Accessibility of

Bilinear Interconnected Systems

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To Christa and Gudrun.

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Chapter 1

Introduction

The area of network science is an interdisciplinary academic field. The study of interconnected networks, random and small-world networks appears in a wide variety of research fields, for instance in the study of social networks, sensor networks, economics and clearly multi-agent coordination and control - just to name some. Two main reasons exist for the increasing interest [42]:

(i) Particularly in biological and material sciences, it has become fundamental to get a deeper understanding of how inter-agent interactions affect the collective functionality of systems that evolve from networks.

(ii) Networked engineering systems which resemble their natural counterparts in terms of their functional and operational complexity can be synthesized due to the technological advances of the past years.

Clearly, being able to control natural or technological systems requires our ultimate understanding of them. Control theory provides mathematical tools for steering dynamical systems towards a desired state through the appropriate input manipulation. However, there is still a general framework missing for the controllability of networks of systems. We would like to understand how a network of interacting dynamical systems will behave collectively, given their individual dynamics and the network's coupling architecture.

The interested reader can find a dense overview of developments in the field of networks in [43, 51]. Moreover, focusing on graph theoretic methods for the analysis and synthesis of dynamic multi-agent networks Mesbahi and Egerstedt provide in [42] an introduction to the analysis and design of dynamic multi-agent networks.

Controllability of networks

One of the most important and basic problems in control theory is finding necessary and sufficient conditions for deciding when a system is controllable. Roughly speaking, controllability of a system refers to its property of being able to be driven from any initial state to any desired final state within finite time by using an appropriate control function [31]. For linear systems of the form

$$\dot{x} = Ax + Bu(t)$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$ for all t this is possible if and only if the controllability matrix

$$C = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}$$

has full rank, i.e. $\operatorname{rank} C = n$. This is called Kalman's controllability rank condition [40]. Today, the study of controllability of linear systems is well established and the emphasis is more on broadening the theory to nonlinear control problems. Nonlinear control systems are either systems, where the underlying dynamics are nonlinear, or the systems are forced by controls which are applied in a non-additive way. In general, a nonlinear control system can be written as

$$\dot{x} = f(x, u(t)),$$

where f is a differentiable vector field and $u(t)$ is the control function. A particular subclass is given by bilinear control systems which have the form

$$\dot{x} = (A + u(t)B)\, x$$

with matrices A and B in $\mathbb{R}^{n\times n}$. In general, controllability of a nonlinear control system is not trivial to check. Therefore, a weaker version of controllability was introduced, which is denoted by accessibility [24, 52]. It describes the system's property of being able to reach an open set of the state space from a given initial state. There is a Lie algebraic rank condition to decide if the system is accessible [44]. If this Lie algebraic rank condition is not satisfied, all trajectories must remain in a lower dimensional submanifold of the state space. For bilinear systems, the Lie algebraic rank condition states that accessibility of the control system is equivalent to whether the Lie algebra generated by the coefficient matrices A and B is big enough [52].

To define a network of N interconnected systems we fix for a set of N agents an interconnection structure given by a directed graph Γ with edge set E. The time-invariant dynamics of the set of agents can be described as a nonlinear dynamical system

$$\begin{aligned} \dot{x} &= f(x, v(t)) \\ y &= g(x), \end{aligned}$$

where the vector $x = (x_1, \ldots, x_N)^\top$ captures the state of a system of N nodes. The differentiable vector field f describes the network's dynamics when no interconnection exists and hence, all agents are controlled independently from the others. Clearly,

$$f(x, v(t)) = (f_1(x_1, v_1(t)), \ldots, f_N(x_N, v_n(t)))^\top,$$

where $f_i(x_i, v_i(t))$ are the dynamics of the i-th agent and $g(x) = (g_1(x_1), \ldots, g_N(x_N))$ is the collective output. Dependent on the interconnection structure, we now fix an inter-agent interaction strategy. An intuitive approach is to use the output of an agent as a control input via the interconnections for the other agents. This inter-agent interaction structure is known as output feedback. The resulting dynamical system has the form

$$\dot{x} = f\left(x, g(x)\right).$$

This is illustrated in Figure 1.1.

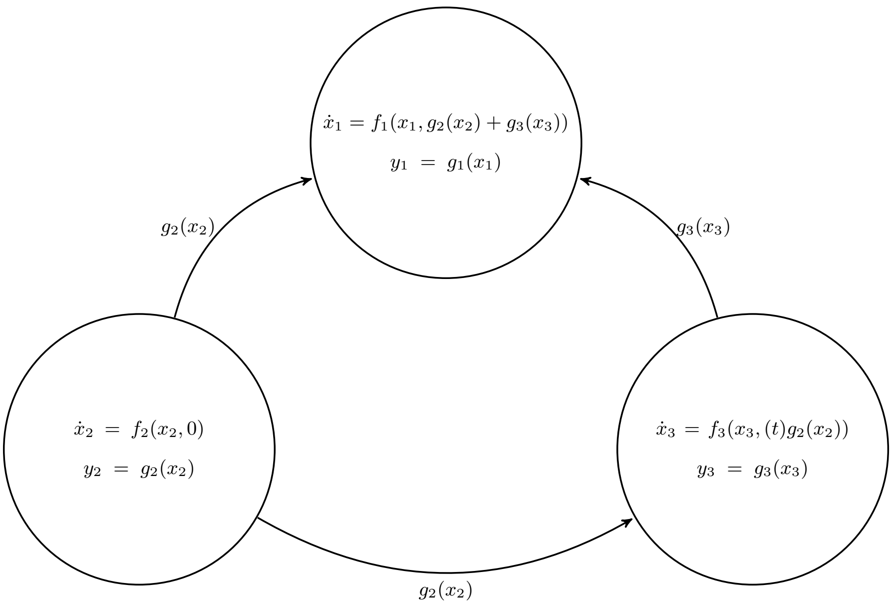

FIGURE 1.1: Network of interconnected systems under output feedback

Apparently, the dynamical system of the network is not a control system anymore since we prescribed fixed control inputs. Therefore, we introduce controls by assuming the interconnection strength to be controllable. Hence, we can steer the strength of the communication between agents. To do so, we include scalar-valued control functions $u_{ij}(t)$ to steer the output feedback. The network control system can then be written as

$$\dot{x} = f\left(x, u(t)g(x)\right).$$

This is illustrated in Figure 1.2. Now, the control function $u(t)$ is matrix-valued and depends on the interconnection structure of the network.

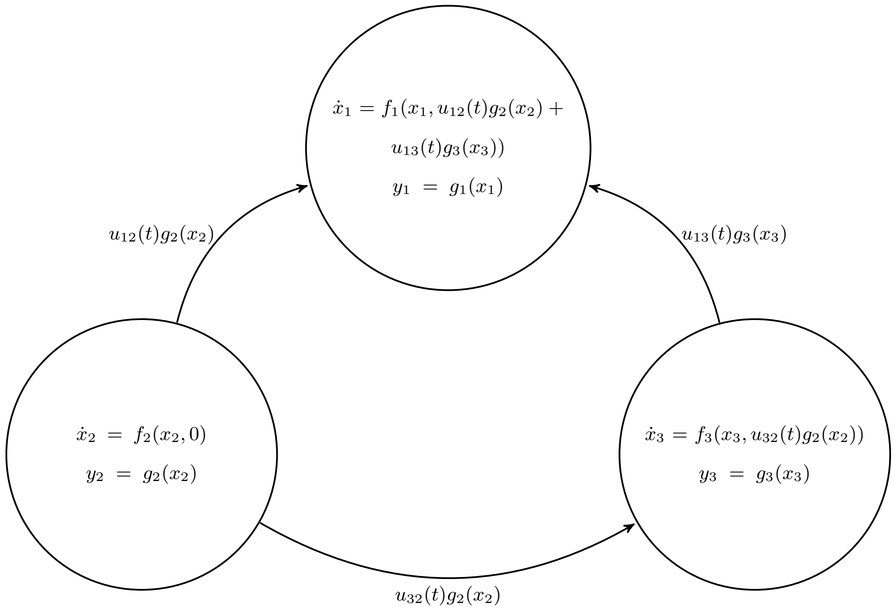

FIGURE 1.2: Network of interconnected systems under controllable output feedback

The characteristics of networks of interconnected systems are affected by the following factors [39]:

(i) the inter-agent interconnection architecture, which is represented by the underlying graph;

(ii) the dynamical rules, which describe the time-dependent interactions between the agents;

(iii) the dynamical rules, which describe the dynamics of the agents.

Clearly, each of these factors contributes to the network's accessibility or, respectively, controllability. In this thesis, we are particularly interested in which interconnection architecture assures accessibility of the overall network system. The center of our interest are networks with linear agent's dynamics. The overall control system of the network is then bilinear and hence, the question for accessibility and controllability is not trivial. We develop accessibility conditions in terms of the interconnection structure, i.e. we ask for the graph structure which guarantees the network to be accessible. However, the interconnection structure of the network is assumed to be static. Thus, we only study the influence of factor (i) on accessibility of the network control system.
The study of networks which consist of agents with nonlinear dynamics is beyond the scope of this work.

Certainly, this work is not the first to study controllability of networks. There exists even work on network controllability in the direction of studying the network architecture. For example in [39], the controllability of networks was studied with regard to driver nodes, i.e. nodes which can offer full control over the network. The authors are in particular interested in finding the minimum number of driver nodes. In contrast to this work, the network control system is linear. Another approach was made by Tanner [53] to study the network architecture. In this work, the author establishes necessary and sufficient conditions for a group of interconnected systems via nearest neighbor rules to be controllable by one of them acting as a leader. The underlying graphs of the networks are not directed and the nodes are not equal. But these are just two examples from the wide variety of research done on network controllability.

Problem Statement

Let the network system be composed of N controllable and observable linear subsystems, each represented in the following form

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i v_i(t)\\ y_i &= C_i x_i,\end{aligned}$$

where $x_i \in \mathbb{R}^n$ is the state vector of the i-th node system, $v_i(t)$ is the control input and y is the output. Let the triples (A_i, B_i, C_i) be all of the same size and fixed. Furthermore, we suppose that the interconnections of the network is given in terms of a directed graph Γ with edge set E. According to the interconnection structure of the network, we apply time-dependent output feedback of the form

$$v_i(t) = \sum_{(j,i)\in E} u_{ij}(t) C_j x_j,$$

where the sum is over all nodes j such that there exists an interconnection from j to i. The resulting control system of node i is of the form

$$\dot{x}_i = A_i x_i + B_i \left(\sum_{(i,j)\in E} u_{ij}(t) C_j x_j \right)$$

and the overall control system of the complete network can be described by

$$\dot{x} = (\mathcal{A} + \mathcal{B} U(t) \mathcal{C})\, x. \tag{1.1}$$

where

$$\mathcal{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix} \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_N \end{pmatrix}.$$

The matrix valued control function $U(t)$ reflects the interconnection structure of the network and we assume for the sake of simplicity that it is piecewise constant. Therefore, it can be considered as a controlled adjacency matrix and written as

$$U(t) = \sum_{(i,j)\in E} u_{ij} E_{ij},$$

where $E_{ij} = (e_{kl})_{kl}$ is a matrix with $e_{ij} = 1$ and 0 else. The arising control system is bilinear. Considering the bilinear control system (1.1) we now study the question which network architecture ensures accessibility or, respectively, controllability.

Accessibility of Bilinear Interconnected Systems

The subject of this thesis is to generalize Brockett's results from [5, 6, 7] to networks of linear systems, where output feedback is applied according to the interconnection structure of the network. To do so, we start with a brief overview on control of bilinear systems and state known accessibility and controllability conditions in Chapter 2. In Chapter 3 we present the entire proofs of Brockett's results from [5, 6, 7].
In Chapter 4 and 5 we tackle the problem stated before. We examine the bilinear control system of the network of the form

$$\dot{x} = (\mathcal{A} + \mathcal{B}U(t)\mathcal{C})\, x \tag{1.2}$$

where the vector $x(t) = (x_1(t), \ldots, x_N(t))^\top$ represents the state of the system of N nodes at time t. Here, the matrix-valued control function $U(t)$ reflects the interconnection structure of the network. We distinguish between the case, when all interconnections are independently controllable (Chapter 4) and thus, $U(t)$ has the structure of a controlled adjacency matrix of the underlying graph, and the case, when linear dependencies between the interconnections are allowed (Chapter 5) and thus, $U(t)$ varies through a subspace of $\mathbb{R}^{N\times N}$.

In detail, this thesis is structured as follows:

Chapter 2 - Preliminaries

We give a brief overview on the control of bilinear systems on Lie groups of the form

$$\dot{x} = (A + u(t)B)\, x,$$

where $u(t)$ is a scalar-valued function. To do so, we recall the basic concepts of bilinear control systems and give the definitions of accessibility and controllability. Additionally, we introduce the related notion of bilinear control systems on homogeneous spaces.
Exploiting that accessibility of bilinear control systems is equivalent to that the coefficient matrices A and B generate a certain linear Lie algebra, we state some known necessary and sufficient conditions on A and B to generate the special linear Lie algebra and, respectively, the general linear Lie algebra.
Since the stated results from the literature assume the eigenvalues of B to be real, we proof a further result, which does not need the assumption that the eigenvalues of B are real. Concluding, we present a reformulation of the known accessibility and controllability results for sets of matrices. This we use throughout this thesis.

Chapter 3 - System Lie Algebras of Linear Feedback Systems

In [5], [6] and [7] Brockett studied the behavior of real single-input, single-output systems (SISO systems) of the form

$$\begin{aligned} \dot{x} &= Ax + u(t)b \\ y &= cx \end{aligned}$$

with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$ under time-dependent feedback. Assuming that the inital linear control system is controllable and observable he examined the system Lie algebra of the resulting output feedback systems

$$\dot{x} = (A + u(t)bc)\, x$$

and showed that there are only four possibilities up to conjugation for the system Lie algebra depending on the symmetry properties of the transfer function

$$g(s) = c(sI - A)^{-1}b.$$

In [5] he proved this by determining the dimension of the Lie algebra as a vector space. Moreover in [7], he used techniques from Galois theory and related the study of the generated Lie algebra of A and bc to the study of the polynomial $det(sI - A - ubc)$. Since the proof of the mentioned result in [7] seems to capture only a special case, we close the small gaps and present the entire proof in Section 3.1 and 3.2.
Furthermore in [6], Brockett allowed A and bc to be complex matrices and considered the real generated Lie algebra, where he showed that in this particular case 11 different conjugation types of Lie algebras can occur. The proof in [6] is, as the author describes it, a sketch of proof and the result has a minor defect. Therefore, we present the correct result and the entire proof in Section 3.3.
Concluding in Section 3.4, we give the analogous results for multiple-input, multiple-output systems (MIMO system) of the form

$$\dot{x} = (A + BU(t)C)\, x$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$ in case the triple (A, B, C) is controllable and observable. First, we consider control by output feedback, i.e. we regard the matrix valued control function $U(t)$ to take its values in the vector space of matrices $\mathbb{R}^{p\times p}$. The result was proven in [5]. Second, we consider control by restricted output feedback, i.e. we assume the control function to be of the form $U(t) = u(t)K$ with $K \in \mathbb{R}^{p\times p}$ being a fixed matrix and $u(t)$ being a scalar valued control function. In this setting we can adapt the results from the SISO case for the class of cyclic matrices.

Chapter 4 - Bilinear Control of Networks by Interconnections

We start with networks of single-input, single-output systems (SISO systems) and assume that the interconnections of the network are all independently controllable. Hence, the control function $U(t)$ of the bilinear control system

$$\dot{x} = (\mathcal{A} + \mathcal{B}U(t)\mathcal{C})\, x$$

is precisely the controlled adjacency matrix of the graph which represents the interconnection structure of the network. To study the associated system Lie algebra we first give some basic definitions and results on graphs and relate the graph structure to the structure of its adjacency matrix. To simplify our examinations we restrict our investigations to graphs without self-loops.
The main result of this chapter, Theorem 4.6, gives necessary and sufficient conditions for the control system (1.2) to be accessible in terms of the connectivity of the underlying graph. We use that the accessibility of bilinear control systems is equivalent to that the by the coefficient matrices generated system Lie algebra is big enough. When the network consists of at least 3 nodes we compute the system Lie algebra in case every node is reachable by a path from every other node. This connectivity property is called strong connectedness. An example for a strongly connected graph is shown in Figure 1.3.

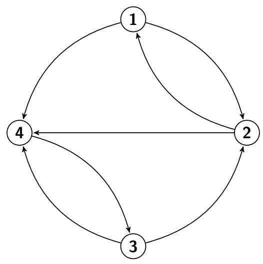

FIGURE 1.3: Example for a strongly connected graph Γ

In this context we prove one controllability result for system (1.2) in case the matrix $\mathcal{A}$ is skew-symmetric (Corollary 4.13). Networks with two nodes are not captured by these proofs. Thus, we give results for this particular case under additional assumption either on the graph structure or on the dynamics of the node systems. We show that the additionally made assumptions are not necessary by an example and pose the conjecture that Theorem 4.6 still holds true for networks with two nodes.
In Section 4.4 we classify the system Lie algebra of networks when the underlying graph is not strongly connected. Preliminary, we study the connection between the Lie algebra generated by all possible values of the controlled adjacency matrix and the existence of directed paths between nodes (Lemma 4.18). In case a network is disconnected the bilinear control system is never accessible and the system Lie algebra is conjugated to the direct sum of the system Lie algebras of the connected components. Therefore, we can

restrict our considerations to connected networks and clarify the structure of the system Lie algebra from a not strongly connected network.
For homogeneous networks, i.e. networks where all node systems are equal, the control system simplifies to

$$\dot{x} = (I_N \otimes A + U(t) \otimes bc)\, x$$

and we can generalize Brockett's result on real system Lie algebras in case the node dynamics are complex. Here, contrary to the preceding chapter, we resort to Brockett's result to compute the network's system Lie algebra.
In the chapter's last section we show exemplarily how to adapt the ideas of the preceding sections to networks of multi-input, multi-output systems (MIMO systems). We derive results for the system Lie algebra on networks of MIMO systems both under output feedback and under restricted output feedback.

Chapter 5 - Bilinear Control of Networks restricted by Subspaces

Contrary to Chapter 4, we allow for linear dependencies between interconnections, self-loops and multiple edges. In terms of the control function $U(t)$ of system (1.2) this means that it is allowed to vary through a subspace of $\mathbb{R}^{N\times N}$. Preliminary, we formulate the main result of Chapter 4 in terms of subspaces.
The new setting demands that we distinguish between homogeneous and heterogeneous networks. Intuitively, we consider the Lie algebra generated by the subspace through which the control function varies. We obtain that, if this generated Lie algebra equals the general linear Lie algebra, accessibility of (1.2) is guaranteed for homogeneous networks. Using results of Chapter 2 on generators of Lie algebras, we derive explicit requirements on the subspace to assure accessibility of the bilinear control system.
By tackling the case of heterogeneous networks we obtain that, in contrast to Chapter 4, the node systems have an influence on the system Lie algebra of (1.2). We introduce the notion of a T-Lie algebra, which is closely related to a Lie algebra. Instead of the Lie bracket, a T-Lie algebra is closed under the commutator

$$[A, B]_T := ATB - BTA$$

for all matrices A, B of the generated Lie algebra, where T is a diagonal matrix. The matrix T depends on all node's dynamics. Substituting Lie algebra through T-Lie algebra we can now adapt the results for homogeneous networks to heterogeneous networks and derive sufficient accessibility conditions.
The last section is dedicated to study networks with a particular interconnection structure. This we do by investigating controllability properties of bilinear control systems defined by classes of Toeplitz matrices. To start we introduce the class of circulant matrices for which the inverse eigenvalue problem is solved, i.e. it is known that for every set of eigenvalues

there exists a corresponding circulant matrix. For instance, the bug problem asks for the behavior of N identical acting agents which pursue each other. Here, the interconnection matrix would be of circulant type. Inspired by the bug problem we consider in the last section control systems with circulant coefficient matrices. Since circulant matrices commute, bilinear systems with circulant coefficient matrices cannot be accessible and hence are never controllable. This leads us to the more general class of pseudo-circulant matrices, which have similar properties like circulant matrices but do not commute. Using results from Chapter 2 we compute the Lie algebra of pseudo-circulant matrices, which coincides with the general linear Lie algebra. This implies accessibility and controllability of the associated driftless bilinear control system. Since pseudo-circulant matrices are a special case of Toeplitz matrices we can immediately transfer the results to Toeplitz matrices and the associated bilinear control systems. When we restrict ourselves to unitary pseudo-circulant matrices we obtain similar results since the Lie algebra generated by all unitary pseudo-circulant matrices is the unitary Lie algebra.

As a by-product we deduce that every complex invertible matrix is the finite product of invertible Toeplitz matrices. Moreover, every complex unitary matrix is shown to be a finite product of complex unitary Toeplitz matrices.

Applying these results to networks we deduce that homogeneous networks with a certain Toeplitz interconnection structure are accessible.

We mention that the results of Section 5.4 are published in [45].

Chapter 2

Preliminaries

In this chapter, we give an overview of the basic definitions and results on bilinear control system with the main focus on accessibility and controllability of bilinear systems. In Section 2.1 we define what we mean by bilinear systems on matrix Lie groups and present the definitions of accessibility and controllability for this particular case. Furthermore, we explain the connection to bilinear control systems on manifolds. The remaining chapter is devoted to give an overview of known accessibility and controllability conditions for bilinear systems.

2.1 Bilinear systems on Lie groups and homogeneous spaces

For the sake of simplicity we assume that G is a connected *matrix Lie group*, i.e. a connected Lie group of the group $GL_n(\mathbb{K})$ of real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) invertible $n \times n$ matrices. Let $\mathcal{L}(G)$ denote its Lie algebra. Clearly, this is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K}) = \mathbb{K}^{n\times n}$ with Lie bracket $[A, B] := AB - BA$. The basic definitions and examples for Lie groups and Lie algebras can be found in Appendix A.

A *bilinear control system on a matrix Lie group* G is defined by

$$\dot{X} = \left(A + \sum_{i=1}^{m} u_i(t) B_i \right) X,\ \ X \in G \tag{2.1}$$

where the constant matrices A and $B_1, \ldots, B_m \in \mathcal{L}(G)$ are given. The controls $u_1, \ldots, u_m \in U$ are piecewise constant functions with values in $\mathbb{K}$. The term AX is called the *drift term* of (2.1) and the matrices $B_1, \ldots, B_m$ are called the *control directions*. In this thesis, we only consider bilinear systems where the control set U is not restricted, i.e. the controls can take arbitrary values in $\mathbb{R}$ and $\mathbb{C}$, respectively. For the sake of simplicity, we assume that the set U contains at least all piecewise constant control functions. The

control system (2.1) is called *right-invariant* since the vector fields AX and B_iX are right-invariant. For more details on right-invariant vector fields see [28].
A *trajectory* of the bilinear system (2.1) on G is a continuous curve $X(t)$ in G defined on an interval $[a,b] \subset \mathbb{R}$ such that there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ where the restriction of $X(t)$ to each open interval (t_{i-1}, t_i) is differentiable and $\dot{X}(t) = A_iX(t)$ for $t \in (t_{i-1}, t_i)$ for all $i = 1, \ldots, n$. Here, A_i is the linear combination of $A, B_1, \ldots, B_m$ corresponding to the control $u \in \mathbb{K}^m$, which is constant for $t \in (t_{i-1}, t_i)$. For any $T \geq 0$ and any $X \in G$ the *reachable set at time* T of the bilinear system (2.1) from the point X_0 is the set $\mathcal{R}_T(X_0)$ of all points $X \in G$, which can be reached from X_0 in exactly T units of time, i.e.

$$\mathcal{R}_T(X_0) := \left\{ X(T) \;\middle|\; X(\cdot) \text{ is a trajectory of (2.1)}, X(0) = X_0 \right\}.$$

The *reachable set* of system (2.1) from a point $X_0 \in G$ is the set $\mathcal{R}(X_0)$ of all terminal points $X(T)$ with $T \geq 0$ of all trajectories of (2.1) starting at X_0, i.e.

$$\mathcal{R}(X_0) := \left\{ X(T) \;\middle|\; X(\cdot) \text{ is a trajectory of (2.1)}, X(0) = X_0, T \geq 0 \right\} = \bigcup_{T \geq 0} \mathcal{R}_T(X_0).$$

The right invariance of (2.1) implies $\mathcal{R}(X_0) = \mathcal{R}(I)X_0$ and it is easily seen that $\mathcal{S} := \mathcal{R}(I)$ defines a subsemigroup of G, which we refer to as the *system semigroup associated to the bilinear system* (2.1). Moreover, the smallest subgroup $\mathcal{G}$ of G that contains $\mathcal{S}$ is called the *system group* of (2.1) and

$$\mathcal{G} = \left\langle \exp\left(t(A + \sum_{i=1}^{m} u_i B_i) \right) \;\middle|\; u \in U, t \in \mathbb{R} \right\rangle$$

holds. Here, $\langle A, B \rangle$ denotes the smallest group which contains the matrices A and B. Clearly, we get

$$\mathcal{S} = \left\langle \exp\left(t(A + \sum_{i=1}^{m} u_i B_i) \right) \;\middle|\; u \in U, t \geq 0 \right\rangle.$$

Associated to the system group we define the *system Lie algebra* $\mathfrak{g}$ of (2.1) as the smallest Lie subalgebra of $\mathcal{L}(G)$ containing A and $B_1, \ldots, B_m$, i.e.

$$\mathfrak{g} := \{A, B_1, \ldots, B_m\}_{LA}.$$

A system (2.1) is called *accessible at a point* $X \in G$ if the reachable set $\mathcal{R}(X)$ has nonempty interior in G. The system (2.1) is called *accessible* on G if it is accessible for every $X \in G$. For bilinear systems on Lie groups it is equivalent that the system semigroup has an interior point in G.
A bilinear system (2.1) is called *controllable on* G if, given any pair of points $X_0, X_1 \in G$, the point X_1 can be reached from X_0 along a trajectory of (2.1) for a non-negative time,

i.e.

$$X_1 \in \mathcal{R}(X_0) \text{ for any } X_0, X_1 \in G.$$

This condition is equivalent to $\mathcal{R}(X) = G$ for all $X \in G$.

Now, let M be a connected smooth manifold and $\theta : G \times M \to M, (X,p) \mapsto X \cdot p$ be a smooth group action of G on M.

Definition 2.1. A Lie group G *acts on a manifold* M if there exists an analytic smooth mapping $\theta : G \times M \to M$ that fulfills the following conditions:

(i) $\theta(X_2X_1, x) = \theta(X_2, \theta(X_1, p))$ for any $X_1, X_2 \in G$ and any $p \in M$;

(ii) $\theta(e, p) = p$ for any $p \in M$.

The map θ is then called the *group action of* G *on* M and the set $\theta(p) := \{\theta(X,p) \mid X \in G\}$ is called the *orbit of* p. A Lie group G *acts transitively* on a manifold M if $\theta(p) = M$ holds for any $p \in M$. A *homogeneous space* of a Lie group G is a manifold which admits a transitive action of G. For example, the euclidean space $\mathbb{R}^n \backslash \{0\}$ is a homogeneous space of $GL_n^+(\mathbb{R})$ or $SL_n(\mathbb{R})$. Via the Lie group action θ, the one-parameter group $\exp(tA(u))$ of (2.1) on G with $A(u) := A_0 + \sum_{i=1}^m u_i B_i$ naturally induces a flow $p_0 \mapsto \exp(tA(u)) \cdot p_0$ on M. Then θ induces a bilinear control system on M of the form

$$\dot{p} = D_1\theta(I,p)\left(A + \sum_{i=1}^m u_i(t)B_i\right) = \left(A + \sum_{i=1}^m u_i(t)B_i\right)p, \ p \in M \tag{2.2}$$

where $D_1\theta$ denotes the tangent map of θ with respect to the first component. Conversely, system (2.1) is called the *group lift* of system (2.2). One has the following relation between solutions of (2.1) and (2.2).

Lemma 2.2 ([16]). *Let $u : [0,T] \to U$ be any piecewise constant control and let $X : [0,T] \to G$ be the corresponding unique solution of (2.1). Then $p(t) := \theta(X(t), p_0)$ is a solution of (2.2). Moreover, any trajectory of (2.2) with piecewise constant controls can by obtained in this way.*

By the Lie group action θ we get for the reachable sets of (2.2)

$$\mathcal{R}(p_0) = \mathcal{S} \cdot p_0$$

for $p_0 \in M$. The definition of accessibility and controllability can be immediately assigned to systems of the form (2.2): System (2.2) is called *accessible* if all reachable sets have non-empty interior and *controllable* if all reachable sets coincide with the entire state space.

Theorem 2.3 ([16]). *Let $\mathcal{G}$ be the system group of system* (2.1) *and M a connected smooth manifold. Then the following statements are equivalent:*

(i) The induced system (2.2) *on M is accessible.*

(ii) The group $\mathcal{G}$ acts transitively on M.

In the sequel we concentrate on induced bilinear systems on the manifold $\mathbb{R}^n \setminus \{0\}$ of the form

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i(t) B_i \right) x, \ \ x \in \mathbb{R}^n \setminus \{0\}. \tag{2.3}$$

One has the following accessibility criterion for bilinear systems on $\mathbb{R}^n \setminus \{0\}$.

Theorem 2.4 ([3, 36]). *A bilinear system (2.3) on $\mathbb{R}^n \setminus \{0\}$ is accessible if and only if its system Lie algebra $\{A, B_1, \dots, B_m\}_{LA}$ is conjugated to one of the following types:*

(1) $\mathfrak{so}(n) \oplus \mathbb{R}$ *if* $n \geq 2$.

(2) $\mathfrak{su}(n/2) \oplus \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}$ *or* $\mathfrak{su}(n/2) \oplus \mathbb{C}$ *if n is even and* $n \geq 3$.

(3) $\mathfrak{sp}(n/4) \oplus \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}$, $\mathfrak{sp}(n/4) \oplus \mathbb{C}$ *or* $\mathfrak{sp}(n/4) \oplus \mathbb{H}$ *if* $n = 4k$.

(4) $\mathfrak{g}_2 \oplus \mathbb{R}$ *if* $n = 7$.

(5) $\mathfrak{spin}(7) \oplus \mathbb{R}$ *if* $n = 8$.

(6) $\mathfrak{spin}(9) \oplus \mathbb{R}$ *if* $n = 16$.

(7) $\mathfrak{sl}(n, \mathbb{R})$ *or* $\mathfrak{gl}(n, \mathbb{R})$ *if* $n \geq 2$.

(8) $\mathfrak{sl}(n/2, \mathbb{C})$, $\mathfrak{sl}(n/2, \mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ *or* $\mathfrak{gl}(n/2, \mathbb{C})$ *if* $n = 2k$.

(9) $\mathfrak{sl}(n/4, \mathbb{H})$, $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ *or* $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathbb{C}$ *if* $n = 4k$.

(10) $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathfrak{sp}(1)$ *or* $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathbb{H}$ *if* $n = 4k$.

(11) $\mathfrak{sp}(n/2, \mathbb{R})$ *or* $\mathfrak{sp}(n/2, \mathbb{R}) \oplus \mathbb{R}$ *if n is even and* $n \geq 3$.

(12) $\mathfrak{sp}(n/4, \mathbb{C})$, $\mathfrak{sp}(n/4, \mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ *or* $\mathfrak{sp}(n/4, \mathbb{C}) \oplus \mathbb{C}$ *if* $n = 4k$.

(13) $\mathfrak{spin}(9, 1, \mathbb{R})$ *or* $\mathfrak{spin}(9, 1, \mathbb{R}) \oplus \mathbb{R}$ *if* $n = 16$.

Here, α and β have to satisfy $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

A sufficient criterion for controllability of system (2.2) is the following.

Theorem 2.5 ([29]). *If a right invariant system*

$$\dot{X} = \left(A + \sum_{i=1}^{m} u_i B_i\right) X, \quad X \in G$$

is controllable on a linear group G that acts transitively on $\mathbb{R}^n \backslash \{0\}$, then the bilinear system

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i B_i\right) x, \quad x \in \mathbb{R}^n \backslash \{0\} \tag{2.4}$$

is controllable on $\mathbb{R}^n \backslash \{0\}$.

2.2 Controllability and accessibility conditions for bilinear systems

The most fundamental characterization for accessibility of a bilinear system was found by Jurdjevic and Sussmann in 1972.

Theorem 2.6 ([52]). *A bilinear system of the form*

$$\dot{X} = \left(A + \sum_{i=1}^{m} u_i(t) B_i\right) X, \quad X \in G \tag{2.5}$$

is accessible if and only if

$$\{A, B_1, \ldots, B_m\}_{LA} = \mathcal{L}(G).$$

In general, Theorem 2.6 is only a necessary condition for controllability. But in some cases the condition is sufficient.

Theorem 2.7 ([30]). *A necessary condition for the control system (2.5) to be controllable is that G is connected and $\{A, B_1, \ldots, B_m\}_{LA} = \mathcal{L}(G)$. If G is compact or if $A = 0$, the condition is also sufficient.*

Since the condition of Theorem 2.6 is not sufficient for controllability, we have to add another assumption for the general case.

Theorem 2.8 ([28]). *The system (2.5) is controllable if and only if*

(i) $\mathcal{R}(I) = G$ *is a group and*

(ii) $\{A, B_1, \ldots, B_m\}_{LA} = \mathcal{L}(G)$.

A sufficient condition to guarantee condition (i) is the following.

Theorem 2.9 ([30]). *Let*

$$\dot{X} = \left(A + \sum_{i=1}^{m} u_i B_i \right) X$$

be a right-invariant control system on the Lie group G, which is accessible. If there exists a constant control u and a sequence of positive numbers $\{t_n\}$ with $t_n \geq \epsilon > 0$ for some $\epsilon > 0$, with the property that $\lim x(t)$ exists and belongs to $\bar{S}$, where $x(t)$ is the associated trajectory to u with $x(0) = I$, then $\mathcal{R}(I) = G$.
Here, S denotes the Lie group associated to the Lie algebra $\{B_1, \ldots, B_m\}_{LA}$ and the closure is taken relative to the system group $\mathcal{G}$.

We have characterized accessibility of bilinear control systems both on Lie groups and on manifolds by checking if the coefficient matrices of the bilinear system generate the whole system Lie algebra. Since computing the Lie algebra can be very time-consuming we now present necessary and sufficient conditions on the coefficient matrices $A, B_1, \ldots, B_m$ to generate the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$.

First, we need the following two definitions.

Definition 2.10. A matrix $A \in \mathbb{C}^{n \times n}$ is called *permutation-reducible* if there exists a permutation matrix P such that

$$PAP^\top = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a $r \times r$ square matrix with $0 < r < n$. A $n \times n$ matrix A is called *permutation-irreducible* if it is not permutation-reducible.

For a permutation-irreducible matrix we have the following result. The matrix E_{ij} denotes the single-entry matrix with entry 1 at (i, j) and all other entries zero.

Theorem 2.11. *Let $A = A_0 + \sum_{i \neq j} a_{ij} E_{ij} \in \mathbb{C}^{n \times n}$ be permutation-irreducible with A_0 diagonal matrix and $a_{ij} \in \mathbb{C}$. Then*

$$\{E_{ij} \mid a_{ij} \neq 0\}_{LA} = \mathfrak{sl}_n(\mathbb{R}).$$

Proof. This is Theorem 2 of [47] since $\mathfrak{sl}_n(\mathbb{R})$ is the normal real form of $\mathfrak{sl}_n(\mathbb{C})$ (cf. p. 353 [23]). □

Definition 2.12. A matrix $A \in \mathbb{C}^{n \times n}$ is called *strongly regular* if the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ of A are distinct and satisfy

$$\lambda_i - \lambda_j \neq \lambda_k - \lambda_l$$

for all pairs of distinct indices $(i, j) \neq (k, l)$ with $i \neq j, k \neq l$.

2.2.1 Controllability and accessibility conditions for bilinear systems on $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$

Due to Theorem 2.3 accessibility of bilinear systems on the homogeneous space $\mathbb{R}^n \backslash \{0\}$ is equivalent to the existence of a transitive action of the system group on the manifold. Controllability of (2.2) follows from the existence of a transitive action of the system group on the manifold and that the group lift is controllable due to Theorem 2.8. Hence, the following results can be immediately assigned to systems of the form (2.3) on $\mathbb{R}^n \backslash \{0\}$.

We distinguish between real bilinear system with drift term

$$\dot{X} = AX + uBX \tag{2.6}$$

and real bilinear systems without drift term

$$\dot{X} = uAX + vBX, \tag{2.7}$$

where $u, v \in \mathbb{R}$ and $A, B \in \mathbb{R}^{n\times n}$, i.e. we consider bilinear control systems with either one or two controls in the following. Bilinear systems without drift term are often called *homogeneous* systems in the literature. From Theorem 2.7 it follows that accessibility of system (2.6) is equivalent to controllability of system (2.7).

One of the first results on A and B was proven by Jurdjevic and Kupka.

Theorem 2.13 ([29]). *Assume that* $\operatorname{tr} A = \operatorname{tr} B = 0$ *and* B *is a strongly regular matrix in diagonal form. If* A *satisfies*

(i) $a_{ij} \neq 0$ *for all* $1 \leq i, j \leq n$ *such that* $|i - j| = 1$ *and*

(ii) $a_{1n} \cdot a_{n1} < 0$.

Then the system

$$\dot{X} = (A + uB)\, X$$

is controllable on $SL_n(\mathbb{R})$.

Since the matrix Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ can be written as $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{sl}_n(\mathbb{R}) \oplus \mathbb{R}I$, we can write a matrix B with $\operatorname{tr} B =: \beta \neq 0$ as $B = B_0 + \beta I$, where $\operatorname{tr} B_0 = 0$ and thus $B_0 \in \mathfrak{sl}_n(\mathbb{R})$. Further, we have $[A, B] = [A, B_0]$ for every matrix $A \in \mathfrak{gl}_n(\mathbb{R})$. Hence, the subalgebra $\mathfrak{k} := \{A, [A, B]\}_{LA}$, which is closed under taking the commutator with B, is an ideal in $\{A, B_0\}_{LA}$. Suppose, A and B_0 satisfy the conditions of Theorem 2.13. Then $\{A, B_0\}_{LA} = \mathfrak{sl}_n(\mathbb{R})$ and since $\mathfrak{sl}_n(\mathbb{R})$ is simple we derive $\mathfrak{k} = \mathfrak{sl}_n(\mathbb{R})$. Thus, the Lie algebra $\{A, B\}_{LA}$ differs only by multiples of the identity from the Lie algebra $\{A, B_0\}_{LA}$ since $\operatorname{tr} B \neq 0$. In case B_0 is strongly regular, it immediately follows that B is strongly regular,

too, since for the characteristic polynomial it holds $\chi_B(\lambda) = \chi_{B_0}(\lambda - \beta)$ and therefore all eigenvalues are just shifted by β.
Therefore, Theorem 2.13 yields a controllability result for control systems on $GL_n^+(\mathbb{R})$.

Corollary 2.14 ([29]). *Assume that the diagonal matrix B is strongly regular with $\operatorname{tr} B \neq 0$. If A satisfies conditions (i) and (ii) of Theorem 2.13, then system*

$$\dot{X} = (A + uB)\, X$$

is controllable on $GL_n^+(\mathbb{R})$.

In case we consider homogeneous control systems of the form (2.7) less assumptions are necessary to obtain controllability.

Theorem 2.15 ([29]). *Assume that $\operatorname{tr} A = \operatorname{tr} B = 0$ and B is a strongly regular diagonal matrix. If A satisfies $a_{ij} \neq 0$ for all i, j with $|i - j| = 1$ then system*

$$\dot{X} = (uA + vB)\, X$$

is controllable on $SL_n(\mathbb{R})$.

In [18] Gauthier and Bornard assumed that $B \in \mathbb{R}^{n\times n}$ is a diagonal matrix, which is strongly regular. Then they obtained necessary and sufficient conditions for A such that system (2.6) is controllable on $SL_n(\mathbb{R})$, which are more general than the conditions of Theorem 2.13.

Theorem 2.16 ([18]). *Suppose $\operatorname{tr} B = \operatorname{tr} A = 0$, B is a strongly regular diagonal matrix and the entries of the matrix A satisfy $a_{1n} \cdot a_{n1} > 0$. Then system*

$$\dot{X} = (A + uB)\, X$$

is controllable on $SL_n(\mathbb{R})$ if and only if A is permutation-irreducible.

Again, for the driftless control system less assumptions are sufficient.

Theorem 2.17 ([18]). *Suppose $\operatorname{tr} B = \operatorname{tr} A = 0$ and B is a strongly regular diagonal matrix. Then system*

$$\dot{X} = (uA + vB)\, X$$

is controllable on $SL_n(\mathbb{R})$ if and only if A is permutation-irreducible.

With Theorem 2.8 the following is an immediate consequence from Theorem 2.17.

Corollary 2.18. *Suppose $\operatorname{tr} A = \operatorname{tr} B = 0$ and B is a strongly regular diagonal matrix. Then, the system Lie algebra*

$$\{A, B\}_{LA} = \mathfrak{sl}_n(\mathbb{R})$$

if and only if A is permutation-irreducible.

The assumption that the matrix B is in diagonal form puts a restriction on the eigenvalues of B since all have to be real. We prove a more general result which allows the eigenvalues of B to be complex.

Theorem 2.19. *Let $A, B \in \mathbb{R}^{n\times n}$ with B strongly regular. If there exists a matrix $S \in GL_n(\mathbb{C})$ such that*

(i) SAS^{-1} is permutation-irreducible and

(ii) SBS^{-1} is a diagonal matrix,

then the real system Lie algebra $\{A, B\}_{LA}$ is either equal to $\mathfrak{sl}_n(\mathbb{R})$ or equal to $\mathfrak{gl}_n(\mathbb{R})$.

Proof. Let $SAS^{-1} = A_0 + \sum_{i,j} a_{ij}E_{ij}$ with A_0 diagonal and $a_{ij} \in \mathbb{C}$. Since B is strongly regular, SBS^{-1} is strongly regular as well and therefore

$$\operatorname{span}_{\mathbb{R}}\{a_{ij}E_{ij} \mid i \neq j\} \subseteq \operatorname{span}_{\mathbb{R}}\{SAS^{-1}, \operatorname{ad}_{SBS^{-1}}(SAS^{-1}), \dots, \operatorname{ad}^k_{SBS^{-1}}(SAS^{-1})\}$$

for some $k \in \mathbb{N}$. Clearly,

$$\begin{aligned}\operatorname{span}_{\mathbb{R}} S^{-1}\{a_{ij}E_{ij} \mid i \neq j\}S &\\ &\subseteq \operatorname{span}_{\mathbb{R}} S^{-1}\{SAS^{-1}, \operatorname{ad}_{SBS^{-1}}(SAS^{-1}), \dots, \operatorname{ad}^k_{SBS^{-1}}(SAS^{-1})\}S \\ &= \operatorname{span}_{\mathbb{R}}\{A, \operatorname{ad}_B(A), \dots, \operatorname{ad}^k_B(A)\} \subseteq \mathfrak{gl}_n(\mathbb{R}).\end{aligned}$$

Hence, the real subspaces $\operatorname{span}_{\mathbb{R}}\{a_{ij}S^{-1}E_{ij}S\}$ are subspaces of $\mathbb{R}^{n\times n}$ for $i \neq j$ and have real dimension 1 if $a_{ij} \neq 0$. By Theorem 2.11 we get

$$\{E_{ij} \mid a_{ij} \neq 0\}_{LA} = \mathfrak{sl}_n(\mathbb{R})$$

since SAS^{-1} is permutation-irreducible. Therefore,

$$\dim_{\mathbb{R}}\{a_{ij}E_{ij} \mid i \neq j\}_{LA} \geq n^2 - 1$$

and hence,

$$\dim_{\mathbb{R}}\{a_{ij}S^{-1}E_{ij}S \mid i \neq j\}_{LA} \geq n^2 - 1.$$

But this implies

$$\mathfrak{sl}_n(\mathbb{R}) \subseteq \{a_{ij}S^{-1}E_{ij}S \mid i \neq j\}_{LA} \subseteq \{A, B\}_{LA}$$

and the result follows. □

In the sequel of this work we will use the following reformulation of Theorem 2.19.

Theorem 2.20. *Suppose that the real matrices $A_1, \dots, A_d$ satisfy the following conditions:*

(i) there exist $u_1, \dots, u_d \in \mathbb{R}$ such that $\sum_{j=1}^d u_j A_j$ is strongly regular;

(ii) $A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$.

Then the real system Lie algebra $\{A_1, \dots, A_d\}_{LA}$ is either equal to $\mathfrak{sl}_n(\mathbb{R})$ or equal to $\mathfrak{gl}_n(\mathbb{R})$.

We need the following lemma for the proof.

Lemma 2.21. *Let the matrices $A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$. Then there exists a linear combination*

$$A := \sum_{i=1}^d u_i A_i,$$

with $u_1, \dots, u_d \in \mathbb{C}$ such that A is permutation-irreducible.

Proof. Let $\mathcal{P}$ be the set of all $n \times n$ permutation matrices. Clearly, $|\mathcal{P}| = n!$ holds. For $P_k \in \mathcal{P}$ we consider the mappings $B_k : \mathbb{R}^d \to \mathbb{R}^{n \times n}$ defined by

$$\Big(u_1, \dots, u_d\Big) \mapsto P_k \left(\sum_{j=1}^d u_j A_j \right) P_k^\top$$

and for $1 \le i, j \le n$ the mappings $l_{ij} : \mathbb{R}^{n \times n} \to \mathbb{R}$ defined by

$$\begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \mapsto a_{ij}.$$

Now, examine for $1 \le r \le n-1$ and $1 \le k \le n!$ the subspaces of $\mathbb{R}^d$

$$\mathrm{Ker}_{r,k} := \bigcap_{\substack{i=n-r+1,\dots,n, \\ j=1,\dots,n-r}} \mathrm{Ker}\, l_{ij}(B_k(\cdot)).$$

Clearly, we have $\dim \mathrm{Ker}_{r,k} < d$ since $A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$. Hence,

$$\bigcup_{r=1}^{n-1} \bigcup_{k=1}^{n!} \mathrm{Ker}_{r,k} \neq \mathbb{R}^d$$

as a finite union of hyperplanes cannot be the whole vector space. Hence, there exists a linear combination of $A_1, \dots, A_d$, which is permutation-irreducible. □

We now give the proof of Theorem 2.20.

Proof. Let $u_1, \dots, u_d \in \mathbb{R}$ as in (i) and let $S \in GL_n(\mathbb{C})$ such that

$$B := S \left(\sum_{j=1}^{d} u_j A_j \right) S^{-1}$$

is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$. Now, we choose A to be a linear combination of $SA_1S^{-1}, \dots, SA_dS^{-1}$ such that A is permutation-irreducible. By Lemma 2.21 such a linear combination exists. With Theorem 2.19 it is shown that the Lie algebra generated by A and B equals $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$. This proves the result. □

2.2.2 Controllability and accessibility conditions for bilinear systems on $SL_n(\mathbb{C})$ and $GL_n(\mathbb{C})$

Complex systems can be regarded as real systems with twice the number of controls and coefficient matrices since we can write every complex matrices uniquely as the sum of a matrix with only real entries and a matrix with only pure imaginary entries. Clearly, we have

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{R}) \oplus \mathrm{i}\mathfrak{gl}_n(\mathbb{R}),$$

where $\mathrm{i} = \sqrt{-1}$ and we can transfer all results on real systems of section 2.2.1 to complex bilinear systems of the form

$$\dot{x} = (A + uB)X, \tag{2.8}$$

where $u \in \mathbb{C}$ and X is in $GL_n(\mathbb{C})$. The same applies to systems on $\mathbb{C}^n \setminus \{0\}$ or $SL_n(\mathbb{C})$.

Theorem 2.22. *Let B be a diagonal matrix, which is strongly regular. Let A be a permutation-irreducible matrix. Then system (2.8) is accessible*

(i) on $SL_n(\mathbb{C})$ if and only if $\operatorname{tr} A = \operatorname{tr} B = 0$;

(ii) on $GL_n(\mathbb{C})$ else.

The proof works in the same manner as in the real case (cf. [18]).

With the remarks from the beginning of this section, we obtain the complex version of Theorem 2.20 from Theorem 2.22. Clearly, Theorem 2.19 still holds true for complex matrices $A, B \in \mathbb{C}^{n \times n}$ when we consider the complex Lie algebra $\{A, B\}_{LA}^{\mathbb{C}}$.

Theorem 2.23. *Suppose that complex matrices $A_1, \dots, A_d$ satisfy*

(i) there exist $u_1, \dots, u_d \in \mathbb{C}$ such that $\sum_{j=1}^{d} u_j A_j$ is strongly regular.

(ii) $A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$.

Then the complex system Lie algebra $\{A_1, \dots, A_d\}_{LA}^{\mathbb{C}}$ is either equal to $\mathfrak{sl}_n(\mathbb{C})$ or equal to $\mathfrak{gl}_n(\mathbb{C})$.

2.2.3 Controllability and accessibility conditions for bilinear systems on general matrix Lie groups

Since we want to present a similar result to Theorem 2.20 for the real Lie algebras $\mathfrak{su}_n(\mathbb{C})$ and $\mathfrak{u}_n(\mathbb{C})$, we examine bilinear control systems on general matrix Lie groups. In [47] Silva Leite and Crouch are concerned with the study of controllability of bilinear system which evolve on certain semisimple Lie groups G. For the definitions of the terms (fundamental) roots, Weyl basis and Cartan subalgebra we refer the reader to [23].

Again, Silva Leite and Crouch distinguish in [47] between systems of the form

$$\dot{X} = (uA + vB)\, X,\ \ X \in G, \tag{2.9}$$

and

$$\dot{x} = (A + uB)\, X,\ \ X \in G, \tag{2.10}$$

with $u, v \in \mathbb{R}$ where G is a connected matrix Lie group $G \subset GL_n(\mathbb{R})$ and A, B are elements of the corresponding matrix Lie algebra $\mathcal{L}(G)$. Now, let $\mathcal{L}(G)$ be a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathcal{L}(G)$ and φ the set of nonzero roots of $\mathcal{L}(G)$ with respect to $\mathfrak{h}$. Every semisimple Lie algebra over $\mathbb{C}$ contains a Cartan subalgebra (Theorem III 4.1. [23]) and all Cartan subalgebras are isomorphic (Theorem II.2.15. [32]). Hence, we can assume without loss of generality that all the matrices in $\mathfrak{h}$ are diagonal. For each $a \in \varphi$ there exists a unique $H_\alpha \in \mathfrak{h}$ with $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, where $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathcal{L}(G)$ (cf. Appendix A). With

$$\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \varphi} \mathbb{R} H_\alpha$$

we get $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{h}_{\mathbb{R}}$ where $\mathrm{i} = \sqrt{-1}$.
For each $\alpha \in \varphi$ there exists an element $E_\alpha \in \mathcal{L}(G)$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$. Let Δ denote the set of fundamental roots. Let the set

$$\{H_\alpha, \alpha \in \Delta\} \cup \{E_\alpha, \alpha \in \Phi\}$$

be a *Weyl basis* of $\mathcal{L}(G)$ (corresponding to $\mathfrak{h}$) with structure constants

$$\begin{aligned}[E_\alpha, E_{-\alpha}] =& H_\alpha, \\ [H, E_\alpha] =& \alpha(H)E_\alpha \quad \forall H \in \mathfrak{h} \\ [E_\alpha, E_\beta] =& \begin{cases} 0, & \text{if } \alpha+\beta \notin \varphi \\ N_{\alpha,\beta}E_{\alpha+\beta}, & \text{if } \alpha+\beta \in \varphi, \end{cases}\end{aligned}$$

where $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ are constants. The subspace

$$\mathcal{L} = \sum_{\alpha\in\varphi} \mathbb{R}H_\alpha \oplus \sum_{\alpha\in\varphi} \mathbb{R}E_\alpha \tag{2.11}$$

is a normal real form of $\mathcal{L}(G)$, which is unique up to isomorphism (Theorem IX 5.10 [23]) and it can be described in terms of the Weyl basis (cf. (2.11)). For the definition of a normal real form see Chapter IX.5 [23]. Define

$$\mathfrak{h}_\mathbb{R} := \sum_{\alpha\in\varphi} \mathbb{R}H_\alpha. \tag{2.12}$$

Since the sum of (2.11) is direct every $A \in \mathcal{L}$ admits a unique decomposition of the form

$$A = A_0 + \sum_{\alpha\in\varphi_a} k_\alpha E_\alpha, \tag{2.13}$$

where $\varphi_a \subset \varphi$, $k_a \in \mathbb{R}\backslash\{0\}$ and $A_0 \in \mathfrak{h}_\mathbb{R}$. The subset $\varphi_a \subset \varphi$ is chosen in the way such that $k_\alpha \neq 0$ in (2.13) for all $\alpha \in \varphi_a$. The notion of strongly regular matrices (cf. Definition 2.12) can be extended to elements of general Lie algebras.

Definition 2.24. An element $B \in \mathfrak{h}$ is called *strongly regular* if

(i) B is regular, i.e. the elements $\alpha(B)$ are nonzero for all $\alpha \in \varphi$;

(ii) every nonzero eigenvalue of $\mathrm{ad}_B(\cdot)$ is simple.

Leite and Crouch introduced the notion of A-strong regularity in [47].

Definition 2.25. Given $A \in \mathcal{L}$ with $A \notin \mathfrak{h}_\mathbb{R}$ and $B \in \mathfrak{h}_\mathbb{R}$, then the element B is called *A-strongly regular* if the elements $\alpha(B)$, $\alpha \in \varphi_a$ are nonzero and distinct.

The distinction between *strongly regular* and *A-strongly regular* is that an element $B \in \mathfrak{h}_\mathbb{R}$ is called A-strongly regular if only the elements $\alpha(B)$ are nonzero and distinct for all roots $\alpha \in \varphi_a$. Hence, every strongly regular element $B \in \mathfrak{h}_\mathbb{R}$ is A-strongly regular.
With the notion of an A-strongly regular element Silva Leite and Crouch weaken the assumptions of Corollary 2.18 for certain systems and replace it by one depending on the other generator A.

Theorem 2.26 ([47]). *Let $A = A_0 + \sum_{\alpha\in\varphi_a} k_\alpha E_\alpha \in \mathcal{L}$ and $B \in \mathfrak{h}_\mathbb{R}$ be A-strongly regular. Then the real Lie algebra $\{A, B\}_{LA} = \mathcal{L}$ if and only if A is permutation-irreducible.*

The Lie algebra $\mathcal{L}$ is defined by (2.11) and $\mathfrak{h}_\mathbb{R}$ is defined by (2.12). Note that in this setting the claim $B \in \mathfrak{h}_\mathbb{R}$ implies that B is diagonal. Silva-Leite and Crouch broaden the results of Gauthier and Bornard in [47] to normal forms of classical complex Lie algebras.

Corollary 2.27. *Let $\mathcal{L}$ be a normal real form of any complex Lie algebra of type $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ or $\mathcal{D}_n$, $A = A_0 + \sum_{\alpha\in\varphi_a} k_\alpha E_\alpha \in \mathcal{L}$ and $B \in \mathfrak{h}_\mathbb{R}$ be A-strongly regular. Then system (2.9) is controllable if and only if A is permutation-irreducible.*

This result immediately follows from Theorem 2.26 since for driftless systems (2.9) controllability is equivalent to $\{A, B\}_{LA} = \mathcal{L}(G)$ and that the classical Lie algebras have a normal real form since they are semisimple. For the definitions of the classical Lie algebras of type $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ or $\mathcal{D}_n$ see [32].

For systems of the form (2.10) they were able to prove the following condition.

Theorem 2.28 ([47]). *Let $\mathcal{L}$ be a normal real form of a complex simple Lie algebra of type $\mathcal{A}_n$ or $\mathcal{D}_n$. Let $A = A_0 + \sum_{\alpha\in\varphi_a} k_\alpha E_\alpha$ satisfy $\langle k_s E_s, k_{-s} E_{-s}\rangle < 0$ for $s = \sup\{\alpha : \alpha \in \varphi\}$ and $B \in \mathfrak{h}_\mathbb{R}$ be A-strongly regular. Then system (2.10) is controllable if and only if A is permutation-irreducible.*

Here, $\langle\cdot,\cdot\rangle$ is the Killing form of $\mathcal{L}(G)$.

Silva Leite extended in [46] the results to real forms of classical semisimple Lie algebras, which are compact. Every classical Lie algebra $\mathcal{L}(G)$ over $\mathbb{C}$ contains a compact real form (Theorem III 6.3 of [23]), which is unique up to isomorphisms. In terms of the Weyl basis the compact real form can be written as

$$\mathcal{L} = \sum_{\alpha\in\varphi} \mathbb{R}(iH_\alpha) \oplus \sum_{\alpha\in\varphi} \mathbb{R}X_\alpha \oplus \sum_{\alpha\in\varphi} \mathbb{R}Y_\alpha, \tag{2.14}$$

where $X_\alpha = E_\alpha - E_{-\alpha}$ and $Y_\alpha = i(E_\alpha + E_{-\alpha})$. For the definition of a compact real form see Chapter III.6. of [23]. Since Cartan subalgebras are isomorphic the sum $\sum_{\alpha\in\varphi} \mathbb{R}(iH_\alpha) = i\mathfrak{h}_\mathbb{R}$ is a Cartan subalgebra of $\mathcal{L}$, too.

Theorem 2.29 ([47]). *Let $\mathcal{L}$ be a compact real form of any classical Lie algebra $\mathcal{L}(G)$ over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathcal{L}(G)$, φ the set of nonzero roots of $\mathcal{L}(G)$ with respect to $\mathfrak{h}$. If*

$$A = A_0 + \sum_{\alpha\in\varphi_r} e_\alpha X_\alpha + \sum_{a\in\varphi_c} f_\alpha Y_\alpha \in \mathcal{L}$$

($A_0 \in i\mathfrak{h}, e_\alpha, f_\alpha \in \mathbb{R}^n\backslash\{0\}$) is permutation-irreducible, then

$$\{X_\alpha, Y_\alpha, \alpha \in \varphi_a = \varphi_r \cup \varphi_c\}_{LA} = \mathcal{L}. \tag{2.15}$$

Here, φ_r and φ_c are defined such that $e_\alpha \neq 0$ for $\alpha \in \varphi_r$ and $f_\alpha \neq 0$ for $\alpha \in \varphi_c$.

Since $\mathfrak{su}_n(\mathbb{C})$ is the compact real form of $\mathfrak{sl}_n(\mathbb{C})$ we can reformulate Theorem 2.29 similar to Theorem 2.20 to the following theorem.

Theorem 2.30. *Suppose that the skew-Hermitian matrices $A_1, \dots, A_d$ satisfy*

(i) there exist $u_1, \dots, u_d \in \mathbb{R}$ such that $\sum_{j=1}^d u_j A_j$ is strongly regular;

(ii) $A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$.

Then the real system Lie algebra $\{A_1, \dots, A_d\}_{LA}$ is either equal to $\mathfrak{su}_n(\mathbb{C})$ or equal to $\mathfrak{u}_n(\mathbb{C})$.

We demand that the linear combination $\sum_{j=1}^d u_j A_j$ is strongly regular in the sense of Definition 2.12.

Chapter 3

System Lie Algebras of Linear Feedback Systems

System Lie algebras of control systems are important for the study in many areas. For instance, the controllability of linear feedback systems of the form

$$\dot{x} = (A + u(t)bc)\,x,$$

where A is a $n \times n$ matrix, c and b are row and column vectors, respectively, and u is a real-valued control function, depends in a crucial way on the Lie algebra generated by the matrices A and bc. Here, A and bc can be either real or complex. The generated matrix Lie algebra depends - up to a conjugation - only on the transfer function $g(s) = c(sI - A)^{-1}b$ of the control system in case the triple (A, b, c) is controllable and observable. Therefore, it is essential for understanding controllability of the control system above to get an idea which transfer function corresponds to which Lie algebra.

In [5] Brockett studied the behavior of a single-input single-output linear system (SISO system) under feedback and showed that there are only four possibilities for the resulting system Lie algebra up to conjugation. In [7] Brockett proved the same result with techniques of Galois theory. Moreover, in [6] Brockett allowed A and bc to be complex and considered the real Lie algebra $\{A, bc\}_{LA}$, where he showed that in this particular case 11 types of Lie algebras can occur. Since the proof of the mentioned results of [7] has small gaps, in [6] the result has a small defect and the proof is, as the author describes, only a sketch of proof, we give the complete proofs in this chapter.

Chapter 3 is organized as follows: In Section 3.1 we discuss basic properties of real linear feedback systems. Subsequently, we show in Section 3.2 that we can use the results from Appendix B on irreducible polynomials to determine the system Lie algebra of SISO feedback systems. In 3.3 we consider the generated real system Lie algebra for the case that the SISO system has complex coefficient matrices. Concluding, we extend the results to MIMO systems (multiple-input, multiple-output) in Section 3.4.

3.1 Real linear feedback systems

In this section we deal with linear feedback systems of the form

$$\dot{x} = (Ax + u(t)bc)\, x$$

with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$ and u a real valued control function, which is piecewise constant. We relate these systems to the results of Appendix B.

Definition 3.1. An n-dimensional triple (A, b, c) with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$ is called *controllable and observable* if (A, b) is controllable and (A, c) is observable.

Here, the pair (A, b) is called *controllable* if

$$\operatorname{rank}\begin{pmatrix} b & Ab & \dots & A^{n-1}b \end{pmatrix} = n$$

and the pair (A, c) is called *observable* if

$$\operatorname{rank}\begin{pmatrix} c \\ cA \\ \dots \\ cA^{n-1} \end{pmatrix} = n.$$

For the sequel of the chapter, we assume that the triple (A, b, c) is in the so called *controller form* (cf. [49])

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ -q_0 & -q_1 & -q_2 & \dots & -q_{n-1} \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} p_0 & p_1 & \dots & p_{n-1} \end{pmatrix}. \tag{3.1}$$

The following theorem allows us to limit our investigations to systems of this form (3.1).

Theorem 3.2 ([49]). *Every controllable system* $(A, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$ *is similar to*

$$(A^\star, b^\star) := \left(\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & \ddots & \\ \vdots & & & 1 \\ -q_0 & -q_1 & \cdots & -q_{n-1} \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$$

with $\det(sI - A) = s^n + \sum_{i=0}^{n-1} q_i s^i$.

For all values of $q_0, \ldots, q_{n-1}$ we see that (A, b) as in (3.1) is controllable. Let $p_0, \ldots, p_{n-1}$ be such that (A, c) is observable. The state-space isomorphism theorem ([49]) yields in case (A', b', c') is another controllable and observable triple with $c'(sI - A')^{-1}b' = c(sI - A)^{-1}b$, then there exists a nonsingular matrix P such that $A = PA'P^{-1}$, $b' = Pb$ and $c' = cP^{-1}$. Hence, the transfer function determines the Lie algebra $\{A, bc\}_{LA}$ up to conjugation.

For the *transfer function* $g(s) = c(sI - A)^{-1}b$ of a controllable and observable triple (A, b, c) we get

$$g(s) = \frac{p(s)}{q(s)},$$

where $q(s) = s^n + q_{n-1}s^{n-1} + q_{n-2}s^{n-2} + \ldots + q_0$ and $p(s) = p_{n-1}s^{n-1} + \ldots + p_0$ with $q_{n-1}, \ldots, q_0$ and $p_{n-1}, \ldots, p_0$ defined in the controller form (3.1). The characteristic polynomial of $A + ubc$ is given by

$$\det(sI - A + ubc) = q(s) + up(s),$$

where q and p are defined as above.

Definition 3.3. The *relative degree* α of a transfer function

$$g(s) = \frac{p(s)}{q(s)}$$

is defined as the difference $\alpha := n - m$ with $n := \deg q(s)$ and $m := \deg p(s)$.

Let $p(s) \in F[s]$ be a monic polynomial of the form

$$p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_0,$$

where F denotes a field of characteristic 0 and denote all roots of $p(s)$ by s_i for $1 \le i \le n$ counted with multiplicity. Dependent on $p(s)$ we define the two following polynomials

$$[p(s)]_{(r)} := \prod_{1\le i_1 < i_2 < \ldots < i_r \le n} (s - (s_{i_1} + s_{i_2} + \ldots + s_{i_r}))$$

and

$$\mathrm{ad}[p(s)] := \prod_{1\leq i,j\leq n} (s - (s_i - s_j))\,.$$

Properties of the polynomials $[p(s)]_{(r)}$ and $\mathrm{ad}[p(s)]$ can be found in Appendix B.

Remark 3.4. When $q(s)+up(s) = q(-s)+up(-s)$ holds for all $u \in \mathbb{R}$, we derive that $q(s)$ and $p(s)$ have to be even. Hence, $g(s) = p(s)/q(s)$ is an even function as well. In general, the converse is not true: If $p(s), q(s)$ are odd polynomials, then $g(s)$ is clearly an even function in s. But, by Lemma B.6, we know that if $[q(s) + up(s)]_{(2)}$ has repeated roots, it follows that $q(s + \sigma/2) + up(s + \sigma/2)$ is an even polynomial, too. Therefore, the case $q(s) + up(s) = -q(-s + \alpha) - up(-s + \alpha)$ cannot occur for any $\alpha \in \mathbb{R}$. Consequently, for our setting applies that

$$g(s) = \frac{p(s)}{q(s)} = g(-s)$$

is equivalent to $q(s) + up(s) = q(-s) + up(-s)$.

Note that it is equivalent to consider the equation

$$g(s) = g(-s + \alpha_1) \tag{3.2}$$

for one $\alpha_1 \in \mathbb{R}$ or the equation

$$g(s + \alpha_2) = g(-s + \alpha_2) \tag{3.3}$$

for one $\alpha_2 \in \mathbb{R}$: Shift s in (3.2) to $s+\alpha_1/2$, then we obtain equality (3.3) with $\alpha_2 = \alpha_1/2$.

Remark 3.5. Since we only consider triples (A, b, c) which are controllable and observable, we derive from (A, c) being observable that either q_0 or p_0 or both in (3.1) have to be unequal zero. As

$$\det(A + ubc) = q(0) + up(0) = q_0 + up_0,$$

the matrix $A + ubc$ is invertible for all $u \in \mathbb{R}$ with maximum one exception.

Let $F(u)$ denote a field of rational functions in u over F with $F = \mathbb{R}$ or $F = \mathbb{C}$ and let $F(u)[s]$ denote the polynomial in s having coefficients in $F(u)$. The monic polynomial

$$\det(sI - A + ubc) = q(s) + up(s)$$

belongs to $F(u)[s]$.

Now, we apply the results from Appendix B. We start with some well-known results on controllable and observable triples (A, b, c), which we use without comment in what follows.

Lemma 3.6 ([13])**.** *Let (A, b, c) be an n-dimensional triple such that*

$$\det(sI - A + ubc) = q(s) + up(s).$$

Then (A, b, c) is controllable and observable if and only if $p(s)$ and $q(s)$ are coprime.

Lemma 3.7 ([8])**.** *Let (A, b, c) be an n-dimensional triple and $p(s)$ and $q(s)$ coprime. Then (A, b, c) is controllable and observable if and only if A and bc act irreducibly on $\mathbb{R}^n$.*

We prove an interesting result for the multiplicity of the eigenvalues of the matrix $A+ubc$.

Lemma 3.8. *Let (A, b, c) be controllable and observable. Then the matrix $A + ubc$ is diagonalizable for almost all $u \in \mathbb{R}$.*

Proof. The characteristic polynomial of $A + ubc$ is of the form

$$\chi_{A+ubc}(s) = q(s) + up(s) = s^n + q_{n-1}s^{n-1} + \ldots + q_o + u(p_{n-1}s^{n-1} + \ldots + p_0).$$

Due to the controllability and observability of (A, b, c), the polynomials $p(s)$ and $q(s)$ are coprime as elements of $F[s]$. Assume that $\chi_{A+ubc}(s)$ has a multiple root s_0. Hence,

$$q(s_0) + up(s_0) = 0 \tag{3.4}$$

$$\text{and} \quad q'(s_0) + up'(s_0) = 0. \tag{3.5}$$

Because of the coprimeness of p and q, s_0 cannot be a zero of $p(s)$ or $q(s)$. Therefore, we can solve (3.4) for u and insert it in (3.5). We get

$$q(s_0)p'(s_0) - q'(s_0)p(s_0) = 0.$$

The degree of the polynomial $q(s_0)p'(s_0) - q'(s_0)p(s_0)$ is smaller or equal than $2n - 2$ and so there are maximal $2n - 2$ possible values for s_0 and hence for $u = -\frac{q(s_0)}{p(s_0)}$. Thus, $\chi_{A+ubc}(s)$ has for almost all values of u distinct zeros and the matrix $A+ubc$ is for almost all values of u diagonalizable. □

An important but simple property of controllable and observable triples is the following, whereof we will make extensive use in the subsequent chapters. It is a direct consequence of (A, b, c) being controllable and observable. The proof is obvious.

Lemma 3.9. *Let (A, b, c) be controllable and observable. Then there exists an integer $i \in \{0, \ldots, n-1\}$ such that*

$$cA^ib \neq 0.$$

The smallest integer $m \geq 0$ with this property equals the relative degree of the transfer function $g(s) = c(sI - A)^{-1}b$ of the controllable and observable triple (A, b, c) (cf. Definition 3.3) and is denoted by m^* in the sequel.

Lemma 3.10. *Let (A, b, c) be controllable and observable. Then the following statements hold:*

(i) if A is invertible, then (A, Ab) is controllable;

(ii) $(A + ubc, b)$ is controllable for generic $u \in \mathbb{R}$;

(iii) $(A + ubc, (A + ubc)b)$ is controllable for generic $u \in \mathbb{R}$;

(iv) if $cb = 0$, then $(A + ubc, Ab)$ is controllable for generic $u \in \mathbb{R}$.

Proof. Let A be invertible. Then we get

$$\begin{aligned}\dim \operatorname{span}\{b, Ab, \ldots, A^{n-1}b\} &= \dim A \operatorname{span}\{b, Ab, \ldots, A^{n-1}b\} \\ &= \dim \operatorname{span}\{Ab, A^2b, \ldots, A^n b\} = n.\end{aligned}$$

This proves (i). Clearly, (A, b) being controllable is defined as

$$\operatorname{rank}\begin{pmatrix} b & Ab & \ldots & A^{n-1}b \end{pmatrix} = n$$

and therefore, $\det \begin{pmatrix} b & Ab & \ldots & A^{n-1}b \end{pmatrix} \neq 0$. The mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$u \mapsto \det \begin{pmatrix} b & (A + ubc)b & \ldots & (A + ubc)^{n-1}b \end{pmatrix}$$

is a polynomial in $u \in \mathbb{R}$ with maximum degree $(n - 1)!$. Apparently, $\varphi \not\equiv 0$ as $\varphi(0) \neq 0$. Using the fundamental theorem of algebra φ has maximum $(n - 1)!$ zeros in $\mathbb{R}$. Hence, $\varphi(u) \neq 0$ for generic $u \in \mathbb{R}$ and $(A + ubc, b)$ is controllable for generic $u \in \mathbb{R}$. This proves (ii). Since $A + ubc$ is invertible for almost all $u \in \mathbb{R}$ (Remark 3.5), statement (iii) directly follows with (i) from (ii). For $cb = 0$ we derive

$$(A + ubc, (A + ubc)b) = (A + ubc, Ab).$$

This proves the result. □

The dual version holds as well.

Lemma 3.11. *Let (A, b, c) be controllable and observable. Then the following statements hold:*

(i) if A is invertible, then (A, cA) is observable;

(ii) $(A + ubc, c)$ is observable for generic $u \in \mathbb{R}$;

(iii) $(A + ubc, c(A + ubc))$ *is observable for generic* $u \in \mathbb{R}$*;*

(iv) *if* $cb = 0$*, then* $(A + ubc, cA)$ *is observable for generic* $u \in \mathbb{R}$*.*

3.2 Lie algebras of real SISO feedback systems

This section is devoted to determine the system Lie algebra of control systems of the form

$$\dot{x} = (A + u(t)bc)x.$$

We denote by $\{A, bc\}_{LA}$ the real Lie algebra generated by A and bc. It is conspicuous that the system Lie algebras depends in a crucial way on if the relative degree of the transfer function

$$g(s) = \sum_{i=0}^{\infty} \left(cA^i b\right) s^{-i-1}$$

is even or odd. Later in this section we show that in case the relative degree is at least 1, i.e. $cb = 0$, we can limit our observations to Lie algebras $\{A, bc\}_{LA}$ with $cb = 0$ and $cAb \neq 0$, i.e. relative degree 1.

In case $cb \neq 0$ the system Lie algebra is easy to compute as the following theorem shows.

Theorem 3.12. *Let* (A, b, c) *be a controllable and observable triple with* $cb \neq 0$*. Then,*

$$\{A, bc\}_{LA} = \mathfrak{gl}_n(\mathbb{R}).$$

We first prove a lemma which provides us with a necessary and sufficient condition for the Lie algebra $\{A, bc\}_{LA}$ to be $\mathfrak{gl}_n(\mathbb{R})$.

Lemma 3.13. *Let* (A, b, c) *be controllable and observable. Then*

$$\{A, bc\}_{LA} = \mathfrak{gl}_n(\mathbb{R})$$

if and only if $A^i bcA^j \in \{A, bc\}_{LA}$ *for* $i, j = 0, \ldots, n-1$*.*

Proof. Let $A^i bcA^j \in \{A, bc\}_{LA}$ for $i, j = 0, \ldots, n-1$. Since (A, b, c) is controllable and observable, $b, Ab, \ldots, A^{n-1}b$ is a basis of $\mathbb{R}^n$ and $c, cA, \ldots, cA^{n-1}$ is a basis of $\mathbb{R}^{1\times n}$. Hence, there exist linear combinations of $b, Ab, \ldots, A^{n-1}b$ for the unit vectors $e_1, \ldots, e_n \in \mathbb{R}^n$ and of $c, cA, \ldots, cA^{n-1}$ for the transposed unit vectors $e_1^\top, \ldots, e_n^\top \in \mathbb{R}^{1\times n}$, respectively. With $e_i e_j^\top = E_{ij}$ we derive

$$E_{ij} \in \operatorname{span}\{A^k bcA^l \mid k, l = 0, \ldots, n-1\}.$$

for all $i, j = 1, \dots, n$. Since $\{E_{ij} \mid i, j = 1, \dots, n\}$ constitutes a basis of $\mathfrak{gl}_n(\mathbb{R})$ we get $\mathfrak{gl}_n(\mathbb{R}) \subset \{A, bc\}_{LA}$ and consequently $\mathfrak{gl}_n(\mathbb{R}) = \{A, bc\}_{LA}$ holds.
The other direction is obvious. □

We now give the proof for Theorem 3.12.

Proof. We proof by induction on $i + j$ that all matrices of the form $A^i bcA^j$ are elements of the Lie algebra $\{A, bc\}_{LA}$ for all $i, j \in \mathbb{N}$. It is clear that $A^0 bcA^0 = bc \in \{A, bc\}_{LA}$. We assume that $A^i bcA^j \in \{A, bc\}_{LA}$ for $i + j \leq p - 1$. Then, we have a closer look at all elements $A^i bcA^j$ with $i + j = p - 1$. By taking the commutator with A we obtain the p linearly independent equations

$$[A^i bcA^{p-1-i}, A] = A^i bcA^{p-i} - A^{i+1} bcA^{p-1-i} \tag{3.6}$$

for $i = 0, \dots, p - 1$. Summing up all equations we get

$$\sum_{i=0}^{p-1} [A^i bcA^{p-1-i}, A] = bcA^p - A^p bc \in \{A, bc\}_{LA}. \tag{3.7}$$

Hence, from

$$[A^p bc + (-1)^p bcA^p, bc] = (cb)A^p bc + (-1)^p (cA^p b)bc - (cA^p b)bc - (-1)^p (cb)bcA^p$$

we get

$$A^p bc + bcA^p \in \{A, bc\}_{LA}$$

due to $cb \neq 0$. Together with (3.7) it follows $A^p bc, bcA^p \in \{A, bc\}_{LA}$ and successively, with (3.6) we obtain that all $A^i bcA^j$ with $i + j = p$ are elements of the Lie algebra $\{A, bc\}_{LA}$. With Lemma 3.13 we obtain

$$\{A, bc\}_{LA} = \mathfrak{gl}_n(\mathbb{R}).$$

□

Now, we assume $cb = 0$ for the sequel of this section.

In the following we show that under the assumption $cb = 0$, it is sufficient to consider Lie algebras $\{A, bc\}_{LA}$ with relative degree 1 and $cA^2 b = 0$: Firstly, it is always sufficient to assume that the transfer function satisfies $cA^{m^*} b \neq 0$ and $cA^{m^*+1} b = 0$, where m^* is the relative degree of (A, b, c). If $cA^{m^*+1} b \neq 0$, we can consider the Lie algebra

$$\mathfrak{g}' := \{A - \alpha I, bc\}_{LA}$$

with $\alpha := \frac{cA^{m^*+1}b}{(m^*+1)cA^{m^*}b}$. Clearly, we get

$$\mathfrak{g} \subseteq \mathfrak{g}' + \mathbb{R}I$$

and

$$\mathfrak{g}' \subseteq \mathfrak{g} + \mathbb{R}I.$$

Therefore, the so generated Lie algebra $\mathfrak{g}'$ only differs by multiples of the identity matrix I from $\{A, bc\}_{LA}$. But we have

$$c\left(A - \frac{cA^{m^*+1}b}{(m^*+1)cA^{m^*}b}I\right)^k b = \sum_{i=0}^{k} \binom{k}{i} (-\alpha I)^{k-i} cA^i b = 0$$

for all $0 \leq k \leq m^* - 1$ and

$$c\left(A - \frac{cA^{m^*+1}b}{(m^*+1)cA^{m^*}b}I\right)^{m^*} b = cA^{m^*}b.$$

Further, we obtain

$$\begin{aligned} c\left(A - \frac{cA^{m^*+1}b}{cA^{m^*}b}I\right)^{m^*+1} b =& c\left(\sum_{i=0}^{m^*+1} \binom{m^*+1}{i} A^{m^*+1-i} (-\alpha I)^i\right) b \\ =& cA^{m^*+1}b + (m^*+1)\left(-\frac{cA^{m^*+1}b}{(m^*+1)cA^{m^*}b}\right) cA^{m^*}b = 0. \end{aligned}$$

Hence, we can assume $cA^{m^*+1}b = 0$ in the next two lemmas. We have to distinguish between whether the first nonzero coefficient $cA^{m^*}b$ occurs for m^* even or m^* odd. We start with a lemma for the case that m^* is even.

Lemma 3.14. *Let m^* be even. If $cA^ib = 0$ for $i = 0, 1, \dots, m^* - 1$, $cA^{m^*}b = \beta$ and $cA^{m^*+1}b = 0$, then*

$$[Abc - bcA, \mathrm{ad}_A^{m^*}(bc)] = \beta(m^*+1)(Abc + bcA).$$

In particular, the matrix A^ibcA^j is an element of $\{A, bc\}_{LA}$ for $i + j = 0, 1, \dots, m^* - 1$.

Note that we can expand $\mathrm{ad}_A^{m^*+1}(bc)$ as

$$\mathrm{ad}_A^{m^*+1}(bc) = \sum_{i=0}^{m^*+1} (-1)^i \binom{m^*+1}{i} A^{m^*+1-i}bcA^i. \tag{3.8}$$

Proof. Let $cA^{m^*}b \neq 0$. Then we obtain with (3.8) and $cA^{m^*+1}b = 0$ that

$$[Abc - bcA, \mathrm{ad}_A^{m^*}(bc)]$$

$$
\begin{aligned}
&=\sum_{i=0}^{m^*}(-1)^i\binom{m^*}{i}(Abc-bcA)A^{m^*-i}bcA^i-\sum_{i=0}^{m^*}(-1)^i\binom{m^*}{i}A^{m^*-i}bcA^i(Abc-bcA)\\
&=(cA^{m^*}b)Abc-(-1)(cA^{m^*}b)m^*bcA-(-1)^{m^*-1}m^*(cA^{m^*}b)Abc+(-1)^{m^*}(cA^{m^*}b)bcA\\
&=(cA^{m^*}b)(m^*+1)(Abc+bcA)
\end{aligned}
$$

is an element of $\{A, bc\}_{LA}$. With $[A, bc] = Abc - bcA \in \{A, bc\}_{LA}$ we get

$$Abc, bcA \in \{A, bc\}_{LA}.$$

A similar calculation shows that

$$\left[\mathrm{ad}_A(Abc), \mathrm{ad}_A^{m^*}(bc)\right] = \left(1-\binom{m^*}{2}\right)\beta A^2bc - (cA^{m^*+2}b)bc$$

is an element of $\{A, bc\}_{LA}$, too. Inductively, we obtain by calculating the commutators

$$[\mathrm{ad}_A(A^kbc), \mathrm{ad}_A^{m^*}(bc)]$$

successively for $k = 0, \ldots, m^* - 2$ that all elements of the form $A^{k+1}bc$ are elements of the Lie algebra $\{A, bc\}_{LA}$. With the $k - 1$ matrices

$$[A, A^ibcA^j] = A^{i+1}bcA^j - A^ibcA^{j+1}$$

for all $0 \leq i, j \leq n - 1$ with $i + j = k - 1$ all k matrices of the form A^ibcA^j are elements of $\{A, bc\}_{LA}$. Hence, it inductively follows that $A^ibcA^j \in \{A, bc\}_{LA}$ for $i + j \leq m^* - 1$ in case m^* is even. □

If $cb = 0$, it is crucial for the Lie algebra if m^* is odd or even and a similar statement to Lemma 3.14 cannot be proven for m^* being odd.

Lemma 3.15. *Let m^* be odd. If $cA^ib = 0$ for $i = 0, 1, \ldots, m^* - 1$, $cA^{m^*}b = \beta$ and $cA^{m^*+1}b = 0$, then*

$$
\begin{aligned}
&[Abc - bcA, \mathrm{ad}_A^{m^*+1}(bc)] =\\
&\qquad -\frac{(m^*+1)m^*}{2}\beta[A,[A,bc]] - (m^*+1)(m^*+2)\beta AbcA - 2(cA^{m^*+2}b)bc.
\end{aligned}
$$

In particular, the matrix A^ibcA^i is an element of $\{A, bc\}_{LA}$ for $i = 0, 1, \ldots, (m^* - 1)/2$.

Proof. Let $cA^{m^*}b \neq 0$. Then we obtain with (3.8) and $cA^{m^*+1}b = 0$ that

$$
\begin{aligned}
&[Abc-bcA, \mathrm{ad}_A^{m^*+1}(bc)]\\
&\quad=\sum_{i=0}^{m^*+1}(-1)^i\binom{m^*+1}{i}AbcA^{m^*+1-i}bcA^i-\sum_{i=0}^{m^*+1}(-1)^i\binom{m^*+1}{i}bcAA^{m^*+1-i}bcA^i
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=0}^{m^*+1} (-1)^i \binom{m^*+1}{i} A^{m^*+1-i} bcA^i Abc + \sum_{i=0}^{m^*+1} (-1)^i \binom{m^*+1}{i} A^{m^*+1-i} bcA^i bcA \\
=&\binom{m^*+1}{0}(cA^{m^*+1}b)Abc - \binom{m^*+1}{1}\beta AbcA - \binom{m^*+1}{0}(cA^{m^*+2}b)bc \\
&+ \binom{m^*+1}{1}(cA^{m^*+1}b)bcA - \binom{m^*+1}{2}\beta bcA^2 - (-1)^{m^*-1}\binom{m^*+1}{m^*-1}\beta A^2bc \\
&- (-1)^{m^*}\binom{m^*+1}{m^*}(cA^{m^*+1}b)Abc - (-1)^{m^*+1}\binom{m^*+1}{m^*+1}(cA^{m^*+2}b)bc \\
&+ (-1)^{m^*}\binom{m^*+1}{m^*}\beta AbcA + (-1)^{m^*+1}\binom{m^*+1}{m^*+1}(cA^{m^*+1}b)bcA.
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
=&[1-(-1)^{m^*}(m^*+1)](cA^{m^*+1}b)Abc + [(m^*+1)+(-1)^{m^*+1}](cA^{m^*+1}b)bcA \\
&+ [-1+(-1)^{m^*}](m^*+1)\beta AbcA - \binom{m^*+1}{2}\beta(bcA^2 + (-1)^{m^*-1}A^2bc) \\
&+ \left(-1-(-1)^{m^*+1}\right)(cA^{m^*+2}b)bc.
\end{aligned}
$$

Since m^* is odd and $cA^{m^*+1}b = 0,$ we obtain

$$
= -\frac{(m^*+1)m^*}{2}\beta[A,[A,bc]] - (m^*+1)(m^*+2)\beta AbcA - 2(cA^{m^*+2}b)bc.
$$

Thus, if $\beta \neq 0$, then $AbcA$ is an element of the Lie algebra $\{A, bc\}_{LA}$. A lengthy calculation using that m^* is odd and $cA^{m^*+1}b = 0$ shows

$$
\begin{aligned}
&[\mathrm{ad}_A(AbcA), \mathrm{ad}_A^{m^*-1}(AbcA)] = \\
&\quad - m^*(m^*-1)\beta\, \mathrm{ad}_A^2(AbcA) - (2m^*+2)(m^*-1)\beta A^2bcA^2 - 2(cA^{m^*+2}b)AbcA.
\end{aligned}
$$

Inductively, we obtain that A^ibcA^i is an element of the Lie algebra $\{A, bc\}_{LA}$ for $i = 0, 1, \ldots, (m^*-1)/2$. □

We now show that it is sufficient to consider transfer functions with relative degree 1: First, let m^* be odd and suppose $m^* = 3$. Then we obtain from Lemma 3.15 that $AbcA \in \{A, bc\}_{LA}$. If $A + ubc$ is nonsingular, as it is for all u with maximum one exception (Remark 3.5), then there exist some $u \in \mathbb{R}$ such that $\{A + ubc, AbcA\}_{LA}$ acts irreducibly (Lemma 3.7, 3.10 and 3.11). Clearly, $\{A + ubc, AbcA\}_{LA} \subset \{A, bc\}_{LA}$ and for the associated transfer function $\hat{g}$ of $\{A + ubc, AbcA\}_{LA}$ due to Lemma 3.15 we obtain

$$
\hat{g}(s) = \sum_{i=0}^{\infty} \left(cA(A+ubc)^i Ab\right) s^{-i-1}.
$$

Clearly, we get $cA(A+ubc)^0Ab = cA^2b = 0$ since $m^* = 3$ and $cA(A+ubc)^1Ab = cA^3b + u(cAb)^2 = cA^3b \neq 0$. Hence, the first coefficient of the transfer function $\hat{g}(s)$

equals zero and the second one is unequal zero.
Inductively, we deduce the same by considering the Lie algebra generated by $A^i bcA^i$ and $A + ubc$ with $i = \frac{m^*-1}{2}$ for $m^* = 5, 7, \ldots$.

Second, let m^* be even and suppose $m^* = 2$. Then we obtain from Lemma 3.14 that $Abc \in \{A, bc\}_{LA}$. If $A + ubc$ is nonsingular, as it is for all u with only one exception, then there exist values for $u \in \mathbb{R}$ such that $\{A + ubc, Abc\}_{LA}$ acts irreducibly (Lemma 3.7, 3.10 and 3.11). Furthermore, $\{A + ubc, Abc\}_{LA} \subset \{A, bc\}_{LA}$ and for the associated transfer function $\hat{g}$ of $\{A + ubc, Abc\}_{LA}$ we obtain

$$\hat{g}(s) = \sum_{i=0}^{\infty} \left(c(A + ubc)^i Ab\right) s^{-i-1}.$$

Clearly, $c(A + ubc)^0 Ab = cAb = 0$ and $c(A + ubc)^1 Ab = cA^2 b + u(cb)(cAb) = cA^2 b \neq 0$. Inductively, we deduce the same by considering the Lie algebra generated by $A^i bc$ and $A + ubc$ with $i = m^* - 1$ for $m^* = 4, 6, \ldots$.
In this manner and by using Lemma 3.15 and Lemma 3.14 inductively, we can limit our investigations to transfer functions with $cb = 0$ and $cAb \neq 0$.

The following lemma relates the dimension of the subspace $\operatorname{span}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$ to the number of nonzero distinct eigenvalues of the operator $\operatorname{ad}_{A+ubc}(\cdot)$.

Lemma 3.16. *Let (A, b, c) be a controllable and observable triple. The maximum over $u \in \mathbb{R}$ of the dimension of the subspace*

$$\operatorname{span}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$$

equals the number of distinct nonzero eigenvalues of $\operatorname{ad}_{A+ubc}(\cdot)$.

For the sake of completeness we give Brockett's proof from [7] in Appendix C.

Due to Lemma B.6 and Lemma B.8 we know that we have to distinguish between $g(s) = g(-s + \alpha)$ for some $\alpha \in \mathbb{R}$ or $g(s) \neq g(-s + \alpha)$ for any $\alpha \in \mathbb{R}$. We give some equivalent statements for the triple (A, b, c) in case $g(s) = g(-s + \alpha)$.

Lemma 3.17. *Let (A, b, c) be controllable and observable and $g(s) = c(sI - A)^{-1} b$. Then the following statements are equivalent for one $\alpha \in \mathbb{R}$:*

(i) $g_\alpha(s) := g(s + \alpha)$ *satisfies* $g_\alpha(s) = g_\alpha(-s)$*;*

(ii) $A - \alpha I, b, c$ *are Hamiltonian, i.e.* $J(A - \alpha I)$ *is symmetric and* $cJ = b^\top$ *for a nondegenerate skew-form* J*;*

(iii) $A - \alpha I, bc \in \mathfrak{sp}_{n/2}(J)$ *for some nondegenerate skew-form* J*;*

(iv) $\{A - \alpha I, bc\}_{LA}$ *is isomorphic to a subalgebra of* $\mathfrak{sp}_{n/2}(\mathbb{R})$*.*

The Lie algebra $\mathfrak{sp}_{n/2}(J)$ is defined by

$$\mathfrak{sp}_{n/2}(J) := \left\{X \in \mathbb{R}^{n\times n} \;\middle|\; X^\top J + JX = 0\right\},$$

where $J = -J^\top$ is nondegenerate.

Proof. The equivalence between (i) and (ii) follows from Theorem 1 of [8] and the argumentation from (ii) to (iii) follows immediately.
We show $(iii) \Rightarrow (ii)$: From $A - \alpha I, bc \in \mathfrak{sp}_{n/2}(J)$ we get $bcJ = J^\top c^\top b^\top$ and hence, $b(cJ) = (J^\top c^\top)b^\top$. It follows $(cJ)^\top = \tau b$ for $\tau \in \mathbb{R}\backslash\{0\}$. With $S := \lambda I$ we obtain from the state-space isomorphism theorem that SAS^{-1}, $Sb = \lambda b$ and $cS^{-1} = \frac{1}{\lambda}c$ is another realization. Then, $\tau\lambda b = \frac{1}{\lambda}(cJ)^\top$ and therefore, $\tau\lambda^2 b = (cJ)^\top$. This yields $\tau\lambda^2 = \pm 1$. In case $\tau\lambda^2 = 1$ choose the same J as in statement (iii). In case $\tau\lambda^2 = -1$ the nondegenerate skew-form $\hat{J} := -J$ does it. The same works for $A - \alpha I$ and hence, $(A - \alpha I, b, c)$ are Hamiltonian.
The equivalence between (iv) and one of the other statements follows immediately with Corollary 8.25, [1]. □

Lemma 3.18. *Let* $S_n = \{X \in \mathfrak{sp}_{n/2}(\mathbb{R}) \mid \operatorname{diag} X = 0\}$. *Then,*

$$\operatorname{span}\left\{[X,Y] \;\middle|\; X, Y \in S_n\right\} = \mathfrak{sp}_{n/2}(\mathbb{R}) \quad \textit{for } n \geq 2.$$

Here, $\operatorname{diag} X = 0$ means that diagonal of the matrix X vanishes.

Proof. Every matrix $X \in S_n$ has the form

$$X = \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix},$$

where B, C are symmetric matrices and the diagonal entries of A are all zero. Hence, we obtain for $X, Y \in S_n$

$$\begin{aligned}[X,Y] &= \left[\begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix}, \begin{pmatrix} D & E \\ F & -D^\top \end{pmatrix}\right] \\ &= \begin{pmatrix} [A,D] + BF - EC & AE + EA^\top - (DB + (DB)^\top) \\ CD + D^\top C - (FA + (FA)^\top) & -([A,D] + BF - EC)^\top \end{pmatrix}.\end{aligned}$$

The diagonal of the commutator $[X, Y]$ does not vanish, since B, F, E and C are only assumed to be symmetric. With $A = E_{ik}$ and $D = E_{kj}$ we get $[A, D] = E_{ij}$ and thus,

$$\begin{pmatrix} H & 0 \\ 0 & -H^\top \end{pmatrix} \in \operatorname{span}\left\{[X, Y] \;\middle|\; X, Y \in S_n\right\}$$

for every $H \in \mathbb{R}^{n\times n}$. With $A = E_{ij}$ and $E = I$ one can easily see that every symmetric matrix with vanishing diagonal can be written as $AE + EA^\top$. The same holds for $CD + D^\top C$. Computing $AE + EA^\top$ for $A = E_{ij}$ and $E = E_{ij} + E_{ji}$ yields all symmetric matrices with non vanishing diagonal. The result follows. □

The complex analog holds as well.

Lemma 3.19. *Let* $S_n^{\mathbb{C}} = \{X \in \mathfrak{sp}_{n/2}(\mathbb{C}) \mid \operatorname{diag} X = 0\}$. *Then,*

$$\operatorname{span}\left\{[X, Y] \;\middle|\; X, Y \in S_n^{\mathbb{C}}\right\} = \mathfrak{sp}_{n/2}(\mathbb{C}) \quad \textit{for } n \geq 2.$$

Before we determine the real generated Lie algebra, we examine the complex generated Lie algebra $\{A, bc\}_{LA}$ in case the transfer function satisfies $g(s) = g(-s + \alpha)$ for one $\alpha \in \mathbb{R}$.

Lemma 3.20. *Let* (A, b, c) *be controllable and observable with* $\operatorname{tr} A = cb = 0$ *and* $g(s) = c(sI - A)^{-1}b$ *satisfy*

$$g(-s) = g(s + \alpha)$$

for one $\alpha \in \mathbb{R}$. *Then*

$$S_n^{\mathbb{C}} \subset \operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$$

for some $u \in \mathbb{R}$. *Here,* $S_n^{\mathbb{C}} \subset \operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$ *is in the sense, that* $S_n^{\mathbb{C}}$ *is conjugated to a subspace of* $\operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$.

Proof. For almost all $u \in \mathbb{R}$ the matrix $A + ubc$ is diagonalizable (Lemma 3.8). Let $Q \in GL_n(\mathbb{C})$ denote the matrix, such that

$$Q(A + ubc)Q^{-1}$$

is a diagonal matrix. Clearly, we have

$$Q\left\{\operatorname{ad}^i_{A+ubc}(bc), i \in \mathbb{N}\right\}Q^{-1} = \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\} =: \mathcal{Q}.$$

With Lemma B.8 we get

$$\dim_{\mathbb{C}} \mathcal{Q} = \frac{n^2}{2} = \dim_{\mathbb{C}}\left\{\operatorname{ad}^i_{A+ubc}(bc), i \in \mathbb{N}\right\}.$$

Since $Q(A+ubc)Q^{-1}$ is diagonal, we get for every $X \in \mathcal{Q}$ that $\operatorname{diag} X = 0$, i.e. all diagonal entries vanish. Due to the dimension of $\mathcal{Q}$ we conclude

$$S_n^{\mathbb{C}} \subset \mathcal{Q}.$$

Since $\mathfrak{sp}_{n/2}(\mathbb{C}) = \{S_n^{\mathbb{C}}\}_{LA}$ we get $\mathfrak{sp}_{n/2}(\mathbb{C}) = \{\mathcal{Q}\}_{LA}$. Hence,

$$Q^{-1}\mathfrak{sp}_{n/2}(\mathbb{C})Q = \left\{\operatorname{ad}^i_{A+ubc}(bc), i \in \mathbb{N}\right\}$$

and the result follows. □

Theorem 3.21. *Let* (A, b, c) *be controllable and observable with* $\operatorname{tr} A = cb = 0$ *and let* $g(s) = c(sI - A)^{-1}b$ *satisfy* $g(s) = g(-s)$*. Then*

$$\{A, bc\}^{\mathbb{C}}_{LA} \cong \mathfrak{sp}_{n/2}(\mathbb{C}).$$

Proof. This results with Lemma 3.19 from Lemma 3.20. □

Now, we determine the real generated Lie algebra $\{A, bc\}_{LA}$ in case the transfer function satisfies $g(s) = g(-s + \alpha)$ for one $\alpha \in \mathbb{R}$.

Theorem 3.22. *Let* (A, b, c) *be controllable and observable with* $\operatorname{tr} A = cb = 0$ *and let* $g(s) = c(sI - A)^{-1}b$ *satisfy* $g(s) = g(-s)$*. Then*

$$\{A, bc\}_{LA} \cong \mathfrak{sp}_{n/2}(\mathbb{R}).$$

Proof. Due to Lemma 3.8 the matrix $A + ubc$ is diagonalizable for almost all $u \in \mathbb{R}$. Let $Q \in GL_n(\mathbb{C})$ such that the matrix

$$Q(A + ubc)Q^{-1}$$

is diagonal. Denote

$$\operatorname{span}_{\mathbb{R}} \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\} =: \mathcal{Q}^{\mathbb{R}}.$$

From Lemma B.8 we get

$$\dim_{\mathbb{R}} \mathcal{Q}^{\mathbb{R}} = \frac{n^2}{2}.$$

Clearly, $\mathcal{Q}^{\mathbb{R}}$ is a split real form of $\mathcal{Q} := \operatorname{span}_{\mathbb{C}} \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\}$ and due to Lemma 3.21 we have $\{\mathcal{Q}\}_{LA} = \mathfrak{sp}_{n/2}(\mathbb{C})$. Due to Theorem IX.5.10. [23] all split real forms are isomorphic. Hence,

$$\{\mathcal{Q}^{\mathbb{R}}\}_{LA} \cong \mathfrak{sp}_{n/2}(\mathbb{R})$$

and the result follows. □

Now, we consider the case $g(s) \neq g(-s+\alpha)$ for any $\alpha \in \mathbb{R}$ and state the analogs for Lemma 3.19, Lemma 3.20, Theorem 3.21 and Theorem 3.22.

Lemma 3.23. *Let* $V_n = \{X \in \mathbb{R}^{n\times n} \mid \operatorname{diag} X = 0\}$*. Then*

$$\operatorname{span}\left\{[X,Y] \;\middle|\; X,Y \in V_n\right\} = \mathfrak{sl}_n(\mathbb{R}) \quad \textit{for } n \geq 3.$$

Again, $\operatorname{diag} X = 0$ means that all diagonal entries of the matrix X are 0.

Proof. The matrices $X = E_{ij}$ and $Y = E_{kl}$ are elements of V_n for $i \neq j, k \neq l$ and satisfy

$$[X,Y] = \begin{cases} E_{ii} - E_{jj}, & k = j, i = l \\ E_{il}, & k = j, i \neq l \\ 0, & k \neq j, i \neq l. \end{cases}$$

This completes the proof. ☐

Clearly, the complex analog holds too.

Lemma 3.24. *Let* $V_n^{\mathbb{C}} = \{X \in \mathbb{C}^{n\times n} \mid \operatorname{diag} X = 0\}$*. Then*

$$\operatorname{span}_{\mathbb{C}}\left\{[X,Y] \;\middle|\; X,Y \in V_n^{\mathbb{C}}\right\} = \mathfrak{sl}_n(\mathbb{C}) \quad \textit{for } n \geq 3.$$

Before we determine the real generated Lie algebra, we examine the complex generated Lie algebra $\{A, bc\}_{LA}$ in case the transfer function satisfies $g(s) \neq g(-s+\alpha)$ for any $\alpha \in \mathbb{R}$.

Lemma 3.25. *Let* (A, b, c) *be controllable and observable and* $g(s) = c(sI - A)^{-1}b$ *satisfy*

$$g(-s) \neq g(s+\alpha)$$

for any $\alpha \in \mathbb{R}$*. Then*

$$V_n^{\mathbb{C}} \subset \operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$$

for some $u \in \mathbb{R}$*. Here,* $V_n^{\mathbb{C}} \subset \operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$ *is in the sense, that* $V_n^{\mathbb{C}}$ *is conjugated to a subspace of* $\operatorname{span}_{\mathbb{C}}\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$.

Proof. For almost all $u \in \mathbb{R}$ the matrix $A + ubc$ is diagonalizable (Lemma 3.8). Let $Q \in GL_n(\mathbb{C})$ denote the matrix, such that

$$Q(A+ubc)Q^{-1}$$

is a diagonal matrix. Clearly, we have

$$Q\left\{\operatorname{ad}^i_{A+ubc}(bc), i \in \mathbb{N}\right\} Q^{-1} = \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\} =: \mathcal{Q}.$$

With Lemma B.8 we get

$$\dim_{\mathbb{C}} \mathcal{Q} = n^2 - n = \dim_{\mathbb{C}} \left\{\operatorname{ad}^i_{A+ubc}(bc), i \in \mathbb{N}\right\}.$$

Since $Q(A+ubc)Q^{-1}$ is diagonal, we get for every $X \in \mathcal{Q}$ that $\operatorname{diag} X = 0$, i.e. all diagonal entries vanish. Due to the dimension of $\mathcal{Q}$ we conclude

$$V_n^{\mathbb{C}} \subset \mathcal{Q}$$

and the result follows. □

Theorem 3.26. *Let (A, b, c) be controllable and observable and let $g(s) = c(sI - A)^{-1}b$ satisfy $g(s) \neq g(-s + \alpha)$ for any $\alpha \in \mathbb{R}$. Then*

$$\mathfrak{sl}_n(\mathbb{C}) \subset \{A, bc\}^{\mathbb{C}}_{LA}.$$

Proof. Since $\{V_n^{\mathbb{C}}\}_{LA} = \mathfrak{sl}_n(\mathbb{C})$ we get $\mathfrak{sl}_n(\mathbb{C}) \subset \{\mathcal{Q}\}_{LA}$ and and the result follows. □

Now, we prove the second main result of this section.

Theorem 3.27. *Let (A, b, c) be controllable and observable and let $g(s) = c(sI - A)^{-1}b$ satisfy*

$$g(-s) \neq g(s + \alpha)$$

for all $\alpha \in \mathbb{R}$. Then

$$\mathfrak{sl}_n(\mathbb{R}) \subset \{A, bc\}_{LA}.$$

Proof. Due to Lemma 3.8 the matrix $A + ubc$ is diagonalizable for almost all $u \in \mathbb{R}$. Let $Q \in GL_n(\mathbb{C})$ such that the matrix

$$Q(A + ubc)Q^{-1}$$

is diagonal. Denote

$$\operatorname{span}_{\mathbb{R}} \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\} =: \mathcal{Q}^{\mathbb{R}}.$$

From Lemma B.8 we get

$$\dim_{\mathbb{R}} \mathcal{Q}^{\mathbb{R}} = n^2 - n.$$

Clearly, $\mathcal{Q}^{\mathbb{R}}$ is a split real form of $\mathcal{Q} := \operatorname{span}_{\mathbb{C}} \left\{\operatorname{ad}^i_{Q(A+ubc)Q^{-1}}(QbcQ^{-1}), i \in \mathbb{N}\right\}$ and due to Lemma 3.26 we have $\{\mathcal{Q}\}_{LA} = \mathfrak{sl}_n(\mathbb{C})$. Due to Theorem IX.5.10. [23] all split real forms are isomorphic. Hence,

$$\{\mathcal{Q}^{\mathbb{R}}\}_{LA} = \mathfrak{sl}_n(\mathbb{R})$$

and the result follows. □

We summarize the results on system Lie algebras of linear feedback systems in one theorem.

Theorem 3.28. *Let (A, b, c) be controllable and observable, $g(s) = c(sI - A)^{-1}b$. Then the system Lie algebra $\mathfrak{g} = \{A, bc\}_{LA}$ of*

$$\dot{x} = (A + u(t)bc)x$$

satisfies:

(i) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{R})$ *if and only if* $g(s) = g(-s)$*;*

(ii) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$ *if and only if* $g(s + \alpha) = g(-s + \alpha)$ *with* $\alpha \neq 0$ *suitable;*

(iii) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ *if and only if* $g(s + \alpha) \neq g(-s + \alpha)$ *for all* α *and* $cb = \operatorname{tr} A = 0$*;*

(iv) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ *else.*

Here, $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{R})$ means that the Lie algebras $\mathfrak{g}$ and $\mathfrak{sp}_{n/2}(\mathbb{R})$ are conjugated.

Proof. In case $cb \neq 0$ we immediately obtain $g(s) \neq g(-s + \alpha)$ for any $\alpha \in \mathbb{R}$ and hence, we can apply Theorem 3.12. With the previous remarks we can limit our observations to the case, where the characteristic polynomial of $A + ubc$ has the form

$$q(s) + up(s) = s^n + q_{n-1}s^{n-1} + \ldots + q_0 + u\left(p_{n-2}s^{n-2} + \ldots + p_0\right).$$

Here, $q(s)$ and $p(s)$ are coprime and $p_{n-2} \neq 0$. For $q_{n-1} = 0$ this is Theorem 3.22 and Theorem 3.27. In case $q_{n-1} \neq 0$, consider the Lie algebra

$$\left\{A - \frac{q_{n-1}}{n}I, bc\right\}_{LA}.$$

Then, $\operatorname{tr}\left(A - \frac{q_{n-1}}{n}I\right) = 0$ and the Lie algebra $\{A, bc\}_{LA}$ only differs by adding multiples of the identity matrix I to the Lie algebra $\{A - \frac{q_{n-1}}{n}I, bc\}_{LA}$. The result follows. □

From Theorem 3.28 we deduce immediately with Theorem 2.4 the following corollary.

Corollary 3.29. *Let (A, b, c) be controllable and observable. Then the bilinear control system*

$$\dot{x} = (A + u(t)bc)x$$

is accessible on $\mathbb{R}^n \setminus \{0\}$.

We obtain the same result if we allow A and bc to be complex and consider $\{A, bc\}_{LA}$ as a complex Lie algebra since

$$\mathfrak{sp}_{n/2}(\mathbb{C}) = \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus i\mathfrak{sp}_{n/2}(\mathbb{R})$$

and

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \oplus i\mathfrak{sl}_n(\mathbb{R}).$$

Corollary 3.30. *Let (A, b, c) be controllable and observable, $g(s) = c(sI - A)^{-1}b$. Then the complex system Lie algebra $\mathfrak{g}$ of*

$$\dot{x} = (A + u(t)bc)x$$

satisfies:

(i) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{C})$ *if and only if* $g(s) = g(-s)$*;*

(ii) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{C}I$ *if and only if* $g(s + \alpha) = g(-s + \alpha)$ *for some* $\alpha \in \mathbb{C} \setminus \{0\}$*;*

(iii) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ *if and only if* $g(s + \alpha) \neq g(-s + \alpha)$ *for all* $\alpha \in \mathbb{C}$ *and* $cb = \operatorname{tr} A = 0$*;*

(iv) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ *else.*

3.3 Real Lie algebras of complex SISO feedback systems

In this section we allow A and bc to be complex and consider $\{A, bc\}_{LA}$ as a real Lie algebra. In [6] Brockett classified all possible Lie algebras of linear feedback systems. Before we state the result, we need some introductory definitions and results.

Definition 3.31. The *Cauchy index of a rational function* $g(s)$ is the number of jumps of the function $g(s)$ from $-\infty$ to $+\infty$ minus the number of jumps of the function $g(s)$ from $+\infty$ to $-\infty$ for $s \in \mathbb{R}$.

Let $g(s) = \sum_{i=0}^{\infty} \left(cA^i b\right) s^{-i-1}$. Then the Cauchy index of $g(s)$ equals the signature of the Hankelmatrix

$$\begin{pmatrix} cb & cAb & cA^2b & \\ cAb & cA^2b & & \\ cA^2b & & \ddots & \\ & & & cA^{2n}b \end{pmatrix}$$

(Theorem I.9.4 [34]).

In the following, $g^*(s)$ denotes the function which results from taking the complex conjugate of the coefficients of $g(s)$, i.e.

$$g^*(s) = \overline{g(\overline{s})} = \overline{c}(sI - \overline{A})^{-1}\overline{b}$$

and $\mathrm{i} = \sqrt{-1}$.

The complex analog to Lemma 3.17 is the following.

Theorem 3.32. *Let (A, b, c) be a controllable and observable triple with transfer function $g(s) = c(sI - A)^{-1}b$. Then A and bc leave invariant a non-degenerate Hermitian form Q if and only if $g(s) = g^*(-s)$ holds for $s \in \mathbb{C}$.*

Proof. Suppose that $g(s) = g^*(-s)$, then for $s \in \mathbb{C}$ holds

$$\begin{aligned} c(sI - A)^{-1}b &= \overline{c}(-sI - \overline{A})^{-1}\overline{b} \\ &= -\overline{c}(sI + \overline{A})^{-1}\overline{b} \\ &= -\overline{b^\top}(sI - (-\overline{A^\top}))^{-1}\overline{c^\top}. \end{aligned}$$

Clearly, the triple $(-\overline{A^\top}, \overline{c^\top}, -\overline{b^\top})$ is controllable and observable, too. Then, due to the state-space isomorphism theorem, there exists a nonsingular matrix P such that

$$\begin{aligned} PAP^{-1} &= -\overline{A^\top}, \\ Pb &= \overline{c^\top}, \\ -\overline{b^\top}P &= c. \end{aligned}$$

By simple calculation including taking the conjugate transpose of the equations above we obtain

$$\begin{aligned} (\overline{P})^\top A(\overline{P^\top})^{-1} &= -\overline{A^\top}, \\ \overline{b^\top P^\top} &= c, \\ -\overline{P^\top}b &= \overline{c^\top}. \end{aligned}$$

Hence, P and $-\overline{P}^\top$ satisfy the same set of equations. Since the solution of these equations for P is unique (Remark 6.5.10 [49]), we see $P = -\overline{P}^\top$ and thus, the matrices A and bc leave invariant the skew-Hermitian form P. When A and bc leave the skew-Hermitian form P invariant, they leave the Hermitian form $\mathrm{i}P$ invariant and the other way round. Clearly, $\mathrm{i}P$ is Hermitian. With $Q := \mathrm{i}P$ the proof of the first claim is completed.
Now, let us assume that

$$\overline{A^\top}Q + QA = \overline{(bc)^\top}Q + Qbc = 0$$

for a Hermitian form Q. Then, we obtain for $s \in \mathbb{R}$

$$\begin{aligned} g(s) &= c(sI - A)^{-1}b \\ &= c(sI + Q^{-1}\overline{A^\top}Q)^{-1}b \\ &= cQ^{-1}(sI + \overline{A^\top})^{-1}Qb \\ &= -cQ^{-1}(-sI - \overline{A^\top})^{-1}Qb \\ &= \overline{b^\top}(-sI - \overline{A^\top})^{-1}\overline{c^\top} = g^*(-s). \end{aligned}$$

For the penultimate equality we used $\overline{(bc)^\top} = -QbcQ^{-1}$ and thus

$$\overline{b^\top}(-sI - \overline{A^\top})^{-1}\overline{c^\top} = -cQ^{-1}(-sI - \overline{A^\top})^{-1}Qb.$$

This proves the converse. □

We now present the correct version of Brockett's result on Lie algebras from [7]. The corrected defect will be explained in Example 3.35.

Theorem 3.33. *Let (A, b, c) be a controllable and observable triple with transfer function $g(s) = c(sI - A)^{-1}b$. Then the real Lie algebra $\{A, bc\}_{LA}$ generated by A and bc is isomorphic to*

(1)	$\mathfrak{sp}_{n/2}(\mathbb{R})$	*if*	$g(s) = g^*(s) = g(-s)$;
(2)	$\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \alpha\mathbb{R}I$	*if*	$g(s) = g^*(s) = g(-s+\alpha)$ *for some* $\alpha \in \mathbb{R}\backslash\{0\}$;
(3)	$\mathfrak{sl}_n(\mathbb{R})$	*if*	$g(s) = g^*(s) \neq g(-s+\alpha)$ *for any* $\alpha \in \mathbb{R}$; $\operatorname{tr} A = cb = 0$;
(4)	$\mathfrak{gl}_n(\mathbb{R})$	*if*	$g(s) = g^*(s)$ *and none of the above;*
(5)	$\mathfrak{su}(\mu, \nu)$	*if*	$g(s) = g^*(-s) \neq g(-s)$; *for* $h(s) = g(\mathrm{i}s)$, $\mu - \nu =$ *Cauchy index of* $h(t)$;
(6)	$\mathfrak{u}(\mu, \nu)$	*if*	$g(s) = g^*(-s+\mathrm{i}\alpha) \neq g(-s+\mathrm{i}\alpha)$ *for some* $\alpha \in \mathbb{R}\backslash\{0\}$ *and* $h(s) = g(\mathrm{i}s)$ *with* $\mu - \nu$ *the Cauchy index of* $h(t)$;
(7)	$\mathfrak{sp}_{n/2}(\mathbb{C})$	*if*	$g(s) = g(-s)$ *and none of the above;*
(8)	$\mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \alpha\mathbb{R}I$	*if*	$g(s) = g(-s+\alpha)$ *and none of the above;*
(9)	$\mathfrak{sl}_n(\mathbb{C})$	*if*	*none of the above and* $\operatorname{tr} A = cb = 0$;
(10)	$\mathfrak{sl}_n(\mathbb{C}) \oplus \alpha\mathbb{R}I$	*if*	*none of the above and* $\operatorname{tr} A$ *and* cb *are linearly dependent over* $\mathbb{R}$ *and* $\{\operatorname{tr} A, cb\} \neq \{0\}$; *with* $\alpha \in \mathbb{C}$ *suitable;*
(11)	$\mathfrak{gl}_n(\mathbb{C})$	*if*	*none of the above.*

The definitions of the Lie algebras of cases (1) – (11) can be found in Appendix A.

The cases (1)-(4) are exactly Theorem 3.28 since the condition $g(s) = g^*(s)$ implies that all coefficients of

$$g(s) = \sum_{i=0}^{\infty} (cA^i b) s^{-i-1}$$

are real.

Now, we proof cases (5) and (6).

Theorem 3.34. *Let (A, b, c) be controllable and observable and $g(s) = c(sI - A)^{-1}b$ be the transfer function. If for $g(s)$ holds either $g(s) = g^*(-s) \neq g(-s)$ or $g(s) = g^*(-s + \mathrm{i}\alpha) \neq g(-s + \mathrm{i}\alpha)$ for some $\alpha \in \mathbb{R}\backslash\{0\}$, then the real generated Lie algebra $\{A, bc\}_{LA}$ is $\mathfrak{su}(\mu, \nu)$ or $\mathfrak{u}(\mu, \nu)$, respectively. The number $\mu - \nu$ is the Cauchy index of $h(t)$, where $h(t) := g(\mathrm{i}t)$.*

Proof. From Theorem 3.32 we get that A and bc leave invariant an Hermitian form if and only if $g(s) = g^*(-s)$ for $s \in \mathbb{R}$. From Corollary 3.30 we know that, since $g(s) \neq g(-s)$ and $g(s) \neq g(-s + \alpha)$ for $\alpha \in \mathbb{R}$, the complex Lie algebra $\{A, bc\}_{LA}^{\mathbb{C}}$ is at least $n^2 - 1$ dimensional over $\mathbb{R}$. Because $\mathfrak{u}(\mu, \nu)$ has dimension n^2 this is the maximum dimension. Thus, $\{A, bc\}_{LA}$ is either $\mathfrak{su}(\mu, \nu)$ or $\mathfrak{u}(\mu, \nu)$.
Now, we compute the signature of the Hermitian form Q. Let $h(s) := g(\mathrm{i}s)$. From $g(s) = g^*(-s)$ we obtain for $h(s)$ and $s \in \mathbb{R}$ that

$$h(s) = g(\mathrm{i}s) = g^*(-\mathrm{i}s) = \overline{g(\mathrm{i}\overline{s})} = \overline{h(\overline{s})} = h^*(s).$$

Therefore, the rational function h takes only real values for $s \in \mathbb{R}$, i.e. $h(\mathbb{R}) \subseteq \mathbb{R}$. Since $g(s) = \sum_{i=0}^{\infty} (cA^i b) s^{-i-1}$ we get

$$h(s) = \sum_{j=0}^{\infty} \mathrm{i}^{-j-1} (cA^j b) s^{-j-1}$$

and $\mathrm{i}^{-j-1}(cA^j b) \in \mathbb{R}$ for all $j \in \mathbb{N}_0$. With $c = b^*Q$ we conclude

$$\begin{aligned} \mathrm{i}^{-(k+l)-1} cA^k A^l b &= \mathrm{i}^{-(k+l)-1} b^* Q A^k A^l b \\ &= -\mathrm{i}b^*((-\mathrm{i}A)^*)^k Q (-\mathrm{i}A)^l b. \end{aligned}$$

From the proof of Theorem 3.32 we know that a matrix A leaves invariant a Hermitian form Q if and only if it leaves invariant a skew-Hermitian form, which is given by iQ. Hence, we obtain with $c = b^*(iQ)$ that

$$\begin{aligned} \mathrm{i}^{-(k+l)-1} cA^k A^l b &= \mathrm{i}^{-(k+l)-1} b^* (\mathrm{i}Q) A^k A^l b \\ &= -\mathrm{i}b^*((-\mathrm{i}A)^*)^k (\mathrm{i}Q)(-\mathrm{i}A)^l b \\ &= b^*((-\mathrm{i}A)^*)^k Q (-\mathrm{i}A)^l b. \end{aligned}$$

Thus,

$$(b, -\mathrm{i}Ab, \dots, (-\mathrm{i})^{n-1}A^{n-1}b)^* Q (b, \mathrm{i}Ab, \dots, (-i)^{n-1}A^{n-1}b)$$

$$= \begin{pmatrix} -\mathrm{i}cb & -cAb & \mathrm{i}cA^2b & \\ -cAb & \mathrm{i}cA^2b & & \\ \mathrm{i}cA^2b & & \ddots & \\ & & & (-\mathrm{i})^{2n-2}cA^{2n}b \end{pmatrix} =: H_{2n}.$$

Hence, the Hankelmatrix H_{2n} and the Hermitian matrix Q have the same signature and this is equal to the Cauchy index of $h(s)$ (Theorem I.9.4, [34]). This completes the proof. □

In [6] Brockett suggests that one gets the parameters μ and ν of the Lie algebras $\mathfrak{su}(\mu, \nu)$ and $\mathfrak{u}(\mu, \nu)$ from cases (5) and (6) by calculating the Cauchy index of the function $h(t)$, which is supposed to be defined by $h(s^2) := g(s)/s$. This implies that the function $g(s)/s$ is even in s, but this does not result from the condition $g(s) = g^*(-s) \neq g(-s)$ as the following examples shows.

Example 3.35. *Choose the Hermitian form as*

$$Q = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

which has signature 1, and the triple (A, b, c) as

$$A = \begin{pmatrix} \mathrm{i} & 2 & 2\mathrm{i} \\ -2 & \mathrm{i} & 2 \\ -2\mathrm{i} & 2 & -2\mathrm{i} \end{pmatrix}, \quad b = \begin{pmatrix} \mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix} \quad \text{and } c = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}.$$

A short calculation shows that A and bc leave Q invariant. Then (A, b, c) is controllable and observable since

$$b = \begin{pmatrix} \mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix}, \quad Ab = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \quad \text{and } A^2b = \begin{pmatrix} 5\mathrm{i} \\ 14 \\ -2\mathrm{i} \end{pmatrix}$$

are linearly independent and

$$c = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \; cA = \begin{pmatrix} 3\mathrm{i} & 0 & 4\mathrm{i} \end{pmatrix} \text{ and } cA^2 = \begin{pmatrix} 5 & 14\mathrm{i} & 2 \end{pmatrix},$$

too. Considering $\operatorname{tr} A = 0 = \operatorname{tr} bc$*, the real Lie algebra generated by* A *and* bc *is supposed to be* $\mathfrak{su}(2,1)$ *due to Theorem 3.33. With the subsequent Theorem 3.32 we obtain that, since* A *and* bc *leave invariant the Hermitian form* Q*, for the transfer function* $g(s)$ *it holds* $g(s) = g^*(-s)$ *for* $s \in \mathbb{C}$*. But for the coefficients of* $g(s)$ *we obtain*

$$\begin{aligned} cb &= 0 \\ cAb &= -7 \\ cA^2b &= 7\mathrm{i} \\ cA^3b &= -7 \\ &\vdots \end{aligned}$$

Hence, $\frac{g(s)}{s}$ *cannot be an even function in* s *and so the parameters* μ *and* ν *have to be obtained in a different manner. We show that it is by calculating the Cauchy index of the function* $h(s) := g(\mathrm{i}s)$ *in Theorem 3.34.*

To conclude the proof of Theorem 3.33, we now proof the cases $(7) - (11)$:

Theorem 3.36. *Let* (A, b, c) *be controllable and observable and* $g(s) = c(sI - A)^{-1}b$*. Let the transfer function* $g(s)$ *do not satisfy any condition from Theorem 3.28* or *Theorem 3.34. Then*

(i) if $g(s) = g(-s) \neq g^*(s)$*, then* $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{C})$*.*

(ii) if $g(s) = g(-s + \alpha)$ *with* $\alpha \in \mathbb{C}\backslash\{\mathbb{R}, \mathrm{i}\mathbb{R}\}$*, then* $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \alpha\{I\}$*.*

(iii) if none of the above and $\operatorname{tr} A = cb = 0$*, then* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$*.*

(iv) if none of the above, $\operatorname{tr} A$ *and* cb *are linear dependent over* $\mathbb{R}$ *and* $\{\operatorname{tr} A, cb\} \neq \{0\}$*, then* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \oplus \alpha\mathbb{R}I$ *for some* $\alpha \in \mathbb{C} \setminus \{0\}$*.*

(v) if none of the above, then $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$*.*

Proof. Clearly, we obtain from Corollary 3.30 that the Lie algebra $\mathfrak{g}$ is contained in the algebras of cases $(i)-(v)$. Now we show that the real Lie algebra $\{A, bc\}_{LA}$ is the complete Lie algebra. We know from Theorem 3.28 that for the transfer function $g(s)$ the equality $g(s) \neq g^*(s)$ holds. Therefore, there exists a natural number $l \in \mathbb{N}$ with $cA^lb \in \mathbb{C}\backslash\mathbb{R}$. Denote the minimal l with this property as l^*.

First, let l^* be odd. Then consider

$$[\mathrm{ad}_A^{l^*}(bc), bc] = \sum_{i=0}^{l^*}(-1)^i \binom{l^*}{i} A^{l^*-i}bcA^i bc - \sum_{i=0}^{l^*}(-1)^i \binom{l^*}{i} bcA^{l^*-i}bcA^i$$
$$= \sum_{i=0}^{l^*}(-1)^i \binom{l^*}{i} (cA^i b)A^{l^*-i}bc - \sum_{i=0}^{l^*}(-1)^i \binom{l^*}{i} (cA^{l^*-i}b)bcA^i$$

A lengthy calculation shows that

$$[\mathrm{ad}_A^{l^*}(bc), bc] = \alpha\left[\left[\left[\mathrm{ad}_A^{l^*-1}(bc), bc\right], A\right], bc\right] + \beta bc + \gamma[A, bc] - 2(cA^{l^*}b)bc,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. It follows that

$$(cA^{l^*}b)bc \in \mathfrak{g}.$$

Second, let l^* be even. Then, consider

$$[\mathrm{ad}_A^{l^*-1}(bc), Abc - bcA] = \sum_{i=0}^{l^*-1}(-1)^i \binom{l^*-1}{i} A^{l^*-1-i}bcA^i(Abc - bcA) -$$
$$\sum_{i=0}^{l^*-1}(-1)^i \binom{l^*-1}{i} (Abc - bcA)A^{l^*-1-i}bcA^i.$$

Another tedious calculation yields

$$[\mathrm{ad}_A^{l^*-1}(bc), Abc - bcA] = -[[\mathrm{ad}_A^{l^*-1}(bc), bc], A] - 2(cA^{l^*}b)bc,$$

which shows

$$-2(cA^{l^*}b)bc \in \mathfrak{g}.$$

In both cases there exists an $\alpha \in \mathbb{C}\backslash\mathbb{R}$ such that bc and αbc are in the real generated Lie algebra $\{A, bc\}_{LA}$ and are linearly independent over the real numbers. Hence, we can generate an ideal $\mathfrak{k}$ in $\{A, bc\}_{LA}$ with real codimension not exceeding 1. Since $\mathfrak{sp}_{n/2}(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$ are simple Lie algebras the result follows. □

Taking Theorem 3.28, Theorem 3.34 and Theorem 3.36 we obtain the complete proof of Theorem 3.33.

3.4 Lie algebras of MIMO feedback systems

Since we have only considered SISO systems in the previous sections of this chapter, we now draw our attention to multiple-input multiple-output (MIMO) systems.

Definition 3.37. A triple (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$ is called *controllable and observable* if

$$\operatorname{rank}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = n$$

and

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n,$$

i.e. (A, B) is controllable and (A, C) is observable.

Note that, in contrast to Definition 3.1, this definition does not exclude the case $A = 0$ per se.

In [5] Brockett presents one result on the system Lie algebras of MIMO systems where he uses the following lemma as the main tool to relate MIMO systems to SISO systems.

Lemma 3.38 ([4])**.** *Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{p\times n}$ be a controllable and observable triple. Then there exist vectors $b \in \mathbb{R}^n$, $c \in \mathbb{R}^{1\times n}$ and a matrix $K \in \mathbb{R}^{p\times p}$ such that*

$$(A + BKC, Bb, cC) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$$

is controllable and observable.

Remark 3.39. From the proof of Lemma 3.38 one can easily see that in case the matrix A is cyclic the matrix K can be chosen as $K = 0$.

Control by Output Feedback

There are two possibilities to generalize a SISO system to a MIMO system. In [5] Brockett suggests

$$\dot{x} = (A + BU(t)C)\,x \tag{3.9}$$

as the appropriate multivariable analog of (3), where $U(t)$ is a rectangular matrix of appropriate dimension and takes all values in $\mathbb{R}^{p\times p}$. Then the system Lie algebra is defined by

$$\left\{A, BUC \;\middle|\; U \in \mathbb{R}^{p\times p}\right\}_{LA}, \tag{3.10}$$

which is the smallest Lie algebra consisting of A and BUC for all real U of appropriate dimension. For this setting Brockett proved the following result.

Theorem 3.40. *Let (A, B, C) be a controllable and observable triple and suppose that $G(s) = C(sI - A)^{-1}B$ is of* rk G ≥ 2 *for some s. Then the system Lie algebra $\mathfrak{g}$ of the output-feedback control system*

$$\dot{x} = (A + BU(t)C)\, x$$

satisfies:

(i) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ *if and only if* $CB = 0$ *and* $\operatorname{tr} A = 0$.

(ii) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ *else.*

The proof can be found in Appendix C.

With Theorem 2.4 we deduce from Theorem 3.40 the following.

Corollary 3.41. *Let (A, B, C) be a controllable and observable triple. Then the output-feedback control system with full feedback control*

$$\dot{x} = (A + BU(t)C)\, x$$

is accessible on $\mathbb{R}^n \setminus \{0\}$.

Control by restricted Output feedback

Another possibility for a MIMO analog is

$$\dot{x} = (A + u(t)BKC)x, \tag{3.11}$$

where K is a constant matrix. Clearly, the system Lie algebra is one dimensional in case $A = 0$ and two dimensional if A and BKC commute. In this case, we can only give a result for A being cyclic.

Corollary 3.42. *Let (A, B, C) be a controllable and observable triple and A be cyclic. Then there exists a matrix K of appropriate dimension such that the system Lie algebra $\mathfrak{g}$ of the output-feedback control system*

$$\dot{x} = (A + u(t)BKC)x$$

satisfies $\mathfrak{g} \in \{\mathfrak{sp}_{n/2}(\mathbb{R}), \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I, \mathfrak{sl}_n(\mathbb{R}), \mathfrak{gl}_n(\mathbb{R})\}$.

Proof. From Remark 3.39 we know that for a controllable and observable triple (A, B, C) with A being cyclic there exist vectors $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$ such that (A, Bb, cC) is controllable and observable. Now, we can choose $K = bc$ and consider the bilinear control system

$$\dot{x} = (A + uBbcC)x.$$

With Theorem 3.28 the result follows. □

Again, with Theorem 2.4 we deduce an accessibility condition for MIMO systems.

Corollary 3.43. *Let (A, B, C) be a controllable and observable triple and A be cyclic. Then there exists a matrix K of appropriate dimension such that the output-feedback control system with feedback strength control*

$$\dot{x} = (A + u(t)BKC)x$$

is accessible on $\mathbb{R}^n \setminus \{0\}$.

Chapter 4

Bilinear Control of Networks by Interconnections

Feedback control systems wherein the control loops are closed through an underlying network are denoted as network control systems [48]. For instance, these networks can be considered as multi-agent systems, where the vertices of a graph given by the network represent the agents and the edges represent the inter-agent communication links. Clearly, the structure of the underlying graph plays a central role in representing the information flow between the agents and in analyzing the system's accessibility. In applications where the coupling structure of a network is not fixed apriori and can be therefore considered as a control parameter the problem occurs whether one can steer a given initial formation of the network to a desired final formation by switching or tuning the interconnections suitably. This chapter aims to develop a graph-based framework for the analysis of accessibility properties of networked control systems.
In contrast to Chapter 5 we assume that there exist no linear dependencies between the interconnections, i.e. all interconnections are assumed to be independently controllable.

The structure of this chapter is as follows: In Section 4.1 we give some basics and introductory results on graph theory and in Section 4.2 we describe the problem setting for networks of SISO systems. We derive the bilinear system which arises from the network of SISO systems by applying output feedback based on the interconnections of the underlying graph and regarding the interconnection strength as independently controllable. In the subsequent section, Section 4.3, we compute the system Lie algebra of a network of SISO systems with controllable interconnections for networks with three or more vertices in case the underlying graph is strongly connected. By using that accessibility of bilinear systems is dependent on the system Lie algebra we deduce a necessary and sufficient condition for this particular type of networks to be accessible in terms of the graph structure. By proving some special cases for networks with two vertices, we back up the conjecture

that we can apply the same results. In Section 4.4 we examine the case when the underlying graph is not strongly connected and hence does not satisfy the requirements for the developed accessibility conditions. We show how the system Lie algebra is structured. Until now, all results are given for heterogeneous networks of real SISO systems.
Section 4.5 gives one further result on homogeneous networks, i.e. networks, where the SISO systems on all vertices are equal. Here, we allow the matrices of the SISO systems to be complex but only allow for real controls and compute the real generated system Lie algebra. Again we have to make the constraint that the number of vertices has to be greater than two. In Section 4.6 we adapt the setting to networks of MIMO systems and show how to use the proof ideas of the preceding sections exemplarily.

4.1 Graph basics

In this section we first recall some basic notions from graph theory. We refer the interested reader to [19, 27].

A *graph* $\Gamma = (E, V)$ is the pair of a finite set of vertices V and a set E of edges, which consists of distinct, unordered pairs of vertices. A *directed graph* $\Gamma = (E, V)$ is a graph with a set of ordered edges E. In this thesis, we only consider directed graphs. Additionally, we always take $V = \{1, 2, \dots, N\}$ as given and assume in this chapter that the graph Γ is *simple*, i.e. it has no parallel edges and no self-loops. A *self-loop* is an edge connecting a vertex to itself. The *associated adjacency matrix* γ *of a graph* Γ is defined by

$$(\gamma)_{ij} := \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & else. \end{cases} \tag{4.1}$$

It is clear that the adjacency matrix of an unordered graph is symmetric, but the adjacency matrix of a directed graph is in general not symmetric. An undirected graph Γ is *connected* if there exists a path in Γ between any pair of distinct vertices. Here, a *path* $i_0 i_1 \dots i_S$ in Γ from vertex i to vertex j is a finite sequence of vertices such that $(i_{k-1}, i_k) \in E$ for $k = 1, \dots, S$, $i_0 = i$ and $i_S = j$ or $i_0 = j$ and $i_S = i$. A *directed path* $i_0 i_1 \dots i_S$ from vertex i to vertex j is a finite sequence of vertices such that $(i_{k-1}, i_k) \in E$ for $k = 1, \dots, S$, $i_0 = i$ and $i_S = j$. Two vertices i and j of a directed graph $\Gamma = (E, V)$ are called *strongly connected* if there exists a directed path in Γ from i to vertex j and one from j to i. Clearly, this constitutes an equivalence relation on the vertices. As such, it partitions V into disjoint sets, called the *strongly connected components*, which correspond to the maximal strongly connected subgraphs. A crucial definition for the sequel of this chapter is the notion of a strongly connected graph.

Definition 4.1. A directed graph $\Gamma = (E, V)$ is called *strongly connected* if every pair of vertices (i, j) is strongly connected.

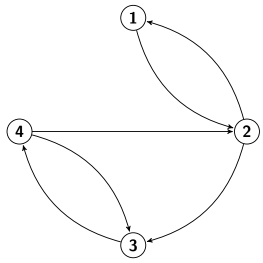

FIGURE 4.1: Example for a strongly connected graph Γ

A directed graph $\Gamma = (E, V)$ is called *weakly connected* if the underlying undirected graph is connected but Γ is not strongly connected.

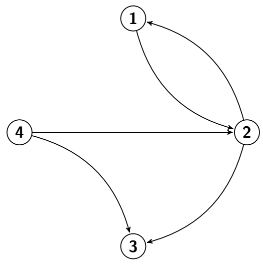

FIGURE 4.2: Example for a weakly connected graph Γ

The *weakly connected components* of a directed graph are the maximal weakly connected subgraphs. The weakly and strongly connected components define unique partitions on the vertices. A graph is called *disconnected* if it has more than one weakly connected component.

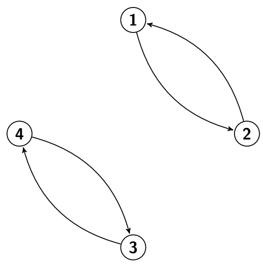

FIGURE 4.3: Example for a disconnected graph Γ with 2 strongly connected components

Definition 4.2. A vertex j of a directed graph is called *isolated* if the strongly connected component, which contains j, only consists of j itself.

The following theorem provides us with a normal form for the adjacency matrix γ of a graph Γ, the so-called *Frobenius normal form* (Theorem 3.2.4. [11]).

Theorem 4.3. *Let A be a matrix of order n. Then there exists a permutation matrix P of order n and an integer $t \geq 1$ such that*

$$PAP^\top = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1t} \\ 0 & A_{22} & \dots & A_{2t} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & A_{tt} \end{pmatrix} \tag{4.2}$$

where $A_{11}, A_{22}, \dots, A_{tt}$ are permutation-irreducible square matrices. The matrices A_{ii} are uniquely determined to within simultaneous permutation of their lines, but their ordering in (4.2) is not necessarily unique.

A matrix A is permutations-reducible, if there exists a permutation matrix P such that

$$PAP^\top = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a $r \times r$ square matrix with $0 < r < n$. It is called permutation-irreducible if it is not permutation-reducible (cf. Definition 2.10).

The following is a well-known result (Theorem 2.7 [2]) relating the structure of a graph to the structure of its adjacency matrix.

Theorem 4.4. *Let $\Gamma = (E, V)$ be a directed graph with N vertices and γ its adjacency matrix. Then*

i) *Γ is strongly connected if and only if γ is permutation-irreducible.*

ii) *Γ is disconnected if and only if there exists a permutation matrix P such that $P\gamma P^\top$ has block diagonal structure, i.e.*

$$P\gamma P^\top = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix},$$

with $A_i \in \mathbb{R}^{N_i \times N_i}$ and $N = \sum_{i=1}^k N_i$.

iii) *Γ is disconnected and consists only of strongly connected components which are not connected among each other if and only if there exists a permutation matrix P*

such that $P\gamma P^\top$ has block diagonal structure and the matrices on the diagonal are permutation-irreducible, i.e.

$$P\gamma P^\top = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$$

with $A_i \in \mathbb{R}^{N_i \times N_i}$ permutation-irreducible for all $i = 1, \ldots, k$ and $N = \sum_{i=1}^{k} N_i$.

4.2 Networks of SISO systems

We consider networks of N not necessarily identical interconnected linear SISO systems. Let the state of each vertex $i \in \{1, \ldots, N\}$ be an n-dimensional vector x_i in $\mathbb{R}^n$. Then the dynamics of every single linear SISO system in this network can be described by

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i v_i, \\ y_i &= c_i x_i, \end{aligned}$$

where each triple (A_i, b_i, c_i) at vertex i is an n-dimensional triple with $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}^{1 \times n}$. To investigate a heterogeneous network of linear systems, we fix an interconnection structure given by the adjacency matrix γ of a graph $\Gamma = (E, V)$. Based on the interconnections of Γ we apply output feedback of the form

$$v_i = \sum_{(j,i) \in E} c_j x_j.$$

Then the dynamics of every single linear system under output feedback can be described by

$$\dot{x}_i = A_i x_i + b_i \sum_{(j,i) \in E} c_j x_j$$

for all $1 \leq i \leq N$. Now, we regard the interconnection strength to be controllable and obtain the control system of every single node as

$$\dot{x}_i = A_i x_i + \sum_{(j,i) \in E} u_{ij}(t) b_i c_j x_j,$$

where u_{ij} are real valued control functions. Then the dynamics of the network forms a bilinear control system of the form

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x. \tag{4.3}$$

Since we assumed that there do not exist linear dependencies between the interconnections of Γ, the so-called *controlled adjacency matrix* $\gamma(u)$ of the graph Γ is given by

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}, \tag{4.4}$$

where E_{ij} denotes the matrix with one 1 at the entry (i, j) and zero elsewhere. This means that we allow the controls of the bilinear control system (4.3) to vary through all linear combinations of the set of matrices

$$\{E_{ij} \,\big|(i,j) \in E\}.$$

The coefficient matrices of (4.3) are given by

$$\mathcal{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{pmatrix} \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_N \end{pmatrix}, \tag{4.5}$$

where $\mathcal{A} \in \mathbb{R}^{nN\times nN}$, $\mathcal{B} \in \mathbb{R}^{nN\times N}$ and $\mathcal{C} \in \mathbb{R}^{N\times Nn}$. When all triples (A_i, b_i, c_i) are equal for all $1 \le i \le N$, the network is called *homogeneous*. Then, the bilinear system simplifies to

$$\dot{x} = (I \otimes A + \gamma(u) \otimes bc)\, x.$$

Otherwise the network is called *heterogeneous*.

Since we want to deduce accessibility conditions for systems of the form (4.3), we are interested in computing the associated system Lie algebra. The real controls u_{ij} in (4.4) can take on every real values and thus one control u_{ij} can be 1 while the other controls are zero. Therefore the system Lie algebra of (4.3) is the Lie algebra generated by the matrices $\mathcal{A}$ and $E_{ij} \otimes b_i c_j$ for all $(i, j) \in E$. Here, the symbol $\otimes$ denotes the *matrix Kronecker product*, which is a bilinear map

$$\otimes : \mathbb{R}^{N\times N} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{nN\times nN}$$

defined by

$$(A, B) \mapsto A \otimes B := \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

The commutator of $\mathcal{A} = \sum_{k=1}^{N} E_{ll} \otimes A_l$ and $E_{ij} \otimes b_i c_j$ for any $(i,j) \in E$ yields

$$[\mathcal{A}, E_{ij} \otimes b_i c_j] = \sum_{l=1}^{N} [E_{ll} \otimes A_l, E_{ij} \otimes b_i c_j] = E_{ij} \otimes (A_i b_i c_j - b_i c_j A_j).$$

Iteratively, we deduce

$$\mathrm{ad}_{\mathcal{A}}^{m} (E_{ij} \otimes b_i c_j) = E_{ij} \otimes \left(\sum_{p=0}^{m} (-1)^p \binom{m}{p} A_i^{m-p} b_i c_j A_j^p \right). \tag{4.6}$$

For later purpose, we define the abbreviation

$$\mathrm{ad}_{A_i,A_j}^{m}(b_i c_j) := \sum_{p=0}^{m} (-1)^p \binom{m}{p} A_i^{m-p} b_i c_j A_j^p \tag{4.7}$$

and therefore, we can write equation (4.6) as

$$\mathrm{ad}_{\mathcal{A}}^{m} (E_{ij} \otimes b_i c_j) = E_{ij} \otimes \mathrm{ad}_{A_i,A_j}^{m}(b_i c_j).$$

The operator ad_{A_i,A_j} can be defined for arbitrary matrices $X \in \mathbb{R}^{n \times n}$ by

$$\mathrm{ad}_{A_i,A_j}(X) = A_i X - X A_j.$$

It is a special case of the *Sylvester operator*

$$X \mapsto AX + XB.$$

Denote by $\sigma(A) := \{\lambda \in \mathbb{C} \mid \mathrm{Ker}(A - \lambda I) \neq \{0\}\}$ the spectrum of a matrix A. Let

$$\mathrm{RE}_\lambda(A) := \left\{ v \in \mathbb{R}^n \mid (A - \lambda I)^k v = 0 \text{ for } k \in \mathbb{N} \right\}$$

be the generalized right eigenspace associated to the eigenvalue λ of A and

$$\mathrm{LE}_\lambda(A) := \left\{ v \in \mathbb{R}^n \mid v^\top (A - \lambda I)^k = 0 \text{ for } k \in \mathbb{N} \right\}$$

the generalized left eigenspace associated to the eigenvalue λ of A.

We state some properties of the operator $\mathrm{ad}_{A_i,A_j}(\cdot)$ in the following theorem.

Theorem 4.5. *Let* $\mathrm{ad}_{A_i,A_j} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ *be defined by* $X \mapsto A_iX - XA_j$. *Then the operator* ad_{A_i,A_j} *has the following properties.*

(i) The spectrum of the Sylvester operator ad_{A_i,A_j} *is*

$$\sigma\left(\mathrm{ad}_{A_i,A_j}\right) = \left\{\lambda - \mu \;\middle|\; \lambda \in \sigma(A_i), \mu \in \sigma(A_j)\right\}.$$

(ii) The spectrum of the iterated Sylvester operator $\mathrm{ad}^k_{A_i,A_j}$ *is*

$$\sigma\left(\mathrm{ad}^k_{A_i,A_j}\right) = \left\{(\lambda - \mu)^k \;\middle|\; \lambda \in \sigma(A_i), \mu \in \sigma(A_j)\right\}.$$

(iii) The Sylvester operator ad_{A_i,A_j} *is nonsingular if and only if* $\sigma(A_i) \cap \sigma(A_j) = \emptyset$.

(iv) The generalized kernel of ad_{A_i,A_j} *is given by*

$$\bigcup_{k\geq 1} \mathrm{Ker}\left(\mathrm{ad}^k_{A_i,A_j}\right) := \mathrm{span}\left\{v_i v_j^\top \;\middle|\; v_i \in \mathrm{RE}_\lambda(A_i), v_j \in \mathrm{LE}_\lambda(A_j) \text{ for } \lambda \in \mathbb{C}\right\},$$

i.e. it is the subspace generated by all rank-one matrices of the form $v_i v_j^\top$, *where* $v_i \in \mathrm{RE}_\lambda(A_i)$ *and* $v_j \in \mathrm{LE}_\lambda(A_j)$.

The proof of Theorem 4.5 can be found in [50].

4.3 Strongly connected networks

By assuming that all interconnections are independently controllable, we want to derive accessibility conditions for the bilinear control system (4.3) in terms of the graph structure. For this purpose, we compute the system Lie algebra in case the graph $\Gamma = (V, E)$ is strongly connected and has $N > 2$ vertices. We show that it is a necessary and sufficient condition for the underlying directed graph to be strongly connected in order to obtain accessibility of the bilinear control system (4.3). Later in this section, we draw our attention to the case $N = 2$. In case the network only consists of one vertex, we need a self-loop in order to obtain a control system. This leads us to control systems of the form

$$\dot{x} = (A + u(t)bc)x,$$

which are addressed by Theorem 3.28.

The main result of this section is the following theorem.

Theorem 4.6. *Let (A_i, b_i, c_i) be controllable and observable for all $1 \leq i \leq N$ with $N > 2$ and*

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

be the controlled adjacency matrix of a strongly connected graph $\Gamma = (E, V)$ with N vertices. Then the system Lie algebra $\mathfrak{G}$ of

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$. In particular, $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$ if and only if $\operatorname{tr}\mathcal{A} \neq 0$.

For the proof we need the following technical lemmas.

Lemma 4.7. *Let $1 \leq i, j, k \leq N$ be fixed and let the triples (A_i, b_i, c_i), (A_j, b_j, c_j) and (A_k, b_k, c_k) be controllable and observable. Then the real vector space generated by elements of the form*

$$X = \operatorname{ad}^m_{A_i,A_j}(b_i c_j) \cdot \operatorname{ad}^v_{A_j,A_k}(b_j c_k)$$

with $m, v \in \mathbb{N}_0$ is $\mathbb{R}^{n\times n}$, i.e.

$$\operatorname{span}\left\{ X = \sum_{p=0}^{m}\sum_{q=0}^{v}(-1)^{p+q}\binom{m}{p}\binom{v}{q}(c_j A_j^{p+v-q} b_j) A_i^{m-p} b_i c_k A_k^q \;\middle|\; m, v \in \mathbb{N}_0 \right\} = \mathbb{R}^{n\times n}.$$

Proof. Let $1 \leq i, j, k \leq N$ be fixed and

$$M := \operatorname{span}\left\{ X = \sum_{p=0}^{m}\sum_{q=0}^{v}(-1)^{p+q}\binom{m}{p}\binom{v}{q}(c_j A_j^{p+v-q} b_j) A_i^{m-p} b_i c_k A_k^q \;\middle|\; m, v \in \mathbb{N}_0 \right\}.$$

In case $A_i^p b_i c_k A_k^q \in M$ holds for all $p, q \in \mathbb{N}_0$, we can apply a similar result to Lemma 3.13 and the result follows immediately.
Due to the controllability and observability of (A_j, b_j, c_j) there exists an integer $m \in \mathbb{N}$ such that $c_j A_j^m b_j \neq 0$. Denote the minimal integer with this property by $m^\star$. First, we consider

$$\begin{aligned}\operatorname{ad}^{m^\star}_{A_i,A_j}(b_i c_j) \cdot \operatorname{ad}^0_{A_j,A_k}(b_j c_k) &= \sum_{p=0}^{m^\star}(-1)^p\binom{m^\star}{p}(c_j A_j^p b_j) A_i^{m^\star-p} b_i c_k \\ &= (-1)^{m^\star}(c_j A_j^{m^\star} b_j) b_i c_k.\end{aligned}$$

Therefore, $b_i c_k \in M$. Now, let $l \in \mathbb{N}$ be fixed. Then we obtain for $h = 0$ the element

$$\operatorname{ad}^{m^\star+l-0}_{A_i,A_j}(b_i c_j) \cdot \operatorname{ad}^0_{A_j,A_k}(b_j c_k) = \sum_{p=0}^{m^\star+l}(-1)^p\binom{m^\star+l}{p}(c_j A_j^p b_j) A_i^{m^\star+l-p} b_i c_k$$

and for $h = 1$ the element

$$\begin{aligned}\mathrm{ad}^{m^\star+l-1}_{A_i,A_j}(b_ic_j)\cdot \mathrm{ad}^{1}_{A_j,A_k}(b_jc_k) &= \sum_{p=0}^{m^\star+l}(-1)^p\binom{m^\star+l-1}{p}(c_jA_j^pb_j)A_i^{m^\star+l-1-p}b_ic_k\\ &+\sum_{p=0}^{m^\star+l}(-1)^p\binom{m^\star+l-1}{p}(c_jA_j^pb_j)A_i^{m^\star+l-2-p}b_ic_kA_k\end{aligned}$$

as elements of M. Under the assumption that $A_i^pb_ic_k$ and $A_i^qb_ic_kA_k$ are elements of M for $p = 0,\dots,l-1$ and $q = 0,\dots,l-2$, we get that $A_i^lb_ic_k$ and $A_i^{l-1}b_ic_kA_k$ are elements of M. In doing so for all $h = 0,\dots,l$, we obtain $A_i^{l-q}b_ic_kA_k^q \in M$ for every $q = 0,\dots,l$. Hence, we can state as a induction hypothesis that $A_j^pb_jc_kA_k^q \in M$ for all $p+q \le l$. We now show for all $l \in \mathbb{N}$, if $A_i^pb_ic_kA_k^q \in M$ for all $p+q \le l$, then $A_i^pb_ic_kA_k^q \in M$ for all $p+q \le l+1$. For this purpose, consider for $0 \le h \le l+1$ the element

$$\begin{aligned}&\mathrm{ad}^{m^\star+l+1-h}_{A_i,A_j}(b_ic_j)\cdot \mathrm{ad}^{h}_{A_j,A_k}(b_jc_k) =\\ &\quad\sum_{p=0}^{m^\star+l+1-h}\sum_{q=0}^{h}(-1)^{p+q}\binom{m^\star+l+1-h}{p}\binom{h}{q}(c_jA_j^{p+h-q}b_j)A_i^{m^\star+l+1-h-p}b_ic_kA_k^q.\end{aligned}$$

Then

$$(c_jA_j^{m^\star}b_j)\sum_{q=0}^{h}\binom{m^\star+l+1-h}{h-q}\binom{h}{q}A_i^{l+1-q}b_ic_kA_k^q \in M$$

for all $0 \le h \le l+1$. Hence, $A_i^{l-p}b_ic_kA_k^p \in M$ and the result follows from Lemma 3.13. □

Lemma 4.8. *Let $\mathfrak{G} \subset \mathfrak{gl}_{nN}(\mathbb{R})$ be a Lie subalgebra. If and only if $E_{ij}\otimes\mathbb{R}^{n\times n} \subset \mathfrak{G}$ for all $i \neq j$, $i,j = 1,\dots,N$, then*

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subseteq \mathfrak{G}.$$

Proof. For $i \neq j$ and $k \neq m$ we have

$$[E_{ij}\otimes E_{kl}, E_{ji}\otimes E_{lm}] = E_{ii}\otimes E_{km} \in \mathfrak{G}.$$

Then the if-direction is an immediate consequence from Lemma 3.23.
The other direction is trivial. □

An obvious lemma is the following, which we do not prove.

Lemma 4.9. *Let $\mathfrak{G} \subseteq \mathfrak{gl}_{nN}(\mathbb{R})$ be a real Lie algebra with $\mathfrak{sl}_{nN}(\mathbb{R}) \subseteq \mathfrak{G}$. Then either $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$.*

Now, we give the proof for Theorem 4.6.

Proof. For $(i,j) \in E$ we obtain $E_{ij} \otimes b_i c_j \in \mathfrak{G}$ and consequently, $E_{ij} \otimes \mathrm{ad}^k_{A_i,A_j}(b_i c_j) \in \mathfrak{G}$ for all $k \in \mathbb{N}$. Let $(i,k) \notin E$. Due to the strong connectedness of Γ we find a directed path on Γ from i to k. Without loss of generality, we us assume that the path consists of two edges, i.e. it exists a vertex j such that (i,j) and $(j,k) \in E$. Thus,

$$[E_{ij} \otimes b_i c_j, E_{jk} \otimes b_j c_k] = (c_j b_j) E_{ik} \otimes b_i c_k \in \mathfrak{G}.$$

In case $c_j b_j = 0$, due to the controllability and observability of (A_i, b_i, c_i), there exists an integer $m \in \mathbb{N}$ such that $c_j A_j^m b_j \neq 0$. Denote the minimal m with this property by $m^\star$. Then

$$\begin{aligned}
[E_{ij} \otimes b_i c_j, E_{jk} \otimes \mathrm{ad}^{m^\star}_{A_j,A_k}(b_j c_k)] =& E_{ik} \otimes \sum_{p=0}^{m^\star} (-1)^p \binom{m^\star}{p} b_i c_j A_i^{m^\star - p} b_i c_k A_k^p \\
=& (c_j A_j^{m^\star} b_j) E_{ik} \otimes b_i c_k \in \mathfrak{G}.
\end{aligned}$$

So, $E_{ij} \otimes \mathrm{ad}^k_{A_i,A_j}(b_i c_j) \in \mathfrak{G}$ for all $i \neq j$ and all $k \in \mathbb{N}$. From

$$[E_{ij} \otimes X, E_{jk} \otimes Y] = E_{ik} \otimes XY$$

we obtain with Lemma 4.7 that $E_{ik} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$. Hence, the result follows from Lemma 4.8. □

Remark 4.10. The requirement that (A_i, b_i, c_i) has to be controllable and observable for all $i = 1, \ldots, N$ is necessary to obtain $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$ as system Lie algebra. To see this we assume that there is one i such that (A_i, b_i, c_i) is not controllable and observable. Hence, either $\det(b_i, A_i b_i, \ldots, A_i^{n-1} b_i) = 0$ or $\det(c_i, c_i A_i, \ldots, c_i A_i^{n-1}) = 0$ holds. Without loss of generality assume $\det(b_i, A_i b_i, \ldots, A_i^{n-1} b_i) = 0$ and denote the rank of $b_i, A_i b_i, \ldots, A_i^{n-1} b_i$ by $r \lneq n$. To get $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$ it is necessary to have $E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$ for all $i \neq j$ (Lemma 4.8). We can easily see that from $E_{ij} \otimes X \in \mathfrak{G}$ it immediately follows that

$$X \in \mathrm{span}\{A_i^p b_i c_j A_j^q \mid p, q \in \mathbb{N}_0\}.$$

But the set $\mathrm{span}\{A_i^p b_i c_j A_j^q, p, q \in \mathbb{N}_0\}$ has dimension $r \cdot n \lneq n^2$ for all $j \neq i$ since (A_i, b_i, c_i) is not controllable and observable. Thus, $E_{ij} \otimes \mathbb{R}^{n \times n} \not\subseteq \mathfrak{G}$ for all $j \neq i$ and therefore,

$$\mathfrak{sl}_{nN}(\mathbb{R}) \not\subseteq \mathfrak{G}.$$

The analog follows in case $\det(c_i, c_i A_i, \ldots, c_i A_i^{n-1}) = 0$.

We deduce the following theorem as a consequence of Theorem 4.6.

Theorem 4.11. *Let $N > 2$, $\Gamma = (E, V)$ be a simple directed graph with N vertices and the triples (A_i, b_i, c_i) be controllable and observable for all $i = 1, \ldots, N$. Then the controlled* *network*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\,x \tag{4.8}$$

is accessible on $\mathbb{R}^{nN}\backslash\{0\}$ *if and only if* Γ *is strongly connected.*

Proof. The if-direction follows from Theorem 4.6 and Theorem 2.3. For the other direction let the graph Γ be not strongly connected. Then, due to Theorem 4.4, its adjacency matrix γ is permutation-reducible and we can find a permutation matrix $P \in GL(N, \mathbb{R})$ such that

$$P^{-1}\gamma P = \begin{pmatrix} d_1 & d_2 \\ 0 & d_3 \end{pmatrix}, \tag{4.9}$$

where d_3 is a $r \times r$ matrix, $0 < r < N$. Hence, we obtain

$$(P^{-1} \otimes I_n)(\mathcal{B}\gamma\mathcal{C})(P \otimes I_n) = \mathcal{B}\begin{pmatrix} d_1 & d_2 \\ 0 & d_3 \end{pmatrix}\mathcal{C}. \tag{4.10}$$

Clearly, a permutation matrix of the form $P \otimes I_n$ leaves $\mathcal{A} = \sum_{i=1}^{N} E_{ii} \otimes A_i$ in its block diagonal form. Hence, $\mathcal{A}$ and $\mathcal{B}\gamma(u)\mathcal{C}$ are conjugated to matrices of the form (4.10). But the set of all matrices of the form (4.10) constitutes a Lie subalgebra $\mathfrak{G}^\star$ such that the group generated by $\exp(\mathfrak{G}^\star)$ does not have an interior point. Hence, system (4.8) is not accessible. □

Since a homogeneous network is a special case of an heterogeneous network, we derive the result for homogeneous networks as a special case of Theorem 4.6.

Corollary 4.12. *Let* (A, b, c) *be controllable and observable,* $N > 2$ *and*

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij}E_{ij}$$

be the controlled adjacency matrix of a strongly connected simple graph $\Gamma = (E, V)$ *with* N *vertices. Then the system Lie algebra of the controlled network*

$$\dot{x} = (I_N \otimes A + \gamma(u) \otimes bc)x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ *or* $\mathfrak{gl}_{nN}(\mathbb{R})$*. In particular,* $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$ *is and only if* $\operatorname{tr} A \neq 0$*.*

As an example, we state the following controllability result.

Corollary 4.13. *Let* $N > 2$*,* $\Gamma = (E, V)$ *be a simple directed graph with* N *vertices, which is strongly connected and the triples* (A_i, b_i, c_i) *are controllable and observable with*

A_i skew-symmetric for all $i = 1, \dots, N$. Then the controlled network

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x$$

is controllable on $\mathbb{R}^{nN}\backslash\{0\}$.

Proof. With Theorem 4.6 we have $\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}$. Now, choose the constant control $u = 0$. Then for the trajectory $x(t)$ of the system

$$\dot{x} = \mathcal{A}x$$

with $x(0) = I$ we get $x(t) \in SO(nN)$ since $\mathcal{A}$ is skew-symmetric. Hence, $x(t)$ is either periodic or almost periodic. Hence, we can find a sequence of positive numbers $\{t_n\}$, such that the limit $\lim x(t_n)$ exists. W.l.o.g., we can assume $\lim t_n = \infty$. Hence, there exists a subsequence $\{t_{n_k}\}$ with $t_{n_{k+1}} - t_{n_k} > k$. Now, consider the sequence $\tau_k := t_{n_{k+1}} - t_{n_k}$. Then, $\lim x(\tau_k) = I$ and therefore $\lim x(\tau_k)$ belongs to any subgroup of the system group $\mathcal{G}$. Now, we can apply Theorem 2.9 and the result follows from Theorem 2.8. □

Since in the proof of Theorem 4.6 we need the graph Γ to consist of three or more vertices to show that $\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}$, we have to make an additional assumption, to prove a similar result for a graph with only two vertices. In the sequel, let $m_i^\star$ denote the smallest integer m_i with $c_i A_i^{m_i} b_i \neq 0$.

Theorem 4.14. *Let $N = 2$, the triples (A_1, b_1, c_1) and (A_2, b_2, c_2) be controllable and observable and $\Gamma = (E, V)$ be a strongly connected simple graph with two vertices. Suppose that $m_1^\star \neq m_2^\star$. Then the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x. \tag{4.11}$$

is either $\mathfrak{sl}_{2n}(\mathbb{R})$ or $\mathfrak{gl}_{2n}(\mathbb{R})$. In particular, $\mathfrak{G} = \mathfrak{gl}_{2n}(\mathbb{R})$ if and only if $\operatorname{tr}\mathcal{A} \neq 0$.

Proof. Since Γ is strongly connected, we have $E_{12} \otimes \operatorname{ad}^m_{A_1,A_2}(b_1c_2), E_{21} \otimes \operatorname{ad}^m_{A_2,A_1}(b_2c_1) \in \mathfrak{G}$ for all $m \in \mathbb{N}_0$. Without loss of generality, let $m_1^\star = \min\{m_1^\star, m_2^\star\}$. Particularly, we get $c_2 A_2^m c_2 = 0$ for $0 \leq m < m_1^\star$ and therefore

$$\begin{aligned}
&[E_{12} \otimes b_1c_2, E_{21} \otimes \operatorname{ad}^{m_1^\star}_{A_2,A_1}(b_2c_1)] \\
&= E_{11} \otimes \sum_{j=0}^{m_1^\star} (-1)^j \binom{m_1^\star}{j} b_1c_2A_2^{m_1^\star - j} b_2c_1A_1^j - E_{22} \otimes \sum_{j=0}^{m_1^\star} (-1)^j \binom{m_1^\star}{j} A_2^{m_1^\star - j} b_2c_1A_1^j b_1c_2 \\
&= E_{11} \otimes \sum_{j=0}^{m_1^\star} (-1)^j \binom{m_1^\star}{j} \left(c_2A_2^{m_1^\star - j} b_2\right) b_1c_1A_1^j - E_{22} \otimes \sum_{j=0}^{m_1^\star} (-1)^j \binom{m_1^\star}{j} \left(c_1A_1^j b_1\right) A_2^{m_1^\star - j} b_2c_2 \\
&= (c_1A_1^{m_1^\star} b_1)E_{22} \otimes b_2c_2 \in \mathfrak{G}.
\end{aligned}$$

As $c_1 A_1^{m_1^\star} b_1 \neq 0$, we obtain $E_{22} \otimes \mathrm{ad}_{A_2}^m(b_2c_2) \in \mathfrak{G}$ for all $m \in \mathbb{N}_0$. With

$$\begin{aligned}[E_{22} \otimes \mathrm{ad}_{A_i}^k(b_ic_i), E_{ij} \otimes \mathrm{ad}_{A_i,A_j}^l(b_ic_j)] =& (-1)^i E_{ij} \otimes \left(\mathrm{ad}_{A_i}^k(b_ic_i) \cdot \mathrm{ad}_{A_i,A_j}^l(b_ic_j)\right) \\ =& (-1)^i E_{ij} \otimes \left(\mathrm{ad}_{A_i,A_i}^k(b_ic_i) \cdot \mathrm{ad}_{A_i,A_j}^l(b_ic_j)\right)\end{aligned}$$

for all $k, l \in \mathbb{N}_0$ we deduce $E_{ij} \otimes \mathbb{R}^{n\times n} \subset \mathfrak{G}$ for $i \neq j$ from Lemma 4.7. Then the result follows by Lemma 4.8. □

Since we assumed $m_1^\star \neq m_2^\star$ the case of a homogeneous network with $(A_1, b_1, c_1) = (A_2, b_2, c_2)$ is excluded in Theorem 4.14. We give another result with a graph theoretical assumption, where we can determine the system Lie algebra of (4.11).

Theorem 4.15. *Let (A_1, b_1, c_1) and (A_2, b_2, c_2) be controllable and observable and let $\Gamma = (E, V)$ be a strongly connected graph with $N = 2$ vertices and at least one self-loop. Then the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x.$$

is either $\mathfrak{sl}_{2n}(\mathbb{R})$ or $\mathfrak{gl}_{2n}(\mathbb{R})$. In particular, $\mathfrak{G} = \mathfrak{gl}_{2n}(\mathbb{R})$ if and only if either $\operatorname{tr}\mathcal{A} \neq 0$ or $c_ib_i \neq 0$, where at vertex i is a self-loop.

Proof. Without loss of generality, let vertex 1 have the self-loop. Then $E_{11} \otimes \mathrm{ad}_{A_1}^m(b_1c_1) \in \mathfrak{G}$ and the remaining part follows as in the proof of Theorem 4.14. □

The following example shows that the assumptions of Theorem 4.14 and Theorem 4.15 are not necessary.

Example 4.16. *Let Γ be a strongly connected simple graph with 2 vertices. Then the controlled adjacency matrix of Γ is given by*

$$\gamma(u) = \begin{pmatrix} 0 & u_{12} \\ u_{21} & 0 \end{pmatrix}.$$

We determine the system Lie algebra for a homogeneous network with the controllable and observable triple

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Thus, we compute the Lie algebra $\mathfrak{G}$ *generated by the three matrices*

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & bc \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ bc & 0 \end{pmatrix}.$$

One sees easily that $E_{ij} \otimes \mathrm{ad}_A(bc)$ *and* $E_{ij} \otimes \mathrm{ad}_A^2(bc) \in \mathfrak{G}$ *for* $i \neq j$*, at which*

$$\mathrm{ad}_A(bc) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_A^2(bc) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}.$$

Since $[E_{12} \otimes bc, E_{21} \otimes bc] = (cb)(E_{11} - E_{22}) \otimes bc$ *we have with* $E_{ij} \otimes X \in \mathfrak{G}$ *that*

$$E_{ij} \otimes (bcX + Xbc) \in \mathfrak{G}.$$

This yields

$$[(E_{11} - E_{22}) \otimes bc, E_{12} \otimes \mathrm{ad}_A^2(bc)] = E_{12} \otimes \begin{pmatrix} 4 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence, $E_{12} \otimes \mathbb{R}^{2\times 2} \subset \mathfrak{G}$ *and analogously* $E_{21} \otimes \mathbb{R}^{2\times 2} \subset \mathfrak{G}$*. With Lemma 4.8 and* $\operatorname{tr} A \neq 0$ *we obtain*

$$\mathfrak{G} = \mathfrak{gl}_4(\mathbb{R}).$$

4.4 Weakly connected networks

Theorem 4.14 and Theorem 4.15 suggest the conjecture that the main result (Theorem 4.6) from Section 4.3 can be transferred to the case $N = 2$. Example 4.16 shows that there exist examples, which support this conjecture. For the sake of simplicity, we assume for the sequel of Section 4.4 that Theorem 4.6 remains true for $N = 2$.

Conjecture 4.17. *Theorem 4.6 remains true for* $N = 2$*.*

In case this conjecture turns out to be not true, all theorems of this section need the additional assumption that the strongly connected components should consist of three or more vertices.

The following lemma gives the connection between the graph structure of Γ and the generated Lie algebra

$$\{E_{kl} \mid (k,l) \in E\}_{LA}.$$

Lemma 4.18. *Let $\Gamma = (E, V)$ be a directed simple graph. Then there exists a directed path in the graph Γ from vertex i to vertex j if and only if*

$$E_{ij} \in \{E_{kl} \mid (k,l) \in E\}_{LA}.$$

This lemma states that E_{ij} can be written as a Lie monomial on $\{E_{kl} \mid (k,l) \in E\}_{LA}$ if and only if there exists a directed path from vertex i to vertex j.

Proof. First, let there exists a directed path from vertex i to j. Then there exists a sequence of vertices

$$(i, i_1), (i_1, i_2), \dots, (i_m, j) \in E$$

and it follows for the associated sequence of matrices that

$$E_{ii_1}, E_{i_1 i_2}, \dots, E_{i_m j} \in \{E_{kl} \mid (k,l) \in E\}.$$

With

$$[E_{ii_1}, [E_{i_1 i_2}, [\dots, E_{i_m j}]]] = E_{ij}$$

we obtain $E_{ij} \in \{E_{kl} \mid (k,l) \in E\}_{LA}$.
Second, let us assume that $E_{ij} \in \{E_{kl} \mid (k,l) \in E\}_{LA}$. Then we have either $E_{ij} \in \{E_{kl} \mid (k,l) \in E\}$, i.e. there exists a vertex $(i,j) = (k,l)$ on Γ, or E_{ij} is a finite linear combination of Lie monomials on $\{E_{kl} \mid (k,l) \in E\}$. Using the Jacobi-identity (cf. Definition A.1) we can show that

$$[[E_{ij}, E_{jk}], [E_{kl}, E_{lm}]] = [E_{ij}, [E_{jk}, [E_{kl}, E_{lm}]]]$$

for all $E_{ij}, E_{jk}, E_{kl}, E_{lm}$. Therefore, we can inductively assume that the Lie monomial for E_{ij} is of the form

$$E_{ij} = [E_{k_1 k_2}, [E_{k_3 k_4}, [\dots, E_{k_{l-1} k_l}]]] \tag{4.12}$$

with $E_{k_i k_{i+1}} \in \{E_{kl} \mid (k,l) \in E\}$. From $[E_{ij}, E_{kl}] \neq 0$ it follows that $j = k$ or $l = i$ and then $[E_{ij}, E_{kl}] = E_{ij}$ if $j = k$ and $[E_{ij}, E_{kl}] = -E_{kj}$ if $l = i$. So, in both cases there exists a directed path on Γ from i to j and we obtain for (4.12) that $k_{2i+1} = k_{2i}$ for all $1 \leq i \leq \lfloor \frac{l}{2} \rfloor$ since $E_{ij} \neq 0$. Now, we get inductively if E_{ij} is a finite linear combination of Lie monomials on $\{E_{kl} \mid (k,l) \in E\}$, there exists a finite sequence of matrices

$$E_{ii_1}, E_{i_1 i_2}, \dots, E_{i_k j} \in \{E_{kl} \mid (k,l) \in E\}_{LA}$$

such that $[E_{ii_1}, [E_{i_1 i_2}, [\dots, E_{i_k j}]]] = E_{ij}$. It follows that there exists a path on Γ from vertex i to j. Hence, the result follows. □

If a directed graph is disconnected, its adjacency matrix has (up to a vertex permutation) block diagonal structure (Theorem 4.4). Since products of block diagonal matrices are again block diagonal matrices, the Lie algebra generated by matrices of block diagonal form consists only of matrices of this particular form. Therefore, the generated system Lie algebra is a proper subalgebra of $\mathfrak{gl}_{nN}(\mathbb{R})$ unequal $\mathfrak{sl}_{nN}(\mathbb{R})$.

Theorem 4.19. *Let (A_i, b_i, c_i) be controllable and observable for $1 \leq i \leq N$ and $\Gamma = (E, V)$ be a directed graph with N vertices. Then if the directed graph Γ is disconnected, the controlled network*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})x \tag{4.13}$$

is never accessible on $\mathbb{R}^{nN} \setminus \{0\}$. The system Lie algebra $\mathfrak{G}$ of (4.13) is the direct sum of the system Lie algebras of the weakly connected components, i.e. the system Lie algebra $\mathfrak{G}$ is a subalgebra of

$$\mathfrak{gl}_{N_1,\dots,N_k}(\mathbb{R}) := \begin{pmatrix} \mathfrak{gl}_{nN_1}(\mathbb{R}) & & \\ & \ddots & \\ & & \mathfrak{gl}_{nN_k}(\mathbb{R}) \end{pmatrix}. \tag{4.14}$$

Here, N_i denotes the number of vertices in the i-th weakly connected component and $\sum_{i=1}^{k} N_i = N$.

Proof. Let $C_1, \dots, C_k$ denote the weakly connected components of the graph Γ. Because the network is disconnected, we get from Theorem 4.4 (ii) that the adjacency matrix γ is similar to a block-diagonal matrix

$$P\gamma P^\top = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{pmatrix}. \tag{4.15}$$

Here, D_i denotes the $N_i \times N_i$ adjacency matrix of the connected component C_i, which consist of N_i vertices. Note that all D_i do not need to have the same size, but $N = \sum_{i=1}^{k} N_i$. Clearly, the structure (4.15) is preserved under the commutator (since products of block diagonal matrices are again block diagonal matrices) and therefore all matrices of the Lie algebra

$$\{\mathcal{A}, \mathcal{B}\gamma(u)\mathcal{C}\}_{LA}$$

are of the form

$$\begin{pmatrix} \tilde{D}_1 & & \\ & \ddots & \\ & & \tilde{D}_k \end{pmatrix}, \tag{4.16}$$

where $\tilde{D}_i$ is a $nN_i \times nN_i$ matrix. But the set of all matrices of the form (4.16) is closed under taking the commutator and thus constitutes a Lie subalgebra $\mathfrak{G}^\star$ such that the associated system Lie group does not have an interior point. Hence, system (4.13) is not accessible on $\mathbb{R}^{n\times n} \setminus \{0\}$. □

The notation (4.14) refers to the set of matrices which have the form

$$\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix}$$

with $X_i \in \mathfrak{gl}_{nN_i}(\mathbb{R})$. Clearly, this set constitutes a Lie algebra.

The following is a direct consequence from Theorem 4.4, Theorem 4.6 and Theorem 4.19.

Corollary 4.20. *Let (A_i, b_i, c_i) be controllable and observable with $\operatorname{tr} A_i = 0$ for $1 \leq i \leq N$ and let $\Gamma = (E, V)$ be a directed graph with N vertices, which decomposes into k strongly connected components. Then the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})x$$

is conjugated by a permutation to

$$\mathfrak{sl}_{N_1,\dots,N_k}(\mathbb{R}) := \begin{pmatrix} \mathfrak{sl}_{nN_1}(\mathbb{R}) & & \\ & \ddots & \\ & & \mathfrak{sl}_{nN_k}(\mathbb{R}) \end{pmatrix}. \tag{4.17}$$

Here, N_i denotes the number of vertices in the i-th strongly connected component and $\sum_{i=1}^k N_i = N$.

In the remaining chapter we concentrate on weakly connected graphs to classify the system Lie algebras of networks. Since the system Lie algebra decomposes into the direct sum of the system Lie algebras of the weakly connected components, this is sufficient to classify the system Lie algebra.

4.4.1 Networks of strongly connected components

Since we can classify with Theorem 4.6 the system Lie algebra in case the underlying graph is strongly connected, our first step to obtain results on weakly connected graphs is to limit ourselves to networks which are weakly connected but not strongly connected

and consist only of strongly connected components.
First, we consider the case, where Γ has two strongly connected components.

Theorem 4.21. *Let Γ be a weakly connected simple graph, which consists of two strongly connected components C_1 and C_2 with number of vertices N_1 and N_2, respectively, and let (A_i, b_i, c_i) be controllable and observable with $\operatorname{tr} A_i = 0$ for $1 \leq i \leq N$. Then the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x \tag{4.18}$$

is conjugated by a permutation to

$$\begin{pmatrix} \mathfrak{sl}_{nN_1}(\mathbb{R}) & \mathbb{R}^{nN_1 \times nN_2} \\ 0 & \mathfrak{sl}_{nN_2}(\mathbb{R}) \end{pmatrix}$$

with $N = N_1 + N_2$.

In case $\operatorname{tr} \mathcal{A} \neq 0$ there exists at least one triple (A_i, b_i, c_i) with $\operatorname{tr} A_i \neq 0$. W.l.o.g. let $i \in C_1$ and $\sum_{j=1}^{N_1} \operatorname{tr} A_j \neq 0$. Then the system Lie algebra is conjugated by a permutation to

$$\begin{pmatrix} \mathfrak{gl}_{nN_1}(\mathbb{R}) & \mathbb{R}^{nN_1 \times nN_2} \\ 0 & \mathfrak{sl}_{nN_2}(\mathbb{R}) \end{pmatrix}.$$

We give the proof of Theorem 4.21.

Proof. By Theorem 4.4 we can assume w.l.o.g. that the matrix $\gamma(u)$ is of the form

$$\begin{pmatrix} B_1 & D \\ 0 & B_2 \end{pmatrix}, \tag{4.19}$$

where $B_1 \in \mathbb{R}^{N_1 \times N_1}$, $B_2 \in \mathbb{R}^{N_2 \times N_2}$, $D \in \mathbb{R}^{N_1 \times N_2}$. With Conjecture 4.17 we can apply Theorem 4.4 to each strongly connected component, since the underlying directed graph of a strongly connected component is strongly connected. Therefore, due to the strong connectedness of the components C_1 and C_2, B_1 and B_2 are permutation-irreducible and $D \neq 0$ due to the connectedness of Γ. Applying Theorem 4.6 to the strongly connected components C_1 and C_2, i.e. to

$$\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \mathfrak{sl}_{nN_1}(\mathbb{R}) & 0 \\ 0 & \mathfrak{sl}_{nN_2}(\mathbb{R}) \end{pmatrix} \subseteq \mathfrak{G}. \tag{4.20}$$

Due to the connectedness of Γ we find a directed path from vertex i to vertex j if and only if they are either in the same strongly connected component or $i < j$ since we assume that the adjacency matrix of Γ is of the form (4.19).
Let $i < j$ be not in the same strongly connected component. Then there exists a directed path from vertex i to vertex j in the graph Γ. We obtain for the entries γ_{kl} of the adjacency matrix γ that

$$\gamma_{ik_1} \neq 0,\ \gamma_{k_1k_2} \neq 0, \ldots, \gamma_{k_{m-1}k_m} \neq 0,\ \gamma_{k_mj} \neq 0$$

holds. For $i \neq k, k \neq j$ regarding

$$[E_{ik} \otimes b_ic_k, E_{kj} \otimes b_kc_j] = E_{ij} \otimes b_ic_kb_kc_j = c_kb_k(E_{ij} \otimes b_ic_j)$$

we have

$$E_{ij} \otimes b_ic_j \in \mathfrak{G} \text{ for all } i < j$$

if $c_kb_k \neq 0$. In case $c_kb_k = 0$ we can show that $E_{ij} \otimes b_ic_j$ is an element of $\mathfrak{G}$ with the same construction as in the proof of Theorem 4.6, i.e. with $c_kA_k^{m^\star}b_k \neq 0$ and $m^\star$ minimal we obtain

$$[E_{ik} \otimes b_ic_k, E_{kj} \otimes \operatorname{ad}^{m^\star}_{A_k,A_j}(b_kc_j)] = \left(c_kA_k^{m^\star}b_k\right) E_{ij} \otimes b_ic_j \in \mathfrak{G}.$$

It follows $E_{ij} \otimes \operatorname{ad}^k_{A_i,A_j}(b_ic_j) \in \mathfrak{G}$ for all $k \in \mathbb{N}_0$ and now, with Lemma 4.7, we obtain

$$E_{ij} \otimes \mathbb{R}^{n\times n} \subset \mathfrak{G} \text{ for all } i < j.$$

Hence,

$$\begin{pmatrix} \mathfrak{sl}_{nN_1}(\mathbb{R}) & \mathbb{R}^{nN_1\times nN_2} \\ 0 & \mathfrak{sl}_{nN_2}(\mathbb{R}) \end{pmatrix} \subseteq \mathfrak{G} \tag{4.21}$$

and with $\operatorname{tr} A_1 = \operatorname{tr} A_2 = 0$ we obtain equality in (4.21). □

We can adapt Theorem 4.21 to a weakly connected graph Γ with finitely many strongly connected components C_i, $1 \leq i \leq k$.

Theorem 4.22. *Let $\Gamma = (E, V)$ be a weakly connected simple graph, which consists of k strongly connected components C_j with number of vertices N_j and let (A_i, b_i, c_i) be controllable and observable with $\operatorname{tr} A_i = 0$ for all $1 \leq i \leq N$. Then the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\,x \tag{4.22}$$

is conjugated by a permutation to

$$\begin{pmatrix} \mathfrak{sl}_{nN_1}(\mathbb{R}) & \mathcal{M}_{12} & \mathcal{M}_{13} & \dots & \mathcal{M}_{1k} \\ 0 & \mathfrak{sl}_{nN_2}(\mathbb{R}) & \mathcal{M}_{23} & \dots & \mathcal{M}_{2k} \\ 0 & 0 & \ddots & & \\ & & 0 & \mathfrak{sl}_{nN_{k-1}}(\mathbb{R}) & \mathcal{M}_{k-1k} \\ 0 & 0 & \dots & 0 & \mathfrak{sl}_{nN_k}(\mathbb{R}) \end{pmatrix} \tag{4.23}$$

with $N = \sum_{j=1}^{k} N_j$*, where* $\mathcal{M}_{ij} = \mathbb{R}^{nN_i \times nN_j}$ *if there exists a directed path on* Γ *from* C_i *to* C_j *and* $\mathcal{M}_{ij} = \{0\}$ *else.*

Proof. By Theorem 4.3 we can assume that γ is in the Frobenius normal form

$$\gamma = \begin{pmatrix} B_1 & D_{12} & \cdots & D_{1k} \\ & B_2 & \ddots & \vdots \\ & & \ddots & D_{k-1k} \\ & & & B_k \end{pmatrix}$$

and since Γ consists only of strongly connected components the matrices B_i are permutation irreducible for $1 \leq i \leq k$. From the weak connectedness of γ we know that not all $D_{ij} = 0$. By Corollary 4.20 we deduce

$$\mathfrak{sl}_{N_1,\dots,N_k}(\mathbb{R}) \subset \mathfrak{G}.$$

Then applying Theorem 4.21 to each pair of strongly connected components C_i and C_j, where a directed path exists from component C_i to C_j, yields

$$\begin{pmatrix} \ddots & & & & \\ & \mathfrak{sl}_{nN_i}(\mathbb{R}) & & \mathbb{R}^{nN_i \times nN_j} & \\ & & \ddots & & \\ & & & \mathfrak{sl}_{nN_j}(\mathbb{R}) & \\ & & & & \ddots \end{pmatrix} \subset \mathfrak{G},$$

where all parts which are left blank are equal to zero. With Lemma 4.18 we obtain $\mathcal{M}_{kl} = \{0\}$ Then the result follows. □

Remark 4.23. The Lie algebra (4.23) can be written as

$$\mathfrak{G} = \mathfrak{sl}_{N_1,\dots,N_k}(\mathbb{R}) \oplus \left(\mathcal{N} \otimes \mathbb{R}^{n\times n}\right),$$

where $\mathfrak{sl}_{N_1,\dots,N_k}(\mathbb{R})$ is defined as in (4.17), $\mathcal{N}$ is a subspace of strictly upper triangular $N \times N$ matrices and $\mathcal{N} \otimes \mathbb{R}^{n\times n}$ is the subspace generated by matrices of the form $N \otimes X$ with $N \in \mathcal{N}$ and $X \in \mathbb{R}^{n\times n}$. From this representation we observe that in case the graph Γ has a self-loop at vertex i and the system Lie algebra $\mathfrak{g} = \{A_i, b_i c_i\}_{LA}$ is either $\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$ or $\mathfrak{gl}_n(\mathbb{R})$ the result only differs by $\mathfrak{gl}_{nN_j}(\mathbb{R})$ instead of $\mathfrak{sl}_{nN_j}(\mathbb{R})$ in (4.17), where C_j is the strongly connected component, which contains the vertex i.

4.4.2 Networks with isolated vertices

Now, we investigate networks in which isolated vertices can appear (Definition 4.2). Again, we discuss systems of the form

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})x,$$

where (A_i, b_i, c_i) is controllable and observable and

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

is the controlled adjacency matrix of the graph $\Gamma = (E, V)$.

Now, we classify all occurring system Lie algebras if the network is weakly connected but not strongly connected.

Theorem 4.24. *Let (A_i, b_i, c_i) be a controllable and observable triple for $1 \le i \le N$, $\Gamma = (V, E)$ be a weakly connected simple graph and*

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

the controlled adjacency matrix of Γ. Denote by $\mathcal{J}$ the set of isolated vertices in Γ. Then the system Lie algebra $\mathfrak{G}$ of

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x.$$

fulfills the following conditions:

a) For $i \neq j$ it holds

(i) $E_{ij} \otimes \mathbb{R}^{n\times n} \subset \mathfrak{G}$ if

- *vertex i lies in a connected component;*

– *there exists a directed path from i to j with more than one edge.*

(ii) $E_{ij} \otimes \mathrm{span}\{\mathrm{ad}^k_{A_i,A_j}(b_i c_j), k \in \mathbb{N}_0\} \subset \mathfrak{G}$ *if $i, j \in \mathcal{J}$ and there exists no directed path from i to j on Γ with more than one edge.*

(iii) $E_{ij} \otimes X \notin \mathfrak{G}$ *for all $X \in \mathbb{R}^{n\times n}$ else.*

b) *(i)* $E_{ii} \otimes X - E_{jj} \otimes Y \in \mathfrak{G}$ *for all matrices $X, Y \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\, X = \mathrm{tr}\, Y$ if i and j are strongly connected;*

(ii) *In case $\mathrm{tr}\,\mathcal{A} \neq 0$ the set*

$$\mathbb{R} \sum_{j=1}^{N} E_{jj} \otimes A_j \subset \mathfrak{G}$$

is an additional Lie subalgebra.

The next three figures show how edges between the vertices i and j should be in order to satisfy (i)-(iii) of condition $a)$.

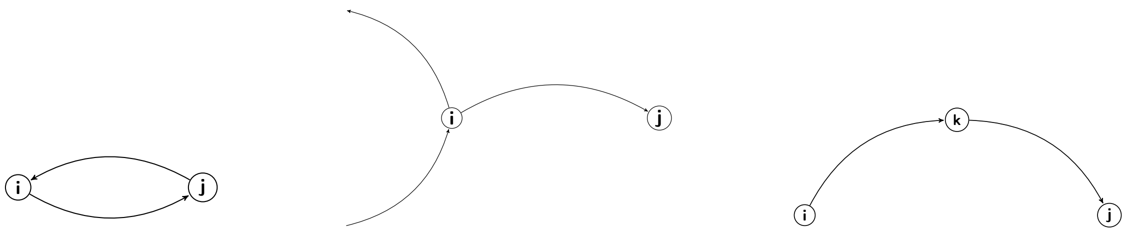

Figure 4.4: Condition $a)$ (i)

Figure 4.5: Condition $a)$ (ii)

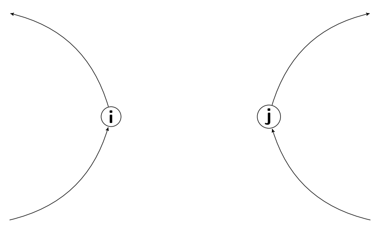

Figure 4.6: Condition $a)$ (iii)

Proof. Let there exist a directed path from vertex i to vertex j in the graph Γ, i.e. there exists a sequence of edges $(i_0, i_1), (i_1, i_2), \ldots, (i_m, i_{m+1}) \in E$ with $i = i_0$ and $j = i_{m+1}$. Hence, we obtain

$$E_{i_{k-1}, i_k} \otimes b_{i_{k-1}} c_{i_k} \in \mathfrak{G}$$

for $k = 1, \dots m+1$. Regarding for $i \neq k, k \neq j$

$$[E_{ik} \otimes b_i c_k, E_{kj} \otimes b_k c_j] = E_{ij} \otimes b_i c_k b_k c_j = c_k b_k (E_{ij} \otimes b_i c_j)$$

we obtain inductively that

$$E_{ij} \otimes b_i c_j \in \mathfrak{G} \text{ for all } i < j$$

if $c_k b_k \neq 0$. In case $c_k b_k = 0$, we can show that $E_{ij} \otimes b_i c_j$ is an element of $\mathfrak{G}$ with the same construction as in the proof of Theorem 4.6, i.e. with $c_k A_k^{m^\star} b_k \neq 0$ and $m^\star$ minimal we obtain

$$[E_{ik} \otimes b_i c_k, E_{kj} \otimes \mathrm{ad}^{m^\star}_{A_k, A_j}(b_k c_j)] = \left(c_k A_k^{m^\star} b_k \right) E_{ij} \otimes b_i c_j \in \mathfrak{G}.$$

It follows $E_{ij} \otimes \mathrm{ad}^k_{A_i, A_j}(b_i c_j) \in \mathfrak{G}$ for all $k \in \mathbb{N}_0$ if there is a directed path from i to j.

Now, we start with proving a). Let $i \neq j$. If vertex i lies in a connected component, we know from Theorem 4.6 that there exists a vertex k with $i \neq k$ and $j \neq k$ such that $E_{kk} \otimes Y - E_{ii} \otimes X \in \mathfrak{G}$ for all $X, Y \in \mathbb{R}^{n \times n}$ with $\operatorname{tr} X = \operatorname{tr} Y$. Therefore, we get

$$\left[(E_{kk} - E_{ii}) \otimes \mathrm{ad}^l_{A_i}(b_i c_i), E_{ij} \otimes \mathrm{ad}^k_{A_i, A_j}(b_i c_j) \} \right] = E_{ij} \otimes \mathrm{ad}^l_{A_i}(b_i c_i) \cdot \mathrm{ad}^k_{A_i, A_j}(b_i c_j) \in \mathfrak{G}$$

for all $k, l \in \mathbb{N}_0$. With Lemma 4.7 we derive

$$E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$$

and the statement follows. Now, let there exist a directed path from i to j with more than one edge. W.l.o.g. we assume that the path consists of two edges, i.e. $(i, k), (k, j) \in E$. Then with the ideas from above, we know for all $l \in \mathbb{N}$ that

$$E_{ik} \otimes \mathrm{ad}^l_{A_i, A_k}(b_i c_k), E_{kj} \otimes \mathrm{ad}^l_{A_k, A_j}(b_k c_j) \in \mathfrak{G}.$$

But then

$$\left[E_{ik} \otimes \mathrm{ad}^l_{A_i, A_k}(b_i c_k), E_{kj} \otimes \mathrm{ad}^m_{A_k, A_j}(b_k c_j) \right] = E_{ij} \otimes \mathrm{ad}^l_{A_i, A_k}(b_i c_k) \cdot \mathrm{ad}^m_{A_k, A_j}(b_k c_j)$$

and with Lemma 4.7 we obtain $E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$. Now, statement $a)(i)$ is proven.
To prove $a)(iii)$, we suppose there does not exist a path from i to j. From Lemma 4.18 we get $E_{ij} \notin \{E_{kl}, (k, l) \in E\}_{LA}$. Therefore, $E_{ij} \otimes X \notin \mathfrak{G}$ for all $X \in \mathbb{R}^{n \times n} \backslash \{0\}$.
Hence, the remaining case is if there only exists a path from i to j with one vertex. Then $a)(ii)$ follows from the comments from the beginning of the proof.
Now, we prove b). If i and j are connected, we know from $a)(i)$ that

$$E_{ij} \otimes \mathbb{R}^{n \times n}, E_{ji} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}.$$

By taking the commutator

$$[E_{ij} \otimes E_{kl}, E_{ji} \otimes E_{pq}] = E_{ii} \otimes E_{kl}E_{pq} - E_{jj} \otimes E_{pq}E_{kl}$$

we deduce that a basis for the subspace

$$\{E_{ii} \otimes X - E_{jj} \otimes Y \mid X, Y \in \mathbb{R}^{n\times n}, \operatorname{tr} X = \operatorname{tr} Y\}$$

is a subset of $\mathfrak{G}$ and the statement follows. Clearly, statement $b)(ii)$ holds. Hence, the result is proven. □

4.5 Control of homogeneous networks

In this section we consider the case, where all (A_i, b_i, c_i) are equal and hence the network is homogeneous. The control system of a homogeneous network can be simplified to

$$\dot{x} = (I_N \otimes A + \gamma(u) \otimes bc)\, x \tag{4.24}$$

and, as before, we denote by $\mathfrak{G}$ its real generated system Lie algebra. Since we know from Theorem 3.28 and Theorem 3.33 the explicit structure of the Lie algebra $\mathfrak{g} = \{A, bc\}_{LA}$, we can exploit the properties of the "small" Lie algebra $\mathfrak{g}$ in the homogeneous case to compute the network Lie algebra.

Similar to Theorem 3.33 we want to determine the associated system Lie algebra of (4.24) if A and bc are not necessarily real but we only allow for real controls. Since we now deal with complex matrices, the real generated Lie algebra $\mathfrak{G}$ is a Lie subalgebra of $\mathfrak{gl}_{nN}(\mathbb{C})$. Lemma 4.7 cannot be adapted to the new setting to determine the Lie algebra: In case for one of the triples $(A_i, b_i, c_i), (A_j, b_j, c_j)$ or (A_k, b_k, c_k) either $A_l \notin \mathbb{R}^{n\times n}$ or $b_l c_l \notin \mathbb{R}^{n\times n}$ holds, Lemma 4.7 only yields that the real vector space

$$\operatorname{span}_{\mathbb{R}}\left\{X = \sum_{p=0}^{m}\sum_{q=0}^{v}(-1)^{p+q}\binom{m}{p}\binom{v}{q}(c_j A_j^{p+v-q} b_j) A_i^{m-p} b_i c_k A_k^q \;\Big|\; m, v \in \mathbb{N}_0\right\}$$

has at least real dimension n^2. Therefore, besides the case when A and bc are real matrices, we need the following lemma.

Lemma 4.25. *The real vector space generated by elements of the set $\mathfrak{su}(p,q) \cdot \mathfrak{su}(p.q)$ is $\mathfrak{gl}_{p+q}(\mathbb{C})$, i.e.*

$$\operatorname{span}_{\mathbb{R}}\{A \cdot B \mid A, B \in \mathfrak{su}(p,q)\} = \mathfrak{gl}_{p+q}(\mathbb{C}).$$

Proof. Let

$$\mathcal{M} := \operatorname{span}_{\mathbb{R}}\{A \cdot B \mid A, B \in \mathfrak{su}(p,q)\}.$$

We show that the real canonical basis $\{E_{ij}, \mathrm{i}E_{ij} \mid 1 \leq i,j \leq p+q\}$ of $\mathfrak{gl}_{p+q}(\mathbb{C})$ is a subset of $\mathcal{M}$. All matrices $X \in \mathfrak{su}(\mathfrak{p},\mathfrak{q})$ have the form

$$X = \begin{pmatrix} A & B \\ \bar{B}^T & C \end{pmatrix},$$

with $A \in \mathbb{R}^{p\times p}, C \in \mathbb{R}^{q\times q}$ quadratic, skew-Hermitian matrices with $\operatorname{tr} A = -\operatorname{tr} C$ and $B \in \mathbb{R}^{p\times q}$. Products of two matrices $X, Y \in \mathfrak{su}(p,q)$ are of the form

$$X \cdot Y = \begin{pmatrix} A & B \\ \bar{B}^T & C \end{pmatrix} \begin{pmatrix} D & E \\ \bar{E}^T & F \end{pmatrix} = \begin{pmatrix} AD + B\bar{E}^T & AE + BF \\ \bar{B}^T D + C\bar{E}^T & \bar{B}^T E + CF \end{pmatrix}. \tag{4.25}$$

Consider $A = E_{ij} - E_{ji}$ and $D = E_{kl} - E_{lk}$ and all other matrices equal to zero. Then for $i \neq j, k \neq l, i \neq l$ and $j = k$ we obtain

$$(E_{ij} - E_{ji})(E_{kl} - E_{lk}) = E_{il}$$

and therefore all matrices of the form

$$\begin{pmatrix} E_{il} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}. \tag{4.26}$$

With $A = \mathrm{i}(E_{ij} + E_{ji})$ and $D = E_{kl} - E_{lk}$ for $i \neq j, k \neq l$ and all other matrices equal to zero we obtain the matrices (4.26) multiplied by i as an element of $\mathcal{M}$. By choosing $C = E_{ij} - E_{ji}$ and $F = E_{kl} - E_{lk}$ and all other matrices equal to zero, we obtain for $i \neq j, k \neq l$ and $i \neq l$

$$\begin{pmatrix} 0 & 0 \\ 0 & E_{il} \end{pmatrix} \in \mathcal{M}. \tag{4.27}$$

Again, $C = \mathrm{i}(E_{ij} + E_{ji})$ and $F = E_{kl} - E_{lk}$ for $i \neq j, k \neq l$ and all other matrices equal to zero gives us the pure imaginary multiples of (4.27) as elements of $\mathcal{M}$. Now, choosing $A = E_{ij} - E_{ji}$, $E = E_{jj}$, $i \neq j$, and all other matrices equal to zero gives us

$$\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} \in \mathcal{M} \tag{4.28}$$

and with $C = E_{ij} - E_{ji}$, $E = E_{jj}$ and the other matrices equal to zero

$$\begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix} \in \mathcal{M}. \tag{4.29}$$

By choosing $A = E_{ij} - E_{ji}$, $E = \mathrm{i}E_{jj}$, $i \neq j$, and the other matrices equal to zero, and $C = E_{ij} - E_{ji}$, $E = -\mathrm{i}E_{jj}$ and the other matrices equal to zero, respectively, we get the pure imaginary multiples of (4.28) and (4.29) as elements of $\mathcal{M}$. For all E_{ii} and $\mathrm{i}E_{ii}$ on the diagonal part consider for $i \neq j$ $A = E_{ij} - E_{ji}$, $E = E_{ji}$ and $C = E_{ij} - E_{ji}$, $E = E_{ji}$ or for the pure imaginary multiples $E = \mathrm{i}E_{ji}$.
Then the real canonical basis $\{E_{ij}, \mathrm{i}E_{ij} \mid 1 \leq i,j \leq p+q\}$ of $\mathfrak{gl}_{p+q}(\mathbb{C})$ is a subset of $\mathcal{M}$ and the result follows. □

Lemma 4.26. *Let $\mathfrak{g}$ be one of the following Lie algebras*

$$\mathfrak{su}(p,q), \mathfrak{u}(p,q), \mathfrak{sp}_{n/2}(\mathbb{C}), \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{gl}_n(\mathbb{C})$$

and let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ with codimension at most one, which is an ideal in $\mathfrak{g}$. Then $\mathfrak{k}$ is one of the following Lie algebras

$$\mathfrak{su}(p,q), \mathfrak{u}(p,q), \mathfrak{sp}_{n/2}(\mathbb{C}), \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{gl}_n(\mathbb{C}).$$

Proof. Let $\mathfrak{g} = \mathfrak{su}(p,q)$, $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then $\mathfrak{k} = \mathfrak{g}$ due to the simplicity of $\mathfrak{g}$ and the statement follows.
Let $\mathfrak{g} = \mathfrak{u}(p,q)$, $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{R}I$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}I$ or $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Then $\mathfrak{g}$ is reductive and a reductive Lie algebra can be written as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus Z_{\mathfrak{g}}$$

with $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ semisimple and $Z_{\mathfrak{g}}$ Abelian (Corollary I.1.56 [32]). In our cases we can draw the conclusion that $\mathfrak{g}_0$ is simple and the real dimension of $Z_{\mathfrak{g}}$ is 1 or 2. Hence, the codimension of $\mathfrak{g}_0$ in $\mathfrak{g}$ is 1 or 2. Therefore, $[\mathfrak{k}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_0$ implies $\mathfrak{g}_0 \subset \mathfrak{k}$ as $\mathfrak{g}_0$ is simple and $\mathfrak{k}$ has only codimension 1 in $\mathfrak{g}$. But we have $\mathfrak{g}_0 = \mathfrak{su}(p,q)$, $\mathfrak{g}_0 = \mathfrak{sp}_{n/2}(\mathbb{C})$ or $\mathfrak{g}_0 = \mathfrak{sl}_n(\mathbb{C})$ and thus, the result follows. □

The complex analog of Lemma 4.9 is the following.

Lemma 4.27. *Let $\mathfrak{G}$ be a real Lie subalgebra of $\mathfrak{gl}_{nN}(\mathbb{C})$. If $E_{ij} \otimes \mathbb{C}^{n \times n} \subset \mathfrak{G}$ for all $i \neq j$, $i,j = 1,\dots,N$, then $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{C})$, $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{C}) \oplus \alpha\mathbb{R}I$ with $\alpha \neq 0$ suitable or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{C})$.*

Now, we can state the main result of the section.

Theorem 4.28. *Let $(A, b, c) \in \mathbb{C}^{n\times n} \times \mathbb{C}^n \times \mathbb{C}^{1\times n}$ be controllable and observable, $N > 2$ and*

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

be the controlled adjacency matrix of the strongly connected graph $\Gamma = (E, V)$ with N vertices. Then the real system Lie algebra of the homogeneous controlled network

$$\dot{x} = (I_N \otimes A + \gamma(u) \otimes bc)x$$

is $\mathfrak{sl}_{nN}(\mathbb{R}), \mathfrak{gl}_{nN}(\mathbb{R})$, $\mathfrak{sl}_{nN}(\mathbb{C})$ or $\mathfrak{sl}_{nN}(\mathbb{C}) \oplus \alpha\mathbb{R}I$ with $\alpha = N \cdot \operatorname{tr} A$.

Proof. In case $A, bc \in \mathbb{R}^{n\times n}$ this is Theorem 4.6. So let either $A \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$ or $bc \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$. Due to the strong connectedness of Γ we obtain, similar as in the proof of Theorem 4.6 that

$$E_{ij} \otimes bc \in \mathfrak{G}$$

for $i \neq j$ and immediately,

$$E_{ij} \otimes \operatorname{ad}^l_A(bc) \in \mathfrak{G} \text{ for } i \neq j \text{ and all } l \in \mathbb{N}.$$

Since (A, b, c) is controllable and observable there exists an integer $m \in \mathbb{N}$ with $cA^m b \neq 0$. Denote by $m^\star$ the minimal integer with this property. Then with

$$[E_{ik} \otimes \operatorname{ad}^{m^\star}_A(bc), E_{ki} \otimes bc] = (cA^{m^\star}b)(E_{ii} - E_{kk}) \otimes bc$$

(for details see the proof of Theorem 4.6) we obtain by

$$[E_{ij} \otimes X, (E_{ii} + E_{jj} - 2E_{kk}) \otimes bc] = (E_{ij} - E_{ij}) \otimes [X, bc]$$

all commutators of the form $E_{ij} \otimes [X, bc]$ for $E_{ij} \otimes X \in \mathfrak{G}$. Hence,

$$E_{ij} \otimes \mathfrak{k} \subset \mathfrak{G},$$

where $\mathfrak{k} := \{\operatorname{ad}^k_A(bc), k \in \mathbb{N}_0\}_{LA}$ is the real Lie algebra generated by all $\operatorname{ad}^k_A(bc)$. Additionally, $\mathfrak{k}$ is an ideal in $\mathfrak{g} = \{A, bc\}_{LA}$. Due to Theorem 3.33 and since we assumed that either $A \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$ or $bc \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$ the Lie algebra $\mathfrak{g}$ is, up to isomorphisms, one of the following

$$\mathfrak{su}(p, q), \mathfrak{u}(p, q), \mathfrak{sp}_{n/2}(\mathbb{C}), \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{gl}_n(\mathbb{C}).$$

By Lemma 4.26 we deduce that $\mathfrak{k}$ is, up to isomorphisms, again one of the following

$$\mathfrak{su}(p, q), \mathfrak{u}(p, q), \mathfrak{sp}_{n/2}(\mathbb{C}), \mathfrak{sp}_{n/2}(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}I, \mathfrak{gl}_n(\mathbb{C}).$$

In case $\mathfrak{g} = \mathfrak{su}(p,q)$ or $\mathfrak{g} = \mathfrak{u}(p,q)$ it follows with Lemma 4.25

$$\sum_{i \neq j} E_{ij} \otimes \mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{G}.$$

Now, let $\mathfrak{g}$ be any other Lie algebra of (4.5). Then either $\mathfrak{sp}_{n/2}(\mathbb{C}) \subseteq \mathfrak{g}$ or $\mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{g}$ holds. With

$$\mathfrak{sp}_{n/2}(\mathbb{C}) = \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus i\mathfrak{sp}_{n/2}(\mathbb{R})$$

and

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \oplus i\mathfrak{sl}_n(\mathbb{R})$$

we can now apply Lemma 4.7. Therefore,

$$\text{span}_{\mathbb{R}}\{[E_{ik} \otimes \mathfrak{g}, E_{kj} \otimes \mathfrak{g}]\} = E_{ij} \otimes \mathfrak{gl}_n(\mathbb{R})$$

and

$$\text{span}_{\mathbb{R}}\{[E_{ik} \otimes \mathfrak{g}, E_{kj} \otimes i\mathfrak{g}]\} = E_{ij} \otimes i\mathfrak{gl}_n(\mathbb{R})$$

for $i \neq j$. Thus, $E_{ij} \otimes \mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{G}$. With a complex analog of Lemma 4.8 we obtain for the system Lie algebra of (4.28)

$$\mathfrak{sl}_{nN}(\mathbb{C}) \subseteq \mathfrak{G}.$$

With Lemma 4.27 we derive $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{C})$, $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{C}) \oplus \alpha\mathbb{R}I$ with α suitable or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{C})$. But it is not possible to get $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{C})$ as Remark 4.29 explains. The result follows. □

Remark 4.29. As long as we assume that Γ is a simple directed graph and has therefore no self-loops there does not exist any example for a heterogeneous network of SISO systems such that the system Lie algebra is $\mathfrak{gl}_{nN}(\mathbb{C})$. This can be understood by the following argumentation: We can write $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{C})$ as the real vector space

$$\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{C}) \oplus \alpha\mathbb{R}I \oplus \beta\mathbb{R}I$$

with $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ and $\alpha/\beta \notin \mathbb{R}$. Now, we can easily see that, in order to obtain $\mathfrak{gl}_{nN}(\mathbb{C})$ as Lie algebra, we need two matrices $A, B \in \mathfrak{G}$ such that $\operatorname{tr} A$ is not a real multiple of $\operatorname{tr} B$. But, since Γ has no self-loops, the matrix $\mathcal{B}\gamma(u)\mathcal{C}$ has always trace zero for all possible controls and $\mathcal{A}$ is the only possibility for a matrix in $\mathfrak{G}$ with trace unequal zero.

With Theorem 4.28 we obtain similar to Corollary 4.11 the following.

Corollary 4.30. *Let (A, b, c) be controllable and observable and either $A \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$ or $bc \in \mathbb{C}^{n\times n}\backslash\mathbb{R}^{n\times n}$. Let*

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

be the controlled adjacency matrix of a simple directed graph $\Gamma = (E, V)$ *with* $N > 2$ *vertices. Then the homogeneous controlled network*

$$\dot{x} = (I_N \otimes A + \gamma(u) \otimes bc)x$$

is accessible on $\mathbb{C}^{nN} \backslash \{0\}$ *if and only if* Γ *is strongly connected.*

4.6 Networks of MIMO systems

In this section we transfer the results we developed for heterogeneous networks of SISO systems to heterogeneous networks of MIMO (multiple-input multiple-output) systems. As already stated in Section 3.4 there are two possibilities to extend the results on SISO systems to MIMO systems. Similar to (3.9) and (3.11) we now introduce the heterogeneous MIMO networks of output feedback type.

As an example, we prove that the system Lie algebra of a network of MIMO systems is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$. For both versions of feedback we reduce the problem to rank one matrices by using Lemma 3.38, i.e. we trace back a network of MIMO systems to a network of SISO systems. With these ideas of proof, we can transfer all results on networks of SISO systems immediately to networks of MIMO systems.

Control by Output Feedback

We start with the network analog of (3.9): Again, the linear system of every vertex i can be described by

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i v_i, \\ y_i &= C_i x_i, \end{aligned}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times p}$ and $C_i \in \mathbb{R}^{p \times n}$. Applying output feedback of the form

$$v_i = \sum_{(j,i) \in E} C_j x_j,$$

the dynamics of an interconnected heterogeneous network with graph $\Gamma = (V, E)$ under output feedback can be described for every vertex $i = 1, \dots, N$ by

$$\dot{x}_i = A_i x_i + B_i \sum_{(j,i) \in E} C_j x_j.$$

Now, we regard the interconnection strength to be controllable and obtain the control system of every single node as

$$\dot{x}_i = A_i x_i + B_i \sum_{(j,i)\in E} U_{ij}(t) C_j x_j,$$

where the feedback matrices U_{ij} are assumed to take every value in $\mathbb{R}^{p\times p}$. For the whole network, we obtain the bilinear control system

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(U)\mathcal{C})\, x,$$

where the adjacency matrix γ is defined in (4.1) and

$$\gamma(U) := \sum_{(i,j)\in E} E_{ij} \otimes U_{ij}(t). \tag{4.30}$$

Similar to the preceding sections we use the notation

$$\mathcal{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix} \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_N \end{pmatrix},$$

where $\mathcal{A} \in \mathbb{R}^{nN\times nN}$, $\mathcal{B} \in \mathbb{R}^{nN\times pN}$ and $\mathcal{C} \in \mathbb{R}^{pN\times nN}$.

Now, we can state the MIMO analog for Theorem 4.6.

Theorem 4.31. *Let (A_i, B_i, C_i) be controllable and observable for all $1 \leq i \leq N$ with $N > 2$. Let $\Gamma = (E, V)$ be a strongly connected graph with N vertices and*

$$\gamma(U) := \sum_{(i,j)\in E} E_{ij} \otimes U_{ij}(t)$$

the controlled adjacency matrix of the simple directed graph Γ. Then the system Lie algebra $\mathfrak{G}$ of

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(U)\mathcal{C})\, x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. With Lemma 3.38 there exists for every controllable and observable triple (A_i, B_i, C_i) a matrix K_i and vectors g_i and h_i such that $(A_i + B_i K_i C_i, B_i g_i, h_i C_i)$ is a controllable and observable triple. Hence, $b_i := B_i g_i$ and $c_i := h_i C_i$ are cyclic vectors of

$A_i + B_iK_iC_i$. It would now be sufficient to show that the matrix

$$\begin{pmatrix} A_1 + B_1K_1C_1 & & \\ & \ddots & \\ & & A_N + B_NK_NC_N \end{pmatrix}$$

is an element of $\mathfrak{G}$ to transfer the proof ideas of the SISO case. But since we excluded self-loops that is not possible. Therefore, we show that

$$E_{ij} \otimes \mathrm{ad}^k_{A_i+B_iK_iC_i, A_j+B_jK_jC_j}(b_ic_j)$$

is an element of $\mathfrak{G}$ for all $i \neq j$. The operator $\mathrm{ad}^k_{A_i+B_iK_iC_i, A_j+B_jK_jC_j}(\cdot)$ is defined by (cf. (4.7))

$$\mathrm{ad}^k_{A_i+B_iK_iC_i, A_j+B_jK_jC_j}(b_ic_j) := \sum_{p=0}^{k}(-1)^p\binom{k}{p}(A_i + B_iK_iC_i)^{k-p}b_ic_j(A_j + B_jK_jC_j)^p.$$

First, by choosing $U_{ij} = g_ih_j$ we obtain for all $i \neq j$

$$E_{ij} \otimes b_ic_j \in \mathfrak{G}$$

due to the strong connectedness of Γ (cf. the proof of Theorem 4.6). Taking commutators of the form $\mathrm{ad}^k_{\mathcal{A}}(E_{ij} \otimes b_ic_j)$ it follows that

$$E_{ij} \otimes \mathrm{ad}^k_{A_i,A_j}(b_ic_j) \in \mathfrak{G}.$$

Since

$$\mathrm{ad}_{A_i+B_iK_iC_i, A_j+B_jK_jC_j}(H) = \mathrm{ad}_{A_i,A_j}(H) + \mathrm{ad}_{B_iK_iC_i, B_jK_jC_j}(H)$$

for every $H \in \mathbb{R}^{n\times n}$ it is now left to show that

$$E_{ij} \otimes \mathrm{ad}_{B_iK_iC_i, B_jK_jC_j}(b_ic_j) \in \mathfrak{G}.$$

To do so, we notice that the matrix $B_iK_iC_i$ can be written as a sum of rank 1 matrices

$$B_iK_iC_i = \sum_{k,l} k^i_{kl}B_iE_{kl}C_i = \sum_{k,l} k^i_{kl}b^i_kc^i_l, \tag{4.31}$$

where k^i_{kl} denotes the entries of K_i, b^i_k denotes the k-th column of B_i and c^i_l denotes the l-th row of C_i. As $B_iE_{pq}C_j = b^i_pc^j_q$, we consider the commutators for $i \neq k, j \neq k$

$$[E_{ik} \otimes b^i_pc^k_q, E_{ki} \otimes b^k_pc^i_q] = (c^k_qb^k_p)E_{ii} \otimes b^i_pc^i_q - (c^i_qb^i_p)E_{kk} \otimes b^k_pc^k_q$$

and

$$[E_{jk} \otimes b_p^j c_q^k, E_{kj} \otimes b_p^k c_q^j] = (c_q^k b_p^k) E_{jj} \otimes b_p^j c_q^j - (c_p^j b_q^j) E_{kk} \otimes b_p^k c_q^k.$$

In case $c_q^k b_p^k = 0$ for some p, q, k a similar construction with $\mathrm{ad}^{m^\star}_{A_k, A_j}(b_p^k c_q^j)$ as in the proof of Theorem 4.6 works. Thus, we deduce that the matrix

$$\begin{pmatrix} \ddots & & & & \\ & A_i + B_i K_i C_i & & & \\ & & \ddots & & \\ & & & A_j + B_j K_j C_j & \\ & & & & \ddots \end{pmatrix} \in \mathfrak{G},$$

where the other diagonal entries are A_l except for $l = k$. Taking commutators we obtain

$$E_{ij} \otimes \mathrm{ad}^l_{A_i + B_i K_i C_i, A_j + B_j K_j C_j}(b_i c_j) \in \mathfrak{G}$$

holds for all $i \neq j$, $l \in \mathbb{N}_0$. The desired result follows and we can apply Lemma 4.7 to

$$\begin{aligned}[E_{ik} \otimes \mathrm{ad}^k_{A_i+B_iK_iC_i, A_k+B_kK_kC_k}(b_i c_k), E_{kj} \otimes \mathrm{ad}^k_{A_k+B_kK_kC_k, A_j+B_jK_jC_j}(b_k c_j)] \\ = E_{ij} \otimes \mathrm{ad}^k_{A_i+B_iK_iC_i, A_k+B_kK_kC_k}(b_i c_k) \cdot \mathrm{ad}^k_{A_k+B_kK_kC_k, A_j+B_jK_jC_j}(b_k c_j).\end{aligned}$$

Thus,

$$E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$$

and with Lemma 4.8 we obtain

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}.$$

Now, the statement results from Lemma 4.9. □

As a consequence, the proof of Theorem 4.11 can be transferred immediately to the MIMO case.

Theorem 4.32. *Let $N > 2$, $\Gamma = (E, V)$ be a directed simple graph with N vertices and (A_i, B_i, C_i) be controllable and observable for $1 \leq i \leq N$. Then the controlled network*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(U)\mathcal{C})\, x$$

is accessible on $\mathbb{R}^{nN} \backslash \{0\}$ if and only if Γ is strongly connected.

Control by restricted Output Feedback

For the network analog of (3.11) we only have to regard (4.30) as

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij}\,(E_{ij} \otimes K_{ij})\,,$$

where the matrices K_{ij} are fixed and the controls u_{ij} take real values. Then the resulting bilinear control system is

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\,x. \tag{4.32}$$

With an additional assumption, we can prove the following:

Theorem 4.33. *Let A_i be cyclic and (A_i, B_i, C_i) be controllable and observable for all $1 \leq i \leq N$ with $N > 2$. Let $\Gamma = (E, V)$ be a strongly connected simple graph with N vertices. Then there exists a set of matrices K_{ij} for $(i,j) \in \Gamma$ such that the system Lie algebra $\mathfrak{G}$ of*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\,x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. Since A_i is cyclic by assumption there exist vectors g_i and h_i by Lemma 3.38 and Remark 3.39 such that $(A_i, B_i g_i, h_i C_i)$ is a controllable and observable triple. When we choose $K_{ij} := g_i h_j$ the result follows from Theorem 4.6. □

Again, we deduce as a consequence a condition for the bilinear system (4.32) to be accessible.

Theorem 4.34. *Let $N > 2$, $\Gamma = (E, V)$ be a directed simple graph with N vertices and (A_i, B_i, C_i) be controllable and observable for $1 \leq i \leq N$. Let A_i be cyclic for all $1 \leq i \leq N$. Then there exists a set of matrices K_{ij} for $(i,j) \in \Gamma$ such that the controlled network*

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\,x$$

is accessible on $\mathbb{R}^{Nn}\backslash\{0\}$ if and only if Γ is strongly connected.

Chapter 5

Bilinear Control of Networks restricted by Subspaces

In Chapter 4, we allowed the interconnections of the network to be independently controllable. This implies that controls of our bilinear system vary through a vector space spanned by certain matrices of the form E_{ij} for some $1 \leq i, j \leq N$. In the present chapter, we generalize the setting by allowing linear dependencies between the interconnections. Hence, the controls of the occurring bilinear system vary through a subspace $\mathcal{S}$, which is not necessarily generated by matrices of the form E_{ij}.

In Section 5.1 we give a formulation for the slightly different setting and present a reformulation of the main result of Chapter 4 (Theorem 4.6) in terms of subspaces. In Section 5.2 we consider homogeneous networks and obtain sufficient conditions for accessibility. Different from Chapter 4 we cannot immediately transfer the results from homogeneous to heterogeneous networks since the generated Lie algebra depends in a crucial way on the scalars $c_i A_i^k b_i$ for all $1 \leq i \leq N$ and $k \in \mathbb{N}_0$. Therefore, in Section 5.3 we develop accessibility conditions for the more general case of heterogeneous networks.
Exemplarily, we study the controllability properties of networks whose interconnection patterns are defined by Toeplitz matrices in Section 5.4.

5.1 The setting

Again, we consider networks of SISO systems and derive the bilinear control system

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C})\, x. \tag{5.1}$$

In Chapter 4 we allowed the controls of the bilinear system to vary through the linear combinations of the adjacency matrix

$$\gamma(u) = \sum_{(i,j)\in E} u_{ij} E_{ij}.$$

Now, we examine the same type of bilinear control systems but allow for linear dependencies in the interconnection structure. This has the consequence that the controls can vary through a finitely generated subspace $\mathcal{S} \subset \mathbb{R}^{N\times N}$, i.e. we can write the control matrix as

$$\gamma_{\mathcal{S}}(u) = \sum_{i=1}^{m} u_i L_i$$

if $\mathcal{S} = \mathrm{span}\{L_1, \ldots, L_m\}$.

Let $\mathcal{S} \subset \mathbb{R}^{N\times N}$ be a subspace which is finitely generated, i.e.

$$\mathcal{S} = \mathrm{span}\left\{L_1, \ldots, L_m \;\middle|\; L_i \in \mathbb{R}^{N\times N}\right\}.$$

Then we denote

$$\gamma_{\mathcal{S}}(u) := \sum_{i=1}^{m} u_i L_i \tag{5.2}$$

as the *control matrix of the subspace* $\mathcal{S}$. Without loss of generality we can assume that the matrices L_i are linearly independent and thus form a basis of $\mathcal{S}$. Similar to Chapter 4, we now consider the bilinear control systems

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma_{\mathcal{S}}(u)\mathcal{C})\, x, \tag{5.3}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined in (4.5) and $\gamma_{\mathcal{S}}(u)$ is defined in (5.2). We develop conditions on the subspace $\mathcal{S} \subset \mathbb{R}^{N\times N}$ such that the system Lie algebra $\mathfrak{G}$ of the control system (5.3) is either $\mathfrak{G} = \mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{G} = \mathfrak{gl}_{nN}(\mathbb{R})$ and hence, system (5.3) is accessible on $\mathbb{R}^{nN}\setminus\{0\}$. We define the subspace

$$\mathcal{BSC} := \mathrm{span}\left\{\mathcal{B}L_i\mathcal{C} \mid i = 1, \ldots, m\right\},$$

which simplifies for the special case of homogeneous networks to

$$\mathcal{S} \otimes bc := \mathrm{span}\{L_i \otimes bc \mid i = 1, \ldots, m\}.$$

In this new setting we can reformulate Theorem 4.6.

Theorem 5.1. *Let (A_i, b_i, c_i) be controllable and observable for $1 \leq i \leq N$ with $N > 2$. Let $\mathcal{S} := \operatorname{span}\{L_1, \dots, L_m\}$ be a subspace of $\mathbb{R}^{N \times N}$ with $L_i \in \{E_{kl} \mid 1 \leq k, l \leq N\}$ and*

$$\gamma_{\mathcal{S}}(u) := \sum_{i=1}^{m} u_i L_i$$

be the control matrix of the subspace $\mathcal{S}$. If $\mathcal{S}$ contains an element $L \in \mathcal{S}$, which is permutation-irreducible, the Lie algebra $\mathfrak{G}$ of the bilinear system

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma_{\mathcal{S}}(u)\mathcal{C})\, x$$

is $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

5.2 Homogeneous networks

First, we consider the easier case of homogeneous networks. The bilinear control system (5.3) simplifies to

$$\dot{x} = (I \otimes A + \gamma_{\mathcal{S}}(u) \otimes bc)\, x$$

and the system Lie algebra is the Lie algebra generated by the matrices $I \otimes A$ and $L \otimes bc$ for all $L \in \mathcal{S}$. We obtain the following result.

Theorem 5.2. *Let (A, b, c) be controllable and observable, $\mathcal{S} := \operatorname{span}\{L_1, \dots L_m\} \subset \mathbb{R}^{N \times N}$ be a finitely generated subspace of matrices with $N > 2$ and let*

$$\gamma_{\mathcal{S}}(u) := \sum_{i=1}^{m} u_i L_i$$

be the control matrix of the subspace $\mathcal{S}$. If

$$\mathfrak{sl}_N(\mathbb{R}) \subseteq \{L_1, \dots, L_m\}_{LA},$$

then the Lie algebra $\mathfrak{G}$ of the bilinear control system

$$\dot{x} = (I \otimes A + \gamma_{\mathcal{S}}(u) \otimes bc)\, x$$

is $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. Let $\mathcal{S} \subset \mathbb{R}^{N \times N}$ be a subspace of $N \times N$ matrices with

$$\mathfrak{sl}_N(\mathbb{R}) \subseteq \{L_1, \dots, L_m\}_{LA}.$$

We consider

$$[L_i \otimes bc, L_j \otimes bc] = (cb)[L_i, L_j] \otimes bc$$

and in case $cb = 0$ we have

$$[L_i \otimes bc, L_j \otimes \mathrm{ad}_A^{m^\star}(bc)] = (cA^{m^\star}b)[L_i, L_j] \otimes bc,$$

where $m^\star$ denotes the smallest integer m such that $cA^m b \neq 0$ (cf. the proof of Theorem 4.6). Thus, for all $L \in \{L_1, \ldots, L_m\}_{LA}$ it follows $L \otimes bc \in \mathfrak{G}$. By assumption the Lie algebra generated by $L_1, \ldots, L_m$ contains $\mathfrak{sl}_N(\mathbb{R})$ and we deduce

$$E_{ij} \otimes bc \in \mathfrak{G}$$

for $i \neq j$. Clearly, we get

$$\mathrm{ad}_{I \otimes A}^k (L \otimes bc) = L \otimes \mathrm{ad}_A^k(bc) \in \mathfrak{G}$$

for every $k \in \mathbb{N}_0$. Since $N > 2$, we can apply Lemma 4.7 to

$$\left[E_{ih} \otimes \mathrm{ad}_A^k(bc), E_{hj} \otimes \mathrm{ad}_A^l(bc)\right] = E_{ij} \otimes \mathrm{ad}_A^k(bc) \cdot \mathrm{ad}_A^l(bc)$$

for $k, l \in \mathbb{N}_0$, $i \neq h, h \neq j, i \neq j$ and obtain $E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}$. According to Lemma 4.8, it yields

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}$$

and the result follows from Lemma 4.9. □

Thus, we can reduce the problem of finding a subspace $\mathcal{S}$ such that the system Lie algebra of (5.3) is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$ to the problem of finding generators of $\mathfrak{sl}_N(\mathbb{R})$ or $\mathfrak{gl}_N(\mathbb{R})$, respectively. In Chapter 2 we already stated existing results for this problem. The next corollary is an immediate application of those results.

Corollary 5.3. *Let (A, b, c) be controllable and observable and let $\mathcal{S}$ be a finitely generated subspace of $N \times N$ matrices with $N > 2$, which satisfies the following conditions:*

i) there exists a matrix $L \in \mathcal{S}$, which is strongly regular;

ii) there exists a matrix $K \in \mathcal{S}$ such that L and K do not possess a common nontrivial subspace.

Then the Lie algebra $\mathfrak{G}$ generated by $I \otimes A$ and $\mathcal{S} \otimes bc$ is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. Since there exists an element $L \in \mathcal{S}$, which is strongly regular, and another element $K \in \mathcal{S}$ such that L and K do not possess a common invariant subspace, we can apply Theorem 2.20 and deduce

$$\mathfrak{sl}_N(\mathbb{R}) \subseteq \{K, L\}_{LA}.$$

Then the result follows with Theorem 5.2. □

With Theorem 2.6 and 2.3 we deduce sufficient conditions on the subspace $\mathcal{S} \subset \mathbb{R}^{N\times N}$ for accessibility.

Corollary 5.4. *Let (A, b, c) be controllable and observable, $\mathcal{S} := \operatorname{span}\{L_1, \dots, L_m\}$ be a subspace of $N \times N$ matrices with $N > 2$ and*

$$\gamma_{\mathcal{S}}(u) := \sum_{i=1}^{m} u_i L_i.$$

If $\mathfrak{sl}_N(\mathbb{R}) \subset \{L_1, \dots, L_m\}_{LA}$ then the system

$$\dot{x} = (I \otimes A + \gamma_{\mathcal{S}}(u) \otimes bc)x$$

is accessible on $\mathbb{R}^{nN} \setminus \{0\}$.

Remark 5.5. As we will see in Section 5.4, the smallest Lie algebra generated by the vector space of Toeplitz matrices of order N is the Lie algebra $\mathfrak{gl}_N(\mathbb{R})$. Hence, networks where the agents are connected by a certain Toeplitz structure are accessible.

As we can see in the next example, it is not necessary for the generated Lie algebra $\{L_1, \dots, L_m\}_{LA}$ to contain $\mathfrak{sl}_N(\mathbb{R})$.

Example 5.6. *Let (A, b, c) be a controllable and observable n-dimensional triple with $g(s) \neq g(-s + \alpha)$ for all $\alpha \in \mathbb{R}$ and $n > 2$ and let A be a diagonal matrix. Define*

$$\mathcal{M} := \operatorname{span}\{\operatorname{ad}_A^k(bc)\}_{k=1}^{\infty}$$

and, since A is a diagonal matrix, all elements of $\mathcal{M}$ have only zero entries on the diagonal. With

$$\dim \mathcal{M} = n^2 - n$$

(Lemma B.6 and Lemma 3.16) we get $E_{ij} \in \mathcal{M}$ for $i \neq j$. Now, let

$$\mathcal{S} := \operatorname{span}\{E_{12} - E_{21}, E_{32} - E_{23}\} \subset \mathbb{R}^{3\times 3}$$

and we consider the Lie algebra $\mathfrak{G}$ generated by $I \otimes A$ and $\mathcal{S} \otimes bc$. One can easily see that

$$\{\mathcal{S}\}_{LA} = \mathfrak{so}_3.$$

Therefore, we cannot apply Theorem 5.2. Since

$$[L \otimes E_{ij}, K \otimes E_{jk}] = LK \otimes E_{ik} \tag{5.4}$$

for $i \neq k$ and $K, L \in \mathcal{S}$, we examine the vector space $\mathcal{V}$ generated by products of elements of $\mathcal{S}$, i.e.

$$\mathcal{V} := \operatorname{span}\left\{L \cdot K \;\middle|\; K, L \in \mathcal{S}\right\}.$$

For example, $(E_{12} - E_{21})(E_{23} - E_{32}) = E_{13}$ *and* $(E_{23} - E_{32})(E_{12} - E_{21}) = E_{31}$*. With* $E_{31}(E_{12} - E_{21}) = E_{32}$ *and* $E_{13}(E_{23} - E_{32}) = E_{12}$ *it immediately follows that* E_{23} *and* E_{21} *can be represented by linear combinations of products of* $\mathcal{S}$*. Hence,*

$$V_3 \subset \mathcal{V},$$

where V_3 *is defined as in Lemma 3.23. Since*

$$\{\mathcal{V}\}_{LA} \subset \mathcal{V}$$

we obtain with Lemma 3.23 that $\mathfrak{sl}_3(\mathbb{R}) \subset \mathcal{V}$*. Concerning (5.4) we obtain* $E_{ij} \otimes E_{kl} \in \mathfrak{G}$ *for all* $i \neq j$ *and* $k \neq l$*, i.e.*

$$E_{ij} \otimes \mathcal{M} \subset \mathfrak{G}$$

and therefore, we obtain with Lemma 4.8 that

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}.$$

Further Remark

In consideration of Theorem 2.17 the actual setting leads to a pole assignment problem. Apparently, assuming that the matrix $\mathcal{A}$ is strongly regular and analyze the conditions under which the subspace $\mathcal{BSC}$ contains a permutation-irreducible matrix requires too many constraints on the triples (A_i, b_i, c_i).
Another ansatz would be to find a strongly regular matrix of the form

$$\mathcal{A} + c \cdot \mathcal{BLC} \tag{5.5}$$

with $c \in \mathbb{R} \setminus \{0\}$. The following theorem would give us necessary and sufficient conditions on the Lie algebra $\{\mathcal{BLC} \mid L \in \mathcal{S}\}_{LA}$ such that we can guarantee the existence of a strongly regular element of the form (5.5).

Theorem 5.7 ([14])**.** *Given a complex Lie algebra* $\mathcal{L} \subset \mathfrak{gl}_n(\mathbb{C})$*. Then the mapping*

$$\chi_A : \mathcal{L} \to \mathbb{C}^n, \quad L \mapsto \det(sI - A - L) = s^n - \sigma_{n-1}s^{n-1} + \ldots + (-1)^n\sigma_0$$

is onto for all matrices $A \in \mathbb{C}^{n\times n}$ *if and only if the following conditions are satisfied*

(i) rank $\mathcal{L} = n$*;*

(ii) there exists a matrix $B \in \mathcal{L}$ *with distinct eigenvalues.*

Clearly, the surjectivity of the characteristic polynomial $\det(sI - A - L)$ implies the existence of a strongly regular element of the form $A + L$ with A fixed and $L \in \mathcal{L}$.

For subspaces the next theorem states necessary and sufficient conditions, but it only holds true for generic matrices A.

Theorem 5.8 ([55])**.** *Let $\mathcal{L} := \{BKC \mid B, C \text{ fixed, } K \text{ arbitrary}\} \subset \mathbb{C}^{n\times n}$ be a linear subspace. Necessary and sufficient conditions for the image of the mapping*

$$\chi_A : \mathcal{L} \to \mathbb{C}^n, \quad L \mapsto \det(sI - A - L) = s^n - \sigma_{n-1}s^{n-1} + \ldots + (-1)^n\sigma_0$$

to contain an open dense set of polynomials for the generic A are

(i) $\dim \mathcal{L} \geq n$*;*

(ii) $\mathcal{L} \not\subset \mathfrak{sl}_n(\mathbb{C})$*.*

Since the proofs of Theorem 5.7 and 5.8 use the Dominant Morphism Theorem, which is not true over $\mathbb{R}$, and Theorem 5.7 only holds true for generic matrices $A \in \mathbb{C}^{n\times n}$, we cannot apply those results to our setting.

5.3 Heterogeneous networks

Let $\mathcal{S} := \operatorname{span}\{L_1, \ldots, L_m\}$ be a subspace of $N \times N$ matrices. We cannot transfer the results from homogeneous networks to heterogeneous networks since the Lie algebra

$$\left\{\mathcal{A}, \mathcal{B}L_i\mathcal{C} \mid i = 1, \ldots, m\right\}_{LA}$$

depends in a crucial way on the constants $c_iA_i^kb_i$ for $k \in \mathbb{N}_0$ and $1 \leq i \leq N$. To figure out the influence of the subspace $\mathcal{S}$ on the Lie algebra $\left\{\mathcal{A}, \mathcal{B}L_i\mathcal{C} \mid i = 1, \ldots, m\right\}_{LA}$ let us consider commutators of elements $\mathcal{B}L\mathcal{C}, \mathcal{B}K\mathcal{C} \in \left\{\mathcal{A}, \mathcal{B}L_i\mathcal{C} \mid i = 1, \ldots, m\right\}_{LA}$ with $K, L \in \mathcal{S}$. We get

$$\begin{aligned} [\mathcal{B}L\mathcal{C}, \mathcal{B}K\mathcal{C}] &= \mathcal{B}L\mathcal{C}\mathcal{B}K\mathcal{C} - \mathcal{B}K\mathcal{C}\mathcal{B}L\mathcal{C} \\ &= \mathcal{B}\left(LTK - KTL\right)\mathcal{C}, \end{aligned} \tag{5.6}$$

where $T := \mathcal{C}\mathcal{B} = \operatorname{diag}(c_1b_1, \ldots, c_Nb_N)$. In general, $c_ib_i \neq c_jb_j$ and if we assume $\mathfrak{sl}_N(\mathbb{R}) \subset \{\mathcal{S}\}_{LA}$ it might not yield $\mathfrak{sl}_{nN}(\mathbb{R}) \subset \left\{\mathcal{A}, \mathcal{B}L\mathcal{C} \mid L \in \mathcal{S}\right\}_{LA}$ as the following example shows.

Example 5.9. *Let*

$$\mathcal{S} = \operatorname{span}\left\{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\},$$

$$(A_1, b_1c_1) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \end{pmatrix}\right)$$

and

$$(A_2, b_2c_2) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \right).$$

Clearly, (A_1, b_1, c_1) and (A_2, b_2, c_2) are controllable and observable. The smallest Lie algebra which contains $\mathcal{S}$ is $\mathfrak{gl}_2(\mathbb{R})$ due to Theorem 2.20. But for the Lie algebra generated by $\mathcal{A}$ and $\mathcal{BSC}$ holds $\mathfrak{sl}_4(\mathbb{R}) \not\subseteq \mathfrak{G}$: Since

$$\left[\mathcal{B} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{C}, \mathcal{B} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C} \right] = 0$$

and the number of linearly independent matrices in the set $\{\mathrm{ad}^k_{A_i,A_j}(b_ic_j), k \in \mathbb{N}_0\}$ is 3 since $\mathrm{ad}^3_{A_i,A_j}(b_ic_j) = 0$ for $i \neq j$. Therefore, the Lie algebra $\mathfrak{G}$ is a subset from the vector space

$$\mathcal{V} := \mathrm{span} \left\{ \mathcal{A}, \mathrm{ad}^k_{\mathcal{A}} \left(\mathcal{B} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{C} \right), \mathrm{ad}^k_{\mathcal{A}} \left(\mathcal{B} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C} \right), k = 0, 1, 2. \right\}.$$

Since $\mathcal{V}$ has dimension 7, we get that $\mathfrak{sl}_4(\mathbb{R}) \not\subset \mathfrak{G}$.

On the other hand, there might exist subspaces $\mathcal{S}$ with $\mathfrak{sl}_N(\mathbb{R}) \not\subset \{\mathcal{S}\}_{LA}$, where $\mathfrak{sl}_{nN}(\mathbb{R}) \subset \{\mathcal{A}, \mathcal{BSC}\}_{LA}$ holds. This can be seen by the following example.

Example 5.10. *Let*

$$\mathcal{S} = \mathrm{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$(A_1, b_1c_1) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix} \right)$$

and

$$(A_2, b_2c_2) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \right).$$

Clearly, (A_1, b_1, c_1) and (A_2, b_2, c_2) are controllable and observable. The smallest Lie algebra, which contains $\mathcal{S}$ is $\mathcal{S}$ itself, hence 2-dimensional and $\mathfrak{sl}_2(\mathbb{R}) \not\subset \mathcal{S}$. But the Lie *algebra generated by $\mathcal{A}$ and $\mathcal{BSC}$ is $\mathfrak{gl}_4(\mathbb{R})$: Since*

$$\left[\mathcal{B}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathcal{C}, \mathcal{B}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathcal{C}\right] = \mathcal{B}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathcal{C}$$

we have $E_{12} \otimes b_1c_2$ and $E_{21} \otimes b_2c_1$ as elements in $\mathfrak{G}$. Then

$$\mathrm{ad}_{A_1,A_2}(b_1c_2) = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} \text{ and } \mathrm{ad}^2_{A_1,A_2}(b_1c_2) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

With

$$\left[\left[E_{12} \otimes \mathrm{ad}^2_{A_1,A_2}(b_1c_2), E_{21} \otimes b_2c_1\right], E_{12} \otimes b_1c_2\right] = E_{12} \otimes \begin{pmatrix}2 & 2\\ 0 & 2\end{pmatrix}$$

we get $E_{12} \otimes \mathbb{R}^{2\times 2} \subset \mathfrak{G}$. A similar calculation shows $E_{21} \otimes \mathbb{R}^{2\times 2} \subset \mathfrak{G}$. Completing this to a Lie algebra yields

$$\mathfrak{G} = \mathfrak{gl}_4(\mathbb{R}).$$

With an additional assumption on the controllable and observable triples we can adapt Theorem 5.2 to heterogeneous networks.

Theorem 5.11. *Let (A_i, b_i, c_i) be controllable and observable for $1 \leq i \leq N$ with $N > 2$ and let $c_ib_i = c_jb_j \neq 0$ hold true for all $1 \leq i, j \leq N$. If*

$$\mathfrak{sl}_N(\mathbb{R}) \subseteq \{L_1, \dots, L_m\}_{LA}$$

holds for $\mathcal{S} := \mathrm{span}\{L_1, \dots, L_m\} \subset \mathbb{R}^{N\times N}$, the Lie algebra $\mathfrak{G}$ generated by $\mathcal{A}$ and $\mathcal{BSC}$ is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. As $\mathfrak{sl}_N(\mathbb{R}) \subseteq \{L_1, \dots, L_m\}_{LA}$ we get

$$E_{ij} \in \{L_1, \dots, L_m\}_{LA}$$

for all $i \neq j$. Now, we show that $E_{ij} \otimes b_ic_j \in \mathfrak{G}$ for all $i \neq j$. Since all $c := c_ib_i$ are equal we obtain with $\mathcal{CB} = cI_N$ that

$$[\mathcal{B}L\mathcal{C}, \mathcal{B}K\mathcal{C}] = c \cdot \mathcal{B}[L, K]\mathcal{C}.$$

Therefore, we get from $L \in \{L_1, \dots, L_k\}_{LA}$ that $\mathcal{B}L\mathcal{C} \in \mathfrak{G}$, i.e. $E_{ij} \otimes b_ic_j \in \mathfrak{G}$ for all $i \neq j$. Taking iteratively commutators with $\mathcal{A}$ we derive

$$\mathrm{ad}^k_{\mathcal{A}}(E_{ij} \otimes b_ic_j) = E_{ij} \otimes \mathrm{ad}^k_{A_i,A_j}(b_ic_j) \in \mathfrak{G}$$

for $k \in \mathbb{N}_0$. As

$$[E_{ik} \otimes \mathrm{ad}^p_{A_i,A_k}(b_i c_k), E_{kj} \otimes \mathrm{ad}^q_{A_k,A_j}(b_k c_j)] = E_{ij} \otimes \mathrm{ad}^p_{A_i,A_k} b_i c_k) \cdot \mathrm{ad}^q_{A_k,A_j}(b_k c_j) \in \mathfrak{G}$$

Lemma 4.7 implies $E_{ij} \otimes \mathbb{R}^{n\times n} \subset \mathfrak{G}$. Completing this to a Lie algebra yields

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subseteq \mathfrak{G}$$

and with Lemma 4.9 the result follows. □

Similar to Corollary 5.3 we obtain from Theorem 2.20 the following.

Corollary 5.12. *Let (A_i, b_i, c_i) be controllable and observable for $1 \le i \le N$ with $N > 2$ and let $c_i b_i = c_j b_j \neq 0$ for all $1 \le i, j \le N$. Then the Lie algebra $\mathfrak{G}$ generated by $\mathcal{A}$ and $\mathcal{BSC}$ is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$ if $\mathcal{S}$ satisfies the following two conditions:*

i) there exists at least one element $L \in \mathcal{S}$, which is strongly regular

ii) there exists one element $K \in \mathcal{S}$ such that L and K do not possess a common nontrivial subspace.

Clearly, Theorem 5.11 and Corollary 5.3 hold true as well when $c_i A_i^{m_i^\star} b_i = c_j A_j^{m_j^\star} b_j \neq 0$ for all i, j and all $m_i^\star$ are equal, where $m_i^\star$ is the minimal integer m_i with $c_i A_i^{m_i} b_i \neq 0$.

From (5.6) we get the idea that in case of heterogeneous networks the system Lie algebra is closed under the commutator

$$[\mathcal{B}L\mathcal{C}, \mathcal{B}K\mathcal{C}] = \mathcal{B}(LTK - KTL)\mathcal{C}$$

with $T = \mathcal{CB}$. We define by

$$[L, K]_T := LTK - KTL.$$

the T-commutator and denote by $\{L_1, \ldots, L_k\}_{T-LA}$ the smallest subspace of $\mathbb{R}^{N\times N}$, which is closed under the T-commutator. Hence, if the subspace $\{L_1, \ldots, L_k\}_{T-LA}$ contains $\mathfrak{sl}_{nN}(\mathbb{R})$, it is sufficient to derive $\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}$. This is the following result.

Theorem 5.13. *Let (A_i, b_i, c_i) be controllable and observable for $1 \le i \le N$ with $N > 2$ for all $1 \le i, j \le N$. If*

$$\mathfrak{sl}_N(\mathbb{R}) \subseteq \{L_1, \ldots, L_m\}_{T-LA}$$

holds for $\mathcal{S} := \mathrm{span}\{L_1, \ldots, L_m\} \subset \mathbb{R}^{N\times N}$, the Lie algebra $\mathfrak{G}$ generated by $\mathcal{A}$ and $\mathcal{BSC}$ is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Proof. The proof works analogously to the proof of Theorem 5.2. □

The conditions of Theorem 5.13 are not necessary. Since

$$[K, L]_T = KTL - LTK = [K, TL] + [T, L]K$$

we can give explicit conditions on the subspace $\mathcal{S}$ in case $[T, L] = 0$.

Theorem 5.14. *Let (A_i, b_i, c_i) be an n-dimensional controllable and observable triple for $1 \leq i \leq N$ and $N > 2$. Let $\mathcal{S}$ be a finitely generated subspace of $\mathbb{R}^{N\times N}$, which satisfies the following conditions:*

(i) there exists a matrix $L \in \mathcal{S}$ such that $[L, T] = 0$ and LT is strongly regular, where $T = \mathcal{C}\mathcal{B}$;

(ii) there exists a matrix $K \in \mathcal{S}$ such that LT and K do not possess a common nontrivial subspace.

Then the Lie algebra $\mathfrak{G}$ generated by $\mathcal{A}$ and $\mathcal{B}\mathcal{S}\mathcal{C}$ is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

A necessary requirement for the existence of a matrix $L \in \mathbb{R}^{N\times N}$, which satisfies condition (i) from Theorem 5.14 is that the matrix $\mathcal{C}\mathcal{B}$ has the eigenvalue 0 at most with simple multiplicity.

Proof. Let $L, K \in \mathcal{S}$ be the matrices, which satisfy condition (i) and (ii). Then

$$\begin{aligned}
[\mathcal{B}K\mathcal{C}, \mathcal{B}L\mathcal{C}] =& \mathcal{B}K\mathcal{C}\mathcal{B}L\mathcal{C} - \mathcal{B}L\mathcal{C}\mathcal{B}K\mathcal{C} \\
=& \mathcal{B}KTL\mathcal{C} - \mathcal{B}LTK\mathcal{C} \\
=& \mathcal{B}KTL\mathcal{C} - \mathcal{B}TLK\mathcal{C} \\
=& \mathcal{B}\,[K, TL]\,\mathcal{C}
\end{aligned}$$

due to the assumption $[L, T] = 0$. Taking the commutator with $\mathcal{B}L\mathcal{C}$ iteratively it yields

$$\mathrm{ad}^i_{\mathcal{B}L\mathcal{C}}(\mathcal{B}K\mathcal{C}) = \mathcal{B}\,\mathrm{ad}^i_{TL}(K)\mathcal{C}.$$

Since TL is strongly regular, we can assume without loss of generality that TL is diagonal. Hence,

$$E_{ij} \otimes b_i c_j \in \mathrm{span}\{\mathcal{B}\,\mathrm{ad}^i_{TL}(K)\mathcal{C} \bigm| i \in \mathbb{N}_0\}$$

in case $k_{ij} \neq 0$. As LT is assumed to be diagonal, $K = (k_{ij})$ is permutation-irreducible since K and L are assumed to posses no common nontrivial subspace. Then we obtain with Theorem 4.4 and Lemma 4.18 that

$$E_{ij} \otimes b_i c_j \in \{\mathcal{A}, \mathcal{B}\mathcal{S}\mathcal{C}\}_{LA}$$

for all $i \neq j$. Then with $\mathrm{ad}^k_{\mathcal{A}}(E_{ij} \otimes b_i c_j) = E_{ij} \otimes \mathrm{ad}_{A_i,A_j}(b_i c_j)$ and Lemma 4.7 we obtain

$$E_{ij} \otimes \mathbb{R}^{n \times n} \subset \mathfrak{G}.$$

With Lemma 4.9 the result follows. □

Thus, we get sufficient conditions for the bilinear control system (5.3) to be accessible.

Corollary 5.15. *Let (A_i, b_i, c_i) be controllable and observable for $1 \leq i \leq N$ with $N > 2$. Let $\mathcal{S} := \mathrm{span}\{L_1, \ldots, L_m\}$ be a subspace of $N \times N$ matrices and*

$$\gamma_{\mathcal{S}}(u) := \sum_{i=1}^{m} u_i L_i$$

the control matrix of the subspace $\mathcal{S}$. If $\mathfrak{sl}_N(\mathbb{R}) \subset \{\mathcal{S}\}_{T-LA}$, then the bilinear control system

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma_{\mathcal{S}}(u)\mathcal{C})x \tag{5.7}$$

is accessible on $\mathbb{R}^{nN} \backslash \{0\}$.

In case $\mathcal{CB} = 0$ Theorem 5.14 can immediately transferred to the more general case: Let (A_i, b_i, c_i) be a n-dimensional controllable and observable triple for $1 \leq i \leq N$ with $N > 2$ and let $m^\star$ be the smallest integer m such that $\mathcal{C}\mathcal{A}^m\mathcal{B} \neq 0$. Then system (5.7) is accessible on $\mathbb{R}^{nN} \backslash \{0\}$ if $\mathcal{S}$ contains the following two elements:

(i) there exists a matrix $L \in \mathcal{S}$ such that $[L, T] = 0$ and LT is strongly regular, where $T = \mathcal{C}\mathcal{A}^{m^*}\mathcal{B}$;

(ii) there exists a matrix $K \in \mathcal{S}$ such that L and K do not possess a common nontrivial subspace.

5.4 Bilinear Control of Toeplitz Formations

Networks of linear dynamical systems with special interconnection structures are important in many application areas of control systems. We mention e.g. the analysis of cyclic pursuit strategies via circulant matrices in formation control of multi-agent systems [41]; the analysis of polygonal approximations of the Euclidean curve shortening flow in image processing via discrete-time circulant systems [12], the work of [9] on discretizing partial differential equations via associated algebras of Toeplitz matrices and, e.g., Turings work on morphogenesis [54] that involves the analysis of circulant linear systems. Recent work by Hamilton and Broucke [22] proposes general classes of structured linear systems, defined by circulant matrices, pseudo-circulant matrices, Toeplitz matrices or block versions.

In all this prior work, the interconnection parameters are not regarded as control parameters and thus bear no direct influence on the controllability properties of the system. The situation changes if one intentionally allows for switches in some of the interconnections and therefore considers these interconnection parameters then as control variables. We refer to [10, 38] for some early work in this direction. Results on circulant matrices can be found in [15, 35].

Section 5.4.1 summarizes basic facts on circulant, pseudo-circulant and Toeplitz matrices. For circulant interconnection patterns it is easily seen and probably well-known, that the bilinear system is never controllable. The situation changes if one extends the Abelian class of circulant matrices to the non-Abelian set of pseudo-circulants. Section 5.4.2 contains the main controllability results. Thus, if one extends to the class of pseudo-circulant matrices, we show that the system becomes controllable. This implies controllability of the bilinear system for arbitrary Toeplitz interconnections, as well as for certain mixed "Circulant+Toeplitz" type of interconnections (Section 5.4.3).
We also deduce some pure Linear Algebra results that seem to be of independent interest. For example we show that every invertible matrix can be presented as a finite product of Toeplitz or pseudo-circulant matrices. Similarly, it is shown that every complex unitary matrix is a finite product of unitary Toeplitz or unitary Hankel matrices. It is an open problem to find good bounds on the number of factors.

We mention that the results of Section 5.4 are published in [45].

5.4.1 Basic definitions and properties

In this section we give the basic definitions of Toeplitz, circulant and pseudo-circulant matrices and establish their main properties which we need for the sequel of this chapter. We start with the most general type of matrices.

Definition 5.16. A complex *Toeplitz matrix* of order n is of the form

$$T = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-(n-2)} & \cdots & a_{-1} & a_0 & a_1 \\ a_{-(n-1)} & a_{-(n-2)} & \cdots & a_{-1} & a_0 \end{pmatrix},$$

where $a_{-(n-1)}, \dots, a_{-1}, a_0, \dots, a_{n-1}$ are $2n-1$ arbitrary complex numbers, i.e. the entries of a Toeplitz matrix T satisfy the relation

$$a_{ij} = a_{(i+1)(j+1)}, \text{ for } i, j = 1, 2, \dots, n-1.$$

That is to say, Toeplitz matrices are constant along all diagonals parallel to the principal diagonal. From the definition it follows immediately that Toeplitz matrices of order n form a linear subspace $\mathrm{Toe}(n)$ of $\mathbb{C}^{n\times n}$ of dimension $2n-1$. Define

$$T_1 := \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T_{-1} := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Clearly, the matrices $T_k := (T_1)^k$ with $k = 0, ..., n-1$ and $T_{-k} := (T_{-1})^k$ with $k = 1, ..., n-1$ form a basis of $\mathrm{Toe}(n)$. A type of matrices closely related to the Toeplitz matrices are Hankel matrices.

Definition 5.17. A complex *Hankel matrix* of order n is of the form

$$\begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_{-1} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ a_1 & \iddots & a_{-(n-4)} & a_{-(n-3)} & a_{-(n-2)} \\ a_0 & a_{-1} & \cdots & a_{-(n-2)} & a_{-(n-1)} \end{pmatrix},$$

where $a_{1-n}, \ldots, a_{-1}, a_0, \ldots, a_{n-1}$ are $2n-1$ arbitrary complex numbers.

Remark 5.18. From the definitions we derive that most of the results about Toeplitz matrices can be applied to Hankel matrices, too, as every Toeplitz matrix T can be written as a product $T = HJ$, where H, J are Hankel matrices and

$$J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

We continue with a brief summary of the main properties of circulant matrices. For this purpose, we follow closely [15] and start with the definition.

Definition 5.19. A complex *circulant matrix* C of order n is a special Toeplitz matrices of the form

$$C = \mathrm{Circ}(c_0, ..., c_{n-1}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

with $c_i \in \mathbb{C}$.

It is clear from the definition that each circulant matrix is completely determined by its first row or its first column. The structure of a circulant matrix is preserved under scalar multiplication and addition of other circulant matrices. Thus, the set of complex circulant matrices forms an n-dimensional subspace of $\mathbb{C}^{n\times n}$.

Let us now consider the permutation matrix

$$S = \mathrm{Circ}(0, 1, 0, \dots, 0) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \tag{5.8}$$

which is a special circulant. Then, $S^2 = \mathrm{Circ}(0, 0, 1, 0, \dots, 0)$ and inductively we see that $S^n = I$. As a consequence, every circulant matrix C can be written as a polynomial in S with degree at most $n-1$, i.e. for every circulant matrix C there exists a polynomial $p_C \in \mathbb{R}[z]$ with $\deg p_C < n$ such that $C = p_C(S)$, where

$$p_C(z) := \sum_{j=0}^{n-1} c_j z^j$$

denotes the generating polynomial of C. Apparently, the polynomial p_C is uniquely determined by the first row of C. This holds either for the set of real or the set of complex circulant matrices. The polynomial p_C is called the *representer* of C. The permutation matrix S can be diagonalized by the Fourier matrix

$$\Phi = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2n-2} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{2n-2} & \dots & \omega^{(n-1)^2} \end{pmatrix},$$

where $\omega = e^{2\pi i/n}$ denotes a primitive $n-$th root of unity. Note, that Φ is both unitary and symmetric. Due to the representation of a circulant matrix as a polynomial in S we deduce the following important result.

Theorem 5.20. *All circulant matrices are simultaneously diagonalizable by the Fourier matrix.*

Direct consequences are that matrix multiplication of circulant matrices is Abelian and that every circulant matrices has the same set of orthonormal eigenvectors. Let

$$\Omega := \operatorname{diag}(1, \omega, \dots, \omega^{n-1}). \tag{5.9}$$

A straightforward computation shows $S = \Phi\Omega\Phi^*$. Therefore, we deduce the eigenvalue decomposition of circulants as

$$C = p_C(S) = \Phi \operatorname{diag}(p_C(1), p_C(\omega), \dots, p_C(\omega^{n-1}))\Phi^*. \tag{5.10}$$

We obtain the following identity for the relationship between the coefficients of the representer p_C and the eigenvalues of C:

$$\begin{pmatrix} p_C(1) \\ p_C(\omega) \\ \vdots \\ p_C(\omega^{n-1}) \end{pmatrix} = \Phi \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}. \tag{5.11}$$

The eigenvalues of C are given by the values of the polynomial p_C at the $n-$th roots of unity. Conversely, let $\lambda_1, \dots, \lambda_n$ denote arbitrary complex numbers. Then

$$p(z) = \sum_{j=0}^{n-1} c_j z^j := \sum_{k=1}^{n} \lambda_k \prod_{l \neq k} \frac{z - \omega^l}{\omega^k - \omega^l}$$

is the unique polynomial of degree strictly less than n that interpolates λ_j at ω^j, $j = 1, \ldots, n$. Therefore, $C = \Phi \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\Phi^*$ is a circulant matrix with given eigenvalues $\lambda_1, \ldots, \lambda_n$.

Theorem 5.21. *A matrix $C \in \mathbb{K}^{n\times n}$ is of circulant structure if and only if it can be written as $p_C(S)$, where $p_C \in \mathbb{K}[z]$.*

This shows that the circulant structure of a complex matrix poses no restriction on the eigenvalues. In particular, the inverse eigenvalue problem is always solvable in the class of complex circulants. This is no longer true for real circulants, where the realness of the coefficients of C poses restrictions on the multiplicities of the real eigenvalues, because they have to be even. We summarize the properties of circulant matrices in a theorem.

Theorem 5.22. *Let A, B be circulant matrices and $\alpha_k \in \mathbb{C}$. Then,*

(i) A^T, A^*, $\alpha_1 A + \alpha_2 B$, $A \cdot B$ and $\sum_{k=0}^{r} \alpha_i A^i$ are circulants;

(ii) A and B commute;

(iii) if A is nonsingular, its inverse is a circulant. If

$$A = \Phi^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\Phi$$

with $\lambda_i \neq 0$, then its inverse is given by

$$A^{-1} = \Phi^* \operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})\Phi.$$

Let $\operatorname{circ}_n(\mathbb{C})$ denote the set of all complex circulant matrices of order n, i.e.

$$\operatorname{circ}_n(\mathbb{C}) := \{p(S) \mid p \in \mathbb{C}[z], \deg p < n\}$$

and let $\operatorname{Circ}_n(\mathbb{C})$ denote the subset of all invertible complex circulants, i.e.

$$\operatorname{Circ}_n(\mathbb{C}) := \{C \mid C \in \operatorname{circ}_n(\mathbb{C}), \det C \neq 0\}.$$

Then, $\operatorname{circ}_n(\mathbb{C})$ is a complex n-dimensional Abelian Lie subalgebra of the full matrix Lie algebra $\mathbb{C}^{n\times n}$. Moreover, since the product of two circulant matrices and the inverse of an invertible circulant are again circulant matrices, the set $\operatorname{Circ}_n(\mathbb{C})$ forms an Abelian Lie subgroup of $GL_n(\mathbb{C})$. The same holds for real circulants. The matrix exponential function

$$\exp : \operatorname{circ}_n(\mathbb{C}) \longrightarrow \operatorname{Circ}_n(\mathbb{C}), \quad C \mapsto e^C$$

maps the Lie algebra $\operatorname{circ}_n(\mathbb{C})$ onto the Lie group $\operatorname{Circ}_n(\mathbb{C})$.
Since the eigenvector decomposition (5.10) applies in particular also to unitary circulant matrices, we obtain the following result.

Theorem 5.23. *A circulant matrix C is unitary if and only if the generating polynomial p_C satisfies $|p_C(\omega^i)| = 1$ for $i = 0, \ldots, n-1$. Moreover, any n complex numbers of absolute value one are the eigenvalues of a suitable unitary circulant matrix.*

Furthermore, the set of all unitary circulants is a compact Abelian Lie group

$$\begin{aligned}\mathrm{UCirc}_n(\mathbb{C}) :&= \{C \mid C \text{ circulant and unitary}\} \\ &= \{\Phi\Lambda\Phi^{-1} \mid \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n) \text{ with } |\lambda_i| = 1\},\end{aligned} \tag{5.12}$$

i.e. $\mathrm{UCirc}_n(\mathbb{C})$ can be identified with the n-Torus $S^1 \times ... \times S^1$. Similarly, the set

$$\mathrm{ucirc}_n(\mathbb{C}) := \{C \mid C \text{ circulant and skew-Hermitian}\}$$

of skew-Hermitian circulants is a real Lie subalgebra of $\mathrm{circ}_n(\mathbb{C})$. It is the Lie algebra of $\mathrm{UCirc}_n(\mathbb{C})$; therefore the exponential map

$$\exp : \mathrm{ucirc}_n(\mathbb{C}) \longrightarrow \mathrm{UCirc}_n(\mathbb{C}), \quad C \mapsto e^C$$

is surjective.

We now extend the definitions and basic results of circulant matrices to the more general setting of λ-circulant matrices.

Definition 5.24. Let $\lambda \in \mathbb{C}$ denote an arbitrary nonzero complex number. A complex λ-circulant matrix of order n is of the form

$$\mathrm{Circ}_\lambda(c_0, c_1, ..., c_{n-1}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ \lambda c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \lambda c_{n-2} & \lambda c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & & & \ddots & \vdots \\ \lambda c_1 & \lambda c_2 & \cdots & \lambda c_{n-1} & c_0 \end{pmatrix}$$

with $c_i \in \mathbb{C}$. A pseudo-circulant matrix is a λ-circulant matrix for some complex number $\lambda \neq 0$.

As we will now see, the set of λ-circulant matrices has properties very similar to those of circulants. For any nonzero complex number $\lambda \neq 0$, we pick any of its $n-$th root which we denote by γ, i.e. $\gamma^n = \lambda$. Then, the special λ-circulant

$$S_\lambda = \mathrm{Circ}_\lambda(0,1,0,...,0) := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ \lambda & 0 & \dots & \dots & 0 \end{pmatrix} \tag{5.13}$$

has eigenvalues $\gamma\omega^k$ for $k = 0, \dots, n-1$, $\gamma^n = \lambda$. Note that $S_1 = S$. By conjugating the circulant matrix S with the diagonal matrix $\Lambda_\gamma = \mathrm{diag}(1, \gamma, \gamma^2, \dots, \gamma^{n-1})$, we obtain that

$$S_\lambda = \Lambda_\gamma(\gamma S)\Lambda_\gamma^{-1}.$$

Since any λ-circulant can be written as a polynomial $C_\lambda = p_C(S_\lambda)$ in S_λ, this shows that the eigenvalues of C_λ are $p_C(\gamma\omega^k)$ for $k = 0, \dots, n-1$. Define

$$\Phi_\gamma := \mathrm{diag}(1, \gamma, ..., \gamma^{n-1})\Phi,$$

which is unitary if and only if $|\gamma| = 1$. It follows that all λ-circulants have the form

$$C_\lambda = \mathrm{Circ}_\lambda(c_0, c_1, ..., c_{n-1}) = \Phi_\gamma \,\mathrm{diag}(p_C(\gamma), p_C(\gamma\omega), \dots, p_C(\gamma\omega^{n-1}))\Phi_\gamma^{-1}.$$

This shows that λ-circulants are simultaneously diagonalizable. We state the analogon to Theorem 5.22 for λ-circulant matrices.

Theorem 5.25. *Let A, B be λ-circulant matrices and $\alpha_k \in \mathbb{C}$. Then,*

(i) A^T, A^*, $\alpha_1 A + \alpha_2 B$, $A \cdot B$ and $\sum_{k=0}^{r} \alpha_i A^i$ are λ-circulants;

(ii) A and B commute;

(iii) if A is nonsingular, its inverse is a λ-circulant. Let

$$A = \Phi_\lambda^* \,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\Phi_\lambda$$

with $\lambda_i \neq 0$, then its inverse is given by

$$A^{-1} = \Phi_\lambda^* \,\mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})\Phi_\lambda.$$

Moreover, the set of invertible λ-circulant matrices forms an Abelian Lie group that is isomorphic to the Abelian group of invertible circulant matrices. Note further, that any pseudo-circulant matrix is a Toeplitz matrix. The connection between pseudo-circulant matrices and Toeplitz matrices has been discovered in [20]:

Theorem 5.26. *A nonsingular Toeplitz matrix has a Toeplitz inverse if and only if T is pseudo-circulant.*

Let $\mathrm{circ}_n^\lambda(\mathbb{C})$ denote the set of all complex $n \times n$ λ-circulant matrices

$$\mathrm{circ}_n^\lambda(\mathbb{C}) := \{p(S_\lambda) \mid p \in \mathbb{C}[z], \deg p < n\}$$

and let $\mathrm{Circ}_n^\lambda(\mathbb{C})$ denote the subset of all invertible complex λ-circulants

$$\mathrm{Circ}_n^\lambda(\mathbb{C}) := \{C \mid C \in \mathrm{circ}_n^\lambda(\mathbb{C}), \det C \neq 0\}.$$

Then $\mathrm{circ}_n^\lambda(\mathbb{C})$ is a complex n-dimensional Abelian Lie subalgebra of the full matrix Lie algebra $\mathbb{C}^{n\times n}$. As in the case for circulant matrices, we obtain that the matrix exponential function

$$\exp : \mathrm{circ}_n^\lambda(\mathbb{C}) \longrightarrow \mathrm{Circ}_n^\lambda(\mathbb{C}), \quad C \mapsto e^C$$

defines a surjection onto the Lie group $\mathrm{Circ}_n^\lambda(\mathbb{C})$. Similarly, the set $\mathrm{ucirc}_n^\lambda(\mathbb{C})$ of skew-Hermitian λ-circulants is the Lie algebra of the compact Abelian Lie group

$$\mathrm{UCirc}_n^\lambda(\mathbb{C}) := \{C \mid C \ \lambda\text{-circulant and unitary}\}$$

with a surjective exponential map $\exp : \mathrm{ucirc}_n^\lambda(\mathbb{C}) \longrightarrow \mathrm{UCirc}_n^\lambda(\mathbb{C})$.

5.4.2 Circulant and pseudo-circulant control systems

Before analyzing the controllability properties of Toeplitz formations we consider the more specific type of control systems: circulant formations. Shortly, we will find out that circulant control systems are never controllable. This motivates to consider afterwards more general classes of systems, e.g., those defined by pseudo-circulants, where pseudo-circulant matrices have one degree of freedom more than circulant matrices.

5.4.2.1 Circulant control systems

We consider now bilinear control systems on $\mathbb{C}^n$ of the form

$$\dot{x} = \left(\sum_{j=1}^{d} u_j C_j \right) x, \tag{5.14}$$

defined by circulant matrices $C_1, \ldots, C_d \in \mathbb{C}^{n\times n}$, $d \leq n$. Without loss of generality we can assume that the circulant matrices $C_j := \mathrm{Circ}(c_0^{(j)}, ..., c_{n-1}^{(j)})$ are linearly independent.

Let

$$p_{C_j}(z) = \sum_{k=0}^{n-1} c_k^{(j)} z^k$$

denote the representer of the circulant matrix C_j, $j = 1, \dots, d$, and let V_d denote the vector space of complex polynomials spanned by $p_{C_1}, \dots, p_{C_d}$. Moreover, we identify a vector of coefficients $x = (x_0, \dots, x_{n-1})^\top \in \mathbb{C}^n$ with the associated polynomial $\pi_x(z) := \sum_{k=0}^{n-1} x_k z^k$.

Since circulant matrices permute, every element in the system Lie algebra of (5.14) is of the form $\Phi p(\Omega)\Phi^*$, where p runs through the elements of V_d and Ω is of the form (5.9). Therefore, the elements of the system Lie group are exactly $\Phi e^{p(\Omega)}\Phi^*$, where $p \in V_d$. Thus, for $x \in \mathbb{C}^n$, the reachable sets of (5.14) are

$$\mathcal{R}(x) = \{\Phi e^{p(\Omega)}\Phi^* x \mid p \in V_d\}.$$

Since circulant matrices commute with each other, the system Lie algebra of (5.14) is d-dimensional and at most n-dimensional. Hence, (5.14) cannot be accessible and thus not controllable.

Easy considerations and the currently made observations yield the following theorem.

Theorem 5.27. *Let*

$$\dot{x} = \left(\sum_{j=1}^{d} u_j C_j\right) x \tag{5.15}$$

be a bilinear system on $\mathbb{C}^n$, where C_j are circulant matrices. Then, the following holds for (5.15):

(i) The circulant control system (5.15) is never controllable on $\mathbb{C}^n$.

(ii) For $d = n$ there is a unique reachable set $\mathcal{R}(x)$ that is dense in $\mathbb{C}^n$. $\mathcal{R}(x)$ is characterized by $\pi_x(\omega^j) \neq 0$ for $j = 0, \dots, n-1$.

(iii) For $d < n$ all reachable sets have empty interior.

5.4.2.2 Pseudo-circulant control systems

In Section 5.4.2.1 we have seen that control systems defined by circulant matrices are never controllable on $\mathbb{C}^n$ since circulant matrices commute with each other. Therefore, we change over to a more general case. This leads us to bilinear control systems with pseudo-circulant coefficient matrices of the form

$$\dot{x} = \left(\sum_{j=1}^{d} u_j C_{\lambda_j}\right) x, \tag{5.16}$$

defined by pseudo-circulant matrices $C_{\lambda_1}, \dots, C_{\lambda_d} \in \mathbb{C}^{n\times n}$ and $d \leq n$. Again we can assume that the pseudo-circulant matrices $C_{\lambda_j} := \mathrm{Circ}_{\lambda_j}(c_0^{(j)}, \dots, c_{n-1}^{(j)})$ are linearly independent.

Before we apply Theorem 2.20 or Theorem 2.23 to pseudo-circulant systems, we first set up some notation. Let $\omega := e^{2\pi i/n}$ denote a primitive $n-$th root of unity and consider the unitary diagonal matrix

$$\Omega(\omega) := \mathrm{diag}(1, \omega, \dots, \omega^{n-1}).$$

For $i < j$ consider the proper linear subspaces

$$V_{ij} := \ker\left(I_n - \Omega(\omega^{j-i})\right) \subset \mathbb{C}^n$$

and for $(i, j) \neq (k, l)$ with $i < j, k \neq l$ consider

$$W_{ijkl} := \ker\Big(I_n - \Omega(\omega^{j-i}) + \Omega(\omega^{l-i}) - \Omega(\omega^{k-i})\Big) \subset \mathbb{C}^n.$$

All the subspaces contain the first standard basis vector e_1 and therefore have codimension at least 1. We have the following lemma, whose proof follows from a short calculation.

Lemma 5.28. *Let $p(z) = \sum_{p=0}^{n-1} c_p z^p \in \mathbb{C}[z]$ and $c = (c_0, \dots, c_{n-1}) \in \mathbb{C}^n$ denote the associated coefficient vector. Then,*

(i) $p(\omega^i z) \neq p(\omega^j z)$ if and only if $c \notin V_{ij}$.

(ii) $p(\omega^i z) - p(\omega^j z) \neq p(\omega^k z) - p(\omega^l z)$ if and only if $c \notin W_{ijkl}$.

Now, consider the union of subspaces $V := \bigcup_{ij} V_{ij}$ and $W := \bigcup_{ijkl} W_{ijkl}$. Then, their complements $\mathbb{C}^n \backslash V$ and $\mathbb{C}^n \backslash W$ are non-empty Zariski-open subsets of $\mathbb{C}^n$. Here, the term *Zariski-open* specifies subsets whose complement is a closed algebraic variety. By construction, $V \subset W$. The next result gives sufficient conditions for V, W being small.

Lemma 5.29. *The inclusion $\mathbb{C}e_1 \subset V \subset W \subset \mathbb{C}^n$ holds.*

(i) $V = \mathbb{C}e_1$ if and only if n is a prime number.

(ii) $V = W$ if (and only if) either n is odd or $n = 2$.

Proof. The inclusion $\mathbb{C}e_1 \subset V \subset W$ is obvious. By definition, the set V differs from $\mathbb{C}e_1$ if and only if there exist $0 < j - i < n$ and $0 < k < n$ such that $\omega^{k(j-i)} = 1$. This is equivalent to the condition that n divides $k(j-i)$. If n is a prime this is clearly impossible, as n would have to divide k or $j - i$, which are both smaller than n. If $n = n_1 n_2$ is not a prime, choose $k := n_1, i = 1, j = n_2 + 1$. Then $\omega^{k(j-i)} = 1$ and $V \neq \mathbb{C}e_1$. This completes the proof of the first claim.

If $n = 2$, one immediately verifies $V = W$. Thus assume, that $n = 2m + 1$ is odd. By construction, the spaces V, W differ from each other if and only if there exist nontrivial, distinct integers $0 < r, s, t < n$

$$1 + \omega^s = \omega^r + \omega^t. \tag{5.17}$$

Without loss of generality, $0 < s \leq m$ and $t \geq r$. By taking absolute values on both sides we obtain the equality of real parts $\Re(\omega^s) = \Re(\omega^{t-r})$, i.e. $\cos(2\pi s/n) = \cos(2\pi(t-r)/n)$, which implies $t = r + s$. Since n is odd, there exists no s with $\omega^s = -1$. Thus $1 + \omega^s \neq 0$. From $1 + \omega^s = \omega^r(1 + \omega^s)$, we therefore conclude $\omega^r = 1$, which is impossible, since $r < n$. Thus there exists no non-trivial solution to equation (5.17) and we are done. □

Now, we can prove the following theorem.

Theorem 5.30. *Let $c \in \mathbb{C}^n$ be given and $\lambda \in \mathbb{C}$ denote any nonzero complex number with $C_\lambda := \mathrm{Circ}_\lambda(c_0, \ldots, c_{n-1})$ the associated λ-circulant matrix. Then,*

(i) there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that C_λ is strongly regular if and only if $c \in \mathbb{C}^n\backslash W$;

(ii) there exists no nontrivial invariant subspace for $C_\lambda, \lambda \in \mathbb{C}$, if $c \in \mathbb{C}^n\backslash V$.

Proof. Let $D(z) := \mathrm{diag}(1, z, \ldots, z^{n-1})$ and substitute $\lambda = z^n$ for $z \in \mathbb{C}\backslash\{0\}$. Define $S := S_1$ the special circulant matrix. Then

$$C_{z^n} = D(z)p(zS)D(z)^{-1}$$

has eigenvalues $p(\omega^i z)$ with $i = 0, \ldots, n-1$. By Lemma 5.28, the vector of coefficients c satisfies $c \notin W_{ijkl}$ if and only if C_{z^n} is strongly regular for some (and hence almost all) $z \in \mathbb{C}$. This proves (i).
For (ii), note that a nontrivial linear subspace $U \subset \mathbb{C}^n$ is invariant under all matrices C_{z^n} if and only if $D(z)^{-1}U$ is an invariant subspace of the circulant matrix $p(zS)$. Condition $c \in \mathbb{C}^n\backslash V$ is equivalent to $p(zS)$ having distinct eigenvalues for almost all z. Thus, for almost all complex numbers z, the circulant matrix $p(zS)$ has only finitely many invariant subspaces and these are independent of z. So, the invariant $D(z)^{-1}U$ is independent of z and, in particular, therefore $D(z)^{-1}U = D(1)^{-1}U = U$ holds for all $z \neq 0$. This shows that U is an invariant subspace of $D(z)$ and is therefore spanned by standard basis vectors $e_{i_1}, \ldots, e_{i_k}$. But the standard basis vectors do not span an invariant subspace of $p(zS)$, as otherwise the column vectors of the Fourier matrix $\Phi e_{i_1}, \ldots, \Phi e_{i_k}$ would have to span an invariant subspace of $\mathrm{diag}(p(z), \ldots, p(\omega^{n-1}z))$, which is impossible.
The result follows. □

Now, we can apply Theorem 2.23.

Theorem 5.31. *Let $c \in \mathbb{C}^n - W$, $c_0 \neq 0$. Then, the bilinear control system on $GL_n(\mathbb{C})$*

$$\dot{X} = \mathrm{Circ}_{u(t)}(c_0, \ldots, c_{n-1})X \tag{5.18}$$

is accessible.

Proof. By Theorem 5.30 the bilinear control system (5.18) with system matrices

$$\mathrm{Circ}_u(c_0, \ldots, c_{n-1}), \quad u \in \mathbb{C}$$

satisfies the assumptions of Theorem 2.23, since $c \in \mathbb{C}^n \backslash W$. Thus, the system Lie algebra is either equal to $\mathfrak{sl}_n(\mathbb{C})$ or equal to $\mathfrak{gl}_n(\mathbb{C})$. But $c_0 \neq 0$ by assumption. Thus, $\mathcal{L} = \mathfrak{gl}_n(\mathbb{C})$. The result follows. □

With Lemma 5.29 we obtain an immediate consequence from Theorem 5.31.

Corollary 5.32. *Let n be a prime number. The bilinear pseudo-circulant control system*

$$\dot{X} = \begin{pmatrix} c_0 & c_1 & & \cdots & c_{n-1} \\ u(t)c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ u(t)c_2 & \cdots & \ddots & \ddots & c_1 \\ u(t)c_1 & \cdots & \cdots & u(t)c_{n-1} & c_0 \end{pmatrix} X$$

is accessible on $GL_n(\mathbb{C})$ if and only if $c_0 \neq 0, (c_1, \ldots, c_{n-1}) \neq (0, \ldots, 0)$.

It is immediately seen by inspection that the polynomial $p(z) = z - 1$ satisfies the genericity condition $c = (-e_1 + e_2)^\top \in \mathbb{C}^n \backslash W$. This shows

Corollary 5.33. *The bilinear pseudo-circulant control system on $GL_n(\mathbb{C})$*

$$\dot{X} = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & & & 1 \\ 1 + u(t) & 0 & \cdots & 0 & -1 \end{pmatrix} X \tag{5.19}$$

is accessible.

We now describe a generalizations of the above mentioned results.

Theorem 5.34. *$GL_n(\mathbb{C})$ is the smallest Lie group that contains all invertible pseudo-circulant matrices. In particular, the pseudo-circulant control system*

$$\dot{X} = \mathrm{Circ}_{u_n(t)}(u_0(t), \dots, u_{n-1}(t))X, \quad X(0) = I_n,$$

is controllable on $GL_n(\mathbb{C})$. Here, $u_0(t), \dots, u_n(t)$ denote arbitrary control sequences such that $\sum_{j=0}^{n-1} u_j z^j \neq 0$ on all $n-$th roots of $u_n \neq 0$.

Proof. By Theorem 5.31, the Lie algebra generated by pseudo-circulant matrices is equal to the full matrix Lie algebra $\mathbb{C}^{n\times n}$. Therefore, the products of exponentials of pseudo-circulant matrices generate $GL_n(\mathbb{C})$. The result follows, as every matrix exponential of a pseudo-circulant is an invertible pseudo-circulant. □

Since the elements of the reachable sets of a symmetric bilinear control system on a Lie group are always reached in finite time, this has an interesting consequence in Linear Algebra.

Corollary 5.35. *Any complex invertible matrix is a finite product of invertible pseudo-circulant matrices.*

For real circulants we can strengthen this result a bit.

Theorem 5.36. *Let $C, C' \in \mathrm{circ}_n(\mathbb{R})$ be circulant matrices. Then:*

(i) There exist real numbers $c_0, \dots, c_{n-1}$ such that C is strongly regular.

(ii) Let $C := \mathrm{Circ}(c_0, \dots, c_{n-1})$ and $C' := \mathrm{Circ}(c'_0, \dots, c'_{n-1})$ be two real circulant matrices such that C and C' have both distinct eigenvalues, respectively. Let $D = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$ be a diagonal matrix. Then C and $D^{-1}C'D$ have no nontrivial common invariant subspace if

$$\det(\mathrm{Circ}(\gamma_1, \dots, \gamma_n)) \neq 0.$$

Proof. From identity (5.11) it is easily seen that the linear map $\Phi : \mathbb{R}^n \longrightarrow \mathbb{C}^n, c \mapsto \Phi c$, has image

$$\{\xi \in \mathbb{C}^n \mid \xi_0 \in \mathbb{R}, \xi_{n-i} = \bar{\xi}_i, i = 1, \dots, n-1\}$$

since the representer of a real circulant matrix $C := \mathrm{Circ}(c_0, \dots, c_{n-1})$ is a polynomial with real coefficients. To prove (i), note, that the sequence of real numbers $2^j, j = 0, \dots, n-1$ is strongly regular for any $n \in \mathbb{N}$.
Let $n = 2m+1$ be odd. Define $\xi_0 = 1$, $\xi_j := 2^j(1+i), j = 1, \dots, m$ and $\xi_j = 2^{n-j}(1-i), j =$

$m+1,\dots,n-1$. Then the pairwise differences $\xi_j - \xi_k = (2^j - 2^k)(1+i)$ are distinct for $1 \le j,k \le m$ and similarly for $\xi_r - \xi_s = (2^r - 2^s)(1-i)$ and $m+1 \le r,s \le n-1$, respectively. In the mixed case, $1 \le j,k \le m$, $m+1 \le j,k \le n-1$, the differences are $\xi_j - \xi_r = (2^j - 2^r) + (2^j + 2^r)i$. Again, these are seen to be pairwise different. Thus $\xi_0, \dots \xi_{n-1}$ are strongly regular. A similar construction works for $n = 2m$ even. This shows (i).

For (ii), note that $C = \Phi\Lambda\Phi^*$ and $C' = \Phi\Lambda'\Phi^*$ hold for diagonal matrices Λ, Λ' with distinct eigenvalues, respectively. Let $\mathbb{C}^I := < e_{i_1}, \dots, e_{i_k} >$ denote the complex k-dimensional subspace, spanned by the standard basis vectors $e_{i_1}, \dots, e_{i_k}$, $1 \le i_k \le n-1$. By assumption, the k-dimensional invariant subspaces of C are of the form $U := \Phi\mathbb{C}^I$. Assume, that one such subspace is also invariant under $D^{-1}C'D$. Equivalently, $\Phi^* D \Phi \mathbb{C}^I$ is an invariant subspace of Λ'. Since Λ' has distinct eigenvalues and is a diagonal matrix, the invariant subspaces are spanned by sets of standard basis vectors. Thus, it follows that $D\Phi\mathbb{C}^I = \Phi\mathbb{C}^J$ for some subset $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ of cardinality k. Let Φ_i denote the i-th column vector of Φ. This leads to $\Phi_{j_r}^* D \Phi_{i_s} = 0$ for all suitable sets $\{j_{k+1}, \dots, j_n\}$, $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Hence, we get the equality

$$\sum_{k,l=1}^{n} \gamma_{kl} \omega^{i_s(l-1) - j_r(k-1)} = 0,$$

where γ_{kl} are the entries of the matrix D. For the diagonal matrix $D = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$, this equation reduces to

$$\sum_{k=1}^{n} \gamma_k (\omega^{i_r - j_s})^{k-1} = 0.$$

But this is impossible, since by assumption $\det(\operatorname{Circ}(\gamma_1, \dots, \gamma_n)) \ne 0$. Thus, the polynomial $\sum_{k=1}^{n} \gamma_k z^{k-1}$ is coprime to $z^n - 1$. □

Corollary 5.37. *Let $c \in \mathbb{R}^n$ be given and $\lambda \in \mathbb{R}$ denote any nonzero real number with $C_\lambda := \operatorname{Circ}_\lambda(c_0, \dots, c_{n-1})$ the associated λ-circulant matrix. Then:*

(i) There exist real numbers $c_0, \dots, c_{n-1}$ such that C_λ is strongly regular.

(ii) Let n be odd and $\lambda = -1$. Let $C := \operatorname{Circ}(c_0, \dots, c_{n-1})$ be any real circulant matrix such that C and C_{-1} have both distinct eigenvalues, respectively. Then, C and C_{-1} have no nontrivial common invariant subspace.

Proof. Statement (i) follows from Theorem 5.36 (i).

To prove (ii), note that Theorem 5.36 applies with $D = \operatorname{diag}(1, -1, \dots, (-1)^{n-1})$. Thus, it is sufficient to check that $\sum_{k=0}^{n-1} (-1)^k z^k$ and $z^n - 1$ are coprime, since then it follows

$$\det(\operatorname{Circ}(1, -1, \dots, (-1)^{n-1})) \ne 0.$$

This is obvious, as for any n-th root of unity ξ the equality

$$2 = 1 + \xi^n = (1+\xi)(1 - \xi + \xi^2 + \ldots + (-\xi)^{n-1})$$

holds. □

Apparently, Theorem 5.34 also holds for real pseudo-circulant matrices.

Theorem 5.38. *$GL_n(\mathbb{R})$ is the smallest Lie group that contains all real invertible pseudo-circulant matrices. In particular, the pseudo-circulant control system*

$$\dot{X} = \mathrm{Circ}_{u_n(t)}(u_0(t), \ldots, u_{n-1}(t))X, \quad X(0) = I_n,$$

is controllable on $GL_n^+(\mathbb{R})$. Here, $u_0(t), \ldots, u_n(t)$ denote real control sequences such that $\sum_{j=0}^{n-1} u_j z^j \neq 0$ on all $n-$th roots of $u_n \neq 0$.

Proof. By Theorem 5.36, the Lie algebra generated by pseudo-circulant matrices is equal to the full matrix Lie algebra $\mathbb{R}^{n \times n}$. Therefore, the products of exponentials of pseudo-circulant matrices generate $GL_n(\mathbb{R})$. The result follows, as every matrix exponential of a pseudo-circulant is an invertible pseudo-circulant. □

Now we can state the real analogon to Theorem 5.31.

Theorem 5.39. *There exists real numbers $c_0, \ldots, c_{n-1}$ such that the bilinear control system on $GL_n^+(\mathbb{R})$*

$$\dot{X} = \mathrm{Circ}_{u(t)}(c_0, \ldots, c_{n-1})X \tag{5.20}$$

is accessible.

Proof. From Corollary 5.37 there exists $c_0, \ldots, c_{n-1}$ such that the system matrices $C_u, u \in \mathbb{R}$ satisfies the assumptions of Theorem 2.20. Thus, the system Lie algebra is either equal to $\mathfrak{sl}_n\mathbb{R}$ or $\mathfrak{gl}_n\mathbb{R}$. Hence, the result follows. □

5.4.3 Toeplitz and Hankel control systems

Since Toeplitz matrices are a generalization of pseudo-circulant matrices, we immediately get as a generalization the following theorem.

Theorem 5.40. *The Lie algebra generated by $n \times n$ complex Toeplitz matrices is equal to the full matrix Lie algebra $\mathbb{C}^{n \times n}$. In particular, the bilinear Toeplitz control system*

$$\dot{X} = \left(\sum_{k=-(n-1)}^{n-1} u_k T_k \right) X$$

is controllable on $GL_n(\mathbb{C})$*.*

Proof. This follows with Theorem 2.6 from Theorem 5.31. □

The above results were stated for complex matrices only. It is possible to extend the theory to real matrices.

Theorem 5.41. *The Lie algebra generated by* $n \times n$ *real Toeplitz matrices is equal to the full matrix Lie algebra* $\mathbb{R}^{n\times n}$*. In particular, the bilinear Toeplitz control system*

$$\dot{X} = \left(\sum_{k=-(n-1)}^{n-1} u_k T_k \right) X$$

is controllable on $GL_n^+(\mathbb{R})$*.*

The proof works analogously to the complex case with Theorem 5.39. We give an sample of what is possible for real Toeplitz matrices.

Theorem 5.42. *Let* C *denote any real* $n \times n$ *circulant matrix with distinct eigenvalues. Then, for a generic class of real symmetric Toeplitz matrices* T*, the bilinear control system*

$$\dot{X} = (uC + vT)\, X$$

is controllable on $GL_n(\mathbb{R})$*. As a special case, the system with Toeplitz matrix* $T_* = T_1 + T_{-1}$ *is controllable.*

Proof. We apply Theorem 5.39 for real matrices. It has been shown by Landau [37] that the inverse eigenvalue problem for real symmetric Toeplitz matrices is always solvable. Thus there exists a Zariski open subset $\mathcal{T}$ of real symmetric Toeplitz matrices T that are strongly regular.
We claim that there exists a Zariski-open subset $\mathcal{T}' \subset \mathcal{T}$ such that, for any $T \in \mathcal{T}'$, the matrices T, C have no non-trivial joint invariant subspaces. By assumption, the circulant C has pairwise different eigenvalues. Therefore, each invariant subspace V of C is spanned by the column vectors of the Fourier matrix Φ. Let $U_1, \ldots, U_N$ denote these non-trivial invariant subspaces of C. Consider the real algebraic varieties

$$M_i := \{T \in \mathcal{T} \mid TU_i \subset U_i\}, \quad i = 1, \ldots, N.$$

Then, $M := M_1 \cup \ldots \cup M_N$ is a real algebraic variety, too, and therefore the complement $\mathcal{T}' := \mathcal{T} - M$ is Zariski open.
We have to show that $\mathcal{T}'$ is nonempty. Suppose it is not, then $M = \mathcal{T}$ and therefore $M_i = \mathcal{T}$ for some $i = 1, \ldots, N$. But this means that every symmetric Toeplitz matrix has

a fixed invariant subspace U_i, spanned by the column vectors of Φ. But this is impossible. Choose for instance the symmetric Toeplitz matrix $T = T_*$. It is well-known that T_* has eigenvalues $\lambda_k = 2\cos(\frac{k\pi}{n+1}), k = 1, \ldots, n$. These eigenvalues are pairwise distinct, but T_* is not always strongly regular. Note that T_* differs from the circulant matrix $\text{Circ}(0, 1, \cdots, 0, 1)$ by the rank two symmetric matrix $J = e_1 e_n^\top + e_n e_1^\top$. Thus, T_* has a common invariant subspace with C if and only if J does.
But $\Phi^* J \Phi$ has entries $a_{ij} = \omega^{-j}(1+\omega^{i+j})$, which is zero if and only if $n = 2m$ is even and $i + j = m$ or $i + j = 3m$. Hence, the zero entries of this matrix are exactly at positions $(m, 0), (m-1, 1), \ldots, (0, m)$ and $(2m-1, m+1), (2m-2, m+2), \ldots, (m+1, 2m-1)$. In particular, J is permutation irreducible and does not have any subspace of the form $\mathbb{C}^I :=< e_{i_1}, \ldots, e_{i_k} >$ as an invariant subspace. This completes the proof. □

As an immediate consequence of Corollary 5.35 we deduce

Corollary 5.43. *Any complex invertible matrix is a finite product of invertible Toeplitz matrices.*

We now extend the above controllability analysis to the situation, where the Toeplitz matrices that define the dynamics are confined to be unitary. This depends on a characterization of unitary Toeplitz matrices via unitary pseudo-circulants. More precisely, we have the following characterization of unitary Toeplitz matrices:

Theorem 5.44. *A unitary matrix T is a Toeplitz matrix if and only if there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $a_0, ..., a_{n-1} \in \mathbb{C}$ with*

$$T = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \lambda a_{n-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ \lambda a_1 & \cdots & \lambda a_{n-1} & a_0 \end{pmatrix},$$

Proof. This is Theorem 3.6. of [21]. □

The result shows that the class of unitary Toeplitz matrices coincides with the class of unitary pseudo-circulant matrices.

Theorem 5.45. *The unitary Toeplitz matrices generate $U_n(\mathbb{C})$. Every unitary matrix is a finite product of unitary Toeplitz matrices and every unitary matrix is a finite product of unitary Hankel matrices.*

Proof. With Theorem 5.44 we obtain that the set of unitary Toeplitz matrices coincides with the set of unitary λ-circulant matrices. Hence, it suffices to prove the theorem for

the class of unitary pseudo-circulant matrices.
Consider the Abelian Lie algebra $\mathrm{circ}_n^{\lambda}(\mathbb{C})$ of skew-Hermitian λ-circulant matrices. The elements in this real vector space can have arbitrary purely imaginary eigenvalues. Thus, there exist strongly regular skew-Hermitian pseudo-circulant matrices.
Now, suppose that the skew-Hermitian pseudo-circulants have a common nontrivial invariant subspace V. Then V is invariant for all skew-Hermitian λ-circulants $A_1, \dots, A_d$, $\lambda \neq 0$ arbitrary. Nevertheless, V is also invariant for $iA_1, \dots, iA_d$. Therefore, V is an invariant subspace for all complex pseudo-circulant matrices. Since the Lie algebra generated by pseudo-circulant matrices equals $gl_n(\mathbb{C})$, we obtain a contradiction. Thus, we can apply Theorem 2.20 and conclude that the Lie algebra generated by the unitary Toeplitz matrices is equal to $gl_n(\mathbb{C})$.
It follows, that every unitary matrix is a finite product of matrix exponentials of skew-Hermitian pseudo-circulants, i.e. a finite product of unitary pseudo-circulants. The result about Hankel matrices follows, cf. Remark 5.18. □

Chapter 6

Conclusions

System Lie algebras of control systems are important for the study in many areas. For example, accessibility of a control system is closely related to it. In 1976 Brockett classified all conjugation types of system Lie algebras which can appear for simple feedback systems of the form

$$\dot{x} = (A + u(t)bc)\, x$$

under the assumption that the linear system is controllable and observable.
In this thesis we consider networks of linear dynamical systems, where output feedback is applied due to the interconnection structure. The resulting bilinear control system of the network can be described as

$$\dot{x} = (\mathcal{A} + \mathcal{B}U(t)\mathcal{C})\, x,$$

where $U(t)$ is a matrix-valued control function somehow dependent on the interconnection structure of the network. Based on the interconnection structure, we developed conditions which guarantee accessibility for the control system above. We discussed two different scenarios for the interconnections:

First, we assumed that all interconnections are independently controllable and hence, the underlying graph is simple. Under these assumptions the control function $U(t)$ has the form of a controlled adjacency matrix. Then a certain connectivity of the underlying graph is necessary and sufficient for the control system to be accessible in case the network has 3 or more nodes. Contrary to expectations we were able to prove the same result both for homogeneous and for heterogeneous networks. For the proofs we did not use Brockett's result on simple feedback systems and computed the Lie algebra directly. Further, applying Brockett's result on simple feedback systems with complex dynamics but only real controls we were able to generalize it to homogeneous networks with complex dynamics.

Second, we allowed for linear dependencies between interconnections, self-loops and multiple vertices. According to expectations, we had to distinguish between homogeneous and heterogeneous in this setting. We only found sufficient conditions for accessibility since the dynamics of the node systems have a big impact on the generated Lie algebra. To obtain accessibility conditions which are either necessary and sufficient one need additional assumptions on the node dynamics.
Concluding, we studied the class of circulant matrices, pseudo-circulant matrices and Toeplitz matrices in view of getting insight in networks with special structured interconnections. Since both pseudo-circulant and Toeplitz matrices generate the general Lie algebra $\mathfrak{gl}_N(\mathbb{R})$, we deduce accessibility results for homogeneous networks.

Open Problem

In this thesis we act on the assumption that the dynamics of each node are linear. Clearly, a more general assumption would be that each node dynamics are nonlinear of the form

$$\begin{aligned} \dot{x}_i &= f_i(x_i, v_i(t)) \\ y_i &= g_i(x_i), \end{aligned}$$

where u is a scalar-valued control function, f and g are functions in C^∞ and x_i are elements of a manifold M. Applying output feedback of the form

$$v_i(t) = \sum_{(j,i)\in E} u_{ij}(t) g_j(x_j).$$

where E is the set of all vertices of the network, yields the affine nonlinear control system

$$\dot{x}_i = f_i\left(x_i, \sum_{(j,i)\in E} u_{ij}(t) g_j(x_j)\right)$$

for every $1 \leq i \leq N$. Considering a network of N agents we deduce the nonlinear control system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} f_1\left(x_1, \sum_{(j,1)\in E} u_{1j}(t) g_j(x_j)\right) \\ f_2\left(x_2, \sum_{(j,2)\in E} u_{2j}(t) g_j(x_j)\right) \\ \vdots \\ f_N\left(x_N, \sum_{(j,N)\in E} u_{Nj}(t) g_j(x_j)\right) \end{pmatrix}, \tag{6.1}$$

where u_{ij} are scalar-valued control functions. Then

$$\hat{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix}^\top$$

is an element of $\hat{M} := M \times M \times \ldots \times M$ which is again a manifold.

For bilinear systems accessibility at one $p \in M$ is equivalent to accessibility at every $p \in M$ due to the right-invariance. Since system (6.1) is no longer right-invariant, its accessibility has to be checked in every point $p \in M$.
The set of all C^∞ vector fields on a manifold M is an infinite dimensional real vector space $\mathcal{X}(M)$ [44] and a Lie algebra with the Lie bracket defined by

$$[f, g](x) := \frac{dg}{dx}(x) \cdot f(x) - \frac{df}{dx}(x) \cdot g(x),$$

where f, g and $[f, g] \in \mathcal{X}(M)$ (cf. A.1). Let $\mathcal{C}$ denote the accessibility Lie algebra of the system (6.1), i.e. the smallest subalgebra of $\mathcal{X}(M)$ that contains $f(x)$ and $U \cdot g(x)$ for all piecewise constant controls U. Then the *accessibility distribution* C is defined as

$$C(p) = \operatorname{span}\{X(p) \mid X \in \mathcal{C}\}$$

for every $p \in M$. If $\dim C(p) = n$ for every $p \in M$, system (6.1) is accessible (Corollary 3.11 [44]). The equivalence does not hold.

The question again arises which network architecture guarantees accessibility of system (6.1). But now, one has to determine the dimension of the Lie algebras $C(p)$ for every $p \in \hat{M}$.

Appendix A

Lie theory

In this appendix we recall some basic facts and definitions in Lie theory. For more details we refer the reader to [23, 25, 32]. According to the content of this thesis, we focus on matrix Lie groups and matrix Lie algebras.

Definition A.1. A $\mathbb{K}$-*Lie algebra* is a $\mathbb{K}$-vector space L together with a bilinear map $[\cdot,\cdot] : L \times L \to L$, which satisfies

(i) $[X, Y] = -[Y, X]$ for all $X, Y \in L$

(ii) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ for all $X, Y, Z \in L$ (Jacobi identity).

The following examples present the matrix Lie algebras which appear in this thesis.

Example A.2. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ the following list gives some examples for Lie algebras, which all appear in this thesis:*

- $\mathfrak{gl}_n(\mathbb{K}) := \mathbb{K}^{n\times n}$
- $\mathfrak{sl}_n(\mathbb{K}) := \{X \in \mathfrak{gl}_n(\mathbb{K}) \mid \operatorname{tr}(X) = 0\}$
- $\mathfrak{sp}_n(\mathbb{K}) := \{X \in \mathfrak{gl}_n(\mathbb{K}) \mid X^\top J_{n,n} + J_{n,n} X = 0,\ \operatorname{tr} X = 0\}$, *where*

  $$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{A.1}$$

  with dimension $n(n+1)/2$. From the definition it is obvious that the Lie algebra only exists for even $n \in \mathbb{N}$.
- $\mathfrak{su}_n(\mathbb{K}) := \{X \in \mathfrak{gl}_n(\mathbb{K}) \mid X + X^\top = 0, \operatorname{tr} X = 0\}$

- $\mathfrak{su}(\mu, \nu) := \{X \in \mathfrak{sl}_{\mu+\nu}(\mathbb{C}) \mid X^* I_{\mu,\nu} + I_{\mu,\nu} X = 0\}$ *for* $\mu + \nu \geq 2$, *where*

$$I_{\mu,\nu} := \begin{pmatrix} I_\mu & 0 \\ 0 & -I_\nu \end{pmatrix}. \tag{A.2}$$

A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called *real*, if $\mathfrak{g}$ is over $\mathbb{C}$ it is called a *complex Lie algebra.* A *Lie subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, whereat

$$[\mathfrak{h}, \mathfrak{h}] := \{[H_1, H_2] | H_1, H_2 \in \mathfrak{h}\}.$$

A Lie subalgebra $\mathfrak{h}$ is called an *ideal*, if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. A Lie algebra $\mathfrak{g}$ is *Abelian* if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.

Definition A.3. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Then $\mathfrak{g}$ is called *semisimple* if the biggest solvable ideal of $\mathfrak{g}$ is $\{0\}$. The Lie algebra $\mathfrak{g}$ is called *simple* if it is not Abelian and has no other ideals than $\mathfrak{g}$ and $\{0\}$.

For any $X \in \mathfrak{g}$, the *adjoint transformation* is the linear map

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad Y \to [X, Y]$$

and

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad Y \mapsto \mathrm{ad}_Y$$

is called the *adjoint representation* of $\mathfrak{g}$. The set of linear mappings $\mathrm{End}(\mathfrak{g})$ consists of all endomorphisms $\mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$. For finite dimensional Lie algebras the symmetric bilinear form

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}, \quad \kappa(X, Y) \mapsto \mathrm{tr}(\mathrm{ad}_X\, \mathrm{ad}_Y)$$

is called the *Killing form* of $\mathfrak{g}$.

Definition A.4. If G is a group and at the same time an analytic manifold, then G is called a *Lie group.*

For an arbitrary Lie group the tangent space $T_1 G$ at the unit element $1 \in G$ has the structure of a Lie algebra. Therefore, the tangent space $T_1 G$ is referred to as the associated Lie algebra to the Lie group G denoted by $\mathcal{L}(G)$.

Example A.5. *Examples of Lie groups*

- *the general linear group*

$$GL_n(\mathbb{K}) = \{A \in \mathbb{K}^{n \times n} : \det A \neq 0\}$$

with associated Lie algebra $\mathfrak{gl}_n(\mathbb{R})$

- *the special linear group*

$$SL_n(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \det A = 1\}$$

with associated Lie algebra $\mathfrak{sl}_n(\mathbb{R})$

- *the symplectic group*

$$SP_n(\mathbb{K}) := \left\{ A \in \mathbb{K}^{n\times n} \;\middle|\; A^\top J A = J \right\},$$

where J *is defined in* (A.1), *with associated Lie algebra* $\mathfrak{sp}_n(\mathbb{K})$

- *the special unitary group*

$$SU(n) = \{A \in GL_n(\mathbb{C}) \mid A^* A = 1 \textit{ and } \det g = 1\}$$

with associated Lie algebra $\mathfrak{su}_n(\mathbb{C})$

- *the special unitary group of signature* μ, ν *is*

$$SU(\mu,\nu) = \left\{ A \in GL_{\mu+\nu}(\mathbb{C}) \;\middle|\; A^* I_{\mu,\nu} A = I_{\mu,\nu} \right\},$$

where $I_{\mu,\nu}$ *is defined in* (A.2), *with associated Lie algebra* $\mathfrak{su}(\mu,\nu)$

Appendix B

Properties of irreducible polynomials

In this section we give some results on irreducible polynomials which we need for Chapter 3. The proofs of all results stated here can be found in [7].

Let $p(s) \in F[s]$ be a monic polynomial of the form

$$p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_0, \tag{B.1}$$

where F denotes a field of characteristic 0 and denote all roots of $p(s)$ by s_i for $1 \leq i \leq n$ counted with multiplicity. Dependent on $p(s)$ we define the two following polynomials

$$[p(s)]_{(r)} := \prod_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} (s - (s_{i_1} + s_{i_2} + \ldots + s_{i_r}))$$

and

$$\mathrm{ad}[p(s)] := \prod_{1 \leq i,j \leq n} (s - (s_i - s_j)) \,.$$

The zeros of $[p(s)]_{(2)}$ and $\mathrm{ad}[p(s)]$ are closely related.

Lemma B.1. *Let $p(s)$ be a polynomial over F of degree n and assume that all roots of $p(s)$ are distinct. Then $p(s/2)[p(s)]_{(2)}$ has a repeated root if and only if $s^{-n}\,\mathrm{ad}[p(s)]$ has a repeated root.*

Proof. Let us assume that $s^{-n}\,\mathrm{ad}[p(s)]$ has a repeated root. Then, in terms of the roots $\{s_i\}$ of $p(s)$ we get

$$s_i - s_j = s_k - s_l$$

for some $i \neq j, k \neq l$ and $\{i,j\} \neq \{k,l\}$. This implies $\{i,l\} \neq \{j,k\}$ and hence

$$s_i + s_l = s_j + s_k.$$

Thus, $p(s/2)[p(s)]_{(2)}$ has a repeated root as well.
Now, suppose $p(s/2)[p(s)]_{(2)}$ has a repeated root, i.e., for $\{i,j\} \neq \{k,l\}$ we have

$$s_i + s_j = s_k + s_l.$$

Due to the assumption that all zeros of $p(s)$ are distinct, we obtain $i \neq k, i \neq l$ and $j \neq k, j \neq l$. Thus,

$$s_i - s_l = s_k - s_j$$

with $i \neq l$ and $k \neq j$ and $s^{-n}\,\mathrm{ad}[p(s)]$ has a repeated root. □

We show that the splitting fields $E_r|F$ of $[p(s)]_{(r)}$ coincide for all $r = 1, \ldots n-1$ and that the multiplicity of repeated roots of $[p(s)]_{(r)}$ is bounded by n/r for $s \in \mathbb{R}$ if the polynomial $[p(s)]_{(r-1)}$ has distinct zeros. Here, a splitting field $E|F$ of a polynomial p with coefficients in the field F is the smallest field extension of F over which the polynomial decomposes into linear factors.

Lemma B.2. *Let F be a field of characteristic zero and $p(s) \in F[s]$ be monic and of degree n. Given r such that $1 \leq r \leq n-1$, then the splitting field $E|F$ of $p(s)$ and the splitting field $E_r|F$ of $[p(s)]_{(r)}$ are the same. Moreover, if $[p(s)]_{(r)} = [q(s)]_{(r)}$ holds for one r with $1 \leq r \leq n-1$ and q monic, this implies $p(s) = q(s)$.*

Proof. Firstly, the zeros of $[p(s)]_{(r)}$ are of the form $s_{i_1} + s_{i_2} + \ldots + s_{i_r}$ with s_{i_j} zero of $p(s)$. Hence, the splitting field $E_r|F$ of $[p(s)]_{(r)}$ is contained in the splitting field $E|F$ of $p(s)$.
Secondly, since $s_1 + s_2 + \ldots + s_{r-1} + s_r$, $s_1 + s_2 + \ldots + s_{r-1} + s_{r+1}$ etc. are elements of the splitting field $E_r|F$ of $[p(s)]_{(r)}$, adding up $n-r-1$ of these and taking advantage of $s_1 + s_2 + \ldots + s_n = p_{n-1} \in F$, where p_{n-1} is defined in (B.1) and therefore $p_{n-1} \in F$, we obtain that $(n-r)(s_1 + s_2 + \ldots + s_{r-1}) \in E_r|F$. Because $n - r \neq 0$, we can show by induction that all sums of zeros of $p(s)$ with arbitrary length are elements of $E_r|F$. Thus, $E|F = E_r|F$ for $1 \leq r \leq n-1$ and the zeros of $p(s)$ can be expressed in terms of the roots of $[p(s)]_{(r)}$. Thus, $[p(s)]_{(r)}$ determines $p(s)$ uniquely as a monic polynomial. □

In the following lemmas we make use of a substantial result in Galois theory: The Galois group of a polynomial acts transitively on the zeros of every irreducible factor of it ([17]). The Galois group G of a field extension $E|F$ is defined as the group of automorphisms of E, which leaves the field F fixed. This Galois group acts transitively on the zeros of a polynomial $p(s)$ if for every pair of zeros s_i, s_j there exists an automorphism $g \in G$ such that $g(s_i) = s_j$. A polynomial $p(s) \in F[s]$ is said to be *irreducible* if it cannot be factored in a product of two or more non-trivial polynomials which are in $F[s]$. For an overview on Galois theory we refer the interested reader to [17].

Lemma B.3. *Let F be a field of characteristic zero and $p(s) \in F[s]$ be irreducible, monic and of degree n. Let $1 < r \leq n/2$ and suppose that $[p(s)]_{(r-1)}$ has no repeated roots. Then no root of $[p(s)]_{(r)}$ is of multiplicity greater than n/r. Moreover, $[p(s)]_{(r)}$ has a root in F if and only if the two conditions are satisfied:*

(i) r divides n;

(ii) $[p(s)]_{(r)}$ has a root of multiplicity n/r.

Proof. Assume that $[p(s)]_{(r)}$ has a root with multiplicity greater than n/r. Then, we have $s_{i_1} + \ldots + s_{i_r} = s_{j_1} + \ldots + s_{j_r}$, where s_k denote the zeros of $p(s)$. Necessarily, there is one root appearing on both sides of the equation, which contradicts our assumption that $[p(s)]_{(r-1)}$ does not have repeated roots.
Now, suppose $[p(s)]_{(r)}$ has a root in F. Then, by renumbering the roots, we can write it as

$$f := s_1 + s_2 + \ldots + s_r \in F.$$

As $p(s)$ is irreducible, the Galois group acts transitively on the zeros and since $f \in F$ it belongs to the fixed field of the Galois group. Therefore, for a suitable numbering of the roots of $p(s)$ we get

$$s_1 + s_2 + \ldots + s_r = s_{r+1} + s_{r+2} + \ldots + s_{2r} = s_{2r+1} + s_{2r+2} + \ldots + s_{3r} = \ldots.$$

Since repetition on a subscript would give us a repeated root of $[p(s)]_{(r-1)}$, this contradicts our assumption. Hence, r divides n and the multiplicity of the root f is n/r.
Let us now assume that r divides n and $[p(s)]_{(r)}$ has a root of multiplicity n/r. It follows with a suitable numbering of the roots of $p(s)$ that

$$s_1 + s_2 + \ldots + s_r = s_{r+1} + s_{r+2} + \ldots + s_{2r} = s_{2r+1} + s_{2r+2} + \ldots + s_{3r} = \ldots.$$

Taking advantage of $[p(s)]_{(r-1)}$ having no repeated roots, we get $(n-r)(s_1+s_2+\ldots+s_r) = p_{n-1} \in F$. Hence, $s_1 + s_2 + \ldots + s_r \in F$ and the result follows. □

Lemma B.4 (Gauss's lemma)**.** *Let F be a field of characteristic zero. If $p(s) + uq(s) = \psi_1(s,u)\psi_2(s,u)$ has a solution with ψ_1 and ψ_2 polynomials in s with coefficients in $F(u)$, then it has a solution with ψ_1, ψ_2 polynomials in s and u.*

Proof. This is a consequence from Theorem 8.18 in [33]. □

Lemma B.5. *A polynomial $q(s) + up(s)$ is irreducible as a polynomial over $F(u)$ if and only if q and p are coprime as elements of $F[s]$.*

Proof. Let us assume that $q(s) + up(s)$ is reducible with factorization $q(s) + up(s) = \psi_1(s,u) \cdot \psi_2(s,u)$, where ψ_1 and ψ_2 are polynomials in s over $F(u)$. Then, we obtain with Lemma B.4 that ψ_1 and ψ_2 are polynomials in s and u. But $q(s) + up(s)$ is linear in u, thus, at most one ψ_i can depend on u. The other ψ_i is a polynomial in s, independent of u and hence a common factor of $q(s)$ and $p(s)$. The other direction is obvious. □

One of the main tools for later purpose is the following lemma.

Lemma B.6. *Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and $q(s) + up(s) \in F(u)[s]$ be of the form*

$$q(s) + up(s) = s^n + q_{n-1}s^{n-1} + \ldots + q_0 + u\left(p_{n-2}s^{n-2} + \ldots + p_0\right).$$

Assume that $q(s)$ and $p(s)$ are coprime and that $p_{n-2} \neq 0$. If $[q(s) + up(s)]_{(2)} \in F(u)[s]$ has a repeated root, then this root is $\sigma := -2q_{n-1}/n$, it is of multiplicity $n/2$ and $\phi(s) := q(s + \sigma/2) + up(s + \sigma/2) \in F(u)[s]$ is an even polynomial.

Proof. Let us assume that the polynomial

$$[q(s) + up(s)]_{(2)} = \prod_{1 \leq i_1 < i_2 \leq n} (s - (s_{i_1} + s_{i_2}))$$

has a repeated root, i.e., $s_i + s_{i'} = s_j + s_{j'}$ for $\{i, i'\} \neq \{j, j'\}$ with $s_i, s_{i'}, s_j, s_{j'}$ being roots of $q(s) + up(s)$. From the coprimeness of $q(s)$ and $p(s)$, we obtain with Lemma B.5 that $q(s) + up(s)$ is irreducible. Hence, the Galois group $\mathrm{Gal}(E|F(u))$ of its splitting field $E|F(u)$ acts transitively on the roots of $q(s) + up(s)$. For two arbitrary zeros s_k, s_l of $q(s) + up(s)$ there always exists an automorphism $g \in \mathrm{Gal}(E|F(u))$ with $g(s_k) = s_l$. In case of a repeated root $s_i + s_{i'} = s_j + s_{j'}$ of $[q(s) + up(s)]_{(2)}$ this gives us

$$g(s_i) + g(s_{i'}) = g(s_i + s_{i'}) = g(s_j + s_{j'}) = g(s_j) + g(s_{j'})$$

for all $g \in \mathrm{Gal}(E/F(u))$. As the Galois group $\mathrm{Gal}(E/F(u))$ acts transitively on the zeros, we get that for every root s_k of $q(s) + up(s)$ a relation exists of the form $s_k + s_{k'} = s_l + s_{l'}$ with $k \neq l$. As $p_{n-2} \neq 0$ two of the roots of $q(s) + up(s)$ can be expressed for u near infinity as

$$s_1 = \sqrt{p_{n-2}u} - \frac{q_{n-1}}{2} + \ldots \tag{B.2}$$

$$s_2 = -\sqrt{p_{n-2}u} - \frac{q_{n-1}}{2} + \ldots, \tag{B.3}$$

where the remaining terms are powers of $1/\sqrt{u}$ and the other $n-2$ roots can be expressed as power series in $1/u$ (Proposition 4.1.3 [26]). As a result, from

$$s_1 + s_i = s_j + s_k,$$

it follows $i = 2$ and from $s_2 + s_i = s_j + s_k$ that $i = 1$ since all roots s_l are distinct. Therefore, if $s_i + s_{i'} = s_j + s_{j'}$, then $s_i + s_{i'}$ is bounded for $u \to \infty$.
When the polynomial $[q(s) + up(s)]_{(2)}$ has repeated roots, it is therefore reducible. Let $\phi(s)$ be an irreducible factor of $[q(s)+up(s)]_{(2)}$, which contains the linear factor $s-s_1-s_2$. Since s_1+s_2 is a repeated root, this irreducible factor is not unique. Cleary, $\phi(s) \in F(u)[s]$ but it divides $[q(s) + up(s)]_{(2)}$. By Lemma B.4 it immediately follows that $\phi(s) \in F[u, s]$. The coefficients of $\phi(s)$ can be written as symmetric functions of its roots. Since all roots of $\phi(s)$ are bounded for $u \in \mathbb{C}\backslash\{0\}$ and the coefficients of $\phi(s)$ are polynomials in u, the coefficients have to be constants. Therefore, $\phi(s) \in F[s]$ and the root $s_1 + s_2$ belongs to the algebraic closure of F, i.e., $s_1 + s_2 \in \mathbb{C}$. With Lemma B.3 we get that 2 divides n and $[q(s) + up(s)]_{(2)}$ has no root with multiplicity greater than $n/2$. Hence, n is even. So we found a root in $\mathbb{C}$, which is of multiplicity $n/2$ and therefore, we get $s_1 + s_2 = s_3 + s_4 = \ldots = s_{n-1} + s_n$. With $s_1 + s_2 + \ldots + s_n = p_{n-1}$ it follows that $s_1 + s_2 = 2p_{n-1}/n$.
When we shift s to $\bar{s} = s + \sigma/2$ the roots of $q(\bar{s}) + up(\bar{s})$ appear with their negatives. As n is even it follows that $q(\bar{s}) + up(\bar{s})$ is even in s. This proves the result. □

The other way around, we get from Lemma B.6 that if $q(s + \sigma) + up(s + \sigma)$ is not even for any $\sigma \in \mathbb{R}$ with p and q coprime, it follows that the polynomial $[q(s) + up(s)]_{(2)}$ does not have any repeated roots. Hence, by Lemma B.1 the polynomial $\mathrm{ad}[q(s) + up(s)]$ has maximum number of distinct roots, which is $n^2 - n + 1$. Note that $s^{-n}\,\mathrm{ad}[q(s) + up(s)]$ does not have the root $s = 0$ since all roots of $q(s) + up(s)$ are assumed to be distinct. Hence, the maximum number of distinct roots of $\mathrm{ad}[q(s)+up(s)]$ is the maximum number of distinct roots of $s^{-n}\,\mathrm{ad}[q(s) + up(s)]$ plus one. Thus, it is left to determine, how many repeated roots the polynomial $\mathrm{ad}[q(s) + up(s)]$ has in case there exists a number $\sigma \in \mathbb{R}$ such that $q(s + \sigma) + up(s + \sigma)$ is an even polynomial.

Lemma B.7. *Let $q(s) + up(s) \in F(u)[s]$ be as in Lemma B.6. Then the repeated roots of*

$$((q + up)(s/2))\,[q(s) + up(s)]_{(2)} \in F(u)[s]$$

are the repeated roots of $[q(s) + up(s)]_{(2)} \in F(u)[s]$ with the same multiplicity.

Proof. Since the zeros of $q(s)+up(s)$ are all distinct, the factor $(q+up)(s/2)$ cannot have a repeated root. Suppose that there exists a repeated root of $(q+up)(s/2)[q(s)+up(s)]_{(2)}$, which has the form $2s_i = s_l + s_k$. Then, we get $i \neq k, i \neq l, l \neq k$. Again, we use that the Galois group G of p acts transitively and we can find an automorphism $g \in G$ such that $g(s_i) = s_1$, where s_1 is the root defined in (B.2). Then we obtain

$$2s_1 = g(2s_i) = g(s_l) + g(s_k). \tag{B.4}$$

Since g is an automorphism $g(s_l) \neq s_1$ and $g(s_k) \neq s_1$ holds. Therefore, the equation (B.4) yields a contradiction. Hence, the repeated roots of the polynomial $(q+up)(s/2)[q(s)+up(s)]_{(2)}$ are only the repeated roots of $[q(s)+up(s)]_{(2)}$. □

Lemma B.8. *Let $q(s)+up(s) \in F(u)[s]$ be as in Lemma B.6 and assume that it is an even polynomial in s. Then $\mathrm{ad}[q(s)+up(s)]$ has $1+n^2/2$ distinct zeros.*

Proof. Since $q(s)+up(s)$ is an even polynomial in s, i.e., $q(s)+up(s) = q(-s)+up(-s)$, the zeros of $q(s)+up(s)$ appear with their negatives. Therefore, $s_i - s_j = (-s_j) - (-s_i)$ is a repeated root of $\mathrm{ad}[q(s)+up(s)]$. Hence, $n^2 - n + 1 - (n^2/2 - n) = 1 + n^2/2$ is an upper bound for the number of distinct zeros of $\mathrm{ad}[q(s)+up(s)]$. We assume that not all zeros of $\mathrm{ad}[q(s)+up(s)]$ are distinct. Then, in case we have a repeated root,

$$s_i - s_j = s_k - s_l \quad \text{or} \quad s_i + s_l = s_j + s_k.$$

From Lemma B.6 we know that this only holds if $s_l = -s_i$ and $s_k = -s_j$ since $q(s)+up(s)$ is assumed to be even. But the polynomial $((q+up)(s/2))\,[q(s)+up(s)]_{(2)}$ has maximum $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ distinct zeros. Then, from Lemma B.6 and Lemma B.7 we get that the only repeated root is 0 with multiplicity $n/2$. Hence, the number of distinct nonzero roots of $((q+up)(s/2))\,[q(s)+up(s)]_{(2)}$ is $\frac{n(n+1)}{2} - \frac{n}{2} = \frac{n^2}{2}$. It follows, $\mathrm{ad}[q(s)+up(s)]$ has $n^2/2$ distinct nonzero roots plus the root 0. □

We know from Lemma B.1 that the polynomial $s^{-n}\,\mathrm{ad}[q(s)+up(s)]$ has a repeated root if and only if $(q+up)(s/2)[q(s)+up(s)]_{(2)}$ has a repeated root. Consequently, if the polynomial $(q+up)(s/2)[q(s)+up(s)]_{(2)}$ has no repeated roots, the polynomial $\mathrm{ad}[q(s)+up(s)]$ has $n^2 - n + 1$ distinct roots. Therefore, it results from Lemma B.6 and Lemma B.8 that $s^{-n}\,\mathrm{ad}[q(s)+up(s)]$ has either no repeated root or $[q(s)+up(s)]_{(2)}$ is an even polynomial and then $\mathrm{ad}[q(s)+up(s)]$ has $1+n^2/2$ distinct roots.

Appendix C

Proofs of Lemma 3.16 and Theorem 3.40

We give the proof of Lemma 3.16.

Proof. If there exists a linear relation between matrices in $\{\operatorname{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$, we have $\sum_{i=1}^k \alpha_i \operatorname{ad}^i_{A+ubc}(bc) = 0$ for some $k \in \mathbb{N}$. With

$$\frac{d^i}{dt^i}\left(e^{(A+ubc)t}bce^{-(A+ubc)t}\right)|_{t=0} = \operatorname{ad}^i_{A+ubc}(bc)$$

for all $i \in \mathbb{N}$ we get

$$\sum_{i=1}^k \alpha_i \frac{d^i}{dt^i}\left(e^{(A+ubc)t}bce^{-(A+ubc)t}\right)|_{t=0} = 0$$

and hence for all $h, g \in \mathbb{R}^n$

$$\sum_{i=1}^k \alpha_i \frac{d^i}{dt^i}\left(he^{(A+ubc)t}bce^{-(A+ubc)t}g\right)|_{t=0} = 0. \tag{C.1}$$

Since (A, b, c) is controllable and observable we know that for any sets of real numbers $\{\beta_i\}$ and $\{\gamma_i\}$ we can find vectors h and g such that

$$he^{(A+ubc)t}bce^{-(A+ubc)t}g = \sum_{i,j} \beta_i \gamma_j e^{(\lambda_i - \lambda_j)t}, \tag{C.2}$$

where λ_i, λ_j denote the eigenvalues of $A + ubc$ and hence $\lambda_i - \lambda_j$ are the eigenvalues of $\operatorname{ad}_{A+ubc}$. This can be seen as follows: Due to Lemma 3.8 we can w.l.o.g. assume that the matrix $A + ubc$ is diagonal. Hence,

$$he^{(A+ubc)t}bce^{-(A+ubc)t}g = \sum_{i,j} h_i b_i c_j g_j e^{(\lambda_i - \lambda_j)t}.$$

Due to the controllability and observability of (A, b, c) we know that b and c have a nonzero component in every invariant subspace of A and therefore, $b_i \neq 0$ and $c_j \neq 0$ for $1 \leq i, j \leq n$. Clearly, we can choose h_i and g_j such that $h_i b_i = \beta_i$ and $c_j g_j = \gamma_j$.
Therefore, we can insert (C.2) into (C.1). Clearly, the minimal polynomial of $\mathrm{ad}_{A+ubc}(\cdot)$ divides $p(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \ldots + \alpha_1 s$. Thus, the minimal polynomial of $\mathrm{ad}_{A+ubc}(\cdot)$ regarded as an operator on the real span of $\{\mathrm{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$ has each factor of $\mathrm{ad}[p(s) + uq(s)]$ as a factor. Therefore, the number of distinct nonzero zeros of $\mathrm{ad}[p(s) + uq(s)]$ is a lower bound for the number of linearly independent matrices in the set $\mathrm{span}\{\mathrm{ad}_{A+ubc}(bc) \mid i \in \mathbb{N}\}$. Taking the maximum we obtain that either $n^2/2$ or $n^2 - n$ are lower bounds for the dimension of the vector space $\mathrm{span}\{\mathrm{ad}_{A+ubc}(bc) \mid i \in \mathbb{N}\}$ from the observations of this chapter. Clearly, the eigenvalues of $\mathrm{ad}_{A+ubc}(\cdot)$ are just the zeros of the polynomial $\mathrm{ad}[p(s) + uq(s)]$.

On the other side, due to Lemma 3.8, the matrix $A + ubc$ is diagonalizable for almost all $u \in F$ and thus has a complete set of eigenvectors. So, $\mathrm{ad}_{A+ubc}(\cdot)$ has a complete set of eigenvectors and is diagonalizable for almost all values of $u \in F$, too. Clearly, there cannot exist a cyclic subspace generated by a finite-dimensional operator acting on bc with a dimension which exceeds the number of distinct eigenvalues of $\mathrm{ad}_{A+ubc}(\cdot)$ and therefore, $n^2 - n + 1$ and $n^2/2 + 1$, respectively, are upper bounds for the number of linearly independent matrices in $\{\mathrm{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}_0\}$.

Let $\mathrm{diag}\, bc \neq 0$, i.e., not all diagonal entries of bc vanish. Since $A + ubc$ can be assumed to be diagonal, the matrices $\mathrm{ad}^i_{A+ubc}(bc)$ have vanishing diagonal. Therefore, $bc \notin \mathrm{span}\{\mathrm{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$. Hence, 0 is an eigenvalue of

$$\mathrm{ad}_{A+ubc}(\cdot)\big|_{\mathrm{span}\{\mathrm{ad}^i_{A+bc}(bc) \mid i \in \mathbb{N}_0\}}$$

and the result holds.
Otherwise, let $\mathrm{diag}\, bc = 0$, i.e., the diagonal entries of bc vanish. With Lemma 3.8 we obtain $\mathrm{ad}^i_{A+ubc}(bc) \neq 0$ for all $i \in \mathbb{N}$ for almost all u. Hence, 0 is not an eigenvalue of $\mathrm{ad}_{A+ubc}(\cdot)$ restricted to the cyclic subspace $\mathrm{span}\{\mathrm{ad}^i_{A+ubc}(bc) \mid i \in \mathbb{N}\}$. Thus, bc is an element of $\mathrm{span}\{\mathrm{ad}_{A+ubc}(bc) \mid i \in \mathbb{N}\}$ and the result follows. □

Before we give the proof of Theorem 3.40, we state a useful lemma.

Lemma C.1. *If $\mathcal{L}$ is a Lie algebra of $n \times n$ matrices, which properly contains $\mathfrak{sp}_{n/2}(\mathbb{R})$, then $\mathcal{L}$ is either equal to $\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$, $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$.*

Proof. If $\mathcal{L}$ differs only by multiples of the identity from $\mathfrak{sp}_{n/2}(\mathbb{R})$, then $\mathcal{L} = \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$. Alternatively, there exists a matrix M in $\mathcal{L}$ of the form

$$M = \begin{pmatrix} F & R \\ S & F^\top \end{pmatrix},$$

where R and S are skew-symmetric, i.e., $R = -R^\top$ and $S = -S^\top$, and F is not a nonzero multiple of the identity, since every matrix can be written as the sum of a symmetric and a skew-symmetric matrix. Clearly, $\mathfrak{sp}_{n/2}(\mathbb{R})$ contains the matrices

$$N = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ P = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \text{ and } P^\top = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

thus the following elements lie in $\mathcal{L}$:

- $[N, M] = \begin{pmatrix} 0 & 2R \\ -2S & 0 \end{pmatrix}$
- $[P, \frac{1}{2}[N, M]] = \begin{pmatrix} -R & 0 \\ 0 & R \end{pmatrix}$ and therefore $\tilde{R} := \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}$ since $R = -R^\top$
- $[P^\top, -\frac{1}{2}[N, M]] = \begin{pmatrix} S & 0 \\ 0 & S^\top \end{pmatrix}$ and therefore $\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}$ since $S = -S^\top$
- $M + \frac{1}{2}[N, M] - 2[\tilde{R}, P^\top] = \begin{pmatrix} F & 0 \\ 0 & F^\top \end{pmatrix}$ and therefore $\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}$

At least one of these 3 preceding matrices is unequal zero and not a multiple of the identity. By calculating

$$\left[\begin{pmatrix} A & 0 \\ 0 & -A^\top \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [A, X] & 0 \\ 0 & 0 \end{pmatrix}$$

for $X = R$, $X = S$ or $X = F$, one can easily see that

$$\begin{pmatrix} \mathfrak{k} & 0 \\ 0 & 0 \end{pmatrix} \subset \mathcal{L},$$

where $\mathfrak{k} := \{R, S, F\}_{Ideal}$ is an ideal of $\mathfrak{gl}_{n/2}(\mathbb{R})$ generated by R, S or F. Hence, $\mathfrak{sl}_{n/2}(\mathbb{R}) \subset \mathfrak{k}$ since $\mathfrak{sl}_{n/2}(\mathbb{R})$ is simple. With the same argumentation we get

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{k} \end{pmatrix} \subset \mathcal{L},$$

with $\mathfrak{sl}_{n/2}(\mathbb{R}) \subset \mathfrak{k}$. With

$$\left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & S_i \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & AS_i \\ 0 & 0 \end{pmatrix} \text{ and } \left[\begin{pmatrix} 0 & S_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right] = \begin{pmatrix} 0 & S_iA \\ 0 & 0 \end{pmatrix}$$

we get

$$\begin{pmatrix} 0 & \mathfrak{k} \\ 0 & 0 \end{pmatrix} \subset \mathcal{L},$$

where $\mathfrak{k} := \{S_i\}_{Ideal}$ is an ideal of $\mathfrak{gl}_{n/2}(\mathbb{R})$ generated by some symmetric matrices S_i and hence $\mathfrak{sl}_{n/2}(\mathbb{R}) \subset \mathfrak{k}$. Similarly,

$$\begin{pmatrix} 0 & 0 \\ \mathfrak{k} & 0 \end{pmatrix} \subset \mathcal{L}.$$

Completing this to a Lie algebra yields

$$\mathfrak{sl}_n(\mathbb{R}) \subset \mathfrak{g}.$$

This proves the result. □

For the sake of completeness we give the proof of Theorem 3.40 which can be found in [5].

Proof. By Lemma 3.38 we know there exist K, b and c such that $(A + BKC, Bb, cC)$ is a controllable and observable triple. Thus, by Theorem 3.28 we know that the Lie algebra (3.10) contains $\mathfrak{sp}_{n/2}(\mathbb{R})$. From Theorem 3.32 we know that it is $\mathfrak{sp}_{n/2}(\mathbb{R})$ or $\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$ if and only if there exists an α and some nonsingular matrix $Q = Q^\top$ such that

$$QUG(s + \alpha) = G^\top(-s)U^\top Q$$

holds for all matrices U of appropriate dimension. This is impossible if $G(s)$ is of rank 2 or more. Thus, the Lie algebra (3.10) is either $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$. Clearly, we have

$$\left\{A, BUC \mid U \in \mathbb{R}^{p\times p}\right\}_{LA} = \mathfrak{sl}_n(\mathbb{R})$$

if and only if all matrices in $\left\{A, BUC \;\middle|\; U \in \mathbb{R}^{p\times p}\right\}_{LA}$ have trace zero. This is only satisfied when $\operatorname{tr} A = 0$ and $CB = 0$. □

List of Notations

$\mathrm{ad}_A(bc)$	adjoint transformation $[A, bc]$
γ	adjacency matrix of the graph Γ
$\gamma(u)$	controlled adjacency matrix of the graph Γ
$\mathrm{Circ}(c_0, ..., c_{n-1})$	Circulant matrix with first row $(c_0, ..., c_{n-1})$
$\mathrm{Circ}_\lambda(c_0, c_1, ..., c_{n-1})$	λ-circulant matrix with first row $(c_0, c_1, ..., c_{n-1})$
$\mathrm{Circ}_n(\mathbb{C})$	Lie group of all invertible circulant matrices
$\mathrm{Circ}_n^\lambda(\mathbb{C})$	Group of all nonsingular λ-circulant matrices
$\mathrm{circ}_n(\mathbb{C})$	Lie algebra of all circulant matrices
$\mathrm{circ}_n^\lambda(\mathbb{C})$	Set of all λ-circulant matrices
$\mathrm{diag}\, X = 0$	The diagonal entries of X are all zero
$\mathfrak{g} \cong \mathfrak{k}$	The Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ are conjugated
$\mathrm{LE}_\lambda(A)$	The generalized left eigenspace associated to the eigenvalue λ of the matrix A
$\mathfrak{gl}_n(\mathbb{K})$	General linear Lie algebra over $\mathbb{K}$
$\mathfrak{sl}_n(\mathbb{K})$	Special linear Lie algebra over $\mathbb{K}$
$\mathfrak{sp}_n(\mathbb{K})$	Symplectic Lie algebra over $\mathbb{K}$
$\mathfrak{su}(\mu, \nu)$	Special unitary Lie algebra of signature (p, q)
$\mathfrak{su}_n(\mathbb{C})$	Special unitary Lie algebra of $n \times n$ matrices
$\mathfrak{u}(\mu, \nu)$	Unitary Lie algebra of signature (p, q)
$\mathfrak{u}_n(\mathbb{C})$	Unitary Lie algebra of $n \times n$ matrices
Ω	$\mathrm{diag}(1, \omega, \dots, \omega^{n-1})$

Φ	Fourier matrix
$\mathrm{RE}_\lambda(A)$	The generalized right eigenspace associated to the eigenvalue λ of the matrix A
$\sigma(A)$	The spectrum of the matrix A
$\mathrm{Toe}(n)$	Vector space of all $n \times n$ Toeplitz matrices
$\{A, bc\}_{LA}^{\mathbb{C}}$	The complex matrix Lie algebra generated by A and bc
$\{X, Y\}_{LA}$	Lie algebra generated by X and Y
A^*	the conjugate transpose $\overline{A}^\top$
E_{ij}	Single-entry matrix with entry 1 at (i, j) and the other entries are zero
$G(s)$	Transfer function of a MIMO system
$g(s)$	Transfer function of a SISO system
$g^*(s)$	$g(s)$ with coefficients complex conjugated
$GL_n(\mathbb{K})$	Lie group of all invertible $n \times n$ matrices
$GL_n^+(\mathbb{R})$	Lie group of all invertible $n \times n$ matrices with positive determinant
$p_C(z)$	Representer of circulant matrix C
S	$\mathrm{Circ}(0, 1, ..., 0)$
S_λ	$\mathrm{Circ}_\lambda(0, 1, 0, ..., 0)$

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