Contributions to Extreme Value Theory in the Space C[0,1]

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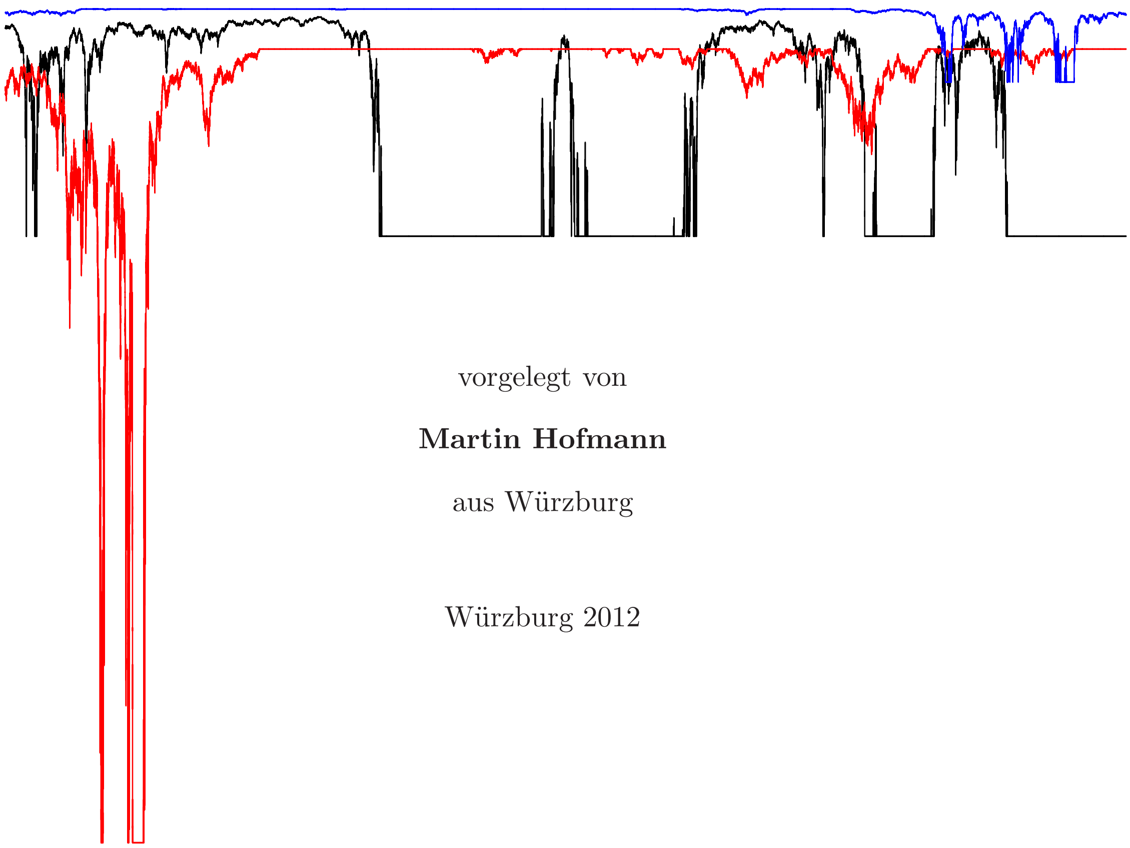

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Preface and Acknowledgments

What is “Extreme Value Theory” and what does “C[0,1]” mean?

Imagine your spouse goes out shopping – perhaps at a hardware or shoe store, depending on his or her preferences. Then, at least if you have been happily married for some time, you can calculate how much your wife or husband spent on average on similar shopping tours in the past and you get an estimate on the amount of money which will leave your bank account soon. All of your experience and calculations may be worthless, however, if, this time, your loved one sees something, he or she has always wanted – and buys it without regard to the costs.

You know that the probability for your account to be overdrawn is very small. But you also know that it could happen and if it did, it would have severe consequences. Even though you have (hopefully) not experienced such an event before, you may ask yourself whether you could estimate a certain amount of money that would not be exceeded by the expenses of your spouse with a particular, very high certainty.

The answer to this question can be given with the help of Extreme Value Theory: motivated by questions as the one above (but in a much more serious context!), first results in this field were published, roughly speaking, in the mid-twentieth century. Extreme Value Theory can be classified as a part of probability theory in a wider sense.

Now, imagine you are out shopping at the same time as your spouse. You have put so much money into your bank account, that you can be almost certain it will not be overdrawn, even if one of you gets infected by shopping fever. But if both of you find something you most urgently need, however, your account is once again in great danger. The probability of such a scenario is of course even smaller than that of the situation described above, but the consequences are more severe. This second issue concerns Multivariate Extreme Value Theory, wich focuses on (finitely many) extreme observations which occur “at the same time”.

This dissertation goes one step further and examines “infinitely many extreme observations occurring simultaneously”. This is quite abstract and lies – admittedly and fortunately – outside the scope of the shopping scenario (infinite number of spouses, all going shopping at the same time!). In general, infinitely many observations are not easy to handle and require additional structure: in the following, the “objects” under consideration are continuous functions on the compact interval [0,1] and the “space” containing all these functions is commonly denoted by C[0,1] in the field of mathematics.

Acknowledgments

The preceding lines were written for all those who do not share my profession but are close to me: this is what I did about the last three and a half years when I said: “I’m going to go think”.

I am very grateful to my family, especially to my wife and my daughter, and to all my friends for standing by my side and for their support, not only concerning this thesis.

Moreover, I am very thankful to my supervisor, Michael Falk, who taught me the basics of probability theory when I was an undergraduate student and who introduced me into this very interesting field of Extreme Value Theory. Especially to him, but also to all my colleagues at the University of Würzburg, I express my gratitude for the great support and all the constructive discussions during the last years.

Würzburg, July 2012, *Martin Hofmann*

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1 Introduction

Background

A max-stable distribution (MSD) on the real line with distribution function G is characterized by the property, that there exists for every $n \in \mathbb{N}$ some numbers $a_n > 0, b_n \in \mathbb{R}$, such that $(G(a_n x + b_n))^n = G(x)$ for every $x \in \mathbb{R}$. It immediately turns out, that "The class of MSD coincides with the class of all (non-degenerate) limit laws for (properly normalized) maxima of iid rvs"; this sentence is exactly Theorem 3.2.2 in Embrechts et al. [14]. Thereby, a "(non-degenerate) limit law" of "(properly normalized) maxima of iid rvs" is obtained, if we have independent and identically distributed (iid) random variables (rv) $X_1, \dots, X_n$ with distribution function F and there are numbers $c_n > 0, d_n \in \mathbb{R}$, such that

$$\lim_{n\to\infty} P\left(\frac{\max_{1\le i\le n} X_i - d_n}{c_n} \le x\right) = \lim_{n\to\infty} \left(F(c_n x + d_n)\right)^n = G(x), \quad x \in \mathbb{R}.$$

This limit behavior of normalized iid maxima of random variables is the starting point of extreme value theory (EVT). The answers to associated questions as on the characterization of those MSD G and on the requirements on the distribution functions F where given in the first papers on that issue until, roughly speaking, the middle of the last century. They can be found in any textbook on EVT since then; just Embrechts et al. [14] should be mentioned here, because this book contains a lot of "Notes and Comments" with all crucial references on the historical development given therein.

We start with a closer look at multivariate EVT, i.e. the random elements are elements of $\mathbb{R}^d$. The well-known de Haan-Resnick representation (cf. de Haan and Resnick [13], Falk et al. [17]) of MSD in $\mathbb{R}^d$, $d \ge 2$, can be reformulated in the following way.

A distribution function G on $\mathbb{R}^d$ is an MSD with standard negative exponential margins if and only if there exists a real number $m \ge 1$ and a random vector $Z = (Z_1, \dots, Z_d)$ which satisfies

$$\min(Z_1, \dots, Z_d) \ge 0 \text{ a.s.}, \quad \max(Z_1, \dots, Z_d) = m \text{ a.s.} \quad \text{and } E(Z_i) = 1,\ i = 1, \dots, d, \tag{1.1}$$

and the representation

$$G(x) = \exp\left(-E\left(\max_{i\le d}(|x_i| Z_i)\right)\right) \tag{1.2}$$

holds for $x = (x_1, \dots, x_d) \in (-\infty, 0]^d$.

Note that the connection between the angular measure ϕ of the MSD G in the commonly known de Haan-Resnick representation and the distribution $P * Z$ of Z is given by

$$\phi(A) = m(P * Z)(mA)$$

for any Borel set $A \subset \{x \in [0, \infty)^d : \max_{1\le i\le d} x_i = 1\} =: S_E$; here we used the notation in Falk et al. [17, Theorem 4.2.5] and we have chosen the maximum norm for $\|\cdot\|$.

We call Z a generator and $m = -\log G(-1, \dots, -1) = E\left(\max_{1\le i\le d} Z_i\right)$ the generator constant of the MSD G.

As the dependence structure within the components of the $d-$variate MSD is solely affected by the angular measure, in our rewritten setup the generator Z now is responsible for the dependence structure of G. Following the argumentation in Falk et al. [17, Section 4.4], we identify the D-norm of the MSD, $\|x\|_D = E\left(\max_{i\le d}(|x_i|Z_i)\right)$ for $x \in \mathbb{R}^d$. Immediately, $1 \le m \le d$ now follows from Falk et al. [17, Proposition 4.4.5], and Takahashi's Theorem (cf. Falk et al. [17, Theorem 4.3.2 and 4.4.1] yields

$$m = 1 \iff G \text{ has complete dependent margins;}$$

$$m = d \iff G \text{ has independent margins.}$$

Let us first have a closer inspection of the latter case of independent margins: for a generator $Z = (Z_1, \dots, Z_d)$ of an MSD G with standard negative exponential and independent margins we get

$$E\left(\sum_{i=1}^d Z_i\right) = d = m = E\left(\max_{1\le i\le d} Z_i\right),$$

where we have used the properties of a generator in (1.1). Because of $\sum_{i=1}^d Z_i \ge \max_{1\le i\le d} Z_i$, this is equivalent to $\sum_{i=1}^d Z_i = \max_{1\le i\le d} Z_i$ with probability one.

On the one hand, immediately $P(\min(Z_i, Z_j) = 0) = 1$ for each $1 \le i \ne j \le d$ follows. On the other hand, if $\min(Z_i, Z_j) = 0$ a.s. is true for each pair of different indices $i \ne j$, then we get by the identity $\max(a_1, \dots, a_d) = \sum_{\emptyset \ne T \subset \{1,\dots,d\}} (-1)^{|T|-1} \min(a_i,\, i \in T)$, which is true for arbitrary numbers $a_1, \dots, a_d$ and can be seen by induction, that

$$\max_{1\le i\le d} Z_i = \sum_{\emptyset \ne T \subset \{1,\dots,d\}} (-1)^{|T|-1} \min(Z_i,\, i \in T) = \sum_{i=1}^d Z_i$$

almost surely. Thus, we have shown the following:

If G is a MSD on $\mathbb{R}^d$ with standard negative exponential univariate margins and $Z = (Z_1, \dots, Z_d)$ is a generator of G, then:

$$G \text{ has independent margins} \iff P(\min(Z_i, Z_j) = 0) = 1, \quad 1 \le i \ne j \le d. \tag{1.3}$$

Note that this has the following reformulation in terms of the angular measure: an MSD has independent margins if and only if its angular measure has all its mass on the axes, i.e. it is the discrete measure with mass 1 on the unit vectors in $\mathbb{R}^d$.

Furthermore, if we have $m = E\left(\max_{i\le d}(Z_i)\right) = 1$, we get $\max_{1\le i\le d} Z_i = Z_j$ a.s., for all $1 \le j \le d$, again because of the second property of a generator in (1.1). But this implies $Z_1 = \dots = Z_d$ a.s. and we have shown:

If G is a MSD on $\mathbb{R}^d$ with standard negative exponential univariate margins and $Z = (Z_1, \dots, Z_d)$ is a generator of G, then:

$$G \text{ has complete dependent margins} \iff P(Z_1 = Z_i) = 1, \quad 1 \le i \le d. \tag{1.4}$$

Note that the latter arguments are in particular a complete proof of the "complete dependence part" of Takahashi's Theorem.

The foregoing considerations show, that proofs of interesting and useful results in finite dimensional EVT may get short and clear if one uses random elements (i.e. those generator random vectors Z) instead of angular measures.

Another great advantage of this approach is that we can adapt it to the infinite dimensional case which is the content of this thesis.

Aims and scope of this work

In the present work elements of functional EVT are considered: we study random elements which are continuous functions on a compact interval (we use $[0, 1]$ all along, but all assertions should hold for arbitrary compact subsets of $\mathbb{R}$) and which have a max-stable distribution on function space. This issue was treated earlier, for example in de Haan [12], de Haan and Pickands [11], Giné et al. [20], de Haan and Lin [10], see also the monograph de Haan and Ferreira [9].

There is an analogon to the de Haan-Resnick representation in $C[0, 1]$, cf. Giné et al. [20], characterizing all continuous max-stable processes (MSP) by means of an "angular measure", this time this measure is defined on the space of continuous functions. We start in this work again by "rewriting" this result in terms of a generator $\boldsymbol{Z}$, now, of course, the realizations of $\boldsymbol{Z}$ are continuous functions. It turns out (cf. Lemma 2.7), that a process $\boldsymbol{\eta}$, which has all its sample paths in $C[0, 1]$ is an MSP with standard negative exponential distributed one-dimensional margins (i.e. $P(\eta_t \le x) = \exp(x),\ x \le 0, t \in [0, 1]$), if, and only if, its functional distribution function can be represented as

$$P(\boldsymbol{\eta} \le f) = \exp\left(-E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right)\right), \quad \text{for all } f \in \bar{E}^-[0, 1],$$

where $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ is a process whose sample paths all are in $\bar{C}^+[0, 1] := \{f : [0, 1] \to [0, \infty),\ f \text{ is continuous}\}$ and it fulfills

$$\sup_{t\in[0,1]} Z_t = m \in [1, \infty) \text{ a.s. and } E(Z_t) = 1, \quad t \in [0, 1] \tag{1.5}$$

(here $\bar{E}^-[0, 1]$ denotes the set of those functions on $f : [0, 1] \to (-\infty, 0]$ which are bounded and have at most a finite set of discontinuities).

This is in high accordance to the finite-dimensional case and this setup enables us to carry over several assertions from the multivariate theory to the functional case in a (somehow) straightforward way. Although some of the results in the Sections 2.1 and 2.2 are already known (and were established, essentially, in Giné et al. [20]) we state them for the sake of completeness and consistency.

But there are new problems (and answers) arising from the infinite dimension of the considered space and the continuity of the processes: Does any continuous MSP hit every value x in its image set with positive probability (cf. Section 2.3.1)? Is there any MSP which hits every $x < 0$ twice with positive probability but the event "hitting the same value three or more times" has probability zero (cf. Section 2.4 for Examples on various issues)? Which MSP with standard negative exponential margins can itself be a generator process by multiplying it by -1 – and are there some MSP which coincide with their generator processes (apart from the sign; cf. Section 2.5)?

In Chapter 3 we turn our focus to the limit of normalized maxima of iid copies of continuous processes. In de Haan and Lin [10], convergence of those maxima towards an MSP is considered in terms of weak convergence on $C[0,1]$. This is, of course, consistent to the finite dimensional setup, as in $\mathbb{R}^d$ weak convergence of measures is equivalent to convergence of the corresponding distribution functions. But as weak convergence on function space is in general quite difficult to handle (cf. Billingsley [7]), we introduce a type of convergence which is formulated in term of distribution functions on function space: we say that a stochastic process $\boldsymbol{Y}$ in $C[0,1]$ is in the functional domain of attraction of a standard MSP $\boldsymbol{\eta}$ ("standard" says, that all univariate margins are standard negative exponential distributions), if there are functions $a_n \in C^+[0,1] := \{f \in C[0,1] : f > 0\}$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} P\left(\frac{\boldsymbol{Y}-b_n}{a_n} \le f\right)^n = P(\boldsymbol{\eta} \le f)$$

for any $f \in \bar{E}^-[0,1]$.

This approach is in great accordance to the finite dimensional case and it is more general than the usual one based on weak convergence, which is shown in Section 3.1. We bring this type of convergence towards a (standard) MSP in line with convergence of the finite dimensional distributions and hypoconvergence (see Molchanov [22]).

Moreover, in Section 3.2, we suggest another type of convergence for continuous processes: Let $\boldsymbol{X}$ be a stochastic process in $\bar{C}^-[0,1]$ and put for $f \in \bar{E}^-[0,1]$

$$S_{\boldsymbol{X}}(f) := \int_0^1 \mathbf{1}\left(X_t > f(t)\right)\, dt,$$

which is the sojourn time of $\boldsymbol{X}$ above the function f, i.e. the random "time" which the process $\boldsymbol{X}$ spends above the "threshold function" f (cf. Berman [6]). We say that a sequence of stochastic processes $\boldsymbol{X}^{(n)}$ in $\bar{C}^-[0,1]$, $n \in \mathbb{N}$, converges with respect to the sojourn time transformation to $\boldsymbol{X}$ in $\bar{C}^-[0,1]$, denoted by $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{X}$, if

$$S_{\boldsymbol{X}^{(n)}}(f) \to_D S_{\boldsymbol{X}}(f), \qquad f \in \bar{E}^-[0,1],\ n \to \infty;$$

note that this is convergence of univariate rv. We compare this type of convergence with the convergence of functional distribution functions introduced before (Lemma 3.8).

In Section 3.4 we introduce "copula processes", which are defined by $\boldsymbol{U} = (U_t)_{t\in[0,1]} := (F_t(Y_t))_{t\in[0,1]}$ if $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ is a continuous stochastic process with continuous marginal df $F_t,\ t \in [0,1]$. Note that the sample paths of $\boldsymbol{U}$ are in $C[0,1]$. The foregoing results on functional domain of attraction are applied to those copula processes.

It was established in de Haan and Lin [10] that univariate weak convergence of the marginal maxima to a univariate MSD together with weak convergence of the corresponding copula process towards a standard MSP in the function space is equivalent to the assertion that a process is in the domain of attraction (in the sense of weak convergence) of an MSP with arbitrary marginal distributions, compare also de Haan and Ferreira [9, Chapter 9]. We examine this issue in case of convergence of functional distribution functions in Section 3.5.

The idea of functional generalized Pareto distributions is introduced in Section 3.6, where the foregoing results allow representations and argumentation in great accordance to the multivariate case.

Chapter 4 is about sojourn times of continuous stochastic processes $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ which have identical continuous marginal distribution functions (df) F, say. Sojourn times of stochastic processes have been extensively studied in the literature, with emphasis on Gaussian processes and Markov random fields, we refer to Berman [6] and the literature given therein. A more general approach is the excursion random measure as investigated by Hsing and Leadbetter [21] for stationary processes. Different to that, we will investigate the sojourn time under the condition that the copula process $\boldsymbol{C} := (F(Y_t))_{t\in[0,1]}$ corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of a max-stable process $\boldsymbol{\eta}$, say.

Denote by $N_s := \sum_{i=1}^n 1_{(s,\infty)}(Y_{i/n})$ the number of exceedances among $(Y_{i/n})_{1\le i\le n}$ above the threshold s. The *fragility index* (FI) corresponding to $(Y_{i/n})_{1\le i\le n}$ is defined as the asymptotic expectation of the number of exceedances given that there is at least one exceedance:

$$FI := \lim_{s\nearrow\omega(F)} E(N_s \mid N_s > 0),$$

where $\omega(F) := \sup\{t\in\mathbb{R} : F(t) < 1\}$. The FI was introduced in Geluk et al. [19] to measure the stability of a stochastic system. The system is called stable if $FI = 1$, otherwise it is called fragile.

We consider the analogon for continuous processes: let $S_{\boldsymbol{Y}}(s)$ be the sojourn time of $\boldsymbol{Y}$ above the constant threshold function $f \equiv s \in \mathbb{R}$. It turns out that the limit $\lim_{s\nearrow\omega(F)} E(S(s) \mid S(s) > 0)$, of the expected sojourn time given that it is positive, exists if the copula process corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of a max-stable process. This limit coincides with the limit of the FI corresponding to $(Y_{i/n})_{1\le i\le n}$ as n and the threshold increase, cf. Section 4.1. Moreover, by defining

$$I(s) = \int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt,$$

we get the total "sum" of excesses above the threshold s; so the idea of the (cumulative) expected shortfall at level s pertaining to $\boldsymbol{Y}$ as the expectation of the total sum of excesses, given that there is at least one exceedance, can be carried over to the function space:

$$\mathrm{ES}(s) := E(I(s) \mid S(s) > 0),$$

see Lemma 4.6 and Proposition 4.7.

For such processes, which are in a certain neighborhood of a generalized Pareto process, we can replace the constant threshold by a threshold function and we can compute the (asymptotic) conditional sojourn time distribution above a high threshold function; max-stable processes are prominent examples, cf. Section 4.2.

Given that there is an exceedance $Y_{t_0} > s$ above the threshold s at t_0, we can also compute the asymptotic distribution of the remaining excursion time, that the process spends above the threshold function without cease; Section 4.3 contains these considerations.

There is, finally, an Appendix on “Random Closed Sets and Hypoconvergence of Continuous Processes” (from p. 79 on). This is more or less a collection of results taken from the book of Molchanov [22]: therein, all results are given for (upper or lower) semi-continuous processes in great generality, so this Appendix should show the connections to our setup more clearly.

Notes and Conventions

One may wonder what those graphs on the cover page stand for: they are realizations of a (standard) generalized Pareto process, cf. Section 3.6, where the pertaining generator process is a modified Brownian Motion, cf. Example 2.31. All graphs pictured in the several figures within this work were produced by using `R` (version 2.10.1).

To improve the readability we use in the sequel bold face, such as $\boldsymbol{\xi}$, $\boldsymbol{Y}$, for stochastic processes and default font, f, a_n etc., for non-stochastic functions. Operations on functions such as $\boldsymbol{\xi} < f$ or $(\boldsymbol{\xi} - b_n)/a_n$ are always meant pointwise. The usual abbreviations *df, fidis, iid, a.s.* and *rv* for the terms *distribution function, finite dimensional distributions, independent and identically distributed, almost surely* and *random variable*, respectively, are used.

Furthermore, we use the terms “weak convergence” and “convergence in distribution” side by side and with the usual inconsistency in denotation. By definition, a sequence of random elements converges “in distribution” if the sequence of the corresponding distributions converges “weakly”, see Billingsley [7] for a detailed introduction to this issue. So if we say that a sequence of random elements converges weakly, this is to be understood as weak convergence of the sequence of the pertaining distributions. The notation for this type of convergence is “$\to_D$”.

2 Max-Stable Processes in $C[0,1]$

A max-stable process (MSP) $\boldsymbol{\zeta} = (\zeta_t)_{t\in[0,1]}$ which has all its sample paths in $C[0,1] := \{f : [0,1] \to \mathbb{R} : \ f \text{ continuous}\}$, equipped with the sup-norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, is a stochastic process with the characteristic property that its distribution is max-stable, i.e., $\boldsymbol{\zeta}$ has the same distribution as $\max_{1\le i\le n}(\boldsymbol{\zeta}^{(i)} - b_n)/a_n$ for independent copies $\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}, \dots$ of $\boldsymbol{\zeta}$ and some $a_n, b_n \in C[0,1]$, $a_n > 0$, $n \in \mathbb{N}$ (c.f. de Haan and Ferreira [9]), i.e.,

$$\boldsymbol{\zeta} \overset{d}{=} \max_{1\le i\le n} (\boldsymbol{\zeta}^{(i)} - b_n)/a_n, \tag{2.1}$$

the maxima being taken pointwise. In particular, ζ_t is a max-stable real valued rv for every $t \in [0,1]$, i.e. its distribution has for some $a(t) > 0$ and $b(t), \gamma(t) \in \mathbb{R}$ a von Mises representation (cf. Falk et al. [17], de Haan and Ferreira [9])

$$P\left(\frac{\zeta_t - b(t)}{a(t)} \le x\right) =: F_{\gamma(t)}(x) = \exp\left(-(1+\gamma(t)x)^{-1/\gamma(t)}\right), \quad \gamma(t)x \ge -1, \tag{2.2}$$

$t \in [0,1]$, where

$$F_{\gamma(t)}(x) = \begin{cases} 0 & \text{for } \gamma(t) > 0 \text{ and } x \le -1/\gamma(t), \\ 1 & \text{for } \gamma(t) < 0 \text{ and } x \ge -1/\gamma(t), \\ \exp(-\exp(-x)) & \text{for } \gamma(t) = 0 \text{ and } x \in \mathbb{R}. \end{cases}$$

It was shown in Giné et al. [20] that there is a straightforward relationship between the continuous norming functions $a_n > 0$, b_n in (2.1) and $a(t) > 0$ and $b(t), \gamma(t) \in \mathbb{R}$ in (2.2) as for every $n \in \mathbb{N}$ and $t \in [0,1]$

$$a_n(t) = n^{\gamma(t)}, \qquad b_n(t) = \begin{cases} \left(n^{\gamma(t)} - 1\right)\left(\frac{a(t)}{\gamma(t)} - b(t)\right) & \text{for } \gamma(t) \ne 0 \\ a(t)\ln n & \text{for } \gamma(t) = 0. \end{cases} \tag{2.3}$$

This can be seen by elementary calculations as follows: given an arbitrary MSP $\boldsymbol{\zeta}$ in $C[0,1]$ and the setup as before, we get for $n \in \mathbb{N}$

$$\begin{aligned} P\left(\frac{\zeta_t - b(t)}{a(t)} \le x\right) &= P\left(\zeta_t \le a(t)x + b(t)\right) \\ &= P\left(\max_{1\le i\le n} (\zeta_t^{(i)} - b_n(t))/a_n(t) \le a(t)x + b(t)\right) \\ &= \left[P\left(\zeta_t \le a_n(t)(a(t)x + b(t)) + b_n(t)\right)\right]^n \\ &= \left[P\left(\frac{\zeta_t - b(t)}{a(t)} \le a_n(t)x + \frac{(a_n(t)-1)b(t) + b_n(t)}{a(t)}\right)\right]^n. \end{aligned} \tag{2.4}$$

For $\gamma(t) = 0$ this reads as

$$\exp(-\exp(-x)) = \exp\left(-n\exp\left(-a_n(t)x + \frac{(a_n(t)-1)b(t) + b_n(t)}{a(t)}\right)\right),$$

which is true for $a_n \equiv 1$ and $b_n(t) = a(t) \cdot \ln n,\ t \in [0,1]$.

In case of $\gamma(t) \neq 0$ equation (2.4) leads to

$$(1+\gamma(t)x)\, n^{\gamma(t)} = \left(1 + \gamma(t)a_n(t)x + \frac{(a_n(t)-1)b(t) + b_n(t)}{a(t)}\right),$$

which is, in turn, true for $a_n(t) = n^{\gamma(t)}$ and $b_n(t) = \left(n^{\gamma(t)} - 1\right)\left(\frac{a(t)}{\gamma(t)} - b(t)\right),\ t \in [0,1]$.

The next assertion was established in Lemma 3.5.(i) in Giné et al. [20], it can be proven by (2.3) together with some analytic arguments; because it is crucial for what follows and for the sake of completeness we state the result here.

Lemma 2.1. *Let $\boldsymbol{\zeta}$ in $C[0,1]$ be an MSP and for $t \in [0,1]$ let $a(t)$, $b(t)$, $\gamma(t)$ be the norming numbers from equation* (2.2)*. Then the functions $a : [0,1] \to (0,\infty),\ t \mapsto a(t),\ b : [0,1] \to \mathbb{R},\ t \mapsto b(t)$ and $\gamma : [0,1] \to \mathbb{R},\ t \mapsto \gamma(t)$ are continuous in $t \in [0,1]$.*

In finite-dimensional case, characterization of all max-stable distributions is typically done by characterizing some standard case (with certain margin restrictions) and reaching all other cases by (margin) transformation. We will proceed in an analoguous way by characterizing a class of standard MSP and stating thereafter, that an arbitrary MSP can always be transformed to such a standard MSP.

2.1 Standard Max-Stable Processes and D-Norm on Function Spaces

We call a process $\boldsymbol{\eta}$ which has all its sample paths in $C[0,1]$ a standard MSP, if it is an MSP with standard negative exponential (one-dimensional) margins, $P(\eta_t \leq x) = \exp(x)$, $x \leq 0$, $t \in [0,1]$.

According to Giné et al. [20] and de Haan and Ferreira [9], a process $\boldsymbol{\xi}$ in $C[0,1]$ is called a simple MSP, if it is an MSP with standard Fréchet (one-dimensional) margins, $P(\xi_t \leq x) = \exp(-1/x)$, $x > 0$, $t \in [0,1]$. We will see that each simple MSP $\boldsymbol{\xi}$ can be transformed to a standard MSP $\boldsymbol{\eta}$ by just transforming the univariate margins $\eta_t := -1/\xi_t$, $0 \leq t \leq 1$, and, vice versa, $\xi_t := -1/\eta_t$. With this one-to-one correspondence one might consider the spaces of simple MSP and standard MSP as dual spaces.

Remark 1. Note a first easy consequence for standard MSP: as the standard negative exponential distribution has the parameters $a(t) = 1,\ b(t) = \gamma(t) = -1,\ t \in [0,1]$ in the von Mises representation (2.2), we immediately get by (2.3) the norming functions $a_n(t) = 1/n,\ b_n(t) = 0$ for all $t \in [0,1]$, i.e. for independent standard MSP $\boldsymbol{\eta}^{(i)}, i = 1, \dots, n$, we have

$$\boldsymbol{\eta} \stackrel{d}{=} n \max_{1 \leq i \leq n} \boldsymbol{\eta}^{(i)}, \quad n \in \mathbb{N}.$$

This is, of course, in complete accordance with multivariate EVT: it is well-known that the one-dimensional margins solely determine the norming "constants" $a_n,\ b_n$ of higher dimensional max-stable random elements.

A crucial observation is the fact that neither a simple MSP $\boldsymbol{\xi}$ nor a standard MSP $\boldsymbol{\eta}$ attains the value 0 with probability one. First we consider the case of standard MSP and we will see, that $P(\eta_t = 0 \text{ for some } t \in [0,1]) = 0$ for some standard MSP $\boldsymbol{\eta}$ follows directly by using the max-stability and the marginal distribution. In Section 2.3 we get a much more general result which contains the following lemma as an easy consequence, compare Remark 5.

Lemma 2.2. *Let K be a compact subset of $[0,1]$ and let $\boldsymbol{\eta}_K = (\eta_t)_{t\in K}$ be an MSP on K with standard negative exponential margins, which has all its sample paths in the space of continuous functions $\bar{C}^-(K) := \{f : K \to (-\infty, 0],\ f \text{ is continuous}\}$. Then we have*

$$P\left(\sup_{t\in K} \eta_t < 0\right) = 1.$$

Proof. First we state

$$P\left(\sup_{t\in K} \eta_t < 0\right) \in \{0,1\}, \tag{2.5}$$

which can be seen as follows.

From the max-stability of $\boldsymbol{\eta}$ (cf. Remark 1) we obtain

$$\begin{aligned} P\left(\sup_{t\in K} \eta_t \le \frac{-\varepsilon}{n}\right) &= P\left(\boldsymbol{\eta}_K \le -\varepsilon/n\right) \\ &= \left(P\left(\boldsymbol{\eta}_K \le -\varepsilon/n\right)^n\right)^{1/n} \\ &= P\left(n \max_{1\le i\le n} \boldsymbol{\eta}_K^{(i)} \le -\varepsilon\right)^{1/n} \\ &= P\left(\sup_{t\in K} \eta_t \le -\varepsilon\right)^{1/n} \\ &\to_{n\to\infty} 1 \end{aligned}$$

unless $P\left(\sup_{t\in K} \eta_t \le -\varepsilon\right) = 0$. Equation (2.5) now follows from the continuity from above of a probability measure:

$$\begin{aligned} P\left(\sup_{t\in K} \eta_t < 0\right) &= P\left(\bigcup_{n\in\mathbb{N}} \left\{\sup_{t\in K} \eta_t \le \frac{-\varepsilon}{n}\right\}\right) \\ &= \lim_{n\to\infty} P\left(\sup_{t\in K} \eta_t \le \frac{-\varepsilon}{n}\right). \end{aligned}$$

We show by contradiction that this probability is actually zero. Assume that it is 1. We divide the interval $[0,1]$ into the two subintervals $[0,1/2]$, $[1/2,1]$. Now we obtain from equation (2.5) that $P\big(\sup_{t\in[0,1/2]\cap K} \eta_t = 0\big) = 1$ or $P\big(\sup_{t\in[1/2,1]\cap K} \eta_t = 0\big) = 1$ (the supremum taken over the empty set is set to $-\infty$). Suppose without loss of generality that the first probability is 1. Then we divide the interval $[0,1/2]$ into the two subintervals $[0,1/4]$, $[1/4,1/2]$ and repeat the preceding arguments. By iterating, this generates a sequence of nested intervals $I_n = [t_n, \tilde{t}_n]$ in $[0,1]$ with $P\left(\sup_{t\in I_n\cap K} \eta_t = 0\right) = 1$, $\tilde{t}_n - t_n = 2^{-n}$, $n \in \mathbb{N}$, and $t_n \uparrow t_0$, $\tilde{t}_n \downarrow t_0$

as $n \to \infty$ for some $t_0 \in K$. From the lower continuity of a probability measure and the fact, that η_{t_0} is negative exponential distributed, we conclude

$$\begin{aligned}
0 &= P\left(\eta_{t_0} = 0\right) \\
&= P\left(\bigcap_{n\in\mathbb{N}}\left\{\sup_{t\in I_n\cap K} \eta_t = 0\right\}\right) \\
&= \lim_{n\to\infty} P\left(\left\{\sup_{t\in I_n\cap K} \eta_t = 0\right\}\right) \\
&= 1,
\end{aligned}$$

which is a contradiction. □

Lemma 2.3 was already established by Giné et al. [20] using the theory of random sets. Furthermore, the proof of Theorem 9.4.1 in de Haan and Ferreira [9] contains this assertion, too, proven by elementary probabilistic arguments.

Lemma 2.3. *Let K be a compact subset of $[0,1]$ and let $\boldsymbol{\xi}_K = (\xi_t)_{t\in K}$ be an MSP on K with standard Fréchet margins, which has all its sample paths in the space of continuous functions $\bar{C}^+(K) := \{f : K \to [0,\infty),\ f \text{ is continuous}\}$. Then we have*

$$P\left(\inf_{t\in K} \xi_t > 0\right) = 1.$$

The following crucial characterization of continuous MSP is a consequence of Giné et al. [20, Proposition 3.2]; we refer also to de Haan and Ferreira [9, Theorem 9.4.1].

Proposition 2.4. *Let $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ be a stochastic process whose sample paths all pertain to $\bar{C}^+[0,1]$ with the properties*

$$\sup_{t\in[0,1]} Z_t = m \in [1,\infty) \text{ a.s. and } E(Z_t) = 1, \quad t \in [0,1]. \tag{2.6}$$

(i) A process $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$ in $\bar{C}^+[0,1]$ is a simple MSP if there exists a stochastic process $\boldsymbol{Z}$ as above such that for compact subsets $K_1,\dots,K_d$ of $[0,1]$ and $x_1,\dots,x_d > 0$, $d \in \mathbb{N}$,

$$\begin{aligned}
&P\left(\max_{t\in K_j} \xi_t \le x_j, 1 \le j \le d\right) \\
&= \exp\left(-E\left(\max_{1\le j\le d}\left(\frac{\max_{t\in K_j} Z_t}{x_j}\right)\right)\right).
\end{aligned} \tag{2.7}$$

(ii) A process $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ in $\bar{C}^-[0,1] = \{f \in C[0,1] : f \le 0\}$ is a standard MSP if there exists a stochastic process $\boldsymbol{Z}$ as above such that for compact subsets $K_1,\dots,K_d$ of $[0,1]$ and $x_1,\dots,x_d \le 0$, $d \in \mathbb{N}$,

$$P\left(\max_{t\in K_j} \eta_t \le x_j, 1 \le j \le d\right)$$

$$= \exp\left(-E\left(\max_{1\le j\le d}\left(|x_j| \max_{t\in K_j} Z_t\right)\right)\right). \tag{2.8}$$

Conversely, every stochastic process $\boldsymbol{Z}$ *in* $\bar{C}^+[0,1]$ *satisfying* (2.6) *gives rise to a simple and to a standard MSP. The connection is via* (2.7) *and* (2.8)*, respectively. We call* $\boldsymbol{Z}$ *a* generator *of* $\boldsymbol{\eta}$ *and* $\boldsymbol{\xi}$*.*

Proof. Identify the finite measure σ in Giné et al. [20, Proposition 3.2] with $m(P * \tilde{\boldsymbol{Z}})$, where $(P * \tilde{\boldsymbol{Z}})$ denotes the distribution of some process $\tilde{\boldsymbol{Z}}$ in $\bar{C}_1^+ := \{f \in C[0,1] : f \ge 0, \|f\|_\infty = 1\}$ and set $\boldsymbol{Z} = m\tilde{\boldsymbol{Z}}$, where m is the total mass of the measure σ. Assertion *(ii)* now follows by setting $\boldsymbol{\eta} = -1/\boldsymbol{\xi}$, which is well defined by Lemma 2.3. □

While a generator $\boldsymbol{Z}$ is in general not uniquely determined, the number $m = E\left(\sup_{t\in[0,1]} Z_t\right)$ is, see Remark 2 below. We, therefore, call m the generator constant of $\boldsymbol{\eta}$.

According to de Haan and Ferreira [9, Corollary 9.4.5], condition (2.6) can be weakened to the condition $E(Z_t) = 1$, $t \in [0,1]$, together with $E\left(\sup_{t\in[0,1]} Z_t\right) < \infty$, see Remark 4 below.

The characterization in Proposition 2.4 implies in particular that the fidis of $\boldsymbol{\eta}$ are multivariate negative EVD with standard negative exponential margins: We have for $0 \le t_1 < t_2 < \dots < t_d \le 1$

$$-\log(G_{t_1,\dots,t_d}(\boldsymbol{x})) = E\left(\max_{1\le i\le d}(|x_i|\, Z_{t_i})\right) =: \|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}, \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d, \tag{2.9}$$

where $\|\cdot\|_{D_{t_1,\dots,t_d}}$ is a D-norm on $\mathbb{R}^d$ (see Falk et al. [17]).

Denote by $E[0,1]$ the set of all functions on $[0,1]$ that are bounded, and which have only a finite number of discontinuities. Furthermore, denote by $E^-[0,1]$ those functions in $E[0,1]$ which do not attain positive values.

Definition 2.5. For a generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ in $\bar{C}^+[0,1]$ with properties (2.6) and all $f \in E[0,1]$ set

$$\|f\|_D := E\left(\sup_{t\in[0,1]}(|f(t)|\, Z_t)\right).$$

Then $\|\cdot\|_D$ obviously defines a norm on $E[0,1]$, called a D-norm with generator $\boldsymbol{Z}$.

Proof. $\|\cdot\|_D$ is, actually, a norm:

As the generator fulfills $\boldsymbol{Z} \ge 0$ a.s, we have $\|f\|_D \ge 0$ for every $f \in E[0,1]$. If $f \in E[0,1]$ is not the constant zero function, then there is some $t_0 \in [0,1]$ with $|f(t_0)| > 0$. Thus,

$$\|f\|_D = E\left(\sup_{t\in[0,1]}(|f(t)|\, Z_t)\right) \ge E\left(|f(t_0)|\, Z_{t_0}\right) = |f(t_0)| > 0,$$

so $\|f\|_D = 0 \iff f \equiv 0$.

The homogeneity $\|cf\|_D = |c| \|f\|_D$ is clear and for the triangle inequalitity take some $f, g \in E[0,1]$ and observe

$$\|f+g\|_D = E\left(\sup_{t\in[0,1]} (|f(t)+g(t)| Z_t)\right) \le E\left(\sup_{t\in[0,1]} (|f(t)| Z_t + |g(t)| Z_t)\right) \le \|f\|_D + \|g\|_D .$$

□

The sup-norm

$$\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|, \qquad f \in E[0,1],$$

is a particular D-norm with generator $Z_t = Z, t \in [0,1]$, for an arbitrary random variable Z, which has the properties $P(Z \ge 0) = 1$ and $E(Z) = 1$ (in particular, $Z = 1$ a.s. is a possible choice). It is, moreover, the least D-norm, as the next Lemma shows.

Lemma 2.6. *For any D-norm* $\|\cdot\|_D$ *whose generator satisfies* $E\left(\sup_{t\in[0,1]} Z_t\right) = m$ *we have*

$$\|f\|_\infty \le \|f\|_D \le m \|f\|_\infty , \qquad f \in E[0,1]. \tag{2.10}$$

Proof. The inequality on the right hand side is true because of

$$\|f\|_D = E\left(\sup_{t\in[0,1]} (|f(t)| Z_t)\right) \le E\left(\sup_{t\in[0,1]} Z_t\right) \sup_{t\in[0,1]} |f(t)| = m \|f\|_\infty .$$

For the inequality on the left hand side observe, that $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$ may not be attained for $f \in E[0,1]$. But there will always be a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to t_0 \in [0,1]$ and $|f(t_n)| \to \|f\|_\infty$, $n \to \infty$. We get, thus,

$$\|f\|_\infty = \lim_{n\to\infty} |f(t_n)| = \lim_{n\to\infty} E(|f(t_n)| Z_{t_n}) \le E\left(\sup_{t\in[0,1]} (|f(t)| Z_t)\right) = \|f\|_D .$$

□

Note that inequality (2.10) implies that each functional D-norm is equivalent to the sup-norm on $\bar{E}^-[0,1]$.

The next Lemma introduces a closed-form expression of the distribution function of a standard MSP in terms of the D-norm on $\bar{E}^-[0,1]$.

Lemma 2.7. *Let* $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ *be a standard MSP with continuous sample paths and generator* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. *Then we have*

$$P(\boldsymbol{\eta} \le f) = \exp(-\|f\|_D) = \exp\left(-E\left(\sup_{t\in[0,1]} (|f(t)| Z_t)\right)\right), \quad \text{for all } f \in \bar{E}^-[0,1]. \tag{2.11}$$

Conversely, if there is some $\boldsymbol{Z}$ *with properties* (2.6) *and a sample continuous process* $\boldsymbol{\eta}$ *in* $C^-[0,1]$ *fulfills* (2.11), *then* $\boldsymbol{\eta}$ *is standard max-stable with generator* $\boldsymbol{Z}$.

Proof. Let $Q = \{q_1, q_2 \dots\}$ be a denumerable and dense subset of $[0,1]$ which contains the finitely many points, at which $f \in \bar{E}^-[0,1]$ has a discontinuity. We identify

$$\{\boldsymbol{\eta} \le f\} = \{\eta_t \le f(t),\ \text{for all } t \in [0,1]\} = \bigcap_{d\in\mathbb{N}} \left\{\eta_{q_j} \le f(q_j),\, 1 \le j \le d\right\},$$

so we obtain from the continuity of $\boldsymbol{\eta}$, the continuity from above of each probability measure and equation (2.9)

$$\begin{aligned}
P(\boldsymbol{\eta} \le f) &= P\left(\bigcap_{d\in\mathbb{N}} \{\eta_{q_j} \le f(q_j),\, 1 \le j \le d\}\right)\\
&= \lim_{d\to\infty} P\left(\eta_{q_j} \le f(q_j),\, 1 \le j \le d\right)\\
&= \lim_{d\to\infty} \exp\left(-E\left(\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-\lim_{d\to\infty} E\left(\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-E\left(\lim_{d\to\infty}\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)\right)\\
&= \exp\left(-\|f\|_D\right),
\end{aligned}$$

where the third to last equation follows from the dominated convergence theorem.

If some $\boldsymbol{Z}$ has properties (2.6) it gives rise to some standard MSP $\widehat{\boldsymbol{\eta}}$ due to Proposition 2.4. But since by (2.8) and (2.11) the fidis of $\widehat{\boldsymbol{\eta}}$ and those of $\boldsymbol{\eta}$ coincide, $\widehat{\boldsymbol{\eta}} \overset{d}{=} \boldsymbol{\eta}$ follows. □

The representation

$$P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right), \qquad f \in \bar{E}^-[0,1],$$

of a standard MSP is in complete accordance with the df of a multivariate EVD with standard negative exponential margins via a D-norm on $\mathbb{R}^d$ as developed in [17, Section 4.4].

Remark 2. Oberserve $P(\boldsymbol{\eta} \le 1) = \exp(-\|1\|_D) = \exp(-m)$, so it is obvious, that the generator constant $m = E(\sup_{t\in[0,1]} Z_t)$ is unique for every standard MSP $\boldsymbol{\eta}$.

The space $\bar{E}^-[0,1]$ allows the incorporation of the finite-dimensional marginal distributions of $\boldsymbol{\eta}$ into the preceding representation: for some subset $I \subset [0,1]$ denote by $\mathbf{1}_I : [0,1] \to \{0,1\}$ the indicator function of I, i.e. $\mathbf{1}_I(t) = 1$, if $t \in I$, and $\mathbf{1}_I(t) = 0$, if $t \notin I$. Now choose indices $0 \le t_1 < \dots < t_d \le 1$ and numbers $x_i < 0$, $1 \le i \le d$ for $d \in \mathbb{N}$. Then the function

$$f(t) = \sum_{i=1}^{d} x_i \mathbf{1}_{\{t_i\}}(t)$$

is an element of $\bar{E}^-[0,1]$ with the property

$$\begin{aligned} P(\boldsymbol{\eta} \le f) &= \exp\left(-\|f\|_D\right) \\ &= \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)\right) \\ &= \exp\left(-E\left(\max_{1\le i\le d}\left(|x_i|\, Z_{t_i}\right)\right)\right) \\ &= \exp\left(-\|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}\right) \\ &= P(\eta_{t_1} \le x_1, \dots, \eta_{t_d} \le x_d). \end{aligned}$$

This is one of the reasons, why we prefer standard MSPs (with standard negative exponential margins), whereas de Haan and Ferreira [9], for instance, consider simple MSPs (with standard Fréchet margins).

The next Lemma states basic properties of the distribution function of a standard MSP $\boldsymbol{\eta}$.

Lemma 2.8. *Let $\boldsymbol{\eta}$ in $\bar{C}^-[0,1]$ be a standard MSP and consider its distribution function*

$$G(f) = P\left(\boldsymbol{\eta} \le f\right) = \exp\left(-\|f\|_D\right), \quad f \in \bar{E}^-[0,1].$$

Then we have

(i) $G(\cdot)$ is continuous with respect to the sup-norm.

(ii) For every $f \in \bar{E}^-[0,1]$ we have $P\left(\boldsymbol{\eta} \le f\right) = P\left(\boldsymbol{\eta} < f\right)$.

Proof. The first assertion follows by the triangle inequality $(|\|f_n\|_D - \|f\|_D| \le \|f_n - f\|_D)$, Lemma 2.6 ($\|f_n - f\|_D \to 0$ as $\|f_n - f\|_\infty \to 0$) and the continuity of the exponential function.

The continuity from below of an arbitrary probability measure implies

$$\begin{aligned} P\left(\boldsymbol{\eta} < f\right) &= P\left(\boldsymbol{\eta} \in \bigcup_{k\in\mathbb{N}} \left\{g \in \bar{C}^-[0,1] : g \le f - 1/k\right\}\right) \\ &= \lim_{k\to\infty} P\left(\boldsymbol{\eta} \le f - \tfrac{1}{k}\right) \\ &= P(\boldsymbol{\eta} \le f), \end{aligned}$$

where the last equality is due to the first part of the Lemma. □

Remark 3. Bringing together the assertions of Proposition 2.4 and Lemma 2.7, the distribution function of a simple MSP $\boldsymbol{\xi}$ is for $f \in E^+[0,1]$, $f > 0$, given by

$$P(\boldsymbol{\xi} \le f) = \exp\left(-E\left(\sup_{s\in[0,1]} \frac{\boldsymbol{Z}(s)}{f(s)}\right)\right),$$

and the distribution of $\boldsymbol{\xi}$ is completely determined by this representation.

Remark 4. Condition (2.6) in Proposition 2.4 can be weakened to the condition $E(Z_t) = 1$, $t \in [0,1]$, together with $E\left(\sup_{t\in[0,1]} Z_t\right) < \infty$. This can be seen by applying Corollary 9.4.5 in de Haan and Ferreira [9]; therefore, we recall this result slightly adjusted to our notation:

> *A simple MSP $\boldsymbol{\xi}$ in $C^+[0,1]$ can be generated in the following way: let $\tilde{N}$ be a Poisson point process on $(0,\infty]$ with mean measure $\tilde{\nu}(\cdot) = \int_\cdot dr/r^2$ and let $\{Y_i\}_{i\in\mathbb{N}}$ be the points of this process. Further consider iid stochastic processes $\boldsymbol{Z}$, $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2, \dots$ in $\bar{C}^+[0,1]$ with the properties $E(Z_t) = 1$ for all $t \in [0,1]$ and $E\left(\sup_{t\in[0,1]} Z_t\right) < \infty$. Let the point process N and the sequence $\boldsymbol{Z}$, $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2, \dots$ be independent. Then*
>
> $$\boldsymbol{\xi} \stackrel{d}{=} \bigvee_{i\in\mathbb{N}} Y_i Z_i$$

It remains to show, that the processes Z_i in this corollary coincide with the generator process in Proposition 2.4. We deduce from de Haan and Ferreira [9, Lemma 9.4.7] that the collection $\{(Y_i, \boldsymbol{Z}_i)\}_{i\in\mathbb{N}}$ are the points of a Poisson point process N on $(0,\infty] \times \bar{C}^+[0,1]$ with mean measure $\nu = \tilde{\nu} \times (P * \boldsymbol{Z})$, where $(P * \boldsymbol{Z})$ denotes the distribution of $\boldsymbol{Z}$. Thus, for arbitrary $f \in E^+[0,1]$,

$$\begin{aligned}
P(\boldsymbol{\xi} \le f) &= P(\bigvee_{i\in\mathbb{N}} Y_i Z_i \le f) \\
&= P\left(N \text{ has no points } (Y_i, \boldsymbol{Z}_i), \text{ for which } \exists s \in [0,1] : Y_i \boldsymbol{Z}_i(s) > f(s)\right) \\
&= P\left(N \text{ has no points in } \left\{(y,z) \in (0,\infty] \times \bar{C}^+[0,1] : y > \inf_{s\in[0,1]} (f(s)/z(s))\right\}\right) \\
&= \exp(-\nu\left(\left\{(y,z) \in (0,\infty] \times \bar{C}^+[0,1] : y > \inf_{s\in[0,1]} (f(s)/z(s))\right\}\right)) \\
&= \exp\left(-\int_{\bar{C}^+[0,1]} \int_{\inf_{s\in[0,1]}(f(s)/z(s))}^{\infty} r^{-2}\, dr\ (P * \boldsymbol{Z})(dz)\right) \\
&= \exp\left(-E\left(\sup_{s\in[0,1]} \frac{\boldsymbol{Z}(s)}{f(s)}\right)\right).
\end{aligned}$$

The assertion now follows by Remark 3.

2.2 Transformation to Arbitrary Margins

Having introduced the concept of the distribution functions of standard MSP by means of the D-norm on function space, we show in the following, that these considerations are already sufficient to characterize all MSP.

In accordance to the finite dimensional case, this is a question of margin transformation. The additional difficulty to state the required transformations in function space is to ensure that all processes under consideration are well-defined – more precisely: to ensure that the resulting processes of specific continuous transformations of arbitrary MSP never take the value zero with probability one.

Hence, the main result in this section is the following Lemma 2.9; its assertion was already shown by Giné et al. [20] using the theory of random closed sets. Nevertheless, we state the result in our setup, giving an alternative point of view to the theory as we use only elementary probabilistic arguments.

Lemma 2.9. *Let $\boldsymbol{\zeta}$ be an arbitrary MSP in $C[0,1]$ and $a>0$, b,γ the continuous functions which fulfill*

$$P\left(\frac{\zeta_t-b(t)}{a(t)}\le x\right)=F_{\gamma(t)}(x),\qquad t\in[0,1],$$

cf. equation (2.2) and Lemma 2.1. Then the (sample continuous) process $\widehat{\boldsymbol{\zeta}}:=1+\frac{\gamma}{a}(\boldsymbol{\zeta}-b)$ will not take the value zero with probability one, i.e.

$$P(\widehat{\zeta}_t\neq 0 \text{ for all } t\in[0,1])=1$$

Proof. First observe, that for every $t\in[0,1]$ we have

$$P(\widehat{\zeta}_t\ge 0)=\left\{\begin{array}{ll}1-P\big((\zeta_t-b(t))/a(t)\le -1/\gamma(t)\big) & \text{for } \gamma(t)>0\\ P\big(1\ge 0\big) & \text{for } \gamma(t)=0\\ P\big((\zeta_t-b(t))/a(t)\le -1/\gamma(t)\big) & \text{for } \gamma(t)<0\end{array}\right\}=1 \tag{2.12}$$

because of (2.2) and the explanation thereafter.

Now define

$$K_1:=\{t\in[0,1]:\gamma(t)\ge 0\},\quad K_2:=\{t\in[0,1]:\gamma(t)\le 0\},$$

which are compact subsets of $[0,1]$ by the continuity of γ. Because of

$$P(\widehat{\zeta}_t=0 \text{ for some } t\in[0,1])\le\sum_{i=1}^{2}P(\widehat{\zeta}_t=0 \text{ for some } t\in K_i)$$

the assertion is shown, if both summands on the right hand side are equal to zero.

Now define for $t\in K_2$

$$\widehat{\eta}_t\ :=\ -\left(1+\frac{\gamma(t)}{a(t)}(\zeta_t-b(t))\right)^{-1/\gamma(t)}\ =\ -(\widehat{\zeta}_t)^{-1/\gamma(t)},$$

where for $\gamma(t)=0$ this is meant to be $\widehat{\eta}_t=-\exp(-(\zeta_t-b(t))/a(t))$.

Then $\widehat{\boldsymbol{\eta}}=(\widehat{\eta}_t)_{t\in[0,1]}$ is well-defined and a standard MSP on K_2, which can be seen by elementary computations as follows: for arbitrary $t\in K_2$ and $x\le 0$, we get

– in case $\gamma(t)=0$:

$$P\left(\widehat{\eta}_t\le x\right)=P\Big(-\exp(-(\zeta_t-b(t))/a(t))\le x\Big)=P\Big((\zeta_t-b(t))/a(t)\le -\ln(-x)\Big)=\exp(x);$$

– in case $\gamma(t) \neq 0$:

$$P(\widehat{\eta}_t \le x) = P\Big(-\Big(1 + \frac{\gamma(t)}{a(t)}(\zeta_t - b(t))\Big)^{-1/\gamma(t)} \le x\Big)$$
$$= P\Big((\zeta_t - b(t))/a(t) \le \Big((-x)^{-\gamma(t)} - 1\Big)/\gamma(t)\Big) = \exp(x);$$

so all one-dimensional margins are standard negative exponential distributions. For the max-stability one has to show, that all fidis are max-stable. Since the one-dimensional margins determine the norming functions $a_n \equiv 1/n$ and $b_n \equiv 0$, $n \in \mathbb{N}$, we get, thus, for arbitrary $d, n \in \mathbb{N}$, $t_1, \dots, t_d \in K_2$ and $x_1, \dots, x_d \le 0$, with the relations in (2.3) and $\widehat{\boldsymbol{\eta}}^{(k)}$ iid copies of $\widehat{\boldsymbol{\eta}}$, $k = 1, \dots, n$:

– in case $\gamma(t) = 0$:

$$\begin{aligned}
&P\left(n \cdot \max_{1\le k\le n} \widehat{\eta}_{t_i}^{(k)} \le x_i,\ i = 1, \dots, d\right)\\
&= P\left(n \cdot \max_{1\le k\le n} -\exp(-(\zeta_{t_i}^{(k)} - b(t_i))/a(t_i)) \le x_i,\ i = 1, \dots, d\right)\\
&= P\left(-\exp(-(\max_{1\le k\le n} \zeta_{t_i}^{(k)} - b(t_i))/a(t_i)) \le x_i/n,\ i = 1, \dots, d\right)\\
&= P\left(\frac{\max_{1\le k\le n} \zeta_{t_i}^{(k)} - b_n(t_i)}{a_n(t_i)} \le \frac{-a(t_i)\ln(|x_i|/n) + b(t_i) - b_n(t_i)}{a_n(t_i)},\ i = 1, \dots, d\right)\\
&= P\left(\zeta_{t_i} \le -a(t_i)\ln(|x_i|/n) + b(t_i) - a(t_i)\ln(n),\ i = 1, \dots, d\right)\\
&= P\left(-\exp(-(\zeta_{t_i} - b(t_i))/a(t_i)) \le x_i,\ i = 1, \dots, d\right)\\
&= P\left(\widehat{\eta}_{t_i} \le x_i,\ i = 1, \dots, d\right);
\end{aligned}$$

– in case $\gamma(t) \neq 0$:

$$\begin{aligned}
&P\left(n \cdot \max_{1\le k\le n} \widehat{\eta}_{t_i}^{(k)} \le x_i,\ i = 1, \dots, d\right)\\
&= P\left(n \cdot \max_{1\le k\le n} \left(-\left(1 + \frac{\gamma(t_i)}{a(t_i)}(\zeta_{t_i}^{(k)} - b(t_i))\right)^{-1/\gamma(t_i)}\right) \le x_i,\ i = 1, \dots, d\right)\\
&= P\left(-\left(1 + \frac{\gamma(t_i)}{a(t_i)}\left(\max_{1\le k\le n} \zeta_{t_i}^{(k)} - b(t_i)\right)\right)^{-1/\gamma(t_i)} \le x_i/n,\ i = 1, \dots, d\right)\\
&= P\Big(\Big(\max_{1\le k\le n} \zeta_{t_i}^{(k)} - b_n(t_i)\Big)/a_n(t_i) \le \frac{1}{a_n(t_i)}\Big[\frac{a(t_i)}{\gamma(t_i)}\Big((|x_i|/n)^{-\gamma(t_i)} - 1\Big) +\\
&\qquad +b(t_i) - b_n(t_i)\Big],\ i = 1, \dots, d\Big)
\end{aligned}$$

$$= P\Big(\zeta_{t_i} \le n^{-\gamma(t_i)}\Big[\frac{a(t_i)}{\gamma(t_i)}\Big((|x_i|/n)^{-\gamma(t_i)} - 1\Big) + b(t_i) - \\ -n^{\gamma(t_i)}\Big(\frac{a(t_i)}{\gamma(t_i)} - b(t_i)\Big) + \frac{a(t_i)}{\gamma(t_i)} - b(t_i)\Big],\ i = 1,\dots,d\Big)$$

$$= P\Big(\zeta_{t_i} - b(t_i) \le n^{-\gamma(t_i)}\frac{a(t_i)}{\gamma(t_i)}(|x_i|/n)^{-\gamma(t_i)} - \frac{a(t_i)}{\gamma(t_i)},\ i = 1,\dots,d\Big)$$

$$= P\left(-\left(1 + \frac{\gamma(t_i)}{a(t_i)}(\zeta_{t_i} - b(t_i))\right)^{-1/\gamma(t_i)} \le x_i,\ i = 1,\dots,d\right)$$

$$= P\left(\widehat{\eta}_{t_i} \le x_i,\ i = 1,\dots,d\right).$$

We have shown that $\widehat{\boldsymbol{\eta}}$ is a standard MSP on K_2, and, thus, Lemma 2.2 yields $P(\widehat{\eta}_t = 0 \text{ for some } t \in K_2) = 0$ and this implies $P(\widehat{\zeta}_t = 0 \text{ for some } t \in K_2) = 0$.

Now define for $t \in K_1$

$$\widehat{\xi}_t := \left(1 + \frac{\gamma(t)}{a(t)}(\zeta_t - b(t))\right)^{1/\gamma(t)} = (\widehat{\zeta}_t)^{1/\gamma(t)}$$

and, as before, $\widehat{\xi}_t = \exp((\zeta_t - b(t))/a(t))$ for $\gamma(t) = 0$.

Then $\widehat{\boldsymbol{\xi}} = (\widehat{\xi}_t)_{t\in[0,1]}$ is a well-defined MSP on K_2 with standard Fréchet margins, which can be seen by analogue computations as before. So $P(\widehat{\boldsymbol{\xi}} \neq 0) = 1$ by Lemma 2.3 and the result follows. ☐

Proposition 2.10. *Let $\boldsymbol{\zeta}$ an arbitrary MSP in $C[0,1]$ and $a > 0$, b, γ the continuous functions for which*

$$P\left(\frac{\zeta_t - b(t)}{a(t)} \le x\right) = F_{\gamma(t)}(x), \qquad t \in [0,1],$$

holds. Define

$$\eta_t := \begin{cases} -\left(1 + \frac{\gamma(t)}{a(t)}(\zeta_t - b(t))\right)^{-1/\gamma(t)} & \text{for } \gamma(t) \neq 0 \\ -\exp\left(-(\zeta_t - b(t))/a(t)\right) & \text{for } \gamma(t) = 0. \end{cases} \tag{2.13}$$

Then $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ is a standard MSP in $C[0,1]$.

Proof. Because of Lemma 2.9 the process $\boldsymbol{\eta}$ is well defined and it is sample continuous because of Lemma 2.1. Moreover, elementary computations as before (cf. proof of Lemma 2.9) show that the one-dimensional distributions are negative exponential distributions and that the fidis of $n\max_{1\le i\le n}\boldsymbol{\eta}^{(i)}$ for iid copies $\boldsymbol{\eta}^{(i)}$ of $\boldsymbol{\eta}$, $n \in \mathbb{N}$, are the same as those of $\boldsymbol{\eta}$, so the process is standard max-stable. ☐

Inverting equation (2.13) yields for an arbitrary MSP $\boldsymbol{\zeta} = (\zeta_t)_{t\in[0,1]}$ in $C[0,1]$ – coming along with its norming functions $a > 0$, b, γ as before – some standard MSP $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ with

$$\zeta_t := \begin{cases} \frac{-a(t)}{\gamma(t)}\left(1 - (-\eta_t)^{-\gamma(t)}\right) + b(t) & \text{for } \gamma(t) \neq 0; \\ -a(t)\ln(-\eta_t) + b(t) & \text{for } \gamma(t) = 0; \end{cases} \qquad t \in [0,1].$$

Thus, the functional df of an arbitrary MSP $\boldsymbol{\zeta}$ in $C[0,1]$ can be written by means of the D-norm: we get for $f \in E[0,1]$

$$\begin{aligned} P(\boldsymbol{\zeta} \le f) &= P\left(\boldsymbol{\eta} \le \Psi(f)\right) \\ &= \exp\left(-\left\|\Psi(f)\right\|_D\right), \end{aligned} \tag{2.14}$$

where we define for functions $f \in E[0,1]$

$$\Psi(f) = \Psi(f(t)) := \begin{cases} -\left(1 + \frac{\gamma(t)}{a(t)}(f(t) - b(t))\right)^{-1/\gamma(t)} & \text{for } \gamma(t) \neq 0; \\ -\exp\left(-(f(t) - b(t))/a(t)\right) & \text{for } \gamma(t) = 0; \end{cases} \qquad t \in [0,1].$$

2.3 Further Properties of Standard MSP and Their Distributions

So far, all MSP were characterized by means of the D-norm on function space. It will turn out in this section, that the characterization of the max-stable distributions on $C[0,1]$ via a distribution function (cf. (2.14)) allows a deeper inside into dependence structures and sample paths properties of (standard) MSP: we give characterizations of the complete dependence MSP and consider hitting probabilities of standard MSP (cf. Section 2.3.1). Moreover, we study the "survivor function", i.e. $P(\boldsymbol{\eta} > f)$ for $f \in \bar{E}^-[0,1]$, on function space (cf. Section 2.3.2), which will play a crucial role in the sequel.

Some results within this section request examples: We collect in Section 2.4 some examples of generator process and standard MSP with several special properties.

First, we turn to the well-known characterization of Takahashi (cf. Takahashi [26], Falk et al. [17, Theorem 4.4.1]) of the complete dependence case of the marginal distributions, which can be extended to the function space $E[0,1]$. To this end, we state the following assertion, which will be useful in Section 2.5, too.

Lemma 2.11. *Let $\boldsymbol{\eta}^{(n)}$, $n \in \mathbb{N}$, be a sequence of standard MSP with pertaining D-norms $\|\cdot\|_{D_n}$, $n \in \mathbb{N}$. Then we have*

$$\left\|\mathbf{1}_{[0,1]}\right\|_{D_n} \to_{n\to\infty} 1 \iff \|f\|_{D_n} \to \|f\|_\infty, \quad f \in E[0,1].$$

Proof. It suffices to establish the implication ' $\Longrightarrow$ '. Let $\boldsymbol{Z}^{(n)}$ be the generator process corresponding to $\boldsymbol{\eta}^{(n)}$. For each $f \in E[0,1]$ and each $\varepsilon > 0$ there exists $t_0 \in [0,1]$ such that $|f(t_0)| + \varepsilon \ge \|f\|_\infty$. We have

$$\begin{aligned} \|f\|_\infty \left\|\mathbf{1}_{[0,1]}\right\|_{D_n} &= \|f\|_\infty E\left(\sup_{t\in[0,1]} Z_t^{(n)}\right) \\ &\ge E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t^{(n)}\right)\right) \\ &= \|f\|_{D_n} \\ &\ge |f(t_0)|\, E\left(Z_{t_0}^{(n)}\right) \end{aligned}$$

$$= |f(t_0)| \\ \geq \|f\|_\infty - \varepsilon.$$

The assertion now follows from the convergence $\left\|\mathbf{1}_{[0,1]}\right\|_{D_n} \to 1$ as $n \to \infty$. □ □

The following result is a consequence of Lemma 2.11 by putting $\boldsymbol{\eta}^{(n)} = \boldsymbol{\eta}$, $n \in \mathbb{N}$.

Lemma 2.12 (Functional Takahashi). *Let $\boldsymbol{\eta}$ be a standard MSP with generator constant $m = E(\sup_{t\in[0,1]} Z_t) = \|1\|_D$ and some generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. Then:*

$$\|\cdot\|_D = \|\cdot\|_\infty \quad \Longleftrightarrow \quad \|1\|_D = 1.$$

In this case, the generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ is for all $t \in [0,1]$ equal to some non-negative rv Z with $E(Z) = 1$ and all sample paths of $\boldsymbol{\eta}$ are constant in t, i.e. $\eta_t = \eta$ for all $t \in [0,1]$, where η is negative exponential distributed.

2.3.1 Hitting Probabilities

We turn to the analysis of sample path properties of standard MSP $\boldsymbol{\eta}$, especially we ask the question:

What is the probability of the event that the sample paths of $\boldsymbol{\eta}$ hit a certain $x_0 \leq 0$?

On the one hand, this question was already partially answered by Lemma 2.2: the value $x_0 = 0$ is not attained by sample paths of any $\boldsymbol{\eta}$ with probability one.

On the other hand one may wonder whether $P(\eta_t = x_0 \text{ for some } t \in [0,1]) = 0$ is true for all $x_0 \leq 0$, in accordance to the finite dimensional case: as a negative exponential distributed rv $X = (X_1, \dots, X_d)$ in $\mathbb{R}^d$ has a continuous distribution, there is $P(X_i = x_0, \text{ for some } i \in \{1,\dots,d\}) = 0$ for every $x_0 \leq 0$.

The somehow surprising answer is that $P(\eta_t = x_0 \text{ for some } t \in [0,1]) > 0$ for all $x_0 < 0$, namely for every standard MSP $\boldsymbol{\eta}$, unless the generator constant of $\boldsymbol{\eta}$ is equal to one: this yields, in particular, an alternative characterization of the complete dependence case.

Going on we ask which conditions have to be fulfilled, that $\boldsymbol{\eta}$ hits some values $x_0 < 0$ more than once with positive probability. The answers are given by Proposition 2.18 (sufficient condition that the sample paths of $\boldsymbol{\eta}$ hit every $x < 0$ (at least) two times) and Example 2.29, which is an example of an MSP, which does not hit any $x_0 < 0$ more than twice.

We start by recalling Lemma 2.8: we know that for every standard MSP $\boldsymbol{\eta}$ and every $f \in \bar{E}^-[0,1]$ we have $P(\boldsymbol{\eta} \leq f) = P(\boldsymbol{\eta} < f)$. In other words: for all $f \in \bar{E}^-[0,1]$

$$\begin{aligned} &P\left(\eta_t = f(t) \text{ for some } t \in [0,1] \middle| \eta_t \leq f(t) \text{ for all } t \in [0,1]\right) \\ &= \left[P\left(\{\eta_t = f(t) \text{ for some } t \in [0,1]\} \cap \{\eta_t \leq f(t) \text{ for all } t \in [0,1]\}\right)\right] \exp(\|f\|_D) \\ &= \left[P(\boldsymbol{\eta} \leq f) - P(\boldsymbol{\eta} < f)\right] \exp(\|f\|_D) \\ &= 0. \end{aligned} \tag{2.15}$$

Remark 5. Note that (2.15) immediately implies Lemma 2.2 by setting $f \equiv 0$.

Remark 6. Note that the preceding considerations imply in particular, that for $f \in \bar{E}^-[0,1]$ the sets $\{g \in \bar{C}^-[0,1] : g(t) \le f(t), \text{ for all } t \in [0,1]\}$ are continuity sets with respect to the distribution of the standard MSP $\boldsymbol{\eta}$ on $(\bar{C}^-[0,1], \|\cdot\|_\infty)$: the boundary ∂B of a set B "consists of those points, which are limits of sequences of points in B and are also limits of sequences of points outside B", cf. Billingsley [7, p. 2]. So we have

$$\begin{aligned}&\partial \{g \in \bar{C}^-[0,1] : g(t) \le f(t), \text{ for all } t \in [0,1]\}\\ &= \{g \in \bar{C}^-[0,1] : g(t) = f(t) \text{ for some } t \in [0,1]\} \cap \{g \in \bar{C}^-[0,1] : g(t) \le f(t) \text{ for all } t \in [0,1]\},\end{aligned}$$

and, thus, (2.15) yields $P\left(\boldsymbol{\eta} \in \partial \{g \in \bar{C}^-[0,1] : g(t) \le f(t), \text{ for all } t \in [0,1]\}\right) = 0$.

Hence, conditioned on $\{\boldsymbol{\eta} \le f\}$, the probability that some $\boldsymbol{\eta}$ hits f is in fact alwas equal to zero. From now on we will consider the unconditioned hitting probability $P(\eta_t = f(t) \text{ for some } t \in [0,1])$.

Example 2.13. Consider the (discontinuous) function $f \in \bar{E}^-[0,1]$ defined by $f(t) = \sum_{i=1}^n x_i \mathbf{1}_{\{t_i\}}(t), t \in [0,1]$ for $t_i \in [0,1], x_i < 0, i = 1,\dots,n, n \in \mathbb{N}$. Then

$$P(\eta_t = f(t) \text{ for some } t \in [0,1]) = P\left(\bigcup_{i=1}^n \{\eta_{t_i} = f(t_i)\}\right) = 0,$$

as η_{t_i} is standard negative exponential distributed, $i = 1,\dots,n$.

Example 2.14. Consider the complete dependence case, i.e. the standard MSP $\boldsymbol{\eta}$ with generator constant $m = 1$, cf. Corollary 2.12.

We immediately get

$$P(\eta_t < x_0 \text{ for all } t \in [0,1]) = P(\eta_t < x_0 \text{ for some } t \in [0,1]) = \exp(x_0),$$

for $x_0 \le 0$ and, thus, $P(\eta_t = x_0 \text{ for some } t \in [0,1]) = 0$ follows.

On the other hand, for arbitrary non-constant continuous functions $f \in \bar{C}^-[0,1]$,

$$P(\eta_t = f(t) \text{ for some } t \in [0,1]) = P(\eta_0 \in \mathfrak{im}(f)) > 0,$$

as the image $\mathfrak{im}(f)$ of f is an interval of positive length.

Having seen some special cases of what could happen, we now give the complete answer to the foregoing question.

Proposition 2.15. *Let $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ be a standard MSP with generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. Suppose that there exists $x_0 < 0$ and a subinterval $I \subset [0,1]$ with positive length such that*

$$P(\eta_t = x_0 \text{ for some } t \in I) = 0.$$

Then $Z_t = Z_s$ almost surely for $t, s \in I$.

In other words: if $\|\mathbf{1}_I\|_D > 1$ for a subinterval $I \subset [0,1]$ with positive length, then

$$P(\eta_t = x_0 \text{ for some } t \in I) > 0 \quad \text{for all } x_0 < 0.$$

Proof. Assume $P(\eta_t = x_0 \text{ for some } t \in I) = 0$ for some $x_0 < 0$ and an interval $I \subset [0,1]$ with positive length. Define for $k \in \mathbb{N}$ and arbitrary $t_0 \in I$ the functions $g, g_k \in \bar{E}^-[0,1]$ by

$$g(t) := x_0 \mathbf{1}_I(t); \qquad g_{t_0,k}(t) := (x_0 - 1/k)\mathbf{1}_{t_0}(t).$$

Then, by Lemma 2.2,

$$\begin{aligned} P(\eta_t < x_0 \text{ for all } t \in I) &= P(\eta_t < g(t) \text{ for all } t \in [0,1]) \\ &= \exp\left(x_0 \|\mathbf{1}_I\|_D\right) = \exp\left(x_0 E\left(\sup_{t\in I} Z_t\right)\right). \end{aligned}$$

By assumption, we get on the other hand

$$\begin{aligned} \exp(x_0 - 1/k) &= P(\eta_t \le g_{t_0,k}(t) \text{ for all } t \in [0,1]) \\ &= P(\eta_t \le g_{t_0,k}(t) \text{ for all } t \in [0,1], \eta_t < x_0 \text{ for all } t \in I), \end{aligned}$$

and, thus,

$$\begin{aligned} \exp(x_0) &= \lim_{k\to\infty} P(\eta_t \le g_{t_0,k}(t) \text{ for all } t \in [0,1], \eta_t < x_0 \text{ for all } t \in I) \\ &= P\left(\bigcup_{k\in\mathbb{N}} \{\eta_t \le g_{t_0,k}(t) \text{ for all } t \in [0,1], \eta_t < x_0 \text{ for all } t \in I\}\right) \\ &= P(\eta_t < x_0 \text{ for all } t \in I\}) \\ &= \exp\left(x_0 \|\mathbf{1}_I\|_D\right), \end{aligned}$$

i.e. $\|\mathbf{1}_I\|_D = E(\sup_{t\in I} Z_t) = 1$.

As $\boldsymbol{Z}$ is a generator process fulfilling the conditions (2.6), we get for every $s \in I$

$$E\left(\sup_{t\in I} Z_t - Z_s\right) = 0 \iff \sup_{t\in I} Z_t = Z_s \text{ a.s.},$$

and, thus, $Z_t = Z_s$ for all $s, t \in I$ with probability one. □

An example of a standard MSP $\boldsymbol{\eta}$ with generator $\boldsymbol{Z}$ and $m > 1$ but $Z_t = Z_s$ for all $s, t \in I$ with probability one for some interval $I \subset [0,1]$ of positive length is given in Example 2.24 below.

The following assertion is an immediate consequence of Proposition 2.15 and Example 2.14.

Corollary 2.16. *A standard MSP $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ has completely dependent margins, i.e. its D-norm is equal to the* sup-norm, *if, and only if there exists some $x_0 < 0$ such that*

$$P(\eta_t = x_0 \text{ for some } t \in [0,1]) = 0. \tag{2.16}$$

In this case (2.16) *holds for* e v e r y $x_0 < 0$.

Now the question arises how often the paths of a standard MSP hit some $x_0 < 0$. We give a sufficient condition on the generator $\boldsymbol{Z}$ of a standard MSP $\boldsymbol{\eta}$ such that the probability of the event that the paths of $\boldsymbol{\eta}$ hit some $x_0 < 0$ (at least) two times in some interval $[t', t''] \subset [0,1]$ is positive for every $x_0 < 0$. We need the following Lemma which is of interest of its own.

Lemma 2.17. *Take $0 \le t' < t'' \le 1$ and consider a standard MSP $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ with generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$.*

If, for every $t_0 \in (t', t'')$ and some $x_0 < 0$,

$$P(\eta_{t'} \le x_0, \eta_{t_0} > x_0, \eta_{t''} \le x_0) = 0 \tag{2.17}$$

holds, then we get

$$E\left(\sup_{t\in[t',t'']} Z_t\right) = E(\max(Z_{t'}, Z_{t''})) \tag{2.18}$$

and (2.17) *is true for* e v e r y $x_0 < 0$.

Conversely, (2.18) *implies* (2.17) *for all $t_0 \in (t', t'')$ and all $x_0 < 0$.*

Proof. Let $x_0 < 0$ be given such that

$$\begin{aligned} &P(\eta_{t'} \le x_0, \eta_{t_0} > x_0, \eta_{t''} \le x_0)\\ &= P(\eta_{t'} \le x_0, \eta_{t''} \le x_0) - P(\eta_{t'} \le x_0, \eta_{t_0} \le x_0, \eta_{t''} \le x_0)\\ &= \exp(x_0 E(\max(Z_{t'}, Z_{t''}))) - \exp(x_0 E(\max(Z_{t'}, Z_{t_0}, Z_{t''}))) = 0, \end{aligned} \tag{2.17'}$$

for all $t_0 \in (t', t'')$. Then

$$\begin{aligned} &E(\max(Z_{t'}, Z_{t''})) = E(\max(Z_{t'}, Z_{t_0}, Z_{t''}))\\ \iff\quad &P(\max(Z_{t'}, Z_{t''}) = \max(Z_{t'}, Z_{t_0}, Z_{t''})) = 1 \end{aligned}$$

for all $t_0 \in (t', t'')$, and, thus, we get for finitely many $t_1, \dots, t_n \in (t', t'')$, $n \in \mathbb{N}$,

$$\max(Z_{t'}, Z_{t_1}, \dots, Z_{t_n}, Z_{t''}) = \max(Z_{t'}, Z_{t''}) \quad \text{a.s..}$$

Hence, the continuity of $\boldsymbol{Z}$ implies for $\{t_1, t_2, \dots\} := (t', t'') \cap \mathbb{Q}$

$$\sup_{t\in[t',t'']} Z_t = \sup_{t\in\{t',t'',t_1,t_2,\dots\}} Z_t = \lim_{n\to\infty} \max_{t\in\{t',t_1,\dots,t_n,t''\}} Z_t = \max(Z_{t'}, Z_{t''}) \tag{2.19}$$

with probability one, which is equivalent to (2.18).

On the other hand, (2.19) implies $P(\max(Z_{t'}, Z_{t''}) = \max(Z_{t'}, Z_{t_0}, Z_{t''})) = 1$ for all $t_0 \in (t', t'')$, so (2.17') follows for all $t_0 \in (t', t'')$ and all $x_0 < 0$. □

Now the following assertion on the hitting probability follows easily from the foregoing Lemma.

Proposition 2.18. *Let $t', t'' \in [0,1]$ be arbitrary with $0 \le t' < t'' \le 1$ and consider a standard MSP $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ with generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. If we have*

$$E\left(\sup_{t\in[t',t'']} Z_t\right) > E\left(\max\left(Z_{t'}, Z_{t''}\right)\right),$$

then there is some $t_0 \in (t', t'')$ with

$$P\left(\eta_t = x_0 \text{ for some } t \in [t', t_0], \eta_t = x_0 \text{ for some } t \in [t_0, t'']\right) > 0,$$

for every $x_0 < 0$.

Proof. Note that

$$P\left(\eta_t = x_0 \text{ for some } t \in [t', t_0], \eta_t = x_0 \text{ for some } t \in [t_0, t'']\right) \ge P\left(\eta_{t'} \le x_0, \eta_{t_0} > x_0, \eta_{t''} \le x_0\right),$$

so Lemma 2.17 implies the assertion.

□

In the proof of Lemma 2.17, the property $\sup_{t\in[t',t'']} Z_t = \max(Z_{t'}, Z_{t''})$ almost surely of a generator process $\boldsymbol{Z}$ plays a crucial role, cf. equation (2.19). It is clear that a generator process which is pathwise linear on $[t', t'']$, i.e. $Z_t := \frac{t''-t}{t''-t'} Z_{t'} + \frac{t-t'}{t''-t'} Z_{t''}, t \in [t', t'']$ a.s., obviously fulfills (2.19). All paths of a generator $\boldsymbol{Z}$ fulfilling (2.19) have to be either strictly monotone or convex on $[t', t'']$ and one may ask if there are other examples than pathwise linear processes: the answer is "yes", see Example 2.28.

Nevertheless, equation (2.19) has some further implications.

Corollary 2.19. *Let $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ be a standard MSP with generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ and fix $0 \le t' < t'' \le 1$. Then the following conditions are equivalent:*

(i) $P\left(\eta_{t'} \le x_0, \eta_{t_0} > x_0, \eta_{t''} \le x_0\right) = 0$, *for all $x_0 < 0$ and every $t_0 \in (t', t'')$;*

(ii) $P(\sup_{t\in[t',t'']} Z_t = \max(Z_{t'}, Z_{t''})) = 1$;

(iii) $E(\sup_{t\in[t',t'']} Z_t) = E(\max(Z_{t'}, Z_{t''}))$;

(iv) $P(\eta_t \le x_0 \text{ for all } t \in [t', t'']) = P(\eta_{t'} \le x_0, \eta_{t''} \le x_0)$, *for all $x_0 < 0$;*

(v) $P(\eta_t \le x_0 \text{ for all } t \in [t', t'']) - P(\eta_{t'} > x_0, \eta_{t''} > x_0) = 2\exp(x_0) - 1$, *for all $x_0 < 0$.*

Proof. The proof of Proposition 2.18 already contains *(i)* $\iff$ *(ii)* $\iff$ *(iii)*. Moreover, *(iii)* is true, if, and only if,

$$P(\eta_t \le x_0 \mathbf{1}_{[t',t'']}(t) \text{ for all } t \in [0,1]) \quad = \quad \exp\left(x_0 E\left(\sup_{t\in[t',t'']} Z_t\right)\right)$$

$$\begin{aligned} &= \exp\left(x_0 E\left(\max(Z_{t'}, Z_{t''})\right)\right) \\ &= P(\eta_{t'} \le x_0, \eta_{t''} \le x_0), \end{aligned}$$

for all $x_0 < 0$, i.e. *(iii)* $\iff$ *(iv)*. Finally, we have for arbitrary $x_0 < 0$

$$\begin{aligned} &P(\eta_{t'} \le x_0, \eta_{t''} \le x_0) = \\ &= P(\eta_t \le x_0 \mathbf{1}_{[t',t'']}(t) \text{ for all } t \in [0,1]) \\ &\quad + P(\{\eta_{t'} \le x_0, \eta_{t''} \le x_0\} \cap \{\eta_t > x_0 \text{ for some } t \in t', t''\}), \end{aligned}$$

and, thus,

$$\begin{aligned} \textit{(iv)} &\iff P(\{\eta_{t'} \le x_0, \eta_{t''} \le x_0\} \cap \{\eta_t > x_0 \text{ for some } t \in t', t''\}) = 0 \\ &\iff P(\{\eta_{t'} > x_0\} \cup \{\eta_{t''} > x_0\} \cup \{\eta_t \le x_0 \text{ for all } t \in t', t''\}) = 1, \end{aligned}$$

which is *(v)* by using the inclusion-exclusion formula. □

2.3.2 The Survivor Function of Standard MSP

In the finite-dimensional setup, survivor functions $x \mapsto P(X > x)$ for a rv X play a fundamental role in EVT. In this section we give some properties of the survivor function $P(\boldsymbol{\eta} > f) = P(\eta_t > f(t) \text{ for all } t \in [0,1]), f \in \bar{E}^-[0,1]$, of a standard MSP $\boldsymbol{\eta}$.

Proposition 2.20. *Let $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ be a standard MSP with generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. Then we obtain for $f \in \bar{E}^-[0,1]$:*

(i) $P(\boldsymbol{\eta} > f) \ge 1 - \exp\left(-E\left(\inf_{0\le t\le 1}(|f(t)|\, Z_t)\right)\right)$;

(ii) $\lim_{s\downarrow 0} \dfrac{P(\boldsymbol{\eta} > sf)}{s} = E\left(\inf_{0\le t\le 1}(|f(t)|\, Z_t)\right)$.

Proof. Due to the continuity of $\boldsymbol{\eta}$ and $\boldsymbol{Z}$ it is sufficient to consider $f \in \bar{E}^-[0,1]$ with $\sup_{0\le t\le 1} f(t) < 0$. From Subsection 2.2 below we know that

$$\xi_t := -\frac{1}{\eta_t}, \qquad 0 \le t \le 1,$$

defines a continuous MSP $\boldsymbol{\xi} = (\xi_t)_{0\le t\le 1}$ on $[0,1]$ with standard Fréchet margins and Proposition 3.2 in Giné et. al [20] yields

$$\boldsymbol{\xi} \overset{d}{=} \max_i \boldsymbol{Y}_i$$

in $\bar{C}^+[0,1]$, where $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots$ are the points (functions in $\bar{C}^+[0,1]$) of a Poisson process N with intensity measure ν given by $d\nu = d\sigma \times dr/r^2$ on $\bar{C}_1^+[0,1] \times (0,\infty) =: C[0,1]^+ = \{h \in C[0,1] : h \ge 0, h \ne 0\}$. By $\bar{C}_1^+[0,1]$ we denote the space of those functions h in $\bar{C}^+[0,1]$ with $\|h\|_\infty = \sup_{0\le t\le 1}|h(t)| = 1$. The (finite) measure σ is given by $\sigma(\cdot) = mP(\tilde{\boldsymbol{Z}} \in \cdot)$, where $\tilde{\boldsymbol{Z}} := \boldsymbol{Z}/m$ and m is the generator constant pertaining to $\boldsymbol{Z}$. Note that m coincides with the

total mass of σ.
Observe $P\left(\eta_t > f(t),\ \text{for all } t \in [0,1]\right) = 1 - P\left(\eta_t \le f(t),\ \text{for some } t \in [0,1]\right)$, and we obtain

$$
\begin{aligned}
&P\left(\eta_t \le f(t),\ \text{for some } t \in [0,1]\right)\\
&= P\left(\xi_t \le \frac{1}{|f(t)|},\ \text{for some } t \in [0,1]\right)\\
&= P\left(\text{for some } t \in [0,1],\ \forall i \in \mathbb{N} : \boldsymbol{Y}_i(t) \le \frac{1}{|f(t)|}\right)\\
&\le P\left(\forall i \in \mathbb{N},\ \text{for some } t \in [0,1] : \boldsymbol{Y}_i(t) \le \frac{1}{|f(t)|}\right)\\
&= P\left(N\left(\left\{g \in C[0,1]^+ :\ g(t) > \frac{1}{|f(t)|},\ t \in [0,1]\right\}\right) = 0\right)\\
&= \exp\left(-\nu\left(\left\{g \in C[0,1]^+ :\ g(t)\,|f(t)| > 1,\ t \in [0,1]\right\}\right)\right)\\
&= \exp\left(-\nu\left(\left\{(h,r) \in \bar{C}_1^+[0,1] \times (0,\infty) :\ rh(t)\,|f(t)| > 1,\ t \in [0,1]\right\}\right)\right)\\
&= \exp\left(-\int_{\left\{(h,r)\in\bar{C}_1^+[0,1]\times(0,\infty):\, rh(t)|f(t)|>1,\, t\in[0,1]\right\}} \frac{1}{r^2}\, dr\, \sigma(dh)\right)\\
&= \exp\left(-\int_{\bar{C}_1^+[0,1]} \int_{1/\inf_{t\in[0,1]}(h(t)|f(t)|)}^{\infty} \frac{1}{r^2}\, dr\, \sigma(dh)\right)\\
&= \exp\left(-\int_{\bar{C}_1^+[0,1]} \inf_{t\in[0,1]} \left(h(t)\,|f(t)|\right) \sigma(dh)\right)\\
&= \exp\left(-E\left(\inf_{t\in[0,1]} \left(|f(t)|\, Z_t\right)\right)\right).
\end{aligned}
$$

which is assertion *(i)*. Next we establish the inequality

$$
\limsup_{s\downarrow 0} \frac{P(\boldsymbol{\eta} > sf)}{s} \le E\left(\min_{1\le j\le m} \left(|f(t_j)|\, Z_{t_j}\right)\right), \tag{2.20}
$$

where $\{t_1, t_2, \ldots\}$ is a denumerable dense subset of [0,1], which contains those finitely many points t_i at which the function f is discontinuous.

The inclusion-exclusion theorem implies

$$
\begin{aligned}
P(\boldsymbol{\eta} > sf) &\le P\left(\bigcap_{j=1}^{m} \left\{\eta_{t_j} > sf(t_j)\right\}\right)\\
&= 1 - P\left(\bigcup_{j=1}^{m} \left\{\eta_{t_j} \le sf(t_j)\right\}\right)\\
&= 1 - \sum_{\emptyset \ne T \subset \{1,\ldots,m\}} (-1)^{|T|-1} P\left(\bigcap_{j\in T} \left\{\eta_{t_j} \le sf(t_j)\right\}\right)\\
&= 1 - \sum_{\emptyset \ne T \subset \{1,\ldots,m\}} (-1)^{|T|-1} \exp\left(-sE\left(\max_{j\in T}\left(|f(t_j)|\, Z_{t_j}\right)\right)\right)\\
&=: 1 - H(s)
\end{aligned}
$$

$$= H(0) - H(s),$$

where the function H is differentiable and, thus,

$$\begin{aligned}
\limsup_{s\downarrow 0} \frac{P(\boldsymbol{\eta} > sf)}{s} &\le -\lim_{s\downarrow 0} \frac{H(s)-H(0)}{s} \\
&= -H'(0) \\
&= \sum_{\emptyset \ne T \subset \{1,\dots,m\}} (-1)^{|T|-1} E\left(\max_{j\in T}\left(|f(t_j)|\, Z_{t_j}\right)\right) \\
&= E\left(\min_{1\le j\le m}\left(|f(t_j)|\, Z_{t_j}\right)\right),
\end{aligned}$$

since $\sum_{\emptyset \ne T\subset\{1,\dots,m\}} (-1)^{|T|-1} \max_{j\in T} a_j = \min_{1\le j\le m} a_j$ for arbitrary numbers $a_1,\dots,a_m \in \mathbb{R}$, which can be seen by induction.

As we have $\lim_{m\to\infty} E\left(\min_{1\le j\le m}\left(|f(t_j)|\, Z_{t_j}\right)\right) = E\left(\min_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)$ by the dominated convergence theorem, we get,

$$\limsup_{s\downarrow 0} \frac{P(\boldsymbol{\eta} > sf)}{s} \le E\left(\min_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right).$$

On the other hand, the inequality $\liminf_{s\downarrow 0} P(\boldsymbol{\eta} > sf)/s \ge E\left(\min_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)$ follows immediately from *(i)* and Taylor expansion of the exponential function and, thus, the proof is complete. □

Example 2.25 below shows, that in general we do not have equality in part *(i)* of the preceding lemma.

A modification of the preceding result is needed to prove Proposition 2.33 below.

Lemma 2.21. *Let* $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ *be a standard MSP with generator process* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. *Then, for* $0 \le t_1 < \dots < t_m \le 1$, $m \in \mathbb{N}$, $x \le 0$ *and* $f \in \bar{E}^-[0,1]$:

(i) $P\left(\min_{1\le i\le m} \eta_{t_i} > x\right) \ge 1 - \exp\left(xE\left(\min_{1\le i\le m} Z_{t_i}\right)\right)$, *if* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ *satisfies in addition* $P\left(\inf_{t\in[0,1]} Z_t > 0\right) = 1$;

(ii) $P\left(\eta_{t_i} > sf(t_i),\ 1 \le i \le k\right) = sE\left(\min_{1\le i\le k}\left(|f(t_i)|\, Z_{t_i}\right)\right) + o(s)$, *as* $s \downarrow 0$.

Proof. Choose $x < 0$ and define for $n \in \mathbb{N}$ the function $f_n \in \bar{E}^-[0,1]$ by

$$f_n(t) := \begin{cases} -n & \text{if } t \notin \{t_1,\dots,t_m\} \\ x & \text{otherwise.} \end{cases}$$

Then we have

$$P(\boldsymbol{\eta} > f_n) \uparrow_{n\in\mathbb{N}} P\left(\min_{1\le i\le m} \eta_{t_i} > x\right)$$

and

$$\inf_{t\in[0,1]} \left(|f_n(t)|\, Z_t\right) \uparrow_{n\in\mathbb{N}} \min_{1\le i\le m} \left(|x|\, Z_{t_i}\right),$$

due to the condition $P\left(\inf_{t\in[0,1]} Z_t > 0\right) = 1$.

From part *(i)* of Proposition 2.20 we obtain

$$P(\boldsymbol{\eta} > f_n) \geq 1 - \exp\left(-E\left(\inf_{0\leq t\leq 1}(|f_n(t)|\, Z_t)\right)\right)$$

and, thus, by the preceding considerations and the monotone convergence theorem

$$P\left(\min_{1\leq i\leq m} \eta_{t_i} > x\right) \geq 1 - \exp\left(x \min_{1\leq i\leq m} Z_{t_i}\right).$$

The assertion of part *(ii)* follows by repeating the arguments in the proof of part *(ii)* of Proposition 2.20 □

By the help of Proposition 2.20 we summarize some considerations on the hitting probability $h_\eta(x)$ of $\boldsymbol{\eta}$ and x, defined by $h_\eta(x) := P\left(\eta_t = x \text{ for some } t \in [0,1]\right),\ x \leq 0$

Proposition 2.22. *Let* $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ *a standard MSP with generator* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$, *generator constant* $m = E(\sup_{t\in[0,1]} Z_t) > 1$ *and with the additional property that* $\widetilde{m} := E(\inf_{t\in[0,1]} Z_t) > 0$.

Then the hitting probability h_η *of* $\boldsymbol{\eta}$ *and* x *has the properties*

$$h_\eta(0) = 0, \qquad h_\eta(x) > 0 \text{ for } x < 0 \quad \text{and} \quad \lim_{x\to-\infty} h_\eta(x) = 0.$$

Moreover,

$$0 < \int_{-\infty}^{0} h_\eta(x)\, dx \leq \frac{m - \widetilde{m}}{m\widetilde{m}}.$$

Proof. The assertion follows from Proposition 2.15 and the inequality

$$\begin{aligned} &P\left(\eta_t = x \text{ for some } t \in [0,1]\right) \\ &= 1 - P\left(\eta_t \neq x \text{ for all } t \in [0,1]\right) \\ &= 1 - \left[P\left(\eta_t > x \text{ for all } t \in [0,1]\right) + P\left(\eta_t < x \text{ for all } t \in [0,1]\right)\right] \\ &= P\left(\eta_t \leq x \text{ for some } t \in [0,1]\right) - \exp(xm) \\ &\leq \exp(x\widetilde{m}) - \exp(xm), \end{aligned}$$

which holds for all $x \in (-\infty, 0]$ by Proposition 2.20. □

In the setup of the preceding proposition, the term $\frac{m-\widetilde{m}}{m\widetilde{m}}$ can be interpreted as a measure of the dependence structure of $\boldsymbol{\eta}$. In case of complete dependence we have $m = \widetilde{m} = 1$, and, thus, $\frac{m-\widetilde{m}}{m\widetilde{m}} = 0$. Because of

$$\begin{aligned} E(\inf_{s\in[0,1]} Z_s) = 1 \quad &\iff \quad E(Z_t - \inf_{s\in[0,1]} Z_s) = 0,\ \forall t \in [0,1] \\ &\iff \quad P(Z_t = \inf_{s\in[0,1]} Z_s) = 1,\ \forall t \in [0,1] \\ &\iff \quad P(Z_t = \sup_{s\in[0,1]} Z_s) = 1,\ \forall t \in [0,1] \end{aligned}$$

$$\iff \quad E(\sup_{s\in[0,1]} Z_s) = 1,$$

we get $m > 1 \iff \widetilde{m} < 1$ and, therefore, $\frac{m-\widetilde{m}}{m\widetilde{m}} > 0$ in this case.

We close this section with an observation on the univariate rv $\inf_{t\in[0,1]} \eta_t$, where $\boldsymbol{\eta}$ is a standard MSP.

Lemma 2.23. *Let $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ be a standard MSP with generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ with the additional property $E(\inf_{t\in[0,1]} Z_t) > 0$. Consider the (univariate) rv $\inf_{t\in[0,1]} \eta_t$ and its df $F(x) := P\left(\inf_{t\in[0,1]} \eta_t \le x\right),\ x < 0$.*

Then:

(i) the function $f : (0,\infty) \to [0,1],\ x \mapsto 1 - F(-1/x)$ is regularly varying with index $\alpha = -1$.

(ii) for every $x < 0$

$$F^n(-\gamma_n x) \to \exp(x),\ n \to \infty,$$

where $\gamma_n = F^{-1}(1-1/n)$ and $F^{-1}(q) = \inf\{t \in \mathbb{R} : F(t) \ge q\},\ q \in (0,1)$ is the generalized inverse of F. That is, $\inf_{t\in[0,1]} \eta_t$ is in the (univariate) maximum domain of attraction of a standard negative exponential distribution.

Proof. Note that part *(ii)* follows immediately from part *(i)* and Proposition 1.13 in Resnick [24]. We know by Lemma 2.20 that for all $y > 0$

$$\lim_{s\downarrow 0} P(\boldsymbol{\eta} > -ys)/s = \lim_{s\downarrow 0} P(\inf_{t\in[0,1]} \eta_t > -ys)/s = yE(\inf_{t\in[0,1]} Z_t).$$

This implies that the function $f(x) = P(\inf_{t\in[0,1]} \eta_t > -1/x)$ fulfills for all $y > 0$

$$\lim_{x\to\infty} \frac{f(xy)}{f(x)} = \lim_{x\to\infty} \frac{xf(xy)}{xf(x)} = y^{-1}.$$

□

The latter assertions use the additional assumption $E(\inf_{t\in[0,1]} Z_t) > 0$ on the generator process $\boldsymbol{Z}$. Example 2.30 shows, that this assumption is not too restrictive.

2.4 Examples

This section contains several examples of standard MSP and generator processes, each of them illustrates a specific issue appearing in one of the assertions of the previous section.

We start with a standard MSP $\boldsymbol{\eta}$, which has a generator constant $m > 1$ but there is a interval I on which its generator $\boldsymbol{Z}$ fulfills $Z_t = Z_s$ for all $s, t \in I$ a.s., cf. Proposition 2.15.

Example 2.24. Let Z_0, Z_1 be iid rv with

$$P(Z_i = \frac{1}{n}) = \frac{n}{n+1} = 1 - P(Z_i = n);$$

in particular $E(Z_i) = 1,\ i = 0,1$ for $n \in \mathbb{N}$. For some $0 < a < b < 1$ define

$$Z_t := \begin{cases} \frac{a-t}{a} Z_0 + \frac{t}{a} & \text{for } t \in [0,a); \\ 1 & \text{for } t \in [a,b]; \\ \frac{1-t}{1-b} + \frac{t-b}{1-b} Z_1 & \text{for } t \in (b,1], \end{cases}$$

and $\boldsymbol{Z}$ is obviously a generator process with properties (2.6) and the corresponding standard MSP $\boldsymbol{\eta}$ is pathwise a constant function on $[a,b]$ with probability one. But $\boldsymbol{\eta}$ is for $n > 1$ not the complete dependence MSP as

$$m = E(\sup_{t\in[0,1]} Z_t) = \left(\frac{n}{n+1}\right)^2 + n\left(1 - \left(\frac{n}{n+1}\right)^2\right) = \frac{3n^2+n}{(n+1)^2} > 1.$$

The following example of a standard MSP $\boldsymbol{\eta}$ shows, that there is in general not equality in assertion *(i)* of Lemma 2.20.

Example 2.25. Let η_0, η_1 independent max-stable rv with standard negative exponential distribution, i.e. $P(\eta_i \le x) = \exp(x),\ x \le 0, i = 1,2$. Their joint (bivariate) distribution is a standard max-stable distribution, so it can written by means of a D-norm on $\mathbb{R}^2$ (actually, the $D-$norm is the L_1-norm because of the independent margins) and this D-norm is generated by some Z_0, Z_1 with $E(Z_i) = 1, i = 0,1$:

$$P(\eta_0 \le x_0, \eta_1 \le x_1) = \exp\left(-\|x_0, x_1\|_D\right) = \exp\left(-E\left(\max_{i=0,1}(|x_i|Z_i)\right)\right).$$

Note that in this setup $E(\max(Z_0,Z_1)) = 2 = E(Z_0 + Z_1)$ which implies $\min(Z_0,Z_1) = Z_0 + Z_1 - \max(Z_0,Z_1) = 0$ almost surely.

Define $Z_t := tZ_0 + (1-t)Z_1$ for $t \in [0,1]$, which is a generator process, so there is some standard MSP $\boldsymbol{\eta}$ with

$$P(\boldsymbol{\eta} \le f) = \exp\left(-E\left(\sup_{t\in[0,1]} |f(t)|Z_t\right)\right).$$

Now consider the function $f_n \in \bar{E}^-[0,1]$ defined for $n \in \mathbb{N}$ and $c < 0$ by

$$f_n(t) := \begin{cases} c & \text{for } t = 0 \text{ and } t = 1; \\ -n & \text{elsewhere.} \end{cases}$$

Then we get on the one hand

$$\lim_{n\to\infty} P(\boldsymbol{\eta} > f_n) \quad = \quad P(\bigcup_{n\in\mathbb{N}} \{\boldsymbol{\eta} > f_n\})$$

$$= \quad P(\eta_0 > c, \eta_1 > c) = (1 - \exp(c))^2 > 0$$

due to the independence of η_0, η_1.

Suppose the equality $P(\boldsymbol{\eta} > f_n) = 1 - \exp\left(-E\left(\inf_{0\le t\le 1}(|f_n(t)|\, Z_t)\right)\right)$ to be true for all $n \in \mathbb{N}$. Then, on the other hand, we get

$$\begin{aligned} \lim_{n\to\infty} P(\boldsymbol{\eta} > f_n) &= 1 - \lim_{n\to\infty} \exp\left(-E\left(\inf_{0\le t\le 1}(|f_n(t)|\, Z_t)\right)\right) \\ &= 1 - \exp\left(-E\left(\min(|c|\, Z_0, |c|\, Z_1)\right)\right) = 1 - \exp\left(c \cdot 0\right) = 0. \end{aligned}$$

Thus, $P(\boldsymbol{\eta} > f_n) = 1 - \exp\left(-E\left(\inf_{0\le t\le 1}(|f_n(t)|\, Z_t)\right)\right)$ cannot be true for all $n \in \mathbb{N}$.

The next two examples concern the upper boundary $(m - \widetilde{m})/(m\widetilde{m})$ of the hitting probability h_η in Proposition 2.22: while the family of standard MSP in Example 2.26 converges to the total dependence MSP, and, therefore, $(m^{(n)} - \widetilde{m}^{(n)})/m^{(n)}\widetilde{m}^{(n)}$ runs to zero for $n \to \infty$, Example 2.27 shows a diverging boundary $(m^{(n)} - \widetilde{m}^{(n)})/m^{(n)}\widetilde{m}^{(n)}$.

Example 2.26. Let $Z_0^{(n)}, Z_1^{(n)}$ be i.i.d. rv with distribution $P(Z_0^{(n)} = 1 + 1/n) = \frac{1}{2} = P(Z_0^{(n)} = 1 - 1/n)$ for $n \ge 2$ and define the generator processes $\boldsymbol{Z}^{(n)}$ by linear interpolation, i.e.

$$Z_t^{(n)} := (1-t)Z_0^{(n)} + tZ_1^{(n)}, \quad t \in [0,1].$$

Then $\boldsymbol{Z}^{(n)}$ gets for large n arbitrarily close to the constant function $\mathbf{1}_{[0,1]}$, which is a generator of the complete dependence MSP.

Furthermore,

$$m^{(n)} = E(\sup_{t\in[0,1]} Z_t) = 1 + \frac{1}{2n}, \qquad \widetilde{m}^{(n)} = E(\inf_{t\in[0,1]} Z_t) = 1 - \frac{1}{2n},$$

and this implies $(m^{(n)} - \widetilde{m}^{(n)})/m^{(n)}\widetilde{m}^{(n)} = 4n/(4n^2 - 1)$, which converges to zero for $n \to \infty$.

Example 2.27. Let $Z_0^{(n)}, Z_1^{(n)}$ be i.i.d. rv with distribution $P(Z_0^{(n)} = 1/n) = \frac{n}{n+1} = 1 - P(Z_0^{(n)} = n)$ for $n \ge 2$ and define the generator processes $\boldsymbol{Z}^{(n)}$ by linear interpolation, i.e.

$$Z_t^{(n)} := (1-t)Z_0^{(n)} + tZ_1^{(n)}, \quad t \in [0,1].$$

Then

$$m^{(n)} = E(\sup_{t\in[0,1]} Z_t) = \frac{2n}{n+1}, \qquad \widetilde{m}^{(n)} = E(\inf_{t\in[0,1]} Z_t) = \frac{2}{n+1},$$

and this implies $(m^{(n)} - \widetilde{m}^{(n)})/m^{(n)}\widetilde{m}^{(n)} = n^2/2n$, which converges to infinity for $n \to \infty$ (note that $m^{(n)} \to 2$ for $n \to \infty$).

The upcoming Example 2.28 shows, that there are generator processes, which fulfill $P\left(\sup_{t\in[t',t'']} Z_t = \max(Z_{t'}, Z_{t''})\right) = 1$ but which are not pathwise linear; cf. Lemma 2.17 and Corollary 2.19.

Example 2.28. Take real numbers $a, b, c, d, e > 0$ with the following properties:

$$1 < a; \quad b < 1; \quad 1 < c < \frac{a-b}{a-1}; \quad (1 <)\frac{a-b}{a-b-c(a-1)} < d; \quad e < 1;$$

and define

$$p := \frac{1-b}{a-b}; \quad \widetilde{p} := \frac{1-e}{d-e}.$$

Let $Y, \widetilde{Y}$ be some independent Bernoulli rvs with $P(Y = 1) = p = 1 - P(Y = 0)$ and $P(\widetilde{Y} = 1) = \widetilde{p} = 1 - P(\widetilde{Y} = 0)$. Now define

$$Z_0 := Ya + (1-Y)b; \quad Z_{1/2} := 1; \quad Z_1 := (1-Y)c + \left(1 - \frac{a-1}{a-b}c\right)\left(\widetilde{Y}d + (1-\widetilde{Y})e\right).$$

Elementary computations show that $E(Z_0) = E(Z_{1/2}) = E(Z_1) = 1$, so the linear interpolation process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ defined by

$$Z_t := \begin{cases} 2(\frac{1}{2} - t)Z_0 + 2tZ_{1/2} & \text{for } t \in [0, 1/2] \\ 2(1-t)Z_{1/2} + 2(t - \frac{1}{2})Z_1 & \text{for } t \in [1/2, 1]. \end{cases}$$

is a proper generator process. We have

$$P\left(\sup_{t\in[t',t'']} Z_t = \max(Z_{t'}, Z_{t''})\right) = 1$$

for arbitrary $0 \le t' < t'' \le 1$, as three of the possible four paths are (strictly) monotone, and there is (with probability $p \cdot \widetilde{p}$) one path which is (strictly) convex, see Figure 2.1.

Note that the numbers a, b, c, d, e can be substituted by appropriate rvs (independent of each other and of $Y, \widetilde{Y}$), which have those values as their expectation, respectively, and that $Z_{1/2}$ can also be chosen to be random.

The next example shows in particular, that there are standard MSP, which hit every $x_0 < 0$ twice with positive probability, but the probability of hitting any $x_0 < 0$ three or more times is equal to zero.

Example 2.29. Let η_0, η_1 independent negative exponential distributed rv and define the continuous process $\boldsymbol{\eta}$ by

$$\eta_t := \max(\frac{1}{1-t}\eta_0, \frac{1}{t}\eta_1), \quad t \in [0,1].$$

Elementary computations show that all fidis of $\boldsymbol{\eta}$ are max-stable and that the one-dimensional marginal distributions are standard negative exponential, so $\boldsymbol{\eta}$ is a standard MSP.

We have $P(\eta_t < x \text{ for all } t \in [0,1]) = P(\max(\eta_0, \eta_1) < x) = \exp(2x)$ for $x < 0$, so the generator constant of $\boldsymbol{\eta}$ is given by $m = 2$. Furthermore, note that

$$\eta_0 + \eta_1 \le \eta_t \le \max(\eta_0, \eta_1), \text{ for all } t \in [0,1].$$

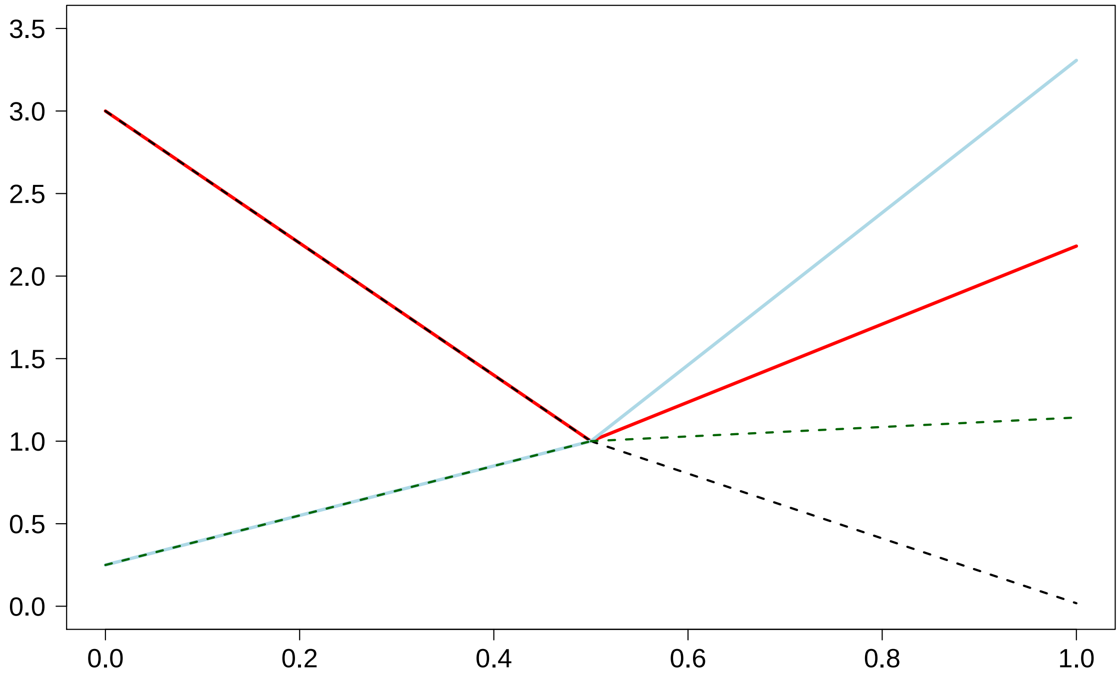

Figure 2.1: All possible paths of the process $\boldsymbol{Z}$ in Example 2.28. The values are $a = 3, b = 0.25, c = 9/8, d = 12, e = 0.1$.

Hence, for abitrary $x < 0$:

$$\begin{aligned}
h(x) &= P(\eta_t = x \text{ for some } t \in [0,1]) \\
&= P(\eta_0 + \eta_1 < x < \max(\eta_0, \eta_1)) \\
&= \int_{-\infty}^{0} P(\eta_0 + y < x < \max(\eta_0, y)) \exp(y)\, dy \\
&= \int_{-\infty}^{x} P(\eta_0 + y < x < \max(\eta_0, y)) \exp(y)\, dy \\
&\qquad + \int_{x}^{0} P(\eta_0 + y < x < \max(\eta_0, y)) \exp(y)\, dy \\
&= \int_{-\infty}^{x} P(\eta_0 > x) \exp(y)\, dy + \int_{x}^{0} P(\eta_0 < x - y) \exp(y)\, dy \\
&= (1 - \exp(x)) \int_{-\infty}^{x} \exp(y)\, dy - x \exp(x) \\
&= (1 - \exp(x) - x) \exp(x),
\end{aligned}$$

and this implies $\int_{-\infty}^{0} h(x)\, dx = 3/2$.

Moreover, elementary computations yield for every $t_0 \in (0,1)$ and arbitrary $x < 0$

$$P\big(\{\eta_t = x \text{ for some } t \in [0, t_0)\} \cap \{\eta_t = x \text{ for some } t \in [t_0, 1]\}\big)$$

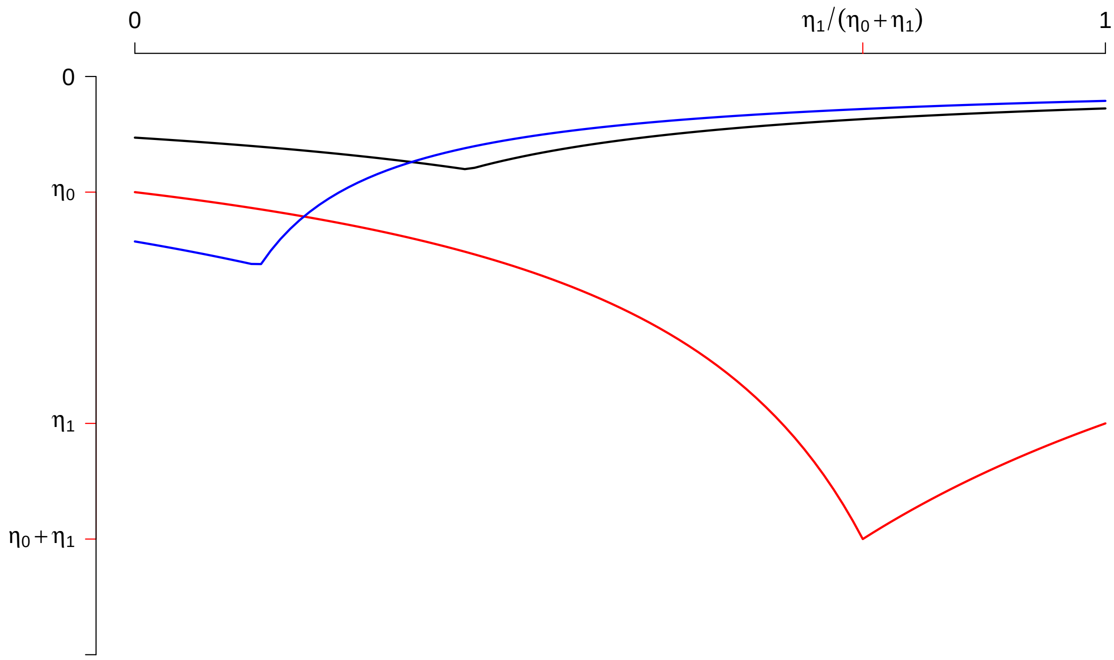

Figure 2.2: Some sample paths of $\boldsymbol{\eta}$ in Example 2.29.

$$= \quad (\exp(x(1-t_0)) - \exp(x))(\exp(xt_0) - \exp(x)) > 0,$$

so all paths of $\boldsymbol{\eta}$ hit every $x < 0$ two times with positive probability.

On the other hand, we get for disjoint intervals $I_1, I_2, I_3 \subset [0,1]$

$$P\big(\bigcap_{k=1,2,3} \{\eta_t = x \text{ for some } t \in I_k\}\big) = 0, \ \ x < 0,$$

so every path of $\boldsymbol{\eta}$ does not hit any $x < 0$ three times (or more often).

Proof. We carry out the elementary computations of the last assertions.

$$\begin{aligned}
&P\big(\{\eta_t = x \text{ for some } t \in [0,t_0)\} \cap \{\eta_t = x \text{ for some } t \in [t_0,1]\}\big) = \\
&\quad = P\Big(\{\eta_t = x \text{ for some } t \in [0,t_0)\} \cap \{\eta_t = x \text{ for some } t \in [t_0,1]\}, \eta_1/(\eta_0+\eta_1) < t_0\Big) \\
&\qquad + P\Big(\{\eta_t = x \text{ for some } t \in [0,t_0)\} \cap \{\eta_t = x \text{ for some } t \in [t_0,1]\}, \eta_1/(\eta_0+\eta_1) \geq t_0\Big) \\
&\quad = P\Big(\max(\eta_0, \eta_{t_0}) \geq x \geq \eta_0+\eta_1, \eta_{t_0} \leq x \leq \eta_1, \eta_1/(\eta_0+\eta_1) < t_0\Big) \\
&\qquad + P\Big(\eta_0 \geq x \geq \eta_{t_0}, \eta_0+\eta_1 \leq x \leq \max(\eta_1, \eta_{t_0}), \eta_1/(\eta_0+\eta_1) \geq t_0\Big) \\
&\quad = P\Big(\{\eta_0 \geq x \geq \eta_0+\eta_1\} \cup \{\eta_{t_0} \geq x \geq \eta_0+\eta_1\}, \eta_{t_0} \leq x \leq \eta_1, \eta_1/(\eta_0+\eta_1) < t_0\Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + P\Big(\eta_0 \ge x \ge \eta_{t_0}, \{\eta_0+\eta_1 \le x \le \eta_1\} \cup \{\eta_0+\eta_1 \le x \le \eta_{t_0}\}, \eta_1/(\eta_0+\eta_1) \ge t_0\Big)\\
&= P\Big(\eta_0 \ge x \ge \eta_0+\eta_1, \max(\eta_0/(1-t_0), \eta_1/t_0) \le x \le \eta_1, \eta_1/(\eta_0+\eta_1) < t_0\Big)\\
&\quad + P\Big(\eta_0 \ge x \ge \max(\eta_0/(1-t_0), \eta_1/t_0), \eta_0+\eta_1 \le x \le \eta_1, \eta_1/(\eta_0+\eta_1) \ge t_0\Big)\\
&= P\Big(x \le \eta_0 \le (1-t_0)x, x \le \eta_1 \le t_0 x, \eta_0+\eta_1 \le x, \eta_1(1-t_0)/t_0 > \eta_0\Big)\\
&\quad + P\Big(x \le \eta_0 \le (1-t_0)x, x \le \eta_1 \le t_0 x, \eta_0+\eta_1 \le x, \eta_1(1-t_0)/t_0 \le \eta_0\Big)\\
&= P\Big(x \le \eta_0 \le (1-t_0)x, x \le \eta_1 \le t_0 x, \eta_0+\eta_1 \le x\Big)\\
&= P\Big(x \le \eta_0 \le (1-t_0)x, x \le \eta_1 \le t_0 x\Big)\\
&= P\Big(x \le \eta_0 \le (1-t_0)x\Big) \cdot P\Big(x \le \eta_1 \le t_0 x\Big)\\
&= (\exp(x(1-t_0)) - \exp(x))(\exp(xt_0) - \exp(x)).
\end{aligned}
$$

In order to show that $P\big(\bigcap_{k=1,2,3}\{\eta_t = x \text{ for some } t \in I_k\}\big) = 0$ for all $x < 0$ and arbitrary disjoint intervals $I_k \subset [0,1], k = 1,2,3$, we show that for arbitrary $0 < t_1 < t_2 < 1$

$$
\begin{aligned}
P\Big(\{\eta_t = x \text{ for some } t \in [0,t_1)\} &\cap \{\eta_t = x \text{ for some } t \in [t_1,t_2)\}\\
&\cap \{\eta_t = x \text{ for some } t \in [t_2,1]\}\Big) = 0. \qquad (2.21)
\end{aligned}
$$

Having seen this, we get the assertion as follows: assume those arbitrary disjoint intervals $I_k \subset [0,1], k = 1,2,3$ without loss of generality to be "ordered", i.e. $\sup(x : x \in I_1) \le \inf(x : x \in I_2)$ and $\sup(x : x \in I_2) \le \inf(x : x \in I_3)$. Then choose $0 < t_1 < t_2 < 1$ such that $I_1 \subset [0,t_1], I_2 \subset [t_1,t_2]$ and $I_3 \subset [t_2,1]$ (which is possible if all $I_k, k = 1,2,3$, are of positive length, otherwise the assertion is trivial anyway) and we get

$$
\begin{aligned}
&P\big(\bigcap_{k=1,2,3}\{\eta_t = x \text{ for some } t \in I_k\}\big)\\
&\le P\Big(\{\eta_t = x \text{ for some } t \in [0,t_1]\} \cap \{\eta_t = x \text{ for some } t \in [t_1,t_2]\}\\
&\qquad\qquad \cap \{\eta_t = x \text{ for some } t \in [t_2,1]\}\Big)\\
&= P\Big(\{\eta_t = x \text{ for some } t \in [0,t_1)\} \cap \{\eta_t = x \text{ for some } t \in [t_1,t_2)\}\\
&\qquad\qquad \cap \{\eta_t = x \text{ for some } t \in [t_2,1]\}\Big) = 0.
\end{aligned}
$$

It remains to show, that (2.21) is true. For ease of notation we write $A_{t_1,t_2} := \{\eta_t = x \text{ for some } t \in [0,t_1)\} \cap \{\eta_t = x \text{ for some } t \in [t_1,t_2)\} \cap \{\eta_t = x \text{ for some } t \in [t_2,1]\}$, and we get

$$
\begin{aligned}
&P\Big(A_{t_1,t_2}\Big)\\
&= P\Big(A_{t_1,t_2} \cap \{\eta_1/(\eta_1+\eta_0) \in [0,t_1)\}\Big) + P\Big(A_{t_1,t_2} \cap \{\eta_1/(\eta_1+\eta_0) \in (t_1,t_2]\}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+P\Big(A_{t_1,t_2}\cap\{\eta_1/(\eta_1+\eta_0)\in[t_2,1]\}\Big)\\
=\;& P\Big(A_{t_1,t_2}\cap\{\eta_1/(\eta_1+\eta_0)\in(t_1,t_2]\}\Big).
\end{aligned}
$$

This is immediate from the construction of $\boldsymbol{\eta}$: if the random time $t=\eta_1/(\eta_1+\eta_0)$ at which every path of $\boldsymbol{\eta}$ attains its global minimum lies in $[0,t_1)$ then all paths with this property are strictly increasing on $[t_1,1]$, so it is impossible that those paths hit any $x<0$ on both subintervals, $[t_1,t_2)$ and $[t_2,1]$; an analogous argument yields that the last term in the foregoing sum is also equal to zero.

Now, in turn,

$$
\begin{aligned}
&P\Big(A_{t_1,t_2}\Big)\\
=\;& P\Big(A_{t_1,t_2}\cap\{\eta_1/(\eta_1+\eta_0)\in(t_1,t_2]\}\Big)\\
=\;& P\Big(\{\eta_0>x>\eta_{t_1},\max(\eta_{t_1},\eta_{t_2})>x>\eta_0+\eta_1,\eta_{t_2}<x<\eta_1\}\\
&\qquad\cap\{\eta_1/(\eta_1+\eta_0)\in(t_1,t_2]\}\Big)\\
\le\;& P\Big(\{x<\eta_0<(1-t_1)x,x<\eta_1<t_2x\}\\
&\qquad\cap\big\{\{\eta_0>(1-t_1)x\}\cup\{\eta_1>t_2x\}\big\}\Big)\\
=\;& 0,
\end{aligned}
$$

and the assertions are completely proven. □

Some of the assertions in Section 2.3 made the additional assumption $E\left(\inf_{0\le t\le 1}Z_t\right)>0$ on a generator process $\boldsymbol{Z}$ of a standard MSP $\boldsymbol{\eta}$. The following considerations show, that there is for every generator constant $m\in[1,\infty)$ some generator process $\boldsymbol{Z}$ which fulfills $E\left(\inf_{0\le t\le 1}Z_t\right)>0$ (note that this is equivalent to $P\left(\min_{t\in[0,1]}Z_t>0\right)>0$).

Example 2.30. Let $Z_{t_i}, i=1,2,\dots$ be iid rv with discrete distribution

$$P\left(Z_{t_i}=1/p\right)=p;\qquad P\left(Z_{t_i}=0\right)=1-p,\ \text{for some } p\in(0,1].$$

Set $\boldsymbol{Z}=(Z_t)_{t\in[0,1]}$ as the linear interpolation of $n\in\mathbb{N}$ of those Z_{t_i}, i.e.

$$Z_t=\begin{cases} Z_{t_i} \text{ for } t=t_i,\ i=1,\dots n;\\ \text{linearly interpolated elsewhere.}\end{cases}$$

Obviously, the process $\boldsymbol{Z}:=(Z_t)_{t\in[0,1]}$ is continuous, fulfills $P(\boldsymbol{Z}\ge 0)=1$, $E(Z_t)=1$ for every $t\in[0,1]$ and there is

$$m=E\left(\sup_{t\in[0,1]}Z_t\right)=\frac{1}{p}(1-(1-p)^n)\in[1,n)\text{ for } p\in(0,1],$$

and $E\left(\min_{t\in[0,1]}Z_t\right)=\frac{1}{p}p^n=p^{n-1}>0$.

As the Brownian motion is quite popular and easy to simulate, we give an example of a generator process, which is an appropriate modification of the Brownian motion.

Example 2.31. Denote by $\boldsymbol{B} = (B_t)_{t\in[0,1]}$ a standard Brownian Motion (restricted to $[0,1]$), i.e. $B_0 = 0$ a.s., $E(B_t) = 0$ and $Var(B_t) = t$ for $t \in [0,1]$. For some $\delta \in (0,1]$ set

$$\boldsymbol{Z}^{(BM)} = (Z_t^{(BM)})_{t\in[0,1]} := 1 + \max(\min(B_t, \delta), -\delta), \quad t \in [0,1].$$

Then $\boldsymbol{Z}^{(BM)}$ has continuous sample paths with $P(Z_t^{(BM)} \in [1-\delta, 1+\delta]$ for all $t \in [0,1]) = 1$ and, for every $t \in [0,1]$, we have

$$\begin{aligned} &P(Z_t^{(BM)} - 1 \le x) = \Phi(x/\sqrt{t}) \quad \text{for } x \in [-\delta, \delta), \\ \text{and} \qquad &P(Z_t^{(BM)} - 1 = -\delta) = \Phi(-\delta/\sqrt{t}) = P(Z_t - 1 = \delta), \end{aligned}$$

where Φ denotes the distribution function of the standard normal distribution, i.e. $\Phi(x) = \int_{-\infty}^{x} \exp(-y^2/2)\, dy/\sqrt{2\pi}$. This implies in particular $E(Z_t^{(BM)}) = 1, t \in [0,1]$, and, thus, $\boldsymbol{Z}$ is a proper generator, see Figure 2.3.

Note that it is possible to modify a Brownian Bridge in an analogous manner to get a generator process: let $\boldsymbol{BB} = (BB_t)_{t\in[0,1]}$ defined by

$$BB_t := B_t - t \cdot B_1, \quad t \in [0,1],$$

where $(B_t)_{t\in[0,1]}$ is a standard Brownian Motion as before. Then, with similar arguments as before,

$$\boldsymbol{Z}^{(BB)} = (Z_t^{(BB)})_{t\in[0,1]} := 1 + \max(\min(BB_t, \delta), -\delta), \quad t \in [0,1].$$

is a generator process for every $\delta \in (0,1]$; see Figure 2.4 for some sample paths.

2.5 Standard MSP as Generator Processes

It is conspicuous that we have $E(\eta_t) = -1,\ t \in [0,1]$, for a standard MSP $\boldsymbol{\eta}$ and that one of the crucial requirements on a generator process is $E(Z_t) = 1,\ t \in [0,1]$. So we consider in this section standard MSP $\boldsymbol{\eta}$ for which $-\boldsymbol{\eta} =: \boldsymbol{Z}'$ is, in turn, a generator processes.

Proposition 2.32. *Let $\boldsymbol{\eta}$ be a standard MSP with some generator $\boldsymbol{Z}$, which fulfills in addition $E\left(\inf_{0\le t\le 1} Z_t\right) > 0$. Then the process $\boldsymbol{Z}' := -\boldsymbol{\eta}$ is a generator process of some standard MSP $\boldsymbol{\eta}'$.*

Proof. We have to show that $P(\boldsymbol{Z}' \ge 0) = 1$, $E(Z_t') = 1$ for every $t \in [0,1]$ and $E\left(\sup_{0\le t\le 1} Z_t'\right) < \infty$ holds. As $\boldsymbol{\eta}$ is a standard MSP, Lemma 2.2 immediately implies $P(\boldsymbol{\eta} \le 0) = P(\boldsymbol{Z}' \ge 0) = 1$. In addition, as η_t is standard negative exponential distributed, $E(\eta_t) = -1$, so it is clear that $E(Z_t') = E(-\eta_t) = 1$ for every $t \in [0,1]$. Note that these properties are fulfilled without any further assumption.

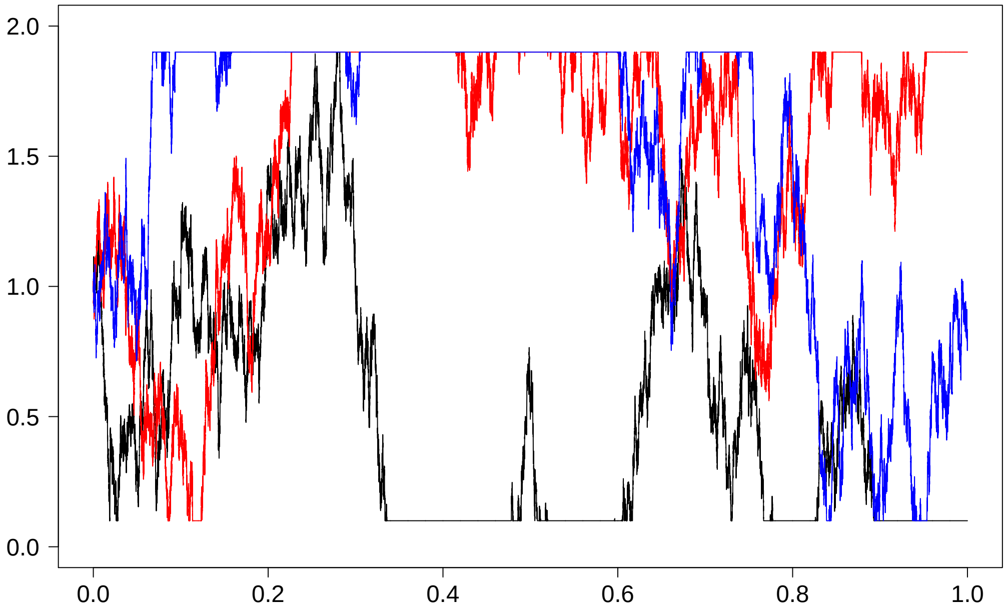

Figure 2.3: Three sample paths of $\boldsymbol{Z}^{(BM)}$ in Example 2.31 with $\delta = 0.9$.

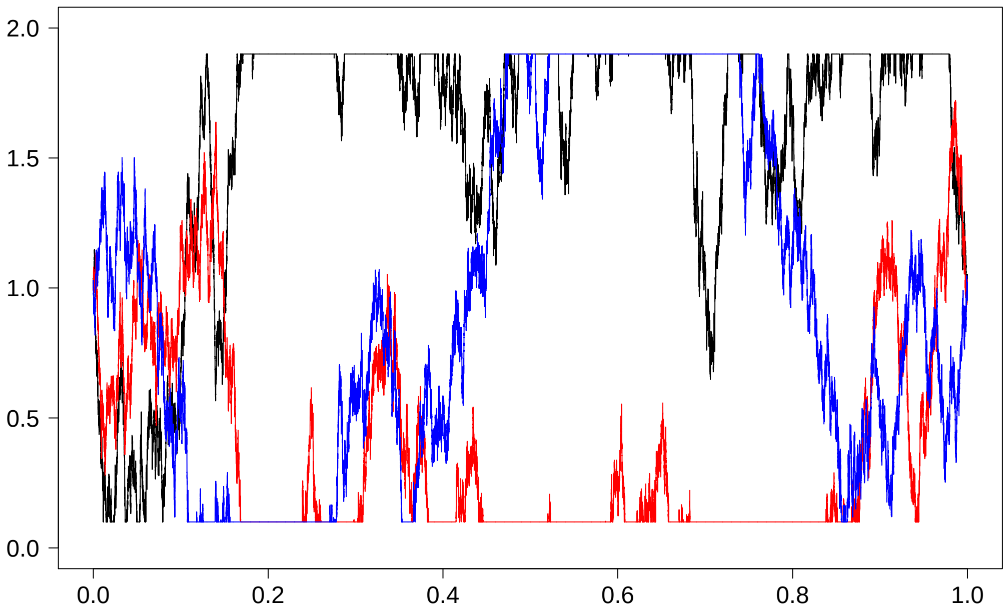

Figure 2.4: Three sample paths of $\boldsymbol{Z}^{(BB)}$ in Example 2.31 with $\delta = 0.9$.

Set $E\left(\inf_{0\le t\le 1} Z_t\right) =: c$, so $c > 0$ by assumption. From Proposition 2.20 we get for every $x \in (0,\infty)$

$$P\left(\sup_{t\in[0,1]} Z_t' > x\right) \le \exp(-xc),$$

which implies

$$E\left(\sup_{0\le t\le 1} Z_t'\right) = \int_0^\infty P\left(\sup_{0\le t\le 1} Z_t' > x\right) dx \le \int_0^\infty \exp(-xc)\, dx = c^{-1}.$$

□

The next naturally arising question is, if self-generating standard MSP exist, i.e., if it is possible, that

$$P(\eta_t \le f(t),\, t\in[0,1]) = \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)\right), \qquad f\in \bar{E}^-[0,1],$$

with $Z_t = |\eta_t|$, $t\in[0,1]$.

Consider the standard MSP $\boldsymbol{\eta}$, whose margins are completely dependent, i.e., $\eta_t = \eta_0, t \in [0,1]$. Then the corresponding D-norm is equal to the sup-norm, $\|\cdot\|_D = \|\cdot\|_\infty$, and, therefore,

$$P(\boldsymbol{\eta}\le f) = \exp(-\|f\|_\infty) = \exp\left(-E\left(\sup_{t\in[0,1]} |f(t)||\eta_0|\right)\right) = \exp\left(-E\left(\sup_{t\in[0,1]} |f(t)||\eta_t|\right)\right),$$

$f\in\bar{E}^-[0,1]$, i.e., the complete dependence standard MSP $\boldsymbol{\eta}$ is indeed self-generating. Moreover, it is the only standard MSP with this property. This is established in the next result.

Proposition 2.33. *The standard MSP $\boldsymbol{\eta}$ with complete dependence of the margins is the only self-generating standard MSP.*

Proof. Let $\boldsymbol{\eta}$ be a self-generating standard MSP. Choose $0\le t_1 < \cdots < t_m \le 1$. Then we have

$$\begin{aligned} E\left(\min_{1\le i\le m} |\eta_{t_i}|\right) &= \int_0^\infty P\left(\min_{1\le i\le m}|\eta_{t_i}| \ge x\right) dx \\ &= \int_0^\infty P\left(\max_{1\le i\le m} \eta_{t_i} \le -x\right) dx \\ &= \int_0^\infty \exp\left(-xE\left(\max_{1\le i\le m} Z_{t_i}\right)\right) dx \\ &= \frac{1}{E\left(\max_{1\le i\le m} Z_{t_i}\right)}, \end{aligned}$$

where $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ is the generator corresponding to $\boldsymbol{\eta}$, i.e., $\boldsymbol{Z} = |\boldsymbol{\eta}|$, and, thus, we have established

$$E\left(\min_{1\le i\le m} |\eta_{t_i}|\right) = \frac{1}{E\left(\max_{1\le i\le m} |\eta_{t_i}|\right)}. \tag{2.22}$$

We have, moreover, by Lemma 2.21

$$\begin{aligned} E\left(\max_{1\le i\le m}|\eta_{t_i}|\right) &= \int_0^\infty P\left(\max_{1\le i\le m}|\eta_{t_i}|>x\right)dx \\ &= \int_0^\infty P\left(\min_{1\le i\le m}\eta_{t_i}<-x\right)dx \\ &\le \int_0^\infty \exp\left(-xE\left(\min_{1\le i\le m}Z_{t_i}\right)\right)dx \\ &= \frac{1}{E\left(\min_{1\le i\le m}Z_{t_i}\right)} \\ &= \frac{1}{E\left(\min_{1\le i\le m}|\eta_{t_i}|\right)} \\ &= E\left(\max_{1\le i\le m}|\eta_{t_i}|\right) \end{aligned}$$

by equation (2.22) and this implies

$$P\left(\min_{1\le i\le m}\eta_{t_i}\le y\right) = \exp\left(yE\left(\min_{1\le i\le m}|\eta_{t_i}|\right)\right), \qquad y\le 0. \tag{2.23}$$

We claim that

$$E\left(\min_{1\le i\le m}|\eta_{t_i}|\right)=1, \qquad t_1,\dots,t_m\in[0,1],\ m\in\mathbb{N}. \tag{2.24}$$

This can be seen by induction over m. The case $m=1$ is trivial. For $m=2$ we obtain

$$\begin{aligned} P\left(\min_{i=1,2}\eta_{t_i}\le y\right) &= P\left(\eta_{t_1}\le y\right)+P\left(\eta_{t_2}\le y\right)-P\left(\eta_{t_1}\le y,\eta_{t_2}\le y\right) \\ &= 2\exp(y)-\exp\left(yE\left(\max_{i=1,2}|\eta_{t_i}|\right)\right) \end{aligned}$$

as well as

$$P\left(\min_{i=1,2}\eta_{t_i}\le y\right)=\exp\left(yE\left(\min_{i=1,2}|\eta_{t_i}|\right)\right)$$

by (2.23), i.e., we have

$$\exp\left(yE\left(\min_{i=1,2}|\eta_{t_i}|\right)\right)=2\exp(y)-\exp\left(yE\left(\max_{i=1,2}|\eta_{t_i}|\right)\right), \qquad y\le 0,$$

or, by (2.22),

$$\exp(ay)=2\exp(y)-\exp\left(\frac{y}{a}\right), \qquad y\le 0,$$

with $a=E\left(\min_{i=1,2}|\eta_{t_i}|\right)$. Differentiating this equation on both sides and setting $y=0$ yields

$$a=2-\frac{1}{a}$$

which implies $a=1$. We, thus, have established equation (2.24) for $m=2$. Suppose now that equation (2.24) is true for $m\in\mathbb{N}$, i.e., $E\left(\max_{1\le i\le m}|\eta_{t_i}|\right)=1$ by (2.22) as well. We have by the inclusion-exclusion formula

$$P\left(\min_{1\le i\le m+1}\eta_{t_i}\le y\right)$$

$$
\begin{aligned}
&= \sum_{\emptyset \neq T \subset \{1,\dots,m+1\}} (-1)^{|T|-1} P\left(\eta_{t_i} \le y,\, i \in T\right) \\
&= \sum_{\emptyset \neq T \subset \{1,\dots,m+1\}} (-1)^{|T|-1} \exp\left(yE\left(\max_{i\in T} |\eta_{t_i}|\right)\right) \\
&= \sum_{k=1}^{m} (-1)^{k-1} \binom{m+1}{k} \exp(y) + (-1)^m \exp\left(yE\left(\max_{1\le i\le m+1} |\eta_{t_i}|\right)\right) \\
&= \exp(y)\left(1-(-1)^m\right) + (-1)^m \exp\left(yE\left(\max_{1\le i\le m+1} |\eta_{t_i}|\right)\right), \qquad y \le 0,
\end{aligned}
$$

as well as

$$
P\left(\min_{1\le i\le m+1} \eta_{t_i} \le y\right) = \exp\left(yE\left(\min_{1\le i\le m+1} |\eta_{t_i}|\right)\right), \qquad y \le 0,
$$

by (2.23) and, thus,

$$
\exp(ay) = \exp(y)\left(1-(-1)^m\right) + \exp\left(\frac{y}{a}\right), \qquad y \le 0,
$$

with $a = E\left(\min_{1\le i\le m+1} |\eta_{t_i}|\right)$ by (2.22). For m even we immediately obtain $a = 1/a$ and, thus, $a = 1$, whereas for m odd we obtain $\exp(ay) = 2\exp(y) + \exp(y/a)$, which also implies $a = 1$ as before. We, thus, have established equation (2.24).

Let $\{t_1, t_2, \dots\}$ be an enumeration of the set of rationals in $[0,1]$. The continuity of $(\eta_t)_{t\in[0,1]}$ together with equation (2.22), (2.24) and the monotone convergence theorem implies

$$
E\left(\sup_{t\in[0,1]} |\eta_t|\right) = E\left(\sup_{t\in\mathbb{Q}\cap[0,1]} |\eta_t|\right) = \lim_{m\in\mathbb{N}} E\left(\max_{1\le i\le m} |\eta_{t_i}|\right) = 1.
$$

We, thus, have shown that the generator constant corresponding to the standard MSP $\boldsymbol{\eta}$ is one, which by Lemma 2.12 implies complete dependence of the margins of $\boldsymbol{\eta}$. □

Let $\boldsymbol{\eta}^{(0)}$ be a standard MSP with generator $\boldsymbol{Z}^{(0)}$ satisfying $P(\inf_{t\in[0,1]} Z_t^{(0)} > 0) = 1$ and generator constant $m^{(0)} = E(\sup_{t\in[0,1]} Z_t^{(0)}) \ge 1$. By Proposition 2.32 the process $\boldsymbol{Z}^{(1)} := |\boldsymbol{\eta}^{(0)}|$ is a generator process of some standard MSP $\boldsymbol{\eta}^{(1)}$ with generator constant $m^{(1)} = E(\sup_{t\in[0,1]} Z_t^{(1)})$. Furthermore, by Lemma 2.2, $P(\sup_{t\in[0,1]} \eta_t < 0) = P(\inf_{t\in[0,1]} |\eta_t| > 0) = 1$ holds for every standard MSP $\boldsymbol{\eta}$, so $\boldsymbol{Z}^{(2)} := |\boldsymbol{\eta}^{(1)}|$ is, in turn, a generator of some standard MSP $\boldsymbol{\eta}^{(2)}$ with generator constant $m^{(2)}$ and so on. Does this iteration have some limit?

Lemma 2.34. *Let $(\boldsymbol{\eta}^{(i)}, \boldsymbol{Z}^{(i)}, m^{(i)})_{i\ge 0}$ be a sequence of standard MSPs with corresponding generator processes and constants, respectively. Consider the iteration $\boldsymbol{Z}^{(i)} := |\boldsymbol{\eta}^{(i-1)}|, i \in \mathbb{N}$, starting with some $\boldsymbol{Z}^{(0)}$ satisfying $P(\inf_{t\in[0,1]} Z_t^{(0)} > 0) = 1$. Then the sequence of generator constants $(m^{(i)})_{i\ge 0}$ fulfills*

$$
m^{(i)} \ge m^{(i+2)}, \qquad i = 0, 1, \dots
$$

Proof. First observe that we have for $i \ge 1$

$$
E\left(\inf_{t\in[0,1]} Z_t^{(i)}\right) \;=\; E\left(\inf_{t\in[0,1]} |\eta_t^{(i-1)}|\right)
$$

$$
\begin{aligned}
&= \int_0^\infty P\left(\inf_{t\in[0,1]} |\eta_t^{(i-1)}| > x\right) dx \\
&= \int_0^\infty P\left(\sup_{t\in[0,1]} \eta_t^{(i-1)} < -x\right) dx \\
&= \int_0^\infty \exp\left(-xE\left(\sup_{t\in[0,1]} Z_t^{(i-1)}\right)\right) dx \\
&= \left(m^{(i-1)}\right)^{-1}. \qquad (2.25)
\end{aligned}
$$

Furthermore, by Lemma 2.20, there is for $i \geq 1$

$$
\begin{aligned}
m^{(i)} = E\left(\sup_{t\in[0,1]} Z_t^{(i)}\right) &= E\left(\sup_{t\in[0,1]} |\eta_t|^{(i-1)}\right) \\
&= \int_0^\infty P\left(\sup_{t\in[0,1]} |\eta_t^{(i-1)}| > x\right) dx \\
&= \int_0^\infty P\left(\inf_{t\in[0,1]} \eta_t^{(i-1)} < -x\right) dx \\
&\leq \int_0^\infty \exp\left(-xE\left(\inf_{t\in[0,1]} Z_t^{(i-1)}\right)\right) dx \\
&= \left(E\left(\inf_{t\in[0,1]} Z_t^{(i-1)}\right)\right)^{-1}. \qquad (2.26)
\end{aligned}
$$

Bringing these results together we get

$$
m^{(0)} \geq m^{(2)} \geq m^{(4)} \geq m^{(6)} \geq \cdots,
$$

and

$$
\left(E\left(\inf_{t\in[0,1]} Z_t^{(0)}\right)\right)^{-1} \geq m^{(1)} \geq m^{(3)} \geq m^{(5)} \geq \cdots.
$$

□

Because $m \geq 1$ holds for every generator constant m, we know by the preceding Lemma, that the sequences $m^{(2n)}$ and $m^{(2n+1)}$ converge to a limit $m, m' \geq 1$, respectively. Define in the following

$$
\begin{aligned}
m &:= \lim_{n\to\infty} m^{(2n)} = \lim_{n\to\infty} E\left(\sup_{t\in[0,1]} Z_t^{(2n)}\right); \\
m' &:= \lim_{n\to\infty} m^{(2n+1)} = \lim_{n\to\infty} E\left(\sup_{t\in[0,1]} Z_t^{(2n+1)}\right),
\end{aligned}
$$

which implies by (2.25)

$$
\lim_{n\to\infty} E\left(\inf_{t\in[0,1]} Z_t^{(2n+1)}\right) = 1/m; \quad \lim_{n\to\infty} E\left(\inf_{t\in[0,1]} Z_t^{(2n)}\right) = 1/m'.
$$

We, thus, have for $y < 0$

$$P\left(\inf_{t\in[0,1]} \eta_t^{(2n)} < y\right) \geq P\left(\boldsymbol{\eta}^{(2n)} < y\right) \to_{n\to\infty} \exp(ym)$$

by Lemma 2.8 as well as by part (i) of Lemma 2.20

$$\begin{aligned} P\left(\inf_{t\in[0,1]} \eta_t^{(2n)} < y\right) &= 1 - P\left(\boldsymbol{\eta}^{(2n)} \geq y\right) \\ &\leq 1 - P\left(\boldsymbol{\eta}^{(2n)} > y\right) \\ &\leq 1 - \left(1 - \exp\left(yE\left(\inf_{t\in[0,1]} Z_t^{(2n)}\right)\right)\right) \\ &\to_{n\to\infty} \exp\left(y\frac{1}{m'}\right). \end{aligned}$$

But this implies $\exp(ym) \leq \exp(y/m')$ and, thus, $m \geq 1/m'$. As both $m, m' \geq 1$, we obtain $m = 1 = m'$. We, consequently, have established that the D-norms pertaining to the sequence $\boldsymbol{\eta}^{(n)}$, $n \in \mathbb{N}$, converge to one as n increases. The next result is, thus, a consequence of Lemma 2.11.

Proposition 2.35. *Let $(\boldsymbol{\eta}^{(n)}, \boldsymbol{Z}^{(n)}, m^{(n)})_{n\geq 0}$ be the sequence of standard MSP with corresponding generator processes and constants, respectively, with $\boldsymbol{Z}^{(n)} := |\boldsymbol{\eta}^{(n-1)}|$, $n \in \mathbb{N}$, starting with some $\boldsymbol{Z}^{(0)}$ satisfying $P\left(\inf_{t\in[0,1]} Z_t^{(0)} > 0\right) = 1$. Then*

$$\lim_{n\to\infty} P\left(\boldsymbol{\eta}^{(n)} \leq f\right) = P(\boldsymbol{\eta} \leq f), \qquad f \in \bar{E}^-[0,1],$$

where $\boldsymbol{\eta}$ is a standard max-stable process in $C[0,1]$ with completely dependent margins.

3 Functional Domain of Attraction

It is well-known that the class of MSP coincides with the class of possible limit processes of the normalized maxima of i.i.d. processes $\boldsymbol{Y}_i$ in $C[0,1]$, $i = 1, \dots, n$, $n \in \mathbb{N}$, cf. de Haan and Ferreira [9, Chapter 9], where the limit is taken with respect to weak convergence in $C[0,1]$. As weak convergence on function space is in general quite difficult to handle, we introduce in this section a type of convergence, which is in great accordance to the finite dimensional case. We give some characterizations, in particular we will see that our approach is more general than the usual one based on weak convergence.

First, we restrict our considerations on convergence (in some sense) towards standard MSP. In Section 3.4 we turn our focus on the special case of copula processes, which are continuous processes in $C[0,1]$, whose marginal one-dimensional distributions are all uniform distributions on $(0,1)$. If the normed maxima of i.i.d. copies of such a copula process converges to a max-stable process, then the limit process is necessarily a standard MSP (which will follow immediately from the fact, that the uniform distribution is in the domain of attraction of the standard negative exponential distribution in the usual univariate sense). Finally, we use the obtained results to examine the case of MSP with arbitrary marginal distributions as limit processes in Section 3.5.

3.1 Functional Domain of Attraction of a Standard MSP

We say that a stochastic process $\boldsymbol{Y}$ in $C[0,1]$ is in the functional domain of attraction of a standard MSP $\boldsymbol{\eta}$, denoted by $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$, if there are functions $a_n \in C^+[0,1] := \{f \in C[0,1] : f > 0\}$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} P\left(\frac{\boldsymbol{Y} - b_n}{a_n} \leq f\right)^n = P(\boldsymbol{\eta} \leq f) = \exp\left(-\|f\|_D\right) \tag{3.1}$$

for any $f \in \bar{E}^-[0,1]$. Note that this condition is equivalent with

$$\lim_{n\to\infty} P\left(\max_{1\leq i\leq n} \frac{\boldsymbol{Y}^{(i)} - b_n}{a_n} \leq f\right) = P(\boldsymbol{\eta} \leq f) \tag{3.1'}$$

for any $f \in \bar{E}^-[0,1]$, where $\boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(2)}, \dots$ are independent copies of $\boldsymbol{Y}$.

In the following, this domain of attraction condition will be studied; we state several properties and implications. We also compare condition 3.1 with other modes of convergence in function space.

Due to the continuity of the functional df of $\boldsymbol{\eta}$, we get immediately the following assertion.

Lemma 3.1. *We have $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$ for some standard MSP $\boldsymbol{\eta}$, i.e.* (3.1') *holds, if, and only if*

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n}(\boldsymbol{Y}^{(i)} - b_n)/a_n < f\right) = P\left(\boldsymbol{\eta} < f\right) \tag{3.2}$$

for every $f \in \bar{E}^-[0,1]$, with $\boldsymbol{Y}^{(i)}$, a_n, b_n as before.

Proof. Set $\boldsymbol{X}^{(n)} := \max_{1\le i\le n}(\boldsymbol{Y}^{(i)} - b_n)/a_n$. If (3.1') holds, we get the inequality

$$\limsup_{n\to\infty} P(\boldsymbol{X}^{(n)} < f) \le \lim_{n\to\infty} P(\boldsymbol{X}^{(n)} \le f) = P(\boldsymbol{\eta} \le f) = P\left(\boldsymbol{\eta} < f\right)$$

for every $f \in \bar{E}^-[0,1]$, see Lemma 2.8.

On the other hand, for all $f \in \bar{E}^-[0,1]$ and every $\varepsilon > 0$:

$$P(\boldsymbol{\eta} \le f - \varepsilon) = \lim_{n\to\infty} P(\boldsymbol{X}^{(n)} \le f - \varepsilon) \le \liminf_{n\to\infty} P(\boldsymbol{X}^{(n)} < f).$$

As $G(f) = P(\boldsymbol{\eta} \le f)$ is continuous in f with respect to the sup $-$norm, cf. Lemma 2.8, (3.2) follows. The reverse implication follows with analogous arguments. □

Again by the help of Lemma 2.8 we deduce the following corollary.

Corollary 3.2. *Let $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$ for some standard MSP $\boldsymbol{\eta}$, and $\boldsymbol{X}^{(n)} := \max_{1\le i\le n}(\boldsymbol{Y}^{(i)} - b_n)/a_n$ as in Lemma 3.1. Then, for $f \in \bar{E}^-[0,1]$:*

$$\begin{aligned}
&\lim_{n\to\infty} P\left(X_t^{(n)} = f(t) \text{ for some } t \in [0,1] \,\middle|\, X_t^{(n)} \le f(t) \text{ for all } t \in [0,1]\right)\\
&= \lim_{n\to\infty} \left[P\left(\boldsymbol{X}^{(n)} \le f\right) - P\left(\boldsymbol{X}^{(n)} < f\right)\right] / P\left(\boldsymbol{X}^{(n)} \le f\right)\\
&= \left[P\left(\boldsymbol{\eta} \le f\right) - P\left(\boldsymbol{\eta} < f\right)\right] / P\left(\boldsymbol{\eta} \le f\right)\\
&= 0
\end{aligned}$$

In particular, we have

$$\lim_{n\to\infty} P\left(X_t^{(n)} = f(t) \text{ for all } t \in [0,1]\right) = 0.$$

There should be no risk of confusion with the notation of domain of attraction in the sense of weak convergence of stochastic processes as investigated in de Haan and Lin [10]. But to distinguish between these two approaches we will consistently speak of f u n c t i o n a l domain of attraction in this paper, if the above definition is meant. Actually, functional domain of attraction is less restrictive as the next lemma shows.

Proposition 3.3. *Suppose that $\boldsymbol{Y}$ in $\bar{C}^-[0,1]$ and let $\boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(2)}, \ldots$ be independent copies of $\boldsymbol{Y}$. If the sequence $\boldsymbol{X}^{(n)} = \max_{1\le i\le n}\left((\boldsymbol{Y}^{(i)} - b_n)/a_n\right)$ of continuous processes converges weakly in $\bar{C}^-[0,1]$, equipped with the sup-norm $\|\cdot\|_\infty$, to the standard MSP $\boldsymbol{\eta}$, then $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$.*

Proof. The Portmanteau Theorem (see, e.g., Billingsley [7]) characterizes weak convergence in particular in terms of convergence of the mass of all continuity sets of the limit measure, which reads in our case

$$\lim_{n\to\infty} P\left(\{\boldsymbol{X}^{(n)} \in C\}\right) \;=\; P\left(\{\boldsymbol{\eta} \in C\}\right)$$

for all Borel-measurable sets C in $(\bar{C}^-[0,1], \|\cdot\|_\infty)$ with $P\left(\{\boldsymbol{\eta} \in \partial C\}\right) = 0$, where ∂C denotes the boundary of the set C. But as we have seen in Remark 6, the sets $\{g \in \bar{C}^-[0,1] : g(t) \leq f(t) \text{ for all } t \in [0,1]\}$ for $f \in \bar{E}^-[0,1]$ are continuity sets of the distribution of $\boldsymbol{\eta}$, and, thus, the assertion follows. □

Examples of continuous processes in $\bar{C}^-[0,1]$, whose properly normed maxima of iid copies converge weakly to an MSP and which obviously satisfy condition (3.1), are the GPD-processes introduced by Buishand et al. [8]. We consider these generalized Pareto processes in Section 3.6 below.

The following example shows, that convergence of a sequence of functional df of some continuous processes $\boldsymbol{\eta}^{(n)}$ to the functional df of some standard MSP $\boldsymbol{\eta}$ does in general not imply weak convergence in $C[0,1]$.

Example 3.4. Let $\boldsymbol{\eta}$ be a standard MSP with generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ satisfying $1 > E\left(\min_{t\in[0,1]} Z_t\right) > 0$; note that the first inequality of the foregoing condition ensures that $\boldsymbol{\eta}$ is not the complete dependence MSP.

Let U be a uniformly on $(0,1)$ distributed rv, which is independent of $\boldsymbol{\eta}$. Define for $u \in [0,1]$ the triangle shaped continuous function $\Delta_n^u : [0,1] \to [0,1]$ by

$$\Delta_n^u(t) := \begin{cases} 1, & \text{if } t = u, \\ 0, & \text{if } t \notin [u - 2^{-n}, u + 2^{-n}], \\ \text{linearly interpolated elsewhere.} & \end{cases}$$

Set

$$\boldsymbol{\eta}^{(n)} := \boldsymbol{\eta} - \Delta_n^U, \qquad n \in \mathbb{N}.$$

Note that $\boldsymbol{\eta}^{(n)} \leq \boldsymbol{\eta}$. We get on the one hand

$$\begin{aligned} P(\boldsymbol{\eta} \leq f) &\leq P(\boldsymbol{\eta}^{(n)} \leq f) \\ &= P\left(\boldsymbol{\eta} \leq f + \Delta_n^U\right) \\ &= \int_0^1 P\left(\eta_t \leq f(t) + \Delta_n^u(t) \text{ for all } t \in [0,1]\right) du \\ &\leq \int_0^1 P(\eta_t \leq f(t), \text{ for all } t \notin [u - 2^{-n}, u + 2^{-n}])\, du \\ &= \int_0^1 \exp\left(-E\left(\sup_{t\notin[u-2^{-n},u+2^{-n}]} (|f(t)|\, Z_t)\right)\right) du \\ &\to_{n\to\infty} \int_0^1 \exp\left(-E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right)\right) du \end{aligned}$$

$$= P(\boldsymbol{\eta} \leq f)$$

by the continuity of $\boldsymbol{Z}$, the continuity up to finitely many points of f and the dominated convergence theorem.

On the other hand, $\boldsymbol{\eta}^{(n)}$ does not converge weakly to $\boldsymbol{\eta}$ in $C[0,1]$: If that would be the case, by the Portmanteau theorem,

$$\liminf_{n\to\infty} P(\boldsymbol{\eta}^{(n)} \in \mathcal{O}) \geq P(\boldsymbol{\eta} \in \mathcal{O})$$

should hold for every open subset $\mathcal{O}$ of $C[0,1]$ (with respect to the maximum distance $\|f-g\|_\infty = \max_{t\in[0,1]} |f(t)-g(t)|$).

Choose a constant c with $-\left(1 - E\left(\min_{t\in[0,1]} Z_t\right)\right)^{-1} < c < -1$. The set $\{g \in C[0,1] : g > c\}$ is an open subset of $C[0,1]$ and, hence, we should have

$$\liminf_{n\to\infty} P(\boldsymbol{\eta}^{(n)} > c) \geq P(\boldsymbol{\eta} > c). \tag{3.3}$$

We know from Lemma 2.20 that

$$P(\boldsymbol{\eta} > c) \geq 1 - \exp\left(cE\left(\min_{t\in[0,1]} Z_t\right)\right),$$

and we get

$$\begin{aligned} P(\boldsymbol{\eta}^{(n)} > c) &= P(\boldsymbol{\eta} - \Delta_n^U > c) \\ &= \int_0^1 P(\boldsymbol{\eta} > c + \Delta_n^u)\, du \\ &\leq \int_0^1 P(\eta_u > c+1)\, du \\ &= \int_0^1 1 - \exp(c+1)\, du \\ &= 1 - \exp(c+1) \\ &< 1 - \exp\left(cE\left(\min_{t\in[0,1]} Z_t\right)\right) \\ &\leq P(\boldsymbol{\eta} > c), \end{aligned}$$

as we have chosen c properly. But this contradicts equation (3.3).

By now, we have shown that functional domain of attraction is less restrictive than the domain of attraction in the sense of weak convergence. In turn, functional domain of attraction obviously implies convergence of the fidis, and, moreover, hypoconvergence of the normed maximum-process to the standard MSP in the sense of Molchanov [22] is implied. For the latter see the Appendix on “Random Closed Sets and Hypoconvergence of Continuous Processes”, p. 79, where we give a summary of – for this work at hand – crucial parts of random set theory, including a rewritten characterization of hypoconvergence. It will turn out, that this type of convergence is indeed strictly weaker than the functional domain of attraction convergence.

3.2 The Sojourn Time Transformation

In Chapter 4 below we study the random sojourn time which a continuous process spends above some deterministic function. The setup enables in particular considerations of a type of convergence for random elements in $\bar{C}^-[0,1]$: we introduce the "convergence with respect to the sojourn time transformation", $\to_{STR}$ for short, and examine the relations to other types of convergence, in particular to the functional df convergence as defined in (3.1). To simplify notation we write in the sequel for example $\mathbf{1}(X_t > f(t))$ for $\mathbf{1}_{\{x:x>f(t)\}}(X_t)$.

Let $\boldsymbol{X}$ be a stochastic process in $\bar{C}^-[0,1]$ and put for $f \in \bar{E}^-[0,1]$

$$S_{\boldsymbol{X}}(f) := \int_0^1 \mathbf{1}(X_t > f(t))\, dt,$$

which is the sojourn time of $\boldsymbol{X}$ above the function f; moreover, set

$$\bar{S}_{\boldsymbol{X}}(f) := 1 - S_{\boldsymbol{X}}(f) = \int_0^1 \mathbf{1}(X_t \le f(t))\, dt.$$

Lemma 3.5. *The* sojourn time transformation, *which maps the distribution of a stochastic process $\boldsymbol{X}$ with sample paths in $\bar{C}^-[0,1]$ onto the family of (one dimensional) sojourn time distributions*

$$P * \boldsymbol{X} \mapsto \left\{ P * S_{\boldsymbol{X}}(f) :\ f \in \bar{E}^-[0,1] \right\}$$

is one-to-one.

Proof. This is trivial from the fact that the distribution of $\boldsymbol{X}$ is determined by its fidis together with the equation

$$P\left(S_{\boldsymbol{X}}(f) = 0\right) = P\left(\boldsymbol{X} \le f\right), \qquad f \in \bar{E}^-[0,1].$$

□

We say that a sequence of stochastic processes $\boldsymbol{X}^{(n)}$ in $\bar{C}^-[0,1]$, $n \in \mathbb{N}$, converges with respect to the sojourn time transformation to $\boldsymbol{X}$ in $\bar{C}^-[0,1]$, denoted by $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{X}$, if

$$S_{\boldsymbol{X}^{(n)}}(f) \to_D S_{\boldsymbol{X}}(f), \qquad f \in \bar{E}^-[0,1]. \tag{3.4}$$

Note that (3.4) is equivalent with

$$\bar{S}_{\boldsymbol{X}^{(n)}}(f) \to_D \bar{S}_{\boldsymbol{X}}(f), \qquad f \in \bar{E}^-[0,1], \tag{3.5}$$

and recall that $\to_D$ denotes usual convergence in distribution of a sequence of rv.

Proposition 3.6. *Let $\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, and $\boldsymbol{X}$ be stochastic processes in $\bar{C}^-[0,1]$ and suppose that $\boldsymbol{X}$ has continuous marginal df. If the fidis of the sequence $\boldsymbol{X}^{(n)}$ converge weakly to those of $\boldsymbol{X}$, then $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{X}$.*

Proof. We prove the assertion by verifying (3.5), which will be shown via convergence of characteristic functions. We have for $s \in \mathbb{R}$ and $f \in \bar{E}^-[0,1]$

$$E\left(\exp\left(is\bar{S}_{\boldsymbol{X}^{(n)}}(f)\right)\right) = E\left(\sum_{k=0}^{\infty} \frac{(is)^k}{k!} \bar{S}^k_{\boldsymbol{X}^{(n)}}(f)\right) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!} E\left(\bar{S}^k_{\boldsymbol{X}^{(n)}}(f)\right),$$

where we can interchange the sum and the expectation by the dominated convergence theorem, as $P(S_{\boldsymbol{X}^{(n)}}(f) \in [0,1]) = 1$.

From Fubini's theorem we obtain for $k \geq 1$

$$\begin{aligned}
E\left(\bar{S}^k_{\boldsymbol{X}^{(n)}}(f)\right) &= E\left(\int_0^1 \dots \int_0^1 \prod_{i=1}^k \mathbf{1}\left(X^{(n)}_{t_i} \leq f(t_i)\right) dt_1 \dots dt_k\right) \\
&= \int_0^1 \dots \int_0^1 E\left(\prod_{i=1}^k \mathbf{1}\left(X^{(n)}_{t_i} \leq f(t_i)\right)\right) dt_1 \dots dt_k \\
&= \int_0^1 \dots \int_0^1 P\left(X^{(n)}_{t_1} \leq f(t_1), \dots, X^{(n)}_{t_k} \leq f(t_k)\right) dt_1 \dots dt_k \\
&\to_{n\to\infty} \int_0^1 \dots \int_0^1 P\left(X_{t_1} \leq f(t_1), \dots, X_{t_k} \leq f(t_k)\right) dt_1 \dots dt_k \\
&= E\left(\bar{S}^k_{\boldsymbol{X}}(f)\right)
\end{aligned}$$

by the dominated convergence theorem. But this implies

$$E\left(\exp\left(is\bar{S}_{\boldsymbol{X}^{(n)}}(f)\right)\right) \to_{n\to\infty} E\left(\exp\left(is\bar{S}_{\boldsymbol{X}}(f)\right)\right).$$

□

The reverse implication in Proposition 3.6 does not hold, i.e., STR-convergence does not imply weak convergence of the fidis. This is shown in the following example.

Example 3.7. Take independent rv U and η, where U is uniformly on $(-1,0)$ distributed and η follows a standard negative exponential distribution. Put for $n \in \mathbb{N}$

$$X^{(n)}_t := \begin{cases} U & \text{if } t = 0, \\ \eta & \text{if } t \in [1/n, 1], \\ (1-nt)U + nt\eta & \text{if } t \in [0, 1/n], \end{cases}$$

and set

$$\eta_t = \eta, \qquad t \in [0,1].$$

Then $\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]}$ are processes in $\bar{C}^-[0,1]$, and we have $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{\eta}$, as with $f \in \bar{E}^-[0,1]$

$$\begin{aligned}
&\left|\int_0^1 \mathbf{1}(X^{(n)}_t > f(t))\, dt - \int_0^1 \mathbf{1}(\eta_t > f(t))\, dt\right| \\
&= \left|\int_0^{1/n} \mathbf{1}(X^{(n)}_t > f(t))\, dt - \int_0^{1/n} \mathbf{1}(\eta_t > f(t))\, dt\right|
\end{aligned}$$

$$\leq \frac{1}{n} \to_{n\to\infty} 0,$$

but $U = X_0^{(n)} \not\to_D \eta_0 = \eta$.

Lemma 3.8. *Let $\boldsymbol{X}^{(n)}$ in $\bar{C}^{-}[0,1]$, $n \in \mathbb{N}$, satisfy*

$$P\left(\boldsymbol{X}^{(n)} \leq f\right) \to_{n\to\infty} P\left(\boldsymbol{X} \leq f\right), \qquad f \in \bar{E}^{-}[0,1],$$

where $\boldsymbol{X} \in \bar{C}^{-}[0,1]$ has continuous marginal df. Then $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{X}$.

Proof. The assumption implies that the fidis of $\boldsymbol{X}^{(n)}$ converge weakly to those of $\boldsymbol{X}$ and, thus, Proposition 3.6 applies. □

Note that zero is typically not a continuity point of the df of $S_{\boldsymbol{X}}(f)$, i.e., we typically have

$$0 < P\left(S_{\boldsymbol{X}}(f) = 0\right) = P\left(\boldsymbol{X} \leq f\right).$$

We, therefore, sharpen the definition of $\boldsymbol{X}^{(n)} \to_{STR} \boldsymbol{X}$ by requiring in addition convergence at zero, i.e.,

$$P\left(S_{\boldsymbol{X}^{(n)}}(f) = 0\right) \to_{n\to\infty} P\left(S_{\boldsymbol{X}}(f) = 0\right).$$

We denote this more restrictive definition by $\boldsymbol{X}^{(n)} \to_{STR^*} \boldsymbol{X}$. The (double star-convergence) STR^*-convergence characterizes convergence of the functional df, and, thus, functional domain of attraction.

Lemma 3.9. *Let $\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, and $\boldsymbol{X}$ be stochastic processes in $\bar{C}^{-}[0,1]$ and suppose that $\boldsymbol{X}$ has continuous marginal df. Then*

$$P\left(\boldsymbol{X}^{(n)} \leq f\right) \to_{n\to\infty} P\left(\boldsymbol{X} \leq f\right), \quad f \in \bar{E}^{-}[0,1] \iff \boldsymbol{X}^{(n)} \to_{STR^*} \boldsymbol{X}.$$

As an example recall the iteration problem of Section 2.5.

Example 3.10. Let $\boldsymbol{\eta}^{(0)}$ be a standard MSP with generator process $\boldsymbol{Z}^{(0)}$ satisfying $P\left(\inf_{t\in[0,1]} Z_t^{(n)} > 0\right) = 1$, and consider the sequence $\boldsymbol{\eta}^{(n)}$, $n \geq 1$, of standard MSP defined by iteration, i.e., $\boldsymbol{Z}^{(n)} := |\boldsymbol{\eta}^{(n-1)}|$, $\boldsymbol{\eta}^{(n)}$ a standard MSP with generator $\boldsymbol{Z}^{(n)}$, $n \in \mathbb{N}$. Let $\boldsymbol{\eta}$ be a standard MSP with completely dependent margins, i.e., $\eta_t = \eta_0$, $t \in [0,1]$.

Bringing together the foregoing Lemma and Proposition 2.35 we get

$$\boldsymbol{\eta}^{(n)} \to_{STR^*} \boldsymbol{\eta}.$$

To summarize the foregoing considerations, the following picture gives an overview of the implications of the different types of convergence towards standard MSP ("FDA" stands for the "Functional Domain of Attraction"-convergence as defined in (3.1)):

$$
\begin{array}{ccccc}
 & & \text{hypo} & & \\
 & & \Uparrow & & \\
\text{weak} & \Rightarrow & \text{FDA} & \Leftrightarrow & STR^* \\
 & & \Downarrow & & \\
 & & \text{fidis} & & \\
 & & \Downarrow & & \\
 & & STR & &
\end{array}
$$

3.3 A Sufficient Condition for Weak Convergence in C[0,1]

We have seen, that weak convergence of a sequence of continuous processes towards some standard MSP is not implied by any of the types of convergence considered so far. However, it is possible to give a sufficient condition for weak convergence in terms of an extension of the sojourn time transformation.

First recall a useful condition for weak convergence on a separable metric space $(S, \mathcal{S})$, where $\mathcal{S}$ is the Borel $\sigma-$field on S, i.e. $\mathcal{S}$ is generated by the open balls $B(x, \varepsilon)$ with respect to the metric, cf. Billingsley [7, Theorem 2.3]: let $\mathcal{A}$ be a subclass of $\mathcal{S}$ which is closed under the formation of finite intersections and for every $x \in S$ and every $\varepsilon > 0$ there is some $A \in \mathcal{A}$, such that $x \in A^\circ \subset A \subset B(x, \varepsilon)$ (A° denotes the interior of A). Then $\mathcal{A}$ is a "convergence determining class", that is, $P_n(A) \to P(A)$ for $n \to \infty$ implies weak convergence of this sequence of measures $(P_n)_{n\in\mathbb{N}}$ towards the measure P.

Furthermore, recall that the space $(C[0,1], \|\cdot\|_\infty)$ is separable (and complete), cf. Billingsley [7, p. 11], and that the closed ball $\bar{B}(f, \varepsilon)$ with center $f \in C[0,1]$ and radius $\varepsilon > 0$ is given by

$$
\begin{aligned}
\bar{B}(f, \varepsilon) &= \left\{ g \in C[0,1] : \sup_{t\in[0,1]} |f(t) - g(t)| \le \varepsilon \right\} \\
&= \{ g \in C[0,1] : f(t) - \varepsilon \le g(t) \le f(t) + \varepsilon \text{ for all } t \in [0,1] \} .
\end{aligned}
$$

Consider the class $\mathcal{A}$ of finite intersections of such closed balls $\bar{B}(f, \varepsilon)$. Then $\mathcal{A}$ obviously fulfills the requirements on a convergence determining class listed above and an arbitrary element $A \in \mathcal{A}$ is given by

$$
\begin{aligned}
A &= \bigcap_{1\le i\le n} \bar{B}(f_i, \varepsilon_i) \\
&= \left\{ g \in C[0,1] : \max_{1\le i\le n} (f_i - \varepsilon_i)(t) \le g(t) \le \min_{1\le i\le n} (f_i + \varepsilon_i)(t) \text{ for all } t \in [0,1] \right\},
\end{aligned}
$$

for some $n \in \mathbb{N}$ and appropriate $f_i \in C[0,1],\ \varepsilon_i > 0,\ i = 1, \ldots, n$.

As both, $\max_{1\le i\le n}(f_i - \varepsilon_i)$ and $\min_{1\le i\le n}(f_i + \varepsilon_i)$ are continuous functions, we get due to the foregoing arguments:

$$
\lim_{n\to\infty} P\left(f_1 \le \boldsymbol{X}^{(n)} \le f_2\right) = P\left(f_1 \le \boldsymbol{X} \le f_2\right), \text{ for all } f_1 \le f_2 \in \bar{C}^-[0,1] \implies \boldsymbol{X}^{(n)} \to_D \boldsymbol{X}, \tag{3.6}
$$

with $\boldsymbol{X}_n, \boldsymbol{X}$ being processes with sample paths in C[0,1], $n \in \mathbb{N}$; note that in this case "$\to_D$" stands for weak convergence in $(C[0,1], \|\cdot\|_\infty)$.

Keeping this in mind, let $\boldsymbol{X}$ be a stochastic process in $\bar{C}^-[0,1]$ and $f_1, f_2 \in \bar{E}^-[0,1]$ with $f_1 \le f_2$. Then

$$S_{\boldsymbol{X}}(f_1, f_2) := \int_0^1 \mathbf{1}(f_1(t) \le X_t \le f_2(t))\, dt$$

is the sojourn time which the process $\boldsymbol{X}$ spends between f_1 and f_2. With $f_2 = 0$, $S_f(\boldsymbol{X}) = S_{f,0}(\boldsymbol{X})$ is the sojourn time of $\boldsymbol{X}$ above the function $f \in \bar{E}^-[0,1]$.

Due to (3.6) we immediately get the following assertion.

Proposition 3.11. *Let $\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, and $\boldsymbol{X}$ be stochastic processes in $\bar{C}^-[0,1]$. Then*

$$P\left(S_{\boldsymbol{X}^{(n)}}(f_1, f_2) = 1\right) \to_{n\to\infty} P\left(S_{\boldsymbol{X}}(f_1, f_2) = 1\right) \text{ for all } f_1 \le f_2 \in \bar{C}^-[0,1] \implies \boldsymbol{X}^{(n)} \to_D \boldsymbol{X}.$$

Remark 7. It can be shown with completely analogous arguments as in the proof of Proposition 3.6 that weak convergence of the fidis of a sequence $\boldsymbol{X}^{(n)}$ towards those of $\boldsymbol{X}$ is sufficient for

$$S_{\boldsymbol{X}^{(n)}}(f_1, f_2) \to_D S_{\boldsymbol{X}}(f_1, f_2) \text{ for all } f_1 \le f_2 \in \bar{E}^-[0,1],$$

$\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, and $\boldsymbol{X}$ being stochastic processes in $\bar{C}^-[0,1]$ and $\boldsymbol{X}$ has continuous marginal df. Note that one is typically not a continuity point of the df of $S_{\boldsymbol{X}}(f_1, f_2)$, as we have

$$P\left(S_{\boldsymbol{X}}(f_1, f_2) = 1\right) = P\left(f_1 \le \boldsymbol{X} \le f_2\right).$$

Remark 8. Suppose a sequence of functional distribution functions of stochastic processes $\boldsymbol{X}^{(n)}$ in $\bar{C}^-[0,1]$ converges to the functional distribution function of some $\boldsymbol{X}$ in $\bar{C}^-[0,1]$ with continuous marginal df, i.e.

$$\lim_{n\to\infty} P\left(\boldsymbol{X}^{(n)} \le f\right) = P(\boldsymbol{X} \le f), \qquad f \in \bar{E}^-[0,1] \tag{3.7}$$

(in particular, $\boldsymbol{X} = \boldsymbol{\eta}$ with some standard MSP $\boldsymbol{\eta}$ is possible). As (3.7) immediately implies weak convergence of the fidis we get $S_{\boldsymbol{X}^{(n)}}(f_1, f_2) \to_D S_{\boldsymbol{X}}(f_1, f_2)$ for all $f_1 \le f_2 \in \bar{E}^-[0,1]$.

The Portmanteau Theorem (e.g. Billingsley [7, Theorem 2.1]) implies that

$$\limsup_{n\to\infty} P\left(S_{\boldsymbol{X}^{(n)}}(f_1, f_2) = 1\right) \le P\left(S_{\boldsymbol{X}}(f_1, f_2) = 1\right).$$

But we have seen in Example 3.4, that in presence of (3.7) it is possible that

$$\liminf_{n\to\infty} P\left(S_{\boldsymbol{X}^{(n)}}(f_1, f_2) = 1\right) < P\left(S_{\boldsymbol{X}}(f_1, f_2) = 1\right).$$

3.4 Functional Domain of Attraction for Copula Processes

Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ in $C[0,1]$ be a stochastic process with continuous marginal df F_t, $t \in [0,1]$. Set

$$\boldsymbol{U} = (U_t)_{t\in[0,1]} := (F_t(Y_t))_{t\in[0,1]}, \tag{3.8}$$

which is the copula process corresponding to $\boldsymbol{Y}$. Note that each one-dimensional marginal distribution of $\boldsymbol{U}$ is the uniform distribution on $[0,1]$. Moreover, $\boldsymbol{U}$ has continuous sample paths, as can be seen as follows: the continuous sample paths of the process $\boldsymbol{Y}$ imply $F_{t_n}(x) \to F_t(x)$ for $t_n \to t$ and as we assume continuous marginal df F_t for every $t \in [0,1]$, the convergence of the df is in fact uniformly in $x \in \mathbb{R}$. But this implies $U_{t_n} = F_{t_n}(Y_{t_n}) \to F_t(Y_t) = U_t$, i.e. the paths of $\boldsymbol{U} = (F_t(Y_t))_{t\in[0,1]}$ are continuous.

Suppose that the copula process corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of a standard MSP $\boldsymbol{\eta}$, representable as in Proposition 2.4. Then we know from Aulbach et al. [3] that for $d \in \mathbb{N}$ the copula C_d corresponding to the rv $(Y_{i/d})_{i=1}^d$ satisfies the equation

$$C_d(\boldsymbol{y}) = 1 - \|\mathbf{1} - \boldsymbol{y}\|_{D_d} + o\left(\|\mathbf{1} - \boldsymbol{y}\|_\infty\right), \tag{3.9}$$

as $\|\mathbf{1} - \boldsymbol{y}\|_\infty \to \mathbf{0}$, uniformly in $\boldsymbol{y} \in [0,1]^d$, where the D-norm is given by

$$\|\boldsymbol{x}\|_{D_d} = E\left(\max_{1\le i\le d}\left(|x_i|\, Z_{i/d}\right)\right), \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

We are going to establish an analogous result for the functional domain of attraction.

Let $\boldsymbol{\eta}$ be a standard MSP and let $\boldsymbol{Y}$ be an arbitrary stochastic process in $C[0,1]$. By taking logarithms, we obtain the following equivalences with some norming functions $a_n \in C^+[0,1]$, $b_n \in C[0,1]$, $n \in \mathbb{N}$:

$$\begin{aligned}
&\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta}) \text{ in the sense of condition (3.1)}\\
&\iff P\left(\frac{\boldsymbol{Y} - b_n}{a_n} \le f\right)^n = \exp\left(-\|f\|_D\right) + o(1),\ f \in \bar{E}^-[0,1],\ \text{as } n \to \infty,\\
&\iff P\left(\frac{\boldsymbol{Y} - b_n}{a_n} \le f\right) = 1 - \frac{1}{n}\|f\|_D + o\left(\frac{1}{n}\right),\ f \in \bar{E}^-[0,1],\ \text{as } n \to \infty.
\end{aligned}$$

Let $\boldsymbol{U}$ be a copula-process as defined in (3.8) and set $H_f(s) := P(\boldsymbol{U} - 1 \le s\,|f|)$, $s \le 0$, $f \in \bar{E}^-[0,1]$. Note that $H_f(\cdot)$ defines a univariate df on $(-\infty, 0]$. The family $\mathcal{P} := \left\{H_f : f \in \bar{E}^-[0,1]\right\}$ of univariate df is the spectral decomposition of the df $H(f) = P(\boldsymbol{U} - 1 \le f)$, $f \in \bar{E}^-[0,1]$ of $\boldsymbol{U} - 1$. This extends the spectral decomposition of a multivariate df in Falk et al. [17, Section 5.4]. Note that the norming functions $a_n \in C^+[0,1]$, $b_n \in C[0,1]$ are in case of copula processes necessarily given by $a_n \equiv n$, $b_n = 1$ for $n \in \mathbb{N}$, due to the uniformly distributed one-dimensional margins of $\boldsymbol{U}$; so standard arguments yield the next result.

Proposition 3.12. *The following equivalences hold:*

$$\begin{aligned}
&\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta}) \textit{ in the sense of condition (3.1)}\\
&\Leftrightarrow\ P\left(\boldsymbol{U} - 1 \le \frac{f}{n}\right) = 1 - \left\|\frac{f}{n}\right\|_D + o\left(\frac{1}{n}\right),\ f \in \bar{E}^-[0,1], \textit{as } n \to \infty,\\
&\Leftrightarrow\ H_f(s) = 1 + s\,\|f\|_D + o(s),\ f \in \bar{E}^-[0,1], \textit{as } s \uparrow 0,
\end{aligned} \tag{3.10}$$

Remark 9. Characterization (3.10) entails in particular that $H_f(s)$ is differentiable from the left in $s = 0$ with derivative $h_f(0) := \frac{d}{ds}H_f(s)|_{s=0} = \|f\|_D$, $f \in \bar{E}^-[0,1]$.

Remark 10. A sufficient condition for $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ is given by

$$P(\boldsymbol{U} - 1 \leq g) = 1 - \|g\|_D + o(\|g\|_\infty) \tag{3.11}$$

as $\|g\|_\infty \to 0$, uniformly for all $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \leq 1$, i.e., for all g in the unit ball of $\bar{E}^-[0,1]$.

Example 3.13. Take $\boldsymbol{U} = \exp(\boldsymbol{\eta})$. Then $\boldsymbol{U}$ is a copula process, and we obtain uniformly for $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \leq 1 - \varepsilon$ by using the approximation $\log(1+x) = x + O\left(x^2\right)$ as $x \to 0$

$$\begin{aligned}
P(\boldsymbol{U} - 1 \leq g) &= P(\boldsymbol{\eta} \leq \log(1+g)) \\
&= \exp\left(-E\left(\sup_{t\in[0,1]} \left(\left|\log(1+g(t))\right| Z_t\right)\right)\right) \\
&= \exp\left(-E\left(\sup_{t\in[0,1]} \left(\left|g(t) + O\left(g(t)^2\right)\right| Z_t\right)\right)\right) \\
&= \exp\left(-E\left(\sup_{t\in[0,1]} \left(\left|g(t)\right| Z_t\right)\right) + O\left(\|g\|_\infty^2\right)\right) \\
&= 1 - \|g\|_D + O\left(\|g\|_\infty^2\right),
\end{aligned} \tag{3.12}$$

i.e., the copula process $\boldsymbol{U} = \exp(\boldsymbol{\eta})$ satisfies condition (3.11).

We have seen in Lemma 2.2 that a standard MSP $\boldsymbol{\eta}$ satisfies $P(\boldsymbol{\eta} < 0) = 1$. We add two corresponding results for copula processes, which are in the domain of attraction of some standard MSP $\boldsymbol{\eta}$.

Lemma 3.14. *Suppose that the copula process $\boldsymbol{U}$ with sample paths in $C[0,1]$ satisfies $-\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is a standard MSP in $C[0,1]$. Then*

$$P\left(\inf_{t\in[0,1]} U_t > 0\right) = 1.$$

Proof. The continuity from below of a probability measure implies

$$\begin{aligned}
P\left(\inf_{t\in[0,1]} U_t > 0\right) &= P\left(\sup_{t\in[0,1]} (-U_t) < 0\right) \\
&= P\left(\bigcup_{n\in\mathbb{N}} \left\{\sup_{t\in[0,1]} (-U_t) \leq -\frac{1}{n}\right\}\right) \\
&= \lim_{n\to\infty} P\left(\sup_{t\in[0,1]} (-U_t) \leq -\frac{1}{n}\right) \\
&= \lim_{n\to\infty} P(n(-\boldsymbol{U}) \leq -1)
\end{aligned}$$

$$= \lim_{n\to\infty} (P(n(-\boldsymbol{U}) \le -1)^n)^{1/n}$$
$$= 1$$

as $P(n(-\boldsymbol{U}) \le -1)^n \to_{n\to\infty} P(\boldsymbol{\eta} \le -1) = \exp(-\|1\|_D) > 0$. □

Corollary 3.15. *Suppose that the copula process $\boldsymbol{U}$ with sample paths in $C[0,1]$ satisfies $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is a standard MSP in $C[0,1]$. Then*

$$P\left(\sup_{t\in[0,1]} U_t < 1\right) = 1.$$

Proof. Note that $\widetilde{\boldsymbol{U}} := \boldsymbol{U} - 1 \in \mathcal{D}(\boldsymbol{\eta})$ and that

$$P\left(\sup_{t\in[0,1]} U_t < 1\right) = P\left(\inf_{t\in[0,1]} -(\widetilde{U}_t) > 0\right).$$

The assertion is now a consequence of Lemma 3.14. □

We close this section by giving an example of a copula process which fulfills the requirements of the foregoing assertions and which will be usefull in the next section.

Example 3.16. Let $\boldsymbol{\zeta} = (\zeta_t)_{t\in[0,1]}$ an MSP in $C[0,1]$ with arbitrary (max-stable) one-dimensional marginal distributions. That is, for every $t \in [0,1]$, there are $a(t) > 0, b(t), \gamma(t) \in \mathbb{R}$ with

$$G_t(x) := P(\zeta_t \le x) = \exp\left(-\left(1+\gamma(t)\frac{x-b(t)}{a(t)}\right)^{-1/\gamma(t)}\right),$$

compare the considerations at the beginning of Chapter 2. Moreover, we have

$$P(\boldsymbol{\zeta} \le f) = \exp(-\|\Psi(f)\|_D), \quad f \in E[0,1]$$

for some D-norm $\|\cdot\|_D$ and with $\Psi(\cdot)$ defined as on page 19.

Define $\boldsymbol{U} := (U_t)_{t\in[0,1]} := (G_t(\zeta_t))_{t\in[0,1]}$ which is the (continuous) copula process of $\boldsymbol{\zeta}$. Then we get for every $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \le 1$

$$\begin{aligned} P(\boldsymbol{U} - 1 \le g) &= P(\zeta_t \le G_t^{-1}(g(t)+1) \text{ for all } t \in [0,1]) \\ &= \exp(-\left\|\Psi(G_t^{-1}(g(t)+1))\right\|_D) \\ &= \exp(-\|\log(g(t)+1)\|_D). \end{aligned}$$

Thus, the same arguments as in Example 3.13 yield that $\boldsymbol{U}$ fulfills (3.11), so $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ for some standard MSP $\boldsymbol{\eta}$.

Moreover, Lemma 3.14 yields $P(\boldsymbol{U} > 0) = 1$.

3.5 A Characterization of Functional Domain of Attraction via Copula Processes

We conclude from de Haan and Lin [10] that a process $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ with continuous marginal df $F_t,\ t \in [0,1]$ is in the domain of attraction (in the sense of weak convergence of probability measures on $C[0,1]$) of an MSP if, and only if Y_t is in the domain of attraction of a univariate extreme value distribution for each $t \in [0,1]$ together with the condition that the pertaining copula process $\boldsymbol{U} = (U_t)_{t\in[0,1]} = (F_t(Y_t))_{t\in[0,1]}$ converges weakly to a standard MSP $\boldsymbol{\eta}$, that is

$$\left(\max_{1\le i\le n} n(U_t^{(i)} - 1)\right)_{t\in[0,1]} \to_D \boldsymbol{\eta} \tag{3.13}$$

in $C[0,1]$, where $\boldsymbol{U}^{(i)}$, $i \in \mathbb{N}$, are independent copies of $\boldsymbol{U}$.

The question arises, if it possible to characterize functional domain of attraction of an MSP $\boldsymbol{\xi}$ with arbitrary univariate (max-stable) margins in an analogous manner. This is the issue of this section.

Let $\boldsymbol{X} = (X_t)_{t\in[0,1]}$ in $C[0,1]$ be a stochastic process with continuous marginal df $F_t(x) = P(X_t \le x)$, $x \in \mathbb{R}$, $t \in [0,1]$, and let $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$ in $C[0,1]$ be an MSP with marginal df G_t, $t \in [0,1]$. Suppose that there exist norming functions $a_n > 0$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\sup_{t\in[0,1]} \left| n\big(F_t(a_n(t)x + b_n(t)) - 1\big) - \log\left(G_t(x)\right)\right| \to_{n\to\infty} 0 \tag{3.14}$$

for each $x \in \mathbb{R}$ with $G_t(x) > 0$, $t \in [0,1]$. This condition is condition (3.11) in de Haan and Lin [10]. Using Taylor expansion $\log(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$, condition (3.14) implies in particular weak convergence of the univariate margins, i.e.,

$$F_t(a_n(t)x + b_n(t))^n \to_{n\to\infty} G_t(x), \qquad x \in \mathbb{R},\ t \in [0,1].$$

Proposition 3.17. *Put $\boldsymbol{U} := (U_t)_{t\in[0,1]} := (F_t(X_t))_{t\in[0,1]}$, which is the* copula process *corresponding to $\boldsymbol{X}$. Let $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \dots$ and $\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \dots$ be independent copies of $\boldsymbol{U}$ and $\boldsymbol{X}$, respectively.*

Then, in the presence of condition (3.14),

$$P\left(\max_{1\le i\le n} \frac{\boldsymbol{X}^{(i)} - b_n}{a_n} \le f\right) \to_{n\to\infty} P(\boldsymbol{\xi} \le f), \qquad f \in E[0,1], \tag{3.15}$$

if and only if

$$P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)} - 1\right) \le g\right) \to_{n\to\infty} P(\boldsymbol{\eta} \le g), \qquad g \in E^-[0,1], \tag{3.16}$$

where for the implication (3.15) $\Longrightarrow$ (3.16) *we set $\eta_t := \log(G_t(\xi_t))$, $t \in [0,1]$, and for the reverse conclusion $\xi_t := G_t^{-1}(\exp(\eta_t))$, $t \in [0,1]$. In both cases the processes $\boldsymbol{\xi} := (\xi_t)_{t\in[0,1]}$, $\boldsymbol{\eta} := (\eta_t)_{t\in[0,1]}$ are max-stable processes in $C[0,1]$, $\boldsymbol{\eta}$ being an standard MSP.*

Note that $P(G_t(\xi_t) = 0 \text{ for some } t \in [0,1]) = 0$ for an MSP $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$, see Example 3.16, and, thus, $\eta_t := \log(G_t(\xi_t)), t \in [0,1]$ is well-defined.

Proof. As $\boldsymbol{X}$ has continuous sample paths, we have continuity of the function $[0,1] \ni t \mapsto G_t(x)$ for each $x \in \mathbb{R}$ and, thus, continuity of the function $[0,1] \times \mathbb{R} \ni (t,x) \mapsto G_t(x)$ as well as its monotonicity in x for a fixed t. Elementary arguments now imply that condition (3.14) is equivalent with

$$\sup_{x\in[x_1,x_2]} \sup_{t\in[0,1]} \left| n\big(F_t(a_n(t)x + b_n(t)) - 1\big) - \log\left(G_t(x)\right)\right| \to_{n\to\infty} 0 \tag{3.17}$$

for each $x_1 \le x_2 \in \mathbb{R}$ with $G_t(x_1) > 0$, $t \in [0,1]$. As $G_t(x_1)$ is a continuous function in $t \in [0,1]$, this condition on x_1 is equivalent with $\inf_{t\in[0,1]} G_t(x_1) > 0$. Condition (3.17) is, therefore, equivalent with

$$\sup_{t\in[0,1]} \left| n\big(F_t(a_n(t)f(t) + b_n(t)) - 1\big) - \log\left(G_t(f(t))\right)\right| \to_{n\to\infty} 0 \tag{3.18}$$

for each $f \in E[0,1]$ with $\inf_{t\in[0,1]} G_t(f(t)) > 0$.

We first establish the implication (3.15) $\implies$ (3.16). Choose $g \in E^-[0,1]$ with $\sup_{t\in[0,1]} g(t) < 0$, and put $f(t) := G_t^{-1}(\exp(g(t)) \in E[0,1]$. From assumption (3.15) we obtain

$$P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right) \to_{n\to\infty} P(\boldsymbol{\xi} \le f) = P(\boldsymbol{\eta} \le g) = \exp\left(-\left\|g\right\|_D\right), \tag{3.19}$$

where $\|\cdot\|_D$ is the D-norm corresponding to the SMSP $\boldsymbol{\eta}$.

We have, on the other hand, by condition (3.18)

$$\begin{aligned}
&P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right)\\
&= P\left(n \max_{1\le i\le n}\left(U_t^{(i)} - 1\right) \le n\big(F_t(a_n(t)f(t) + b_n(t)) - 1\big),\ t \in [0,1]\right)\\
&= P\left(n \max_{1\le i\le n}\left(U_t^{(i)} - 1\right) \le g(t) + r_n(t),\ t \in [0,1]\right),
\end{aligned}$$

where $r_n(t) = o(1)$ as $n \to \infty$, uniformly for $t \in [0,1]$. We claim that

$$P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)} - 1\right) \le g\right) \to_{n\to\infty} P(\boldsymbol{\eta} \le g). \tag{3.20}$$

Replace g by $g + \varepsilon$ and $g - \varepsilon$ for $\varepsilon > 0$ small enough such that $g + \varepsilon < 0$, and put

$$f_\varepsilon(t) := G_t^{-1}(\exp(g(t) + \varepsilon)), \quad f_{-\varepsilon}(t) := G_t^{-1}(\exp(g(t) - \varepsilon)), \qquad t \in [0,1].$$

Then $f_\varepsilon, f_{-\varepsilon} \in E[0,1]$, and we obtain from condition (3.18) and equation (3.19) for $n \ge n_0$

$$\begin{aligned}
&P\left(n \max_{1\le i\le n}\left(U_t^{(i)} - 1\right) \le n\big(F_t(a_n(t)f_\varepsilon(t) + b_n(t)) - 1\big),\ t \in [0,1]\right)\\
&\ge P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)} - 1\right) \le g\right)\\
&\ge P\left(n \max_{1\le i\le n}\left(U_t^{(i)} - 1\right) \le n\big(F_t(a_n(t)f_{-\varepsilon}(t) + b_n(t)) - 1\big),\ t \in [0,1]\right),
\end{aligned}$$

where the upper bound converges to $\exp\left(-\|g+\varepsilon\|_D\right)$ and the lower bound to $\exp\left(-\|g-\varepsilon\|_D\right)$. As both converge to $\exp\left(-\|g\|_D\right)$ as $\varepsilon \to 0$, we have established (3.20).

Next we claim that (3.20) is true for each $g \in E^-[0,1]$, i.e., we drop the assumption $\sup_{t\in[0,1]} g(t) < 0$. We prove this by a contradiction. Suppose first that there exists $g \in E^-[0,1]$ such that

$$\liminf_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g\right) \le \exp\left(-\|g\|_D\right)-\delta$$

for some $\delta > 0$. From (3.20) we deduce that for each $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g-\varepsilon\right) = \exp\left(-\|g-\varepsilon\|_D\right)$$

and, thus,

$$\begin{aligned}
&\exp\left(-\|g\|_D\right)-\delta\\
&\ge \liminf_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g\right)\\
&\ge \liminf_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g-\varepsilon\right)\\
&= \exp\left(-\|g-\varepsilon\|_D\right).
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we have reached a contradiction and, thus, we have established that

$$\liminf_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g\right) \ge \exp\left(-\|g\|_D\right), \qquad g \in E^-[0,1].$$

Suppose next that there exists $g \in E^-[0,1]$ such that

$$\limsup_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g\right) \ge \exp\left(-\|g\|_D\right)+\delta$$

for some $\delta > 0$. We have by (3.20) for $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le -\varepsilon\right) = \exp\left(-\varepsilon\|1\|_D\right) \to_{\varepsilon\downarrow 0} 1,$$

and, thus,

$$\begin{aligned}
&\exp\left(-\|g\|_D\right)+\delta\\
&\le \limsup_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g\right)\\
&\le \limsup_{n\to\infty}\Bigg(P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g,\, n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le -\varepsilon\right)\\
&\qquad\qquad + P\left(\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le -\varepsilon\right)^{\complement}\right)\Bigg)\\
&= \exp\left(-\left\|(\min(g(t),-\varepsilon)_{t\in[0,1]}\right\|_D\right)+1-\exp\left(-\varepsilon\|1\|_D\right)
\end{aligned}$$

by (3.20). As the first term in the final line above converges to $\exp(-\|g\|_D)$ as $\varepsilon \downarrow 0$ and the second one to zero, we have established another contradiction and, thus,

$$\limsup_{n\to\infty} P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le g\right)\le \exp\left(-\|g\|_D\right),\qquad g\in E^-[0,1].$$

This proves equation (3.20) for arbitrary $g \in E^-[0,1]$ and completes the proof of the conclusion (3.15) $\Longrightarrow$ (3.16).

Next we establish the implication (3.16) $\Longrightarrow$ (3.15). Choose $f \in E[0,1]$ with $\inf_{t\in[0,1]} G_t(f(t)) > 0$ and put $g(t) := \log(G_t(f(t)))$, $t \in [0,1]$. From the assumption (3.16) we obtain

$$P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le g\right)\to_{n\to\infty} P(\boldsymbol{\eta}\le g)=P(\boldsymbol{\xi}\le f)=\exp\left(-\|g\|_D\right).$$

On the other hand, we have by condition (3.14)

$$\begin{aligned}&P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le g\right)\\&=P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le n\big(F_t(a_n(t)f(t)+b_n(t))-1\big)+r_n(t),\, t\in[0,1]\right),\end{aligned}$$

where $r_n(t) = o(1)$ as $n \to \infty$, uniformly for $t \in [0,1]$. We claim that

$$\begin{aligned}&P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le n\big(F_t(a_n(t)f(t)+b_n(t))-1\big),\, t\in[0,1]\right)\\&\to_{n\to\infty} P\left(\boldsymbol{\eta}\le g\right)\\&=\exp\left(-\|g\|_D\right).\end{aligned}\tag{3.21}$$

Replace g by $\min(g+\varepsilon,0)$ and $g-\varepsilon$, where $\varepsilon>0$ is arbitrary. Then we obtain from (3.16) and condition (3.14) for $n \ge n_0 = n_0(\varepsilon)$

$$\begin{aligned}&P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le \min(g+\varepsilon,0)\right)\\&=P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le g+\varepsilon\right)\\&\ge P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le n\big(F_t(a_n(t)f(t)+b_n(t))-1\big),\, t\in[0,1]\right)\\&\ge P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le g-\varepsilon\right).\end{aligned}$$

As

$$P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right)\le \min(g+\varepsilon,0)\right)\to_{n\to\infty}\exp\left(-\|\min(g+\varepsilon,0)\|_D\right)$$

and

$$P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le g-\varepsilon\right) \to_{n\to\infty} \exp\left(-\|g-\varepsilon\|_D\right),$$

and $\varepsilon > 0$ was arbitrary, (3.21) follows.

From the equality

$$\begin{aligned}&P\left(n \max_{1\le i\le n}\left(\boldsymbol{U}^{(i)}-1\right) \le n\big(F_t(a_n(t)f(t)+b_n(t))-1\big),\, t\in[0,1]\right)\\&= P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right)\end{aligned}$$

we obtain from (3.21) that

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right) = P(\boldsymbol{\xi} \le f) \tag{3.22}$$

for each $f \in E[0,1]$ with $\inf_{t\in[0,1]} G_t(f(t)) > 0$. If $\inf_{t\in[0,1]} G_t(f(t)) = 0$, then, for $\varepsilon > 0$, there exists $t_0 \in [0,1]$ such that $G_{t_0}(f(t_0)) \le \varepsilon$. We, thus, have $P(\boldsymbol{\xi} \le f) \le P\left(\xi_{t_0} \le f(t_0)\right) = G_{t_0}(f(t_0)) \le \varepsilon$ and, by condition (3.14)

$$\begin{aligned}P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right) &\le P\left(\max_{1\le i\le n} X_{t_0}^{(i)} \le a_n(t_0)f(t_0)+b_n(t_0)\right)\\&\to_{n\to\infty} G_{t_0}(f(t_0))\\&\le \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we have established

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \boldsymbol{X}^{(i)} \le a_n f + b_n\right) = 0 = P(\boldsymbol{\xi} \le f)$$

in that case, where $\inf_{t\in[0,1]} G_t(f(t)) = 0$ and, thus, (3.22) for each $f \in E[0,1]$.

□

We close this section by considering the case of identical marginal distributions: under this restriction a characterization as in (3.13) is possible for functional domain of attraction. It is subject of current research whether this condition of identical marginal distribution can be dropped.

Corollary 3.18. *Let $\boldsymbol{X} = (X_t)_{t\in[0,1]}$ in $C[0,1]$ be a stochastic process with identical continuous marginal df $F(x) = P(X_t \le x)$, $x \in \mathbb{R}$, $t \in [0,1]$, and let $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$ in $C[0,1]$ be an MSP with identical marginal df G. Denote by $\boldsymbol{U} = (U_t)_{t\in[0,1]} := (F(U_t))_{t\in[0,1]}$ the copula process pertaining to $\boldsymbol{X}$. Then we have $\boldsymbol{X} \in \mathcal{D}(\boldsymbol{\eta})$ if and only if $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ together with the condition $F \in \mathcal{D}(G)$.*

Proof. The assumption $F \in \mathcal{D}(G)$ yields $\sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - G(x)| \to_{n\to\infty} 0$ for some sequence of norming constants $a_n > 0$, $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Taking logarithms and using Taylor expansion of $\log(1+x)$ for $x \in [x_0, x_1]$ with $x_0 > 0$ implies

$$\sup_{x \in [x_0,x_1]} |n(F(a_n x + b_n) - 1) - \log(G(x))| \to_{n\to\infty} 0$$

and, thus, condition (3.14) is satisfied. Corollary 3.18 is now an immediate consequence of Theorem 3.17 together with the fact that the assumption $\boldsymbol{X} \in \mathcal{D}(\boldsymbol{\xi})$ implies in particular that $F \in \mathcal{D}(G)$. □

3.6 Generalized Pareto Processes

A univariate GPD W is simply given by $W(x) = 1 + \log(G(x))$, $G(x) \geq 1/e$, where G is a univariate EVD (see, e.g., Falk et al. [17]). It was, roughly, established by Pickands [23] and Balkema and de Haan [5] that the maximum of n iid univariate observations, linearly standardized, converges in distribution to an EVD as n increases if, and only if, the exceedances above an increasing threshold follow a generalized Pareto distribution (GPD). The multivariate analogon is due to Rootzén and Tajvidi [25]. It was observed by Buishand et al. [8] that a d-dimensional GPD W with ultimately standard Pareto margins can be represented in its upper tail as $W(\boldsymbol{x}) = P(U^{-1}\boldsymbol{Z} \leq \boldsymbol{x})$, $\boldsymbol{x}_0 \leq \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, where the rv U is uniformly on $(0,1)$ distributed and independent of the rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $0 \leq Z_i \leq c$ for some $c \geq 1$ and $E(Z_i) = 1$, $1 \leq i \leq d$.

In infinite dimensional space, the investigation of generalized Pareto *processes* is at the very beginning: in Buishand et al. [8] and as a sketch in exercise 9.5 in de Haan and Ferreira [9], the foregoing constructive approach on $\mathbb{R}^d$ was carried over to continuous processes on compact intervals:

Let U be a uniformly on $[0,1]$ distributed rv, which is independent of a generator process $\boldsymbol{Z} \in \bar{C}^+[0,1]$ with properties (2.6). Then the stochastic process

$$\boldsymbol{Y} := \frac{1}{U}\boldsymbol{Z} \ \in \bar{C}^+[0,1].$$

is called a *simple GPD-process* (cf. Buishand et al. [8]).

The one-dimensional margins Y_t of $\boldsymbol{Y}$ have ultimately standard Pareto tails:

$$\begin{aligned} P(Y_t \leq x) &= P\left(\frac{1}{x} Z_t \leq U\right) \\ &= \int_0^m P\left(\frac{1}{x} z \leq U\right) (P * Z_t)(dz) \\ &= 1 - \frac{1}{x}\int_0^m z\,(P * Z_t)(dz) \\ &= 1 - \frac{1}{x} E(Z_t) \\ &= 1 - \frac{1}{x}, \qquad x \geq m,\ 0 \leq t \leq 1. \end{aligned}$$

Put $\boldsymbol{V} := -1/\boldsymbol{Y}$. Then we get

$$P(\boldsymbol{V} \leq f) = P\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t) \leq U\right)$$

$$= 1 - \int_0^1 P\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t) > u\right) du$$
$$= 1 - E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right)$$
$$= 1 - \|f\|_D$$

for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le 1/m$, i.e., $\boldsymbol{V}$ has the property that its distribution function is in its upper tail equal to

$$\begin{aligned} W(f) &:= P(\boldsymbol{V} \le f) \\ &= 1 - \|f\|_D \\ &= 1 + \log(\exp(-\|f\|_D)) \\ &= 1 + \log(G(f)), \qquad f \in \bar{E}^-[0,1],\ \|f\|_\infty \le 1/m, \end{aligned} \tag{3.23}$$

where $G(f) = P(\boldsymbol{\eta} \le f)$ is the functional df of the MSP $\boldsymbol{\eta}$ with D-norm $\|\cdot\|_D$ and generator $\boldsymbol{Z}$.

The preceding representation of the upper tail of a functional GPD in terms of $1 + \log(G)$ is in complete accordance with the unit- and multivariate case (see, for example, Falk et al. [17, Chapter 5]). We write $W = 1 + \log(G)$ in short notation and call $\boldsymbol{V}$ a GPD-process as well.

Remark 11. One of the most recent considerations on the issue of generalized Pareto processes is the paper of Ferreira and de Haan [18]. Therein, a simple Pareto process $\boldsymbol{W}$ is defined as given above, but the generator process $\boldsymbol{Z} \equiv \boldsymbol{V}$ (this is the notation within the referenced work) has slightly modified properties: the requirement $E(Z(s)) = 1,\ s \in [0,1]$ in our setup is weakened to $E(V(s)) > 0,\ s \in [0,1]$, but there has to be some $\omega_0 > 0$ with $P(\sup_{s\in[0,1]} V(s) = \omega_0) = 1$. It was shown, that

$$P(\boldsymbol{W} \le f) = 1 - E\left(\sup_{t\in[0,1]} \frac{V(s)}{f(s)}\right), \quad \text{and} \quad P(\boldsymbol{W} > f) = E\left(\inf_{t\in[0,1]} \frac{V(s)}{f(s)}\right),$$

for $\boldsymbol{W} = (W(s))_{s\in[0,1]} := (V(s)/U)_{s\in[0,1]}$ and $f \in C[0,1]$ with $\inf_{t\in[0,1]} f(t) \ge \omega_0$. This is in accordance with the setup presented here (compare also Lemma 3.22 below).

Remark 12. Due to representation (3.23), the GPD process $\boldsymbol{V}$ is clearly in the functional domain of attraction of the standard MSP $\boldsymbol{\eta}$ with D-norm $\|\cdot\|_D$ and generator $\boldsymbol{Z}$ (in the sense of equation (3.1); take $a_n \equiv 1/n$ and $b_n \equiv 0$).

Remark 13. As already mentioned by Buishand et. al [8], the GPD-process $\boldsymbol{Y}$ is in the domain of attraction of a simple max-stable process $\boldsymbol{\xi}$ in the sense of weak convergence on $C[0,1]$: for $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ independent copies of $\boldsymbol{Y}$ we have

$$\frac{1}{n} \max_{1\le i\le n} \boldsymbol{Y}_i \ \to_D \boldsymbol{\xi} \qquad \text{in } C[0,1].$$

We deduce in particular a functional version of the well-known fact that the spectral df of a GPD random vector is equal to a uniform df in a neighborhood of 0.

Lemma 3.19. *We have for $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le m$ and some $t_0 < 0$*

$$W_f(t) := P\left(\boldsymbol{V} \le t \,|f|\right) = 1 + t\,\|f\|_D\,, \qquad t_0 \le t \le 0.$$

Let $\boldsymbol{U}$ be a copula process. Then the following extension of Proposition 3.12 holds.

Proposition 3.20. *The property $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ in the sense of condition* (3.1) *is equivalent to*

$$\lim_{t\uparrow 0} \frac{1 - H_f(t)}{1 - W_f(t)} = 1, \qquad f \in \bar{E}^-[0,1], \tag{3.24}$$

i.e., the spectral df $H_f(t) = P(\boldsymbol{U} - 1 \le t\,|f|)$, $t \le 0$, of $\boldsymbol{U} - 1$ is tail equivalent *with that of the GPD $W = 1 + \log(G)$.*

Definition 3.21. A stochastic process $\boldsymbol{V}$ with sample paths in $\bar{C}^-[0,1]$ is a standard generalized Pareto process (GPP), if there exists a D-norm $\|\cdot\|_D$ on $E[0,1]$ and some $c_0 > 0$ such that

$$P(\boldsymbol{V} \le f) = 1 - \|f\|_D$$

for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le c_0$.

Equivalently, $\boldsymbol{V}$ is a standard GPP, if and only if there is $\varepsilon_0 > 0$, $M < 0$ and some generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ with $P(\boldsymbol{V} \le f) = P\left((\max(-U/Z_t, M))_{0\le t\le 1} \le f\right)$ for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le \varepsilon_0$. As $\boldsymbol{Z}$ may attain the value zero, we introduce the constant M to ensure finite values of the process.

The following assertion will be useful in Section 4.2.

Lemma 3.22. *For each standard GPP $\boldsymbol{V}$ there exists $s_0 > 0$ such that for $0 \le s \le s_0$ and for each $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le 1$*

(i)

$$P(\boldsymbol{V} \le sf) = 1 - sE\left(\sup_{t\in[0,1]} \left(|f(t)|\, Z_t\right)\right) = 1 - s\,\|f\|_D\,,$$

(ii)

$$P(\boldsymbol{V} > sf) = sE\left(\inf_{t\in[0,1]} \left(|f(t)|\, Z_t\right)\right),$$

(iii)

$$P\left(V_{t_i} > sf(t_i),\, 1 \le i \le k\right) = sE\left(\min_{1\le i\le k}\left(|f(t_i)|\, Z_{t_i}\right)\right)$$

for each set $0 \le t_1 < \cdots < t_k \le 1$, $k \in \mathbb{N}$.

Proof. Assertion *(i)* is immediately from the definition. For *(ii)* take a generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ and an uniformly distributed rv U, independent of $\boldsymbol{Z}$, with $\boldsymbol{V} = (V_t)_{t\in[0,1]} = (\max(-U/Z_t, M))_{t\in[0,1]}$ in the upper tail. Then, for some $s_0 > 0$ small enough, we get

$$\begin{aligned} P(\boldsymbol{V} > sf) &= P(-U/\boldsymbol{Z} > sf) \\ &= P(U < s \inf_{t\in[0,1]}(|f(t)|\, Z_t)) \\ &= sE(\inf_{t\in[0,1]}(|f(t)|\, Z_t)), \end{aligned}$$

for $0 \le s \le s_0$ and for each $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le 1$ by the independence of U and $\boldsymbol{Z}$. With completely analogous arguments, *(iii)* follows. □

Examples of Standard GPP

Due to the representation $\boldsymbol{V} = (V_t)_{t\in[0,1]} = (-U/Z_t)_{t\in[0,1]}$ for standard GPP with some generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$, we can take the exemplary generator processes of Section 2.4 to get a visual idea of possible sample paths of standard GPP.

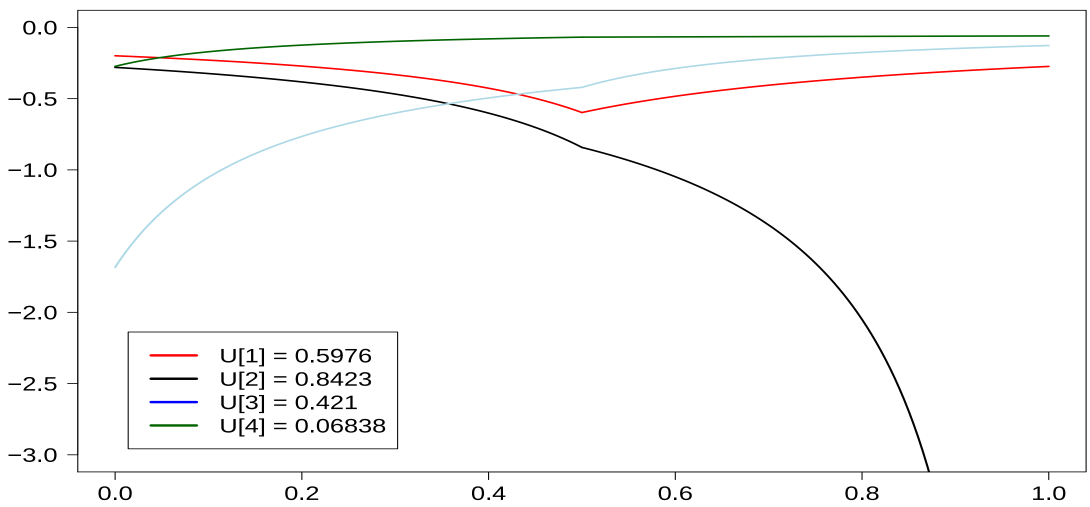

Figure 3.1: Some sample paths of the standard GPP $\boldsymbol{V} = -U/\boldsymbol{Z}$ resulting from the sample paths pictured in Figure 2.1 with different realizations of the uniformly distributed rv U.

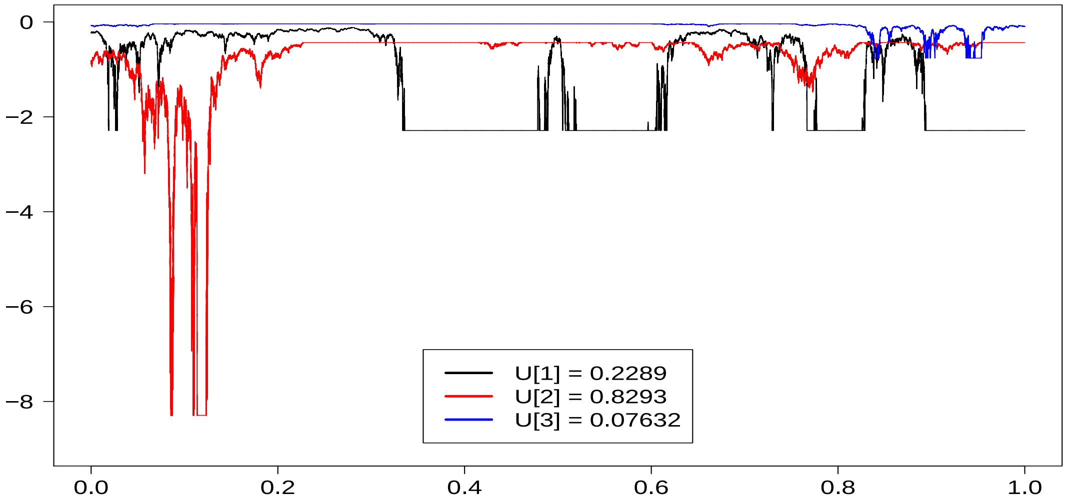

Figure 3.2: Some sample paths of the standard GPP $\boldsymbol{V} = -U/\boldsymbol{Z}^{(BM)}$ resulting from the sample paths pictured in Figure 2.3 with different realizations of the uniformly distributed rv U.

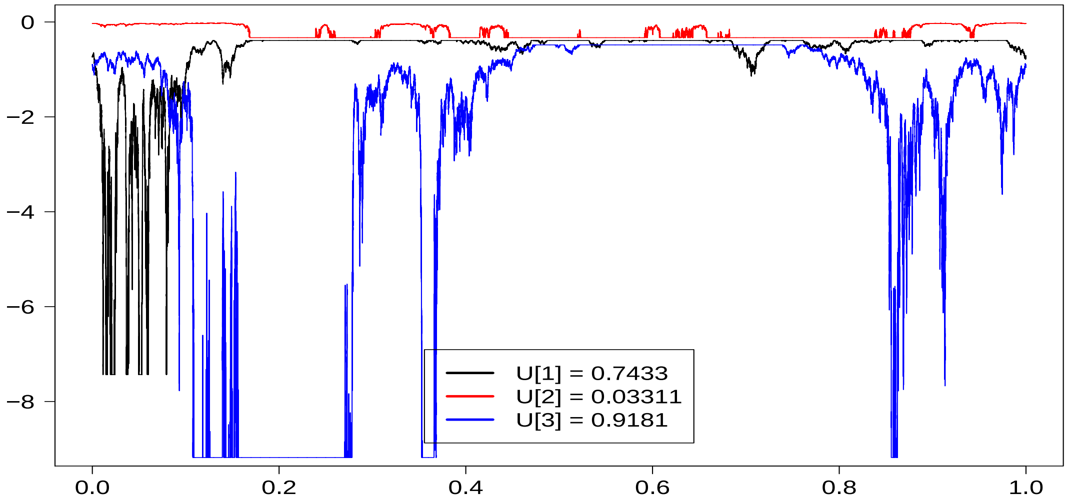

Figure 3.3: Some sample paths of the standard GPP $\boldsymbol{V} = -U/\boldsymbol{Z}^{(BB)}$ resulting from the sample paths pictured in Figure 2.4 with different realizations of the uniformly distributed rv U.

4 Sojourn Times of Continuous Processes

We already introduced in Subsection 3.2 the random sojourn time $S_{\boldsymbol{X}}(f)$ of a stochastic process $\boldsymbol{X}$ with sample paths in $\bar{C}^-[0,1]$ above some function $f \in \bar{E}^-[0,1]$ and we defined a type of convergence for continuous processes based on that. In this chapter we study properties of

$$S_{\boldsymbol{Y}}(f) := \int_0^1 \mathbf{1}(Y_t > f(t))\, dt,$$

where $\boldsymbol{Y}$ is a sample continuous process with – in most cases – identical continuous marginal distribution. We first investigate in Section 4.1 the expectation of the sojourn time $S_{\boldsymbol{Y}}(s)$ of $\boldsymbol{Y}$ above a constant threshold function $s \in \mathbb{R}$, i.e., $S_{\boldsymbol{Y}}(s) = \int_0^1 \mathbf{1}(Y_t > s)\, dt$, under the condition that there is an exceedance, i.e., $S(s) > 0$. In particular we establish its asymptotic equality with the limit of the so-called fragility index (FI; cf. Geluk et al. [19]) corresponding to $(Y_{i/n})_{1 \le i \le n}$. Moreover, our setup enables an appealing representation of the cumulative expected shortfall for continuous stochastic processes.

For processes, which are in a certain neighborhood of a generalized Pareto process (e.g. MSP), we can replace the constant threshold $s \in \mathbb{R}$ by a threshold function and we can compute the asymptotic sojourn time distribution above a high threshold function, see Section 4.2. Given that there is an exceedance $Y_{t_0} > s$ above the threshold s at t_0, we compute in Section 4.3 the asymptotic distribution of the remaining excursion time, that the process spends above the threshold function without cease.

4.1 Sojourn Times, Fragility Index and Expected Shortfall

Let $\boldsymbol{Y} = (Y_t)_{t \in [0,1]}$ be a continuous stochastic process with identical continuous marginal df F.

Before we present the main results of this section we need some auxiliary results. Put for $n \in \mathbb{N}$

$$S^{(n)}(s) := S^{(n)}_{\boldsymbol{Y}}(s) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(Y_{i/n} > s),$$

which is a Riemann sum of the integral $S(s) := S_{\boldsymbol{Y}}(s)$. We have

$$S^{(n)}(s) \to_{n \to \infty} S(s)$$

and, thus,

$$P(S^{(n)}(s) \le x) \to_{n \to \infty} P(S(s) \le x)$$

for each $x \ge 0$ such that $P(S(s) = x) = 0$. Additionally, we have always convergence for $x = 0$, although $P(S(s) = 0) = P(Y_t \le s \text{ for all } t \in [0,1]) > 0$ in general.

Lemma 4.1. *We have*

$$P(S^{(n)}(s) = 0) \to_{n\to\infty} P(S(s) = 0),$$

Proof. We have

$$P(S^{(n)}(s) = 0) \le P(S^{(n)}(s) \le \varepsilon) \to_{n\to\infty} P(S(s) \le \varepsilon) = P(S(s) = 0) + \delta,$$

where $\varepsilon, \delta > 0$ can be made arbitrarily small. This implies $\limsup_{n\to\infty} P(S^{(n)}(s) = 0) \le P(S(s) = 0)$. We have, on the other hand,

$$P(S(s) = 0) = P\left(\bigcap_{n\in\mathbb{N}} \left\{S^{(n)}(s) = 0\right\}\right) \le \liminf_{n\to\infty} P(S^{(n)}(s) = 0),$$

which implies the assertion. □

As a consequence we obtain

$$\begin{aligned} P(S^{(n)}(s) \le x \mid S^{(n)}(s) > 0) &= \frac{P(0 < S^{(n)}(s) \le x)}{P(S^{(n)}(s) > 0)} \\ &\to_{n\to\infty} \frac{P(0 < S(s) \le x)}{P(S(s) > 0)} \\ &= P(S(s) \le x \mid S(s) > 0) \end{aligned}$$

for each $x > 0$ such that $P(S(s) = x) = 0$, if $P(S(s) > 0)$.

Due to the assumption that all one dimensional margins of $\boldsymbol{Y}$ have identical continuous df F, we get

$$S^{(n)}(s) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(F(Y_{i/n}) > F(s)) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(U_{i/n} > c)$$

with probability one, where $U_{i/n} := F(Y_{i/n})$ is uniformly distributed on $(0,1)$, $i = 1, \dots, n$, and $c := F(s)$.

Denote by $N_s := \sum_{i=1}^{n} 1_{(s,\infty)}(Y_{i/n})$ the number of exceedances among $(Y_{i/n})_{1\le i\le n}$ above the threshold s. The fragility index (FI) corresponding to $(Y_{i/n})_{1\le i\le n}$ is defined as the asymptotic expectation of the number of exceedances given that there is at least one exceedance:

$$FI := \lim_{s\nearrow\omega(F)} E(N_s \mid N_s > 0),$$

where $\omega(F) := \sup\{t \in \mathbb{R} : F(t) < 1\}$. The FI was introduced in Geluk et al. [19] to measure the stability of a stochastic system. The system is called stable if $FI = 1$, otherwise it is called fragile. The collapse of a bank, symbolized by an exceedance, would be a typical example, illustrating the FI as a measure of joint stability among a portfolio of banks. For an extensive investigation and extension of the FI we refer to Falk and Tichy [16, 15].

Note that

$$FI^{(n)}(s) := E(nS^{(n)}(s) \mid S^{(n)}(s) > 0)$$

$$
\begin{aligned}
&= E\left(\sum_{i=1}^{n} \mathbf{1}(U_{i/n} > c) \mid S^{(n)}(s) > 0\right)\\
&= \sum_{i=1}^{n} P\left(U_{i/n} > c \mid S^{(n)}(s) > 0\right)\\
&= \sum_{i=1}^{n} \frac{P(U_{i/n} > c)}{P\left(S^{(n)}(s) > 0\right)}\\
&= n\frac{1-c}{1-P\left(S^{(n)}(s) = 0\right)}
\end{aligned}
$$

is the FI of level s corresponding to $Y_{i/n}$, $1 \le i \le n$. The following theorem is the first main result of this section.

Theorem 4.2. *Let $\boldsymbol{Y}$ be a stochastic process in $C[0,1]$ with identical continuous marginal df F. Suppose that the copula process $\boldsymbol{U} = (F(Y_t))_{t\in[0,1]}$ corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of a MSP $\boldsymbol{\eta}$ with generator constant $m \ge 1$, cf. Proposition 2.4. Then we have*

$$
\lim_{n\to\infty} \lim_{s\nearrow\omega(F)} \frac{FI^{(n)}(s)}{n} = \lim_{s\nearrow\omega(F)} \lim_{n\to\infty} \frac{FI^{(n)}(s)}{n} = \lim_{s\nearrow\omega(F)} E(S(s) \mid S(s) > 0) = \frac{1}{m}.
$$

Proof. Expansion (3.9) implies for $n \in \mathbb{N}$

$$
\begin{aligned}
&P(S^{(n)}(s) > 0)\\
&= 1 - P\left(\sum_{i=1}^{n} \mathbf{1}(U_{i/n} > c) = 0\right)\\
&= 1 - P(U_{i/n} \le c,\ 1 \le i \le n)\\
&= 1 - C_n(c,\dots,c)\\
&= (1-c)\left\|(1,\dots,1)\right\|_{D_n} + o\left((1-c)\left\|(1,\dots,1)\right\|_{D_n}\right)\\
&= (1-c)E\left(\max_{1\le i\le n} Z_{i/n}\right) + o\left((1-c)E\left(\max_{1\le i\le n} Z_{i/n}\right)\right)
\end{aligned}
$$

as $c \uparrow 1$ and, thus,

$$
\begin{aligned}
\frac{FI^{(n)}(s)}{n} &= \frac{1-c}{P(S^{(n)}(s) > 0)}\\
&= \frac{1}{E\left(\max_{1\le i\le n} Z_{i/n}\right) + o\left(E\left(\max_{1\le i\le n} Z_{i/n}\right)\right)}
\end{aligned}
$$

as $c \uparrow 1$. This yields, by the dominated convergence theorem,

$$
\lim_{n\to\infty} \lim_{s\nearrow\omega(F)} \frac{FI^{(n)}(s)}{n} = \lim_{n\to\infty} \frac{1}{E\left(\max_{1\le i\le n} Z_{i/n}\right)} = \frac{1}{E\left(\sup_{t\in[0,1]} Z_t\right)} = \frac{1}{m}.
$$

We have, on the other hand,

$$\lim_{n\to\infty}\frac{FI^{(n)}(s)}{n}=\lim_{n\to\infty}\frac{1-c}{1-P\left(S^{(n)}(s)=0\right)}=\frac{1-c}{1-P\left(S(s)=0\right)},$$

due to Lemma 4.1. Since $\boldsymbol{U}\in\mathcal{D}(\boldsymbol{\eta})$, we obtain from condition (3.10)

$$\begin{aligned}\lim_{s\nearrow\omega(F)}\lim_{n\to\infty}\frac{FI^{(n)}(s)}{n}&=\lim_{s\nearrow\omega(F)}\frac{1-c}{1-P\left(S(s)=0\right)}\\&=\lim_{s\nearrow\omega(F)}\frac{1-c}{1-P\left(\boldsymbol{Y}\le s\right)}\\&=\lim_{s\nearrow\omega(F)}\frac{1-c}{1-P\left(\boldsymbol{U}\le c\right)}\\&=\lim_{s\nearrow\omega(F)}\frac{1-c}{1-(1-(1-c)\left\|\mathbf{1}_{[0,1]}\right\|_D+o(1-c))}\\&=\frac{1}{\left\|\mathbf{1}_{[0,1]}\right\|_D}\\&=\frac{1}{E\left(\sup_{0\le t\le 1}Z_t\right)}\\&=\frac{1}{m}.\end{aligned}$$

Finally, the dominated convergence theorem implies

$$\begin{aligned}\frac{FI^{(n)}(s)}{n}&=E(S^{(n)}(s)\mid S^{(n)}(s)>0)\\&=\frac{E(S^{(n)}(s))}{P(S^{(n)}(s)>0)}\\&\to_{n\to\infty}\frac{E(S(s))}{P(S(s)>0)}\\&=E(S(s)\mid S(s)>0).\end{aligned}$$

□

Remark 14. Under the conditions of Theorem 4.2 we have

$$P(S(s)>0)=(1-c)m+o(1-c)\quad\text{as } c\nearrow 1\quad\text{and}\quad E(S(s))=1-F(s).$$

To apply the preceding result to generalized Pareto processes, we add an extension of Theorem 4.2. It is shown by repeating the preceding arguments.

We call a copula process $\boldsymbol{U}=(U_t)_{t\in[0,1]}$ (upper) tail continuous, if the process $\boldsymbol{U}_{c_0}:=(\max(c_0,U_t))_{t\in[0,1]}$ is a.s. continuous for some $c_0<1$. Note that in this case $\boldsymbol{U}_c$ is a.s. continuous for each $c\ge c_0$.

A stochastic process $\boldsymbol{Y}=(Y_t)_{t\in[0,1]}$ is said to have ultimately identical and continuous marginal df F_t, $t\in[0,1]$, if $F_t(x)=F_s(x)$, $0\le s,t\le 1$, $x\ge x_0$ with $F_1(x_0)<1$, and $F_1(x)$ is continuous for $x\ge x_0$.

Theorem 4.3. *Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ be a stochastic process with ultimately identical and continuous marginal df. Suppose that the copula process pertaining to $\boldsymbol{Y}$ is tail continuous and that it is in the functional domain of attraction of a MSP $\boldsymbol{\eta}$, whose corresponding D-norm is generated by a sample continuous generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ with $0 \le Z_t \le m$ a.s., $E(Z_t) = 1$, $t \in [0,1]$, for some $m \ge 1$. Then we have*

$$\begin{aligned}\lim_{n\to\infty}\lim_{s\nearrow\omega(F)} \frac{FI^{(n)}(s)}{n} &= \lim_{s\nearrow\omega(F)}\lim_{n\to\infty} \frac{FI^{(n)}(s)}{n}\\ &= \lim_{s\nearrow\omega(F)} E(S(s) \mid S(s) > 0)\\ &= \frac{1}{E\left(\sup_{0\le t\le 1} Z_t\right)}.\end{aligned}$$

Example 4.4. Consider the d-dimensional EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_p)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, $d \ge 2$, where the D-norm is the usual p-norm $\|\boldsymbol{x}\|_D = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} = \|\boldsymbol{x}\|_p$, $\boldsymbol{x} \in \mathbb{R}^d$, with $1 \le p \le \infty$. This is known as the Gumbel-Hougaard or logistic model. The case $p = \infty$ yields the maximum-norm $\|\boldsymbol{x}\|_\infty$. Let the rv $(Z_1, \dots, Z_d)$ be a generator of $\|\cdot\|_p$, i.e., $0 \le Z_i \le c$ a.s., $E(Z_i) = 1$, $1 \le i \le d$ with some $c \ge 1$, and $\|\boldsymbol{x}\|_p = E\left(\max_{1\le i\le d}(|x_i|\, Z_i)\right)$, $\boldsymbol{x} \in \mathbb{R}^d$. The rv $(Z_1, \dots, Z_d)$ can be extended by linear interpolation to a generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ of a standard MSP $\boldsymbol{\eta}$: Put for $i = 1, \dots, d-1$

$$Z_{(1-\vartheta)\frac{i-1}{d-1}+\vartheta\frac{i}{d-1}} := (1-\vartheta)Z_i + \vartheta Z_{i+1}, \qquad 0 \le \vartheta \le 1,$$

which yields a continuous generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. In this case we have

$$\frac{1}{E\left(\sup_{0\le t\le 1} Z_t\right)} = \frac{1}{E\left(\max_{1\le i\le d} Z_i\right)} = \frac{1}{\|(1,\dots,1)\|_p} = \frac{1}{d^{1/p}},$$

i.e., the generator constant is $d^{1/p}$.

Note that a standard MSP $\boldsymbol{\eta}$, whose finite dimensional marginal distributions $G_{t_1,\dots,t_d}$ are for each set set of indices $0 \le t_1 < t_2 < \dots < t_d \le 1$ and each $d \ge 1$ given by $G_{t_1,\dots,t_d}(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_p)$, $\boldsymbol{x} \le 0 \in \mathbb{R}^d$, does not exist for $p \in [1,\infty)$. This follows from the fact that in this case the generator constant would be infinite. The case $p = \infty$ leads to complete dependent margins; as a generator one can choose the constant function $Z_t = 1$, $t \in [0,1]$.

Example 4.5. Note that the copula process pertaining to the GPP $\boldsymbol{Z}/U$ is in its upper tail given by the shifted standard GPP $1 + \boldsymbol{V}$, which satisfies the conditions of Theorem 4.3. We, therefore, obtain for the GPP process $\boldsymbol{Z}/U$

$$\lim_{n\to\infty}\lim_{s\nearrow\omega(F)} \frac{FI^{(n)}(s)}{n} = \lim_{s\nearrow\omega(F)} E(S(s) \mid S(s) > 0) = \frac{1}{E\left(\sup_{0\le t\le 1} Z_t\right)}.$$

The mathematical tools from Section 3.4 enable also the computation of the (cumulative) expected shortfall corresponding to a stochastic process as defined below.

Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ be a stochastic process in $C[0,1]$ with identical and continuous univariate marginal df F and put

$$I(s) = \int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt.$$

The number $I(s)$ can be interpreted as the total sum of excesses above the threshold s. The *expected shortfall* at level s pertaining to $\boldsymbol{Y}$ is the expectation of the total sum of excesses, given that there is at least one exceedance:

$$\mathrm{ES}(s) := E(I(s) \mid S(s) > 0).$$

Lemma 4.6. *Let $\boldsymbol{U} = (U_t)_{t\in[0,1]} = (F(Y_t))_{t\in[0,1]}$ be the copula process pertaining to $\boldsymbol{Y}$. Then we have*

$$\mathrm{ES}(s) = \frac{\int_s^{\omega(F)} 1 - F(x)\, dx}{1 - P\left(U_t \le F(s) \text{ for all } t \in [0,1]\right)}.$$

Proof. We have

$$\begin{aligned}
E(I(s) \mid S(s) > 0) &= E\left(\int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt \mid \int_0^1 \mathbf{1}(Y_t > s)\, dt > 0\right)\\
&= E\left(\int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt \mid \sup_{t\in[0,1]} Y_t > s\right)\\
&= \frac{E\left(\left(\int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt\right)\mathbf{1}\left(\sup_{t\in[0,1]} Y_t > s\right)\right)}{P\left(\sup_{t\in[0,1]} Y_t > s\right)}\\
&= \frac{E\left(\int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt\right)}{P\left(\sup_{t\in[0,1]} Y_t > s\right)},
\end{aligned}$$

where by Fubini's theorem

$$\begin{aligned}
E\left(\int_0^1 (Y_t - s)\mathbf{1}(Y_t > s)\, dt\right) &= \int_0^1 E((Y_t - s)\mathbf{1}(Y_t > s))\, dt\\
&= \int_0^1 \int_0^{\omega(F)} 1 - P(Y_t - s \le x)\, dx\, dt\\
&= \int_0^1 \int_0^{\omega(F)} 1 - F(x + s)\, dx\, dt\\
&= \int_s^{\omega(F)} 1 - F(x)\, dx
\end{aligned}$$

and

$$P\left(\sup_{t\in[0,1]} Y_t > s\right) = 1 - P\left(\sup_{t\in[0,1]} Y_t \le s\right) = 1 - P\left(U_t \le F(s) \text{ for all } t \in [0,1]\right).$$

□

Suppose in addition that the copula process $\boldsymbol{U}$ is in the domain of attraction in the sense of condition (3.10) of a standard MSP with generator constant m. Then there exists a D-norm $\|\cdot\|_D$ on $C[0,1]$ with $\left\|\mathbf{1}_{[0,1]}\right\|_D = m$ such that

$$P(U_t \le F(s),\, t \in [0,1]) = 1 - (1 - F(s)) \left\|\mathbf{1}_{[0,1]}\right\|_D + o(1 - F(s))$$

as $s \nearrow \omega(F)$. The next result is, therefore, an obvious consequence of Lemma 4.6.

Proposition 4.7. *If in addition the copula process $\boldsymbol{U}$ is in the domain of attraction of a standard MSP with generator constant m, then we obtain*

$$\mathrm{ES}(s) = \frac{\int_s^{\omega(F)} 1 - F(x)\, dx}{1 - F(s)} \left(\frac{1}{m} + o(1 - F(s))\right)$$

as $s \nearrow \omega(F)$.

Proposition 4.7 precisely separates the contribution of the dependence structure of the stochastic process $\boldsymbol{Y}$ on the expected shortfall as the threshold increases, which is $1/\left\|\mathbf{1}_{[0,1]}\right\|_D = 1/m$, from that of the marginal distribution, which is the first factor. In particular we obtain that the expected shortfall converges in $[0,\infty)$ as $s \nearrow \omega(F)$ if and only if $\lim_{s \nearrow \omega(F)} \int_s^{\omega(F)} 1 - F(t)\, dt/(1 - F(s)) := c \in [0,\infty)$. And in this case its limit is c/m.

4.2 Conditional Sojourn Time Distribution

In this section we compute the asymptotic distribution of the sojourn time, under the condition that it is positive, of such processes, which are in a certain neighborhood of a standard GPP. A standard MSP is a prominent example. In this setup we can replace the constant threshold s by a threshold function.

The conditional sojourn time distribution of a standard GPP is easily computed as the following lemma shows. This distribution is independent of the threshold level s, which reveals an exceedance stability of a GPP. Note that we replace the constant threshold line s in what follows by a threshold function $sf(t)$, where $f \in \bar{E}^-[0,1]$ is fixed and s is the variable threshold level.

Lemma 4.8. *Let $\boldsymbol{V}$ in $\bar{C}^-[0,1]$ be a standard GPP, i.e. there is an $\varepsilon_0 > 0$ such that $P(\boldsymbol{V} \le g) = P(-U/\boldsymbol{Z} \le g)$ for all $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \le \varepsilon_0$, where U is uniformly on $(0,1)$ distributed and independent of the generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$, which has the properties (2.6). Choose $f \in \bar{E}^-[0,1]$. Then there is an $s_0 > 0$ such that the sojourn time df H_f of $\boldsymbol{V}$ above sf is given by*

$$P\left(\int_0^1 \mathbf{1}\left(V_t > sf(t)\right)\, dt > y \mid \int_0^1 \mathbf{1}\left(V_t > sf(t)\right)\, dt > 0\right)$$
$$= \frac{\int_0^{m\|f\|_\infty} P\left(\int_0^1 \mathbf{1}\left(|f(t)|\, Z_t > u\right)\, dt > y\right) du}{\int_0^{m\|f\|_\infty} P\left(\int_0^1 \mathbf{1}\left(|f(t)|\, Z_t > u\right)\, dt > 0\right) du}$$

$$=: 1 - H_f(y), \qquad 0 \le y \le 1,\, 0 < s \le s_0,$$

provided the denominator is greater than zero. Note that $H_f(0) = 0$, $H_f(1) = 1$.

Proof. The assertion is an immediate consequence of standard rules of integration together with conditioning on $U = u$:

$$\begin{aligned}
&P\left(\int_0^1 \mathbf{1}\left(V_t > sf(t)\right) dt > y\right)\\
&= P\left(\int_0^1 \mathbf{1}\left(U < s\left|f(t)\right| Z_t\right) dt > y\right)\\
&= \int_0^1 P\left(\int_0^1 \mathbf{1}\left(u < s\left|f(t)\right| Z_t\right) dt > y\right) du,
\end{aligned}$$

where substituting u by su yields

$$\begin{aligned}
&= s\int_0^{1/s} P\left(\int_0^1 \mathbf{1}\left(\left|f(t)\right| Z_t > u\right) dt > y\right) du\\
&= s\int_0^{m\|f\|_\infty} P\left(\int_0^1 \mathbf{1}\left(\left|f(t)\right| Z_t > u\right) dt > y\right) du
\end{aligned}$$

if $s \le 1/(m\left\|f\right\|_\infty)$. This implies the assertion. □

Example 4.9. Any continuous df F on $[0,1]$ can occur as a conditional sojourn time df. Take $Z_t = 1$, $0 \le t \le 1$, which provides the case of complete dependence of the margins of the corresponding standard MSP $\boldsymbol{\eta}$. Choose a continuous df $F : [0,1] \to [0,1]$ and put $f(t) = F(t) - 1$, $0 \le t \le 1$. Then the conditional sojourn time df equals F, $H_f(y) = F(y)$, $y \in [0,1]$.

If we take, on the other hand, $f(t) = -1$, $t \in [0,1]$, then H_f has all its mass at 1, i.e., $H_f(y) = 0$, $y < 1$, and $H_f(1) = 1$. These examples show in particular that the sojourn time df H_f can be continuous as well as discrete.

Next we will extend the preceding lemma to processes $\boldsymbol{\xi}$ in $\bar{C}^-[0,1]$ which are in certain neighborhoods of a standard GPP $\boldsymbol{V}$. Precisely, we require that for a given function $f \in \bar{E}_1^-[0,1] := \left\{f \in \bar{E}^-[0,1] : \left\|f\right\|_\infty \le 1\right\}$

$$P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right) = P\left(V_{t_i} > sf(t_i),\, 1 \le i \le k\right) + o(s) \tag{4.1}$$

for each set $0 \le t_1 < \dots < t_k \le 1$, $k \in \mathbb{N}$, and

$$P(\boldsymbol{\xi} \le sf) = P(\boldsymbol{V} \le sf) + o(s) \tag{4.2}$$

as $s \downarrow 0$.

Recall that for a standard GPP $\boldsymbol{V}$ there exists $s_0 > 0$ such that for $0 \le s \le s_0$ and for each $f \in \bar{E}^-[0,1]$ with $\left\|f\right\|_\infty \le 1$ we have $P(\boldsymbol{V} \le sf) = 1 - sE\left(\sup_{t\in[0,1]}\left(\left|f(t)\right| Z_t\right)\right)$

and $P\left(V_{t_i} > sf(t_i),\, 1 \le i \le k\right) = sE\left(\min_{1\le i\le k}\left(\left|f(t_i)\right| Z_{t_i}\right)\right)$ for some generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$, cf. Lemma 3.22. Therefore, Lemma 2.21 and equation (2.11) (together with Taylor expansion of exp) implies, that all standard MSP $\boldsymbol{\eta}$ are examples of processes which satisfy conditions (4.1) and (4.2).

The next result extends Lemma 4.8 to processes which satisfy condition (4.1) and (4.2).

Proposition 4.10. *Suppose that $\boldsymbol{\xi} \in \bar{C}^-[0,1]$ has identical univariate margins and that it satisfies condition (4.1) as well as (4.2). Choose $f \in \bar{E}_1^-[0,1]$. Then the asymptotic sojourn time distribution of $\boldsymbol{\xi}$, conditional on the assumption that it is positive, is given by*

$$P\left(\int_0^1 \mathbf{1}\left(\xi_t > sf(t)\right)\, dt > y \mid \int_0^1 \mathbf{1}\left(\xi_t > sf(t)\right)\, dt > 0\right) \to_{s\downarrow 0} 1 - H_f(y),$$

where the sojourn time df H_f is given in Lemma 4.8.

Proof. We establish this result by establishing convergence of characteristic functions. Put $I_s := \int_0^1 \mathbf{1}(\xi_t > sf(t))\, dt$, $s > 0$. The characteristic function of the rv I_s, conditional on the event that it is positive, is

$$E\left(\exp\left(itI_s\right) \mid I_s > 0\right) = \frac{\int_{\{I_s>0\}} \exp(itI_s)\, dP}{P(I_s > 0)}.$$

Note that $0 \le I_s \le 1$. By the dominated convergence theorem we have

$$\begin{aligned}
\int_{\{I_s>0\}} \exp(itI_s)\, dP &= \int_{\{I_s>0\}} \sum_{k=0}^{\infty} \frac{(itI_s)^k}{k!}\, dP \\
&= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \int_{\{I_s>0\}} I_s^k\, dP \\
&= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \int_{\Omega} I_s^k\, dP \\
&= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} E\left(I_s^k\right). \qquad (4.3)
\end{aligned}$$

From condition (4.2) we obtain

$$\begin{aligned}
P(I_s > 0) &= 1 - P(I_s = 0) \\
&= 1 - P(\boldsymbol{\xi} \le sf) \\
&= 1 - P(\boldsymbol{V} \le sf) + o(s) \\
&= s\left(E\left(\sup_{t\in[0,1]} \left|f(t)Z_t\right|\right) + o(1)\right) \qquad (4.4)
\end{aligned}$$

as $s \downarrow 0$.

From Fubini's theorem we obtain for $k \in \mathbb{N}$

$$\begin{aligned}
E(I_s^k) &= E\left(\left(\int_0^1 \mathbf{1}(\xi_t > sf(t))\, dt\right)^k\right) \\
&= E\left(\int_0^1 \dots \int_0^1 \prod_{i=1}^k \mathbf{1}\left(\xi_{t_i} > sf(t_i)\right)\, dt_1 \dots dt_k\right) \\
&= \int_0^1 \dots \int_0^1 E\left(\prod_{i=1}^k \mathbf{1}\left(\xi_{t_i} > sf(t_i)\right)\right)\, dt_1 \dots dt_k \\
&= \int_0^1 \dots \int_0^1 P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)\, dt_1 \dots dt_k.
\end{aligned}$$

We have by condition (4.1)

$$P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right) \le P(\xi_{t_1} > -s) = P(\xi_0 > -s) = P(V_0 > -s) + o(s)$$

uniformly for $t_1, \dots, t_k \in [0,1]$ and, thus, $P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)/s$ is uniformly bounded. Condition (4.1) together with the dominated convergence theorem now implies

$$\begin{aligned}
\frac{E(I_s^k)}{s} &= \int_0^1 \dots \int_0^1 \frac{P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)}{s}\, dt_1 \dots dt_k \\
&= \int_0^1 \dots \int_0^1 \frac{P\left(V_{t_i} > sf(t_i),\, 1 \le i \le k\right) + o(s)}{s}\, dt_1 \dots dt_k \\
&\to_{s \downarrow 0} \int_0^1 \dots \int_0^1 E\left(\min_{1 \le i \le k} |f(t_i) Z_{t_i}|\right)\, dt_1 \dots dt_k. \qquad (4.5)
\end{aligned}$$

From equations (4.3)-(4.5) we obtain

$$\begin{aligned}
&\int_{\{I_s > 0\}} \exp(itI_s)\, dP \\
&= s(1 + o(1)) \left(E\left(\sup_{t \in [0,1]} |f(t) Z_t|\right) \right. \\
&\qquad \left. + \sum_{k=1}^n \frac{(it)^k}{k!} \left(\int_0^1 \dots \int_0^1 E\left(\min_{1 \le i \le k} |f(t_i) Z_{t_i}|\right)\, dt_1 \dots dt_k\right)\right) \\
&\quad + \sum_{k=n+1}^{\infty} \frac{(it)^k}{k!} E(I_s^k),
\end{aligned}$$

where $n \in \mathbb{N}$ is chosen such that for a given $\varepsilon > 0$ we have $\sum_{k=m+1}^{\infty} 1/k! \le \varepsilon$. As $I_s \in [0,1]$, we obtain

$$\begin{aligned}
E(I_s^k) &\le E(I_s) \\
&= \int_0^1 P(\xi_t > sf(t))\, dt \\
&= \int_0^1 P(\xi_0 > sf(t))\, dt
\end{aligned}$$

$$\le P\left(\xi_0 > s \inf_{t\in[0,1]} f(t)\right)$$
$$= s \inf_{t\in[0,1]} |f(t)| + o(s)$$

by condition (4.1) and, thus,

$$\begin{aligned}
&\int_{\{I_s>0\}} \exp(itI_s)\, dP \\
&= s(1+o(1))\left(E\left(\sup_{t\in[0,1]} |f(t)Z_t|\right)\right. \\
&\qquad \left. + \sum_{k=1}^{n} \frac{(it)^k}{k!}\left(\int_0^1 \dots \int_0^1 E\left(\min_{1\le i\le k} |f(t_i)Z_{t_i}|\right) dt_1 \dots dt_k\right) + O(\varepsilon)\right)
\end{aligned}$$

as $s \downarrow 0$. Since $\varepsilon > 0$ was arbitrary we obtain

$$\begin{aligned}
&\lim_{s\downarrow 0} \frac{\int_{\{I_s>0\}} \exp(itI_s)\, dP}{P(I_s>0)} \\
&= 1 + \frac{\sum_{k=1}^{\infty} \frac{(it)^k}{k!}\left(\int_0^1 \dots \int_0^1 E\left(\min_{1\le i\le k} |f(t_i)Z_{t_i}|\right) dt_1 \dots dt_k\right)}{E\left(\sup_{t\in[0,1]} |f(t)Z_t|\right)} \\
&=: \varphi(t), \qquad t \in \mathbb{R}.
\end{aligned}$$

An inspection of the preceding arguments shows that φ is the characteristic function of the sojourn time df H_f, which completes the proof. □

4.3 Remaining Excursion Time

The considerations in the previous section enable us also to compute the limit distribution of the "remaining" excursion time above the threshold sf of a process $\boldsymbol{X}$ in $\bar{C}^-[0,1]$, which is in a neighborhood of a standard GPP. Precisely, we require the following condition. Choose $0 \le a \le b \le 1$, and denote by $\bar{C}^-[a,b]$ the set of continuous functions $f : [a,b] \to (-\infty, 0]$. We suppose that for $f \in \bar{C}^-[a,b]$

$$P\left(X_t > sf(t),\, t \in [a,b]\right) = P\left(V_t > sf(t),\, t \in [a,b]\right) + o(s) \tag{4.6}$$

as $s \downarrow 0$, where $\boldsymbol{V} = (V_t)_{t\in[0,1]}$ is a standard GPP. Note that

$$P(V_t > sf(t),\, t \in [a,b]) = sE\left(\min_{a\le t\le b} (|f(t)|\, Z_t)\right) + o(s), \quad s \in (0, s_0), \tag{4.7}$$

and that we allow the case $a = b$. We do not require $\boldsymbol{X}$ to have identical marginal distributions.

A standard MSP $\boldsymbol{\eta}$ satisfies condition (4.6), see Proposition 2.20. Another example is provided by the following class of processes. Substitute the rv U in the GPP $\boldsymbol{V} = (\max(-U/Z_t, M))_{t\in[0,1]}$ by a rv $W \ge 0$, which is independent of $\boldsymbol{Z}$ as well and whose df H satisfies

$$H(x) = x + o(x), \qquad \text{as } x \to 0.$$

The standard exponential df $H(x) = 1 - \exp(-x)$, $x > 0$, is a typical example. Then the process

$$\boldsymbol{X} := \left(\max\left(-\frac{W}{Z_t}, M\right)\right)_{t\in[0,1]}$$

satisfies condition (4.7) as well.

The *remaining excursion time* above sf of the process $\boldsymbol{X}$ with inspection point $t_0 \in [0,1)$ is the remaining time that the process spends above sf, under the condition that $X_{t_0} > sf(t_0)$, i.e., it is defined by

$$\tau_{t_0}(s) := \sup\{L \in (0, 1-t_0] : X_t > sf(t), t \in [t_0, t_0 + L)\}$$

under the condition that $X_{t_0} > sf(t_0)$.

Proposition 4.11. *Suppose that* $\boldsymbol{X}$ *in* $\bar{C}^-[0,1]$ *satisfies condition* (4.6). *Then we have for* $u \in [0, 1-t_0)$ *and* $f \in \bar{C}^-[a,b]$ *with* $f(t_0) < 0$

$$\lim_{s\downarrow 0} P\left(\tau_{t_0}(s) > u \mid X_{t_0} > sf(t_0)\right) = \frac{E\left(\min_{t_0\le t\le t_0+u}(|f(t)|\, Z_t)\right)}{|f(t_0)|}.$$

Proof. We have for $u \in [0, 1-t_0)$

$$\begin{aligned} P\left(\tau_{t_0}(s) > u \mid X_{t_0} > sf(t_0)\right) &= \frac{P(X_t > sf(t),\, t \in [t_0, t_0+u])}{P\left(X_{t_0} > sf(t_0)\right)} \\ &= \frac{P(V_t > sf(t),\, t \in [t_0, t_0+u]) + o(s)}{P\left(V_{t_0} > sf(t_0)\right) + o(s)} \\ &= \frac{E\left(\min_{t_0\le t\le t_0+u}(|f(t)|\, Z_t)\right)}{|f(t_0)|} + o(1) \end{aligned}$$

as $s \downarrow 0$. □

The *asymptotic* remaining excursion time T_{t_0}, as $s \downarrow 0$, with inspection point $t_0 \in [0,1)$ has, consequently, the continuous df

$$P\left(T_{t_0} \le u\right) = 1 - \frac{E\left(\min_{t_0\le t\le t_0+u}(|f(t)|\, Z_t)\right)}{|f(t_0)|}$$

for $0 \le u < 1 - t_0$, and possibly positive mass at $1 - t_0$:

$$P\left(T_{t_0} = 1 - t_0\right) = \frac{E\left(\min_{t_0\le t\le 1}(|f(t)|\, Z_t)\right)}{|f(t_0)|}.$$

Its expected value is, therefore, given by

$$\begin{aligned} E\left(T_{t_0}\right) &= \int_0^{1-t_0} P\left(T_{t_0} > u\right)\, du \\ &= \frac{1}{|f(t_0)|}\int_0^{1-t_0} E\left(\min_{t_0\le t\le t_0+u}(|f(t)|\, Z_t)\right)\, du \\ &= \frac{1}{|f(t_0)|} E\left(\int_{t_0}^1 \min_{t_0\le t\le u}(|f(t)|\, Z_t)\, du\right). \end{aligned}$$

5 Reflection and Outlook

We have started to think about "functional" Extreme Value Theory when we have realized, that reformulating the analogon to the de Haan-Resnick representation in function space due to Giné et al. [20, Proposition 3.2] in terms of those generator processes Z makes things a bit more "elementary": we have been able to consider random functions instead of "angular" measures to examine the dependence structure within MSP. Many interesting questions immediately posed themselves and some answers have been given in Chapter 2. But, of course, there are several open problems which are subject to current and future research. For example, properties like stationarity in some sense or analytic path properties (beyond the content of Section 2.3.1) have not been considered yet. Moreover, we do not know so far, whether sets like $\{g \in \bar{C}^-[0,1] : g \geq f\}$ for some $f \in \bar{E}^-[0,1]$ are continuity sets with respect to the distribution of a (standard) MSP (recall, that the sets $\{g \in \bar{C}^-[0,1] : g \leq f\}$ are continuity sets for all standard MSP, cf. Remark 6).

The answer to this question would also affect another, perhaps the most crucial part of this dissertation: in Chapter 3 we have introduced a type of convergence in function space, which is "weaker" than weak convergence of probability measures. Until now, we did not succeed to give the exact difference between these two types of convergence. It is not clear, whether condition (3.6) is also necessary for weak convergence on function space – and the answer to the foregoing question concerning the continuity sets could help to solve the latter problem.

It can be seen in Chapter 4, that the concept of convergence of functional distribution functions is useful. Nice results on terms which are in touching distance to applications have been obtained, such as sojourn times above high thresholds, remaining excursion times and the extensions of the fragility index and the expected shortfall to function space. "Real" applications are not considered within this work, which would also be an interesting issue of future research.

Finally, it should be mentioned that there are already some answers to further questions based on the theory introduced within this work, see Aulbach et al. [4] and Aulbach and Falk [1, 2].

Appendix: Random Closed Sets and Hypoconvergence of Continuous Processes

We collect some basic theory on random sets, following the book of Molchanov [22].

Let $\mathbb{E}$ be a locally compact Hausdorff second countable topological space and denote by $\mathcal{F}, \mathcal{G}, \mathcal{K}$ the family of closed/open/compact subsets of $\mathbb{E}$, respectively.

Moreover, let $(\Omega, \mathcal{A}, P)$ be a complete probability space, i.e., $B \subset A \in \mathcal{A}$ with $P(A) = 0$ implies that $B \in \mathcal{A}$.

Definition 5.1 (Molchanov [22, Definition 1.1.1]). A map $X : \Omega \mapsto \mathcal{F}$ is called a random closed set, if

$$\{\omega \in \Omega : X \cap K \neq \emptyset\} \in \mathcal{A} \quad \text{for all } K \in \mathcal{K}.$$

Set $\mathcal{F}_K := \{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ for $K \in \mathcal{K}$ and denote by $\mathbb{B}(\mathcal{F})$ the σ-algebra generated by $(\mathcal{F}_K)_{K \in \mathcal{K}}$, which is the so-called Effros-σ-algebra on $\mathcal{F}$ (compare Molchanov [22, Section 1.2.1]). However, $\mathbb{B}(\mathcal{F})$ is the Borel-σ-algebra of the so-called Fell-topology, see Molchanov [22, p. 2 and Appendix B]. In the sequel, therefore, the boundary $\partial\mathcal{X}$, the interior $\mathcal{X}^\circ$ and the closure $\bar{\mathcal{X}}$ of sets $\mathcal{X} \in \mathcal{F}$ should be interpreted with respect to the Fell-topology.

The defining condition of a random closed set can be reformulated: $X : \Omega \mapsto \mathcal{F}$ is a random closed set, if

$$X^{-1}(\mathcal{X}) = \{\omega \in \Omega : X(\omega) \in \mathcal{X}\} \in \mathcal{A} \quad \text{for all } \mathcal{X} \in \mathbb{B}(\mathcal{F}).$$

The values of the distribution $P(X \in \mathcal{F}_K)$ of a random closed set X on $\mathcal{F}_K,\ K \in \mathcal{K}$, play a crucial role in the following.

Definition 5.2 (Molchanov [22, Definition 1.1.4]). The functional $T_X : \mathcal{K} \mapsto [0, 1]$ given by

$$T_X(K) = P(X \cap K \neq \emptyset) = P(X \in \mathcal{F}_K), \quad K \in \mathcal{K},$$

is said to be the capacity functional of X.

The capacity functional can be extended onto the family of all subsets $\mathcal{P}$ of $\mathbb{E}$ by putting

$$T^*(G) := \sup\{T(K) : K \in \mathcal{K}, K \subset G\}, \quad G \in \mathcal{G},$$

and

$$T^*(M) := \inf\{T^*(G) : G \in \mathcal{G}, M \subset G\}, \quad M \in \mathcal{P}.$$

Then, in particular, $T^*(K) = T(K)$ for $K \in \mathcal{K}$ (cf. Molchanov [22, p. 9]).

The concept of weak convergence of probability measures is defined for random closed sets in the usual way (cf. Billingsley [7]).

Definition 5.3 (Molchanov [22, Definition 1.6.1]). A sequence of random closed sets $(X_n)_{n\in\mathbb{N}}$ is said to converge weakly to a random closed set X, if

$$P(X_n \in \mathcal{X}) \to P(X \in \mathcal{X}) \quad \text{for } n \to \infty$$

for each $\mathcal{X} \in \mathbb{B}(\mathcal{F})$ such that $P(X \in \partial\mathcal{X}) = 0$.

There is a useful result which helps to identify some special but crucial continuity sets; for a proof see Molchanov [22, Lemma 1.6.3].

Lemma 5.4 (Molchanov [22, p. 85]). *For every $K \in \mathcal{K}$, each of the equivalent conditions*

$$\begin{aligned} P(X \in \mathcal{F}_K) &= P(X \in \mathcal{F}_{K^\circ}) \\ P(X \cap K \neq \emptyset, X \cap K^\circ = \emptyset) &= 0 \\ T_X(K) &= T_X(K^\circ) \end{aligned}$$

implies $P(X \in \partial\mathcal{F}_K) = 0$.

Definition 5.5 (Molchanov [22, Definition 1.6.4]). The family consisting of elements of the relatively compact sets $\{B \in \mathbb{B}(\mathcal{F}) : \bar{B} \in \mathcal{K}\}$ satisfying

$$T_X(\bar{B}) = T_X(B^\circ)$$

is called the continuity family of a random closed set X and will be denoted by $\mathcal{C}_X$.

Weak convergence of random closed sets can be characterized by convergence of the corresponding capacity functionals.

Theorem 5.6 (Molchanov [22, Theorem 1.6.5]). *A sequence of random closed sets X_n converges weakly to a random closed set X if and only if*

$$T_{X_n}(K) \to T_X(K), \quad n \to \infty, \tag{5.1}$$

for each $K \in \mathcal{K} \cap \mathcal{C}_X$.

It is sufficient to check condition (5.1) on a possibly smaller family of sets.

Definition 5.7 (Molchanov [22, Definition 1.1.25]). A class $\mathcal{B}$ of relatively compact sets is called separating if

(i) $\emptyset \in \mathcal{B}$

(ii) for every $K \in \mathcal{K}$ and $G \in \mathcal{G}$ with $K \subset G$, there exists a set $B \in \mathcal{B}$ such that $K \subset B \subset G$.

Proposition 5.8 (Molchanov [22, Corollary 1.6.9]). *Weak convergence of X_n to X for $n \to \infty$ follows, if*

$$T_{X_n}(B) \to T_X(B), \quad n \to \infty,$$

for each $B \in \mathcal{B} \cap \mathcal{C}_X$, where $\mathcal{B}$ is a separating class.

Note that Proposition 5.8 implies Theorem 5.6 for $\mathcal{B} = \mathcal{K}$. According to a remark in Molchanov [22, p. 87] typically used separating classes in metric spaces $\mathbb{E}$ are the class of finite unions of balls of positive radii or, if $\mathbb{E}$ is a subspace of $\mathbb{R}^d$, the countable class of finite unions of balls with rational midpoints and positive rational radii.

Weak Epi- and Hypoconvergence

The preceding setup of random closed sets can be used to define a type of convergence for stochastic processes, cf. Molchanov [22, Section 5.3].

Let $\mathbb{E}$ be a locally compact Hausdorff second countable topological space and $(\Omega, \mathcal{A}, P)$ a complete probability space. Consider for a function $\boldsymbol{\zeta} : \mathbb{E} \times \Omega \mapsto \bar{\mathbb{R}}$ its epigraph

$$\operatorname{epi} \boldsymbol{\zeta}(\omega) := \{(x,t) \in \mathbb{E} \times \mathbb{R} : t \geq \boldsymbol{\zeta}(x;\omega)\}$$

and its hypograph

$$\operatorname{hypo} \boldsymbol{\zeta}(\omega) := \{(x,t) \in \mathbb{E} \times \mathbb{R} : t \leq \boldsymbol{\zeta}(x;\omega)\},$$

for $\omega \in \Omega$.

Definition 5.9 (Molchanov [22, Definition 5.3.5]). A function $\boldsymbol{\zeta} : \mathbb{E} \times \Omega \mapsto \bar{\mathbb{R}}$ is called a normal integrand if its epigraph $\operatorname{epi} \boldsymbol{\zeta}$ is a random closed set.

It is necessary for $\boldsymbol{\zeta}$ to be a normal integrand that its epigraph $\operatorname{epi} \boldsymbol{\zeta}(\omega)$ is a closed set in $\mathbb{E} \times \mathbb{R}$. At this point, the concept of semicontinuity should be introduced (cf. Molchanov [22, pp. 391]).

Definition 5.10. A function $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ is called upper semicontinuous at $x \in \mathbb{E}$, if

$$\limsup_{y \to x} f(y) \leq f(x),$$

and lower semicontinuous at $x \in \mathbb{E}$, if

$$\liminf_{y \to x} f(y) \geq f(x).$$

If f is upper (lower) semicontinuous at every $x \in \mathbb{E}$, then f is called upper (lower) semicontinuous.

The link between semicontinuous functions and the theory of random closed sets is the following characterization of semicontinuity in terms of epi-/hypographs.

Proposition 5.11 (Molchanov [22, Proposition A.2]). *Let $f : \mathbb{E} \mapsto \mathbb{R}$ be a real valued function. Then:*

$$f \text{ is upper semicontinuous} \iff \operatorname{hypo} f = \{(x,t) : t \leq f(x)\} \text{ is closed in } \mathbb{E} \times \mathbb{R};$$

and

$$f \text{ is lower semicontinuous} \iff \operatorname{epi} f = \{(x,t) : t \geq f(x)\} \text{ is closed in } \mathbb{E} \times \mathbb{R}.$$

Thus, as also mentioned in Molchanov [22, p. 339], it is necessary for a stochastic process $\boldsymbol{\zeta}$ to be a normal integrand that $\boldsymbol{\zeta}$ is lower semicontinuous almost surely. But this is not sufficient because of the specific requirements on measurability for random closed sets.

Proposition 5.12 (Molchanov [22, Proposition 5.3.6]). *Let $\boldsymbol{\zeta}(\cdot, \omega)$ be a lower semicontinuous function for almost all $\omega \in \Omega$ and let $\boldsymbol{\zeta}$ be jointly measurable in (x, ω), i.e. $\boldsymbol{\zeta}^{-1}(B) \in (\mathbb{B}(\mathbb{E}) \times \mathcal{A})$ for every $B \in \mathbb{B}(\mathbb{R})$. Then $\boldsymbol{\zeta}$ is a normal integrand.*

It is now possible to define the so-called weak epiconvergence which is based on weak convergence of the epigraphs of normal integrands in the sense of Definition 5.3.

Definition 5.13 (Molchanov [22, Definition 5.3.14]). A sequence $(\boldsymbol{\zeta}_n)_{n \in \mathbb{N}}$ of normal integrands weakly epiconverges to a normal integrand $\boldsymbol{\zeta}$ if the sequence $X_n := \operatorname{epi} \boldsymbol{\zeta}_n$ converges weakly to $X = \operatorname{epi} \boldsymbol{\zeta}$ as random closed sets in $\mathbb{E} \times \mathbb{R}$.

There is an immediate characterization of weak epiconvergence in terms of the distribution of $\boldsymbol{\zeta}_n, \boldsymbol{\zeta}$, essentially due to Corollary 5.8.

Proposition 5.14 (Molchanov [22, Proposition 5.3.15]). *A sequence $(\boldsymbol{\zeta}_n)_{n \in \mathbb{N}}$ of normal integrands weakly epiconverges to a normal integrand $\boldsymbol{\zeta}$ if and only if*

$$P\left(\inf_{x \in K_i} \boldsymbol{\zeta}_n(x) > t_i, i = 1, \ldots, m\right) \to P\left(\inf_{x \in K_i} \boldsymbol{\zeta}(x) > t_i, i = 1, \ldots, m\right), n \to \infty,$$

for all $m \in \mathbb{N}, t_1, \ldots, t_m \in \mathbb{R}$ *and* $K_1, \ldots, K_m$ *belonging to a separating class* $\mathcal{B}$ *on* $\mathbb{E}$ *and satisfying the continuity condition*

$$P\left(\inf_{x\in K_i} \boldsymbol{\zeta}(x) > t_i\right) = P\left(\inf_{x\in K_i^\circ} \boldsymbol{\zeta}(x) \geq t_i\right), \quad i = 1, \ldots, m.$$

A note in Molchanov [22, p. 384] says, that the introduced concepts can be reformulated for upper semicontinuous functions and their hypographs.

Definition 5.15. Let $\boldsymbol{\zeta}_n, \boldsymbol{\zeta}$ be jointly measurable and upper semi-continuous processes. Then the sequence $\boldsymbol{\zeta}_n$ is said to weakly hyper-converge to $\boldsymbol{\zeta}$, if the hypographs hypo $\boldsymbol{\zeta}_n$ converge weakly to hypo $\boldsymbol{\zeta}$ as random closed sets in $\mathbb{E} \times \mathbb{R}$.

The Specific Case of Processes with Continuous Sample Paths

The preceding theory is in particular applicable to continuous processes $\boldsymbol{\zeta} : [0, 1] \times \Omega \mapsto \mathbb{R}$, i.e $\mathbb{E} = [0, 1]$ with the usual (Euclidean) topology.

As $\boldsymbol{\zeta}$ is continuous for all $\omega \in \Omega$, it is lower and upper semicontinuous, so epi $\boldsymbol{\zeta}$ and hypo $\boldsymbol{\zeta}$ are both closed sets. For every $x \in [0, 1]$, $\boldsymbol{\zeta}(x)$ is a random variable on $\mathbb{R}$, but by continuity, $\boldsymbol{\zeta}(x, \omega)$ is also jointly measurable (i.e., $\boldsymbol{\zeta}^{-1}(B) \in (\mathbb{B}(\mathbb{E}) \times \mathcal{A})$ for every $B \in \mathbb{B}(\mathbb{R})$):

Define for $n \in \mathbb{N}$

$$\boldsymbol{\zeta}^{(n)}(t, \omega) := \boldsymbol{\zeta}(t, \omega) \text{ for } k/n \leq t < (k + 1)/n,\ k = 0, 1, \ldots, n - 1, \quad \boldsymbol{\zeta}^{(n)}(1, \omega) := \boldsymbol{\zeta}(1, \omega).$$

Then, for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$(\boldsymbol{\zeta}^{(n)})^{-1}((-\infty, x]) = \bigcup_{k=1}^{n-1} \{[k/n, (k+1)/n) \times \boldsymbol{\zeta}_{k/n}^{-1}((-\infty, x])\} \cup \{\{1\} \times \boldsymbol{\zeta}_1^{-1}((-\infty, x])\} \in \mathbb{B}([0, 1]) \times \mathcal{A},$$

so $\boldsymbol{\zeta}^{(n)}$ is jointly measurable for all n, and, because of the continuity of $\boldsymbol{\zeta}$ for all ω, so is $\lim_{n\to\infty} \boldsymbol{\zeta}^{(n)} = \boldsymbol{\zeta}$.

Thus, for $\boldsymbol{\zeta}$ in $C[0, 1]$, epi $\boldsymbol{\zeta}$ as well as hypo $\boldsymbol{\zeta}$ are random closed sets and weak epi- and hypoconvergence is well defined.

The following result is a reformulation of Proposition 5.14 in the specific case of stochastic processes $\boldsymbol{\zeta}$ whose sample paths lie in $C[0, 1]$ and hypoconvergence. Note that on $\mathbb{R}$ and on $\mathbb{E} = [0, 1]$ the set of all finite unions of (closed) intervals form a separating class.

Proposition 5.16. *A sequence of stochastic processes* $(\boldsymbol{\zeta}_n)_{n\in\mathbb{N}}$ *with sample paths in* $C[0, 1]$ *weakly hypoconverges to a sample continuous* $\boldsymbol{\zeta}$ *if and only if*

$$P\left(\sup_{x\in K_i} \boldsymbol{\zeta}_n(x) < t_i, i = 1, \ldots, m\right) \to P\left(\sup_{x\in K_i} \boldsymbol{\zeta}(x) < t_i, i = 1, \ldots, m\right), n \to \infty, \qquad (5.2)$$

for all $m \in \mathbb{N}, t_1, \dots, t_m \in \mathbb{R}$ *and* $K_1, \dots, K_m \subset [0,1]$ *finite unions of (closed) intervals satisfying the continuity condition*

$$P\left(\sup_{x \in \bar{K}_i} \boldsymbol{\zeta}(x) < t_i,\ i = 1, \dots, m\right) = P\left(\sup_{x \in K_i^\circ} \boldsymbol{\zeta}(x) \le t_i,\ i = 1, \dots, m\right). \tag{5.3}$$

Proof. Let $K_i \subset [0,1]$ be finite unions of intervals, $i = 1, \dots, m$, then $\bigcup_{i=1}^m K_i \times [a_i, b_i]$ with $a_i \le b_i \in \mathbb{R}, i = 1, \dots, m$ forms a separating class in $[0,1] \times \mathbb{R}$ (recall that $(K_i \times [a_i, b_i])_i$ is a base of the product topology in $[0,1] \times \mathbb{R}$, so $(\bigcup_{i=1}^m K_i \times [a_i, b_i])_{m \in \mathbb{N}}$ is a separating class in the product space). Furthermore, for a lower semi-continuous function f,

$$\operatorname{hypo} f \cap K_i \times [a, b] \neq \emptyset \iff \operatorname{hypo} f \cap K_i \times [a, \infty) \neq \emptyset$$

by construction of the hypograph.

Thus, by Corollary 5.8, $\boldsymbol{\zeta}_n$ weakly hypoconverges if and only if

$$T_{\operatorname{hypo} \zeta_n}(K) \to T_{\operatorname{hypo} \zeta}(K), \quad n \to \infty, \tag{5.4}$$

for all $K = \bigcup_{i=1}^m K_i \times [t_i, \infty)$ with the additional property

$$T_{\operatorname{hypo} \zeta}(\bar{K}) = T_{\operatorname{hypo} \zeta}(K^\circ), \tag{5.5}$$

where $m \in \mathbb{N}, t_1, \dots, t_m \in \mathbb{R}$ and $K_1, \dots, K_m \subset [0,1]$ are finite unions of (closed) intervals.

It remains to show, that (5.4) corresponds to (5.2) and (5.5) is (5.3).

To this end, note that the capacity functional $T_{\operatorname{hypo} \zeta}(K)$ can be rewritten as

$$\begin{aligned} T_{\operatorname{hypo} \zeta}(K) &= P\left(\operatorname{hypo} \boldsymbol{\zeta} \cap \bigcup_{i=1}^m K_i \times [t_i, \infty) \neq \emptyset\right) \\ &= P\left(\operatorname{hypo} \boldsymbol{\zeta} \text{ hits some element of } \{K_i \times [t_i, \infty), 1 \le i \le m\}\right) \\ &= 1 - P\left(\operatorname{hypo} \boldsymbol{\zeta} \text{ hits no element of } \{K_i \times [t_i, \infty), 1 \le i \le m\}\right) \\ &= 1 - P\left(\sup_{x \in K_i} \boldsymbol{\zeta}(x) < t_i, i = 1, \dots, m\right), \end{aligned}$$

so an application of this calculation to (5.4) and (5.5) yields the desired correspondence.

□

Functional Domain of Attraction and Hypoconvergence

After recalling the theory of random closed sets and adjusting the results in Molchanov [22] to our purposes, we show in this section that the functional domain of attraction condition implies hypoconvergence of the normalized maximum-processes to the standard MSP $\boldsymbol{\eta}$.

Recall that a stochastic process $\boldsymbol{Y}$ with continuous sample paths is said to be in the functional domain of attraction of a standard MSP $\boldsymbol{\eta}$ if there are functions $a_n \in C^+[0,1] := \{f \in C[0,1] : f > 0\}$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} P\left(\frac{\boldsymbol{Y} - b_n}{a_n} \le f\right)^n = P(\boldsymbol{\eta} \le f) \tag{3.1}$$

for any $f \in \bar{E}^-[0,1]$. Moreover, recall that there is for every standard MSP $\boldsymbol{\eta}$ some sample continuous generator process $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ with

$$P(\boldsymbol{\eta} \le f) = P(\boldsymbol{\eta} < f) = \exp\left(-E\left(\sup_{t\in[0,1]} f(t)Z_t\right)\right)$$

for $f \in \bar{E}^-[0,1]$, cf. Lemmata 2.7 and 2.8.

Let for the sake of simplicity and without loss of generality $K_i \subset [0,1]$ be disjoint intervals, $x_i \in (-\infty, 0], i = 1,\dots,m$, and define for $t \in [0,1]$ the functions

$$g(t) := \sum_{i=1}^m x_i \mathbf{1}_{K_i}(t); \qquad \bar{g}(t) := \sum_{i=1}^m x_i \mathbf{1}_{\bar{K}_i}(t); \qquad g^\circ(t) := \sum_{i=1}^m x_i \mathbf{1}_{K_i^\circ}(t).$$

Then $g, \bar{g}, g^\circ \in \bar{E}^-[0,1]$, and we have by the continuity of the generator process $\boldsymbol{Z}$

$$\begin{aligned}
P\left(\sup_{t\in K_i^\circ} \eta(t) \le x_i,\ i = 1,\dots,m\right) &= P(\boldsymbol{\eta} \le g^\circ) \\
&= \exp\left(-E\left(\max_{1\le i\le m}\left(|x_i| \sup_{t\in K_i^\circ} Z_t\right)\right)\right) \\
&= \exp\left(-E\left(\max_{1\le i\le m}\left(|x_i| \sup_{t\in \bar{K}_i} Z_t\right)\right)\right) \\
&= P(\boldsymbol{\eta} \le \bar{g}) \\
&= P(\boldsymbol{\eta} < \bar{g}) \\
&= P\left(\sup_{t\in \bar{K}_i} \eta(t) < x_i,\ i = 1,\dots,m\right).
\end{aligned}$$

This shows, that, as the limit process is a standard MSP, the continuity condition (5.3) is satisfied for all $m \in \mathbb{N}, x_1,\dots,x_m \le 0$ and $K_1,\dots,K_m \subset [0,1]$ finite unions of (closed) intervals. Hence, must hold for all those $x_i,\ K_i$ as before and this is, indeed, the case:

In Lemma 3.1 it was shown, that (3.1) is equivalent to

$$\lim_{n\to\infty} P\left(\frac{\boldsymbol{Y} - b_n}{a_n} < f\right) = P(\boldsymbol{\eta} < f), \quad f \in \bar{E}^-[0,1]. \tag{3.2}$$

With g defined as before, this reads

$$\lim_{n\to\infty} P\left(\sup_{t\in K_i} \boldsymbol{X}_n(t) < x_i, i = 1,\dots,m\right) = P\left(\sup_{t\in K_i} \boldsymbol{\eta}(t) < x_i, i = 1,\dots,m\right),$$

which is (5.2).

Note that hypoconvergence of the normalized maximum process does in general not imply convergence in the sense of (3.2), since the continuity condition in Proposition 5.16 excludes convergence for closed subsets $K \subset [0,1]$ of the form $K = \{t\}, t \in [0,1]$.

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