+} + \bar{\Psi}^{i_+}_\nu \gamma^\nu{}_\rho \Phi_{\mu i_+} - \frac{2}{3}g_{\mu\rho}\bar{\Psi}^{i_+}_\nu \gamma^{\nu\lambda}\Phi_{\lambda i_+}\big) + \text{c.c.} \;, \end{aligned} \tag{4.99}$$

where the '+ c.c.' applies to the complete line that it appears in. $\mathcal{R}^{\mathrm{cov}}_{\mu\nu}(M)$ was defined in (4.86), $\mathcal{R}_{\mu\nu}(\mathcal{A})$ in (4.84) and $\Phi_{\mu i_+}$ in (4.78a). Similarly to (4.44) the anomalous Ward identity for Weyl invariance reads

$$\langle T^a_\mu\rangle e^\mu_a - \frac{1}{2}\bar{\Psi}_{\mu i_+}\langle S^{\mu i_+}\rangle = \mathcal{A} \;. \tag{4.100}$$

In the anomaly $\mathcal{A}$ in (4.99) all dependences on the subdominant components which are not determined by the near-boundary analysis have dropped out, as expected and seen also in Sec. 4.1.3. We have thus obtained holographically the full trace anomaly in generic backgrounds for the $\mathcal{N}$=1 SCFTs which admit a dual description in terms of the $\mathcal{N}$=2 gauged supergravity, allowing again for a test of the AdS/CFT conjecture. Comparing to [38] the anomaly $\mathcal{A}$ corresponds to the action of $\mathcal{N}$=1 conformal supergravity up to sign and normalization conventions and four-fermion terms. This confirms the expectation that the anomaly is given by the supersymmetrization of the squared Weyl tensor.

4.3 Dynamical Gravity on the Boundary

Employing the results of Sec. 4.1, 4.2 we now promote the boundary gravity multiplets arising as boundary values of the bulk fields to dynamical quantities. To this end we show that Neumann and mixed boundary conditions can be imposed on the full supergravity theories, extending the results of [52] for pure gravity (see also [131] for an earlier study) and [130] for a four-dimensional supergravity. Deviating from the standard Dirichlet boundary conditions usually results in only non-normalizable solutions and thus in trivial theories. However, with the discussion of the holographic renormalization in the previous sections we are in a position to refine this discussion for the $\mathcal{N}$=2 and $\mathcal{N}$=4 gauged supergravities, by taking into account the counterterm contributions. These are crucial not only for a well-defined AdS/CFT prescription and to obtain the correct anomalies, but also for the discussion of normalizability [52]. We find that the bulk excitations with Neumann boundary conditions are indeed rendered finite by the counterterm contributions, such that the boundary conformal supergravity multiplets can be dynamical.

Once the more abstract formulation for deriving the conserved current as outlined in Sec. 2.4 is given and used to obtain the general results discussed there, one may reformulate it in a way more directly suited for applications and in terms of the one-form current $j := \star \mathsf{u}$ as follows. Given an action functional $S = \int \mathcal{L}[\phi]$, where ϕ denotes a collection of fields, the variation yields $\delta L = \mathrm{EOM} + D_\mu \theta^\mu[\phi, \delta\phi]$. The one-form current is now defined by

$$j^\mu[\phi, \delta_1\phi, \delta_2\phi] := \delta_1\theta[\phi, \delta_2\phi] - \delta_2\theta[\phi, \delta_1\phi] \ . \tag{4.101}$$

The inner product of fluctuations $\delta_1\phi$, $\delta_2\phi$ around a background ϕ is then given by $\langle \delta_1\phi, \delta_2\phi\rangle = \int_\Sigma n_\mu j^\mu[\phi, \delta_1\phi, \delta_2\phi]$, where $n = n^\mu\partial_\mu$ denotes the unit-normal vector field to the spacelike hypersurface Σ. Here we are dealing with an action $S = S_0 + S_{\mathrm{ct}}$ consisting of a bare part and counterterms on the boundary. The procedure just discussed yields the two currents $j_0^{\hat\mu}$ and j_{ct}^μ for the bulk and boundary actions, respectively, with our standard use of Fefferman-Graham coordinates and the hat to distinguish bulk from boundary quantities. Choosing Σ constant along the radial direction, such that the normal vector field has vanishing radial component, $\hat n = \hat n^\mu \partial_\mu$, we find the inner product

$$\langle \hat\phi_1, \hat\phi_2\rangle_{\mathrm{ren}} = \langle \hat\phi_1, \hat\phi_2\rangle_0 + \langle \hat\phi_1, \hat\phi_2\rangle_{\mathrm{ct}} \ , \qquad \langle \hat\phi_1, \hat\phi_2\rangle_{0/\mathrm{ct}} = \int_{\Sigma/\partial\Sigma} \hat n_{\hat\mu} j_{0/\mathrm{ct}}^{\hat\mu}[\hat\phi_1, \hat\phi_2] \ , \tag{4.102}$$

where $\partial\Sigma = \Sigma|_{r=\epsilon}$. In the following Sec. 4.3.1 we discuss the inner products for the $\mathcal{N}$=2 and $\mathcal{N}$=4 supergravities using that notation, and turn to the dual boundary dynamics resulting from non-Dirichlet boundary conditions in Sec. 4.3.2.

4.3.1 Neumann boundary conditions for the $\mathcal{N}$=2, 4 supergravities

That the holographic counterterms indeed render the Neumann modes normalizable has been shown for the metric in [52] for a number of dimensions, including the five-dimensional case. The pure-metric part of our counterterms agrees with the standard results, so that discussion applies here and we focus on the remaining fields. For the $\mathcal{N}$=2 gauged supergravity discussed in Sec. 4.2 it thus remains to consider the gravitinos and the U(1) gauge field. A discussion of the boundary conditions allowed by the bare inner product for the U(1) gauge field can be found in [132]. The current corresponding to the bulk action is

$$j_0^\mu[\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2] = \hat{\mathcal{A}}_{1\nu}\big(\partial^\mu \hat{\mathcal{A}}_2^\nu - \partial^\nu \hat{\mathcal{A}}_2^\mu\big) - (1 \leftrightarrow 2)\ , \tag{4.103}$$

and the associated bare symplectic product after performing the radial integration reads

$$\langle \hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2\rangle_0 = -\int_{\partial\Sigma} d^3x\, e^* n_\mu \left[\mathcal{A}_{1\nu}\big(\partial^\mu \mathcal{A}_2^\nu - \partial^\nu \mathcal{A}_2^\mu\big) - (1 \leftrightarrow 2)\right] \log\epsilon + \mathcal{O}(\epsilon^0)\ . \tag{4.104}$$

We used $\hat{n}^\mu = r n^\mu$ such that $n = n^\mu\partial_\mu$ is normalized with respect to $g_{\mu\nu}$. The counterterms (4.98), (4.99) yield $j_{\rm ct}^\mu[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_{1\nu}\big(\partial^\mu \mathcal{A}_2^\nu - \partial^\nu \mathcal{A}_2^\mu\big)\log\epsilon - (1 \leftrightarrow 2)$ such that the divergent part is exactly cancelled by the counterterm contribution. In contrast to the bare inner product, which on AdS_5 allows to impose Neumann boundary conditions only on certain components of the gauge field [132], this renormalized inner product thus yields a complete space of normalizable solutions also for Neumann boundary conditions on the entire gauge field.

To establish finiteness of the inner product for the gravitinos we fix a rigid AdS background geometry with the other fields vanishing. Although this is not the most general case it is certainly a particularly relevant one and suggests that finiteness also holds for more generic asymptotically-AdS spaces. The currents $j_0^{\hat{\mu}}$ associated to the bulk action, $j_{\rm ct}^\mu$ to the counterterm action (4.97) and $j_{\rm ct,log}^\mu$ to the logarithmic counterterms (4.98) are

$$\begin{aligned} j_0^{\hat{\mu}}[\hat{\bar{\psi}}_1, \hat{\psi}_2] &= \frac{1}{2} i \hat{\bar{\psi}}_{\hat{\nu}}^{i_+} \hat{\gamma}^{\hat{\nu}\hat{\mu}\hat{\rho}} \hat{\psi}_{\hat{\rho} i_+}\ , \qquad j_{\rm ct}^\mu[\hat{\bar{\psi}}_1, \hat{\psi}_2] = -\frac{1}{4} i \hat{\bar{\psi}}_\nu^{\mathrm{L}i_+} \hat{\gamma}^{\nu\mu\rho} \hat{\psi}^{\mathrm{L}}_{\rho i_+}\ , \\ j_{\rm ct,log}^\mu[\hat{\bar{\psi}}_1, \hat{\psi}_2] &= \frac{1}{2}\Big(i \hat{\bar{\psi}}_\nu^{\mathrm{R}i_+} \hat{\gamma}^{\nu\mu\rho} \hat{\psi}^{\mathrm{R}}_{\rho i_+} + \hat{\bar{\psi}}_\tau^{\mathrm{L}i_+}\big(\gamma^{\tau\mu}{}_\nu - \frac{2}{3}\gamma^{\tau\mu}\gamma_\nu\big)\gamma^{\nu\alpha\beta} D_\alpha \psi^{\mathrm{R}}_{\beta i_+}\Big) \log\epsilon\ . \end{aligned} \tag{4.105}$$

The contribution $j_{\rm ct}^\mu[\hat{\bar{\psi}}_1, \hat{\psi}_2]$ to the inner product constructed according to (4.102) cancels the ϵ^{-2}-divergence of the bare inner product without contributing a logarithmic divergence. It thus remains to show that the logarithmic divergence of the bare inner product constructed from $j_0^{\hat{\mu}}$ is cancelled by the contribution from $j_{\rm ct,log}^\mu[\hat{\bar{\psi}}_1, \hat{\psi}_2]$. From (4.105) we find

$$\begin{aligned} \langle \hat{\psi}_1, \hat{\psi}_2\rangle_{\rm ren}\Big|_{\mathcal{O}(\log\epsilon)} = \int_{\partial\Sigma} \frac{1}{2}\, n_\mu \Big(&\bar{\Psi}_{\tau i_+}\big(\gamma^{\tau\mu}{}_\nu - \frac{2}{3}\gamma^{\tau\mu}\gamma_\nu\big)\gamma^{\nu\alpha\beta} D_\alpha \Phi_{\beta i_+} \\ &- i\bar{\Psi}_\nu^{i_+}\gamma^{\nu\mu\rho}\psi^{\mathrm{L}(2)}_{\rho i_+} - i\bar{\psi}_\nu^{\mathrm{L}(2)i_+}\gamma^{\nu\mu\rho}\Psi_{\rho i_+}\Big) \log\epsilon\ . \end{aligned} \tag{4.106}$$

To further evaluate that expression we split off the γ-traceless part of $\Psi_{\mu i_+}$ by defining

$$\Psi_{\mu i_+} = \Psi^{\mathrm{T}}_{\mu i_+} + \gamma_\mu \lambda_{i_+} \ , \qquad \gamma \cdot \Psi^{\mathrm{T}}_{i_+} = 0 \ . \tag{4.107}$$

Starting from $\gamma_{\mu\nu}{}^{\rho}\Psi^{\mathrm{T}}_{\rho i_+}$ and extracting γ^μ and γ^ν to the left in the two possible orders we find $\gamma_\mu \Psi^{\mathrm{T}}_{\nu i_+} = -\gamma_\nu \Psi^{\mathrm{T}}_{\mu i_+}$. Furthermore, with $D \cdot \Psi_{i_+} := D^\mu \Psi_{\mu i_+}$ we have

$$\Phi_{\mu i_+} = \frac{2}{3} i \gamma_\mu \, D \cdot \Psi_{i_+} + i D_\mu \lambda_{i_+} \ , \qquad \psi^{\mathrm{L}(2)}_{\mu i_+} = -\frac{2}{3} D_\mu D \cdot \Psi_{i_+} \ . \tag{4.108}$$

Using $\gamma_{\mu\nu}{}^{\rho}\Psi^{\mathrm{T}}_{\rho i_+} = -2\gamma_\mu \Psi^{\mathrm{T}}_{\nu i_+}$ and integration by parts, (4.106) then evaluates to

$$\langle \hat\psi_1, \hat\psi_2 \rangle_{\mathrm{ren}} \Big|_{\mathcal{O}(\log \epsilon)} = \int_{\partial\Sigma} \frac{2}{3} i \, n_\mu \Big(\bar\lambda^{i_+} \gamma^{\alpha\mu} D_\alpha D \cdot \Psi^{\mathrm{T}}_{i_+} + \big(D_\alpha D \cdot \bar\Psi^{\mathrm{T}}_{i_+} \big) \gamma^{\alpha\mu} \lambda_{i_+} \Big) \log \epsilon \ . \tag{4.109}$$

Extracting from $\gamma_{\mu\nu\rho}{}^{\sigma}\Psi^{\mathrm{T}}_{\sigma i_+}$ the γ-matrices to the left in the orders $(\mu\nu\rho)$ and $(\mu\rho\nu)$, one finds $\gamma_{\mu\nu}\Psi^{\mathrm{T}}_{\rho i_+} = -\gamma_{\mu\rho}\Psi^{\mathrm{T}}_{\nu i_+}$. This in particular implies $\gamma^{\mu\alpha} D_\alpha D \cdot \Psi^{\mathrm{T}}_{i_+} = 0$ on rigid AdS. The logarithmic divergence therefore vanishes and the inner product is indeed finite in the sense that the asymptotic expansion in ϵ does not yield poles. This extends the results of [130] to the five-dimensional case where only a symplectic Majorana condition is available. We have thus established the availability of Neumann and mixed boundary conditions for the fields of the $\mathcal{N}$=2 supergravity, which allows to impose Neumann boundary conditions and to couple the theory to explicit gravitational dynamics on the boundary.

Extension to $\mathcal{N}$=4 gauged supergravity

We now discuss the $\mathcal{N}$=4 theory, starting with the bosonic sector for which we have determined the counterterms in Sec. 4.1.3. The calculation for the gauge fields is fully analogous to that for the U(1) field of the $\mathcal{N}$=2 theory, so finiteness of the inner product as discussed there applies here just as well and it remains to consider the scalar and the two-form fields $\hat B^\alpha_{\hat\mu\hat\nu}$. The scalar has a mass saturating the Breitenlohner-Freedman bound and thus both of the solution sets corresponding to Dirichlet and Neumann boundary conditions are normalizable [22]. Indeed, the counterterms (4.54), (4.55) do not give an additional contribution to the bare inner product which is finite already. It remains to consider the two-form field for which we find

$$\begin{aligned} j_0^{\hat\mu}[\hat B_1, \hat B_2] &= \frac{i}{2g_1} \hat e^{-1} \epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau} \hat{\bar{C}}_{1\hat\nu\hat\rho} \hat C_{2\hat\sigma\hat\tau} + \text{c.c.} \ , \\ j_{\mathrm{ct}}^{\mu}[\hat B_1, \hat B_2] &= \hat{\bar{C}}{}_1^{\rho\mu} D_\lambda \hat C_{2\rho}{}^{\lambda} \log \epsilon + \text{c.c.} - (1 \leftrightarrow 2) \ . \end{aligned} \tag{4.110}$$

The divergence of the bare part is only a logarithmic one since one of the components of $\hat C_{1/2}$ necessarily is in the radial direction. It is cancelled by the counterterm contribution arising from (4.55) due to (4.33).

We now turn to the fermionic part. Although we have not obtained the counterterms explicitly, we note that the discussion of the gravitino inner product is fully analogous to the $\mathcal{N}$=2 theory, as the derivative terms contributing to the inner product are the same up to the fact that we have twice as many gravitinos. It remains to discuss the spin-$\frac{1}{2}$ fermions. The bare inner product is

$$\langle \chi_1, \chi_2 \rangle_0 = -\frac{1}{2} \int_\Sigma i \hat{\bar{\chi}}_1^{i_+} (\hat{n} \cdot \hat{\gamma}) \hat{\chi}_{2 i_+} \ , \tag{4.111}$$

which is logarithmically divergent. We can infer the derivative term in the full counterterm action (4.55) from the supersymmetry transformations derived in Sec. 4.1.2.3 as follows. The full Weyl anomaly of which we have derived the bosonic part in (4.59) arises upon variation of the CFT generating functional with respect to a Weyl transformation. Thus, it is expected to be invariant under the non-anomalous Q-supersymmetry transformations. The full expression contains the fermionic parts in addition to (4.59), and for the inner product of the spin-$\frac{1}{2}$ fermions we are particularly interested in the terms which are quadratic in the fields and involve derivatives. The only possible term which can contribute to the inner product is $\bar{\chi}^{i_+ \mathrm{L}} \not{D} \chi^{\mathrm{L}}_{i_+}$, and to fix the coefficient we consider a supersymmetry variation of the full anomaly $\mathcal{A}^{\text{full}}$. Since the full variation has to vanish, this in particular applies to the restriction to just the terms involving each of ϕ and χ along with a derivative. Due to the structure of the supersymmetry transformations derived in Sec. 4.1.2.3 only the terms in $\mathcal{A}^{\text{full}}$ which are $\propto \phi^2$ or $\propto \bar{\chi}^{i_+ \mathrm{L}} \not{D} \chi^{\mathrm{L}}_{i_+}$ can contribute to these terms. We thus find

$$0 \overset{!}{=} \delta_\zeta \mathcal{A}^{\text{full}}\big|_{\phi D \chi} = \delta_\zeta \Big[i\alpha \bar{\chi}^{i_+ \mathrm{L}} \not{D} \chi^{\mathrm{L}}_{i_+} + \text{c.c.} - \frac{1}{2}\varphi^2 \Big] = -\frac{1}{\sqrt{2}}(1+2\alpha)\varphi \bar{\zeta}^{i_+} \not{D} \chi^{\mathrm{L}}_{i_+} + \text{c.c.}\,, \tag{4.112}$$

where the last equality holds up to total-derivative terms. This fixes $\alpha = -\frac{1}{2}$ and the bosonic counterterm action discussed in Sec. 4.1.3.2 therefore has to be augmented by

$$S^{\log}_{\text{ct},\chi} = \int_{r=\epsilon} d^4x \frac{1}{2} \hat{e}^* \big(i \hat{\bar{\chi}}^{i_+ \mathrm{L}} \not{D}^* \hat{\chi}^{\mathrm{L}}_{i_+} + \text{c.c.} \big) \log \epsilon \ , \tag{4.113}$$

to reproduce that term in the holographic Weyl anomaly. The associated contribution to the inner product exactly cancels the logarithmic divergence in (4.111). This completes the discussion also for the $\mathcal{N}{=}4$ gauged supergravity, showing that the inner products are again finite also for Neumann and mixed boundary conditions.

4.3.2 The gravity dynamics on the boundary

We have shown that for the $\mathcal{N}{=}2,4$ gauged supergravity fields the Neumann modes are rendered normalizable by the counterterm contributions, such that Neumann or arbitrary mixed boundary conditions can be imposed as long as they lead to a conserved symplectic structure. In this section we discuss in some more detail the corresponding dual boundary theory. Switching to Neumann or mixed boundary conditions can be understood as a deformation of the original AdS/CFT duality with Dirichlet boundary conditions, such that one still expects a holographic duality to hold [52]. For the argument one considers a Euclidean bulk gravity with partition function $\mathcal{Z}_{\text{Dir}}[g^{(0)}] := \int (\mathcal{D}\hat{g})_{g^{(0)}} e^{-S_{\text{ren}}}$, where $(\mathcal{D}\hat{g})_{g^{(0)}}$ is the measure for integration over metrics with fixed boundary value $g^{(0)}$ and S_{ren} denotes the renormalized bulk action. The AdS/CFT correspondence relates $\mathcal{Z}_{\text{Dir}}[g^{(0)}]$ to the CFT partition function in the background metric $g^{(0)}$ via $\mathcal{Z}_{\text{CFT}}[g^{(0)}] = \mathcal{Z}_{\text{Dir}}[g^{(0)}]$. For the induced gravity [133] obtained by integrating out the CFT fields, $\mathcal{Z}_{\text{induced}} := \int \mathcal{D}g^{(0)} \mathcal{Z}_{\text{CFT}}[g^{(0)}]$, one then finds

$$\mathcal{Z}_{\text{induced}} = \int \mathcal{D}g^{(0)} \mathcal{Z}_{\text{Dir}}[g^{(0)}] = \int \mathcal{D}g^{(0)} (\mathcal{D}\hat{g})_{g^{(0)}} e^{-S_{\text{ren}}} = \int \mathcal{D}\hat{g}\, e^{-S_{\text{ren}}} =: \mathcal{Z}_{\text{Neu}} \,. \tag{4.114}$$

This bulk partition function $\mathcal{Z}_{\text{Neu}}$ is interpreted by passing to the semi-classical limit. The variation of S_{ren} yields

$$\delta S_{\text{ren}} = \text{EOM} + \frac{1}{2} \int_{\partial\mathcal{M}} T^{\mu\nu} \delta g^{(0)}_{\mu\nu} \,, \tag{4.115}$$

with a finite $T_{\mu\nu}$. Due to the integration over $g^{(0)}$ stationarity of the action implies the Neumann boundary condition $T^{\mu\nu} = 0$. Thus, the bulk theory with that Neumann boundary condition is expected to be dual to the induced gravity theory on the boundary.

Among other Weyl-invariant contributions the induced gravity contains the non-local effective action necessary to reproduce the Weyl anomaly, which we have calculated in Sec. 4.1.3 and 4.2.2. For the explicit form of the pure-gravity part of this non-local action see e.g. (4.21) of [134]. Moreover, as shown in [135] at least a subsector of the theory describing the dynamics of the conformal factor does not yield ghosts on the Einstein cylinder $\mathbb{R} \times S^3$, despite its higher-derivative nature. Direct calculations e.g. in [106] suggest that the dynamics for the non-anomalous degrees of freedom, which are present already classically, is described by conformal supergravity. An analysis of pure Neumann boundary conditions for perturbative gravity on the Poincaré patch of AdS in [52] revealed tachyons and ghosts in the spectrum. More precisely, while for odd boundary dimensions the boundary theory is free of ghosts and the Weyl transformations are gauge transformations, for even boundary dimensions, where Weyl invariance is spoiled by the conformal anomaly, there are ghosts and tachyons.

Two interesting issues are raised by these results. For one thing, one would certainly like to formulate dualities with variants of conformal gravity on the boundary which are free of pathologies to begin with. The other line of research is to take the perturbative ghosts on Minkowski space seriously – they are in fact expected due to the higher-derivative character of conformal supergravity – and study e.g. their role beyond perturbation theory. We discuss approaches to these topics suggested by our previous investigations below.

On a technical level, the appearance of tachyons and ghosts can be traced back to the logarithmic terms in the asymptotic expansion of the bulk metric for even boundary dimension. In that sense the situation is very similar to the scalar field saturating the unitarity bound discussed in Sec. 3.2. As found in Sec. 3.1 and for global AdS in [69], such scalar fields similarly exhibit tachyons and ghosts due to the log-terms. However, in Sec. 3.2 we have shown how to obtain the singleton theory as a particular limit of the Klein-Gordon field with renormalized inner product, and that this allows for the formulation of a unitary theory. Speculations on a possible singleton theory for gravity can be found in [104], but to our knowledge there has been no formulation of a gravity singleton theory yet. Let us consider for metric perturbations on AdS a construction similar to the one discussed in Sec. 3.2 for the scalar field, which led to the singleton theory. This involved a field rescaling by the coefficient of a particular finite combination of boundary terms and the subsequent limit where this parameter is sent to infinity. The counterterms for gravity in the bulk have an analogous structure, as seen e.g. by replacing in (4.55) $\log\epsilon \to \log\epsilon + \kappa$. Drawing from the discussion of the scalar field we would similarly expect for gravity that bulk interaction terms become irrelevant in this limit, while on the boundary only a particular term quadratic in the field survives. We therefore expect to obtain a free spin-2 field with boundary conditions suppressing the logarithmic term, as found in Sec. 3.2 for the scalar field. Thus, we expect a unitary theory without tachyons. Interactions could be reintroduced by adding explicit κ-dependent terms, but to preserve unitarity they would have to be of a very specific form, see also [104]. Since the logarithmic terms appear in the asymptotic expansions of all the fields of the bulk supergravities discussed here, we expect the possibility to formulate a unitary singleton field theory for the full supergravity multiplets. The construction of the appropriate supersingleton representations can be found in [136].

Another route to formulating a duality with ghost-free boundary gravity is suggested by the recent observation that conformal gravity on AdS can be truncated to Einstein gravity by a suitable choice of boundary conditions [137]. A similar mechanism allows to render certain critical gravities ghost-free [138], where the Lagrangians are constructed from the cosmological Einstein-Hilbert part augmented by higher-order curvature terms with specific coefficients. Like conformal gravity alone, for suitable parameter choices the spectrum on AdS contains only the benign tachyons familiar for AdS spaces and it can be truncated to positive-norm states by appropriately choosing the boundary conditions. It may therefore also be possible to render the boundary gravities arising from our bulk supergravities with Neumann boundary conditions ghost-free by considering perturbations around AdS on the boundary instead of perturbations around Minkowski space. It would be interesting to study the truncations of [137, 138] holographically by basically repeating the analysis of [52] for the geometry with AdS on the boundary, as discussed in Sec. 3.1.1. To this end one certainly should improve on the renormalization prescription to cope with the normalizability issues discussed in Sec. 3.1.2.4.

As for studying the boundary theory directly on Minkowski space, we note that the results of [52] discussed above suggest that the appearance of the tachyons and ghosts is tied to the anomalous Weyl invariance[12]. As discussed in [41] for supersymmetric Yang-Mills theory coupled to conformal supergravity, demanding the conformal anomaly to cancel restricts the gauge group of the SYM sector to one of those with a four-dimensional Lie algebra. Thus, to establish a duality of gravity theories without pathologies for a four-dimensional boundary one would like to study the holographic dual of e.g. SU(2)⊗U(1) SYM theory coupled to conformal supergravity. Coupling the boundary CFT to explicit gravitational dynamics is achieved by adding to the bulk action gravitational boundary terms S_{bndy}, resulting in the boundary condition $T_{\mu\nu} = \delta S_{\text{bndy}}/\delta g^{\mu\nu}$ as described already in Sec. 3.3.3. Choosing an appropriate S_{bndy} should result in preservation of the radial diffeomorphisms/boundary Weyl transformations as follows. The failure of invariance under radial diffeomorphisms for the pure Neumann boundary condition can be seen from the fact that the transformation of $T_{\mu\nu}$ defined in (4.115) receives an anomalous contribution due to the log-terms in the asymptotic expansion of the metric. Thus, the condition $T_{\mu\nu} = 0$ is not invariant. For an appropriate S_{bndy} one then recovers the cancellation of the conformal anomaly from the holographic perspective as the fact that the boundary condition $\delta S_{\text{bndy}}/\delta g^{\mu\nu} = T_{\mu\nu}$ becomes invariant

[12] The appearance of ghosts due to a symmetry being anomalous is also familiar e.g. from the axial anomaly.

under radial diffeomorphisms/boundary Weyl transformations since both sides produce the same anomalous contribution. However, studying our specific setup requires the Yang-Mills theory at finite N and hence quantization of the bulk theory, to be discussed in Sec. 5.

Employing the dual description in terms of string theory one could also investigate possible effects restoring unitarity and whether the instability signaled by the tachyons is eliminated by supersymmetry or a mechanism like tachyon condensation (see e.g. [139]). As mentioned before, the SYM multiplet is coupled to gravity in AdS/CFT beyond the strict large-N and large-λ limits [106]. To study the gravitational boundary theory in string theory one has to uplift the configurations we have studied to the corresponding brane configurations, e.g. to the elliptic brane configuration in IIA string theory discussed in Sec. 4.1.3, and then study the brane dynamics within string theory. As string theory is expected to be ghost-free and the brane configurations being BPS states are stable, this should shed light on possible mechanisms curing the pathologies. It might be helpful here to better understand the switch from Dirichlet to Neumann boundary conditions directly in the dual string theory. For Abelian vector fields on AdS_4 switching the boundary conditions can be understood as follows [59]: Studying an Abelian vector field on AdS_4 is equivalent to studying it on $\mathbb{R}^{1,2} \times \mathbb{R}_+$, where it corresponds to the worldvolume theory of a D3-brane ending on a 5-brane. Dirichlet/Neumann boundary conditions now correspond to the D3-brane ending on a D5-brane/an NS5-brane, and the setups are related by an S-duality transformation. A step towards a similar string-theory description of the change of boundary conditions for the metric may be provided by the duality transformations discussed in [140].

4.4 Discussion and Outlook

In Sec. 4.1 we have studied the asymptotic structure of SU(2)⊗U(1) gauged $\mathcal{N}$=4 supergravity on asymptotically-AdS_5 backgrounds. We have shown that the $\mathcal{N}$=2 Weyl multiplet is induced on the conformal boundary with the complete local $\mathcal{N}$=2 superconformal transformations, see Tab. 4.1 and (4.37), (4.38). The gauge fixings we have employed for the bulk symmetries were chosen such that they do not cause a fixing of the symmetries induced on the boundary. Furthermore, the rescaled boundary limit of the bulk fields agrees with their rescaled pullback to the boundary. Different gauge fixings are expected to yield the same boundary fields and symmetries, possibly gauge fixed and/or with additional gauge degrees of freedom. We have then studied the four-dimensional $\mathcal{N}$=2 SCFTs dual to Romans' theory, e.g. the worldvolume theory on the D4-branes of the elliptic brane configuration discussed in [120], and carried out the holographic renormalization of the bosonic sector of the bulk theory. The boundary terms (4.54), (4.55) ensure finiteness of the action evaluated on solutions of the classical field equations, and for $\alpha = \frac{1}{4}$ also of the variations of the action evaluated on the classical solutions. In particular, we found a finite SCFT energy-momentum tensor which is obtained as the rescaled boundary limit of the Brown York energy-momentum tensor of the bulk theory (4.58). The boundary terms (4.55) break part of the bulk diffeomorphism invariance, which leads to the anomalous contribution (4.59) to the boundary Ward identity for Weyl invariance (4.44). Thus, we have obtained the Weyl anomaly for the dual SCFTs in a generic bosonic $\mathcal{N}$=2 conformal supergravity background, including the matter-field contributions. The renormalized action and Brown-York energy-momentum tensor (4.58) may also be useful for characterizing solutions of Romans' theory involving matter fields, e.g. for the solutions with non-Abelian gauge fields discussed in [141].

The discussion of holographic renormalization and the calculation of the Weyl anomaly of the dual theories in Sec. 4.1 was restricted to the bosonic sector of the bulk theory, as the natural backgrounds to consider the dual SCFTs in are the bosonic ones. With regard to establishing the availability of Neumann boundary conditions for the bulk theory we have extended these calculations in Sec. 4.2 to also include the fermionic sector of a truncation of the bulk theory to $\mathcal{N}$=2 gauged supergravity. We have calculated the full expression for the Weyl anomaly of the dual $\mathcal{N}$=1 SCFTs in generic, possibly fermionic conformal supergravity backgrounds, and the result confirms the expectations from the field-theory side.

The discussion of the anomalies also offers an interesting perspective on the results of [142]. As found there, for lower-dimensional theories the holographic counterterms coincide with the boundary terms required to preserve half of the supersymmetry in the presence of a boundary. The interesting question is whether this is a general pattern, i.e. whether demanding supersymmetry is sufficient to reproduce the holographic counterterms also in higher dimensions. The effect of the counterterms is to render the variations of the action finite, i.e. $\delta S_{\text{ren}} = \int_{\partial\mathcal{M}} X^I \delta\phi_I$ is finite on shell, where I is a multi-index labeling the various bulk fields ϕ_I of different spins. As discussed in Sec. 4.1.2 the bulk supersymmetry transformations split into two sets corresponding to Q- and S-supersymmetries on the boundary. Choosing the bulk variations such that they correspond to a Q-supersymmetry on the boundary yields for the boundary theory the corresponding Ward identity. Since the boundary Q-supersymmetry does not suffer from anomalies this variation should vanish, and if AdS/CFT holds the holographic counterterms therefore ensure the preservation of half of the supersymmetries. By this argument we expect that the holographic counterterms include a minimal set of boundary terms to preserve supersymmetry, but they may still contain an additional set of separately supersymmetry-invariant boundary terms.

Building on the previously obtained results on the asymptotic structure and the holographic renormalization, we have then established in Sec. 4.3 the existence of the bulk $\mathcal{N}$=2, 4 gauged supergravities with Neumann or mixed boundary conditions in the sense that the respective solutions are normalizable. This offers the possibility to investigate a duality of the specific supergravities arising from string theory to gravitational theories on the boundary, extending the results of [130] to the physically interesting case of five-dimensional bulk theories with four-dimensional boundary. The gravitational boundary theory contains pathologies like ghosts and tachyons found already before for the pure-gravity case. We have discussed possible approaches to formulating dualities with ghost-free gravity on the boundary, e.g. employing the geometries discussed in Sec. 3.1 or the singleton representation discussed in Sec. 3.2, and to studying the pathologies directly via the uplifts of the supergravity configurations to string theory.

A promising route available specifically for the case of a four-dimensional boundary is to study the boundary theory using twistor string theory [143]. The basic idea is that, as the descriptions of different asymptotic regimes of a single theory may well differ drastically, the type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ may also look vastly different in the limit of small $g_s^2 N$. The proposal of [143] is that this limit may be described by the topological B-model on $\mathbb{CP}^{3|4}$. The spectrum of open strings indeed reproduces the $\mathcal{N}$=4 SYM multiplet, and

SYM scattering amplitudes at small coupling can be calculated from that theory. However, from the closed-string sector the theory inevitably includes also conformal supergravity, the contributions of which can not be removed by a straightforward scaling limit [144]. This result, although not welcome in the first place, nicely fits into our discussion of AdS/CFT generalized by allowing also for Neumann-like boundary conditions and may even be given a deeper meaning in that context. One can then attempt to study the gravity sector of the boundary theory using twistor string theory. As discussed in [144] also from the twistor-string perspective, demanding the conformal anomaly to cancel restricts the gauge group of the SYM sector to one of those with a four-dimensional Lie algebra. The holographic description then requires AdS/CFT beyond the $N \to \infty$ limit. As $1/N$ corrections on the boundary correspond to quantum corrections in the bulk this entails a well-defined quantization prescription for supergravity in the first place, which is also called for on much more general grounds. We study that issue in the next section.

5 AdS/CFT beyond $N \to \infty$: Gravitino Quantization on Curved Spacetimes

The AdS/CFT prescription relates CFTs in the limit of infinite rank N of the gauge group to classical bulk supergravities. These inevitably contain as superpartner for the graviton a number of gravitinos, which are spin 3/2 Rarita-Schwinger fields. Corrections to the large-N limit in the boundary theory are related to quantum corrections in the bulk, such that a consistent quantization prescription for the bulk supergravities is crucial. This is required not only on AdS but also on generic non-Einstein spaces like the Klebanov-Strassler background, employed for the holographic description of non-conformal field theories with chiral symmetry breaking and confinement [145], or Lifshitz geometries for the description of quantum critical points [146]. On the other hand, supergravity as an extension of Einstein's general relativity may also play a role for a direct description of nature with interesting consequences for cosmology and particle physics. Specific issues like production mechanisms and properties of gravitino dark matter or the analysis of scattering experiments are best addressed in terms of effective QFT for the fluctuations around appropriate solutions of classical supergravity. While this means Minkowski space for collider physics, the less symmetric Friedmann-Robertson-Walker (FRW) backgrounds are of particular interest for studies of the early universe. A consistent QFT for the gravitino linearized around such non-Einstein solutions is therefore of great importance for both, the holographic description of strongly-coupled field theories and also for direct physical applications. Interestingly enough, the two fields are joined by the proposal to understand the cosmological FRW spacetimes from a holographic perspective [147]. The basic idea is to to understand the physics of an open FRW universe, where the constant-time slices are three-dimensional hyperbolic spaces, as encoded holographically in a two-dimensional Euclidean CFT on the conformal boundary of the hyperbolic slices[1], see Fig. 5.1.

The gravitino field equation is not of a form where the existence of a consistent quantization prescription is guaranteed by general results like [149]. The quantization on generic curved spacetimes has been discussed in detail recently [150], arriving at the conclusion that a consistent quantization is only possible on Einstein spaces, i.e. when the Einstein tensor is proportional to the metric. This statement is based on the non-conservation of the specific gravitino current proposed there, which would lead to inconsistencies when imposing canonical anticommutation relations (CAR). In this section we show that a consistent quantization of the gravitino on FRW spacetimes is indeed possible, raising this conclusion to question. This work was done in collaboration with Alexander Schenkel and published in [16]. In the sequel [150] has been revised and partial results have been obtained towards a treatment of $\mathcal{N}$=1 supergravity [151].

[1] A similar codimension-2 holography is discussed in [148] for asymptotically flat spaces.

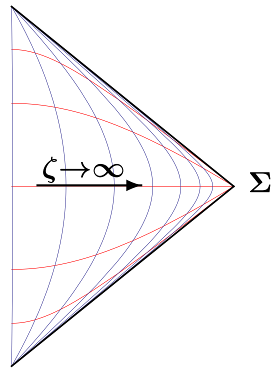


Figure 5.1: Penrose diagram of a four-dimensional open FRW universe with metric $ds^2 = -dt^2 + a(t)^2 d\mathcal{H}_3^2$, where $d\mathcal{H}_3^2 = d\zeta^2 + \sinh^2\zeta d\Omega_2^2$. The Ω_2 part is suppressed. The red and blue curves correspond to constant t and ζ, respectively. The conformal boundary of the hyperbolic slices $\mathcal{H}_3$ corresponds to spacelike infinity Σ.

We consider the Rarita-Schwinger field in d dimensions without assuming a specific model, but with properties general enough to capture the relevant supergravity cases. In particular, we allow for a spacetime-dependent mass as it arises in linearizations of supergravity around FRW [152], but do not fix the dependence a priori. We construct a canonical conserved current, and specializing to the case of d-dimensional spatially flat FRW spacetimes we prove that the associated inner product is positive definite on solutions of the Rarita-Schwinger equation in Sec. 5.1. In particular, it satisfies non-negativity, the necessary condition for a consistent implementation of CAR emphasized in [150], and we carry out the quantization in terms of a CAR-algebra. In Sec. 5.2 we discuss causality and the role of supergravity in that respect. We find that the propagation is in general non-standard, yet completely causal on a wide class of FRW spacetimes. Specifically, for the trace part of the Rarita-Schwinger field the domains of dependence on these spacetimes are in general more narrow than naïvely expected. Time-variations of the mass stretch these domains, eventually arriving at the standard light cones in the supergravity model of [152]. Examples of cosmological spacetimes allowing for a causal propagation are then studied in Sec. 5.3.

5.1 Consistent Quantization

We consider a Dirac Rarita-Schwinger field ψ_μ on a spacetime of dimension $d \geq 3$ with metric signature mostly minus. The action reads

$$S = \int d^d x\, e\, \overline{\psi}_\mu \mathcal{R}^\mu[\psi] \;, \tag{5.1}$$

with the Rarita-Schwinger operator

$$\mathcal{R}^\mu[\psi] := i\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + m\gamma^{\mu\nu}\psi_\nu \;. \tag{5.2}$$

The covariant derivative is $\mathcal{D}_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho$ and we assume a torsion-free background configuration. The mass m may be spacetime-dependent, but is assumed to be real and positive. This action with d=4 is the quadratic gravitino part of the matter-coupled $\mathcal{N}$=1 supergravity discussed in [152], up to metric conventions and the Majorana condition. The mass m in [152] is related to the Kähler and superpotential via $m = e^{K/2}\, W/M_{\mathrm{P}}^2$. The action (5.1) is real up to a boundary term and the Rarita-Schwinger operator (5.2) is formally self-adjoint with respect to $\big(\psi_1, \psi_2\big) := \int d^dx\, e\; \overline{\psi}{}_1^{\mu}\psi_{2\mu}$. That is, for all ψ_1 and ψ_2 with supports of compact overlap we have

$$\big(\psi_1, \mathcal{R}[\psi_2]\big) = \big(\mathcal{R}[\psi_1], \psi_2\big)\ . \tag{5.3}$$

Note that due to the vielbein postulate we have $\mathcal{D}_\mu \gamma^{\nu_1 \cdots \nu_n} = 0$. Contracting the equation of motion $\mathcal{R}^\mu[\psi] = 0$ with γ_μ leads to the on-shell constraint

$$i\,\not{\mathcal{D}}\,\gamma\cdot\psi - i\,\mathcal{D}_\mu\psi^\mu + \frac{d-1}{d-2}\,m\;\gamma\cdot\psi = 0\ , \tag{5.4a}$$

where $\gamma\cdot\psi := \gamma^\mu\,\psi_\mu$. Acting with $\mathcal{D}_\mu$ on $\mathcal{R}^\mu[\psi] = 0$ and using (5.4a) yields the second constraint

$$\frac{i}{2}G^{\mu\nu}\gamma_\mu\psi_\nu + (\partial_\mu m)\,\gamma^{\mu\nu}\psi_\nu + \frac{d-1}{d-2}i\,m^2\,\gamma\cdot\psi = 0\ , \tag{5.4b}$$

where $G^{\mu\nu} := \mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\,\mathcal{R}$ is the Einstein tensor. Using (5.4a), the Rarita-Schwinger equation $\mathcal{R}^\mu[\psi] = 0$ can be written as

$$\big(i\,\not{\mathcal{D}} - m\big)\psi_\mu - \big(i\,\mathcal{D}_\mu + \frac{m}{d-2}\,\gamma_\mu\big)\gamma\cdot\psi = 0\ . \tag{5.5}$$

Due to the derivative in the second term, (5.5) is not of Dirac-type [149] and the causal propagation of the Rarita-Schwinger field on a generic spacetime is not guaranteed a priori. We will discuss this point further in Sec. 5.2 and 5.3.

5.1.1 Conserved current

We construct Zuckerman's universal conserved current [53, 54, 55] for the Rarita-Schwinger field using the variational bicomplex reviewed in Sec. 2.4. We will verify its conservation explicitly, so the reader may also pass directly to (5.11).

The basic idea of the variational bicomplex is to consider functions and differential forms on the product space $\mathcal{M}\times\mathcal{S}$, with $\mathcal{M}$ being spacetime and $\mathcal{S}$ the space of field configurations. The differential forms on $\mathcal{M}\times\mathcal{S}$ can be decomposed into subspaces of a definite horizontal (i.e. spacetime) and vertical (i.e. field space) degree. Likewise, the exterior differential on $\mathcal{M}\times\mathcal{S}$ splits into a horizontal differential d and a vertical differential δ, increasing the horizontal/vertical degree by one.

The starting point of the construction is a Lagrangian described by a $(d, 0)$-form, i.e. of maximal horizontal degree, on $\mathcal{M}\times\mathcal{S}$. The Lagrangian form corresponding to the Dirac Rarita-Schwinger action (5.1) reads

$$L = i\,\overline{\psi}\wedge\star V^3\wedge\mathcal{D}\psi + (-1)^d m\,\overline{\psi}\wedge\star V^2\wedge\psi\ , \tag{5.6}$$

where $\psi := \psi_\mu dx^\mu$, $\mathcal{D}\psi := \mathcal{D}_\mu \psi_\nu \, dx^\mu \wedge dx^\nu$ and $\star$ denotes the Hodge operator defined by $\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}) = \frac{e}{(d-r)!}\epsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_d} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_d}$. Furthermore, $V := \gamma_\mu dx^\mu$ and the normalized n-fold product is denoted by $V^n := \frac{1}{n!} V \wedge .. \wedge V$. The vertical exterior derivative of the Lagrangian admits a decomposition

$$\delta L = E + \mathrm{d}\Theta \ , \tag{5.7}$$

with a unique source form E of degree $(d, 1)$ yielding the equations of motion and Θ of degree $(d{-}1, 1)$, which is unique up to horizontally exact parts. For the Lagrangian (5.6) we find

$$\Theta = -i \, \overline{\psi} \wedge \star V^3 \wedge \delta\psi \ . \tag{5.8}$$

Zuckerman's universal current is defined as the contraction of the $(d{-}1, 2)$-form $\mathsf{u} := \delta\Theta$ with two Jacobi fields, i.e. solutions of the linearized equations of motion. Since we are considering a linear theory, the Jacobi fields coincide with solutions of the Rarita-Schwinger equation $\mathcal{R}^\mu[\psi] = 0$. From (5.8) we find

$$\mathsf{u} = -i\delta\overline{\psi} \wedge \star V^3 \wedge \delta\psi \ , \tag{5.9}$$

and contracting with the two Jacobi fields $\overline{\psi}_1$ and ψ_2 we obtain the $(d{-}1, 0)$-form current

$$\mathsf{u}[\overline{\psi}_1, \psi_2] = i(-1)^d \, \overline{\psi}_1 \wedge \star V^3 \wedge \psi_2 \ . \tag{5.10}$$

Note that (5.10) does not depend on the field space coordinates. We pull back (5.10) to $\mathcal{M}$ to obtain a $d{-}1$-form current denoted by the same symbol on spacetime. From that current on $\mathcal{M}$ we define the more familiar one-form current $j[\overline{\psi}_1, \psi_2] := i \star \mathsf{u}[\overline{\psi}_1, \psi_2]$, which reads

$$j_\mu[\overline{\psi}_1, \psi_2] = -\overline{\psi}{}^\nu_1 \gamma_{\nu\mu\rho} \psi^\rho_2 \ . \tag{5.11}$$

Conservation of the $d{-}1$-form current $\mathsf{u}[\overline{\psi}_1, \psi_2]$, i.e. $\mathrm{d}\mathsf{u}[\overline{\psi}_1, \psi_2] = 0$, is equivalent to the condition $\nabla_\mu j^\mu[\overline{\psi}_1, \psi_2] = 0$, with ∇_μ being the covariant derivative on vector fields. We obtain

$$\begin{aligned} -\nabla_\mu j^\mu[\overline{\psi}_1, \psi_2] &= \overline{\mathcal{D}_\mu \psi_{1\nu}} \gamma^{\nu\mu\rho} \psi_{2\rho} + \overline{\psi}_{1\nu} \mathcal{D}_\mu \big(\gamma^{\nu\mu\rho} \psi_{2\rho}\big) \\ &= \overline{\gamma^{\rho\mu\nu} \mathcal{D}_\mu \psi_{1\nu}} \psi_{2\rho} + \overline{\psi}_{1\nu} \gamma^{\nu\mu\rho} \mathcal{D}_\mu \psi_{2\rho} \\ &= i \, \overline{\mathcal{R}^\rho[\psi_1]} \psi_{2\rho} - i \, \overline{\psi}_{1\nu} \mathcal{R}^\nu[\psi_2] \ . \end{aligned} \tag{5.12}$$

In the first line we have used the Leibniz rule for the covariant derivative and $\mathcal{D}_\mu \overline{\psi}_\nu = \overline{\mathcal{D}_\mu \psi_\nu}$, and in line two that due to the vielbein postulate $\mathcal{D}_\mu \gamma^{\nu\mu\rho} = 0$. Thus, the current is conserved when evaluated on solutions.

5.1.2 Positivity of the inner product

As noted in [150], non-negativity of the inner product constructed from the current (5.11) is a necessary condition for a consistent implementation of CAR. This is due to the anticommutator of the smeared quantum fields $\Psi_\mu(\bar{f}^\mu)$ and $\overline{\Psi}^\mu(f_\mu)$ being an expression of the form

$A^\dagger A + AA^\dagger$, which has a non-negative expectation value in any normalized state, see also Sec. 5.1.3 for more details.

We define the inner product associated to (5.11) by

$$\langle \psi_1, \psi_2 \rangle := \int_\Sigma n^\mu j_\mu[\overline{\psi}_1, \psi_2] \ , \tag{5.13}$$

where Σ is a Cauchy surface with future-directed unit normal vector field n^μ. Splitting $\mu = (0, m)$ we choose coordinates such that $ds^2 = g_{00} d\tau^2 + g_{mn} dx^m dx^n$ and likewise fix $e_0^a = \sqrt{g_{00}} \delta_0^a$. With a choice of Σ such that $n = \sqrt{g^{00}} \partial_\tau$ the integrand evaluates to

$$n^\mu j_\mu[\overline{\psi}_1, \psi_2] = -\Big(\psi_{1m}^\dagger \psi_2^m + \left(\gamma^m \psi_{1m}\right)^\dagger \left(\gamma^n \psi_{2n}\right) \Big) \ . \tag{5.14}$$

We verify non-negativity of the inner product (5.13) evaluated on solutions of the Rarita-Schwinger equation for d-dimensional FRW spacetimes, $e_\mu^a = a(\tau) \delta_\mu^a$. For compatibility with the FRW symmetries we assume that the mass depends on time only, $m = m(\tau)$. The spin connection is given by $\omega_{\mu ab} = 2a' a^{-1} e_{\mu[a} e_{b]}^0$, where prime denotes the derivative with respect to τ. The constraints (5.4) read for the FRW background

$$i\gamma^{\mu\nu} \partial_\mu \psi_\nu + \Big[\frac{i}{2}(d-2) \frac{a'}{a} \gamma^0 + \frac{d-1}{d-2} m \Big] \gamma \cdot \psi - i \Big(d - \frac{3}{2} \Big) \frac{a'}{a} \psi^0 = 0 \ , \tag{5.15a}$$

$$\gamma^0 \psi_0 = \frac{p - 2m^2 \frac{d-1}{d-2} + 2im' \gamma^0}{\rho + 2m^2 \frac{d-1}{d-2}} \gamma^m \psi_m =: \mathcal{A} \gamma^m \psi_m \ , \tag{5.15b}$$

where for the second equation we have used Friedmann's equations $G_0^0 = \rho$ and $G_m^n = -p\, \delta_m^n$ (in units $M_{\mathrm{P}} = 1$). These expressions in d=4 have been obtained in [152], up to metric conventions. Combining the $\mu = 0$ component of (5.5) with (5.15a) yields

$$i\gamma^{mn} \partial_m \psi_n = -\Big(m + \frac{i}{2}(d-2) \frac{a'}{a} \gamma^0 \Big) \gamma^m \psi_m =: -\mathcal{B} \gamma^m \psi_m \ . \tag{5.16}$$

Due to the constraints (5.16), (5.15b), only $(d-2) \cdot 2^{\lfloor d/2 \rfloor}$ of the $d \cdot 2^{\lfloor d/2 \rfloor}$ complex degrees of freedom of the Rarita-Schwinger field are independent. It is convenient to transform to spatial Fourier space via $\psi_m(\tau, x) = (2\pi)^{1-d} \int d^{d-1}k \; e^{ik_n x^n} \, \widetilde{\psi}_m(\tau, k)$. As in [129, 152] we separate the spatial part of the Rarita-Schwinger field $\widetilde{\psi}_m$ into the γ_m and k_m traceless part $\widetilde{\psi}_m^{\mathrm{T}}$ and the traces

$$\widetilde{\chi} := \gamma^n \widetilde{\psi}_n \ , \qquad \widetilde{\zeta} := k^n \widetilde{\psi}_n \ . \tag{5.17}$$

With $\hat{k}_m := k_m / |k|$, $|k| := \sqrt{-k_n k^n}$ and $\hat{\not{k}} := \hat{k}_m \gamma^m$ we find

$$\widetilde{\psi}_m = \widetilde{\psi}_m^{\mathrm{T}} + \frac{\gamma_m + \hat{k}_m \hat{\not{k}}}{d-2} \widetilde{\chi} + \frac{\gamma_m \hat{\not{k}} - (d-1) \hat{k}_m}{(d-2)|k|} \widetilde{\zeta} \ . \tag{5.18}$$

The constraint (5.16) in k-space yields the following relation between the traces

$$\widetilde{\zeta} = \big(\not{k} - \mathcal{B} \big) \widetilde{\chi} \ . \tag{5.19}$$

With (5.19) the decomposition (5.18) becomes

$$\widetilde{\psi}_m = \widetilde{\psi}_m^{\mathrm{T}} - \left(\hat{k}_m \hat{\not{k}} - \frac{(d-1)\hat{k}_m - \gamma_m \hat{\not{k}}}{(d-2)|k|} \mathcal{B} \right) \widetilde{\chi} \ . \tag{5.20}$$

The traceless part Ψ_m^{T} comprises $(d-3) \cdot 2^{\lfloor d/2 \rfloor}$ degrees of freedom and the trace part $\gamma^n \Psi_n$ the remaining $2^{\lfloor d/2 \rfloor}$. Using this on-shell decomposition, the inner product in Fourier space evaluates to

$$\langle \psi_1, \psi_2 \rangle = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \, a^{d-1} \left(-\widetilde{\psi}_{1m}^{\mathrm{T}\,\dagger} \widetilde{\psi}_2^{\mathrm{T}m} + \mathcal{C} \, \widetilde{\chi}_1^\dagger \widetilde{\chi}_2 \right) \ , \tag{5.21}$$

where

$$\mathcal{C} = \frac{d-1}{(d-2) \ |k|^2} \left(m^2 + \frac{1}{4}(d-2)^2 \, \frac{a'^2}{a^4} \right) \tag{5.22}$$

is positive. The integrand is pointwise (in k-space) non-negative, since in our conventions the spatial metric is negative definite. Thus, for any nonzero solution $\psi \not\equiv 0$ the norm is positive, $\langle \psi, \psi \rangle > 0$.

5.1.3 Quantization

Using the inner product (5.13), we can quantize the Dirac Rarita-Schwinger field on d-dimensional FRW spacetimes analogously to the spin 1/2 Dirac field [153]. We briefly outline the construction of the CAR-algebra and refer for details on fermionic quantization to [153, 154, 149] and references therein. We denote by Sol the space of spinor solutions of the Rarita-Schwinger equation which are of compact support when restricted to any Cauchy surface. The space $\overline{\mathsf{Sol}}$ of cospinor solutions is defined as the image of Sol under the map $\mathsf{Sol} \ni f_\mu \mapsto \overline{f}^\mu$. To the spinor/cospinor solutions we associate smeared field operators via $\mathbb{C}$-linear maps $f_\mu \mapsto \overline{\Psi}^\mu(f_\mu)$ and $\overline{f}^\mu \mapsto \Psi_\mu(\overline{f}^\mu)$. The CAR-algebra is defined as the $*$-algebra with unit $\mathbb{1}$ generated by these operators, subject to the relations

$$\Psi_\mu(\overline{f}^\mu)^\dagger = \overline{\Psi}^\mu(f_\mu) \ , \tag{5.23a}$$

$$\left\{ \Psi_\mu(\overline{f}^\mu), \overline{\Psi}^\nu(h_\nu) \right\} = \langle f, h \rangle \, \mathbb{1} \ , \tag{5.23b}$$

$$\left\{ \Psi_\mu(\overline{f}^\mu), \Psi_\nu(\overline{h}^\nu) \right\} = \left\{ \overline{\Psi}^\mu(f_\mu), \overline{\Psi}^\nu(h_\nu) \right\} = 0 \ , \tag{5.23c}$$

with $\dagger$ denoting the involution in the algebra. As pointed out in [150], non-negativity of the inner product is essential for the CAR (5.23): Assume any Hilbert space representation of the algebra above. Let $f_\mu \in \mathsf{Sol}$ be arbitrary and define $A := \overline{\Psi}^\mu(f_\mu)$, then (5.23a) and (5.23b) imply

$$A^\dagger A + A A^\dagger = \langle f, f \rangle \, \mathbb{1} \ . \tag{5.24}$$

From the expectation value in any normalized Hilbert space state $|\varphi\rangle$ one concludes

$$\langle f, f \rangle = \langle A\varphi | A\varphi \rangle + \langle A^\dagger \varphi | A^\dagger \varphi \rangle \geq 0 \ , \tag{5.25}$$

completing the argument. As we have shown in Sec. 5.1.2, for d-dimensional FRW spacetimes the inner product (5.13) indeed satisfies this necessary condition for a consistent quantization of the Rarita-Schwinger field.

The Dirac Rarita-Schwinger field as discussed above amounts to the generic case in d dimensions without imposing restrictions on d. However, if a Majorana condition $\overline{\psi}_\mu = \psi_\mu^{\mathsf{T}} C$ is available and used to reduce the Dirac spinor (e.g. in d=4 minimal supergravity), the quantization proceeds in a similar way: We restrict Sol to Majorana solutions $\mathsf{Sol}_{\mathsf{maj}}$ satisfying $\overline{f}_\mu = f_\mu^{\mathsf{T}} C$. The inner product (5.13) for Majorana solutions $f_\mu, h_\mu \in \mathsf{Sol}_{\mathsf{maj}}$ reads

$$\langle f, h\rangle = -\int_\Sigma n^\mu f^{\nu\mathsf{T}}\, C\, \gamma_{\nu\mu\rho}\, h^\rho \ . \tag{5.26}$$

It is symmetric since $C\gamma_{\nu\mu\rho}$ is anti-symmetric in the cases where a Majorana condition is available [155], such that

$$h^{\nu\mathsf{T}} C\, \gamma_{\nu\mu\rho}\, f^\rho = f^{\rho\mathsf{T}} \big(C\gamma_{\nu\mu\rho}\big)^{\mathsf{T}} h^\nu = f^{\rho\mathsf{T}} C\, \gamma_{\rho\mu\nu} h^\nu \ , \tag{5.27}$$

and it is also real

$$\langle f, h\rangle^* = \langle h, f\rangle = \langle f, h\rangle \ . \tag{5.28}$$

We quantize the Majorana Rarita-Schwinger field in terms of a self-dual CAR-algebra [156, 149]: We associate to the Majorana solutions hermitian smeared field operators via the $\mathbb{R}$-linear map $f_\mu \mapsto \overline{\Psi}^\mu_{\mathrm{maj}}(f_\mu)$. The self-dual CAR-algebra is defined as the $*$-algebra with unit 1 generated by these operators, subject to the relations

$$\big\{\overline{\Psi}^\mu_{\mathsf{maj}}(f_\mu), \overline{\Psi}^\nu_{\mathsf{maj}}(h_\nu)\big\} = \langle f, h\rangle\, \mathsf{1} \ . \tag{5.29}$$

5.2 Causality and the Role of Supergravity

In this section we discuss the propagation of the transversal and longitudinal parts of the Rarita-Schwinger field on d-dimensional FRW spacetimes[2]. The relevant equations are the constraints (5.15) and the equation of motion (5.5), or equivalently (5.5), (5.15b) and (5.16). The non-dynamical ψ_0 can be eliminated by solving (5.15b), $\psi_0 = \gamma_0\, \mathcal{A}\, \chi$, and (5.16) is manifestly implemented in the decomposition (5.20). It thus remains to solve (5.5). The $\mu = 0$ component yields the equation of motion for χ

$$i\big(\gamma^0 \partial_0 + \gamma^m \mathcal{A}\, \partial_m\big)\chi + \frac{2\mathcal{B} - m}{d-2}\, \chi + \frac{d-1}{d-2}\, \mathcal{B}^\dagger \mathcal{A}\, \chi = 0 \ . \tag{5.30}$$

The spatial components of (5.5) give – after using (5.30) – the equation for the transversal polarizations

$$i\gamma^\nu \partial_\nu \psi_m^{\mathrm{T}} + \left(\frac{i\, a'}{2a}(d-3)\gamma^0 - m\right) \psi_m^{\mathrm{T}} = 0 \ . \tag{5.31}$$

[2] We focus on the transversal and trace parts, ψ_m^{T} and χ, which are the degrees of freedom entering the inner product (5.21). The explicit reconstruction of ψ_m via (5.20) might be problematic due to the inverse powers of k. We thank Thomas-Paul Hack for useful discussions on this issue.

Thus, the transversal and longitudinal parts decouple.

Note that (5.31) is a Dirac-type operator and therefore ψ_m^{T} propagates causally, see e.g. [149]. In order to understand the causal properties of the longitudinal part χ, we define the 'effective gamma matrices'

$$\gamma^0_{\text{eff}} := \gamma^0 \;, \qquad \gamma^m_{\text{eff}} := \gamma^m \mathcal{A} \;. \tag{5.32}$$

They form a Clifford algebra $\{\gamma^\mu_{\text{eff}}, \gamma^\nu_{\text{eff}}\} = 2\, g^{\mu\nu}_{\text{eff}}$ with an 'effective metric' g_{eff} with components $g^{0\mu}_{\text{eff}} = g^{0\mu} = a^{-2}\delta^{0\mu}$ and

$$g^{mn}_{\text{eff}} = \left(\mathcal{A}_1^2 + a^{-2}\,\mathcal{A}_2^2\right) g^{mn} =: c^2_{\text{eff}}(\tau)\, g^{mn} \;. \tag{5.33}$$

The numerical coefficients $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined by $\mathcal{A} =: \mathcal{A}_1 + i\,\mathcal{A}_2\,\gamma^0$, see (5.15b). Thus, (5.30) is a Dirac-type operator on the spacetime with 'effective metric' $g^{\mu\nu}_{\text{eff}}$ and χ propagates causally with respect to $g^{\mu\nu}_{\text{eff}}$. Interpreted with respect to the original metric it propagates with a time-dependent speed of light $c^2_{\text{eff}}(\tau)$, as can be seen from (5.33). The propagation is therefore causal with respect to $g_{\mu\nu}$ as long as $c^2_{\text{eff}}(\tau) \leq 1$ for all times τ. Note that any time-dependence of the mass leads to a positive contribution to the effective speed of light since $\mathcal{A}_2^2 \propto (m')^2$ while $\mathcal{A}_1$ does not depend on m'. Thus, time-variations in the mass can never reconcile an otherwise acausal propagation of the longitudinal part with causality.

It is remarkable that the supergravity model discussed in [152] leads to $c^2_{\text{eff}}(\tau) \equiv 1$ for all d=4 FRW solutions. This ensures causal propagation of the gravitino on the one hand, but on the other hand also means that the time-varying mass $m = e^{K/2}W$ exactly compensates the deficit in the effective speed of light from being one, thus leading to standard causal properties. As we will discuss in the next section, also for the Rarita-Schwinger field alone, without the restrictions imposed by the supergravity model, a causal propagation is possible on a variety of FRW spacetimes. This, however, generically involves a time-dependent speed of light $c^2_{\text{eff}}(\tau) \leq 1$.

5.3 Cosmological Spacetimes

Within the class of matter models described by the equation of state $p = \omega\rho$, $\omega \in \mathbb{R}$, we identify the d=4 FRW spacetimes on which the Rarita-Schwinger field can propagate causally, and study the time-dependence of the effective speed of light. For a clearer physical interpretation we work in cosmological time t defined by $dt = a(\tau)d\tau$, such that the FRW metric reads $ds^2 = dt^2 - a(t)^2\, d\vec{x}^{\,2}$. As discussed in the previous section, time-variations of the mass give a positive contribution to the effective speed of light and thus can only tighten the restrictions on the background spacetime. We therefore focus on a constant mass $m > 0$ and identify the spacetimes for which $c^2_{\text{eff}}(t) \leq 1$ for all t.

Solving Friedmann's equations for the Hubble rate H yields

$$H(t) := \frac{\dot{a}(t)}{a(t)} = \frac{2}{3t(\omega+1) + 2\alpha} \;, \tag{5.34}$$

with constant of integration $\alpha = H(0)^{-1}$. The energy density is given by $\rho = 3H^2$ and the effective speed of light (5.33) for the longitudinal part becomes

$$c_{\text{eff}}^2(t) = \left(\frac{m^2 - \omega\, H(t)^2}{m^2 + H(t)^2} \right)^2 . \tag{5.35}$$

For the special case $\omega = -1$, i.e. de Sitter space, we find $c_{\text{eff}}^2(t) \equiv 1$ such that the Rarita-Schwinger field propagates with the standard speed of light, as expected. Consider now the case $\omega \neq -1$, where we set $\alpha = 0$ such that the cosmological singularity is at $t = 0$. For $t \to \pm\infty$ the Hubble rate vanishes and the speed of light c_{eff}^2 approaches 1. Thus, we find standard causal properties at late times. On the other hand, for $t \to 0$ the Hubble rate diverges, such that $c_{\text{eff}}^2 \to \omega^2$. For a causal propagation at times close to $t = 0$ we have to require $\omega \in [-1, 1]$. In fact, from (5.35) this condition is necessary and sufficient for causal propagation

$$c_{\text{eff}}^2(t) \leq 1 \quad \text{for all } t \qquad \Longleftrightarrow \qquad \omega \in [-1, 1] \; . \tag{5.36}$$

Interestingly, the matter models used in standard cosmology satisfy $\omega \in [-1, 1]$ and hence allow for a causal propagation of the Rarita-Schwinger field. We plot the effective speed of light for the cases $\omega = -1$ (cosmological constant), $\omega = 0$ (dust) and $\omega = 1/3$ (radiation) in Fig. 5.2. Note that for $\omega > 0$ the effective speed of light vanishes at $t = \pm 2\sqrt{\omega}/(3m\,|\omega + 1|)$.

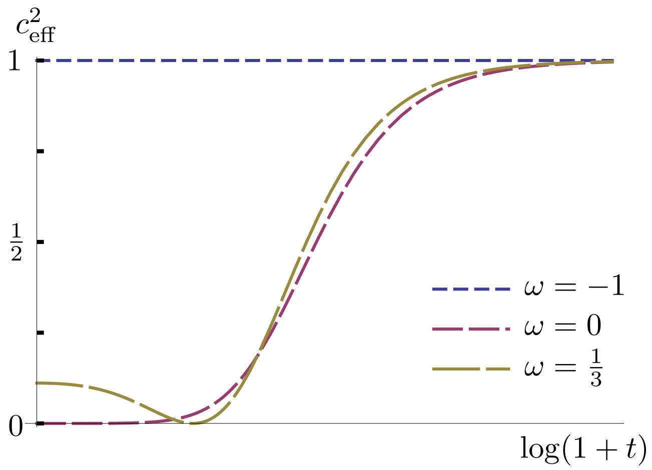


Figure 5.2: Cosmological-time dependence of the effective speed of light for the longitudinal gravitino components. The plot shows with increasing dash length $\omega \in \{-1, 0, \frac{1}{3}\}$, corresponding to a cosmological constant, dust and radiation dominated FRW universe, respectively.

This means that the longitudinal part of the Rarita-Schwinger field effectively does not propagate over extended spatial distances around these times.

5.4 Discussion

We have investigated the massive spin 3/2 Rarita-Schwinger field focusing on the properties relevant for quantization to show that a consistent quantization is possible also on non-Einstein spaces. Using the variational bicomplex we have constructed for generic spacetimes of dimension $d \geq 3$ a current which is conserved on solutions of the equation of motion. For d-dimensional FRW spacetimes we have also shown that the associated inner product is positive definite and therefore allows for a consistent quantization of the Rarita-Schwinger field in terms of a CAR-algebra. We then have studied the propagation of the transversal and longitudinal parts of the Rarita-Schwinger field and found that, while the transversal polarizations propagate causally on all FRW spacetimes, the propagation of the longitudinal part has quite distinct features. Its propagation is characterized by a time-dependent effective speed of light, and demanding causality imposes restrictions on the background spacetime and on time-variations of the mass. This discussion offered an interesting perspective on the role of the time-dependent mass in the supergravity model [152]. For a constant mass we have found that the propagation is causal for d=4 FRW spacetimes with a matter model described by the equation of state $p = \omega\,\rho$, if and only if $\omega \in [-1, 1]$. This in particular includes cosmological constant, dust and radiation dominated universes. Comparing this result to the weak-field condition found for the electromagnetic background in [157] which singles out preferred frames, we note that our condition is invariant under the FRW isometries.

The positive results of this detailed investigation of the gravitino on FRW spacetimes have allayed the fear that a consistent quantization could be generally impossible on non-Einstein spaces, which would otherwise challenge applications of supergravity in AdS/CFT or particle physics and cosmology. Moreover, the distinct features of the propagation of the longitudinal modes with time-dependent speed of light may also be relevant for models with explicit supersymmetry breaking, e.g. the MSSM. Interesting physical consequences may therefore be expected e.g. for bounds on gravitino dark matter.

6 Discussion and Outlook

We have focussed in this work on generalizations of the celebrated AdS/CFT dualities, which started off as a correspondence between certain supersymmetric conformal quantum field theories on the one hand and certain supersymmetric gravitational theories on specific string-theory backgrounds on the other. These dualities have been generalized in numerous ways to describe holographically phenomena like chiral symmetry breaking or confinement, the spectrum of mesons or QFTs at finite temperature. Moreover, there are even applications to solid-state physics to describe superconductors or quantum phase transitions. The generalizations we were after here involve on the lower-dimensional side of the duality, which corresponds to the conformal boundary of the asymptotically-AdS spaces, QFTs defined on curved spacetimes and as a further generalization also gravity. Attempting to holographically study these subjects is well motivated. QFT on curved spacetimes is non-trivial already for free theories with interesting effects like particle production due to spacetime curvature. Thus, alternative descriptions are even more called for than in the case of flat-space QFT. The properties of gravity as a quantum theory on the other hand are a ubiquitous subject and strong-coupling effects may specifically be relevant for restoring unitarity of conformal supergravities, which would then be valid candidates for renormalizable quantum theories of gravity.

Geometries where the conformal boundary is identified with the maximally symmetric dS or AdS spaces, rather than Minkowski space, have been considered in the literature. They can be employed for a dual description of CFTs defined on these spacetimes. This may in particular be used to shed light on the dynamics of strongly-coupled field theories on inflating cosmological spacetimes, see e.g. [61] where possibilities for phase transitions during cosmological evolution were explored. Also CFTs on manifolds with boundary are of particular interest, e.g. in the context of string-theory configurations with branes ending on branes. See e.g. [60] for a recent discussion of D3-branes ending on 5-branes and the connection to $\mathcal{N}=4$ SYM theory on AdS_4. As global/Poincaré AdS is conformal to half of the Einstein static universe/Minkowski space, such BCFTs can be studied equivalently on AdS in a fully covariant way, as pointed out in [62]. The geometries with AdS on the boundary thus allow for a holographic study of BCFTs. In the first part we have set up the framework of holographic renormalization for the geometries with dS and AdS on the boundary. We have then discussed the unitarity properties of the dual CFTs from the holographic perspective. Although this revealed subtleties in the renormalization for the setup with AdS on the boundary, a careful discussion has shown that generically the properties of the bulk theory are consistent with the expectations from the CFT perspective for both, dS and AdS on the boundary. The only exception is the specific case where the unitarity bound on the scaling dimension for gauge-invariant operators is saturated by an operator on the CFT side, corresponding to a specific choice of mass in the bulk theory. We have resolved the tension found for this specific case by employing the singleton representation, which allows for the formulation of a gauge theory for scalar fields on AdS. This has also shown that the singleton avoids some of the subtleties mentioned above for the AdS boundary.

The setup with AdS on the boundary in particular offers the possibility of implementing multi-layered AdS/CFT-type dualities, a logical possibility which was speculated to play a role for appropriate boundary geometries already in [52]. We have studied $\langle n \rangle$-manifolds as an appropriate geometric setting, where the boundary of a boundary is a well-defined concept, and discussed extensions of the notion of conformal compactness to that setting. The geometry with AdS on the boundary is recovered as a $\langle 2 \rangle$-manifold. We have then discussed specific $\langle n \rangle$-manifolds which are conformally $\langle n \rangle$-compact with boundary, meaning that they in principle allow for n-fold nestings of AdS/CFT dualities. However, building on our previous results we could conclude that nesting more than two instances of AdS/CFT does – at least in the framework we have discussed – not lead to non-trivial relations. An interesting concrete realization of double-layered AdS/CFT has been speculated to exist for M-theory setups involving M2-branes [101]. It is built around ABJM theory [97], the worldvolume theory of a stack of M2-branes which also provides a realization of three-algebras [158]. A version of that theory is expected to be dual to a two-dimensional CFT on the one hand and to a four-dimensional gravitational theory on the other hand, and we have discussed in some detail the particular $\langle 2 \rangle$-manifold which is appropriate in that context and the role of its isometries.

This discussion of multi-layered dualities already involved gravitational theories on the boundary of AdS, an issue to which we turned in more detail in the second part. Boundary conditions are known to play a crucial role for (quantum) field theory on AdS already since the early work [21] and the studies of gauged supergravities in [22]. Generalizations of Neumann and Dirichlet boundary conditions and also certain mixtures turn out to be appropriate, but e.g. for a scalar only Dirichlet boundary conditions are admissible for the full range of allowed masses. In a window above the Breitenlohner-Freedman stability bound also more general boundary conditions are possible and play a dedicated role in AdS/CFT [68]. Likewise, the discussion of allowed boundary conditions for vector fields reveals an intricate dependence on the spacetime dimension [132]. For the metric a similar analysis reveals that generically only Dirichlet boundary conditions are available, thus fixing the boundary geometry. However, the refined discussion in [52], taking into account the holographic-counterterm contributions to the symplectic structure, has shown that Neumann or mixed boundary conditions can in fact also be imposed on metric fluctuations, such that the boundary theory can be coupled to gravity. Actually, the possibility of a string-theory description of the $\mathcal{N}=4$ SYM theory coupled to conformal supergravity has been discussed already in [106]. The second part of this thesis was aimed at a concrete realization of such a duality within string theory. We considered five-dimensional gauged supergravities whose solutions can be lifted to describe actual string-theory setups. Restricting the configuration space to asymptotically-AdS geometries we have constructed the asymptotic multiplets of fields induced on the conformal boundary along with the symmetry transformations induced from the bulk symmetries. Building on these results we performed the holographic renormalization and calculated the Weyl anomaly of the dual CFTs in generic conformal supergravity backgrounds, providing a non-trivial test of the AdS/CFT conjecture. Employing the holographic counterterms we could then proceed to establish that Neumann and mixed boundary conditions can be imposed on the full five-dimensional supergravities, generalizing the results of [52] and [130] appropriately. We have thus established a dual description of a gravitational theory on the boundary in terms of a bulk theory which can be directly connected to string-theory setups. However, the boundary theory obtained this way possesses pathologies like perturbative ghosts and tachyons, as already known from [52]. These features do not come as a surprise for conformal gravity, and in fact their possible resolution e.g. by strong-coupling effects is a long-standing issue. We have discussed for instance how a singleton-type formulation of the bulk theory can lead to a unitary boundary theory. On the other hand, possible effects restoring unitarity directly for the boundary conformal gravity may be studied by means of a dual string-theory description. A promising route is the description in terms of twistor string theory, and a first investigation led back to the anomaly-free combinations of conformal supergravity with matter considered already in [41]. This in particular involves the $\mathcal{N}=4$ SYM theory with a gauge group of finite rank, and therefore calls for AdS/CFT beyond the limit of large N where the bulk theory is classical.

A consistent quantization of supergravity is certainly needed for holographic applications, as $1/N$ corrections to the large-N limit in the boundary theory of AdS/CFT are related to quantum corrections in the bulk. In particular, the quantization prescription is required not only on AdS but also on generic, possibly non-Einstein backgrounds. Moreover, it is also crucial for direct physical applications in particle physics or cosmology, e.g. to describe the dynamics in the early universe. Supergravities inevitably contain a number of spin 3/2 fields, the gravitinos, as superpartners for the graviton. In linearizations of supergravity around backgrounds like dS or FRW these gravitinos yield massive spin 3/2 fields, which should be describable in the framework of effective QFT. However, the field equations for such massive Rarita-Schwinger fields are not of the type where general results like [149] guarantee the existence of a consistent quantization prescription on all globally hyperbolic spacetimes. In the third part we have explicitly carried out the quantization of massive Rarita-Schwinger fields on FRW spacetimes and studied the causality properties of the resulting theories. We indeed obtained a consistent and causal quantization on a wide class of these FRW spacetimes as examples of non-Einstein spaces which are of interest for cosmology. Thus, we have shown that – in contrast to claims in the literature – a consistent quantization is possible at least on certain classes of non-Einstein spacetimes. This also revealed interesting causal features of the trace part. On dust or radiation dominated FRW universes, for example, it propagates in regions which are actually more narrow than the standard light cones. The propagation in particular is causal, and in the supergravity model [152] these regions are stretched to the standard light cones by a specific time dependence of the mass arising there. The non-standard features may still be of interest for particle physics models with explicit supersymmetry breaking like the MSSM.

In summary, we have on the one hand expanded on the holographic description of CFTs defined on curved spacetimes. In concrete applications we found results which – although partly in a somewhat intricate way – confirm the expectation that a duality also holds in that more general setting. We have also resolved a unitarity puzzle present already in the more common AdS/CFT setup involving global or Poincaré AdS and performed geometric constructions towards realizing multi-layered dualities. On the other hand, we have also expanded on the holographic description of gravity on the boundary. As an intermediate step we have calculated from concrete low-energy string theories the Weyl anomaly of the dual

CFTs in generic but fixed conformal supergravity backgrounds, and then promoted these gravitational background fields to actual dynamical supergravities. Both of these points offer interesting prospects for future research.

The discussion of the geometries with AdS on the boundary offered the possibility of multi-layered dualities. An interesting point is to study in more detail the concrete realization in M-theory. Specifically, the steps outlined in Sec. 3.3.3 for understanding from the M-theory perspective the coupling of ABJM theory to conformal supergravity with the subsequent Higgsing should be worked out in detail, to obtain a bulk description of the ABJM-type boundary theory on AdS_3. This boundary theory is itself supposed to have a dual description in terms of a two-dimensional CFT, and the construction would therefore provide a major step towards establishing double-layered holography in a concrete setting. For the implementation of further nestings of AdS/CFT-type dualities we found an obstruction as pointed out in Sec. 3.3.2. However, these renormalizability issues for nested Neumann boundary conditions discussed in Sec. 3.1.2 apparently are rather technical in nature and not so much of conceptual type. It would be interesting to see whether they can be resolved e.g. by employing the total boundary blow-up mentioned in the context of $\langle n \rangle$-manifolds in Sec. 3.3. Regarding the gravitational boundary theories the main point deserving further investigation is the fate of the ghosts discussed in Sec. 4.3.2. The truncation of conformal supergravity on AdS to a unitary subsector by imposing suitable boundary conditions [137, 138] provides an interesting starting point. It would certainly be worthwhile to study whether this mechanism can be applied to the concrete boundary theories arising from the bulk supergravities of Sec. 4 with Neumann boundary conditions, which in particular also contain the non-local action to reproduce the Weyl anomaly. To this end one would like to study the Neumann bulk theory perturbatively on the geometry with AdS on the boundary, which was discussed in Sec. 3. Mechanisms to restore unitarity directly on Minkowski space could be investigated by means of a possible ghost-free dual string theory, as discussed in Sec. 4.4. In particular, for the four-dimensional boundary it would be interesting to study the relation of the conformal boundary gravity to twistor string theory, which not only yields a description of $\mathcal{N}$=4 SYM theory but inevitably also contains conformal supergravity.

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List of Figures

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B Publications

Peer reviewed publications of PhD research:

[1] T. Ohl and **C. F. Uhlemann**
‘Saturating the Unitarity Bound in AdS/CFT$_{(\mathrm{AdS})}$’
JHEP 1205, 161 (2012), [arXiv:1204.2054 [hep-th]]

[2] T. Andrade and **C. F. Uhlemann**
‘Beyond the unitarity bound in AdS/CFT$_{(\mathrm{A})\mathrm{dS}}$’
JHEP 1201, 123 (2012), [arXiv:1111.2553 [hep-th]]

[3] A. Schenkel and **C. F. Uhlemann**
‘Quantization of the massive gravitino on FRW spacetimes’
Phys. Rev. D 85, 024011 (2012), [arXiv:1109.2951 [hep-th]]

[4] T. Ohl and **C. F. Uhlemann**
‘The Boundary Multiplet of N=4 SU(2)xU(1) Gauged Supergravity on Asymptotically-AdS$_5$’
JHEP 1106, 086 (2011), [arXiv:1011.3533 [hep-th]]

Earlier publications:

[5] A. Schenkel and **C. F. Uhlemann**
‘Field Theory on Curved Noncommutative Spacetimes’
SIGMA 6, 061 (2010), [arXiv:1003.3190 [hep-th]]

[6] A. Schenkel and **C. F. Uhlemann**
‘High energy improved scalar quantum field theory from noncommutative geometry without UV/IR-mixing’
Phys. Lett. B 694, 258 (2010), [arXiv:1002.4191 [hep-th]]

[7] T. Ohl, A. Schenkel and **C. F. Uhlemann**
‘Spacetime Noncommutativity in Models with Warped Extradimensions’
JHEP 1007, 029 (2010), [arXiv:1002.2884 [hep-th]]

[8] **C. F. Uhlemann** and N. Kauer
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[9] M.M. Nojiri, N. Kauer, **C.F. Uhlemann** et al.
‘Physics Beyond the Standard Model: Supersymmetry’
Les Houches 2007, Physics at TeV colliders, arXiv:0802.3672 [hep-ph]

Erklärung

Diese Doktorarbeit wurde am Lehrstuhl für Theoretische Physik II, Theoretische Elementarteilchenphysik, der Julius-Maximilians-Universität Würzburg angefertigt. Ich erkläre hiermit, dass ich diese Dissertation selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Ich erkläre außerdem, dass diese Dissertation weder in gleicher noch in anderer Form bereits in einem anderen Prüfungsverfahren vorgelegen hat. Ich habe früher außer den mit dem Zulassungsgesuch urkundlich vorgelegten Graden keine weiteren akademischen Grade erworben oder zu erwerben versucht.

Christoph Uhlemann
Würzburg, 19.09.2012