

## The role of abduction in realizing geometric invariants

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The purpose of this chapter is to scrutinize the role of abduction during students' proving processes. The research has revealed that a dynamic geometry environment fosters the 'observed fact' to breed abductive argumentation and amplifies the potential of realizing geometric invariants in order to generate ideas of proof. Realizing geometric invariants is an activity that has brought to the fore a proof scheme and promoted a smooth transition from invariance recognition to abductive argumentation and thereafter deductive proof. I have also proposed five levels of recognizing geometric invariants using GeoGebra software: *no invariant*, *static invariants*, *moving invariants*, *invariants of a geometric transformation* and *invariants of different geometries*. This approach aimed at clarifying how students explain a hypothesis corresponding with their levels and which factors foster the production of conjecture and its validation. A Toulmin's model of argumentation was used to analyze the role of abduction in different phases of proving processes: realizing invariants, formulating conjectures, producing arguments, validating conjectures and writing a formal proof.

**Keywords** abduction; abductive argumentation; geometric invariant; levels of realizing invariants; dynamic geometry environment; dynamic visual thinking; proof and proving

### 1. Introduction

Realizing invariants plays an essential role in the process of 'flashing' the idea of geometric proof. It makes a contribution to a smooth transition from elementary to advanced mathematical thinking, especially *dynamic visual thinking*. During the process of discovering invariants, arguments are produced. These supportive arguments serve the conjectures production and subsequently proof construction. In my research, therefore, I have encouraged the students using dynamic geometry software (GeoGebra) to realize geometric invariants and formulate their conjectures. In these processes, the students used visualization and abduction to analyze the situation because proof in the context of geometric transformations requires manipulating consecutive visual images. For that reason, I believe that visual dynamic thinking might be an important component of the students' proving development and the levels of this geometric thinking based on the ability of recognizing invariants. This research have also analyzed dragging modality from a cognitive point of view, focusing on the way dragging may effect the students' invariants recognition and argumentation. The bridge connecting a structural gap between abductive argumentation and deductive proof is also analyzed in this chapter.

Dynamic visual thinking in proving process is a part of the way the students producing arguments by extracting information and data from a *drawing*<sup>1</sup>, dynamic diagram, *figure*<sup>2</sup> and representing them in mathematical language. These arguments are not only based on words, figures but also on drawings and visual mental pictures. Students tend to conceive of the objects in a figure as being in motion, and use the *dynamic visualization*<sup>3</sup> in geometric thinking. There are four key aspects of geometric thinking: invariance, geometric language and points of view, reasoning, visualizing and representing (Johnston-Wilder & Mason, 2005). My chief concern is to make visualization apt as a means of realizing invariants, explanation and in the service of developing proving ability. During the process of proving, therefore, the students used diagrammatic and visual forms to communicate, explain, validate and show the steps involved in reaching a formal proof. These mental pictures depend on the experience of seeing from a drawing including the object is not actually being observed. This process forms a sequence of images stored in long-term memory in a hierarchical organization, and then modifies the depicted images in mind from different perspectives. This approach sows the seed of developing the sense of invariant recognition and conjecture formulation. During the conjecturing phase, some unknown properties, relationships between objects, hidden invariants were gradually evoked. As a result, some inchoate arguments were produced aimed at validating the true conjectures and disproving the false ones. These arguments can serve as supported arguments in the chain of deductive reasoning and provide a rich opportunity to write proofs. However, in order to achieve this goal, the students must use abduction to analyze and seek geometric invariants. Advantages of this procedure offer the students a deep understanding about structure of *proof scheme*<sup>4</sup> (Harel & Sowder, 1998) and reverse it to write a deductive proof. This method also facilitates the students overcoming the difficulties in constructing a formal proof at the tertiary level.

<sup>1</sup> drawing refers to the material entity. In a dynamic geometry environment, a drawing can be a juxtaposition of geometrical objects resembling closely the intended construction (Laborde, 1993).

<sup>2</sup> a figure refers to a theoretical object. It additionally captures the geometric relationships between the objects used in the construction. In such a way, the figure is invariant when any basic object used in the construction is dragged (Holzl, 1995).

<sup>3</sup> process of producing images in mind based on drawings and dynamic diagrams.

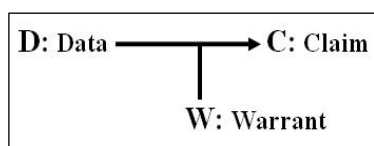
<sup>4</sup> Harel and Sowder (1998) argue that proving or justifying a mathematical conjecture involves ascertaining (convincing oneself) and persuading (convincing others). An individual's „proof scheme“ consists of what constitutes ascertaining and persuading for that person.

## 2. Abduction in Toulmin's model of argumentation

The term “abduction” was coined by Peirce (1960) to differentiate this type of reasoning from deduction and induction. Abduction is an inference which allows the construction of a claim starting from an observed fact (Magnani, 2001; Peirce, 1960; Polya, 1962). It has often been considered as a kind of ‘backwards’ reasoning and as an ‘inference to the best explanation’ because it starts from the observed facts and probes backwards into the reasons or explanations for these facts (Douglas Walton, 2001). Moreover, abduction is crucial in introducing new ideas and supports the transition to the proving modality (Peirce, 1960; Arzarello et al., 1998b). Therefore, it supports explanatory conjectures and the subsequent related proof. Using this type of inference, I can analyze students’ proving styles while they formulating conjectures and generating the ideas of proof. In a dynamic geometry environment, the strength of abduction depends on all evidence and data which are collected by dragging, observing, measuring, and checking the relationship between objects. The produced data in this environment may spur the students to generate abductive argumentation for realizing invariants.

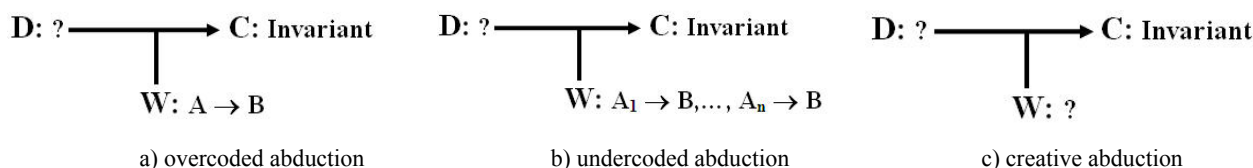
*In mathematics, proof is deductive, but the discovering and conjecturing processes are often characterized by abductive steps. When students are engaged in the mathematical practice of proving, they often “come up” with an idea. To analyze what students are doing when this happens, one can refer to abduction (Pedemonte & Reid, 2010).*

To understand the nature of abduction it is necessary to investigate the relationship between *conjectures construction* and *selection*. The purpose of constructing conjectures is to propose and explain some collected facts. The conjectures selection provides the students the way to move from naïve conjectures to the known premises and then turn back again to delete the naïve conjectures and replace them with mathematical theorems (Lakatos, 1976). Therefore, the students should make the conjectures as much as possible to supply proof construction with data and supported premises. In geometry, in order to arrive at a conjecture, the students need to realize invariants. The cognitive relation between invariance phase and conjecture phase makes a contribution to interpret the role of abduction in realizing geometric invariants. Toulmin’s basic model of argumentation (including three key elements of arguments) was used to scrutinize the process of producing arguments (Toulmin, 1958):



**Fig. 1** Toulmin’s basic model of argumentation. In the context of dynamic geometry environment, the first element (C: Claim) is the statement obtained by observing invariants in dragging modality. The second element (D: Data) is a set of data justifying the claim and the third element (W: Warrant) is the inference rule that allows data to be connected to the claim.

This model may be suitable to represent a deductive structure (data and warrants lead to a claim) but it is also a potent tool to represent an abductive step (Pedemonte & Reid, 2010). In dynamic geometry environment, dragging modality enables the students to engage in searching for a new invariant. This invariant appears in the form of a claim. Subsequently, the students tend to seek the data and select (or invent) new warrants for validating the claim:



**Fig. 2** Three kinds of abduction in Toulmin’s basic model of argumentation: overcoded abduction occurs when the arguer is aware of only one rule from which that case would follow, undercoded abduction occurs when the arguer is aware of more than one rule to follow, and creative abduction occurs when the arguer is aware of no rule to follow and she/he has to invent a new rule (Eco, 1983).

In tandem with the approach to teaching proof and proving through abduction, the students’ invariants recognition was analyzed according to Toulmin’s basic model in order to highlight and understand the continuity between realizing invariants and producing arguments. This relation seems to be natural because the students have a great need for explanation of discovered invariants. The arguments which are produced during the process of explanation may contribute to a set of plausible arguments for a valid proof. That is why teachers should encourage their students to make explanatory hypothesis so that they can accumulate the data and warrants. It is also easier to explicate the origin of invariants in the case of overcoded and undercoded abduction because the students must merely find the data and select a known rule. But in the case of creative abduction, they obligate to create new rule as a bridge connecting the found data and the realized invariants. This is cognitive obstacle in realizing new invariants and validating conjectures. There are a lot of students hence can not overcome this difficulty and get out of the way of searching for the fundamental ideas of proof.

### 3. The role of abduction in proving processes

#### 3.1 The role of abduction in realizing geometric invariants

Invariant is a central concept in geometry and is preserved under a transformation. It plays an important role in proving process using geometric transformations approach. One of the breakthroughs in modern mathematics was to characterize transformations in terms of what they leave invariant, rather than thinking about what they change (Johnston-Wilder & Mason, 2005). To understand this seminal idea, I have concentrated on the powers of dynamic geometry environment in filtering invariants of isometries from a compound shape. *Isometries* are transformations that preserve distance between points, so a figure and its image are also congruent. In plane Euclidean geometry, there are three main isometries: translation, rotation (which preserve orientation) and reflection (which reverse orientation). They produced an equivalent relation 'is congruent to' in the class of shapes which have the same size. Nevertheless, there are also some other transformations (similarities) which preserve shape but not necessarily size and they produced the relation 'is similar to' in the mathematical sense. In order to retain the peculiarity of isometry, the students should draw their attention to some basic invariants and commit them to memory such as: equality (length of a segment, measurement of angle), linearity, concurrency, perpendicularity, parallelism, congruence, etc. For instance, if two straight lines are parallel then their images under an isometry are also parallel because parallelism is preserved.

In the process of realizing invariants, the students 'read' the dynamic figure in order to seek for invariants. The stream of thought goes from the figure manipulation to the arguments production. This phenomenon is called *ascending control* (Saada-Robert, 1989). Subsequently, the students use *abduction* method to choose or invent a rule that connects the data and supported arguments with those invariants. The results of this process are conjectures. Finally, the students seek a validation for produced conjectures. They refer to the arguments in order to justify what they have previously 'read' in the figure and validate their conjectures. This phenomenon is called *descending control*. Therefore, abduction offers a smooth transition from ascending control to descending control in realizing geometric invariants. Furthermore, in this research, I have shown the essential role of other kinds of inferences in different consecutive phases of proving processes: realizing invariants, formulating conjectures, producing arguments, validating conjectures and writing deductive proofs.

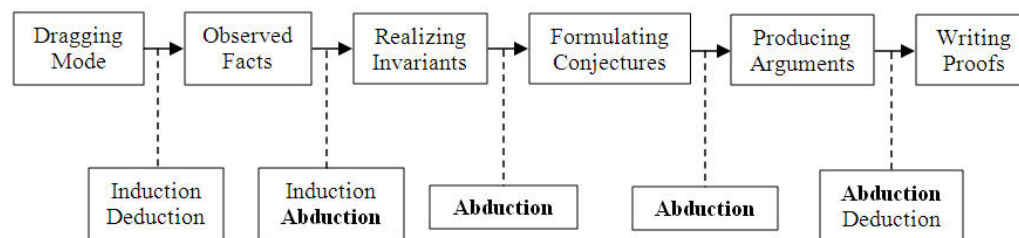


Fig. 3 Three kinds of inferences in proving processes

In order to monitor and track students' proving processes, I have recorded the students' working frame<sup>5</sup> in a dynamic geometry environment by using the screen-casting Wink® software<sup>6</sup> (Kumar, 2007; Reis & Karadag, 2008). The students were required to find invariants, form conjectures and write proofs of two real-life problems. The following protocols analysis based on the students' snapshots and audio clips aimed at interpreting the role of abduction in realizing geometric invariants:

**School Problem.** *People living in the neighbourhood of the town A and working at the company B are to drive their children to school on their way to work. Where on highway l should they build the school C in order to minimize their driving? (When the site C for the school is chosen, the roads AC and CB will be built).*

The students used GeoGebra software to model the situation. Supposed the town A and the company B are situated on the same side of the highway l. The students created an arbitrary point C on the line l and measured the length of the broken line ACB. They dragged point C slowly on the line until the sum (AC + CB) is minimal.

10. L: Now drag point C and observe what's occurred with the figure?

11. T: But firstly you have to measure the length of broken line ACB.

15. L: Drag point C more slowly please! This position may satisfy the length is minimal, can you try it again?

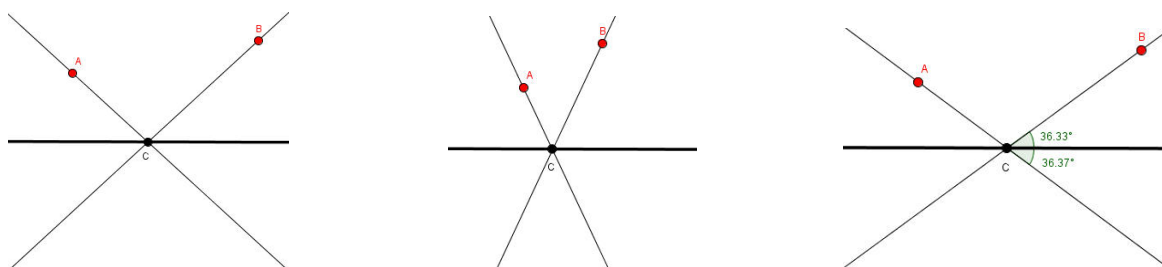
17. T: Yes, this school should be built here, but what are special characteristics at this position? I see nothing!

<sup>5</sup> A frame is defined as the snapshots of the computer screen at a specified moment.

<sup>6</sup> This software also allowed me to zoom into any frame recorded and to annotate it. This feature delivered my messages and jotted my notes down on the desired frames. It also made the communication easier because I can easily navigate the frames, describe the moment of action, and deliver the message in order to provide opportunity of just-in-time commenting.

The students could not see invariants at the first glance. Thus, they have altered their initial strategy by changing the positions of point  $A$ , point  $B$  or both of them in order to realize invariants. In each case where the sum is minimal, the students saved the pictures and simultaneously committed them to the memory (in the form of mental pictures). The effect of this strategy depends on the students' level of dynamic visual thinking. As a result, they arrived at the first conjecture after measuring some angles by GeoGebra.

19. H: Save the picture in this case. Change the position of point  $A$  or point  $B$ , drag point  $C$  again and observe!  
 24. L: Hey, wait! I think the angle between the line  $CB$  and the highway seems to be equal to the angle between the line  $CA$  and the highway. Can you measure these angles!  
 25. T: That's right! One angle is  $36^{\circ}33'$  and the other is  $36^{\circ}37'$ !  
 27. L: We change the position of points  $A$ ,  $B$  and measure again!

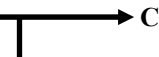


**Fig. 4** Three saved pictures (including mental pictures in other senses) where the sum  $(AC + CB)$  is minimal

28. H: Yes, they are almost equal! But if they were equal, so what would be happened? How can we explain these facts?  
 29. L: What data we have got until now? Which transformation can we use to solve this problem?  
 30. H: A fixed line represents the highway;  $A$  and  $B$  are also two fixed points because they are presenting two cities, and perhaps the angle between the line  $CB$  and the highway is equal to the angle between the line  $CA$  and the highway.  
 31. T: Exactly! We have the following plane transformations: line reflection, point reflection, translation, rotation, dilation, etc. Which transformations can we choose?  
 32. L: Which transformation preserves the measurements of angles?  
 33. H: All of above transformations preserve the measurements of angles, but I think, in this case, there is a fixed line, so we will probably use a line reflection to tackle this problem?  
 38. T: Suitable reasoning! It means that the line  $CB$  is image of the line  $CA$  under a reflection in the line that representing the highway?

After making the conjectures, the students used abduction to seek for explanatory data by posing some questions like “if they were..., what would be happened?”, “how can we explain these facts?” and followed by selecting a supported warrant to explicate the origin of the invariant “which transformation can we use to solve this problem?” or “which transformation preserves the measurements of angles”, etc. The following Toulmin’s model describes aforementioned abductive processes:

**C:** The angle between the line  $CB$  and the highway is equal to the angle between the line  $CA$  and the highway.

**D = ?**   
**W: Property of Line Reflection**

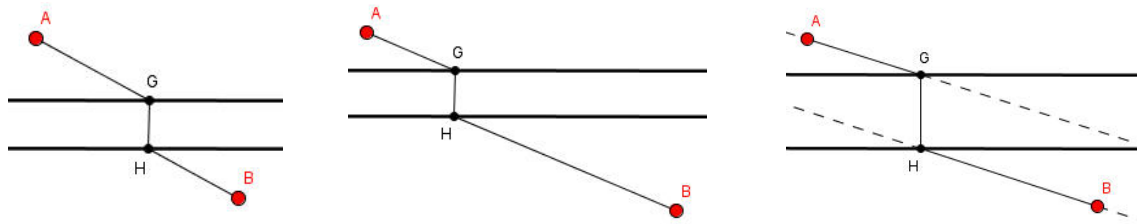
**D:** The line  $CB$  is image of the line  $CA$  under a reflection in the line representing the highway.

In the second problem, the students used the same strategy but it is more difficult to realize invariants than in the first one. They had to draw two auxiliary parallel lines after ‘flashing’ a mental picture about the invariant in mind. Then they used GeoGebra to check the initial conjecture.

**One-Bridge Problem.** A river has straight parallel sides and cities  $A$  and  $B$  lie on opposite sides of the river. Where should we build a bridge in order to minimize the travelling distance between  $A$  and  $B$  (a bridge, of course, must be perpendicular to the sides of the river)?

8. L: How can we know where point  $G$  should be situated?  
 10. H: We can measure the length of sum the  $(AG + GH + HB)$  and observe the position of point  $G$  until the sum is minimal.  
 13. L: Hey, perhaps the sum is minimal at this position!  
 14. T: Yes, that's right! We save this picture and change the position of two points  $A$ ,  $B$  or even the distance between two banks of the river in order to realize some special characteristics.  
 19. L: Look! Maybe the line  $AG$  is parallel to the line  $HB$ ?

20. T: Wow, it is very good! We will draw these parallel lines and check it in other cases by moving point  $A$  (or point  $B$ ) to the new position again.  
23. H: Exactly, the situation keeps the same characteristics! If two lines  $AG$  and  $HB$  are parallel then the length of broken line  $AGHB$  is minimal.



**Fig. 5** Three saved pictures (including mental pictures in other senses) where the sum  $(AG + GH + HB)$  is minimal

24. T: That's right! But in this situation, what transformations are you thinking about?  
25. H: We have two fixed parallel lines representing two banks of the river and the distance between them is also a constant, etc.  
29. L: A translation! You can image that if the first line move towards the second line and they will coincide. From that, we can realize that vector  $\overrightarrow{GH}$  is vector of the translation.  
30. H: Yes, it means that the line  $AG$  is image of the line  $HB$  under a translation in the vector  $\overrightarrow{GH}$  direction.

Similarly, in this situation, the students realized the first invariant (sub-invariant) “two lines  $AG$  and  $HB$  are always parallel when the sum  $AG + GH + HB$  is minimal”. Based on this sub-invariant, the students used undercoded abduction combine with their imagination in order to discover the key invariant “the line  $AG$  is image of the line  $HB$  under a translation in the vector  $\overrightarrow{GH}$  direction”. They used some words such as “image”, “move towards ... coincide”. This accomplishment shows a high level of dynamic visual thinking in geometry. They could create a lot of ‘dynamic’ mental pictures in mind in order to realize a geometric transformation (in this case, a translation in the vector  $\overrightarrow{GH}$  direction). An important abductive step is represented as follows:

**C:** The line  $AG$  is parallel to the line  $HB$ .

**D = ?**  $\xrightarrow{\quad}$  **C**  
**W: Property of Translation**

**D:** The line  $AG$  is image of the line  $HB$  under a translation in the vector  $\overrightarrow{GH}$  direction.

In the proving processes, some realized invariants are the birth of the ideas of proof and the explanation of these invariants produced some arguments. The students’ remaining work is to select plausible arguments and connect them into a logical chain in order to form a deductive proof. However, there were a lot of students who could not write their formal proof; even they could realize the key invariant. This obstacle explains the structural gap between argumentation and proof (Pedemonte, 2007) and will be discussed in the next section.

### 3.2 Transition from abductive argumentation to deductive proof

‘The proof of the pudding is in the eating’, therefore, teachers should encourage their students to formulate conjectures during the proving process. This activity was set on a par with the proving itself because the production of conjectures motivates the students producing arguments and constructing proofs on their own. Argumentation structure is often abductive but proof is deductive. Hence, the structural gap between abductive argumentation and deductive proof is not always covered by the students. In the one-bridge problem, abduction not only plays an essential role in realizing invariants, but also in connecting the ascending control with the descending control in proving process. It also contributes to a transition from abductive structure of argumentation to deductive structure of proof. This smooth transition is described as follows:

Abductive Argumentation	Deductive Proof
<p>After realizing geometric transformation, the students produced abductive argumentation and reserved this structure to construct a deductive proof.</p> <p>By measuring and validating based on the property of a translation, the students had some initial data:</p> $GH = B'B; HB = GB'; DE = B'B; EB = DB'$ <p>By measuring, the students discovered the following inequality (claim <math>C_1</math>):</p> $C_1: AG + GH + HB \leq AD + DE + EB \quad (1)$ <p><math>D_1 = ?</math> <math>\xrightarrow{\quad}</math> <math>C_1</math></p> <p><math>W_1: GH = B'B; HB = GB'</math>  <math>DE = B'B; EB = DB'</math></p> <p><math>D_1 = C_2: AG + GB' + \underline{B'B} \leq AD + DB' + \underline{B'B} \quad (2)</math></p> <p>After finding out the data <math>D_1</math>, the students continued using abductive argumentation in order to establish the new data and claims <math>D_2 = C_3</math>, <math>D_3 = C_4</math>, and <math>D_4</math>:</p> <p><math>D_2 = ?</math> <math>\xrightarrow{\quad}</math> <math>C_2</math></p> <p><math>W_2: B'B \text{ is common summand}</math></p> <p><math>D_2 = C_3: AG + GB' \leq AD + DB' \quad (3)</math></p> <p><math>D_3 = ?</math> <math>\xrightarrow{\quad}</math> <math>C_3</math></p> <p><math>W_3: A, G, B' \text{ are collinear}</math></p> <p><math>D_3 = C_4: AB' \leq AD + DB' \quad (4)</math></p> <p><math>D_4 = ?</math> <math>\xrightarrow{\quad}</math> <math>C_4</math></p> <p><math>W_4: \text{Triangle Inequality}</math></p> <p>The final claim <math>D_4</math> is a theorem.</p>	<p>Let <math>D</math> be an arbitrary point on the line <math>l_1</math>. Let <math>B'</math> be image of point <math>B</math> under the translation of vector <math>\overrightarrow{HG}</math>. Let <math>G</math> be the intersection of the line <math>AB'</math> and the line <math>l_1</math> and <math>G</math> is the position where we can situate the bridge.</p> <p><i>“Based on the properties of the translation, the students gathered initial data. They wrote:”</i></p> <p>From the properties of a translation, we derive that:</p> $GH = BB'; HB = GB'$ $DE = B'B; EB = DB'$ <p><i>“The students reversed abductive structure in order to write the formal proof as follows: (4) <math>\rightarrow</math> (3) <math>\rightarrow</math> (2) <math>\rightarrow</math> (1)”.</i></p> <p>We have obviously the following inequality:</p> $AB' \leq AD + DB$ <p>Since three points <math>A, G, B'</math> are collinear, so we derive:</p> $AB' = AG + GB' \leq AD + DB$ <p>Add <math>B'B</math> to both side of previous inequality, we obtain:</p> $AG + GB' + B'B \leq AD + DB + B'B$ <p>From above inequality, we substitute <math>GH, HB, DE, EB</math> for <math>BB', GB', B'B, DB'</math> respectively, we obtain the following inequality:</p> $AG + GH + HB \leq AD + DE + EB$ <p>This inequality shows that point <math>G</math> is the position we can build the bridge.</p>

The bulk of this research has revealed that a dynamic geometry environment amplifies the potential of providing the students with claims (observed facts). They must look for data (based on dragging modality) and warrants (based on abduction) to justify the claims. Therefore, if the data are collected, a certain rule is selected and the abductive structure is reserved, there will be a smooth transition from abductive argumentation to deductive proof.

### 3.3 Classifying levels of realizing invariants

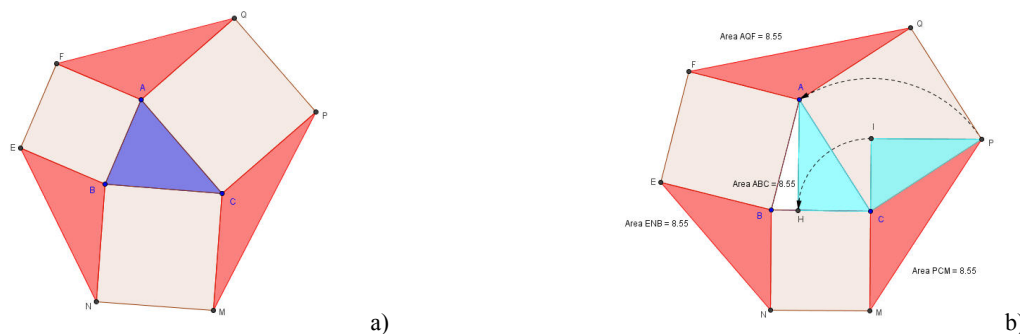
Realizing invariants is crucial phase of proving process in geometry because the students must know invariants before formulating conjectures. This process includes two transformational steps: transformations on objects (involving manipulations of objects via dragging or mental objects) and transformations on statements (shifts from observed facts and experiences to logical statements of the form ‘if...then’). In order to ensure that the conjecture is valid, the students need to produce arguments on the basis of accepted properties. It means that they give the reasons to explain why some invariants are preserved (Bishop, 1991). The performance of this activity relied on the students’ levels of realizing invariants. In my research, I have classified five different levels of realizing invariants according to the solutions of three tasks as follows:

- Level 0: Realize no invariant
- Level 1: Realize static invariants
- Level 2: Realize moving invariants
- Level 3: Realize invariants of a transformation
- Level 4: Realize invariants of different geometries



**Task 1.** Let  $ABC$  be a triangle. Construct three squares  $ABEF$ ,  $BCMN$ ,  $ACPQ$  outwards the triangle. Prove that the areas of four triangles  $ABC$ ,  $BNE$ ,  $CMP$ , and  $AFQ$  are equal.

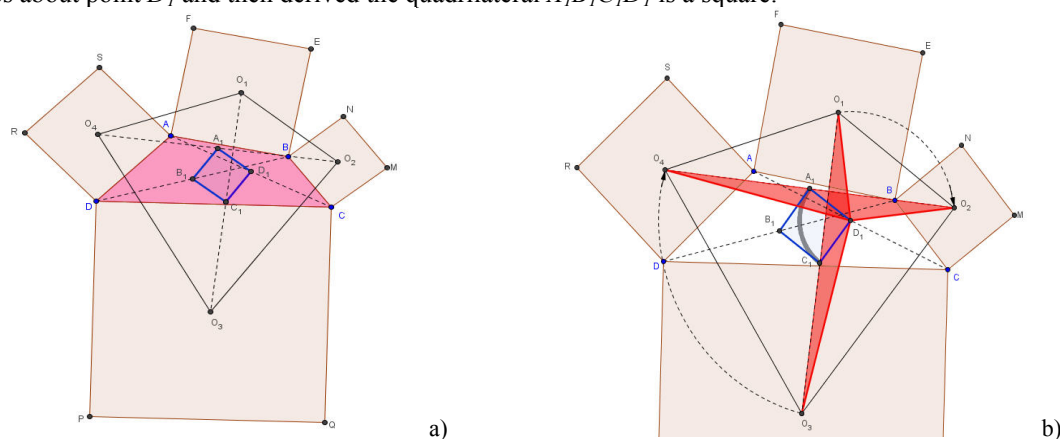
Students who have level 0 in realizing invariants could not see any invariant, especially some hidden invariants, for example  $BC = CM$  (sides of the square  $BCMN$ ). The reason is that they forgot the properties of a square or perhaps did not think about these buried data. Students attained level 1 knew the static invariant  $BC = CM$  but could not see that the altitude  $AH$  and the altitude  $PI$  should be equal. However, the students attained level 2 could realize this ‘moving’ invariant  $AH = PI$  and then  $\triangle AHC = \triangle PIC$ . This is the condition to show that the area of triangle  $ABC$  is equal to the area of triangle  $CMP$ . But these students did not know how to prove  $\triangle AHC = \triangle PIC$  because they did not see a transformation in this situation. Students attained level 3 realized that the triangle  $AHC$  is image of the triangle  $PIC$  under a rotation of 90 degrees about point  $C$ . As a result, they could write proofs for this problem.



**Fig. 6** Figures in task 1 before and after realizing invariants

**Task 2.** Let  $ABCD$  be a quadrilateral. Construct four squares  $ABEF$ ,  $BCMN$ ,  $CDPQ$ , and  $ADRS$  outwards the quadrilateral. Let  $O_1, O_2, O_3, O_4$  be the centers of these squares. Prove that four midpoints of the diagonals of two quadrilaterals  $ABCD$  and  $O_1O_2O_3O_4$  forming a square  $A_1B_1C_1D_1$ .

Students at level 0 and level 1 could not tackle this problem. Students at level 2 could produce some arguments because they realize some ‘moving’ invariants such as  $\triangle D_1O_1 = \triangle D_1O_2$ ,  $\triangle D_1O_3 = \triangle D_1O_4$  and  $\triangle D_1O_1O_2 = \triangle D_1O_3O_4$  but they could not realize that a rotation of 90 degrees about point  $D_1$  preserving the shapes of two triangles  $\triangle D_1O_1O_2$  and  $\triangle D_1O_3O_4$ . Students attained level 3 could see this transformation and showed that point  $C_1$  is image of point  $A_1$  under a rotation of 90 degrees about point  $D_1$  and then derived the quadrilateral  $A_1B_1C_1D_1$  is a square.

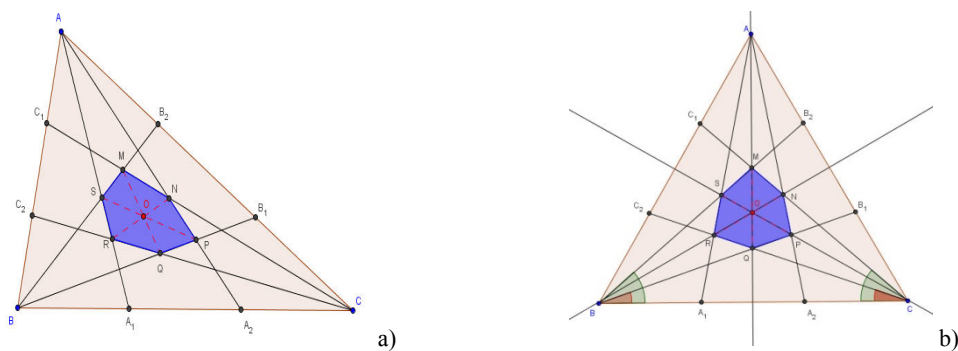


**Fig. 7** Figures in task 2 before and after realizing invariants

**Task 3.** Let  $ABC$  be a triangle. Take six points  $A_1, A_2$  on the side  $BC$ ;  $B_1, B_2$  on the side  $CA$  and  $C_1, C_2$  on the side  $AB$  such that the condition:  $BA_1 = A_1A_2 = A_2C$ ,  $CB_1 = B_1B_2 = B_2A$ ,  $AC_1 = C_1C_2 = C_2B$ . Six straight lines  $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$  intersect each other forming a hexagon  $MNPQRS$ . Prove that three diagonals of this hexagon are concurrent.

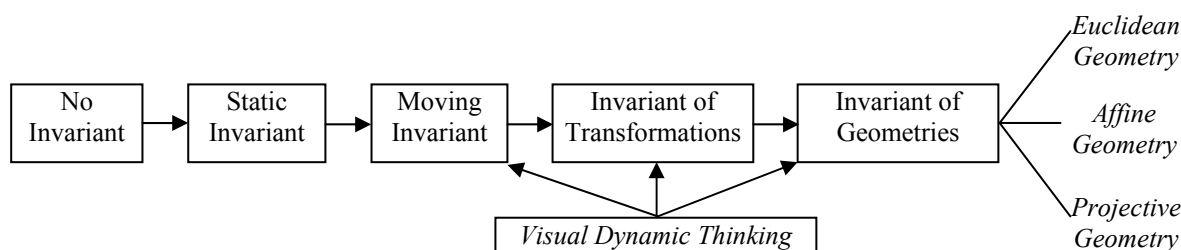
In this task, only the students who attained level 4 of realizing invariants could solve the problem. Students at level 4 could reveal affine properties in the figure because they perceived that no matter how triangle  $ABC$  changed, diagonals of the hexagon are always concurrent. Concurrence of three diagonals is affine invariants, so they could prove this

property with an equilateral triangle  $ABC$  in plane Euclidean geometry. If  $ABC$  is equilateral triangle, of course, three diagonals of the hexagon are concurrent because they coincide with three perpendicular bisectors of triangle  $ABC$ .



**Fig. 8** Figures in task 3 for arbitrary triangle and equilateral triangle

In general, the students' recognizing invariants ability has improved from level 1 to level 3 (or level 4). But the relationship between level 3 and level 4 is not necessarily hierarchical. The diagram (in Fig.9 below) shows the close relationship between these levels and visual dynamic thinking in dynamic geometry environment. This environment also provides a rich opportunity to develop the ability of realizing invariants in paper-and-pencil format.



**Fig. 9** Levels of realizing invariants in geometry

During the process of invariants recognition, students must produce some arguments in order to validate their hypothesis. These arguments have possibly contributed to reduce the gap between conjecture and proof. However, the plausibility of these reasons depends on the students' level of realizing invariants. Therefore, the students who attained low level must put more effort into their ability of writing proofs.

## 4. Conclusions

This chapter takes the possibilities of using dynamic geometry software (GeoGebra) into consideration. This tool supports the students catching the invariance of geometric transformation and constructing a formal proof with respect to their difficulties. Abduction is a type of inference supported the students throughout proving processes: realizing invariants, producing arguments, and writing proofs. Nevertheless, there are some cognitive and structural gaps between different phases of proving processes. These gaps will be covered if an abductive argumentation activity is developed for the construction of a conjecture. From this activity, the students seize an opportunity to modify their understanding about the role of invariants in devising new ideas of proving. For that reason, the development of the students' abductive argumentation should be a crucial part in mathematics education.

Students at the tertiary level tend to search for invariants of different geometries in reaching the solution of a geometric problem. Their arguments usually stem from the analysis and synthesis activities and then abduction will be used to reserve the structure of the solution. It means that abduction may assist the students in realizing invariants but not necessarily ensure their proofs writing. Indeed, a conjecture could be provided without any supported arguments and may be derived directly from a drawing and explain why most of students do not understand the necessity of abductive argumentation for the generation of ideas in the mathematical classroom. Generally speaking, abduction provides insight into the process of realizing geometric invariants and provides useful arguments to bridge the distance between conjecture and proof in geometry.



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