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UNDERSTANDING THE DEVELOPMENT OF THE PROVING PROCESS WITHIN A DYNAMIC GEOMETRY ENVIRONMENT

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DECLARATION

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I hereby declare that this dissertation represents my own work and that is has not been previously submitted to this university or any other institution in application for admission for a degree, diploma, or other qualifications.

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VIETNAMESE MATHEMATICAL SIGNS

Vietnamese Style	*Mathematical Meaning*
Segment *AB*	Segment *AB*
Straight line *AB*	Straight line *AB*
Straight line *l*	Straight line *l*
$\overrightarrow{AB}$	Vector *AB*
AB	The length of segment *AB*
AB = *CD*	The length of segment *AB* is equal to the length of segment *CD*
AB > *CD (AB* < *CD)*	The length of segment *AB* is greater (smaller) than the length of segment *CD*
AB = *CD*	Segment *AB* is congruent to segment *CD*
AB // *CD*	Segment (straight line) *AB* is parallel to segment (straight line) *CD*
$AB \perp CD$	Segment (straight line) *AB* is perpendicular to segment (straight line) *CD*
AB =// *CD*	Segment *AB* is equal and parallel to segment *CD*
$AB \equiv CD$	Segment (straight line) *AB* is coincident with segment (straight line) *CD*
$l_1 \equiv l_2$	Straight line l_1 is coincident with straight line l_2
$l_1 \perp l_2$	Straight line l_1 is perpendicular to straight line l_2
$l_1 // l_2$	Straight line l_1 is parallel to straight line l_2
$\angle A$; $\hat{A}$ $\angle ABC$;$\widehat{ABC}$	Angle *A* Angle *ABC*
$\angle A = \angle B$	Angle *A* is congruent to angle *B*
$\angle A = 60^0$	The measure of angle *A* is 60^0
$\angle A = \angle B$	The measure of angle *A* is equal to the measure of angle *B*
$\Delta ABC = \Delta A'B'C'$	Triangle *ABC* is congruent to triangle *A'B'C'*
$(O; R)$	A circle with center *O* and radius *R*

$A \in BC$	Point *A* lies on the straight line *BC*
$A \in [BC]$	Point *A* lies on the segment *BC*
$A = l_1 \cap l_2$	Point *A* is the intersection of the straight line l_1 and the straight line l_2
$A' = r_O(A)$	Point *A'* is an image of point *A* under the point reflection (central symmetry) with respect to point *O*
$a' = r_O(a)$	The straight line *a'* is an image of the straight line *a* under the point reflection with respect to point *O*
$A' = r_l(A)$	Point *A'* is an image of point *A* under the line reflection with respect to the straight line *l*
$A' = r_{BC}(A)$	Point *A'* is an image of point *A* under the line reflection with respect to the straight line *BC*
$A' = T_{\vec{v}}(A)$	Point *A'* is an image of point *A* under the translation through vector $\vec{v}$
$A' = T_{\overrightarrow{BC}}(A)$	Point *A'* is an image of point *A* under the translation through vector $\overrightarrow{BC}$
$a' = T_{\vec{v}}(a)$	The straight line *a'* is an image of the straight line *a* under the translation through vector $\vec{v}$
$a' = T_{\overrightarrow{BC}}(a)$	The straight line *a'* is an image of the straight line *a* under the translation through vector $\overrightarrow{BC}$
$A' = R_{C,\alpha^0}(A)$	Point *A'* is an image of point *A* under the rotation through angle α^0 about point *C*
$a' = R_{C,\alpha^0}(a)$	The straight line *a'* is an image of the straight line *a* under the rotation through angle α^0 about point *C*
$p_{\Delta ABC}$	The perimeter of triangle *ABC*
$S_{\Delta ABC}$	The area of triangle *ABC*

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ABSTRACT

Argumentation and proof have played a fundamental role in mathematics education in recent years. Much of the research that has been conducted on the proving process has been aimed at clarifying the functions and the need of proofs in teaching and learning mathematics, especially its role in the current mathematical curriculum. In particular, a strand of the research has thoroughly studied the impact new technologies on supporting students overcoming their difficulties in proof-related problems. The author of this dissertation would like to investigate the development of the proving process within a dynamic geometry environment in order to support tertiary students understanding the proving process. The strengths of this environment stimulate students to formulate conjectures and produce arguments during the proving process. Nevertheless, there are many tertiary students who are not able to write a formal proof. This barrier may stem from the lack of understanding in the proving strategy using geometric transformations which was considered in this dissertation. Through empirical research, we classified different levels of proving and proposed a methodological model for proving. Based on this model, we designed an interactive HELP SYSTEM in order to bridge the gaps between different phases of the proving process. This methodological model makes a contribution to improve students' levels of proving and develop their dynamic visual thinking. The findings of the research have also revealed that a dynamic geometry environment provides data and 'observed facts' for formulating conjectures. As a result, students can realize some geometric invariants by using dragging mode and these invariants would be a key factor in generating new ideas for proofs. Then students can use previously produced arguments and reverse the abductive structure to write a deductive proof. We used TOULMIN model of argumentation as a theoretical model to analyze the relationship between argumentation and proof. This research also offers some possible explanation so as to why students have cognitive difficulties in constructing proofs and provides mathematics educators with a deeper understanding on the proving process at the tertiary level. Moreover, the research may open a valuable discussion on the cognitive development of the proving process among mathematics teachers. In particular, we have also analyzed the role of abduction in transition from conjecturing to proving modality within a dynamic geometry environment.

Chapter 1

INTRODUCTION

1.1. STATEMENT OF THE PROBLEM

Proving is a crucial activity within mathematical classrooms at the different educational levels. It provides a way of thinking that deepens mathematical understanding, broadens the nature of human reasoning, and fosters students' creativity. The NCTM standards (2000) emphasized in a particular section the importance of developing students' reasoning and proving abilities, formulating conjectures, producing arguments, and using various methods of approaching proofs. POLYA (1954) has also claimed that understanding is a necessary condition for proving because when students have reassured themselves that a theorem is true, they will start proving it. However, many researchers have argued that teaching students the key idea of proofs is not an easy task (e.g. HANNA, 2000; MARIOTTI, 2007). Therefore, mathematics teachers are usually faced with the difficult task of teaching students how to understand the proving process in mathematics classroom. In addition, the different forms of proofs (such as verbal, visual, formal, informal, etc) may also directly influence the students' understanding of proofs and the proving process. In this dissertation, we concentrate on a formal proof at the tertiary level. In order to offer a situation for the construction of a formal proof, EDWARDS (1997) proposed the term "conceptual territory before proof". It was defined by demonstrating that conjecturing, reasoning, exploration, explanation, and validation constitute the essential elements (or steps) before formulating a proof. It also considered the basis of understanding the proving process. Thus, in this research, we propose a methodological model concerning this area in order to support students in realizing each element or step before a formal proof. This model was utilized to design an interactive HELP SYSTEM that was embedded in a dynamic geometry environment aimed at improving the student's level of proving. In order to achieve this goal, we must classify student's level of proving and build the corresponding level of hints in the interactive HELP SYSTEM with the purpose of guiding the student in constructing diagrams, realizing geometric invariants, formulating and validating conjectures, producing arguments, and writing a formal proof.

Working within a dynamic geometry environment, such as GeoGebra, students would gain their understanding through verifying conjectures and transforming understanding into an explanation as to "why" the statement is valid. Moreover, the students are able to see and accept the truth of the conjectures easily by dragging in this environment; thereby they may have no need for further verification (see e.g. HÖLZL, 2001). As a result, a dynamic geometry environment may prevent students from understanding the need and function of proofs in school mathematics (see e.g. YEUSHALMY, CHAZAN & GORDON, 1990). Therefore, our task is to provide enough open-ended questions and explorative tasks in the interactive HELP SYSTEM so that students can construct proofs on their own. Simultaneously, this system should also provide students with an opportunity to develop a sense of proof and improve their geometric intuition during the proving process.

In the mathematics teacher training universities, it is important to improve students' proving skills within a dynamic geometry environment. These students also need to understand the development of the proving process in order to provide their students at a secondary school with a suitable strategy in approaching proofs. This means that the prospective teachers should learn how to design instructional and methodological interventions to support their students in overcoming cognitive difficulties, enhancing proof techniques, and properly understanding mathematical proofs. For that reason, our interactive HELP SYSTEM should also provide tertiary students with some strategies to bridge the cognitive and structural gaps between the different phases of the proving process such as conjecture and argumentation, argumentation and proof. At the end of the experimental teaching, we evaluated the effects of the interactive HELP SYSTEM on the improvement of proving levels and the development of geometric thinking. In particular, we also studied the influence of dynamic visual thinking on enhancing students' geometrical intuition and revealing geometric invariants. Furthermore, we investigated the discussion among students while they used the interactive HELP SYSTEM to support proof-related problems. Through group discussion, we analyzed the students' structure of argumentation and examined the role of abduction during the process of realizing geometric invariants and writing a formal proof. The relationship between the students' level of proving and level of realizing geometric invariants was also taken into consideration aimed at illuminating the impact of realizing geometric invariants in the proving process.

1.2. PURPOSE OF THE RESEARCH

Mathematics educators have shown increasing interest in improving students' levels of proving in mathematics classroom. Proving activities exist in a variety of contexts in mathematics and proof construction involves some techniques where conjectures arise through the combination of exploration, argumentation, and validation. The main purpose of this research is to design a methodological model within a dynamic geometry environment. This model was utilized to design the interactive HELP SYSTEM and investigate cognitive processes while students are constructing proofs. In order to employ this orientation in the period of experimental teaching, the specific purposes of this research are:

a) to classify student's proving level in transformational geometry;
b) to develop an interactive HELP SYSTEM in which seven developmental phases of the proving process are contained with respect to the student's proving level;
c) to investigate the effect of using the interactive HELP SYSTEM on different phases of the proving process;
d) to provide a deeper understanding of the development of the proving process within a dynamic geometry environment;
e) to investigate students' behaviors and cognitive difficulties in constructing a formal proof;
f) to use TOULMIN basic model of argumentation which represents the structural gap between abductive argumentation and deductive proof;
g) to study the methods of constructing cognitive unity in the process of validating conjectures;
h) to classify students' levels of realizing geometric invariants within a dynamic geometry environment;
i) to examine the role of abduction in realizing geometric invariants and generating the ideas of proofs;
j) to examine the concept of dynamic visual thinking and its role in the proving process.

At the tertiary level, proofs involve understanding and the use of formal definitions in combination with previously established theorems. Proofs also tend to be longer, more complex, and more rigorous than those at earlier educational levels (SELDEN, 2010). Therefore, teachers should increasingly use the students' original proof constructions as a means of assessing their understanding of the proving process, especially the transition from realizing geometric invariants and formulating conjectures to writing a deductive proof. Therefore, in this research, we have designed task-based activities to encourage students to produce arguments and write formal proofs on a piece of paper. Throughout this research, we provided students with opportunities to think visually and dynamically, to look for geometric invariants, to formulate conjectures, to produce arguments, and to pose and answer the explorative questions on their own. A dynamic geometry environment, such as GeoGebra (HOHENWARTER & JONES, 2007), can serve as a context for realizing geometric invariants, formulating conjectures about geometric objects, and consequently lead to proof-generating situations. In particular, the dragging mode can play the role of a mediator in the transition from argumentation to proof. We also utilized the concept of cognitive unity (e.g. BOERO et al., 1996) to reveal students' difficulties in bridging the gap between conjecture and argumentation, also the gap between argumentation and proof.

1.3. RESEARCH QUESTIONS

Proofs and proving activities are important in mathematics education. However, there have been some difficulties in the teaching and learning of proofs in schools. Therefore, mathematics educators have carried out lots of research aimed at supporting students in overcoming these difficulties. In our research, we focused on the difficulties of producing valid arguments and writing a formal proof. Some researchers see argumentation[1] and mathematical proof[2] as parts of a continuum, whereas the others see this relation as a dichotomy (see e.g. BALACHEFF, 1991). The aim of argumentation is to obtain an agreement of the partner of the interaction, but not to necessarily establish the truth of some statement; whereas the aim of a mathematical proof is to fit the requirement for the use of knowledge taken from a body of knowledge that a

[1] Argumentation is a reasoned discourse that is not necessarily deductive but uses arguments of plausibility.

[2] Mathematical proof is a chain of well-organized deductive inferences that uses arguments of necessity.

community of mathematicians agrees upon. Therefore, argumentation and mathematical proofs are not of the same nature. As a result, understanding the relationship between them may help students manage effectively their produced arguments in writing proofs. Therefore, in the period of experimental teaching, we took some challenges in constructing a formal proof into consideration. For instance, HEALY & HOYLES (2000) claimed that there exist students' divergent perceptions of what constitutes as a proof. For that reason, we designed the interactive HELP SYSTEM in a dynamic geometry environment to provide students with some elements which are used to construct proofs. Moreover, a dynamic geometry environment has an enormous potential to encourage exploration, explanation, argumentation, and proof because this environment makes it easier to pose and test conjectures (HANNA, 2000). These activities also make a noteworthy contribution to the connection between explorations with deductive reasoning. On the basis of the specific purpose of our research, the interactive HELP SYSTEM was utilized to conduct experimental teaching and answer the following leading research questions:

- **Question 1.** What is the role of the interactive HELP SYSTEM in constructing a formal proof?
- **Question 2.** Does the interactive HELP SYSTEM improve the student's level of proving?
- **Question 3.** What are a student's difficulties in constructing a formal proof at the tertiary level?
- **Question 4.** How can the gap be bridged between conjecture and proof in the proving process?
- **Question 5.** What is the role of abduction in the proving process within a dynamic geometry environment?
- **Question 6.** What is the relationship between a student's level of realizing geometric invariants and level of proving?
- **Question 7.** What is the role of dynamic visual thinking in realizing geometric invariants?

In order to clarify and answer these research questions, we collected the empirical data during the period of experimental teaching and then analyzed all of these materials by using qualitative and quantitative methods such as observations, questionnaires, semi-structured interviews, and hypotheses testing. The main results of these analyses will be reported in chapter three. The research questions will be clearly answered throughout the empirical data analyses.

1.4. SIGNIFICANCE OF THE RESEARCH

The results of this research have both theoretical and practical significance. At the theoretical level, the methodological model can be used to *improve students' level of proving* and *develop their dynamic visual thinking*. It also provides students with a strategy for solving proof-related problems and illuminates the basic ideas for proofs. At the practical level, some task-based activities were designed to support students in producing arguments and constructing a deductive proof. In order to analyze student's arguments, TOULMIN model (TOULMIN, 1958) was used to represent the abductive structure in realizing geometric invariants and producing supportive arguments for writing formal proofs. This research also lays a theoretical framework for understanding the development of the proving process within a dynamic geometry environment in a non-traditional (computer-supported) teaching context. Furthermore, it can change the goal of teaching proofs in the classroom with explorative group-based activities such as experimenting, visualizing, measuring, reasoning, and writing a formal proof. The outcomes of this research may amplify the scholarly discussion on strategic intervention while students are proving and can inspire enthusiasm for exploring new knowledge in mathematics students. This research also classifies a student's level of proving and level of realizing geometric invariants. Based on this classification, mathematics teachers can easily evaluate students' proving abilities and geometric thinking before making an appropriate teaching plan in geometry courses.

The prospective mathematics teachers may also enormously benefit from the methodological model and some cognitive strategies gained throughout this research. It broadens the students' scope of proof understanding, and provides some detailed explanations on how they can overcome the cognitive difficulties in constructing a formal proof at the tertiary level. The outcomes of the research have also an ample amount of evidence to suggest that existing knowledge and beliefs of authoritarian (like textbooks, teachers) can hinder students' motivation in validating conjectures which lead to proof-generating situations. However, the 'observed facts' in a dynamic geometry environment and reasonable suggestions in the interactive HELP SYSTEM may stimulate students to find data and evidence for verification. This behavior lays the foundation for possible connections between conjecture and proof in the proving process. In the secondary level mathematical textbooks in some countries, proofs were dogmatically presented and students were not given the opportunity to appreciate the proving process, especially the development of logical reasoning. Thus, in order to enhance the quality of teaching this topic in secondary schools, proofs should be taught as a process of guided reinvention, where students can perceive actively and recapitulate the learning process of the mathematicians. That is also a strategy for prospective mathematics teachers to understand and then support their students in solving proof-related tasks in the secondary school classroom.

1.5. STRUCTURE OF THE RESEARCH

There are four chapters in the dissertation. Chapter one states the problem, the purpose, the significance, and the structure of the research. It also poses some leading research questions aimed at looking for a means of understanding the proving process within a dynamic geometry environment. These research questions were used to design an experimental teaching plan and to collect empirical data for qualitative and quantitative analyses.

Chapter two provides a literature review of the recent research on proof and proving within a dynamic geometry environment from different perspectives. A historical perspective presents the functions and the role of proofs in teaching and learning mathematics. The cognitive perspective delineates some difficulties in realizing invariants, validating conjectures, and writing a formal proof. The pedagogical perspective introduces some strategies and methods using geometric transformation within the GeoGebra environment to formulate and validate conjectures. These strategies are represented in the methodological model as well as in the interactive HELP SYSTEM during the proving process. This chapter also presents the role of abduction in realizing geometric invariants and writing a formal proof. A TOULMIN basic model of argumentation was also introduced to analyze a structural gap between argumentation and proof. This chapter proposes four basic conditions for understanding the development of the proving process within a dynamic geometry environment. These conditions are the criteria in determining whether tertiary students understand the proving process. This chapter also addresses some cognitive difficulties in teaching proofs using geometric transformations and provides some suggestions for learning and teaching proofs at the tertiary level.

Chapter three determines and describes comprehensively methodologies of analyzing the collected data from the period of experimental teaching, which consist of typical methods usually used in educational researches such as observations, questionnaires, semi-structured interviews, and hypotheses testing. This chapter also presents comprehensive qualitative and quantitative analyses of the leading research questions in chapter one. The methodological model and its role in assisting students constructing formal proofs were also thoroughly analyzed in this chapter.

Chapter four draws the final conclusions from a thorough analysis in chapter three. The main part of this chapter is to present the findings of the research which give implications for prospective mathematics teachers in Vietnam and recommend some issues for further research. This chapter deals also with the discussion of the preliminary findings in which the development of the proving process within a dynamic geometry environment has been brought to light.

1.6. SUMMARY

There has been growing concern about teaching mathematical proof as a process, not as a product with the support of new technologies such as dynamic geometry software. Thus the findings of this research can shed light on how students learn mathematical proofs at the tertiary level and secondary level. Indeed, a better understanding of the development of the proving process may assist prospective mathematics teachers in identifying ways to improve their problem-solving abilities. The research has also revealed that a suitable interference of proving activities will encourage students to make conjectures and produce arguments on their own. Results from the qualitative analysis show students' positive attitudes towards the interactive HELP SYSTEM and its far-reaching effect on the processes of exploration and validation. Furthermore, this research has addressed the problem of applying abductive argumentation to support students in shifting between *ascending control* and *descending control* (see e.g. GALLO, 1994) in order to write a deductive proof. It also represents the role of dynamic visual thinking in realizing geometric invariants and generating the ideas of a proof. In general, this research has revealed some necessary aspects such as invariance, conjecture, cognitive unity, argumentation, and dynamic visual thinking in relation to formal proofs in order to support tertiary students in understanding the development of the proving process.

Chapter 2

LITERATURE REVIEW

2.1. THE FUNCTIONS OF PROOFS

In mathematics, proofs contain logical chains of inference that follow rules of deduction by using formal notation, syntax, and laws of manipulation. They also provide a long-term link with the discipline of proofs shared by mathematicians and play an important role both in mathematics and mathematics education. Therefore, it is useful to consider the vast range of functions that proofs play in the mathematical practice. DE VILLIERS (1990; 2003; 2007) has proposed and extended the different functions of proofs as follows: *verification* (concerned with the truth of a statement), *explanation* (providing insight into why it is true), *systematization* (the organization of various results into a deductive system of axioms, major concepts, and theorems), *discovery* (the discovery or invention of new results), *communication* (the transmission of mathematical knowledge), and *intellectual challenge* (the self-realization/fulfillment derived from constructing a proof). These functions of proofs can potentially be utilized in the mathematics classroom to make proofs a more meaningful activity. They also provide students with a chance to communicate mathematical knowledge and systematize mathematical statements into an axiomatic system (e.g. KNUTH, 2002a; 2002b). Traditionally from a strict logical viewpoint, the function of proofs has been seen almost exclusively in terms of its verification function, but in mathematics education the verification function is far less important than other functions. For instance, HERSH (1993) and HANNA & JAHNKE (1993; 1996) claimed that the functions of proofs should promote students' understanding by explaining. From this point of view, they differentiated between two kinds of proofs: *proofs that convince* and *proofs that explain*. They also distinguished the differences between the functions of proofs in mathematics (justification and verification) and those in mathematics education (exploration and explanation). Therefore, some functions of proofs (especially, verification and explanation) should be taught when students start facing the concept of proofs because these functions can promote a way of thinking (e.g. HANNA, 2000; HANNA & BARBEAU, 2008). Proofs also provided students with an opportunity to shift from being dependent upon a teacher's authority to trusting the authority of mathematical reasoning. This is an important transformation in students' thinking, such as ways of thinking and perceiving, ways of reasoning, etc.

In this research, we consider proofs as the final product of the proving process. Hence, this process has been used to explicate the functions of proofs in mathematics education. Specifically, HAREL & SOWDER (1998) defined proving as a process that is employed by an individual to remove or create doubts about the truth of an observation. It contains two sub-processes: *ascertaining* and *persuading*. Ascertaining is a process that an individual employs to remove her or his own doubts (refers to proofs that convince). Persuading is a process that an individual employs to remove others' doubts about the truth of an observation (refers to proofs that explain). Therefore, understanding the development of the proving process contributes in gaining insight into the understanding of the invention of mathematical ideas and the nature of proofs. That is the reason why tertiary students should learn how to write, read, understand, and construct proofs, even though the functions of proofs are not fulfilled in the teaching of proofs in schools and remain hidden in some mathematics textbooks (see e.g. HANNA, 2000; DE VILLIERS, 2003).

2.2. THE TEACHING OF PROOFS

At the secondary school level, numerous researchers have recently approached proving with different forms and teaching methods (ICMI[3] 19). The reasons for teaching proofs are to develop logical and intuitive abilities; to recognize logical concepts and rules in argumentative and proving contexts; to prove mathematical properties and theorems; and to make sense of formal mathematical symbols and notations. NCTM[4] (2000) has also laid stress on proofs and proving in mathematics curriculum: "Reasoning and proof are not special activities reserved for special times or special

[3] ICMI (International Commission on Mathematical Instruction) study concentrates on an issue of prominent current interest in mathematics education. Based on an international conference, it is directed towards the preparation of a published volume intended to promote large discussion and research action at the international, regional or institutional level.

[4] NCTM (The National Council of Teachers of Mathematics) was founded in 1920. It has supported teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research. In 2000, NCTM released the new *Principles and Standards for School Mathematics*.

topics in the curriculum but should be a natural, ongoing part of classroom discussion, no matter what topic is being studied". Therefore, proofs at the secondary school level should be taught in a way that develops students' thinking and reasoning. Particularly, in geometry, teachers should offer their students a chance to 'see' a dynamic object, to recognize relationships between objects, and to construct a logically connected chain of reasoning. For that reason, a reformation of mathematics curricula has recently paid attention to the teaching of proofs and its relation to other forms of explanation, illustration, reasoning, and justification. For instance, MOGETTA, OLIVERO & JONES (1999) designed some classroom tasks that stimulate the proving process, like activating mental processes involved in the 'dynamic' exploration of a problem, provide opportunities for students to explain why they obtain a particular result, and stimulate the different kinds of reasoning processes associated with the transition from argumentation to proof. In geometry, JONES & RODD (2001) implemented experimental research on the teaching and learning of geometric proofs. They have revealed that proofs and proving can provide a way of thinking that deepens mathematical understanding and broadens the nature of human reasoning. Additionally, it is also worthwhile to examine the design of textbooks with the intentions of uncovering the 'opportunities for proofs'. For example, the exercises predominately in mathematics textbooks in England, France, and Germany made few connections between the practiced concepts and students were encouraged to explore, question, and autonomize (e.g. PEPIN & HAGGARTY, 2001); contemporary textbooks in Scotland and Japan also provided opportunities for the development of students' deductive reasoning through teaching proofs using various approaches (see e.g. FUJITA & JONES, 2003). However, there were some mathematics textbooks that did not concentrate on the role of proofs and reasoning. For instance, the exercises in a selection of ongoing mathematics textbooks in Australia have a low procedural complexity with considerable repetition and they are also absent of deductive reasoning (VINCENT & STACEY, 2008). Therefore, it is necessary to reform mathematics curriculum and textbooks with a rich opportunity for proving so that students can learn mathematics through more meaningful activities.

At the tertiary level, a proof involves creativity and insight as well as knowledge understanding and using formal definitions. The teaching of a formal proof at secondary school was also attempted but was not successful (see CLEMENTS & BATTISTA, 1992). However, tertiary students are able to understand the meaning of a proving activity and the approach to formal proof. Proofs at the tertiary level tend to be longer, more complex, and more rigorous than those at earlier educational levels. Hence, learning to understand proofs is actually a daunting task (SELDEN, 2010). Moreover, the transition from the experimental and intuitive habits of school mathematical reasoning to the requirements of a formal proof is also not smooth. This obstacle originates from the divergence of perceptions of what constitutes as a proof that students made before entering university (e.g. HEALY & HOYLES, 2000; RECIO & GODINO, 2001). Therefore, to understand the tertiary students' writing of a formal proof, teachers should use their original proof constructions as a means of assessing their understanding as well as arrange individual interviews with their students about the proof writing. Other problems involving the teaching of proofs include tertiary students sometimes mistakenly thinking proofs are constructed from the "top down" because they usually have seen this structure presented to them in a university lecture. Students usually transit to proofs by evaluating and judging the correctness of a conjecture (SELDEN & SELDEN, 2003). Consequently, the normative way to construct proofs is to always give a reason for each statement in a two-column proof before continuing with the next rows in the process of writing proofs (WEISS, HERBST & CHEN, 2009). Moreover, the traditional definition-theorem-proof style of lecture presentation may not convey the content in the most efficient way because it does not enable students to gain more insight into proofs. Some students do not even correctly differentiate the meaning of the required words in each proof-related problem such as: "explain", "demonstrate", "show", "justify", and "prove". In particular, at the tertiary level, students are required to move flexibly between representations (e.g., a function can be given symbolically; it can also be described as a graph, as a table or even as an element of an algebraic structure). In other words, students often use multiple representations in order to imagine and describe a mathematical object. This is also an indication of the richness of students' understanding of a concept (e.g. EVEN, 1998). Furthermore, tertiary students need to approach the axiomatic method so that they can interpret some abstract mathematical concepts (such as algebraic structures, vector space, different geometries structure, topological space, measurable space, etc) and develop their own advanced mathematical thinking (see e.g. TALL, 2002).

In mathematics education, it is important to investigate how the teaching of proofs in school and at the tertiary level supports students in perceiving mathematical knowledge and developing mathematical thinking. In this research, we concentrated on the students' understanding of the development of the proving process and proposed some basic conditions for this understanding as well. Mathematics teachers should provide their students with opportunities, both in a traditional classroom and online courses that aim to realize these conditions for approaching formal proofs.

2.3. BASIC CONDITIONS FOR UNDERSTANDING THE DEVELOPMENT OF THE PROVING PROCESS

The proving process is a sequence of mental and physical actions, such as writing or thinking a line of a proof, drawing or visualizing a diagram, reflecting on the results of some earlier actions, or trying to remember an example. This process in mathematics education is more than validation (e.g. DE VILLIERS, 1990; HANNA & JAHNKE, 1996). Therefore, understanding the development of the proving process is very important, but it is also elusive. For instance, the process of proving a theorem may take years and include various approaches but the final product is written in the form of formal proofs. For that reason, it is not sufficient to show only the final product while learning proofs. Students should be focused on the proving process itself and realize the different phases in the transition from formulating conjectures to writing a formal proof. Moreover, students have to determine the conceptual territory before a proof (see e.g. EDWARDS, 1997) such as conjecturing, verification, exploration, and explanation. These activities constitute necessary components that precede a formal proof. It also refers to a "space" of exploring for proofs, such as a way of thinking, communicating, producing arguments, and acting that support students in looking for mathematical certainty. In other words, it encompasses a range of activities, which students may be engaged in, that are aimed at constructing a formal proof. On the basis of this territory, we proposed four basic conditions in determining whether students understand the development of the proving process:

- *Realizing the geometric invariants for generating ideas for proofs.* This is an important phase in the development of the proving process. Realizing geometric invariants supports students in getting more data for proving and searching for the ideas of proofs by using geometric transformations.
- *Constructing a cognitive unity in the transition from conjecture to proof.* This construction produces arguments for validating conjectures and writing proofs. Therefore, students need to know how to construct an unbroken cognitive unity in the transition from conjecture to proof so that they can collect and select some valuable arguments for writing formal proofs.
- *Understanding the relationship between argumentation and proof.* Students have difficulties in constructing proofs and fail in writing proofs because they do not determine the continuity and gap between argumentation and proof. So understanding this relationship helps students bridge the gap and effectively utilize the continuity between them. This understanding includes the ability of using different kinds of inferences during the proving process such as deduction, induction, and abduction.
- *Organizing arguments in order to write a formal proof.* This is one of the most difficult phases in the proving process because students have to use formal language, symbols, and notations. They also need to organize produced arguments as a chain of logical arguments to form a formal proof.

Throughout this dissertation some basic conditions for understanding the development of the proving process will be clarified and thoroughly analyzed under mathematics teacher's perspective.

2.3.1. Realizing geometric invariants for generating ideas for proofs

Invariance is a key concept in geometry, especially in modern geometry that is taught at the tertiary level. It is also used in other areas of mathematics such as topology and algebra. In this research, we define the *geometric invariant* of a geometric transformation as follows: "A geometric invariant is a property or relation of a class of mathematical objects that does not change under a geometric transformation". For example, the sum of the internal angles of a planar triangle is an invariant as the triangle changes shape. In geometry, there are transformations that keep shapes the same size, known as *isometries*. The characteristic of an isometry is distance-preserving. Invariant under an isometry produces an equivalent relation 'is congruent to'. It means that a geometric object is always congruent to its image under an isometry such as reflection, translation, rotation, glide-reflection, etc. In contrast, there are some transformations which preserve shape but not necessarily size, known as *similarities* (dilation, for instance). Invariance under a similarity also produces an equivalent relation 'is similar to' (in the mathematical sense). A figure and its image under a similarity are similar figures. Therefore, the invariance is a crucial idea in the process of classifying mathematical objects and looking for ideas for proofs.

One of the difficulties with geometric proving in school mathematics is due to the fact that some geometric invariants are easily recognizable for mathematics teachers but are not so intuitive for students because intuition depends on previous experience (e.g. TALL, 2002). Moreover, one of the breakthroughs in modern mathematics was to characterize geometric transformations in terms of what they leave invariant, rather than thinking about what they change (see e.g. JOHNSTON-WILDER & MASON, 2005). Consequently, teachers should provide students with a rich opportunity to realize invariants of a geometric transformation, especially at the upper secondary level and tertiary level. The properties are preserved under both isometry and similarity such as collinearity, parallelism, perpendicularity, concurrence, measurement of angles, etc. Parallelism, for instance, is called an invariant that is preserved under a transformation if two lines are parallel then their images are also parallel. A collection[5] of geometric transformation is a group under the operation of composition of transformation, called *group of transformations.* This conception was proposed in the Erlangen Program by KLEIN (1872). The Erlangen program aimed at classifying geometries by their underlying symmetry groups and studying the properties of a space whose invariants are under a given group of transformation. He claimed that the essential properties of a given geometry could be represented by the group of transformations that preserve those properties. The extended notion of invariant developed by KLEIN has become one of

[5] This set of transformation under the operation of composition satisfies three axioms: a) the composition of transformations is closed and associative; b) there is an identity element (identity transformation); c) each geometric transformation has an inverse because it is always a bijection.

the fundamental concepts in mathematical thinking: “to every group of transformations there corresponds a kind of geometry or theory of invariants, dealing with those properties of geometrical or analytical configurations, which are unaltered by the group”.

In short, students need to realize invariants before determining the geometric transformations that are used as the ideas for proofs. Therefore, students must concentrate on some geometric invariants of a specific group of transformation with the support of a dynamic geometry environment. The following examples describe how students realize geometric invariants by using a dynamic geometry software and the role of these invariants in the proving process (see e.g. NGUYEN, 2011).

> **Example 2.1. (School Problem)** *People living in the neighborhood of town A and working at company B are to drive their children to school on their way to work. Where on highway l should they build school C in order to minimize their driving?*

In this problem, firstly, students used the GeoGebra software to construct lots of drawings with different corresponding positions for point *A* and for point *B*. Secondly, they dragged point *C* until the sum (*AC* + *CB*) is minimal. Thirdly, students made observation and made conjectures about invariants by generating different cases as the figures below. After that, students realized that if the sum (*AC* + *CB*) is minimal then two angles at point *C* are equal. Finally, they made a guess about a geometric transformation which preserves angle measurement. The existence of a fixed line *l* (the highway) leads students to argue that the realized invariant in this case is an invariant of a line reflection and also suggested them to use a line reflection as an idea of proof.

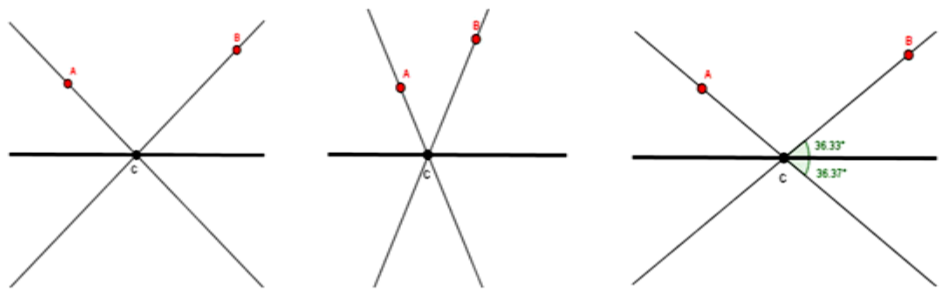


Figure 2.1: *Realizing geometric invariants for generating ideas for proof*

In fact, different students might discern different aspects of the same geometrical figure. Some of them focus on the shapes (two equal angles, two equal sides, two congruent shapes, and two similar shapes), some students pay attention to perceivable properties (two lines are parallel, three points are collinear, three lines are concurrent, two lines are perpendicular, etc) and the others tend to use properties to reason (an isosceles triangle has two equal sides in length, so it has two equal angles, for instance). Therefore, teachers should offer students various strategies to 'see' geometric invariants both in traditional (the use of paper and pencil) and non-traditional (the use of a computer) environment.

Example 2.2. (One-Bridge Problem) *A river has straight parallel sides and cities A and B lie on opposite sides of the river. Where should we build a bridge in order to minimize the travelling distance between A and B (a bridge, of course, must be perpendicular to the sides of the river)?*

In this problem, students used the GeoGebra software to model two cities A, B and construct a movable point G on the upper line, representing the bank of the river. Then the students moved point G until the sum ($AG + GH + HB$) is minimal. They realized that if the sum ($AG + GH + HB$) is minimal then two straight lines AG and HB are parallel. Finally, they made a conjecture about a geometric transformation which preserves parallelism. The existence of a fixed vector $\overrightarrow{GH}$ suggested that the students realized the invariant of a translation and they argued that the straight line HB is an image of the straight line AG under a translation in the vector $\overrightarrow{GH}$ direction. Therefore, it can also be said that the revealed invariant makes a contribution by generating another idea for a proof.

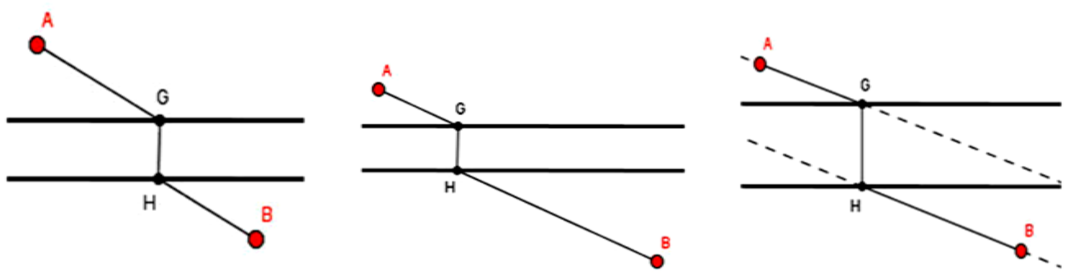


Figure 2.2: *Realizing invariants for determining geometric transformations*

In this problem, the students realized the first invariant (sub-invariant) “two straight lines AG and HB are parallel when the sum ($AG + GH + HB$) is minimal”. Based on this sub-invariant, they continued to realize the key invariant, “the straight line HB is an image of the straight line AG under a translation in the vector $\overrightarrow{GH}$ direction”. In the interview protocol (see Section 3.2.3), the students said that they created a lot of ‘dynamic’ mental pictures in their minds in order to realize this geometric transformation. Their final work is to select plausible arguments and combine these arguments into a logical chain of reasoning for writing a deductive proof.

2.3.2. Constructing a cognitive unity in the transition from conjecture to proof

In the development of the proving process, the transition from argumentation to proof was taken into special consideration. BOERO, GARUTI & MARIOTTI (1996) underlined that the proving process should be started with argumentative activities and the validation of produced arguments for the conjecture. Therefore, it is necessary to consider the connection between conjecturing and proving in the process of approaching proofs. For that reason, the concept of “cognitive unity” was introduced with the aim of interpreting the transition from conjecture to proof and the students’ cognitive difficulties in the proving process:

> “*During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain*” (BOERO et al., 1996, p.124).

In other words, cognitive unity is a situation or phenomenon where some arguments, which are produced for the plausibility of the conjecture during the conjecture production phase, become ingredients for the construction of a proof. The following diagram shows the cognitive unity as a crucial factor in the process of constructing proofs:

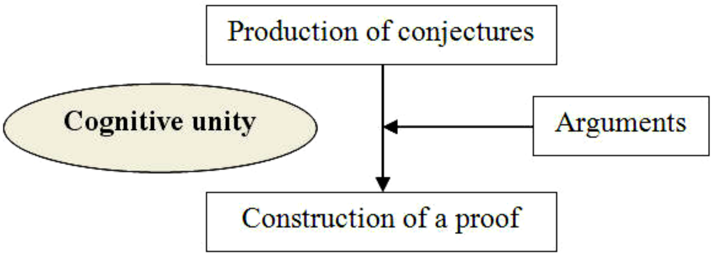


Figure 2.3: *Constructing a cognitive unity in the proving process*

In a traditional school, students often struggle with theorems in mathematics textbooks because they do not know how to construct a cognitive unity (BOERO et al., 1996). Moreover, theorems in textbooks are introduced in the form of a "prove that" kind of task. In this case the process of conjecturing is not demanded and the cognitive unity is broken. The unity can only be restored by reconstructing the following cycle: *exploring*, *producing a conjecture*, *coming back to the exploration*, and *reorganizing it into a formal proof*. The following examples accompanied with discussions from a group of students might illuminate whether a cognitive unity exists or is broken.

Example 2.3. (Parallelogram Problem) *Let ABCD be a parallelogram. The bisectors of angles ∠A, ∠B, ∠C, and ∠D intersect each other forming a quadrilateral MNPQ. What are special characteristics of this quadrilateral?*

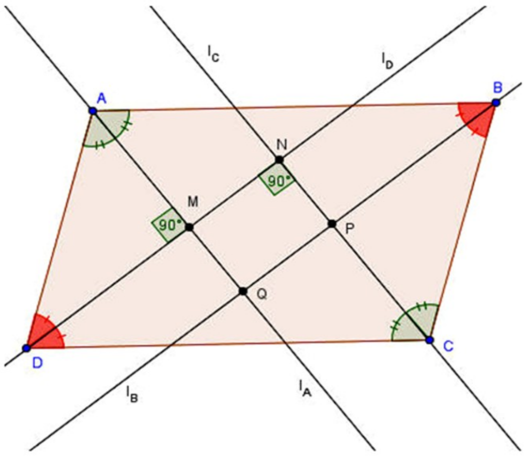


Figure 2.4: *A cognitive unity in the parallelogram problem*

The following dialogue was extracted from students' audio recording in the group discussion during the process of producing arguments:

♣12. Student 1: I think this quadrilateral is a rectangle [*posing a conjecture*].

♣14. Student 2: It means that we have to prove this quadrilateral has four 90-degree angles [*seeking for the way to validate the conjecture*].

♣15. Student 3: That's right! Suppose that we prove $\angle AMD = 90^0$. This requirement is equivalent to proving the sum $\angle DAM + \angle ADM = 90^0$ [*producing arguments*].

♣26. Student 2: In a parallelogram we have $\angle BAD + \angle CDA = 180^0$. Moreover, according to supposition, we derive that $\angle DAM + \angle ADM = (\angle BAD + \angle CDA)/2 = 180^0/2 = 90^0$ [*producing arguments*].

Some produced statements support students in producing arguments like "a rectangle has four 90-degree angles", "this requirement is equivalent to proving the sum $\angle DAM + \angle ADM = 90^0$", "in a parallelogram we have $\angle BAD + \angle CDA = 180^0$", "we derive that $\angle DAM + \angle ADM = (\angle BAD + \angle CDA)/2 = 180^0/2 = 90^0$", etc. Therefore, in this case, we interpret this phenomenon as the existence of a cognitive unity. In general, the concept of cognitive unity, which addresses the link between spontaneous arguments and mathematically acceptable arguments, may provide a powerful tool for understanding different phases of the proving process. We speak of a broken cognitive unity if:

- *Changing mathematical frame or external representation.* Students formulate a conjecture within a synthetic geometry frame and validate it within an analytic geometry frame (using algebraic language). It means that most arguments produced in the conjecturing phase are not recyclable (unavailable in the case of changing external representation) in the proving phase.

Example 2.4. (Right Triangle Problem) *Let ABC be a right triangle at point A. Let M be the midpoint of segment BC. Compare the lengths of three segments AM, CM, and BM.*

In this example, students argued that the circle with center M and radius MB goes through point A, therefore $MA = MB = MC$ (radius of the circle). However, in order to prove this conjecture, students used algebraic language and did not refer to the circle in the proving process at all. They used the following arguments in order to write a formal proof:

"Suppose that point $A = (0, 0)$, point $B = (b, 0)$, and point $C = (0, c)$. According to the formula of calculating the distance between two points, we have the length of segment $BC = \sqrt{b^2 + c^2}$ $\Rightarrow MB = MC = \frac{1}{2}BC = \frac{1}{2}\sqrt{b^2 + c^2}$. Since M is the midpoint of segment BC then we derived that $M = (\frac{b}{2}, \frac{c}{2})$ and then we obtained the length of the segment $MA = \sqrt{\frac{b^2+c^2}{4}} = \frac{1}{2}\sqrt{b^2 + c^2} = MB = MC$".

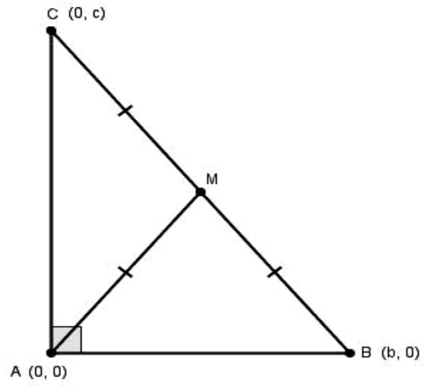


Figure 2.5: *A cognitive unity is broken (case 1)*

This proof did not use the arguments from production of initial conjectures that related to the circle. Therefore, in this case, we see the cognitive unity as broken.

- *Changing explorative strategies and heuristics.* Produced arguments which are relevant in a given exploration during the conjecturing phase may become useless, or even forgotten, in another kind of exploration:

Example 2.5. (Hexagon Problem) *Let ABC be a triangle. We take six points on the sides of the triangle $A_1, A_2 \in BC$; $B_1, B_2 \in CA$; $C_1, C_2 \in AB$ such that $BA_1 = A_1A_2 = A_2C$, $CB_1 = B_1B_2 = B_2A$, $AC_1 = C_1C_2 = C_2B$. Six straight lines AA_1, AA_2, BB_1, BB_2, CC_1, and CC_2 intersect each other forming a hexagon MNPQRS. What is characteristic of three diagonals of this hexagon?*

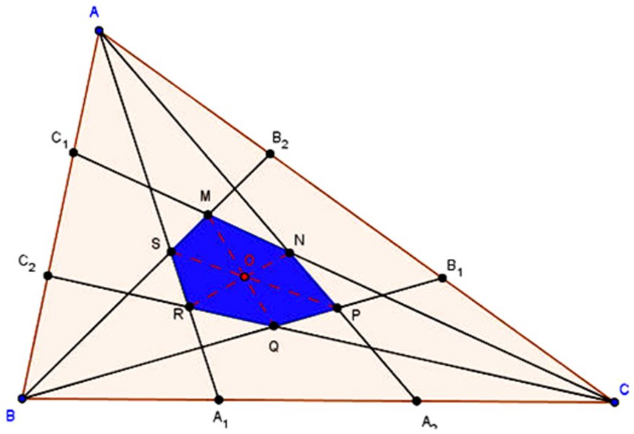


Figure 2.6: *A cognitive unity is broken (case 2)*

♣27. Student 2: I think three diagonals of this hexagon are concurrent [*posing a conjecture*].

♣31. Student 3: I realize that no matter how triangle *ABC* changes, three diagonals of the hexagon are still concurrent.

♣38. Student 1: Concurrency is an affine property (concurrency is preserved under an affine transformation). Therefore, our hypothesis is also true in the case of an equilateral triangle [*producing arguments*].

♣44. Student 2: It means that we only need to prove this property for the case of an equilateral triangle and then that result implies the initial requirement [*producing arguments*].

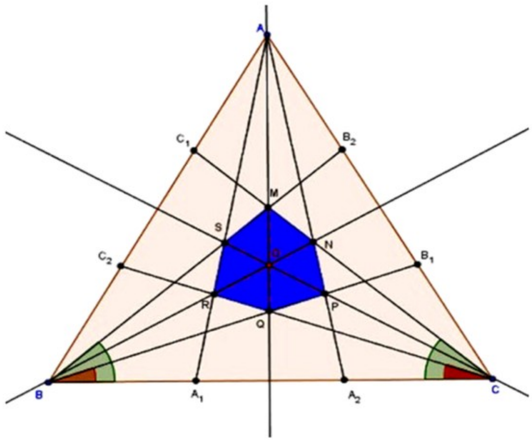


Figure 2.7: *A cognitive unity is broken when changing explorative strategy*

According to above dialogue, to confirm that three diagonals are concurrent, students must show that these diagonals coincide with three perpendicular bisectors of triangle *ABC*. Therefore, we see the cognitive unity as broken in this case because produced arguments support only in revealing the idea of proofs, not in writing formal proofs.

To sum up, cognitive unity concerns the arguments that create the continuity in the transition from conjecturing to proving. During this transition, an argumentation activity is developed for validating a conjecture. When the statement expressing the conjecture is made true in a mathematical theory, it is said that a proof is produced. Therefore, cognitive unity is also a concept that can clarify the difference between argumentation activity in the conjecturing phase and the proving phase, which also supports students in understanding the development of the proving process.

2.3.3. Understanding the relationship between argumentation and proof

An argument is defined as a sequence of mathematical statements with the chief aim of convincing, whereas an argumentation[6] consists of one or more logically connected arguments. KRUMMHEUER (1995) views an argument as either a specific sub-structure within a complex argumentation or the outcome of an argumentation. In this dissertation, we consider *argumentation as a process* and *argument as a product*. During the proving process, an argumentation activity is developed by producing arguments for the validation of a conjecture. The connective arguments produced in a conjecture become proofs if they are valid in a mathematical theory. It is also said that proofs are particular arguments but arguments are not necessarily proofs. Moreover, argumentation and mathematical proofs are not of the same nature because the aim of argumentation is to obtain the agreement of the partner of the interaction. However, the aim of a mathematical proof is to obtain the agreement of the mathematics community. To understand this complex relationship, PEDEMONTE (2001) analyzed and compared argumentation supporting a conjecture and its subsequent proof in solving practical problems in geometry. Departing from some initial results of these empirical analyses,

[6] TOULMIN (1958) differentiated three terms *argumentation*, *reasoning*, and *argument*. Argumentation refers to the whole activity of making claims, challenging them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms. Reasoning has more narrow meaning than argumentation and it is used for the central activity of presenting the reasons in support of a claim, so as to show how those reasons succeed in giving strength to the claim. An argument, in the sense of a train of reasoning, is the sequence of interlinked claims and reasons that, between them; establish the content and force of the position for which a participating in an argument shows its rationality.

PEDEMONTE (2007; 2008) concentrated on the continuity and the gap (or distance) between argumentation and proof. She considered the relationship between two points of view: the *referential system* and the *structure*. The referential system contains the representation system (the language, the heuristic, and the drawing) and the knowledge system (conceptions and theorems). The structure considers different kinds of inferences between statements (abduction, induction, and deduction).

At first glance there seems to be no natural mediator between argumentation and proof, however PEDEMONTE (2007) has showed that there is "natural" continuity between them in the referential system. Indeed, some words, arguments, drawings, theorems, etc. used in the proof have also been used in the proof-supporting argumentation. There is also a structural continuity between argumentation and proof if some deductive or abductive steps used in the argumentation are present once again in the proof. However, according to DUVAL (1991), there is also a gap between argumentation and proof even if they use very similar linguistic forms and connective propositions. He also claimed that the conclusion of a step, in a proof, serves as an input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. For that reason, the gap between proof and argumentation is not only logical but is also cognitive. PEDEMONTE (2007) also confirmed that this gap appears when argumentation is abductive and proof is deductive. Because of the existence of this "distance", even tertiary students sometimes do not cover it and produce incorrect proofs. It also means that students are not able to transform the structure of argumentation into the structure of proof. Concentrating on some elements of connection, BALACHEFF (1991) realized that the relationship between argumentation and proof is strictly connected to the relationship between conjecture and valid statement. Experimental research on cognitive unity (BOERO et al., 1996; GARUTI et al., 1998; MARIOTTI, 2000) also showed that a proof is more "accessible" to students if an argumentation activity is developed for the construction of a conjecture. Therefore, mathematics teachers should stimulate their students to formulate and validate conjectures during argumentation. Nevertheless, the teachers must also pay attention to the process of producing arguments because sometimes a conjecture could be provided without any argumentation. A conjecture can be an "observed fact", derived directly from a dynamic/static drawing, from an intuition, and the like.

Another point of view, BALACHEFF (1991) & DUVAL (1992) showed that argumentation as an obstacle to mathematical proof learning, since argumentation is used in everyday life. Therefore, students might show high degrees of ability in reasoning in social situations but they do not naturally grasp the concept of mathematical proof and deductive reasoning. For that reason, teachers must help students reason deductively and to recognize some proof-generating situations, especially with the support of a dynamic geometry software. This environment can play the role of a mediator in the transition between argumentation and proof through the dragging mode, thanks to the drawings that appeared on the screen as a result of the dragging movements. Moreover, while dragging, students switch back and forth from drawings to concepts which helps them transition from the empirical to the theoretical level (see e.g. DUVAL, 1992; MARIOTTI, 2000). In general, doing experiments, producing deductive arguments towards a proof, writing proofs based on experimental results will be helpful in bridging students' perceptual gaps between proof in mathematics and argumentation in everyday life.

2.3.4. Organizing arguments in order to write a formal proof

In order to investigate the students' work of writing proofs and explicate the cognitive development of proofs, BALACHEFF (1998) determined four levels of proofs: *naive empiricism*, *crucial experiment*, *generic example*, and *thought experiment.* Naive empiricism (level 1) consists of asserting the truth of a result after verifying several cases. Crucial experiment (level 2) is the process of verifying a proposition on an instance and validating by choosing some particular cases. Generic example (level 3) involves making explicit the reasons for the truth of an assertion by choosing representatives of its class. Thought experiment (level 4) invokes action by internalizing it and detaching itself from a particular representation.

One of the difficulties in writing a formal proof is that students do not understand the meaning of a formal proof (e.g. SOWDER & HAREL, 1998). As a result, many students request for the reason why they need to do formal proofs in the classroom. They consider doing proofs just as verifying and restating theorems or statements that are obviously true. Therefore, they have made some mistakes in writing proofs. For instance, a lot of students believed the truth of a statement on the basis of a few examples (they attained level 1 according to BALACHEFF's classification) and did not understand the function of a counterexample. This result was supported by GALBRAITH (1981). He has also contended that a single exception disproves a generalization is not accepted by some students. This misunderstanding stems from the fact that they often confuse the conditions with the conclusion of a mathematical statement. To elucidate these real situations, WEBER (2001) determined students' difficulties with proofs, especially in reading given information from diagrams. Students always find confusion and difficulties in distinguishing embedded or overlapping figures in a static/dynamic diagram. As a result, diagrams sometimes inaccurately provide the content and the relationship between objects and then mislead students in writing a proof. Moreover, students are sometimes unable to write down what they have in their minds and organize produced arguments in a logical way. This situation was also recognized by HEALY & HOYLES (1998). They argued that although students are capable of conjecturing and arguing using everyday language, and most of them realize that an empirical justification is not enough, but they still do not know how to provide a formal argument.

HANNA (1995) showed that students would realize that writing proofs are difficult when they get used to working with different patterns of arguments and with the formal structure of proofs. To interpret possible insights gained from proofs, HAREL & SOWDER (1998) mapped students' cognitive schemes of mathematical proof and offered developmental models on the concept of proofs within mathematics. The notion of a proof scheme was proposed and aimed at representing a cognitive stage in the proving process: "A person's proof scheme consists of what constitutes ascertaining and persuading for that person". It is the collection of arguments that students use to convince themselves or others of the validity of a mathematical statement. The class of a proof scheme is constituted by three main categories: *external conviction proof schemes* (justification depends on the word of an authority such as a mathematician, a teacher or a textbook, on the of the strictly argument presentation or on the symbolic form of the argument), *empirical proof schemes* (conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences), and *analytical proof schemes* (conjectures are validated by means of logical deductions). Therefore, in order to write a formal proof, students need to understand proof schemes and be flexibly in transition from one schema to another by using reasonable language, symbols, notations, and rules of logical reasoning. It means that they must know how to select and combine produced arguments (the products of constructing a cognitive unity) in order to organize/arrange these arguments forming a formal proof.

2.4. SUGGESTIONS FOR TEACHING A FORMAL PROOF

2.4.1. Using TOULMIN basic model of argumentation

TOULMIN (1958) argued that the abstract and formal criteria of mathematical logic have little applicability to the methods of assessing arguments. Therefore, he built a model to represent the structures of arguments in different fields. According to TOULMIN, in any argumentation the first step is expressed by a *claim* (an assertion, an opinion or a conjecture). The second step consists of the production of *data* supporting it. It is important to provide justification or warrant for using the concerned data as support for the data-claim relationships. The *warrant* can be expressed as a principle, a rule or a theorem. The warrant acts as a bridge which connects the data and the claim. Claim, data, and warrant form the basic structure of argumentation. Besides these basic elements, TOULMIN supplemented three auxiliary elements of argumentation; they are *qualifier*, *rebuttal*, and *backing*. Qualifier is a word or phrase that expresses the speaker's degree of force or certainty concerning the claim such as 'necessarily', 'probably', 'possible' or 'presumably'. Rebuttal is a statement that recognizes the restrictions which may legitimately be applied to the claim. Backing must be introduced when the warrant itself is not convincing enough to the readers or the listeners.

This dissertation uses the following basic model of argumentation to represent the structure of arguments (e.g. deductive and abductive structure) that are produced in the proving process. This model was utilized throughout our observation analyses in order to interpret students' reasoning and proving strategies. It consists of the following elements:

C (claim): the statement of the speaker

D (data): data justifying the claim **C**

W (warrant): the inference rule that allows data to be connected to the claim

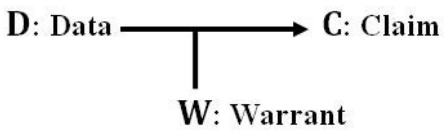
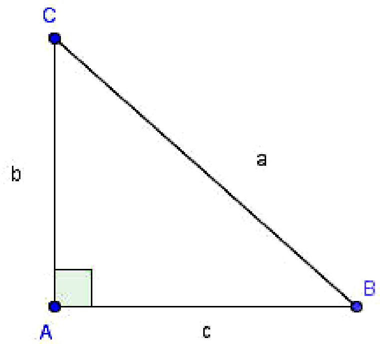


Figure 2.8: *TOULMIN basic model of argumentation*

For example, the following TOULMIN model describes the structure of argumentation which proves that ΔABC is a right triangle by using the converse of Pythagorean Theorem. Based on this model, students may know how to find the data in order to validate the claim (the given conclusion).



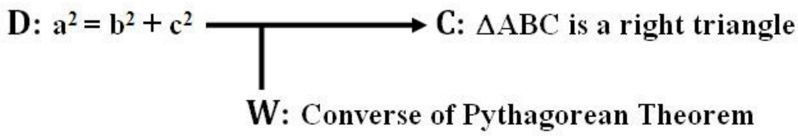


Figure 2.9: *TOULMIN model describes how to prove a right triangle*

The most important role of TOULMIN model is to analyze the relationship between argumentation and proof, especially the gap between them. In TOULMIN model, a step appears as a deductive structure because data and warrants lead to a claim. Therefore, it is useful to represent a chain of logical deduction. However, it is also a powerful tool to represent an abductive structure, which can be used to explicate the role of abduction in the proving process (PEDEMONTE & REID, 2011). In particular, the students can reverse abductive structure in order to write a deductive proof and understand logical reasoning produced in the proving process. Using this model mathematics teacher can understand the argumentation structure in students' thinking as well as the root of their proof ideas. On the basis of this understanding, teachers can also provide their students with suitable strategy during the proving process.

2.4.2. Exploring proof-related problems within a dynamic geometry environment

Learning geometry within a dynamic geometry environment involves transitions in the learning process between figures and concepts, between perceptual activity and mathematical knowledge. Typically, a geometrical problem cannot be solved while remaining only at the perceptual level of figures on the screen. Conceptual control is needed and this requires explicit knowledge. The use of the dragging function validates procedures and is the crucial instrument of mediation between figure and concepts, perception and knowledge. In particular, ARZARELLO et al (1998) presented some features of such transitions in the move from conjecturing to proofs in geometry when using a dynamic geometry software. He also reports on students' elevating the use of the language of mathematical argumentation, particularly when justifying constructions, and when students were working with the dragging mode.

LABORDE & STRÄSSER (1990) proclaimed that a dynamic geometry software provides an interactive learning environment and a helpful instrument for proving geometric theorems. As a role of an instrument, it offers students the means to articulate and test hypotheses (e.g. CONNELL, 1998) and also provide students with opportunities to discover uncertainties. These suspicious results require students to search for explanations. However, students easily believe in the truth of a conjecture by visually seeing geometric objects and their relationship on the screen (see e.g. DE VILLIERS, 1990; 2003; 2007). For instance, PANDISCIO (2002) has showed that prospective teachers believe that high school students may not see the need for proofs after using a dynamic geometry software. This conviction might also be the source of explanation as to why some educational researchers claimed that this software could lead to the "further dilution of the role of proofs in the high school geometry" (see CHAZAN, 1993). However, in some cases, despite obtaining a conviction by dragging, students still have a strong cognitive need to explain the results. Furthermore, this need was also re-confirmed by HOYLES & HEALY (1999). They indicated that using a dynamic geometry software to explore geometrical concepts could motivate students to discover geometrical properties, explain their empirical conjectures, produce arguments for a formal proof, and then improve students' proof writing abilities (e.g. MARRADES &

GUTIÉRREZ, 2000; JONES, 2000a). It also promotes the link between empirical and deductive reasoning that is supported by "what if" and "what if not" questions (HOYLES & JOHNES, 1998). Exploring a proof-related problem within a dynamic geometry environment helps students construct a figure and trace the trajectory of an object for conjecturing, gain insight into proof and intuition, discover new patterns and relationships, and graph to expose mathematics principles (see e.g. BORWEIN & BAILEY, 2004).

There is a lot of dynamic geometry software that encourage students to actively participate in the proving activities. However, the GeoGebra software[7] was chosen for our research because it is freely available on-line, and supplemented with a variety of dynamic worksheets. This software also allows students to make and test assertions and prepare for more formal proof writing (see e.g. EDWARDS & JONES, 2006; HOHENWARTER & JONES, 2007). Students can build a geometric construction and simultaneously observe how changes in a formula in the algebra window are affected by the manipulation of the construction and vice versa. Teachers can use this software to construct interactive applets on the internet to improve students' proving abilities. By participating in these explorative tasks, the student will engage in realizing geometric invariants and formulating conjectures activities. As a result, students take produced arguments for granted that support to construct formal proofs.

In short, the development of dynamic geometry software provides students with many opportunities to explore and discover mathematics concepts according to their own individual needs and pace (LABORDE et al., 2006). This dynamic environment could motivate students to explain their empirical conjectures using formal proofs and provide an opportunity to link empirical and deductive reasoning together. A dynamic geometry software can also be utilized to gain insight into a deductive argument, support experimentation, and thus lead to conviction. Therefore, the main objective of our research is to find effective ways to use a dynamic geometry software in order to give students a chance to verify, formulate conjectures, generalize, communicate, prove, and make connections between the properties of the drawings.

[7] This software can be downloaded freely at the following website: http://www.geogebra.org.

2.4.3. Using abduction during the proving process

The term "abduction" was introduced by PEIRCE (1960) to differentiate this type of reasoning from deduction and induction. Abduction is an inference which allows the construction of a claim starting from an observed fact (see e.g. PEIRCE, 1960; POLYA, 1962; MAGNANI, 2001). Deduction is an inference allowing the construction of a claim starting with some data and a rule. Induction is an inference which allows the construction of a claim generalized from some particular cases (see e.g. POLYA, 1954; FANN, 1970). In other words, abduction plays the role of generating new ideas or hypotheses; deduction functions as evaluating the hypotheses; and induction is justifying the hypothesis with empirical data (e.g. STAAT, 1993). At the tertiary level, students tend to use abduction in producing arguments and searching for ideas for proofs because the logic of abduction contributes to the conceptual understanding of a phenomenon. Therefore, our research will also focus on the role of abduction in the proving process:

> "*By using abduction as a tool for a better understanding and for the reconstruction of the generation of ideas in the mathematical classroom, the social processes of knowledge construction became analyzable. Nevertheless the consideration of the abduction cannot capture the individual cognitive processes of constructing new knowledge, but it provides insight into those processes*" (MEYER, 2008, p.50).

ECO (1983) identified three kinds of abduction (overcoded, undercoded, and creative) based on the research of PEIRCE (1878). Overcoded abduction occurs when the arguer is aware of only one rule from which the conclusion would follow. Undercoded abduction occurs when the arguer is aware of more than one selectable rule. MAGNANI (2001) combined these two kinds of abduction and called it "selective abduction". He defined selective abduction as the process of finding the right explanatory hypothesis from a given set of possible explanations. Creative abduction occurs when there is no general rule known to the arguer that would imply the conclusion. Thus, the arguer must invent a new rule. Based on the scheme of abduction as a plausible reasoning[8] (see POLYA, 1962) and the classification of abduction (ECO,

[8] As a plausible reasoning, abduction can be modeled as follows: If A then B, B true → A more credible.

1983), PEDEMONTE & REID (2011) presented three kinds of abduction in TOULMIN basic model of argumentation as follows:

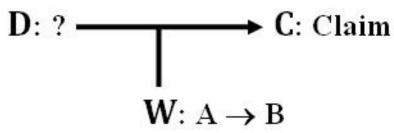


Figure 2.10: *Overcoded abduction in TOULMIN model*

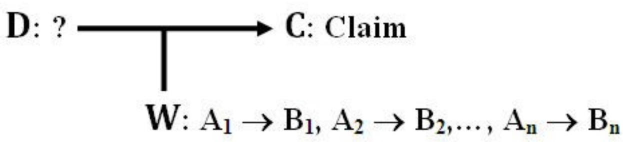


Figure 2.11: *Undercoded abduction in TOULMIN model*

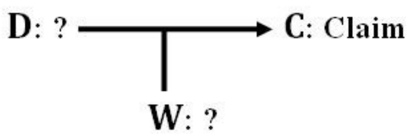


Figure 2.12: *Creative abduction in TOULMIN model*

For example, overcoded abduction in TOULMIN model is used to represent the following situation: students need to validate the claim (**C**: ΔABC is a right triangle) and they know only one rule or a theorem (**W**: Converse of Pythagorean Theorem) to justify the claim. The rest of students' work is to determine the supported data which need to be collected or found in this case ($a^2 = b^2 + c^2$ where a is the length of the hypotenuse; b and c are the lengths of the remaining two sides of that right triangle, for instance).

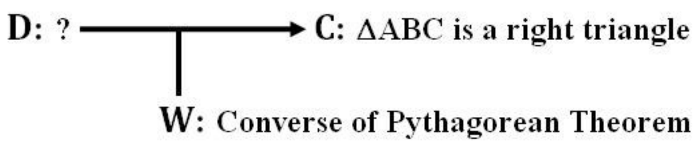


Figure 2.13: *Overcoded abduction for proving a right triangle*

In the case if students know more than one rule to follow such as converse of Pythagorean Theorem, measure of angle A is equal 90^0, and scalar product of two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is equal 0; they have to choose the most appropriate rule to solve the problem. Using the following TOULMIN model for undercoded abduction, students might know how to find suitable data in order to validate the claim (**C**).

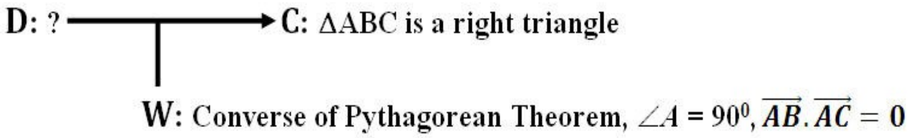
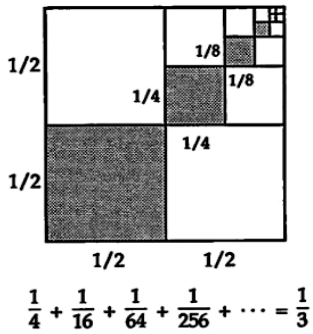


Figure 2.14: *Undercoded abduction for proving a right triangle*

In the case if students do not know any rule to follow; they have to invent a new means. For example, students are required to calculate the following infinite sum $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$ and they have not yet learned to apply the basic formula $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$.



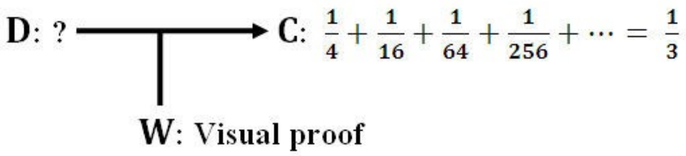


Figure 2.15: *Creative abduction for calculating the sum of infinite series*

To tackle this problem, students have to 'invent' a new method (see the unit square in Fig. 2.15 above). They might think the number $\frac{1}{4}$ is the same to $\frac{1}{4}$ of unit square, and they divide the square into four squares and repeat this process forever. They can realize that this infinite sum is equal to $\frac{1}{3}$, which is the total areas of the black unit squares and verify this result by sketching the construction of these squares (or using visual proof).

The term ‘abductive argumentation’ is also frequently used in this dissertation. This kind of argumentation originated from abduction. It has been considered as a type of ‘backwards’ reasoning and as an ‘inference to the best explanation’ because it starts from the observed facts and probes backwards into the reasons or explanations for these facts (WALTON, 2001). Abductive argumentation also supports the transition from conjecturing to proving modality (PEIRCE, 1960; ARZARELLO et al., 1998). It was used to analyze students’ interactions and proving styles while they were discovering mathematical knowledge or generating ideas of a proof. Therefore, it supports explanatory conjectures and the subsequent related proof. In geometry, proofs are normally deductive, but the discovering and conjecturing processes is often characterized by abductive argumentation. Particularly, in a dynamic geometry environment like GeoGebra, the produced data might sow the seeds of generating abductive argumentation. Its strength depends on all evidence and data which are collected by dragging, observing, measuring, conjecturing, and checking the relationship between the objects. However, abductive argumentation is sometimes an obstacle for students in constructing a deductive proof because they are not able to convert these argumentations into deductive proofs (see e.g. PEDEMONTE, 2007).

According to PEDEMONTE & REID (2011), in the case of using overcoded abduction, the structural gap between abductive argumentation and deductive proof is shorter because students must merely look for data to justify the claim; the rule and the claim are already present. In the case of undercoded abduction where several plausible rules are known, it is important to select a useful and correct rule in order to produce a proof. Using this kind of abduction, students have to take more time to choose a right rule for proving but this obstacle makes students more flexible in thinking and solving problems. Creative abduction is probably the most difficult kind of abduction to use when writing deductive proofs because a great deal of irrelevant information may be confused and then creates disorder in the student’s thought process. In this case, the student could construct an incorrect “proof” if they are not able to invent a new rule for validating the claim. However, this kind of abduction encourages students to think actively and creatively while solving a difficult problem. Another approach examining the role of abduction in the proving process, ARZARELLO et al. (1998; 2011) showed that abduction plays an essential role in the process of transitioning from ascending control[9] to descending control[10]. In other words, abduction might conduct the transition from exploring-conjecturing to validating-proving modality. In this process, the conjectures are produced and written in a logical form 'if ... then'. It means that the results of exploration are transformed into conjectures (e.g. PEIRCE, 1960; MAGNANI, 2001). Then abduction is used to explore data, find and choose a pattern, and explain plausible hypotheses, which aim to determine the methods of solving problems and producing arguments for proofs. Students are also used this kind of inference to recognize and understand the structure of writing formal proofs. Therefore, mathematics teachers should provide their students with an opportunity to use this strategy during the proving process.

2.4.4. Developing dynamic visual thinking in geometry

"Geometry takes place in a world of forms and images" (see JOHNSTON-WILDER & MASON, 2005, p.111). Therefore, it is a prosperous territory for students to improve their geometric thinking through the power of mental pictures, reasoning based on dynamic images, and communication with the others. There are four key aspects of geometric thinking: *invariance*, *language and points of view*, *reasoning*, *visualizing and representing*. In past two decades, with the appearance of dynamic geometry software, the role of visualization and analysis of visual information as steps to formal argumentation and as means of proof have been a significant trend in teaching and learning mathematics (see e.g. BLUM & KIRSH, 1991; NOSS, HEALY & HOYLES, 1997; PINTO & TALL, 2002). CHAZAN (1993) identified students' view of empirical and deductive reasoning which are associated with given diagrams or figures. Moreover, dynamic geometry software (such as GeoGebra, Geometry Cabri, Cabri 3D, Geometer's Sketchpad, Cinderella, etc) provides new possibilities for visual experiences. This visualization helps students transcend the limitations of the mind (e.g. dynamic visual imagery, short-term memory span) in geometric thinking activities (e.g. PEA, 1987; MARIOTTI, 2000; STRÄSSE, 2001). For example, when seeing a dynamic diagram,

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[9] *Ascending control* is the modality according to which the solver 'read' the figure in order to make conjectures. The stream of thought goes from the figure to the supporting theory that related to the initial situation (GALLO, 1994).

[10] *Descending control* occurs when conjectures have already been produced and the solver seeks for a validation. The solver refers to the supporting theory in order to justify what she/he has previously 'read' in the figure and validate the produced conjectures (GALLO, 1994).

students may conjure up a 'dynamic' picture and be engaged in mental activities (imagining and expressing, conjecturing, and reasoning). These activities produce mental images that express what has been visualized in the students' thoughts. These pictures also create a sense of geometric reasoning because reasoning in geometry is not only based on words or symbols, but also on drawings and students' mental images. However, they only encourage students to produce arguments and find the ideas of proofs, but not to substitute the proof itself.

According to ARNHEIM (1970), visual thinking is "an active exploration, selection, grasping of essentials, simplification, specialization, visualization, abstraction, generalization, analysis and synthesis, completion, correction, comparison, analogy, problem solving, as well as arguing, combining, separating, differentiating, representing, imagining, recognizing, putting in a certain context,…", i.e., visual thinking[11] is a powerful tool which can be applied in many situations. It involves visual imagination or visual perception of external diagrams (see e.g. GIAQUINTO, 2007; PREIßING, 2008). Visualization is also an important process supporting visual thinking in mathematics. SENECHAL (1991) proposed the definition of visualization and its relationship to the general framework of visual thinking: visualization is any process producing images (pictures, objects, graphs, diagrams, etc) in the service of developing visual thinking. In particular, the mathematicians EULER and VENN are well-known for their development of diagrammatic tools for solving mathematical problems, and the logician PEIRCE (1878) developed an extensive diagrammatic calculus to generate a reasoning tool for comprehension. BARWISE & ETCHEMENDY (1999) also developed the *Hyperproof* computer program which allows students to solve deductive reasoning tasks using an integrated combination of sentences and diagrams.

Additionally, visualization, diagrammatic tools, and visual thinking have been understood more widely when dynamic geometry software appeared. This environment is the most powerful tool for helping students develop their power to imagine and

[11] In other words, visual thinking is a way to organize your thoughts and improve your ability to think and communicate. It is a way to expand your range and capacity by going beyond the linear world of the written word, list and spreadsheet, and entering the non-linear world of complex spatial relationships, networks, maps, and diagrams (e.g. ARNHEIM, 1970; SENECHAL, 1991; NELSEN, 1993; GIAQUINTO, 2007).

express what they imagine in movement, pictures, and words. In order to enhance visualizing ability, students should try to imagine a figure before sketching it, to sketch it before using dynamic geometry software, to make a conjecture as to what will happen before actually doing it (dragging, constructing a new figure), thus creating an imagination. This imagination may give students a strong grasp of properties and relationships between objects and support reasoning on the basis of perceived properties. These are features of dynamic visual thinking, a type of visual thinking that usually is referred in a dynamic geometry environment. It can be defined as follows:

> *Dynamic visual thinking is a mental manipulation that involves visual imagination of dynamic diagrams in dynamic geometry environment. It supports students to see the 'movement' of geometric objects with different flexible perspectives, recognizing relationships, differentiating static and moving invariants, especially in the context of determining geometric transformations for proving. It also refers to the ability of seeing and visualizing the 'movement' of geometric objects without a dynamic geometry environment.*

In general, developing this type of thinking may help students not only see geometric invariants in a 'dynamic' drawing or diagram, but also in a 'static' one of paper-and-pencil environment. Therefore, dynamic visual thinking plays a facilitating role in realizing geometric invariants during the proving process. Especially, at the tertiary level, students need to differentiate between geometric invariants of various transformations (isometries and similarities) as well as invariants of different geometries[12] (Euclidean geometry, affine geometry, and projective geometry). Therefore, this type of thinking should be taken into consideration in the mathematics classroom with the aim of providing students a flexible way of thinking in problem solving as well as the proving process.

[12] In the teacher training universities in Vietnam, students are required to differentiate the distinction between invariants of different geometries such as Euclidean geometry, affine geometry, and projective geometry.

2.5. AN INTERACTIVE HELP SYSTEM FOR PROVING

2.5.1. Introduction

BALACHEFF (1998) classified four levels of a proof to understand students' cognitive development in writing proofs such as naive empiricism, crucial experiment, generic example, and thought experiment (see Section 2.3.4). These levels are not enough to determine how students understand the development of the proving process. For that reason, in this research we classified seven levels of proving that represent the developmental phases in the proving process. These level are proposed as follows: *information* (level 0), *construction* (level 1), *invariance* (level 2), *conjecture* (level 3), *argumentation* (level 4), *proof* (level 5), and *delving* (level 6). In order to check the validity of this classification and support students in understanding the development of the proving process, we built an interactive HELP SYSTEM that is embedded in the GeoGebra environment. The system contains *seven levels of proving* which provides students with hints from understanding to delving into the problem. Based on the interactive HELP SYSTEM and students' solution of tasks, we can estimate their knowledge understanding and proving levels as well.

Research on the help system was conducted several decades ago. For instance, SELZ (1935) formulated the principle of minimal help in assisting students with problem solving. It implies a gradual increase in the level of specificity of the help needed, thereby aiming at the greatest mental activity possible. POLYA (1973) & WICKELGREN (1974) also conducted research on hints and gave a listing of general heuristic hints. They offered no empirical data about the effectiveness of different types of general heuristic hints. Therefore, extensive research in application was done by TRISMEN (1981; 1982). He experimented with various forms of hints and used two main formats, the open-ended and the multiple-choice form. He developed hints intuitively; afterwards the relation to general principles. In order to determine how effective a stimulating hint is on the process of problem solving, TRISMEN showed that hints have two functions: to *direct thought* and to *convey information*; the first is of primary importance and the form of hint best suited for the directing of thought is the question-form. He also claimed that initial misconceptions about a problem by students are extremely difficult to correct with hints and the particular nature of a problem dictates the kinds of hints which will be most relevant.

During the process of working with the interactive HELP SYSTEM, we always keep the well-known heuristic principles of POLYA in the back of our mind. The interactive HELP SYSTEM consists of open-ended questions and explorative tasks which were designed so that students can use them as they needed. These questions (or tasks) should be direct thought and suitable for students to explore the problem on their own. An *open-ended question* (to direct thought) is used to find geometric invariants and connect arguments forming a formal proof. An *explorative task* (to convey information) is used to help students explore the problem on their own. By answering open-ended questions or completing explorative tasks, the idea of proofs may also emerge gradually. After apparently unsuccessful trials and a period of hesitation, it may occur suddenly, in a flash, as a "bright idea" or "seeing the light". Bright idea is a colloquial expression describing a sudden advance towards the problem solution. Each proof-related problem needs some auxiliary or supplementary elements such as diagrams, auxiliary lines, geometric transformations, supporting theorems, deductive rules, etc. Sometimes, we consider a static/dynamic diagram as a hint for generating ideas of proofs. For instance, the following diagram might be a hint/suggestion for the solution of the orthic problem: "The orthic triangle[13] of an acute-angled triangle has the minimum perimeter among all triangles inscribed within the given triangle". In reality, Let *ABC* be a triangle with altitudes *AQ*, *BR*, and *CP*. The triangle *PQR* formed by the feet of the altitudes is the orthic triangle. In this problem, a diagram (see Fig. 2.16 below) reveals the translation that used to solve this problem. We may realize that the length of segment *PP'* is twice the perimeter of orthic triangle *PQR*. Moreover, the length of broken line *MM'* is twice the perimeter of arbitrary triangle *MJK* that inscribed about a given triangle *ABC*. Clearly, the segment *PP'* is smaller than the broken line *MM'*. This diagram is also called to illustrate a visual proof. In mathematics classroom, teachers should sometimes provide their students with a visual proof (as a hint towards the solution of the problem) aimed at developing students' logical reasoning based on figures or diagrams.

[13] An orthic triangle has three vertices that coincide with three feet of the altitudes of a certain triangle. It has the smallest perimeter among all triangles inscribed within a given triangle.

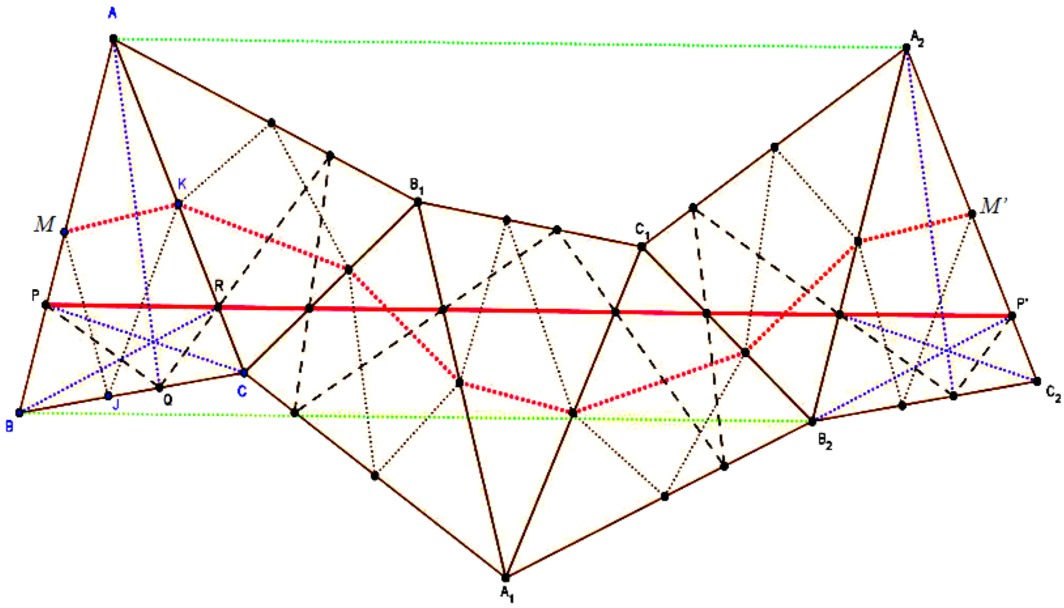


Figure 2.16: *A hint for generating proof ideas in the orthic problem*

In order to highly motivate students when they are working with the interactive HELP SYSTEM, open-ended questions or explorative tasks should satisfy the following characteristics:

1. *Should be suitable to students' understanding level:* The question or task is not too easy, not too difficult, and stimulates the students' thinking. It means that the hints must be suitable to students' level and are given as they needed.
2. *Should be heuristic:* There should be a gap between the help and solution of the problem. If the help is understood, it gives the whole secret away, very little remains for the student to do.
3. *Should be instructive and natural:* Student may perceive naturally the ideas of proof through exploring the problem with help and get into the habit of using these methods. In other words, students may definitely profit by using the methodological model from this system.

We consider the following example as a way to describe our interactive HELP SYSTEM to support students in transition from the lowest proving level to the highest proving level by using some appropriate hints. It also makes a contribution to bridge the students' cognitive gaps during the proving process:

Example 2.6. (Parallelogram Problem) *Let ABCD be a parallelogram. Draw four internal bisectors of angles of the parallelogram. These lines cut each other forming a new quadrilateral EFGH. Determine the shape of the quadrilateral EFGH.*

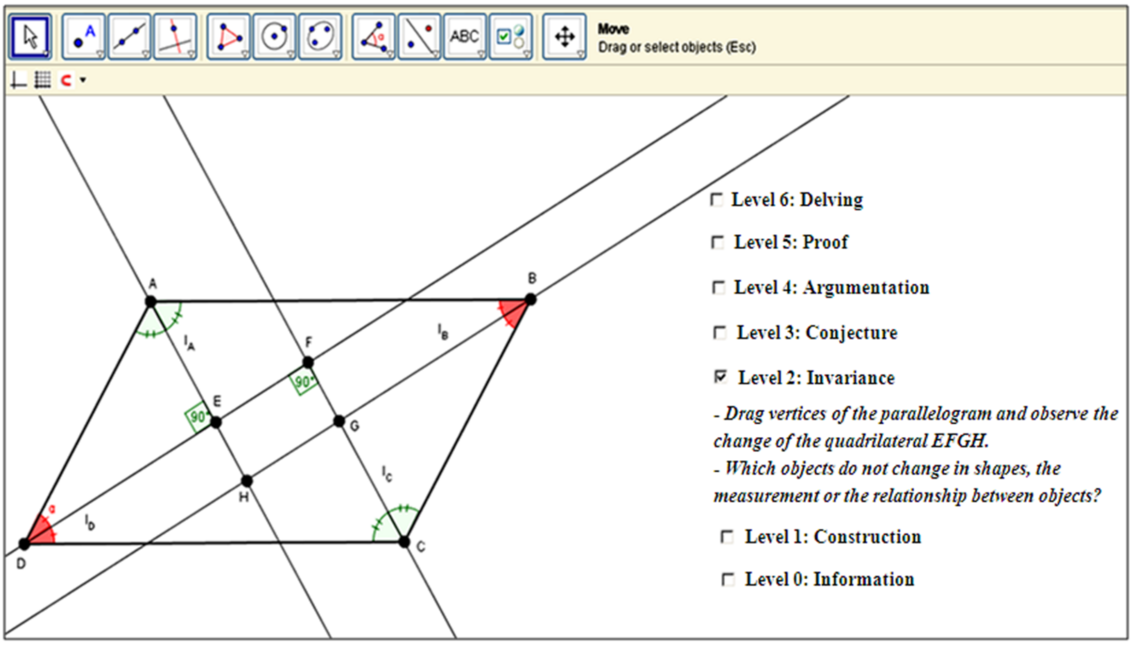


Figure 2.17: *An interactive HELP SYSTEM in the parallelogram problem (level 2)*

2.5.2. Level 0. Information

In order to solve a problem, students should *understand it*. If they are lacking understanding or interest, they are not motivated and cannot successfully tackle the problem. Thus the interactive HELP SYSTEM should give information clearly aimed at pointing out the principal parts of the problem, the unknown, the data, and the condition. At this level, the interactive HELP SYSTEM also introduces suitable notation, gives names to the objects, and guides students in knowing what they need to do next. For instance, in example 2.6, the help system provides students some information of the problem as follows:

Given data.

(1) *ABCD* is a parallelogram: $AD = a$; $AB = b$; $\angle ADC = 2\alpha$; $(0 < \alpha \leq 90^0)$;

(2) l_A, l_B, l_C, l_D are internal bisectors of the respective angles $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$;

(3) Let $E = l_A \cap l_D$, $F = l_C \cap l_B$, $G = l_B \cap l_C$, and $H = l_B \cap l_A$ be intersections of internal bisectors;

Exploration. Determine the shape of the quadrilateral *EFGH* and prove it.

2.5.3. Level 1. Construction

Students should *construct the figure on their own* by using the GeoGebra software. In this case, the interactive HELP SYSTEM shows students how to construct the drawing[14] step by step. For instance, in example 2.6, students have to:

- Construct a parallelogram *ABCD*;
- Draw internal bisectors of the angles $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$;
- Construct the intersections *E*, *F*, *G*, and *H* of four internal bisectors.

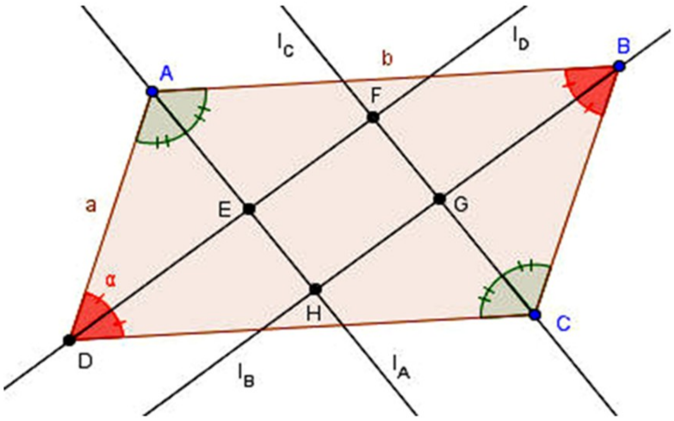


Figure 2.18: *Construction level in the parallelogram problem*

To attain this level, students need some basic construction skills. Thanks to construction functions of the GeoGebra software, students can construct their drawings easily such as: intersect of two objects, midpoint of a segment, a line through two points, parallel/perpendicular lines, angle bisector, perpendicular bisector, tangents, segment with given length, angle with given measure, polygon, circle, conic, etc. However, students often have some difficulties in constructing drawings as well as some auxiliary figures because of the lack of basic construction skills and knowledge.

[14] In our research, we differentiate two terms 'figure' and 'drawing'. Drawing is a figure which can change its shape by dragging. It is a dynamic figure, not only a static figure.

2.5.4. Level 2. Invariance

This phases aims to *search for invariants* supporting the proving process. To attain this level, students need only to apply an invariance principle to realize geometric invariants. Some helpful questions can be used to support students throughout this phase: What property is preserved by dragging? Which figures do not change their shapes as they move? Which figures are congruent or similar while moving? When the students get stuck on a geometrical problem, they usually try to use a familiar problem or specialize it in some way and then look for something familiar. However, the interactive HELP SYSTEM will help students consider the problem from different sides and aspects. It contains two phases of looking for invariants. Firstly, the students guess the transformations appearing in the problem. These can be realized by some signs: there exist constant angles, constant distances, constant directions, equal distances, equal angles, equal figures, regular polygon, fixed lines, fixed points, parallel lines, and so on. Secondly, students need to find geometric invariants by using the dragging mode. Invariants of a geometric transformation may be: measurement of angle, length of segment, parallelism, concurrency, perpendicularity, betweenness, collinearity, ratio of two segments, shape of a figure, etc. Realizing geometric invariants when dragging will provide students with more data for the proving process. For instance, in example 2.6, the interactive HELP SYSTEM will guide the students to discover the problem on their own:

- *An explorative task.* Drag vertices of the parallelogram and observe the change of the quadrilateral *EFGH*.
- *An open-ended question.* Which objects do not change in shapes, the measurement or the relationship between objects?

With the support of dragging mode, students realize some geometric invariants such as the straight lines *EF* and *GH* are parallel, the straight lines *EH* and *FG* are also parallel, and four angles of the quadrilateral *EFGH* are right angles. They might realize some other static invariants such as $AB = CD$, $AD = BC$, $\angle A + \angle D = 180^0$, etc. From this realization of the invariants, students will formulate conjectures and move on to the next level of proving.

2.5.5. Level 3. Conjecture

A conjecture is a statement strictly connected with an argumentation and a set of conceptions (see BALACHEFF, 1994) where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it. It is the postulation that something ought to be true or false. Conjectures often originate from experimentation, numerical investigations, and measurements. During the process of formulating conjectures, students work with arguments to construct a proof. As mentioned above, after revealing the invariants, students *make a conjecture* with the support of some GeoGebra functions such as measure the area; check the relation between two objects; calculate the angle measure, the distance between two objects; and check the locus of the moving objects. For instance, in example 2.6, the students explore the problem by:

- Calculating the lengths of the segments *EF*, *FG*, *GH*, and *HE*;
- Measuring the measurement of the angle ∠*HGF*;
- Using an open-ended question in the interactive HELP SYSTEM: *What are special characteristics of the quadrilateral EFGH, e.g. (the measurements of angles, the length of sides, etc)?*

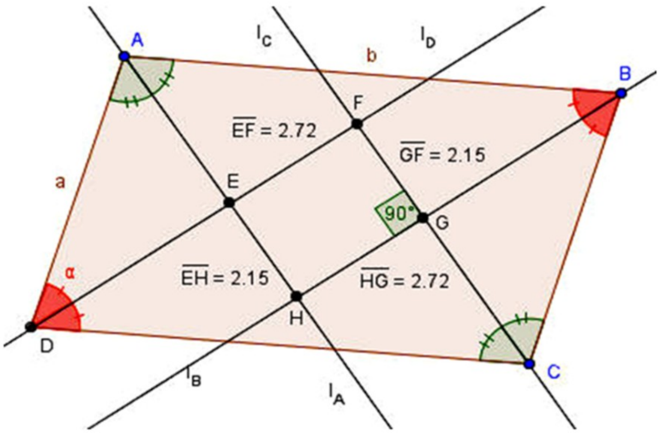


Figure 2.19: *Conjecture level in the parallelogram problem*

Then they formulated a conjecture "*The quadrilateral EFGH is a rectangle*". In order to formulate the conjecture, the students need to realize that *EF* = *HG*, *EH* = *FG*, and ∠*HGF* = 90^0 by measuring and using some initial reasoning. As soon as the conjecture is formulated, some arguments are produced for validating this conjecture. When students are able to provide some valid arguments during the process of formulating and validating conjectures, they may attain a new level of proving.

2.5.6. Level 4. Argumentation

At this level, the interactive HELP SYSTEM supports students in *producing and collecting arguments* by guiding them to answer some open-ended questions or explore some tasks. The data and arguments, which are not necessary for a proof writing, have also been reduced or omitted. For instance, in example 2.6, the following spontaneous arguments were produced by some students in the experimental group:

Data – Warrants.

(4) $AB = CD = b$; $AD = BC = a$; (1)

(5) $\angle ADE = \angle CDE = \alpha$; $\angle ABG = \angle CBG = \alpha$; (1)

(6) $\angle ADC + \angle DAB = 180^0$; (1)

(7) $\angle ADE + \angle DAE + \angle AED = 180^0$ (the sum of angles in a triangle)

(8) $EF = DF - DE$; $HG = BH - BG$;
$EH = AH - AE$; $FG = CF - CG$; (3)

Claims – Warrants.

(9) $\angle AED = 180^0 - (\angle ADE + \angle DAE)$ (7)

(10) $\angle ADE + \angle DAE = \frac{1}{2}(\angle ADC + \angle DAB) = 90^0$ ((2), (6))

(11) $\angle AED = \angle BGC = \angle DFC = \angle AHB = 90^0$ ((9), (10))

(12) $DE = BG = a\cos\frac{\alpha}{2}$; $DF = BH = b\cos\frac{\alpha}{2}$ ((4), (5), (11))

(13) $AE = CG = a\sin\frac{\alpha}{2}$; $AH = CF = b\sin\frac{\alpha}{2}$ ((4), (5), (11))

(14) $EH = FG = (b-a)\sin\frac{\alpha}{2}$ ((8), (12))

(15) $EF = GH = (b-a)\cos\frac{\alpha}{2}$ ((8), (13))

(16) *EFGH* is a rectangle ((11), (14), (15)

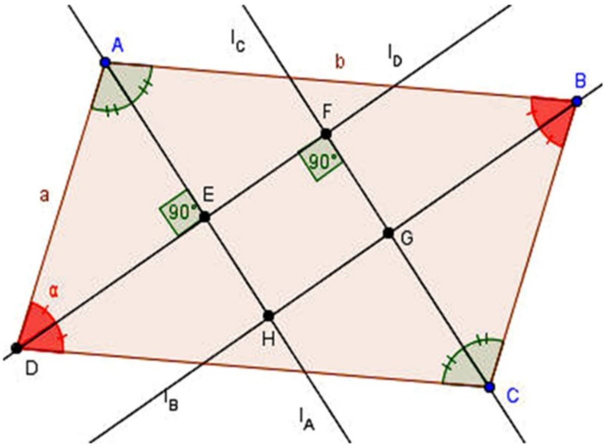


Figure 2.20: *Argumentation level in the parallelogram problem*

The argument is a major part of proofs, so it is necessary to produce it. For teaching and learning purposes, argumentation is a fruitful means to control the validity of reasoning. There are two levels of argumentation: as a part of the proving tasks, especially for producing and organizing arguments; and in discussing procedures, as a means to assimilate and master the elements of the proving process. Therefore, argumentation is the most important phase of the proving process and it provides valuable data for writing proofs at the next level.

2.5.7. Level 5. Proof

Based on produced arguments, at this level, the interactive HELP SYSTEM guides students in writing proofs. Students have to *select some helpful arguments*, *connect them to form a chain of reasoning*. The use of mathematical language and logical laws are essential for students in this phase of the proving process. Therefore, the interactive HELP SYSTEM will provide a logical rule to connect arguments or suggest some open-ended questions aimed at producing deductive arguments. In some cases, this system can provide students with a solution diagram aimed at engaging them in writing a logical chain of arguments. In general, at this level, students will be guided in writing their formal proof and the interactive HELP SYSTEM takes the responsibility for supporting students in organizing their arguments and encourages them to overcome some difficulties in writing proofs such as choosing valid arguments; using statements, notations, logical rules, etc.

2.5.8. Level 6. Delving

Delving into a problem by reconsidering, expanding the result, students could not only consolidate their knowledge but also develop their ability to solve problems. At this level, the interactive HELP SYSTEM suggests that students use some mathematical thinking strategies in the process of delving, such as generalization, expansion, specialization, analogy, decomposing and recombining, etc. Delving into a problem also means that the students should try to make their proofs as simple as possible. For instance, in example 2.6, when the students drag vertices of the parallelogram, in some cases, the quadrilateral *EFGH* is a square or a point. So the interactive HELP SYSTEM suggests to students to specialize the problem as follows:

- If *ABCD* is a rectangle then $\alpha = 90^0$:

  $$\frac{\alpha}{2} = 45^0 \Rightarrow cos\frac{\alpha}{2} = sin\frac{\alpha}{2} \Leftrightarrow (b-a)cos\frac{\alpha}{2} = (b-a)sin\frac{\alpha}{2} \Leftrightarrow EF = EH$$

  Hence, the quadrilateral *EFGH* is a square.
- If *ABCD* is a rhombus then $a = b \Rightarrow EF = EH = 0$ or the quadrilateral *EFGH* is degenerated into a point.

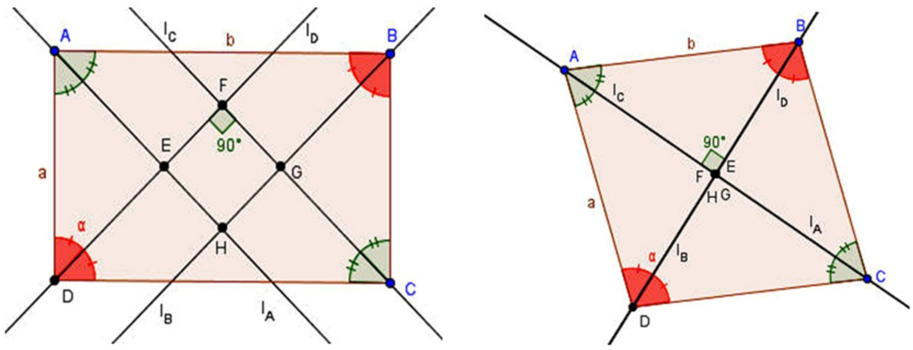


Figure 2.21: *Delving level in the parallelogram problem*

At this level, the interactive HELP SYSTEM will give students some open-ended questions or explorative tasks in order to discover new problems or find other shorter solutions. Sometimes, it will also propose a related problem aimed at helping students form a habit of thinking whenever they finish solving a proof-related problem. In general, each of abovementioned levels has its own role in the proving process. Some students can ignore one of these levels if the idea of a proof suddenly appears. Then students can jump and go straight to the solution of the problem. Based on the levels of proving, we can also estimate the students' knowledge understanding and proof writing ability.

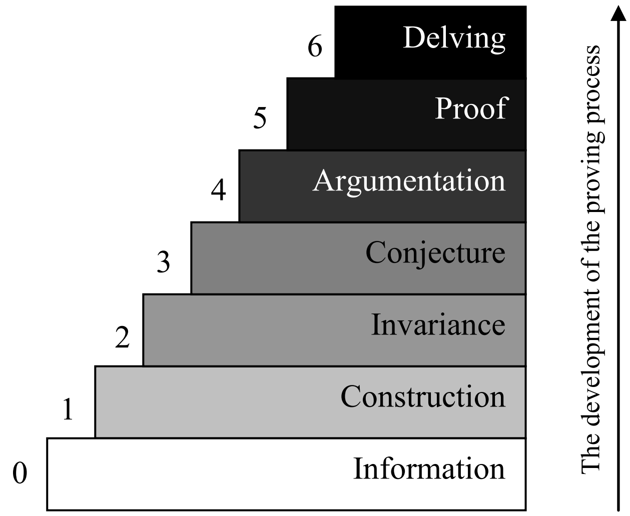
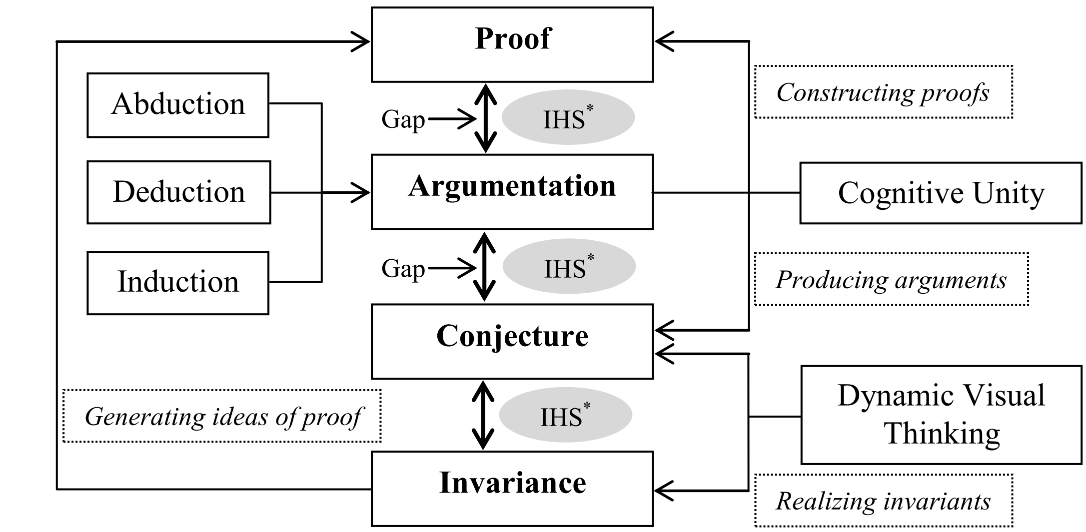


Figure 2.22: *The interactive HELP SYSTEM as a methodological model*

On the basis of these levels of proving we realized that students spend most of their time in four phases of the proving process such as *invariance*, *conjecture*, *argumentation*, and *proof*. Therefore, throughout this dissertation, we concentrate on the basic aspects of these phases. In the variance phase, students realize sub-invariants as well as a key invariant so as to generate the ideas for proofs. The ability to realize these invariants depends on the students' level of dynamic visual thinking. In the transition from the conjecture to the argumentation phase, there is a cognitive gap between them. Students must produce valuable arguments or construct a cognitive unity in the process of validating conjectures. Moreover, in the argumentation phase, students use different strategies in explaining 'observed facts' and constructing proofs. We also used abduction to analyze students' explaining methods. Our hypothesis is that students have difficulties in writing a formal proof because of the existence of the cognitive and structural gaps between argumentation and proof. Therefore, this research reveals the gaps and determines some fundamental aspects that influence the proving process. These aspects are described in the following model:



(*) The interactive HELP SYSTEM

Figure 2.23: *Some fundamental aspects that influence the proving process*

2.6. SUMMARY

This chapter summarizes some research and studies involving proofs and proving in mathematics education. Most of these works have confirmed that proving activities give students the opportunity to enhance their better understanding of mathematics knowledge. In this chapter, we propose a methodological model in accordance with seven levels of proving, on which the interactive HELP SYSTEM based. Through students' activities with this system and the original solutions of tasks, we can estimate their knowledge understanding and develop their 'habits of mind' in solving a problem. We also give some suggestions for teaching a formal proof at the tertiary level such as using TOULMIN model to analyze the structure of argumentation, applying abduction to interpret the way students think, using a dynamic geometry software to realize geometric invariants, developing dynamic visual thinking to support in generating idea of proof, etc. From these suggestions, mathematics teachers should provide students with explorative strategies to assist them during the proving process. This chapter also determines some basic conditions for understanding the development of the proving process and fundamental aspects that influence the proving process within a dynamic geometry environment. We will clarify each condition and analyze the role of each aspect during the process of constructing a formal proof in chapter three.

Chapter 3
DATA COLLECTION AND ANALYSIS

3.1. DATA COLLECTION PROCEDURES

3.1.1. Research design

The data were collected during the summer semester 2010. The participants were enrolled in a required elementary geometry classes for a teacher training course. The raw materials were firstly checked, coded, edited, entered into a computer, and subsequently analyzed. These materials consist of transcripts from the video and audio recordings, students' solution worksheets, questionnaires, semi-structured interviews, hypotheses, and teacher's field-notes. Because of the large amount of data during the period of experimental teaching, we have used the method of data reduction. It refers to the process of selecting, focusing, simplifying, abstracting, and transforming the data that appears in the transcriptions, students' worksheets, and teacher's field-notes. The process of data analysis was oriented by research questions and hypotheses formulated at the beginning of the research from the students' perspective. The aim of this analysis is to understand the development of students' proving processes and the interactions between students in the group and the interactive HELP SYSTEM (see Section 2.5). It also determines students' difficulties during the proving process and proposes some pedagogical strategies for improving the way of teaching proofs in both secondary level and tertiary level.

Participants. Proofs and proving are crucial issues in mathematics curricula and textbooks in the secondary schools in Vietnam. Therefore, these topics are also important contents for students at the teacher training universities. Students who become mathematics teachers at the secondary school level need to be trained on how to sow the seeds of enhancing problem-solving and proving skills in different mathematics subjects. In particular, understanding the development of the proving process is one of the most fundamental parts of mathematics teacher training programs because it helps students better understand mathematical ideas and mathematics itself. In order to enter the mathematics teacher training university, after graduating the upper secondary school, students are required to pass the entrance examination (including three subjects: mathematics, physics, and chemistry). The prerequisite for this entrance differs from university to university and is based on the total points of three subjects. Once entering university, one of the difficulties which first-year students face is methods of teaching and learning mathematics at the tertiary level because these methods are different from those at the secondary level. According to NGUYEN (2005) and LE (2006), the methods in a secondary school in Vietnam emphasize lecturing, memorization, and preparation for final examinations. The examinations are highly competitive with a low cognitive goal and stress the students' achievement. Therefore, students usually take solving-problem strategies and rote learning into their consideration. As a result, they may not possess a deep level of knowledge understanding and do not necessarily understand the development of the proving process as well. To improve this real situation, the methods and strategies of teaching proofs at the teacher training universities should encourage students to find their own solution to the problems by exploring and conjecturing. That is a reason we chose mathematics teacher students in our empirical data collection. These difficulties in teaching and learning proofs at the secondary level and tertiary level are also presented in Section 2.2 and Section 2.4.

All of the 132 participants involved in our research were second-year students of Thai Nguyen University of Education[15]. The participants were divided randomly into two sample groups (i.e. an experimental group with 67 students and a control group with 65 students). At the start of summer semester 2009/2010 the experimental group and control group took a preliminary test (pre-test) to determine the equivalency of the two groups' background and at the end of the semester there was a post-test to check the formulated hypotheses. All the tests were taken using the paper and pencil format (the use of computer was not allowed). The students' test worksheets were used to evaluate the increase in the students' proving levels and levels of realizing geometric invariants in the experimental group. These worksheets were also utilized in a semi-structured interview in order to understand how tertiary students tackle proof-related problems within a dynamic geometry environment.

[15] The author of this dissertation has been teaching elementary geometry at mathematics faculty of Thai Nguyen University of Education since 2004. This is one of largest teacher training universities in the north of Vietnam. Website: http://www.dhsptn.edu.vn.

Elementary Geometry Course. Elementary geometry is a fundamental subject which is integrated into the mathematics curriculum at the teacher training universities in Vietnam. This subject contains rich contents for improving a student's proving level. The reason for choosing this subject in the period of experimental teaching is due to our work experience with elementary geometry in both secondary schools and universities. In particular, the topic of geometric transformation and its application in solving geometric problems could be exploited effectively within a dynamic geometry environment like GeoGebra or Geometer's Sketchpad. Furthermore, in all branches of mathematics, transformation is a key concept and provides a powerful tool for both discovering and proving new mathematical theorems.

The elementary geometry course is taught within four credit contact hours per week and includes the following contents: the basis for Euclidean geometry; some methods, strategies and techniques for proving; geometric transformations and their applications; locus and construction. The objectives of this course provide students with some methods for proving, improving proving skills, developing geometric thinking, and enhancing the ability to use advanced geometry for solving problems in elementary geometry. This course was taught by the author of this dissertation in a period of four months during the summer semester 2009/2010. In particular, in this course we also referenced groups of geometric transformations that preserved some geometric concepts, axioms, and theorems, for example, a group of similartities which contains *angle-preserving* transformations. An important case of similarities is isometry. This is a *length-preserving* transformation such as a point reflection, line reflection, rotation, and translation. These transformations are at the basis of the familiar idea of congruence, "two figures are congruent if and only if one can be transformed into the other by an isometry". One of crucial objectives in teaching geometry at the upper secondary school levels in Vietnam is to introduce students with the idea of congruence and thereafter they can easily understand some congruent objects in their real-world life. Moreover, through this topic, we offer students the opportunity to study the geometric invariants in groups of transformations. These invariants provide important insight for a deeper understanding of geometry and the structure of transformation groups as well as the differentiation between various types of geometries such as Euclidean geometry, affine geometry, and projective geometry (as shown in Section 2.3.1).

Computer Laboratory. There are two computer laboratories at the Thai Nguyen University of Education. They facilitate approximately 60 computers connected to the internet and serve as a space for students learning computer science as well as doing 'mathematics experiments'. In a computer laboratory, students were divided into groups of three, who sat together at one computer. Each computer was installed with the GeoGebra software in order to create a dynamic environment for group-based activities. The reason for this division is the fact that working in groups positively affected the development of the proving process (see e.g. OLIVERO, 2002). Especially, in the group-based activities, open-ended questions and explorative tasks in the interactive HELP SYSTEM were sought out and were jointly considered. These may have been challenged and counter-challenged, but these challenges were justified and alternative conjectures were offered. Throughout the group-based discussion, a debate is publicly arisen and arguments are produced. During the process of using the computer, groups of students drag the point, measure the length, check the relationship, and formulate conjectures. These activities could take a step toward addressing discussion and reasoning. We also installed Wink® software on each computer in order to capture and audio-record of all the group discussions.

Moreover, in order to provide rich opportunities for students outside of the computer laboratory, we have also designed an online corresponsive course using the Moodle platform at the following website: http://www.daotaotructuyen.org. The students' participation in the e-lessons was not mandatory, it was optional and selective. We offered some further exercises in the form of GeoGebra applets with the interactive HELP SYSTEM aimed at helping students gain experience in proving activities while creating a center of debate on the online forums of the course. With online courses, students could deepen their knowledge beyond the traditional classroom and manipulate some applets with a dynamic geometry environment. These manipulative-based exercises provide students with an opportunity to grow accustom to using of the interactive HELP SYSTEM throughout the proving process and effectively strengthen applet-student interaction.

Variables. During the period of experimental teaching, we differentiated dependent and independent variables aimed at investigating the cause-effect relationship between them in the hypotheses. Environmental factors (including a dynamic geometry environment in which students could use the GeoGebra software to support proving activities and the paper-and-pencil environment) are the independent variables while the students' scores on tests are the dependent variables. Thus, at the beginning of the experimental research design, we determined the following dependent variables which were collected from each group in various environments: *mean scores* for the pre-test and post-test, *students' level of proving*, *proof-writing ability*, and *students' level of realizing geometric invariants*. We also considered *gender* as another independent variable for the purpose of comparing the attained results of male and female students. Before experimental teaching, we were concerned about the *students' priori knowledge* as a variable and how this can influence the results of the two groups. However, by formulating and testing the null hypothesis, we can conclude that two groups have equivalent backgrounds. Thus, the data from the dependent variables were collected from each environment, as well as the gender factor, in order to test the hypotheses that were formulated at the beginning of the empirical research.

Instrumentation. In order to produce data for the testing hypotheses, we need to measure students' levels of proving and levels of realizing geometric invariants. Therefore, we have determined two ordinal-data scales as follows: seven levels of proving (information, construction, invariance, conjecture, argumentation, proof, and delving); five levels of realizing geometric invariants (no invariant, static invariants, dynamic or 'moving' invariants, invariants of geometric transformation, and invariants of different geometries). We also designed exercises in the pre-test and post-test (see Appendix E and F) as the tools for investigating the differences between 'inputs' and 'outputs'. Some group-based and individual-based tasks (see Appendix A, B, and C) were also designed to determine and evaluate the level of proving, level of realizing geometric invariants and proof-writing ability as well. The solutions to the problems in these tasks also were used in the semi-structured interviews. Based on the students' explanations about their work, we classified levels of proving and determined students' difficulties during the proving process. A questionnaire was designed and modified to investigate students' attitudes towards the dynamic environment in support for proving and classifying levels of realizing geometric invariants. All the students' responses were coded and then analyzed using SPSS 17.0 statistics packages. Comments on issues of the open-ended questions were selected and summarized to supplement the findings of the SPSS outputs. Furthermore, we used teacher's field notes which recorded some remarkable activities and reasonable arguments that emerge from discussions in order to interpret the students' thinking and behavior during the process of constructing a cognitive unity.

3.1.2. Methodology

With the purpose of understanding the students' proving process, we used both qualitative and quantitative methods in analyzing empirical data. Each method has its own strengths and weaknesses. Thus, in order to effectively utilize these methods, we have triangulated the gathered data. Firstly, we observed students' behaviors associated with audio and snapshot video recordings. This method of observation is a powerful tool that offers us the chance to gather live data from the students' discussion, get inside situations and observe directly what is happening, thus collect more valid and authentic data. However, observations might have difficulties and weaknesses as well. On the one hand, observations demand a lot of time, effort, resources, and they are vulnerable to the observer's bias. In particular, audio and video recordings would reduce the validity since the informants might behave differently in the presence of such devices. As a supplement, we would clarify the ambiguity in the semi-structured interviews with the students in order to increase the validity and illuminate the students' understanding of the proving process.

A semi-structured interview schedule is an instrument that can be used to gather in-depth personal information about participants' thoughts, knowledge, reasoning, perceptions, and experiences about a certain topic. The advantages of a semi-structured interview are that they are flexible and applicable to many different types of questions. This kind of interview involves a number of pre-determined questions or several topics, which are typically asked in a systematic and consistent order, but to some degree allow freedom for modification. A semi-structured interview permits us to examine far beyond the answer to our prepared standardized questions and obtain more information. Specifically, throughout the interview, we asked reasons or explanations for the students' solutions based on five open-ended questions. These questions were designed to gather in-depth information and conducted by us after completing the task-based activities. Each interview lasted from 20 to 30 minutes. It was conducted in the students' mother tongue (Vietnamese) and recorded using an audio recorder and later transcribed. The participants in the interview were the students from the experimental group, selected using purposeful sampling (15 students were divided into five groups of three). The selection was based on their levels of proving and regular attendance during the period of experimental teaching. The purpose of the interview was to investigate the influence of the use of dynamic geometry software on the students' proof-writing abilities. During five individual interviews we also took field notes aimed at providing more detailed information for the data analysis. A questionnaire survey was also designed to investigate the students' attitudes towards the role of the interactive HELP SYSTEM in the proving process and their perceptions on visual proofs within a dynamic geometry environment. The reason we used a questionnaire is that it is time-saving, low cost, and a deeper understanding, because some students find it more difficult to express their private opinion in the semi-structured interview.

In addition, in order to answer the quantitative questions, we used hypotheses testing, correlation between variables, and ANOVA analysis. The experimental research design attempted to investigate the cause-effect relationship between the use of the interactive HELP SYSTEM during the proving process and students' test scores and proving abilities. All statistical analyses were conducted using the statistical package SPSS 17.0. Firstly, determined independent and dependent variables that influence on students' test scores (test performance) as well as their proving levels. Then we formulated a null hypothesis in order to confirm that two groups of students are equivalent backgrounds. Consequently, the same pre-test was given to the two groups before conducting the experiment teaching. This was followed by a teaching course on proving strategies (approximately 20 credit contact hours), geometric transformations and their applications (approximately 20 credit contact hours). The experimental group was taught these topics in the computer lab and control group was taught the same content but with the paper and pencil format. After the experimental teaching course, the same post-test was given to the two groups in order to verify the effect of using a dynamic geometry environment with the interactive HELP SYSTEM in supporting students constructing a formal proof.

3.2. DATA ANALYSIS

3.2.1. Observations

In order to analyze students' thinking and behavior during the proving process, we used an effective analysis method, called "frame analysis method" (see e.g. MCDOUGALL & KARAGAD, 2008) combined with an audio-taped method, to monitor and track the students' proving process without distracting the students while they were working on their tasks with the interactive HELP SYSTEM. The method was based on recording students' manipulation and discussion in the computer environment by using a screen-casting Wink® software[16] and allowed us to capture not only what, but how they did on the GeoGebra worksheets. During the observation we chose to adopt the role of the observers as participants. Adopting this role was particularly helpful for the data collection. It was possible for us to observe from a distance, take notes and help students who faced difficult problems. A camera was also used to videotape the whole classroom scene.

In this research, we divided 67 students in experimental group into twenty one groups of three[17] and two groups of two. They were required to formulate conjectures and write a formal proof. The interactive HELP SYSTEM provided students with a scaffold to bridge the gap between argumentation and proof (see Section 2.3.3) and to realize geometric invariants. We analyzed students' task-based activities by comparing each frame[18] of record and tracking every movement of mouse and entry of keyboard. The recording software was set to record one frame per two seconds. After collecting the data, all of the videos, audio clips and snapshots were watched and listened to several times, so as to understand the students' thinking and behavior while they used the interactive HELP SYSTEM to support proving activities. Students' discussions in these materials were annotated, transcribed on paper and finally translated into English.

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[16] This software also allowed us to zoom into any frame recorded and to annotate it. This feature delivered our messages and jotted our notes down on the desired frames. It also made the communication easier because we can easily navigate the frames, describe the moment of action, and deliver the message in order to provide opportunity of just-in-time commenting. You can free download this application at the following website: http://www.debugmode.com.

[17] The students were grouped in threes because they can produce arguments better in the process of argumentation and proof. Furthermore, participant observation is particularly useful in studying small groups for a short time. The data, which is derived from the discussion, is 'strong on reality'.

[18] A frame is defined as the snapshots of the computer screen at a specified moment.

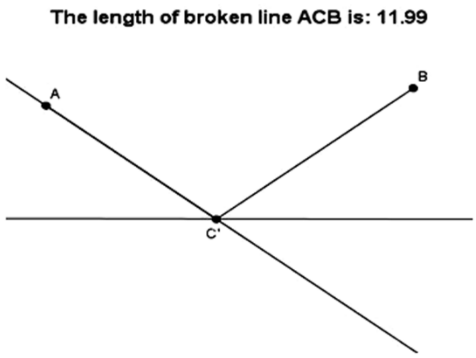
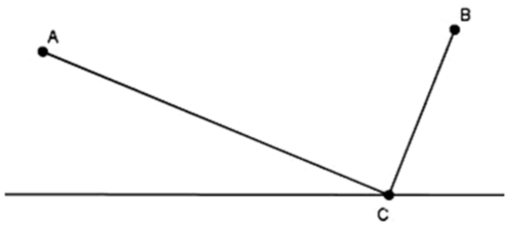
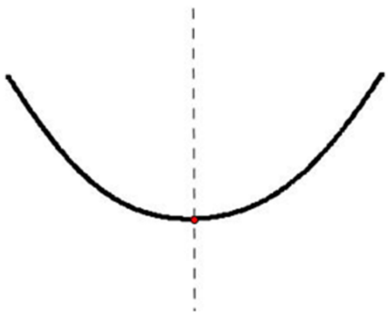
We also used TOULMIN model of argumentation as a tool to analyze continuity and structural gaps between argumentation and proof (see Section 2.4.1). On the basis of the functions of proofs (see Section 2.1) in the mathematics classroom and investigation on how tertiary students understood the development of the proving process within a dynamic geometry environment, some following questions were involved during our observation:

1. How do students realize geometric invariants?
2. How do students formulate a conjecture?
3. What do they use to validate their conjectures?
4. How do students produce and collect arguments for the proving process?
5. How do students combine their arguments to form a valid proof?
6. What would students like to do after finishing their proof?
7. How do students interact with the interactive HELP SYSTEM and with each other in their groups?

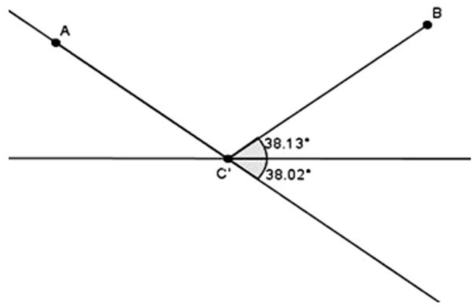
We have designed four group-based tasks to observe some acts of proving, students' behavior as well as their interactions when they were using the interactive HELP SYSTEM.

Task 1. (School Problem) *People living in the neighborhood of town A and working at company B are to drive their children to school on their way to work. Where on highway l should they build school C in order to minimize their driving? (When the site C for the school is chosen, the roads AC and CB will be built).*

For task 1, we chose the discussion of group 4 to analyze the way students used abductive argumentation and also to reveal the *role of abduction during the proving process*. In addition, overcoded abudction in TOULMIN model (see Section 2.4.1) was also used to interpret the gap between abductive argumentation and deductive proof. We may see how students realize geometric invariants, formulate conjectures, validate conjectures, produce arguments, collect arguments, and organize these arguments in order to write formal proof. The discussion was transcribed based on captured snapshots and audio clips as follows:

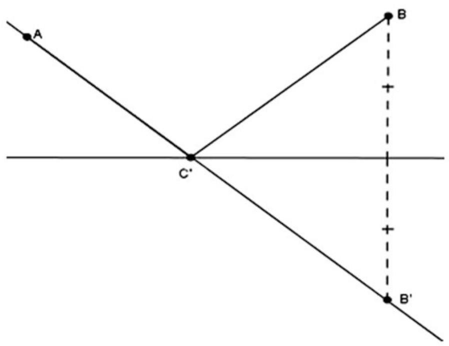
♣03. <u>Student 1:</u> The problem requires finding the minimal value. Therefore, we can firstly use the functions in GeoGebra to measure the length. Now we have to construct the figure. ♣04. <u>Student 2:</u> Yes, we can try it. But how we can draw the figure? ♣05. <u>Student 3:</u> Why don't you use construction level in interactive HELP SYSTEM? ♣06. <u>Student 2:</u> That's right! ♣10. <u>Student 2:</u> Now drag point *C* and observe what's happening with the sum and the figure? ♣11. <u>Student 3:</u> But you have to measure the length of broken line *ACB*! ♣12. <u>Student 1:</u> Exactly! ♣15. <u>Student 2:</u> Drag point *C* more slowly please! I realize that this position satisfies the broken line *ACB* is minimal. Oh this point... can you do it again!  ♣16. <u>Student 1:</u> That's right! I find that at this position the sum (*AC* + *CB*) is minimal because the red point representing this sum is a minimum point of a parabola. ♣17. <u>Student 3:</u> Yes, the school should be built at this point. But is there anything special at this position? I cannot see anything! ♣18. <u>Student 1:</u> Yeah, I cannot see anything either! We can use invariance level in the interactive HELP SYSTEM in order to get more	Students read information and the requirements of the task. Firstly, they faced difficulties in modeling the situation and they used the *Construction level* in the interactive HELP SYSTEM as follows: *- Draw a straight line l that representing the highway.* *- Take two points A and B on the same side of the straight line l.* *- Draw a movable point, C, on the straight line l which represents position of the school.* *- Construct two segments AC and CB.*  Students measured the length of broken line *ACB* and dragged point *C*, slowly moving on the line *l*. They determined the position that the length of broken line *ACB* is minimal but they could not realize any invariants.  This is the students' difficulty in realizing geometric invariants. Therefore, they used more support from the *Invariance level* in the interactive HELP SYSTEM as follows: *- Determine point C', which would be a position to build the school.* *- Draw a straight line passing through the two points*

<table>
<tr>
<td>support.
♣21. <u>Student 3:</u> What are the invariants in this case?
♣24. <u>Student 2:</u> Wait a moment! Two angles ∠(C'B; l) and ∠(C'A; l) seem to be equal? Measure two angles, please!
♣25. <u>Student 3:</u> That's right! They may be equal because measure of one angle is $38^0$13' and measure of the other is $38^0$02'.
♣28. <u>Student 1:</u> They are almost equal! I guess that they really are equal. This error may be due to the fact that we cannot move point C on the exact position! But if they were equal, what would be happened?
♣29. <u>Student 2:</u> Please be calm! Which data we have got until now? I mean which invariants, do we have: a fixed line representing the highway; two fixed points A, B; and perhaps two angles C'_1, C'_2 are equal. Therefore, the problem is which geometric transformation should we use in this case?
♣30. <u>Student 1:</u> I think there are more invariants such as: distances from points A and B to the line l; distance between these points and the like.
♣31. <u>Student 3:</u> Exactly! But we know the following geometric transformations: line reflection, point reflection, translation, rotation, glide translation, and maybe the product of these transformations. So which transformation can we choose?
♣33. <u>Student 1:</u> In my opinion, there exists a fixed line l, so probably in this case we use line reflection to tackle this problem? What do you think?
♣35. <u>Student 2:</u> Yes, suppose that we use line</td>
<td>A and C' and take notes.
- Change the positions of points A, B and repeat this process.



Students changed the position of points A and B in order to affirm this invariant. They realize that this phenomenon is the same. As a result, they realized a sub-invariant. Students made the first conjecture after checking the hypothesis by using GeoGebra. This conjecture is constructed by generalization based on inductive observation[19].

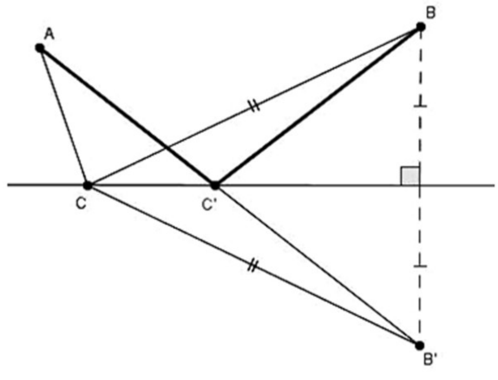
Students started collecting their invariants. They determined some invariants of geometric transformations in order to choose a suitable one. From this phase, students started using abductive argumentation.

"Which data ← Line reflection"


Students formulated the second conjecture about the</td>
</tr>
</table>

[19] Inductive observation is a process of discovering based on several different cases while observation of movements within a dynamic geometry environment.

reflection to solve this problem, what kind of data may we receive in the next step? ♣36. <u>Student 1:</u> That's right! We need to consider it thoroughly. Look! Because two angles C'_1 and C'_2 seem to be equal, probably the line *C'A* is image of the line *C'B* under a line reflection r_l. What do you think? ♣38. <u>Student 3:</u> It is suitable reasoning! It means that in order to construct point *C'* we have to construct the image line of *C'B*, but we do not know point *C'* yet! ♣39. <u>Student 2:</u> We have only two fixed points *A* and *B*. Hey, since the line *C'A* is image of *C'B* so *C'A* must contain image of point *B*! Thus, we can construct point $B' = r_l\ (B)$ and point *C'* is the intersection of the line *AB'* and the line *l*. ♣40. <u>Student 1:</u> That great! That is a good idea! Now we can click on conjecture to get more guidance. ♣41. <u>Student 3:</u> We are on the right track! Now we have to construct the point *C'* and prove that *C'* is the position to build the school. Until now we have the data: $r_l(B) = B'$ and $\angle C_1 = \angle C_2$. ♣44. <u>Student 1:</u> We can measure to get more data by using GeoGebra. I found that two segments *C'B* and *C'B'* are also equal! ♣46. <u>Student 2:</u> I agree with you. Since $r_l(B) = B'$ so we could derive directly the following equalities: $C'B = C'B'$ and $CB = CB'$. ♣49. <u>Student 3:</u> Yeah, but I think the most important thing now is that we must show that *C'* is best position to build the school! ♣50. <u>Student 2:</u> How can we prove it? ♣51. <u>Student 1:</u> I think we have to compare the length of sum $(C'A + C'B)$ with the length of sum $(CA + CB)$ by measuring them. This result	transformation used to solve the problem and utilized deductive argumentation in order to construct point *C'*. As a result, students could have made a conjecture in this case before they used the support from *Conjecture level* in the interactive HELP SYSTEM: - *Investigate the relationship between two lines C'A and C'B.* - *Make a conjecture about this relationship.* Students collected the arguments and wrote them on a piece of paper. Then they used abductive argumentation in order to validate conjectures. **C_1**: $\angle C'_1 = \angle C'_2$ The structure of the argumentative step is an abduction: **D_1** = ? ⟶ **C_1** **W_1**: Property of line reflection **D_1**: $r_l\ (C'B) = C'A$ **C_2**: $C'B = C'B'$ and $CB = CB'$ **D_2 = ?** ⟶ **C_2** **W_2:** Property of line reflection **D_2:** $r_l(B) = B'$ Students had a need to proof but they got stuck in this stage and used *Argumentation level* in the interactive HELP SYSTEM: - *Compare the length of the sum (C'A + C'B) with the length of the sum (CA + CB).* - *Write your arguments on a piece of paper.* Students collected empirical data by measuring two broken lines *AC'B* and *ACB*: **C_3:** $C'A + C'B \leq CA + CB$ Students began collecting the data and combining the

will lead us to answer the question. ♣53. <u>Student 1:</u> Generally, we can realize that the sum ($C'A + C'B$) is always smaller than the sum ($CA + CB$) and we can conclude again that point C' is the position where we can build the school. ♣54. <u>Student 2:</u> Now I think we should list all our collected data as follows: i) $r_l(B) = B'$ so we have $CB' = C'B'$ and $CB = CB'$. ii) $C'A + C'B = C'A + C'B'$ (because $C'B = C'B'$). iii) $CA + CB = CA + CB'$ (because $CB = CB'$). Finally, we must prove the following inequality: $C'A + C'B' \leq CA + CB'$. ♣56. <u>Student 3:</u> Because A, C', B' are collinear, we derive $C'A + C'B' = AB'$. It means that we have to prove the inequality: $AB' \leq CA + CB'$! ♣57. <u>Student 1:</u> This will lead us to the end of this problem! ♣58. <u>Student 3:</u> Why? Can you explain more details to me! ♣59. <u>Student 1:</u> Because we can use triangle inequality for the last inequality! This inequality is always true, it is a theorem. ♣60. <u>Student 2:</u> Yes, so we have strategy for tackling this problem already. Now we have to write the valid proof. Where should we start? ♣63. <u>Student 1:</u> Let me look back all of the process of analyzing. I think we have to prove the sequence of inequalities: $AB' \leq CA + CB'$ $\leftarrow C'A + C'B' \leq CA + CB'$ $\leftarrow C'A + C'B' \leq CA + CB$. Therefore, we can start proving from the last inequality!	data by using deduction to produce sub-arguments. Students continued using abductive argumentation in the next step of the proving process: **D_3 = ?** ⟶ **C_3** **W_3:** C_2 **D_3 = C_4:** $C'A + C'B' \leq CA + CB'$ **D_4 = ?** ⟶ **C_4** **W_4:** A, C', B' are collinear **D_4 = C_5:** $AB' \leq CA + CB'$ **D_5 = ?** ⟶ **C_5** **W_5:** Triangle inequality **D_5** is a mathematical theorem. Students had difficulties in writing a proof and they used abductive argumentation in order to combine produced arguments into a formal proof by reversing the structure of abduction. 

Students wrote a formal proof on their paper as follows:

Let B' =r_l *(B) and C' = AB'* $\cap$ *l. From that we have equalities CB = CB' and C'B = C'B'.*

We derive: CA + CB = CA + CB'

C'A + C'B' = C'A + C'B

Therefore, CA + CB = CA + CB' $\geq$ *B'A = C'A + C'B (since three points A, C', B' are collinear).*

Equality occurs when and only when C $\equiv$ *C'.*

Task 2 is an expansion on task 1 as two points *A* and *B* lie on different sides of the highway. This task was designed to understand students' arguments with their priori knowledge from the previous task.

Task 2. (One-Bridge Problem) *A river has straight parallel sides and cities A and B lie on opposite sides of the river. Where should we build a bridge in order to minimize the traveling distance between A and B (a bridge, of course, must be perpendicular to the sides of the river)?*

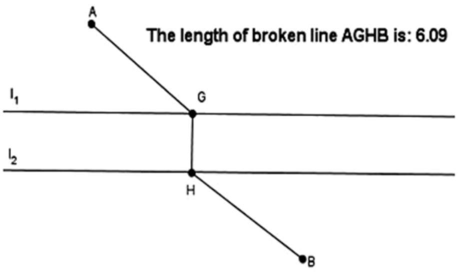
For the task 2, we chose the discussion of group 9. It was transcribed based on their snapshots and audio clips as follows:

♣03. <u>Student 2:</u> Now we draw two parallel lines representing two banks of the rivers and then determine the position where we can build the bridge! ♣05. <u>Student 1:</u> Hey, these lines are not parallel! You move a point on the one line and look two lines. I think they are no longer parallel! ♣06. <u>Student 3:</u> That's right! I think you must use the parallel function of GeoGebra to construct these lines. ♣08. <u>Student 2:</u> But how can we know where point *D* should be situated? ♣10. <u>Student 3:</u> You can measure the length of sum the (*AD* + *DE* + *EB*) and observe the figure until the sum has a minimal value! ♣11. <u>Student 1:</u> I agree with you. ♣13. <u>Student 2:</u> Drag the point slowly please! In my opinion, this point is the position where we	Students read information and requirements of the task. This group had an idea to model the situation but they had a habit of drawing a figure in the paper and pencil environment. Thus, they did not use the parallel function of GeoGebra and they drew two arbitrary lines that seem to be parallel. After moving a point they realized that they failed in modeling the situation and they had to use the *Construction level* in the interactive HELP SYSTEM as follows: --- *- Draw two parallel lines representing two banks of the river using the parallel function of GeoGebra.* *- Draw two points A and B representing two cities.* *- Draw movable point D on the straight line* l_1*.* *- Draw a straight line passing through point D and perpendicular to the straight line* l_1*, cut the straight line* l_2 *at a point E. Construct three*

can build the bridge!

♣14. <u>Student 1:</u> Let me see. Yes, the red point which represents the sum (*AG* + *GH* + *HB*) is in the minimum point of the parabola. But what are the special characteristics in this case?

♣16. <u>Student 3:</u> Yeah, it is too difficult to see anything at the moment!



♣19. <u>Student 2:</u> I think these two lines seem to be parallel? Look at the figure!

♣20. <u>Student 3:</u> You can check it again by moving point *A*, point *B* or both to the new positions!

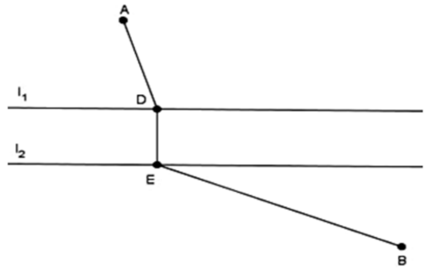
♣23. <u>Student 1:</u> Yes, the situation is the same in the new case! I think we have one more sub-invariant in this problem: When the length of the broken line *AGHB* is minimal, two straight lines *AG* and *HB* are always parallel.

♣24. <u>Student 2:</u> That's right! We have also two parallel lines representing two banks of the river, they are fixed lines; the points *A* and *B* are also fixed, therefore the distances from *A* and *B* to the lines l_1, l_2 are constant numbers. Can you see something more?

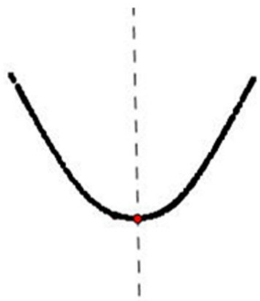
♣25. <u>Student 1:</u> But is it more important now, to show what kind of geometric transformations we use to solve this problem based on realized invariants? They are line reflection, point reflection, translation or rotation?

♣26. <u>Student 3:</u> I think the transformation in this case is line reflection but which line is chosen as a line of reflection? And how can we explain the two parallel lines l_1 and l_2 under a

segments AD, DE, and EB.



Students dragged point *D* but cannot determine where the distance from city *A* to city *B* minimal is.



In order to justify this hypothesis, students used the *Invariance level* in the interactive HELP SYSTEM:

- Draw two straight lines passing through A, G and H, B. You can change the position of points A, B and realize the invariants.

- Write your realized invariants on a piece of paper.

Students named the points, whose lengths are minimal, *G* and *H*. They moved point *G* to and fro many times but they could not see anything. Finally, they decided to use the *Conjecture level* in the interactive HELP SYSTEM:

- What is the relationship between two lines AG and HB?

- Write your conjectures on a piece of paper.

---

line reflection?

♣27. Student 2: Exactly! These two lines cannot be images of each other under a line reflection. But they also can be images of each other under a translation!

♣28. Student 1: That is a reasonable argument but how we can determine the vector of this translation?

♣29. Student 2: You can imagine that if the line l_1 moved a distance towards the line l_2, you will realize the vector of translation. In my opinion, this vector has a length equal to the distance between two banks of the river (vector $\overrightarrow{HG}$).

♣30. Student 3: So we have: The line l_1 is an image of the line l_2 under the translation of vector $\overrightarrow{HG}$. But how can we construct point *G*?

♣31. Student 1: Since $T_{\overrightarrow{HG}}(HB) = AG$ point *G* must have lain on the line $T_{\overrightarrow{HG}}(AG)$ passing through point *A* and point $B' = T_{\overrightarrow{HG}}(B)$.

♣32. Student 2: So now we have to prove that *G*, *H* are two points we can build two ends of the bridge. It means that the distance from *A* to *B* passing through the points *G*, *H* is minimal.

♣33. Student 3: It is obvious that the length of the broken line *AGHB* is smaller the length of broken line *ADEB*. How can we prove this inequality?

♣34. Student 1: We already have the following data: *HG* = *BB'* = *ED* = constant; *HB* = *GB'*, *DB'* = *ED*, and *HB* // *AB'*.

♣38. Student 3: So we may start from the following inequality:

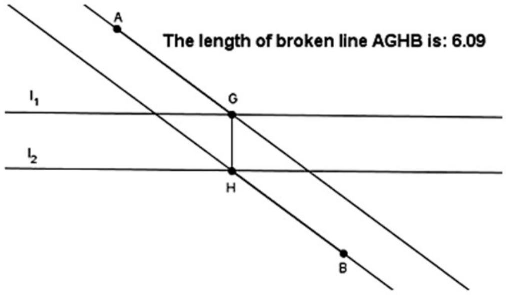
$$AG + GH + HB \leq AD + DE + EB \qquad (1)$$

But how can we prove this inequality?

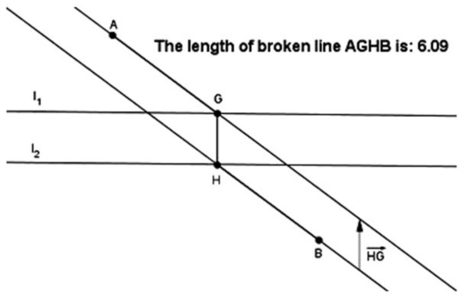
♣40. Student 1: We use the collected data to derive that:

$$AF + FG + GB = AF + BB' + FB' \qquad (2)$$

$$AD + DE + EB = AD + DB' + BB' \qquad (3)$$



Students formulate a conjecture: *If two lines AG and GB are parallel then the sum of broken line AGHB is minimal.*



We realized that some groups of students could not discover key invariant, so they could not solve the problem. Therefore, recognizing invariant is one of the most important phases in the proving process. Students formulated one more conjecture: *The line l_1 is image of the line l_2 under the translation of vector $\overrightarrow{HG}$*. Then, they used abductive argumentation so as to find a way to construct point *G*. In the next step, students had difficulties in validating a conjecture. Thus, they use the *Argumentation level* in the interactive HELP SYSTEM as follows:

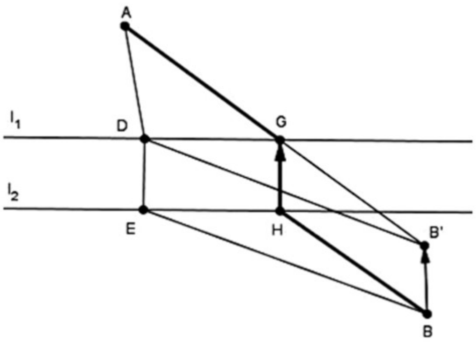
- Compare the length of broken line ADEB and broken line AGHB.

- Write all of your arguments on a piece of paper.

The following steps are abductive argumentation:

<table>
<tr>
<td>♣41. <u>Student 2:</u> Look! We have BB’ as a common summand, so we need only to prove that:
$AF + FB' = AB' \leq AD + DB'$ (4)

♣44. <u>Student 3:</u> That is a triangle inequality! So now we can write a formal proof.
♣47. <u>Student 2:</u> But where can we start to prove this problem?
♣48. <u>Student 1:</u> I think we must construct point B’, point G and point H. After that we can derive the target inequality (4) from the departing inequality (1).


</td>
<td>C_1: $ED = HG = BB'$; $HB = GB'$ and $EB = DB'$
D_1 = ? ⟶ C_1
W_1: Property of translation
D_1: $E = T_{\overrightarrow{HG}}(D)$; $B' = T_{\overrightarrow{HG}}(B)$; and $H = T_{\overrightarrow{HG}}(G)$
C_2: $AG + GH + HB \leq AD + DE + EB$
D_2 = ? ⟶ C_2
W_2: C_1
D_2 = C_3: $AG + GB' + B'B$
$\leq AD + DB' + B'B$
D_3 = ? ⟶ C_3
W_3: BB’ is common summand
D_3 = C_4: $AG + GB' \leq AD + DB'$
D_4 = ? ⟶ C_4
W_4: A, G, B’ are collinear

The conclusion C_4 of the previous step is the data needed to apply the inference to the next step.
D_4 = C_5: $AB' \leq AD + DB'$

D_5 = ? ⟶ C_5
W_5: Triangle inequality
D_5 is a mathematical theorem.</td>
</tr>
<tr>
<td colspan="2">Students wrote a formal proof as follows:
Let $B' = T_{\overrightarrow{HG}}(B)$ and $G = AB' \cap l_1$. We have following equalities: $ED = HG = BB'$;
$EB = DB'$, $HB = GB'$
Therefore: $AD + DE + EB = (AD + DB') + BB' \geq AB' + BB' = AF + FG + GB$
Equality occurs when only when three points A, D, B’ are collinear.</td>
</tr>
</table>

Task 3. (Two-Bridge Problem) *Where would you build two bridges over the two sleeves of a river with parallel straight sides to minimize the length of the path between the cities A and B? (Bridges have to perpendicular to the sides of the river).*

The problem in task 3 is an expansion on task 2. The purpose of designing task 3 is to increase the difficulty of the proving activities such as realizing geometric invariants, formulating conjectures, producing arguments, and writing proofs. We chose the discussion of group 9 in order to investigate how these students deal with more difficult tasks in the proving process. This discussion was transcribed based on their snapshots and audio clips as follows:

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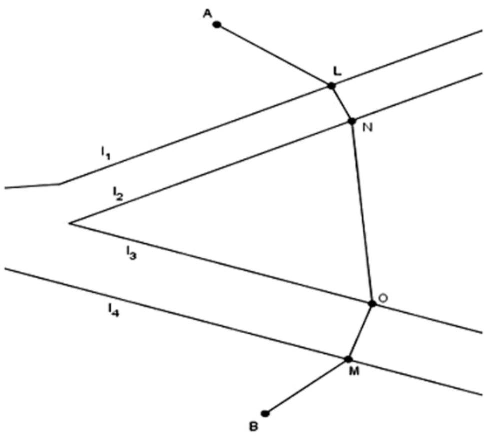
<td>
♣02. <u>Student 2:</u> Firstly, we have to draw four lines representing the banks of two rivers. Take two arbitrary points A, B representing two cities.


♣03. <u>Student 3:</u> It looks like problem in task 3.2. Can we apply something from the previous problem to this case?
♣04. <u>Student 1:</u> Exactly! I think we should draw the figure and find some invariants.
♣05. <u>Student 1:</u> Now we move point L and point O in order to find the best position. But which point moves first?
♣07. <u>Student 2:</u> Now we measure the length of the sum (AL + LN + NO + OM + MB) and move the points!
♣09. <u>Student 3:</u> Look! I think these points satisfy the condition where the sum of the broken line ALNOMB is minimal (the sum is equal to 11.22) and the red point lies on the minimal position of the parabola.
♣13. <u>Student 2:</u> Yes, but what are the special characteristics in this situation? Or what are invariants in this case?
♣14. <u>Student 1:</u> I cannot see anything that is special here!
♣20. <u>Student 2:</u> They are probably parallel!
</td>
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Students read the information and requirements of the task in the interactive HELP SYSTEM. They also realized that this situation is the same as previous problem but in a higher level. They used the Construction level in the interactive HELP SYSTEM as follows:
<hr>
- Draw two pairs of parallel lines (l_1 // l_2) & (l_3 // l_4) representing two sleeves of the river.
- Draw two points A, B representing two cities. Construct two points L, O moving on the line l_1 and the line l_3.
- Construct two straight lines: one passing through point L, perpendicular to l_1, cut l_2 at point N and another passing through point O, perpendicular to l_3, cut l_4 at point M.
- Draw four segments: AL, LN, NO, OM, and MB.
<hr>
Students moved point L and point O but they do not know how to determine the positions of L and O so as to distance between two cities is minimal.
They used Invariance level in the interactive HELP SYSTEM as follows:
<hr>
- Fix one point and move the other. Repeat this process until the length of broken line ALNOMB is minimal. Find the invariants in this case. You can change the position of point A (or point B) and do it again.
- Write realized invariants on a piece of paper.
<hr>
</td>
</tr>
</table>

Three lines *AL'*, *BM'* and *N'O'* are parallel.

♣23. <u>Student 3:</u> Yes, may be! Now we can change the position of point *A* (or point *B*) in order to confirm this invariant. What do you think?

♣24. <u>Student 1:</u> I think it is true! They are parallel again!

♣25. <u>Student 2:</u> Suppose that they are parallel, so which geometric transformation can we use in this case?

♣26. <u>Student 3:</u> Let's consider thoroughly! We have more two pairs of parallel lines (l_1 // l_2 and l_3 // l_4); two fixed points *A*, *B*; and the distances between two banks of each river are also constant numbers.

♣27. <u>Student 1:</u> But point reflection, line reflection, translation and even rotation are preserved parallel lines.

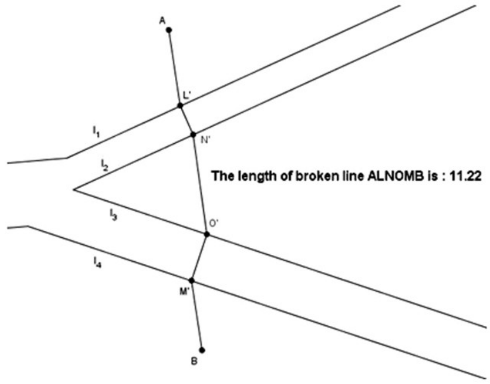
♣28. <u>Student 2:</u> In my opinion, like the previous problem (task 2), we also have two fixed vectors $\overrightarrow{L'N'}$ and $\overrightarrow{M'O'}$. So we can use translation in this case?

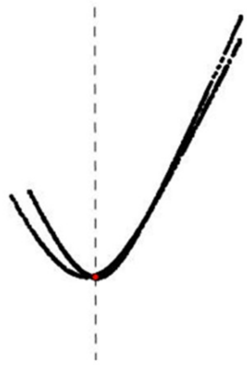
♣29. <u>Student 1:</u> We can derive that $T_{\overrightarrow{L'N'}}(AL') = N'O'$ and $T_{\overrightarrow{M'O'}}(BM') = O'N'$.

♣33. <u>Student 3:</u> What I think is important now is how we can construct points *L'* and *M'* so that we can determine the positions of two bridges.

♣34. <u>Student 2:</u> You mean that we have to construct two points *N'* and *O'*?

♣35. <u>Student 1:</u> That's right! We can construct a straight line passing through *N'* and *O'* because it also contains two points $T_{\overrightarrow{L'N'}}(A) = A'$ and $T_{\overrightarrow{M'O'}}(B) = B'$.





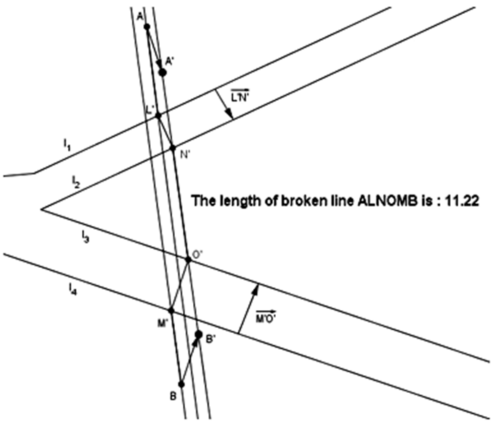
Students had difficulty in recognizing invariants so they could not make any conjecture. They used the *Conjecture level* in the interactive HELP SYSTEM:

- What is the relationship between the three lines AL', BM' and N'O'?

- Write your conjectures on a piece of paper.

They formulated a conjecture: *If three straight lines AL', BM' and N'O' are parallel then the length of the broken line ALNOMB is minimal.*

They used abductive argumentation to find the transformation used to solve this problem.



♣41. Student 1: We find that the broken line *AL'N'O'M'B* is always smaller than the broken line *ALNOMB*.

♣42. Student 3: So we need to prove following inequality:

AL' + L'N' + N'O' + O'M' + M'B
≤ *AL + LN + NO + OM + MB* (1)

♣44. Student 2: We will use some properties of geometric translation such as: *AA' = L'N' = LN*, *BB' = M'O' = MO*, *AL = A'N*, *AL' = A'N'*, *BM = B'O*, *BM' = B'O'*.

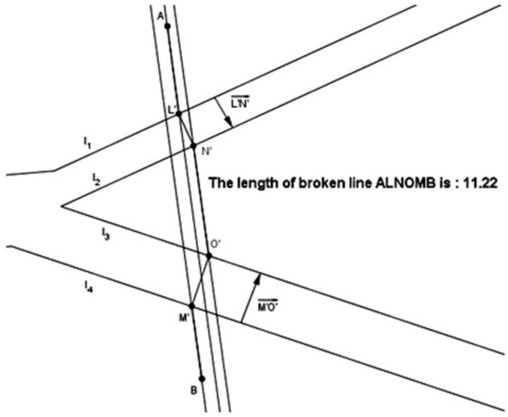
♣48. Student 2: We have:

(*AL' + L'N' + N'O' + O'M' + M'B*)
= (*A'N' + AA' + N'O' + B'B + O'B'*)
= (*AA' + A'B' + B'B*) (2)

(*AL + LN + NO + OM + MB*)
= (*A'N + AA' + NO + B'B + OB'*)
= (*AA' + A'N + NO + OB' + B'B*) (3)

♣51. Student 1: From (2) and (3) we have to prove this inequality: *A'B'* ≤ *A'N + NO + OB'*. But I think this inequality is always true!

♣52. Student 2: I agree with you. So now we have to write a formal proof. First, I think we have to construct four points *L'*, *N'*, *O'*, *M'*. Then we start from the inequality (1) combination with the inequalities (2), and (3).



They made the second conjecture and started collecting more invariants. Students used abductive argumentation in order to determine the geometric transformation which can be used to tackle this problem. Students also used abductive argumentation to construct points which satisfy the requirement of problem.

They used the *Argumentation level* in the interactive HELP SYSTEM:

- Compare the length of broken line AL'N'O'M'B and the length of broken line ALNOMB.
- Write all your arguments on a piece of paper.

Students started collecting arguments and used abductive argumentation in order to produce a proof.

C_1: *AA' = L'N' = LN*, *BB' = M'O' = MO*, *AL = A'N*, *AL' = A'N'*, *BM = B'O*, and *BM' = B'O'*

D_1 = ? ——→ **C_1**
W_1: Property of translation

D_1: $T_{\overrightarrow{L'N'}}(A) = A'$ and $T_{\overrightarrow{M'O'}}(B) = B'$

C_2: *AL' + L'N' + N'O' + O'M' + M'B*
≤ *AL + LN + NO + OM + MB*

D_2 = ? ——→ **C_2**
W_2: C_1

D_2 = C_3: *AA' + A'N' + N'O' + O'B' + B'B*

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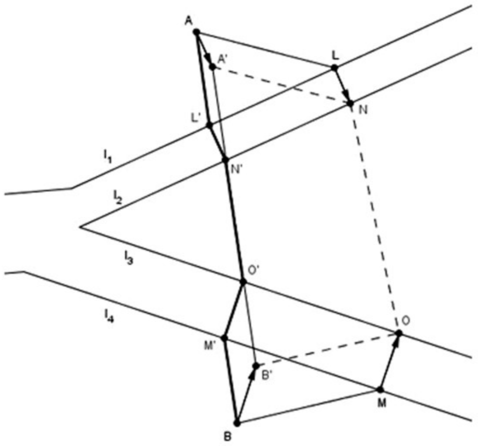
<td>
</td>
<td>$\leq AA' + A'N + NO + OB' + B'B$
D_3 = ? ⟶ **C_3**
W_3: $(AA' + B'B)$ are common summand
$D_3 = C_4$: $A'N' + N'O' + O'B'$
$\leq A'N + NO + OB'$
D_4 = ? ⟶ **C_4**
W_4: A', N', O', B' are collinear
$D_4 = C_5$: $A'B' \leq A'N + NO + OB'$
D_5 = ? ⟶ **C_5**
W_5: Theorem
D_5 is a true statement.</td>
</tr>
<tr>
<td colspan="2">Students wrote their formal proof as follows:
Let point A' be an image of point A under the translation of vector $\overrightarrow{L'N'}$ *and point B' be image of point B under the translation of vector* $\overrightarrow{M'O'}$ *and connect A' and B' by a segment.*
Let N', O' be the intersections of A'B' and l_2, l_3, respectively. We derive that:
$AA' = L'N' = LN,\ BB' = M'O' = MO;$
$AL = A'N,\ AL' = A'N',\ BM = B'O, \text{ and } BM' = B'O'.$
Therefore: $AL + LN + NO + OM + MB = AA' + A'N + NO + OB'$
$\geq AA' + A'B' + B'B$
$= AA' + A'N + N'O' + O'B' + B'B$
Equality occurs when and only when four points A', N', O', B' are collinear.</td>
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During the process of working with three tasks, the students try to transform abductive argumentation (in invariance and conjecture phases) into deductive argumentation in the proof-writing phase. The extractions of the proof in three tasks described the abductive argumentation made by the students. A structural gap seems to be evident in constructing a deductive proof. But these students did not cover this gap. All of the arguments are still an abductive step, so we can observe a structural continuity between argumentation and proof (see PEDEMONTE, 2001). There are some students who solved the problem but they were not able to construct the proof. The strength of the deductive chain seems to be so strong that they are not able to construct continuity in the referential system (as shown in Section 2.3.3) with the argumentation. It means that they lost the connection with the referential system. We have also found that there are continuities or gaps between argumentation and proof and the use of these conceptions relates to the ability of realizing geometric invariants, especially some key geometric invariants[20].

A transition from an abductive argumentation into a proof needs to be reversed in the structure and may demand relevant changes concerning structures. In our research, we tried to clarify the nature of abductive argumentation (particularly in the invariance and conjecture phases) in order to find other analogies or differences in proofs. Indeed, students use sub-arguments both in the argumentation and proofs in order to combine the statements. Words and expressions used in the two processes are often of the same format. But looking more carefully through three tasks, we can observe a gap between the structures of the two processes: sometimes students collected enough arguments but they did not know how to combine them logically. In this case it seems that the students have not faced difficulties in the passage from an abductive argumentation to a deductive proof. This is also difficult for students in writing a formal proof. But with the support of the interactive HELP SYSTEM, they can bridge the gap on their own (see Section 2.3.3).

In the following situation, the combination of inductive and deductive argumentation supports the students in producing valid arguments. This situation also showed a link of three main components of proof writing such as *observation, conjecture*, and then *deduction*. We chose the discussion of group 4 in task 2 to investigate students' behavior and the acts of proving. This group tends to use deduction instead of abduction during the proving process. Students in this group also attained a high proving level. This discussion was transcribed based on their snapshots and audio clips as follows:

♣02. <u>Student 2:</u> Now we need to draw two parallel lines representing two banks of the rivers. Then determine two points, *A* and *B*. ♣03. <u>Student 1:</u> We have to find the position of point *D* (or point *E*) such that the length of broken line *ADEB* is minimal. If we leave the line l_1 out, this problem is the same situation	In this example, they did not use the interactive HELP SYSTEM to solve this problem. Instead, they used a lot of deductive argumentation during their proving process. They used some ideas of proofs from the previous problem in task 1.

[20] *Key geometric invariants* play an crucial role in generating ideas for proofs. Almost students cannot tackle the problem because they do not realize these invariants.

in problem in task 3.1.

♣05. <u>Student 3:</u> Now we need to find a point *A'* such that *A'E* = *AD*.

♣06. <u>Student 2:</u> You mean that we have to find a geometric transformation that can move segment *A'E* to segment *AD*?

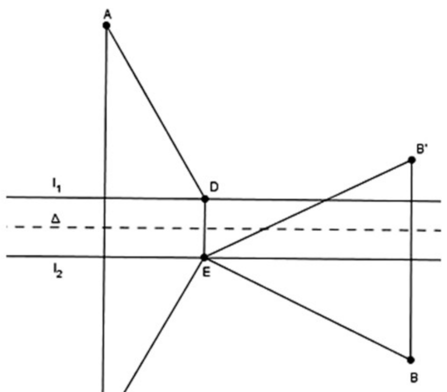
♣07. <u>Student 1:</u> I think we can use translation of vector $\overrightarrow{DE}$ and line reflection l_1 or l_2.

♣08. <u>Student 3:</u> But I want the two points *A'* and *B* to lie on the same side of the line l_2, so I can use the result of problem in task 1.

♣09. <u>Student 2:</u> Yeah, you can draw the line Δ equidistant from the two lines l_1 and l_2. We can construct point $A' = r_\Delta(A)$.

♣10. <u>Student 1:</u> That's right! We have also $r_\Delta(D) = E$.

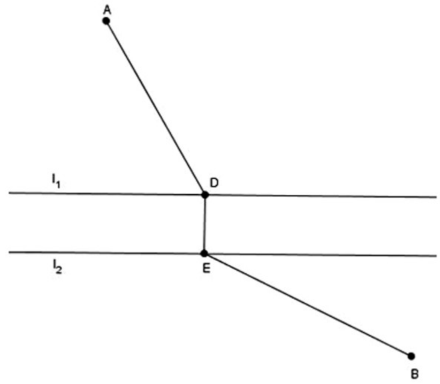
♣13. <u>Student 3:</u> $r_\Delta(A) = A'$ and $r_\Delta(D) = E$ so we have *ADEA'* is an isosceles trapezoid. Therefore, we have *AD* = *A'E*.

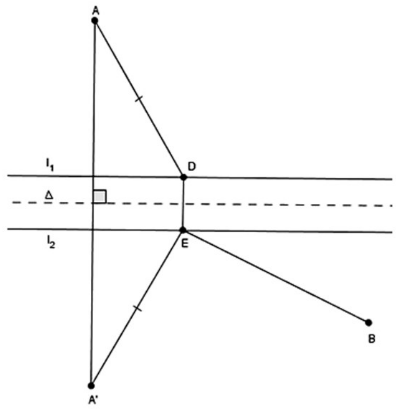


♣15. <u>Student 2:</u> Is the sum (*AD* + *DE* + *EB*) equal to the sum (*A'E* + *DE* + *EB*)?

♣16. <u>Student 1:</u> That's right! Since the length of the segment *DE* is a constant number so we need to find the position of point *E* such that *A'E* + *EB* is minimal.

♣18. <u>Student 3:</u> Now we can use the result





Students used deductive argumentation.

D₁: $r_\Delta(A) = A'$ ⟶ **C₁:** *AD* =*A'E*
W₁: Property of line reflection

They put *t* = *AD* + *DE* + *EB*

D₂: *t* ⟶ **C₂:** *t* = *A'E* + *DE* + *EB*
W₂: C₁

D₃: $r_{l_2}(B) = B'$ ⟶ **C₃:** *EB* = *EB'*
W₃: Property of line reflection

D₄ = C₂: ⟶ **C₄:** *t* = *A'E* + *DE* + *EB'*
W₄: C₃

D₅ = C₄: ⟶ C₅: *t* ≥ *A'B'* + *DE*
W₅: Triangle inequality

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from the problem in task 1!
♣19. <u>Student 1:</u> That is true! We construct point $B' = r_{l_2}(B)$ and we have $EB = EB'$. Then we derive that $A'E + EB = A'E + EB'$.
♣22. <u>Student 3:</u> Exactly! This sum is always smaller than $A'B'$.
♣23. <u>Student 2:</u> But how can we determine the position where we can build a bridge?
♣25. <u>Student 1:</u> The sum ($A'E + EB$) is minimal when and only when A', E, B' are collinear.
♣29. <u>Student 2:</u> It means that the first place to build the bridge is at the intersection of the line $A'B'$ and the line of reflection l_2?
♣32. <u>Student 1:</u> That's right! We can say that C_2 is intersection of the line $A'B'$ and the line l_2. We must show that C_2 is the best position to build the bridge. From that we can construct point C_1 (C_1 is the intersection of the line l_1 and the line which passing through C_2, perpendicular to l_1).
♣36. <u>Student 3:</u> We put $t = AD + DE + EB$. Because three points A', C_2, B' are collinear so we have:
$t \geq A'C_2 + C_2B' + DE$.
♣39. <u>Student 2:</u> Since $r_{l_2}(B) = B'$ so we have $C_2B = C_2B'$. From that, we derive:
$t \geq A'C_2 + C_2B + C_1C_2$ ($C_1C_2 = DE$).
♣42. <u>Student 1:</u> $t \geq AC_1 + C_1C_2 + C_2B$ (since $A'C_2 = AC_1$). Therefore, C_1 and C_2 are two points that we can build the bridge.
♣43. <u>Student 3:</u> Now we must to deductively write a formal proof.
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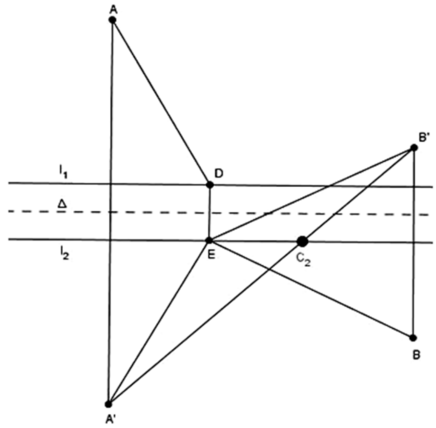
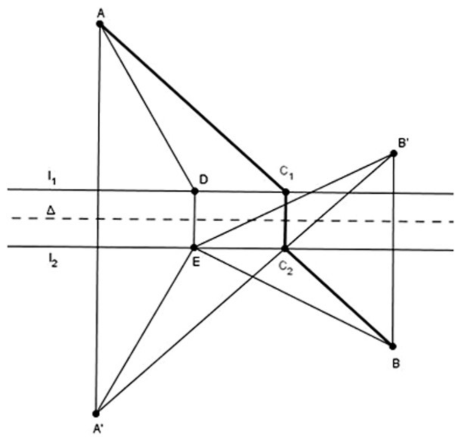


The argumentation's step is an abduction:
$D_6 = C_5$: ——→ **C_6:** $t \geq A'C_2 + C_2B' + DE$
W_6: A', C_2, B' are collinear
D_7: $r_{l_2}(B) = B'$ ——→ **C_7:** $C_2B = C_2B'$
W_7: Property of line reflection


$D_8 = C_6$: ——→ **C_8:** $t \geq A'C_2 + C_2B + C_1C_2$
W_8: $DE = C_1C_2$ and C_7
$D_9 = C_8$: ——→ **C_9:** $t \geq AC_1 + C_1C_2 + C_2B$
W_9: AC_1C_2A' is isosceles trapezoid
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<td colspan="2">
Students wrote a formal proof as follows:
Let Δ be a line equidistant from two lines l_1 and l_2. Let be $r_\Delta(A) = A'$ and $r_{l_2}(B) = B'$, we have:
$AD + DE + EB = A'E + DE + EB'$ (since $AD = A'E$ and $EB = EB'$) (1)
On the one hand, $(A'E + EB') + DE \geq A'B' + DE = A'C_2 + C_2B' + C_1C_2$ (2)
</td>
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(since A', C_2, B' are collinear and $DE = C_1C_2$).

On the other hand, $A'C_2 + C_2B' + C_1C_2 = AC_1 + C_1C_2 + C_2B$ (3)

(since $A'C_2 = AC_1$ and $C_2B' = C_2B$).

Finally, from (1), (2), and (3), we derive that:

$$AD + DE + EB \geq AC_1 + C_1C_2 + C_2B$$

Equality occurs when and only when $D \equiv C_1$ and $E \equiv C_2$.

In this example, we have also found that high proving-level students do not want to use the computer with a simple problem, but they effectively use the computer to tackle another difficult problem. During the process of realizing geometric invariants, an argumentation activity is developed in order to produce a conjecture. After the statement expressing the conjecture is validated, a valid proof is produced partly, thus laying the groundwork for deductive proof. This proof is a particular kind of argumentation[21] based on the process of producing a conjecture and proof. It can be said that, students constructed cognitive unity (as shown in Section 2.3.2) during the process of validating their conjectures. The produced arguments are used to write proofs. Some groups who could not construct a cognitive unity could also write a formal proof. Therefore, constructing this unity is one condition for understanding the proving process. Furthermore, in order to prove a problem using geometric transformations, dynamic visual thinking (see Section 2.4.4) is also a very important factor within a dynamic geometry environment. It helps students by producing more relevant conjectures, fewer false conjectures, and by allowing the students' convictions in all conjectures to be significantly greater when compared to the conjectures formed in the static environment (e.g. GILLIS, 2005).

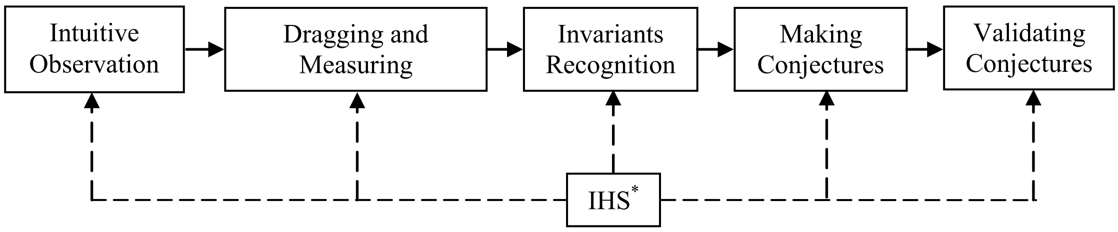
ROTH (2005) has also affirmed the crucial role of dynamic visual thinking in argumentation and proof. This kind of thinking also supports students in solving problems and exploring for generalization:

Welche Fähigkeit war notwendig, um diesen Beweis führen zu können? THALES VON MILET musste in ein statisches Phänomen, hier den Kreis mit eingezeichnetem Durchmesser, eine Bewegung hineinsehen und damit

[21] In this dissertation, we distinguish the differences between argumentation as producing a conjecture (that produces spontaneous arguments) and argumentation as forming a proof (that produces helpful/valid arguments for writing proofs).

argumentieren können. Allein die Fähigkeit, sich eine Bewegung vorstellen zu können, reicht offensichlich nicht aus, um Problem zu können, Zusammenhänge zu entdecken und ganz allgemein (mathematische) Phänomene zu erforschen. Man muss vielmehr mit der hineingesehenen Bewegung auch argumentieren können (see ROTH, 2005, p.21).

Mathematical "conjectures"" are formed by observing data, recognizing patterns, and making generalizations. These generalizations are unproven statements based on inductive reasoning (see e.g. SERRA, 2008). It was also found that most students could not make conjectures after they constructed the figure; they needed to drag the dynamic figure and measure some distances, angles, or check some relationships before they could formulate their conjectures. However, they could make a conjecture after they realized some initial geometric invariants. So we can use the following schema:



(*) The interactive HELP SYSTEM

Figure 3.1: *The role of the interactive HELP SYSTEM in the proving process*

We think abduction may bring the structure of constructing a cognitive unity to light and simultaneously illuminate the importance of initial arguments which are generated during the process of modifying the understanding and the constructing of a proof. In our research, we have shown that three kinds of inferences play an essential role in different consecutive phases of the proving process: *realizing invariants*, *formulating conjectures*, *producing arguments*, *validating conjectures*, and *writing deductive proofs*. Each kind of inference is used in a certain phase of the proving process, but sometimes the combination of these kinds of inferences (induction, deduction, and abduction) plays a fundamental role in realizing geometric invariants and formulating conjectures from 'observed facts' or writing a formal proof from produced arguments. This combination can be described in the following diagram:

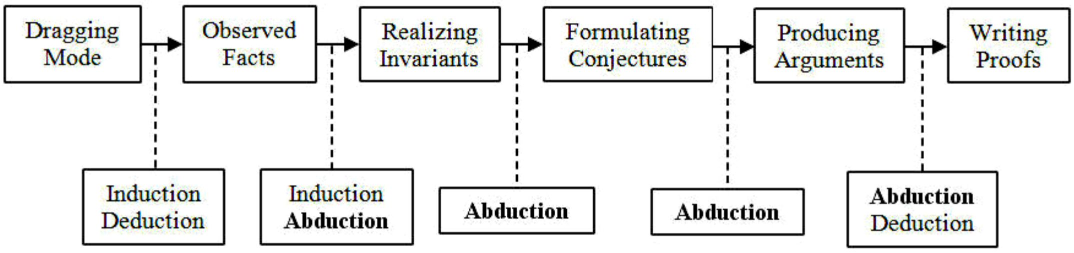


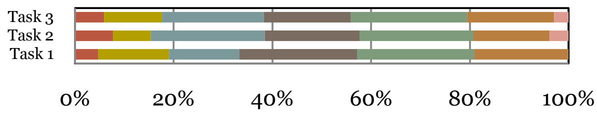
Figure 3.2: *Three kinds of inferences in the proving process*

From our observations, the interactive HELP SYSTEM supported students in formulating and checking their conjectures. Their behaviors during the process of constructing their mathematical knowledge using GeoGebra software are described below:

- *Drag some points to and fro for a short distance and distort the given figures;*
- *Modify the given figures by adding some auxiliary figures;*
- *Try to find the relationships between the figures by reading diagrams and using the support from the interactive HELP SYSTEM;*
- *Sometimes they make some mistakes when they manipulate with GeoGebra software;*
- *Write down some sub-arguments during the proving process on a piece of paper;*
- *Interact (making a plan, looking for geometric invariants, making conjectures, collecting arguments, combining produced arguments, and writing a formal proof) with partners in their groups. They also discuss with others after using the interactive HELP SYSTEM.*

In general, deforming the figure by dragging allows students to directly observe how various components of geometric figures and their measures are affected by dynamic changes. By generalizing the patterns that emerge during these explorations and observing changes in the figures and their measures, students may be able to form their own mathematical conjectures (see e.g. GLASS et al., 2001). The use of dynamic software enables students to examine many cases, thus extending their ability to formulate and explore conjectures. Although students seemed to have gone through a similar process of making conjectures, there were some differences in their behavior. It seems that students were quite hesitant to drag the figure because most of them only dragged some points to and fro for a short distance. It seems that they are not used to having an exploratory activity in a dynamic environment. It was also found that many students could not prove their conjectures on their own or even in collaboration with their partners. They all needed some hints or guidelines from the interactive HELP SYSTEM, especially in the invariance phase. We also found that, realizing geometric invariants plays a crucial role in solving the problem in the three tasks because it not only supports in formulating conjectures but also in generating ideas of proof. Thus realizing invariants is also a condition for understanding the development of proving process (as shown in Section 2.3).

We can see more clearly this development through the following proving profile charts. They facilitate the efficiency of the students' *invariance recognition*, *data collection*, *conjecturing*, *deductive reasoning*, *arguments combining*, *proof writing*, and *the level of proving* required by students. By focusing on invariance and the overall structure of a proof, that is, on conjecturing and the proving process, the proving profile can provide valuable insight into how students approach problem-solving tasks by using geometric transformations:



	Task 1	Task 2	Task 3
Information	1	2	2
Construction	3	2	4
Invariance	3	6	7
Conjecture	5	5	6
Argumentation	5	6	8
Proof	4	4	6
Delving	0	1	1

Figure 3.3: *Time distribution during the proving process (in minutes)*

In this research, students followed the open-ended questions and explorative tasks of the interactive HELP SYSTEM. It is a necessary factor in the proving process, e.g., for the recognition of geometric invariants, the exploration of the problem situation, and the collection of additional data (especially valid arguments). In the snapshots, the first exploration phase (including finding invariants, formulating conjectures) consists mainly of constructing drawings, measuring the segments or angles, checking the relationships between figures, lines, angles, etc. The students spent almost all of their time on *invariance*, *conjecture*, *argumentation*, and *proof* phases. In this analysis, the proving process was separated into chucks, which include different levels of proving. We analyzed based on students' discussion and movements on the screen. We considered any period of time with "no change" or "no sound" as silence time (or thinking).

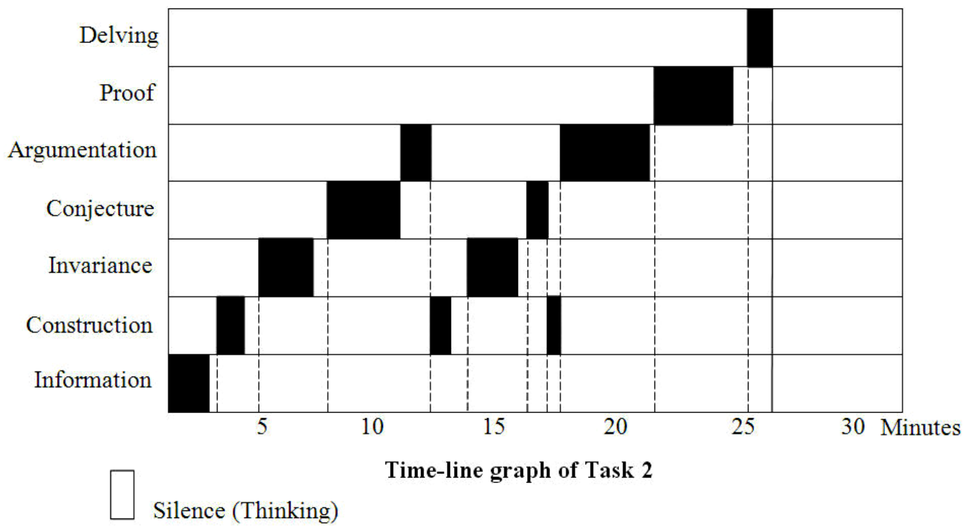
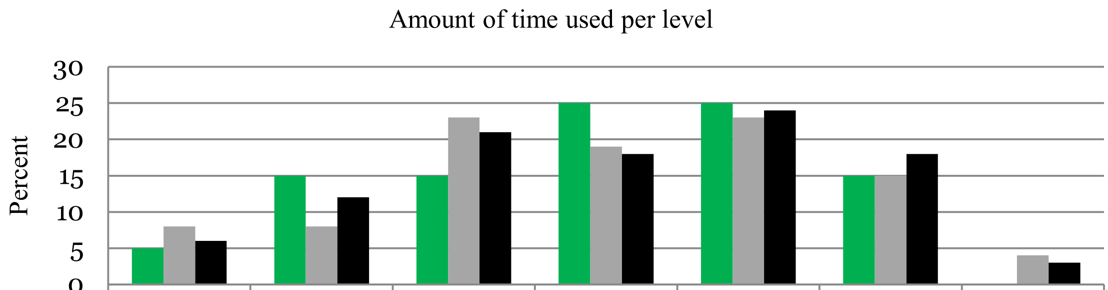


Figure 3.4: *Typical time-line graph in the proving process (task 2)*

The graph showed that, in the proving process, students' level of proving moved from one to another continuously. After they found a geometric invariant, they made a conjecture and produced sub-arguments. They continued drawing some auxiliary figures and found more invariants, and formulated new conjectures, and so on. Finally, they combined the arguments which were collected in previous phases to write a formal proof. The graph also presented a surprised situation that students spent little time delving into the problem. This phenomenon explained that students were satisfied with their solution or lack of time.

We will use time-based analysis to clearly understand the time during the proving process. Fig. 3.4 gives an overview of the time portion (in percent) of each level in the proving process. One can see that most of the time is spent on the third, fourth and fifth level, which is the *recognition invariants, formulation conjectures*, and *organization of arguments* in a deductive chain. The second and third level, the experimental stage, which mainly consists of drawing, measurements and check conjectures, takes about one third of the time. The third phase, in which the exploration of the invariants and producing of sub-arguments take place, comprises of 15 to 23% of the proof time. Therefore, we can argue that students used a lot of time to construct a cognitive unity or produce arguments (as shown in Section 2.3.2) in the transition from the conjecture to the proof phase.



	Information	Construction	Invariance	Conjecture	Argumentation	Proof	Delving
Task 1	5	15	15	25	25	15	0
Task 2	8	8	23	19	23	15	4
Task 3	6	12	21	18	24	18	3

Figure 3.5: *Amount of time during the proving process (in percent)*

This diagram reflects the typical process of dealing with proof problems in the captured snapshots lessons. First of all, the students have to draw a geometrical figure and make some observations on an invariance level. Afterwards a conjecture is discussed and formulated (level 3). If the students are not able to generate a proof idea, then they can use the assistance of the interactive HELP SYSTEM. The writing on the piece of paper is frequently a collection of sub-arguments as expected in fourth level of proving. For students who attain level 2, 3 of proving, this help system may bridge the gap between conjecture and argumentation. These students do not know how to produce arguments until they answer open-ended questions or discover some explorative tasks in the system.

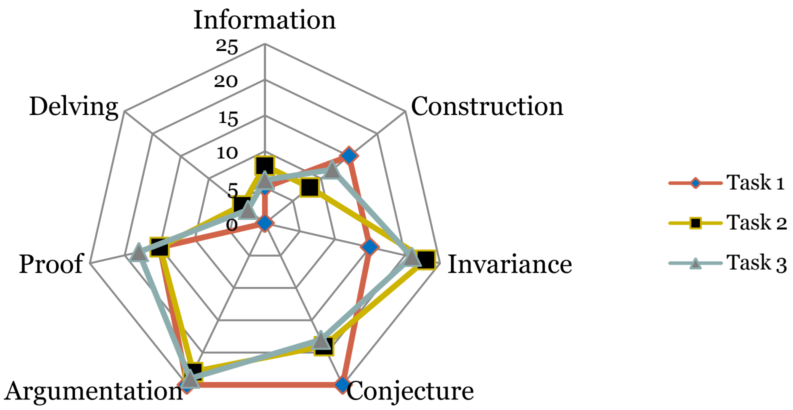


Figure 3.6: *Time distribution with the support of the interactive HELP SYSTEM*

Based on this distribution of time, we can affirm that *invariance*, *conjecture*, *argumentation*, and *proof* are important phases in the proving process and for the learning of geometric proofs. However, realizing geometric invariants seems to be the most crucial phase for generating ideas for proofs in the proving process. Students with a high proving level can easily recognize invariant elements when they are observing or dragging within a dynamic geometry environment. Finally, through our observation analysis, we can partly answer research question 1, research question 4, and research question 5 as follows:

- *The interactive HELP SYSTEM supports students in realizing geometric invariants, producing and collecting arguments, and combining arguments to write a formal proof;*
- *Students realize geometric invariants by dragging some points, measuring segments or angles, checking relationships, or even using dynamic visual thinking in a static environment;*
- *Students formulate a conjecture after they discover invariants. They usually use 'backwards' strategy (like abduction) to analyze the way to make conjectures. This strategy also helps students write a deductive proof by reversing the abductive structure, which is sketched when producing arguments. Therefore, abduction is considered as a tool in bridging the structural gap between argumentation and proof;*

- *Students validate their conjectures by using sub-arguments that were produced as they were dragging and discovering geometric invariants. Thus, dragging modality in a dynamic geometry software and hints in the help system may bridge the gap between conjecture and argumentation. They usually discuss in their groups, take notes to remember their arguments and write proofs;*
- *Normally, students produce, collect, and combine their arguments by using abductive argumentation. However, some students, who have a high level of proving, tend to use deductive argumentation during the proving process. This different behaviors show that it is easier for high level students to connect the structural gap between argumentation and proof.*

3.2.2. Questionnaires

The first questionnaire survey was designed and administered only to the experimental group as a whole after the period of experimental teaching. The survey includes twenty close-ended questions and two open-ended questions. The main purpose of this questionnaire was an investigation on *the influence of the use of the interactive HELP SYSTEM on the students' proving process* in the learning of geometric transformations. Respondents were required to indicate their choices (the extent to which they agree or disagree) to each item in a 5-point LIKERT type scale: strongly disagree, disagree, not sure, agree, and strongly agree. For coding purposes, the response options were coded 1, 2, 3, 4, or 5 from strongly disagree to strongly agree. This survey was divided into seven categories: attitudes towards the interactive HELP SYSTEM, difficulty in constructing drawings, realizing geometric invariants, formulating conjectures, collecting and organizing arguments, writing formal proofs, and delving into the problem.

The second questionnaire survey consists of ten items aimed at *classifying different levels of realizing geometric invariants* while proving. The same questionnaire was administered during the pre- and post-test to reveal whether the interactive HELP SYSTEM impacts the development of the proving process during the period of experimental teaching or not. The first version of the questionnaire was conducted after two weeks of experimental teaching to access the suitability and clarity of items. The feedback obtained from students in this survey was used to modify identified weaknesses in the items. The reliability of the questionnaires was also considered. It means consistency of the research instruments used to measure particular variables. The same results, as the instruments are administered again in a stable condition, guarantee reliable instruments (see e.g. MAKGATO, 2003). Therefore, reliability coefficients (*Cronbach's alpha*) were calculated using the SPSS 17.0 statistical software to determine the reliability of the measuring instruments in the pre-test and in the post-test. Finally, four questions (3, 6, 7, and 8) are preserved and the remaining questions are modified with more reliability to form the final version of the questionnaire. In this questionnaire, students had to fill their name and self-assess about their corresponding levels of realizing geometric invariants. The data obtained from the respondents were analyzed statistically:

Table 3.1: *Descriptive statistics of questionnaire*

No	Mean	Std. Deviation	N
Question 1	3.79	.946	67
Question 2	3.30	.718	67
Question 3	3.72	.966	67
Question 4	3.76	.818	67
Question 5	3.55	1.063	67
Question 6	3.36	.980	67
Question 7	4.06	.795	67
Question 8	3.28	.901	67
Question 9	3.67	.877	67
Question 10	4.07	.942	67
Question 11	4.01	.826	67
Question 12	3.60	1.045	67
Question 13	4.04	.944	67
Question 14	3.39	.920	67
Question 15	3.21	.946	67
Question 16	3.64	.933	67
Question 17	4.03	.904	67
Question 18	3.72	.918	67
Question 19	3.33	.824	67
Question 20	3.21	.845	67

The first part of the output gives a summary of the responses by the participants to individual questions, and provides information regarding the *Mean* and *Standard*

Deviations for responses to each question, and a report stating how many participants completed the question. We can see that 67 participants answer all the questions. We can also see which questions elicited a wide variety of responses, as shown by the larger standard deviations. Higher mean scores therefore indicate questions where participants were at the agreement end of the rating scale.

The data from question 1 have showed that students enjoy working with the interactive HELP SYSTEM since the mean values of students' responses was close to '4' (Mean = 3.79). Responses for question 7 were relatively high (Mean = 4.06) showed that the students have difficulty in constructing auxiliary figures in the proving process. Auxiliary figures also play a very crucial role in finding the relationships among figures. In question 3, the students slightly agreed (Mean = 3.72) that working with the interactive HELP SYSTEM makes them believe more in the success of the proving process. In this case, the system played the role of an authoritative factor. These findings are also supported by many researchers who found that the authoritative factor helps students become more confident with their problem-solving success.

Students also highly agreed that discovering invariants helps them find more ideas for proofs (Mean = 4.07). It means that after they realized invariants and perhaps the relationship, they could 'flash' the ideas of proofs in their minds. Therefore, in their opinion, realizing geometric invariants is the most crucial phase in the proving process (Mean = 4.01). In fact, almost all students, who could not solve the problem, did not recognize these crucial invariants. During the proving process, the students produced sub-arguments when formulating a conjecture (Mean = 4.04). As a result, it is important for argumentation and proof phases to collect all sub-arguments and combine them in a logical way. There have been a lot of students who could not combine these arguments to form a proof without the interactive HELP SYSTEM (Mean = 4.03). Hence, this system supported them in arranging their arguments and using reasonable rules of inference. This difficulty[22] was confirmed in question 18 (Mean = 3.72) "writing a formal proof is the most difficult phase of the proving process" and showed the

[22] Students' difficulties in the proving process were classified into different categories in the semi-structured interviews analyses.

cognitive gap between argumentation and proof in the proving process. This gap can be connected if students know how to use their arguments in order to write proofs.

Table 3.2: *Summary item statistics*

	Mean	Minimum	Maximum	Range	Maximum/ Minimum	Variance	N of Items
Item Means	3.637	3.209	4.075	.866	1.270	.091	20
Item Variances	.827	.516	1.130	.614	2.191	.023	20

Table 3.3: *Scale statistics*

Mean	Variance	Std. Deviation	N of Items
72.75	88.435	9.404	20

The total scores of the questionnaire were examined. Participants scored a mean of 72.75, with a variance 88.435, and a standard deviation of 9.404. This small standard deviation thus indicates that there are no wide variations in the scores of our participants for the overall total score on the questionnaire. The *Item Means* row details descriptive statistics for a response on individual questions. As we can see from table 3.2, the mean score of the items is 3.637. This mean score shows a positive attitude towards the interactive HELP SYSTEM in the proving process. The *Minimum* and *Maximum* values are the two most extreme scores selected by participants; these are 3.209 and 4.075, which indicates that no questions selected by all of respondents were from the most extreme ends of the scale. The *Item Variances* row shows the variance in scores when looking at individual items.

Table 3.4: *Item-total statistics*

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Squared Multiple Correlation	Cronbach's Alpha if Item Deleted
Question 1	68.96	76.377	.675	.714	.839
Question 2	69.45	79.281	.676	.785	.842
Question 3	69.03	77.999	.557	.763	.844

Question 4	68.99	80.803	.473	.557	.848
Question 5	69.19	79.795	.395	.528	.852
Question 6	69.39	79.908	.432	.464	.850
Question 7	68.69	81.643	.428	.469	.850
Question 8	69.46	79.919	.478	.621	.848
Question 9	69.07	78.131	.615	.614	.843
Question 10	68.67	78.557	.538	.491	.845
Question 11	68.73	77.836	.681	.644	.841
Question 12	69.15	78.856	.457	.444	.849
Question 13	68.70	78.910	.515	.472	.846
Question 14	69.36	79.658	.483	.546	.848
Question 15	69.54	77.252	.619	.595	.842
Question 16	69.10	85.580	.115	.275	.862
Question 17	68.72	84.327	.198	.202	.859
Question 18	69.03	85.575	.119	.223	.862
Question 19	69.42	84.883	.189	.297	.858
Question 20	69.54	83.071	.302	.388	.854

The *Corrected Item-Total Correlation* column shows the relationship between the responses on individual questions and the overall total score of the questionnaire. All of questions in the questionnaire are reliable questions because they have a positive relationship with the overall total (ideally being above 0.3), and only four questions (16, 17, 18, 19), with positive relationship, were smaller than 0.3. The effects that individual questions can have on the overall reliability of the questionnaire are highlighted by the inverse relationship between the *Corrected Item-Total Correlation* and the *Alpha If Item Deleted* columns. The importance of the weak relationship, for example, between question 16 and the overall total score on the questionnaire, is reflected by an increase in the *Alpha* score for the questionnaire if this item is omitted. *Cronbach's Alpha* is the most popular method of examining reliability. The calculation of *Cronbach's Alpha* is based on the number of items and the average inter-item correlation (see e.g. HINTON, 2004). A high correlation between different items in this questionnaire indicates that it would measure the same thing if there would have been only small values for the error.

Table 3.5: *Reliability statistics*

Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
.856	.858	20

Table 3.5 shows the overall *Alpha* value for our questionnaire. The reliability coefficient for all twenty items in the questionnaire is displayed as a simple *Alpha*, and a *Standardized Alpha*. The *Alpha* score of this questionnaire is 0.856. It is generally taken to indicate a scale of high reliability. In general, from the responses of the 20 questions we realized that the help system provides a strategy for students during the proving process. They would like to work with this system in order to approach proof-related problem easier. As a supplement, two open-ended questions were given at the end of the questionnaire aimed at gaining validity for our report and determining students' barriers in the proving process.

Open-ended questions analysis. There are two open-ended questions in the first questionnaire. The aims of these questions are to investigate the students' difficulties and the role of interactive HELP SYSTEM in the proving process:

1. Where do you usually meet difficulties in the proving process?
2. Which stages in the proving process does the interactive HELP SYSTEM support you?

The comments from the students indicate that most of them had a positive view towards the interactive HELP SYSTEM. They also suggested that the teacher provided more tasks with this system on the official website for the online course so that they can do these exercises at home. Most of the students also perceived that they usually met difficulties in *realizing invariants*, *validating conjectures*, and *writing a formal proof*. It means that they did not know what to do until they discovered some invariants. More importantly, they had to draw auxiliary lines in order to establish new relationship, new conjectures. They also did not 'flash' the idea of proof until they revealed some crucial invariants, especially 'moving' invariants[23]. Some students wrote that they could have not realized invariants, made conjectures, collected arguments, and combined arguments to form a formal proof without the interactive HELP SYSTEM. It depends on the students' levels of proving. Students also said that this system made them believe more and could be a strategic scaffolding in the proving process (Mean = 3.76). They also

[23] In this dissertation, we differentiate between 'static' invariants and 'moving/dynamic' invariants. Moving/dynamic invariants required high level of visualization and dynamic visual thinking. We will explain more clearly in the second survey.

suggested that the system was a tool which extended their zone of proximal development[24] (see e.g. NEWMAN & HOLZMAN, 1993). The students also commented on the methodological structure of this system. It helped them get used to a method of proof or proving strategy while solving a geometric problem by using geometric transformations:

Invariance → Conjecture → Argumentation → Proof

The second survey is the ten-item questionnaire. It aimed at classifying different levels of realizing geometric invariants while proving. The same questionnaire was administered during the pre- and post-test in order to investigate whether or not the interactive HELP SYSTEM has an impact on the increase of these levels during the period of experimental teaching. The first version of the questionnaire was conducted after two weeks of experimental teaching to access the suitability and clarity of the items. The feedback obtained from the students in this survey was used to modify identified weaknesses in the items. Finally, four questions (3, 6, 7, and 8) are preserved; the remaining questions were modified with more reliability in order to form the final version of the questionnaire. In this questionnaire, students had to fill their name and self-assess about their corresponding levels of realizing geometric invariants.

Table 3.6: *Levels of realizing geometric invariants*

	Mean	Std. Deviation
Question 1	4.14	.714
Question 2	3.46	.628
Question 3	3.31	.719
Question 4	3.58	.920
Question 5	3.86	1.046
Question 6	3.17	.890
Question 7	2.31	.908
Question 8	2.19	.926
Question 9	3.17	1.120
Question 10	3.66	.843
Post-test	7.19	1.719

[24] The zone of proximal development in this case was defined as a conceptual distance between what the students can do on their own and what they can grasp with the support of the interactive HELP SYSTEM.

Table 6 shows the results of the students' responses to all items which were categorized into five levels. In terms of level 1, most of the students agreed that they had a high rank in level 1 because the mean values for several items (like question 1, question 2, and question 3) for this level are also high. These findings are supported by the increased mean values in level 2 (question 4, question 5). This also indicates that almost all of the students attained level 2. This means that they can realize static invariants (especially some given invariants) and some moving invariants by using the dragging mode in dynamic geometry environment. Table 6 also suggested that a lot of students attained level 4 (question 9), but only some students got a high rank in level 3 (question 6, question 7, and question 8). This is also their difficulty in the proving process. Data from question 10 show that students slightly agree that dynamic visualization played an important role in developing a sense of proof in geometric transformations (Mean = 3.66).

Table 3.7: *Inter-item correlation matrix of the 10 questions*

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Post-test
Q1	1.000	.645	.550	.583	.474	.649	.647	.655	.559	.535	.706
Q2	.645	1.000	.672	.727	.668	.740	.720	.754	.700	.611	.765
Q3	.550	.672	1.000	.650	.659	.650	.753	.744	.653	.629	.765
Q4	.583	.727	.650	1.000	.705	.820	.731	.740	.785	.607	.772
Q5	.474	.668	.659	.705	1.000	.835	.710	.731	.770	.574	.709
Q6	.649	.740	.650	.820	.835	1.000	.816	.815	.841	.596	.809
Q7	.647	.720	.753	.731	.710	.816	1.000	.822	.797	.716	.840
Q8	.655	.754	.744	.740	.731	.815	.822	1.000	.854	.632	.830
Q9	.559	.700	.653	.785	.770	.841	.797	.854	1.000	.733	.820
Q10	.535	.611	.629	.607	.574	.596	.716	.632	.733	1.000	.698
Post-test	.706	.765	.765	.772	.709	.809	.840	.830	.820	.698	1.000

Data from table 7 also indicate that the items in the questionnaire have a relatively high correlation, especially in question 8 and question 9. It shows that students can realize a geometric transformation by visualizing the movement of the picture or figure in their minds, they can better differentiate invariants in different types of geometry (coefficient correlation $r = 0.854$, at $p < 0.05$). It also indicates that if students can realize moving invariants (level 2), they can also easily realize a geometric transformation (level 3) in a proof-related problem (r = 0.835). Coefficient correlation in the last row indicates that levels of realizing geometric invariants highly affect students' achievements in the post-test, especially students who attained level 3 (r > 0.8). For the question 3, question 6, question 7, and question 8, the data showed that mean values for all items significantly increased in the post-test differently. This data indicates that the interactive HELP SYSTEM did support students in improving the students' levels of realizing geometric invariants.

Table 3.8: *Ranks in question 3 of questionnaire*

		N	Mean Rank	Sum of Ranks
Question 3 Pre-test – Question 3 Post-test	Negative Ranks	31[a]	17.03	528.00
	Positive Ranks	2[b]	16.50	33.00
	Ties	34[c]		
	Total	67		

Table 3.9: *Test statistics[b]*

	Question 3 Pre-test – Question 3 Post-test
Z	-5.000[a]
Asymp. Sig. (2-tailed)	.000

a. Based on positive ranks.
b. Wilcoxon Signed Ranks Test.

Table 3.8 summarizes the changes in the students' levels after the post-test. It shows that 31 students increased their levels while only 2 students had a decrease in their levels. From table 3.9 it can be seen that Z = -5.000. A two-tailed analysis was carried out by default, which is significant at $p < 0.01$. From this we can conclude that a student's level rates before and after the experimental teaching significantly differed and the level on the post-test being significantly higher than one on the pre-test.

Table 3.10: *Ranks in question 6 of questionnaire*

		N	Mean Rank	Sum of Ranks
Question 6 Pre-test – Question 6 Post-test	Negative Ranks	39[a]	20.53	800.50
	Positive Ranks	1[b]	19.50	19.50
	Ties	27[c]		
	Total	67		

Table 3.11: *Ranks in question 7 of questionnaire*

		N	Mean Rank	Sum of Ranks
Question 7 Pre-test – Question 7 Post-test	Negative Ranks	33[a]	17.52	578.00
	Positive Ranks	1[b]	17.00	17.00
	Ties	33[c]		
	Total	67		

Table 3.12: *Ranks in question 8 of questionnaire*

		N	Mean Rank	Sum of Ranks
Question 8 Pre-test – Question 8 Post-test	Negative Ranks	22[a]	13.00	286.00
	Positive Ranks	3[b]	13.00	39.00
	Ties	42[c]		
	Total	67		

The findings of the Wilcoxon test continued to validate the above results: *student's level of realizing geometric invariants significantly differed after a period of the experimental teaching* (specifically, question 6: $Z = -5.892$, $N = 67$, $p < 0.01$; question 7: $Z = -5.425$, $N = 67$, $p < 0.01$; and question 8: $Z = -3.800$, $N = 67$, $p < 0.01$).

Students' levels of realizing geometric invariants. On the basis of obtained results from the second questionnaire, five students (S_0, S_1, S_2, S_3, and S_4) who attained different levels of realizing geometric invariants were chosen for the next study. The researcher required these students to do four explorative tasks. While the students tried tackling the tasks, the researcher observed and made notes of their workings. At last, their solutions were classified into the five described levels of realizing geometric invariants (as shown in Section 2.3.1). Their behaviors and solutions of the tasks were summarized as follows:

Task 4. (Parallelogram Problem) *Let ABCD be a parallelogram. The bisectors of four angles $\angle A$, $\angle B$, $\angle C$, and $\angle D$ intersect each other forming a quadrilateral MNPQ. What are special characteristics of this quadrilateral?*

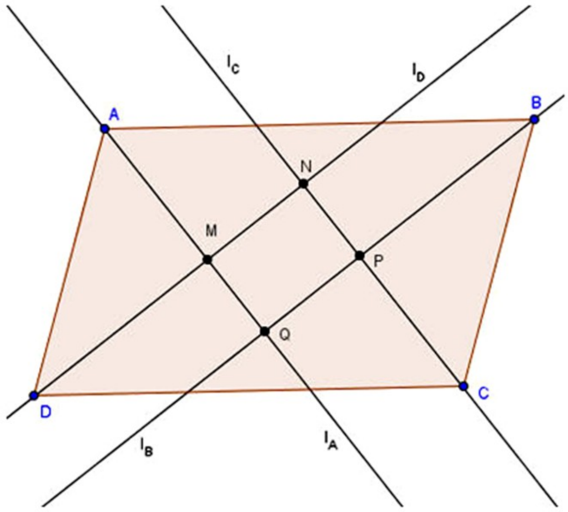


Figure 3.7: *Parallelogram problem in task 4*

Task 5. (Area Comparison) *Let ABC be a triangle. Construct three squares ABEF, BCMN, ACPQ outwards the triangle. Compare the areas of four triangles ABC, BNE, CMP, and AFQ.*

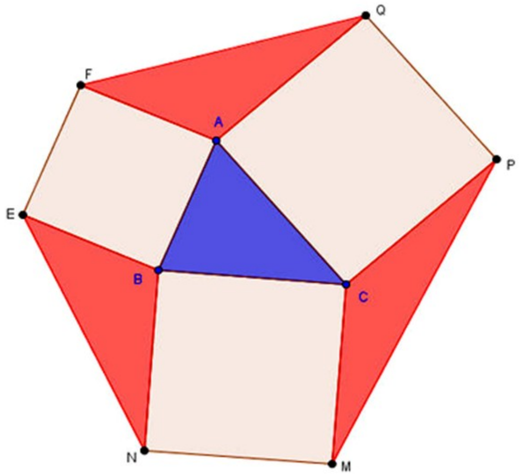


Figure 3.8: *Area comparison problem in task 5*

First of all, we consider some following invariants in geometry: equal segments, equal angles, equal figures, collinearity of points, parallelism of lines, concurrency, orthogonality, constant measurement, measurement ratio of segments, similar figures, other fixed factors, and so on. Some of them are also invariants of geometric transformations such as: line reflection, point reflection, translation, rotation, and dilation. We described different levels of realizing these invariants as follows:

Level 0. Cannot realize any geometric invariants

Student S_0 drew the figures but could not realize any invariant, even given invariants or some static invariants (properties of parallelogram in task 4, for example) because he said that he did not remember these properties. Finally, he concluded that the quadrilateral was a square, but he gave no explanation. In task 5, he also could not realize that the areas of the four triangles are equal even though he used GeoGebra software to interact with the figures. Of course, he also could not solve other tasks. We concluded that he attained level 0 of realizing geometric invariants.

Task 6. (Square Problem) *Let ABCD be a quadrilateral. Construct four squares ABEF, BCMN, CDPQ, ADRS outwards the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of these four squares. Prove that four midpoints of the diagonals of two quadrilaterals ABCD and $O_1O_2O_3O_4$ forming a square $A_1B_1C_1D_1$.*

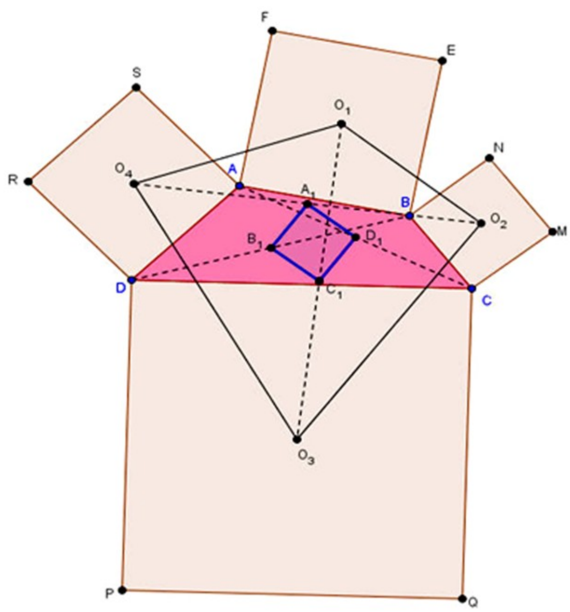


Figure 3.9: *Square problem in task 6*

Task 7. (Hexagon Problem) *Let ABC be a triangle. We take six points A_1, $A_2 \in BC$; B_1, $B_2 \in CA$; C_1, $C_2 \in AB$ such that: $BA_1 = A_1A_2 = A_2C$, $CB_1 = B_1B_2 = B_2A$,* and *$AC_1 = C_1C_2 = C_2B$. Suppose that six straight lines AA_1, AA_2, BB_1, BB_2, CC_1, CC_2 intersect each other forming a hexagon MNPQRS. Prove that three diagonals of this hexagon are concurrent.*

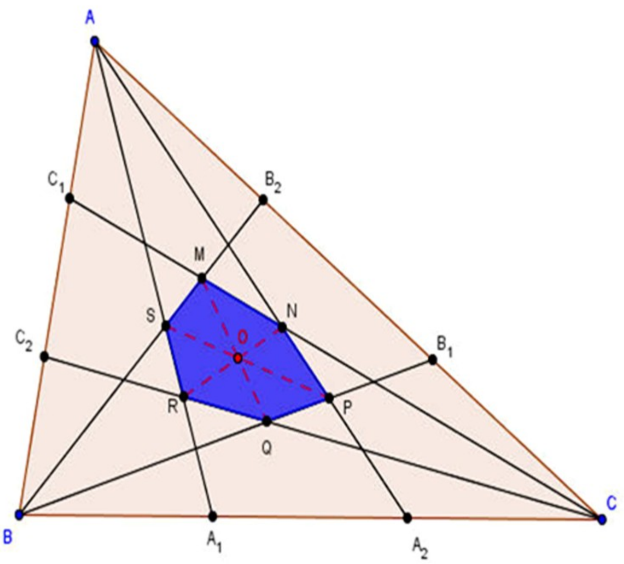


Figure 3.10: *Hexagon problem in task 7*

Level 1. Realize static invariants

Before dragging some points student S_1 realized some static invariants such as: $AB =// CD$, $AD =// BC$, $\angle A = \angle C$, $\angle B = \angle D$ and $\angle A + \angle D = \angle B + \angle C = 180^0$. He dragged to change the shape of parallelogram and realized that the quadrilateral is a rectangle. He checked the measures of angle $\angle M$ and $\angle N$. Finally, he claimed that:

$$\angle M = 180^0 - (\angle ADM + \angle DAM)$$
$$= 180^0 - 90^0 = 90^0$$
$$\angle N = 180^0 - (\angle CDN + \angle DCN)$$
$$= 180^0 - 90^0 = 90^0$$

Clearly, his ability to recognize static invariants helped him to tackle this problem easily step by step. The reason he realized static invariants is that he remembered some properties of the parallelogram. Therefore, when he drew the figure he could write some initial data on a piece of paper. Then he measured the angles of the new quadrilateral to make conjectures and write proofs.

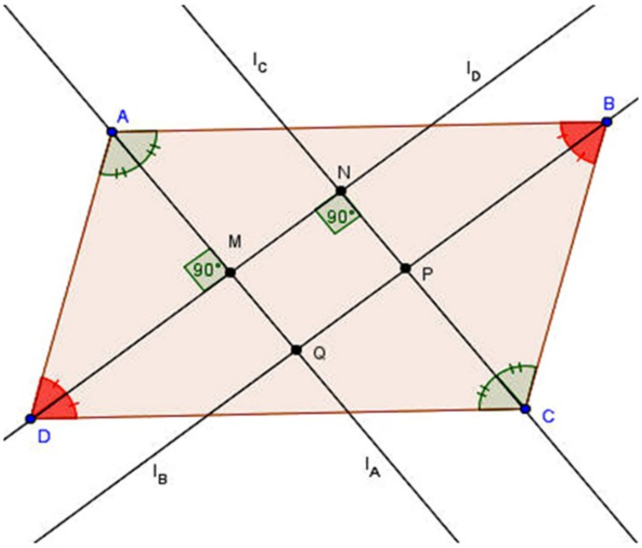


Figure 3.11: *Realizing static invariants in the parallelogram problem*

In task 5, he saw that the areas of four triangles are equal. He also knew the equality $BC = CM$, but he could not realize that $AH = PI$ or $\Delta AHC = \Delta PIC$. He could not see these 'moving/dynamic' invariants and therefore not tackle the problem in task 6 and task 7. Therefore, we can conclude that he only attained level 1 of realizing geometric invariants.

Level 2. Realize moving/dynamic invariants

Student S_2 drew the figure and dragged to change the shape of triangle *ABC*. She tried to look for invariants. After a short time for thinking, she guessed that the areas of these four triangles are equal. She checked this conjecture by measuring them. She knew that $BC = CM$ so it is necessary to show that $AH = PI$ in order to derive the area of triangle *ABC* equal to the area of triangle *CMP*. From that she realized the moving invariant: *ΔACH is always congruent to ΔPCI*. Moreover, she knew that ΔACH is an image of ΔPCI under a rotation of 90 degrees about point *C* but she could not prove it. It means that she realized these invariants but could not prove her conjectures.

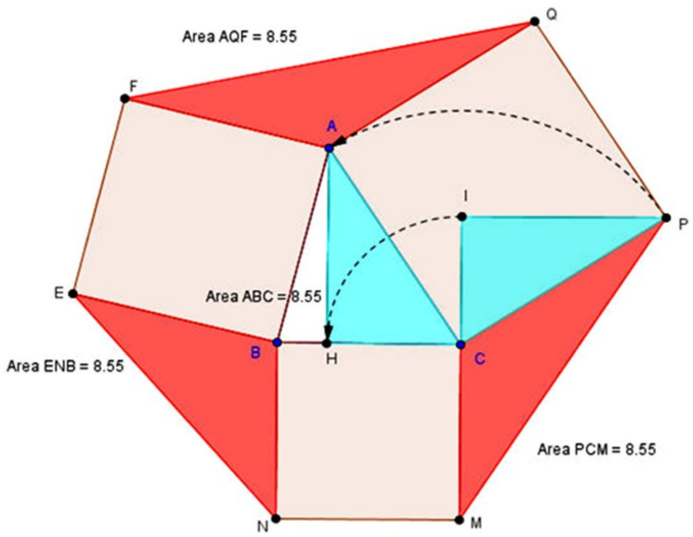


Figure 3.12: *Realizing moving invariants in the area comparison problem*

A similar situation occurred in task 6 when she knew that $D_1O_2 = D_1O_1$, $D_1O_4 = D_1O_3$ and $\Delta D_1O_3O_1 = \Delta D_1O_4O_2$ but she could not show that there exists a rotation of 90^0 degrees about point D_1, preserving the shape of these triangles. We say that this student could not realize invariants of a geometric transformation and she attained level 2 of realizing geometric invariants.

Level 3. Realize invariants of a geometric transformation

In task 5, student S_3 showed that $R_{C,90^0}(\Delta PIC) = \Delta ACH \Rightarrow AH = PI \Rightarrow$ the area of triangle ABC is equal to area of triangle PCM. Similarly, she proved for the other cases. In task 6, she started solving the problem by drawing the figure, adding some auxiliary lines and measuring some segments. She used $R_{P,\alpha}$ as a symbol for the image under a rotation of α degrees about point P. She noted that $R_{B,90^0}(A) = E$, $R_{B,90^0}(N) = C$ so she finds the following geometric invariants: $AN = CE, AN \perp CE$ and derived at $D_1O_2 = D_1O_1$, and $D_1O_2 \perp D_1O_1$. It also means that $R_{D_1,90^0}(O_1) = O_2$. Similarly, she proved that $R_{D_1,90^0}(O_3) = O_4$.

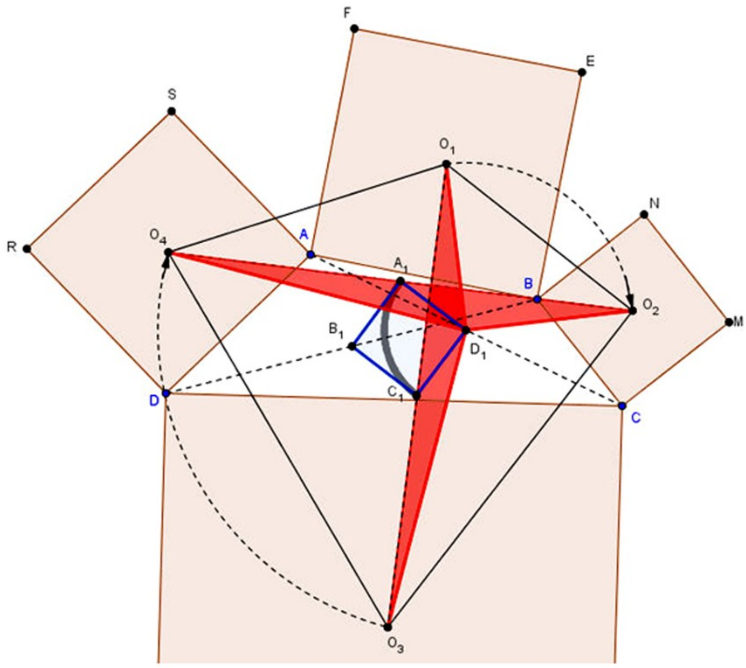


Figure 3.13: *Realizing invariants of a geometric transformation*

Student S_3 was able to explain her arguments in written words and also realized that $R_{D_1,90^0}(\Delta D_1O_3O_1)=\Delta D_1O_4O_2 \Rightarrow R_{D_1,90^0}(C_1)=A_1$ (1). Then she continued using similar arguments to show that $R_{B_1,90^0}(A_1)=C_1$ (2). From (1) and (2) she concluded that the quadrilateral $A_1B_1C_1D_1$ is a square. However, in task 7, she could not see affine properties in the problem. So she tried to prove them with an arbitrary triangle *ABC* but she failed at her final challenge. It means that she could not realize the invariants of affine geometry (and invariants of projective geometry). We say that she attained level 3 of realizing geometric invariants.

Level 4. Realize invariants of different geometries

Student S_4 dragged the triangle *ABC* and focused on the hexagon. He seemed to perceive that no matter how triangle *ABC* changed, diagonals of the hexagon are always concurrent. From his notes, he realized that the concurrence of the three diagonals is an affine invariant, so this conclusion can easily prove with the case of an equilateral triangle in Euclidean geometry.

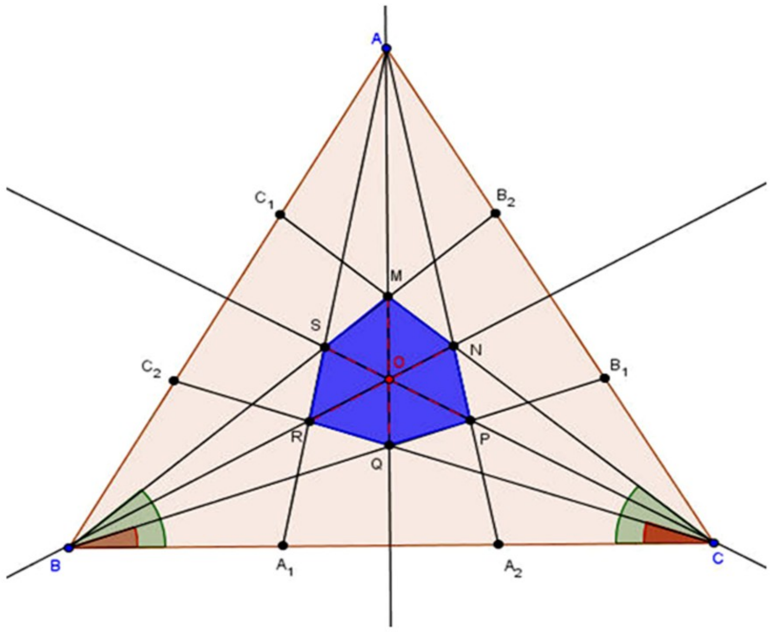


Figure 3.14: *Realizing invariants of different geometries*

He supposed that *ABC* was an equilateral triangle. First of all, he realized the following invariants: $\Delta BB_1C = \Delta CC_2B$ and $\Delta BB_2C = \Delta CC_1B \Rightarrow \angle QBC = \angle QCB$, $\angle MBC = \angle MCB \Rightarrow QB = QC$, $MB = MC \Rightarrow Q$, *M* lay on perpendicular bisector of segment *BC*. Similarly, he proved that *R*, *N* and *P*, *S* lay on a perpendicular bisector of segments *AC*, *AB*, respectively. Finally, he concluded that *MQ*, *NR*, *PS* are concurrent at point *O*. In this problem, he proved a special case in Euclidean geometry and came to a general case in affine geometry. We conclude that he attained level 4 of realizing geometric invariants. However, there was a surprise; he could not completely solve the problem in task 6. He failed to show that $R_{D_1,90^0}(\Delta D_1O_3O_1) = \Delta D_1O_4O_2$. It means that he did not have a high rank in level 3. This finding is also represented in table 6, the mean values of question 9 (Mean = 3.17) is higher than the mean values of question 6, 7, 8 (level 3). It shows that some students attain a high rank of level 4 but a lower rank of level 3.

To sum up, the students' realizing geometric invariants improved from level 1 to level 3 (or level 4). But the relationship between level 3 and level 4 is not necessarily a hierarchical order. Students can attain a high rank in level 4 but a low rank in level 3. Progressing to level 3 (or level 4), the students were able to produce arguments by transforming from visualizing into writing of a formal proof. This ability is a characteristic of dynamic visual thinking (see Section 2.4.4). This kind of thinking plays an important role in the process of realizing geometric invariants, especially in realizing 'moving' invariants. Students who have high levels of dynamic visual thinking can easily see particular movements, equal figures, moving directions, geometric transformations, etc. Furthermore, these students can imagine 'the movement' within a paper-and-pencil environment. It means that they are able to manipulate dynamically with different objects (like segments, lines, angles, figures, diagrams, symbols, and words) in their minds and then write out their thoughts. In our research, we only refer to the role of dynamic visual thinking during the proving process such as realizing geometric invariants, looking for ideas of proofs, and manipulating by thoughts with produced arguments.

3.2.3. Semi-structured interviews

The first part of the interview was conducted after classifying the students' level of proving[25]. We aimed at understanding how students proved the problems from the original solutions they provided. The purpose of this interview was to investigate the influence of the interactive HELP SYSTEM on the students' proving process such as: finding the invariants, making conjectures, validating conjectures, collecting arguments, combining arguments, writing a formal proof and delving into the geometric problems. From this interview, we noticed the differences in students' abilities in writing proofs.

The second part of the interview identified the students' difficulties in writing a formal proof. It was based mainly on their conception of proofs, acquired knowledge and strategies employed in the proving process. The revealed difficulties were compared with the checklist prepared by us. They also continued exploring and investigating. When suggesting students to answer interview questions, we took field notes on what they said about their difficulties. If a mistake was found, we offered hints to help them recognize and correct it. After the interview, we made an analysis of the students' understanding as well as misunderstandings, which depended on the response of some on-going controlling activities. Subsequently, we tried to identify their difficulties

[25] Levels of proving that are proposed in this dissertation: *information* (level 0), *construction* (level 1), *invariance* (level 2), *conjecture* (level 3), *argumentation* (level 4), *proof* (level 5), and *delving* (level 6).

which lead to mistakes and summarize the difficulties into different categories. Modifications that had been made to the categories originated from a better understanding of the students' barriers in writing proofs.

The third part of the interview was to determine the students' perception of visual proofs. Some researchers have pointed out the danger using a dynamic geometry software, saying it may limit the mathematical work of the majority to only empirical arguments and pattern spotting. This conviction can be obtained easily because of the dragging mode in the dynamic geometry environment. Therefore, this environment may prevent students from understanding the need and functions of proofs (see e.g. HADAS et al., 2000). On the contrary, in our empirical research, a suitable strategy with thought-provoking situations might motivate the students to prove the problem aimed at explaining their valuable queries. The different behaviors of students after a visual proof were also analyzed in order to recognize the role of visualization in producing a deductive proof.

In this research, we chose five groups of students who attained different levels of proving respectively for a semi-structured interview: construction level, invariance level, conjecture level, argumentation level, and proof level. These interviews are based on the student's solutions of three tasks 1, 2, and 3 (as shown in Section 3.2.2).

Group A. Construction level

The students in this group could not see the invariants when they dragged the figures. Even with the interactive HELP SYSTEM, they could not realize the static or dynamic invariants. They could construct the figures or model the situation by using construction functions of GeoGebra. However, they could not make a conjecture even although they used dragging mode to check the relationship between geometric objects. This group could also not produce any arguments because they only wrote down some given information and considered this information as their arguments. It means that in this case, these students could not read any information/data from the static/dynamic diagrams or figures.

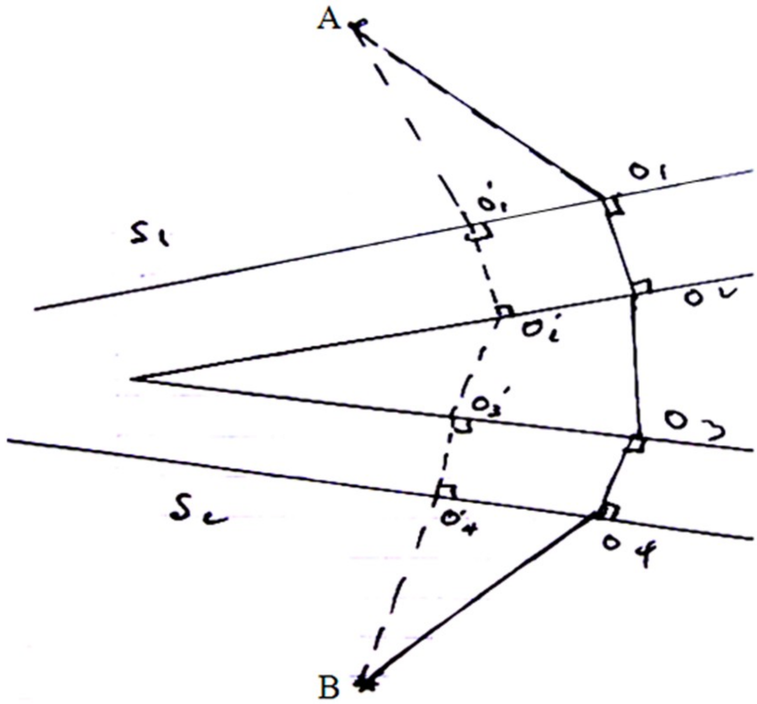


Figure 3.15: *Construction level in task 3*

In task 3, these students also could not determine the position of the bridges. It means that they were unable to realize any geometric invariant and could not formulate a conjecture:

> *"We dragged point O_1 on the line s_1 and point O_4 on the line s_2. By measuring the length of the broken line $AO_1O_2O_3O_4B$, we knew the position (only relatively on the screen) but we could not realize its special characteristics of the figure and therefore could also not realize geometric invariants".*

> *"We used support from the interactive HELP SYSTEM but could not find any invariant. As a result, we were not able to make any conjecture and failed in looking ideas for proofs".*

In general, the students in this group said that they could not find geometric invariants because they did not remember some characteristics of geometric transformations as well as their invariants. They also could not differentiate the geometric invariants of different transformations like isometry and similarity. As a result, they could not realize the appearance of two translations in this problem.

Group B. Invariance level

In this interview, we asked students in this group in some ways to recognize geometric invariants. They said that by experimenting, the peculiarities of the figures were retained and they gradually realized these invariants:

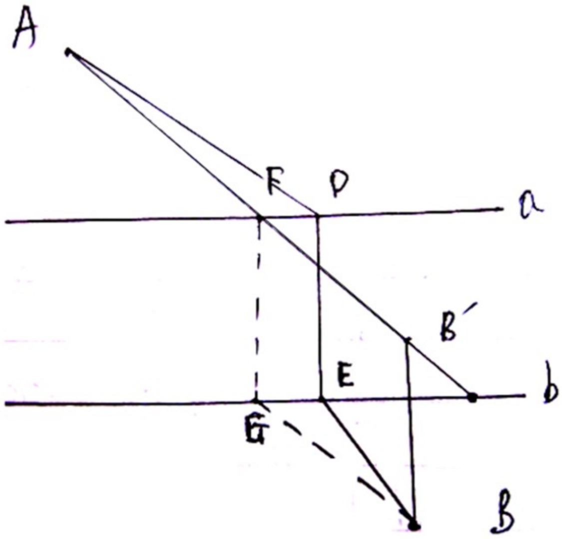


Figure 3.16: *Invariance level in task 2*

"We dragged point D on the line a and we observed the change of measurement of the broken line ADEB at the same time. Then we changed the positions A, B, as well as the distance between the banks of the river. We repeated the same procedure and realized that this measurement is minimal when the straight line AD is parallel to the straight line EB".

During the proving process, the students tried to understand the problem in different viewpoints. All of them said that it would be very difficult for them to find geometric invariants and new geometric properties without the interactive HELP SYSTEM. Sometimes they realized some initial invariants but not the key invariant which generates the idea of a proof. So they could not formulate a suitable conjecture for proving.

"After constructing the figure, we tried to find geometric invariants by using the interactive HELP SYSTEM. We have also used the GeoGebra software to check some invariants but we really did not know their role in the proving process. We

could only formulate some initial conjectures until the use of the help-system. However, we could not validate our conjectures. For that reason we left the problem and moved on to the next one".

"In order to realize the eclipsed properties, we scrutinized the diagram. It might be faster to manipulate the figures because we were also well-versed in using the GeoGebra software. By dragging we recognized some properties which did not change while moving to a certain point. These 'non-change' properties supported us by eliciting some thinking directions on how to get geometric invariants".

The availability of dynamic geometry software has enabled the realization of the vision of proving. It allows students to do experiments (the ability to drag objects and manipulate them dynamically) as well as control visualization. The dragging mode enables them to alter the conditions by maintaining the invariants (e.g. GOLDENBERG & CUOCO, 1998). The strength of dynamic geometry software facilitates conjecturing and more inductive approaches to geometric knowledge, as students can argue about the generality of their hypotheses for several cases (e.g. KAPUT, 1992). Also by dragging, the students utilized their epistemological sense by discovering 'dependent' or 'independent' factors in a certain diagram. The dynamic diagram enlightens where the key invariant might be. These discoveries also brought 'the idea of proof' to light and sowed the seeds of proof completion. Students said that geometric pictures also played an important role in discovering geometric invariants and provided leverage for their on-going effort.

Group C. Conjecture level

Before this group made the conjectures, they had already discovered the invariants. The students said that they looked deeper into the diagram and visualized the properties of the figures to overcome the initial challenges:

"We were certain that if the straight line AE and the straight line BF are parallel then the length of the broken line AEFB is minimal. These positions were named A" and B". After that, we imagined that the line BB" moved towards the line AA" until they coincide with the same line. We also recognized this displacement was a geometric transformation. Moreover, the distance between two banks of the river is

also a constant. So we thought that there was a translation in the vector $\vec{v}$ direction".

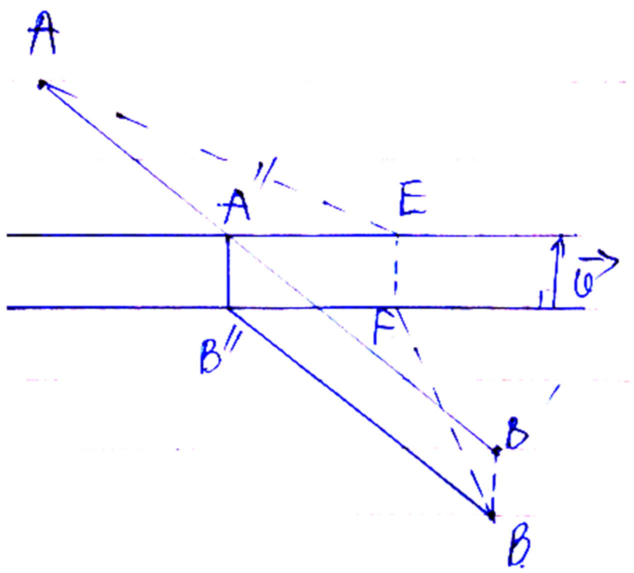


Figure 3.17: *Conjecture level in task 2*

Clearly, in the above-mentioned situation, the students used visualization (iconic images) to generate some speculative situations and formulated a crucial conjecture. We also realized that group C made less mistakes and encountered fewer difficulties than group B. They would consider a proof problem difficult if auxiliary figures (lines, segments, etc) were needed in order to tackle the problem. Writing an assumption was another difficult task for them and they did not know how to do it from start. Sometimes they jumped to the conclusion at once without providing the reason for it. However, when the interactive HELP SYSTEM appeared, the students could follow the steps and validate conjectures. Yet again they insisted that if the interactive HELP SYSTEM was not given, the problem would have been difficult (to understand some difficulties in teaching formal proofs, see Section 2.2).

It was found that almost all of students could not make their own conjectures right after constructing the figure, but they usually dragged some points, measured some segments or angles and thoroughly observed the dynamic diagram. The following processes describe students' development in the process of formulating conjectures:

intuitive observation → dragging, measuring, and checking the relationships → making inductive hypothesis → formulating conjectures.

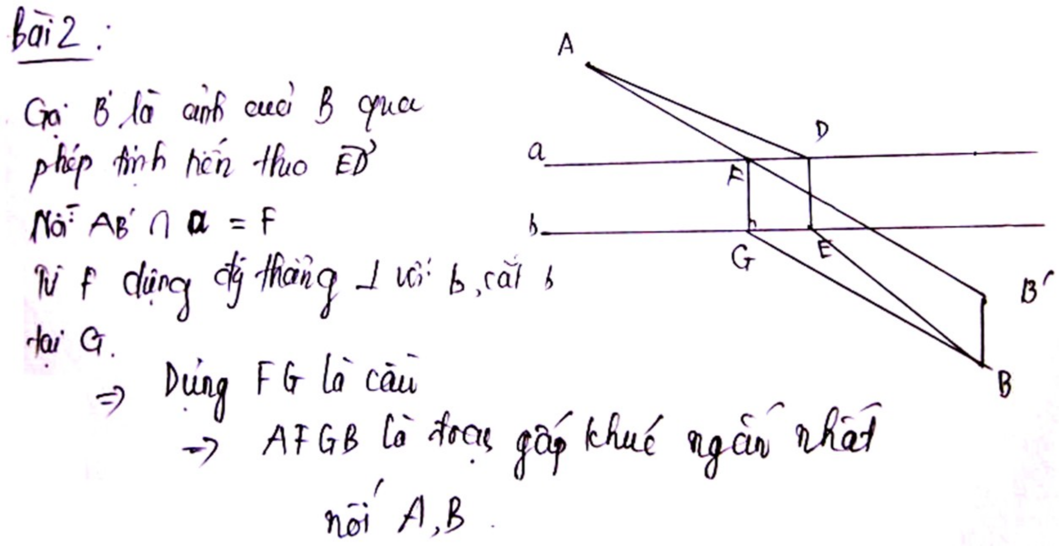


Figure 3.18: *Formulating a conjecture in task 2*

"Formulization of conjectures" is a good thing for mathematics, and an inevitable thing, as mathematical theories grow larger (e.g. MAZUR, 1997). It might take a step toward addressing the need to prove. Hence, in a drill-and-practice approach of geometric teaching, more attention is paid to make conjectures before constructing a formal proof by means of analysis and synthesis.

> *"The hints of the interactive HELP SYSTEM allowed us to construct figures easily and measure angles, segments accurately, so we could concentrate our efforts on finding the common properties between the figures in order to make and prove conjectures".*

Although some students seemed to have gone through a similar process of formulating a conjecture, there were still a lot of differences in their thinking and behavior. They said that they were quite hesitant to drag the figure because most of them only dragged some points to and fro for a short distance. This was also a center of debate on how the students should validate their conjectures. But we found that there were a few students who could prove their conjectures on their own or in collaboration with other partners. Most of them needed some hints from the interactive HELP

SYSTEM, so their proving process was more or less the same among groups. We have also taken conjecture formulization into account for the sake of collecting reasonable arguments and writing a formal proof.

Another result from this interview is that strategies the students used in validating the conjectures were slightly different. They often added and modified some lines in the figure; found new relationships according to some theorems, axioms or definitions; and wrote down the proofs systematically. Validating conjecture is also an explanation-building process. Throughout this process, the students have a chance at to deepen the understanding of their tasks, which might appear to be ideas for proofs because each conjecture might provide students with a certain proof method. As DE VILLIERS (2003) points out, it is of utmost importance that while students are investigating a geometric conjecture through continuous variation with a dynamic geometry software, they are asked *why* they think a particular result is true and challenge them to *explain*:

> *"We got a sense of satisfaction whenever we proved our conjectures. This result also increased our confidence and motivated us to prove other problems".*

Group D. Argumentation level

This group made minor mistakes such as giving the wrong reasons and getting confused with properties of different figures, invariants of different transformations or geometries. They realized that sometimes they had to use a 'backward' strategy to explain some 'observed facts' and then approach a formal proof.

> *"We sometimes used the strategy: suppose the results were proved, what would we have to find to support the inference. In order to make the proving process more clear, we drew a solution diagram. From that we could understand what needed to be proved and sometimes we followed an inverse procedure".*

Group D could solve most of the problems in the test except problems in task 3. They knew how to construct a proof and could write statements with reasons *in a logical way* by using a 'backwards' strategy (in this dissertation, we considered this strategy as an abductive argumentation). In reality, during the process of realizing geometric invariants and formulating conjectures, they used abductive argumentation to look for data or rules for validating their conjectures (see more how students use abduction in the proving process in Section 3.2). However, this group could not reverse the abductive structure in order to write a formal proof.

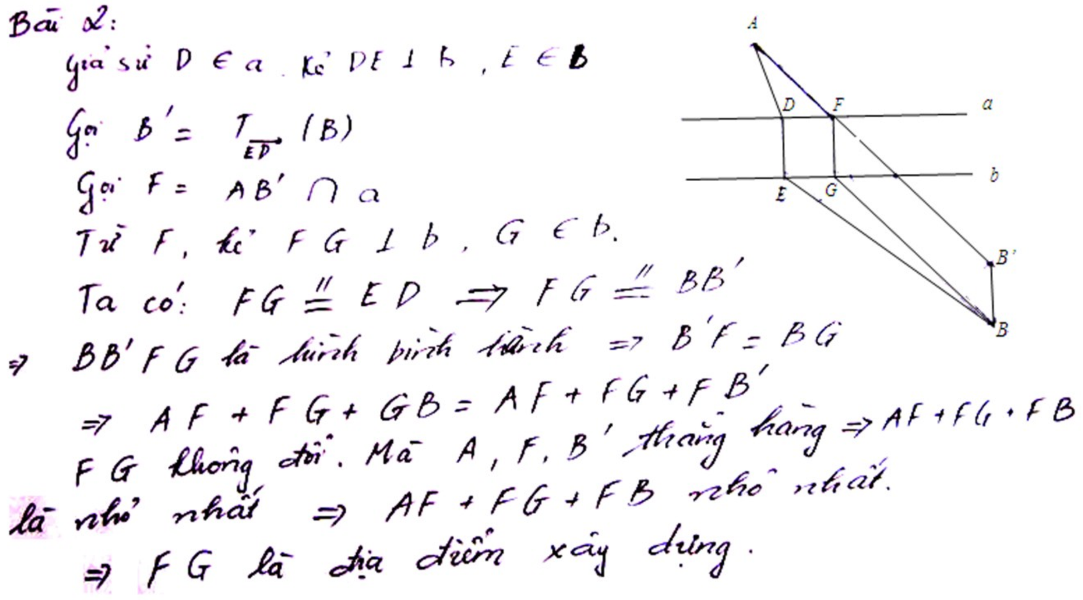


Figure 3.19: *Argumentation level in task 2*

Through argumentation, the students had to modify their understanding. The students collect their valid arguments and these collective arguments were used in writing a formal proof. But sometimes they did not realize the arguments obtained in the previous tasks could be used to solve the next. The discussion within the groups also improved the students' reasoning ability and cemented their relationship. They had an amalgamation of ideas for proving during an active disputation even though there were lots of arguments pros and cons. When two members had an irreconcilable disagreement, the third person had to supplement the tenable arguments as to why they can produce valid arguments. Finally they came to an individually-accepted consensus with well-understood argumentation.

The objective of the interview with this group was to know how deep the arguments develop through interaction in the group-based activities. The students said that they usually discussed with other partners while proving. Especially, throughout this stage they may improve their proving skills rather than understanding. They also explained that:

> *"Because we had two partners to work with, if one made a mistake or did not know how to do it, the others could help, so we were not so afraid to try. In particular, each member of the group can supplement and correct other arguments. We think it is the best way to collect and modify our arguments for the proving process".*

In fact, some other groups had their own computer to utilize, but they tended to work in the paper and pencil environment. This difference in students' behavior may be due to their background, personalities and habits of solving geometric problems. In this research, we see argumentation and proof as parts of a continuum rather than as a dichotomy because we focus on the production of arguments in the context of problem solving, experimentation, and exploration. In order to breed a formal proof, the students must organize collected arguments in an appropriately logical order. This strategy makes proofs more meaningful and constructive.

The interactive HELP SYSTEM (with carefully designed tasks and motivated questions) also supports the inductive and experimental approaches (like conjecture formulation and deductive justification) that are so prevalent in the activity of mathematicians. HOYLES & JONES (1998) cautioned that when students can generate their own empirical evidence, it means that they have motivation to appreciate the importance of the logical argument and to produce a proof. Some empirical studies also suggested that a dynamic geometry environment could play an important role in enabling students to develop deductive reasoning (see e.g. JONES, 2000a; HEALY & HOYLES, 2000; MARIOTTI, 2000; 2007). In our research, we have found that the role of dynamic environment in supporting students' reasoning may depend upon their proving levels. For instance, the groups, who attained argumentation level, showed a preferred use of empirical argument over deductive reasoning. However, the groups, who attained proof level, tended to use deductive reasoning in the process of producing arguments. Therefore, in the teaching of geometry, mathematics teachers should predict or classify students' levels of proving in order to effectively support them in producing empirical arguments or deductive reasoning.

Group E. Proof level

EDWARDS (1997) wrote that proving a theorem is often a difficult task which requires mathematics teachers to give some guidance to the learners. HOYLES & JONES (1998) also stated that: "It is central that teachers design novel activities that enable students to make links between empirical and deductive reasoning while investigating a given problem". Therefore, during the interview, we tried to discover *how students collect and combine their arguments* with the support of the interactive HELP SYSTEM. This group performed well in the test and had in-depth understanding in the given data. They could precisely explain their steps in writing the proofs. They said that some problems were easy and straightforward such as task 1, task 2. However, task 3 was quite difficult. It needed more time and effort. Based on the proof idea and selected arguments, they combined these arguments into a deductive chain that constituted a formal proof. It means that they made a transition from empirical reasoning to a deductive proof. However, they sometimes gave wrong reasons for a statement and mixed up the invariants of different transformations. They tried to break down the problem into basic elements of the problem and found the relation between the given data and the target. The problem was simplified or reformulated. In our research, organizing arguments in order to write a formal proof is one basic condition for understand the development of the proving process (see Section 2.3.4). Therefore, examining how students choose and connect their arguments in a logical way was taken into consideration during the interview:

> *"If we have some new data or sometimes if we must explain some 'facts' we usually write them down as our arguments. We check again the validity of these arguments, and then draw some diagram to link the discovered data to the conclusion. In particular, we sometimes start from the conclusion in order to find the supported data for validating. Finally, we combine these valid arguments in order to write a formal proof".*

In general, their discussion was also directed by any new idea and strategy popping up through trial-and-error method. They validated conjectures by using some data they had found before. The emphasis was on the strategy that combined the steps to solve the problem. In fact, students used the 'backwards' strategy (abductive argumentation) during the process of getting new data and then reversed the abductive structure for writing proofs. For instance, in the two-bridge problem, students reversed abductive structure (Fig. 3.20 below) in order to write formal proofs. They started from a mathematical theorem (see Section 3.2.1, Task 3) and find data D_5 for validating claim C_5, find data D_4 for validating claim C_4, and so on. This strategy would be the best way to support the students in understanding the meaning of proving activities and understanding formal proofs as well.

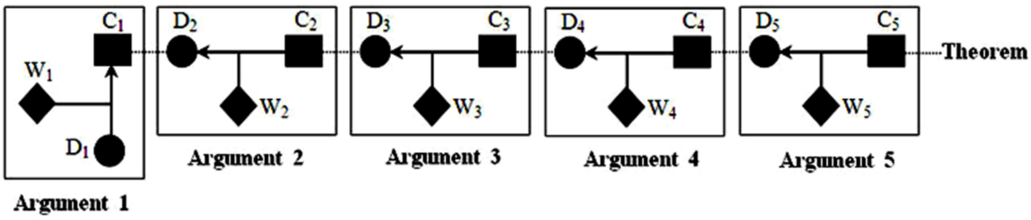


Figure 3.20: *Abductive structure of argumentation in the two-bridge problem*

In the second interview, we determined and classified students' difficulties in the proving process. Their solutions of the problems in the three tasks were kept and used as a source to identify their difficulties, and as the tool for interviewing students. There are a lot of difficulties in learning a formal proof, so even the students who excel at geometry could not solve all of the proof-related problems (as shown in Section 2.2). Through observations (using teacher's field notes) and semi-structured interviews (using students' original solutions of four tasks), we classified students' difficulties in the proving process into the following different categories:

Category 1. Explorative strategy

Students have a poor conceptual understanding and an ineffective explorative strategy. They did not see the relationship between exploration and proof as well as did not know how to exploit the data collecting from this relationship. As a result, whenever they solved the problems, some students tended to find familiar problems or similar methods for proving. They looked for priori knowledge or previous results which could apply in the present situation:

"When we read the requirement of the task, we always thought about a familiar problem or found the similar idea for proving. If we could not search for any further information we always got stuck on the next strategy. The problem is that we usually did not use explorative strategies because we reckoned there was not enough time to carry it out in the classroom".

"We read the information and analyzed the problem in the task. But at that time, we had no strategy to solve the problem. Finally, we used some hints from the interactive HELP SYSTEM".

They also did not recognize the role of exploration in the proving process. In the explorative approach, the students must formulate conjectures, verification, refutation, and they have to gain insight into the problem. They used inductive exploration which was appropriate to the deductive structure of geometrical proofs. These processes produced arguments and generated the ideas for proofs. However, a lot of students could not bridge the gap between inductive argumentation and deductive proof. They failed in solving the problem since they had an inappropriate exploration strategy. This difficulty usually leads to divergent interpretations while proving.

Some tasks required the students to use previous knowledge and add auxiliary lines to solve the problem. In such cases, the students usually spent quite a lot of time to think about an auxiliary factor, but they could not handle it. Therefore, the strategy of the interactive HELP SYSTEM should intrigue students into thinking and helping them internalize the methods of proof into their memories. To sum up, students could not determine the territory before proof (as shown in Section 2.3.1). As a result, they did not know how to start with each proof-related problem, especially as they worked with a dynamic geometry environment. That is the reason why they could not solve an open problem which often originates from their real-world life.

Category 2. Reading diagram ability

Students have difficulties in reading a dynamic diagram. This ability depends on the students' levels of realizing geometric invariants and the levels of diagrammatic thinking (see e.g. DÖRFLER, 2005). Through this interview, we can affirm that some students, who had high levels of realizing geometric invariants could understand the diagram better than lower level ones, especially with an immature version of the diagram. Otherwise, some students could not create appropriate diagrams because they did not know how to model the real-life situation in an open problem:

> *"When we read the information in task 3, we did not know how to draw the figure with two rivers and two bridges. That was a really embarrassing situation. We only had a clear picture when we referred to the interactive* HELP SYSTEM*".*

> *"We usually faced some mistakes and confusion when the figures contained a lot of lines, segments, or angles. We could not realize equal figures, similar figures, or invariants in the diagram. Because of this embarrassment we failed to produce arguments as well as discover new data for the proving process".*

In fact, a dynamic diagram makes something more intuitive and helps students seize the opportunity to explore and do experiments. Students also said that it is easier to see the simple invariants such as equal sides, angles or triangles from the static diagram than moving a diagram (when dragging). Students found a diagram confusing if there were lots of lines and angles. For that reason, they found difficulties in distinguishing embedded or overlapping figures and they preferred separate diagrams for separate parts of the task. Actually, not all of the information from the problem is shown on the diagram. Therefore, auxiliary lines have to be added to solve the problem. Using this strategy was very difficult for many students unless a hint[26] was given. Besides, students could not know exactly what was required for proving just by looking at the diagram. If students used the perceived information from a diagram to write a proof, it might be misleading. Notwithstanding, some students accepted a figure in only one particular way, e.g. the vector of translation has a horizontal direction, the angle of rotation moved in a counterclockwise way, etc. Visualization also plays an important role in argumentation and proof because there is a link between visualization and argumentation. Thus students have to look deep into the diagram from different points of view in order to produce arguments for proving. In addition, 'dynamic' diagram[27] reading abilities contribute to the realization of 'moving' invariants and geometric

[26] A hint in the interactive HELP SYSTEM is an open-ended question or an explorative task which motivates and suggests students to think about the problem.

[27] Dynamic diagram is a diagram in a dynamic geometry environment. It contains some 'moving' objects which hide some invariants and geometric relationships.

transformation. Developing dynamic visual thinking (see Section 2.4.4) would be a suitable approach to improve students' ability of reading such diagrams.

Category 3. Transition from invariance to argumentation

The restriction of reading the diagram makes it too difficult for students to realize geometric invariants. This barrier also made students more unconfident of making new conjectures:

> *"When we looked into the diagram and even dragged the figure, we could not realize any geometric invariants. Only after referring to the interactive HELP SYSTEM and checking lots of information; finally, we could make some conjectures but we could not validate them".*

During the process of discovering geometric invariants, students also blossomed out the idea of formulating their conjectures. These conjectures lead to proof-generating situations and play the role of mediator in the transition between argumentation and proof through the "dragging" mode. The students used this mode to switch from abductive to deductive argumentation and then deductive proof. According to ARZARELLO at el. (1998), this process includes the three following stages: *ascending control* (read the figure to make conjectures), *abduction* (exploration is transformed into conjectures), *descending control* (validate conjectures). Abduction also plays an essential role in the process of transition from ascending to descending control, from exploring-conjecturing to proving. Therefore, explorative conjectures and abduction are the necessary factors in transitioning from invariance to argumentation. We have also found that conjecture-formulating and conjecture-validating activities bridge the gap between conjecture and argumentation in the proving process. These activities lead to argument-generating situations and these arguments are used to write proofs. However, some students also got confused with the argumentation and proof. Thus, they did not see the connection between argumentation in an empirical situation and mathematical proof. Some students also made a mistake in the logic of writing an argument. On the one hand, they got confused with properties of different geometric transformations and the conditions for existence of corresponding transformation; while on the other hand, they could not memorize the implication of some geometric invariants:

> *"We could not make conjectures after realizing geometric invariants. It was very difficult to us. As a result, we could not formulate any conjecture as well as produce arguments for proving. But sometimes we found that it was not necessary to validate conjectures because it was obviously true or an indubitable truth".*

Therefore, we have found that students were able to realize geometric invariants but they could not realize the role of these invariants or could not produce arguments for validating conjectures. It means that students had difficulty in shifting from a priori knowledge (non-empirical) to a posteriori knowledge (empirical) or they could not generate reasoning based on realized invariants.

Category 4. Reasoning in writing proofs

Some researchers have demonstrated that students at all levels have great difficulty in constructing formal proofs (see e.g. MOORE, 1994; HAREL & SOWDER, 1998; SELDEN & SELDEN, 2003). Some students did not understand the concept and the meaning of proofs. They had a frequent obstacle in the conception of procept[28] with an inadequate cognitive development. They also lacked the knowledge and skills in writing proofs. They usually had to grapple with difficulties during the proving process.

> *"We could validate our conjectures but we wrote these arguments separately and we did not know how to combine them to write proofs. We could not differentiate necessary arguments for proving from a set of arguments".*

There seems to be no mediator between argumentation and proof because students were able to produce arguments but they could not write a formal proof. It means that students did not know how to combine the selected valid arguments for proving. As a result, they just wrote some arguments down in an inappropriate way. Some others could write the correct statement but not the reasons. They said that it was easy to get the invariants from the diagram but difficult to explain with reasons. Most of the time, they just put down 'given' as the reason.

[28] A procept is an amalgam of three components: a *pro*cess which produces a mathematical *object* and a *symbol* which is used to represent either process or object. It derives from the work of E. GRAY and D. TALL, and is a much recently used construct in mathematics education research. The notion was first published in a paper in the Journal for Research in Mathematics Education in 1994, and is part of the process-object literature.

"We could make some conjectures but then we could not realize the ideas of proofs. Therefore, we only wrote the given information as our arguments. When we wrote a statement down, we tried to reason it as true or false but we really did not know about the validity of the proof".

Students are also unfamiliar with the terms used in the proving process such as assumption, conjecture, statement, generalization, transformation, argument, and formal proof. Besides, they may not have a full understanding of the theorems used in constructing proofs. They just memorize the theorems by rote learning and have difficulty in applying the correct theorems in writing proofs. Other difficulties encountered by students are that they were unable to write down verbal reasoning and what they had in mind. Students could explain some steps to us verbally in the interview but claimed that it was difficult to write the statement and reason in words. The reason for this difficulty is due to the fact that giving reasons for statements needs to use a lot of symbols, vocabularies, abbreviations, and terms in a strict way which was not an easy task for some students. It can be said that the language used in proof writing is difficult for students. We think that a mediator between argumentation and proof within a dynamic geometry environment might be the 'dragging' mode in the following procedure:

Exploring → Dragging → Conjecturing → Validating (Arguing) → Proving

Using logical chain of reasoning in writing a proof has also been a student's barrier. Some of them know the procedures of writing a proof yet cannot present the steps in a logical way. They may write down all the correct information and steps but in an illogical order. Another common weakness of students was that they also got confused with the condition and conclusion of a statement. Some students use the concluding statement to write a proof, or cite the definitions, properties or theorems to be proved in the steps of proving or they did not understand the deductive method of writing a proof. Students said that it is easier to write statements than arguments and it seems meaningless to prove something that was obviously true. That is the reason why they could omit some steps and jump to the conclusion. This problem is involved in the concept of visual proof that will be considered in the next part.

The third part of the interview investigated student's perception of visual proofs as well as the role of dynamic visualization in the proving process. In dynamic geometry environment, a conviction can be obtained easily by using dragging mode. Therefore, it may prevent students from understanding the need and function of proof (see e.g. HADAS et al., 2000). However, in our empirical research, we have found that with an appropriate strategy in thought-provoking situations might motivate the students to prove the problem aimed at explaining their conjectures. They could realize geometric invariants and make some conjectures by using dynamic geometry software, but they did not know how to verify these 'phenomena'. This means that some 'observed facts' in dynamic geometry environment inspire students with a need of proof. For that reason, we suggested that *motivation* would also be a function of proof (see Section 2.1):

> *"Before this task, we felt it was not necessary to prove some theorems, since mathematicians had done that already! We should apply them to solve problems by rote learning like formula using. So we preferred spending more time doing exercises or easier problems. But in this task (e.g. task 2), we become very eager to find the proof because we have a need to find an explanation for our conjectures. Moreover, visualization within a dynamic geometry environment plays an important role in producing arguments for proof".*

Furthermore, we have also realized that the students who attain low levels of proving would find it easier to accept a visual proof than those who attain high levels. However, some high-level students tend to use deductive argumentation in the proving process instead of abductive argumentation. The following example illustrates two different approaches in tackling a problem:

> **Equilateral Triangle Game.** *Let ABC be an equilateral triangle with side length 5. Mark and Mike play the following game under the control of a referee: The referee chooses an arbitrary point, X, on the side AC, then Mark chooses a point, Y, on the side BC, and finally Mike chooses a point, Z, on the side AB. Suppose that Mike's aim is to obtain a triangle XYZ with the smallest possible perimeter, while Mark's aim is to get triangle XYZ with the largest possible perimeter. Find a strategy helping them achieve their goal?*

We have organized this game for three-student groups (they took in turn the part of Mark, Mike, and a referee). Each group played the game in 15 minutes. The first group that found the solution (a strategy for achieving the goal) would be the winner. There were two groups with different strategies as follows:

Group 1. Using deductive argumentation and visual proof

Firstly, a student (as a referee) chose an arbitrary point X on the segment AC. By experience, they said that the best strategy for Mark is to choose $Y \equiv B$ or $Y \equiv C$. Put $X' = r_{AB}(X)$ and $X'' = r_{BC}(X')$. Clearly, if X and Y are already chosen, Mike has to choose Z as the intersection point of AB and $X'Y$. For such a choice of Z they derived the equality:

$$XY + YZ + ZX = XY + YX' = XY + YX''$$

Since Y lies on BC, the latter sum will be a maximum when $Y \equiv B$ or $Y \equiv C$, depending on the position of the segment XX''. After that they gave a new proposition:

There exists point E on AC such that such that 2BE = CE + C'E (*)

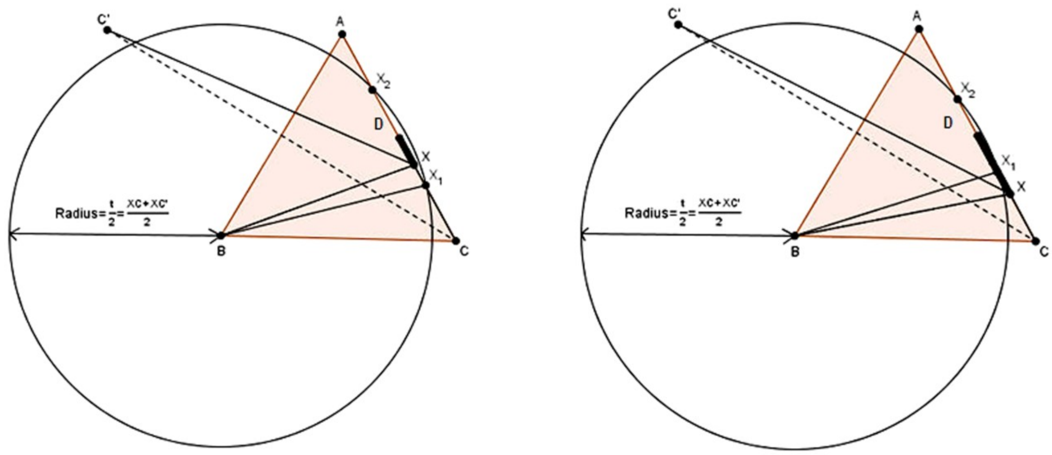


Figure 3.21: *Visual proof in the equilateral triangle game*

They proved the proposition (*) by the following arguments:

"Let $t = XC + XC' (C' = r_{AB}(C))$. Construct a circle $r = (B; t/2)$. Let X_1, X_2 be intersections of the circle r and the line AC. Let D be the midpoint of the segment AC. When we drag point X on the line AC, we can see that the circle r changes and two points X_1, X_2 also moving. We consider two following cases $X \equiv X_1$ and $X \equiv X_2$:

Clearly, when $X \equiv X_1$ or $X \equiv X_2$ then we have $t = XC + XC' = 2BX$. They explained the above result as follows ($\alpha \rightarrow \beta$: α is approaching β when dragging):

- $X \in [DC]$ and $X \rightarrow C$: $\begin{cases} BD \leq BX \leq BC \text{ and } BX \rightarrow BC \\ BD \leq BX_1 \leq BC \text{ and } BX_1 \rightarrow BD \end{cases} \Longrightarrow \exists E \in [DC]$ such that $BE = BX = BX_1$.
- $X \in [DA]$ and $X \rightarrow A$: $\begin{cases} BD \leq BX \leq BA \text{ and } BX \rightarrow BA \\ BD \leq BX_2 \leq BA \text{ and } BX_1 \rightarrow BA \end{cases} \Longrightarrow \exists E \in [DC]$ such that $BE = BX = BX_1$ $(E \equiv A)$ ".

If the referee chose the point *X* on the segment *AE*, Mark must choose *Y* ≡ *C*; if the referee chose point *X* on the segment *CE*, Mark must choose *Y* ≡ *B*.

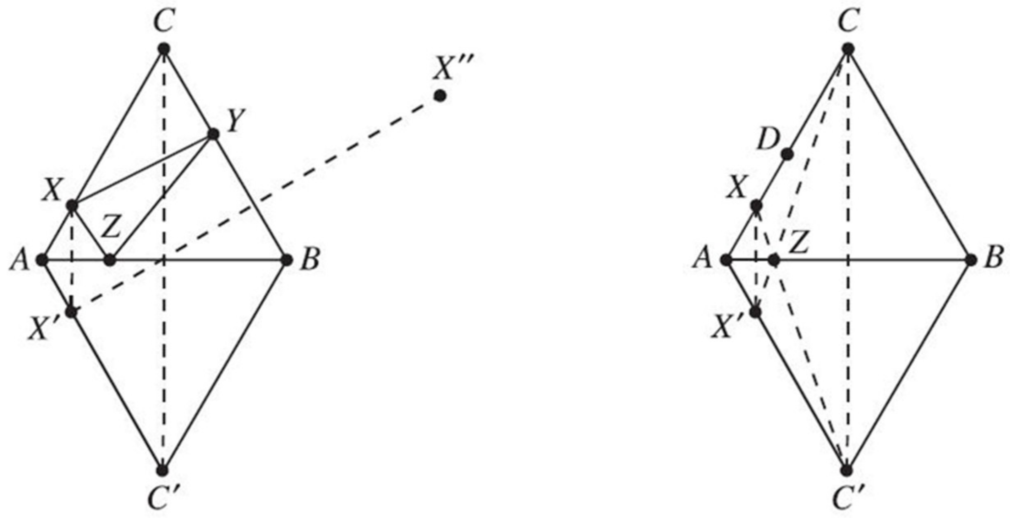


Figure 3.22: *Deductive argumentation in the equilateral triangle game*

Next, if Mark chooses *Y* ≡ *B*, then Mike will choose *Z* ≡ *B*, and the perimeter of Δ*XYZ* will be 2*XB*. In case Mark takes *Y* ≡ *C*, John will put *Z* at the intersection point of *X'C* and *AB*, and then the perimeter of Δ*XYZ* will be *XC* + *X'C* = *XC* + *XC'*, as desired.

Group 2. Using abductive argumentation and experimentation

* Case 1 (Mike's strategy): Suppose points *D* and *E* are chosen. The problem is determining point *F* on segment *AB* such that the perimeter of triangle *DEF* ($p_{\Delta DEF}$) is minimal. Let be $D' = r_{AB}(D)$. From that we have following equalities $DF' = DF$, and $DF = D'F$. Therefore:

$$p_{\Delta DEF} = DE + EF + FD = DE + EF + D'F \geq DE + ED' = DE + EF' + F'D'$$
$$= DE + EF' + F'D = p_{\Delta DEF'} \Rightarrow p_{\Delta DEF} \geq p_{\Delta DEF'}$$

The equality occurs: $p_{\Delta DEF} = p_{\Delta DEF'} \Leftrightarrow F \equiv F'$.

* Case 2 (Mark's strategy): By experimenting they realized that if $E \equiv B$ or $E \equiv C$ then $p_{\Delta DEF}$ is maximal. Suppose that $AF > AD$. Let FM be a parallel line to the line BC. We realized that the quadrilateral *BFMC* is isosceles trapezoid. Then derived that *BF* = *MC* and *CF* = *BM*. Therefore:

$$BF + BD = MC + BD \leq MC + DM + BM = DC + BM = DC + FC$$

$$\Rightarrow \; p_{\Delta BFD} < p_{\Delta CFD} \;\; (1)$$

Now we must show that $p_{\Delta EDF} < p_{\Delta CFD}$ (2)

We have the inequality: $EF + ED \leq EF + ME + DM$ (3). On the one hand, we have $S_{\Delta EFM} = S_{\Delta CFM}$ (where $S_{\Delta EFM}$ denoted by area of triangle *EFM*).

Since point E lies on the segment BC, therefore:

$\Rightarrow p_{\Delta EFM} < p_{\Delta FBM} = p_{\Delta FCM} \;\Rightarrow EF + EM \leq FC + MC$

$\Rightarrow EF + EM + MD \leq FC + MC + DM = \; FC + CD$ (4)

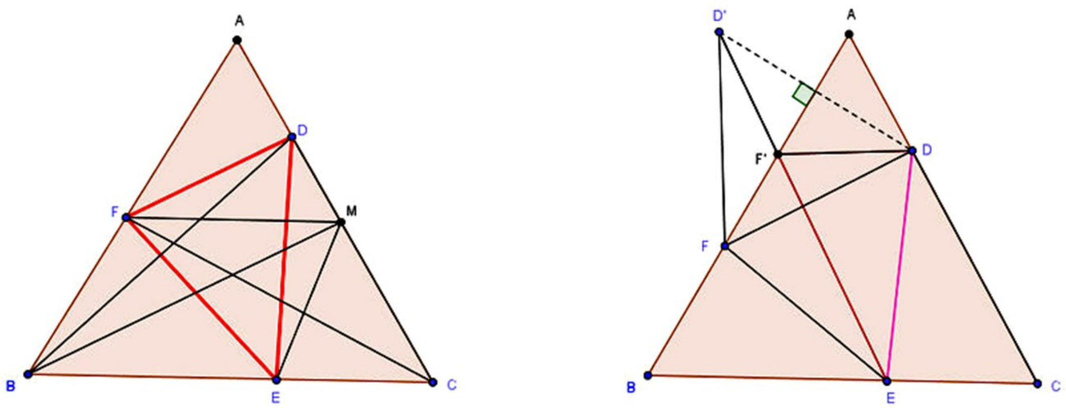


Figure 3.23: *Abductive argumentation in the equilateral triangle game*

From (3) and (4) we derive: $EF + ED \leq FC + CD \Rightarrow$ (2). The equality occurs when and only when point $E \equiv C$. Similarly, we prove if $AF < AD$ then we choose point $E \equiv B$. From the third part of the interview, we have revealed so far that some high proving-level students, who have intimate knowledge, tend to use deductive argumentation. However, they sometimes accepted visual proof as a means of producing arguments for deductive proof. On the contrary, some medium proving-level students, who would like to carry out explorative experiments within a dynamic geometry environment, tend to use abductive argumentation for explaining and interpreting the organization of the proving process. These students could not write a formal proof until they reversed the abductive structure of argumentation.

In general, a lot of students in these interviews said that it would be very difficult for them to find the invariants without the interactive HELP SYSTEM. Since it allowed students to construct figures easily and measure angles accurately, so they could concentrate their efforts on finding the common properties between figures. We have also posed lots of thought-experiment tasks in order to encourage students to prove in an active way. This process permeates gradually through the students' proving ability and creates a didactical cycle for proving (see Fig. 3.24 below), which should be implanted in students' thought. Therefore, our methodological model should be a well-established framework as well as an articulation between theory and practice. It will be a better-developed instrument in order to orchestrate didactic situations in the teaching of geometry. A hint in the interactive HELP SYSTEM must set students' curiosity and effectively support them during the proving process. After the interviews, we have a basic understanding of the difficulties encountered by each of the five groups of students. Comparing students' difficulties with their proving levels, it was obvious that students with lower proving levels were less capable in writing geometric formal proofs and faced more difficulties. Furthermore, some categories of difficulties, such as discovering invariants, reasoning in writing proofs, were common to all students. Other categories, such as reading diagram ability and showing logic in writing proofs, were only encountered by students of higher proving levels.

Finally, all of the students in the interviews have also said that some explorative tasks and open-ended questions in the interactive HELP SYSTEM stimulated them to think about the ideas of proofs. After discovering some initial invariants, the students had a chance to gain insight into invariant elements (sub-invariants, key invariants). Some arguments were produced based on these realized invariants and these arguments were used to write a formal proof. This process seems to be a circle of proving activities: *understanding*, *constructing*, *conjecturing*, *arguing*, *proving*, *delving*, and *new problem*. Therefore, we described the students' proving process corresponding with seven levels of proving in a didactical circle. This circle may provide tertiary students with a strategy for proving within a dynamic geometry environment:

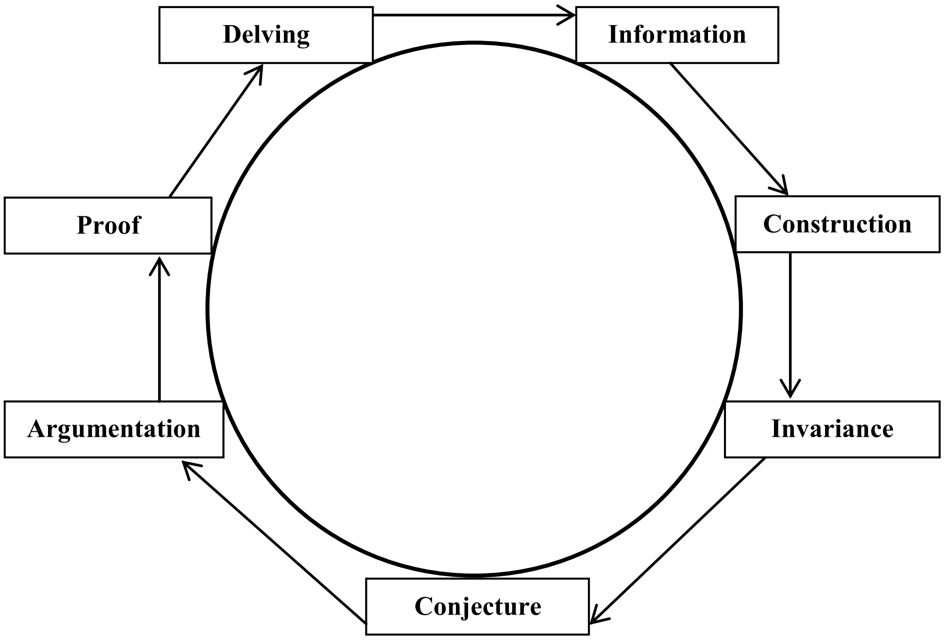


Figure 3.24: *Didactical circle for the proving process*

3.2.4. Hypotheses testing

A pre-test was designed to investigate the equivalence of the experimental and control groups. This was administered to the students in both the experimental and control group prior to the experiment. If the means of the performances of the two groups do not differ significantly, it can be assumed that the two groups are comparable. A post-test was also designed and administered at the end of the experiment to students in both the experimental and control groups. If the mean performance of the experimental group is significantly different from the mean performance of the control group, it can be assumed that students' performance must have been influenced by the methodological model within a dynamic geometry environment.

The same pre-test was given to the two groups before conducting a treatment[29] in order to investigate the equivalence of the experimental and control groups. The null hypothesis H_0- was formulated:

(H_0-): *There is no significant difference between the students' mean scores in the experimental and control groups* *before* *the treatment.*

During the four-month teaching course, the experimental group was taught in the computer laboratory and the control group was taught using the paper-and-pencil format with the same content. At the end of the period of experimental teaching, the same post-test was designed and administered to students in both groups. In order to determine whether the use of the interactive HELP SYSTEM has an effect on the students' performance and the improvement of proving levels, the null hypothesis H_0+ was formulated:

(H_0+): *The students' mean scores in the experimental group are higher than ones in the control groups* *after* *the treatment.*

The data collected in the pre-test and post-test were analyzed using a t-test for independent groups to determine whether there is a significant difference between the mean scores and variance of the experimental and the control groups. The following table shows the mean scores and standard deviations of the two groups before and after the treatment:

Table 3.13: *Group statistics for pre-test and post-test*

Group		N	Mean	Std. Deviation	Std. Error Mean
Pre-test	Control Group	65	5.97	1.131	.140
	Experimental Group	67	5.93	1.418	.173
Post-test	Control Group	65	6.11	1.715	.213
	Experimental Group	67	6.58	1.519	.186

[29] To support tertiary students writing a formal proof by using the GeoGebra software with the interactive HELP SYSTEM.

Table 3.14: *Independent samples t-test*

Dependent variables		Statistics								
		Levene's Test for Equality of Variances		t-test for Equality of Means						
		95% Confidence Interval of the Difference								
		F	Sig.	t	df	Sig. (1-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
Pre-test	Equal variances assumed	2.273	.134	.196	130	.423	.044	.224	-.399	.486
Post -test	Equal variances assumed	.534	.466	-1.684	130	.048	-.474	.282	-1.032	.083

The criterion for using a parametric t test is to have both samples with equal variances. The Levene's test was used to verify whether two variances do differ significantly. It can be seen that (Table 3.14): $F = 2.273$, $p > 0.05$. Therefore, we can accept the equal variances assumption for the independent samples t-test as the variances are not significantly different. For the pre-test, we have the following values: $t(130) = 0.196$, $p > 0.05$. It means that there is no significant difference between experimental and control group. However, for the post-test, we have the following values: $t(130) = -1.684$, $p < 0.05$. So we reject the null hypothesis and conclude that two groups are significantly different. In Table 3.13, we can also see that the experimental group attaining a higher mean score and smaller standard deviation (Mean = 6.58; SD = 1.519) than the control group (Mean = 6.11; SD = 1.715). This data also indicates that the experimental group was performed better than the control one after the treatment.

Boxplots were used to illustrate normal distribution of students' scores in the pre- and post-test. In Fig. 3.25, we can realize mean scores, extreme scores, and variances in two groups. It seems to be symmetrical in the pre-test. The median lines are in the center of the boxes and the whiskers extending from the top and bottom half of the boxes have equal lengths. But the scores have a little change in the post-test. The number of students with high scores in the experimental group is more than that in the control one. At the same time, the number of students with low scores in the experimental group has actually decreased.

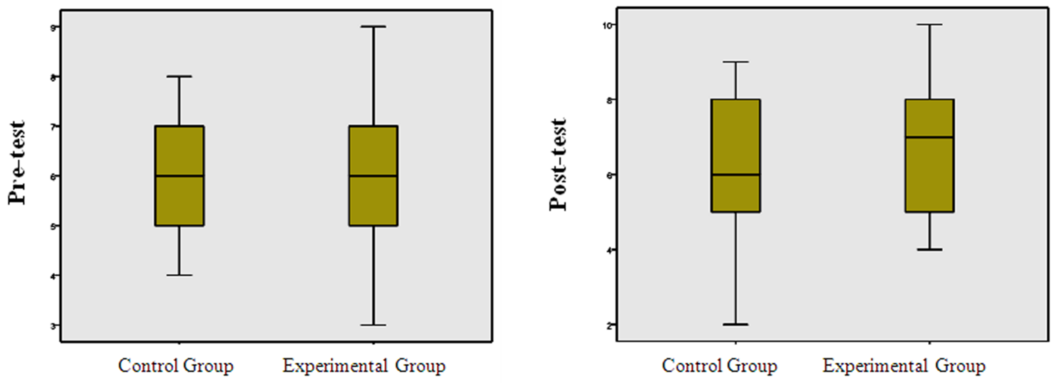


Figure 3.25: *Boxplots for independent samples t-test*

In general, we have found evidence that our experimental treatment has a statistically significant effect. The means of the performances of the experimental group do differ significantly after the treatment.

(H_1): *There is no significant difference between the students' mean scores in the experimental group* ***before*** *and* ***after*** *the treatment.*

In order to test this hypothesis, a paired sample t-test was used to test for significant achievement differences of the experimental group with $p < 0.05$.

Table 3.15: *Paired sample statistics for experimental group*

		Mean	N	Std. Deviation	Std. Error Mean
Sample	Experimental Group Pre-test	5.93	67	1.418	.173
	Experimental Group Post-test	6.58	67	1.519	.186

Table 3.16: *Paired sample correlations for experimental group*

		N	Correlation	Sig.
Sample	Experimental Group Pre-test & Experimental Group Post-test	67	.330	.006

Table 3.16 shows the Pearson correlation coefficient and its significance value. We have found that $r = 0.330$, $p < 0.05$, which is found to be significant. This test is conducted to show if the results found are consistent. We are predicting that the treatment have the same effect on all students. There will be a consistent effect on them.

It means that a student who performed better than average on the pre-test would still do better than average in the post-test and someone near the bottom of the group in the pre-test would still be near the bottom of the group in the post-test). Therefore, the students are behaving consistently as their scores in the pre-test indicated because they are significantly correlated with their scores in the post-test. The results of paired samples t test (see Table 3.17 below) $t(66) = -3.159$, $p < 0.05$ shows that there is significant difference between the students' mean scores and variance in the experimental groups before and after the treatment.

Table 3.17: *Paired samples test for experimental group*

	Paired Differences					t	df	Sig. (2-tailed)
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference				
				Lower	Upper			
Experimental Group Pre-test & Experimental Group Post-test	-.657	1.702	.208	-1.072	-.242	-3.159	66	.002

Besides the change in the students' mean scores and variance, we have also concentrated on the change of students' level of proving after period of experimental teaching. Hence, the following hypothesis was formulated:

> (H_2): *There is no significant difference between the students' level of proving in the experimental group and control group after the treatment.*

The student's level of proving is a rating scale. This scale includes seven levels of proving based on the students' solution of a problem. It is usually treated as ordinal data. Therefore, we used the Mann-Whitney U test which is a nonparametric equivalent of the independent samples t test to check this hypothesis. The following data describe the levels of proving ranks between experimental and control group. Test statistics are also presented to clarify the improvement of proving levels in the experimental group in comparison with the control one:

Table 3.18: *Levels of proving ranks in the post-test*

	Group	N	Mean Rank	Sum of Ranks
Level of Proving	Control Group	65	52.09	3386.00
	Experimental Group	67	80.48	5392.00
	Total	132		

Table 3.19: *Test statistics in the post-test*

	Level of Proving
Mann-Whitney U	1241.000
Wilcoxon W	3386.000
Z	-4.437
Asymp. Sig. (2-tailed)	.000

The results of the Mann-Whitney U test are presented as follows: U = 1241.000, Z = -4.437, $p < 0.01$. We can conclude that there is a high significant difference between the level of proving in the control and experimental group after the treatment. The students in the experimental group (Mean Rank = 80.48) have attained higher level of proving than the control one (Mean Rank = 52.09). However, there is also another question that needs to be answered: Have the students' levels of proving in the experimental group really changed after the treatment?

(H_3)*: There is no significant difference between students' levels of proving in the experimental group* *before* *and* *after* *the treatment.*

Firstly, we have recorded the students' level of proving in the pre-test and post-test and determined whether the two sets of levels of proving come from the same distribution. To carry out this work, the one sample Kolmogorov-Smirnov test (more commonly known as the K-S test) was used. It took the observed cumulative distribution of levels of proving and compared them to the theoretical cumulative distribution for a normally-distributed population.

Table 3.20: *One-sample Kolmogorov-Smirnov test*

		Pre-test Level of Proving	Post-test Level of Proving
N		67	67
Normal Parameters[a,,b]	Mean	3.73	4.27
	Std. Deviation	.963	1.009
Most Extreme Differences	Absolute	.207	.214
	Positive	.179	.164
	Negative	-.207	-.214
Kolmogorov-Smirnov Z		1.694	1.748
Asymp. Sig. (2-tailed)		.006	.004

a. Test distribution is Normal.

b. Calculated from data.

From Table 3.20, we have obtained the following results: $Z = 1.694$, $p < 0.05$ (in the pre-test) and $Z = 1.748$, $p < 0.05$ (in the post-test). This indicates the observed distribution corresponds to a theoretical distribution. That is, the data are not significantly different to a normal distribution at the $p < 0.05$ level of significance. Secondly, we used the Wilcoxon signed-ranks test to analyze the initial situation. This test is the nonparametric equivalent of the related t test.

Table 3.21: *Levels of proving ranks in the experimental group*

		N	Mean Rank	Sum of Ranks
Post-test Levels of Proving & Pre-test Levels of Proving	Negative Ranks	15[a]	21.80	327.00
	Positive Ranks	36[b]	27.75	999.00
	Ties	16[c]		
	Total	67		

a. Post-test Levels of Proving < Pre-test Levels of Proving.

b. Post-test Levels of Proving > Pre-test Levels of Proving.

c. Post-test Levels of Proving = Pre-test Levels of Proving.

Table 3.22: *Wilcoxon test statistics*[b]

	Post-test Levels of Proving & Pre-test Levels of Proving
Z	-3.269[a]
Asymp. Sig. (2-tailed)	.001

a. Based on negative ranks.

b. Wilcoxon Signed Ranks Test.

The results of the Wilcoxon test are as follows: Z = -3.269, N = 67, $p < 0.01$. Therefore, we can conclude that the students' levels of proving do differ significantly after the treatment. There are 36 students increasing their levels of proving, 15 students decreasing their levels while 16 students having the same levels in the post-test. To sum up, this result shows the positive effect of the interactive HELP SYSTEM in the proving process. The data has also showed that *using a methodological model like the interactive HELP SYSTEM during the proving process really improves the student's level of proving*. Moreover, we have also taken gender as an independent variable into account by the following hypothesis:

(H_4): *Males and females do not differ in students' mean scores and variance in the experimental group after the treatment.*

An independent samples t test was used to see whether there is a difference between the performances of the two groups.

Table 3.23: *Group statistics in the post-test*

	Sex	N	Mean	Std. Deviation	Std. Error Mean
Experimental Group	Male	17	5.94	1.478	.358
	Female	50	6.80	1.485	.210

Table 3.24: *Independent samples t test in the post-test*

	Levene's Test for Equality of Variances	t-test for Equality of Means
	95% Confidence Interval of the Difference	

		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
Experimental Group	Equal variances assumed	.018	.894	-2.063	65	.043	-.859	.416	-1.690	-.027
	Equal variances not assumed			-2.068	27.796	.048	-.859	.415	-1.710	-.008

From the Levene's test for equality of variances: F = 0.18, $p > 0.05$, we can accept the equal variances assumption for the independent samples t test. The results are as follows: $t(65) = -2.063$, $p < 0.05$. It has revealed that, there is significant difference between males and females after the treatment. However, the results of the crosstabulation and Pearson Chi-Square test have shown that there is no difference in the males and females' attitude towards the interactive HELP SYSTEM:

$$\chi^2 = 0.405, df = 2, p > 0.05$$

It means that the students would like to use this system during the proving process, but the females have better strategies in solving the problem than males. This behavior stems from different factors that need further research.

Table 3.25: *Towards to the interactive HELP SYSTEM by crosstabulation*

			Sex		Total
			Male	Female	
Towards to the interactive HELP SYSTEM	Disagree	Count	2	4	6
		Expected Count	1.5	4.5	6.0
		% within Sex	11.8%	8.0%	9.0%
	Neutral	Count	3	7	10
		Expected Count	2.5	7.5	10.0
		% within Sex	17.6%	14.0%	14.9%
	Agree	Count	12	39	51
		Expected Count	12.9	38.1	51.0
		% within Sex	70.6%	78.0%	76.1%
Total		Count	17	50	67
		Expected Count	17.0	50.0	67.0
		% within Sex	100.0%	100.0%	100.0%

Table 3.26: *Chi-Square tests*

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	.405[a]	2	.817
Likelihood Ratio	.391	2	.822
Linear-by-Linear Association	.391	1	.532
N of Valid Cases	67		

a. 3 cells (50.0%) have expected count less than 5. The minimum expected count is 1.52.

In order to deeply interpret the students' satisfaction with the interactive HELP SYSTEM, we have divided 30 students in the experimental group into 3 groups with different performance in the post-test. Then they were asked to rate their satisfaction on a 10-point scale while they worked with this system. We chose the Kruskal-Wallis test because there are more than two independent samples (three groups: *poor*, *average*, and *good* performance in the post-test).

Table 3.27: *Kruskal-Wallis ranks test*

	Group of Student	N	Mean Rank
Level of Satisfaction	Poor	10	11.80
	Average	10	19.10
	Good	10	15.60
	Total	30	

Table 3.28: *Kruskal-Wallis test statistics[a,b]*

	Levels of Satisfaction
Chi-Square	3.564
df	2
Asymp. Sig.	.168

a. Kruskal Wallis Test.

b. Grouping Variable: Group of Student.

The results of the Kruskal-Wallis Chi-Square are shortly presented as follows: $\chi^2 = 3.564, df = 2, p > 0.05$. So we can conclude that the difference between the ratings of the three groups is not significant.

In our research, one factor that may influence the proving process is the *levels of realizing geometric invariants*. This plays a crucial role in invariance phase and we would like to understand the relationship between the students' levels of realizing geometric invariants and their levels of proving by checking the following hypothesis:

(H_5): *There is no relationship between the students' levels of realizing geometric invariants and their levels of proving after the treatment.*

The Spearman correlation (r_s) was used to correlate data because two variables are ordinal. The Spearman correlation uses exactly the same calculations as the Pearson but performs the analysis on the tied ranks of the scores instead of on the actual data values.

Table 3.29: *Spearman correlation in the post-test*

			Score	Level of Proving
Spearman's rho	Score	Correlation Coefficient	1.000	.850**
		Sig. (2-tailed)	.	.000
		N	67	67
	Level of Proving	Correlation Coefficient	.850**	1.000
		Sig. (2-tailed)	.000	.
		N	67	67

**. Correlation is significant at the 0.01 level (2-tailed).

The results of the Spearman correlation are as follows: $r_s = 0.85, p < 0.01$. This finding shows that there is a strong relationship between the students' levels of realizing geometric invariants and their levels of proving. This relation is highly statistically significant. We have continued using Kendall tau-b correlation to examine more thoroughly the above-mentioned relationship. The Kendall tau-b correlation is a measure of the association between two ordinal variables and takes ranks into account, so it can be used for small data sets with a large number of tied ranks. It assesses how well the rank ordering on the second variable matches the rank ordering on the first variable.

Table 3.30: *Kendall's tau-b correlations in the post-test*

			Levels of Realizing Invariants	Levels of Proving
Kendall's tau_b	Levels of Realizing Invariants	Correlation Coefficient	1.000	.544**
		Sig. (1-tailed)	.	.000
		N	67	67
	Levels of Proving	Correlation Coefficient	.544**	1.000
		Sig. (1-tailed)	.000	.
		N	67	67

**. Correlation is significant at the 0.01 level (1-tailed).

The Kendall tau-b correlation output: Kendall tau-b = 0.544, N = 67, $p < 0.01$. This value shows that the data is positively correlated. It indicates the students who have high levels of realizing geometric invariants, also high levels of proving and vice versa. The results of the post-test show a significant correlation between students' proving levels and their test scores. However, we have decided a third variable, *level of realizing geometric invariants*, could be influencing the correlation. To answer this question we used partial correlations.

Table 3.31: *Partial correlations in the post-test*

Control Variables			Levels of Proving	Score
Levels of Realizing Geometric Invariants	Levels of Proving	Correlation	1.000	.748
		Significance (1-tailed)	.	.000
		df	0	64
	Score	Correlation	.748	1.000
		Significance (1-tailed)	.000	.
		df	64	0

The correlation test statistic r = 0.748, df = 64, $p < 0.01$. These results indicate that as the level of proving increases, scores on test also increases, when the effects of realizing geometric invariants have been controlled for. This is a positive correlation. So the relationship between levels of proving and test scores (performance) is not the result of the levels of realizing geometric invariants.

Finally, we have investigated the effects of open-ended questions and explorative tasks in the interactive HELP SYSTEM on a student's proof-writing ability. The same problem was given to 20 students in four conditions. These students were chosen and five randomly allocated to each condition (*variance*, *conjecture*, *argumentation*, and *proof*). We used ANOVA test to support our verification.

Table 3.32: *Descriptive statistics*

Group	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean		Minimum	Maximum
					Lower Bound	Upper Bound		
Invariance	5	6.00	.791	.354	5.02	6.98	5	7
Conjecture	5	6.80	.570	.255	6.09	7.51	6	8
Argumentation	5	7.10	.742	.332	6.18	8.02	6	8
Proof	5	7.40	.652	.292	6.59	8.21	7	8
Total	20	6.83	.832	.186	6.44	7.21	5	8

Levene's test of homogeneity of variances results are as follows: Levene's test = 0.153, df1 = 3, df2 = 16, $p > 0.05$. This result is not significant so we can assume that the variances of the groups are approximately equal.

Table 3.33: *Test of homogeneity of variances*

Levene Statistic	df1	df2	Sig.
.153	3	16	.926

Table 3.34: *ANOVA test*

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	5.438	3	1.813	3.766	.032
Within Groups	7.700	16	.481		
Total	13.138	19			

ANOVA test result as follows: $F(3,16) = 3.766$; $p < 0.05$. This indicates that there is a significant difference between the four groups. The following post hoc test shows all the possible pairwise comparisons for our four groups of students:

Table 3.35: *Post hoc test multiple comparisons*

Dependent Variable: Score.
Tukey HSD.

(I) Group of Student	(J) Group of Student	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Invariance	Conjecture	-.800	.439	.299	-2.06	.46
	Argumentation	-1.100	.439	.097	-2.36	.16
	Proof	-1.400*	.439	.026	-2.66	-.14
Conjecture	Invariance	.800	.439	.299	-.46	2.06
	Argumentation	-.300	.439	.902	-1.56	.96
	Proof	-.600	.439	.536	-1.86	.66
Argumentation	Invariance	1.100	.439	.097	-.16	2.36
	Conjecture	.300	.439	.902	-.96	1.56
	Proof	-.300	.439	.902	-1.56	.96
Proof	Invariance	1.400*	.439	.026	.14	2.66
	Conjecture	.600	.439	.536	-.66	1.86
	Argumentation	.300	.439	.902	-.96	1.56

*. The mean difference is significant at the 0.05 level.

In each comparison, one group is given the identifier 'I' and the second 'J'. This is evident in the *Mean Difference* column, which gives the resulting figure when the mean of one group (J) has been subtracted from the mean of another group (I). We can see that the difference between invariance and proof is significant at the 0.05 level of significance. This indicates the gap between realizing geometric invariants and writing a formal proof in the proving process.

The results of this research provide a strategy for using the interactive HELP SYSTEM in the proving process. The treatment in this research has a positive effect and increases students' levels of proving. From the results of testing the hypotheses, it can be concluded that there is positive correlation between students' levels of proving and their performance. This correlation is actually influenced by the students' levels of realizing geometric invariants. It was also found that there is a difference performance among groups which attained different levels of proving. This distinction shows the structural gap between argumentation and proof. Furthermore, it has also been found the difficulty in transitioning from the invariance phase to the proof phase in learning geometry at the tertiary level (as shown in Section 3.2.3).

3.3. SUMMARY

This chapter deals with the design of the research, methods by which the data was gathered and analyzed. It also reports briefly some results obtained from the analyses of observations, questionnaires, semi-structured interviews, and hypotheses testing. These analyses contribute in answering some research questions which were posed at the beginning of the research. Through these analyses, we can understand how students use the interactive HELP SYSTEM as well as their behaviors during the proving process. The students' positive attitude towards this system represents its role of support for students in different phases of the proving process. Furthermore, through this investigation, the gap between conjecture and argumentation as well as the gap argumentation and proof were also revealed. Some appropriate strategies used to cover these gaps were recognized in this chapter. We also examined students' difficulties and concentrated on some aspects which related to generating ideas of proofs, especially the role of realizing geometric invariants in the proving process. In particular, some results from this chapter showed that our methodological model improves students' levels of proving as well as levels of realizing geometric invariants. In general, these results have not only statistically but also pedagogically significance in the proving process. These findings will be reported concisely in chapter four.

Chapter 4
FINDINGS AND RECOMMENDATIONS

4.1. FINDINGS OF THE RESEARCH

A proof as a final product of the proving process in mathematics textbooks does not feel satisfactory to students because this proof quite often does not provide understanding of the process of proving itself. The essence of learning proofs is to understand the proving process and use appropriate strategies and tools as a means of exploration, discovery, and invention. For that reason, there should also be a distinction between understanding a proof as product and a proof as process. In other words, there should be a distinction between understanding proofs logically and understanding the central ideas of a proof. For tertiary students, in order to realize the ideas of proof in solving open problems, they need to understand the development of the proving process. This understanding does not solely consist of knowing how each phase logically follows the previous phases. It includes the process of constructing cognitive unity during conjecture validation and the transition from abductive argumentation to deductive proof. Therefore, organizing the proving process in terms of different phases was a useful tool for building an understanding of it. Based on these arguments, we have determined *students' levels of proving* with respect to the corresponsive phases of the proving process. According to our classification, there are seven levels of proving: level 0 (information), level 1 (construction), level 2 (invariance), level 3 (conjecture), level 4 (argumentation), level 5 (proof), and level 6 (delving). In order to verify the validity and reliability of this classification, we built an interactive HELP SYSTEM which was embedded in a dynamic geometry environment (GeoGebra). This help system includes *open-ended questions* and *explorative tasks* that correspond with different levels of proving. On the basis of the students' performance at the end of experimental teaching we have found that this system *improves students' level of proving* and makes a contribution in developing their geometric thinking.

In addition, throughout our experimental teaching, we *classified different difficulty categories* that students met during the proving process such as: lack of explorative strategy, difficulty in reading diagram, producing arguments, and organizing valid arguments to write a formal proof. These difficulties also reveal *the gap between conjecture and argumentation* and *the gap between argumentation and proof*. It means that some students could not the construct cognitive unity during the process of validating a conjecture. In other words, they have difficulty in producing 'valuable arguments' for proofs. The gaps between argumentation and proof are structural and cognitive. After producing arguments, because of these gaps students could not differentiate valuable arguments from a set of collected arguments and could also not reverse the structure of abductive argumentation in writing a deductive proof. However, with the support of the interactive HELP SYSTEM, some students can bridge these gaps in writing a formal proof. They tend to use abductive argumentation during the process of realizing invariants, validating conjectures and writing proofs. This research has also revealed that *realizing geometric invariants is a crucial element in generating ideas for proofs*. To interpret this element, we have also *classified levels of realizing geometric invariants* and investigated the relationship between the level of proving and the level of realizing geometric invariants. Based upon the empirical results, we proposed five levels of realizing geometric invariants: level 0 (realize no invariant), level 1 (realize static invariants), level 2 (realize moving invariants), level 3 (realize invariants of a geometric transformation), and level 4 (realize invariants of different geometries). Hypothesis testing has also confirmed that there is *a 'positive linear' relationship between level of proving and level of realizing geometric invariants*. It means that students who attain high levels of realizing geometric invariants also attain high levels of proving and vice versa.

In this research, we have also defined the concept of *dynamic visual thinking* and realized *its role in the proving process*. This kind of thinking supports students in improving the level of realizing geometric invariants and revealing the ideas of proof. Developing students' dynamic visual thinking would provide them with the ability to observe the static and moving invariants, realize the properties of shapes, interpret the diagram, and transform from the diagram into a chain of arguments. Dynamic visual thinking has also *improved students' logical skills* by linking the geometric objects. In particular, a link between dynamic visual diagrams and formal arguments is an essential aspect in the transition from argumentation to proof.

The most significant contribution to improvements in proof teaching will be made by the development of good models of methodology, supported by carefully designed activities and resources. For that reason, our methodological model – the interactive HELP SYSTEM – would *provide tertiary students with a strategy to solve proof-related problems and enhance the level of provin*g. Working with this model, students could clearly understand the development of the proving process within a dynamic geometry environment. During our experimental teaching, we also realized and *proposed basic conditions for understanding the proving process*. These conditions determine the territory before a proof, realize different levels of proving, realize geometric invariants for generating ideas for proofs, construct a cognitive unity in the transition from conjecture to proof, understand the relationship between argumentation and proof, use different kinds of argumentation during the proving process, and organize arguments to write a formal proof. It means that to understand the proving process, every student needs to know how to work and experience these conditions during the proving process with the support of dynamic geometry software. This dynamic environment creates collaborative activities to explore open problems and provides useful elements to explain why and how this tool can be a support for proving activities. It also appears the interplay between *the spatio-graphical field* (including geometric objects, paper drawings, etc) and *the theoretical field* (including geometrical properties, relationship, theorems, etc) occurs as students interact with a dynamic geometry environment. Therefore, the findings of this research can be used to *enhance the quality of learning and teaching proof* both at the tertiary level and the secondary level.

4.2. RECOMMENDATIONS

In some countries, a lot of crucial aspects of mathematics including proofs and proving have been reduced in importance or eliminated from the mathematics curriculum and basic requirements of secondary school. However, proofs and proving have also played an important role in the Vietnamese curriculum of mathematics in secondary schools, especially in upper secondary school (as shown in Section 2.2). Through proving activities, students can realize the meaning of mathematics in their real-world life. Therefore, they should understand the development of the proving process not only in the geometry field but also in other fields of mathematics, especially in algebra. The further research question is *whether or not the classification of proving levels in this dissertation is suitable for the proving process in algebra*. Thus, it is important to determine the differences between the development of the proving process in geometry and algebra. This means we have to consider the *appropriation of the basic conditions for understanding the development of the proving process in algebra*. The meaning of the invariant concept in the interactive HELP SYSTEM model needs to be redefined and evaluated concerning its role in algebraic problems. In particular, the relationship between argumentation and proof in the algebra is different from that in geometry. The reason for this difference in the algebra field is due to the fact that "it often appears unbroken cognitive unity in the transition from conjecture to proof". Therefore, it is also necessary to examine how to construct a cognitive unity in algebraic proof-related problems. Finally, it should also be investigated the role of computer algebra systems during the proving process and scrutinize how to use this environment to support students in understanding algebraic proofs.

In our research, the participants who took part in the experimental teaching are second-year students from the teacher training university. We have found that our interactive HELP SYSTEM, as well as basic conditions for understanding the development of proving process, is suitable for tertiary students. However, a problem that may arise from this research is how to apply this model to support students in learning of proof and *whether or not the basic conditions have appropriately remained* for understanding the proving process at the secondary school. It should also determine which level of proving the students need to be supported from dynamic geometry software. For instance, the student attains level 2 of proving needs to use dynamic geometry software to realize moving geometric invariants. In addition, in the teaching of proof in secondary schools, mathematics teachers should focus on the pedagogical tasks which contain typical mathematical processes to work in depth with. Teachers should also provide students with a rich opportunity to make a conjecture during the proving process because once students have formulated a conjecture; the conflict is not that they do not know how to prove it, but to understand that they need a proof. Otherwise, in geometry, when students validate their conjectures, they often visualize and collect data from diagrams and figures to produce arguments. These arguments may provide a few clues for the proving process, although do not produce an instant proof. Therefore, teachers should design visualizing activities to support students at the secondary school in transforming from figural into conceptual aspect to produce arguments during the proving process.

Dynamic visual thinking is also an important concept which was introduced in this research. We have realized that it plays a crucial role in proofs using geometric transformations. Indeed, it provides students with 'a vision' of realizing geometrical facts, internalizing specific facts, learning to reason, shifting attention from specific relationships to properties, and then reasoning on the basis of realized and then perceiving properties. Through these activities students' powers are engaged and developed, such as the power to imagine, to express what is imagined in figures, diagrams, movements, invariants, and symbolic objects. These powers are emphasized both before and after using dynamic geometry software; especially the ability of imagining or re-imagining what changed and what invariants remained the same without dynamic geometry software. It is also important to describe (to 'say what you see') accurately what objects are related after imagining. For instance, to imagine a parallelogram, students' attention may be attracted to many different features: vertices, sides, angles, parallelism, equality of length, equality of angles, but also size and orientation. Some features are useful mathematically, others are not; some features are to be discerned, others are relationships to be recognized; others are properties that are being exemplified. Therefore, in order to work with images and use effective visual dynamic thinking during the proving process, teachers should encourage their students to experience reasoning by using the strategy "say what you see and write what you imagine". To achieve this goal, the teachers' job in setting up this situation is to provide the scaffolding for discussing what is seen, but also to fade this scaffolding as the discussion, aimed at enhancing students' sense of geometric seeing and reasoning, progresses. The further research on dynamic visual thinking would make a significant contribution to improve students' proving levels and develop geometric thinking.

As a result, the mathematics curriculum at the secondary school should pay special attention to *the role of geometric transformations* in solving geometric problems and developing students' dynamic visual thinking. Students need to realize invariants for different transformations such as point reflection, line reflection, rotation, translation, and similarity. This is a basis for approaching the group of transformations, which is an important concept in building geometry in a modern point of view. Therefore, the group of transformation should be focused on solving geometric problems at the teacher training universities. Mathematics students should be trained to develop the sense of realizing invariants of transformations and the ability of reasoning based on geometric invariants. A further research on this issue within a dynamic geometry environment would make a contribution to develop students' reasoning and proof. This topic also relates to logical thinking and dynamic visual thinking in geometry. Thus, we would like to recommend this topic to other researchers aimed at creating a discussion forum about learning and teaching geometric invariants at the secondary school.

4.3. FINAL CONCLUSIONS

The current trend of mathematics education is towards mathematical problem solving and non-routine mathematical tasks because the use of drill-and-practice alone might not be sufficient. Therefore, *a mathematics teacher needs to be trained both with the pedagogical skills and sufficient mathematical content knowledge*. In order to achieve this goal, more effort should be taken by the teacher training universities to allow opportunities for trainee teachers to improve their mathematical content knowledge. With our experiences in teaching elementary geometry for second-year students at the Thai Nguyen University of Education in Vietnam, we did our research on geometric proving process aimed at *improving student's proving level and the quality of teaching proofs at the secondary school as well*. Some proofs in the Vietnamese mathematics textbooks are usually presented as already-made products. In this situation, students are only 'users' and not allowed to participate in the process of the construction of knowledge. Therefore, it is necessary to make students explain why a conjecture holds and how to validate the conjecture. There is also a need to introduce proofs in the classroom in a way that allows students to face and overcome the difficulties, especially the difficulty of distinguishing and using appropriate empirical and deductive arguments. It means that students need to know how to transform from empirical arguments into deductive arguments and then a deductive proof. This transition shows the relationship between proof and argumentation on which is focused in our research. However, the transition to proof is abrupt when students enter university because they need not understand only proof as product but proof as process at the tertiary level. Therefore, our objectives in this research is providing prospective mathematics teachers strategic and pedagogical techniques both in tackling proving-related problems and teaching geometric proofs in the secondary school.

The general objectives of the mathematics curriculum at the tertiary level related to proving are: developing logical and intuitive abilities; recognizing logical concepts and rules in argumentative and proving contexts; proving geometric properties; making sense of formal mathematics symbols. Therefore, the proving process relies on a range of 'habits of mind' such as looking for geometric invariants, formulating and validating conjectures, organizing logical arguments, and writing a formal proof. However, students usually do not understand the meaning of these acts of proving and when they are asked to prove a theorem; they cannot really make sense of it. These cognitive difficulties lead to the fact that students prefer empirical arguments over deductive arguments when presented the results of the act of proving. As a result, students are capable of conjecturing and arguing using everyday language and most of them recognize that an empirical justification is not enough, but they do not know how to provide a formal proof. For that reason, we have designed a methodological model, on which the interactive HELP SYSTEM is based, for supporting tertiary students in understanding the proving process. This model encourages students to formulate conjectures and to engage them in abductive argumentation activities. The findings of this research have also confirmed that this model makes a contribution to *improving the students' levels of proving and developing their dynamic visual thinking*. In particular, this methodological model within a dynamic geometry environment also promotes that students realize geometric invariants so as to 'flash' ideas of proofs and encourages them to construct a proof for what they discover. It also acts as a potential environment for exploring and promoting links between empirical and deductive arguments. Therefore, we suggest two different approaches in using a dynamic geometry environment: one is moving, related to invariants, and the other is static, related to diagrams, words, and symbols. Students need to focus on invariants rather than focus on details which suppress the overall impression of a drawing in its concentration on local relationships between parts of a figure. The idea of focusing on geometric invariants during the proving process relates well to the idea of developing the 'geometrical eye'[30]. It is the power of seeing geometrical properties detaching themselves from a figure and supporting students in making new conjectures. Some spontaneous arguments which are produced during the process of validating these conjectures may be utilized to write a formal proof.

Teachers should also provide students *TOULMIN model of argumentation as a means of producing and analyzing arguments* during the proving process. It supports students in the transition from abductive argumentation to deductive proofs by reversing the abductive structure in the process of validation of conjectures. Moreover, during the proving process, lots of interplay between, transitions from-to, transformations of-into would take place. Therefore, in order to understand the proving process, students must consider the relationships between different activities which are related to proofs, i.e. experimenting, realizing geometric invariants, conjecturing, arguing, and proving. That is the reason why we proposed some basic conditions for understanding the proving process within a dynamic geometry environment. On the basis of these conditions, mathematics teachers could evaluate their students' understanding of proofs and the proving process in the mathematics classroom. Consequently, they could provide each student with appropriate learning methods and strategies to construct a formal proof.

[30] GODFREY (1910) defined the 'geometrical eye' as the power of seeing geometrical properties detach themselves from a figure. This notion might be a potent tool for building effectively on geometrical intuition and solving geometrical problems.

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APPENDICES

Appendix A (*Students' attitudes towards the interactive HELP SYSTEM in the proving process*)

Below are a series of statements. There are no correct answers to these statements. They are designed to permit you to indicate the extent to which you agree or disagree with the ideas expressed. Place a checkmark in the space under the label indicating which is closest to your agreement or disagreement with the statements.

Table A.1: *Questionnaire for investigating students' attitudes*

Item No.	*Items*	*Strongly Disagree*	*Disagree*	*Unsure*	*Agree*	*Strongly Agree*
1	I would enjoy working with the interactive HELP SYSTEM					
2	Task-based activities in the interactive HELP SYSTEM stimulate my proving					
3	Working with the interactive HELP SYSTEM make me believe more in the proving process					
4	I think the interactive HELP SYSTEM could be a strategic scaffolding in the proving process					
5	The interactive HELP SYSTEM offers opportunities to discuss with the others while proving					
6	I make less mistakes when constructing the figures with the interactive HELP SYSTEM					
7	It is difficult for me to draw auxiliary figures without the interactive HELP SYSTEM					
8	It takes shorter time when constructing a figure with the interactive HELP SYSTEM					
9	I can recognize geometric invariants when dragging the figures with the interactive HELP SYSTEM					
10	The recognized invariants help me discover the ideas for proving					
11	I think realizing geometric invariants					

	is the most crucial phase for the proving process					
12	I cannot make a conjecture on my own until I use the interactive HELP SYSTEM					
13	I produce small arguments when formulating a conjecture					
14	I do not know how to produce arguments without the interactive HELP SYSTEM					
15	I can collect all arguments after validating conjectures with the interactive HELP SYSTEM					
16	I usually use wrong rules of inference in argumentation					
17	I do not know how to combine arguments in order to construct a proof without the interactive HELP SYSTEM					
18	I think writing a formal proof is the most difficult phase of the proving process					
19	I do not know how to delve into the problem without the interactive HELP SYSTEM					
20	The interactive HELP SYSTEM stimulates me to look back upon the problem after finishing proof writing					

Open-ended Questions.

1. Where do you usually meet difficulties in the proving process?
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2. *Which stages of the proving process does the interactive HELP SYSTEM support you?*
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Appendix B (*Students' levels of realizing geometric invariants*)

Below are a series of statements. They are levels of realizing geometric invariants in the proving process. Place a checkmark in the space under the label which is closest to your levels (from the lowest level 1 to the highest level 5). Statement 10 aims at investigating your assessment on the role of dynamic visualization in developing a sense of proof, especially proof by using geometric transformations method:

1 = Strongly Disagree

2 = Disagree

3 = Unsure

4 = Agree

5 = Strongly Agree

Note. Mark the number that you think of first, do not spend long time thinking about any one statement. It is important that you answer each statement. Do not worry about what you think your teachers or anyone else might want you to say. Your answers are CONFIDENTIAL. Thank you for your cooperation!

Table A.2: *Questions for classifying levels of realizing geometric invariants*

Item No.	*Items*	*1*	*2*	*3*	*4*	*5*
1	I can see given geometric invariants					
2	I can distinguish embedded or overlapping figures in a diagram					
3	I can realize static invariants					
4	I can differentiate invariants of different geometric transformations					
5	I can realize moving invariants by dragging, measuring, and checking the relationships in a dynamic geometry environment					
6	I can realize an image of the figure under a certain geometric transformation					
7	I can realize moving invariants in a static environment					
8	I can realize a geometric transformation by visualizing the movement with the picture in mind					
9	I can differentiate geometric invariants in different types of geometry					
10	I think dynamic visualization helps me develop a sense of proof in geometric transformations					

Appendix C (*Student interview questions*)

You will need about 30 minutes to finish this interview. The aim of this interview is to try and find out how you solve a problem using geometric transformations and not to know about your achievement in the subject. It will not be counted on any of your examination. Therefore, you can relax when replying to all the following questions:

Table A.3: *Questions for understanding the proving process*

No.	*Questions*	*Supplementary Questions*
1	How did you start solving the problem?	✓ *How did you get the information?* ✓ *What was your proof strategy?*
2	How did you realize the geometric invariants?	✓ *Where did you get a key invariant, from previous experiences or from the diagram?* ✓ *What did you do in order to discover the invariants (dragging, measuring, checking the relationships, discussing with the others or working alone with paper and pencil)?* ✓ *Did you produce some preliminary arguments while discovering invariants?*
3	How did you formulate a conjecture?	✓ *Did you make a conjecture as soon as recognizing invariants?* ✓ *Did you enjoy making conjectures? Were you interested in proving your conjectures?*
4	How did you validate your conjectures?	✓ *Did you gather some arguments while discovering invariants and formulating conjectures?* ✓ *Why did you need to prove your conjectures?* ✓ *What was in your mind when you proved the conjectures?*
5	How did you collect your arguments?	✓ *Did you usually justify every reason in the chains of reasoning?* ✓ *Did you take notes while discussing with the others?* ✓ *Did you look into your arguments as a*

		process?
6	How did you arrive at the formal proof?	✓ *Where did your ideas of the proof come from?* ✓ *Did you combine your arguments into a formal proof?*
7	How did you get a generalization in the proving process?	✓ *Did you have enough time to do the tasks?* ✓ *What kind of thinking manipulation did you use to delve into the problem?* ✓ *What kind of inferences did you use?*
8	What did you think about the group-of-three interactions while proving?	✓ *Did it raise your argumentation?* ✓ *What did you do when you had difficulty in the proving process?* ✓ *Did you come to agreement during the discussion? Why did you change your mind/insist your own point of view?*
9	What are your difficulties in the proving process?	
10	Could you pick out embedded or overlapping figures from the given diagram?	✓ *How did you read your diagram?* ✓ *Could you see any invariant? Static or moving invariants?*
11	Did you know what constitutes as a formal proof?	
12	Could you write down what you had in mind?	✓ *How did you validate your conjectures?* ✓ *Could you sort out the arguments in a logical way?* ✓ *Did you discuss with your group members?*

Appendix D (*Tasks for Semi-structured Interviews*)

Task D_1. *(School Problem)* People living in the neighborhood of town A and working at company B are to drive their children to school on their way to work. Where on highway l should they build school C in order to minimize their driving? (When the site C for the school is chosen, the roads AC and CB will be built).

Task D_2. *(One-Bridge Problem)* A river has straight parallel sides and cities A and B lie on opposite sides of the river. Where should we build a bridge in order to minimize the traveling distance between A and B (A bridge, of course, must be perpendicular to the sides of the river)?

Task D_3. *(Two-Bridge Problem)* Where would you build two bridges over the two sleeves of a river with parallel straight sides to minimize the length of the path between the cities A and B? (Bridges have to perpendicular to the sides of the river).

Task D_4. *(Equilateral Triangle Game)* Let *ABC* be an equilateral triangle with side length 5. Mark and Mike play the following game under the control of a referee: The referee chooses an any point, *X*, on the side *AC*; then Mark chooses a point, *Y*, on the side *BC*; and finally Mike chooses a point, *Z*, on the side *AB*. Suppose that Mike's aim is to obtain a triangle *XYZ* with the smallest possible perimeter, while Mark's aim is to get triangle *XYZ* with the largest possible perimeter. Find a strategy helping them achieve their goal?

Solutions.

Task D_1. Draw the symmetric image point *B'* of point *B* with respect to the straight line *l* (representing the highway) and the straight line *B'A*.

We have the following equality: $CA + CB = CA + CB'$ (since $CB = CB'$)

$\geq B'A = C'A + C'B'$

$= C'A + C'B$ (since we apply the triangle inequality $CA + CB \geq B'A$). Therefore, the intersection *C'* of the straight line *B'A* and the straight line *l* is the site for the school S.

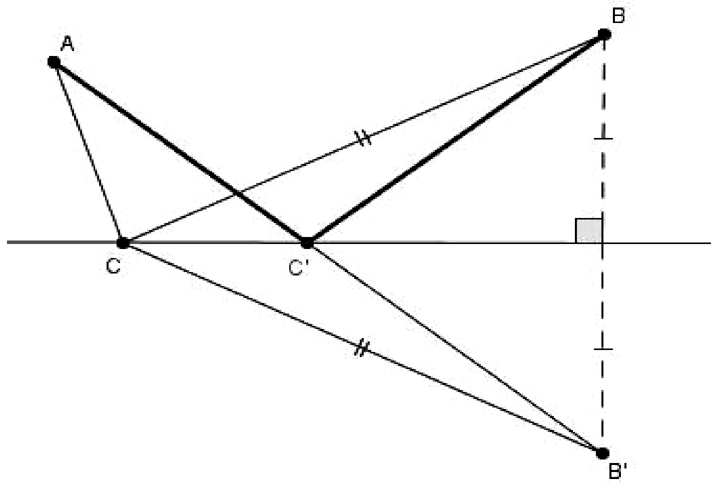

Figure A.1: *Determining a place for building a school*

Task D_2. Draw the image point *B'* of point *B* under the translation through vector $\overrightarrow{ED}$ toward the river and the perpendicular to it, and connect point *A* and point *B'* by a segment.

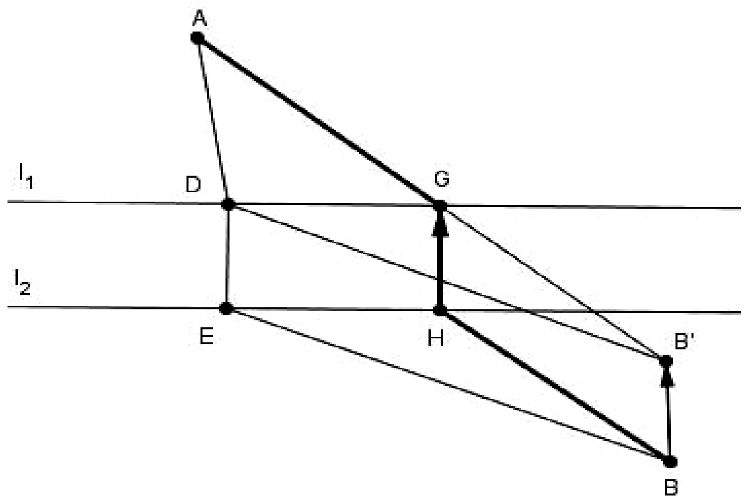

Figure A.2: *Determining a place for building one bridge*

The intersection *G* of the straight line *AB'* with the upper side of the river (representing by the line l_1) is the best place to build the bridge, since:

$AD + DE + EB = (AD + DB') + B'B$ (since $DE = BB'$, and $EB = DB'$)

$\geq AB' + BB' = AG + GH + HB$ (inequality $AD + DB' \geq AB'$).

Therefore, we always have the following inequality:

$$AD + DE + EB \geq AG + GH + HB$$

Task D₃. Draw the image point *A'* of point *A* under the translation through vector $\overrightarrow{LN}$ and image point *B'* of point *B* under the translation through vector $\overrightarrow{MO}$ (toward the rivers and the perpendicular to them), and connect point *A'* and point *B'* by a segment.

The intersection *N'* and *O'* of the straight line *A'B'* with the sides of the river (the lines l_2 and l_3) are the best places to build the bridges, since:

$$\begin{aligned} AL + LN + NO + OM + MB &= AA' + A'N + NO + OB' \\ &\geq AA' + A'B' + B'B \\ &= AA' + A'N + N'O' + O'B' + B'B \end{aligned}$$

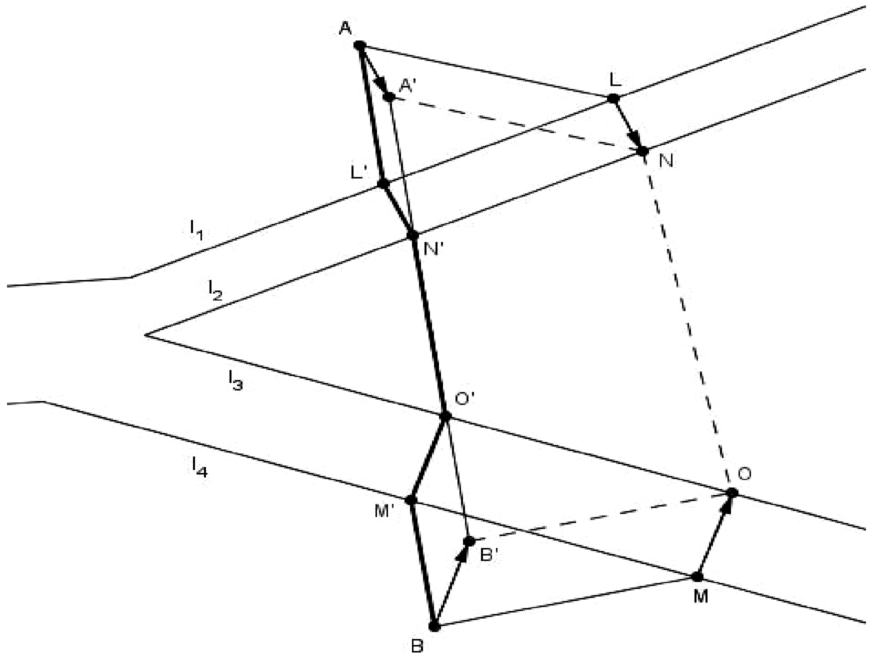

Figure A.3: *Determining two places for building two bridges*

Task D₄. Firstly, we will show that the best strategy for Mark is to choose point $Y \equiv B$ or point $Y \equiv C$. If the referee chooses a point, *X*, on the segment *AC*, we consider its reflection *X'* in the line *AB* and the reflection *X"* of point *X'* in the line *BC*. Clearly, if *X* and *Y* are already chosen, Mike has to choose *Z* as the intersection point of *AB* and *X'Y*. For such a choice of *Z* we have:

$$XY + YZ + ZX = XY + YX' = XY + YX''$$

Since Y lies on BC, the latter sum will be a maximum when $Y \equiv B$ or $Y \equiv C$, depending on the position of the segment XX''.

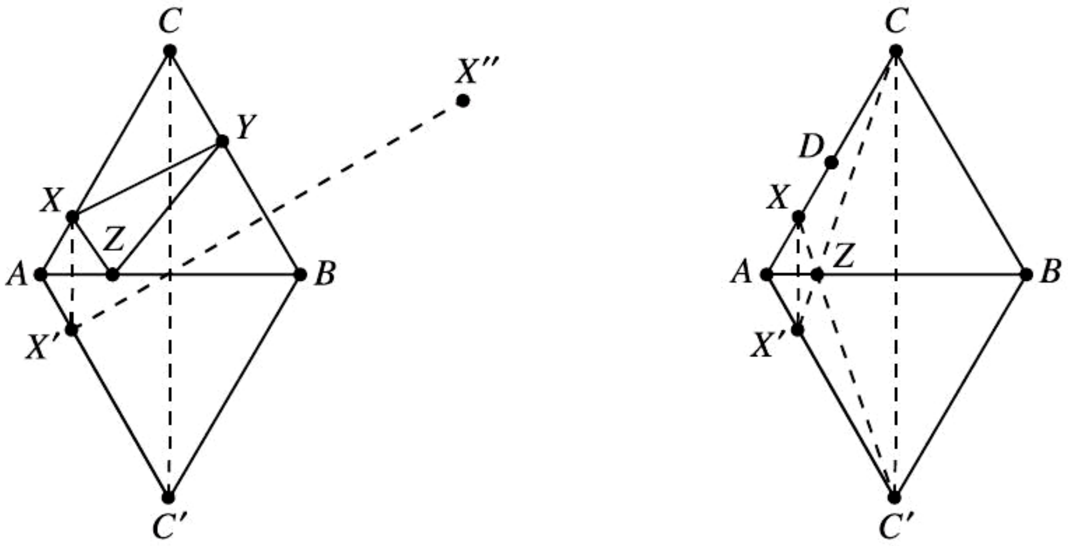

Figure A.4: *Mark's and Mike's strategies in the equilateral triangle game*

Next, if Mark chooses $Y \equiv B$, then Mike will choose $Z \equiv B$, and the perimeter of ΔXYZ will be $2XB$. Let point C' be the reflection of point C in the line AB. In case Mark takes $Y \equiv C$, Mike will put Z at the intersection point of $X'C$ and AB, and then the perimeter of ΔXYZ will be $XC + X'C = XC + XC'$. Let D be midpoint of the segment AC. Clearly, Mike has to choose X on the segment DC. There exists a point, E, on DC such that $2BE = CE + C'E$ and then Mike has to choose $X \equiv E$.

Appendix E (*Tasks for classifying levels of realizing geometric invariants*)

Task E$_1$. *(Parallelogram Problem)* Let *ABCD* be a parallelogram. The bisectors of four angles $\angle A$, $\angle B$, $\angle C$, and $\angle D$ intersect each other and forming a quadrilateral *MNPQ*. What are special characteristics of this quadrilateral?

Task E$_2$. *(Area Comparison Problem)* Let *ABC* be a triangle. Construct three squares *ABEF*, *BCMN*, *ACPQ* outwards of the triangle. Compare the areas of four triangles *ABC*, *BNE*, *CMP*, and *AFQ*.

Task E$_3$. *(Square Problem)* Let *ABCD* be a quadrilateral. Construct four squares *ABEF*, *BCMN*, *CDPQ*, *ADRS* outwards of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of these four squares. Prove that four midpoints of diagonals of two quadrilateral *ABCD* and $O_1O_2O_3O_4$ form a square $A_1B_1C_1D_1$.

Task E$_4$. *(Hexagon Problem)* Let *ABC* be a triangle. We take six points $A_1, A_2 \in BC$; $B_1, B_2 \in CA$; $C_1, C_2 \in AB$ such that: $BA_1 = A_1A_2 = A_2C$, $CB_1 = B_1B_2 = B_2A$, and $AC_1 = C_1C_2 = C_2B$. Six straight lines AA_1, AA_2, BB_1, BB_2, CC_1, CC_2 intersect each other forming a hexagon *MNPQRS*. Prove that three diagonals of this hexagon are concurrent at a point.

Solutions.

Task E$_1$. Since *ABCD* is a parallelogram, we derive that:

$$\angle A + \angle D = \angle D + \angle C = 180^0$$

We calculate the measure of two angles $\angle M$ and $\angle N$ as follows:

$$\angle M = 180^0 - (\angle ADM + \angle DAM) = 180^0 - \tfrac{1}{2}(\angle A + \angle D) = 180^0 - 90^0 = 90^0$$

$$\angle N = 180^0 - (\angle CDN + \angle DCN) = 180^0 - \tfrac{1}{2}(\angle D + \angle C) = 180^0 - 90^0 = 90^0$$

Similarly, we can prove that $\angle P = \angle Q = 90^0$. In general, we can conclude that the quadrilateral *MNPQ* is a rectangle.

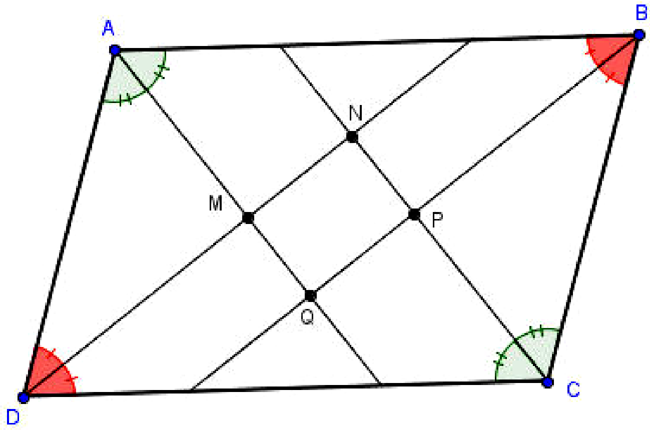

Figure A.5: *A constructed rectangle in the parallelogram problem*

Task E$_2$. We denoted $S_{\Delta ABC}$ by the area of the triangle ABC. We have:

$$S_{\Delta ABC} = \frac{1}{2}AH.BC,\ S_{\Delta CMP} = \frac{1}{2}PI.CM$$

Moreover, we realize that point H is an image of point I under the rotation of 90^0 about point C or $R_{C,90^0}(I) = H$. Therefore, we have $AH = PI$. Since $BC = CM$, so we can conclude that $S_{\Delta ABC} = \frac{1}{2}AH.BC = \frac{1}{2}PI.CM = S_{\Delta CMP}$.

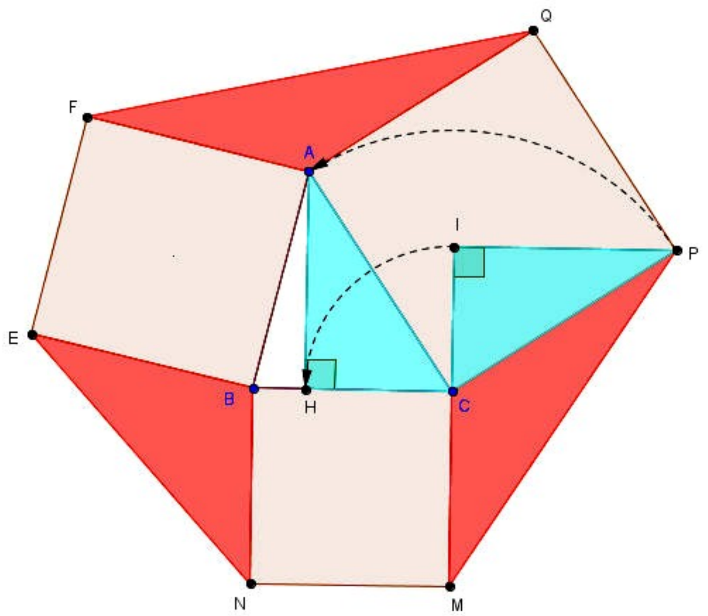

Figure A.6: *A rotation in the area comparison problem*

Similarly, we prove that $S_{\Delta ABC} = S_{\Delta BNE} = S_{\Delta AFQ}$. It follows that the areas of four triangles *ABC*, *BNE*, *CMP*, and *AFQ* are equal.

Task E$_3$. We denoted $R_{B,90^0}$ by the rotation of angle 90^0 about point *B*. It is evident that $R_{B,90^0}(A) = E,\ R_{B,90^0}(N) = C$, so it follows that $AN = CE,\ AN \perp CE$.

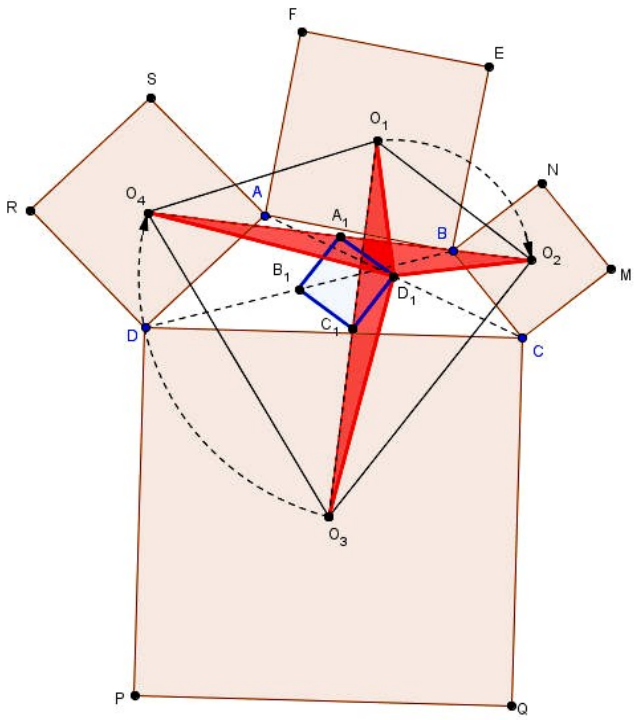

Figure A.7: *A rotation in the square problem*

From this assertion, we obtain $D_1O_2 = D_1O_1,\ D_1O_2 \perp D_1O_1$. It is easily verified that $R_{D_1,90^0}(O_1) = O_2$. Additionally, from the above reasoning we can also find that $R_{D_1,90^0}(O_3) = O_4$. It is also clear that $R_{D_1,90^0}(\Delta D_1O_3O_1) = \Delta D_1O_4O_2$ and $R_{D_1,90^0}(C_1) = A_1$ (1). We repeat the same reason as before, we can show that $R_{B_1,90^0}(A_1) = C_1$ (2). From (1) and (2), it is straightforward to verify that the quadrilateral $A_1B_1C_1D_1$ is a square, as claimed.

Task E$_4$. Firstly, we can realize that this problem contains affine concepts such as concurrency, triangle, hexagon, division of a segment, and so on. Therefore, we can conclude that this problem is an affine problem. It means that if the assertion of the problem is valid in the case of equilateral triangle then it is also valid in the case of (an) arbitrary triangle which is affine equivalent to an equilateral triangle.

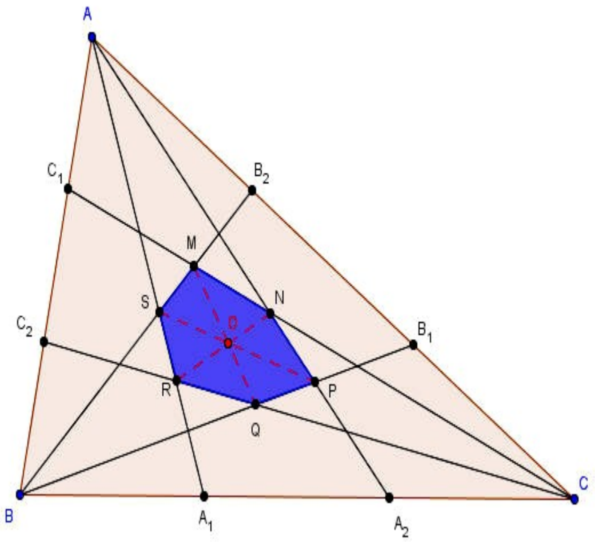

Figure A.8: *Affine invariants in the hexagon problem*

For simplicity we consider the case of an equilateral triangle *ABC*. It is a simple matter to show that $\Delta BB_1C = \Delta CC_2B$ and $\Delta BB_2C = \Delta CC_1B \Rightarrow \angle QBC = \angle QCB$, $\angle MBC = \angle MCB \Rightarrow QB = QC$, $MB = MC \Rightarrow Q$, M lay on a perpendicular bisector of the segment *BC*.

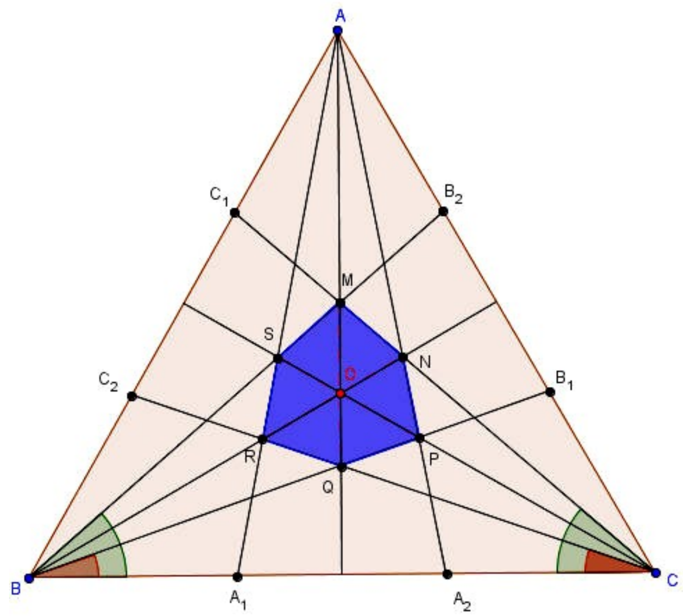

Figure A.9: *Equilateral triangle in the hexagon problem*

In such a manner, we proved that *R*, *N* and *P*, *S* lay on a perpendicular bisector of segments *AC*, *AB*, respectively. This implies that *MQ*, *NR*, *PS* are concurrent at point *O* which is the center of circumcircle of triangle *ABC*, as desired.

Appendix F (*Pre-test problems and solutions*)

Exercise F_1. Let *ABCD* is a parallelogram, and suppose that squares are constructed externally on the four sides of the parallelogram. Then the centers of these squares also form a convex quadrilateral. What are special characteristics of this quadrilateral?

Exercise F_2. *(Triangle Translation Problem)* Let *D*, *E*, and *F* be the midpoints of sides *AB*, *BC*, and *CA*, respectively, of triangle *ABC*. Let O_1, O_2, and O_3 denote the centers of the circles circumscribed about triangles *ADF*, *BDE*, and *CEF*, respectively, and let Q_1, Q_2, and Q_3 be the centers of the circles inscribed in these same triangles. What is the relationship between the triangle $O_1O_2O_3$ and the triangle $Q_1Q_2Q_3$?

Exercise F_3. *(Inscribed Quadrilateral Problem)* Let *ABCD* be a quadrilateral inscribed about a circle with the center *O*. Let *M*, *N*, *P*, *Q* be the midpoints of the sides *AB*, *BC*, *CD*, and *DA* of the quadrilateral. Prove that the four straight lines, which pass through *M*, *N*, *P*, *Q*, respectively and are perpendicular to the opposite sides of the quadrilateral, are concurrent.

Solutions.

Task F_1. Suppose the centers of the squares that constructed externally on the four sides of the parallelogram form the quadrilateral $M_1M_2M_3M_4$. We can easily show that the diagonals M_1M_3 and M_2M_4 are equal and mutually perpendicular (in a manner entirely analogous to the result of the previous task E_3).

Further, since the point *O* of the intersection of the diagonals of the parallelogram *ABCD* is its center of symmetry. In particular, it is the center of symmetry for the quadrilateral $M_1M_2M_3M_4$ (which must, therefore, be a parallelogram – since the parallelogram is the only quadrilateral that has a center of symmetry). But a parallelogram whose diagonals are equal and perpendicular must be a square.

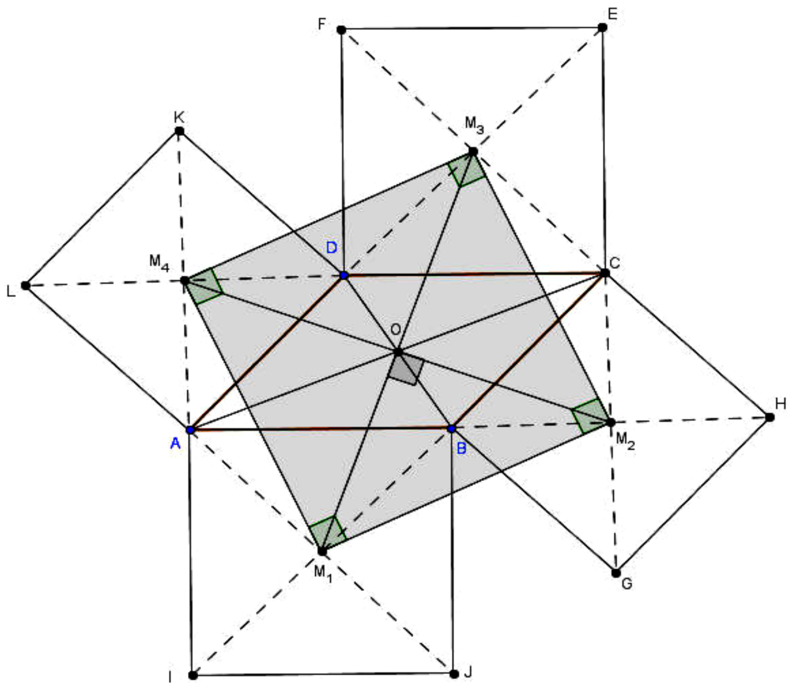

Figure A.10: *The quadrilateral that has a center of symmetry*

Task F_2. Observe that triangle ΔBDE is obtained from triangle ΔDAF by a translation (in the direction *AB* through a distance *AD*); thus the line segments joining pairs of corresponding points in these two figures are equal and parallel to one another. Therefore, we obtain $O_1O_2 = Q_1Q_2$ and $O_1O_2 \ // \ Q_1Q_2$.

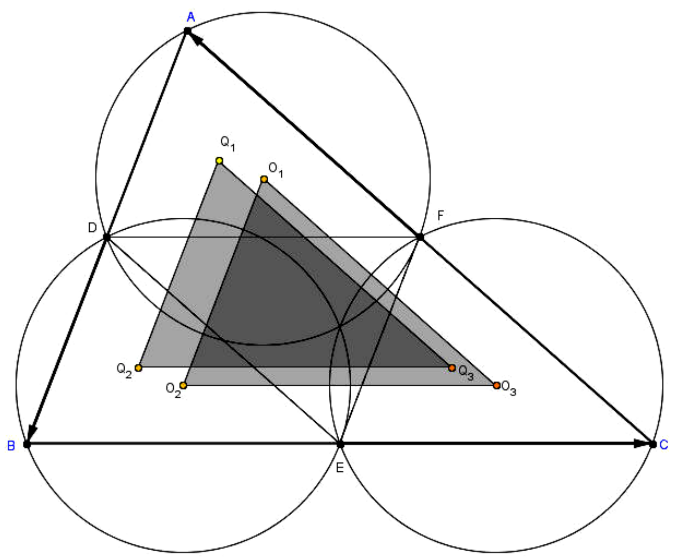

Figure A.11: *Illustrating a translation of a triangle*

Similarly, we indicate that $O_2O_3 = Q_2Q_3$ and $O_2O_3 \,//\, Q_2Q_3$; $O_3O_1 = Q_3Q_1$ and $O_3O_1 \,//\, Q_3Q_1$. Therefore, two triangles $\Delta O_1O_2O_3$ and $\Delta Q_1Q_2Q_3$ are congruent (in fact, their corresponding sides are parallel, that is, the triangles are obtained from the other by a translation).

Task F_3. Since O is the center of circle then we have four straight lines OM, ON, OP, OQ perpendicular to four sides of the quadrilateral AB, BC, CD, DA, respectively. Otherwise, M, N, P, Q are midpoints of the sides of the quadrilateral, we derive that $MNPQ$ is a parallelogram. Let point I be the intersection of two diagonals of the parallelogram $MNPQ$. We denoted r_I by the central symmetry (point reflection) I.

Clearly, $r_I(M) = P$, $r_I(N) = Q$, $r_I(P) = M$, $r_I(Q) = N$. Therefore, four straight lines MM', NN', PP', QQ', which pass through M, N, P, Q and perpendicular to the opposite sides of the quadrilateral, are the images of the lines OM, ON, OP, OQ, respectively by r_I.

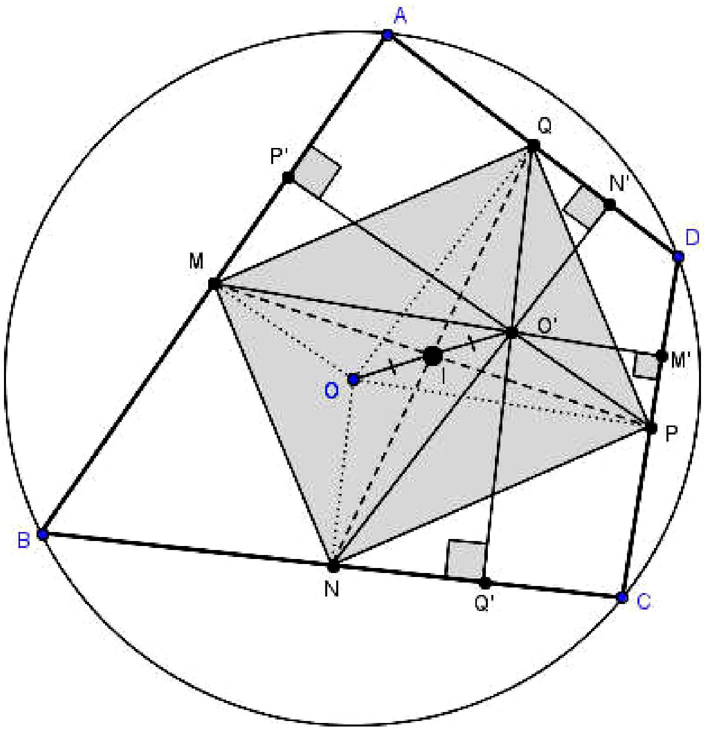

Figure A.12: *Central symmetry in the inscribed quadrilateral problem*

Since OM, ON, OP, OQ are concurrent at point O, yielding four straight lines MM', NN', PP', QQ' that are concurrent at point $O' = r_I(O)$, as claimed.

Appendix G (*Post-test problems and solutions*)

Exercise G_1. *(Orthic Triangle Problem)* Inscribe a triangle with a minimal perimeter in a given acute-angled triangle ΔABC.

Exercise G_2. *(Electronic Transformer Station)* There are three villages *A*, *B*, *C* forming a triangle *ABC*. Find the position *X*, lying inside the triangle, in order to build an electronic transformer station such that the sum of distance from the station to the villages is minimal.

Exercise G_3. *(Mine-Finding Problem)* A solider has to check for mines in a region having the form of an equilateral triangle. The radius of activity of the mine detector is half the altitude of the triangle. Assuming that the solider starts at one of the vertices of the triangle, find the shortest path he could use to carry out his task.

Solutions.

Exercise G_1. Let *ABC* be the given triangle. We want to find points *M*, *N*, and *P* on the sides *BC*, *CA*, and *AB*, respectively, such that the perimeter of ΔMNP is minimal.

First, we consider a simpler version of this problem. Fix an arbitrary point, *P*, on the side *AB*. We are now going to find points *M* and *N* on the sides *BC* and *CA*, respectively, such that ΔMNP has a minimal perimeter (this minimum, of course, will depend on the choice of point *P*). Let *P'* be the reflection of the point *P* in the line *BC* and *P"* the reflection of *P* in the line *AC*. Then $CP' = CP = CP''$, $\angle P'CB = \angle PCB$, and $\angle P''CA = \angle PCA$. Setting $\gamma = \angle BCA$, we then have $\angle P'CP'' = \angle 2\gamma$. Furthermore, we obtain $2\gamma < 180^0$, since $\gamma < 90^0$ by the assumption. Consequently, the line segment *P'P"* intersects the sides *BC* and *AC* of ΔABC at some points, *M* and *N*, respectively, and the perimeter of ΔMNP is equal to *P'P"*.

In a similar way, if *X* is any point on *BC* and *Y* is any point on *AC*, the perimeter of ΔXPY equals the length of the broken line *P'XYP"*, which is greater than or equal to *P'P"*. So, the perimeter of ΔPXY is greater than or equal to perimeter of ΔPMN, and equality holds precisely when $X \equiv M$ and $Y \equiv N$.

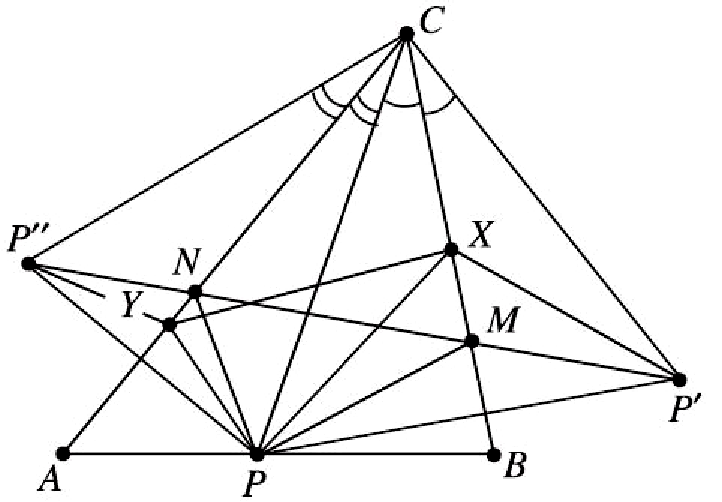

Figure A.13: *Minimal perimeter in the orthic triangle problem*

Thus, we have to find a point, P, on the side AB such that the line segment $P'P''$ has minimal length. Notice that this line segment is the base of an isosceles triangle $P''P'C$ with constant angle 2γ at point C and sides $CP' = CP'' = CP$. So, we have to choose P on AB such that $CP' = CP$ is minimal. Obviously, for this to happen, P must be the foot of the altitude through C in ΔABC.

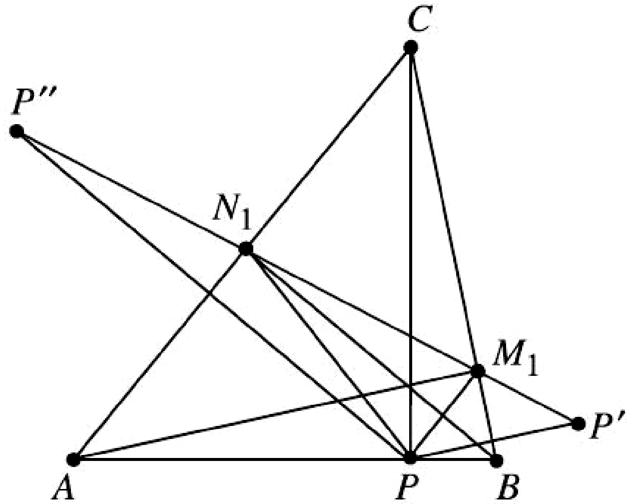

Figure A.14: *An orthic triangle with the smallest perimeter*

Note now that if P is the foot of the altitude of ΔABC through point C, then point M and point N are the feet of the other two altitudes. To prove this, denote by M_1 and N_1

the feet of the altitudes of ΔABC through A and B, respectively. Then $\angle BM_1P' = \angle BM_1P = \angle BAC = \angle CM_1N_1$, which shows that the point P' lies on the line M_1N_1. Similarly, point P'' lies on the line M_1N_1 and therefore $M \equiv M_1$, $N \equiv N_1$. Hence, of all triangles inscribed in ΔABC, the one with vertices at the feet of the altitudes of ΔABC has a minimal perimeter. The triangle has its vertices at the feet of the altitudes of a certain triangle is called an orthic triangle.

Exercise G_2. We consider the reflection point X' of point X on the line AB. We have $AX' = AX$, $BX' = BX$. Also, the line segment CX intersects the line AB at some point Y, and $XY = X'Y'$. Putting $t(X) = AX + BX + CX$. Now the triangle inequality is given:

$$CX' < CY + X'Y = CY + XY = CX \rightarrow t(X') < t(X)$$

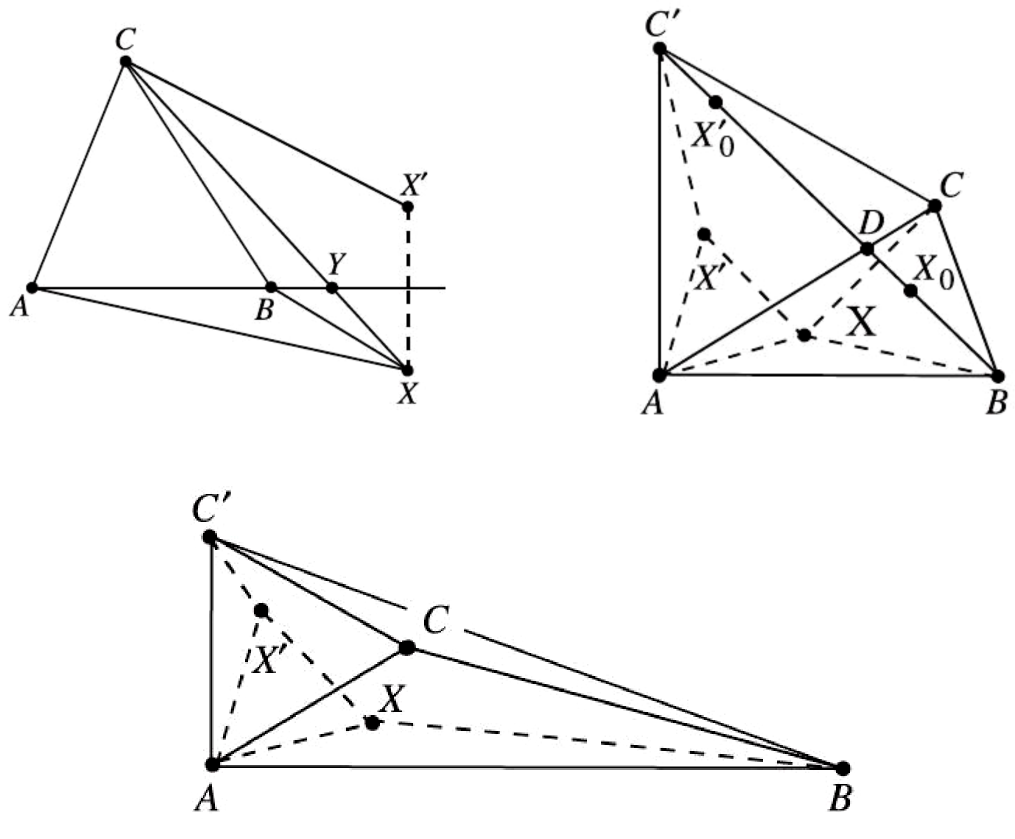

Figure A.15: *Determining a place for building an electronic transformer station*

Let α, β, and γ be the angles of ΔABC. Without the loss of generality we will assume that $\gamma \geq \alpha \geq \beta$. Then α and β are both acute angles.

Denote by $R_{A,-60^0}$ the rotation through 60^0 counterclockwise about A. For any point M in the plane let $M' = R_{A,-60^0}(M)$. Then AMM' is an equilateral triangle. In particular, $\Delta ACC'$ is equilateral.

Consider an arbitrary point, X, in ΔABC. Then $AX = XX'$, while $R_{A,-60^0}(X) = X'$, and $R_{A,-60^0}(C) = C'$, imply $CX = C'X'$. Consequently, $t(X) = BX + XX' + X'C'$, i.e., $t(X)$ equals the length of the broken line $BXX'C'$. We now consider three cases:

Case 1. $\gamma \geq 20^0$. Then we obtain $\angle BCC' = \gamma + 60^0 < 180^0$. Since, we have $\angle BAC' < 180^0$, the line segment BC' intersects the side AC at some point D. Denoted by X_0 the intersection point of BC' with circumcircle of $\Delta ACC'$. Then X_0 lies in the interior of the line segment BD and X_0' lies on $C'X_0$ since $\angle AX_0C' = \angle ACC' = 60^0$.

Moreover, we have:

$$t(X_0) = BX_0 + X_0X_0' + X_0'C' = BC' \rightarrow t(X_0) \leq t(X)$$

Equality occurs only of both X and X' lie on BC', which is possible only when $X \equiv X_0$. Notice that the constructed point X_0 above satisfies:

$$\angle AX_0C = \angle AX_0B = \angle BX_0C = 120^0$$

Case 2. $\gamma = 20^0$. In this case the line segment BC' contains C and:

$$t(X) = BX + XX' + X'C' = BC', \text{ precisely when } X \equiv C.$$

Case 3. $\gamma > 20^0$. Then the line BC' has no common points with the side AC. If $AX \geq AC$ then the triangle inequality gives:

$$t(X) = AX + BX + CX \geq AC + BC,$$

If $AX < AC$ then X' lies in $\Delta ACC'$ and $t(X) = BX + XX' + X'C' \geq AC + BC$, since C lies in the rectangle $BC'X'X$. In both cases equality occurs precisely when $X \equiv C$.

Exercise G$_3$. Let h be the length of the altitude of the given equilateral ΔABC. Assume that the soldier's path starts at the point A. We consider the circles k_1 and k_2 with centers B and C, respectively, both with radius $\frac{h}{2}$. In order to check the points B and C, the soldier's path must have common points with both k_1 and k_2. Assume that the total length of the path is t and it has a common point M with k_2 first and then a common point N with k_1. Denoted by D, the common point of k_2 and the altitude through C in ΔABC and by l the line through D parallel to AB. Adding the constant $\frac{h}{2}$ to t and using the triangle inequality, one gets:

$$t + \frac{h}{2} \geq AM + MN + NB = AM + MP + PN + NB \geq AP + PB,$$

where P is the intersection point of the straight line MN and the straight line l. On the other hand, Heron's problem shows that $AP + PB \geq AD + DB$, where equality occurs precisely when $P \equiv D$. This implies that $t + \frac{h}{2} \geq AD + DB$, i.e., $t \geq AD + DE$, where E is the point of intersection of DB and k_1.

The above argument shows that the shortest path of the soldier that starts at point A and has common points first with k_2 and then with k_1 is the broken line ADE. It remains to show that moving along this path, the soldier will be able to check the whole region bounded by ΔABC.

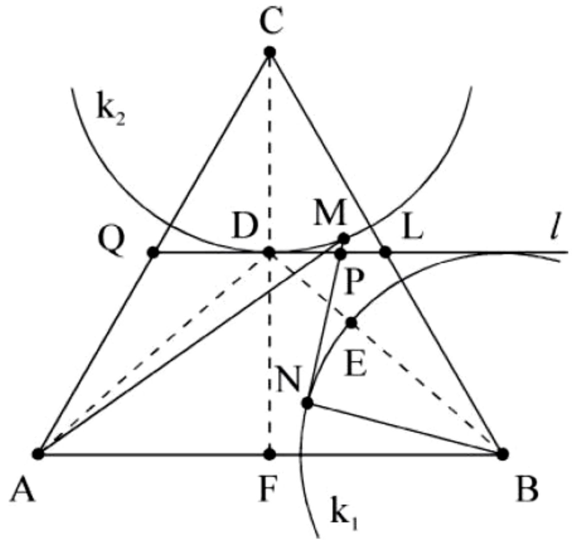

Figure A.16: *Determining a path for finding mine*

Let F, Q, and L be the midpoints of the sides AB, AC, and BC, respectively. Since $DL < \frac{h}{2}$, it follows that the disk with the center D and radius $\frac{h}{2}$ contains the whole ΔQLC. In other words, from position D the soldier will be able to check the whole region bounded by ΔQLC. When the soldier moves along the line segment AD he will check all points in the region bounded by the quadrilateral *AFDQ*; while moving along *DE*, he will check all points in the region bounded by *FBLD*.

Thus, moving along the path *ADE*, the soldier will be able to check the whole region bounded by ΔABC. Therefore, *ADE* is one solution of the problem. Another solution is given by the path symmetric to *ADE* with respect to the line *CD*. The above arguments also show that there are no other solutions starting at point *A*.

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