On some Maximal Convergence Theorems for Real Analytic Functions in $\mathbb{R}^N$

Dissertation zur Erlangung des
naturwissenschaftlichen Doktorgrades
der Bayerischen Julius–Maximilians–Universität Würzburg

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Christiane Kraus

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1. Gutachter: Prof. Dr. Stephan Ruscheweyh
2. Gutachter: Prof. Dr. Jo´zef Siciak

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Contents

Chapter 1

Introduction

1.1 Maximal convergence

A central theme in constructive approximation theory is the relation between the smoothness of a function and the speed at which it can be approximated by polynomials. Classical one dimensional results in this context are for instance Jackson theorems and maximal convergence theorems of Bernstein and Walsh. Both kind of theorems have attracted much attention and some endeavor has recently been made to extend them to higher dimensions. Our work deals with maximal convergence theorems in several variables. To state some results of this type we first need to define an approximation measure.

Definition 1.1

(i) Let $K \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be compact and let $F : K \to \mathbb{R}$ be a continuous function. Then we define

$$E_n(K, F) := \inf\{\|F - P_n\|_K : \ P_n : \mathbb{R}^N \to \mathbb{R},\ P_n \text{ a polynomial of degree} \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K.

(ii) Let $K \subset \mathbb{C}^N$, $N \in \mathbb{N}$, be compact and let $f : K \to \mathbb{C}$ be a continuous function. Then we define analogously

$$e_n(K, f) := \inf\{\|f - p_n\|_K : \ p_n : \mathbb{C}^N \to \mathbb{C},\ p_n \text{ a polynomial of degree} \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K.

Now let $\rho \in (1, \infty]$ and $F : K \to \mathbb{R}$ be a continuous function on the compact set $K \subset \mathbb{R}^N$, $N \in \mathbb{N}$, such that

$$\limsup_{n \to \infty} \sqrt[n]{E_n(K, F)} = \frac{1}{\rho}. \tag{1.1}$$

Then we say a sequence $\{P_n\}_{n \in \mathbb{N}}$ of polynomials P_n of degree $\leq n$ converges maximally to F, if for every $R \in (1, \rho)$ the estimate

$$\|F - P_n\|_K \leq \frac{M}{R^n}, \qquad n \in \mathbb{N},$$

holds, where $M > 0$ is some constant independent of n.
Theorems which describe the connection between ρ and F as in equation (1.1) are called

maximal convergence theorems. Analogously, we use this terminology for complex–valued functions f defined on compact sets in $\mathbb{C}^N$.
A famous result that marks the beginning of a series of studies on maximal convergence is the Bernstein theorem:

Theorem 1.2 ([Ber52], 1912)
Let $F : [-1, 1] \to \mathbb{R}$ be continuous and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1],F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to the set

$$\{z \in \mathbb{C} : \ |h(z)| < \rho\},$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\{z \in \mathbb{C} : |z| < 1\}$ is defined by $h(z) = z + \sqrt{z^2 - 1}$[1].

In the year 1934 Walsh (and Russell) discovered an outstanding extension of Theorem 1.2. The interval $[-1, 1]$ in Theorem 1.2 can be replaced by compact sets $K \subset \mathbb{C}$ whose complement is connected and regular in the sense that for $\hat{\mathbb{C}}\backslash K$, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, Green's function g_K with pole at infinity exists[2].
We recall, Green's function g_K is the uniquely determined function which has a logarithmic singularity at infinity, is continuous in $\mathbb{C}$, harmonic in $\mathbb{C}\backslash K$ and identically zero on K.

Theorem 1.3 ([Wal35], 1934)
Let K be a compact subset of $\mathbb{C}$ such that $\hat{\mathbb{C}}\backslash K$ is connected and regular. Furthermore, let $f : K \to \mathbb{C}$ be continuous and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,f)} \leq \frac{1}{\rho}$$

if and only if $f \equiv \tilde{f}|_K$, where $\tilde{f}$ is a holomorphic function in

$$L_\rho = \{z \in \mathbb{C} : \ e^{g_K(z)} < \rho\}.$$

A first step to an extension of the Bernstein–Walsh theorems to higher dimensions was taken by Sapagov in 1956. He stated the following analogous result to the Bernstein theorem.

Theorem 1.4 ([Sap56], 1956)
Let $F : K \subset \mathbb{R}^N \to \mathbb{R}$ be a continuous function, where $K := K_1 \times K_2 \times \cdots \times K_N$, $K_j = [-1, 1]$, $1 \leq j \leq N$, and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_\rho = L_{\rho_1} \times L_{\rho_2} \times \cdots \times L_{\rho_N},$$

where $L_{\rho_j} = \{z \in \mathbb{C} : |h(z)| < \rho\}$, $1 \leq j \leq N$, and h is defined as in Theorem 1.2.

[1] The branch of the square root is chosen such that $h(x) > 1$ for $x > 1$.

[2] The generalization of Theorem 1.2 is due to Walsh [Wal26] in the case that $\hat{\mathbb{C}}\backslash K$ is simply connected in $\hat{\mathbb{C}}$ and due to Walsh and Russell [WR34] in the case that $\hat{\mathbb{C}}\backslash K$ is connected and regular. However in the literature Theorem 1.2 and Theorem 1.3 are just called the Bernstein-Walsh theorems.

The proof of Theorem 1.4 uses concepts of the proof of Bernstein's theorem. The function F is considered on the intervals K_j, $j = 1, 2, \ldots, N$, separately. Similarly, Theorem 1.3 can be generalized if the compact set $K \subset \mathbb{C}^N$ can be expressed as a Cartesian product of compact subsets of the complex plane. However, for an arbitrary (sufficiently nice) compact set $K \subset \mathbb{C}^N$ the situation is much more involved. Siciak [Sic62] was the first who managed to extend Theorem 1.3 to appropriate compact sets $K \subset \mathbb{C}^N$, see Theorem 1.5. His key to this result was the introduction of an extremal function Φ for compact sets K in $\mathbb{C}^N$, which behaves in many ways like the (generalized) Green's function for $\hat{\mathbb{C}} \backslash K$ with pole at infinity. Later Zaharjuta found a different approach to Theorem 1.5, using the technique of Hilbert scales, compare [Zah76]. A refinement of Siciak's proof [Sic62] of Theorem 1.5 can be found in [Sic81]. We also refer to Bloom [Blo89] for an ingenious modification of Siciak's latter proof.

Theorem 1.5 ([Sic62], 1962)
Let $K \subset \mathbb{C}^N$ be a compact set such that the extremal function $\Phi(z, K)$ is continuous in $\mathbb{C}^N$. Further, let $f : K \to \mathbb{C}$ be continuous and $\rho > 1$. Then

$$\limsup_{n \to \infty} \sqrt[n]{e_n(K, f)} \le \frac{1}{\rho}$$

if and only if f has a holomorphic extension to

$$\{z \in \mathbb{C}^N : \Phi(z, K) < \rho\}.$$

Apart from Siciak's and Zaharjuta's generalizations of Theorem 1.3 there are also many other methods to extend the Bernstein–Walsh theorems to higher dimensions, e.g. for harmonic functions in $\mathbb{R}^N$ ([And93], [BL91], [SZ01]), pluriharmonic functions in $\mathbb{C}^N$ ([Sic96]) and solutions of elliptic equations ([BL94]) in $\mathbb{R}^N$.

1.2 Results for real analytic functions in $\mathbb{R}^N$

In our work we prove maximal convergence theorems for real analytic functions[3] in $\mathbb{R}^N$, especially for functions of squared modulus holomorphic type. In addition, we will give various applications of our results.
Our starting point for this paper is a question raised by Braess. In [Bra01] he proved the following theorem, which may be regarded as a maximal convergence theorem for some real analytic functions defined on the closed unit disk in $\mathbb{R}^2$.

Theorem 1.6 ([Bra01], 2001)
Let the function $F : \overline{B}_2 \to \mathbb{R}$, $\overline{B}_2 = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \le 1\}$, be given by

$$F(x, y) = \frac{1}{\left((x - x_0)^2 + (y - y_0)^2\right)^s},$$

[3] A function F defined on an open set $U \subset \mathbb{R}^N$ with range $\mathbb{R}$ or $\mathbb{C}$ is said to be real analytic in U, if for each $x \in U$ the function F may be represented by a convergent power series in some non–empty neighborhood of x in U.

where $s \in (0,\infty)$ *and* $(x_0, y_0) \in \mathbb{R}^2$ *such that* $\rho := \sqrt{x_0^2 + y_0^2} > 1$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \le \frac{1}{\rho}. \tag{1.2}$$

Furthermore, if $\rho \in [3,\infty)$ *or* $s \in (0,1)$ *then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}. \tag{1.3}$$

Braess's method to prove equation (1.3) of Theorem 1.6 only works if $\rho \in [3,\infty)$ or $s \in (0,1)$. For that reason Braess posed as an open problem whether equation (1.3) of Theorem 1.6 is also valid for $\rho \in (1,3)$ and $s \in [1,\infty)$. In Section 2.1 we examine Theorem 1.6 from a different point of view which allows an extension to the remaining cases by proving the opposite inequality of equation (1.2) for $\rho \in (1,\infty)$ and $s \in (0,\infty)$.

Clearly, the function F in Theorem 1.6 can be expressed as the squared modulus of a holomorphic function in some neighborhood of the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$. If we set $g(z) := 1/(z-z_0)^s$, where $z_0 = x_0 + iy_0$, then F can be written as

$$F(x,y) = \frac{1}{\left((x-x_0)^2 + (y-y_0)^2\right)^s} = g(z)\overline{g(z)}.$$

Further, g is holomorphic in $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ but in no neighborhood containing $\overline{\mathbb{D}}_\rho$. These facts indicate that the maximal convergence number ρ for a function of squared holomorphic type on the closed unit disk in $\mathbb{R}^2$ could be in accordance with the maximal convergence number for the corresponding holomorphic function on the closed unit disk in $\mathbb{C}$.

Our observation leads to the question whether Theorem 1.6, in particular equation (1.3), can be extended to any function F which may be written as the squared modulus of a holomorphic function. Indeed, in Section 2.2 we will prove

Theorem 1.7

Let $g \in \mathcal{H}(\overline{\mathbb{D}})$ *and* $F : \overline{B}_2 \to \mathbb{R}$ *be defined by*

$$F(x,y) = |g(x+iy)|^2.$$

If $\rho \in (1,\infty)$ *then the following conditions are equivalent:*

(i) $\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}$.

(ii) $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$.

Furthermore,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = 0$$

if and only if g *has a holomorphic extension to* $\mathbb{C}$.

In light of Theorem 1.7, Braess's problem reduces to a special case of it. However, the methods we are going to work out to prove Theorem 1.7 are far from being necessary to establish Theorem 1.6. Therefore we first give a quick and direct proof of Theorem 1.6.
In Section 2.3 we discuss several applications and consequences of Theorem 1.7 and the results we have developed in order to prove Theorem 1.7. To mention only one example we show that for functions F of squared modulus type the relation

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)}$$

is valid if F has no zeros in $B_2 := \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}$, whereas the statement fails if F has zeros in B_2.

As the function F in Theorem 1.6 can be continued analytically to some open neighborhood of $\overline{B}_2$ in $\mathbb{C}^2$, Theorem 1.5 gives rise to ask if there exists a similar result for real–valued continuous functions defined on compact sets $K \subset \mathbb{R}^N$. For that reason we explore in Chapter 3 the machinery Siciak used to prove Theorem 1.5. Many results in this context are extracted from [Sic62] and [Sic81]. Nevertheless some exertion has to be put in to present a clear and complete proof of Theorem 1.5. After this establishment an analogue of Theorem 1.5 to non–empty compact sets $K \subset \mathbb{R}^N = \mathbb{R}^N + i0 \subset \mathbb{C}^N$ is quite natural:

Theorem 1.8
Let $K \subset \mathbb{R}^N$ be a compact set such that the extremal function $\Phi(z, K)$ is continuous in $\mathbb{C}^N$. Furthermore, let $F : K \to \mathbb{R}$ be continuous and $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K, F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_\rho = \big\{ z \in \mathbb{C}^N : \Phi(z, K) < \rho \big\}.$$

Now, from some "theoretical" point of view the maximal convergence problem in $\mathbb{R}^N$ is solved. However, bearing for example Braess's problem in mind, we also would like to determine the maximal convergence number ρ for a given function F defined on a compact set $K \subset \mathbb{R}^N$. Consequently, we need the explicit formula of Φ, which requires even for simple compact sets, e.g. that of a closed unit ball in $\mathbb{R}^N$, much effort, as we will see in Chapter 4. There we introduce an explicit representation of Φ for compact, convex and symmetric sets S of $\mathbb{R}^N$ with non–empty interior $\mathrm{Int}S$, which is due to Lundin [Lun85][4]:

Theorem 1.9
Let S be a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ with $\mathrm{Int}S \neq \emptyset$ in $\mathbb{R}^N$. Then

$$\Phi(z, S) = \max_{y\in\partial B_N} \big| h\big(a(y)\langle z, y\rangle\big)\big| \quad \textit{for } z \in \mathbb{C}^N, \tag{1.4}$$

[4]For a different approach to this formula see [BT86]. A generalization of Lundin's formula for some special classes of compact, convex and symmetric subsets of $\mathbb{C}^N$ was discovered by Baran [Bar88]. It was achieved by considering various properties of a function of Joukowski type and making use of equation (1.5).

where $h : \mathbb{C} \to \mathbb{C}\backslash\{z \in \mathbb{C} : |z| < 1\}$, $h(z) = z + \sqrt{z^2 - 1}$ *and* $a(y) := 1/\max_{x\in S} \langle x, y \rangle$ *for* $y \in \partial B_N := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \left(\sum_{j=1}^N |x_j|^2\right)^{1/2} = 1\}$. *The symbol* $\langle \cdot, \cdot \rangle$ *means the standard scalar product in* $\mathbb{R}^N$ *and* $\mathbb{C}^N$ *respectively.*

The formula (1.4) was obtained by a representation of Φ in terms of plurisubharmonic functions. In fact, a great deal of work has been devoted to the identity

$$\log \Phi(z, K) = \sup\{u(z) : u \in \mathcal{L}, u|_K \le 0\}, \qquad z \in \mathbb{C}^N, \tag{1.5}$$

for compact sets $K \subset \mathbb{C}^N$, where $\mathcal{L}$ denotes the set of all plurisubharmonic functions v in $\mathbb{C}^N$ which satisfy the condition $\sup_{z\in\mathbb{C}^N} |v(z) - \log(1 + |z|)| < \infty$.
It was Zaharjuta [Zah76] who first showed this identity on the assumption that Φ is continuous. He studied various properties of Hilbert spaces of analytic functions in this context. For the general case Siciak provides two different proofs, see [Sic81] and [Sic82]. His first proof is based on an approximation theorem by means of spectral theory. The latter proof, we are going to discuss in Section 4.2, was obtained by deep classical results of several complex variables.

Chapter 3 and Chapter 4 have apart from the aspects we already brought into focus preparatory character for Chapter 5. In Section 5.1 we deal with the computation of the maximal convergence number ρ for given functions on the closed unit ball in $\mathbb{R}^N$. In this setting we point out a different solution of Braess's problem. Section 5.1 also reveals that in general ρ can't be determined explicitly, even if the explicit formula of Φ is known. Regarded from this point of view it is even more desirable to extend Theorem 1.7 to higher dimensions, because then ρ can be easily calculated for that kind of functions. This problem is the gist of Section 5.2. Of course, it is much more intricate as in the one dimensional case. We need a quite lengthy preparation with several rather technical auxiliary results to prove

Theorem 1.10
Let $g \in \mathcal{H}(\overline{\mathcal{B}}_N)$ *and* $F : \overline{B}_{2N} \to \mathbb{R}$ *be defined by*

$$F(x, y) = |g(x + iy)|^2, \quad (x, y) \in \overline{B}_{2N}, \quad x, y \in \mathbb{R}^N,$$

where $\overline{\mathcal{B}}_N$ *and* $\overline{B}_{2N}$ *are the closed unit balls in* $\mathbb{C}^N$ *and* $\mathbb{R}^{2N}$ *with respect to the Euclidean norm. Further, let* $\rho > 1$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \le \frac{1}{\rho}$$

if and only if g *has a holomorphic extension to* $\mathcal{B}_{N,\rho} = \{z \in \mathbb{C}^N : \left(\sum_{j=1}^N |z_j|^2\right)^{1/2} < \rho\}$.

Moreover, we will see that Theorem 1.10 can be extended to

Theorem 1.11
Let $g, h \in \mathcal{H}(\overline{\mathcal{B}}_N)$, $g \not\equiv 0, h \not\equiv 0$, *and* $F : \overline{B}_{2N} \to \mathbb{R}$ *defined by*

$$F(x, y) = g(x + iy)\overline{h(x + iy)}, \quad (x, y) \in \overline{B}_{2N}, \quad x, y \in \mathbb{R}^N.$$

Further, let $\rho > 1$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \le \frac{1}{\rho}$$

if and only if g *and* h *have holomorphic extensions to* $\mathcal{B}_{N,\rho}$.

In Section 5.2 we also verify Theorem 1.11 for closed unit balls in $\mathbb{R}^N$ with respect to the maximum norm.
Our work ends with maximal convergence considerations on closed polysquares P in $\mathbb{R}^{2N}$. We will see that for these compact sets there exists no analogue of Theorem 1.10 in the sense that $\mathcal{B}_{N,\rho} = \{z \in \mathbb{C}^N : \Phi(z, \overline{\mathcal{B}}_N) < \rho\}$ may be replaced by $\{z \in \mathbb{C}^N : \Phi(z, K) < \rho\}$, where $K = \{x + iy \in \mathbb{C}^N : (x, y) \in P\}$.

This, in consequence, leads to the question for which compact sets, except for closed balls in $\mathbb{R}^{2N}$, is Theorem 1.10 and Theorem 1.11 respectively valid, as for the general case only the "if"–direction of Theorem 1.11 holds:

Remark 1.12
Let $K \subset \mathbb{C}^N$ be a compact set such that Siciak's extremal function Φ is continuous in $\mathbb{C}^N$ and define $L = \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$. Moreover, let g, h be holomorphic functions in an open connected neighborhood of K and let $F : L \to \mathbb{R}$ be given by

$$F(x, y) = g(x + iy)\overline{h(x + iy)}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L, F)} \le \frac{1}{\rho}$$

if g and h have holomorphic extensions to $\{z \in \mathbb{C}^N : \Phi(z, K) < \rho\}$.

Chapter 2

Maximal convergence on the closed unit disk in $\mathbb{R}^2$

2.1 An extension of a theorem of Braess

The point of departure for this section is Braess's problem we introduced in Chapter 1. Here, we will extend Theorem 1.6 to the remaining cases by proving the opposite inequality of (1.2). Thus we will obtain

Theorem 2.1
Let the function $F : \overline{B}_2 \to \mathbb{R}$ *be given by*

$$F(x,y) = \frac{1}{\big((x-x_0)^2 + (y-y_0)^2\big)^s},$$

where $s \in (0,\infty)$ *and* $(x_0, y_0) \in \mathbb{R}^2$ *such that* $\rho := \sqrt{x_0^2 + y_0^2} > 1$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}. \tag{2.1}$$

Before we however prove Theorem 2.1 let us introduce some basic notations.

We abbreviate the disk of radius r and center 0 in $\mathbb{R}^2$ by

$$B_{2,r} := \big\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < r\big\}, \quad \text{in particular} \quad B_2 := B_{2,1},$$

and denote the open disk of radius r about 0 in $\mathbb{C}$ by

$$\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}, \quad \text{especially} \quad \mathbb{D} := \mathbb{D}_1.$$

The annulus A_{r_1,r_2} with center 0 and radii r_1, r_2 is the set

$$A_{r_1,r_2} = \big\{z \in \mathbb{C} : r_1 < |z| < r_2\big\}.$$

$\mathcal{H}(G)$ stands for the set of holomorphic functions defined on a domain $G \subset \mathbb{C}$ and $\mathcal{H}(\overline{G})$ for the set of functions holomorphic in some neighborhood of $\overline{G}$, where $\overline{G}$ is the closure of G. By ∂G we mean the boundary of G. The symbols E_n and e_n are used as in Chapter 1. Since we consider functions F defined on sets in $\mathbb{R}^2$ and functions f defined on sets in $\mathbb{C}$

simultaneously, we distinguish them by capital and small letters.

Our proof of Theorem 2.1 requires some elementary facts of the theory of best approximation which we point out in our next remark.

Remark 2.2
Let X be the Banach space of all real–valued continuous functions defined on a compact subset $K \subset \mathbb{R}^2$ and let X_n be the subspace of all real–valued polynomials P_n, $P_n : K \to \mathbb{R}$, of degree $\leq n$. Then the following statements are valid:

(i) There exists a best approximation $P_n \in X_n$ to $F \in X$, that is

$$E_n(K, F) = \|F - P_n\|_K.$$

(ii) The set of best approximations to F is convex. More precisely, if

$$E_n(K, F) = \|F - P_{n,1}\|_K = \|F - P_{n,2}\|_K \quad \text{for } P_{n,1}, P_{n,2} \in X_n$$

then

$$E_n(K, F) = \|F - \alpha_1 P_{n,1} - \alpha_2 P_{n,2}\|_K,$$

where $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

Taken (i) and (ii) together shows:

(iii) If, in addition, K is symmetric with respect to the y–axis[1], then an even function F in y, i.e. $F(x, y) = F(x, -y)$ for $(x, y) \in K$, has an even best approximation P_n in y.

Besides Remark 2.2 Bernstein's Theorem 1.2 plays a vital role to verify equation (2.1). A quick proof of Theorem 1.2 can be found in [DL93], p. 229–231. Note, the function $h(z) = z + \sqrt{z^2 - 1}$ defined in Theorem 1.2 is the inverse of the Joukowski function with domain $\mathbb{C}$ and range $\mathbb{C}\backslash\mathbb{D}$.

Remark 2.3
The Joukowski function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is defined by

$$f(z) = \frac{1}{2}\Big(z + \frac{1}{z}\Big) = w.$$

It is a second order rational function satisfying the condition $f(z) = f(1/z)$. This means that every point of the w–plane save for $w = \pm 1$ has exactly two distinct inverse images z_1 and z_2 such that $z_1 z_2 = 1$. Its regions of univalence are $\mathbb{D}\backslash\{0\}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$. Both domains are mapped conformally onto $\mathbb{C}\backslash[-1, 1]$. The image of the boundary $\partial\mathbb{D}$ is $[-1, 1]$. Moreover, f maps concentric circles $|z| = r$ conformally onto confocal ellipses with semiaxes $(r \pm r^{-1})/2$ and foci at ± 1, see Figure 2.1. These mapping properties will accompany us throughout the whole work.

[1] We call a set $K \subset \mathbb{R}^2$ symmetric with respect to the y–axis, if $(x, y) \in K$ implies $(x, -y) \in K$.

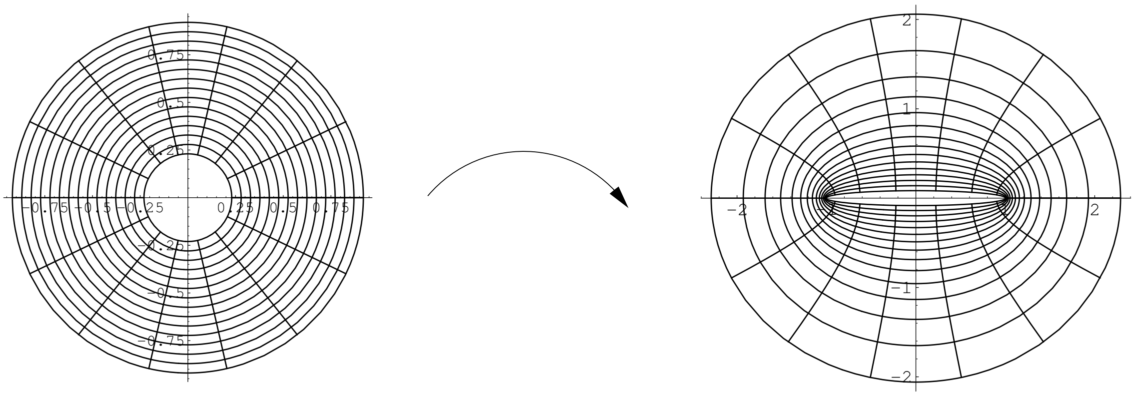

Figure 2.1 $f(z)=\frac{1}{2}\left(z+\frac{1}{z}\right)$

Proof of Theorem 2.1:

We will show the inequality

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \frac{1}{\rho}$$

for $\rho \in (1,\infty)$ and $s \in (0,\infty)$. Then the claim follows in conjunction with equation (1.2) of Theorem 1.6.

After rotating and translating coordinates we may assume that F takes the form

$$F(x,y) = \frac{1}{\left((x-\rho)^2+y^2\right)^s}.$$

By Remark 2.2 there exists a polynomial $P_{n,b}$ of degree $\leq n$ which is a best approximation to F on $\overline{B}_2$. Since F is an even function in y we may assume that $P_{n,b}$ is also even in y:

$$P_{n,b}(x,y) = \sum_{\substack{j,k=0\\ 0\leq j+2k\leq n}}^{n} a_{jk} x^j y^{2k}.$$

As $x^2+y^2=1$ for $(x,y)\in\partial B_2$ we obtain the estimate

$$\begin{aligned}\|F-P_{n,b}\|_{\overline{B}_2} \geq \|F-P_{n,b}\|_{\partial B_2} &= \max_{x\in[-1,1]} \left| \frac{1}{(\rho^2-2x\rho+1)^s} - \sum_{\substack{j,k=0\\ 0\leq j+2k\leq n}}^{n} a_{jk} x^j (1-x^2)^k \right| \\ &= \max_{x\in[-1,1]} |\hat{F}(x)-\hat{P}_n(x)| \geq E_n([-1,1],\hat{F}),\end{aligned}$$

where $\hat{F}(x) = 1/(\rho^2-2x\rho+1)^s$ and $\hat{P}_n(x) = \sum_{\substack{j,k=0\\ 0\leq j+2k\leq n}}^{n} a_{jk} x^j (1-x^2)^k$.

Consequently,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \limsup_{n\to\infty} \sqrt[n]{E_n([-1,1],\hat{F})}. \tag{2.2}$$

We next apply Theorem 1.2 to $\hat{F} : [-1,1] \to \mathbb{R}$. Note that $\tilde{F}(z) = 1/(\rho^2 - 2z\rho + 1)^s$ is a holomorphic extension of $\hat{F}$ to the set $L_\rho = \{z \in \mathbb{C} : |h(z)| < \rho\}$, where h is defined as in Theorem 1.2. Clearly, the function $\tilde{F}$ cannot be continued analytically to any neighborhood of the point $z = 1/2(\rho + 1/\rho)$. In other words, $\tilde{F}$ has no holomorphic extension to any domain containing $\overline{L}_\rho$. Thus Theorem 1.2 implies

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1],\hat{F})} \geq \frac{1}{\rho}.$$

The latter, combined with (2.2), shows

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2,F)} \geq \frac{1}{\rho}.$$

■

2.2 Real analytic functions of squared modulus holomorphic type

The plan for this section is to establish Theorem 1.7 in several steps.

First of all we verify the estimate $\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2,F)} \leq 1/\rho$, if $F = |g|^2$ and $g \in \mathcal{H}(\mathbb{D}_\rho)$.

Lemma 2.4
Let $F : \overline{B}_2 \to \mathbb{R}$ be given by

$$F(x,y) = |g(x+iy)|^2,$$

where $g \in \mathcal{H}(\mathbb{D}_\rho)$ with $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2,F)} \leq \frac{1}{\rho}.$$

Proof:
Choose $R \in (1,\rho)$. Since g is a holomorphic function in $\mathbb{D}_R$, we can expand g in its power series $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and obtain

$$|a_k| \leq \frac{M}{R^k}, \qquad M := \sup\{|g(z)| : |z| \leq R\}, \quad k \in \mathbb{N},$$

by Cauchy's estimates.
Further, let us denote the n-th Taylor polynomial of g by $p_n(z) = \sum_{k=0}^{n} a_k z^k$.
It follows

$$|g(z) - p_n(z)| \leq \frac{M}{R^{n+1}(R-1)} = \frac{M_1}{R^n}, \qquad M_1 := \frac{M}{R(R-1)}, \quad z \in \overline{\mathbb{D}}. \tag{2.3}$$

The last estimate implies for $z \in \overline{\mathbb{D}}$ and n sufficiently large

$$\begin{aligned}|g(z)\overline{g(z)} - p_n(z)\overline{p_n(z)}| &\leq |g(z)\overline{g(z)} - g(z)\overline{p_n(z)}| + |g(z)\overline{p_n(z)} - p_n(z)\overline{p_n(z)}| \\ &\leq |g(z)|\ |\overline{g(z)} - \overline{p_n(z)}| + |\overline{p_n(z)}|\ |g(z) - p_n(z)| \\ &\leq \Big(2|g(z)| + \frac{M_1}{R^n}\Big)|g(z) - p_n(z)| \\ &\leq 3\,\|g\|_{\overline{\mathbb{D}}}\frac{M_1}{R^n}.\end{aligned}$$

Similarly to [Bra01] we put

$$q_0(z) := p_0(z), \quad q_k(z) := p_k(z) - p_{k-1}(z), \qquad k \in \mathbb{N},$$

and define the real–valued polynomials

$$Q_n(x,y) := \sum_{\substack{k,l=0\\ k+l\leq n}}^{n} q_k(z)\overline{q_l(z)}, \quad z = x+iy, \quad n \in \mathbb{N},$$

as well as

$$P_{2n}(x,y) := \sum_{k,l=0}^{n} q_k(z)\,\overline{q_l(z)} = p_n(z)\overline{p_n(z)}, \quad z = x+iy, \quad n \in \mathbb{N}.$$

Notice,

$$P_{2n}(x,y) - Q_n(x,y) = \sum_{\substack{k,l=0\\ k+l>n}}^{n} q_k(z)\overline{q_l(z)} = \sum_{k=1}^{n} q_k(z)\big(\overline{p_n(z) - p_{n-k}(z)}\big).$$

Because of (2.3) we get

$$|p_l(z) - p_k(z)| \leq |g(z) - p_l(z)| + |g(z) - p_k(z)| \leq \frac{2M_1}{R^k} \qquad \text{for} \quad k < l, \quad z \in \overline{\mathbb{D}}.$$

Owing to the definition of q_k and the last estimate we achieve

$$|q_k(z)\big(\overline{p_n(z) - p_{n-k}(z)}\big)| \leq \frac{2M_1}{R^{k-1}}\frac{2M_1}{R^{n-k}} = \frac{4M_1^2}{R^{n-1}} \qquad \text{for} \quad z \in \overline{\mathbb{D}},$$

which gives

$$|P_{2n}(x,y) - Q_n(x,y)| \leq \frac{4nM_1^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in \overline{B}_2.$$

Finally we obtain

$$\begin{aligned}|F(x,y) - Q_n(x,y)| &\leq |F(x,y) - P_{2n}(x,y)| + |P_{2n}(x,y) - Q_n(x,y)| \\ &\leq 3\|g\|_{\overline{\mathbb{D}}}\frac{M_1}{R^n} + \frac{4nM_1^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in \overline{B}_2.\end{aligned}$$

This yields

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\rho},$$

as $R < \rho$ was arbitrary. ■

Since in Lemma 2.4 the shape of the domain on which F is defined is only pertinent for the estimate of (2.3), we can extract the following statement from the proof of Lemma 2.4.

Remark 2.5
Let $K \subset \mathbb{C}$ be a compact set and g be a holomorphic function in some open connected neighborhood of K. Consider the function F defined by

$$F(x,y) = |g(x+iy)|^2 \quad \text{for } (x,y) \in L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}.$$

If there exists a sequence of complex–valued polynomials of degree $\leq n$ such that

$$|g(z) - p_n(z)| \leq \frac{M}{R^n}, \qquad z \in K, \quad n \in \mathbb{N},$$

for some $R \in (1,\infty)$ and some constant $M > 0$ independent of n, then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L,F)} \leq \frac{1}{R}.$$

Now, taking Theorem 1.3 into account allows the following generalization of Lemma 2.4.

Lemma 2.6
Let K be a compact subset of $\mathbb{C}$, such that $\hat{\mathbb{C}}\backslash K$ is connected and regular. Further, let F be given by

$$F(x,y) = |g(x+iy)|^2 \qquad \text{for} \quad (x,y) \in L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\},$$

where $g \in \mathcal{H}(L_\rho)$, $L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\}$ and g_K is Green's function for $\hat{\mathbb{C}}\backslash K$ with pole at infinity. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L,F)} \leq \frac{1}{\rho}.$$

Our next lemma will serve as a basis for the estimate $\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2,F)} \geq 1/\rho$, if $F = |g|^2$ and $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$.

Lemma 2.7
Let $F : \partial B_{2,r} \to \mathbb{R}$ be a continuous function, $r \in (0,\infty)$ and $\rho > 1$.
If

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_{2,r},F)} \leq \frac{1}{\rho},$$

then the function

$$h_r(z) = F\left(\frac{r}{2}\left(z+\frac{1}{z}\right), \frac{r}{2i}\left(z-\frac{1}{z}\right)\right), \quad z \in \partial\mathbb{D},$$

has a holomorphic extension $\tilde{h}_r$ to the annulus $A_{1/\rho,\rho}$.

Proof:
Let $R_1 \in (1,\rho)$. Then there exists a constant $M > 0$ such that $E_n(\partial B_{2,r}, F) \le M/R_1^n$ for all $n \in \mathbb{N}$. Consequently we can find to each $n \in \mathbb{N}$ a polynomial P_n of degree $\le n$, $P_n : \mathbb{R}^2 \to \mathbb{R}$, satisfying

$$|F(x,y) - P_n(x,y)| \le \frac{M}{R_1^n} \qquad \text{for} \quad (x,y) \in \partial B_{2,r}. \tag{2.4}$$

Now we define

$$p_{r,n}(z) := P_n\left(r\frac{1}{2}\Big(z + \frac{1}{z}\Big), r\frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right) \qquad \text{for} \quad z \in \mathbb{C}\backslash\{0\}.$$

As

$$p_{r,n}(e^{it}) = P_n(r\cos t, r\sin t) \qquad \text{for} \quad t \in [0,2\pi],$$

we can write (2.4) in the form

$$|F(r\cos t, r\sin t) - p_{r,n}(e^{it})| \le \frac{M}{R_1^n} \qquad \text{for} \quad t \in [0,2\pi].$$

Therefore we get

$$|p_{r,n+1}(z) - p_{r,n}(z)| \le \frac{2M}{R_1^n} \qquad \text{for} \quad z \in \partial\mathbb{D},$$

which implies

$$|z^{n+1}\big(p_{r,n+1}(z) - p_{r,n}(z)\big)| \le \frac{2M}{R_1^n} \qquad \text{for} \quad z \in \partial\mathbb{D}.$$

Note, the expression on the left side of the last estimate is a complex–valued polynomial of degree $\le 2(n+1)$. By the maximum principle we deduce

$$|z^{n+1}\big(p_{r,n+1}(z) - p_{r,n}(z)\big)| \le \frac{2M}{R_1^n} \qquad \text{for} \quad z \in \overline{\mathbb{D}}.$$

Now, let R_2 be an arbitrary number of $(1, R_1)$. Then it follows

$$|p_{r,n+1}(z) - p_{r,n}(z)| \le \frac{2M}{R_1^n}R_2^{n+1} \quad \text{for} \quad \frac{1}{R_2} \le |z| \le 1.$$

Consequently, as $R_1 \in (1,\rho)$ was arbitrary, the series

$$p_{r,0} + \sum_{n=1}^{\infty}\big(p_{r,n} - p_{r,n-1}\big)$$

converges locally uniformly in $\tilde{A}_{1/\rho,1} := \big\{z \in \mathbb{C} : 1/\rho < |z| \le 1\big\}$ to a function $\tilde{h}_r$, which is holomorphic in $A_{1/\rho,1}$ and continuous on $\tilde{A}_{1/\rho,1}$.

Combining now

$$p_{r,n}(e^{it}) = P_n(r\cos t, r\sin t) \to F(r\cos t, r\sin t) \quad \text{for} \quad n \to \infty, \quad t \in [0,2\pi],$$

with

$$p_{r,n}(z) \to \tilde{h}_r(z) \qquad \text{for} \quad n \to \infty, \quad z \in \tilde{A}_{\frac{1}{\rho},1},$$

gives

$$\tilde{h}_r(e^{it}) = F(r\cos t, r\sin t) = h_r(e^{it}) \qquad \text{for} \quad t \in [0, 2\pi].$$

Hence $\tilde{h}_r$ is the holomorphic extension of h_r to $A_{1/\rho,1}$. Further, since the function h_r is continuous on $\tilde{A}_{1/\rho,1}$ and real–valued on $\partial\mathbb{D}$ we can apply Schwarz's reflection principle and see that h_r has even a holomorphic extension to $A_{1/\rho,\rho}$. This completes the proof. ■

For technical reasons we next bring in the following notation.

Definition 2.8
If $h(z) = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic function in $\mathbb{D}_r$, $r > 0$, then we define $h_- \in \mathcal{H}(\mathbb{D}_r)$ by

$$h_-(z) := \sum_{k=0}^{\infty} \overline{a_k} z^k, \quad z \in \mathbb{D}_r. \tag{2.5}$$

Now we have all the necessary ingredients together to prove the main result of this section.

Theorem 2.9
Let $g \in \mathcal{H}(\overline{\mathbb{D}})$ and $F : \overline{B}_2 \to \mathbb{R}$ be defined by

$$F(x, y) = |g(x + iy)|^2.$$

If $\rho \in (1, \infty)$ then the following conditions are equivalent:

(i) $\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \dfrac{1}{\rho}$.

(ii) $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$.

Furthermore, it holds

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = 0$$

if and only if g has a holomorphic extension to $\mathbb{C}$.

Proof:
Ad (i) ⇔ (ii): Because of Lemma 2.4 we only need to verify the inequality

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \ge \frac{1}{\rho}$$

if $F(x, y) = |g(x + iy)|^2$ and $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$. We split this estimate into two steps.
In step (1) we handle the case that g has only zeros on $\partial\mathbb{D}$ whereas in step (2) g obliges no "zero"–restriction.

Step (1): Our goal is to show that even the stronger estimate

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)} \ge \frac{1}{\rho}$$

holds, if $F(x, y) = |g(x + iy)|^2$, $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$ and g is zero-free in $\mathbb{D}$. In order to show this, we assume

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)} \le \frac{1}{R_1} < \frac{1}{\rho}$$

for some $R_1 > \rho$. Then by Lemma 2.4 the function

$$h_1(z) := F\left(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right), \quad z \in \partial\mathbb{D},$$

has a holomorphic extension $\tilde{h}_1$ to the annulus $A_{1/R_1, R_1}$.
Next, let $a_1, \ldots, a_m$ be the finitely many zeros of g on $\partial\mathbb{D}$. Hence we can rewrite g as

$$g(z) = \tilde{g}(z) \prod_{j=1}^{m} (z - a_j),$$

where $\tilde{g} \in \mathcal{H}(\mathbb{D}_\rho)$. Further, for $z \in \partial\mathbb{D}$ we get

$$\tilde{h}_1(z) = |g(z)|^2 = \tilde{g}(z)\tilde{g}_{-}\Big(\frac{1}{z}\Big) \prod_{j=1}^{m} (z - a_j)\Big(\frac{1}{z} - \overline{a_j}\Big),$$

where $\tilde{g}_{-}$ is specified in equation (2.5). Thus

$$\tilde{h}(z) := \frac{\tilde{h}_1(z)}{\prod_{j=1}^{m}\Big(\frac{1}{z} - \overline{a_j}\Big)} = \tilde{g}(z)\tilde{g}_{-}\Big(\frac{1}{z}\Big) \prod_{j=1}^{m}(z - a_j)$$

is holomorphic in $A_{1/R_1, R_1}$ because $\tilde{h}_1 \in \mathcal{H}(A_{1/R_1, R_1})$.
Since g is zero-free in $\mathbb{D}$ and $\tilde{g}$ is zero-free in a neighborhood of $\overline{\mathbb{D}}$, we can find an $\varepsilon > 0$ with $1/R_1 < 1 - \varepsilon$ such that $\tilde{g}_{-}(1/z) \neq 0$ for $z \in A_{1-\varepsilon, R_1}$. Consequently,

$$\hat{h}(z) = \frac{\tilde{h}(z)}{\tilde{g}_{-}(\frac{1}{z})}$$

is holomorphic in $A_{1-\varepsilon, R_1}$. The fact that $\hat{h}$ coincides with g on $\partial\mathbb{D}$ implies that g has a holomorphic extension to $\mathbb{D}_{R_1}$, $R_1 > \rho$, and the contradiction is apparent.
Step (2): Here, we represent g in the form

$$g(z) = B(z)\hat{g}(z), \qquad B(z) = \prod_{j=1}^{m} \frac{z_j - z}{1 - \overline{z_j} z}, \quad z_j \in \mathbb{D},$$

where $\hat{g} \in \mathcal{H}(\overline{\mathbb{D}})$ is zero-free in $\mathbb{D}$.
Now there are two possibilities for the holomorphic behavior of $\hat{g}$:
(a) $\hat{g} \in \mathcal{H}(\mathbb{D}_\rho) \backslash \mathcal{H}(\overline{\mathbb{D}}_\rho)$ and (b) $\hat{g} \in \mathcal{H}(\mathbb{D}_{\rho_1})$ for some $\rho_1 > \rho$.

To (a): Let $\hat{F}(x, y) := |\hat{g}(x + iy)|^2, (x, y) \in \overline{B}_2$. Then $F = \hat{F}$ on ∂B_2, so

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, \hat{F})} \geq \frac{1}{\rho} \tag{2.6}$$

by Step (1).
To (b): In this case, g is meromorphic in $\mathbb{D}_{\rho_1}$ with possible poles at $1/\overline{z_1}, \ldots, 1/\overline{z_m}$. As

$g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$ we infer that $|z_l| = 1/\rho$ for at least one $l \in \{1, \ldots, m\}$ such that g has a pole at $1/\overline{z_l}$. Similarly as before assume

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{R_2} < \frac{1}{\rho}$$

for some $R_2 > \rho$. Next, choose $\hat{r} \in (\rho/R_2, 1)$ such that

$$\frac{z_k}{z_l} \neq \hat{r}^2 \quad \text{for all } k \in \{1, 2, \ldots, m\}.$$

Thus g doesn't vanish at $z = \hat{r}^2 z_l$. This entails that the function

$$h_{\hat{r}}(z) := g(\hat{r}z)g_-\Big(\frac{\hat{r}}{z}\Big)$$

has a pole at the point $z = 1/(\overline{z_l}\hat{r}) \in A_{1,R_2}$.
On the other hand, Lemma 2.7 shows that $h_{\hat{r}}$ has a holomorphic extension to $A_{1/R_2,R_2}$. This contradiction completes part (b) and therefore step (2).
The additional statement of Theorem 2.9 is quite obvious if we regard it as the limiting case "$\rho = \infty$". ■

In Theorem 2.9 the maximal convergence number ρ for F was obtained by determining the largest disk $\mathbb{D}_r$ to which g has an analytic continuation. A different but equivalent method to find the maximal convergence number is described in our next theorem.

Theorem 2.10
Let $F : \overline{B}_2 \to \mathbb{R}$ be a function with the representation

$$F(x, y) = |g(x + iy)|^2, \quad \textit{where} \quad g \in \mathcal{H}(\overline{\mathbb{D}}),$$

and let $\rho \in (1, \infty)$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}$$

if and only if for every $r \in (0, 1]$ the function $h_r : \partial\mathbb{D} \to \mathbb{C}$,

$$h_r(z) := F\left(r\frac{1}{2}\Big(z + \frac{1}{z}\Big), r\frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right),$$

has a holomorphic extension $\tilde{h}_r$ to $A_{r/\rho,\rho/r}$, and at least one of these extensions is not holomorphic in a neighborhood of $\overline{A}_{r/\rho,\rho/r}$.
In particular, to each $\varepsilon > 0$ there exists a number $\hat{r} \in (1 - \varepsilon, 1]$ such that $h_{\hat{r}}$ has no holomorphic extension to a neighborhood of $\overline{A}_{\hat{r}/\rho,\rho/\hat{r}}$.

Proof:
By Theorem 2.9 it suffices to check the following equivalence:

$$g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$$

if and only if for every $r \in (0, 1]$ the function $h_r : \partial\mathbb{D} \to \mathbb{C}$,

$$h_r(z) = F\left(r\frac{1}{2}\Big(z + \frac{1}{z}\Big), r\frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right),$$

has a holomorphic extension $\tilde{h}_r$ to $A_{r/\rho,\rho/r}$, and at least one of these extensions is not holomorphic in a neighborhood of $\overline{A}_{r/\rho,\rho/r}$.
To prove the "if"–direction, let $\tilde{h}_r$ be the holomorphic extension of h_r to $A_{\frac{r}{\rho},\frac{\rho}{r}}$, $r \in (0,1]$. Then, for $z \in \partial\mathbb{D}$, we have

$$|g(rz)|^2 = F(r\operatorname{Re} z, r\operatorname{Im} z) = F\left(\frac{r}{2}\Big(z+\frac{1}{z}\Big), \frac{r}{2i}\Big(z-\frac{1}{z}\Big)\right) = \tilde{h}_r(z), \quad r \in (0,1].$$

Therefore $g(rz)$ can be represented by

$$g(rz) = \frac{\tilde{h}_r(z)}{\overline{g(\frac{r}{\overline{z}})}}$$

for all $r \in (0,1]$ and $z \in \partial\mathbb{D}$. Since $z \mapsto \overline{g(r/\overline{z})}$ is holomorphic in $A_{r,\rho/r}$, we see that $z \mapsto g(rz)$ is for sure meromorphic in $A_{r,\rho/r}$ for each $r \in (0,1]$. If now $z_0 \in \mathbb{D}_\rho$ is a pole of g, then $g(rz)$ has a pole at $z = z_0/r$, so $\overline{g(r/\overline{z})}$ has a zero at $z = z_0/r$ for each $r \in (0,1]$. Thus $g(z)$ would have a zero at $z = r^2/\overline{z_0}$ for each $r \in (0,1]$, which is clearly impossible. Consequently, $g \in \mathcal{H}(\mathbb{D}_\rho)$.
To finish the proof of the "if"–statement and to prove the "only if"–assertion let $g \in \mathcal{H}(\mathbb{D}_\rho)$. Then

$$\tilde{h}_r(z) = g(rz)g_-\left(\frac{r}{z}\right)$$

is the holomorphic extension of h_r to $A_{r/\rho,\rho/r}$ for each $r \in (0,1]$.

A closer look at the proof of Step (2) of Theorem 2.9 shows that we find to each ε, $0 < \varepsilon < 1$, an $\hat{r} \in (1-\varepsilon, 1]$ such that $h_{\hat{r}}$ has no holomorphic extension to any domain containing $\overline{A}_{\hat{r}/\rho,\rho/\hat{r}}$, if $g \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$. ■

2.3 Applications

By the maximum principle we clearly have $\|g-p\|_{\overline{\mathbb{D}}} = \|g-p\|_{\partial\mathbb{D}}$ if g and p are holomorphic functions in a neighborhood of $\overline{\mathbb{D}}$. Thus, for determining the maximal convergence number ρ for g on $\overline{\mathbb{D}}$ we can draw back to $\partial\mathbb{D}$. If now $F = |g|^2$, $g \in \mathcal{H}(\overline{\mathbb{D}})$, and g has no zeros in $\mathbb{D}$ we can also restrict to the boundary of $\overline{B}_2$ in order to work out the maximal convergence number ρ for F on $\overline{B}_2$. However in the case that F has zeros in B_2, Example 2.12 unveils some disparity.

Theorem 2.11
Let $F(x,y) = |g(x+iy)|^2$, where $g \in \mathcal{H}(\overline{\mathbb{D}})$.
(i) If g has either no zeros on $\overline{\mathbb{D}}$ or only zeros on $\partial\mathbb{D}$, then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)} = \frac{1}{\rho}, \tag{2.7}$$

where $\rho > 1$ is the largest number such that g has a holomorphic extension to $\mathbb{D}_\rho$.
(ii) If g has zeros in $\mathbb{D}$, choose the representation

$$g(z) := \hat{g}(z)\prod_{j=1}^{m}(z-z_j), \quad z_j \in \mathbb{D}, \quad m \in \mathbb{N},$$

such that $\hat{g}$ *is a zero-free holomorphic function in* $\mathbb{D}$*. Further, define* $\hat{F} : \overline{B}_2 \to \mathbb{R}$ *by*

$$\hat{F}(x, y) = |\hat{g}(x + iy)|^2.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, \hat{F})} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, \hat{F})} = \frac{1}{\rho},$$

where $\rho > 1$ *is the largest number such that* g *has a holomorphic extension to* $\mathbb{D}_\rho$*.*

Proof:
To (i): Since g is holomorphic in $\mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$ we derive from Theorem 2.9

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}.$$

By Step (1) of Theorem 2.9 we have the relation

$$\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)}.$$

To (ii): Again from Theorem 2.9 we conclude

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho}.$$

Since $\hat{F} = |\hat{g}|^2$, $\hat{g}$ is zero–free in $\mathbb{D}$ and $\hat{g} \in \mathcal{H}(\mathbb{D}_\rho)\backslash\mathcal{H}(\overline{\mathbb{D}}_\rho)$, we obtain

$$\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, \hat{F})} = \limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, \hat{F})}.$$

■

Our next example illustrates that equation (2.7) goes wrong if g has zeros in $\mathbb{D}$.

Example 2.12
Let F be the squared modulus of a Blaschke product, i.e.

$$F(x, y) = \left| \prod_{j=1}^{m} \frac{z_j - z}{1 - \overline{z_j} z} \right|^2, \quad (x, y) \in \overline{B}_2, \quad z = x + iy, \quad z_j \in \mathbb{D}, \quad m \in \mathbb{N}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\partial B_2, F)} = 0,$$

but

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \max_{1\le j\le m} |z_j|.$$

A natural question which may arise is, whether a similar result like Theorem 2.9 also holds for functions of the form

$$F(x,y) = |g(x+iy)|, \quad g \in \mathcal{H}(\overline{\mathbb{D}}).$$

Obviously, if F is a zero-free function defined on $\overline{B}_2$ it doesn't make any difference. However, the situation changes if F has zeros on $\overline{B}_2$. Indeed, there exists no sequence of polynomials which converges maximally to F. This is content of our next result.

Theorem 2.13
Let $g \in \mathcal{H}(\overline{\mathbb{D}})$ and $F : \overline{B}_2 \to \mathbb{R}$ defined by

$$F(x,y) = |g(x+iy)|^2 \prod_{j=1}^{m} |x+iy-a_j|, \quad a_j \in \overline{\mathbb{D}}, \quad a_l \neq a_k \quad \textit{for} \quad l \neq k.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} > \frac{1}{\rho} \tag{2.8}$$

for any $\rho > 1$.

Proof:
The statement is proved by contradiction. We distinguish the cases (a) $a_k \neq 0$ for some $k \in \{1,\dots,m\}$ and (b) $m = 1$ and $a_m = 0$.
To (a): We assume there exists a function of the form

$$F(x,y) = |g(x+iy)|^2 \prod_{j=1}^{m} |x+iy-a_j|, \qquad g \in \mathcal{H}(\overline{\mathbb{D}}), \quad a_j \in \overline{\mathbb{D}}, \quad a_l \neq a_k \quad \text{for} \quad l \neq k,$$

which can be approximated maximally by polynomials. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\tilde{\rho}}$$

for some $\tilde{\rho} \in (1, \hat{\rho})$, where $\hat{\rho} > 1$ is chosen so small that g is also holomorphic in $\mathbb{D}_{\hat{\rho}}$.
Due to Lemma 2.7 the function $h_r : \partial\mathbb{D} \to \mathbb{C}$, $h_r(z) = |g(rz)|^2 \prod_{j=1}^{m} |rz - a_j|$, has for each $r \in (0,1]$ a holomorphic extension $\tilde{h}_r$ to $A_{1/\tilde{\rho},\tilde{\rho}}$.
For $z \in \partial\mathbb{D}$ we have

$$\begin{aligned}
\tilde{h}_r(z) &= F\left(r\frac{1}{2}\Big(z+\frac{1}{z}\Big), r\frac{1}{2i}\Big(z-\frac{1}{z}\Big)\right) \\
&= g(rz)g_-\Big(r\frac{1}{z}\Big) \prod_{j=1}^{m} \sqrt{\left(r\frac{1}{2}\Big(z+\frac{1}{z}\Big) - \operatorname{Re} a_j\right)^2 + \left(r\frac{1}{2i}\Big(z-\frac{1}{z}\Big) - \operatorname{Im} a_j\right)^2} \\
&= \tilde{g}(rz)\tilde{g}_-\Big(r\frac{1}{z}\Big) \prod_{k=1}^{l} \left(\Big(rz-b_k\Big)\Big(r\frac{1}{z}-\overline{b}_k\Big)\right) \times \\
&\qquad \prod_{j=1}^{m} \sqrt{\left(r\frac{1}{2}\Big(z+\frac{1}{z}\Big) - \operatorname{Re} a_j\right)^2 + \left(r\frac{1}{2i}\Big(z-\frac{1}{z}\Big) - \operatorname{Im} a_j\right)^2},
\end{aligned}$$

where $g(z) = \tilde{g}(z) \prod_{k=1}^{l}(z - b_k)$, $b_k \in \overline{\mathbb{D}}$, $\tilde{g}(z) \neq 0$ for $z \in \overline{\mathbb{D}}$ and $\tilde{g}_-, g_-$ are defined as in equation (2.5). If $\varepsilon > 0$ is sufficiently small, then $\tilde{g} \in \mathcal{H}(\mathbb{D}_{1+\varepsilon})$ and $A_{1-\varepsilon,1+\varepsilon} \subset A_{1/\tilde{\rho},\tilde{\rho}}$. We also may assume that $\tilde{g}(z) \neq 0$ and $\tilde{g}_-(1/z) \neq 0$ for $z \in A_{1-\varepsilon,1+\varepsilon}$.
Thus the function $l_r : \partial\mathbb{D} \to \mathbb{C}$, $r \in (0,1]$, defined by

$$l_r(z) = \prod_{k=1}^{l} \Big(\Big(rz - b_k\Big) \Big(r\frac{1}{z} - \bar{b}_k\Big) \Big) \prod_{j=1}^{m} \sqrt{\Big(r\frac{1}{2}\Big(z + \frac{1}{z}\Big) - \operatorname{Re} a_j\Big)^2 + \Big(r\frac{1}{2i}\Big(z - \frac{1}{z}\Big) - \operatorname{Im} a_j\Big)^2}$$

has a holomorphic extension to $A_{1-\varepsilon,1+\varepsilon}$, because the functions

$$\tilde{l}_r(z) := \frac{\tilde{h}_r(z)}{\tilde{g}(rz)\tilde{g}_-\Big(r\frac{1}{z}\Big)}, \qquad r \in (0,1],$$

are holomorphic in $A_{1-\varepsilon,1+\varepsilon}$ and $\tilde{l}_r \equiv l_r$ on $\partial\mathbb{D}$.
However, if $r_k = |a_k|$ and $|a_k| > 0$, $k \in \{1, \dots, m\}$, then the function l_{r_k} has a branch point on $\partial\mathbb{D}$. Hence it can't be holomorphic in $A_{1-\varepsilon,1+\varepsilon}$ and the result follows.
To (b): In this case F has the representation

$$F(x,y) = |g(z)|^2 |z|, \qquad g \in \mathcal{H}(\overline{\mathbb{D}}), \quad z = x + iy.$$

Therefore let us consider a restriction of F. We define $\tilde{F} : \overline{B}_{2,1-a} \to \mathbb{R}$, $0 < a < 1/2$, by

$$\tilde{F}(x,y) = |g(z-a)|^2 |z - a|.$$

For $\tilde{F}$ we can apply similar arguments as in (a) if we replace $\overline{B}_2$ by $\overline{B}_{2,1-a}$ and $r \in (0,1]$ by $r \in (0, 1-a)$. We obtain finally that

$$l_a(z) = |a| \sqrt{\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big) - 1\Big)^2 + \Big(\frac{1}{2i}\Big(z - \frac{1}{z}\Big)\Big)^2}, \qquad z \in \mathbb{D},$$

has a holomorphic extension to $A_{1-\varepsilon,1+\varepsilon}$, which is absurd.
As $\tilde{F}$ is a restriction of F to a subset of $\overline{B}_2$ our proof is finished. ■

Lemma 2.7 may be a useful tool for determining upper bounds for E_n – also in the case of non squared modulus holomorphic functions – as we will see in our next example.

Example 2.14
Let

$$F(x,y) = \frac{1}{a - xy}, \qquad a \in \mathbb{R}\backslash[-1,1], \quad (x,y) \in \overline{B}_2,$$

then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho},$$

where $\rho > 1$ is uniquely determined by

$$2|a| = \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big).$$

Proof:
"$\geq$": We plug $x = \frac{1}{2}(z + \frac{1}{z})$ and $y = \frac{1}{2i}(z - \frac{1}{z})$ in F and define the function

$$\tilde{h}_1(z) := \frac{1}{a - \frac{1}{4i}\Big(z^2 - \frac{1}{z^2}\Big)},$$

which is holomorphic in $\mathbb{C}$ except at the points z_j, $j = 1, 2, 3, 4$, where

$$a = \frac{1}{4i}\Big(z_j^2 - \frac{1}{z_j^2}\Big).$$

Now let us set $\rho := \min\{|z_j| : |z_j| > 1, j = 1, 2, 3, 4\}$. Then Lemma 2.7 implies

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \frac{1}{\rho}.$$

"$\leq$": Let us consider the function $G : [-1, 1] \to \mathbb{R}$ defined by

$$G(u) = \frac{2}{2a - u}.$$

By Theorem 1.2 there exists a sequence of polynomials P_n of degree $\leq n$ satisfying

$$|G(u) - P_n(u)| \leq \frac{M}{R^n} \qquad \text{for all } n \in \mathbb{N}, \quad u \in [-1, 1],$$

where $M > 0$ is some constant independent of n, R is any number of the interval $(1, \rho_1)$ and $\rho_1 > 1$ is uniquely determined by

$$2|a| = \frac{1}{2}\Big(\rho_1 + \frac{1}{\rho_1}\Big).$$

Notice, if $(x, y) \in \overline{B}_2$ we have $2xy \in [-1, 1]$ and therefore

$$F(x, y) = \frac{1}{a - xy} = \frac{2}{2a - 2xy} = \frac{2}{2a - u} = G(u) \quad \text{for} \quad u = 2xy, \quad (x, y) \in \overline{B}_2.$$

Hence we get

$$|F(x, y) - \tilde{P}_{2n}(x, y)| \leq \frac{M}{R^n} \quad \text{for} \quad (x, y) \in \overline{B}_2,$$

where $\tilde{P}_{2n}(x, y) := P_n(2xy)$. As $\tilde{P}_{2n}$ is a polynomial of degree $\leq 2n$ we achieve

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\sqrt{\rho_1}}.$$

Because of $\Big(\rho_1 + \frac{1}{\rho_1}\Big) = \frac{1}{i}\Big(i\rho_1 - \frac{1}{i\rho_1}\Big)$ we obtain $\sqrt{\rho_1} = \rho$, which completes the proof. ∎

Lemma 2.6 and Theorem 2.9 give rise to

Problem 2.15
For which compact sets $K \subset \mathbb{C}$ can the following statement be confirmed.
Let K be a compact subset of $\mathbb{C}$, such that $\hat{\mathbb{C}} \backslash K$ is connected and regular, and let $L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$. Furthermore, let $F : L \to \mathbb{R}$ be defined by

$$F(x, y) = |g(x + iy)|^2,$$

where g is holomorphic in a neighborhood of K. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L, F)} = \frac{1}{\rho}$$

if and only if $g \in \mathcal{H}(L_\rho) \backslash \mathcal{H}(\overline{L}_\rho)$, where $L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\}$ and g_K is Green's function for $\hat{\mathbb{C}} \backslash K$ with pole at infinity.

Clearly, by Theorem 2.9 the statement is true for $K = \overline{\mathbb{D}}$. However we will see in Section 5.3 that it fails, if K is a closed square in $\mathbb{C}$.

Chapter 3

Maximal convergence in $\mathbb{C}^N$ and $\mathbb{R}^N$

3.1 Introduction

The main aim of this chapter is a discussion of Siciak's machinery used to prove Theorem 1.5. For that reason we divide the chapter into several parts. First of all we like to give a rough sketch of Walsh's original proof of Theorem 1.3 to recall the situation in the complex plane and to get a better understanding for the generalized situation in several complex variables. Then we set the stage for Siciak's method to prove Theorem 1.5. We introduce interpolation formulas, unisolvent and extremal sets as well as Siciak's extremal function Φ. After that we verify Theorem 1.5 and prove Theorem 1.8. Finally we close this chapter with some remarks on the condition "Φ is continuous in $\mathbb{C}^N$" in Theorem 1.5 and Theorem 1.8.

Now, let us become acquainted with some notations and definitions in $\mathbb{R}^N$ and $\mathbb{C}^N$ which we need throughout our work.
An element of $\mathbb{R}^N$ is denoted by $x = (x_1, x_2, \ldots, x_N)$ and an element of $\mathbb{C}^N$ by $z = (z_1, z_2, \ldots, z_N)$. We equip the space $\mathbb{C}^N$ with the Euclidean norm

$$\|z\| = \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_N\overline{z_N}}$$

and the maximum norm

$$|z| = \max\{|z_1|, \ldots, |z_N|\},$$

where we regard $\mathbb{R}^N$ as a subset of $\mathbb{C}^N$. The *open polydisc* in $\mathbb{C}^N$ with center at $a \in \mathbb{C}^N$ and radius $r > 0$ is abbreviated by

$$\mathcal{D}_N(a, r) := \big\{z \in \mathbb{C}^N : |z - a| < r\big\}.$$

In particular, we denote for simplification

$$\mathcal{D}_{N,r} := \mathcal{D}_N(0, r) \qquad \text{and} \qquad \mathcal{D}_N := \mathcal{D}_N(0, 1).$$

The *closed polydisc* in $\mathbb{C}^N$ with center $a \in \mathbb{C}^N$ and radius $r > 0$ is defined by

$$\overline{\mathcal{D}}_N(a, r) := \big\{z \in \mathbb{C}^N : |z - a| \le r\big\}.$$

Similar as before, we put

$$\overline{\mathcal{D}}_{N,r} := \overline{\mathcal{D}}_N(0, r) \qquad \text{and} \qquad \overline{\mathcal{D}}_N := \overline{\mathcal{D}}_N(0, 1).$$

The symbols $D_N(a,r)$ and $\overline{D}_N(a,r)$ are used for the sets

$$D_N(a,r) = \{x \in \mathbb{R}^N : |x-a| < r\} \qquad \text{and} \qquad \overline{D}_N(a,r) = \{x \in \mathbb{R}^N : |x-a| \le r\},$$

where $a \in \mathbb{R}^N$ and $r > 0$.
Polydiscs are balls with respect to the maximum norm. Open and closed balls in $\mathbb{C}^N$ with respect to the Euclidean norm are abbreviated by $\mathcal{B}_N(a,r)$ and $\overline{\mathcal{B}}_N(a,r)$, whereas $B_N(a,r)$ and $\overline{B}_N(a,r)$ stand for the open and closed balls in $\mathbb{R}^N$.

3.2 Comparison of maximal convergence in $\mathbb{C}$ and $\mathbb{C}^N$

The "only if"–part of Theorem 1.3 is based on the so–called Bernstein-Walsh property:

Lemma 3.1
Let K be a compact subset of $\mathbb{C}$ such that $\hat{\mathbb{C}}\backslash K$ is connected and possesses a Green's function g_K with pole at infinity[1]*. Then g_K has the representation*

$$g_K(z) = \max\left\{0, \sup\left\{\frac{1}{\deg p}\log|p(z)|\right\}\right\}, \quad z \in \mathbb{C}, \quad \deg p: \textit{ degree of } p, \tag{3.1}$$

where the supremum is taken over all non-constant polynomials p satisfying $\|p\|_K \le 1$. Furthermore, if

$$L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\},$$

where $\rho > 1$, then

$$|p(z)| \le \|p\|_K \rho^{\deg p} \quad \textit{for } z \in L_\rho.$$

As we will see in Section 3.5 and 4.2 there exists an extension of the Bernstein–Walsh property to several complex variables. Using this generalization the "only if"–part of Theorem 1.5 can be proved quite similar to the one dimensional case:
One shows in the complex plane that there exists a sequence of polynomials $p_n : \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$, such that the series $p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$ converges uniformly on compact subsets of L_ρ to a holomorphic function $\tilde{f}$ which agrees with f on K.
Green's function which plays the central role in approximating and interpolating holomorphic functions by polynomials in the complex plane is replaced by $\log \Phi$, where Φ is Siciak's extremal function introduced in [Sic62]. Φ is defined for compact sets $K \subset \mathbb{C}^N$ in connection with a bounded function $b : K \to \mathbb{R}$. For our purpose we are interested in the extremal function Φ for the case $b \equiv 0$ on K. Therefore we restrict ourselves to this special case in Section 3.5 and discuss only few of the many beautiful properties of Φ, exactly these we need for the proof of Theorem 1.5.

The "if"–part of Walsh's theorem can be established by using series expansions for holomorphic functions f in the region L_ρ, which can be approximated by lemniscates. To be more precisely, one can construct a sequence of lemniscates

$$\Omega_n := \{z \in \mathbb{C} : |p_n(z)| < r_n\}, \quad n \in \mathbb{N},$$

[1]Let G be a domain in $\hat{\mathbb{C}}$. Then there exists a unique Green's function for G if and only if ∂G is non–polar.

where p_n is a polynomial of degree $\leq n$ and r_n is a positive number, such that Ω_n increases up to L_ρ and contains K for n sufficiently large. Within the lemniscates Ω_n, $n \in \mathbb{N}$, the function f can be expanded into a Jacobi–series of the form

$$f(z) = \sum_{j=0}^{\infty} q_j(z)\,[p_n(z)]^j ,$$

where q_j is a polynomial of degree $\leq n-1$. The Jacobi–series of f converges uniformly on $\overline{\Omega}'_n := \{z \in \mathbb{C} : |p_n(z)| \leq r'_n\}$, $0 < r'_n < r_n$. Truncating the series appropriately we get a suitable polynomial approximant to f. For details see Chapter III and Chapter IV of [Wal35]. We will see in Section 3.6 that the lemniscates can be replaced by polynomial polyhedra in several complex variables. To expand a holomorphic function f in $\mathbb{C}^N$ into a series of polynomials analogously to the one dimensional case we fall back on a deep theorem in several complex variables, namely the Oka–Weil extension theorem.

3.3 On holomorphic functions and interpolation formulas in $\mathbb{C}^N$

3.3.1 Holomorphic functions in $\mathbb{C}^N$

As in the one dimensional case there are several possibilities to define a holomorphic function in $\mathbb{C}^N$, see e.g. [Kra01] for a comprehensive discussion of this topic. The one we are interested in is

Definition 3.2
Let $G \subset \mathbb{C}^N$ be a domain. A function $f : G \to \mathbb{C}$ is called holomorphic if to each $\hat{z} \in G$ there is an $r > 0$ such that f can be written as a convergent power series of the form

$$\begin{aligned} f(z) &= \sum_{\alpha} a_\alpha (z-\hat{z})^\alpha \\ &= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N=0}^{\infty} a_{\alpha_1\ldots\alpha_N} (z_1-\hat{z}_1)^{\alpha_1} \ldots (z_N-\hat{z}_N)^{\alpha_N} \end{aligned}$$

for all $z \in \mathcal{D}_N(\hat{z}, r) \subset G$, where $\alpha = (\alpha_1, \ldots, \alpha_N), \alpha_j \in \mathbb{Z}_+ := \{0,1,2,\ldots\}$ for $j = 1, \ldots, N$.

It turns out that a number of fundamental theorems for holomorphic functions in the complex plane may be lifted to several complex variables. Our forthcoming survey serves as a short overview.
For that let G be a domain in $\mathbb{C}^N$ and $\mathcal{H}(G)$ be the set of all holomorphic functions defined on G.

Survey:

(i) *The identity principle*: Let $f, g \in \mathcal{H}(G)$ and $f = g$ on a non–empty open subset of G. Then $f = g$ in G.

(ii) *Liouville*: If $f \in \mathcal{H}(\mathbb{C}^N)$ and $|f|$ is bounded, then f is constant.

(iii) *The maximum principle*: Let $f : G \to \mathbb{C}$ be a holomorphic mapping. If $|f|$ attains a local maximum at a point $a \in G$. Then f is constant.

(iv) *Weierstrass*: If the sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}(G)$ is locally uniformly convergent to a function $f : G \to \mathbb{C}$, then $f \in \mathcal{H}(G)$. Furthermore, the sequence $\left\{\frac{\partial f_j}{\partial z_k}\right\}_{j\in\mathbb{N}}$ converges locally uniformly to $\left\{\frac{\partial f}{\partial z_k}\right\}$ for $k = 1, 2, \dots, N$.

(v) *Montel*: Any locally uniformly bounded family in $\mathcal{H}(G)$ is normal.

One phenomenon which only occurs in the theory of several complex variables is discussed in Section 4.1.2.

3.3.2 Interpolation formulas

Here we will observe that in contrast to the Hermite reminder formula the well-known Lagrange's and Newton's interpolation formula can be carried over to higher dimensions, see [Sic62]. This, of course, requires some preliminary terminology.
We define the vector space of all polynomials of degree $\le n$ by

$$\mathcal{P}_n^c := \Big\{p : p(z) = \sum_{\substack{k_1+k_2+\dots\\ +k_N \le n}} a_{k_1k_2\dots k_N} z_1^{k_1} z_2^{k_2} \dots z_N^{k_N}, \quad z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N, \quad a_{k_1k_2\dots k_N} \in \mathbb{C}, \quad k_l \in \mathbb{Z}_+, \quad 1 \le l \le N\Big\},$$

where $N \in \mathbb{N}$ and $n \in \mathbb{Z}_+$.
Notice, there are

$$m_n := \binom{N+n}{n} \tag{3.2}$$

different solutions of the inequality

$$k_1 + k_2 + \dots + k_N \le n, \tag{3.3}$$

where $k_l \in \mathbb{Z}_+$, $1 \le l \le N$. Consequently, if we denote by $k(j) = \big(k_1(j), k_2(j), \dots, k_N(j)\big)$, $j = 1, \dots, m_n$, the m_n different solutions of (3.3), then the set of monomials

$$e_j(z) := z^{k(j)} = z_1^{k_1(j)} z_2^{k_2(j)} \dots z_N^{k_N(j)}, \quad j = 1, \dots, m_n,$$

is a basis of the vector space $\mathcal{P}_n^c$. Thus any $p \in \mathcal{P}_n^c$ can be represented by

$$p(z) = \sum_{j=1}^{m_n} a_j\, e_j(z), \quad a_j \in \mathbb{C}.$$

Now we like to introduce the Vandermondian determinant in higher dimensions.

Definition 3.3
Let $\zeta^{(n)} = \{\zeta_1, \dots, \zeta_{m_n}\}$ be a system of $m_n = \binom{N+n}{n}$ points $\zeta_1, \dots, \zeta_{m_n} \in \mathbb{C}^N$, where $N \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Then the determinant

$$V(\zeta^{(n)}) = V(\zeta_1, \zeta_2, \dots, \zeta_{m_n}) := \begin{vmatrix} e_1(\zeta_1) & e_2(\zeta_1) & \dots & e_{m_n}(\zeta_1) \\ e_1(\zeta_2) & e_2(\zeta_2) & \dots & e_{m_n}(\zeta_2) \\ \vdots & \vdots & & \vdots \\ e_1(\zeta_{m_n}) & e_2(\zeta_{m_n}) & \dots & e_{m_n}(\zeta_{m_n}) \end{vmatrix} \tag{3.4}$$

is called the Vandermondian of the system $\zeta^{(n)}$.

Remark 3.4
$V(\zeta^{(n)})$ is the determinant to the system of linear equations

$$\sum_{j=1}^{m_n} a_{k_1(j)\,k_2(j)\dots k_N(j)} \zeta_{i\,1}^{k_1(j)} \zeta_{i\,2}^{k_2(j)} \dots \zeta_{i\,N}^{k_N(j)} = b_i, \;\; b_i \in \mathbb{C}, \quad i = 1, \dots, m_n,$$

where $a_{k_1(j)k_2(j)\dots k_N(j)} \in \mathbb{C}$, $j = 1, \dots, m_n$, are the unknowns.
Therefore, there is exactly one polynomial p_n of degree $\leq n$ which takes the values b_i, $i = 1, \dots, m_n$, at the points ζ_i, $i = 1, \dots, m_n$, of the system $\zeta^{(n)}$ if $V(\zeta^{(n)}) \neq 0$.

Definition 3.5
If $V(\zeta^{(n)}) \neq 0$, then the Lagrange interpolation polynomials in higher dimensions are defined by

$$L^{(j)}(z, \zeta^{(n)}) := \frac{V(\zeta_1, \dots, \zeta_{j-1}, z, \zeta_{j+1}, \dots, \zeta_{m_n})}{V(\zeta_1, \dots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \dots, \zeta_{m_n})} \qquad \text{for } z \in \mathbb{C}^N, \quad j = 1, \dots, m_n.$$

Observe, the Lagrange interpolation polynomials $L^{(j)}$ satisfy the relation

$$L^{(j)}(\zeta_l, \zeta^{(n)}) = \delta_{jl}, \quad j, l = 1, 2, \dots, m_n, \quad \delta_{jl} := \begin{cases} 1 & \text{for} \quad j = l \\ 0 & \text{for} \quad j \neq l \end{cases}.$$

Hence we have just verified the *Lagrange interpolation formula:*

Theorem 3.6
If $\zeta^{(n)}$ is any system of $m_n = \binom{N+n}{n}$ points $\zeta_1, \zeta_2, \dots, \zeta_{m_n} \in \mathbb{C}^N$ such that $V(\zeta^{(n)}) \neq 0$ and p_n is any polynomial of degree $\leq n$, then

$$p_n(z) = \sum_{j=1}^{m_n} p_n(\zeta_j) L^{(j)}(z, \zeta^{(n)}), \quad z \in \mathbb{C}^N. \tag{3.5}$$

Let us now state the generalized version of Newton's interpolation formula. We shall omit the proof of this result as a quick substantiation based on induction can be found for example in [Sic62].

Theorem 3.7
Let $k(j) = \big(k_1(j), k_2(j), \dots, k_N(j)\big)$, $j = 1, \dots, m_n$, be the m_n different solutions of

$$k_1(j) + k_2(j) + \dots + k_N(j) \leq n,$$

where $k_l(j) \in \mathbb{Z}_+$ for $l = 1, 2, \dots, N$. If

$$\zeta_{k_1(j)k_2(j)\dots k_N(j)} = (\zeta_{k_1(j)1}, \zeta_{k_2(j)2}, \dots, \zeta_{k_N(j)N}) \in \mathbb{C}^N, \qquad j = 1, \dots, m_n,$$

are m_n points such that

$$\zeta_{0l},\ \zeta_{1l},\ \dots,\ \zeta_{nl} \in \mathbb{C}$$

are $n+1$ distinct points of the complex plane for each $l = 1, 2, \dots, N$. Then there exists a unique polynomial p_n of degree $\leq n$ satisfying

$$p_n(\zeta_{k_1(j)k_2(j)\dots k_N(j)}) = a_{k_1(j)k_2(j)\dots k_N(j)} \qquad \text{for} \quad j = 1, \dots, m_n,$$

where $a_{k_1(j)k_2(j)\dots k_N(j)} \in \mathbb{C}$, $j = 1, \dots, m_n$, are m_n arbitrary numbers.

3.4 Unisolvent sets and extremal points

Unisolvent sets and extremal points play a key role in constructing sequences which converge to the extremal function Φ. Let us start with the definition of unisolvent sets.

Definition 3.8
A set $K \subset \mathbb{C}^N$ is called

(i) *unisolvent of order $n \in \mathbb{N}$, if every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ of degree $\leq n$ which vanishes on K is identical zero on $\mathbb{C}^N$.*

(ii) *unisolvent, if K is unisolvent of order n for any $n \in \mathbb{N}$.*

A useful tool to characterize unisolvent sets is the Vandermondian determinant.

Proposition 3.9 (cf. [Sic81])
Let $K \subset \mathbb{C}^N$ and let $V_j(K)$ for $j = 1, 2, \dots, m_n$ be defined by

$$V_j(K) := \sup_{\{z_1,\dots,z_j\}\subset K} |V(z_1, \dots, z_j)|,$$

where $m_n = \binom{N+n}{n}$ and V is the Vandermondian determinant. Then the following conditions are equivalent:

(i) *K is unisolvent of order n.*

(ii) *$V_j(K) \neq 0$ for every $j \in \{1, 2, \dots, m_n\}$.*

(iii) *$V_{m_n}(K) \neq 0$.*

Proof:
(i) $\Rightarrow$ (ii) We prove this implication by induction. Clearly, we have $V_1(K) = 1$. Let us now suppose that $V_j(K) \neq 0$ for $j < m_n$ and let $\{\zeta_1, \dots, \zeta_j\} \subset K$ be any system of j points of K satisfying $V(\zeta_1, \zeta_2, \dots, \zeta_j) \neq 0$. Then the function

$$h(z) := V(\zeta_1, \dots, \zeta_j, z), \qquad z \in \mathbb{C}^N,$$

can be written in the form

$$h(z) = V(\zeta_1, \dots, \zeta_j) e_{j+1}(z) + \sum_{l=1}^{j} c_l e_l(z), \quad c_l \in \mathbb{C}.$$

Thus h can't be identically zero on K. Otherwise we would have $h \equiv 0$ in $\mathbb{C}^N$, in contrast to the fact that the functions e_j, $j = 1, 2, \dots, m_n$, are linearly independent. This implies

$$V_{j+1}(K) \geq \sup_{z \in K} |V(\zeta_1, \dots, \zeta_j, z)| > 0.$$

(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i) follows from Theorem 3.6. ■

Observe, in the one dimensional case any set $K \subset \mathbb{C}$, containing $n+1$ different points, is a unisolvent set as then equation (3.4) is the standard Vandermondian determinant in $\mathbb{C}$. In several complex variables the situation is not that easy. An auxiliary result in this context is Theorem 3.7 which gives us a first glimpse about unisolvent sets.

Example 3.10

(i) Let $K_j \subset \mathbb{C}$, $j = 1, \dots, N$, be an arbitrary set consisting of at least $n+1$ different points. Then $K := K_1 \times \cdots \times K_N$ is unisolvent of order n.

(ii) A compact set $K \subset \mathbb{R}^N$ with non–empty interior in $\mathbb{R}^N$ is unisolvent.

(iii) If $K \subset \mathbb{C}^N$ is unisolvent, then $\tilde{K} \supset K$ is also unisolvent.

In the next definition we get to know extremal points of a compact set $K \subset \mathbb{C}^N$. These points coincide with the well known Fekete points if $K \subset \mathbb{C}$, see e.g. [Gai80].

Definition 3.11

Let $K \subset \mathbb{C}^N$ be a compact set. A system $\zeta^{(n)}$ of m_n points $\zeta_1, \zeta_2, \dots, \zeta_{m_n} \in K$, where m_n is defined as in equation (3.2), is said to be a unisolvent system of order n (with respect to K) if

$$|V(\zeta^{(n)})| \neq 0,$$

and any unisolvent system $\xi^{(n)}$ of m_n points $\xi_1, \dots, \xi_{m_n} \in K$ is called a system of extremal points of order n (with respect to K) if

$$|V(\xi^{(n)})| = \max_{\zeta^{(n)} \subset K} |V(\zeta^{(n)})|.$$

3.5 The extremal function Φ

Now we are well prepared to define the extremal function Φ and to become acquainted with some of its representations. We shall also explore several fundamental properties of Φ in this section.

Let K be a compact subset of $\mathbb{C}^N$. We set for every $n \in \mathbb{Z}_+$

$$\mathcal{P}_n^c(K) := \left\{ p \in \mathcal{P}_n^c : |p(z)| \leq 1 \text{ for } z \in K \right\}$$

and define for every $n \in \mathbb{Z}_0$ the function $\Phi_n : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$ by

$$\Phi_n(z, K) := \sup \left\{ |p(z)| : p \in \mathcal{P}_n^c(K) \right\}. \tag{3.6}$$

The extremal function Φ may be defined by means of Φ_n.

Definition 3.12
The function $\Phi : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$ given by

$$\Phi(z, K) = \sup_{n \in \mathbb{N}} \sqrt[n]{\Phi_n(z, K)} \tag{3.7}$$

is called the extremal function on K.

A wide variety of polynomial estimates can be derived from the extremal function Φ. The cause depends upon the different ways to express Φ. Remark 3.13 and Lemma 3.14 show diverse representations of Φ. Another important representation of Φ in terms of plurisubharmonic functions will be conducted in Section 4.2.

Remark 3.13

(i) As the inequality

$$\Phi_k(z, K)\Phi_l(z, K) \leq \Phi_{k+l}(z, K), \qquad z \in \mathbb{C}^N, \tag{3.8}$$

is ensured for all $k, l \in \mathbb{N}$, we conclude by a well-known convergence theorem that

$$\Phi(z, K) = \lim_{n \to \infty} \sqrt[n]{\Phi_n(z, K)}, \qquad z \in \mathbb{C}^N.$$

(ii) Since $p \equiv 1 \in \mathcal{P}_n^c(K)$, $n \in \mathbb{N}$, we see $\Phi(z, K) \geq 1$ for all $z \in \mathbb{C}^N$. Thus we obtain an analogous form to equation (3.1):

$$\log \Phi(z, K) = \max \left\{ 0, \sup \left\{ \frac{1}{\deg p} \log |p(z)| \right\} \right\}, \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all non-constant polynomials $p : \mathbb{C}^N \to \mathbb{C}$ satisfying $\|p\|_K \leq 1$.

Lemma 3.14 ([Sic62])
Let $K \subset \mathbb{C}^N$ be a unisolvent compact set and let $\xi^{(n)} = \{\xi_1, \ldots, \xi_{m_n}\}$ be an arbitrary system of extremal points of order n with respect to K. Consider the following functions $\Phi_n^{(l)}(z, K) : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$, $l = 1, 2, 3, 4$, defined by

$$\Phi_n^{(1)}(z, K) := \max_{1 \leq j \leq m_n} |L^{(j)}(z, \xi^{(n)})|,$$

$$\Phi_n^{(2)}(z, K) := \sum_{j=1}^{m_n} |L^{(j)}(z, \xi^{(n)})|,$$

$$\Phi_n^{(3)}(z, K) := \inf_{\zeta^{(n)} \subset K} \max_{1 \leq j \leq m_n} |L^{(j)}(z, \zeta^{(n)})|, \quad \zeta^{(n)} \text{ an arbitrary unisolvent system of } K,$$

$$\Phi_n^{(4)}(z, K) := \inf_{\zeta^{(n)} \subset K} \sum_{j=1}^{m_n} |L^{(j)}(z, \zeta^{(n)})|, \qquad \zeta^{(n)} \text{ an arbitrary unisolvent system of } K.$$

Then

$$\Phi(z, K) = \lim_{n \to \infty} \sqrt[n]{\Phi_n^{(l)}(z, K)}, \quad z \in \mathbb{C}^N,$$

for $l = 1, 2, 3, 4$.

Proof:
We verify for $z \in \mathbb{C}^N$ the following chain of inequalities:

$$\begin{aligned}\Phi_n(z,K) &\leq \Phi_n^{(4)}(z,K) \leq m_n\Phi_n^{(3)}(z,K) \leq m_n\Phi_n^{(1)}(z,K)\\ &\leq m_n\Phi_n^{(2)}(z,K) \leq m_n^2\Phi_n(z,K),\end{aligned} \tag{3.9}$$

where m_n is defined like in equation (3.2).
The first inequality is due to the Lagrange interpolation formula, as we will see in a moment. Let $p \in \mathcal{P}_n^c(K)$ be arbitrary and let $\zeta^{(n)}$ be any unisolvent system of order n with respect to K. Then

$$p(z) = \sum_{j=1}^{m_n} p(\zeta_j)L^{(j)}(z,\zeta^{(n)}), \quad z \in \mathbb{C}^N.$$

As $|p(z)| \leq 1$ on K, we get

$$|p(z)| \leq \sum_{j=1}^{m_n} |L^{(j)}(z,\zeta^{(n)})|, \quad z \in \mathbb{C}^N.$$

This guarantees

$$\Phi_n(z,K) = \sup_{p\in\mathcal{P}_n^c(K)} |p(z)| \leq \inf_{\zeta^{(n)}\subset K} \sum_{j=1}^{m_n} |L^{(j)}(z,\zeta^{(n)})| = \Phi_n^{(4)}(z,K), \quad z \in \mathbb{C}^N,$$

since $p \in \mathcal{P}_n^c(K)$ and $\zeta^{(n)}$ were arbitrary.
The next three inequalities in (3.9) follow immediately from the definition of the functions $\Phi_n^{(l)}$, $l = 1,\ 2,\ 3,\ 4$, whereas the last inequality of the chain is obtained by the fact that

$$L^{(j)}(z,\xi^{(n)}) \in \mathcal{P}_n^c(K), \quad j = 1,2,\dots,m_n.$$

Now, since $\lim_{n\to\infty} \sqrt[n]{m_n} = 1$ and $\lim_{n\to\infty} \sqrt[n]{\Phi_n(z,K)} = \Phi(z,K)$ for $z \in \mathbb{C}^N$, we conclude

$$\Phi(z,K) = \lim_{n\to\infty} \sqrt[n]{\Phi_n^{(l)}(z,K)}, \quad z \in \mathbb{C}^N, \quad l = 1,\ 2,\ 3,\ 4.$$

■

Now we carry on with the Bernstein–Walsh inequality in higher dimensions.

Theorem 3.15
Let $K \subset \mathbb{C}^N$ be a unisolvent compact set and $p_n \in \mathcal{P}_n^c$. Then

$$|p_n(z)| \leq \|p_n\|_K[\Phi(z,K)]^n \quad \textit{for } z \in \mathbb{C}^N. \tag{3.10}$$

Proof:
As $\|p_n\|_K = 0$ implies $p_n \equiv 0$ if K is unisolvent, we may assume $\|p_n\|_K \neq 0$. By definition of Φ we then have

$$\frac{|p_n(z)|}{\|p_n\|_K} \leq [\Phi(z,K)]^n \quad \text{for } z \in \mathbb{C}^N$$

and the result follows immediately. ■

Up to now we are only able to prove the "only if"–part of Theorem 1.5. To show the opposite direction we need some more basic properties of Φ_n and Φ. Let us have a look at them.

Proposition 3.16
Let $K \subset \mathbb{C}^N$ be a unisolvent compact set and let $\xi^{(n)} = \{\xi_1, \dots, \xi_{m_n}\}$ be an arbitrary system of extremal points with respect to K. Then

(i) $\sqrt[n]{\Phi_n^{(1)}(z,K)} \le \Phi(z,K)$ *for every $n \in \mathbb{N}$ and $z \in \mathbb{C}^N$.*

(ii) The sequence $\left\{ \sqrt[2^k]{\Phi_{2^k}(z,K)} \right\}_{k\in\mathbb{N}}$ *increases monotonically for $k \in \mathbb{N}$ and fixed $z \in \mathbb{C}^N$.*

(iii) Let Φ be continuous in $\mathbb{C}^N$. If

$$l_{2^k}(z) := \sqrt[2^k]{\max_{1\le j\le m_{2^k}} |L^{(j)}(z, \xi^{(2^k)})|}, \quad z \in \mathbb{C}^N,$$

then the sequence $\{l_{2^k}\}_{k\in\mathbb{N}}$ converges locally uniformly to Φ in $\mathbb{C}^N$.

(iv) If $\hat{K}$ is the polynomial convex hull of K, that is

$$\hat{K} := \left\{ z \in \mathbb{C}^N : |p(z)| \le \|p\|_K \text{ for all } p \in \mathcal{P}_n^c,\ n \in \mathbb{N} \right\},$$

then

$$\hat{K} = \left\{ z \in \mathbb{C}^N : \Phi(z,K) = 1 \right\}.$$

Proof:
(i) Let an arbitrary $z \in \mathbb{C}^N$ and an arbitrary $n \in \mathbb{N}$ be given. Further let $\xi^{(n)}$ and $\xi^{(m)}$ be extremal systems of order n and m respectively, where m is an arbitrary integer greater or equal than n. Then there exist two uniquely determined integers k and l such that $m = kn + l$ and $0 \le l < n$. By the interpolation formula (3.5) we have

$$|L^{(i)}(z, \xi^{(n)})|^k \le \sum_{j=1}^{m_m} |L^{(j)}(z, \xi^{(m)})| \le m_m \Phi_m^{(1)}(z,K), \quad i = 1, \dots, m_n, \quad z \in \mathbb{C}^N.$$

Therefore we get

$$\left(\sqrt[n]{\Phi_n^{(1)}(z,K)} \right)^{\frac{kn}{m}} \le \sqrt[m]{m_m}\ \sqrt[m]{\Phi_m^{(1)}(z,K)}, \quad z \in \mathbb{C}^N.$$

Since $\frac{kn}{m} \to 1$ and $\sqrt[m]{m_m} \to 1$ as $m \to \infty$, we conclude

$$\sqrt[n]{\Phi_n^{(1)}(z,K)} \le \Phi(z,K), \quad z \in \mathbb{C}^N.$$

Thus the result follows as $z \in \mathbb{C}^N$ and $n \in \mathbb{N}$ were arbitrary.
(ii) The inequality (3.8) of Remark 3.13 shows

$$\left(\Phi_{2^k}(z,K)\right)^2 \le \Phi_{2^{k+1}}(z,K), \quad z \in \mathbb{C}^N,$$

for $k \in \mathbb{N}$, which implies

$$\sqrt[2^k]{\Phi_{2^k}(z,K)} \le \sqrt[2^{k+1}]{\Phi_{2^{k+1}}(z,K)}, \quad z \in \mathbb{C}^N,$$

for $k \in \mathbb{N}$.

(iii) By inequality (3.9) of the proof of Lemma 3.14 and (i) of this proposition we obtain for every $n \in \mathbb{N}$ and $z \in \mathbb{C}^N$ the estimate

$$h_n(z) := \frac{1}{\sqrt[n]{m_n}} \sqrt[n]{\Phi_n(z,K)} \leq \sqrt[n]{\max_{1 \leq j \leq m_n} |L^{(j)}(z, \xi^{(n)})|} \leq \Phi(z,K).$$

Notice, h_n is for every $n \in \mathbb{N}$ lower semicontinuous in $\mathbb{C}^N$ as ϕ_n, $n \in \mathbb{N}$, is the supremum of a family of continuous functions. Further, $\{h_{2^k}\}_{k \in \mathbb{N}}$ increases monotonically to Φ in $\mathbb{C}^N$. Hence Dini's theorem for lower semicontinuous functions guarantees that the sequence $\{h_{2^k}\}_{k\in\mathbb{N}}$ and consequently $\{l_{2^k}\}_{k\in\mathbb{N}}$ converge locally uniformly to Φ in $\mathbb{C}^N$.

(iv) We first prove $\hat{K} \subset \{z \in \mathbb{C}^N : \Phi(z,K) = 1\}$. If this were not true, we would find a point $z_0 \in \hat{K}$ with

$$\Phi(z_0, K) > 1$$

and hence

$$\Phi_n^{(1)}(z_0, K) > 1$$

for $n \in \mathbb{N}$ sufficiently large. This ensures that there exists some i_0, $1 \leq i_0 \leq m_n$, $n \in \mathbb{N}$ sufficiently large, such that

$$|L^{(i_0)}(z_0, \xi^{(n)})| > 1.$$

However, $L^{(i_0)}(z, \xi^{(n)})$ is a polynomial of degree $\leq n$ and $|L^{(i_0)}(z, \xi^{(n)})| \leq 1$ for $z \in K$. Hence by the definition of $\hat{K}$ this would lead to $|L^{(i_0)}(z_0, \xi^{(n)})| \leq 1$. ↯

Now we move on proving $M := \{z \in \mathbb{C}^N : \Phi(z,K) = 1\} \subset \hat{K}$. If $z_0 \in M$ and p is an arbitrary polynomial of degree $\leq n$, then due to the Bernstein-Walsh inequality we have

$$|p(z_0)| \leq \|p\|_K [\Phi(z_0, K)]^n = \|p\|_K,$$

which completes the proof. ■

A useful tool for our work in Chapter 5 is the next theorem due to Siciak [Sic62], which describes the extremal function Φ for Cartesian products of compact sets. Observe, Theorem 1.4 is then just an application of Theorem 3.17. Moreover, we see that Cartesian products of compact intervals with non–empty interior have a continuous extremal function Φ.

Theorem 3.17
Let $K_1 \subset \mathbb{C}^{N_1}$ and $K_2 \subset \mathbb{C}^{N_2}$ be compact sets. Then the extremal function Φ for $K_1 \times K_2$ is given by

$$\Phi((z,w), K_1 \times K_2) = \max\{\Phi(z, K_1), \Phi(w, K_2)\}, \qquad (z,w) \in \mathbb{C}^{N_1+N_2}.$$

We refer the reader to [Kli91] for a nice proof of Theorem 3.17.

3.6 Maximal convergence theorems

As already mentioned in Section 3.2 we wish to expand a holomorphic function f into a series of appropriate polynomials for which we need the Oka-Weil extension theorem, cf. [Hoe66]:

Theorem 3.18
Let Ω_δ be a polynomial polyhedron, that is

$$\Omega_\delta := \big\{z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N : |z_i| < \delta,\ |p_j(z)| < \delta, \quad i = 1, \dots, N,\ j = 1, \dots, k\big\},$$

where $\delta > 0$, $p_j \in \mathcal{P}_n^c$ and $n, k \in \mathbb{N}$, and let $f : \Omega_\delta \to \mathbb{C}$ be a holomorphic function. Then there exists a function g holomorphic in the polycylinder

$$\triangle_\delta := \big\{(z, w) \in \mathbb{C}^N \times \mathbb{C}^k : |z| < \delta, |w| < \delta\big\}$$

such that f can be expressed as

$$f(z) = g(z, p(z)), \quad z \in \Omega_\delta,$$

where $p(z) = (p_1(z), \dots, p_k(z))$.

As we want to stay in reasonable bounds we skip the proof of Theorem 3.18.
Our next step is to construct suitable polynomial polyhedra on which we can expand f in a polynomial series. This leads us to the following lemma, cf. [Sic82].

Lemma 3.19
Let $p_j \in \mathcal{P}_l^c$ be non–constant polynomials for $j = 1, \dots, k$. Furthermore, let $R > 1$ and $t > 0$. If f is a holomorphic function on the set

$$L_{R^t} := \Big\{z \in \mathbb{C}^N : \max_{1 \le j \le k} \sqrt[l]{|p_j(z)|} < R^t\Big\},$$

then the estimate

$$\limsup_{n \to \infty} \sqrt[n]{e_n(\overline{L}_{R^s}, f)} \le \frac{1}{R^{t-s}}$$

holds for any $s \in (0, t)$.

Proof:
Since by Liouville's theorem L_{R^t} is bounded we can choose $\delta > 0$ so large that for a fixed $s \in (0, t)$ the set L_{R^t} is contained in the polydisc

$$\big\{z \in \mathbb{C}^N : |z| < \delta R^{sl}\big\}. \tag{3.11}$$

Then, in turn, L_{R^t} can be represented as a polynomial polyhedron:

$$L_{R^t} = \big\{z \in \mathbb{C}^N : |z| < \delta R^{tl},\ |\delta p_j(z)| < \delta R^{tl},\ j = 1, \dots, k\big\}.$$

Applying Theorem 3.18 gives

$$f(z) = g(z, \delta p(z)) = \sum_{\alpha \in \mathbb{Z}_+^N, \beta \in \mathbb{Z}_+^k} c_{\alpha\beta} z^\alpha (\delta p(z))^\beta\ , \quad z \in L_{R^t},$$

where $p(z) = (p_1(z), \dots, p_k(z))$ and $g(z, w) = \sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha w^\beta$ is holomorphic in the polycylinder

$$\triangle = \big\{(z, w) \in \mathbb{C}^N \times \mathbb{C}^k :\ |z| < \delta R^{tl},\ |w| < \delta R^{tl}\big\}.$$

Pick $\theta < 1$ so close to 1 that

$$\{(z, \delta p(z)) \in \mathbb{C}^{N+k} : z \in \overline{L}_{R^s}\} \subset \theta \triangle.$$

Since $\theta\triangle$ is a relatively compact subset of $\triangle$ and g is holomorphic in $\triangle$, we obtain

$$|c_{\alpha\beta}| \le \frac{M}{(\theta\delta R^{tl})^{|\alpha|+|\beta|}}$$

for some constant $M > 0$. Let us now define the polynomials

$$\tilde{p}_{nl}(z) := \sum_{|\alpha|+|\beta|\le n} c_{\alpha\beta} z^\alpha (\delta p(z))^\beta, \quad n \in \mathbb{N},$$

which are of degree $\le nl$. As by equation (3.11) $|z| \le \delta R^{sl}$ for $z \in \overline{L}_{R^s}$ and $|\delta p(z)| \le \delta R^{sl}$ for $z \in \overline{L}_{R^s}$, we get

$$\|f - \tilde{p}_{nl}\|_{\overline{L}_{R^s}} \le M_1 \frac{(\delta R^{sl})^n}{(\theta\delta R^{tl})^n} = M_1 \theta^{-n} R^{(s-t)ln}$$

for some constant $M_1 > 0$. Finally we set

$$p_m := \tilde{p}_{kl} \qquad \text{for every } m \in \mathbb{N},$$

where $k \in \mathbb{N}$ is the uniquely determined integer such that $m = kl + d$, $0 \le d < l$.
Since $(kl)/m \to 1$ as $m \to \infty$ and $\theta < 1$ can be chosen arbitrarily close to 1, we obtain the desired result

$$\limsup_{m\to\infty} \sqrt[m]{e_m(\overline{L}_{R^s}, f)} \le R^{s-t} \quad \text{for } \; 0 < s < t.$$

■

As a preparation for the proof of Theorem 1.5, we will see that the extremal function $\Phi(z, K)$ can only be continuous in $\mathbb{C}^N$ if K is unisolvent.

Lemma 3.20
If $K \subset \mathbb{C}^N$ is a compact set such that $\Phi(z, K)$ is bounded in some closed proper neighborhood $\overline{U}$ of K, then K is unisolvent. In particular, if $\Phi(z, K)$ is continuous in $\mathbb{C}^N$ then K is unisolvent.

Proof:
We assume K is not unisolvent. Then there exists a polynomial $\hat{p}_n \in \mathcal{P}_n^c$ for some $n \in \mathbb{N}$, such that

$$\|\hat{p}_n\|_K = 0 \qquad \text{but} \qquad \|\hat{p}_n\|_{\overline{U}} = t, \quad t > 0.$$

Now, since $\Phi(z, K)$ is bounded in $\overline{U}$, there exists some constant $M > 0$ such that

$$|\Phi(z, K)| < M, \qquad z \in \overline{U}.$$

As $(M^n \cdot \hat{p}_n)/t \in \mathcal{P}_n^c(K)$ we obtain due to the definition of Φ the inequality

$$\left| \frac{M^n}{t} \hat{p}_n(z) \right| \le [\Phi(z)]^n, \qquad z \in \mathbb{C}^N.$$

In particular,

$$\left|\frac{M^n}{t}\hat{p}_n(z)\right| < M^n \qquad \text{for } z \in \overline{U}.$$

The latter is clearly impossible, since we have $|\hat{p}(\hat{z})| = t$ for some $\hat{z} \in \overline{U}$. Hence K is unisolvent. ∎

Finally we are able to prove the maximal convergence theorem we yearned for.

Theorem 3.21 (cf. [Sic82])
Let $K \subset \mathbb{C}^N$ be a compact set such that the extremal function $\Phi(z, K)$ is continuous in $\mathbb{C}^N$. Further, let $f : K \to \mathbb{C}$ be a continuous function and $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K, f)} \leq \frac{1}{\rho}$$

if and only if f has a holomorphic extension $\tilde{f}$ to

$$L_\rho := \left\{z \in \mathbb{C}^N : \Phi(z, K) < \rho\right\}.$$

Proof:
"$\Leftarrow$": Let the function l_{2^k}, $k \in \mathbb{N}$, defined like in Proposition 3.16 (iii). Then the sequence $\{l_{2^k}\}_{k\in\mathbb{N}}$ converges locally uniformly to Φ in $\mathbb{C}^N$. Therefore, as $\overline{L}_\rho$ is bounded, we can find to each $\varepsilon > 0$ an integer $\tilde{k}$ such that for every $k \geq \tilde{k}$ we have

$$\Phi(z, K)\frac{1}{\rho^\varepsilon} \leq l_{2^k}(z) \leq \Phi(z, K), \qquad z \in \overline{L}_\rho.$$

Setting

$$L_{\rho^t} := \left\{z \in \mathbb{C}^N : \Phi(z, K) < \rho^t\right\} \quad \text{and} \quad L'_{\rho^t} := \left\{z \in \mathbb{C}^N : l_{2^k}(z) < \rho^t\right\}$$

for $t > 0$ and $k > \tilde{k}$ gives

$$K \subset L_{\rho^t} \subset L'_{\rho^t} \subset L_{\rho^{t+\varepsilon}}.$$

Since f is holomorphic in L_ρ and therefore in $L'_{\rho^{1-\varepsilon}}$ if $\varepsilon \in (0, 1/2)$, we get by Lemma 3.19

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K, f)} \leq \limsup_{n\to\infty} \sqrt[n]{e_n(\overline{L}'_{\rho^\varepsilon}, f)} \leq \frac{1}{\rho^{1-2\varepsilon}}.$$

This yields

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K, f)} \leq \frac{1}{\rho},$$

as $\varepsilon \in (0, \frac{1}{2})$ was arbitrary.
"$\Rightarrow$": We choose an arbitrary $R \in (1, \rho)$. By hypothesis there exists to every $n \in \mathbb{N}$ a polynomial $p_n \in \mathcal{P}_n^c$ such that

$$\|f - p_n\|_K \leq \frac{M}{R^n} \tag{3.12}$$

for some constant $M > 0$ independent of n. Since K is unisolvent we may apply the Bernstein-Walsh inequality to the polynomial $p_n - p_{n-1}$ and obtain

$$\begin{aligned}|p_n(z) - p_{n-1}(z)| &\leq \|p_n - p_{n-1}\|_K [\Phi(z,K)]^n \leq (\|f - p_{n-1}\|_K + \|f - p_n\|_K)[\Phi(z,K)]^n \\ &\leq \frac{M(R+1)}{R^n}[\Phi(z,K)]^n, \quad z \in \mathbb{C}^N.\end{aligned}$$

As $R \in (1,\rho)$ was arbitrary, we conclude that the series

$$p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$$

is uniformly convergent on compact subsets of L_ρ. Hence it converges to a holomorphic function $\tilde{f}$ in L_ρ. From equation (3.12) we derive that $\tilde{f} \equiv f$ on K and the theorem is proved. ■

Theorem 3.21 sheds some light on the maximal convergence behavior of real–valued polynomials on compact sets in $\mathbb{R}^N$.

Theorem 3.22
Let $K \subset \mathbb{R}^N$ be a compact set such that the extremal function Φ is continuous in $\mathbb{C}^N$. Furthermore, let $F : K \to \mathbb{R}$ be continuous and $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_\rho := \{z \in \mathbb{C}^N : \Phi(z,K) < \rho\}.$$

Proof:
"$\Leftarrow$": Let us associate to each complex–valued polynomial $p_n : \mathbb{C}^N \to \mathbb{C}$ of degree $\leq n$,

$$p_n(z) = \sum_{\alpha\in\mathbb{Z}_+^N, |\alpha|\leq n} a_\alpha z^\alpha,$$

the real–valued polynomial $P_n : \mathbb{R}^N \to \mathbb{R}$,

$$P_n(x) = \sum_{\alpha\in\mathbb{Z}_+^N, |\alpha|\leq n} \operatorname{Re}(a_\alpha) x^\alpha .$$

Notice, $P_n(x) = \operatorname{Re} p_n(x)$ for $x \in \mathbb{R}^N$. Therefore we obtain the inequality

$$\|F - P_n\|_K = \|F - \operatorname{Re} p_n\|_K \leq \|F - p_n\|_K$$

and in view of Theorem 3.21 we achieve

$$\limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \leq \limsup_{n\to\infty} \sqrt[n]{e_n(K,F)} \leq \frac{1}{\rho}.$$

"⇒": For an arbitrary real polynomial $\hat{P}_n : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq n$ we define

$$\hat{p}_n(z) := \hat{P}_n(z), \quad z \in \mathbb{C}^N.$$

Since $\hat{p}_n \in P_n^c$ we derive the estimate

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,F)} \leq \limsup_{n\to\infty} \sqrt[n]{E_n(K,F)} \leq \frac{1}{\rho}.$$

Theorem 3.21 now implies that F has a holomorphic extension to

$$L_\rho := \left\{z \in \mathbb{C}^N : \Phi(z,K) < \rho\right\}.$$

■

3.7 Some remarks on the maximal convergence theorems

Theorems 3.21 and 3.22 are based on the assumption "Φ is continuous in $\mathbb{C}^N$". For that reason we like to point out some equivalent conditions to this prerequisite.

Remark 3.23 ([Sic82])
Let K be a compact subset of $\mathbb{C}^N$. Then the following conditions are equivalent:

(i) *Φ is continuous at every point $z \in K$, that is* $\lim\limits_{\substack{z_k\to z\in K,\\ z_k\in\mathbb{C}^N}} \Phi(z_k,K) = \Phi(z,K)$;

(ii) *Φ is continuous in $\mathbb{C}^N$;*

(iii) *To each real number $R > 1$ there exist an open neighborhood U of K and a constant $M > 0$ such that*

$$\|p\|_U \leq M\|p\|_K R^n$$

for every $p \in \mathcal{P}_n^c$, $n \in \mathbb{N}$.

Remark 3.24
Baouendi and Goulaouic [BG74] as well as Siciak and Nguyen Thanh Van [SN74] provided an additional equivalent condition in Remark 3.23 in the case that the compact set K is not too "small". The requirement "K should not be too small" means that any holomorphic function defined on a connected open neighborhood of K with $f|_K \equiv 0$ is identical zero.
Now, let K be such a set. Then (i), (ii) and (iii) of Remark 3.23 are equivalent to the statement:
If f is continuous and $\limsup_{n\to\infty} \sqrt[n]{e_n(f,K)} < 1$, then f extends to a uniquely determined holomorphic function in a neighborhood of K.

In view of Chapter 5 we show that the closed unit balls $\overline{B}_N$ in $\mathbb{R}^N$ are not too "small" compact sets.

Remark 3.25

Let $F : \overline{B}_N \to \mathbb{R}$ be continuous. If $F : \overline{B}_N \to \mathbb{R}$ has a holomorphic extension $\tilde{F}$ to some neighborhood of $\overline{B}_N$ in $\mathbb{C}^N$, then it is uniquely determined.

Suppose this were not true, then there exist two different holomorphic extensions $\tilde{F}_1 : G_1 \to \mathbb{C}$ and $\tilde{F}_2 : G_2 \to \mathbb{C}$, where G_1 and G_2 are appropriate chosen neighborhoods of B_N in $\mathbb{C}^N$. In particular, these extensions are holomorphic in $\mathcal{D}_N(0,\varepsilon) = \{z \in \mathbb{C}^N : |z| < \varepsilon\} \subset G := G_1 \cap G_2$ for $\varepsilon > 0$ sufficiently small. There we may expand $\tilde{F}_1$ and $\tilde{F}_2$ into their power series

$$\tilde{F}_1(z) = \sum_{\alpha \in \mathbb{Z}_+^N} a_\alpha z^\alpha \quad \text{and} \quad \tilde{F}_2(z) = \sum_{\alpha \in \mathbb{Z}_+^N} b_\alpha z^\alpha, \quad z \in \mathcal{D}_N(0,\varepsilon).$$

As we have

$$\sum_{\alpha \in \mathbb{Z}_+^N} a_\alpha z^\alpha = \sum_{\alpha \in \mathbb{Z}_+^N} b_\alpha z^\alpha$$

for $z \in D_N(0,\varepsilon) = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$, we get by the identity principle of power series

$$a_\alpha = b_\alpha, \qquad \alpha \in \mathbb{Z}_+^N,$$

and therefore

$$\tilde{F}_1(z) = \tilde{F}_2(z) \quad \text{for } z \in \mathcal{D}_N(0,\varepsilon).$$

Since $\mathcal{D}_N(0,\varepsilon)$ is a non-empty open set of G we can apply the identity principle and obtain

$$\tilde{F}_1|_G \equiv \tilde{F}_2|_G.$$

In order to apply Theorem 3.21 or Theorem 3.22 to a given maximal convergence problem on a compact set $K \subset \mathbb{C}^N$ we must find out if the assumption "$\Phi(z,K)$ is continuous in $\mathbb{C}^N$" is satisfied. A useful geometric criterion to check the continuity of $\Phi(z,K)$ in $\mathbb{C}^N$ goes back to Plesniak. We also refer the reader [Sic97].

Theorem 3.26 ([Ple84])

Let Ω be a bounded open subset of $\mathbb{C}^N$ with C^1–boundary. Then the extremal function Φ for $\overline{\Omega}$ is continuous in $\mathbb{C}^N$.

At the end of this chapter we like to show that in analogy to the complex plane the sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials p_n, $p_n \in \mathcal{P}_n^c$, constructed by interpolation at extremal points converges maximally to the corresponding holomorphic function, cf. [Sic62].

Theorem 3.27

Let $K \subset \mathbb{C}^N$ be compact, $\Phi(z,K)$ continuous in $\mathbb{C}^N$ and $\xi^{(n)} = \{\xi_1, \ldots, \xi_{m_n}\}$ an extremal system of K. Furthermore, let f be holomorphic in an open connected neighborhood of K. Then the sequence $\{l_n\}_{n\in\mathbb{N}}$ of interpolation polynomials

$$l_n(z) = \sum_{j=1}^{m_n} f(\xi_j) L^{(j)}(z, \xi^{(n)})$$

converges maximally to f.

Proof:
Due to Theorem 3.21 there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials p_n of degree $\leq n$ converging maximally to f, that is

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,f)} = \frac{1}{\rho},$$

where $\rho > 1$ is the greatest number such that f can be extended holomorphically to L_ρ. Therefore we can choose an arbitrary $R \in (1,\rho)$ such that

$$\|f - p_n\|_K \leq \frac{M}{R^n}$$

for all $n \in \mathbb{N}$, where $M > 0$ is some constant independent of n. Now, observe the identity

$$l_n(z) - p_n(z) = \sum_{j=1}^{m_n} \big(f(\xi_j) - p_n(\xi_j)\big) L^{(j)}(z, \xi^{(n)}), \quad z \in \mathbb{C}^N,$$

for every $n \in \mathbb{N}$. Since by the definition of $L^{(j)}$, $j = 1, 2, \ldots, m_n$, the inequality

$$\max_{1\leq j\leq m_n} |L^{(j)}(z, \xi^{(n)})| \leq 1, \qquad z \in K,$$

holds, we obtain

$$|l_n(z) - p_n(z)| \leq \frac{m_n M}{R^n}, \qquad z \in K,$$

for every $n \in \mathbb{N}$. This implies

$$|f(z) - l_n(z)| \leq \frac{M}{R^n}(m_n + 1), \qquad z \in K,$$

for every $n \in \mathbb{N}$. Finally, we end up with

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,f)} \leq \frac{1}{\rho}$$

as $R \in (1,\rho)$ was arbitrary. ■

Chapter 4

On some representations of Φ

4.1 Preliminaries

In Section 4.3 we will focus on Baran's method [Bar88] to prove Lundin's formula for the extremal function Φ for compact, convex and symmetric sets $S \subset \mathbb{R}^N$ whose interior is not empty. Baran exploited like Lundin the fact that Φ can be represented as

$$\log \Phi(z, K) = \sup\{u(z) : u \in \mathcal{L}, u|_K \leq 0\}, \qquad z \in \mathbb{C}^N, \tag{4.1}$$

for compact sets $K \subset \mathbb{C}^N$, where $\mathcal{L}$ denotes the set of all plurisubharmonic functions v in $\mathbb{C}^N$ which satisfy the condition $\sup_{z\in\mathbb{C}^N} |v(z) - \log(1 + |z|)| < \infty$[1]. For that reason we introduce in Section 4.2 Siciak's second proof of this identity, which is based on classical results of several complex variables. For the convenience of the reader we take a "pluricomplex interlude" before progressing to a study of equation (4.1). However, we will only outline the necessary equipment as to go into further details would be beyond the scope of this manuscript. The reader interested in these aspects is referred in particular to [Hoe66], [Kli91] and [Kra01].

4.1.1 Plurisubharmonicity and smoothing

Let us first recall the definitions of subharmonic and plurisubharmonic functions.

Definition 4.1
Let $\Omega \subset \mathbb{C}$ be an open set. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic, if

(i) u is upper semicontinuous;

(ii) the local submean inequality holds, i.e. for every $z_0 \in \Omega$ there exists an $\rho > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt$$

for any $r \in (0, \rho)$.

[1] Observe, this representation resembles the situation in the complex plane. In fact, if K is a compact subset of $\mathbb{C}$ such that $\hat{\mathbb{C}}\backslash K$ is connected and regular, then Green's function for $\hat{\mathbb{C}}\backslash K$ with pole at infinity is equal to $\log \Phi$.

Definition 4.2
Let $\Omega \subset \mathbb{C}^N$ be an open set. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic, if

(i) *u is upper semicontinuous;*
(ii) *to each $z \in \Omega$ and $w \in \mathbb{C}^N$ correspond a neighborhood U of 0 in $\mathbb{C}$ such that the function*

$$\tau \mapsto u(z + \tau w)$$

is subharmonic in U.

The set of all plurisubharmonic functions defined on an open set $\Omega \subset \mathbb{C}^N$ is denoted by $PSH(\Omega)$.
Typical examples of plurisubharmonic functions are $\log|f|$ and $|f|^\alpha$ for $\alpha > 0$, if f is holomorphic. Further candidates of plurisubharmonic functions are constructed in Proposition 4.4 and Theorem 4.5. Item (i) of Proposition 4.4 is a direct consequence of the definition of plurisubharmonic functions, whereas the items (ii) to (iv) are applications of

Theorem 4.3
Let Ω be an open subset of $\mathbb{C}^N$. If $u, -v \in PSH(\Omega)$, $u \geq 0$ in Ω, $v > 0$ in Ω and $\psi : [0,\infty) \to [0,\infty)$ is a convex function with $\psi(0) = 0$, then $v\psi(u/v)$ belongs to $PSH(\Omega)$.

Proposition 4.4
Let Ω be an open subset of $\mathbb{C}^N$.

(i) *If $u_1, u_2 \in PSH(\Omega)$ and $\alpha, \beta \in [0,\infty)$, then $\alpha u_1 + \beta u_2 \in PSH(\Omega)$.*
(ii) *If $u \in PSH(\Omega)$, then $e^u \in PSH(\Omega)$.*
(iii) *If $u \in PSH(\Omega)$ is non–negative, then $u^\alpha \in PSH(\Omega)$ for $\alpha \geq 1$.*
(iv) *If u_1, u_2 are non–negative functions in Ω such that $\log u_1, \log u_2 \in PSH(\Omega)$, then $u_1 u_2$ and $\log(u_1 + u_2)$ belong to $PSH(\Omega)$.*

Theorem 4.5
Let Ω be an open subset of $\mathbb{C}^N$, $v \in PSH(\Omega)$ and $S := \{z \in \Omega : v(z) = -\infty\}$ be a closed subset of Ω. If $u \in PSH(\Omega \backslash S)$ is bounded above, then the function

$$\tilde{u}(z) := \begin{cases} u(z) & \text{if } z \in \Omega \backslash S \\ \limsup\limits_{\substack{w \to z \\ w \notin S}} u(w) & \text{if } z \in S \end{cases} \tag{4.2}$$

is plurisubharmonic in Ω.

For our further considerations it is indispensable to provide a method which smoothes plurisubharmonic functions. This leads us to the *standard smoothing kernel.*

Definition 4.6
The standard smoothing kernel is defined for $z \in \mathbb{C}^N$ by

$$\chi_\varepsilon(z) = \frac{1}{\varepsilon^{2N}} \chi\Big(\frac{z}{\varepsilon}\Big), \qquad \varepsilon > 0, \tag{4.3}$$

where

$$\chi(z) := \begin{cases} Ce^{-\frac{1}{1-\|z\|^2}} & \text{if } \|z\| < 1 \\ 0 & \text{if } \|z\| \geq 1 \end{cases} \quad \text{with} \quad C := \Big(\int_{\mathcal{B}_N(0,1)} e^{-\frac{1}{1-\|z\|^2}} \, dz \Big)^{-1}$$

and the integration is taken with respect to the $2N$–dimensional Lebesgue measure in $\mathbb{C}^N$.

Note that $\int_{\mathbb{C}^N} \chi_\varepsilon(z)dz = 1$ and the support of χ_ε is $\overline{\mathcal{B}}_N(0,\varepsilon)$.

The next theorem clarifies why the functions χ_ε are called the standard smoothing kernels. In view of this theorem we define for every proper open subset Ω of $\mathbb{C}^N$ the set

$$\Omega_\varepsilon := \left\{ z \in \Omega : \operatorname{dist}(z, \partial\Omega) = \inf_{w\in\mathbb{C}^N\setminus\Omega} |z-w| > \varepsilon \right\} \quad \text{for } \varepsilon > 0.$$

If $\Omega = \mathbb{C}^N$, we put $\Omega_\varepsilon := \mathbb{C}^N$ for $\varepsilon > 0$.
The symbol $C^k(\Omega)$ stands for the set of all k–times continuously differentiable functions defined on an open subset $\Omega \subset \mathbb{C}^N$.

Theorem 4.7
Let Ω be an open subset of $\mathbb{C}^N$ and let $u \in PSH(\Omega)$, $u \not\equiv -\infty$.
If $\varepsilon \in (0,1)$ and $\Omega_\varepsilon \neq \emptyset$ then

$$u_\varepsilon(z) := \int u(z - \varepsilon\zeta)\chi_\varepsilon(\zeta)d\zeta, \qquad z \in \Omega, \tag{4.4}$$

belongs to $C^\infty(\Omega_\varepsilon) \cap PSH(\Omega_\varepsilon)$. Moreover, u_ε decreases pointwise to u in Ω as ε tends to zero, i.e.

$$u_\varepsilon \searrow u \quad \text{for} \quad \varepsilon \searrow 0.$$

The following result on sequences of plurisubharmonic functions is known as *Hartogs's Lemma.*

Lemma 4.8
Let $\Omega \subset \mathbb{C}^N$ be an open set and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of plurisubharmonic functions in Ω, which is locally uniformly bounded above. If there exists some constant $M > 0$ such that

$$\limsup_{n\to\infty} u_n(z) \le M$$

for every $z \in \Omega$, then there exists for every compact set $K \subset \Omega$ and every $\varepsilon > 0$ a natural number n_0 such that

$$\sup_{z\in K} u_n(z) \le M + \varepsilon$$

for $n > n_0$.

Our next theorem can be proved by means of *Hartogs's Lemma.* It treats homogeneous expansions of holomorphic functions which are an essential tool of our work, see also Section 5.2.
We recall, a polynomial p in $\mathbb{C}^N$ is *homogeneous of degree s*, $s \in \mathbb{N}$, if

$$p(\lambda z) = \lambda^s p(z)$$

for $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^N$.
Now, assume f is a holomorphic function in $\mathcal{D}_N(0,\varepsilon)$. Then f can be expanded into a power series

$$f(z) = \sum_{\alpha\in\mathbb{Z}_+^N} a_\alpha z^\alpha,$$

which is absolutely and uniformly convergent on compact subsets of $\mathcal{D}_N(0,\varepsilon)$. Thus the power series can be written in the grouped form

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha z^\alpha, \qquad z \in \mathcal{D}_N(0,\varepsilon), \quad \alpha \in \mathbb{Z}_+^N.$$

This is known to be the *homogeneous expansion* of f.
We recall, a set $S \subset \mathbb{C}^N$ is said to be *balanced* if for every $z \in S$ and $\eta \in \overline{\mathbb{D}}$ also $\eta z \in S$.

Theorem 4.9
Let $\Omega \subset \mathbb{C}^N$ be a balanced domain in $\mathbb{C}^N$ and suppose f is holomorphic in Ω. Then f has a locally uniformly convergent homogeneous expansion in Ω. An expansion of this form is unique.

4.1.2 Domains of holomorphy

The intention of this subsection is to provide some characterizations of domains of holomorphy in several variables. Here, we get a first glimpse that the situation is decidedly different to the one dimensional case.

Definition 4.10
An open set $\Omega \subset \mathbb{C}^N$ is called a domain of holomorphy if there are no open sets Ω_1 and Ω_2 in $\mathbb{C}^N$ with the following properties:
(i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$;
(ii) Ω_2 *is connected and not contained in* Ω;
(iii) For every $f \in \mathcal{H}(\Omega)$ *there is a function* $\tilde{f} \in \mathcal{H}(\Omega_2)$ *such that* $f = \tilde{f}|_{\Omega_1}$;

Roughly speaking, Ω is a domain of holomorphy if there is no part of the boundary across which every holomorphic function in Ω can be holomorphically extended.
In contrast to the theory of holomorphic functions of one complex variable not every open connected set in $\mathbb{C}^N$, $N > 1$, is a domain of holomorphy. For example let

$$\Omega = \mathcal{D}_2(0,2)\backslash\overline{\mathcal{D}}_2(0,1) \subset \mathbb{C}^2.$$

Then every holomorphic function in Ω continues analytically to the domain $\mathcal{D}_2(0,2)$. This fact, known as *Hartogs extension phenomenon*, is a special case of a much more powerful result, the so–called *Kugelsatz*:

Theorem 4.11
Let $\Omega \subset \mathbb{C}^N$, $N \geq 2$, be a domain, and $K \subset \Omega$ a compact set such that $\Omega\backslash K$ is connected. If f is holomorphic in $\Omega\backslash K$ then f has a unique holomorphic extension to Ω.

Great effort has been put in to characterize domains of holomorphy in terms of some geometric properties of the boundary of the domain. It turns out that every convex domain is a domain of holomorphy. As convexity is not preserved under holomorphic mappings some less rigid geometric condition is required to describe domains of holomorphy, known as pseudoconvexity[2]. This highly nontrivial problem was first solved by Oka [Oka37] for $N = 2$ and by Bremermann [Bre54], Norguet [Nor54] and Oka [Oka53] for $N \geq 3$.
Before we state more results about domains of holomorphy let us get familiar with three types of convex sets.

[2]We refer the reader [Kra01] for an excellent discussion of this topic.

Definition 4.12
Let Ω be an open subset of $\mathbb{C}^N$ and let $K \subset \Omega$ be compact.
The polynomially convex hull of K is the set

$$\hat{K} := \left\{ z \in \mathbb{C}^N : |p(z)| \leq \|p\|_K \text{ for all } p \in \mathcal{P}_n^c, n \in \mathbb{N} \right\}.$$

Similarly we define the holomorphically convex hull of K in Ω by

$$\hat{K}_{\mathcal{H}(\Omega)} := \left\{ z \in \Omega : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{H}(\Omega) \right\},$$

and the plurisubharmonically convex hull of K in Ω by

$$\hat{K}_{PSH(\Omega)} := \left\{ z \in \mathbb{C}^N : |u(z)| \leq \|u\|_K \text{ for all } u \in PSH(\Omega) \right\}.$$

Ω itself is said to be polynomially convex, holomorphically convex and pseudoconvex respectively, if for every compact set $K \subset \Omega$, the sets $\hat{K}$, $\hat{K}_{\mathcal{H}(\Omega)}$ and $\hat{K}_{PSH(\Omega)}$ respectively are relatively compact in Ω.

As $|f| \in PSH(\Omega)$ for $f \in \mathcal{H}(\Omega)$ we obtain the following chain of inclusion:

$$\hat{K}_{PSH(\Omega)} \subset \hat{K}_{\mathcal{H}(\Omega)} \subset \hat{K}.$$

Thus every holomorphically convex set is pseudoconvex. As already indicated the converse is also true. Our next lemma describes in some sense pseudoconvex sets.

Lemma 4.13
Let $h : \mathbb{C}^N \to [0, \infty)$, $h \not\equiv 0$, be an upper semicontinuous function which is non-negative homogeneous, i.e. $h(tz) = |t|h(z)$ for all $t \in \mathbb{C}$ and $z \in \mathbb{C}^N$. Then the following conditions are equivalent:
- *(i) h is plurisubharmonic;*
- *(ii) the set $\{z \in \mathbb{C}^N : h(z) < 1\}$ is pseudoconvex.*

There are several different characterizations of domains of holomorphy, which are quite subtle. The one we are interested in is due to Cartan and Thullen [CT32]:

Theorem 4.14
Let Ω be an open subset of $\mathbb{C}^N$. Then the following conditions are equivalent.
- *(i) Ω is a domain of holomorphy;*
- *(ii) Ω is holomorphically convex.*

If, in addition, $\Omega \subset \mathbb{C}^N$ is a *Runge domain*, that means Ω is a domain of holomorphy on which $f \in \mathcal{H}(\Omega)$ can be locally uniformly approximated by complex polynomials, then we have the next characterization.

Theorem 4.15
The following conditions for a domain of holomorphy $\Omega \subset \mathbb{C}^N$ are equivalent:
- *(i) Ω is a Runge domain;*
- *(ii) Ω is polynomially convex.*

4.2 A representation of Φ by means of plurisubharmonic functions

Equation (4.1) suggests itself to introduce a multi-dimensional counterpart of Green's function with pole at infinity. This, in turn, requires some terminology.

A function $u \in PSH(\mathbb{C}^N)$ is said to be of *minimal growth at infinity* if

$$u(z) - \log(1 + |z|) \leq O(1) \quad \text{as} \quad |z| \to \infty.$$

The family of all such functions will be denoted by

$$\mathcal{L} := \left\{ u \in PSH(\mathbb{C}^N) : u(z) \leq \beta + \log(1 + |z|) \text{ for } z \in \mathbb{C}^N \right\}, \tag{4.5}$$

where $\beta \in \mathbb{R}$ may depend on u.
Further, we put for any set $S \subset \mathbb{C}^N$

$$\mathcal{L}(S) := \{ u \in \mathcal{L} : u(z) \leq 0 \quad \text{for } z \in S \},$$

and define for every $z \in \mathbb{C}^N$ the function

$$V(z, S) := \sup\{ u(z) : u \in \mathcal{L}(S) \}.$$

The function V is called *the pluricomplex Green's function* to emphasize the analogy to the one-dimensional case.

Remark 4.16
If $p \in \mathcal{P}_n^c$, then $1/n \cdot \log |p| \in \mathcal{L}$.
This can be easily checked if we set $M := \sup\{|p(z)| : |z| \leq 1\}$. Then by Cauchy's inequalities we get

$$|p(z)| \leq M(1 + |z| + |z|^2 + \cdots + |z|^n) \leq M(1 + |z|)^n$$

and the result follows, as $1/n \cdot \log |p| \in PSH(\mathbb{C}^N)$.

An attractive feature of plurisubharmonic functions with minimal growth at infinity is the full description by polynomials, see [Sic82]. In our next theorem we will show how a special class of non–negative homogeneous plurisubharmonic functions can be expressed in terms of homogeneous polynomials. Our presentation is based on [Sic82].

Theorem 4.17
Let $h \in PSH(\mathbb{C}^N) \cap C(\mathbb{C}^N)$, $h \not\equiv 0$, be a non–negative homogeneous function, i.e. $h(tz) = |t| h(z)$ for all $t \in \mathbb{C}$ and $z \in \mathbb{C}^N$, which satisfies the condition $h(z) \geq M\|z\|$ for all $z \in \mathbb{C}^N$ and some constant $M > 0$. Then

$$h(z) = \sup \left\{ |q(z)|^{\frac{1}{\deg q}} : q : \mathbb{C}^N \to \mathbb{C} \text{ is a homogeneous polynomial with } |q|^{\frac{1}{\deg q}} \leq h \text{ in } \mathbb{C}^N \right\}$$

for every $z \in \mathbb{C}^N$.

Proof:
Let us define for $z \in \mathbb{C}^N$ the function

$$\psi(z) := \sup\Big\{|q(z)|^{\frac{1}{\deg q}} : q \text{ is a homogeneous polynomial with } |q|^{\frac{1}{\deg q}} \leq h \text{ in } \mathbb{C}^N\Big\}.$$

Then we clearly have $\psi(z) \leq h(z)$ for $z \in \mathbb{C}^N$.
To prove the opposite inequality we only need to show that $h(z_0) = 1$ implies $\psi(z_0) \geq 1$. Therefore let $z_0 \in \mathbb{C}^N$ be a fixed point such that $h(z_0) = 1$ and define

$$S := \big\{z \in \mathbb{C}^N : h(z) < 1\big\}.$$

Then S is a domain of holomorphy by Lemma 4.13. Further S is balanced. From Theorem 4.9 we conclude that S is a Runge domain. Hence S is also polynomially convex. This in turn means that the polynomially convex hull $\hat{K}_t$ of the compact set

$$K_t := \big\{z \in \mathbb{C}^N : h(z) \leq t\big\}, \quad t \in (0,1),$$

is a subset of S.
Next, we verify the assertion that $\hat{K}_t$ is also a convex hull of K_t with respect to homogeneous polynomials, i.e.

$$\hat{K}_t = \{z \in \mathbb{C}^N : |q(z)| \leq \|q\|_{K_t} \text{ for all } q \in Q\}, \tag{4.6}$$

where Q is the set of all homogeneous polynomials $q : \mathbb{C}^N \to \mathbb{C}$.
The inclusion

$$\hat{K}_t \subset \{z \in \mathbb{C}^N : |q(z)| \leq \|q\|_{K_t} \text{ for all } q \in Q\} =: T$$

is obvious. Therefore let us show that an arbitrary point $\hat{z} \in T$ is also an element of $\hat{K}_t$. As every polynomial p_n of degree $\leq n$ can be represented by homogeneous polynomials in the form

$$p_n(z) = \sum_{k=0}^{n} q_k(z), \quad q_k \in Q, \quad \deg q_k = k, \quad z \in \mathbb{C}^N,$$

we have

$$p_n(\lambda z) = \sum_{k=0}^{n} \lambda^k q_k(z), \quad \text{for } \lambda \in \mathbb{C}, \quad z \in \mathbb{C}^N.$$

Now, we choose $\lambda \in \partial\mathbb{D}$. Since K_t is balanced we get by Cauchy's estimates

$$|q_k(\tilde{z})| \leq \|p_n\|_{K_t} \quad \text{for } \tilde{z} \in K_t, \quad k = 0, 1, \ldots, n,$$

and therefore

$$|p_n(\hat{z})| \leq \sum_{k=0}^{n} |q_k(\hat{z})| \leq \sum_{k=0}^{n} \|q_k\|_{K_t} \leq (n+1)\|p_n\|_{K_t}.$$

Consequently,

$$|p_n(\hat{z})|^{1/n} \leq (n+1)^{1/n}\|p_n\|_{K_t}^{1/n}.$$

If we replace p_n by p_n^j, $j = 1, 2, \ldots$, in the last inequality we obtain

$$|p_n(\hat{z})|^{1/n} \leq (jn+1)^{1/(jn)}\|p\|_{K_t}^{1/n}.$$

Letting $j \to \infty$ gives

$$|p_n(\hat{z})|^{1/n} \leq \|p\|_{K_t}^{1/n}$$

and hence $T \subset \hat{K}_t$, as required.

Since $\hat{K}_t \subset S$ for $t \in (0,1)$ and $z_0 \notin S$ we can find to each $t \in (0,1)$ a point $s \in (t,1)$ and a homogeneous polynomial $\hat{q}$ which satisfy

$$1 = \|\hat{q}\|_{K_t} \leq \hat{q}(sz_0).$$

Then it holds

$$K_t = \{z \in \mathbb{C}^N : h(z) \leq t\} \subset \{z \in \mathbb{C}^N : |t^{\deg \hat{q}} \hat{q}(z)|^{\frac{1}{\deg \hat{q}}} \leq t\}. \tag{4.7}$$

Our aim is now to show that

$$\left(t^{\deg \hat{q}} |\hat{q}(z)|\right)^{1/\deg \hat{q}} \leq h(z) \quad \text{for } z \in \mathbb{C}^N.$$

If there were some $\tilde{z} \in K_t$ such that

$$h(\tilde{z}) < \left(t^{\deg \hat{q}} |\hat{q}(\tilde{z})|\right)^{1/\deg \hat{q}},$$

then the inequality would also be true in a sufficiently small open neighborhood $U_\varepsilon(\tilde{z})$ of $\tilde{z}$ and due to the homogeneity of h and $t|\hat{q}|^{1/\deg \hat{q}}$ also in $\mathbb{C}^N$. However this is impossible because of equation (4.7). Thus we get

$$\left(t^{\deg \hat{q}} |\hat{q}(z)|\right)^{1/\deg \hat{q}} \leq h(z), \qquad z \in K_t.$$

Again, from the homogeneity of h and $t|\hat{q}|^{1/\deg \hat{q}}$ we obtain

$$\left(t^{\deg \hat{q}} |\hat{q}(z)|\right)^{1/\deg \hat{q}} \leq h(z), \qquad z \in \mathbb{C}^N.$$

All in all we have

$$t \leq \left| t^{\deg \hat{q}} \hat{q}(sz_0) \right|^{1/\deg \hat{q}} \leq \psi(sz_0) = s\psi(z_0).$$

As $s \to 1$ if $t \to 1$, we end up with

$$1 \leq \psi(z_0),$$

which completes the proof. ■

For simplification we introduce the set

$$\mathcal{L}_0 := \left\{ e^u : \ u \in \mathcal{L}, \ e^{u(z)} = \sup_{n \in \mathbb{N}} |p_n(z)|^{1/k_n}, \quad z \in \mathbb{C}^N \right\},$$

where p_n are appropriate polynomials of degree $\leq k_n$, $k_n \in \mathbb{N}$.
Observe, every function $v \in \mathcal{L}_0$ is continuous. This follows from the fact that on the one hand v is upper continuous by definition and that on the other hand v is the supremum of a family of continuous functions.
Now, at this point the various components of our considerations lead to the representation of equation (4.1).

Theorem 4.18 (cf. [Sic82])
Let $K \subset \mathbb{C}^N$ be compact. Then

$$V(z,K) = \log \Phi(z,K) \quad \textit{for } z \in \mathbb{C}^N.$$

Proof:
"≥": It follows from Remark 4.16 that the estimate

$$V(z,K) \geq \sup_{p \in \mathcal{P}_n^c(K)} \left\{ \log\left(\sqrt[n]{|p(z)|} \right) \right\} = \log\left(\sqrt[n]{\Phi_n(z)} \right), \qquad z \in \mathbb{C}^N,$$

is true for every $n \in \mathbb{N}$, where Φ_n is defined as in equation (3.6). Hence we have

$$V(z,K) \geq \log \Phi(z,K), \qquad z \in \mathbb{C}^N.$$

"≤": To show this inequality we first prove the following claim:
Let u be an element of $\mathcal{L}$. Then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{L}_0$ such that

$$e^{u(z)} = \lim_{n\to\infty} \varphi_n(z), \quad z \in \mathbb{C}^N.$$

Proof: Let $u \in \mathcal{L}$ be given. We put for $z \in \mathbb{C}^N$ and $n \in \mathbb{N}$

$$\psi_n(z_0, z) := \begin{cases} |z_0| \left(\frac{1}{n} + e^{u_{1/n}(z/z_0)}\right)^{1-\frac{1}{n}} + \frac{1}{n}\left(|z_0|^2 + \|z\|^2\right)^{\frac{1}{2}} & \text{if } z_0 \in \mathbb{C}\backslash\{0\}, \\ \frac{1}{n}\|z\| \quad \text{if } z_0 = 0, & \end{cases}$$

where $u_{\frac{1}{n}}(z) = \int_{\mathbb{C}^N} u(z - \frac{1}{n}\zeta)\chi_{\frac{1}{n}}(\zeta)d\zeta$ is defined as in equation (4.4).
Obviously, ψ_n, $n \in \mathbb{N}$, is a non–negative homogeneous function and $\psi_n(\eta) \geq \frac{1}{n}||\eta||$ for $\eta \in \mathbb{C}^{N+1}$.
Because of the estimate

$$u(z) \leq u_{\frac{1}{n}}(z) \leq \max_{\eta \in B(z,\frac{1}{n^2})} u(\eta) < \beta + \log\left(1 + |z| + \frac{1}{n^2}\right) \quad \text{for } z \in \mathbb{C}^N, \tag{4.8}$$

where $\beta \in \mathbb{R}$ is a fixed number, the functions ψ_n are also continuous in $\mathbb{C}^{N+1}$.
We now show $\psi_n \in PSH(\mathbb{C}^{N+1})$, $n \in \mathbb{N}$. From Proposition 4.4 (iv) we conclude

$$\log\left(\frac{1}{n} + e^{u_{1/n}(z/z_0)}\right) \in PSH(\mathbb{C}^{N+1}\backslash\{0\}),$$

and therefore by Proposition 4.4 (i) $\log(1/n + e^{u_{1/n}(z/z_0)})^{1-1/n} \in PSH(\mathbb{C}^{N+1}\backslash\{0\})$. Now applying Proposition 4.4 (iv) again gives

$$|z_0| \left(\frac{1}{n} + e^{u_{1/n}(z/z_0)}\right)^{1-\frac{1}{n}} \in PSH(\mathbb{C}^{N+1}\backslash\{0\})$$

and in consequence by Proposition 4.4 (i)

$$|z_0| \left(\frac{1}{n} + e^{u_{1/n}(z/z_0)}\right)^{1-\frac{1}{n}} + \frac{1}{n}\left(|z_0|^2 + \|z\|^2\right)^{\frac{1}{2}} \in PSH(\mathbb{C}^{N+1}\backslash\{0\}).$$

Finally, as ψ_n is bounded in $\mathbb{C}^{N+1}\backslash\{0\}$, we derive from Theorem 4.5 that $\psi_n \in PSH(\mathbb{C}^{N+1})$ for $n \in \mathbb{N}$. Thus for each ψ_n, $n \in \mathbb{N}$, the hypotheses of Theorem 4.17 are satisfied and we can represent ψ_n in the form

$$\psi_n(\eta) = \sup\left\{ |q(\eta)|^{\frac{1}{\deg q}} \right\} \quad \text{for } \eta \in \mathbb{C}^{N+1},$$

where the supremum is taken over all homogeneous polynomials $q : \mathbb{C}^{N+1} \to \mathbb{C}^N$ with $|q|^{1/\deg q} \leq \psi_n$ in $\mathbb{C}^{N+1}$. Moreover, we have

$$\lim_{n\to\infty} \psi_n(1, z) = e^{u(z)}, \qquad z \in \mathbb{C}^N.$$

Due to the inequality (4.8) we conclude that

$$\varphi_n(z) := \psi_n(1, z), \qquad z \in \mathbb{C}^N,$$

is an element of $\mathcal{L}_0$ for every $n \in \mathbb{N}$. This completes the proof of the claim.

Next let $u \in \mathcal{L}(K)$. Then $S := \{z \in \mathbb{C}^N : e^{u(z)} < 1 + \varepsilon\}$ is an open set which contains K. The latter, combined with the fact that the sequence $\{\varphi_n\}$ is locally uniformly bounded from above in $\mathbb{C}^N$, allows us to apply Hartogs Lemma. Thus we get for every $\varepsilon > 0$ the estimate

$$\varphi_n(z) \leq 1 + 2\varepsilon, \quad z \in K,$$

if n is sufficiently large. Due to the definition of $\mathcal{L}_0$ we have

$$\varphi_n(z) \leq (1 + 2\varepsilon)\Phi(z, K), \quad z \in \mathbb{C}^N,$$

for n sufficiently large. Letting $\varepsilon \to 0$ and consequently $n \to \infty$ gives the desired result

$$e^{u(z)} \leq \Phi(z, K), \quad z \in \mathbb{C}^N.$$

■

4.3 An explicit representation of Φ for compact, convex and symmetric sets in $\mathbb{R}^N$

Theorem 4.18 enables us to present Lundin's formula for the extremal function Φ for compact, convex and symmetric (with respect to 0) subsets S of $\mathbb{R}^N$ whose interior Int$S$ is not empty. These sets have the nice property that they can be described by a continuous function with range in $[-1, 1]$. More precisely:
If $S \subset \mathbb{R}^N$ is a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ and Int$S \neq \emptyset$ in $\mathbb{R}^N$, then S can be described by

$$S = \big\{x \in \mathbb{R}^N : a(y)\langle x, y\rangle \in [-1, 1] \quad \text{for every } y \in \partial B_N\big\},$$

where $a(y) := 1/\max_{x\in S}\langle x, y\rangle$ is a continuous function defined on ∂B_N and $\langle\cdot,\cdot\rangle$ means the standard scalar product in $\mathbb{R}^N$ and $\mathbb{C}^N$ respectively.

The key role for the explicit formula for Φ plays the following lemma, cf. [Bar88].

Lemma 4.19
Let K be a compact set in $\mathbb{C}^N$. Assume there exist a domain $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, and a function $f : \overline{G} \to \mathbb{C}^N$, which is holomorphic in G, continuous in $\overline{G}$ and $f(\partial G) \subset K$. Furthermore, let w be a continuous function $w : f(\overline{G}) \to [1, \infty)$ satisfying the conditions:

(i) $w(z) = 1$ *for* $z \in f(\partial G)$;

(ii) $\log(w \circ f)$ *is a harmonic function in* G.

Then

$$\Phi(z, K) \leq w(z) \quad \text{for} \quad z \in f(G).$$

Proof:
We choose an arbitrary function $u \in \mathcal{L}(K)$ and define for $z \in \overline{G}$ the function

$$v(z) := u(f(z)) - \log w(f(z)).$$

Then we obtain for any $z_0 \in \partial G$

$$\limsup_{\substack{z \to z_0 \\ z \in G}} v(z) = \limsup_{\substack{z \to z_0 \\ z \in G}} u(f(z)) \leq \limsup_{\substack{z \to f(z_0) \\ z \in \mathbb{C}}} u(z) \leq u(f(z_0)) \leq 0,$$

where the last two inequalities follow from the fact that u is upper semicontinuous in $\mathbb{C}^N$ and $u|_K \leq 0$. Since v is a subharmonic function in G we have

$$v(z) \leq 0 \quad \text{for} \;\; z \in G$$

by the maximum principle. This implies

$$u(z) \leq \log w(z) \quad \text{for} \quad z \in f(G),$$

and from Theorem 4.18 we achieve

$$\Phi(z, K) \leq w(z) \quad \text{for} \quad z \in f(G).$$

■

Now we have reached the point to verify Lundin's formula. The proof we give differs in some parts from the original one, cf. [Bar88].

Theorem 4.20
Let S be a compact, convex and symmetric (with respect to 0) subset of $\mathbb{R}^N$ with $\mathrm{Int}S \neq \emptyset$ in $\mathbb{R}^N$. Then

$$\Phi(z, S) = \max_{y \in \partial B_N} \big| h\big(a(y)\langle\, z, y\,\rangle\big)\big| \quad \textit{for } z \in \mathbb{C}^N,$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\mathbb{D}$ is defined by $h(\eta) = \eta + \sqrt{\eta^2 - 1}$ and $a(y) := 1/\max_{x \in S} \langle x, y\,\rangle$ for $y \in \partial B_N$.

Proof:
We will verify for $z \in \mathbb{C}^N$ the inequalities

$$\Phi(z, S) \geq \max_{y \in \partial B_N} \big| h\big(a(y)\langle\, z, y\,\rangle\big)\big| \quad \text{and} \quad \Phi(z, S) \leq \max_{y \in \partial B_N} \big| h\big(a(y)\langle\, z, y\,\rangle\big)\big|.$$

"$\geq$": For an arbitrary point $y \in \partial B_N$ let us define the function $u : \mathbb{C}^N \to \mathbb{R}$ by

$$u(z) = \log |h(a(y)\langle\, z, y\,\rangle)|.$$

Then u belongs to $\mathcal{L}$. Moreover, we have $u|_S \equiv 0$, which means that $u \in \mathcal{L}(S)$. Hence we get

$$\Phi(z, S) \geq \max_{y \in \partial B_N} |h(a(y)\langle\, z, y\,\rangle)|$$

since $y \in \partial B_N$ was arbitrary.
"$\leq$": In order to prove this inequality we construct two functions f and w which satisfy the hypotheses of Lemma 4.19 for $G = \mathbb{C}\backslash\overline{\mathbb{D}}$ and $K = S$.
Let us fix a point $z_0 \in \mathbb{C}^N\backslash S$. Further, as h is a continuous function in $\mathbb{C}$ we may choose $y_0 \in \partial B_N$ such that

$$|h(a(y_0)\langle z_0, y_0\rangle)| = \max_{y\in\partial B_N} |h(a(y)\langle z_0, y\rangle)| > 1.$$

Next, we set $\eta_0 := h(a(y_0)\langle z_0, y_0\rangle)$ and define the function

$$f : \overline{G} \to \mathbb{C}^N, \qquad \eta \mapsto \frac{1}{2}\left(c\eta + \overline{c}\frac{1}{\eta}\right),$$

where $c \in \mathbb{C}^N$ is uniquely determined by the condition

$$\frac{1}{2}\left(c\eta_0 + \overline{c}\frac{1}{\eta_0}\right) = z_0. \tag{4.9}$$

We now compute c. Multiplying equation (4.9) with $\overline{\eta}_0$ and $1/\overline{\eta}_0$ respectively gives

$$\frac{1}{2}\left(c|\eta_0|^2 + \overline{c}\frac{\overline{\eta}_0}{\eta_0}\right) = z_0\overline{\eta}_0 \quad \text{and} \quad \frac{1}{2}\left(c\frac{\eta_0}{\overline{\eta}_0} + \overline{c}\frac{1}{|\eta_0|^2}\right) = z_0\frac{1}{\overline{\eta}_0}.$$

Thus we obtain

$$\frac{1}{2}c\left(|\eta_0|^2 - \frac{1}{|\eta_0|^2}\right) = z_0\overline{\eta_0} - \overline{z_0}\frac{1}{\eta_0},$$

and in consequence

$$c = \frac{z_0\overline{\eta_0} - \overline{z_0}\frac{1}{\eta_0}}{\frac{1}{2}(r^2 - \frac{1}{r^2})}, \quad \text{where} \quad r := |\eta_0|. \tag{4.10}$$

As $f(\eta_0) = z_0$ by definition of f, we clearly have $z_0 \in f(G)$.
Note, the Joukowski function

$$g(\eta) := \frac{1}{2}\Big(\eta + \frac{1}{\eta}\Big) \quad \text{for} \quad \eta \in \mathbb{C}\backslash\mathbb{D}$$

satisfies the relation

$$\overline{\eta}g(\eta) - \frac{1}{\eta}\overline{g(\eta)} = \frac{1}{2}\Big(|\eta|^2 - \frac{1}{|\eta|^2}\Big). \tag{4.11}$$

In addition, as $1 \leq |h(a(y)\langle z_0, y\rangle)| \leq r$ for $y \in \partial B_N$, we acquire

$$a(y)\langle z_0, y\rangle \in g(\overline{A}_{1,r}) \qquad \text{for} \quad y \in \partial B_N,$$

where $\overline{A}_{1,r} := \{z \in \mathbb{C} : 1 \leq |z| \leq r\}$. The latter gives

$$\frac{|ra(y)\langle z_0, y\rangle - \frac{1}{r}a(y)\overline{\langle z_0, y\rangle}|}{\frac{1}{2}(r^2 - \frac{1}{r^2})} = \left|\frac{\operatorname{Re}(a(y)\langle z_0, y\rangle)}{\frac{1}{2}(r + \frac{1}{r})} + i\frac{\operatorname{Im}(a(y)\langle z_0, y\rangle)}{\frac{1}{2}(r - \frac{1}{r})}\right| \leq 1$$

for $y \in \partial B_N$. Thus we obtain

$$|a(y)\langle c, y\rangle| = \frac{|ra(y)\langle z_0, y\rangle - \frac{1}{r}a(y)\overline{\langle z_0, y\rangle}|}{\frac{1}{2}(r^2 - \frac{1}{r^2})} \leq 1 \quad \text{for} \quad y \in \partial B_N. \tag{4.12}$$

Because of the relation

$$a(y)\langle f(e^{it}), y\rangle = a(y)\frac{1}{2}\langle ce^{it} + \overline{c}e^{-it}, y\rangle$$
$$= a(y)\frac{1}{2}\big(\langle ce^{it}, y\rangle + \overline{\langle ce^{it}, y\rangle}\big) = a(y)\,\mathrm{Re}\,\langle ce^{it}, y\rangle$$

for $y \in \partial B_N$ and $t \in [0, 2\pi]$, we conclude

$$a(y)\langle f(e^{it}), y\rangle = a(y)\,\mathrm{Re}\,\langle e^{it}c, y\rangle \in [-1, 1] \tag{4.13}$$

for $y \in \partial B_N$ and $t \in [0, 2\pi]$. The last equation implies

$$f(\partial G) \subset S.$$

Now, since f is holomorphic in G, continuous in $\overline{G}$ and $f(\partial G) \subset S$, we see f fulfills the hypotheses of Lemma 4.19.
Next, we will show that the function $w : f(\overline{G}) \to [1, \infty)$ defined by

$$w(z) := |h(a(y_0)\langle z, y_0\rangle)|$$

satisfies the conditions of Lemma 4.19. Since

$$a(y_0)\langle z_0, y_0\rangle = g(h(a(y_0)\langle z_0, y_0\rangle)) = g(\eta_0),$$

we derive from equations (4.10) and (4.11)

$$a(y_0)\langle c, y_0\rangle = \frac{\overline{\eta}_0 a(y_0)\langle z_0, y_0\rangle - \frac{1}{\eta_0}a(y_0)\overline{\langle z_0, y_0\rangle}}{\frac{1}{2}\left(r^2 - \frac{1}{r^2}\right)} = \frac{\overline{\eta}_0 g(\eta_0) - \frac{1}{\eta_0}\overline{g(\eta_0)}}{\frac{1}{2}\left(r^2 - \frac{1}{r^2}\right)} = 1.$$

This yields

$$a(y_0)\langle f(\eta), y_0\rangle = g(\eta)$$

and in turn

$$w(f(\eta)) = |h(a(y_0)\langle f(\eta), y_0\rangle)| = |\eta|, \quad \eta \in \overline{G}.$$

As the function w meets the conditions (i) and (ii) of Lemma 4.19, all hypotheses of Lemma 4.19 are fulfilled and we obtain

$$\Phi(z_0, S) \leq w(z_0).$$

Since $z_0 \in \mathbb{C}^N \backslash S$ was an arbitrary point we deduce

$$\Phi(z, S) \leq w(z) \quad \text{for} \quad z \in \mathbb{C}^N \backslash S.$$

Now, bearing in mind that $\Phi(z, S) = 1$ for $z \in S$, we conclude

$$\Phi(z, S) = |h(a(y)\langle z, y\rangle)| \quad \text{for} \quad z \in S,$$

and therefore

$$\Phi(z, S) \leq \max_{y \in \partial B_N} |h(a(y)\langle z, y\rangle)| \quad \text{for} \quad z \in \mathbb{C}^N.$$

■

In particular, if S is the closed unit ball in $\mathbb{R}^N$ the formula for Φ can be refined. This is the content of our next corollary. We follow largely Baran's proof [Bar88].

Corollary 4.21
Let $h : \mathbb{C} \to \mathbb{C}\backslash\mathbb{D}$ be defined by $h(\eta) = \eta + \sqrt{\eta^2 - 1}$. Then

$$\Phi(z, \overline{B}_N) = \sqrt{h(\|z\|^2 + |\langle z, \overline{z}\rangle - 1|)} \quad \text{for} \quad z \in \mathbb{C}^N.$$

Proof:
Note, in this special case we have $a(y) = 1$ for $y \in \partial B_N$. Now, let us fix a point $z_0 \in \mathbb{C}^N \backslash \overline{B}_N$. We define

$$\eta_0 := h(\langle z_0, y_0 \rangle) \quad \text{and} \quad r := |h(\langle z_0, y_0 \rangle)|,$$

where $y_0 \in \partial B_N$ is an arbitrary point satisfying

$$|h(\langle z_0, y_0 \rangle)| = \max_{y \in \partial B_N} |h(\langle z_0, y \rangle)|.$$

As in proof of Theorem 4.20, let

$$c = c_0 + ic_1, \quad c_0, c_1 \in \mathbb{R}^N,$$

be given by

$$\frac{1}{2}\left(c\eta_0 + \overline{c}\frac{1}{\eta_0}\right) = z_0.$$

Then z_0 may be written as

$$z_0 = \frac{1}{2}\left(\eta_0 + \frac{1}{\eta_0}\right)c_0 + i\frac{1}{2}\left(\eta_0 - \frac{1}{\eta_0}\right)c_1.$$

In the proof of Theorem 4.20 we saw that

$$\langle c, y_0 \rangle = 1 \quad \text{and} \quad |\langle c, y \rangle| \leq 1 \quad \text{for} \quad y \in \partial B_N.$$

Thus, as y_0 is real, we obtain

$$\langle c_0, y_0 \rangle = 1 \quad \text{and} \quad \langle c_1, y_0 \rangle = 0.$$

Now we define

$$y_1 := \frac{c_0}{\|c_0\|} \quad \text{and} \quad y_2 := \frac{c_1}{\|c_1\|}.$$

Then $y_1, y_2 \in \partial B_N$. Moreover, since $|\langle c, y \rangle| \leq 1$ for $y \in \partial B_N$, we infer

$$|\langle c_0, y_1 \rangle| = \|c_0\| \leq 1 \quad \text{and} \quad |\langle c_1, y_1 \rangle| = \|c_1\| \leq 1.$$

By the Cauchy-Schwarz inequality we estimate

$$1 = \langle c_0, y_0 \rangle \leq \|c_0\| \|y_0\| \leq 1,$$

which means

$$c_0 = y_0 \quad \text{and} \quad \langle c_0, c_1 \rangle = 0.$$

Further, the terms $\|z_0\|^2$ and $\langle z_0, \overline{z_0} \rangle - 1$ can be expressed as

$$\begin{aligned}
\|z_0\|^2 = \langle z_0, z_0 \rangle &= \frac{1}{2}\left(\eta_0 + \frac{1}{\eta_0}\right)\langle c_0, z_0 \rangle + \frac{1}{2}\left(\eta_0 - \frac{1}{\eta_0}\right)i\langle c_1, z_0 \rangle \\
&= \left|\frac{1}{2}\left(\eta_0 + \frac{1}{\eta_0}\right)\right|^2 \|c_0\|^2 + \left|\frac{1}{2}\left(\eta_0 - \frac{1}{\eta_0}\right)\right|^2 \|c_1\|^2 \\
&= \left|\frac{1}{2}\left(\eta_0 + \frac{1}{\eta_0}\right)\right|^2 + \left|\frac{1}{2}\left(\eta_0 - \frac{1}{\eta_0}\right)\right|^2 \|c_1\|^2
\end{aligned}$$

and

$$\langle z_0, \overline{z}_0 \rangle - 1 = \left(\frac{1}{2}\Big(\eta_0 + \frac{1}{\eta_0}\Big)\right)^2 \|c_0\|^2 + \left(i\frac{1}{2}\Big(\eta_0 - \frac{1}{\eta_0}\Big)\right)^2 \|c_1\|^2 - 1$$
$$= \left(\frac{1}{2}\Big(\eta_0 - \frac{1}{\eta_0}\Big)\right)^2 (1 - \|c_1\|^2).$$

Thus we get

$$\|z_0\|^2 + |\langle z_0, \overline{z_0} \rangle - 1| = \frac{1}{2}\Big(r^2 + \frac{1}{r^2}\Big).$$

Therefore we deduce

$$h(\|z_0\|^2 + |\langle z_0, \overline{z_0} \rangle - 1|) = r^2$$

and this yields

$$\Phi(z_0, \overline{B}_N) = \sqrt{h(\|z_0\|^2 + |\langle z_0, \overline{z_0} \rangle - 1|)}.$$

As

$$\|z\|^2 + |\langle z, \overline{z} \rangle - 1| = 1 \qquad \text{for} \quad z \in \overline{B}_N$$

and $z_0 \in \mathbb{C}^N \backslash \overline{B}_N$ was an arbitrary point, we obtain

$$\Phi(z, \overline{B}_N) = \sqrt{h(\|z\|^2 + |\langle z, \overline{z} \rangle - 1|)} \quad \text{for all} \quad z \in \mathbb{C}^N,$$

as required. ■

Chapter 5

Real analytic functions of squared modulus holomorphic type in $\mathbb{R}^{2N}$

5.1 Computation of ρ

As we have already seen Theorem 3.22 and Theorem 4.20 are efficient tools to solve maximal convergence problems in higher dimensions. In this section we like to illustrate on Braess's problem and Example 2.14 how we can apply these theorems to calculate the maximal convergence number ρ for a given continuous function on the closed unit ball $\overline{B}_{2N}$ in $\mathbb{R}^{2N}$. For that purpose we first combine Theorem 3.22 and Corollary 4.21 to the following

Lemma 5.1
Let $F : \overline{B}_{2N} \to \mathbb{C}$ be a continuous function and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} = \frac{1}{\rho}$$

if and only if F has a holomorphic extension $\tilde{F}$ to

$$L_{2N,\rho} := \left\{ z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \|z\|^2 + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \right\}$$

but to no larger domain containing $\overline{L}_{2N,\rho}$.

Thus in view of Lemma 5.1 the number ρ is the largest root of the equation

$$\gamma := \inf_{z\in P} \left\{ \|z\|^2 + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| \right\} = \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big), \qquad z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N}, \tag{5.1}$$

where P denotes the set of all non–removable singularities[1] of $\tilde{F}$ in $\mathbb{C}^{2N}$.

Now we will see that the solution of Braess's problem and therefore Theorem 2.1 can also be obtained by giving an upper bound for γ.

[1]A point $\tilde{z} \in \mathbb{C}^N$ is called a non–removable singularity of $\tilde{F}$ if the function $\tilde{F}$ has no analytic continuation to a non–empty open neighborhood of $\tilde{z}$.

Remark 5.2
Recall, the function F in Theorem 1.6 was defined by

$$F(x,y) = \frac{1}{\left((x-x_0)^2+(y-y_0)^2\right)^s},$$

where $s \in (0,\infty)$ and $(x_0,y_0) \in \mathbb{R}^2$ with $\rho := \sqrt{x_0^2+y_0^2} > 1$.
Then

$$\tilde{F}(z_1,z_2) = \frac{1}{\left((z_1-x_0)^2+(z_2-y_0)^2\right)^s}$$

is the uniquely determined holomorphic extension of F to some appropriate chosen neighborhood of $\overline{B}_2$ in $\mathbb{C}^2$, see Remark 3.25 for the argument of uniqueness.
Our goal is now to find the smallest upper bound for γ in equation (5.1). Here, the clue is to consider the non–removable singularity

$$(z_1,z_2) = \left(\frac{1}{2}\left(x_0+iy_0+\frac{1}{x_0+iy_0}\right), \frac{1}{2i}\left(x_0+iy_0-\frac{1}{x_0+iy_0}\right)\right)$$

of $\tilde{F}$. Therefore we have

$$\inf_{(z_1,z_2)\in P}\left(|z_1|^2+|z_2|^2+|z_1^2+z_2^2-1|\right) \le \left|\frac{1}{2}\left(x_0+iy_0+\frac{1}{x_0+iy_0}\right)\right|^2+\left|\frac{1}{2i}\left(x_0+iy_0-\frac{1}{x_0+iy_0}\right)\right|^2,$$

where P is the set of all non–removable singularities of $\tilde{F}$. An easy computation shows

$$\left|\frac{1}{2}\left(x_0+iy_0+\frac{1}{x_0+iy_0}\right)\right|^2+\left|\frac{1}{2i}\left(x_0+iy_0-\frac{1}{x_0+iy_0}\right)\right|^2 = \frac{1}{2}\left(\rho^2+\frac{1}{\rho^2}\right)$$

as $\rho^2 = x_0^2+y_0^2$. Hence we obtain

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N},F)} \ge \frac{1}{\rho}$$

and consequently another proof for the opposite inequality of (1.2).

Example 5.3
Let us consider the function

$$F(x,y) = \frac{1}{a-xy}, \qquad a \in \mathbb{R}\backslash[-1,1], \quad (x,y) \in \overline{B}_2,$$

of Example 2.14.
Then $\tilde{F}(z_1,z_2) := 1/(a-z_1z_2)$ is the unique holomorphic extension of F to $\mathbb{C}^2\backslash P$, where $P := \{(z_1,z_2) \in \mathbb{C}^2 : z_1z_2 = a\}$. Therefore ρ is the largest number which satisfies

$$\frac{1}{2}\left(\rho^2+\frac{1}{\rho^2}\right) = \inf_{z_1\in\mathbb{C}\backslash\{0\}}\left(|z_1|^2+\left|\frac{a}{z_1}\right|^2+\left|z_1^2+\left(\frac{a}{z_1}\right)^2-1\right|\right).$$

To find ρ we evaluate the right hand side of this equation numerically, see Figure 5.1 for $a=5$. (From Example 2.14 we know that $1/2(\rho^2+1/\rho^2)=10$.)

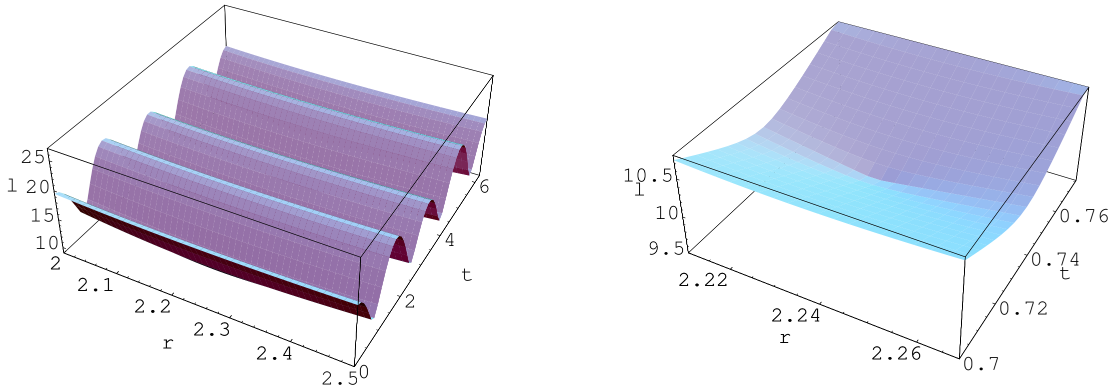

Figure 5.1 $l(z_1):=|z_1|^2+\left|\frac{5}{z_1}\right|^2+\left|z_1^2+\left(\frac{5}{z_1}\right)^2-1\right|$, $z_1=re^{it}$;

So the way we proposed to calculate ρ in Section 2.2 has the advantage that it can be done explicitly. Here, in contrast, we have the possibility to find an upper bound for ρ by evaluating $\|\hat{z}\|^2 + \left|\sum_{j=1}^{2N} \hat{z}_j^2 - 1\right|$ for a given non-removable singularity $\hat{z}$ of F. We can also calculate ρ numerically. However, we have to bear in mind that the numerical computation of γ gets more involved if N increases.

5.2 Maximal convergence on closed unit balls

As mentioned before we would like to show that Theorem 2.9 has a natural counterpart in several complex variables. This section is devoted to that problem. Let us start with stating the generalized version of this theorem.

Theorem 5.4
Let $g \in \mathcal{H}(\overline{\mathcal{B}}_N)$ and $F : \overline{B}_{2N} \to \mathbb{R}$ be defined by

$$F(x_1, y_1, x_2, \ldots, y_N) = |g(x_1 + iy_1, x_2 + iy_2, \ldots, x_N + iy_N)|^2.$$

Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

if and only if g has a holomorphic extension to $\mathcal{B}_{N,\rho}$.

Remember, the proof of this theorem in one complex variable is based on some special factorizations of holomorphic functions g like $g = \tilde{g}B$, where B is a Blaschke product. As inner functions of that form lack in several complex variables we have to choose a different approach in order to establish Theorem 5.4. Here, a useful tool is Lemma 5.1. We will see that

$$F \in \mathcal{H}(L_{2N,\rho}) \quad \text{if and only if} \quad g \in \mathcal{H}(\mathcal{B}_{N,\rho}).$$

In this context an additional change of variables is of crucial importance.
Now, let us set off on proving Theorem 5.4.

Lemma 5.5
Consider the map $h : \mathbb{C}^2\backslash\{(\xi,\eta) \in \mathbb{C}^2 : \xi = 0 \vee \eta = 0\} \to \mathbb{C}^2\backslash\{(z_1, z_2) \in \mathbb{C}^2 : z_1 = \pm i z_2\}$,

$$(\xi,\eta) \mapsto \left(\xi\frac{1}{2}\Big(\eta + \frac{1}{\eta}\Big), \xi\frac{1}{2i}\Big(\eta - \frac{1}{\eta}\Big)\right).$$

Then h *is surjective.*

Proof:
Observe, h maps $\mathbb{C}^2\backslash\{(\xi,\eta) \in \mathbb{C}^2 : \xi = 0 \vee \eta = 0\}$ in $\mathbb{C}^2\backslash\{(z_1, z_2) \in \mathbb{C}^2 : z_1 = \pm i z_2\}$.
Now let $(z_1, z_2) \in \mathbb{C}^2\backslash\{(z_1, z_2) \in \mathbb{C}^2 : z_1 = \pm i z_2\}$ be an arbitrary point. Then

$$(\xi,\eta) = \left(\sqrt{z_1^2 + z_2^2}, \frac{z_1 + i z_2}{\sqrt{z_1^2 + z_2^2}}\right)$$

is an element of $\mathbb{C}^2\backslash\{(\xi,\eta) \in \mathbb{C}^2 : \xi = 0 \vee \eta = 0\}$ and we calculate

$$\begin{aligned} h(\xi,\eta) &= \left(\xi\frac{1}{2}\Big(\frac{\eta^2+1}{\eta}\Big), \xi\frac{1}{2i}\Big(\frac{\eta^2-1}{\eta}\Big)\right) \\ &= \left(\xi\frac{1}{2}\frac{(z_1+iz_2)^2 + z_1^2 + z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}, \xi\frac{1}{2i}\frac{(z_1+iz_2)^2 - z_1^2 - z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}\right) \\ &= \left(\xi\frac{1}{2}\frac{2z_1^2 + 2iz_1z_2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}, \xi\frac{1}{2i}\frac{2iz_1z_2 - 2z_2^2}{(z_1+iz_2)\sqrt{z_1^2+z_2^2}}\right) \\ &= \left(z_1, -\frac{-z_1z_2 - iz_2^2}{z_1 + iz_2}\right) = (z_1, z_2). \end{aligned}$$

■

Important for our forthcoming work is the following slightly modified version of Lemma 5.5.

Remark 5.6
Any point $(z_1, z_2) \in \mathbb{C}^2\backslash\{(z_1, z_2) \in \mathbb{C}^2 : z_1 = \pm i z_2\}$ *can be expressed as*

$$(z_1, z_2) = \left(\xi\frac{1}{2}\Big(\eta + \frac{1}{\eta}\Big), \xi\frac{1}{2i}\Big(\eta - \frac{1}{\eta}\Big)\right),$$

if $\xi, \eta \in \mathbb{C}\backslash\{0\}$ *are chosen appropriately.*

The next two lemmata may be regarded as the nub for determining non–removable singularities of functions F of the type as in Lemma 5.1.

Lemma 5.7
Let $\rho_j \in (0,\infty)$, $j = 1, 2, \ldots, N$, *be arbitrary real numbers such that* $\rho := \sqrt{\sum_{j=1}^{N} \rho_j^2} > 1$.

Then the function $h : \mathbb{C}^N \to \mathbb{R}$ *defined by*

$$h(w) = \sum_{j=1}^{N} \frac{|w_j|^4}{2\rho_j^2} + \left|\sum_{j=1}^{N} w_j^2 - 1\right|, \qquad w = (w_1, w_2, \ldots, w_N) \in \mathbb{C}^N,$$

attains its minimum at the points $\hat{w} = (\pm\rho_1/\rho, \pm\rho_2/\rho, \ldots, \pm\rho_N/\rho)$. *In particular,*

$$h(w) > \frac{1}{2\rho^2} \qquad \text{for} \quad w \in \mathbb{C}^N \backslash \{\hat{w}\}.$$

Proof:
Note, we may consider h as a function of $2N$ real variables. For that reason we define the function $\hat{h} : \mathbb{R}^{2N} \to \mathbb{R}$ by $\hat{h}(x,y) = h(x+iy)$, where $x, y \in \mathbb{R}^N$. Then the function $\hat{h}$ can be written as

$$\begin{aligned}\hat{h}(x,y) &= \sum_{j=1}^{N} \frac{(x_j^2+y_j^2)^2}{2\rho_j^2} + \left| \sum_{j=1}^{N} (x_j + iy_j)^2 - 1 \right| \\ &= \sum_{j=1}^{N} \frac{(x_j^2+y_j^2)^2}{2\rho_j^2} + \sqrt{\left(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \right)^2 + \left(2 \sum_{j=1}^{N} x_j y_j \right)^2}.\end{aligned}$$

Observe, for $\hat{x} = (\pm\rho_1/\rho, \pm\rho_2/\rho, \ldots, \pm\rho_N/\rho)$ and $\hat{y} = (0, 0, \ldots, 0)$ the function $\hat{h}$ takes the value

$$\hat{h}(\hat{x}, \hat{y}) = \sum_{j=1}^{N} \frac{\rho_j^2}{2\rho^4} + \left| \sum_{j=1}^{N} \frac{\rho_j^2}{\rho^2} - 1 \right| = \frac{1}{2\rho^2}.$$

Since our intention is to show that $\hat{h}$ assumes its minimum at the points $(\hat{x}, \hat{y})$, we compute the first partial derivatives of $\hat{h}$ which necessarily vanish at critical points:

$$\frac{\partial \hat{h}(x,y)}{\partial x_j} = \frac{2(x_j^2+y_j^2)2x_j}{2\rho_j^2} + \frac{2\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)2x_j + 2\Big(2\sum_{j=1}^{N} x_j y_j\Big)2y_j}{2\sqrt{\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)^2 + \Big(2\sum_{j=1}^{N} x_j y_j\Big)^2}} = 0 \tag{5.2}$$

$$\frac{\partial \hat{h}(x,y)}{\partial y_j} = \frac{2(x_j^2+y_j^2)2y_j}{2\rho_j^2} + \frac{2\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)(-2y_j) + 2\Big(2\sum_{j=1}^{N} x_j y_j\Big)2x_j}{2\sqrt{\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)^2 + \Big(2\sum_{j=1}^{N} x_j y_j\Big)^2}} = 0 \tag{5.3}$$

for $j = 1, 2, \ldots, N$. Now $(5.2)y_j - (5.3)x_j$ gives

$$\frac{8\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)x_j y_j + 4\Big(\sum_{j=1}^{N} 2x_j y_j\Big)(y_j^2 - x_j^2)}{2\sqrt{\Big(\sum_{j=1}^{N}(x_j^2-y_j^2)-1\Big)^2 + \Big(2\sum_{j=1}^{N} x_j y_j\Big)^2}} = 0$$

for $j = 1, 2, \ldots, N$.
For simplification we set $A := \sum_{j=1}^{N}(x_j^2 - y_j^2) - 1$ and $B := \sum_{j=1}^{N} 2x_j y_j$. Hence the last equation assumes the form

$$\frac{8Ax_j y_j + 4B(y_j^2 - x_j^2)}{2\sqrt{A^2+B^2}} = 0$$

for $j = 1, 2, \ldots, N$.
Note, the assumption $A \neq 0$ implies $B = 0$ and $A = 0$ entails $B = 0$ since

$$2x_j y_j A = B(x_j^2 - y_j^2), \quad j = 1, 2, \ldots, N, \tag{5.4}$$

and therefore

$$AB = A\left(\sum_{j=1}^{N} 2x_j y_j\right) = B\left(\sum_{j=1}^{N} x_j^2 - y_j^2\right) = B(A+1).$$

Thus, we only have to distinguish the two cases:
(i) $A \neq 0$ and $B = 0$
(ii) $A = 0$ and $B = 0$

Case (i): By equation (5.4) we obtain

$$x_j = 0 \quad \text{or} \quad y_j = 0 \qquad \text{for } j = 1, 2, \ldots, N.$$

If $x_j = y_j = 0$ for $j = 1, 2, \ldots, N$, we have $\hat{h}(0,0) = 1$. Hence $\hat{h}$ can't have a absolute minimum at $(0,0)$. Therefore we may assume that there exists at least one x_k or y_k, $k \in \{1, 2, \ldots, N\}$, which is not zero. Thus equations (5.2) and (5.3) simplify to

$$\frac{x_k^3}{\rho_k^2} + \frac{x_k A}{|A|} = 0 \tag{5.5}$$

if $x_k \neq 0$, and

$$\frac{y_k^3}{\rho_j^2} - \frac{y_k A}{|A|} = 0 \tag{5.6}$$

if $y_k \neq 0$. Equation (5.5) implies $A < 0$ and (5.6) shows $A > 0$. Thus either $x_j = 0$ for all $j = 1, 2, \ldots, N$, or $y_j = 0$ for all $j = 1, 2, \ldots, N$. Since $\hat{h}(0, y) \geq 1$ for $y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$, we only have to study the case $y_j = 0$ for all $j = 1, 2, \ldots, N$. Because of (5.5) we get

$$x_j = 0 \quad \text{or} \qquad \text{or} \quad x_j = \pm\rho_j, \qquad j = 1, 2, \ldots, N.$$

Without loss of generality we may assume

$$x_j = \pm\rho_j \qquad \text{for} \quad j = 1, 2, \ldots, m, \quad m \leq N,$$

and

$$x_j = 0 \qquad \text{for} \quad j = m+1, m+2, \ldots, N.$$

Then we obtain for such a point the estimate

$$\hat{h}(\pm\rho_1, \ldots, \pm\rho_m, 0, \ldots, 0) = \frac{1}{2}\sum_{j=1}^{m} \rho_j^2 + \left|\sum_{j=1}^{m} \rho_j^2 - 1\right| \geq \frac{1}{2} > \frac{1}{2\rho^2}$$

as

$$\frac{1}{2}\sum_{j=1}^{m} \rho_j^2 + \left|\sum_{j=1}^{m} \rho_j^2 - 1\right| = \begin{cases} \frac{1}{2}\sum_{j=1}^{m} \rho_j^2 + \sum_{j=1}^{m} \rho_j^2 - 1 & \text{for } \sum_{j=1}^{m} \rho_j^2 \geq 1 \\ \frac{1}{2}\sum_{j=1}^{m} \rho_j^2 + 1 - \sum_{j=1}^{m} \rho_j^2 & \text{for } \sum_{j=1}^{m} \rho_j^2 < 1 \end{cases}$$

$$\geq \begin{cases} \frac{1}{2} & \text{for } \sum_{j=1}^{m} \rho_j^2 \geq 1 \\ 1 - \frac{1}{2}\sum_{j=1}^{m} \rho_j^2 > \frac{1}{2} & \text{for } \sum_{j=1}^{m} \rho_j^2 < 1. \end{cases}$$

Consequently, $\hat{h}$ has a chance to take on a absolute minimum only if $A = 0$. This fact leads us indispensably to the second case:
Case (ii): Here our minimum problem consists in the following extrema problem with side conditions:
Find the minimum of

$$\hat{h}(x,y) = \sum_{j=1}^{N} \frac{(x_j^2 + y_j^2)^2}{2\rho_j^2}$$

under the conditions

$$g_1(x,y) = \sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 = 0 \quad \text{and} \quad g_2(x,y) = \sum_{j=1}^{N} x_j y_j = 0.$$

We shall solve this problem by the Lagrange multiplication formalism as the hypotheses for this machinery are fulfilled.
Consequently, we have to determine the minimum of the function

$$\tilde{h}(x,y,\lambda_1,\lambda_2) = \sum_{j=1}^{N} \frac{(x_j^2 + y_j^2)^2}{2\rho_j^2} + \lambda_1 \left(\sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 \right) + \lambda_2 \sum_{j=1}^{N} x_j y_j$$

for $x, y \in \mathbb{R}^N$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
Thus the following conditions must be fulfilled:

$$\frac{\partial \tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial x_j} = \frac{2x_j(x_j^2 + y_j^2)}{\rho_j^2} + \lambda_1 2x_j + \lambda_2 y_j = 0, \quad j = 1, 2, \ldots, N, \tag{5.7}$$

$$\frac{\partial \tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial y_j} = \frac{2y_j(x_j^2 + y_j^2)}{\rho_j^2} + \lambda_1 2(-y_j) + \lambda_2 x_j = 0, \quad j = 1, 2, \ldots, N, \tag{5.8}$$

$$\frac{\partial \tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial \lambda_1} = \sum_{j=1}^{N} (x_j^2 - y_j^2) - 1 = 0, \tag{5.9}$$

$$\frac{\partial \tilde{h}(x,y,\lambda_1,\lambda_2)}{\partial \lambda_2} = \sum_{j=1}^{N} x_j y_j = 0. \tag{5.10}$$

In order to determine λ_1 and λ_2, we consider the equations

$$(5.7)x_j - (5.8)y_j = \frac{2(x_j^2 - y_j^2)(x_j^2 + y_j^2)}{\rho_j^2} + \lambda_1 2(x_j^2 + y_j^2) = 0, \qquad j = 1, 2, \ldots, N, \tag{5.11}$$

and

$$(5.7)y_j + (5.8)x_j = \frac{4x_j y_j (x_j^2 + y_j^2)}{\rho_j^2} + \lambda_2 (x_j^2 + y_j^2) = 0, \qquad j = 1, 2, \ldots, N. \tag{5.12}$$

Now, equations (5.11) and (5.12) imply for $j = 1, 2, \ldots, N$,

$$(x_j^2 - y_j^2) = -\lambda_1 \rho_j^2 \quad \text{or} \quad x_j = y_j = 0 \tag{5.13}$$

and

$$4x_j y_j = -\lambda_2 \rho_j^2 \quad \text{or} \quad x_j = y_j = 0. \tag{5.14}$$

Without loss of generality we may assume $x_j \neq 0$ or $y_j \neq 0$ for $j = 1, 2, \ldots, m$, $m \leq N$, and $x_j = y_j = 0$ for $j = m+1, \ldots, N$.
As the case $x_j = y_j = 0$ for $j = 1, 2, \ldots, N$, does not meet the side conditions we can exclude it. Therefore we obtain by (5.13)

$$\sum_{j=1}^{N}(x_j^2 - y_j^2) - 1 = \sum_{j=1}^{m}(-\lambda_1 \rho_j^2) - 1 = 0$$

and by (5.14)

$$\sum_{j=1}^{N} x_j y_j = -\frac{1}{4}\lambda_2 \sum_{j=1}^{m} \rho_j^2 = 0.$$

Thus it follows

$$\lambda_1 = -\frac{1}{\sum\limits_{j=1}^{m} \rho_j^2} \quad \text{and} \quad \lambda_2 = 0.$$

Further, we receive from equations (5.7) and (5.8)

$$x_j = \pm\sqrt{-\lambda_1}\rho_j \quad \text{and} \quad y_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, m.$$

Inserting x_j and y_j, $j = 1, 2, \ldots, N$, into $\hat{h}$ gives

$$\hat{h}(x, y) = \frac{1}{2\Big(\sum\limits_{j=1}^{m} \rho_j^2\Big)^2} \sum_{j=1}^{m} \rho_j^2 = \frac{1}{2\sum\limits_{j=1}^{m} \rho_j^2}.$$

Since

$$\sum_{j=1}^{m} \rho_j^2 < \sum_{j=1}^{N} \rho_j^2$$

we conclude that $\hat{h}$ assumes its minimum if

$$x_j = \pm\frac{\rho_j}{\sqrt{\sum\limits_{j=1}^{N} \rho_j^2}} \quad \text{and} \quad y_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, N,$$

and we are done. ■

Lemma 5.8
Let $\rho > 1$ be arbitrary.
(i) Consider the sets

$$L_{2N,\rho} := \left\{ z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \sum_{j=1}^{2N} |z_j|^2 + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \right\}$$

and

$$T_{2N,\rho} := \left\{ z=(z_1,...,z_{2N}) \in \mathbb{C}^{2N}: \left(\sum_{j=1}^{N} |z_{2j-1}+ iz_{2j}|^2 \right)^{\frac{1}{2}} < \rho \ \wedge \ \left(\sum_{j=1}^{N} |z_{2j-1}- iz_{2j}|^2 \right)^{\frac{1}{2}} < \rho \right\}.$$

Then

$$L_{2N,\rho} \subset T_{2N,\rho}.$$

(ii) Analogously, the sets

$$\mathcal{L}_{2N,\rho} := \left\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} \left(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1| \right) < \frac{1}{2}\left(\rho^2 + \frac{1}{\rho^2}\right) \right\}$$

and

$$\mathcal{T}_{2N,\rho} := \left\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} |z_{2j-1} + iz_{2j}| < \rho \ \wedge \max_{1\le j\le N} |z_{2j-1}- iz_{2j}| < \rho \right\}$$

satisfy the inclusion

$$\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}.$$

Proof:
To (i): To prove this inclusion let an arbitrary point $z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N}\backslash T_{2N,\rho}$ be given. Without loss of generality we may assume that

$$z_{2j-1} = \pm\, iz_{2j} \qquad \text{for} \quad j = 1, \ldots, m, \quad m \in \mathbb{Z}_+,$$

and

$$z_{2j-1} \neq \pm\, iz_{2j} \qquad \text{for} \quad j = m+1, \ldots, N.$$

Now, for $j = m + 1, \ldots, N$ we choose the representation of Remark 5.6

$$z_{2j-1} = \xi_j \frac{1}{2}\left(\eta_j + \frac{1}{\eta_j}\right) \qquad \text{and} \qquad z_{2j} = \xi_j \frac{1}{2i}\left(\eta_j - \frac{1}{\eta_j}\right), \qquad \xi_j, \eta_j \in \mathbb{C}\backslash\{0\}.$$

In addition, we set

$$\tilde{\rho}_j := |\xi_j \eta_j| \qquad \text{as well as} \qquad \hat{\rho}_j := \left|\xi_j \frac{1}{\eta_j}\right|, \qquad j = m+1, \ldots, N.$$

Thus we obtain

$$\begin{aligned}\sum_{j=1}^{2N} |z_j|^2 + \left|\sum_{j=1}^{2N} z_j^2 - 1\right| &= \sum_{j=1}^{m} 2|z_{2j}|^2 + \sum_{j=m+1}^{N} \left(\left|\frac{\xi_j}{2}\left(\eta_j + \frac{1}{\eta_j}\right)\right|^2 + \left|\frac{\xi_j}{2i}\left(\eta_j - \frac{1}{\eta_j}\right)\right|^2 \right) + \\ &\quad \left| \sum_{j=m+1}^{N} \left(\left(\frac{\xi_j}{2}\left(\eta_j + \frac{1}{\eta_j}\right)\right)^2 + \left(\frac{\xi_j}{2i}\left(\eta_j - \frac{1}{\eta_j}\right)\right)^2 \right) - 1 \right| \\ &= \sum_{j=1}^{m} 2|z_{2j}|^2 + \sum_{j=m+1}^{N} |\xi_j|^2 \frac{1}{2}\left(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\right) + \left| \sum_{j=m+1}^{N} \xi_j^2 - 1 \right|\end{aligned}$$

$$\underset{\left|\frac{1}{\eta_j}\right|=\frac{|\xi_j|}{\tilde\rho_j}}{=} \sum_{j=1}^{m} 2|z_{2j}|^2 + \frac{1}{2}\sum_{j=m+1}^{N} \tilde\rho_j^2 + \frac{1}{2}\sum_{j=m+1}^{N} \frac{|\xi_j|^4}{\tilde\rho_j^2} + \left|\sum_{j=m+1}^{N} \xi_j^2 - 1\right|$$
$$= \sum_{j=1}^{m} 2|z_{2j}|^2 + \frac{1}{2}\tilde\rho^2 + \frac{1}{2}\sum_{j=m+1}^{N} \frac{|\xi_j|^4}{\tilde\rho_j^2} + \left|\sum_{j=m+1}^{N} \xi_j^2 - 1\right|$$

and

$$\sum_{j=1}^{2N}|z_j|^2 + \left|\sum_{j=1}^{2N} z_j^2 - 1\right| = \sum_{j=1}^{m} 2|z_{2j}|^2 + \sum_{j=m+1}^{N} |\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \left|\sum_{j=m+1}^{N} \xi_j^2 - 1\right|$$
$$\underset{|\eta_j|=\frac{|\xi_j|}{\hat\rho_j}}{=} \sum_{j=1}^{m} 2|z_{2j}|^2 + \frac{1}{2}\sum_{j=m+1}^{N} \hat\rho_j^2 + \frac{1}{2}\sum_{j=m+1}^{N} \frac{|\xi_j|^4}{\hat\rho_j^2} + \left|\sum_{j=m+1}^{N} \xi_j^2 - 1\right|$$
$$= \sum_{j=1}^{m} 2|z_{2j}|^2 + \frac{1}{2}\hat\rho^2 + \frac{1}{2}\sum_{j=m+1}^{N} \frac{|\xi_j|^4}{\hat\rho_j^2} + \left|\sum_{j=m+1}^{N} \xi_j^2 - 1\right|,$$

where $\tilde\rho = \Big(\sum_{j=m+1}^{N} \tilde\rho_j^2\Big)^{\frac{1}{2}}$ and $\hat\rho = \Big(\sum_{j=m+1}^{N} \hat\rho_j^2\Big)^{\frac{1}{2}}$.

By the definition of $T_{2N,\rho}$ we have for the chosen element $z = (z_1, \dots, z_{2N})$ either the estimate

$$\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 = \sum_{j=1}^{m} |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2$$

or the estimate

$$\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 = \sum_{j=1}^{m} |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2.$$

Therefore we conclude

$$\sum_{j=1}^{m} 2|z_{2j}|^2 \geq \begin{cases} \max\{\frac{1}{2}(\rho^2 - \tilde\rho^2), 0\} & \text{if } \sum_{j=1}^{m} |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2. \\ \max\{\frac{1}{2}(\rho^2 - \hat\rho^2), 0\} & \text{if } \sum_{j=1}^{m} |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2. \end{cases}$$

Lemma 5.7 implies now

$$\sum_{j=1}^{2N}|z_j|^2 + \left|\sum_{j=1}^{2N} z_j^2 - 1\right|$$
$$\geq \begin{cases} \max\{\frac{1}{2}(\rho^2 - \tilde\rho^2), 0\} + \frac{1}{2}\big(\tilde\rho^2 + \frac{1}{\tilde\rho^2}\big) & \text{if } \sum_{j=1}^{m} |z_{2j-1} + iz_{2j}|^2 + \tilde\rho^2 \geq \rho^2. \\ \max\{\frac{1}{2}(\rho^2 - \hat\rho^2), 0\} + \frac{1}{2}\big(\hat\rho^2 + \frac{1}{\hat\rho^2}\big) & \text{if } \sum_{j=1}^{m} |z_{2j-1} - iz_{2j}|^2 + \hat\rho^2 \geq \rho^2. \end{cases}$$

This guarantees

$$\sum_{j=1}^{2N}|z_j|^2 + \left|\sum_{j=1}^{2N} z_j^2 - 1\right| \geq \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big).$$

As $z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} \backslash T_{2N,\rho}$ was arbitrary we derive

$$L_{2N,\rho} \subset T_{2N,\rho}$$

as required.

To (ii): Let $z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} \backslash \mathcal{T}_{2N,\rho}$ be an arbitrary point. We may assume that

$$z_{2j-1} = \pm\, i z_{2j} \qquad \text{for} \quad j = 1, \ldots, m, \quad m \in \mathbb{Z}_+,$$

and

$$z_{2j-1} \neq \pm\, i z_{2j} \qquad \text{for} \quad j = m+1, \ldots, N.$$

If $j = m+1, \ldots, N$, then we take the representation of Remark 5.6

$$z_{2j-1} = \xi_j \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \qquad \text{and} \qquad z_{2j} = \xi_j \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big), \qquad \xi_j, \eta_j \in \mathbb{C}\backslash\{0\}$$

and set

$$\tilde{r}_j := |\xi_j \eta_j| \qquad \text{as well as} \qquad \hat{r}_j := \left|\xi_j \frac{1}{\eta_j}\right|, \qquad j = m+1, \ldots, N.$$

Hence we obtain either the estimate

$$\max_{1\le j\le m} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) \ge \max_{1\le j\le m} \Big(\frac{1}{2}|z_{2j-1} \pm i z_{2j}|^2 + 1\Big) \ge$$
$$\ge \frac{1}{2}\rho^2 + 1 > \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \qquad \text{if} \max_{1\le j\le m}\{|z_{2j-1} + i z_{2j}|, |z_{2j-1} - i z_{2j}|\} \ge \rho$$

or the estimate

$$\max_{m+1\le j\le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) = \max_{m+1\le j\le N}\Big(|\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \Big|\xi_j^2 - 1\Big|\Big)$$
$$\ge \begin{cases} \max\limits_{m+1\le j\le N} \Big(\frac{1}{2}\tilde{r}_j^2 + \frac{|\xi_j|^4}{2\tilde{r}_j^2} + \Big|\xi_j^2 - 1\Big|\Big) \ge \frac{1}{2}\big(\rho^2 + \frac{1}{\rho^2}\big) & \text{if} \max\limits_{m+1\le j\le N} |z_{2j-1} + i z_{2j}| \ge \rho. \\ \max\limits_{m+1\le j\le N} \Big(\frac{1}{2}\hat{r}_j^2 + \frac{|\xi_j|^4}{2\hat{r}_j^2} + \Big|\xi_j^2 - 1\Big|\Big) \ge \frac{1}{2}\big(\rho^2 + \frac{1}{\rho^2}\big) & \text{if} \max\limits_{m+1\le j\le N} |z_{2j-1} - i z_{2j}| \ge \rho. \end{cases}$$

Putting all things together gives

$$\max_{1\le j\le N} \big(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\big) \ge \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big)$$

but this means that

$$\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}.$$

■

As a preparation for the proof of Theorem 5.4 let us state the next lemma.

Lemma 5.9
Let $\rho \in (1,\infty)$ and $\hat\rho_j \in (0,\infty)$, $j=1,2,\dots,N$, be arbitrary numbers such that $\hat\rho := \sqrt{\sum_{j=1}^N \hat\rho_j^2} \in (1,\rho)$. Furthermore, let $\varepsilon > 0$ be any real number satisfying

$$\varepsilon < \min\left\{\frac{\rho^2+\frac{1}{\rho^2}-\hat\rho^2-\frac{1}{\hat\rho^2}}{28N}, \min_{j=1,2,\dots,N}\frac{\hat\rho_j}{\hat\rho}\right\}.$$

Then

$$\frac{1}{2}\hat\rho^2 + \frac{1}{2}\sum_{j=1}^N \frac{|\xi_j|^4}{\hat\rho_j^2} + \left|\sum_{j=1}^N \xi_j^2 - 1\right| < \frac{1}{2}\Big(\rho^2+\frac{1}{\rho^2}\Big)$$

for $\xi = (\xi_1,\dots,\xi_N) \in U_{N,\varepsilon} := \Big\{z=(z_1,\dots,z_N)\in\mathbb{C}^N : \max_{1\le j\le N}\Big|z_j - \frac{\hat\rho_j}{\hat\rho}\Big| < \varepsilon\Big\}$.

Proof:
We first determine an upper bound for the expression

$$\left|\sum_{j=1}^N \xi_j^2 - 1\right|^2 = \left(\sum_{j=1}^N\big((\operatorname{Re}\xi_j)^2-(\operatorname{Im}\xi_j)^2\big)-1\right)^2 + 4\left(\sum_{j=1}^N \operatorname{Re}\xi_j\operatorname{Im}\xi_j\right)^2.$$

If $\xi=(\xi_1,\xi_2,\dots,\xi_N)\in U_{N,\varepsilon}$ we may estimate

$$\begin{aligned}\left|\sum_{j=1}^N \xi_j^2-1\right|^2 &\le \left(\left|\sum_{j=1}^N(\operatorname{Re}\xi_j)^2-1\right| + \sum_{j=1}^N(\operatorname{Im}\xi_j)^2\right)^2 + 4\left(\sum_{j=1}^N(\operatorname{Re}\xi_j)^2\right)\left(\sum_{j=1}^N(\operatorname{Im}\xi_j)^2\right)\\
&< \left(\max\left\{\sum_{j=1}^N\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^2-1, 1-\sum_{j=1}^N\Big(\frac{\hat\rho_j}{\hat\rho}-\varepsilon\Big)^2\right\}+N\varepsilon^2\right)^2 + 4\sum_{j=1}^N\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^2N\varepsilon^2\\
&< \left(\frac{2\varepsilon}{\hat\rho}\sum_{j=1}^N\hat\rho_j+2N\varepsilon^2\right)^2+16N^2\varepsilon^2 < (2\varepsilon N+2N\varepsilon^2)^2+16N^2\varepsilon^2 \le 32N^2\varepsilon^2.\end{aligned}$$

Thus for $\xi=(\xi_1,\xi_2,\dots,\xi_N)\in U_{N,\varepsilon}$ we obtain

$$\begin{aligned}\frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{|\xi_j|^4}{\hat\rho_j^2}+\left|\sum_{j=1}^N\xi_j^2-1\right| &< \frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{\Big(\frac{\hat\rho_j}{\hat\rho}+\varepsilon\Big)^4}{\hat\rho_j^2}+6N\varepsilon\\
&= \frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{(\hat\rho_j+\varepsilon\hat\rho)^4}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon = \frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{(\hat\rho_j^2+2\varepsilon\hat\rho\hat\rho_j+\varepsilon^2\hat\rho^2)^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon\\
&< \frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{(\hat\rho_j^2+3\varepsilon\hat\rho\hat\rho_j)^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon = \frac{1}{2}\hat\rho^2+\frac{1}{2}\sum_{j=1}^N\frac{\hat\rho_j^4+6\varepsilon\hat\rho\hat\rho_j^3+9\varepsilon^2\hat\rho^2\hat\rho_j^2}{\hat\rho^4\hat\rho_j^2}+6N\varepsilon\\
&< \frac{1}{2}\hat\rho^2+\frac{1}{2\hat\rho^2}+\frac{15}{2}\varepsilon\sum_{j=1}^N\frac{\hat\rho_j}{\hat\rho^3}+6N\varepsilon < \frac{1}{2}\hat\rho^2+\frac{1}{2\hat\rho^2}+14N\varepsilon\\
&< \frac{1}{2}\hat\rho^2+\frac{1}{2\hat\rho^2}+\frac{1}{2}\Big(\rho^2+\frac{1}{\rho^2}-\hat\rho^2-\frac{1}{\hat\rho^2}\Big) = \frac{1}{2}\Big(\rho^2+\frac{1}{\rho^2}\Big).\end{aligned}$$

■

Definition 5.10
Let $\Omega \subset \mathbb{C}^N$ be a Reinhardt domain, where a Reinhardt domain is characterized by the property that $(z_1, \ldots, z_N) \in \Omega$ implies $(e^{i\theta_1}z_1, \ldots, e^{i\theta_N}z_N) \in \Omega$ for all $\theta_j \in [0, 2\pi]$, $j = 1, \ldots, N$. Furthermore, let $0 \in \Omega$ and a function $f \in \mathcal{H}(\Omega)$ with its homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha z^\alpha, \qquad z \in \Omega, \quad \alpha \in \mathbb{Z}_+^N,$$

be given[2]*. Then we define the function*

$$f_-(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \overline{a_\alpha} z^\alpha, \qquad z \in \Omega.$$

Clearly, we have $f_- \in \mathcal{H}(\Omega)$.
After this lengthy preparation all basic tools are now available to prove our main result of this section.

Proof of Theorem 5.4:

Let us begin the proof with some notes which we need for the verification of both directions. We define for $\rho \in (1, \infty)$ the sets

$$S_{2N,\rho} = \Big\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : z_{2j-1} = \xi_j \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big), z_{2j} = \xi_j \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big),$$
$$\xi_j, \eta_j \in \mathbb{C}\backslash\{0\},\ j = 1, \ldots, N,\ \Big(\sum_{j=1}^{N} |\xi_j\eta_j|^2\Big)^{\frac{1}{2}} < \rho \ \wedge\ \Big(\sum_{j=1}^{N} \Big|\xi_j \frac{1}{\eta_j}\Big|^2\Big)^{\frac{1}{2}} < \rho \Big\}$$

and

$$T_{2N,\rho} = \Big\{ z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} \colon \Big(\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2\Big)^{\frac{1}{2}} < \rho \ \wedge\ \Big(\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2\Big)^{\frac{1}{2}} < \rho \Big\}.$$

Then $S_{2N,\rho} \subset T_{2N,\rho}$. To see this inclusion, let us choose for any element $z = (z_1, z_2, \ldots, z_{2N})$ of $S_{2N,\rho}$ the representation

$$(z_1, z_2, z_3, \ldots, z_{2N}) = \Big(\xi_1 \frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1 \frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2 \frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \xi_N \frac{1}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\Big).$$

We receive

$$\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 = \sum_{j=1}^{N} |\xi_j\eta_j|^2 < \rho^2$$

and

$$\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 = \sum_{j=1}^{N} \Big|\xi_j \frac{1}{\eta_j}\Big|^2 < \rho^2.$$

[2] The following result in several complex variables is well-known: Let $\Omega \subset \mathbb{C}^N$ be a Reinhardt domain with $0 \in \Omega$. If f is a holomorphic function in Ω, then f can be expanded into a series of homogeneous polynomials converging locally uniformly on Ω.

Consequently, $S_{2N,\rho} \subset T_{2N,\rho}$.
"$\Leftarrow$": By hypothesis we have $g \in \mathcal{H}(\mathcal{B}_{N,\rho})$. We now show that F has a holomorphic extension to $L_{2N,\rho}$, where $L_{2N,\rho}$ is defined as in Lemma 5.1, because then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

follows from Lemma 5.1. Therefore we define the function $f_1 : T_{2N,\rho} \to \mathbb{C}$ by

$$f_1(z_1, z_2, ..., z_{2N-1}, z_{2N}) = g(z_1 + iz_2, ..., z_{2N-1} + iz_{2N})g_{\text{-}}(z_1 - iz_2, ..., z_{2N-1} - iz_{2N}), \tag{5.15}$$

where g_- is specified in Definition 5.10. Since g is holomorphic in $\mathcal{B}_{N,\rho}$ we deduce from the definition of $T_{2N,\rho}$ that f_1 is holomorphic in $T_{2N,\rho}$. Moreover, f_1 is a holomorphic extension of F to $T_{2N,\rho}$ as $f_1 = F|_{\overline{B}_{2N}}$ and $\overline{B}_{2N} \subset T_{2N,\rho}$. Since $L_{2N,\rho} \subset T_{2N,\rho}$ by Lemma 5.14 we are done.
"$\Rightarrow$": This direction is proved by contradiction. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

but g has no holomorphic extension to $\mathcal{B}_{N,\rho}$.
Then there exists a number $\tilde{\rho} \in (1, \rho)$ such that $g \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}}) \setminus \mathcal{H}(\overline{\mathcal{B}}_{N,\tilde{\rho}})$. Hence we can find to an arbitrary $\varepsilon_0 > 0$ a non–removable singularity $\hat{z} = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N)$ of g with

$$\hat{\rho} := \|\hat{z}\| \in [\tilde{\rho}, \rho) \cap [\tilde{\rho}, \tilde{\rho} + \varepsilon_0). \tag{5.16}$$

Further, let us set $\hat{\rho}_j := |\hat{z}_j|$, $j = 1, 2, \ldots, N$.
Now, for more clarity we consider the cases (i) $\hat{z}_j \neq 0$ for $j = 1, 2, \ldots, N$ and (ii) $\hat{z}_k = 0$ for at least one $k \in \{1, \ldots, N\}$.
To (i): We define the function $f_1 : T_{2N,\tilde{\rho}} \to \mathbb{C}$ as in the "if"–direction. Then f_1 can be expressed by

$$\begin{aligned} f_1(z_1, z_2, z_3, \ldots, z_{2N}) &= f_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \frac{\xi_2}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \frac{\xi_N}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\right) \\ &= g\Big(\xi_1\eta_1, \ldots, \xi_N\eta_N\Big)\, g_{\text{-}}\Big(\xi_1\frac{1}{\eta_1}, \ldots, \xi_N\frac{1}{\eta_N}\Big), \end{aligned}$$

if $(z_1, z_2, \ldots, z_{2N}) \in S_{2N,\tilde{\rho}}$, $z_{2j-1} = \xi_j\frac{1}{2}\big(\eta_j + \frac{1}{\eta_j}\big), z_{2j} = \xi_j\frac{1}{2i}\big(\eta_j - \frac{1}{\eta_j}\big)$, $\xi_j, \eta_j \in \mathbb{C}\setminus\{0\}$, $j = 1, 2, \ldots, N$. From the "if"–direction we know that f_1 is holomorphic in $T_{2N,\tilde{\rho}}$ and $f_1 = F|_{\overline{B}_{2N}}$. In addition, we infer from Lemma 5.1 that F has a holomorphic extension $\tilde{F}$ to $L_{2N,\rho}$. Thus, in view of Remark 3.25 and by the identity principle we obtain that f_1 has a unique holomorphic extension to $L_{2N,\rho}$ and that $f_1|_{L_{2N,\rho}} \equiv \tilde{F}|_{L_{2N,\rho}}$.
Let us now define the set

$$U_{N,\tilde{\varepsilon}} := \left\{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : \max_{1\leq j\leq N} \left| z_j - \frac{|\hat{z}_j|}{\hat{\rho}} \right| < \tilde{\varepsilon} \right\},$$

where $\tilde{\varepsilon} = \min\Big\{(\rho^2 + 1/\rho^2 - \hat{\rho}^2 - 1/\hat{\rho}^2)/(28N), \min\limits_{1\leq j\leq N}(|\hat{z}_j|/\hat{\rho}), (\sqrt{\hat{\rho}} - 1)\min\limits_{1\leq j\leq N}(|\hat{z}_j|/\hat{\rho})\Big\}$.

Then for $\eta_j = \hat{z}_j/\xi_j$, $j = 1, 2, \ldots, N$, and $\xi = (\xi_1, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}$ we may express the non–removable singularity $\hat{z}$ in the form

$$\hat{z} = (\hat{z}_1, \ldots, \hat{z}_N) = (\xi_1\eta_1, \ldots, \xi_N\eta_N)$$

and obtain

$$\left(\xi_1\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1\frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2\frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \ldots, \xi_N\frac{1}{2i}\Big(\eta_N - \frac{1}{\eta_N}\Big)\right) \in L_{2N,\rho},$$

since for $\xi \in U_{N,\tilde{\varepsilon}}$ the inequality

$$\begin{aligned}\sum_{j=1}^{N} |\xi_j|^2 \left(\left|\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\right|^2 + \left|\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\right|^2\right) + \\ \left|\sum_{j=1}^{N} \xi_j^2 \left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2\right) - 1\right| = \\ = \sum_{j=1}^{N} |\xi_j|^2 \frac{1}{2}\Big(|\eta_j|^2 + \frac{1}{|\eta_j|^2}\Big) + \Big|\sum_{j=1}^{N} \xi_j^2 - 1\Big| \\ = \frac{1}{2}\hat{\rho}^2 + \frac{1}{2}\sum_{j=1}^{N} \frac{|\xi_j|^4}{\hat{\rho}_j^2} + \Big|\sum_{j=1}^{N} \xi_j^2 - 1\Big| < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big)\end{aligned} \tag{5.17}$$

is valid, where the upper bound follows from Lemma 5.9.
In addition, if $\eta_j = \hat{z}_j/\xi_j$, $j = 1, 2, \ldots, N$, and $\xi = (\xi_1, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}$, we derive from the estimate

$$\sum_{j=1}^{N} \left|\xi_j \frac{1}{\eta_j}\right|^2 = \sum_{j=1}^{N} \left|\xi_j^2 \frac{1}{\hat{z}_j}\right|^2 < \sum_{j=1}^{N} \frac{\left(\frac{|\hat{z}_j|}{\hat{\rho}} + \tilde{\varepsilon}\right)^4}{|\hat{z}_j|^2} \le \sum_{j=1}^{N} \frac{|\hat{z}_j|^4}{\hat{\rho}^4} \frac{(1 + \sqrt{\hat{\rho}} - 1)^4}{|\hat{z}_j|^2} = 1, \tag{5.18}$$

that

$$\left(\xi_1\frac{1}{\eta_1}, \xi_2\frac{1}{\eta_2}, \ldots, \xi_N\frac{1}{\eta_N}\right) \in \mathcal{B}_N.$$

We now claim

$$g_-\Big(\xi_1^2\frac{1}{\hat{z}_1}, \xi_2^2\frac{1}{\hat{z}_2}, \ldots, \xi_N^2\frac{1}{\hat{z}_N}\Big) = 0 \quad \text{for } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in U_{N,\tilde{\varepsilon}}.$$

Proof of the claim: This is done by contradiction. Therefore we assume there exists some $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_N) \in U_{N,\tilde{\varepsilon}}$ such that

$$g_-\left(\hat{\xi}_1^2\frac{1}{\hat{z}_1}, \hat{\xi}_2^2\frac{1}{\hat{z}_2}, \ldots, \hat{\xi}_N^2\frac{1}{\hat{z}_N}\right) \neq 0.$$

Then we also have

$$g_-\left(\hat{\xi}_1^2\frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2\frac{1}{\hat{z}_2} + w_2, \ldots, \hat{\xi}_N^2\frac{1}{\hat{z}_N} + w_N\right) \neq 0$$

for $w = (w_1, w_2, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$ if $\varepsilon > 0$ is sufficiently small. Further, in view of the inequalities (5.17) and (5.18) we may suppose that for $w = (w_1, w_2, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$

$$\left(\frac{\hat{\xi}_1}{2}\left(\frac{\hat{z}_1}{\hat{\xi}_1} + \frac{\hat{\xi}_1}{\hat{z}_1}\right) + w_1, \frac{\hat{\xi}_1}{2i}\left(\frac{\hat{z}_1}{\hat{\xi}_1} - \frac{\hat{\xi}_1}{\hat{z}_1}\right), \frac{\hat{\xi}_2}{2}\left(\frac{\hat{z}_2}{\hat{\xi}_2} + \frac{\hat{\xi}_2}{\hat{z}_2}\right) + w_2, \dots, \frac{\hat{\xi}_N}{2i}\left(\frac{\hat{z}_N}{\hat{\xi}_N} - \frac{\hat{\xi}_N}{\hat{z}_N}\right)\right) \in L_{2N,\rho}$$

and

$$\left(\hat{\xi}_1^2 \frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2 \frac{1}{\hat{z}_2} + w_2, \dots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N} + w_N\right) \in \mathcal{B}_N.$$

Next, we consider the functions

$$h(w_1, w_2, \dots, w_N) := f_1\left(\frac{\hat{\xi}_1}{2}\left(\frac{\hat{z}_1}{\hat{\xi}_1} + \frac{\hat{\xi}_1}{\hat{z}_1}\right) + w_1, \frac{\hat{\xi}_1}{2i}\left(\frac{\hat{z}_1}{\hat{\xi}_1} - \frac{\hat{\xi}_1}{\hat{z}_1}\right), \frac{\hat{\xi}_2}{2}\left(\frac{\hat{z}_2}{\hat{\xi}_2} + \frac{\hat{\xi}_2}{\hat{z}_2}\right) + w_2, \dots, \frac{\hat{\xi}_N}{2i}\left(\frac{\hat{z}_N}{\hat{\xi}_N} - \frac{\hat{\xi}_N}{\hat{z}_N}\right)\right)$$

and

$$\hat{g}_-(w_1, w_2, \dots, w_N) := g_-\left(\hat{\xi}_1^2 \frac{1}{\hat{z}_1} + w_1, \hat{\xi}_2^2 \frac{1}{\hat{z}_2} + w_2, \dots, \hat{\xi}_N^2 \frac{1}{\hat{z}_N} + w_N\right)$$

for $w = (w_1, w_2, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Then h and $\hat{g}_-$ are holomorphic in $\mathcal{D}_N(0, \varepsilon)$. Furthermore, $\hat{g}_-(w_1, w_2, \dots, w_N) \neq 0$ for $w = (w_1, w_2, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Thus the function

$$l(w_1, w_2, \dots, w_N) := \frac{h(w_1, w_2, \dots, w_N)}{\hat{g}_-(w_1, w_2, \dots, w_N)}$$

is holomorphic for $w = (w_1, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$. Since for $\varepsilon_0 > 0$ sufficiently small $\mathcal{D}_N(0, \varepsilon) \cap \{w = (w_1, \dots, w_N) \in \mathbb{C}^N : \left(\sum_{j=1}^N |\hat{z}_j + w_j|^2\right)^{1/2} < \tilde{\rho}\}$ is certainly a non–empty open set in $\mathbb{C}^N$ (see (5.16)), we obtain that g has a holomorphic extension $\tilde{g}$ to some non–empty neighborhood of $\hat{z}$. To be more precisely,

$$\tilde{g}(\hat{z}_1 + w_1, \hat{z}_2 + w_2, \dots, \hat{z}_N + w_N) = l(w_1, w_2, \dots, w_N)$$

for $w = (w_1, w_2, \dots, w_N) \in \mathcal{D}_N(0, \varepsilon)$ which contradicts the hypothesis that g has a non–removable singularity at $\hat{z}$. These aspects show

$$g_-\left(\xi_1^2 \frac{1}{\hat{z}_1}, \xi_2^2 \frac{1}{\hat{z}_2}, \dots, \xi_N^2 \frac{1}{\hat{z}_N}\right) = 0 \quad \text{for } \xi = (\xi_1, \xi_2, \dots, \xi_N) \in U_{N,\tilde{\varepsilon}}$$

and the claim is proved.

By the identity principle we conclude

$$g_-(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}}$$

and therefore

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}},$$

which is clearly a contradiction to the assumption that g has no holomorphic extension to some neighborhood of $\overline{\mathcal{B}}_{N,\tilde{\rho}}$.

To (ii): Now let $\hat{z}_k = 0$ for some $k \in \{1, 2, \ldots, N\}$. Without loss of generality we may assume that

$$\hat{z}_j \neq 0 \qquad \text{for } j = 1, 2, \ldots, m, \quad m < N,$$

and

$$\hat{z}_{m+l} = 0 \qquad \text{for } l = 1, 2, \ldots, N-m.$$

Next, we consider instead of $S_{2N,\tilde{\rho}}$ the set

$$\tilde{S}_{2N,\tilde{\rho}} := \Big\{ z = (z_1,\ldots, z_{2m}, w_1,\ldots, w_{2N-2m}) \in \mathbb{C}^{2N}\colon z_{2j-1} = \frac{\xi_j}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big), z_{2j} = \frac{\xi_j}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big),$$
$$\xi_j, \eta_j \in \mathbb{C}\backslash\{0\}, j = 1,\ldots, m, \left(\sum_{j=1}^{m} |\xi_j\eta_j|^2 + \sum_{j=1}^{N-m} |w_{2j-1} + iw_{2j}|\right)^{\frac{1}{2}} < \tilde{\rho} \wedge$$
$$\left(\sum_{j=1}^{m} \left|\xi_j \frac{1}{\eta_j}\right|^2 + \sum_{j=1}^{N-m} |w_{2j-1} - iw_{2j}|\right)^{\frac{1}{2}} < \tilde{\rho} \Big\}.$$

Now let us define the function $\tilde{f}_1 : T_{2N,\tilde{\rho}} \to \mathbb{C}$, like in equation (5.15). Then $\tilde{f}_1$ takes the form

$$\begin{aligned}
&\tilde{f}_1(z_1, z_2, \ldots, z_{2m}, w_1, w_2, \ldots, w_{2N-2m}) \\
&\quad = \tilde{f}_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \ldots, \frac{\xi_m}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-2m}\right) \\
&\quad = g\Big(\xi_1\eta_1, \ldots, \xi_m\eta_m, w_1 + iw_2, w_3 + iw_4, \ldots, w_{2N-2m-1} + iw_{2N-2m}\Big) \\
&\qquad\qquad g_-\Big(\xi_1\frac{1}{\eta_1}, \ldots, \xi_m\frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \ldots, w_{2N-2m-1} - iw_{2N-2m}\Big),
\end{aligned}$$

if $z = (z_1, z_2, \ldots, z_{2m}, w_1, w_2, \ldots, w_{2N-2m}) \in \tilde{S}_{2N,\tilde{\rho}}$.
As $\tilde{f}_1 \equiv F|_{\overline{B}_{2N}}$ Lemma 5.1 and Remark 3.25 ensure that $\tilde{f}_1$ can be continued analytically to $L_{2N,\rho}$. Hence we may proceed quite similar to the case $\hat{z}_j \neq 0$ for $j = 1, 2, \ldots, N$.
We define for $\varepsilon_1 > 0$ the set

$$U_{N,\varepsilon_1} := \Big\{ (z, w) = (z_1, z_2, \ldots, z_m, w_2, w_4, \ldots, w_{2N-2m}) \in \mathbb{C}^N : z_j \neq 0, j = 1, \ldots, m,$$
$$\max_{1\le j\le m} \left| z_j - \frac{|\hat{z}_j|}{\hat{\rho}} \right| < \varepsilon_1 \ \wedge \max_{1\le j\le N-m} |w_{2j}| < \varepsilon_1 \Big\}.$$

If now $\eta_j = \hat{z}_j/\xi_j$ for $j = 1, 2, \ldots, m$ and $w_{2l-1} = -iw_{2l}$ for $l = 1, \ldots, N-m$, where $(\xi_1, \xi_2, \ldots, \xi_m, w_2, w_4, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$, we will see that for $\varepsilon_1 > 0$ sufficiently small

$$\hat{z} = (\hat{z}_1, \ldots, \hat{z}_m, 0, \ldots, 0) = (\xi_1\eta_1, \ldots, \xi_m\eta_m, w_1 + iw_2, \ldots, w_{2N-2m-1} + iw_{2N-2m})$$

is a non–removable singularity of g such that the following conditions are fulfilled:

(a) $$\left(\xi_1\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1\frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \ldots, \xi_m\frac{1}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \ldots, w_{2N-2m}\right) \in L_{2N,\rho}$$

(b) $$\left(\xi_1\frac{1}{\eta_1}, \xi_2\frac{1}{\eta_2}, \ldots, \xi_m\frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \ldots, w_{2N-2m-1} - iw_{2N-2m}\right) \in \mathcal{B}_{N,\tilde{\rho}}$$

To (a): By Lemma 5.9 we obtain for $(\xi_1, \xi_2, \ldots, \xi_m, w_2, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$

$$\sum_{j=1}^{m} |\xi_j|^2 \left(\left| \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \right|^2 + \left| \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big) \right|^2 \right) + \sum_{j=1}^{2N-2m} |w_j|^2 +$$
$$\left| \sum_{j=1}^{m} \xi_j^2 \left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2 \right) + \sum_{j=1}^{2N-2m} w_j^2 - 1 \right| \leq$$
$$\sum_{j=1}^{m} |\xi_j|^2 \left(\left| \frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big) \right|^2 + \left| \frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big) \right|^2 \right) + 2 \sum_{j=1}^{2N-2m} |w_j|^2 +$$
$$\left| \sum_{j=1}^{m} \xi_j^2 \left(\Big(\frac{1}{2}\Big(\eta_j + \frac{1}{\eta_j}\Big)\Big)^2 + \Big(\frac{1}{2i}\Big(\eta_j - \frac{1}{\eta_j}\Big)\Big)^2 \right) - 1 \right| \leq$$
$$\frac{1}{2}\hat{\rho}^2 + \frac{1}{2} \sum_{j=1}^{m} \frac{|\xi_j|^4}{\hat{\rho}_j^2} + \left| \sum_{j=1}^{m} \xi_j^2 - 1 \right| + 4(N-m)\varepsilon_1^2 < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big),$$

if $\varepsilon_1 > 0$ is sufficiently small.
To (b): For $(\xi_1, \xi_2, \ldots, \xi_m, w_2, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$ and $\varepsilon_1 > 0$ sufficiently small we estimate as in equation (5.18)

$$\sum_{j=1}^{m} \left| \xi_j \frac{1}{\eta_j} \right|^2 + \sum_{j=1}^{N-m} |w_{2j-1} - i w_{2j}|^2 < 1 + 4(N-m)\varepsilon_1^2 < \tilde{\rho}^2.$$

Now, item (a) and (b) combined with the proof of the claim in case (i), yield

$$g_-\Big(\xi_1^2 \frac{1}{\hat{z}_1}, \xi_2^2 \frac{1}{\hat{z}_2}, \ldots, \xi_m^2 \frac{1}{\hat{z}_m}, -i2w_2, -i2w_4, \ldots, -i2w_{2N-2m} \Big) = 0$$

for $(\xi_1, \xi_2, \ldots, \xi_m, w_2, \ldots, w_{2N-2m}) \in U_{N,\varepsilon_1}$ if $\varepsilon_1 > 0$ sufficiently small. However this would imply

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{B}_{N,\tilde{\rho}},$$

which is clearly impossible. ■

Theorem 5.4 also holds for unit balls with respect to the maximum norm. To prove this analogous result we make use of Siciak's Theorem 3.17, which describes the extremal function Φ for Cartesian products of compact sets.

Theorem 5.11
Let $g \in \mathcal{H}(\overline{\mathcal{D}}_N)$ and $F : \overline{D}_{2N} \to \mathbb{R}$ defined by

$$F(x_1, y_1, x_2, \ldots, y_N) = |g(x_1 + iy_1, x_2 + iy_2, \ldots, x_N + iy_N)|^2$$

and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \leq \frac{1}{\rho}$$

if and only if g has a holomorphic extension to $\mathcal{D}_{N,\rho}$.

Proof:
Observe, Lemma 5.1 and Theorem 3.17 ensure

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}$$

if and only if F has an analytic continuation to

$$\mathcal{L}_{2N,\rho} = \Big\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} \left(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\right) < \frac{1}{2}\Big(\rho^2 + \frac{1}{\rho^2}\Big) \Big\}.$$

Next, we define the set

$$\mathcal{T}_{2N,\tilde\rho} = \Big\{ z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} |z_{2j-1} + iz_{2j}| < \tilde\rho \ \wedge \max_{1\le j\le N} |z_{2j-1} - iz_{2j}| < \tilde\rho \Big\},$$

where $\tilde\rho \in (1,\infty)$ is so chosen that $g \in \mathcal{H}(\mathcal{D}_{N,\tilde\rho}) \backslash \mathcal{H}(\overline{\mathcal{D}}_{N,\tilde\rho})$. If g is holomorphic in $\mathbb{C}^N$ we set $\tilde\rho = \infty$ and consider $\mathcal{T}_{2N,\infty} = \mathbb{C}^{2N}$.
Then the function $f_1 : \mathcal{T}_{2N,\tilde\rho} \to \mathbb{C}$ defined by

$$f_1(z_1, z_2, \ldots, z_{2N}) := g(z_1 + iz_2, z_3 + iz_4, \ldots, z_{2N-1} + iz_{2N}) \\ g_-(z_1 - iz_2, z_3 - iz_4, \ldots, z_{2N-1} - iz_{2N})$$

is holomorphic in $\mathcal{T}_{2N,\tilde\rho}$ and

$$f_1(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) = F(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N)$$

for $(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) \in \overline{D}_{2N}$.
"$\Leftarrow$": By hypothesis we have $g \in \mathcal{H}(\mathcal{D}_{N,\rho})$. Consequently, $\rho \le \tilde\rho$ and f_1 is holomorphic in $\mathcal{T}_{2N,\rho}$. Since $\mathcal{L}_{2N,\rho} \subset \mathcal{T}_{2N,\rho}$ by Lemma 5.14, we see that f_1 is a holomorphic extension of F to $\mathcal{L}_{2N,\rho}$ which implies

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}.$$

"$\Rightarrow$": This direction is proved by contradiction. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \le \frac{1}{\rho}$$

but g has no holomorphic extension to $\mathcal{D}_{N,\rho}$. Then f_1 has a uniquely determined holomorphic extension to $L_{2N,\rho}$ and $\tilde\rho$ is a number of the interval $(1,\rho)$. Hence there exists a non–removable singularity $\hat z$ of g such that $\hat\rho := |\hat z| \in [\tilde\rho, \rho) \cap [\tilde\rho, \tilde\rho + \varepsilon_0)$, where $\varepsilon_0 > 0$ is an arbitrary number. Without loss of generality we may assume

$$\hat z_j \ne 0 \qquad \text{for} \quad j = 1, 2, \ldots, m, \quad m \le N,$$

and

$$\hat z_j = 0 \qquad \text{for} \quad j = m+1, m+2, \ldots, N.$$

Now, we write the non–removable singularity $\hat{z}$ of g in the form

$$\hat{z} = \left(\xi_1\frac{\hat{z}_1}{\xi_1}, \xi_2\frac{\hat{z}_2}{\xi_2}, \dots, \xi_m\frac{\hat{z}_m}{\xi_m}, w_1 + iw_2, \dots, w_{2N-2m-1} + iw_{2N-2m}\right),$$

where $\xi_j \in \mathbb{C}\backslash\{0\}$, $j = 1, 2, \dots, m$, and $w_{2j-1} = -iw_{2j}$, $w_j \in \mathbb{C}$, $j = 1, \dots, N-m$. Next we define for $\tilde{\varepsilon} > 0$ the set

$$U_{N,\tilde{\varepsilon}} = \left\{z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N : \max_{1\le j\le m} |z_j - |\hat{z}_j|/\hat{\rho}| < \tilde{\varepsilon},\ \max_{m+1\le j\le N} |z_j| < \tilde{\varepsilon}\right\}.$$

Further, let for the rest of the proof

$$(\xi_1,\dots, \xi_m, w_2, w_4,\dots, w_{2N-2m}) \in U_{N,\tilde{\varepsilon}}\,, \quad \eta_j = \hat{z}_j/\xi_j, \quad \xi_j \neq 0, \quad j = 1, 2, \dots, m,$$

and

$$w_{2j-1} = -iw_{2j}, \qquad j = 1, \dots, N-m.$$

Then we deduce from Lemma 5.7 that for $\tilde{\varepsilon} > 0$ sufficiently small

$$\Big(\xi_1\frac{1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \xi_1\frac{1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \xi_2\frac{1}{2}\Big(\eta_2 + \frac{1}{\eta_2}\Big), \xi_2\frac{1}{2i}\Big(\eta_2 - \frac{1}{\eta_2}\Big), \dots,$$
$$\xi_m\frac{1}{2}\Big(\eta_m + \frac{1}{\eta_m}\Big), \xi_m\frac{1}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \dots, w_{2N-2m}\Big) \in \mathcal{L}_{2N,\rho}.$$

Moreover, we have for $\tilde{\varepsilon} > 0$ sufficiently small

$$\max\left\{\max_{1\le j\le m}\left|\xi_j\frac{1}{\eta_j}\right|, \max_{1\le l\le N-m}|w_{2l-1} - iw_{2l}|\right\} \le \max\left\{\max_{1\le j\le m}\left|\xi_j^2\frac{1}{\hat{z}_j}\right|, \max_{1\le l\le N-m} 2|w_{2l}|\right|\right\} < \tilde{\rho}.$$

Observe, f_1 takes the form

$$\begin{aligned}
&f_1(z_1,z_2, \dots, z_{2m}, w_1, w_2, \dots, w_{2N-2m})\\
&= f_1\left(\frac{\xi_1}{2}\Big(\eta_1 + \frac{1}{\eta_1}\Big), \frac{\xi_1}{2i}\Big(\eta_1 - \frac{1}{\eta_1}\Big), \dots, \frac{\xi_m}{2i}\Big(\eta_m - \frac{1}{\eta_m}\Big), w_1, w_2, \dots, w_{2N-m}\right)\\
&= g\Big(\xi_1\eta_1, \dots, \xi_m\eta_m, w_1 + iw_2, w_3 + iw_4, \dots, w_{2N-2m-1} + iw_{2N-2m}\Big)\\
&\qquad g_-\Big(\xi_1\frac{1}{\eta_1}, \dots, \xi_m\frac{1}{\eta_m}, w_1 - iw_2, w_3 - iw_4, \dots, w_{2N-2m-1} - iw_{2N-2m}\Big)
\end{aligned}$$

if $z_{2j-1} = \frac{\xi_j}{2}\big(\eta_j + \frac{1}{\eta_j}\big), z_{2j} = \frac{\xi_j}{2i}\big(\eta_j - \frac{1}{\eta_j}\big)$, $\xi_j, \eta_j \neq 0$, $j = 1, 2, \dots, m$. Thus, by similar arguments as in the proof of Theorem 5.4, we conclude

$$g(z) = 0 \qquad \text{for} \quad z \in \mathcal{D}_{N,\tilde{\rho}},$$

which is in contrast to our assumption that g has no analytic continuation to $\overline{\mathcal{D}}_{N,\tilde{\rho}}$. ■

Remark 5.12

(i) The proofs of the "if"–directions of Theorem 5.4 and Theorem 5.11 are based on Lemma 5.7 and Lemma 5.8. A different approach, quite similar to the one dimensional case, shows the proof of Lemma 5.15. Hence at a quick glance one might be tempted to think that Lemma 5.7 and Lemma 5.8 are not significant. However, the importance of these lemmata becomes apparent when we prove Theorem 5.14.

(ii) *Clearly, Theorem 5.4 and Theorem 5.11 are not necessarily restricted to the closed unit ball in $\mathbb{R}^{2N}$ with respect to the Euclidean or the maximum norm. They are also valid for an arbitrary closed ball in $\mathbb{R}^{2N}$ with respect to Euclidean or the maximum norm.*

The methods we used to prove Theorems 5.4 and 5.11 can be also applied to prove

Theorem 5.13
(i) Let $g, h \in \mathcal{H}(\overline{\mathcal{B}}_N)$, $g \not\equiv 0, h \not\equiv 0$, and $F : \overline{B}_{2N} \to \mathbb{R}$ defined by

$$F(x_1, y_1, x_2, ..., y_N) = g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}.$$

Further, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

if and only if g and h have holomorphic extensions to $\mathcal{B}_{N,\rho}$.

(ii) Let $g, h \in \mathcal{H}(\overline{\mathcal{D}}_N)$, $g \not\equiv 0, h \not\equiv 0$, and $F : \overline{D}_{2N} \to \mathbb{R}$ defined by

$$F(x_1, y_1, x_2, ..., y_N) = g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}.$$

Moreover, let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{D}_{2N}, F)} \leq \frac{1}{\rho}$$

if and only if g and h have holomorphic extensions to $\mathcal{D}_{N,\rho}$.

Proof:
The proof of this theorem can be established by slight modifications of the proofs of Theorem 5.4 and Theorem 5.11 respectively. Therefore we only give a rough sketch of the proof.
To (i): The "if"–part follows immediately from the proof of the "$\Leftarrow$"–direction of Theorem 5.4 if we replace g_- by h_-, where h_- is specified as in Definition 5.10. Thus let us concentrate on the "if and only if"–part. We suppose

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)} \leq \frac{1}{\rho}$$

but g or h has no holomorphic extension to $\mathcal{B}_{N,\rho}$.
We first consider the case that $g \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}}) \backslash \mathcal{H}(\overline{\mathcal{B}}_{N,\tilde{\rho}})$ and $h \in \mathcal{H}(\mathcal{B}_{N,\tilde{\rho}})$, where $\tilde{\rho} \in (1, \rho)$. Then, we may proceed as in the proof of the "$\Rightarrow$"–direction of Theorem 5.4, if we choose h_- instead of g_-. Hence we obtain that $h \equiv 0$ on $\mathcal{B}_N$ which is a contradiction to the hypothesis.
Now, let $h \in \mathcal{H}(\mathcal{B}_{N,\hat{\rho}}) \backslash \mathcal{H}(\overline{\mathcal{B}}_{N,\hat{\rho}})$ and $g \in \mathcal{H}(\mathcal{B}_{N,\hat{\rho}})$, where $\hat{\rho} \in (1, \rho)$. Then we define

$$G(x_1, y_1, x_2, ..., y_N) := h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}.$$

As

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, G)} = \limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_{2N}, F)}$$

we may argue as above (take G instead of F and h for g). Thus we derive that $g \equiv 0$ on $\mathcal{B}_N$ in contrast to our assumption and we are done.
(ii) This statement can be verified by similar arguments as in item (i). ■

Lemma 5.7 and 5.8 play the key role for the next theorem.

Theorem 5.14
(i) Assume $F \in \mathcal{H}(\overline{B}_{2N})$ has the representation

$$F(x_1, y_1, x_2, ..., y_N) = g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}$$

for $(x_1, y_1, x_2, ..., y_N) \in \overline{B}_{2N}$, where $g, h \in \mathcal{H}(\overline{\mathcal{B}}_N)$. Then F has a holomorphic extension to

$$T_{2N,\rho} = \left\{z = (z_1, ..., z_{2N}) \in \mathbb{C}^{2N}\colon \left(\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2\right)^{\frac{1}{2}} < \rho \ \wedge \ \left(\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2\right)^{\frac{1}{2}} < \rho\right\}$$

if and only if F has no singular points on

$$A_{2N,\rho} := \left\{z = (z_1, ..., z_{2N}) \in \mathbb{C}^{2N}\colon z_{2j-1} = \frac{R_j}{2R}\left(Re^{it_j} + \frac{1}{Re^{it_j}}\right), z_{2j} = \pm\frac{R_j}{2iR}\left(Re^{it_j} - \frac{1}{Re^{it_j}}\right),\right.$$
$$\left.\sum_{j=1}^{N} R_j^2 = R^2,\ R \in (1, \rho),\ R_j \in [0, R],\ t_j \in [0, 2\pi],\ j = 1, ..., N\right\}.$$

In particular, F has a holomorphic extension to

$$L_{2N,\rho} = \left\{z = (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \sum_{j=1}^{2N} |z_j|^2 + \left|\sum_{j=1}^{2N} z_j^2 - 1\right| < \frac{1}{2}\left(\rho^2 + \frac{1}{\rho^2}\right)\right\}$$

if and only if F has no singular points on $A_{2N,\rho}$.
(ii) Let $F \in \mathcal{H}(\overline{D}_{2N})$ be of the form

$$F(x_1, y_1, x_2, ..., y_N) = g(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)\overline{h(x_1 + iy_1, x_2 + iy_2, ..., x_N + iy_N)}$$

for $(x_1, y_1, x_2, ..., y_N) \in \overline{D}_{2N}$, where $g, h \in \mathcal{H}(\overline{\mathcal{D}}_N)$. Then F has a holomorphic extension to

$$\mathcal{T}_{2N,\rho} = \left\{z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} |z_{2j-1} + iz_{2j}| < \rho \ \wedge \ \max_{1\le j\le N} |z_{2j-1} - iz_{2j}| < \rho\right\}$$

if and only if F has a no singular points on

$$\mathcal{A}_{2N,\rho} := \left\{z = (z_1, ..., z_{2N}) \in \mathbb{C}^{2N}\colon z_{2j-1} = \frac{R_j}{2R}\left(Re^{it_j} + \frac{1}{Re^{it_j}}\right), z_{2j} = \pm\frac{R_j}{2iR}\left(Re^{it_j} - \frac{1}{Re^{it_j}}\right),\right.$$
$$\left.\max_{1\le j\le N} R_j = R,\ R \in (1, \rho),\ R_j \in [0, R],\ t_j \in [0, 2\pi],\ j = 1, ..., N\right\}.$$

Moreover, F has a holomorphic extension to

$$\mathcal{L}_{2N,\rho} = \left\{z = (z_1, z_2, \ldots, z_{2N}) \in \mathbb{C}^{2N} : \max_{1\le j\le N} \left(|z_{2j-1}|^2 + |z_{2j}|^2 + |z_{2j-1}^2 + z_{2j}^2 - 1|\right) < \frac{1}{2}\left(\rho^2 + \frac{1}{\rho^2}\right)\right\}$$

if and only if F has no singular points on $\mathcal{A}_{2N,\rho}$.

Proof:
To (i): Firstly, we assume that F has no holomorphic extension to $\mathbb{C}^{2N}$. Therefore we can choose $\rho \in (1,\infty)$ such that $F \in \mathcal{H}(T_{2N,\rho})\backslash\mathcal{H}(\overline{T}_{2N,\rho})$. Corollary 5.1 and the proof of the "if"–part of Theorem 5.4 combined, shows

$$F \in \mathcal{H}(T_{2N,\rho})\backslash\mathcal{H}(\overline{T}_{2N,\rho}) \qquad \text{if and only if} \qquad F \in \mathcal{H}(L_{2N,\rho})\backslash\mathcal{H}(\overline{L}_{2N,\rho}).$$

Hence, since $T_{2N,\rho} \supset L_{2N,\rho}$ by Lemma 5.14, there exists a singular point $\hat{z}$ of F satisfying

$$\hat{z} \in \partial T_{2N,\rho} \qquad \text{if and only if} \qquad \hat{z} \in \partial L_{2N,\rho}.$$

Consequently, we obtain that F has a holomorphic extension to $T_{2N,\rho}$ if F has no singular points on $M_{2N,\rho}$ (and vice versa), where

$$\begin{aligned} M_{2N,\rho} := \Bigg\{ & z=(z_1,\dots,z_{2N}) \in \mathbb{C}^{2N} : \\ & \left(\|z\|^2 + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| = \frac{1}{2}\left(R^2 + \frac{1}{R^2}\right) \wedge \sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 = R^2 \right) \ \vee \\ & \left(\|z\|^2 + \left| \sum_{j=1}^{2N} z_j^2 - 1 \right| = \frac{1}{2}\left(R^2 + \frac{1}{R^2}\right) \wedge \sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 = R^2 \right), \quad R \in (1,\rho) \Bigg\}. \end{aligned}$$

Now, from the proof of Lemma 5.14 we conclude $z=(z_1,\dots,z_{2N}) \in M_{2N,\rho}$ if and only if $z=(z_1,\dots,z_{2N})$ has the form

$$z_{2j-1} = \frac{R_j}{R}\frac{1}{2}\left(Re^{it_j} + \frac{1}{Re^{it_j}}\right), \qquad z_{2j} = \pm\frac{R_j}{R}\frac{1}{2i}\left(Re^{it_j} - \frac{1}{Re^{it_j}}\right), \quad j=1,\dots,N,$$

where $\sum_{j=1}^{N} R_j^2 = R^2$, $R_j \in [0,R]$, $t_j \in [0,2\pi]$, $j=1,\dots,N$, and $R \in (1,\rho)$.
If F has a holomorphic extension to $\mathbb{C}^{2N}$ then the statement is quite obvious, if we regard it as the limiting case "$\rho=\infty$". This finishes item (i).
To (ii): We skip the proof of this result as it can be obtained quite similar to (i). ■

5.3 Further results

The purpose of this section is to demonstrate that the "if"–direction of Theorem 5.13 can be extended to a much larger class of domains than closed balls in $\mathbb{R}^{2N}$, whereas the "if and only if"–direction fails to be true in general.

Lemma 5.15
Let $K \subset \mathbb{C}^N$ be a compact set such that Siciak's extremal function Φ is continuous in $\mathbb{C}^N$ and define $L = \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$. Further, let g, h be holomorphic functions in an open connected neighborhood of K and let $F : L \to \mathbb{R}$ be given by

$$F(x,y) = g(x+iy)\overline{h(x+iy)}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L,F)} \le \frac{1}{\rho}$$

if g and h have holomorphic extensions to

$$L_\rho = \left\{z \in \mathbb{C}^N : \Phi(z,K) < \rho\right\}.$$

Proof:
Due to Theorem 3.21 we can choose two sequences $\{p_{1,n}\}_{n\in\mathbb{N}}$ and $\{p_{2,n}\}_{n\in\mathbb{N}}$ of polynomials $p_{1,n}, p_{2,n}$ of degree $\le n$ such that for an arbitrary $R \in (1,\rho)$ and all $n \in \mathbb{N}$ the estimate

$$\max\left\{\|g - p_{1,n}\|_L, \|h - p_{2,n}\|_L\right\} \le \frac{M}{R^n} \tag{5.19}$$

holds, where $M > 0$ is some constant independent of n. Consequently, we have

$$\begin{aligned} \|g\overline{h} - p_{1,n}\overline{p_{2,n}}\|_L &\le \|g\overline{h} - p_{1,n}\overline{h}\|_L + \|p_{1,n}\overline{h} - p_{1,n}\overline{p_{2,n}}\|_L \\ &\le \|h\|_L\|g - p_{1,n}\|_L + \|p_{1,n}\|_L\|\overline{h} - \overline{p_{2,n}}\|_L \\ &\le \frac{M_1}{R^n}, \qquad M_1 := 3M\max\{\|g\|_L, \|h\|_L\}. \end{aligned}$$

Next we put

$$q_{1,0}(z) := p_{1,0}(z), \quad q_{1,k}(z) := p_{1,k}(z) - p_{1,k-1}(z), \qquad k \in \mathbb{N}, \quad z \in \mathbb{C}^N,$$

and

$$q_{2,0}(z) := p_{2,0}(z), \quad q_{2,k}(z) := p_{2,k}(z) - p_{2,k-1}(z), \qquad k \in \mathbb{N}, \quad z \in \mathbb{C}^N,$$

Further, let us define the real–valued polynomials

$$Q_n(x,y) := \sum_{\substack{k,l=0\\k+l\le n}}^{n} q_{1,k}(z)\overline{q_{2,l}(z)}, \quad z = x+iy, \quad x,y \in \mathbb{R}^N, \quad n \in \mathbb{N},$$

as well as

$$P_{2n}(x,y) := \sum_{k,l=0}^{n} q_{1,k}(z)\,\overline{q_{2,l}(z)} = p_{1,n}(z)\overline{p_{2,n}(z)}, \quad z = x+iy, \quad x,y \in \mathbb{R}^N, \quad n \in \mathbb{N}.$$

Then we get

$$P_{2n}(x,y) - Q_n(x,y) = \sum_{\substack{k,l=0\\k+l>n}}^{n} q_{1,k}(z)\overline{q_{2,l}(z)} = \sum_{k=1}^{n} q_{1,k}(z)\left(\overline{p_{2,n}(z) - p_{2,n-k}(z)}\right).$$

In view of equation (5.19) we obtain

$$|p_{2,l}(z) - p_{2,k}(z)| \le |h(z) - p_{2,l}(z)| + |h(z) - p_{2,k}(z)| \le \frac{2M}{R^k} \qquad \text{for} \quad k < l, \quad z \in K.$$

From the definition of $q_{1,k}$ and the last estimate we conclude

$$|q_{1,k}(z)\,\overline{\left(p_{2,n}(z)-p_{2,n-k}(z)\right)}| \le \frac{2M}{R^{k-1}}\frac{2M}{R^{n-k}} = \frac{4M^2}{R^{n-1}} \qquad \text{for} \quad z \in K.$$

This gives

$$|P_{2n}(x,y)-Q_n(x,y)| \le \frac{4nM^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in L$$

and in consequence,

$$\begin{aligned} |F(x,y)-Q_n(x,y)| &\le |F(x,y)-P_{2n}(x,y)| + |P_{2n}(x,y)-Q_n(x,y)| \\ &\le \frac{M_1}{R^n} + \frac{4nM^2}{R^{n-1}} \qquad \text{for} \quad (x,y) \in L. \end{aligned}$$

Thus the result

$$\limsup_{n\to\infty} \sqrt[n]{E_n(L,F)} \le \frac{1}{\rho}$$

follows as $R < \rho$ was arbitrary. ∎

Our next theorem reveals that if K is a closed square or a closed polysquare in $\mathbb{C}^N$ then the sufficient condition of Lemma 5.15 can't be necessary.

Theorem 5.16
Consider the function

$$F(x,y) = \frac{1}{\left((x-\rho_1)^2+y^2\right)^s} \qquad \text{for} \quad (x,y) \in [-1,1]\times[-1,1],$$

where $s \in (0,\infty)$ and $\rho_1 \in (1,\infty)$. Further, define the function

$$g(z) = \frac{1}{(z-\rho_1)^s} \qquad \text{for} \quad z \in K := \{z \in \mathbb{C} : z = x+iy, x,y \in [-1,1]\}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1]\times[-1,1],F)} = \frac{1}{\rho_1} \tag{5.20}$$

but

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,g)} = \frac{1}{|\psi(\rho_1)|} > \frac{1}{\rho_1}, \tag{5.21}$$

where ψ maps $\hat{\mathbb{C}}\backslash K$ univalently onto $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ such that $\psi(\infty)=\infty$[3].

To prove this theorem let us show the next two auxiliary results.

Lemma 5.17
Let $K \subset \mathbb{C}$, $K \neq \emptyset$, be compact such that $\hat{\mathbb{C}}\backslash K$ is simply connected and let $g: K \to \mathbb{C}$ be a continuous function. Then

$$\limsup_{n\to\infty} \sqrt[n]{e_n(K,g)} \le \frac{1}{\rho}$$

if and only if g has a holomorphic extension to the set

$$K \cup \{z \in \mathbb{C} : 1 < |\psi(z)| < \rho\},$$

where ψ is a function which maps $\hat{\mathbb{C}}\backslash K$ univalently onto $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ such that $\psi(\infty)=\infty$.

[3]The conformal mapping ψ is up to a rotation uniquely determined.

Proof:
This result is an immediate consequence of Theorem 1.3. We only have to take into account that for simply connected proper subsets of $\hat{\mathbb{C}}$ Green's function for $\hat{\mathbb{C}}\backslash K$ with pole at infinity coincides with $\log|\psi|$ on $\hat{\mathbb{C}}\backslash K$. ■

Lemma 5.18
Let $K \subset \mathbb{C}$ be a compact set with $\overline{\mathbb{D}} \subsetneq K$. If there exists a function ψ which maps $\hat{\mathbb{C}}\backslash K$ univalently onto $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ such that $\psi(\infty) = \infty$, then

$$|\psi(z)| < |z| \qquad \textit{for} \quad z \in \mathbb{C}\backslash K.$$

Proof:
Consider the function $h : \mathbb{D} \to \mathbb{C}$ defined by

$$h(z) = \frac{1}{\psi^{-1}\left(\frac{1}{z}\right)},$$

where ψ^{-1} is the inverse function of ψ. Then h is holomorphic in $\mathbb{D}$. Moreover, we have $h(\mathbb{D}) \subset \mathbb{D}$ and $h(0) = 0$. Hence by Schwarz's Lemma we obtain that

$$\left|\psi^{-1}\left(\frac{1}{z}\right)\right| > \left|\frac{1}{z}\right| \qquad \text{for} \quad z \in \mathbb{D}\backslash\{0\},$$

and therefore

$$|\psi(z)| < |z| \qquad \text{for} \quad z \in \mathbb{C}\backslash K.$$

■

Proof of Theorem 5.16:

Equation (5.21) is an immediate consequence of Lemma 5.17 and Lemma 5.18. Thus it remains to prove equation (5.20).
"$\leq$": Combining Theorem 1.2 and Theorem 3.17 yields

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1]\times[-1,1],F)} \leq \frac{1}{\rho}$$

if and only if F has an analytic continuation to

$$L_{2,\rho} = \left\{(z_1,z_2) \in \mathbb{C}^2 : \max\left\{\left|z_1 + \sqrt{z_1^2-1}\right|, \left|z_2 + \sqrt{z_2^2-1}\right|\right\} < \rho\right\},$$

where the branch of the square root is chosen such that $\sqrt{z} > 0$ for $z > 0$.
Now, F is holomorphic in $\mathbb{C}^2\backslash\{(z_1,z_2) \in \mathbb{C}^2 : z_2 = \pm i(z_1 - \rho_1)\}$ and has non–removable singularities if and only if $z_2 = \pm i(z_1 - \rho_1)$, where $z_1 \in \mathbb{C}$ is arbitrary.
Therefore we have to show that these singularities fulfill the condition

$$\max\left\{\left|z_1 + \sqrt{z_1^2-1}\right|, \left|z_2 + \sqrt{z_2^2-1}\right|\right\} \geq \rho_1. \tag{5.22}$$

For that reason we write z_1 in the form

$$z_1 = \frac{1}{2}\Big(R + \frac{1}{R}\Big)\cos t + i\frac{1}{2}\Big(R - \frac{1}{R}\Big)\sin t, \qquad R \in [1, \infty), \quad t \in [0, 2\pi]. \tag{5.23}$$

If $R \geq \rho_1$ then inequality (5.22) is obviously satisfied, as $|z_1 + \sqrt{z_1^2 - 1}| = R$. Hence we only have to show that

$$\Big|z_2 + \sqrt{z_2^2 - 1}\Big| \geq \rho_1$$

if $R \in [1, \rho_1)$. Because of equation (5.23) we have for z_2 the representation

$$z_2 = \pm\Big(- \frac{1}{2}\Big(R - \frac{1}{R}\Big)\sin t + i\Big(\frac{1}{2}\Big(R + \frac{1}{R}\Big)\cos t - \rho_1\Big)\Big).$$

The modulus of the imaginary part of z_2 can be estimated by

$$\Big|\frac{1}{2}\Big(R + \frac{1}{R}\Big)\cos t - \rho_1\Big| \geq \rho_1 - \frac{1}{2}\Big(R + \frac{1}{R}\Big) > \frac{1}{2}\Big(\rho_1 - \frac{1}{\rho_1}\Big).$$

Now, bearing the mapping properties of the inverse Joukowski function in mind, we obtain

$$\Big|z_2 + \sqrt{z_2^2 - 1}\Big| > \rho_1.$$

Consequently,

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1]\times[-1,1], F)} \leq \frac{1}{\rho_1}.$$

"$\geq$": This direction follows from Theorem 2.9. Since $[-1,1]\times[-1,1] \supset B_2$ we have

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1]\times[-1,1], F)} \geq \limsup_{n\to\infty} \sqrt[n]{E_n(B_2, F)} \geq \frac{1}{\rho_1}.$$

■

Remark 5.19

Notice, Theorem 3.17 and Theorem 5.16 imply that the "if"–condition in Lemma 5.15 can't be necessary if K is a closed polysquare in $\mathbb{C}^N$.

Let us end this section with an illustration of Theorem 5.16.

Remark 5.20

The conformal mapping ψ of Theorem 5.16 takes the form

$$\psi(z) = e^{i\varphi}\Big(\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{1}{4n-1}\Big)^{-1}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{1}{4n-1}z^{-4n+1}, \quad \varphi \in [0, 2\pi],$$

see for example [Gai64] for the construction of ψ.

If $\rho_1 = 1.4$ *then we compute* $\rho = |\psi(\rho_1)| = 1.26540$ *up to five digits accuracy*[4].

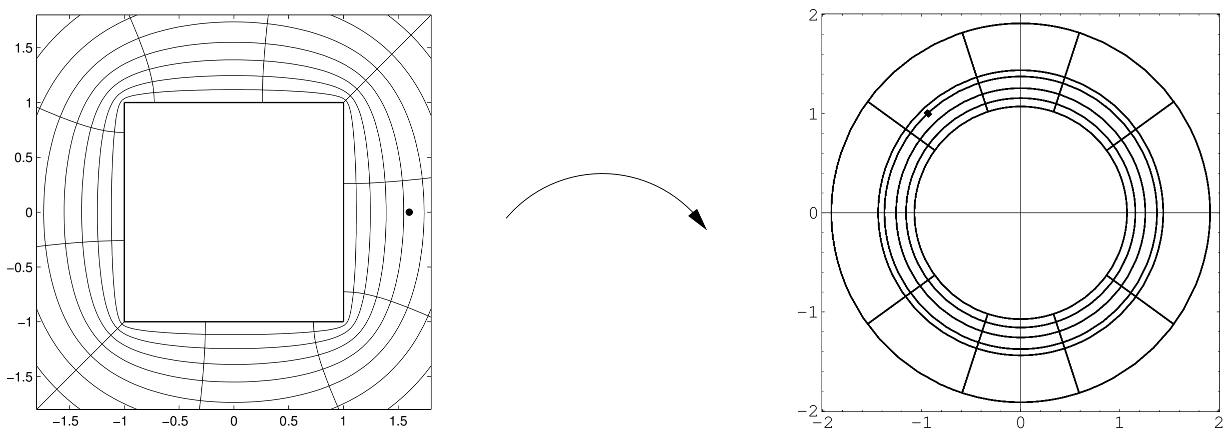

Figure 5.2 $\psi(z)$

[4]The plot of the mapping was produced by using the Schwarz-Christoffel Toolbox for Matlab. This toolbox is especially then helpful if we are interested in the maximal convergence number ρ for a holomorphic function defined on a polygon in $\mathbb{C}$ as it can compute Schwarz-Christoffel mappings to eight digits accuracy if crowding doesn't become severe. We refer the reader to [DT02] for more details on this matter.

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