Conformal Pseudo–metrics and some Applications

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– Preface –

Wer kann was Dummes, wer was Kluges denken,
das nicht die Vorwelt schon gedacht?
(J.W. v. Goethe: Faust II)

The point of departure for the present work has been the following free boundary value problem for analytic functions f which are defined on a domain $G \subset \mathbb{C}$ and map into the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 1
Let $z_1, \ldots, z_n$ be finitely many points in a bounded simply connected domain $G \subset \mathbb{C}$. Show that there exists a holomorphic function $f : G \longrightarrow \mathbb{D}$ with critical points z_j (counted with multiplicities) and no others such that

$$\lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = 1$$

for all $\xi \in \partial G$.

If $G = \mathbb{D}$, Problem 1 was solved by Kühnau [32] in case of one critical point, which is sufficiently close to the origin, and for more than one critical point by Fournier and Ruscheweyh [17]. The method employed by Kühnau, Fournier and Ruscheweyh easily extends to more general domains G, say bounded by a Dini–smooth Jordan curve, but does not work for arbitrary bounded simply connected domains.

In this paper we present a completely new approach to Problem 1, which shows that this boundary value problem is not an isolated question in complex analysis, but is intimately connected to a number of basic open problems in conformal geometry and non–linear PDE. One of our results is a solution to Problem 1 for arbitrary simply connected domains. However, we shall see that our approach has also some other ramifications, for instance to a well–known problem due to Rellich and Wittich in PDE.

Roughly speaking, this paper is broken down into two parts. In a first step we construct a conformal metric in a bounded regular domain $G \subset \mathbb{C}$ with prescribed non–positive Gaussian curvature $\kappa(z)$ and prescribed singularities by solving the first boundary value problem for the Gaussian curvature equation $\Delta u = -\kappa(z)e^{2u}$ in G with prescribed singularities and continuous boundary data. This is related to the Berger–Nirenberg problem in Riemannian geometry, that is, the question which functions on a surface R can arise as the Gaussian curvature of a Riemannian metric on R. The special case, where $\kappa(z) \equiv -4$ and the domain G is bounded by finitely many analytic Jordan curves was treated by Heins [22]. In a second step we show every conformal pseudo–metric on a simply connected domain $G \subseteq \mathbb{C}$ with constant negative Gaussian curvature and isolated zeros of integer order is the pullback of the hyperbolic metric on $\mathbb{D}$ under an analytic map $f : G \longrightarrow \mathbb{D}$. This extends a theorem of Liouville which deals with the case that the pseudo–metric has no zeros at all. These two steps together allow in particular a quick and complete solution of Problem 1.

Contents

Chapter I contains the statement of the main results and connects them with some old and new problems in complex analysis, conformal geometry and PDE: the Uniformization Theorem for Riemann surfaces, the problem of Schwarz–Picard, the Berger–Nirenberg problem, Wittich's problem, etc.. Chapter II and Chapter III have preparatory character. In Chapter II we recall some basic results about ordinary differential equations in the complex plane. In our presentation we largely follow Laine [33], but we have completely reorganized the material and present a self–contained account of the basic features of Riccati, Schwarzian and second order differential equations. In Chapter III we discuss the first boundary value problem for the Poisson equation. We shall need to consider this problem in the most general situation, which does not seem to be covered in a satisfactory way in the existing literature, see [10, 11]. To this end it is unavoidable to prove some rather technical and painstaking potential theoretic lemmas. In Chapter IV we turn to a discussion of conformal pseudo–metrics in planar domains. We focus on conformal metrics with prescribed singularities and prescribed non–positive Gaussian curvature. In particular, we shall establish the existence of such metrics, that is, we solve the corresponding Gaussian curvature equation by making heavy use of the results of Chapter III. In Chapter V we show that every constantly curved pseudo–metric can be represented as the pullback of either the hyperbolic, the euclidean or the spherical metric under an analytic map. This is proved by using the results of Chapter II about complex differential equations. After these lengthy preparations the proofs for the applications in Chapter VI become comfortably short.

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To Oli, my parents and Chrisi

– Contents –

– Chapter I –

Introduction and Main Results

I.1 Representation of constant curvature metrics

By the Uniformization Theorem every simply connected Riemann surface is conformally equivalent to either the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the finite complex plane $\mathbb{C}$ or the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$. Each of these canonical Riemann surfaces $X = \mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$ carries a canonical conformal metric

$$\lambda_X(w) = \begin{cases} \dfrac{1}{1-|w|^2} & X = \mathbb{D}, \\ 1 & \text{if} \quad X = \mathbb{C}, \\ \dfrac{1}{1+|w|^2} & X = \mathbb{P}, \end{cases}$$

of constant Gaussian curvature

$$\kappa = \begin{cases} -4 & X = \mathbb{D}, \\ 0 & \text{if} \quad X = \mathbb{C}, \\ +4 & X = \mathbb{P}, \end{cases} \tag{I.1}$$

which induces hyperbolic geometry on $\mathbb{D}$, euclidean geometry on $\mathbb{C}$, and spherical geometry on $\mathbb{P}$.

Now, let G be a domain in $\mathbb{C}$ and let $f : G \longrightarrow X$ be an analytic map, where $X = \mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$. Then the canonical geometry on X can be pulled back from X to G via f by defining the pseudo–metric

$$\lambda(z) := \lambda_X(f(z))\,|f'(z)|$$

on G, which has constant curvature $\kappa \in \{-4, 0, +4\}$ in $G \backslash \{z \in G : f'(z) = 0\}$. Note, the zeros of the pseudo–metric $\lambda(z)$ are exactly the critical points of the analytic map f. More precisely, λ has a zero of order α at a point $\xi \in G$, i.e. the limit

$$\lim_{z \longrightarrow \xi} \frac{\lambda(z)}{|z-\xi|^{\alpha}}$$

exists and $\neq 0$, if and only if f' has a zero of order α at the point ξ.

In order to represent a pseudo–metric λ on G as the pullback $\lambda_X(f(z))\,|f'(z)|$ of the canonical metric λ_X under an analytic map $f : G \longrightarrow X$ it is therefore necessary that λ has constant curvature $\kappa \in \{-4, 0, +4\}$ and that the zeros of λ are discrete in G and of integer order. If the domain G is simply connected, then these two conditions are also sufficient for λ to be the pullback of the canonical metric under an analytic map. This is the content of the following result, which we are going to prove in Chapter V.

Theorem I.1
Let $E = \{z_1, z_2, \ldots\}$ be a discrete set in a simply connected domain $G \subseteq \mathbb{C}$, let $\alpha_1, \alpha_2, \ldots$ be positive integers, and let $\lambda : G \longrightarrow [0, \infty)$ be a pseudo–metric of constant curvature $\kappa \in \{-4, 0, +4\}$ in $G \backslash E$ with zeros of orders α_j at z_j and no others, that is, the limits

$$\lim_{z \longrightarrow z_j} \frac{\lambda(z)}{|z - z_j|^{\alpha_j}} \qquad \text{exist and } \neq 0 \, .$$

(a) *If $\kappa = -4$, then λ is the pullback of the hyperbolic metric under a holomorphic function $f : G \longrightarrow \mathbb{D}$, i.e.*

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \, , \qquad z \in G \, .$$

If $g : G \longrightarrow \mathbb{D}$ is another holomorphic function such that

$$\lambda(z) = \frac{|g'(z)|}{1 - |g(z)|^2} \, , \qquad z \in G \, ,$$

then $g = T \circ f$, where T is a conformal automorphism of the unit disk $\mathbb{D}$.

(b) *If $\kappa = 0$, then λ is the pullback of the euclidean metric under a holomorphic function $f : G \longrightarrow \mathbb{C}$, i.e.*

$$\lambda(z) = |f'(z)| \, , \qquad z \in G \, .$$

If $g : G \longrightarrow \mathbb{C}$ is another holomorphic function such that

$$\lambda(z) = |g'(z)| \, , \qquad z \in G \, ,$$

then $g = T \circ f$, where T is an euclidean motion of the complex plane $\mathbb{C}$, that is $T(z) = az + b$ for some constants $a, b \in \mathbb{C}$ with $|a| = 1$.

(c) *If $\kappa = +4$, then λ is the pullback of the spherical metric under a holomorphic function $f : G \longrightarrow \mathbb{P}$[1], i.e.*

$$\lambda(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \, , \qquad z \in G \, .$$

If $g : G \longrightarrow \mathbb{P}$ is another holomorphic function such that

$$\lambda(z) = \frac{|g'(z)|}{1 + |g(z)|^2} \, , \qquad z \in G \, ,$$

then $g = T \circ f$, where T is a rotation of the Riemann sphere $\mathbb{P}$.

Thus every pseudo–metric of constant Gaussian curvature κ on a simply connected domain G with at most a discrete set of zeros, each of integer order, is the pullback of the canonical metric λ_X under an analytic map $f : G \longrightarrow X$, where $X = \mathbb{D}$ if $\kappa = -4$, $X = \mathbb{C}$ if $\kappa = 0$, and $X = \mathbb{P}$ if $\kappa = 4$.

[1] By a holomorphic function $f : G \longrightarrow \mathbb{P}$ we mean a meromorphic function defined on G.

Remarks I.2

(a) *The condition in Theorem I.1 that G is simply connected cannot be dropped. Indeed, the function*

$$\lambda(z) := \frac{1}{2\sqrt{|z|}\,(1-|z|)}$$

is a conformal metric of constant curvature -4 on the punctured unit disk $G = \mathbb{D}\backslash\{0\}$, but certainly not the pullback of $\lambda_{\mathbb{D}}$ under a holomorphic *function $f : G \longrightarrow \mathbb{D}$.*

(b) *Part (b) of Theorem I.1 is simply the well–known fact that every harmonic function u in a simply connected domain G can be written as the real part of a holomorphic function $H : G \longrightarrow \mathbb{C}$. In fact, $\lambda(z)$ is a conformal metric on G of constant Gaussian curvature $4k$ with no zeros, if and only if*

$$u(z) := \log \lambda(z)$$

is a solution of the partial differential equation

$$\Delta u = -4k\, e^{2u}$$

in G. In particular, u is harmonic in G if and only if $k = 0$. By Theorem I.1 (b), $u(z) = \log|f'(z)|$ for some locally univalent function $f : G \longrightarrow \mathbb{C}$, so $u(z) = \operatorname{Re} H(z)$ for $H(z) = \log f'(z)$.

We point out two special cases of Theorem I.1. If the conformal pseudo–metric λ is strictly positive on G, i.e. $\lambda(z)$ is a conformal *metric*, then Theorem I.1 is Liouville's theorem [34].

Corollary I.3 (Liouville 1853)
Let λ be a strictly positive conformal pseudo–metric on a simply connected domain $G \subseteq \mathbb{C}$ with constant Gaussian curvature $\kappa \in \{-4, 0, +4\}$ in G. Then there exists a locally univalent analytic map $f : G \longrightarrow X$ with $X = \mathbb{D}$ if $\kappa = -4$, $X = \mathbb{C}$ if $\kappa = 0$, and $X = \mathbb{P}$ if $\kappa = +4$, such that $\lambda(z) = \lambda_X(f(z))\,|f'(z)|$.

If the pseudo–metric $\lambda(z)$ in Theorem I.1 is the *maximal* conformal pseudo–metric of constant curvature -4 with prescribed zeros in the unit disk, then we obtain a theorem of Heins.

Corollary I.4 (Heins 1962)
Let $E = \{z_1, z_2, \ldots\}$ be a discrete set in $\mathbb{D}$ and $\alpha_1, \alpha_2, \ldots$ be positive integers. If λ is the maximal conformal pseudo–metric in $\mathbb{D}$ of constant Gaussian curvature -4 in $\mathbb{D}\backslash E$ with zeros of orders α_j at z_j, then $\lambda(z) = \lambda_{\mathbb{D}}(f(z))\,|f'(z)|$ for some holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$.

Some remarks are in order.

Remark I.5
A slightly different version of Corollary I.3 can be found in Nitsche [39]. Corollary I.3 was also proved by Warnecke [53] apparently unaware of Liouville's result. Still another proof was furnished by Bieberbach [6] also without reference to Liouville. Bieberbach's motivation for Corollary I.3 was to give a proof of the Uniformization Theorem for certain algebraic Riemann surfaces. His arguments follow a way proposed by Schwarz [48] and

Poincaré [44]. One can show that Bieberbach's main idea which leads to Corollary I.3 can be used to prove the Uniformization Theorem for every hyperbolic compact *Riemann surface, see Section I.2 below. A further, more geometric proof of Corollary I.3 was offered by Minda [36]. All of these proofs cannot directly be modified to establish the more general Theorem I.1. Here is a short description why.*

If $f : G \longrightarrow X$ is an analytic function such that $\lambda(z) = \lambda_X(f(z))\,|f'(z)|$, then it is easy to see that its Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is given by

$$\frac{1}{2}S_f(z) = \left(\frac{\partial^2}{\partial z^2}\log\lambda\right)(z) - \left[\left(\frac{\partial}{\partial z}\log\lambda\right)(z)\right]^2. \tag{I.2}$$

Conversely, in order to find for a given pseudo–metric λ of constant curvature a function $f : G \longrightarrow X$ such that

$$\lambda(z) = \lambda_X(f(z))\,|f'(z)|,$$

one needs to solve the Schwarzian differential equation (I.2). The condition that λ has constant curvature in $G\backslash E$, E defined as in Theorem I.1, is equivalent to the fact that the right hand side of (I.2) is a holomorphic function in $G\backslash E$.

In particular, in the situation of Liouville's theorem, where λ has no zeros at all, the right hand side of (I.2) is a holomorphic *function in G and it is a classical and elementary fact that every solution to (I.2) is a locally univalent* meromorphic *function. One needs then to find among all solutions to (I.2) a function $f : G \longrightarrow X$ which satisfies $\lambda(z) = \lambda_X(f(z))\,|f'(z)|$. This can be done either by looking at a second order linear differential equation associated to (I.2) (Bieberbach's approach) or by using the local univalence of the solutions of (I.2) to reduce the assertion to the case of vanishing Schwarzian (Minda's proof).*

If, however, the metric λ has zeros in G, then the right hand side of (I.2) has second order poles at these zeros and from the general theory of Schwarzian differential equations it is not clear at all that all solutions to (I.2) are meromorphic functions in G. In order to guarantee that also in this case all solutions of the Schwarzian differential equation are meromorphic in G we will take a closer look at (I.2) and an associated Riccati differential equation. After that we proceed along the lines of Bieberbach's proof of Corollary I.3 to conclude the proof of Theorem I.1, see Chapter V for details.

Remark I.6
Heins's proof [22] of Corollary I.4 gives an ad hoc argument for the existence of the function f using Blaschke products. In particular, the function f is not constructed from the metric λ. As indicated above our way of proving Theorem I.1 and its Corollary I.4 is constructive. We obtain the function f as a solution of the Schwarzian differential equation (I.2).

Remark I.7
Theorem I.1 can also be extracted from the work of Nitsche [39] and Warnecke[2] [53] who provided a classification of the possible isolated singularities of the real valued twice

[2]Warnecke in imitation of Nitsche

continuously differentiable solutions u of the PDE $\Delta u = 4e^{2u}$. We wish to point out that Corollary I.3 is indispensable to their characterization of the singularities. It is used in connection with a general fundamental system of a second order differential equation, where the function of the right hand side of $(I.2)$ comes into consideration. This function is holomorphic except for the singularities of the function u.

Extensive use of Corollary I.3 was also made by Chou and Wan [8, 9] to find a representation formula for real valued twice continuously differentiable solutions u of $\Delta u = -4ke^{2u}$ in $\mathbb{D}\backslash\{0\}$ for $k \in \{-1, 0, +1\}$, which also allows a description of the possible singularities. In brief, they showed that each such function u has the form

$$u(z) = \log \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \textit{for } z \in \mathbb{D}\backslash\{0\}\,,$$

where f is some multi–valued locally univalent function in $\mathbb{D}\backslash\{0\}$. Their key to deduce the above expression for u is the universal covering surface $\mathbb{D}$ of $\mathbb{D}\backslash\{0\}$ combined with Liouville's theorem (Corollary I.3). At first sight it seems that Chou and Wan's and Nitsche's and Warnecke's proof don't have much in common, but a closer look reveals some similar ideas. They all use Liouville's theorem to obtain a local representation of u in terms of a locally univalent function f. Then they study the behaviour of the analytic continuation of f along a path which surrounds the singularity.

Our starting point for Theorem I.1 differs markedly from the purpose Nitsche, Warnecke, Chou and Wan pursued in their papers. We assume a solution of the PDE $\Delta u = -4ke^{2u}$ with prescribed singularities and prescribed behaviour in a neighborhood of these singularities whereas the other authors classify the possible types of singularities. Therefore we examine the problem from a different perspective. The proof of Theorem I.1 we give below is completely different than [39, 53, 9] and shows a strong connection between pseudo–metrics of constant curvature and the Schwarzian derivative of not necessarily locally univalent holomorphic functions. This aspect might be new in this context. We want to emphasize that we don't need Corollary I.3 for our proof. In fact, our method shows that Liouville's theorem is contained in Theorem I.1 as a special case.

Theorem I.1 provides a representation formula for conformal metrics of constant curvature with isolated zeros. The following equivalent statement deals with the case of conformal metrics with variable curvature of a special type.

Theorem I.8

Let $G \subseteq \mathbb{C}$ be a simply connected domain, $h : G \longrightarrow \mathbb{C}$ a holomorphic function $\not\equiv 0$, and $\lambda : G \longrightarrow (0, \infty)$ a conformal metric in G with curvature $4k|h(z)|^2$, $k \in \{-1, 0, +1\}$. Then there exists a holomorphic function $f : G \longrightarrow X$ such that

$$\lambda(z) = \frac{1}{|h(z)|} \frac{|f'(z)|}{1 + k|f(z)|^2}\,, \quad z \in G,$$

where $X = \mathbb{D}$ if $k = -1$, $X = \mathbb{C}$ if $k = 0$, and $X = \mathbb{P}$ if $k = +1$.
If $g : G \longrightarrow X$ is another holomorphic function satisfying

$$\lambda(z) = \frac{1}{|h(z)|} \frac{|g'(z)|}{1 + k|g(z)|^2}\,, \quad z \in G,$$

then $g = T \circ f$, where T is a rigid motion of X, i.e. T is a unit disk automorphism if $X = \mathbb{D}$, a euclidean motion of the complex plane if $X = \mathbb{C}$, and a rotation of the sphere if $X = \mathbb{P}$.

Theorem I.8 can be used to prove a slight extension of a theorem of Wittich [55], see §I.3.2 below.

I.2 Existence and uniqueness of pseudo–metrics of constant curvature

The construction of harmonic functions with prescribed singularities, i.e. the integration of the Laplace equation

$$\Delta u = 0 \tag{I.3}$$

with prescribed singularities for the function u plays an important role in complex function theory. It leads, for instance, to the Poisson–Jensen formula for meromorphic functions and hence to Nevanlinna's first fundamental theorem. In the language of conformal metrics the problem consists in finding conformal pseudo–metrics of constant Gaussian curvature 0 with prescribed zeros.

From the viewpoint of complex analysis the construction of conformal metrics of constant *negative* Gaussian curvature with prescribed zeros is even more important. We may assume Gaussian curvature -4 by renormalization, so the problem is equivalent to find solutions of the non–linear PDE

$$\Delta u = 4e^{2u} \tag{I.4}$$

with prescribed singularities. The integration of this PDE leads for example to a proof of the Uniformization Theorem for Riemann surfaces and also to Nevanlinna's second fundamental theorem.

The partial differential equation (I.4) occurs in the year 1890 in a problem ("Preisaufgabe") of the "Königliche Gesellschaft der Wissenschaften zu Göttingen" posed by Schwarz [48] in connection with the uniformization problem for Riemann surfaces. Note, if there exists a solution u of the PDE (I.4) on a compact Riemann surface R, then $\lambda(z) := e^{u(z)}$ defines a conformal metric of constant Gaussian curvature -4 without zeros on R, so by Liouville's Theorem $\lambda(z) = \lambda_{\mathbb{D}}(f(z))\,|f'(z)|$ for some analytic map $f : U \longrightarrow \mathbb{D}$ on a simply connected subset U of R. The map f is locally univalent and it is not difficult to show that every branch of its inverse map can be continued analytically along any path in $\mathbb{D}$. In this way a locally univalent map $\pi : \mathbb{D} \longrightarrow R$ is obtained, which turns out to be the universal covering map of the Riemann surface R. Thus, solving the PDE (I.4) on R provides a method to uniformize the Riemann surface R. We remark that for algebraic Riemann surfaces the singularities of the metric λ correspond to the branch points of the surface.

The problem of Schwarz was extensively studied by Picard [41, 42, 43], Poincaré [44] and Bieberbach [5, 6]. It was completely solved for compact Riemann surfaces by Heins in [22] (see [35] for an independent proof). Heins obtained a necessary and sufficient condition for the existence of a conformal metric on a compact Riemann surface with constant Gaussian curvature -4 and prescribed singularities. He studied the equation (I.4) using the method of *subsolutions* and developed a theory in analogy to the classical theory of subharmonic functions, i.e. the subsolutions of the Laplace equation (I.3). The problem of Schwarz in an extended form is to prescribe the Gaussian curvature of a metric on a Riemann surface. This is sometimes called the Berger–Nirenberg problem. It is well understood in the compact case (see [29, 50]), but still unsolved in the general case (cf. [26] for partial results).

Here, we focus on the easier problem of studying the partial differential equation (I.4) on plane domains with prescribed singularities and boundary values. More precisely, we consider the following problem. In the sequel we denote by $C(\Omega)$ the set of real valued continuous functions and by $C^2(\Omega)$ the set of real valued twice continuously differentiable functions on $\Omega \subseteq \mathbb{C}$.

Problem I.9 (The planar Schwarz–Picard Problem)
Let $G \subset \mathbb{C}$ be a bounded domain, let $E = \{z_1, \ldots, z_n\}$ be a finite subset of G and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers. Further, let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function. Find a function $u \in C(\overline{G}\backslash E) \cap C^2(G\backslash E)$ such that

$$\begin{aligned} \Delta u &= 4e^{2u} \ \textit{in } G\backslash E, \\ u &= \Phi \quad \textit{on } \partial G, \end{aligned} \tag{I.5}$$

and $\lim\limits_{z \longrightarrow z_j} (u(z) - \alpha_j \log|z - z_j|)$ exists finitely for every $j = 1, \ldots, n$.

For bounded domains $G \subset \mathbb{C} \doteq \mathbb{R}^2$ with sufficiently smooth boundary standard methods in the theory of non–linear partial differential equations can be used to solve Problem I.9, see for instance [19, Chapter 11]. Heins considers the case $E = \emptyset$ in [22, pp. 26–28] and shows that (I.5) has a solution for every continuous boundary function $\Phi : \partial G \longrightarrow \mathbb{R}$ if the domain G is bounded by finitely many mutually disjoint Jordan curves.

We handle the general case in the following result.

Theorem I.10
Let $G \subset \mathbb{C}$ be a bounded and regular[3] domain, let $E = \{z_1, \ldots, z_n\}$ be a finite subset of G and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers. Further, let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function. Then there exists a unique solution $u \in C(\overline{G}\backslash E) \cap C^2(G\backslash E)$ of the boundary value problem (I.5) such that $\lim\limits_{z \longrightarrow z_j} (u(z) - \alpha_j \log|z - z_j|)$ exists for every $j = 1, \ldots, n$.

To prove Theorem I.10 we study in detail the Dirichlet problem for the Poisson equation in a bounded and regular domain $G \subset \mathbb{C}$, i.e. we are looking for a real valued function $u \in C(\overline{G}) \cap C^2(G)$ satisfying

$$\begin{aligned} \Delta u &= f \ \text{ in } G, \\ u &= \Phi \text{ on } \partial G, \end{aligned} \tag{I.6}$$

where $f : G \longrightarrow \mathbb{R}$ is locally Hölder continuous and $\Phi : \partial G \longrightarrow \mathbb{R}$ is continuous on the boundary of G. We follow largely [10, 11] and [19, Chapter 4]. The treatment of (I.6) "only" requires a good deal of classical potential theory, but nevertheless we have gone to some trouble to present a clear and complete proof for the existence of a solution to the boundary value problem (I.6). In [19, Chapter 4 and Chapter 6] the Dirichlet problem (I.6) is discussed for bounded domains $G \subset \mathbb{R}^n$, $n \in \mathbb{N}, n \geq 2$. It turns out that for this more general situation the existence of a solution of (I.6) can be ensured only if either the boundary function Φ obliges some "smoothness" condition or the boundary of G is sufficiently nice, e.g. G satisfies an exterior sphere condition.

[3]See Chapter III for the definition.

The knowledge of the Dirichlet problem (I.6) can't be applied directly to prove Theorem I.10, since the right hand side of (I.5) depends on the solution u itself. In order to conclude the existence of a solution of problem (I.5) from the existence of a solution of problem (I.6) we fall back on a fixed point argument and apply Schauder's fixed point theorem.

The point here is that we don't use Schauder's fixed point theorem in a Banach space setting. This is in contrast to the standard approach in PDE (see [19, Chapter 11]), which only works when the boundary of the domain has additional smoothness properties. Replacing the Banach space by a suitable Fréchet space allows to drop these smoothness conditions and to obtain the quite general Theorem I.10.

In fact, we shall prove the following more general result in Chapter IV.

Theorem I.11
Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, z_2, \ldots, z_n \in G$ be finitely many distinct points and let $\alpha_1, \ldots, \alpha_n \in (0, \infty)$. Let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function and let $\kappa : G \longrightarrow [0, \infty)$ be a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique solution $u \in C(\overline{G} \backslash \{z_1, \ldots, z_n\}) \cap C^2(G \backslash \{z_1, \ldots, z_n\})$ of the boundary value problem

$$\begin{aligned} \Delta u &= \kappa(z) e^{2u} \text{ in } G \backslash \{z_1, \ldots, z_n\}, \\ u &= \Phi \qquad\quad \text{on } \partial G, \end{aligned} \tag{I.7}$$

such that $\lim\limits_{z \longrightarrow z_j} (u(z) - \alpha_j \log |z - z_j|)$ exists for every $j = 1, \ldots, n$.

If we set $\lambda(z) = e^{u(z)}$, where u is the solution of the boundary value problem (I.7) in Theorem I.11, we obtain (see Section IV.1.2):

Theorem I.12
Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, z_2, \ldots, z_n \in G$ be finitely many distinct points and let $\alpha_1, \ldots, \alpha_n \in (0, \infty)$. Let $\phi : \partial G \longrightarrow (0, \infty)$ be a continuous function and $\kappa : G \longrightarrow [0, \infty)$ a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique pseudo–metric $\lambda : G \longrightarrow [0, \infty)$ of curvature $-\kappa(z)$ in $G \backslash \{z_1, z_2, \ldots, z_n\}$ with zeros of orders α_j at z_j such that λ is continuous on $\overline{G}$ with $\lambda(z) = \phi(z)$ for $z \in \partial G$.

The situation for pseudo–metrics of *positive* curvature is much more involved and as a consequence also the situation of pseudo–metrics of variable curvature with alternating sign. In fact, neither the existence nor the uniqueness of a pseudo–metric $\lambda : G \longrightarrow [0, \infty)$ with constant positive curvature can be guaranteed. In general, existence can be ensured only if the domain G is sufficiently small and nice, see [56, Chapter 7]. Further, in case there exists such a pseudo–metric, then it is far from being the only one.

Let us illustrate this by a simple example. The notation $K_r(z)$ will stand for the open disk in $\mathbb{C}$ with radius r and center z.

Example I.13 ([45])
Let $K_r(0)$ be the open disk with radius r about 0.

(a) *If $r > 1/2$ then there exists no metric of constant curvature $+4$ in $K_r(0)$ with boundary values*

$$\lim_{|z|\longrightarrow r} \lambda(z) = 1\,.$$

For a proof see [45].

(b) *Let $r \leq 1/2$ and define $c_1 = (1+\sqrt{1-4r^2})/(2r^2)$ and $c_2 = (1-\sqrt{1-4r^2})/(2r^2)$. Then the metrics*

$$\lambda_1(z) = \frac{c_1}{1+c_1^2|z|^2} \quad \text{and} \quad \lambda_2(z) = \frac{c_2}{1+c_2^2|z|^2}$$

defined for $z \in K_r(0)$ are obviously different. However, both have constant curvature $+4$ and the same boundary values, that is $\lim\limits_{|z|\longrightarrow r} \lambda_{1/2}(z) = 1$.

Recently, a number of papers appeared dealing with metrics of non–negative curvature. For example, existence and uniqueness of metrics with singularities and constant positive curvature on the Riemann sphere are discussed in [15, 52], to mention only one problem. However we won't go into these questions any further.

I.3 Applications

I.3.1 On a free boundary value problem for analytic functions

The following conjecture goes back to Ruscheweyh and has its origin in a multiplier conjecture for univalent functions related to the Bieberbach conjecture [16].

Conjecture I.14
Let F be an analytic function in the closed unit disk $\overline{\mathbb{D}}$ such that

$$F(0) = 0 \qquad \text{and} \qquad |F'(z)| = 1 - |F(z)|^2 \;\; \text{for } z \in \partial\mathbb{D}. \tag{I.8}$$

Does this imply $F(z) = cz^N$ for some constants $c \in \mathbb{C}$ and $N \in \mathbb{N}$?

This conjecture is indeed true under some additional conditions on F (see [1]), but fails to hold in general. The first who established the existence of a function $F(z)$ satisfying (I.8) which is not of the form cz^N was Kühnau [32]. Kühnau's method is constructive and shows that to a given point $z_0 \in \mathbb{D}\backslash\{0\}$ sufficiently close to the origin there exists a function F analytic in $\overline{\mathbb{D}}$ such that $F(0) = 0$ and $F'(z_0) = 0$. Later Kühnau's idea was used by Fournier and Ruscheweyh [17] to prove the following more general result.

Theorem I.15
To every finite Blaschke product $B(z)$ there exists a uniquely determined function F analytic in the closed unit disk with $F(0) = 0$ and $|F'| = 1 - |F|^2$ on the unit circle such that $F' = B\,h$, where h is an analytic non–vanishing function in the closed unit disk with $h(0) > 0$.

In order to prove Theorem I.15 Fournier and Ruscheweyh had to replace the constructive step in Kühnau's method by a non–constructive fixed point argument. Note, Theorem I.15 gives an affirmative answer to the following problem.

Problem I.16
Let $z_1, \dots, z_n$ be finitely many distinct points in $\mathbb{D}$ and $\alpha_1, \dots, \alpha_n$ positive integers. Does there exist a function F analytic in the closed unit disk with critical points of orders α_j at z_j and no others such that $|F'| = 1 - |F|^2$ on $\partial\mathbb{D}$?

In Chapter VI we give a proof of Theorem I.15 (i.e. another solution to Problem I.16) based on Theorem I.10 and Theorem I.1. Actually we derive the following more general result:

Theorem I.17
Let $G \subset \mathbb{C}$ be a bounded simply connected domain, $z_1, \dots, z_n$ finitely many distinct points in G and $\alpha_1, \dots, \alpha_n$ positive integers. Also, let $\phi : \partial G \longrightarrow \mathbb{R}$ be a continuous positive function. Then there exists a holomorphic function $f : G \longrightarrow \mathbb{D}$ with critical points of orders α_j at z_j and no others such that

$$\lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi), \qquad \xi \in \partial G. \tag{I.9}$$

If $g : G \longrightarrow \mathbb{D}$ is another holomorphic function with these properties, then $g = T \circ f$ for some conformal disk automorphism $T : \mathbb{D} \longrightarrow \mathbb{D}$.

Some remarks are in order.

Remarks I.18

(a) *If G is a uniform domain, then the functions f in Theorem I.17 extend continuously to ∂G and $|f| < 1$ on $\overline{G}$, see [45, Lemma 3.2]. This implies f' can have only finitely many zeros in G if the domain G is not "pathological" and the boundary function $\phi : \partial G \longrightarrow \mathbb{R}$ is positive everywhere.*

(b) *If ∂G is an analytic Jordan curve and $\phi = |h|$ for some function h holomorphic in a neighborhood of ∂G, then the functions f in Theorem I.17 can be continued analytically across ∂G. This is a consequence of the extended Schwarz–Carathéodory reflection principle, see [18, 45]. Thus Theorem I.15 is simply a consequence of the special case $G = \mathbb{D}$ and $\phi \equiv 1$ of Theorem I.17.*

(c) *We note Theorem I.17 can also be deduced from the result of Kühnau and Fournier & Ruscheweyh by conformal mapping provided that the boundary of the domain G is sufficiently smooth (for instance if ∂G is Dini–smooth). This method, however, does not work for general bounded simply connected domains, and Kühnau asked in [32] if Theorem I.15 can be generalized to such domains. Theorem I.17 gives an affirmative answer to this question – even in an extended form.*

(d) *In general, if G is not simply connected, Theorem I.17 does not hold. For example, let G be the annulus $\{z \in \mathbb{C} : 1/4 < |z| < 1/2\}$ and let $\phi : \partial G \longrightarrow \mathbb{R}$ be the continuous function*

$$\phi(\xi) = \begin{cases} \dfrac{4}{3} & \text{if} \quad |\xi| = \dfrac{1}{4}\,, \\ \sqrt{2} & \quad |\xi| = \dfrac{1}{2}\,. \end{cases}$$

Now, assume there exists a locally univalent holomorphic function $f : G \longrightarrow \mathbb{D}$ such that

$$\lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi) \qquad \text{for } \xi \in \partial G.$$

Note,

$$\tilde{\lambda}(z) = \frac{1}{2\sqrt{|z|}\,(1-|z|)}$$

is a conformal metric of constant curvature -4 *in* G *and* $\tilde{\lambda}(\xi) = \phi(\xi)$ *for* $\xi \in \partial G$. *Hence, by the uniqueness statement of Theorem I.10, we see*

$$\tilde{\lambda}(z) = \frac{|f'(z)|}{1-|f(z)|^2}, \qquad z \in G.$$

On the other hand, in the simply connected domain $D = G\backslash(-1/4, -1/2)$, *we have*

$$\tilde{\lambda}(z) = \frac{|g'(z)|}{1-|g(z)|^2}$$

for the holomorphic function $g : D \longrightarrow \mathbb{D}$, $g(z) = \sqrt{z}$. *Applying Theorem I.1, we deduce* $g = T \circ f$ *in* D *for some automorphism* T *of* $\mathbb{D}$. *Thus* g *has an analytic extension to* G *given by* $T \circ f$ *which is absurd.*

If we relax the boundary condition (I.9) to non–tangential limits, we can allow infinitely many critical points for f. This is the content of the next theorem, which we are going to prove in Chapter VI.

Theorem I.19
Let (z_j) *be a sequence of points in* $\mathbb{D}$ *satisfying the Blaschke condition*

$$\sum_{j=1}^{\infty} (1-|z_j|) < \infty,$$

and let $\phi : \partial\mathbb{D} \longrightarrow [0,\infty)$ *be a function such that* $\log\phi \in L^\infty(\partial\mathbb{D})$. *Then there exists a holomorphic function* $f : \mathbb{D} \longrightarrow \mathbb{D}$ *with critical points* z_j *(counted with multiplicities) such that*

$$\sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1-|f(z)|^2} < \infty,$$

and

$$n.t. \qquad \lim_{z\longrightarrow\xi} \frac{|f'(z)|}{1-|f(z)|^2} = \phi(\xi) \qquad \text{for a.e. } \xi \in \partial\mathbb{D}.$$

If $g : \mathbb{D} \longrightarrow \mathbb{D}$ *is another holomorphic function with these properties, then* $g = T \circ f$ *for some conformal disk automorphism* $T : \mathbb{D} \longrightarrow \mathbb{D}$.

A partial converse to Theorem I.19 is our next

Proposition I.20
Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ *be a non–constant holomorphic function such that*

$$\sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1-|f(z)|^2} < \infty.$$

Then the non–tangential limit

$$n.t. \qquad \lim_{z\longrightarrow\xi} \frac{|f'(z)|}{1-|f(z)|^2} =: \phi(\xi)$$

exists for a.e. $\xi \in \partial\mathbb{D}$ *and* $\log\phi \in L^1(\partial\mathbb{D})$*. Moreover, if* (z_j) *is the sequence of critical points of* f *(counted with multiplicities), then*

$$\sum_{j=1}^{\infty}(1-|z_j|) < \infty.$$

I.3.2 On a theorem of Wittich

A quite different aspect of Theorem I.1 than the solution of the non–linear boundary value Problem I.16 is its connection to the following theorem of Wittich [55].

Theorem I.21 (Wittich)
There exists no C^2*–function* $w : \mathbb{C} \longrightarrow \mathbb{R}$ *which is a solution of the PDE*

$$\Delta w = e^w$$

in all of $\mathbb{C}$*.*

Wittich's theorem gives rise to the following

Problem I.22
For which (smooth) functions $k : \mathbb{C} \longrightarrow \mathbb{R}$ *does exist a* C^2*–solution of the PDE*

$$\Delta w = k(z)e^w \tag{I.10}$$

in all of $\mathbb{C}$ *?*

Note that Problem I.22 is equivalent to the question which (smooth) functions $k : \mathbb{C} \longrightarrow \mathbb{R}$ can arise as Gaussian curvature of a conformal metric on $\mathbb{C}$. Problem I.22 has been attacked by a number of people. For instance, Sattinger [47] (see also Oleinik [40]) used Wittich's method to prove the following generalization of Wittich's Theorem I.21.

Theorem I.23 (Sattinger & Oleinik)
Let $k : \mathbb{C} \longrightarrow \mathbb{R}$ *be a non–negative smooth function such that*

$$k(z) \geq \frac{c}{|z|^2} \quad \text{for } |z| \geq R\,,$$

where R *is sufficiently large and* c *is some positive constant. Then there exists no* C^2*–solution of (I.10) in all of* $\mathbb{C}$*.*

A converse of Theorem I.23 was obtained by Ni [38], who proved

Theorem I.24 (Ni)
Let $k : \mathbb{C} \longrightarrow \mathbb{R}$ *be a non–negative Hölder continuous function and let*

$$k(z) \leq \frac{c}{|z|^l} \quad \text{for } |z| \geq R\,,$$

where R *is sufficiently large, and* $l > 2$ *and* $c > 0$ *are constants. Then there exist infinitely many* C^2*–solutions of (I.10) in* $\mathbb{C}$*.*

Theorem I.23 and Theorem I.24 do *not* give a complete solution to Problem I.22. In the following extension of Wittich's theorem we deal with a situation which is not covered by Theorem I.23.

Theorem I.25
Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function, $h \not\equiv 0$. Then there exists no C^2–function $w : \mathbb{C} \longrightarrow \mathbb{R}$ which is a solution of the PDE

$$\Delta w = |h(z)|^2 e^w$$

in all of $\mathbb{C}$.

Remark I.26

(a) *Theorem I.25 shows that the condition on $k(z)$ to be positive for large $|z|$ in Theorem I.23 is not necessary to ensure the non–existence of solutions of (I.10). If we set $k(z) = |h(z)|^2$ for some appropriate entire function h then k can very well have zeros in a neighborhood of ∞.*

(b) *If $k(z) = |h(z)|^2$ for some entire function $h(z)$, then the condition $k(z) \leq c/|z|^l$ of Theorem I.24 implies $h \equiv 0$ and the statement of Theorem I.24 reduces to the fact that there exist infinitely many harmonic functions in $\mathbb{C}$.*

(c) *As observed by Nitsche [39] and Warnecke [54], Liouville's theorem (Corollary I.3) combined with another well–known result of Liouville that a bounded entire function is constant gives immediately the statement of Wittich's Theorem I.21. The same argument, but replacing Liouville's theorem by the more general Theorem I.8, leads to a quick proof of Theorem I.25.*

– Chapter II –

Ordinary Differential Equations in the Complex Plane

This chapter is dedicated to a discussion of the following special types of ordinary differential equations in the complex plane:

- second order homogeneous linear differential equations of the form

$$\psi'' + A(z)\psi = 0\,,$$

- the associated Schwarzian differential equations

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 2A(z)\,,$$

- and the corresponding Riccati differential equations

$$w' = A(z) + w^2\,.$$

We will have a look at each of these differential equations separately, but we will put our emphasis on the various connections between them. Before we bring up the details, we give – in Section II.1 – a short summary of some basic results about general ordinary differential equations in the complex plane: the existence and uniqueness theorem and the permanence principle. In Section II.2 we then turn to second order linear differential equations and Schwarzian differential equations. After that we move on to Riccati differential equations in Section II.3. Finally, we close this chapter with a survey on some of the relations between the above mentioned differential equations in Section II.4.

For further details on ordinary differential equations in the complex plane see, for instance, [4, 7, 20, 24, 25, 27, 28, 31, 33].

II.1 The existence and uniqueness theorem and the permanence principle

The theorems of this section shall provide a basis for the considerations in Section II.2 and Section II.3. As most of the statements are well–known we omit all the proofs. Let's begin with the existence and uniqueness theorem for a single complex ordinary differential equation.

Theorem II.1 (cf. [24])
Let $G \subseteq \mathbb{C}^2$ be a domain and let $\Omega := \{(z,w) \in G : |z - z_0| \leq a, |w - w_0| \leq b\} \subseteq G$. Further, let $f : G \longrightarrow \mathbb{C}$ be a holomorphic function and $M := \max_{(z,w)\in\Omega} |f(z,w)|$. Then there exists a unique holomorphic solution of the initial value problem

$$w' = f(z,w)\,,\ w(z_0) = w_0\,, \tag{II.1}$$

in the disk $K_\alpha(z_0) = \{z \in \mathbb{C} : |z - z_0| < \alpha\}$, where $\alpha = \min\{a, b/M\}$.

Note, Theorem II.1 ensures only the local existence of a solution to the initial value problem (II.1). On the other hand, the following permanence principle guarantees that the analytic continuation of a solution of the initial value problem (II.1) remains a solution of this initial value problem.

Theorem II.2 (cf. [24])
Let G and G_j, $j = 1, \ldots, m$, be domains in $\mathbb{C}$, let z_0 be a point in G such that the disk $K(z_0) := \{z \in \mathbb{C} : |z - z_0| < r_0\}$ belongs to G, and let $F : G \times G_1 \times \cdots \times G_m \longrightarrow \mathbb{C}$ and $f_j : K(z_0) \longrightarrow G_j$, $j = 1, \ldots, m$, be holomorphic functions. Finally, let γ be a path in G with initial point z_0 which is covered by finitely many disks $K(z_\nu) \subseteq G$, $\nu = 0, 1, \ldots, k$, such that each function f_j has a holomorphic continuation $f_j(z, z_\nu)$ to $K(z_\nu)$ with $f_j(K(z_\nu), z_\nu) \subseteq G_j$.
If

$$F(z, f_1(z), \ldots, f_m(z)) \equiv 0 \quad \text{for } z \in K(z_0)\,,$$

then

$$F(z, f_1(z, z_\nu), f_2(z, z_\nu), \ldots, f_m(z, z_\nu)) \equiv 0 \quad \text{for } z \in K(z_\nu),$$

for every $\nu = 0, \ldots, k$.

As we will also need a version of the existence and uniqueness Theorem II.1 for first–order linear systems and linear differential equations of order n in our later work, we state the corresponding generalizations of Theorem II.1 next.

Theorem II.3 (cf. [24])
Let the functions $\mathbf{A}(z) \in \mathbb{C}^{n\times n}$ and $\mathbf{a}(z) \in \mathbb{C}^n$ be holomorphic in a disk $K_r(z_0) \subset \mathbb{C}$. Then for every $\mathbf{w_0} \in \mathbb{C}^n$ the initial value problem

$$\mathbf{w}' = \mathbf{A}(z)\mathbf{w} + \mathbf{a}(z)\,, \quad \mathbf{w}(z_0) = \mathbf{w_0}, \tag{II.2}$$

has exactly one holomorphic solution in $K_r(z_0)$.

For the general solution set of a homogeneous system, we have

Theorem II.4 (cf. [24])
Let the function $\mathbf{A}(z) \in \mathbb{C}^{n\times n}$ be holomorphic in the disk $K \subset \mathbb{C}$. Then the solutions of the differential equation

$$\mathbf{w}' = \mathbf{A}(z)\mathbf{w} \tag{II.3}$$

are holomorphic in K and form an n–dimensional complex vector space.

An important consequence of the permanence principle is the fact that the analytic continuations of k linearly independent solutions of a system of first order differential equations are again linearly independent solutions of this system of differential equations. An immediate application of this and Theorem II.3 is the following

Theorem II.5 (cf. [24])
Let $G \subseteq \mathbb{C}$ be a domain and let the functions $\mathbf{A}(z) \in \mathbb{C}^{n\times n}$ and $\mathbf{a}(z) \in \mathbb{C}^n$ be holomorphic in G. Then every local solution[1] *of the differential equation*

$$\mathbf{w}' = \mathbf{A}(z)\mathbf{w} + \mathbf{a}(z) \tag{II.4}$$

has an analytic continuation along any path in G and the function obtained by this analytic continuation is a solution of the differential equation (II.4). In particular, if G is a simply connected domain, then every local solution of the differential equation (II.4) can be extended to a holomorphic solution of (II.4) in G.

II.2 Linear ordinary differential equation and the Schwarzian differential equation

After these preparations we are now going to discuss in some depth the various connections between the linear homogeneous differential equation

$$\psi'' + A(z)\psi = 0$$

and the associated Schwarzian differential equation

$$\mathcal{S}_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 2A(z)\,.$$

The following classical result deals with the case that $A(z)$ is a *holomorphic* function.

Theorem II.6
If $A(z)$ is a holomorphic function in a simply connected domain $G \subseteq \mathbb{C}$, then the quotient $f = g_1/g_2$ of any two linearly independent holomorphic solutions g_1, g_2 of the differential equation

$$\psi'' + A(z)\psi = 0 \tag{II.5}$$

in G is a locally injective meromorphic function and satisfies the Schwarzian differential equation

$$\mathcal{S}_f(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 2A(z) \tag{II.6}$$

in G. Conversely, if f is a locally injective meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$ and $A(z)$ is defined by (II.6), then $A(z)$ is holomorphic in G and the differential equation (II.5) has two linearly independent holomorphic solutions g_1 and g_2 in G, such that $f = g_1/g_2$.

One of the main tools in the proof of Theorem I.1 in Chapter V is a generalized version of Theorem II.6, where $A(z)$ is a *meromorphic* function. Under certain additional conditions on the poles of $A(z)$ it can be shown that Theorem II.6 is still valid in the case of a meromorphic function $A(z)$. In passing from the holomorphic to the meromorphic case, we have to encounter some questions, which arise quite naturally. For instance, what do the solutions of the linear equation (II.5) look like, if A is meromorphic? Do they have poles or other singularities? Under which hypotheses do there exist meromorphic solutions of the corresponding Schwarzian differential equation (II.6)?

Before turning to these questions, we give – for completeness – a proof of Theorem II.6.

[1] i.e. every holomorphic solution in a disk $K \subseteq G$

Proof.
(a) We first suppose $A(z)$ is a holomorphic function in the simply connected domain G. Let g_1, g_2 be two linearly independent holomorphic solutions of $\psi''+A(z)\psi=0$ in G and let $f=g_1/g_2$. We will show f solves $\mathcal{S}_f(z)=2A(z)$, that is equation (II.6).

For this purpose we differentiate f and obtain

$$f'=\frac{g_1'g_2-g_2'g_1}{g_2^2}=-\frac{W(g_1,g_2)}{g_2^2}\,,$$

where

$$W(g_1,g_2):=g_1g_2'-g_1'g_2$$

is the Wronski determinant of the fundamental system $\{g_1,g_2\}$ of the linear equation (II.5). Since g_1 and g_2 are solutions of $\psi''+A(z)\psi=0$ we get

$$\frac{d}{dz}W(g_1,g_2)=g_1g_2''-g_1''g_2=0\,.$$

Hence $W(g_1,g_2)$ is a constant function and f' reduces to

$$f'=-\frac{c}{g_2^2}\quad\text{for some } c\in\mathbb{C}\backslash\{0\}\,.$$

This leads for the Pre–Schwarzian f''/f' of f to

$$\frac{f''}{f'}=-2\frac{g_2'}{g_2}\,.$$

Taking the derivative of the last expression yields

$$\left(\frac{f''}{f'}\right)'=-2\frac{g_2''}{g_2}+2\left(\frac{g_2'}{g_2}\right)^2=2A(z)+\frac{1}{2}\left(\frac{f''}{f'}\right)^2,$$

and this is equivalent to $\mathcal{S}_f(z)=2A(z)$.

Our next goal is to check that f is a locally injective function in G.

Let $z_0\in G$ be an arbitrary point in G. If f is holomorphic at z_0, then f' has at z_0 the Taylor series expansion

$$f'(z)=k(z-z_0)^{\alpha}+\cdots\,,\qquad k\neq 0,\quad \alpha\geq 0\,.$$

Otherwise, f has a pole at z_0 and a Laurent series expansion of the form

$$f(z)=k(z-z_0)^{-\alpha}+\cdots\,,\qquad k\neq 0,\quad \alpha\geq 1\,.$$

Inserting these expansions into $\mathcal{S}_f$ gives

$$\mathcal{S}_f(z)=-\frac{\alpha}{(z-z_0)^2}-\frac{1}{2}\frac{\alpha^2}{(z-z_0)^2}+\cdots$$

in the holomorphic situation, and

$$\mathcal{S}_f(z)=\frac{\alpha+1}{(z-z_0)^2}-\frac{1}{2}\frac{(\alpha+1)^2}{(z-z_0)^2}+\cdots\,,$$

when f has a pole at z_0.

Since $\mathcal{S}_f(z)=2A(z)$ is holomorphic, we deduce the following condition for α:
In the first case

$$2\alpha + \alpha^2 = 0, \text{ which is equivalent to either } \alpha = -2 \text{ or } \alpha = 0,$$

and in the second case

$$2(\alpha + 1) = (\alpha + 1)^2, \text{ which is equivalent to either } \alpha = 1 \text{ or } \alpha = -1.$$

Our assumptions on α imply $\alpha = 0$ and $\alpha = 1$, respectively, and this means f is locally injective in z_0 in both cases. Since z_0 was an arbitrary point in G, we conclude f is a locally injective function in all of G.

(b) Now we turn to the converse statement. Let f be a locally injective meromorphic function in G and $A(z)$ be defined by (II.6). We first prove $A(z)$ is a holomorphic function in G.

Pick an arbitrary point $z_0 \in G$. If f is holomorphic in a neighborhood of $z_0 \in G$, then $A(z)$ is holomorphic in z_0 because $f'(z_0) \neq 0$. If f has a simple pole at z_0, then f can be written as $f(z) = k(z - z_0)^{-1} + \phi(z)$, $k \neq 0$, where $\phi(z)$ is holomorphic in z_0. This shows

$$\frac{f''(z)}{f'(z)} = \frac{-2}{z - z_0} + c_1(z - z_0) + \mathcal{O}(|z - z_0|^2)$$

for the Pre–Schwarzian and yields

$$\mathcal{S}_f(z) = 3c_1 + \mathcal{O}(|z - z_0|)$$

for the Schwarzian derivative. So in any case, $A(z)$ is holomorphic in z_0 and consequently in all of G.

Next, we have to find two linearly independent solutions g_1 and g_2 of the linear ODE (II.5) such that $f = g_1/g_2$.

Choose a point $z_0 \in G$ with $f'(z_0) \neq \infty$. As f' has no zeros, there exists a holomorphic square root φ of $1/f'$ in a neighborhood U of z_0. We will show that φ is in U a solution of the differential equation $\psi'' + A(z)\psi = 0$.

From the representation

$$\varphi^2 = \frac{1}{f'} \quad \text{in } U \tag{II.7}$$

we deduce the following equations:

$$2\varphi\varphi' = -\frac{f''}{(f')^2} \quad \text{and} \quad \frac{\varphi'}{\varphi} = -\frac{f''}{2f'} \quad \text{as well as} \quad \frac{\varphi''}{\varphi} = -\frac{\varphi'}{\varphi}\frac{f''}{2f'} - \frac{f'''}{2f'} + \frac{(f'')^2}{2(f')^2}.$$

Replacing φ'/φ by $-f''/(2f')$ in the last identity yields

$$\frac{\varphi''}{\varphi} = -\frac{f'''}{2f'} + \frac{3}{4}\frac{(f'')^2}{(f')^2} = -A(z).$$

Thus φ is a solution to $\psi'' + A(z)\psi = 0$ in U.

Since G is a simply connected domain, every solution of the differential equation (II.5) has a holomorphic extension to G due to Theorems II.4 and II.5. Consequently, φ has a holomorphic continuation to G, which we call g_2.

Now, we define the function g_1 by $g_1 := f g_2$ and compute

$$g_1'' = f'' g_2 + 2 f' g_2' + f g_2'' = f'' g_2 - f'' g_2 + f g_2'' = -A(z) g_2 f = -A(z) g_1 \,,$$

where we used the identity $2g_2'/g_2 = -f''/f'$. Hence g_1 solves the differential equation $\psi'' + A(z)\psi = 0$. Finally, we note that the functions g_1 and g_2 are linearly independent, since f is locally injective. ■

Let $\{g_1, g_2\}$ be a fundamental system of the linear differential equation (II.5). Then the quotient of any two linearly independent solutions of $\psi'' + A(z)\psi = 0$ has the form $f = (\alpha_1 g_1 + \alpha_2 g_2)/(\beta_1 g_1 + \beta_2 g_2)$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Owing to Theorem II.6 every such f satisfies the Schwarzian differential equation (II.6). The next remark is an immediate consequence of this observation.

Remark II.7
Let f and h be two locally injective meromorphic functions in a simply connected domain $G \subseteq \mathbb{C}$. Then $\mathcal{S}_f = \mathcal{S}_h$ if and only if $f = \sigma \circ h$, where σ is a Möbius transformation.

Proof.
First we suppose

$$\mathcal{S}_f = \mathcal{S}_h =: 2A(z) \,.$$

Then by Theorem II.6 each of the functions f and h is the quotient of a pair, say g_1, g_2 and v_1, v_2, respectively, of linearly independent holomorphic solutions of the linear differential equation $\psi'' + A(z)\psi = 0$ in G, i.e.

$$f(z) = \frac{g_1(z)}{g_2(z)} \quad \text{and} \quad h(z) = \frac{v_1(z)}{v_2(z)} \,.$$

Furthermore, we can find constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$f(z) = \frac{g_1(z)}{g_2(z)} = \frac{\alpha_1 v_1(z) + \alpha_2 v_2(z)}{\beta_1 v_1(z) + \beta_2 v_2(z)} = \frac{\alpha_1 h(z) + \alpha_2}{\beta_1 h(z) + \beta_2} = \sigma \circ h(z) \,,$$

where σ is the Möbius transformation $\sigma(\xi) = (\alpha_1\xi + \alpha_2)/(\beta_1\xi + \beta_2)$. Note, $\alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$ as g_1 and g_2 are linearly independent.

Conversely, assume $f = \sigma \circ h$ for some Möbius transformation σ. Then the following computation shows $\mathcal{S}_{\sigma\circ h} = \mathcal{S}_h$:

Taking the first and second derivative of the function

$$f(z) := \frac{\alpha_1 h(z) + \alpha_2}{\beta_1 h(z) + \beta_2} \,, \quad \text{where} \quad \alpha_1\beta_2 - \beta_1\alpha_2 \neq 0 \,,$$

yields

$$f'(z) = \frac{(\alpha_1\beta_2 - \alpha_2\beta_1) h'(z)}{(\beta_1 h(z) + \beta_2)^2}$$

and

$$f''(z) = \frac{(\alpha_1\beta_2 - \alpha_2\beta_1) h''(z)}{(\beta_1 h(z) + \beta_2)^2} - \frac{2\beta_1(\alpha_1\beta_2 - \alpha_2\beta_1) h'(z)^2}{(\beta_1 h(z) + \beta_2)^3} \,.$$

Thus the Pre–Schwarzian of f takes the form

$$\frac{f''(z)}{f'(z)} = \frac{h''(z)}{h'(z)} - 2\frac{\beta_1 h'(z)}{\beta_1 h(z) + \beta_2},$$

and it follows that

$$\mathcal{S}_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 = \mathcal{S}_h\,.$$

■

In our next theorem we focus on the solutions of the differential equation $\psi'' + A(z)\psi = 0$, when $A(z)$ is a *meromorphic* function. This gives rise to the following

Definition II.8
Let $A(z)$ be a meromorphic function in a domain $G \subseteq \mathbb{C}$. We call a meromorphic function g in G a meromorphic solution of $\psi'' + A(z)\psi = 0$, if $g''(z) + A(z)g(z) = 0$ for $z \in G$. By a local solution of $\psi'' + A(z)\psi = 0$ we mean a meromorphic solution of this differential equation in a subdomain $G' \subseteq G$.

Theorem II.9
Let h be a holomorphic function in the disk $K_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}$ and denote by ρ_1 and ρ_2 the roots of the characteristic equation

$$\rho(\rho - 1) + h(z_0) = 0\,. \tag{II.8}$$

Then the differential equation

$$\psi'' + \frac{h(z)}{(z - z_0)^2}\psi = 0 \tag{II.9}$$

admits in the slit disk $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}\setminus\{z_0 + t : 0 \le t < r\}$ for some $r \in (0, R]$ a fundamental system g_1, g_2 of the following form:

(1) If $\rho_1 - \rho_2 \notin \mathbb{Z}$, then

$$\begin{aligned} g_1(z) &= (z - z_0)^{\rho_1}\sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_0 \neq 0,\\ g_2(z) &= (z - z_0)^{\rho_2}\sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad b_0 \neq 0. \end{aligned} \tag{II.10}$$

(2) If $\rho_1 - \rho_2 \in \mathbb{Z}$ and $\rho_1 - \rho_2 \ge 0$, then

$$\begin{aligned} g_1(z) &= (z - z_0)^{\rho_1}\sum_{k=0}^{\infty} a_k (z - z_0)^k, && a_0 \neq 0,\\ g_2(z) &= \chi g_1(z)\log(z - z_0) + (z - z_0)^{\rho_2}\sum_{k=0}^{\infty} b_k (z - z_0)^k, && b_0 \neq 0\,, \end{aligned} \tag{II.11}$$

where $\chi = 0$ or $\chi = 1$. If $\rho_1 = \rho_2$, then $\chi = 1$.

Proof.
(a) First we will establish that functions of the type

$$g(z) = (z - z_0)^\rho \sum_{k=0}^{\infty} c_k (z - z_0)^k, \quad c_0 \neq 0\,, \tag{II.12}$$

where ρ is suitably chosen, are formal solutions of the differential equation (II.9). As the function $h(z)$ is holomorphic in the disk $K_R(z_0)$, it can be expanded in a power series about z_0:

$$h(z) = \sum_{k=0}^{\infty} \beta_k (z - z_0)^k\,. \tag{II.13}$$

We next plug the formulas (II.12) for g and (II.13) for h in the equation

$$g''(z)(z - z_0)^2 + h(z)g(z) = 0$$

and compare equal powers of $(z - z_0)$. This yields

$$(\rho + n)(\rho + n - 1)c_n + \sum_{k=0}^{n} \beta_k c_{n-k} = 0 \quad \text{for } n = 0, 1, \dots\,. \tag{II.14}$$

By using the definitions

$$\begin{aligned} \varphi_0(\rho) &:= \rho(\rho - 1) + \beta_0 \\ \varphi_k(\rho) &:= \beta_k \qquad \text{for} \quad k \in \mathbb{N}\,, \end{aligned} \tag{II.15}$$

we can rewrite the equations (II.14) in the following system of equations:

$$\begin{aligned} &c_0\varphi_0(\rho) = 0 \\ &c_1\varphi_0(\rho + 1) + c_0\varphi_1(\rho) = 0 \\ &c_2\varphi_0(\rho + 2) + c_1\varphi_1(\rho + 1) + c_0\varphi_2(\rho) = 0 \\ &\qquad\vdots \\ &c_n\varphi_0(\rho + n) + c_{n-1}\varphi_1(\rho + n - 1) + \cdots + c_1\varphi_{n-1}(\rho + 1) + c_0\varphi_n(\rho) = 0 \\ &\qquad\vdots \end{aligned} \tag{II.16}$$

Because of the assumption $c_0 \neq 0$, the number ρ must satisfy the condition

$$\varphi_0(\rho) = \rho(\rho - 1) + \beta_0 = 0\,.$$

This, however, is exactly the characteristic equation (II.8) and their roots are $\rho = \rho_1$ and $\rho = \rho_2$.

In particular, if $\rho_1 - \rho_2 \notin \mathbb{Z}$, then $\varphi_0(\rho + n) \neq 0$ for both $\rho = \rho_1$ and $\rho = \rho_2$, and every $n \in \mathbb{N}$. Hence we can find to $\rho = \rho_1$ and to $\rho = \rho_2$ — after an arbitrary choice of $c_0 \neq 0$ — the coefficients c_k, $k \in \mathbb{N}$, for a formal solution to (II.9) of the form (II.12) by solving the equations (II.16) recursively. Thus, we get to $\rho = \rho_1$ as well as to $\rho = \rho_2$ a formal solution

$g_1(z)$ and $g_2(z)$ of (II.9), respectively. In case we can guarantee the convergence of these power series, the functions g_1 and g_2 will be linearly independent.

In contrast, if $\rho_1 = \rho_2$ we obtain, for each choice of $c_0 \neq 0$, only one formal solution g_1 of (II.9) of the form (II.12) by solving the equations of (II.16) successively.

Finally, if $\rho_1 - \rho_2 \in \mathbb{Z}\backslash\{0\}$, then we may assume $\kappa := \rho_1 - \rho_2 \geq 1$ without loss of generality. Since $\varphi_0(\rho_1 + n) \neq 0$ for every $n \in \mathbb{N}$ we can determine to $\rho = \rho_1$ the numbers c_k, $k \in \mathbb{N}$ as above, provided $c_0 \neq 0$. This gives one formal power series and consequently one formal solution of type (II.12) to (II.9). In general we cannot find to the equation $\rho = \rho_2$ another formal solution of the form (II.12) to (II.9) since $\varphi_0(\rho_2 + \kappa) = \varphi_0(\rho_1) = 0$.

So, if there exists a second formal solution, the coefficients, $c_0 \neq 0$, c_1, c_2, …, $c_{\kappa-1}$, we already computed from (II.16), must fulfill the equation

$$c_{\kappa-1}\varphi_1(\rho_2 + \kappa - 1) + \cdots + c_1\varphi_{\kappa-1}(\rho_2 + 1) + c_0\varphi_\kappa(\rho_2) = 0\,, \tag{II.17}$$

see (II.16). If they do so, we choose c_κ arbitrarily and continue to solve the equations of (II.16) for $n = \kappa+1, \kappa+2, \ldots$. Thus, we obtain the rest of the $c_k's$, i.e. $c_{\kappa+1}, c_{\kappa+2}, \ldots$ and therefore a second formal solution g_2 of (II.9) with a representation of the form (II.12). Note, in this situation we must have $\chi = 0$ in (II.11). Further, g_1 and g_2 are obviously linearly independent, if we can prove the convergence of the constructed power series. So we can find, depending on the validity of equation (II.17), one or two formal power series of type (II.12), which are formal solutions of (II.9).

(b) Now we will focus our attention on proving the convergence of the formal power series we constructed in part (a) – at least in a neighborhood of z_0. As we have seen above the formal solutions are of type (II.12), i.e.

$$g(z) = (z - z_0)^\rho \sum_{k=0}^{\infty} c_k (z - z_0)^k\,.$$

If we can find a number $r \in (0, R)$ and a constant $M > 0$, such that

$$|c_k| r^k \leq M \tag{II.18}$$

for every $k = 0, 1, 2, \ldots$, then

$$\limsup_{k\longrightarrow\infty} |c_k|^{\frac{1}{k}} \leq \frac{1}{r}$$

and hence the "power series" in (II.12) converges in $K_r(z_0)$.

We are now going to prove an estimate of the form (II.18). Since ρ solves the equation (II.8), there exists an integer $p \geq 0$ with the property that $\varphi_0(\rho+p) = 0$ and $\varphi_0(\rho+n) \neq 0$ for every $n > p$. Then, because of $\varphi_0(\rho) = 0$, we get

$$\begin{aligned}\varphi_0(\rho + n) &= \varphi_0(\rho + n) - \varphi_0(\rho) = (\rho + n)(\rho + n - 1) + \beta_0 - \rho(\rho - 1) - \beta_0\\ &= n^2 + n(2\rho - 1) = n^2\left(1 + \frac{2\rho - 1}{n}\right)\end{aligned}$$

for every $n > p$. This, however, means that there is some constant $c > 0$ such that

$$|\varphi_0(\rho + n)| \geq cn^2 \tag{II.19}$$

for every $n > p$. As the function

$$h(z) = \sum_{k=0}^{\infty} \beta_k (z - z_0)^k$$

is holomorphic in a neighborhood of z_0, the sum

$$\sum_{k=1}^{\infty} |\beta_k| (z - z_0)^k$$

is a continuous function in this neighborhood and vanishes at $z = z_0$. Thus there exists an $r \in (0, R)$ such that

$$\sum_{k=1}^{\infty} |\beta_k| r^k \le c(p+1)^2 .$$

Now we define

$$M := \max\{|c_0|, |c_1| r, \dots, |c_p| r^p\} > 0$$

and assume equation (II.18) holds for $k = 0, 1, \dots, n-1$, where $n > p$.

Then the equations (II.15) and (II.16) lead to

$$|\varphi_0(\rho + n)| |c_n| \le \sum_{k=1}^{n} |c_{n-k}| |\varphi_k(\rho + n - k)| = \sum_{k=1}^{n} |c_{n-k}| |\beta_k| ,$$

and thus to

$$|\varphi_0(\rho + n)| |c_n| r^n \le \sum_{k=1}^{n} |c_{n-k}| r^{n-k} |\beta_k| r^k \le M \sum_{k=1}^{n} |\beta_k| r^k \le M c(p+1)^2 .$$

This yields, in view of (II.19),

$$|c_n| r^n \le \frac{M c(p+1)^2}{cn^2} \le M .$$

Therefore the power series part of each formal solution (II.12) is indeed a holomorphic function in the disk $K_r(z_0)$. In summary, we proved case (1) of the theorem, i.e. if $\rho_1 - \rho_2 \notin \mathbb{Z}$, and part (2) for $\rho_1 - \rho_2 \in \mathbb{Z}\backslash\{0\}$, if equation (II.17) is fulfilled, where in this situation $\chi = 0$ has to be chosen in equation (II.11).

It remains to consider the case $\kappa \ge 0$, if (II.17) is not valid. As we already observed there is only one formal solution $g_1(z)$ of type (II.12) in this situation. We write g_1 as

$$g_1(z) = (z - z_0)^{\rho_1} \phi(z) ,$$

where $\phi(z)$ is a holomorphic function in $K_r(z_0)$ and $\phi(z_0) \ne 0$. Thus

$$g_1^{-2}(z) = (z - z_0)^{-2\rho_1} \psi(z) ,$$

where $\psi(z) = \phi(z)^{-2}$ is a holomorphic function in $z = z_0$. Note,

$$-2\rho_1 = \rho_2 - 1 - \rho_1 = -\kappa - 1 ,$$

since $\rho_1 + \rho_2 = 1$. This implies for g_1^{-2} a Laurent series expansion of the form

$$g_1^{-2}(z) = (z - z_0)^{-\kappa - 1} \sum_{k=0}^{\infty} \gamma_k (z - z_0)^k = \frac{\gamma_0}{(z - z_0)^{\kappa+1}} + \cdots + \frac{\gamma_\kappa}{z - z_0} + H(z)$$

with a function $H(z)$ holomorphic in a neighborhood of z_0. Obviously, g_1^{-2} has a meromorphic anti–derivative, say G_1, in a slit disk D about z_0. Without loss of generality we may assume $D = D_r(z_0) = K_r(z_0) \backslash \{z_0 + t \, : \, 0 \le t < r\}$. The function G_1 can then be expressed as

$$G_1(z) = (z - z_0)^{-\kappa} F(z) + \gamma_\kappa \log (z - z_0)$$

with a holomorphic function F in $K_r(z_0)$. As we want to find a fundamental system of (II.9) we define g_2 by

$$g_2(z) := g_1(z) G_1(z) \, .$$

To check that g_2 solves (II.9) in $D_r(z_0)$ we compute

$$g_2'(z) = g_1'(z) G_1(z) + \frac{1}{g_1(z)} \quad \text{and} \quad g_2''(z) = g_1''(z) G_1(z) \, .$$

Hence, we have

$$\frac{g_2''(z)}{g_2(z)} = \frac{g_1''(z)}{g_1(z)} = -\frac{h(z)}{(z - z_0)^2} \, .$$

This shows g_2 is really a solution of (II.9) in the slit disk $D_r(z_0)$. We remark $\gamma_\kappa \neq 0$ since otherwise g_2 would be a solution whose coefficients satisfy equation (II.17). Now we may assume $\gamma_\kappa = 1$ by choosing $c g_1$ instead of g_1 with a suitable constant c, and g_2 takes the form

$$g_2(z) = \chi g_1(z) \log (z - z_0) + (z - z_0)^{\rho_1 - \kappa} G_2(z) = \chi g_1(z) \log (z - z_0) + (z - z_0)^{\rho_2} G_2(z)$$

with $\chi = 1$, where G_2 is holomorphic in $K_r(z_0)$ such that $G_2(z_0) \neq 0$.

Lastly, because of the "log–term", the two functions g_1 and g_2 are obviously linearly independent. This concludes the proof of Theorem II.9. ■

Remark II.10
The power series which occur in the fundamental systems (II.10) or (II.11) are easily seen to converge in the whole disk $K_R(z_0)$.

Theorem II.9 reveals that in contrast to Theorem II.6 the quotient of two linearly independent solutions of the differential equation $\psi'' + A(z)\psi = 0$ is not necessarily meromorphic if $A(z)$ is a meromorphic function. Since we wish to generalize Theorem II.6 to a meromorphic A we are interested in a characterization of equations of the type

$$\psi'' + A(z)\psi = 0$$

for which the quotient of any two linearly independent local solutions has a meromorphic extension. Such a characterization is provided by the following theorem.

Theorem II.11
Let $G \subseteq \mathbb{C}$ be a simply connected domain and let $A(z)$ be a meromorphic function in G. Then the quotient f of any two linearly independent local solutions g_1, g_2 of the differential equation $\psi'' + A(z)\psi = 0$ has a meromorphic extension to G, if and only if at every pole $z_0 \in G$ of A the following two conditions are satisfied:

(1) A has at z_0 a Laurent series expansion of the form

$$A(z) = \frac{1-n^2}{4(z-z_0)^2} + \cdots, \qquad n \in \mathbb{Z}, \ |n| \geq 2 .$$

(2) Let $K_{r(z_0)}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r(z_0)\} \subseteq G$ be a disk such that $A(z)$ is holomorphic in $K_{r(z_0)}(z_0) \backslash \{z_0\}$ and consider the differential equation $\psi''+A(z)\,\psi = 0$ in $K_{r(z_0)}(z_0)$. Then $\chi = 0$ in the local solution base given in the slit disk $D_{r(z_0)}(z_0) = K_{r(z_0)}(z_0)\backslash\{z_0 + t : 0 \leq t < r(z_0)\}$, see Theorem II.9, equation (II.11) and Remark II.10.

Before turning to the proof we like to illustrate Theorem II.11 with the following two examples.

Example II.12
(a) Consider in $\mathbb{C}$ the differential equation

$$\psi'' + \frac{1}{4z^2}\psi = 0 . \tag{II.20}$$

Then the functions

$$\begin{aligned} g_1(z) &= z^{1/2} \\ g_2(z) &= z^{1/2}(\log z + 1) \end{aligned}$$

defined for $\mathbb{C}\backslash[0,\infty)$ form a fundamental system of (II.20).
The quotient

$$f(z) = \frac{a g_1(z) + b g_2(z)}{c g_1(z) + d g_2(z)}, \qquad \text{where } a,b,c,d \in \mathbb{C} \text{ and } ad - bc \neq 0 ,$$

of any two linearly independent solutions of (II.20) is a meromorphic function in $\mathbb{C}\backslash[0,\infty)$ but f has no meromorphic extension to $\mathbb{C}$. Further, every such f solves the Schwarzian differential equation

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2z^2}$$

in $\mathbb{C}\backslash[0,\infty)$. Thus not every solution of the Schwarzian differential equation is meromorphic in $\mathbb{C}$.

(b) Now we have a look at the differential equation

$$\psi'' - \frac{3}{4z^2}\psi = 0 \tag{II.21}$$

in $\mathbb{C}$. A corresponding solution base is given in $\mathbb{C}\backslash[0,\infty)$ by

$$\begin{aligned} g_1(z) &= z^{3/2} \\ g_2(z) &= z^{-1/2}\,. \end{aligned}$$

The quotient of any two linearly independent solutions of (II.21)

$$f(z) = \frac{ag_1(z) + bg_2(z)}{cg_1(z) + dg_2(z)}\,, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0\,,$$

has a meromorphic extension to $\mathbb{C}$ and any such f solves the Schwarzian differential equation $\mathcal{S}_f = -3/(2z^2)$ in $\mathbb{C}$.

Thus, if $A(z)$ is a meromorphic function in a simply connected domain G, there can exist meromorphic functions f in a subdomain G' of G which have no meromorphic extension to G but are solutions to $\mathcal{S}_f(z) = 2A(z)$ in G'. It is therefore necessary to distinguish between *local* meromorphic solutions and solutions meromorphic in all of G of $\mathcal{S}_f(z) = 2A(z)$.

Definition II.13
Let $A(z)$ be a meromorphic function in a domain $G \subseteq \mathbb{C}$. A meromorphic function f is called a solution of $\mathcal{S}_f = 2A(z)$ in G, if

$$\mathcal{S}_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 2A(z)$$

holds for every $z \in G$. By a local solution of $\mathcal{S}_f = 2A(z)$ we mean a meromorphic solution of this differential equation in a subdomain $G' \subseteq G$

The next lemma is the first step of the proof of Theorem II.11.

Lemma II.14
Let $G \subseteq \mathbb{C}$ be a simply connected domain and let $A(z)$ be a meromorphic function in G. If the quotient f of two linearly independent local solutions of

$$\psi'' + A(z)\psi = 0$$

has a meromorphic extension to G, then this extension solves the Schwarzian differential equation

$$\mathcal{S}_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 2A(z)$$

in G.

Proof.
Choose a disk $K \subseteq G$, where $A(z)$ is holomorphic, and let g_1, g_2 be two linearly independent solutions of $\psi'' + A(z)\psi = 0$ in K. Then by assumption, the function $f := g_1/g_2$ has a meromorphic extension to G. According to Theorem II.6, f solves $\mathcal{S}_f = 2A(z)$ first in K and then also in G because of the identity principle. ■

We are now prepared to prove Theorem II.11.

Proof of Theorem II.11.
(a) We start proving the only if part. By Theorem II.6 the quotient (or the meromorphic extension) f of any two linearly independent solutions of the equation (II.5) cannot be locally injective at a pole $z_0 \in G$ of $A(z)$. Therefore $f(z)$ is of the form

$$f(z) = c_0 + \sum_{\substack{k=n \\ k\neq 0}}^{\infty} c_k (z-z_0)^k \,,$$

where $n \in \mathbb{Z}, n \neq 0, \pm 1$, and $c_n \neq 0$. Due to Lemma II.14 we have $2A(z) = \mathcal{S}_f(z)$. Inserting the Laurent series of f about z_0 into this equation yields

$$A(z) = \frac{1-n^2}{4(z-z_0)^2} + \frac{1}{z-z_0}\phi(z)\,,$$

where ϕ is a holomorphic function in a neighborhood of z_0. This shows (1).

To establish (2), we choose a disk $K_{r(z_0)}(z_0) \subseteq G$ such that z_0 is the only pole of A in this disk. Since $A(z) = (1-n^2)/(4(z-z_0)^2) + \cdots$, $n \in \mathbb{Z}$, $|n| \geq 2$, the roots of the characteristic equation of the differential equation $\psi'' + A(z)\psi = 0$ are $(1+n)/2$ and $(1-n)/2$, see equation (II.8). Without loss of generality we may assume $n \geq 2$, so $\rho_1 = (1+n)/2$ and $\rho_2 = (1-n)/2$. A local solution base in the slit disk $D_{r(z_0)}(z_0) = K_{r(z_0)}(z_0)\backslash\{z_0 + t : 0 \leq t < r(z_0)\}$ is then given by

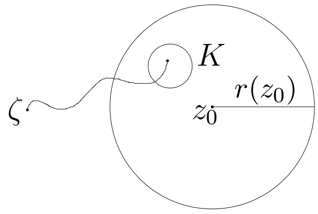
$$\begin{aligned} g_1(z) &= (z-z_0)^{\rho_1}\phi_1(z) \\ g_2(z) &= \chi g_1(z)\log(z-z_0) + (z-z_0)^{\rho_2}\phi_2(z), \end{aligned}$$

where ϕ_1, ϕ_2 are holomorphic functions in $K_{r(z_0)}(z_0)$ with $\phi_1(z_0) \neq 0$ as well as $\phi_2(z_0) \neq 0$, and either $\chi = 0$ or $\chi = 1$. Since g_2/g_1 has by hypothesis a meromorphic extension to all of G we can exclude $\chi = 1$. This gives condition (2).

(b) Now we move on to the if part. Let f be the quotient of two linearly independent local solutions of $\psi'' + A(z)\psi = 0$ at an arbitrary point $\zeta \in G$, where $A(z)$ is holomorphic in the disk $K_{r(\zeta)}(\zeta) \subseteq G$, say $f = \tilde{g}_1/\tilde{g}_2$. By the local existence Theorem II.3 the solutions $\tilde{g}_1$ and $\tilde{g}_2$ of $\psi'' + A(z)\psi = 0$ are holomorphic functions in $K_{r(\zeta)}(\zeta)$ and their continuations along any path in $G\backslash\{z_0 : z_0 \text{ is a pole of } A\}$ are holomorphic, compare Theorem II.5. Thus f has a meromorphic continuation along any path in $G\backslash\{z_0 : z_0 \text{ is a pole of } A\}$. We aim at showing f is meromorphic in G.

To this end let z_0 be an arbitrary pole of $A(z)$ and $K_{r(z_0)}(z_0) \subseteq G$, $r(z_0) > 0$, be a disk such that $A(z)$ is holomorphic in $K_{r(z_0)}(z_0)$ save z_0. Because of condition (1) and (2) there exists a local solution base $\{g_1, g_2\}$ of $\psi'' + A(z)\psi = 0$ in the slit disk $D_{r(z_0)}(z_0)$ with $\chi = 0$ and $\rho_1 = (1+n)/2$, $\rho_2 = (1-n)/2$. This however implies g_1/g_2 is meromorphic in $K_{r(z_0)}(z_0)$.

Now let K be a disk in $D_{r(z_0)}(z_0)$. We choose a non–selfintersecting path γ in G, connecting ζ with an arbitrary point in K and avoiding all the poles of $A(z)$. Further, we may assume f has a meromorphic extension to a domain $\Omega \supset K$ along γ. In K however, f is a locally injective meromorphic function, since f satisfies there the differential equation



$\mathcal{S}_f = 2A(z)$ with a holomorphic A. Thus, by Theorem II.6 there exist two linearly independent holomorphic solutions v_1, v_2 of $\psi'' + A(z)\psi = 0$ in K such that $f = v_1/v_2$ in the disk K. But $v_1 = \alpha_1 g_1(z) + \alpha_2 g_2(z)$ and $v_2(z) = \beta_1 g_1(z) + \beta_2 g_2(z)$ in K for some α_1, α_2, β_1, $\beta_2 \in \mathbb{C}$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Therefore we can write f in K as

$$f(z) = \frac{\alpha_1 g_1(z) + \alpha_2 g_2(z)}{\beta_1 g_1(z) + \beta_2 g_2(z)} \,.$$

Since g_1/g_2 is meromorphic in $K_{r(z_0)}(z_0)$, we deduce that f has a meromorphic extension to $K_{r(z_0)}(z_0)$. In view of the fact that this process is valid for every pole z_0 of $A(z)$ we finally conclude f has a meromorphic extension to G by the Monodromy theorem. ■

In [33, Theorem 6.7 and Corollary 6.8] an explicit algebraic characterization of A such that $\mathcal{S}_f = 2A(z)$ has at least one meromorphic solution is given. This algebraic formula follows readily from equations (II.16) and (II.17). However, it seems difficult to exploit this algebraic condition. Therefore, we prefer the formally different equivalent formulation of this condition in the next theorem.

Theorem II.15
Let $G \subseteq \mathbb{C}$ be a simply connected domain and let $A(z)$ be a meromorphic function in G. Then the function $2A(z)$ is the Schwarzian derivative of a meromorphic function f in G if and only if at every pole z_0 of A the conditions (1) and (2) of Theorem II.11 are satisfied.

Proof.
(a) Suppose first $\mathcal{S}_f = 2A(z)$, where f is a meromorphic function in G. As in the proof to Theorem II.11 it follows that $A(z)$ has a Laurent series expansion at z_0 of the desired form, i.e.

$$A(z) = \frac{1-n^2}{4(z-z_0)^2} + \cdots\,, \qquad n \in \mathbb{Z},\ |n| \geq 2\,.$$

For the second condition let $K \subseteq G$ be a disk, where $A(z)$ is holomorphic. By Theorem II.6 and Remark II.7 there exists a local solution base $\{v_1, v_2\}$ of $\psi'' + A(z)\psi = 0$ in K such that $\mathcal{S}_f = \mathcal{S}_{v_1/v_2}$ in K. Therefore $v_1/v_2 = \sigma \circ f$ in K for some Möbius transformation σ. Since $\sigma \circ f$ is meromorphic in G the quotient v_1/v_2 has a meromorphic extension to G, which we will also call v_1/v_2, and it follows from the identity principle that $\mathcal{S}_{v_1/v_2} = 2A(z)$ in G. Now let $z_0 \in G$ be an arbitrary pole of $A(z)$ and denote by $K_{r(z_0)}(z_0) \subseteq G$ a disk, where $A(z)$ is holomorphic in $K_{r(z_0)}(z_0)$ except for z_0. Note, the roots of the characteristic equation of the differential equation $\psi'' + A(z)\psi = 0$ are $\rho_1 = (1+n)/2$, $\rho_2 = (1-n)/2$, where we assume $n \geq 2$ w.l.o.g.. Thus a local solution base in the slit disk $D_{r(z_0)}(z_0)$ is given by

$$\begin{aligned} g_1(z) &= (z-z_0)^{\rho_1}\phi_1(z) \\ g_2(z) &= \chi g_1(z)\log(z-z_0) + (z-z_0)^{\rho_2}\phi_2(z), \end{aligned} \tag{II.22}$$

where ϕ_1, ϕ_2 are holomorphic in a neighborhood of z_0 with $\phi_1(z_0) \neq 0$, $\phi_2(z_0) \neq 0$, and $\chi = 0$ or $\chi = 1$. So we obtain

$$\frac{v_1(z)}{v_2(z)} = \frac{a g_1(z) + b g_2(z)}{c g_1(z) + d g_2(z)} \quad \text{for } z \in D_{r(z_0)}(z_0)\,,$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. This yields

$$\frac{v_1(z)}{v_2(z)} = \frac{a + bg_2(z)/g_1(z)}{c + dg_2(z)/g_1(z)} \quad \text{in } D_{r(z_0)}(z_0)\,.$$

The function $\sigma(z) = (a + bz)/(c + dz)$ is a Möbius transformation and has consequently an inverse. Thus we get $g_2/g_1 = \sigma^{-1} \circ (v_2/v_1)$ in $D_{r(z_0)}(z_0)$. Since v_2/v_1 is meromorphic in G, g_2/g_1 has a meromorphic extension to $K_{r(z_0)}(z_0)$ and this means $\chi = 0$ in (II.22). This shows condition (2) of Theorem II.11 is fulfilled.

(b) We now turn to the if part. If the condition (1) and (2) are satisfied at each pole of $A(z)$, then by Theorem II.11 the quotient f of any two linearly independent solutions of the linear equation $\psi'' + A(z)\psi = 0$ has a meromorphic extension to G and by Lemma II.14 each of these functions solves $\mathcal{S}_f = 2A(z)$. ■

An obvious generalization of Remark II.7 is

Remark II.16
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$, and let f and g be meromorphic functions in G. Then $\mathcal{S}_f = \mathcal{S}_g = 2A(z)$ if and only if $f = \sigma \circ g$, where σ is a Möbius transformation.

Proof.
If $f = \sigma \circ g$ then $\mathcal{S}_f = \mathcal{S}_g$ follows as in the proof to Remark II.7.

Conversely, let $\mathcal{S}_f = \mathcal{S}_g = 2A(z)$. Then choose a disk $K \subseteq G$, where $A(z)$ is holomorphic. This implies $f = \sigma \circ g$ in K by Remark II.7 and hence we see $f = \sigma \circ g$ in G by the identity principle. ■

The last observation leads to

Remark II.17
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$ and let f be a meromorphic solution of the differential equation $\mathcal{S}_f = 2A(z)$ in G. Then every local solution of $\mathcal{S}_f = 2A(z)$ has a meromorphic extension to G and any two of them, say f and g, are related by $f = \sigma \circ g$, where σ is a Möbius transformation.

Now, if we combine Theorem II.11, Theorem II.15 and Lemma II.14, we arrive at the generalization of Theorem II.6 we have sought.

Theorem II.18
Let $G \subseteq \mathbb{C}$ be a simply connected domain and let $A(z)$ be a meromorphic function in G, which satisfies condition (1) and (2) of Theorem II.11 at every pole z_0 of A. Then the quotient $f = g_1/g_2$ of any two linearly independent local solutions g_1, g_2 of $\psi'' + A(z)\psi = 0$ has a meromorphic extension to G and is a solution of $\mathcal{S}_f = 2A(z)$. Conversely, if f is a (meromorphic) solution of $\mathcal{S}_f = 2A(z)$ in G, then there exist two linearly independent local solutions g_1, g_2 of $\psi'' + A(z)\psi = 0$, such that g_1/g_2 has a meromorphic extension to G and $f = g_1/g_2$ in G.

Proof.
The first part follows directly from Theorem II.11 and Lemma II.14. By Theorem II.15 and Remark II.17 all solutions of $\mathcal{S}_f = 2\,A(z)$ are meromorphic in G. To prove the representation claim, choose a disk $K \subseteq G$, where $A(z)$ is holomorphic. Then, by Theorem II.6, there exist two linearly independent holomorphic solutions g_1, g_2 in K, such that $f = g_1/g_2$ in K. By Theorem II.11 the quotient g_1/g_2 has a meromorphic extension to G and thus the assertion, $f = g_1/g_2$ in G, follows from the identity principle. ■

Theorem II.6 and Theorem II.18 show a strong connection between the linear differential equation $\psi'' + A(z)\psi = 0$ and the Schwarzian differential equation $\mathcal{S}_f = 2A(z)$. Both types of differential equations are also closely related to the Riccati differential equation $w' = A(z) + w^2$. Therefore Riccati differential equations are the topic of our next section.

II.3 Riccati differential equations

The object which attracts now our interest is the normalized Riccati differential equation

$$w' = A(z) + w^2\,.$$

In particular we focus on the interconnection of the Riccati differential equation $w' = A(z) + w^2$ with the second order differential equation $\psi'' + A(z)\psi = 0$. Further, we will study the different types of solutions of $w' = A(z) + w^2$, when A is a holomorphic function or a meromorphic function. However, before having a look at this in greater detail we like to begin with the following definition which fits into the content of Definition II.8 and II.13.

Definition II.19
Let $A(z)$ be a meromorphic function in a domain $G \subseteq \mathbb{C}$. We call a meromorphic function $w(z)$ in G a meromorphic solution of $w' = A(z) + w^2$ in G, if $w(z)$ satisfies $w'(z) = A(z) + w(z)^2$ in G. By a local solution of $w' = A(z) + w^2$ we mean a meromorphic solution of this differential equation in a subdomain $G' \subseteq G$.

We now relate the Riccati equation $w' = A(z) + w^2$ to the linear differential equation $\psi'' + A(z)\psi = 0$ and the Schwarzian differential equation $\mathcal{S}_f = 2A(z)$.

Remark II.20
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$.

(a) Let g be a local solution of $\psi'' + A(z)\psi = 0$ and assume $-g'/g$ has a meromorphic extension to a domain $G' \subseteq G$. Then the function $w := -g'/g$ is a meromorphic solution of $w' = A(z) + w^2$ in G'.

(b) Let f be a local solution of $\mathcal{S}_f = 2A(z)$ and assume f''/f' has a meromorphic extension to a domain $G' \subseteq G$. Then the function $w := f''/(2f')$ is a meromorphic solution of $w' = A(z) + w^2$ in G'.

In view of Remark II.20 (a) and Theorem II.5 we may suspect that the Riccati differential equation $w' = A(z) + w^2$, where $A(z)$ is holomorphic, does not need to have only holomorphic solutions. The following simple example illustrates this phenomenon.

Example II.21
Consider the initial value problem

$$\begin{aligned} w' &= w^2 \\ w(0) &= c, \ c \in \mathbb{C}\,, \end{aligned}$$

in $\mathbb{C}$*. Then the solutions are the meromorphic functions*

$$w(z) = -\frac{1}{z - \frac{1}{c}} \quad \text{if } \ c \in \mathbb{C}\backslash\{0\}\,,$$

and the entire function

$$w \equiv 0 \quad \text{if } \ c = 0\,.$$

In Example II.21 every local solution of $w' = A(z) + w^2$ has a meromorphic extension to any simply connected domain, where A is holomorphic. This is true in general and is the content of the next theorem.

Theorem II.22
Let $A(z)$ *be a holomorphic function in a simply connected domain* $G \subseteq \mathbb{C}$*. Then every local solution* w *of the differential equation* $w' = A(z) + w^2$ *can be continued analytically to a meromorphic solution in* G*.*

Proof.
Let w be a local solution of $w' = A(z) + w^2$ and assume w is meromorphic in the disk $K \subseteq G$. In K we choose a point z_0 such that w is finite, i.e. $w(z_0) = w_0$ with $w_0 \in \mathbb{C}$. Further, let c_1, c_2, c_3 be three distinct complex numbers with $c_j \neq w_0$ for $j = 1, 2, 3$.

Next, we have a look at the system of linear differential equations

$$\begin{aligned} P' &= A(z)Q \\ Q' &= -P. \end{aligned} \tag{II.23}$$

By Theorem II.3 and Theorem II.5 we can find three pairs, say (P_1, Q_1), (P_2, Q_2) and (P_3, Q_3), of holomorphic solutions of (II.23) in G which have the "initial values"

$$P_k(z_0) = c_k \quad \text{and} \quad Q_k(z_0) = 1 \quad \text{for} \quad k = 1, 2, 3\,.$$

The functions

$$w_k(z) := \frac{P_k(z)}{Q_k(z)}, \qquad k = 1, 2, 3,$$

are meromorphic in G and solutions of $w' = A(z) + w^2$ in G. This follows from

$$w_k' = \left(\frac{P_k}{Q_k}\right)' = \frac{P_k' Q_k - Q_k' P_k}{Q_k^2} = \frac{(A(z)Q_k)Q_k + P_k^2}{Q_k^2} = A(z) + w_k^2\,.$$

Because of our choice of the "initial values", the functions w_k are clearly distinct and consequently the cross ratio

$$F(z) = \frac{w_1(z) - w_3(z)}{w_1(z) - w(z)} : \frac{w_2(z) - w_3(z)}{w_2(z) - w(z)}$$

is a well defined meromorphic function in K. Since w_1, w_2, w_3 and w are solutions of $w' = A(z) + w^2$ in K we replace in the formula for F' the expressions $(w_j' - w_k')$ and $(w_j' - w')$ by $(w_j - w_k)(w_j + w_k)$ and $(w_j - w)(w_j + w)$, respectively, and obtain $F' \equiv 0$ in K. Thus F is a constant function in K, say $F \equiv c$, where $c \in \mathbb{C}\backslash\{0\}$, since $F(z_0) \neq 0, \infty$. This leads to

$$w = \frac{(w_1 - w_3)w_2 - cw_1(w_2 - w_3)}{(w_1 - w_3) - c(w_2 - w_3)} \quad \text{in } K,$$

where the right side of this equation is a meromorphic function in G. Therefore the function w has a meromorphic extension to G. Lastly, w solves the Riccati differential equation in G by the identity principle, since $w'(z) - w(z)^2 = A(z)$ in K. This completes the proof. ∎

An immediate consequence of Theorem II.22 is

Corollary II.23
Let $G \subseteq \mathbb{C}$ be a domain, let $A(z)$ be a meromorphic function in G and let γ be a path in G which avoids every pole of A. Then every local solution of $w' = A(z) + w^2$ has a meromorphic continuation along γ and is a solution of the differential equation there.

Now we turn to the more general case, when $A(z)$ is a meromorphic function in the Riccati differential equation $w' = A(z) + w^2$. The next two examples follow Example II.12 and reveal that a Riccati differential equation can have meromorphic and non–meromorphic solutions at the same time, and that also only meromorphic solutions can occur.

Example II.24

(a) *We put $A(z) = 1/(4z^2)$ for $z \in \mathbb{C}$ and consider the differential equation*

$$w' = \frac{1}{4z^2} + w^2 \,. \tag{II.24}$$

Then the meromorphic function

$$w_1(z) = -\frac{1}{2z}$$

is a solution of (II.24) in $\mathbb{C}$.

On the other hand the function

$$w_2(z) = (\log z + 3)/(-2z - 2z \log z)$$

is meromorphic in $\mathbb{C}\backslash[0,\infty)$ without a meromorphic extension to $\mathbb{C}$, but it solves (II.24) in $\mathbb{C}\backslash[0,\infty)$.

(b) *Now let $A(z) = -3/(4z^2)$ in $\mathbb{C}$. Then the solutions of the Riccati differential equation*

$$w' = -\frac{3}{4z^2} + w^2$$

are given by the one–parameter family of meromorphic functions in $\mathbb{C}$:

$$w_1(z) = -\frac{3}{2z} \quad \cup \quad \omega_c(z) = -\frac{3}{2z} + \frac{2}{2cz^3 + z}\,, \ c \in \mathbb{C}\,.$$

Remark II.20 (a) and Example II.24 give rise to assume that under some conditions the Riccati differential equation $w' = A(z) + w^2$ has *only* meromorphic solutions.

The next lemma provides us in some sense with such a criterion.

Lemma II.25
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$, and let w_1, w_2 be two distinct meromorphic solutions of $w' = A(z) + w^2$ in G. If every pole of w_1 is simple and the corresponding residue of $2w_1$ is an integer, then the Riccati differential equation has a one–parameter–family $w_1 \cup (\omega_c)_{c\in\mathbb{C}}$ of meromorphic solutions in G such that to every (local) solution $\omega \neq w_1$ there exists a $c \in \mathbb{C}$ with $\omega = \omega_c$. In particular, every solution of $w' = A(z) + w^2$ is meromorphic in G.

Proof.
Let's define the function v_0 in G by

$$v_0(z) = \frac{1}{w_1(z) - w_2(z)}\,.$$

Then v_0 is a meromorphic solution of the linear differential equation

$$v' + 2w_1(z)v = 1 \tag{II.25}$$

in G, since

$$\begin{aligned} v_0' + 2w_1(z)\, v_0 &= -\frac{w_1'(z) - w_2'(z)}{(w_1(z) - w_2(z))^2} + \frac{2w_1(z)}{w_1(z) - w_2(z)} \\ &= -\frac{w_1(z) + w_2(z)}{w_1(z) - w_2(z)} + \frac{2w_1(z)}{w_1(z) - w_2(z)} \equiv 1\,. \end{aligned}$$

In the last but one step we used the fact that both w_1 and w_2 are solutions of $w' = A(z) + w^2$ and hence the identity $w_1'(z) - w_2'(z) = w_1(z)^2 - w_2(z)^2$ is valid. Now recall that in a simply connected domain a meromorphic function which has only simple poles and integers as residues can be represented by the formal logarithmic derivative of another meromorphic function[2]. Therefore the function $2w_1$ assumes the form

$$2w_1 = \frac{y'}{y}$$

in G, where y is a meromorphic function in G. Thus equation (II.25) reduces to

$$v' + \frac{y'}{y}v = 1\,. \tag{II.26}$$

The functions

$$v_c(z) := v_0(z) + \frac{c}{y(z)}\,, \quad c \in \mathbb{C},$$

form a family of meromorphic solutions of (II.26) as v_0 is a particular solution of (II.26) and the functions $c\,y^{-1}$, $c \in \mathbb{C}$, are solutions to the homogeneous differential equation

$$v' + \frac{y'}{y}v \equiv 0\,.$$

[2]see for instance [49], p. 193

Now we are going back to the Riccati differential equation.
The meromorphic functions

$$\omega_c(z) = w_1(z) - \frac{1}{v_0(z) + \frac{c}{y(z)}}\,, \quad c \in \mathbb{C}\,,$$

form a one–parameter–family in G. As a matter of course the functions ω_c, $c \in \mathbb{C}$, are distinct, since the functions v_c are pairwise distinct. Further, ω_c, $c \in \mathbb{C}$, satisfies the Riccati differential equation $w' = A(z) + w^2$:

$$\begin{aligned}
\omega_c' &= w_1' - \frac{-1}{(v_0 + \frac{c}{y})^2}\left(v_0' - \frac{cy'}{y^2}\right) \\
&= A(z) + w_1^2 + \frac{1}{(v_0 + \frac{c}{y})^2}\left(1 - \frac{y'}{y}\left(v_0 + \frac{c}{y}\right)\right) \\
&= A(z) + w_1^2 + \frac{1}{(v_0 + \frac{c}{y})^2}\left(1 - 2w_1\left(v_0 + \frac{c}{y}\right)\right) \\
&= A(z) + \left(w_1 - \frac{1}{v_0 + \frac{c}{y}}\right)^2 .
\end{aligned}$$

Hence we found a one–parameter–family $(w_1) \cup (\omega_c)_{c\in\mathbb{C}}$ of meromorphic functions in G, which solve the equation $w' = A(z) + w^2$.

For the second assertion, let $\omega \neq w_1$ be a (local) solution of $w' = A(z) + w^2$. Then the function $v(z) = (w_1(z) - \omega(z))^{-1}$ is a (local) solution of (II.25) for some $c \in \mathbb{C}$. Thus we can conclude

$$\frac{1}{w_1(z) - \omega(z)} = v_0(z) + \frac{c}{y(z)}$$

for some $c \in \mathbb{C}$. Now it is obvious that ω is meromorphic in G and

$$\omega(z) = w_1(z) - \frac{1}{v_0(z) + \frac{c}{y(z)}} = \omega_c(z)\,.$$

■

We have as an immediate consequence of Theorem II.22 and Lemma II.25 the following

Proposition II.26
Let $F \subseteq G \subseteq \mathbb{C}$ be simply connected domains and let A be a meromorphic function in G such that the Riccati differential equation $w' = A(z) + w^2$ admits a one–parameter–family of meromorphic solutions in G, say $\mathcal{G} := (w_1) \cup (\omega_c)_{c\in\mathbb{C}}$. Then $\mathcal{G}|_F$ is a family of meromorphic solutions of $w' = A(z) + w^2$ in F and every meromorphic solution of $w' = A(z) + w^2$ in F belongs to $\mathcal{G}|_F$.

Theorem II.9, in particular equations (II.10), (II.11) and (II.8), combined with Remark II.20 (a) gives a necessary condition for a meromorphic A such that all solutions of the differential equation $w' = A(z) + w^2$ are meromorphic, namely condition (a) of Theorem II.11 must be fulfilled at every pole z_0 of A. On the other hand, this condition is not sufficient. The proof of our next theorem clarifies this a bit further.

Theorem II.27
Let $A(z)$ be holomorphic in a simply connected domain $G \subseteq \mathbb{C}$ save a double pole at $z_0 \in G$ and suppose $A(z)$ has at $z = z_0$ the Laurent series expansion

$$A(z) = \frac{1-n^2}{4(z-z_0)^2} + \cdots \;, n \in \mathbb{N},\, n \geq 2\,.$$

Then the Riccati differential equation $w' = A(z) + w^2$ admits either exactly one meromorphic solution in G or a one–parameter–family of meromorphic solutions $w_1 \cup (\omega_c)_{c\in\mathbb{C}}$ such that to every (local) solution $\omega \neq w_1$ there exists a number $c \in \mathbb{C}$ with $\omega = \omega_c$.

Proof.
Let's look at the corresponding linear differential equation $\psi'' + A(z)\,\psi = 0$ of $w' = A(z) + w^2$. We write this differential equation in the form

$$\psi'' + \frac{h(z)}{(z-z_0)^2}\psi = 0\,, \tag{II.27}$$

where $h(z)$ is holomorphic in G and $h(z_0) = (1-n^2)/4$, $n \in \mathbb{N}$, $n \geq 2$.

Going back to Theorem II.9 we compute for the roots of the characteristic equation

$$\rho = \rho_{1/2} = \frac{1 \pm n}{2}\,.$$

Accordingly, there exists a solution base $\{g_1, g_2\}$ of the form

$$\begin{aligned} g_1(z) &= (z-z_0)^{\frac{1+n}{2}}\,\phi_1(z), \\ g_2(z) &= \chi g_1(z)\log(z-z_0) + (z-z_0)^{\frac{1-n}{2}}\,\phi_2(z), \end{aligned} \tag{II.28}$$

in a slit disk $D_r(z_0) \subset G$ for some $r > 0$, where ϕ_1 and ϕ_2 are holomorphic functions in $K_r(z_0)$ with $\phi_1(z_0) \neq 0$ and $\phi_2(z_0) \neq 0$. Further, $\chi = 0$ or $\chi = 1$.

Now we will show that there is in any case at least one meromorphic solution of the Riccati equation $w' = A(z) + w^2$. This meromorphic solution comes from the solution $g_1(z) = (z-z_0)^{\frac{1+n}{2}}\,\phi_1(z)$ of the linear equation $\psi'' + A(z)\psi = 0$. We see the quotient $-g_1'/g_1$ is defined in the slit disk $D_r(z_0)$, but has a meromorphic extension to the whole disk $K_r(z_0)$, since the right side of

$$-\frac{g_1'}{g_1} = -\frac{1+n}{2(z-z_0)} - \frac{\phi_1'}{\phi_1}$$

is meromorphic in this disk. Note, the meromorphic function

$$w_1 = -\frac{1+n}{2(z-z_0)} - \frac{\phi_1'}{\phi_1}$$

is a solution of $w' = A(z) + w^2$ in $K_r(z_0)$. By Corollary II.23 and the Monodromy Theorem w_1 can be extended to a meromorphic solution of $w' = A(z) + w^2$ in all of G.
If there is more than one meromorphic solution depends whether $\chi = 0$ or $\chi = 1$.

If $\chi = 0$ in (II.28) then $-g_2'/g_2$ leads similarly as above to a second meromorphic solution of $w' = A(z) + w^2$. Since w_1 has only simple poles and the residues of $2w_1$ are integers at these poles (this follows from a straightforward pole consideration, see also Remark II.29) the hypotheses of Lemma II.25 are fulfilled, so there exists a one–parameter–family of meromorphic solutions with the desired properties.

If $\chi = 1$ in (II.28), then $-g_2'/g_2$ has at z_0 a logarithmic singularity. Therefore, $-g_2'/g_2$ is a meromorphic solution in the slit disk $D_r(z_0)$ of $w' = A(z) + w^2$, which however cannot be extended to a meromorphic function in the whole disk $K_r(z_0)$. Again by Lemma II.25 and the properties of w_1 we conclude that the Riccati differential equation admits exactly one meromorphic solution in this case. ■

From Theorem II.27 and Corollary II.23 we derive

Corollary II.28
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$ and suppose for every pole z_0 of A there exists a disk $K_{r(z_0)}(z_0) \subseteq G$ such that $w' = A(z) + w^2$ has only meromorphic solutions in $K_{r(z_0)}(z_0)$. Then every local solution of $w' = A(z) + w^2$ has a meromorphic extension to G, in other words every solution of $w' = A(z) + w^2$ is meromorphic in G.

The last remark of this section describes the possible poles and the corresponding residues of meromorphic solutions w of $w' = A(z) + w^2$.

Remark II.29
Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$, and let w be a (local) solution of $w' = A(z) + w^2$ in a neighborhood U of $z_0 \in G$.

(1) If $A(z)$ is holomorphic in $z_0 \in G$, then either w is holomorphic in z_0 or w has a simple pole at z_0 with residue -1.

(2) If $A(z)$ has a pole of order 2 at z_0 such that

$$A(z) = \frac{1-n^2}{4(z-z_0)^2} + \cdots\,,\ n \in \mathbb{N},\ n \geq 2\,,$$

then

(a) w has in z_0 a simple pole with residue $-(1+n)/2$, if there exists only one meromorphic solution w of $w' = A(z) + w^2$ in U.

(b) every solution w of the Riccati equation has in z_0 a simple pole, if every solution of $w' = A(z) + w^2$ is meromorphic. In this case, only one solution has residue $-(n+1)/2$ and all others have residue $(n-1)/2$ at z_0.

II.4 Survey

We have seen in the last two sections that linear differential equations of second order, Schwarzian differential equations and Riccati differential equations are closely related to each other. Therefore it is sometimes an advantage to consider a corresponding differential equation rather than the original equation to get information about the solutions. This is the reason, why we like to give a short survey.

Let $A(z)$ be a meromorphic function in a simply connected domain $G \subseteq \mathbb{C}$ and let $A(z)$ satisfy condition (1) and (2) of Theorem II.11 at every pole z_0 of A. Then we have the following diagram, where g, g_1, g_2 denote solutions of $\psi'' + A(z)\psi = 0$, f stands for a meromorphic solution of $\mathcal{S}_f = 2A(z)$ in G and w for a meromorphic solution of $w' = A(z) + w^2$ in G:

$$
\begin{tikzcd}
\psi'' + A(z)\psi = 0 \arrow[dd, Rightarrow, "w:=-\frac{g'}{g}"'] \arrow[dr, Leftrightarrow, "f=\frac{g_1}{g_2}\quad \text{Theorem II.6 and Theorem II.18}"] & \\
& \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 2A(z) \arrow[dl, Rightarrow, "w=\frac{1}{2}\frac{f''}{f'}"] \\
w' = A(z) + w^2 \arrow[uu, dotted] \arrow[ur, dotted] &
\end{tikzcd}
$$

– Chapter III –

Some Results from Potential Theory

III.1 Introduction

The purpose of this chapter is to discuss some results from potential theory, which will provide a basis for solving Schwarz–Picard's Problem I.9 for plane domains, i.e. for proving the existence of pseudo–metrics of constant curvature -4 with prescribed zeros, see Section IV.2, in particular Theorem IV.18.

Here, we focus on

Problem III.1 (The Dirichlet problem for the Poisson equation)
Let $G \subset \mathbb{C}$ be a bounded regular domain, i.e. a bounded domain with Green's function $g(z,\xi)$, let $q : G \longrightarrow \mathbb{R}$ be a bounded and locally Hölder continuous function, and let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function. Then the Dirichlet problem for the Poisson equation $\Delta v = q$ consists in finding a function $v \in C(\overline{G}) \cap C^2(G)$ such that

$$\begin{aligned} \Delta v &= q \quad \textit{in} \quad G \\ v &= \Phi \quad \textit{on} \quad \partial G\,. \end{aligned} \tag{III.1}$$

In view of the maximum principle for harmonic functions it is clear that a solution to Problem III.1, if there exists one, is uniquely determined. The proof for the existence of a solution to Problem III.1 is more involved and will occupy almost the rest of this chapter.

Suppose for a moment there exists a solution $v \in C(\overline{G}) \cap C^2(G)$ of Problem III.1. Then we can find an explicit formula for v in the following way.

Let h be the solution of the classical Dirichlet boundary value problem for the Laplace equation, i.e. $h : G \longrightarrow \mathbb{R}$ belongs to $C(\overline{G}) \cap C^2(G)$ and is a solution to

$$\begin{aligned} \Delta h &= 0 \quad \text{in} \quad G \\ h &= \Phi \quad \text{on} \quad \partial G\,. \end{aligned}$$

Then the function $\varpi = v - h$ is well–defined and a solution of the boundary value problem

$$\begin{aligned} \Delta \varpi &= q \quad \text{in} \quad G \\ \varpi &= 0 \quad \text{on} \quad \partial G\,. \end{aligned}$$

If we additionally assume $\varpi \in C^1(\overline{G}) \cap C^2(G)$, then Green's second formula[1] gives for ϖ the representation formula

$$\varpi(z) = -\frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi)\, d\sigma_\xi\,, \tag{III.2}$$

[1] $\displaystyle\iint\limits_G (u\Delta\omega - \omega\Delta u)\, d\sigma = \int\limits_{\partial G} \left(u\frac{\partial \omega}{\partial n} - \omega \frac{\partial u}{\partial n}\right) |d\xi|$ for $u, \omega \in C^1(\overline{G}) \cap C^2(G)$.

where $g(z,\xi)$ is Green's function for the domain $G \subset \mathbb{C}$ and $d\sigma_\xi$ denotes euclidean area element with respect to ξ. For a quick proof of this fact, see for instance [23, p.35/36]. Therefore our solution v to Problem III.1 is given by

$$v(z) = h(z) - \frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi) d\sigma_\xi \,. \tag{III.3}$$

As might be expected, this reasoning can be reversed, and the function v defined by formula (III.3) is indeed a solution to Problem III.1.

Theorem III.2
Let $G \subset \mathbb{C}$ a bounded and regular domain, let $q : G \longrightarrow \mathbb{R}$ be a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$, and let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function. If $g(z,\xi)$ denotes Green's function for G and h the solution of the classical Dirichlet problem for G with boundary function Φ, i.e. $\Delta h = 0$ in G and $h = \Phi$ on ∂G, then the function

$$v(z) := h(z) - \frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi) d\sigma_\xi \tag{III.4}$$

belongs to $C(\overline{G}) \cap C^2(G)$ and is a solution of the boundary value problem

$$\begin{aligned} \Delta v &= q \quad \text{in} \quad G \\ v &= \Phi \quad \text{on} \quad \partial G \,. \end{aligned}$$

Remark III.3
We like to point out the function v in (III.4) is well–defined. This follows from the facts that for a bounded regular domain G on the one hand the integral

$$\frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi) d\sigma_\xi$$

exists for every $z \in G$ if $q : G \longrightarrow \mathbb{R}$ is a bounded and integrable function, and on the other hand the Dirichlet problem for the Laplace equation is solvable for every continuous boundary function.

Before we start with the proof of Theorem III.2 in Section III.3, we compile in Section III.2 some notation, definitions and basic facts, which will be helpful throughout this chapter. In Section III.4 we discuss two further rather technical results from potential theory, which will come into play in the next chapter.

III.2 Preliminaries

One may wonder, why we suppose the function q to be locally Hölder continuous in Theorem III.2, since the integral in (III.4) is already well–defined, if q is only assumed to be bounded and integrable on G, see Remark III.3. The main point here is that v should become a C^2–function, which, as we shall see, can be guaranteed only if q is locally Hölder continuous in G.

We now begin by recalling the definition of Hölder continuous functions.

Definition III.4
Let $G \subseteq \mathbb{C}$ be a domain in the complex plane. A function $q : G \longrightarrow \mathbb{R}$ is called locally Hölder continuous in G with exponent α, $0 < \alpha \leq 1$, if

$$[q]_{\alpha;\Omega} = \sup_{\substack{z,\xi\in\Omega \\ z\neq\xi}} \frac{|q(z) - q(\xi)|}{|z - \xi|^\alpha}$$

is finite for every compact subset $\Omega \subset G$.

Example III.5
Let $G \subseteq \mathbb{C}$ be a domain, $z_0 \in G$, and $\alpha \in (0, 1]$. Then the function

$$f(z) = |z - z_0|^\alpha$$

is locally Hölder continuous in G with exponent α.

A simple, but important observation is

Remark III.6
Let $G \subseteq \mathbb{C}$ be a domain and let $f : G \longrightarrow \mathbb{R}$, $g : G \longrightarrow \mathbb{R}$ be locally Hölder continuous functions with exponent α, $0 < \alpha \leq 1$ and β, $0 < \beta \leq 1$, respectively. Then the product $f \cdot g$ is a locally Hölder continuous function in G with exponent $\gamma = \min\{\alpha, \beta\}$.

We next turn to a discussion of Green's function.

Definition III.7
Let $G \subseteq \mathbb{C}$ be a domain. Then we call a function $g : G \times G \longrightarrow [-\infty, \infty]$ Green's function of G, if it has the following properties:

(1) For every $z \in G$ the function $\xi \mapsto g(z, \xi) + \log|z - \xi|$ is harmonic in G.

(2) For every $z \in G$ the function $\xi \mapsto g(z, \xi)$ is harmonic in $G\backslash\{z\}$.

(3) For every $z \in G$ and for every $\tau \in \partial G$ the limit relation $\lim\limits_{\xi\longrightarrow\tau} g(z, \xi) = 0$ holds.

Remark III.8

(a) For a domain $G \subseteq \mathbb{C}$, there doesn't need to exist a Green's function. But if there exists one at all, then it is uniquely determined.

(b) If $G \subsetneq \mathbb{C}$ is a simply connected domain, then Green's function exists and is given by the formula

$$g(z, \xi) = -\log\left|\frac{f(z) - f(\xi)}{1 - \overline{f(\xi)}f(z)}\right|,$$

where $f : G \longrightarrow \mathbb{D}$ is a conformal map onto $\mathbb{D}$.

For the following let $g(z, \xi)$ be Green's function for a domain $G \subseteq \mathbb{C}$. Then

(c) g is non–negative, i.e. $g(z, \xi) \geq 0$ for all $z, \xi \in G$.

(d) g is symmetric, i.e. $g(z, \xi) = g(\xi, z)$ for all $z, \xi \in G$.

(e) *g has the representation $g(z,\xi) = -\log|z-\xi| + \gamma(z,\xi)$, where $z \mapsto \gamma(z,\xi)$ is a harmonic function in G for each fixed $\xi \in G$ and $\xi \mapsto \gamma(z,\xi)$ is a harmonic function in G for each fixed $z \in G$.*

Since we only consider bounded domains $G \subset \mathbb{C}$, the following definition is appropriate.

Definition III.9
The diameter of a bounded set $G \subset \mathbb{C}$ is defined by

$$d_G := \sup_{x,y \in G} |x-y| \,.$$

At the end of this section we like to mention the following elementary inequalities which will be indispensable to the proof of Theorem III.2:

$$\big|\log|x|\big| \le |\log a| + |\log b| \quad \text{if} \quad 0 < a \le |x| \le b\,, \tag{III.5}$$

$$\big|\log|x|\big| \le |\log a| \quad \text{if} \quad 0 < \frac{1}{a} \le |x| \le a\,, \tag{III.6}$$

$$\log|x| \le |x| - 1 \quad \text{if} \quad 1 \le |x|\,, \tag{III.7}$$

$$\log|x| \ge 1 - \frac{1}{|x|} \quad \text{if} \quad 0 < |x| \le 1\,. \tag{III.8}$$

Now we are prepared to set about proving Theorem III.2.

III.3 The solution of the Poisson equation – Proof of Theorem III.2

In view of later applications, we shall prove the following slight extension of Theorem III.2.

Theorem III.10
Let $G \subset \mathbb{C}$ be a bounded regular domain, let $q : G \longrightarrow \mathbb{R}$ be a bounded and integrable function, and let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function. Further, let h be the solution of the classical Dirichlet problem for G with boundary function Φ and denote by $g(z,\xi)$ Green's function for G. Then the function

$$v(z) := h(z) - \frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi) \, d\sigma_\xi$$

belongs to $C(\overline{G}) \cap C^1(G)$ and $v|_{\partial G} \equiv \Phi$. If, in addition, q is locally Hölder continuous with exponent $\alpha \in (0,1]$ in some subdomain $\mathcal{O} \subseteq G$, then $v \in C^2(\mathcal{O})$ and $\Delta v = q$ in $\mathcal{O}$.

The proof of Theorem III.10 will be broken down into several steps.

As h is harmonic in G and continuous on $\overline{G}$ we only need to focus on the function

$$\varpi(z) = -\frac{1}{2\pi} \iint\limits_G g(z,\xi) q(\xi) \, d\sigma_\xi,$$

and have to prove

$$\varpi \in C^1(G) \quad \text{and} \quad \varpi \in C^2(\mathcal{O}) \text{ with } \Delta\varpi = q \text{ in } \mathcal{O}, \tag{III.9}$$

and

$$\lim_{z \longrightarrow \tau} \varpi(z) = 0 \text{ for every } \tau \in \partial G. \tag{III.10}$$

Since $g(z,\xi) = -\log|z-\xi| + \gamma(z,\xi)$, we have

$$\varpi(z) = \frac{1}{2\pi} \iint\limits_G \log|z-\xi| q(\xi) d\sigma_\xi - \frac{1}{2\pi} \iint\limits_G \gamma(z,\xi) q(\xi) d\sigma_\xi \,,$$

and (III.9) follows from the following two results.

Theorem III.11
Let $G \subset \mathbb{C}$ be a bounded regular domain and let $q : G \longrightarrow \mathbb{R}$ be a bounded and integrable function. Then the Newton potential

$$w(z) := \frac{1}{2\pi} \iint\limits_G \log|z-\xi|\, q(\xi)\, d\sigma_\xi \tag{III.11}$$

of q belongs to $C^1(\overline{G})$. If, in addition, q is locally Hölder continuous with exponent α, $0 < \alpha \leq 1$, in a subdomain $\mathcal{O} \subseteq G$, then $w \in C^2(\mathcal{O})$ and $\Delta w = q$ in $\mathcal{O}$.

Theorem III.12
Let $G \subset \mathbb{C}$ be a bounded regular domain with Green's function $g(z,\xi) = -\log|z-\xi| + \gamma(z,\xi)$ and let $q : G \longrightarrow \mathbb{R}$ be a bounded and integrable function. Then the function

$$H(z) := \frac{1}{2\pi} \iint\limits_G \gamma(z,\xi) q(\xi) d\sigma_\xi \tag{III.12}$$

belongs to $C^2(G)$ and $\Delta H = 0$ in G.

Finally, (III.10) is established in

Theorem III.13
Let $G \subset \mathbb{C}$ be a bounded regular domain, let $g(z,\xi)$ be Green's function of G, and let $q : G \longrightarrow \mathbb{R}$ be a bounded and integrable function. Then

$$\lim_{z \longrightarrow \tau} \iint\limits_G g(z,\xi)\, q(\xi)\, d\sigma_\xi = 0$$

for every point $\tau \in \partial G$.

Thus, Theorem III.10 follows from Theorem III.11, Theorem III.12 and Theorem III.13.

For the proof of Theorem III.11 we need the following technical lemma, which we shall not prove here. Notice that the partial derivatives of a function of the two real variables x_1 and x_2 will be denoted by

$$D_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1,2 \quad \text{and} \quad D_{lj} = \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial x_j}\right) \quad \text{for } l,j = 1,2\,.$$

Lemma III.14
Let $z = x_1 + ix_2$ and $\xi = \zeta_1 + i\zeta_2$. Then the first and second derivatives with respect to x_1, x_2 of the function $z \longrightarrow \log|z-\xi|$ are given by

$$D_j \log|z-\xi| = \frac{x_j - \zeta_j}{|z-\xi|^2} \qquad \textit{for} \quad j \in \{1,2\}\,,$$

and

$$D_{lj} \log|z-\xi| = -\frac{2(x_1-\zeta_1)(x_2-\zeta_2)}{|z-\xi|^4} \qquad \textit{for} \quad l,j \in \{1,2\}\,, \quad \textit{if} \quad l \neq j\,,$$

as well as

$$D_{ll} \log|z-\xi| = (-1)^l \frac{(x_1-\zeta_1)^2 - (x_2-\zeta_2)^2}{|z-\xi|^4} \qquad \textit{for} \quad l \in \{1,2\}\,. \tag{III.13}$$

Moreover, we have

$$|D_j \log|z-\xi|| \leq \frac{1}{|z-\xi|} \qquad \textit{for} \quad j \in \{1,2\}\,, \tag{III.14}$$

and

$$|D_{lj} \log|z-\xi|| \leq \frac{2}{|z-\xi|^2} \qquad \textit{for} \quad l,j \in \{1,2\}\,, \tag{III.15}$$

respectively.

Proof of Theorem III.11. (cf. [19])
First of all we note the function w is even well–defined on $\overline{G}$. Since q is bounded on G, we have $M := \sup\limits_{\xi \in G} |q(\xi)| < \infty$.

In a first step we like to show $w \in C^1(\overline{G})$, if q is bounded and integrable on G. This will be achieved by approximating w with $C^1(\overline{G})$–functions in the Banach space $(C^1(\overline{G}), ||\cdot||_{C^1})$. The functions which will do the job are given by

$$w_\varepsilon(z) = \frac{1}{2\pi} \iint\limits_G \log|z-\xi|\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) q(\xi)\, d\sigma_\xi, \qquad \varepsilon > 0,$$

where $\eta : \mathbb{R} \longrightarrow [0,1]$ is a continuously differentiable function, such that

$$\eta(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ 1 & \text{if } t \geq 2, \end{cases} \qquad \text{and} \qquad 0 \leq \eta'(t) \leq 2,\ t \in \mathbb{R}\,.$$

Clearly, every w_ε, $\varepsilon > 0$, is a well–defined function on $\overline{G}$. Moreover,

- $z \mapsto \log|z-\xi|\, \eta\left(\dfrac{|z-\xi|}{\varepsilon}\right) q(\xi) \in C^1(\overline{G})$ for every $\xi \in G$,
- $\xi \mapsto \log|z-\xi|\, \eta\left(\dfrac{|z-\xi|}{\varepsilon}\right) q(\xi)$ is integrable over G for every $z \in \overline{G}$,

- the estimates

  $$\left|\log|z-\xi|\,\eta\left(\frac{|z-\xi|}{\varepsilon}\right)q(\xi)\right| \le M\left(|\log\varepsilon| + |\log d_G|\right)$$

  and

  $$\left|D_j\left(\log|z-\xi|\,\eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right)q(\xi)\right| \le M\left(\frac{1}{\varepsilon}+\frac{2}{\varepsilon}\Big(|\log\varepsilon| + |\log d_G|\Big)\right), \quad j=1,2\,,$$

  are valid for $z,\xi\in G$.

Consequently, $w_\varepsilon \in C^1(\overline{G})$ for every $\varepsilon > 0$, see, for instance, [14].

To prove $w\in C(\overline{G})$ it suffices to show $w_\varepsilon \longrightarrow w$ uniformly in $\overline{G}$ as ε tends to 0. This, in turn, is a consequence of the following estimate for $z\in\overline{G}$:

$$\begin{aligned}|w_\varepsilon(z)-w(z)| &\le \frac{1}{2\pi}\iint\limits_{G\cap K_{2\varepsilon}(z)} \big|\log|z-\xi|\big|\,|q(\xi)|\,d\sigma_\xi \le M\frac{1}{2\pi}\iint\limits_{K_{2\varepsilon}(z)}\big|\log|z-\xi|\big|\,d\sigma_\xi \\ &\le M\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\varepsilon}|\log\rho|\;\rho\,d\rho\,d\varphi \le M\varepsilon^2\left(1+2\,|\log(2\varepsilon)|\right).\end{aligned}$$

Similarly, we can deduce $w\in C^1(\overline{G})$ as soon as we know

$$D_jw_\varepsilon(z) = \frac{1}{2\pi}\iint\limits_G D_j\left(\log|z-\xi|\eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right)q(\xi)\,d\sigma_\xi$$

converges uniformly in $\overline{G}$ to

$$v_j(z) = \frac{1}{2\pi}\iint\limits_G D_j\left(\log|z-\xi|\right)q(\xi)\,d\sigma_\xi{}^2\,,$$

for $j\in\{1,2\}$ as ε goes to 0. This also implies $D_jw(z) = v_j(z)$ for $z\in\overline{G}$ and $j\in\{1,2\}$. To check the above convergence claim we just observe for $z\in\overline{G}$

$$\begin{aligned}&|v_j(z)-D_jw_\varepsilon(z)| \le \frac{1}{2\pi}\iint\limits_{K_{2\varepsilon}(z)}\left|D_j\left\{\log|z-\xi|\left(1-\eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right)\right\}q(\xi)\right|\,d\sigma_\xi \\ &\le M\cdot\frac{1}{2\pi}\iint\limits_{K_{2\varepsilon}(z)}\left(\left|\left(D_j\log|z-\xi|\right)\left(1-\eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right)\right| + \left|\log|z-\xi|\cdot D_j\eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right|\right)\,d\sigma_\xi \\ &\le M\cdot\frac{1}{2\pi}\iint\limits_{K_{2\varepsilon}(z)}\left(\frac{1}{|z-\xi|}+\big|\log|z-\xi|\big|\cdot\frac{2}{\varepsilon}\right)\,d\sigma_\xi = M\int\limits_0^{2\varepsilon}\left(\frac{1}{\rho}+\frac{2}{\varepsilon}\,|\log\rho|\right)\rho\,d\rho \\ &\le M\cdot 4\varepsilon\left(1+|\log(2\varepsilon)|\right).\end{aligned}$$

[2]Note, v_j is a well–defined function on $\overline{G}$ because of estimate (III.14).

In our next step we will draw our attention to the proof of $w \in C^2(\mathcal{O})$ and $\Delta w = q$ in $\mathcal{O}$, if we now suppose q is locally Hölder continuous with exponent $\alpha \in (0,1]$ in $\mathcal{O}$. The path we will follow is similar to the preceding one, i.e. we will find functions $v_{j\varepsilon}$ which converge uniformly to $D_j w = v_j$ in $\overline{G}$ and whose derivatives converge locally uniformly in $\mathcal{O}$.

Since we have to apply the divergence theorem, we choose a disk K so large that $K \supseteq G$ and $\operatorname{dist}(\partial G, \partial K) > 1$ holds. Further, we extend q to K by setting $q(\xi) = 0$ for $\xi \in K \backslash G$.

Now we define functions

$$v_{j\varepsilon}(z) = \frac{1}{2\pi} \iint\limits_G (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) q(\xi)\, d\sigma_\xi\,, \quad \varepsilon > 0\,,$$

where η is as above. Taking into account the easily verified facts that

- $z \mapsto (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) q(\xi) \in C^1(\overline{G})$ for every $\xi \in G$,
- $\xi \mapsto (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) q(\xi)$ is integrable for every $z \in \overline{G}$, and
- the estimates

$$\left|(D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) q(\xi)\right| \le M \cdot \frac{1}{\varepsilon},$$

$$\left|D_l \left\{(D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right\} q(\xi)\right| \le M \cdot \frac{4}{\varepsilon^2}$$

  are valid for every z, ξ in G and $l, j \in \{1,2\}$,

we conclude $v_{j\varepsilon} \in C^1(\overline{G})$.
In view of the estimate

$$\begin{aligned}
|v_j(z) - v_{j\varepsilon}(z)| &\le \frac{1}{2\pi} \iint\limits_{K_{2\varepsilon}(z)} \left|(D_j \log|z-\xi|) \left(1 - \eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right) q(\xi)\right| d\sigma_\xi \\
&\le M \cdot \int_0^{2\varepsilon} \frac{1}{\rho}\, \rho\, d\rho = M \cdot 2\varepsilon
\end{aligned}$$

the functions $v_{j\varepsilon}$ converge uniformly to $v_j = D_j w$ in $\overline{G}$ as ε approaches 0 for $j \in \{1,2\}$.

We will move on by showing that the derivatives of $v_{j\varepsilon}$, namely $D_l v_{j\varepsilon}(z)$, $l, j \in \{1,2\}$, converge locally uniformly in $\mathcal{O}$ to the functions

$$v_{lj}(z) = \frac{1}{2\pi} \iint\limits_K (D_{lj} \log|z-\xi|)\,(q(\xi) - q(z))\, d\sigma_\xi - \frac{1}{2\pi} q(z) \int\limits_{\partial K} (D_j \log|z-\xi|) \cdot n_l(\xi)\, |d\xi|\,,$$

where $(n_1(\xi), n_2(\xi))^T$ is the unit outward normal at the point $\xi \in \partial K$.

Note, v_{lj}, $l, j \in \{1,2\}$, are well–defined functions on $\mathcal{O}$ by estimate (III.15) and the local Hölder continuity of q in $\mathcal{O}$. Now we compute the first derivatives of $v_{j\varepsilon}$, i.e. $D_l v_{j\varepsilon}$ for $l = 1, 2$:

$$\begin{aligned}
D_l v_{j\varepsilon}(z) &= \frac{1}{2\pi} \iint\limits_G D_l \left\{ (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \right\} q(\xi)\, d\sigma_\xi \\
&= \frac{1}{2\pi} \iint\limits_K D_l \left\{ (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \right\} (q(\xi) - q(z))\, d\sigma_\xi \\
&\quad + \frac{1}{2\pi} q(z) \iint\limits_K D_l \left\{ (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \right\} d\sigma_\xi \\
&= \frac{1}{2\pi} \iint\limits_K D_l \left\{ (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \right\} (q(\xi) - q(z))\, d\sigma_\xi \\
&\quad - \frac{1}{2\pi} q(z) \int\limits_{\partial K} (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \cdot n_l(\xi)\, |d\xi|.
\end{aligned}$$

In the last step we applied the divergence theorem. Thus, if $2\varepsilon < 1 < \operatorname{dist}(\partial G, \partial K)$, we get

$$\begin{aligned}
D_l v_{j\varepsilon}(z) &= \frac{1}{2\pi} \iint\limits_K D_l \left\{ (D_j \log|z-\xi|)\, \eta\left(\frac{|z-\xi|}{\varepsilon}\right) \right\} (q(\xi) - q(z))\, d\sigma_\xi \\
&\quad - \frac{q(z)}{2\pi} \int\limits_{\partial K} (D_j \log|z-\xi|) \cdot n_l(\xi)\, |d\xi|\,.
\end{aligned}$$

From the following estimate we can infer that the functions $D_l v_{j\varepsilon}$ converge locally uniformly in $\mathcal{O}$ to v_{lj} as ε goes to 0. This implies $w \in C^2(\mathcal{O})$ and $D_{lj} w = v_{lj}$ for $z \in \mathcal{O}$ and $l, j \in \{1,2\}$ since $D_j w = v_j$ holds.

Let $\Omega \subset \mathcal{O}$ be a compact subset. Choose

$$0 < \delta < \frac{\operatorname{dist}(\Omega, \partial G)}{4}$$

and define the set $\tilde{\Omega} := \Omega \cup \{\xi \in \mathbb{C} : \operatorname{dist}(\xi, \partial\Omega) \le 2\delta\}$. Then $\tilde{\Omega}$ is compactly contained in $\mathcal{O}$. By the local Hölder continuity of q in $\mathcal{O}$ the quantity

$$[q]_{\alpha,\tilde{\Omega}} = \sup_{z,\xi \in \tilde{\Omega}} \frac{|q(z) - q(\xi)|}{|z-\xi|^\alpha}$$

is finite.

This leads for $z \in \Omega$ and $\varepsilon < \min\{\delta, 1/2\}$ to

$$
\begin{aligned}
&|v_{lj}(z) - D_l v_{j\varepsilon}(z)| \\
&\quad = \frac{1}{2\pi}\left| \iint\limits_{K_{2\varepsilon}(z)} D_l \left\{ (D_j \log|z-\xi|) \left(1 - \eta\left(\frac{|z-\xi|}{\varepsilon}\right)\right)\right\} (q(\xi) - q(z))\, d\sigma_\xi \right| \\
&\quad \leq [q]_{\alpha,\tilde{\Omega}} \cdot \frac{1}{2\pi} \iint\limits_{K_{2\varepsilon}(z)} \left(\left|D_{lj} \log|z-\xi|\right| + \left|D_j \log|z-\xi|\right| \frac{2}{\varepsilon}\right) |z-\xi|^\alpha\, d\sigma_\xi \\
&\quad \leq [q]_{\alpha,\tilde{\Omega}} \cdot \frac{1}{2\pi} \iint\limits_{K_{2\varepsilon}(z)} \left(\frac{2}{|z-\xi|^2} + \frac{2}{\varepsilon|z-\xi|}\right) |z-\xi|^\alpha\, d\sigma_\xi \\
&\quad = [q]_{\alpha,\tilde{\Omega}} \cdot 2 \cdot \left\{ \frac{(2\varepsilon)^\alpha}{\alpha} + 2\frac{(2\varepsilon)^\alpha}{\alpha+1}\right\} \leq [q]_{\alpha,\tilde{\Omega}} \cdot \frac{6}{\alpha} \cdot (2\varepsilon)^\alpha,
\end{aligned}
$$

which proves $w \in C^2(\mathcal{O})$ and $D_l v_j = D_{lj} w$.

Finally, we derive $\Delta w = q$ in $\mathcal{O}$. First, we note

$$
\begin{aligned}
\Delta w(z) &= D_{11} w(z) + D_{22} w(z) \\
&= -q(z) \cdot \frac{1}{2\pi} \int\limits_{\partial K} \big((D_1 \log|z-\xi|) n_1(\xi) + (D_2 \log|z-\xi|) n_2(\xi)\big) |d\xi|,
\end{aligned}
$$

for $z \in \mathcal{O}$. Here we have used $D_{jj} w(z) = v_{jj}(z)$ for $z \in \mathcal{O}$ and $j \in \{1,2\}$, and also (III.13). Hence we are left to check that

$$
\frac{1}{2\pi} \int\limits_{\partial K} \big((D_1 \log|z-\xi|) n_1(\xi) + (D_2 \log|z-\xi|) n_2(\xi)\big) |d\xi| = -1.
$$

But this is a simple consequence of the argument principle for the disk $K = K_R(z_0)$:

$$
\begin{aligned}
&\frac{1}{2\pi} \int\limits_{\partial K} \left((D_1 \log|z-\xi|) n_1(\xi) + (D_2 \log|z-\xi|) n_2(\xi)\right) |d\xi| = \\
&\frac{1}{2\pi} \int\limits_{\partial K} \operatorname{Re}\left(\frac{n_1(\xi) + i n_2(\xi)}{z-\xi}\right) |d\xi| = \frac{1}{2\pi} \operatorname{Re} \int\limits_0^{2\pi} \frac{Re^{i\varphi}}{z - (z_0 + Re^{i\varphi})}\, d\varphi = \\
&\frac{1}{2\pi} \operatorname{Re}\left(-i \int\limits_{\partial K} \frac{d\xi}{z-\xi}\right) = -n(z, \partial K) = -1,
\end{aligned}
$$

where $n(z, \partial K)$ denotes the winding number of ∂K with respect to $z \in G$. Thus the proof of Theorem III.11 is complete. ■

Proof of Theorem III.12.
First we see the function $H : G \longrightarrow \mathbb{R}$ is well–defined. This follows from the fact that $\xi \mapsto \gamma(z,\xi)q(\xi)$ is measurable on G for every fixed $z \in G$ and the estimate

$$|\gamma(z,\xi)q(\xi)| \leq M \max_{\xi \in \partial G} |\log|z-\xi||$$

holds for all $z, \xi \in G$. Thus $\xi \mapsto \gamma(z,\xi)q(\xi)$ is an integrable function over G for every $z \in G$.

Our goal is to prove $H \in C^2(G)$ and $\Delta H \equiv 0$ in G. This is a rather delicate matter, as the derivatives of γ are *not* bounded in G and there don't even exist globally integrable majorants for these derivatives. In a first step we will find suitable *locally* integrable majorants for these functions, which are then used in the second step to conclude $H \in C^2(G)$ and $\Delta H \equiv 0$ in G.

Now let $z_0 \in G$ be arbitrary, but fixed, and denote $\delta := \mathrm{dist}(z_0, \partial G)$. Then the functions

$$\begin{aligned} &\xi \mapsto \gamma(z_0,\xi) \\ &\xi \mapsto D_j\gamma(z_0,\xi)\,, \qquad j = 1,2 \\ &\xi \mapsto D_{lj}\gamma(z_0,\xi)\,, \qquad l,j = 1,2 \end{aligned}$$

are harmonic in G. As $z_0 + h \in G$ and $z_0 + ih \in G$ for $0 \leq h < \delta$ we can define for $0 < h < \delta$ and $\xi \in \overline{G}$ the functions

$$\begin{aligned} Q_{x_1}(h,\xi) &= \frac{\gamma(z_0+h,\xi) - \gamma(z_0,\xi)}{h} \\ Q_{x_2}(h,\xi) &= \frac{\gamma(z_0+ih,\xi) - \gamma(z_0,\xi)}{h}\,. \end{aligned}$$

Note, the functions $\xi \mapsto Q_{x_1}(h,\xi)$ and $\xi \mapsto Q_{x_2}(h,\xi)$ are harmonic in G and continuous on $\overline{G}$ for every $0 < h < \delta$. Their boundary values are

$$Q_{x_1}(h,\tau) = \frac{\log|(z_0+h)-\tau| - \log|z_0-\tau|}{h}$$

and

$$Q_{x_2}(h,\tau) = \frac{\log|(z_0+ih)-\tau| - \log|z_0-\tau|}{h}\,,$$

respectively, where $\tau \in \partial G$.

We shall now see that the functions $Q_{x_1}(h,\tau)$ and $Q_{x_2}(h,\tau)$ converge uniformly for $\tau \in \partial G$ to $\mathrm{Re}(1/(z_0-\tau))$ and $-\mathrm{Im}(1/(z_0-\tau))$, respectively, as h tends to 0. For that we keep in mind the estimate

$$\left|\frac{1}{w}\left(\log|1+w| - \mathrm{Re}\,w\right)\right| \leq \left|\frac{1}{w}\left(\log(1+w) - w\right)\right| \leq \frac{1}{2}\frac{|w|}{1-|w|}$$

which is valid for every $w \in \mathbb{D}$. For a given $\varepsilon > 0$ we choose $|h| < 2\varepsilon\delta^2/(1+2\varepsilon\delta)$, where $\delta = \operatorname{dist}(z_0, \partial G)$ as above. This leads for every $\tau \in \partial G$ to

$$\begin{aligned}
&\left|\frac{1}{h}\log\left|1+\frac{h}{z_0-\tau}\right| - \operatorname{Re}\left(\frac{1}{z_0-\tau}\right)\right| \\
&\qquad = \left|\frac{1}{z_0-\tau}\right|\left|\left(\frac{z_0-\tau}{h}\right)\left(\log\left|1+\frac{h}{z_0-\tau}\right| - \operatorname{Re}\left(\frac{h}{z_0-\tau}\right)\right)\right| \\
&\qquad \le \frac{1}{2}\frac{1}{|z_0-\tau|}\frac{|h|}{|z_0-\tau|-|h|} \le \frac{1}{2}\frac{1}{\delta}\frac{|h|}{\delta-|h|} \le \varepsilon\,,
\end{aligned}$$

and means

$$\lim_{h\longrightarrow 0} Q_{x_1}(h,\tau) = \operatorname{Re}\left(\frac{1}{z_0-\tau}\right)$$

uniformly for $\tau \in \partial G$. In a similar way we get

$$\lim_{h\longrightarrow 0} Q_{x_2}(h,\tau) = -\operatorname{Im}\left(\frac{1}{z_0-\tau}\right)$$

uniformly for $\tau \in \partial G$.

From the maximum and minimum principle for harmonic functions and $\lim_{h\longrightarrow 0} Q_{x_j}(h,\xi) = D_j\gamma(z_0,\xi)$ for $\xi \in G$ and $j = 1, 2$, we therefore deduce

$$\min_{\tau\in\partial G}\operatorname{Re}\left(\frac{1}{z_0-\tau}\right) \le D_1\gamma(z_0,\xi) \le \max_{\tau\in\partial G}\operatorname{Re}\left(\frac{1}{z_0-\tau}\right)$$

and

$$\min_{\tau\in\partial G}\operatorname{Im}\left(-\frac{1}{z_0-\tau}\right) \le D_2\gamma(z_0,\xi) \le \max_{\tau\in\partial G}\operatorname{Im}\left(-\frac{1}{z_0-\tau}\right)$$

for all $\xi \in G$. As z_0 is an arbitrary point in G this leads in view of $-|z| \le \operatorname{Re}(z) \le |z|$ and $-|z| \le \operatorname{Im}(z) \le |z|$ to

$$|D_j\gamma(z,\xi)| \le \max_{\tau\in\partial G}\frac{1}{|z-\tau|} \tag{III.16}$$

for all $z, \xi \in G$ and $j \in \{1, 2\}$.

In the following we will play the same game for the second derivatives $D_{lj}\gamma$. Therefore, let $z_0 \in G$ be arbitrary, but fixed, and denote $\delta := \operatorname{dist}(z_0, \partial G)$. We define

$$\tilde{Q}_{x_1,x_j}(h,\xi) = \frac{\gamma_{x_j}(z_0+h,\xi) - \gamma_{x_j}(z_0,\xi)}{h}$$

and

$$\tilde{Q}_{x_2,x_j}(h,\xi) = \frac{\gamma_{x_j}(z_0+ih,\xi) - \gamma_{x_j}(z_0,\xi)}{h}\,,$$

where $\gamma_{x_j} = D_j\gamma$ and $j \in \{1, 2\}$.

The function $\xi \mapsto \tilde{Q}_{x_1,x_1}(h,\xi)$ is harmonic in G and continuous in $\overline{G}$ for each $0 < h < \delta$. Its boundary values are

$$\tilde{Q}_{x_1,x_1}(h,\tau) = \frac{1}{h}\left(\mathrm{Re}\left(\frac{1}{z_0+h-\tau}\right) - \mathrm{Re}\left(\frac{1}{z_0-\tau}\right)\right), \quad \tau \in \partial G .$$

Now

$$\lim_{h\longrightarrow 0} \tilde{Q}_{x_1,x_1}(h,\tau) = -\mathrm{Re}\left(\frac{1}{(z_0-\tau)^2}\right)$$

uniformly for $\tau \in \partial G$. This is a direct consequence of the estimate

$$\left|\frac{1}{h}\left(\mathrm{Re}\left(\frac{1}{z_0+h-\tau}\right) - \mathrm{Re}\left(\frac{1}{z_0-\tau}\right)\right) + \mathrm{Re}\left(\frac{1}{(z_0-\tau)^2}\right)\right| =$$
$$\left|\mathrm{Re}\left(\left(\frac{-1}{z_0+h-\tau} + \frac{1}{z_0-\tau}\right)\frac{1}{z_0-\tau}\right)\right| \le \frac{1}{\delta}\frac{|h|}{\delta(\delta-|h|)} .$$

In a similar vein we obtain

$$\lim_{h\longrightarrow 0} \tilde{Q}_{x_l,x_j}(h,\tau) = \mathrm{Re}\left(\frac{1}{(z_0-\tau)^2}\right), \quad \text{if} \quad l=j=2 ,$$

and

$$\lim_{h\longrightarrow 0} \tilde{Q}_{x_l,x_j}(h,\tau) = \mathrm{Im}\left(\frac{1}{(z_0-\tau)^2}\right), \quad \text{if} \quad l=1,\ j=2 \text{ and } l=2,\ j=1 .$$

Analogously as in the case of the first derivatives we thus find for $j,l \in \{1,2\}$ and all $z,\xi \in G$

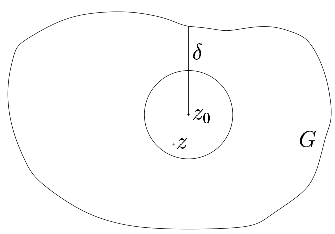
$$|D_{lj}\gamma(z,\xi)| \le \max_{\tau\in\partial G} \frac{1}{|z-\tau|^2} . \tag{III.17}$$

We now use the estimates (III.16) and (III.17) to show $H \in C^2(G)$ and $\Delta H = 0$ in G.

Again, we fix $z_0 \in G$ and define $\delta := \mathrm{dist}(z_0,\partial G)$ as well as $M := \sup_{\xi\in G}|q(\xi)|$.

The function $H(z)$ is continuous in $K_{\delta/4}(z_0)$ by [14, Kap. IV, Satz 5.6] because

- $z \mapsto \gamma(z,\xi)q(\xi)$ is continuous in G for every fixed $\xi \in G$, and
- for every $z \in K_{\frac{\delta}{4}}(z_0) \subset G$ and $\xi \in G$ the estimate



$$\begin{aligned}|\gamma(z,\xi)q(\xi)| \le& M \cdot \max_{\xi\in\partial G} |\log|z-\xi|| \\ \le& M \cdot \max_{\xi\in\partial G}\left\{2 + |z-\xi| + \frac{1}{|z-\xi|}\right\} \\ \le& M \cdot \left\{2 + d_G + \frac{1}{\mathrm{dist}(z,\partial G)}\right\} \le M \cdot \left\{2 + d_G + \frac{4}{3\delta}\right\}\end{aligned}$$

holds.

Now we continue to show $H \in C^1(K_{\delta/4}(z_0))$. Note,

- $D_j\gamma(z,\xi)q(\xi)$ exists for every $z,\xi \in G$ and $j \in \{1,2\}$, $z \mapsto D_j\gamma(z,\xi)q(\xi)$ is continuous in G for every $\xi \in G$, and
- for every $z \in K_{\frac{\delta}{4}}(z_0) \subset G$ and every $\xi \in G$ the estimate

$$|D_j\gamma(z,\xi)q(\xi)| \le M \cdot \max_{\xi\in\partial G} \frac{1}{|z-\xi|} \le \frac{4}{3}\frac{M}{\delta}$$

  is valid for $j \in \{1,2\}$. Here we have used (III.16).

Thus H is continuously differentiable in $K_{\delta/4}(z_0)$, see again [14, Kap. IV, Satz 5.6/5.7].

Now similar arguments as above let us conclude $H \in C^2(K_{\delta/4}(z_0))$. We observe,

- $D_{lj}\gamma(z,\xi)q(\xi)$ exists for every $z,\xi \in G$ and $l,j \in \{1,2\}$, $z \mapsto D_{lj}\gamma(z,\xi)q(\xi)$ is continuous in G for every $\xi \in G$ and
- for every $z \in K_{\frac{\delta}{4}}(z_0) \subset G$ and $\xi \in G$, the estimate

$$|D_{lj}\gamma(z,\xi)q(\xi)| \le M \cdot \left(\frac{4}{3\delta}\right)^2$$

  is valid for $l,j \in \{1,2\}$ because of (III.17).

So we get that the function H is twice continuously differentiable in $K_{\delta/4}(z_0)$ and therefore in all of G.

Finally,

$$\Delta H(z) = \Delta\left(\frac{1}{2\pi}\iint_G \gamma(z,\xi)q(\xi)d\sigma_\xi\right) = \frac{1}{2\pi}\iint_G (\Delta\gamma(z,\xi))\, q(\xi)d\sigma_\xi = 0$$

in G as $z \mapsto \gamma(z,\xi)$ is harmonic in G for every $\xi \in G$. This brings us to the end of the proof of Theorem III.12. ■

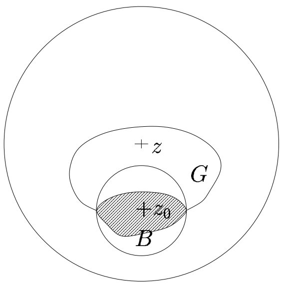
Before we continue to prove Theorem III.13, i.e. $\varpi \in C(\overline{G})$ and $\varpi|_{\partial G} \equiv 0$, we point out that the integral over Green's function gets small, if we integrate over small sets. This is the gist of the next

Lemma III.15
Let $G \subset \mathbb{C}$ be a bounded domain, where Green's function $g(z,\xi)$ exists and let $B \subseteq G$ a subset. Then

$$\iint_B g(z,\xi)d\sigma_\xi < 24\pi \cdot d_G \cdot d_B \tag{III.18}$$

for every $z \in G$.

Proof.
We define $R := 4d_G$ and note $\overline{G}$ is a subset of $K_R(\tilde{z}) = \{\zeta \in \mathbb{C} : |\zeta - \tilde{z}| < R\}$ for every $\tilde{z} \in G$. Choose an arbitrary point $z \in G$. Then the Riemann map f from $K_R(z)$ onto $\mathbb{D}$



with $f(z) = 0$ is given by $f(\xi) = (\xi - z)/R$. Hence Green's function for the disk $K_R(z)$ with singularity in z takes the form

$$k(z,\xi) = -\log\left|\frac{f(\xi) - f(z)}{1 - \overline{f(z)}f(\xi)}\right| = -\log\left|\frac{\xi - z}{R}\right| .$$

Furthermore, the function

$$\xi \mapsto g(z,\xi) - k(z,\xi) ,$$

has a harmonic extension to G and because of

$$\lim_{\xi \longrightarrow \partial G} g(z,\xi) = 0 ,$$

we deduce from the Carleman–principle for harmonic functions

$$0 \le g(z,\xi) \le k(z,\xi) \qquad \text{for } \xi \in G .$$

Finally we can find to the set $B \subseteq G$ a disk $K_r(z_0)$ with radius $r \le d_B$ and center $z_0 \in \mathbb{C}$ such that $B \subseteq \overline{K_r(z_0)}$. The main point here is that $\overline{K_r(z_0)} \subseteq K_R(z)$. This is a consequence of the following few lines. First we see $z_0 \in K_R(z)$ since for $\zeta \in B$

$$|z_0 - z| \le |z_0 - \zeta| + |\zeta - z| \le r + d_G \le d_B + d_G \le 2d_G < R$$

holds. Therefore we get for all $\xi \in \overline{K_r(z_0)}$

$$|\xi - z| \le |\xi - z_0| + |z_0 - z| \le r + 2d_G \le 3d_G < R .$$

The observation $B \subseteq \overline{K_r(z_0)} \subseteq K_R(z)$ leads now to the estimate

$$\iint_B g(z,\xi)d\sigma_\xi \le \iint_B k(z,\xi)d\sigma_\xi \le \iint_{\overline{K_r(z_0)}} k(z,\xi)d\sigma_\xi .$$

As a matter of course we will find an upper bound for the last integral. This works best, if we distinguish the following two cases:

(1) If $|z - z_0| \le 2r$ then $\overline{K_r(z_0)} \subset \overline{K_{3r}(z)} \subset K_R(z)$, which allows the estimate

$$\iint_{\overline{K_r(z_0)}} k(z,\xi)d\sigma_\xi \le \iint_{\overline{K_{3r}(z)}} -\log\left|\frac{z-\xi}{R}\right| d\sigma_\xi \le$$

$$\le \iint_{\overline{K_{3r}(z)}} \frac{R}{|z-\xi|} d\sigma_\xi = \int_0^{2\pi}\int_0^{3r} \frac{R}{\rho}\rho\, d\rho d\varphi = 6\pi \cdot R \cdot r .$$

(2) On the other hand, if $|z - z_0| > 2r$, then $R > |z - \xi| > r$ is valid for every $\xi \in \overline{K_r(z_0)}$, so

$$\iint_{\overline{K_r(z_0)}} k(z,\xi)d\sigma_\xi = \iint_{\overline{K_r(z_0)}} -\log\left|\frac{z-\xi}{R}\right| d\sigma_\xi \le \iint_{\overline{K_r(z_0)}} \frac{R}{|z-\xi|} d\sigma_\xi \le$$

$$\le R \iint_{\overline{K_r(z_0)}} \frac{1}{|z - z_0| - |z_0 - \xi|} d\sigma_\xi \le R \iint_{\overline{K_r(z_0)}} \frac{1}{2r - |z_0 - \xi|} d\sigma_\xi =$$

$$= 2\pi R \int_0^r \frac{\rho}{2r - \rho} d\rho \le 2\pi \log 2 \cdot R \cdot r .$$

As a result of (1) and (2) we have

$$\iint\limits_{K_r(z_0)} k(z,\xi)\,d\sigma_\xi \leq 6\pi \cdot R \cdot r$$

for our chosen point $z \in G$. But this inequality is independent of the point z. All in all we obtain the desired upper bound

$$\iint\limits_{B} g(z,\xi)\,d\sigma_\xi \leq 6\pi \cdot R \cdot r \leq 24\pi \cdot d_G \cdot d_B\,. \qquad \blacksquare$$

Remark III.16
Note, estimate (III.18) does only depend on the diameter of the subset $B \subseteq G$, but not on B itself.

Proof of Theorem III.13.
Let $\tau \in \partial G$ be an arbitrary boundary point of G and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points in G such that $\lim\limits_{n\longrightarrow\infty} z_n = \tau$. Our intention is to prove

$$\lim_{n\longrightarrow\infty} \iint\limits_{G} g(z_n,\xi)\,q(\xi)d\sigma_\xi = 0.$$

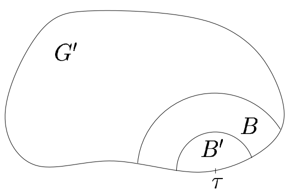
As the sequence of functions $\xi \mapsto g(z_n,\xi)$ does not converge uniformly in G to 0 and also has no integrable majorant in all of G, we cannot make direct profit out of the property that $g(\tau,\xi)$ vanishes for $\tau \in \partial G$. For this reason we divide G into two parts G' and B, where $\tau \in \overline{B}$, and show

$$\iint\limits_{B} g(z_n,\xi)\,q(\xi)d\sigma_\xi$$

can be made arbitrarily small, uniformly with respect to n, by choosing B sufficiently small, and

$$\iint\limits_{G'} g(z_n,\xi)\,q(\xi)d\sigma_\xi$$

approaches 0 as n tends to ∞ for every choice of B.



More precisely, we define the above mentioned sets by $B = G \cap K_r(\tau)$, $B' = G \cap K_{r/2}(\tau)$, where $r > 0$, and $G' = G\backslash B$. Now let $M := \sup\limits_{\xi\in G} |q(\xi)|$ and note

$$\left| \iint\limits_{G} g(z,\xi)\,q(\xi)\,d\sigma_\xi \right| \leq M \left\{ \iint\limits_{G'} g(z,\xi)\,d\sigma_\xi + \iint\limits_{B} g(z,\xi)\,d\sigma_\xi \right\}.$$

Since the integral over G' needs a bit more attention we like to begin with the integral over B, which can be handled very quickly. Without loss of generality we may assume

$z_n \in B' \subseteq B$ for every $n \in \mathbb{N}$. Due to the fact that B is a subset of G with diameter $d_B \leq 2r$, we conclude, based on Lemma III.15,

$$\iint\limits_B g(z_n, \xi)\, d\sigma_\xi \leq C \cdot 2r$$

for every $n \in \mathbb{N}$, where C is some positive constant.

For the integral over G' choose $z_0 \in \mathbb{C}$ and $R > 0$ such that $G \subseteq K_R(z_0)$. By Carleman's principle we have

$$g(z, \xi) \leq k(z, \xi), \qquad (z, \xi) \in G \times G\,,$$

where $k(z, \xi)$ denotes Green's function for $K_R(z_0)$.

We find for $z \in B'$ and $\xi \in G'$ the estimate

$$R \cdot \frac{r}{2} \leq |R(z - \xi)| \leq \left|R^2 - (\overline{\xi} - \overline{z_0})(z - z_0)\right| \leq 2R^2\,. \tag{III.19}$$

Since k is given by the formula

$$k(z, \xi) = -\log\left|\frac{R(z - \xi)}{R^2 - (\overline{\xi} - \overline{z_0})(z - z_0)}\right|\,,$$

we obtain for $z \in B'$ and $\xi \in G'$ because of (III.19)

$$|k(z, \xi)| \leq \Big|\log|R(z - \xi)|\Big| + \Big|\log\left|R^2 - (\overline{\xi} - \overline{z_0})(z - z_0)\right|\Big| \leq 2\left|\log R\frac{r}{2}\right| + 2\left|\log 2R^2\right|.$$

This yields for every $\xi \in G'$ and for every $n \in \mathbb{N}$

$$|g(z_n, \xi)| \leq 2\left(\left|\log R\frac{r}{2}\right| + \left|\log 2R^2\right|\right)\,.$$

Thus by virtue of $g(z_n, \xi) \longrightarrow 0$ for every $\xi \in G'$ and Lebesgue's dominated convergence theorem we get

$$\lim_{n \longrightarrow \infty} \iint\limits_{G'} g(z_n, \xi)\, d\sigma_\xi = \iint\limits_{G'} \lim_{n \longrightarrow \infty} g(z_n, \xi)\, d\sigma_\xi = 0.$$

As B can be chosen arbitrarily small, we conclude

$$\lim_{z \longrightarrow \tau} \iint\limits_G g(z, \xi)\, q(\xi)\, d\sigma_\xi = 0 \qquad \text{for } \tau \in \partial G\,. \qquad \blacksquare$$

III.4 Further results

Real analyticity of solutions of real analytic elliptic PDEs

The next theorem will be one of the keys to Theorem I.1.

Theorem III.17
Let $G \subseteq \mathbb{C}$ be a domain, let $f : G \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and let $u : G \longrightarrow \mathbb{R}$ be a C^2–solution of the partial differential equation

$$\Delta u = f(z, u)$$

in G. If $(x, y, u) \mapsto f(x + iy, u)$ is a real analytic function in a neighborhood of the point $(x_0, y_0, u(x_0 + iy_0)) \in \mathbb{R}^3$, $x_0 + iy_0 \in G$, then $(x, y) \mapsto u(x + iy)$ is real analytic in a neighborhood of (x_0, y_0).

In [11] the authors give a proof for this statement using hyperbolic partial differential equation techniques, which is beyond the scope of this work.

Equicontinuity of Green's function

The last result for this chapter will turn out to be a very important tool to prove the existence of pseudo–metrics of non–positive Gaussian curvature with prescribed zeros, see Theorem IV.14.

Recall

$$F(z) = \iint\limits_G g(z, \xi) d\sigma_\xi$$

belongs to $C(\overline{G}) \cap C^1(G)$. So F is locally uniformly continuous in G, i.e.

$$\iint\limits_G \left(g(z_1, \xi) - g(z_2, \xi)\right) d\sigma_\xi \longrightarrow 0$$

as $|z_1 - z_2|$ tends to 0 locally uniformly for z_1, $z_2 \in G$. We shall need, however, the following stronger statement.

Theorem III.18
Let $G \subset \mathbb{C}$ be a regular and bounded domain and let $g(z, \xi)$ be Green's function for G. Then the integral

$$\iint\limits_G |g(z_1, \xi) - g(z_2, \xi)| d\sigma_\xi$$

tends to 0 as $|z_1 - z_2|$ approaches 0 locally uniformly for $z_1, z_2 \in G$. More precisely, for every compact set $\Omega \subset G$ let $r := dist(\Omega, \partial G)/2$ and let $0 < \delta < r/2$. Then

$$\iint\limits_G |g(z_1, \xi) - g(z_2, \xi)| d\sigma_\xi \leq \left(\frac{9}{2r} \operatorname{area}(G) + 4\pi r\right) \delta \tag{III.20}$$

for all z_1, $z_2 \in \Omega$ with $|z_1 - z_2| < \delta$.

Proof.
Let $\Omega \subset G$ be a compact subset and define $\tilde{r} := \operatorname{dist}(\Omega, \partial G)$ as well as $r := \tilde{r}/2$. Then for every $z_0 \in \Omega$ the disk $K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is a proper subset of G. Now choose $0 < \delta < r/2$ and $z_1, z_2 \in \Omega$, such that $|z_1 - z_2| < \delta$.

First, we direct our attention to the function

$$A(\xi) := g(z_1, \xi) - g(z_2, \xi) + \log|z_1 - \xi| - \log|z_2 - \xi|\,.$$

As the function $A(\xi)$ has a harmonic extension to G, which is even continuous on $\overline{G}$, we infer from the maximum and minimum principle for harmonic functions

$$|A(\xi)| \le \max_{\xi \in \partial G} \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right| =: \hat{A}(z_1, z_2)\,.$$

This gives the estimate

$$|g(z_1, \xi) - g(z_2, \xi)| \le \hat{A}(z_1, z_2) + \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right|\,.$$

In view of (III.20) we will find in a first step an upper bound for $\hat{A}(z_1, z_2)$, which only depends on δ and $\tilde{r}$, and in a second step we will have a look at

$$\iint\limits_G \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right| d\sigma_\xi\,.$$

Since $|z_1 - z_2| < \delta$ we set $z_1 = z_2 + \rho e^{i\varphi}$, where $0 \le \rho < \delta$ and $0 \le \varphi < 2\pi$, and observe

$$\frac{1}{1 + \frac{\delta}{\tilde{r}}} \le \left|\frac{z_1 - \xi}{z_2 - \xi}\right| \le 1 + \frac{\rho}{|z_2 - \xi|} \le 1 + \frac{\delta}{\tilde{r}}\,.$$

for $\xi \in \partial G$. These inequalities imply

$$\hat{A}(z_1, z_2) \le \log\left(1 + \frac{\delta}{\tilde{r}}\right) \le \frac{\delta}{\tilde{r}}\,,$$

which leads to

$$\iint\limits_G \hat{A}(z_1, z_2) d\sigma_\xi \le \delta \cdot \frac{1}{\tilde{r}} \cdot \operatorname{area}(G)\,. \tag{III.21}$$

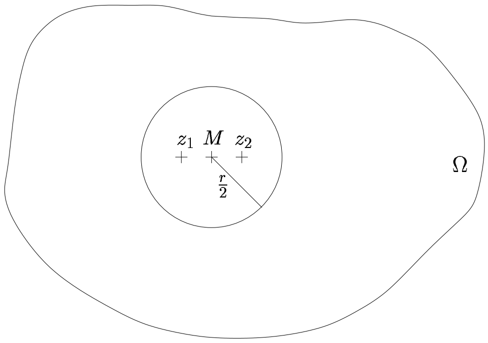
Now we will take care about the integral

$$I := \iint\limits_G \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right| d\sigma_\xi\,.$$

For the purpose of simplification, we split the integral into the following two parts:

$$I_1 = \iint\limits_{G \setminus K_{\frac{r}{2}}(M)} \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right| d\sigma_\xi \quad \text{and} \quad I_2 = \iint\limits_{K_{\frac{r}{2}}(M)} \left|\log\left|\frac{z_1 - \xi}{z_2 - \xi}\right|\right| d\sigma_\xi\,,$$

where $M = (z_1 + z_2)/2$ is the midpoint of z_1 and z_2 and $K_{\frac{r}{2}}(M)$ is the disk about M with radius $r/2$.



We first have a look at I_1. Since $\xi \in G \backslash K_{\frac{r}{2}}(M)$ there is the estimate

$$\frac{1}{1+\frac{\delta}{\frac{r}{4}}} \le \left|\frac{z_1-\xi}{z_2-\xi}\right| \le 1+\frac{|z_1-z_2|}{|z_2-\xi|} \le 1+\frac{\delta}{|z_2-\xi|} \le 1+\frac{\delta}{\frac{r}{4}}\,.$$

Hence we obtain

$$\left|\log\left|\frac{z_1-\xi}{z_2-\xi}\right|\right| \le \log\left(1+\frac{\delta}{\frac{r}{4}}\right) \quad \text{for} \quad \xi \in G\backslash K_{\frac{r}{2}}(M)\,,$$

that is

$$I_1 = \iint\limits_{G\backslash K_{\frac{r}{2}}(M)} \left|\log\left|\frac{z_1-\xi}{z_2-\xi}\right|\right| d\sigma_\xi \le \log\left(1+\frac{\delta}{\frac{r}{4}}\right)\cdot \operatorname{area}(G) \le \frac{4\delta}{r}\cdot \operatorname{area}(G)\,. \tag{III.22}$$

Finally, we move on to I_2. In this case we have for $\xi \in K_{\frac{r}{2}}(M)$ the estimate

$$\frac{1}{1+\frac{\delta}{|z_1-\xi|}} \le \left|\frac{z_1-\xi}{z_2-\xi}\right| \le 1+\frac{\delta}{|z_2-\xi|}$$

and hence

$$\left|\log\left|\frac{z_1-\xi}{z_2-\xi}\right|\right| \le \log\left(1+\frac{\delta}{|z_2-\xi|}\right)+\log\left(1+\frac{\delta}{|z_1-\xi|}\right)\,. \tag{III.23}$$

From the observation that the disk $K_{\frac{r}{2}}(M)$ is a subset of both $K_r(z_1)$ and $K_r(z_2)$ we infer

$$I_2 = \iint\limits_{K_{\frac{r}{2}}(M)} \left|\log\left|\frac{z_1-\xi}{z_2-\xi}\right|\right| d\sigma_\xi \le \iint\limits_{K_r(z_2)} \log\left(1+\frac{\delta}{|z_2-\xi|}\right) d\sigma_\xi + \iint\limits_{K_r(z_1)} \log\left(1+\frac{\delta}{|z_1-\xi|}\right) d\sigma_\xi =$$

$$= 2\int_0^{2\pi}\int_0^r \log\left(1+\frac{\delta}{\rho}\right)\rho\, d\rho\, d\varphi \le 4\pi\delta\cdot r\,. \tag{III.24}$$

We conclude from (III.21), (III.22) and (III.24)

$$\iint\limits_G |g(z_1,\xi)-g(z_2,\xi)|\, d\sigma_\xi \le \left(\frac{9}{2r}\operatorname{area}(G)+4\pi r\right)\delta$$

for all $z_1, z_2 \in \Omega$ with $|z_1-z_2|<\delta$ provided $\delta < r/2 = \operatorname{dist}(\Omega,\partial G)/4$. ■

– Chapter IV –

Conformal Pseudo–metrics

This chapter aims to present, in a relatively self–contained manner, some aspects of the theory of conformal pseudo–metrics for plane domains. We have already discussed the relevance of this theory in complex analysis in our introduction.

We begin by recalling some basic definitions, and introduce the Gaussian curvature of a conformal pseudo–metric. In particular, we discuss how every pseudo–metric is linked with its curvature by a non–linear elliptic PDE, the Gaussian curvature equation. The most important metrics from the viewpoint of function theory are metrics of constant negative Gaussian curvature with isolated singularities. Therefore we will briefly go into a classification of the different types of singularities of such metrics due to Nitsche [39], Heins [22] and Warnecke [53].

We proceed by illustrating the connection between pseudo–metrics with vanishing curvature and harmonic functions. Afterwards we move on to a proof of the existence of conformal metrics with non–positive Hölder continuous curvature in bounded and regular domains by using the Gaussian curvature equation. Then we are well–prepared to turn to the general case of conformal pseudo–metrics and to prove Theorem I.12 and Theorem I.10 which show that the planar Schwarz–Picard Problem I.9 has an affirmative answer for every bounded and regular domain.

IV.1 Basic concepts

IV.1.1 Pseudo–metrics and their curvature

Let us begin with a quick outline of the theory of conformal pseudo–metrics. For more details see [30].

Definition IV.1
Let $G \subseteq \mathbb{C}$ be a domain. A twice continuously differentiable function $\lambda : G \longrightarrow (0,\infty)$ is called a metric on G. A continuous function $\lambda : G \longrightarrow [0,\infty)$ which is twice continuously differentiable in $G \backslash G_\lambda$, where $G_\lambda = \{z \in G \,:\, \lambda(z) = 0\}$, is called pseudo–metric on G. We say a pseudo–metric $\lambda : G \longrightarrow [0,\infty)$ has a zero of order $\alpha_0 \in (0,\infty)$ at a point $z_0 \in G$, if the limit

$$\lim_{z \longrightarrow z_0} \frac{\lambda(z)}{|z-z_0|^{\alpha_0}}$$

exists and is not equal to 0.

Typical examples for conformal metrics are the following canonical metrics.

Examples IV.2

(a) *Let* $\mathbb{C}$ *be the complex plane. Then*

$$\lambda_{\mathbb{C}}(z) = 1$$

is the euclidean metric on $\mathbb{C}$.

(b) *On the unit disk* $\mathbb{D}$ *the so called hyperbolic metric or Poincaré metric is given by*

$$\lambda_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2}.$$

(c) *Let* $\mathbb{P}$ *be the Riemann sphere and let*

$$\lambda_{\mathbb{P}}(z) = \frac{1}{1 + |z|^2}.$$

Then $\lambda_{\mathbb{P}}$ *is the spherical metric on* $\mathbb{P}$ *(given in local coordinates).*

Further examples of pseudo–metrics can be easily constructed in the following way.

Let λ be a pseudo–metric on a domain $G \subseteq \mathbb{C}$ and let $f : D \longrightarrow G$ be a holomorphic function from a domain $D \subseteq \mathbb{C}$ into G, then

$$f^*\lambda(z) = \lambda(f(z))\,|f'(z)|$$

is a pseudo–metric on D. $f^*\lambda$ is called the pullback of λ under the map f. Note, if λ is a metric on G, then the zeros of the pullback $f^*\lambda$ of λ are precisely the zeros of f', i.e. the critical points of the map f.

An important quantity associated with a metric is its curvature.

Definition IV.3

Let λ *be a pseudo–metric on* G. *Then the number*

$$\kappa_\lambda(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}$$

is defined for every $z \in G \backslash G_\lambda$ *and is called the Gaussian curvature of* λ *at the point* z.

We will write κ instead of κ_λ, if it is clear which pseudo–metric is meant. A straightforward computation shows $\kappa(z) = 0$ for the euclidean metric $\lambda_{\mathbb{C}}$, $\kappa(z) = -4$ for the hyperbolic metric $\lambda_{\mathbb{D}}$, and $\kappa(z) = +4$ for the spherical metric $\lambda_{\mathbb{P}}$.

One pleasant aspect of the Gaussian curvature is its absolute conformal invariance. This observation goes back to Gauss and is the content of the following result.

Theorem IV.4

Let λ *be a pseudo–metric on a domain* G *and let* $f : D \longrightarrow G$ *be a holomorphic function. Then*

$$\kappa_{f^*\lambda}(z) = \kappa_\lambda(f(z))$$

for every point $z \in G$ *for which* $f'(z) \neq 0$ *and* $\lambda(f(z)) \neq 0$.

We will focus in this work on pseudo–metrics which are strictly positive except for isolated zeros. Since the curvature of a pseudo–metric $\lambda : G \longrightarrow [0, \infty)$ is only defined at points where λ doesn't vanish, we say λ is a pseudo–metric on G with curvature $\kappa(z)$ in $G \backslash E$ if $\lambda \in C^2(G \backslash E)$, $\lambda > 0$ in $G \backslash E$ and $\lambda(z) = 0$ if and only if $z \in E$.

IV.1.2 Conformal pseudo–metrics and their PDEs

In this section we relate pseudo–metrics to solutions of a corresponding partial differential equation, the so–called Gaussian curvature equation. In particular, we shall show that Theorem I.12 is equivalent to Theorem I.11, and we explain how to reduce Theorem I.11 to a more or less standard boundary value problem for a nonlinear partial differential equation. Let us begin with the easier case of conformal *metrics*.

Let $G \subseteq \mathbb{C}$ be a domain and λ a conformal metric on G with curvature $-\kappa(z)$[1]. Then, by definition,

$$\frac{\Delta \log \lambda(z)}{\lambda(z)^2} = \kappa(z) \quad \text{for } z \in G\,. \tag{IV.1}$$

Since λ is strictly positive on G the function $u(z) = \log \lambda(z)$ is well–defined, belongs to $C^2(G)$ and is a solution of the PDE

$$\Delta u = \kappa(z) e^{2u} \tag{IV.2}$$

in G. We call (IV.2) the Gaussian curvature equation.

Conversely, if u is a C^2–solution of (IV.2) in G, where κ is a continuous function in G, then $\lambda(z) := e^{u(z)}$ defines a conformal metric with curvature $-\kappa(z)$ in G. In particular, in order to show the existence of a conformal metric with prescribed curvature $-\kappa(z)$ one has to find a C^2–solution of the PDE (IV.2).

The situation for conformal *pseudo–metrics* (with finitely many zeros) is only slightly more complicated. Let $G \subseteq \mathbb{C}$ be a domain and let $z_1, \ldots, z_n$ be finitely many points in G. Denote by κ a real valued continuous function on G, and let λ be a pseudo–metric in G with curvature $-\kappa(z)$ in $G\backslash\{z_1, \ldots, z_n\}$ and zeros of orders α_j at z_j. Then the curvature condition consists in

$$\frac{\Delta \log \lambda(z)}{\lambda(z)^2} = \kappa(z) \quad \text{for } z \in G\backslash\{z_1, \ldots, z_n\}\,.$$

The function $u(z) := \log \lambda(z)$ is well–defined and C^2 in $G\backslash\{z_1, \ldots, z_n\}$, and a solution of the Gaussian curvature equation

$$\Delta u = \kappa(z) e^{2u} \tag{IV.3}$$

in $G\backslash\{z_1, \ldots, z_n\}$. Further, u has the property that $\lim_{z \longrightarrow z_j}(u(z) - \alpha_j \log|z - z_j|)$ exists for every $j \in \{1, \ldots, n\}$.

Conversely, let u be a real valued C^2–function in $G\backslash\{z_1, \ldots, z_n\}$ which is a solution to

$$\Delta u = \kappa(z) e^{2u}$$

in $G\backslash\{z_1, \ldots, z_n\}$, where $\kappa : G \longrightarrow \mathbb{R}$ is a continuous function. Furthermore, assume $\lim_{z \longrightarrow z_j}(u(z) - \alpha_j \log|z - z_j|)$ exists finitely for some $\alpha_j \in (0, \infty)$ for every $j \in \{1, \ldots, n\}$. Then $\lambda(z) := e^{u(z)}$ is clearly a pseudo–metric in G with curvature $-\kappa(z)$ in $G\backslash\{z_1, \ldots, z_n\}$ and zeros of orders α_j at z_j.

This shows that in order to find a conformal pseudo–metric in a domain G with prescribed curvature $-\kappa : G \longrightarrow \mathbb{R}$ and prescribed zeros $z_1, \ldots, z_n \in G$ of orders $\alpha_1, \ldots, \alpha_n \in (0, \infty)$,

[1] Note, κ is continuous on G.

it is necessary and sufficient to find a solution $u \in C^2(G\backslash\{z_1,\dots,z_n\})$ of the curvature equation $\Delta u = \kappa(z)e^{2u}$ in $G\backslash\{z_1,\dots,z_n\}$ with singularities $z_1,\dots,z_n$ such that $\lim_{z\longrightarrow z_j}(u(z)-\alpha_j\log|z-z_j|)$ exists for every $j\in\{1,\dots,n\}$. In particular, Theorem I.12 is equivalent to Theorem I.11.

Now our idea to find a singular solution u of $\Delta u = \kappa(z)e^{2u}$ is to pass to an associated curvature equation with regular solutions. To this end, let u be a function as described above, that is $u \in C^2(G\backslash\{z_1,\dots,z_n\})$ with $\Delta u = \kappa(z)\,e^{2u}$ in $G\backslash\{z_1,\dots,z_n\}$ for a continuous function $\kappa : G \longrightarrow \mathbb{R}$ and $\lim_{z\longrightarrow z_j}(u(z)-\alpha_j\log|z-z_j|)$ exists finitely for some $\alpha_j \in (0,\infty)$ for every $j\in\{1,\dots,n\}$. We set $B(z)=\prod_{j=1}^n (z-z_j)^{\alpha_j}$ and consider the function $v(z)=u(z)-\log|B(z)|$. Then v is continuous in G, C^2 in $G\backslash\{z_1,\dots,z_n\}$ and a solution of the PDE

$$\Delta v = \kappa(z)|B(z)|^2 e^{2v} \tag{IV.4}$$

in $G\backslash\{z_1,\dots,z_n\}$. Actually, under some mild additional assumption on κ, the function v will even belong to $C^2(G)$:

Theorem IV.5
Let $G\subseteq\mathbb{C}$ be a domain with $\overline{K_r(z_0)}\subset G$, let $\kappa : G\longrightarrow\mathbb{R}$ be a continuous function which is locally Hölder continuous in $K_r(z_0)$ with exponent α, $0<\alpha\le 1$, and let $v\in C(G)\cap C^2(G\backslash\{z_0\})$ be a solution of the PDE

$$\Delta v = \kappa(z)e^{2v} \quad \textit{in } G\backslash\{z_0\}\,.$$

Then $v\in C^2(G)$.

Thus in order to find a pseudo–metric in a domain G with prescribed Hölder continuous curvature $-\kappa : G\longrightarrow\mathbb{R}$ and prescribed zeros $z_1,\dots,z_n\in G$ it suffices to find a *regular* solution $v\in C^2(G)$ of the curvature equation (IV.4). Indeed, if $v\in C^2(G)$ solves (IV.4) in G, then $\lambda(z):=|B(z)|e^{v(z)}$ is a pseudo–metric in G with curvature $-\kappa(z)$ in $G\backslash\{z_1,\dots,z_n\}$ and zeros of orders α_j at z_j. As a matter of fact, we also need the Hölder continuity of $\kappa(z)$ to solve the PDE (IV.4) in Section IV.3 below.

Proof of Theorem IV.5.
We define a function $v_1 : \overline{K_r(z_0)}\longrightarrow\mathbb{R}$ by

$$v_1(z) = h(z) - \frac{1}{2\pi}\iint\limits_{K_r(z_0)} g(z,\xi)\kappa(\xi)e^{2v(\xi)}d\sigma_\xi\,, \tag{IV.5}$$

where the continuous function $h : \overline{K_r(z_0)}\longrightarrow\mathbb{R}$ is harmonic in $K_r(z_0)$ with boundary values v, and g denotes Green's function for the disk $K_r(z_0)$. Theorem III.10 tells us that $v_1\in C(\overline{K_r(z_0)})\cap C^1(K_r(z_0))\cap C^2(K_r(z_0)\backslash\{z_0\})$ and

$$\Delta v_1 = \kappa(z)e^{2v} \quad \text{in } K_r(z_0)\backslash\{z_0\}\,.$$

In view of the fact that v is a solution of $\Delta v=\kappa(z)e^{2v}$ in $K_r(z_0)\backslash\{z_0\}$ it's natural to consider the function $v-v_1$. We note $v-v_1$ is continuous in the closure of $K_r(z_0)$ and $\Delta(v-v_1)\equiv 0$ in $K_r(z_0)\backslash\{z_0\}$. Consequently, $v-v_1$ has a harmonic extension to $K_r(z_0)$ and since $(v-v_1)|_{\partial K_r(z_0)}\equiv 0$ we conclude $v\equiv v_1$ in $K_r(z_0)$. This shows v is a C^1–function in $K_r(z_0)$ and therefore $\xi\mapsto\kappa(\xi)e^{2v(\xi)}$ is locally Hölder continuous in $K_r(z_0)$ with exponent

α. Now, due to Theorem III.10 the function v_1 belongs to $C^2(K_r(z_0))$ and, since $v_1 \equiv v$ in $K_r(z_0)$, we see $v \in C^2(G)$. ■

Theorem IV.5 shows the solution $v = \log \lambda - \log |B|$ of the PDE (IV.4) is C^2–smooth, if the curvature of the pseudo–metric λ is Hölder continuous. So the differentiability properties of a pseudo–metric depend on the regularity properties of its curvature. The next result, which will be indispensable to the proof of Theorem I.1, is of similar flavor.

Theorem IV.6
Let $G \subseteq \mathbb{C}$ be a domain, let $z_1, \ldots, z_n$ be finitely many distinct points in G, and let $\kappa : G \longrightarrow \mathbb{R}$ be a bounded and real analytic function. Further, let λ be a pseudo–metric in G with curvature $-\kappa(z)$ in $G \backslash \{z_1, \ldots, z_n\}$ and zeros of orders α_j at z_j, where $\alpha_j \in (0, \infty)$. Then λ has the form

$$\lambda(z) = \left| \prod_{j=1}^{n} (z - z_j)^{\alpha_j} \right| e^{v(z)}$$

with a real analytic function $v : G \backslash \{z_1, \ldots, z_n\} \longrightarrow \mathbb{R}$. If α_j is an integer, then v is even real analytic at the point z_j. In particular, if $\alpha_j \in \mathbb{N}$ for every $j = 1, \ldots, n$, then v and also λ are real analytic in G.

Proof.
In view of Theorem IV.5 the function $v(z) = \log \lambda(z) - \log |B(z)|$, $z \in G$, with $B(z) = \prod_{j=1}^{n} (z - z_j)^{\alpha_j}$, is a C^2–solution of the PDE

$$\Delta v = \kappa(z) |B(z)|^2 e^{2v} \tag{IV.6}$$

in G. Since $z \mapsto \kappa(z)|B(z)|^2$ is real analytic in $G \backslash \{z_1, z_2, \ldots, z_n\}$ the solution v of (IV.6) is real analytic in $G \backslash \{z_1, z_2, \ldots, z_n\}$ by Theorem III.17. If $\alpha_j \in \mathbb{N}$ for some $j \in \{1, \ldots, n\}$, then $z \mapsto \kappa(z)|B(z)|^2$ is even real analytic in a neighborhood of the point z_j. Applying again Theorem III.17, we conclude v is real analytic at the point z_j. ■

IV.1.3 Isolated singularities of constantly curved metrics

We can view conformal pseudo–metrics with a discrete set of zeros as conformal metrics with isolated singularities of a special kind. Here we want to deal with more general isolated singularities of conformal metrics, but restrict ourselves to the case of metrics with constant curvature -4, 0 and $+4$. Let's begin with a complete description of the different isolated singularities of metrics of constant curvature -4.

Theorem IV.7
Denote by $K_r(0)$ the disk with radius $0 < r < 1$ and center $z = 0$ and let λ be a metric with constant curvature -4 in $K_r(0) \backslash \{0\}$. Then either λ has a C^2–extension to $K_r(0)$ with $\lambda(0) > 0$ or

$$\lim_{z \longrightarrow 0} \frac{\lambda(z)}{|z|^\alpha} \quad \textit{exists and } \neq 0 \quad \textit{for some } \alpha \in (-1, \infty) \backslash \{0\} \, ,$$

or

$$\lim_{z \longrightarrow 0} \left(\lambda(z) |z| \log \frac{1}{|z|} \right) \quad \textit{exists and } \neq 0.$$

Taking into account the observations we made in Section IV.1.2, Theorem IV.7 is an immediate consequence of the following classification of the possible singularities of solutions to the Gaussian curvature equation $\Delta u = 4e^{2u}$:

Theorem IV.8
Let $K_r(0)$ be the disk with radius $0 < r < 1$ and center $z = 0$. Further, let $u : K_r(0)\backslash\{0\} \longrightarrow \mathbb{R}$ be a C^2–solution of the PDE

$$\Delta u = 4e^{2u} \quad \text{in } K_r(0)\backslash\{0\}\,. \tag{IV.7}$$

Then the function u has either a C^2–extension to $K_r(0)$ or one of the following singularities:

$$\begin{aligned} u(z) &= \alpha \log|z| + \nu(z)\,,\ \alpha \in (-1,\infty)\backslash\{0\}\,,\\ u(z) &= -\log|z| - \log\log\frac{1}{|z|} + \tilde{\nu}(z) \end{aligned}$$

where ν, $\tilde{\nu}$ are continuous function a neighborhood in $K_r(0)$. Moreover, ν and $\tilde{\nu}$ are real analytic in $K_r(0)\backslash\{0\}$. If $\alpha \in \mathbb{N}$ then ν is even real analytic in $K_r(0)$.

Theorem IV.8 was obtained by Nitsche [39], Warnecke [53], and also by Heins [22] who didn't establish the statement about the real analyticity. Heins used techniques involving differential inequalities, whereas Nitsche and Warnecke deduced Theorem IV.8 from Liouville's Theorem, see Corollary I.3. We also want to remark that Theorem IV.8 gives Theorem IV.6 in the case, where $\kappa(z) = +4$.

The next examples shall show that there exist metrics of constant curvature -4 with all possible types of isolated singularities described by Theorem IV.7.

Examples IV.9

(a) *For $\alpha \in (-1,\infty)\backslash\{0\}$*

$$\lambda(z) = \frac{(\alpha+1)|z|^{\alpha}}{1-|z|^{2\alpha+2}}$$

defines a metric of constant curvature -4 in $\mathbb{D}\backslash\{0\}$ with a zero of order α at $z=0$, if $\alpha > 0$ and a "pole" of order $|\alpha|$ at $z = 0$, if $\alpha \in (-1,0)$.

(b) *An example for a metric with curvature -4 in $\mathbb{D}\backslash\{0\}$ and a "log"–singularity is*

$$\lambda(z) = \left(2|z|\log\left(\frac{1}{|z|}\right)\right)^{-1}\,.$$

We now have a look at isolated singularities of metrics with constant curvature 0 and +4.

In these cases we can easily find examples of metrics with prescribed zero or "pole" of almost arbitrary order. Roughly speaking such metrics arise for example as the pullback of the euclidean or spherical metric under the function $f(z) = z^{\alpha+1}$, $(\alpha+1) \in \mathbb{R}\backslash\{0\}$.

Examples IV.10

(a) *In $\mathbb{D}\backslash\{0\}$*

$$\mu(z) = |z|^{\alpha}$$

defines a metric of constant curvature 0 with a zero of order α at $z = 0$ if $\alpha \in (0,\infty)$, and a "pole" of order $|\alpha|$ at $z = 0$ if $\alpha \in (-\infty,0)$.

(b) In $\mathbb{D}\backslash\{0\}$

$$\sigma(z)=\frac{(\alpha+1)|z|^{\alpha}}{1+|z|^{2\alpha+2}}$$

defines a metric of constant curvature $+4$ *with a zero of order* α *at* $z=0$ *if* $\alpha\in(0,\infty)$*, and a "pole" of order* $|\alpha|$ *at* $z=0$ *if* $\alpha\in(-\infty,0)\backslash\{-1\}$*.*

Contrary to metrics of constant curvature -4 metrics of constant curvature 0 and $+4$ cannot have "log"–singularities. This will be shown in Chapter V.3.

The last thing, we wish to point out is that the limit points for metrics of constant curvature -4 coincide if we approach a singularity. This is not the case for metrics of constant curvature 0 or $+4$ as the next examples show. The reason for the difference is that we can define metrics of constant curvature 0 or $+4$ as the pullback of the euclidean or spherical metric under a holomorphic function with an essential singularity. The classification of the possible singularities of metrics with constant curvature 0 and $+4$ is therefore more involved than for metrics of constant curvature -4. See also [8].

Examples IV.11

(a) The function

$$\mu(z)=\exp\left(\exp\left(\frac{\operatorname{Re} z}{|z|^2}\right)\cdot\cos\left(\frac{\operatorname{Im} z}{|z|^2}\right)\right)$$

is a conformal metric with constant curvature 0 *in* $\mathbb{D}\backslash\{0\}$ *and*

$$\lim_{n\longrightarrow\infty}\mu\left(-\frac{1}{n^2}+i\frac{1}{n^4}\right)=1\neq e=\lim_{n\longrightarrow\infty}\mu\left(\frac{i}{2\pi n}\right).$$

(b) The function

$$\sigma(z)=\left(|z|^2\cdot\left(\exp\left(\frac{z+\overline{z}}{2z\overline{z}}\right)+\exp\left(-\frac{z+\overline{z}}{2z\overline{z}}\right)\right)\right)^{-1}$$

is a conformal metric with constant curvature $+4$ *in* $\mathbb{D}\backslash\{0\}$ *and*

$$\lim_{n\longrightarrow\infty}\sigma\left(\frac{1}{n}+i\frac{1}{n}\right)=0\neq\infty=\lim_{n\longrightarrow\infty}\mu\left(\frac{i}{n}\right).$$

IV.2 Pseudo–metrics with vanishing Gaussian curvature

In this section we will restrict the discussion to pseudo–metrics with vanishing Gaussian curvature. As we have observed these pseudo–metrics are closely related to harmonic functions[2]. This immediately allows to deduce a number of properties of pseudo–metrics with vanishing Gaussian curvature from results about harmonic functions. For instance, every pseudo–metric $\lambda(z)$ of vanishing Gaussian curvature is real analytic in $G\backslash G_{\lambda}$, since every harmonic function is real analytic, see also Theorem IV.6.

[2] If λ is a pseudo–metric on a domain G, then $u(z):=\log\lambda(z)$ is a harmonic function in $G\backslash G_{\lambda}$.

Further, every regular domain G, i.e. every domain, which possesses a Green's function $g(z,\xi)$, carries a metric of curvature 0. Indeed, the function $z \mapsto g(z,\xi) + \log|z-\xi|$ is harmonic in G for every fixed $\xi \in G$, so $|z-\xi|e^{g(z,\xi)}$ is a metric of vanishing curvature in G.

It is also possible to prescribe the zeros (including their orders) and the boundary values of zero curvature metrics in bounded regular domains. Equivalently, it is possible to prescribe the singularities and the boundary values of harmonic functions in a bounded regular domain. This is an important and well–known fact in complex analysis:

Theorem IV.12
Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, \ldots, z_n$ be finitely many distinct points in G and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers. Further, let $\phi : \partial G \longrightarrow (0,\infty)$ be a continuous function. Then there exists a unique pseudo–metric $\lambda : G \longrightarrow [0,\infty)$ of constant Gaussian curvature 0 in $G\backslash\{z_1,\ldots,z_n\}$ with zeros of orders α_j at z_j for $j=1,\ldots,n$, such that λ is continuous on $\overline{G}$ with $\lambda(z)=\phi(z)$ for $z \in \partial G$.

Proof.
Let $B(z) = \prod_{j=1}^{n}(z-z_j)^{\alpha_j}$. Then there exists a unique harmonic function $v : G \longrightarrow \mathbb{R}$ with boundary values $\varphi(z) = \log\phi(z) - \log|B(z)|$. Then $u(z) = v(z) + \log|B(z)|$ is a harmonic function in $G\backslash\{z_1,\ldots,z_n\}$, so $\lambda(z) = e^{u(z)} = |B(z)|e^{v(z)}$ is a pseudo–metric in G of constant Gaussian curvature 0 in $G\backslash\{z_1,\ldots,z_n\}$ with zeros of orders α_j at z_j for $j=1,\ldots,n$, and $\lambda(z)=\phi(z)$ for $z \in \partial G$. If $\lambda_1(z)$ and $\lambda_2(z)$ are two such metrics, then $u_1(z) = \log\lambda_1(z) - \log|B(z)|$ and $u_2(z) = \log\lambda_2(z) - \log|B(z)|$ are two harmonic functions in $G\backslash\{z_1,\ldots,z_n\}$, which are bounded there, so they have harmonic extensions to G. But on the boundary these two harmonic functions agree. Thus $\lambda_1(z) = \lambda_2(z)$. ■

We finally note the following strong maximum principle for conformal metrics of constant Gaussian curvature 0.

Theorem IV.13
Let $G \subset \mathbb{C}$ be a bounded domain and let $z_1,\ldots,z_n$ be finitely many points in G. Let λ_1 and λ_2 be two pseudo–metrics in G with curvature 0 in $G\backslash\{z_1,\ldots,z_n\}$. Assume that λ_2 has zeros of orders $\alpha_j \in (0,\infty)$ at z_j for $j \in \{1,\ldots,n\}$ and λ_1 has a zero of order at least α_j at z_j for every $j \in \{1,\ldots,n\}$. If

$$\limsup_{z\longrightarrow\xi} \frac{\lambda_1(z)}{\lambda_2(z)} \le 1, \qquad \xi \in \partial G,$$

then either $\lambda_1(z) < \lambda_2(z)$ in $G\backslash\{z_1,\ldots,z_n\}$ or $\lambda_1(z) = \lambda_2(z)$ in G.

Proof.
We set $u_1(z) = \log\lambda_1(z)$ and $u_2(z) = \log\lambda_2(z)$. Then $u_1 - u_2$ is a harmonic function in $G\backslash\{z_1,z_2,\ldots,z_n\}$ and $\limsup_{z\longrightarrow\xi}(u_1(z)-u_2(z)) \le 0$ for $\xi \in \partial G \cup \{z_1,\ldots,z_n\}$. It follows $u_1 < u_2$ in $G\backslash\{z_1,\ldots,z_n\}$ or $u_1 = u_2$ in $G\backslash\{z_1,\ldots,z_n\}$. This implies $\lambda_1 < \lambda_2$ in $G\backslash\{z_1,\ldots,z_n\}$ or $\lambda_1 \equiv \lambda_2$ in G. ■

IV.3 Negatively curved pseudo–metrics

We now prove Theorem I.12:

Theorem IV.14
Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, z_2, \ldots, z_n \in G$ be finitely many distinct points and let $\alpha_1, \ldots, \alpha_n \in (0, \infty)$. Let $\phi : \partial G \longrightarrow (0, \infty)$ be a continuous function and $\kappa : G \longrightarrow [0, \infty)$ a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique pseudo–metric $\lambda : G \longrightarrow [0, \infty)$ of curvature $-\kappa(z)$ in $G \backslash \{z_1, z_2, \ldots, z_n\}$ with zeros of orders α_j at z_j and boundary values ϕ, i.e. λ is continuous on $\overline{G}$ and $\lambda|_{\partial G} \equiv \phi$.

The proof will be split into several parts. Let's start with the following more general result about the uniqueness of pseudo–metrics with non–positive curvature.

IV.3.1 Uniqueness

Theorem IV.15
Let $G \subset \mathbb{C}$ be a bounded domain, $E = \{z_1, z_2, \ldots\}$ a discrete set of G, and $\kappa : G \longrightarrow [0, \infty)$ a continuous function. Further, denote by λ and μ two pseudo–metrics in G with the following properties:

(i) λ and μ are continuous on $\overline{G}$ and $\lambda \equiv \mu > 0$ on ∂G.

(ii) λ and μ have zeros of the same orders at z_j and no others.

(iii) λ and μ have curvature $-\kappa(z)$ in $G \backslash E$.

Then λ and μ coincide in G, i.e. $\lambda \equiv \mu$ in G.

Proof.
First we note the quotient of the pseudo–metrics λ and μ, λ/μ has a continuous non–vanishing extension to G. Therefore the function

$$h := \left(\log \frac{\lambda}{\mu}\right)^+ = \begin{cases} \log \dfrac{\lambda}{\mu} & \text{if } \dfrac{\lambda}{\mu} \geq 1, \\ 0 & \text{if } \dfrac{\lambda}{\mu} < 1 \end{cases}$$

is well–defined on G. We will show h is a subharmonic function in G. For that it suffices to prove h is subharmonic in $G \backslash E$, see for instance [12, p. 228/229]. To this end let $z_0 \in G \backslash E$ be an arbitrary point. If $\lambda(z_0) > \mu(z_0)$, then $\lambda > \mu$ in a neighborhood of z_0 and we compute

$$\Delta h(z_0) = \Delta \left(\log \frac{\lambda}{\mu}\right)^+ (z_0) = \Delta \left(\log \frac{\lambda}{\mu}\right)(z_0) = \Delta \log \lambda(z_0) - \Delta \log \mu(z_0) =$$
$$= \kappa(z_0)(\lambda(z_0)^2 - \mu(z_0)^2) \geq 0.$$

On the other hand, if $\lambda(z_0) < \mu(z_0)$, then $\lambda < \mu$ in some neighborhood of z_0 and we obtain $\Delta h(z_0) = 0$. Lastly, if $\lambda(z_0) = \mu(z_0)$, then $h(z) \geq h(z_0)$ holds in some neighborhood of z_0.

Thus, we conclude h is a subharmonic function in $G \backslash E$ and consequently in G. The maximum principle for subharmonic functions now implies $h(z) = 0$ in G, because $h \equiv 0$ on ∂G by hypothesis. Therefore we have $\lambda(z) \leq \mu(z)$ on G and by symmetry $\lambda \equiv \mu$. ■

IV.3.2 Existence

First we are going to prove the existence part of Theorem IV.14 for the special case, when λ is a conformal metric, i.e λ has no zeros at all. In view of Section IV.1.2 we need to check:

Theorem IV.16
Let $G \subset \mathbb{C}$ be a bounded and regular domain. Further, let $\Phi : \partial G \longrightarrow \mathbb{R}$ be a continuous function and $\kappa : G \longrightarrow [0, \infty)$ a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique function $v \in C(\overline{G}) \cap C^2(G)$ such that

$$\begin{aligned} \Delta v &= \kappa(z) e^{2v} \quad \text{in } G, \\ v &= \Phi \qquad\qquad \text{on } \partial G. \end{aligned} \tag{IV.8}$$

Proof.
Let's have a quick look at the uniqueness assertion. Assume v and w are two solutions of the boundary value problem (IV.8). Then $\lambda = e^v$ and $\mu = e^w$ are two conformal metrics in G which fulfill the hypotheses of Theorem IV.15. Thus $\lambda \equiv \mu$ in $\overline{G}$ and consequently $v \equiv w$ in $\overline{G}$.

The existence part is much harder. Suppose for a moment $v \in C(\overline{G}) \cap C^2(G)$ is a solution to (IV.8). Since v is uniquely determined, Theorem III.2 indicates that

$$v(z) = h(z) - \frac{1}{2\pi} \iint\limits_G g(z, \xi) \kappa(\xi) e^{2v(\xi)} \, d\sigma_\xi,$$

where $h(z)$ is continuous on $\overline{G}$, harmonic in G and coincides with Φ on ∂G.

This suggests to introduce the operator

$$T[v](z) := h(z) - \frac{1}{2\pi} \iint\limits_G g(z, \xi) \kappa(\xi) e^{2v(\xi)} \, d\sigma_\xi,$$

and to apply Schauder's fixed point theorem.

To set the stage for Schauder's theorem, let X be the Fréchet space of real valued continuous functions in G equipped with the (metriziable) compact–open topology, and let $M := \{v \in X \,:\, m \leq v(z) \leq h(z) \text{ for all } z \in G\}$, where

$$m := \inf_{z \in G} T[h](z).$$

Note that $m > -\infty$. In order to be able to apply Schauder's fixed point theorem we need to verify the following properties of the operator T.

Lemma IV.17
The set M is closed and convex (in X). The operator $T : M \longrightarrow X$ is continuous, maps M into M, and $T(M)$ is precompact.

Proof.
M is clearly closed and convex. Next, we will prove $T[M]$ is precompact, by showing $T[M]$ is a locally equicontinuous family and $T[M]$ is bounded. Let Ω be a compact subset of G, let $2r := \operatorname{dist}(\Omega, \partial G) > 0$, and let $\varepsilon > 0$ be fixed. It is convenient to set

$$C := \frac{1}{2\pi} \sup_{\xi \in G} \left(\kappa(\xi) e^{2h(\xi)} \right) ,$$

and we may assume $C > 0$.
Since h is continuous in $\overline{G}$ there exists a constant $\delta' > 0$ such that $|h(z_1) - h(z_2)| < \varepsilon/2$ for all $z_1, z_2 \in G$ with $|z_1 - z_2| < \delta'$. Further, we define

$$\delta := \min \left\{ \delta', \frac{\varepsilon}{2C \left(\frac{9}{2r} \operatorname{area}(G) + 4\pi r \right)}, \frac{r}{2} \right\} ,$$

a quantity, which depends only on h, κ, G and Ω. Then we have the following estimate for all $z_1, z_2 \in \Omega$ with $|z_1 - z_2| < \delta$ and any $v \in M$:

$$\begin{aligned} |T[v](z_2) - T[v](z_1)| &\le |h(z_2) - h(z_1)| + \frac{1}{2\pi} \iint\limits_G |g(z_2, \xi) - g(z_1, \xi)| \, \kappa(\xi) e^{2v(\xi)} \, d\sigma_\xi \\ &\le |h(z_2) - h(z_1)| + C \cdot \iint\limits_G |g(z_2, \xi) - g(z_1, \xi)| \, d\sigma_\xi \\ &\le \frac{\varepsilon}{2} + C \cdot \left(\frac{9}{2r} \operatorname{area}(G) + 4\pi r \right) \cdot \delta \le \varepsilon . \end{aligned}$$

For the last but one step see (III.20) in Theorem III.18. Thus $T[M]$ is a locally equicontinuous set of functions on G. Moreover, for all $v \in M$ and all $z \in \overline{G}$

$$T[h](z) \le T[v](z) \le h(z) \, , \tag{IV.9}$$

which implies

$$\min_{\zeta \in \overline{G}} T[h](\zeta) \le T[v](z) \le \max_{\zeta \in \overline{G}} |h(\zeta)| \quad \text{for every } v \in M \text{ and for all } z \in \overline{G} \, .$$

This shows $T[M]$ is bounded, so $T[M]$ is a precompact subset of X. Note, estimate (IV.9) also gives $T[M] \subseteq M$.

It remains to prove $T : M \longrightarrow M$ is continuous. Let $(v_k)_k$ be a sequence of functions in M which converges locally uniformly in G to $v \in M$. We have to show the sequence $(T[v_k])_k$ converges locally uniformly in G to $T[v]$.

Let Ω be a compact subset of G and fix $\varepsilon > 0$. We shall deduce that $|T[v_k](z) - T[v](z)| < \varepsilon$ for all $z \in \Omega$ and all $k \ge \tilde{k}$ for some $\tilde{k}$ independent of z.

For the purpose of simplification, let us define

$$\begin{aligned} C_1 &= \frac{1}{\pi} \sup_{\xi \in G} \kappa(\xi) \cdot \max_{\xi \in \overline{G}} e^{2h(\xi)}, \\ C_2 &= 2 \max_{\xi \in \overline{G}} |h(\xi)|, \\ C_3 &= 24\pi {d_G}^2. \end{aligned}$$

Now we choose a compact set K with $\Omega \subseteq K \subseteq G$ such that
$$\operatorname{dist}(\partial\Omega, \partial K) > r$$
and
$$C_1 \cdot C_2 \cdot (|\log r| + |\log d_G| + |\log 2d_G|) \cdot \operatorname{area}(G\backslash K) < \frac{\varepsilon}{2}.$$
Then there exists an index $\tilde{k} \in \mathbb{N}$ such that
$$\max_{z\in K} |v_k(z) - v(z)| < \frac{\varepsilon}{2C_1C_3}, \qquad k \geq \tilde{k}. \tag{IV.10}$$
Consequently, we get for $z \in \Omega$
$$\begin{aligned}
|T[v_k](z) - T[v](z)| &= \left| \frac{1}{2\pi} \iint\limits_G g(z,\xi)\kappa(\xi)(e^{2v_k(\xi)} - e^{2v(\xi)}) d\sigma_\xi \right| \\
&\leq \frac{1}{2\pi} \cdot \sup_{\xi\in G} \kappa(\xi) \cdot 2 \cdot \max_{\xi\in\overline{G}} e^{2h(\xi)} \cdot \iint\limits_G g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi \\
&\leq C_1 \cdot \iint\limits_G g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi\,.
\end{aligned}$$
In order to find an upper bound for the last integral we split it into the following two parts:
$$\iint\limits_{G\backslash K} g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi\,, \tag{IV.11}$$
$$\iint\limits_{K} g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi. \tag{IV.12}$$
The integral over K yields for $k \geq \tilde{k}$
$$\begin{aligned}
\iint\limits_K g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi &\leq \max_{\xi\in K} |v(\xi) - v_k(\xi)| \iint\limits_K g(\xi,z)\, d\sigma_\xi \\
&\overset{\text{(IV.10)}}{\leq} \frac{\varepsilon}{2C_1C_3} \iint\limits_G g(z,\xi)\, d\sigma_\xi \leq \frac{\varepsilon}{2C_1},
\end{aligned}$$
see Lemma III.15. For an estimate of the integral (IV.11) it suffices to consider
$$\iint\limits_{G\backslash K} g(z,\xi)\, d\sigma_\xi\,,$$
since
$$\begin{aligned}
\iint\limits_{G\backslash K} g(z,\xi)\,|v(\xi) - v_k(\xi)|\; d\sigma_\xi &\leq \left(\sup_{\xi\in G} |v(\xi)| + \sup_{\xi\in G} |v_k(\xi)| \right) \iint\limits_{G\backslash K} g(z,\xi)\, d\sigma_\xi \\
&\leq C_2 \iint\limits_{G\backslash K} g(z,\xi)\, d\sigma_\xi\,.
\end{aligned}$$

For that reason define $R := 2d_G$ such that $\overline{G} \subseteq K_R(z)$ for every $z \in G$ and choose an arbitrary point $z \in G$. Then Green's function $k(z,\xi)$ for $K_R(z)$ with singularity at z is given by

$$k(z,\xi) = -\log \frac{|z-\xi|}{R}$$

and the maximum principle for harmonic functions implies $g(z,\xi) \le k(z,\xi)$ for all $\xi \in G$. This leads to

$$\begin{aligned}\iint\limits_{G\backslash K} g(z,\xi)\, d\sigma_\xi \le \iint\limits_{G\backslash K} k(z,\xi)\, d\sigma_\xi &= \iint\limits_{G\backslash K} -\log \frac{|z-\xi|}{R}\, d\sigma_\xi \\ &\le (|\log r| + |\log d_G| + |\log 2d_G|) \cdot \operatorname{area}(G\backslash K)\end{aligned}$$

since $r < |\xi - z| < d_G$ holds for all $\xi \in G\backslash K$ and all $z \in \Omega$. Taking into account that the last estimate is valid for every $z \in \Omega$, we obtain

$$\iint\limits_{G\backslash K} g(z,\xi)\, d\sigma_\xi \le \frac{\varepsilon}{2C_1C_2} \quad \text{for } z \in G\,.$$

So we have $|T[v_k](z) - T[v](z)| < \varepsilon$ for all $k \ge \tilde{k}$ and all $z \in \Omega$ and conclude $T : M \longrightarrow M$ is continuous. □

We are now in a position to continue the actual proof of Theorem IV.16.

Since the hypotheses of Schauder's fixed point theorem[3] are fulfilled, an application to the operator $T : M \longrightarrow M$ gives a fixed point $\tilde{v} \in M$ of T, that is,

$$\tilde{v}(z) = T[\tilde{v}](z) = h(z) - \frac{1}{2\pi} \iint\limits_{G} g(z,\xi)\, \kappa(\xi) e^{2\tilde{v}(\xi)}\, d\sigma_\xi. \tag{IV.13}$$

We claim $\tilde{v}$ belongs to $C(\overline{G}) \cap C^2(G)$ and is a solution of (IV.8).

Indeed, since $\xi \mapsto \kappa(\xi)e^{2\tilde{v}(\xi)}$ is bounded and continuous in G, the function $\tilde{v}$ belongs to $C(\overline{G}) \cap C^1(G)$ by Theorem III.10. This implies that the function $\xi \mapsto \kappa(\xi)e^{2\tilde{v}(\xi)}$ is actually locally Hölder continuous with exponent α as $\tilde{v}$ is a C^1–function. Applying Theorem III.10 once more proves $\tilde{v} \in C(\overline{G}) \cap C^2(G)$ and

$$\begin{aligned}\Delta\tilde{v} &= \kappa(\xi)e^{2\tilde{v}} && \text{in } G,\\ \tilde{v} &= \Phi && \text{on } \partial G.\end{aligned}$$

■

Now we are going to prove Theorem IV.14

[3] compare [13, p. 90] and [51]

Proof of Theorem IV.14.
We define $B(z) = \prod_{j=1}^{n}(z - z_j)^{\alpha_j}$ and set $\Phi(z) = \log\phi(z) - \log|B(z)|$. The problem of finding a pseudo–metric with the desired properties consists in solving the boundary value problem

$$\begin{aligned} \Delta u &= \kappa(z)|B(z)|^2 e^{2v} \text{ in } G, \\ u &= \Phi \qquad\qquad\quad \text{ on } \partial G, \end{aligned} \tag{IV.14}$$

cf. Section IV.1.2. The function $z \mapsto \kappa(z)|B(z)|^2$ is bounded and locally Hölder continuous with exponent $\beta = \min\{\alpha, 2\alpha_1, \ldots, 2\alpha_n, 1\}$ on G, see Remark III.6, and meets therefore the conditions of Theorem IV.16. So we get a unique solution $u \in C(\overline{G}) \cap C^2(G)$ of (IV.14) in G. Thus, $\lambda(z) = |B(z)|e^{u(z)}$ is a pseudo–metric in G with curvature $-\kappa(z)$ in $G\backslash\{z_1, \ldots, z_n\}$, zeros of orders α_j at z_j and $\lambda = \phi$ on ∂G. ■

An immediate consequence of Theorem IV.14 is the next theorem, which is equivalent to Theorem I.10.

Theorem IV.18
Let $G \subset \mathbb{C}$ be a bounded and regular domain, let $z_1, \ldots, z_n$ be finitely many distinct points in G and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers. Further, let $\phi : \partial G \longrightarrow (0, \infty)$ be a continuous function. Then there exists a unique pseudo–metric $\lambda : G \longrightarrow [0, \infty)$ of constant Gaussian curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ with zeros of orders α_j at z_j for $j = 1, \ldots, n$, such that λ is continuous on $\overline{G}$ and $\lambda(z) = \phi(z)$ for $z \in \partial G$.

Note, Theorem IV.18 is the precise analog to Theorem IV.12 for metrics with constant curvature -4.

In the next two corollaries Theorem IV.16 is considered from a different point of view. The first one shows that under some additional assumptions zeros are possible for the boundary functions of pseudo–metrics and the second one deals with non–continuous boundary functions.

Corollary IV.19
Let $G \subset \mathbb{C}$ be a bounded and regular domain. Further, let $h : G \longrightarrow \mathbb{C}$ be a holomorphic function such that $|h| : \overline{G} \longrightarrow [0, \infty)$ is continuous, and let $\kappa : G \longrightarrow [0, \infty)$ be a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique pseudo–metric λ of Gaussian curvature $-\kappa(z)$ in $G\backslash\{z \in G : h(z) = 0\}$, such that

$$\lim_{z \longrightarrow \xi} \lambda(z) = |h(\xi)|, \qquad \xi \in \partial G.$$

Proof.
We consider the boundary value problem

$$\begin{aligned} \Delta v &= \kappa(z)|h(z)|^2 e^{2v} \text{ in } G \\ v &= 0 \qquad\qquad\quad \text{ on } \partial G\,. \end{aligned} \tag{IV.15}$$

By Theorem IV.16 there exists a unique solution $v \in C(\overline{G}) \cap C^2(G)$ of (IV.15) since $z \mapsto \kappa(z)|h(z)|^2$ is bounded and locally Hölder continuous with exponent α on G. The function $\lambda(z) := |h(z)|e^{v(z)}$ is then the desired metric. ■

Corollary IV.20
Let $b : \mathbb{D} \longrightarrow \mathbb{C}$ be a bounded holomorphic function and denote by $\varphi : \partial\mathbb{D} \longrightarrow \mathbb{C}$ its boundary function. Further, let $\kappa : \mathbb{D} \longrightarrow [0, \infty)$ be a bounded and locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there exists a unique pseudo–metric λ of Gaussian curvature $-\kappa(z)$ in $\mathbb{D}\backslash\{z \in \mathbb{D} : b(z) = 0\}$ such that the non–tangential limits for λ exist almost everywhere and coincide with $|\varphi(\xi)|$, that is

$$n.t. \quad \lim_{z \longrightarrow \xi} \lambda(z) = |\varphi(\xi)|$$

for almost every $\xi \in \partial\mathbb{D}$.

Proof.
In this case we make use of the following boundary value problem

$$\begin{aligned} \Delta\omega &= \kappa(z)|b(z)|^2 e^{2\omega} \text{ in } \mathbb{D}\,, \\ \omega &= 0 \qquad\qquad\qquad \text{on } \partial\mathbb{D}\,. \end{aligned} \tag{IV.16}$$

Again, by Theorem IV.16, there exists a unique solution $\omega \in C(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$ of (IV.16), because of the fact that $z \mapsto \kappa(z)|b(z)|^2$ is bounded and locally Hölder continuous with exponent α in $\mathbb{D}$. The metric $\lambda(z) := |b(z)|e^{\omega(z)}$ does the job. ∎

IV.3.3 Maximal conformal pseudo–metrics

In view of the preceding considerations and the great significance of maximal conformal metrics of constant curvature -4 in complex analysis, it is of interest for which domains $G \subseteq \mathbb{C}$ one can find maximal pseudo–metrics of constant curvature -4 with prescribed zeros. Indeed, one can prove the existence of such pseudo–metrics for every bounded and regular domain G, but it turns out that this is a rather delicate matter. Nevertheless, we wish to explain the basic ideas how to construct maximal pseudo–metrics of constant curvature -4 in the next theorem. We need to consider pseudo–metrics which are only supposed to be upper semicontinuous and for which the curvature is defined by replacing the Laplacian with the generalized lower Laplacian.

Theorem IV.21
Let $G \subset \mathbb{C}$ be a regular and bounded domain, let $z_1, z_2, \ldots, z_n \in G$ be finitely many distinct points and let $\alpha_1, \ldots, \alpha_n$ be positive numbers. Further, let

$$\mathcal{F} = \Big\{ \lambda \,|\, \lambda \textit{ is an upper semicontinuous pseudo–metric in } G \textit{ with curvature} \leq -4 \textit{ in } G\backslash\{z_1, z_2, \ldots, z_n\} \textit{ and zeros of orders } \alpha_j \textit{ at } z_j \textit{ for } j = 1, \ldots, n \Big\}$$

Then

$$\tilde{\lambda}(z) := \sup_{\lambda \in \mathcal{F}} \lambda(z)$$

is the maximal pseudo–metric in G, which belongs to $C^2(G\backslash\{z_1, \ldots, z_n\})$, has constant curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ and zeros of orders α_j at z_j for $j \in \{1, \ldots, n\}$.

The statement of Theorem IV.21 is to be reminiscent of Perron's method for the Dirichlet problem for harmonic functions, and in fact the proof given below is patterned after Perron's approach. We will only give a rough sketch of the necessary arguments. The gaps can be filled with little change of Heins's ideas and proofs to this topic, see [22].

It is instructive to keep in mind the following dictionary:

harmonic functions	$\doteq$	conformal pseudo–metrics with curvature -4
subharmonic functions	$\doteq$	conformal pseudo–metrics with curvature less than -4

Proof.

(0) For $\alpha > 1$ the pseudo–metric
$$\frac{\alpha|z|^{\alpha-1}}{1-|z|^{2\alpha}}$$
is the maximal pseudo-metric in $\mathbb{D}$ with curvature -4 in $\mathbb{D}\backslash\{0\}$ and a zero of order $\alpha-1$ at $z=0$.

(1) $\mathcal{F} \neq \emptyset$ by Theorem IV.18.

(2) If $\lambda_1, \lambda_2 \in \mathcal{F}$, then $\max\{\lambda_1, \lambda_2\} \in \mathcal{F}$.

(3) With the help of Theorem IV.14 we can modify a pseudo–metric $\lambda \in \mathcal{F}$ in every disk K which is compactly contained in G such that the new function (i) is upper semi-continuous in G, (ii) has the desired zeros, (iii) is twice continuously differentiable in K except for the zeros, (iv) has constant curvature -4 in K save the zeros in K. Further, this new pseudo–metric is unique, and we call it the modification of λ with respect to the disk K.

The next two points are general statements about pseudo–metrics.

(4) Let $(\lambda_k)_k$ be a sequence of C^2 pseudo–metrics in G with Gaussian curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ and every λ_k has a zero of order α_j at z_j for each $j = 1, \ldots, n$. If $\infty > \lambda_L(z) = \lim_{k\longrightarrow\infty} \lambda_k(z) > 0$ in $G\backslash\{z_1, z_2, \ldots, z_n\}$, then $\lambda_L \in C^2(G\backslash\{z_1, \ldots, z_n\})$ and has constant curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$.

(5) Let λ, μ be two C^2 pseudo–metrics in G with constant curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ and the same zeros (with the same order). Further, let $\lambda \geq \mu$ in G and $\lambda(a) = \mu(a)$ for a point $a \in G\backslash\{z_1, \ldots, z_n\}$. Then $\lambda \equiv \mu$ in G.

Now we return to the family $\mathcal{F}$.

(6) Choose a point $a \in G\backslash\{z_1, \ldots, z_n\}$, a disk K_a about a which is compactly contained in G, and a sequence $(\lambda_k)_k \subseteq \mathcal{F}$ with the following properties: (i) $\lambda_{k+1} \geq \lambda_k$, (ii) every λ_k is its own modification w.r.t. K_a, and (iii) $\lambda_L(a) = \lim_{k\longrightarrow\infty} \lambda_k(a) = \tilde{\lambda}(a)$.

Next, choose a point $b \in K_a\backslash\{a, z_1, \ldots, z_n\}$ and a sequence $(\mu_k)_k \subseteq \mathcal{F}$ such that (i) $\mu_{k+1} \geq \mu_k$ and $\mu_k \geq \lambda_k$, (ii) every μ_k is its own modification w.r.t. K_a, and (iii) $\mu_L(b) = \lim_{k\longrightarrow\infty} \mu_k(b) = \tilde{\lambda}(b)$.

This gives us $\mu_L(z) \geq \lambda_L(z)$ in K_a and $\mu_L(a) = \lambda_L(a)$. Thus we can conclude from (4) and (5) that $\lambda_L = \mu_L$ in K_a. In particular, $\tilde{\lambda}(b) = \mu_L(b) = \lambda_L(b)$. Since b was an arbitrary point in $K_a \backslash \{a\}$, we have $\tilde{\lambda} = \lambda_L$ in $K_a \backslash \{a\}$. As $\tilde{\lambda}(a) = \lambda_L(a)$, we therefore get $\tilde{\lambda} \equiv \lambda_L$ in all of K_a. Because K_a was an arbitrary disk compactly contained in G, we conclude $\tilde{\lambda} \in C^2(G \backslash \{z_1, \ldots, z_n\})$ and $\tilde{\lambda}$ has constant curvature -4 in $G \backslash \{z_1, \ldots, z_n\}$.

(7) The last thing to check is that $\tilde{\lambda}$ has the correct zeros. Since $\lambda \leq \tilde{\lambda}$ for every $\lambda \in \mathcal{F}$ the order β_j of the zero z_j of $\tilde{\lambda}$ is $\leq \alpha_j$. We need to show that $\beta_j < \alpha_j$ is impossible. Choose a disk $K \subseteq G$ with center z_j and radius r, which contains none of the other zeros of $\tilde{\lambda}$. Then define the functions $f(z) = (z - z_j)/r$ and $g(z) = z^{\alpha_j + 1}$. The pseudo-metric

$$\sigma(z) := \frac{|(g \circ f)'(z)|}{1 - |g \circ f(z)|^2}$$

is well-defined in K and the maximal pseudo-metric in K with curvature -4 in $K \backslash \{z_j\}$ and a zero of order α_j in z_j. Therefore $\lambda(z) \leq \sigma(z)$ for $z \in K$ and any $\lambda \in \mathcal{F}$, and this implies $\tilde{\lambda}(z) \leq \sigma(z)$ for $z \in K$. Consequently, $\tilde{\lambda}$ has a zero of order α_j at z_j for any $j \in \{1, \ldots, n\}$. $\blacksquare$

– Chapter V –

Representation of Conformal Pseudo–metrics

V.1 Pseudo–metrics with zeros of integer order

We are now going to prove Theorem I.1 and first note that we can state Theorem I.1 as follows.

Theorem V.1
Let $E = \{z_1, z_2, \ldots\}$ be a discrete subset of a simply connected domain $G \subseteq \mathbb{C}$ and let $\alpha_1, \alpha_2, \ldots$ be positive integers. Further, let $\lambda : G \longrightarrow [0, \infty)$ be a conformal pseudo–metric of constant Gaussian curvature $\kappa = 4k \in \{-4, 0, +4\}$ in $G \backslash E$ with zeros of orders α_j at z_j. Then there exists an analytic function $f : G \longrightarrow X$ such that

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \textit{for } z \in G\,,$$

where

$$X = \begin{cases} \mathbb{D} & \textit{if} \quad k = -1 \quad \textit{or} \quad \kappa = -4, \quad \textit{respectively,} \\ \mathbb{C} & \textit{if} \quad k = 0 \quad \textit{or} \quad \kappa = 0, \quad \textit{respectively,} \\ \mathbb{P} & \textit{if} \quad k = +1 \quad \textit{or} \quad \kappa = +4, \quad \textit{respectively}\,. \end{cases} \tag{V.1}$$

If $g : G \longrightarrow X$ is another analytic map satisfying

$$\lambda(z) = \frac{|g'(z)|}{1 + k|g(z)|^2} \quad \textit{for } z \in G\,,$$

then $g = T \circ f$, where T is a rigid motion of X, i.e. T is a unit disk automorphism if $X = \mathbb{D}$, a euclidean motion of the complex plane if $X = \mathbb{C}$, and a rotation of the sphere if $X = \mathbb{P}$.

We first observe Theorem V.1 follows quite easily from the following local version of Theorem V.1 which allows at most one zero of the pseudo–metric $\lambda(z)$.

Lemma V.2
Let $K_r(z_0)$ denote the disk of radius $r > 0$ about $z_0 \in \mathbb{C}$. Further, let λ be a conformal pseudo–metric in $K_r(z_0)$ such that either λ is a conformal metric of constant curvature $4k \in \{-4, 0, +4\}$ in $K_r(z_0)$, or λ has constant curvature $4k \in \{-4, 0, +4\}$ in $K_r(z_0) \backslash \{z_0\}$ and a zero of order $\alpha \in \mathbb{N}$ at $z = z_0$. Then there exists a holomorphic function $f : K_r(z_0) \longrightarrow X$ such that

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \textit{for } z \in K_r(z_0)\,,$$

where $X = \mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$ if $k = -1$, 0 or $+1$, respectively. If $g : K_r(z_0) \longrightarrow X$ is another holomorphic function with the same properties, i.e.

$$\lambda(z) = \frac{|g'(z)|}{1 + k|g(z)|^2} \quad \text{for } z \in K_r(z_0)\,,$$

then $g = T \circ f$, where T is a rigid motion of X.

Before turning to the proof of Lemma V.2, let us explain why Theorem V.1 can be deduced from its local version, Lemma V.2, using the well–known "Kreiskettenverfahren".

Let $\lambda : G \longrightarrow [0, \infty)$ be the pseudo–metric of Theorem V.1. Then we can find to every point $\xi \in G$ a disk $K_{r(\xi)}(\xi) \subseteq G$ about ξ with radius $r(\xi) > 0$ such that the restriction of λ to $K_{r(\xi)}(\xi)$ is a conformal pseudo–metric with constant curvature $4k$ in $K_{r(\xi)}(\xi)\backslash\{\xi\}$, i.e. λ has no zeros in $K_{r(\xi)}(\xi)\backslash\{\xi\}$. Now, pick a base point, say $\xi_0 \in G$, and apply Lemma V.2 to the corresponding disk $K_{r(\xi_0)}(\xi_0)$. This yields

$$\lambda(z) = \frac{|f_0'(z)|}{1 + k|f_0(z)|^2} \quad \text{for } z \in K_{r(\xi_0)}(\xi_0)\,,$$

where $f_0 : K_{r(\xi_0)}(\xi_0) \longrightarrow X$ is some holomorphic function. We have the intention to show that f_0 has a holomorphic extension to G. For that purpose we claim f_0 can be continued analytically along every path γ in G starting at the base point ξ_0. In order to prove this assertion, let $\gamma : [0, 1] \longrightarrow G$ be an arbitrary path with initial point $\gamma(0) = \xi_0$ and endpoint $\gamma(1) = \xi_e \in G$. Then we can cover γ by finitely many of the disks $K_{r(\xi)}(\xi)$, say $K_{r(\xi_j)}(\xi_j)$, $j = 1, \ldots, N$, such that $\xi_0 \in K_{r(\xi_1)}(\xi_1)$, $\xi_e \in K_{r(\xi_N)}(\xi_N)$ and $K_{r(\xi_j)}(\xi_j) \cap K_{r(\xi_{j+1})}(\xi_{j+1}) \neq \emptyset$ for $j = 1, \ldots, N-1$. On each of these balls $K_{r(\xi_j)}(\xi_j)$ we have

$$\lambda(z) = \frac{|f_j'(z)|}{1 + k|f_j(z)|^2} \quad \text{in } K_{r(\xi_j)}(\xi_j)\,,$$

for some holomorphic function $f_j : K_{r(\xi_j)}(\xi_j) \longrightarrow X$ by Lemma V.2.

Since $K_{r(\xi_j)}(\xi_j) \cap K_{r(\xi_{j+1})}(\xi_{j+1}) \neq \emptyset$ there exists a disk $K_{r_j}(z_j)$ with center $z_j \in G$ and radius $r_j > 0$ such that $K_{r_j}(z_j) \subseteq \big(K_{r(\xi_j)}(\xi_j) \cap K_{r(\xi_{j+1})}(\xi_{j+1})\big) \backslash \{\xi_j, \xi_{j+1}\}$. Applying Lemma V.2 to the disk $K_{r_j}(z_j)$ gives

$$f_j(z) = T(f_{j+1}(z)) \quad \text{for } z \in K_{r_j}(z_j),$$

where $T : X \longrightarrow X$ is some rigid motion of X. Thus $T \circ f_{j+1}$ is the direct analytic continuation of $f_j : K_{r(\xi_j)}(\xi_j) \longrightarrow X$ to the disk $K_{r(\xi_{j+1})}(\xi_{j+1})$.

This process gives an analytic continuation of f_0 along γ to a holomorphic function

$$f : \bigcup_{j=1}^{N} K_{r(\xi_j)}(\xi_j) \longrightarrow X$$

satisfying

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \text{in} \quad \bigcup_{j=1}^{N} K_{r(\xi_j)}(\xi_j).$$

Consequently, f_0 admits unrestricted analytic continuation in the simply connected domain G and therefore there exists a holomorphic function $f : G \longrightarrow X$ such that $\lambda(z)$ is the pullback of the metric λ_X under f.

Now let $g : G \longrightarrow X$ be another holomorphic function which represents λ, that is

$$\frac{|f'(z)|}{1+k|f(z)|^2} = \frac{|g'(z)|}{1+k|g(z)|^2} \quad \text{for } z \in G\,. \tag{V.2}$$

In particular, equation (V.2) holds for some disk $K \subset G$. Thus Lemma V.2 implies $g = T \circ f$ in K for some rigid motion $T : X \longrightarrow X$ and this leads to $g = T \circ f$ in all of G. Consequently, Theorem V.1 is reduced to Lemma V.2.

Now we are buckling down to the

Proof of Lemma V.2.
Since it obviously suffices to consider the special case $K_r(z_0) = \mathbb{D}$ and $z_0 = 0$, we restrict ourselves to this situation. To get an idea of the proof, we first like to give a rough sketch of the argument.

By hypothesis, λ is a pseudo–metric in $\mathbb{D}$ and

$$\lim_{z \longrightarrow 0} \frac{\lambda(z)}{|z|^\alpha}$$

exists and $\neq 0$ for some $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The case $\alpha = 0$ corresponds to the situation where λ is a conformal metric on $\mathbb{D}$.

Since λ has constant curvature $\kappa \in \{-4, 0, +4\}$ in $\mathbb{D}$ or $\mathbb{D}\backslash\{0\}$, respectively, the function

$$u(z) := \log \lambda(z)$$

is a solution of the semi–linear elliptic PDE

$$\Delta u = -4ke^{2u} \qquad \Longleftrightarrow \qquad u_{z\bar{z}} = -ke^{2u} \tag{V.3}$$

in $\mathbb{D}$ or $\mathbb{D}\backslash\{0\}$, respectively, where $k \in \{-1, 0, +1\}$.

Now we are going to work out the following three steps.

Step 1:
We define for $z \in \mathbb{D}$ the function

$$A(z) := u_{zz}(z) - u_z(z)^2\,. \tag{V.4}$$

It will turn out that A is holomorphic in $\mathbb{D}$ if $\alpha = 0$. If $\alpha \geq 1$, then the function A is holomorphic in $\mathbb{D}$ expect for a double pole at $z = 0$, where it has a Laurent expansion of the form

$$A(z) = \frac{1-(\alpha+1)^2}{4z^2} + \cdots\,.$$

Step 2:
In this step we will consider the Schwarzian differential equation

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 2A(z) \tag{V.5}$$

in $\mathbb{D}$, where A is the function defined in (V.4). We shall show *every* solution of the Schwarzian differential equation (V.5) is a *meromorphic* function in *all of* $\mathbb{D}$. Note, this step is trivial in the special case $\alpha = 0$, see Theorem II.6 and Remark II.7.

Step 3:
Finally, we will find a *holomorphic* (for the hyperbolic and euclidean case) and a *meromorphic* (for the spherical case) solution f of (V.5) such that

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \qquad \text{for } z \in \mathbb{D}\,.$$

Let's fill in the details of *Step 1* to *Step 3*, and let $n = \alpha + 1$ for expository reasons, so $n \in \mathbb{N}$. Note, in view of Theorem IV.6, the function $u(z) = \log \lambda(z)$ has the representation

$$u(z) = \frac{n-1}{2} \log |z|^2 + \nu(z, \bar{z})\,, \tag{V.6}$$

where $\nu(z, \bar{z})$ is a real analytic function in $\mathbb{D}$. If $n = 1$, then u itself is real analytic in $\mathbb{D}$.

Proof of Step 1:
First let $n \geq 2$. As $u(z) = \log \lambda(z)$ is real analytic in $\mathbb{D}\backslash\{0\}$ the function $A(z)$ in (V.4) is well–defined and real analytic in $\mathbb{D}\backslash\{0\}$.

Differentiation of A with respect to $\bar{z}$ in $\mathbb{D}\backslash\{0\}$ yields

$$A_{\bar{z}} = u_{zz\bar{z}} - 2u_z u_{z\bar{z}} \overset{(V.3)}{=} u_{z\bar{z}z} + 2ku_z e^{2u} = \left(-ke^{2u}\right)_z + 2ku_z e^{2u} = 0.$$

Hence $A(z)$ is a holomorphic function in $\mathbb{D}\backslash\{0\}$. To check the double pole statement we replace in (V.4) the function u by its representation (V.6). Then $A(z)$ takes in $\mathbb{D}\backslash\{0\}$ the form

$$A(z) = \frac{1-n^2}{4z^2} + \nu_{zz} - \frac{n-1}{z}\nu_z - {\nu_z}^2\,. \tag{V.7}$$

Since ν_z, ${\nu_z}^2$ and ν_{zz} are real analytic in $\mathbb{D}$, the function

$$z \mapsto \left(A(z) - \frac{1-n^2}{4z^2}\right) z^2$$

is bounded in some neighborhood of $z = 0$. Thus $A(z)$ has a double pole at $z = 0$ and the asserted Laurent expansion.
If $n = 1$, then A is well–defined and holomorphic $\mathbb{D}$. This completes the proof of *Step 1.*

Proof of Step 2:
The issue here is to show that every solution of the Schwarzian differential equation

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 2A(z)$$

is a meromorphic function in $\mathbb{D}$. If $n = 1$ there is nothing to do, since in this case A is holomorphic in $\mathbb{D}$ and Theorem II.6 combined with Remark II.7 gives the desired result.

Therefore we assume $n \geq 2$ from now on. Owing to Theorem II.15 and Remark II.17 as well as in view of *Step 1* we only need to establish that $\chi = 0$ in the local solution base

$$\begin{aligned} g_1(z) &= z^{\rho_1} h_1(z), \\ g_2(z) &= \chi g_1(z) \log z + z^{\rho_2} h_2(z), \end{aligned} \tag{V.8}$$

of the differential equation $\psi'' + A(z)\psi = 0$. Here $\rho_1 = (1+n)/2$, $\rho_2 = (1-n)/2$ and h_1, h_2 are holomorphic functions in $\mathbb{D}$ with $h_1(0) \neq 0$ and $h_2(0) \neq 0$.

To justify $\chi = 0$ we consider the corresponding Riccati differential equation

$$w' = A(z) + w^2 \tag{V.9}$$

in $\mathbb{D}$. Assume for a moment we knew that this Riccati equation admits a one parameter family of meromorphic functions $(w_c)_{c \in \mathbb{P}}$ in $\mathbb{D}$. Then $\chi = 0$ in (V.8) follows for the following reason. Suppose to the contrary $\chi = 1$ in (V.8). Then the function

$$w := -\frac{g_2'}{g_2}$$

is well–defined and meromorphic in $\mathbb{D} \backslash [0,1)$, but has no meromorphic extension to $\mathbb{D}$. On the other hand, w is a meromorphic solution of $w' = A(z) + w^2$ in $\mathbb{D} \backslash [0,1)$, so $w = w_c|_{\mathbb{D} \backslash [0,1)}$ for some parameter $c \in \mathbb{P}$ according to Proposition II.26, which means w has a meromorphic extension to $\mathbb{D}$. This contradiction shows $\chi = 0$ in (V.8).

In order to complete the proof of *Step 2* it remains to establish that every solution of the Riccati differential equation (V.9) is a meromorphic function in $\mathbb{D}$.

Since $A(z)$ is holomorphic in $\mathbb{D} \backslash \{0\}$ and its Laurent expansion at $z = 0$ has the form

$$A(z) = \frac{1-n^2}{4z^2} + \cdots, \qquad n \in \mathbb{N},\, n \geq 2,$$

Theorem II.27 guarantees at least one meromorphic solution of the Riccati differential equation (V.9) in $\mathbb{D}$. By Remark II.29 this solution is of the form

$$w_1(z) = -\frac{1+n}{2z} + \gamma_1(z),$$

where $\gamma_1(z)$ is a meromorphic function in $\mathbb{D}$, which is holomorphic in a neighborhood of $z = 0$.

Now our strategy is to find a second meromorphic solution of the Riccati equation $w' = A(z) + w^2$, because then every solution is meromorphic in $\mathbb{D}$, since w_1 fulfills the hypotheses of Lemma II.25, cf. Remark II.29. For that purpose we study the structure of the function $A(z)$. Recall,

$$A(z) = \frac{1-n^2}{4z^2} + \nu_{zz} - \frac{n-1}{z}\nu_z - {\nu_z}^2,$$

where ν is a real analytic function in $\mathbb{D}$. We now will make essential use of the fact that ν is real analytic in $\mathbb{D}$. This allows to expand ν in a power series in a neighborhood U of $z = 0$. Thus we obtain for $z, \bar{z} \in U$

$$\nu(z, \bar{z}) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{jk} z^j \right) \bar{z}^k = \sum_{j=0}^{\infty} a_{j0} z^j + \sum_{k=1}^{\infty} \left(\sum_{j=0}^{\infty} a_{jk} z^j \right) \bar{z}^k,$$

that is

$$\nu(z,\overline{z}) = h(z) + \Lambda(z,\bar{z})\,,$$

if we set

$$h(z) = \sum_{j=0}^{\infty} a_{j0} z^j \qquad \text{and} \qquad \Lambda(z,\bar{z}) = \sum_{k=1}^{\infty}\left(\sum_{j=0}^{\infty} a_{jk} z^j\right)\bar{z}^k.$$

Clearly, $h(z)$ is holomorphic and $\Lambda(z,\bar{z})$ is real analytic in U.

Replacing ν by $h+\Lambda$ in (V.7) yields

$$A(z) = \frac{1-n^2}{4z^2} - \left(\frac{n-1}{z}\right)h'(z) - h'(z)^2 + h''(z) + H(z,\bar{z})$$

with

$$H(z,\bar{z}) = \Lambda_{zz}(z,\bar{z}) - \frac{n-1}{z}\Lambda_z(z,\bar{z}) - 2h'(z)\Lambda_z(z,\bar{z}) - \left(\Lambda_z(z,\bar{z})\right)^2.$$

Since $zH(z,\bar{z})$ is real analytic in U, $H(z,\bar{z})$ can be written as

$$H(z,\bar{z}) = \frac{1}{z}\cdot\left(\sum_{k=1}^{\infty}\left(\sum_{j=0}^{\infty} b_{jk} z^j\right)\bar{z}^k\right).$$

In view of the holomorphy of $A(z)$ in $\mathbb{D}\backslash\{0\}$ one might suspect $H \equiv 0$. Indeed, as the function $z^2A(z)$ is holomorphic in $U \subseteq \mathbb{D}$, we have

$$0 = \left(z^2 H(z,\bar{z})\right)_{\bar{z}} = \sum_{k=1}^{\infty}\left(\sum_{j=0}^{\infty} b_{jk} z^{j+1}\right) k\bar{z}^{k-1}\,,$$

which clearly implies $b_{jk}=0$ for all $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$, and consequently $H(z,\bar{z}) \equiv 0$.

Hence the function $A(z)$ is given by

$$A(z) = \frac{1-n^2}{4z^2} - \left(\frac{n-1}{z}\right)h'(z) - h'(z)^2 + h''(z) \quad \text{for } z \in U.$$

A glance at the right hand side of this equation shows

$$A(z) = \left(\frac{n-1}{2z} + h'(z)\right)' - \left(\frac{n-1}{2z} + h'(z)\right)^2.$$

Therefore the function

$$w_2(z) = \frac{n-1}{2z} + h'(z)$$

is a meromorphic solution of the Riccati differential equation $w' = A(z) + w^2$ in U, which can be extended to a meromorphic solution of (V.9) in the unit disk $\mathbb{D}$, see Corollary II.23. In other words, we have found a second meromorphic solution w_2 of (V.9) in $\mathbb{D}$, which obviously differs from the meromorphic solution w_1. Finally, as we have explained above, *every* solution of the Riccati equation (V.9) is meromorphic in $\mathbb{D}$ and therefore *every* solution of the Schwarzian differential equation (V.5) is a meromorphic function in $\mathbb{D}$. This concludes the proof of *Step 2*.

Proof of Step 3:
In this final step, we will find under all meromorphic solutions of $\mathcal{S}_f = 2A(z)$ one holomorphic function $f : \mathbb{D} \longrightarrow X$ such that

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \text{for } z \in \mathbb{D}\,.$$

Since the function $A(z)$ meets the requirements of Theorem II.6 and Theorem II.18, respectively, there exists to every meromorphic solution of $\mathcal{S}_f = 2A(z)$ a pair g_1, g_2 of linearly independent local solutions of the corresponding differential equation $\psi'' + A(z)\psi = 0$ such that g_2/g_1 is meromorphic in $\mathbb{D}$ and represents f there, i.e. $f = g_2/g_1$ in $\mathbb{D}$. Conversely, the quotient $f = g_2/g_1$ of any two linearly independent local solution g_1, g_2 of $\psi'' + A(z)\psi = 0$ extends to a meromorphic function in $\mathbb{D}$, which fulfills $\mathcal{S}_f = 2A(z)$ in $\mathbb{D}$.

Consequently, all we have to do is to find two linearly independent (local) solutions g_1^*, g_2^* of $\psi'' + A(z)\psi = 0$ such that $f := g_2^*/g_1^*$ satisfies

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \text{for } z \in \mathbb{D}\,.$$

We now follow largely Bieberbach [5, 6][1] and restrict ourselves to the case that $A(z)$ has a double pole at $z = 0$, since the case where A is holomorphic in $z = 0$ can be handled in a similar way.

Let g_1, g_2 be two linearly independent holomorphic solutions of the differential equation $\psi'' + A(z)\psi = 0$ in $\mathbb{D}\backslash[0,1)$. Further, the function e^{-u}, where $u = \log\lambda$, is a formal solution of $\psi'' + A(z)\psi = 0$ in $\mathbb{D}\backslash[0,1)$. This follows from the computation

$$(e^{-u})_{zz} = (u_z^2 - u_{zz})e^{-u} \quad \text{for } z \in \mathbb{D}\backslash[0,1)\,.$$

Thus we expect

$$e^{-u} = \overline{\tau(z)}\,g_1 + \overline{\sigma(z)}\,g_2 \quad \text{in } \mathbb{D}\backslash[0,1)$$

for some functions σ and τ holomorphic in $\mathbb{D}\backslash[0,1)$.
To verify such a representation for e^{-u}, we define the functions $\Upsilon_l : \mathbb{D}\backslash[0,1) \longrightarrow \mathbb{C}$ for $l \in \{1,2\}$ by

$$\Upsilon_l(z) = \overline{\frac{(-1)^l}{c}\big[g_l'(z)\,e^{-u(z)} - (e^{-u(z)})_z\,g_l(z)\big]}\,,$$

where $c := \det\left(\begin{smallmatrix} g_1 & g_2 \\ g_1' & g_2' \end{smallmatrix}\right) \in \mathbb{C}\backslash\{0\}$ is the Wronskian of g_1 and g_2.

Taking the derivative of Υ_1, Υ_2 with respect to $\overline{z}$ gives that both Υ_1 and Υ_2 are holomorphic in $\mathbb{D}\backslash[0,1)$. Now we claim

$$\sigma(z) := \overline{\frac{-1}{c}\big[g_1'(z)\,e^{-u(z)} - (e^{-u(z)})_z\,g_1(z)\big]}$$

and

$$\tau(z) := \overline{\frac{1}{c}\big[g_2'(z)\,e^{-u(z)} - (e^{-u(z)})_z\,g_2(z)\big]}$$

[1] Bieberbach considered only the hyperbolic case $k = -1$.

are the functions we are looking for. To this end we write this system of equations in matrix form, i.e.

$$\begin{pmatrix} \overline{\tau} \\ \overline{\sigma} \end{pmatrix} = \frac{1}{c} \begin{pmatrix} g_2' & -g_2 \\ -g_1' & g_1 \end{pmatrix} \begin{pmatrix} e^{-u} \\ -u_z e^{-u} \end{pmatrix} .$$

Solving for $(e^{-u}, -u_z e^{-u})^T$ yields

$$\begin{pmatrix} e^{-u} \\ -u_z e^{-u} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} \begin{pmatrix} \overline{\tau} \\ \overline{\sigma} \end{pmatrix} ,$$

which shows

$$e^{-u} = \overline{\tau(z)} g_1 + \overline{\sigma(z)} g_2 \quad \text{for } z \in \mathbb{D} \backslash [0,1) .$$

In view of this equation it seems to be advisable to take a closer look at the functions τ and σ.

As the function e^{-u} is real valued in $\mathbb{D}\backslash[0,1)$ we deduce from

$$(e^{-u})_{zz} = -A(z) \cdot e^{-u}$$

the equation

$$\tau''(z)\overline{g_1} + \sigma''(z)\overline{g_2} = -A(z)\left(\tau(z)\overline{g_1} + \sigma(z)\overline{g_2}\right) \quad \text{in } \mathbb{D}\backslash[0,1) .$$

Since g_1 and g_2 are linearly independent, the functions τ and σ are solutions of

$$\psi'' + A(z)\psi = 0$$

in $\mathbb{D}\backslash[0,1)$ and for that reason they are a linear combination of g_1 and g_2, that is

$$\tau(z) = a g_1(z) + b g_2(z) \text{ and } \sigma(z) = c g_1(z) + d g_2(z) \quad \text{for } z \in \mathbb{D}\backslash[0,1) ,$$

where $a, b, c, d \in \mathbb{C}$ are appropriate constants.
Therefore e^{-u} takes the form

$$e^{-u} = \overline{a}\,\overline{g_1} g_1 + \overline{b}\,\overline{g_2} g_1 + \overline{c}\,\overline{g_1} g_2 + \overline{d}\,\overline{g_2} g_2 \quad \text{for } z \in \mathbb{D}\backslash[0,1) .$$

The fact that e^{-u} is real valued in $\mathbb{D}\backslash[0,1)$ leads to

$$a, d \in \mathbb{R} \quad \text{and} \quad \overline{c} = b .$$

Combining this information gives

$$e^{-u} = (\overline{g_1}, \overline{g_2}) \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad \text{for } z \in \mathbb{D}\backslash[0,1) .$$

Since the matrix $\left(\begin{smallmatrix} a & b \\ \overline{b} & d \end{smallmatrix}\right)$ is hermitean there exists a matrix, call it S, with the following two properties:

$$\overline{S}^{-T} M S^{-1} = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} ,$$

where M is one of the following five matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; M_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; M_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; M_5 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$g_1^* g_2^{*\prime} - g_1^{*\prime} g_2^* = 1 \,,$$

if we define

$$\begin{pmatrix} g_1^* \\ g_2^* \end{pmatrix} = S^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} .$$

Accordingly, e^{-u} can be represented in the form

$$e^{-u} = \left(\overline{g_1^*}, \overline{g_2^*}\right) M \begin{pmatrix} g_1^* \\ g_2^* \end{pmatrix} \quad \text{in } \mathbb{D}\backslash[0,1)\,.$$

As the function e^{-u} is positive in $\mathbb{D}\backslash[0,1)$, we can exclude the possibilities M_3 and M_5 for M. To find the correct matrix M for the hyperbolic, euclidean and spherical case, we write u as

$$u = -\log\left(g_1^* \overline{g_1^*} + \delta g_2^* \overline{g_2^*}\right) \quad \text{for } z \in \mathbb{D}\backslash[0,1)\,,$$

where $\delta \in \{-1, 0, +1\}$ and plug this formula for u in the equation $\Delta u = -4ke^{2u}$. This shows, M_2 is the correct choice for the hyperbolic case, M_4 for the euclidean case and M_1 for the spherical case.

In summary

$$e^{-u} = g_1^* \overline{g_1^*} + k g_2^* \overline{g_2^*} = g_1^* \overline{g_1^*} \left(1 + k \frac{g_2^*}{g_1^*} \frac{\overline{g_2^*}}{\overline{g_1^*}}\right) \quad \text{in } \mathbb{D}\backslash[0,1)\,,$$

where $k = -1$, 0 and $+1$ corresponds to the hyperbolic, euclidean and spherical case, respectively. As mentioned earlier the function g_2^*/g_1^* extends to a meromorphic function in $\mathbb{D}$, which we will call f. We like to emphasize that

$$f' = \frac{g_2^{*\prime} g_1^* - g_1^{*\prime} g_2^*}{g_1^{*2}} = \frac{1}{g_1^{*2}}$$

holds for $z \in \mathbb{D}\backslash[0,1)$. Now we treat the hyperbolic, euclidean and spherical situation separately.

- *hyperbolic case*
  Since $f = g_2^*/g_1^*$ we obtain

  $$e^{-u} = \frac{1}{g_1^* \overline{g_1^*}} \left(1 - |f|^2\right) \quad \text{in } \mathbb{D}\backslash[0,1)\,.$$

  The fact that e^{-u} is positive in $\mathbb{D}\backslash[0,1)$ implies that the meromorphic function f is holomorphic in $\mathbb{D}$ and $|f| < 1$ in $\mathbb{D}\backslash[0,1)$. As a result of the maximum principle we see $|f| < 1$ in $\mathbb{D}$. Thus we have

  $$e^{-2u} = \frac{1}{f' \overline{f'}} \left(1 - |f|^2\right)^2 \quad \text{in } \mathbb{D}\backslash[0,1)$$

and by continuity we conclude

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \quad \text{for } z \in \mathbb{D}\,,$$

where $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic.

- *euclidean case*
  Here our starting point is

$$e^{2u} = f'\overline{f'} \quad \text{in } \mathbb{D}\backslash[0,1)\,.$$

  According to the fact that e^{2u} is finite on $\mathbb{D}$ the function f is holomorphic in $\mathbb{D}$. So we get

$$\lambda(z) = |f'(z)| \quad \text{for } z \in \mathbb{D}\,,$$

  where $f : \mathbb{D} \longrightarrow \mathbb{C}$ is holomorphic.

- *spherical case*
  In this case we have

$$e^{-2u} = \frac{1}{f'\overline{f'}}\left(1 + |f|^2\right)^2 \quad \text{in } \mathbb{D}\backslash[0,1)\,.$$

  By continuity it turns out that

$$\lambda(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

  holds for $z \in \mathbb{D}$, where f is a meromorphic function in $\mathbb{D}$.
  Note, f' has at $z = 0$ a zero of order α or a pole of order $\alpha + 2$.

So far we proved for a pseudo–metric λ in $\mathbb{D}$ with curvature $\kappa = 4k \in \{-4, 0, +4\}$ in $\mathbb{D}\backslash\{0\}$ the representation formula

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2}\,, \quad z \in \mathbb{D}\,,$$

for some holomorphic function $f : \mathbb{D} \longrightarrow X$, where $X = \mathbb{D}$, $\mathbb{C}$ and $\mathbb{P}$ belongs to $k = -1$, 0 and $+1$, respectively.

For the second assertion, let $g : \mathbb{D} \longrightarrow X$ be another holomorphic function, such that

$$\lambda(z) = \frac{|g'(z)|}{1 + k|g(z)|^2}\,, \quad z \in \mathbb{D}\,,$$

where $k \in \{-1, 0, 1\}$ corresponds to $X = \mathbb{D}$, $\mathbb{C}$ and $\mathbb{P}$, respectively.
As λ is represented by f and g, i.e.

$$\frac{|f'(z)|}{1 + k|f(z)|^2} = \frac{|g'(z)|}{1 + k|g(z)|^2}\,, \quad z \in \mathbb{D}\,, \tag{V.10}$$

the Schwarzian derivatives of f and g coincide in $\mathbb{D}$, i.e. $\mathcal{S}_f = \mathcal{S}_g$ in $\mathbb{D}$. Thus, by Remark II.16, $g = T \circ f$ with a Möbius transformation T. Therefore we write equation (V.10) in the form

$$\frac{|f'(z)|}{1+k|f(z)|^2} = \frac{|(T\circ f)'(z)|}{1+k|(T\circ f)(z)|^2}\,, \quad z \in \mathbb{D}\,,$$

and replace $f(z)$ by ξ, which then leads to

$$\frac{1}{1+k|\xi|^2} = \frac{|T'(\xi)|}{1+k|T(\xi)|^2}\,, \quad \xi \in f(\mathbb{D})\,. \tag{V.11}$$

Now we will distinguish between the hyperbolic, euclidean and spherical situation.

- *hyperbolic case*
  First we note the functions $\xi \mapsto 1/(1-|\xi|^2)$ and $\xi \mapsto |T'(\xi)|/(1-|T(\xi)|^2)$ are real analytic in $\mathbb{D}$ and $f(\mathbb{D}) \subseteq \mathbb{D}$, respectively. Since they agree on the non empty open set $f(\mathbb{D}) \subseteq \mathbb{D}$, they coincide on $\mathbb{D}$ and (V.11) holds for every $\xi \in \mathbb{D}$. Thus $T(\mathbb{D}) \subseteq \mathbb{D}$ and the Lemma of Schwarz–Pick implies that T is a unit disk automorphism.

- *euclidean case*
  In this situation (V.11) reduces to

  $$|T'(\xi)| = 1$$

  for $\xi \in f(\mathbb{D}) \subseteq \mathbb{C}$. Due to the maximum and minimum principle and the identity principle T assumes the form $T(z) = a\,z + b$, where $|a| = 1$, i.e. T is a euclidean motion of the complex plane.

- *spherical case*
  The equation

  $$\frac{1}{1+|\xi|^2} = \frac{|T'(\xi)|}{1+|T(\xi)|^2}\,, \quad \xi \in f(\mathbb{D}) \subseteq \mathbb{P}\,,$$

  immediately shows T is a rotation of the sphere $\mathbb{P}$. Consequently T is of the form

  $$T(z) = \theta\frac{z-z_0}{1+\overline{z_0}z} \quad \text{or} \quad T(z) = \frac{\theta}{z}\,,$$

  where $\theta \in \partial\mathbb{D}$ and $z_0 \in \mathbb{C}$.

The proof of Lemma V.2 is thereby complete. ■

Remark V.3
To prove Lemma V.2 we took the way which is in a sense naturally suggested by the observation that in the formula

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2}$$

the metric λ and the analytic map f are related via

$$(\log\lambda)_{zz} - ((\log\lambda)_z)^2 = \frac{1}{2}\,\mathcal{S}_f.$$

The consideration of $\psi'' + A(z)\psi = 0$ instead of $\mathcal{S}_f$ would have led to the result a bit more directly.

Theorem V.1 deals with the representation of *constantly* curved pseudo–metrics in terms of analytic functions. It can easily be generalized to conformal pseudo–metrics whose curvature is the square of the modulus of a holomorphic function.

Corollary V.4
Let $G \subseteq \mathbb{C}$ be a simply connected domain, $h : G \longrightarrow \mathbb{C}$ a holomorphic function $\not\equiv 0$, $E = \{z_1, z_2, \ldots\}$ a discrete set in G and $\alpha_1, \alpha_2, \ldots$ positive integers. Further, let $\lambda : G \longrightarrow [0, \infty)$ be a pseudo–metric of curvature $4k|h(z)|^2$ in $G\backslash E$, $k \in \{-1, 0, +1\}$, with zeros of orders α_j at z_j. Then there exists a holomorphic function $f : G \longrightarrow X$ such that

$$\lambda(z) = \frac{1}{|h(z)|} \cdot \frac{|f'(z)|}{1 + k|f(z)|^2}, \quad z \in G,$$

where $X = \mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$ if $k = -1$, 0 or $+1$, respectively. If $g : G \longrightarrow X$ is another analytic map satisfying

$$\lambda(z) = \frac{1}{|h(z)|} \frac{|g'(z)|}{1 + k|g(z)|^2}, \quad z \in G,$$

then $g = T \circ f$ with a rigid motion T of X.

Proof.
If we denote $u := \log \lambda$, then u solves by hypothesis

$$\Delta u = -4k|h(z)|^2 e^{2u} \quad \text{in } G\backslash E.$$

Let's consider the pseudo–metric $\tilde{\lambda}(z) = |h(z)|\lambda(z)$ on G. We compute

$$\Delta \tilde{u} = \Delta u = -4k|h(z)|^2 e^{2u} = -4k e^{2\tilde{u}} \quad \text{in } G\backslash(E \cup G_*),$$

where $\tilde{u} = \log \tilde{\lambda}$ and $G_* = \{z \in G : h(z) = 0\}$. Thus $\tilde{\lambda}$ has constant curvature $4k$ in $G\backslash(E \cup G_*)$ and zeros of integer order exactly at the points of $E \cup G_*$. Applying Theorem V.1 to the pseudo–metric $\tilde{\lambda} : G \longrightarrow [0, \infty)$ yields

$$\tilde{\lambda}(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \text{for } z \in G,$$

where $f : G \longrightarrow X$ is some holomorphic function. Hence we deduce

$$\lambda(z) = \frac{1}{|h(z)|} \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \text{for } z \in G.$$

■

Remark V.5
In fact, Corollary V.4 is equivalent *to Theorem V.1 and also to Theorem I.8. Corollary V.4 clearly implies Theorem I.8. To deduce Theorem V.1 from Theorem I.8 take a non–constant holomorphic function $h : G \longrightarrow \mathbb{C}$ with zeros of orders α_j at z_j and no others. Then $\lambda(z)/|h(z)|$ is a conformal* metric *of curvature $4k|h(z)|^2$ and an application of Theorem I.8 yields the assertion of Theorem V.1.*

V.2 Pseudo–metrics with zeros of non–integer order

We now turn to a representation formula for conformal pseudo–metrics with zeros of non–integer order. Such conformal metrics with constant negative curvature play an important role in the following well–known generalization of Schwarz's Lemma due to Nehari [37].

Theorem V.6 (Nehari)
Let $w = f(z)$ be a non–uniform function, regular for $|z| < 1$ apart from a finite number of algebraic branch points and let $f'(z)$ be finite everywhere in $|z| < 1$; let further, for all determinants of $f(z)$, $|f(z)| \leq 1$ for $|z| < 1$. Then we have

$$|f'(0)| \leq 1$$

for all the different values $f'(0)$ may assume. The case $|f'(0)| = 1$ can only happen for $f(z) \equiv K z$, $|K| = 1$.

In brief we can describe Nehari's proof as follows:
In a first step Nehari studies

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$$

on a Riemann surface R associated with f and proves λ represents a pseudo–metric on R which has constant curvature -4 except for the points, where $f'(z) = 0$. In a second step he applies Ahlfors's Lemma to the metric λ in order to obtain his result.

Note, if f is as in Nehari's theorem, then the metric $\lambda(z)$ has (in local coordinates) a zero of order $\alpha \in (0, \infty)$ at the branch points of f. The following result shows that conversely every constantly curved conformal pseudo–metric with isolated zeros of positive order in a not necessarily simply connected domain G can be represented as the pullback of one of the canonical metrics λ_X under a multi–valued analytic function $f : G \longrightarrow X$. For $G = \mathbb{D}$ and $E = \{0\}$ this result was also obtained by Chou and Wan [8, 9], whose proof is based on Liouville's theorem, our Corollary I.3, applied to the pullback of the metric λ on the universal covering surface of $G \backslash E = \mathbb{D} \backslash \{0\}$. The proof given below is more direct and avoids the use of the universal covering. Instead, it is based on some simple facts from the theory of complex differential equations.

Theorem V.7
Let $E = \{z_1, z_2, \ldots\}$ be a discrete set in a domain $G \subseteq \mathbb{C}$ and let $\alpha_1, \alpha_2, \ldots \in (0, \infty)$. Further, let $\lambda : G \longrightarrow [0, \infty)$ be a conformal pseudo–metric of constant curvature $\kappa = 4k \in \{-4, 0, +4\}$ in $G \backslash E$ with zeros of orders α_j at z_j. Then there exists a possibly multi–valued analytic function f from G into X such that

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \text{for } z \in G,$$

with $X = \mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$ according to $k = -1$, 0 or $+1$. The functions $|f| : G \longrightarrow [0, \infty]$ and $|f'| : G \longrightarrow [0, \infty]^2$ are continuous. If g is another multi–valued analytic function from G into X satisfying

$$\lambda(z) = \frac{|g'(z)|}{1 + k|g(z)|^2} \quad \text{for } z \in G,$$

then $g = T \circ f$, where T is some rigid motion of X.

[2] In the spherical case "∞" may be attained.

Proof.
We may restrict ourselves to the case where λ is a conformal pseudo–metric in $\mathbb{D}$ which has curvature $4k$ in $\mathbb{D}\backslash\{0\}$ and a zero of order $\alpha \not\in \mathbb{N}$ at $z=0$. Then Theorem V.7 follows from the same arguments we used to derive Theorem V.1 from Lemma V.2.

Let's assume λ is such a metric. If we apply Theorem V.1 to the metric λ on the simply connected domain $\mathbb{D}\backslash[0,1)$ we obtain

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \text{in } \mathbb{D}\backslash[0,1)\,,$$

for some holomorphic function $f : \mathbb{D}\backslash[0,1) \longrightarrow X$. Now we define $u(z) := \log\lambda(z)$ and observe u has in $\mathbb{D}\backslash\{0\}$ the representation

$$u(z) = \alpha \log|z| + \nu(z)\,,$$

where $\nu \in C^2(\mathbb{D})$ and ν is real analytic in $\mathbb{D}\backslash\{0\}$, see Theorem IV.5 and Theorem IV.6. Thus the function

$$A(z) = u_{zz}(z) - u_z(z)^2$$

is well–defined and holomorphic in $\mathbb{D}\backslash\{0\}$. At $z=0$ the function $A(z)$ has the Laurent expansion

$$A(z) = \frac{1-\eta^2}{4z^2} + \cdots$$

with $\eta = \alpha+1$. Further, the function f is a solution of the Schwarzian differential equation

$$\mathcal{S}_f = 2A(z)$$

in $\mathbb{D}\backslash[0,1)$. Consequently, by Theorem II.6 and Theorem II.9, the function f has the form

$$f(z) = \frac{a\,z^{\rho_1}\,h_1(z) + b\,z^{\rho_2}\,h_2(z)}{c\,z^{\rho_1}\,h_1(z) + c\,z^{\rho_2}\,h_2(z)} = \frac{a\,z^{\eta}\,h_1(z) + b\,h_2(z)}{c\,z^{\eta}\,h_1(z) + d\,h_2(z)}\,, \tag{V.12}$$

where $\rho_1 = (1+\eta)/2$, $\rho_2 = (1-\eta)/2$, h_1, h_2 are holomorphic functions in $\mathbb{D}$ with $h_1(0) \neq 0$, $h_2(0) \neq 0$ and $a, b, c, d \in \mathbb{C}$ are appropriate constants with $ad - bc \neq 0$.

Since $\eta \not\in \mathbb{N}$ the function f has no meromorphic extension to $\mathbb{D}$, but it can be continued analytically along any path in $\mathbb{D}\backslash\{0\}$. Therefore we conclude with the help of Lemma V.2

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \text{for } z \in \mathbb{D}\backslash\{0\}\,, \tag{V.13}$$

where

$$f : \mathbb{D}\backslash\{0\} \xrightarrow[\text{valued}]{\text{multi}} \begin{cases} \mathbb{D} & \text{if } k=-1 \\ \mathbb{C} & \text{if } k=0 \\ \mathbb{P} & \text{if } k=+1\,. \end{cases}$$

Our next aim is to show

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \text{for } z \in \mathbb{D}\,.$$

This follows from the fact that $|f|$ and $|f'|$ are continuous functions in $\mathbb{D}$. This latter assertion, in turn, is clear from (V.12) and the explicit formula for f', that is

$$f'(z) = z^{\eta-1} \cdot \frac{(ad-bc)\cdot(\eta\, h_1(z)\, h_2(z) + z\cdot(h_1'(z)\, h_2(z) - h_1(z)\, h_2'(z)))}{(c\, z^\eta\, h_1(z) + d\, h_2(z))^2}\,. \tag{V.14}$$

Also, we can infer from (V.14) and (V.13) that f' has at $z=0$ a zero of order $\eta - 1 = \alpha$ if $k=-1$ or $k=0$, or a zero of order $\eta-1=\alpha$ or a "pole" of order $\eta+1=\alpha+2$ if $k=+1$. ■

Remark V.8
If we consider in Theorem V.7 the case $G=\mathbb{D}$ and $k=-1$, then we can conclude from Ahlfors's Lemma $|f'(0)| \le 1$ for all different values $|f'(0)|$ may assume.

V.3 Pseudo–metrics with isolated singularities

For the sake of completeness we discuss the following two representation lemmas for constantly curved conformal metrics with isolated singularities.

Lemma V.9
Let λ be a conformal metric on $\mathbb{D}\backslash\{0\}$ of constant curvature $4k \in \{-4,0,+4\}$ in $\mathbb{D}\backslash\{0\}$, such that

$$\lim_{z\longrightarrow 0} \frac{\lambda(z)}{|z|^\alpha} \quad \textit{exists and } \neq 0$$

for some $\alpha \in (-1,0)$ and $\log\lambda$ has a representation of the form

$$\log\lambda(z) = \alpha\log|z| + \nu(z) \quad \textit{for } z\in\mathbb{D}\backslash\{0\}$$

with some function $\nu \in C^2(\mathbb{D})$. Then there exists a multi–valued analytic function f from $\mathbb{D}$ into X, such that

$$\lambda(z) = \frac{|f'(z)|}{1+k|f(z)|^2} \quad \textit{for } z\in\mathbb{D}\backslash\{0\}\,,$$

where $X=\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$ if $k=-1$, 0 or $+1$, respectively. The functions $|f|:\mathbb{D}\longrightarrow[0,\infty]$ and $|f'|:\mathbb{D}\longrightarrow[0,\infty]$ are continuous and $|f'|$ has in the hyperbolic and euclidean case at $z=0$ the same singularity as λ, that is the function

$$\log|f'(z)| - \alpha\log|z|$$

has a C^2–extension to $\mathbb{D}$. If g is another multi-valued analytic function from $\mathbb{D}$ into X satisfying

$$\lambda(z) = \frac{|g'(z)|}{1+k|g(z)|^2} \quad \textit{for } z\in\mathbb{D}\backslash\{0\}\,,$$

then $g = T\circ f$, where T is some rigid motion of X.

Proof.
The proof runs in principle parallel to the proof of Theorem V.7. We only remark that f' has at $z=0$ a "pole" of order

$$\begin{cases} |\alpha| & \text{if } k=-1 \text{ or } k=0, \\ |\alpha| \text{ or } 2-|\alpha| & \text{if } k=+1. \end{cases}$$

■

The next lemma is only valid for conformal metrics of constant curvature -4, see Remark V.11 below.

Lemma V.10
Let λ be a conformal metric on $\mathbb{D}\backslash\{0\}$ which has constant curvature -4 there. Further suppose

$$\lim_{z\longrightarrow 0} \lambda(z)\cdot\left(|z|\log\frac{1}{|z|}\right) \quad \textit{exists and } \neq 0$$

and $\log\lambda$ has a representation of the form

$$\log\lambda(z) = -\log|z| - \log\log\frac{1}{|z|} + \nu(z) \quad \textit{for } z\in\mathbb{D}\backslash\{0\}$$

with some function $\nu\in C^2(\mathbb{D})$. Then there exists a multi-valued analytic function f in $\mathbb{D}$ such that

$$\lambda(z) = \frac{|f'(z)|}{1-|f(z)|^2} \quad \textit{for } z\in\mathbb{D}\backslash\{0\}$$

and $|f|<1$ in $\mathbb{D}\backslash\{0\}$. The singularity of $|f'|$ does not correspond to the singularity of λ in this situation. If g is another multi-valued analytic function from $\mathbb{D}$ into $\mathbb{D}$ satisfying

$$\lambda(z) = \frac{|g'(z)|}{1-|g(z)|^2} \quad \textit{for } z\in\mathbb{D}\backslash\{0\},$$

then $g = T\circ f$, where T is an automorphism of $\mathbb{D}$.

Proof.
We are going to follow the same lines as in the proof of Theorem V.7.

We consider λ in $\mathbb{D}\backslash(-1,0]$ and obtain by Theorem V.1

$$\lambda(z) = \frac{|f'(z)|}{1-|f(z)|^2} \quad \text{in } \mathbb{D}\backslash(-1,0]$$

for some holomorphic function $f:\mathbb{D}\backslash(-1,0]\longrightarrow\mathbb{D}$. The function

$$A(z) = u_{zz}(z) - u_z(z)^2,$$

where $u:=\log\lambda$, is well–defined and holomorphic in $\mathbb{D}\backslash\{0\}$, and has at $z=0$ a Laurent expansion of the form

$$A(z) = \frac{1}{4z^2} + \cdots .$$

The function f is then a solution of the Schwarzian differential equation

$$\mathcal{S}_f = 2A(z)$$

in $\mathbb{D}\backslash(-1,0]$. Thus by Theorem II.6 and Theorem II.9 it has the form

$$f(z) = \frac{a\,z^{1/2}\,h_1(z) + b\,z^{1/2}\,h_2(z) + b\,z^{1/2}\,h_1(z)\,\log z}{c\,z^{1/2}\,h_1(z) + d\,z^{1/2}\,h_2(z) + d\,z^{1/2}\,h_1(z)\,\log z}, \tag{V.15}$$

where h_1, h_2 are holomorphic functions in $\mathbb{D}$ with $h_1(0) \neq 0$, $h_2(0) \neq 0$ and $a,b,c,d \in \mathbb{C}$ are appropriate constants with $ad - bc \neq 0$.

We see the function f has no holomorphic extension to $\mathbb{D}$, but f can be continued analytically along any path in $\mathbb{D}\backslash\{0\}$. Therefore we have by Lemma V.2

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \quad \text{for } z \in \mathbb{D}\backslash\{0\}\,,$$

and $|f| < 1$ in $\mathbb{D}\backslash\{0\}$.
Now we move on to the singularity statement:
Taking the derivative of f yields

$$f'(z) = \frac{(bc - ad)(h_1(z)^2 - z\,(h_2(z)\,h_1'(z) - h_1(z)\,h_2'(z)))}{z \cdot (c\,h_1(z) + d\,h_2(z) + d\,h_1(z)\,\log z)^2}\,. \tag{V.16}$$

This shows

$$\lim_{z \longrightarrow 0} |f'(z)||z| \log\frac{1}{|z|} = 0 \quad \text{if } d \neq 0$$

and

$$\lim_{z \longrightarrow 0} |f'(z)||z| \log\frac{1}{|z|} = \infty \quad \text{if } d = 0\,,$$

which means $|f'|$ cannot have the same singularity at $z = 0$ as λ. Thus $\lim_{z \longrightarrow 0} |f(z)| = 1$ must hold. ■

Remark V.11
We observe there does not exist a conformal metric in $\mathbb{D}\backslash\{0\}$ of constant curvature $4k \in \{0, +4\}$ with the properties of Lemma V.10.

In fact, if there exists such a metric, then we follow the same lines as in the proof of Lemma V.10 and find a holomorphic function $f : \mathbb{D}\backslash(-1,0] \longrightarrow X$ such that

$$\lambda(z) = \frac{|f'(z)|}{1 + k|f(z)|^2} \quad \textit{for } z \in \mathbb{D}\backslash(-1,0]\,,$$

where $X = \mathbb{C}$ or $X = \mathbb{P}$ corresponds to $k = 0$ or $k = +1$.

The function f has the same form as in (V.15) and therefore f' is given by formula (V.16). Now take a sequence $(z_n) \subseteq (0,1)$ such that $\lim_{n \longrightarrow \infty} z_n = 0$. By hypothesis

$$\lim_{z \longrightarrow 0} \lambda(z)|z| \log\frac{1}{|z|} = L\,,$$

where $L \in (0, \infty)$. But

$$\lim_{n \longrightarrow \infty} |f'(z_n)||z_n| \log \frac{1}{|z_n|} = \begin{cases} \infty & \text{if } d = 0 \\ 0 & \text{if } d \neq 0 \end{cases}$$

and

$$\lim_{n \longrightarrow \infty} \frac{|f'(z_n)|}{1 + |f(z_n)|^2} \cdot |z_n| \log \frac{1}{|z_n|} = 0 \,,$$

which in both cases leads to a contradiction.

We finish up this section with an example which shows that there exists indeed a metric which fulfills the hypotheses of LemmaV.10.

Example V.12
In Example IV.9 (b) we saw that

$$\lambda(z) = \frac{1}{2|z| \log\left(\frac{1}{|z|}\right)}$$

is a conformal metric with curvature -4 on $\mathbb{D} \backslash \{0\}$. The multi–valued function which represents λ is

$$f(z) = \frac{\frac{1}{2} + \log z}{-\frac{1}{2} + \log z} \,,$$

i.e.

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \quad \text{in } \mathbb{D} \backslash \{0\} \,.$$

– Chapter VI –

Applications

VI.1 Proofs of Theorem I.17, Theorem I.19 and Proposition I.20

After the preparations in Chapter IV and Chapter V we are now in a position to provide rapid proofs of Theorem I.17, Theorem I.19 and Proposition I.20. For the convenience of the reader we recall the statements of these results and begin with Theorem I.17.

Theorem VI.1
Let $G \subset \mathbb{C}$ be a bounded simply connected domain, $z_1, \ldots, z_n$ finitely many distinct points in G and $\alpha_1, \ldots, \alpha_n$ positive integers. Also, let $\phi : \partial G \longrightarrow \mathbb{R}$ be a continuous positive function. Then there exists a holomorphic function $f : G \longrightarrow \mathbb{D}$ with critical points of orders α_j at z_j and no others such that

$$\lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi), \qquad \xi \in \partial G. \tag{VI.1}$$

If $g : G \longrightarrow \mathbb{D}$ is another holomorphic function with these properties, then $g = T \circ f$ for some conformal disk automorphism $T : \mathbb{D} \longrightarrow \mathbb{D}$.

Proof.
This is an immediate consequence of Theorem IV.18 and Theorem V.1. In fact, in view of Theorem IV.18 there exists a uniquely determined pseudo–metric λ in G of curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ with zeros of orders α_j at z_j and boundary values ϕ. By Theorem V.1

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2}, \qquad z \in G,$$

for some holomorphic function $f : G \longrightarrow \mathbb{D}$. Thus f has critical points of orders α_j at z_j and no others, and the boundary condition (VI.1) is fulfilled. If g is another holomorphic function $g : G \longrightarrow \mathbb{D}$ with the properties stated in Theorem VI.1, then

$$\tilde{\lambda}(z) := \frac{|g'(z)|}{1 - |g(z)|^2}$$

is a conformal pseudo–metric in G of curvature -4 in $G\backslash\{z_1, \ldots, z_n\}$ with zeros of orders α_j at z_j and boundary values ϕ. From the uniqueness statement of Theorem IV.18 we infer $\tilde{\lambda} = \lambda$ in G, that is

$$\frac{|g'(z)|}{1 - |g(z)|^2} = \frac{|f'(z)|}{1 - |f(z)|^2} \quad \text{for } z \in G.$$

Hence, applying Theorem V.1, we see that $g = T \circ f$ for some conformal automorphism T of $\mathbb{D}$. ■

If we use Theorem IV.16 instead of Theorem IV.18 in the above proof, we obtain Theorem I.19:

Theorem VI.2
Let (z_j) be a sequence of points in $\mathbb{D}$ satisfying the Blaschke condition

$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty,$$

and let $\phi : \partial\mathbb{D} \longrightarrow [0, \infty)$ be a function such that $\log \phi \in L^\infty(\partial\mathbb{D})$. Then there exists a holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ with critical points z_j (counted with multiplicities) such that

$$\sup_{z \in \mathbb{D}} \frac{|f'(z)|}{1 - |f(z)|^2} < \infty,$$

and

$$n.t. \qquad \lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi) \quad \textit{for a.e. } \xi \in \partial\mathbb{D}.$$

If $g : \mathbb{D} \longrightarrow \mathbb{D}$ is another holomorphic function with these properties, then $g = T \circ f$ for some conformal disk automorphism $T : \mathbb{D} \longrightarrow \mathbb{D}$.

Proof.
We only prove the existence part. Let B be a Blaschke product with zeros z_j (counted with multiplicities). Note, $\lim_{r \longrightarrow 1-} |B(r\xi)| = 1$ for a.e. $\xi \in \partial\mathbb{D}$. Next, let h be the harmonic function in $\mathbb{D}$ with boundary values $\log \phi$, so $|h(z)| \leq M$ in $\mathbb{D}$ for some $M > 0$ and

$$n.t. \qquad \lim_{z \longrightarrow \xi} h(z) = \log \phi(\xi) \qquad \text{for a.e. } \xi \in \partial\mathbb{D}.$$

By Theorem IV.16 there exists a unique conformal metric μ with curvature $-4|B(z)|^2 e^{2h(z)}$ in $\mathbb{D}$ and $\mu(\xi) = 1$ for $\xi \in \partial\mathbb{D}$. Then $\lambda(z) = e^{h(z)}|B(z)|\mu(z)$ is a (bounded) conformal pseudo–metric in $\mathbb{D}$ with curvature -4 in $\mathbb{D}\backslash\{z_j : j \in \mathbb{N}\}$ and non–tangential boundary values $\phi(\xi)$ for a.e. $\xi \in \partial\mathbb{D}$. Thus we can apply Theorem V.1 and see there is a holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ with critical points z_j (counted with multiplicities) and

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \quad \text{for } \; z \in \mathbb{D}.$$

By construction,

$$n.t. \qquad \lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = \phi(\xi) \quad \text{for a.e. } \xi \in \partial\mathbb{D}.$$

■

Lastly, we turn to Proposition I.20.

Proposition VI.3
Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ *be a non–constant holomorphic function such that*

$$\sup_{z \in \mathbb{D}} \frac{|f'(z)|}{1 - |f(z)|^2} < \infty.$$

Then the non–tangential limit

$$n.t. \qquad \lim_{z \longrightarrow \xi} \frac{|f'(z)|}{1 - |f(z)|^2} =: \phi(\xi)$$

exists for a.e. $\xi \in \partial\mathbb{D}$ *and* $\log \phi \in L^1(\partial\mathbb{D})$. *Moreover, if* (z_j) *is the sequence of critical points of* f *(counted with multiplicity), then*

$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty.$$

Proof.
As we already noted, see Remark I.18 (a), f extends to a continuous function on $\overline{\mathbb{D}}$ with $f(\overline{\mathbb{D}}) \subseteq \mathbb{D}$. Hence f' is a *bounded* analytic function in $\mathbb{D}$. Therefore the zeros z_j of f' satisfy the Blaschke condition, f' has non–tangential limits a.e. on $\partial\mathbb{D}$ and the boundary function $\log|f'|$ belongs to $L^1(\partial\mathbb{D})$, see [46]. ■

VI.2 Proofs of generalizations of Wittich's theorem

We are now going to prove Theorem I.25 and first note the following result which is also a generalization of Wittich's Theorem I.21.

Theorem VI.4
Let $E = \{z_1, z_2, \ldots\} \subseteq \mathbb{C}$ *be a discrete set. Then there exists no solution* w *of the PDE*

$$\Delta w = e^w \quad \text{in } \mathbb{C} \backslash E$$

with the following properties:

(a) $w : \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\}$ *is* C^2 *in* $\mathbb{C}\backslash E$.

(b) For every $z_j \in E$ *there is a non–negative integer* α_j *such that*

$$\lim_{z \longrightarrow z_j} (w(z) - 2\alpha_j \log|z - z_j|)$$

exists finitely.

Proof.
Assume there is such a solution, say $\tilde{w}$. Then $u = (\tilde{w} - \log 8)/2$ is a C^2–solution of $\Delta u = 4e^{2u}$ in $\mathbb{C}\backslash E$, so $\lambda = e^u$ is a conformal pseudo–metric in $\mathbb{C}$ with curvature -4 in $\mathbb{C}\backslash E$ and zeros of orders α_j at z_j. From Theorem V.1 we see that the function $\tilde{w}$ has the representation

$$\tilde{w}(z) = 2\log\left(\frac{|f'(z)|}{1-|f(z)|^2}\right) + \log 8 \quad \text{for } z \in \mathbb{C},$$

where $f : \mathbb{C} \longrightarrow \mathbb{D}$ is a holomorphic function. This implies f is constant and the contradiction is apparent. ■

Theorem I.25 is now a simple corollary of Theorem VI.4.

Proof of Theorem I.25.
Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function not identically zero. We prove Theorem I.25 by contradiction, and assume therefore there does exist a C^2–function $w : \mathbb{C} \longrightarrow \mathbb{R}$ such that $\Delta w = |h(z)|^2 e^w$ in $\mathbb{C}$. Then the function $\tilde{w} = w + 2\log|h|$ has the following properties:

(a) $\tilde{w} : \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\}$ is C^2 with $\Delta\tilde{w} = e^{\tilde{w}}$ in $\mathbb{C}\backslash\{z \in \mathbb{C} \,:\, h(z) = 0\}$.

(b) If $z_0 \in \mathbb{C}$ is a zero of $h(z)$ of order α, then the limit

$$\lim_{z \longrightarrow z_0} \left(\tilde{w}(z) - 2\alpha \log|z - z_0|\right)$$

exists finitely.

However, since $E := \{z \in \mathbb{C} \,:\, h(z) = 0\}$ is a discrete subset of $\mathbb{C}$ this contradicts Theorem VI.4. ■

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