



# Advances in the stability analysis of large-scale discrete-time systems

Dissertationsschrift zur Erlangung des  
naturwissenschaftlichen Doktorgrades  
der Julius-Maximilians-Universität Würzburg

vorgelegt von  
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aus Ulm

Würzburg 2015

Eingereicht am: 10.12.2014

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Tag der mündlichen Prüfung: 20.04.2015

*To Vanessa, Joela, Robin and Matheo.*



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# Introduction

Some of the early works on the stability analysis of *nonlinear discrete-time systems* described by difference equations are [97, 118]. In these works, conditions are derived under which the asymptotically stable linear part of the dynamics is dominating the nonlinear part to conclude asymptotic stability of the equilibrium point. An alternative method is the usage of Lyapunov functions, [105], whose existence implies asymptotic stability of the equilibrium point. The advantage of using Lyapunov functions is that the conditions are easy to check, in general. In particular, solutions of the system do not have to be known. For discrete-time systems, Lyapunov methods are developed in [54, 73]. A particular observation of [54] is that for asymptotically stable linear discrete-time systems, quadratic Lyapunov functions can always be obtained by solving a matrix equation.

For nonlinear discrete-time systems with asymptotically stable equilibrium point, the necessity of the existence of Lyapunov functions is shown by *converse Lyapunov theorems* such as [71, 81, 116]. However, converse Lyapunov theorems for general nonlinear systems do not lead to constructive procedures. Hence, Lyapunov functions are, in general, hard to find. There are only a few constructive approaches to obtain Lyapunov functions for nonlinear systems such as e.g. construction methods using linear programming [51, 96] or Zubov methods [17]. Consequently, there is a need for developing new tools for constructing Lyapunov functions.

For systems of a significant size in terms of dimension or complexity, direct methods such as the Lyapunov method are often challenging or intractable. Although there is no precise definition, such systems are frequently called *large-scale*, and they receive a lot of attention since the mid 70s, see e.g. the books [109, 127, 140]. One prominent

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method for studying large-scale systems is the *small-gain approach*: The large-scale system is considered as an interconnection of smaller subsystems, and the influence of the subsystems on each other is treated as a disturbance. The *classical small-gain idea* is to assume that the subsystems are in some sense *robust*, and the disturbing influence is in some sense small. Then, asymptotic stability of the equilibrium point of the overall system can be guaranteed.

In [128] the concept of *input-to-state stability* (ISS) was introduced. This concept opened the door to nonlinear extensions [24,69] of the small-gain results in [127,140], where the *gains* describing the disturbing influence of the subsystems were modeled via linear functions. Moreover, as the concept of ISS can also be characterized by the existence of ISS Lyapunov functions, several Lyapunov-based ISS small-gain theorems have been derived, see e.g. [21,25,68,75]. For discrete-time systems, ISS and its Lyapunov characterizations are studied in [46,70,76,93,94], and ISS small-gain theorems in various forms are derived e.g. in [24,48,65,70,89,99].

Most of the above references concerning ISS and ISS small-gain theorems were published during the last 15 years, and the topic still receives a lot of attention. Let us briefly mention some open problems in this area, which are studied in this thesis.

As mentioned before, small-gain theorems, in general, provide *sufficient* conditions to conclude asymptotic stability (or ISS). A natural question is to study *necessity* of the conditions involved. Only few authors studied this question so far. The authors in [19] derive a small-gain theorem for continuous-time systems using the behavioral setting, [120], and show that the conditions of the small-gain result are also necessary for  $\mathcal{L}_p$ -stability. In [64], necessary small-gain conditions are investigated for interconnected continuous-time *integral-input-to-state stable* (iISS) subsystems. Note that for continuous-time systems the concept of iISS is strictly stronger than the 0-GAS property<sup>1</sup>, whereas for discrete-time systems, the notions of iISS and 0-GAS are equivalent, see [3]. The authors in [40] provide necessary and sufficient conditions for interconnected discrete-time systems to conclude *global exponential stability* (GES) of the origin. This work has also been the starting point for the small-gain results that are derived in Chapter 2.

Whereas cascades of ISS systems are again ISS, stability of cascades of iISS is not guaranteed, in general, see [64]. In particular, the authors in [64] show that at least one subsystem has to be ISS to conclude stability of the interconnection. A more general question is deriving conditions on interconnections within the context of small-gain theory, to treat subsystems that do not share the same stability property.

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<sup>1</sup>0-GAS means that the equilibrium point of a system is globally asymptotically stable if the input is set to zero.



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A common assumption in the stability analysis of discrete-time systems is that the dynamics of the system are assumed to be continuous, [65,71,99]. However, there are several recent stabilization schemes that commonly lead to discontinuous feedback control laws (such as e.g. *model predictive control* (MPC) [16,47,90], *event-triggered control* [119,137] or *quantized control* [113]). Other examples, where discontinuities occur, can be found in *networked control systems* (NCS) [57], where channel imperfections are modeled as error dynamics [12,39], which are usually discontinuous at transmission times. In this respect, the interest in tools for studying stability of systems with discontinuous dynamics is high.

We see that there exists a variety of open problem within the context of stability analysis of nonlinear large-scale discrete-time systems. The contribution of this thesis is to answer some of these questions. In the remainder of this introduction, the problems studied in this thesis are explained in more detail.

## Lyapunov methods

Roughly speaking, *stability* of an equilibrium point of a discrete-time system is the property that any trajectory that starts close to the equilibrium point will stay close for all times. Furthermore, *asymptotic stability* of an equilibrium point means that, in addition, trajectories starting close to the equilibrium point eventually converge to the stable equilibrium point.

A powerful tool for establishing stability properties, such as global asymptotic stability (GAS) of the equilibrium point of a system, is the concept of a *Lyapunov function* [105]. A Lyapunov function is a scalar function whose existence shows (global) asymptotic stability of the equilibrium point. The main advantage of this concept is to conclude asymptotic stability of the equilibrium point without knowing any trajectory of the system. Moreover, the existence of a Lyapunov function is not only sufficient, but also necessary for (global) asymptotic stability of the equilibrium point. Theorems stating this converse direction, i.e., the necessity, are thus called *converse Lyapunov theorems*. In general, converse Lyapunov theorems are proved by constructing a Lyapunov function. This abstract construction of a Lyapunov function is usually performed by taking infinite series [71] or the supremum over all trajectories and all times [81,116]. As such, these approaches require the knowledge of trajectories for all positive times.

For linear discrete-time systems quadratic Lyapunov functions can be constructed by solving a matrix equation [54,73]. On the other hand, for nonlinear discrete-time systems constructive approaches to obtain a Lyapunov function are scarce and often limited to certain classes of discrete-time systems, see also the explanation in Section 2.5.

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In this thesis, we aim at deriving a constructive converse Lyapunov theorem for a broader class of systems. In addition, we do not demand regularity assumptions such as continuity that is often assumed in the literature [71,96,145]. In fact, we allow for *discontinuous* dynamics, which recently attracts much attention [46,48,93].

To be more precise, let  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy  $G(0) = 0$ , and consider the discrete-time system

$$x(k+1) = G(x(k)), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^n. \quad (1)$$

Then, see [71], global asymptotic stability of the equilibrium point 0 is ensured by the existence of a Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  satisfying the following two properties:

- (i) There exist  $\mathcal{K}_\infty$ -functions  $\alpha_1, \alpha_2$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|).$$

- (ii) There exists a positive definite function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\rho(s) < s$  for all  $s > 0$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$V(G(\xi)) \leq \rho(V(\xi)).$$

If we denote by  $x(k, \xi)$  the trajectory of system (1) at time instant  $k \in \mathbb{N}$ , starting in the initial value  $x(0, \xi) = \xi \in \mathbb{R}^n$ , the second condition of a Lyapunov function can be written as

$$V(x(1, \xi)) \leq \rho(V(\xi)).$$

In other words, condition (ii) ensures a decrease of the Lyapunov function along trajectories at *each* time step.

A relaxation of the Lyapunov function concept, which was inspired by the results in [1] for time-varying dynamical systems, is the following: the Lyapunov function is allowed to decrease along the system trajectories after a *finite number of time steps*, and not at *every* time step. To be precise, instead of condition (ii), we require the following condition:

- (ii') There exists a finite  $M \in \mathbb{N}$  and a positive definite function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\rho(s) < s$  for all  $s > 0$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$V(x(M, \xi)) \leq \rho(V(\xi)).$$

The idea of this relaxation is sketched in Figure 1. The function  $V$  to the left decreases along the trajectory  $x(k, \xi)$  at any step. On the other hand, the function  $V$  to the right decreases along the trajectory  $x(k, \xi)$  at least any three steps, and is thus allowed to increase in between  $x(k, \xi)$  and  $x(k+3, \xi)$ .

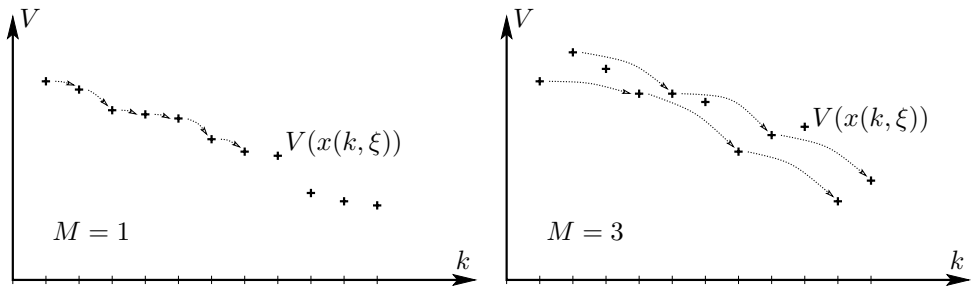


Figure 1: *to the left:* The function  $V$  decreases along  $x(k, \xi)$  at any step. *to the right:* The function  $V$  decreases along  $x(k, \xi)$  any 3 steps.

We will show that this relaxation, named *global finite-step Lyapunov function* in this thesis, still yields sufficient conditions for establishing GAS of the underlying system's origin. Here, we propose a different proof to the one in [1], where we do not require any regularity assumptions on the system's dynamics.

The main achievements by considering global finite-step Lyapunov functions instead of Lyapunov functions that are developed in this thesis are the following:

1. Necessity of the existence of a global finite-step Lyapunov function to conclude GAS of the origin can be ensured using converse Lyapunov theorems as e.g. [46, 71]. Here, we propose an alternative approach by considering norms as candidates for global finite-step Lyapunov functions. In particular, we prove that for systems with *globally exponentially stable* (GES) origin any norm is a global finite-step Lyapunov function, i.e., there exist an  $M \in \mathbb{N}$  and a positive definite function  $\rho < \text{id}$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$\|x(M, \xi)\| \leq \rho(\|\xi\|).$$

In addition, for systems with GAS origin we can show that any norm is an  $(a, b)$  *finite-step Lyapunov function*, where  $0 < a < b < \infty$ , which guarantees a finite-step decay for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| \in [a, b]$ . By picking  $M \in \mathbb{N}$  large enough we can enlarge the interval  $[a, b]$ . Hence, by taking  $M \in \mathbb{N}$  large enough, practical asymptotic stability can be ensured.

Consequently, we can take norms as (global or  $(a, b)$ ) finite-step Lyapunov functions. The difficulty is then to find a number  $M \in \mathbb{N}$  large enough.

2. We present two methods to construct a (global) Lyapunov function for the underlying system, in case a (global) finite-step Lyapunov function and a corresponding step size  $M \in \mathbb{N}$  is known. The first construction is to take the (finite) sum of the finite-step Lyapunov function evaluated at the first  $M$  trajectory values, while the second construction is the maximum of this values,

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but suitably scaled. These constructions, in contrast to the infinite series construction [71] and the supremum construction [81, 116], are implementable.

3. Bringing these two items together, we obtain an alternative converse Lyapunov theorem that provides an explicit construction of a Lyapunov function. In particular, for any (discontinuous) discrete-time system with GES origin, we can always construct a global Lyapunov function as a finite sum of the norm of trajectories. Hence, this construction can be implemented straightforwardly if a suitable number  $M \in \mathbb{N}$  is known.

The Lyapunov function construction hinges on finding a suitable natural number  $M \in \mathbb{N}$  a priori. In this thesis, several systematic ways to find such a suitable number are discussed for certain classes of systems. In particular, we establish necessity of specific types of Lyapunov functions *via* the developed converse Lyapunov theorems. Most notably, it is established that the existence of a conewise linear Lyapunov function is sufficient *and* necessary for GES of conewise linear systems, which is one of the open problems in stability analysis of conewise linear systems [72]. The latter result further yields, as a by-product, a new method to construct polyhedral Lyapunov functions for linear systems, i.e., the Lyapunov function  $V$  is of the form  $V(\xi) = \|P\xi\|$ , where  $P \in \mathbb{R}^{p \times n}$ ,  $p \geq n$ . We give an explicit formula of  $P$  in terms of iterates of  $A$ . Hence, this method is tractable even in state spaces of high dimension.

For discrete-time systems *with inputs* of the form

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N}, \quad (2)$$

where  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , we are interested in providing conditions guaranteeing *input-to-state stability* (ISS) as introduced in [128]. Despite several other characterizations such as e.g. given in [133, Theorem 1], ISS of system (2) is characterized by the following properties:

- *0-GAS*: the origin of system (2) with zero input ( $u = 0$ ) is GAS;
- *asymptotic gain property*: any trajectory converges to a neighborhood of the origin, where the size of the neighborhood depends on the magnitude of the input.

A particular consequence is that “small” perturbations, i.e., inputs with small magnitude, have only “small” effects on the system trajectories. Importantly, ISS of system (2) is equivalent to the existence of a dissipative ISS Lyapunov function, see e.g. [46, 70]. Thus, we introduce the notion of *dissipative finite-step ISS Lyapunov functions* as an extension of global finite-step Lyapunov functions for systems without inputs. In particular, the decrease condition (ii) is now of the form

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(ii') There exist a finite  $M \in \mathbb{N}$ ,  $\sigma \in \mathcal{K}$ , a positive definite function  $\rho$  with  $(\text{id} - \rho) \in \mathcal{K}_\infty$  such that for any  $\xi \in \mathbb{R}^n$  and all  $u(\cdot) \subset \mathbb{R}^m$  we have

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(\|u\|_\infty).$$

As the dissipative finite-step ISS Lyapunov function evaluated at  $x(k, \xi, u(\cdot))$  does now depend on a (worst-case) estimate of the input, the proposed construction from a global finite-step Lyapunov function to a global Lyapunov function fails if we have additional inputs, see Section 5.4 for an explanation.

However, some of the results for systems *without* inputs can be carried over to systems *with* inputs. More specifically, we prove that the existence of a dissipative finite-step ISS Lyapunov function is equivalent to the system being ISS. We consider the *dissipative form* of an ISS Lyapunov function characterization as we do not require continuity of the dynamics map. In particular, as it has been shown in [46], the *implication-form* ISS Lyapunov function characterization is not strong enough to conclude ISS of the system if the dynamics are discontinuous.

In addition, we prove that norms are dissipative finite-step ISS Lyapunov functions for exponentially input-to-state stable (expISS) systems. Again, this result implies a systematic procedure to check the expISS property of a system. Another important application of the concept of global finite-step and dissipative finite-step ISS Lyapunov functions lies in the context of interconnected nonlinear discrete-time systems as outlined next.

## Small-gain results

If we consider “large-scale” systems, e.g. in terms of size or complexity, then direct methods such as Lyapunov methods might be challenging or intractable. A common approach is to treat a *large-scale dynamical system* as an interconnection of smaller subsystems, [50, 109, 127, 140]. The idea is then to derive properties of the overall interconnected system, as e.g. stability properties, from characteristics of the subsystems.

One such appealing approach is the so-called *small-gain approach*, [24, 25, 65, 68, 69, 89]. Loosely speaking, the *classical*<sup>2</sup> idea of small-gain theorems is to assume that all subsystems’ equilibrium points are 0-GAS, and that the (disturbing) influence of the interconnection structure is small enough. Then GAS of the overall system’s equilibrium point can be deduced. However, the requirement that each subsystem’s equilibrium point is 0-GAS is not necessary, even for simple linear interconnected

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<sup>2</sup>We call those small-gain theorems *classical* to better distinguish our proposed relaxations from former small-gain results.

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systems such as e.g.

$$x(k+1) = \begin{pmatrix} 1.5 & 1 \\ -2 & -1 \end{pmatrix} x(k), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^2. \quad (3)$$

The matrix on the right-hand side has spectral radius  $\frac{\sqrt{2}}{2} < 1$ . Thus, the origin of the linear discrete-time system (3) is GAS, see e.g. [54, Satz 5]. However, the origin of the first decoupled subsystem with dynamic  $[x(k+1)]_1 = 1.5[x(k)]_1$  is even unstable. This in turn reveals that classical small-gain theorems come with certain conservatism.

To reduce conservatism in small-gain theory, we relax the *Lyapunov-based* small-gain approach as treated in e.g. [25, 67, 68, 89, 99], where the gains are derived from Lyapunov function estimates of the subsystems, in the following way. Consider an interconnection of  $N$  subsystems of the following form

$$x_i(k+1) = g_i(x_1(k), \dots, x_N(k)), \quad k \in \mathbb{N},$$

where  $x_i \in \mathbb{R}^{n_i}$  and  $g_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \rightarrow \mathbb{R}^{n_i}$  for  $i \in \{1, \dots, N\}$ . We assume that for each subsystem there exists a function  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$  satisfying a Lyapunov-type decrease estimate after a finite number of time steps of the form

$$\mathcal{V}_i(x_i(M, \xi)) \leq \max_{j \in \{1, \dots, N\}} \gamma_{ij}(\mathcal{V}_j(\xi_j)). \quad (4)$$

Here,  $M \in \mathbb{N}$ ,  $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n$  with  $\xi_i \in \mathbb{R}^{n_i}$ , and  $x_i(\cdot, \xi)$ , denotes the trajectory of the  $i$ th subsystem. The estimates also yield a set of  $\mathcal{K}_\infty$ -functions  $\gamma_{ij}$ , called gains. The conclusion of this relaxation is that if a small-gain condition invoking the  $\mathcal{K}_\infty$ -functions  $\gamma_{ij}$  is satisfied then we can construct a global finite-step Lyapunov function of the overall system by using the construction method proposed in [25]. In particular, the existence of the global finite-step Lyapunov function ensures GAS of the overall system's origin.

The advantages of this relaxed small-gain theorem are the following:

1. Although the functions  $\mathcal{V}_i$  look similar to the finite-step Lyapunov functions introduced in this thesis, these are *not* finite-step Lyapunov functions. Indeed, the relaxed small-gain theorem does not imply that the origin of each subsystem has to be 0-GAS. In particular, this relaxation allows the subsystems to be *unstable*, when considered decoupled.

In this sense, the notion *gain* seems to be inappropriate, as gains usually consider the influence of the subsystems on each other as disturbance. Here,  $\gamma_{ij}$  characterizes the (disturbing) effect of then initial state  $\xi_j$  on  $x_i(M, \xi)$ , and is thus called gain. Note that stabilizing feedback effects of the subsystems are implicitly taken into account.

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2. The small-gain theorem is proved by constructing a global finite-step Lyapunov function of the overall system, which shows GAS of the overall system's origin. Moreover, using the Lyapunov functions constructions via finite sums or finite maxima that are established in this thesis, we can compute a global Lyapunov function of the overall system in a straightforward manner. So despite the fact that the subsystems might be unstable, a global Lyapunov function of the overall system can be derived.
  3. Under an additional assumption on the overall system, we can show that norms are always admissible as Lyapunov-type functions  $\mathcal{V}_i$  satisfying a condition of the form (4). A particular class of systems satisfying the additional assumption is the class of systems with GES origin. Hence, if the overall system's origin is GES then we can always find suitable Lyapunov-type estimates by simply taking the norm of the subsystems' trajectories. This implies a systematic procedure to construct Lyapunov functions of the overall interconnected system.
  4. Moreover, under the same additional assumption, the relaxed small-gain theorem is shown to be sufficient *and* necessary, hence non-conservative. This is a distinguishing benefit over former small-gain results such as e.g. [65, 67, 99], which only provide sufficient criteria.

For interconnected subsystems *with* additional inputs of the form

$$x_i(k+1) = g_i(x_1(k), \dots, x_N(k), u(k)), \quad k \in \mathbb{N}, \quad i \in \{1, \dots, N\}$$

we propose a similar strategy. Usually, ISS small-gain results such as [24, 25, 68, 69, 89, 99] assume that all subsystems are ISS, and the gains describing the disturbing influence of the subsystems on each other, are in some sense small. Here, we propose similar Lyapunov-type decrease conditions to the ones (4) in the case without external inputs. Again, the distinguishing difference is that the decrease of the Lyapunov-type functions is considered after a finite number of time steps. Then a small-gain condition, where the gains are derived from these Lyapunov-type inequality conditions, ensures that the overall system is ISS. The advantages of this relaxation are similar to the ones in the case without external inputs:

1. We do not require the subsystems to be ISS. In particular, subsystems may be unstable, when decoupled from the others.
2. For the class of expISS systems, we show that we can always obtain norm estimates of the subsystems trajectories such that the Lyapunov-type inequality conditions of the small-gain theorem that we propose are met. Hence, for the class of expISS systems the relaxed small-gain theorems are sufficient *and* necessary, thus non-conservative. In particular, the proof implies a straightforward procedure to derive expISS of the overall system.

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We emphasize that in contrast to other available discrete-time small-gain results in the literature such as e.g. [65, 67, 99], we do not impose regularity assumptions as continuity on the dynamics. This is a distinguishing extension to former small-gain results. In particular, as there is currently an immense interest in stabilization schemes that commonly lead to discontinuous feedback control laws (such as e.g. *model predictive control* (MPC) [16, 47, 90] or *event-triggered control* [119, 137]) the proposed small-gain results can be applied.

## Gain construction methods

A condition that all small-gain theorems presented in this thesis have in common, is the so-called *small-gain condition*. This condition stems from [24, 25] as  $N$ -dimensional extension to the 2-dimensional versions obtained in [68, 69]. Usually, (trajectory-based or Lyapunov-based) estimates of the subsystems lead to a set of  $\mathcal{K}_\infty$ -functions (the *gains*), which describe the influence of the subsystems on each other as a disturbance, and to a set of functions describing how the gains are aggregated. For instance, in the *summation case* the gains are aggregated via summation and in the *maximization case* the gains are aggregated via maximization. More general cases of aggregating gains can be described by using *monotone aggregation functions*, see [25, 122].

The underlying problem of small-gain theory is that an interconnection of systems is given, and one asks for stability properties of the overall system. On the other hand, instead of considering fixed systems, we might consider parameter-dependent systems, where we can scale the gains. For example, in *networked control systems* (NCS) [57] one approach is to treat the error dynamics as a subsystem, see e.g. [12], where the error gain might be decreased by faster sampling, more bandwidth, and so on. It seems natural to ask how small the gains have to be such that a small-gain condition is satisfied.

A slightly different design question is the following. Assume we are given a large-scale system satisfying a small-gain condition, we wish to add further subsystems, and we are able to design the interconnection gains. How can we derive a priori bounds on the new gains depending on the given gains in order that a small-gain condition holds? Interestingly, this question does not, to the best of the author's knowledge, appear in the literature.

We illustrate the problem via Figure 2. In this figure we see a weighted directed graph, where the vertices **1**, **2**, **3** and **4** correspond to the subsystems. The goal is that we want a small-gain condition to hold. Therefore assume that the gains ( $\mathcal{K}_\infty$ -functions)  $\gamma_{12}, \gamma_{21}$  and  $\gamma_{23}$  are given from estimates of subsystems **1** and **2**. On the other hand, the gains  $(a_{34}\bar{\gamma})$ ,  $(a_{41}\bar{\gamma})$  and  $(a_{42}\bar{\gamma})$  are determined by a positive weight  $a_{ij}$  and the joint  $\mathcal{K}_\infty$ -function  $\bar{\gamma}$ . The positive weights  $a_{34}$ ,  $a_{41}$  and  $a_{42}$  may



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be interpreted as a scaling of the problem. For instance, the relative cost necessary for achieving a gain in an interconnection might be high. In that case choosing a small scaling factor  $a_{ij}$  leads to a relatively small gain in that position. The problem is now to decide if there exists a  $\mathcal{K}_\infty$ -function  $\bar{\gamma}$  such that a small-gain condition for the whole interconnection holds, and to find ways how to compute  $\bar{\gamma}$ .

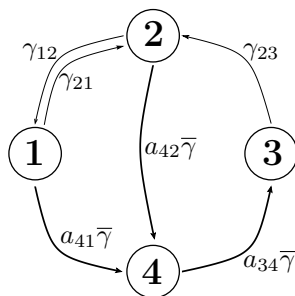


Figure 2: A weighted directed graph with known gains  $\gamma_{12}, \gamma_{21}, \gamma_{23}$ , known weights  $a_{34}, a_{41}, a_{42}$  and  $\bar{\gamma} \in \mathcal{K}_\infty$  to be determined.

Intuitively, if the small-gain condition for the whole interconnection is satisfied for  $\bar{\gamma} \equiv 0$  then the small-gain condition is also satisfied if  $\bar{\gamma} \in \mathcal{K}_\infty$  is “small” enough. We will show in this thesis that this intuition is correct. Roughly speaking, a “small” gain means that the disturbing influence of one subsystem on another subsystem is also small. Hence, we are interested in making  $\bar{\gamma}$  “large” in order to allow preferably much disturbing influence.

For instance, in [12], the authors consider networked control systems, where states of the subsystems are sent over a communication network. At each transmission time, one subsystem is granted access to the communication network and is allowed to send its state. ISS of the networked control system is derived from a small-gain condition in terms of a *maximal allowable transfer interval* (MATI), i.e., an upper bound on the transmission times, at which the communication network has to send the state of a subsystem to the others. Clearly, the smaller the MATI is, the more often the communication network has to send the state of a subsystem. In particular, as it is implied by [12, Equation (34)], the MATI is small if the corresponding gain (here  $\bar{\gamma}$ ) is small.

We pose the following main question:

*How small does  $\bar{\gamma} \in \mathcal{K}_\infty$  have to be in order that a small-gain condition holds?*

For several situations we derive constructive methods for obtaining a *maximal gain*  $\bar{\gamma}$ . Here, a maximal gain is characterized by the property that any (point-wise) smaller

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$\mathcal{K}_\infty$ -function is admissible in the sense that a small-gain condition holds, while any (point-wise) greater  $\mathcal{K}_\infty$ -function violates the small-gain condition.

1. In the *linear summation case*, in which the linear gains are aggregated via summation, the matrix collecting the gains is a nonnegative matrix. In addition, the small-gain condition can be equivalently expressed in terms of the spectral radius of this matrix. By using results from the theory of nonnegative matrices [8] and from the theory of stability radii [60,61], we give constructive methods to compute maximal gains.
2. In the *maximization case*, in which gains are aggregated via maximization, we can compute maximal gains by solving *iterative functional  $\mathcal{K}_\infty$ -equations* as outlined in the next paragraph. Here, we make use of the equivalent characterization of the small-gain condition in terms of weakly contracting cycles of the weighted directed graph.
3. For the general case we cannot derive maximal gains, but at least admissible gains can be constructed. The difficulty in this case is that, in contrast to the linear summation case (spectral radius condition) and the maximization case (cycle condition), there is no equivalent small-gain condition that is easy to check.

## Iterative functional $\mathcal{K}_\infty$ -equations

In the maximization case, the small-gain condition is equivalent to the *cycle condition* [122], which says that the composition of gains along a cycle in the directed graph has to be less than the identity function. A cycle satisfying this property is called *weakly contracting*. So to compute maximal gains in the maximization case, we have to ensure that any cycle is weakly contracting. For instance, in Figure 2, the cycle from vertex 2 to 4 to 3 to 2 has to satisfy the inequality

$$\gamma_{23} \circ (a_{34}\bar{\gamma}) \circ (a_{42}\bar{\gamma}) < \text{id}. \quad (5)$$

If we define  $\alpha_1 := \gamma_{23}(a_{34} \text{id})$  and  $\alpha_2 := a_{42} \text{id}$ , and assume that  $\gamma \in \mathcal{K}_\infty$  satisfies the *iterative functional  $\mathcal{K}_\infty$ -equation*  $\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma = \text{id}$ , then, by monotonicity arguments, every  $\mathcal{K}_\infty$ -function  $\tilde{\gamma} < \gamma$  satisfies (5). Moreover, if  $\tilde{\gamma} \geq \gamma$  then (5) and with it also the small-gain condition is violated. Hence, a solution of the iterative functional  $\mathcal{K}_\infty$ -equation  $\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma = \text{id}$  yields a greatest upper bound for gains  $\bar{\gamma}$  satisfying (5).

More general, we study the question of the existence of solutions of iterative functional  $\mathcal{K}_\infty$ -equations of the form

$$\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma = \text{id}, \quad (6)$$

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where  $\alpha_1, \dots, \alpha_k \in \mathcal{K}_\infty$ . Although iterative functional equations have been widely studied in the literature [6, 138], the problem we address was only pointed out for the special case  $\gamma^k = \alpha$ . In particular, from [87], we derive, as a preliminary result, that solutions of the special case  $\gamma^k = \alpha$  exist, but they are not unique.

The results we develop can be summarized as follows:

1. We establish a subclass of the class  $\mathcal{K}_\infty$ , which extends the class of piecewise linear functions to the class of so-called *right-affine  $\mathcal{K}_\infty$ -functions*, that is, functions that are piecewise affine linear on intervals of a partition of  $[0, \infty)$ , where partition intervals can only accumulate to the right.
2. We prove that for functions  $\alpha_i$  within this class of right-affine  $\mathcal{K}_\infty$ -functions there exists a solution of the iterative functional equation (6) that is unique in the same class.
3. The method of proof leads to a constructive procedure. This is important as we do not only derive an existence result, but we can also numerically compute solutions for applications. In particular, maximal gains in the maximization case can be numerically computed.

## Outline of this thesis

In Chapter 1 we state the necessary preliminaries.

Discrete-time systems without inputs are treated in Chapter 2, where the main sections are concerned with the stability analysis via finite-step Lyapunov functions (Section 2.2), relaxed small-gain theorems (Section 2.3), and application of the results to several system classes (Section 2.4).

Chapter 3 considers discrete-time systems with inputs. Here, we first study stability analysis via dissipative finite-step ISS Lyapunov functions in Section 3.2. Then we present two relaxed small-gain approaches: a Lyapunov-based approach in Section 3.3 and a trajectory-based approach in Section 3.4.

Gain construction methods and iterative functional  $\mathcal{K}_\infty$ -equations are studied in Section 4.1 resp. Section 4.2.

In Chapter 5 we indicate some extensions of the results in this thesis and present ideas for ongoing research.

Finally, some results from the theory of nonnegative matrices and stability radii for linear systems are given in the Appendix.

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## Acknowledgements

This thesis wouldn't be what it is now without the support of many people.

First of all, I would like to thank my advisor Fabian Wirth for giving me the chance to do my PhD in this nice field of mathematics. Fabian, thank you for many fruitful discussions, your trust in me, and giving me the freedom to develop own ideas. You taught me to work rigorously and independently. I deeply enjoyed working with you.

Next, I have to thank Mircea Lazar, Rob Gielen and the control group of the department of electrical engineering of the TU/e for hosting me in Eindhoven. Mircea, thank you for your hospitality. I very much enjoyed our collaboration.

Many thanks go to Rudolf Sailer, Frederike Rüppel and Michael Schönlein for proof-reading some of the chapters, Axel Kunz for helping me with the English language, and Martin Pierzchala for the cover design. I would also like to take the opportunity to thank all my colleagues in Würzburg and the colleagues I met on conferences and workshops. In particular, I would like to thank Chris Kellett for the nice discussions we had and Lars Grüne who agreed to examine this thesis.

This thesis wouldn't have possible without the financial support that was offered me by the Elitenetzwerk Bayern (ENB) and the University of Würzburg.

Finally, I thank my friends and family for their confidence in me, and for supporting and encouraging me trough the last years. Especially, I thank my parents, my brothers and sisters and, most importantly, I thank my wife Vanessa and my children Joela, Robin and Matheo.

# Preliminaries

In this chapter, we give an overview of the basic definitions, notions and preliminary results that are used throughout this thesis.

## 1.1 Notions

By  $\mathbb{N}$  we denote the natural numbers, where we assume  $0 \in \mathbb{N}$ , and by  $\mathbb{C}$  we denote the complex numbers. Let  $\mathbb{R}$  denote the field of real numbers,  $\mathbb{R}_+$  the set of non-negative real numbers and  $\mathbb{R}^n$  the vector space of real column vectors of length  $n$ . For a vector  $v \in \mathbb{R}^n$  we denote by  $[v]_i$  its  $i$ th component. The cone<sup>1</sup>  $\mathbb{R}_+^n$  induces a partial order on  $\mathbb{R}_+^n$ . For vectors  $v, w \in \mathbb{R}_+^n$  we denote

$$\begin{aligned} v \geq w &\iff \forall i \in \{1, \dots, N\} : [v]_i \geq [w]_i \\ v > w &\iff \forall i \in \{1, \dots, N\} : [v]_i > [w]_i \\ v \not\geq w &\iff \exists i \in \{1, \dots, N\} : [v]_i < [w]_i. \end{aligned}$$

Accordingly, for a matrix  $A \in \mathbb{R}^{n \times m}$  we denote by  $[A]_{i,j}$  its  $(i, j)$ th entry. Furthermore, the notation  $[A]_{i,:}$  denotes the  $i$ th row (resp.  $[A]_{:,j}$  denotes the  $j$ th column) of matrix  $A$ . For matrices  $A_1, \dots, A_N \in \mathbb{R}^{n \times m}$  we use the abbreviation

$$(A_1; \dots; A_N) := (A_1^\top \dots A_N^\top)^\top \in \mathbb{R}^{Nn \times m},$$

and for vectors  $v_i \in \mathbb{R}^{n_i}$ ,  $i \in \{1, \dots, N\}$  we write

$$(v_1, \dots, v_N) := (v_1^\top \dots v_N^\top)^\top.$$

<sup>1</sup>for a definition of a cone, see Definition 2.46.

For a given square matrix  $Q \in \mathbb{R}^{n \times n}$  the *spectrum* of  $Q$ , i.e., the set of eigenvalues of  $Q$ , is defined by

$$\sigma(Q) := \{\lambda \in \mathbb{C} : \exists v \in \mathbb{C}^n \setminus \{0\} \text{ such that } Qv = \lambda v\}.$$

Furthermore, the *spectral radius* of  $Q$  is defined as the largest absolute eigenvalue of  $Q$ , i.e.,

$$\rho(Q) := \max_{\lambda \in \sigma(Q)} |\lambda|.$$

By  $I_n$  we denote the  $n \times n$  identity matrix, but we mostly write  $I$  if the dimension is clear from the context.

For the following functions, we use  $\mathbb{R}_+$  as the domain of definition. Clearly, these definitions can be extended to other domains of definition (such as e.g.  $\mathbb{R}$ ).

By  $\text{id} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  we denote the identity function  $\text{id}(s) = s$  for all  $s \in \mathbb{R}_+$ , and by  $0 : \mathbb{R}_+ \rightarrow \{0\}$  we denote the zero function  $0(s) = 0$  for all  $s \in \mathbb{R}_+$ .

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is called

- *increasing* if  $\alpha(s_2) \geq \alpha(s_1)$  for all  $s_2 \geq s_1 \geq 0$ ;
- *strictly increasing* if  $\alpha(s_2) > \alpha(s_1)$  for all  $s_2 > s_1 \geq 0$ ;
- *positive (semi-)definite* if it is continuous, satisfies  $\eta(0) = 0$  and  $\eta(s) > 0$  (resp.  $\eta(s) \geq 0$ ) for all  $s > 0$ ;
- *sub-additive* if for all  $s_1, s_2 \in \mathbb{R}_+$  it holds

$$\alpha(s_1 + s_2) \leq \alpha(s_1) + \alpha(s_2).$$

For two functions  $\alpha_1, \alpha_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we write  $\alpha_1 < \alpha_2$  (resp.  $\alpha_1 \leq \alpha_2$ ) if  $\alpha_2 - \alpha_1$  is positive (semi-)definite. Furthermore,  $\alpha_1 \circ \alpha_2$  denotes the composition of two functions  $\alpha_1, \alpha_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , and  $\alpha_1^k := \alpha_1 \circ \dots \circ \alpha_1$  is the  $k$ th iterate of  $\alpha_1$ .

## 1.2 Norms

In this work we need several different norms. Firstly, we give a formal definition.

**Definition 1.1.** Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  and  $n \in \mathbb{N}$ . A function  $\|\cdot\| : \mathbb{K}^n \rightarrow \mathbb{R}_+$  is called a *norm* on  $\mathbb{K}^n$  if the following holds:

- $\|\cdot\|$  is *positive definite*, i.e.,  $\|x\| \geq 0$  for all  $x \in \mathbb{K}^n$  and  $\|x\| = 0$  iff  $x = 0$ ;
- $\|\cdot\|$  is *absolutely homogenous*, i.e.,  $\|\lambda x\| = |\lambda| \|x\|$  for all  $\lambda \in \mathbb{K}$ ,  $x \in \mathbb{K}^n$ ;

(iii)  $\|\cdot\|$  satisfies the *triangle inequality*, i.e., for all  $x, y \in \mathbb{K}^n$  we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

Let a pair of norms  $\|\cdot\|_{\mathbb{K}^l}$  and  $\|\cdot\|_{\mathbb{K}^n}$  on  $\mathbb{K}^l$  and  $\mathbb{K}^n$ , and a matrix  $P \in \mathbb{K}^{l \times n}$  be given. Then

$$\|P\| := \max_{\|x\|_{\mathbb{K}^n}=1} \|Px\|_{\mathbb{K}^l}$$

denotes the induced *operator norm* of  $P$ .

A particularly relevant norm on  $\mathbb{R}^n$  is the *p-norm*  $\|\cdot\|_p$  with  $p \in [1, \dots, \infty]$ , which is defined for all  $x \in \mathbb{R}^n$  by

$$\|x\|_p := \left( \sum_{j=1}^n |[x]_j|^p \right)^{1/p}, \quad p \in [1, \infty),$$

$$\|x\|_\infty := \max_{j \in \{1, \dots, n\}} |[x]_j|.$$

The latter norm  $\|\cdot\|_\infty$  is also called *infinity norm*. Also often used is the *p-norm* for  $p = 1$  or  $p = 2$ , and thus, we state it explicitly:

$$\|x\|_1 = \sum_{j=1}^n |[x]_j| \quad (1\text{-norm}),$$

$$\|x\|_2 = \left( \sum_{j=1}^n (|[x]_j|)^2 \right)^{1/2} \quad (Euclidean\ norm).$$

It is well-known (see e.g. [82, Appendix A]) that norms on  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ) are *equivalent* in the sense that for any two norms  $\|\cdot\|_{(i)}$ ,  $\|\cdot\|_{(ii)}$  on  $\mathbb{R}^n$  there exist real constants  $c, C > 0$  such that for all  $x \in \mathbb{R}^n$  we have

$$c\|x\|_{(i)} \leq \|x\|_{(ii)} \leq C\|x\|_{(i)}.$$

For instance, if  $1 \leq p_2 \leq p_1 \leq \infty$  then for all  $x \in \mathbb{R}^n$  it holds

$$\|x\|_{p_1} \leq \|x\|_{p_2} \leq n^{\frac{1}{p_2} - \frac{1}{p_1}} \|x\|_{p_1}.$$

A direct consequence of the equivalence of norms is the following inequality, which is used in Chapter 3: For any norm  $\|\cdot\|$  on  $\mathbb{R}^n$  there exists a constant  $\kappa \geq 1$  such that for all  $x = (x_1, \dots, x_N) \in \mathbb{R}^n$  with  $x_i \in \mathbb{R}^{n_i}$  and  $n = \sum_{i=1}^N n_i$ , it holds

$$\|x\| \leq \kappa \max_{i \in \{1, \dots, N\}} \|x_i\|, \quad (1.1)$$

where  $\|x_i\| := \|(0, \dots, 0, x_i, 0, \dots, 0)\|$  is the induced norm on  $\mathbb{R}^{n_i}$ . In particular, if  $\|\cdot\|$  is a  $p$ -norm then  $\kappa = N^{1/p}$  is the smallest constant satisfying (1.1).

Some results stated in Appendix A.2 require the following property of a norm.

**Definition 1.2.** For any  $x = ([x]_1, \dots, [x]_n) \in \mathbb{R}^n$  let  $\text{abs} : \mathbb{R}^n \rightarrow \mathbb{R}_+^n$  be defined by

$$\text{abs}(x) := (|[x]_1|, \dots, |[x]_n|).$$

A norm  $\|\cdot\|$  on  $\mathbb{R}^n$  is said to be *monotonic* if for all  $x, y \in \mathbb{R}^n$  it holds

$$\text{abs}(x) \leq \text{abs}(y) \quad \Rightarrow \quad \|x\| \leq \|y\|.$$

It can be shown that a vector norm  $\|\cdot\|$  is monotonic if and only if it is *absolute*, i.e.,  $\|x\| = \|\text{abs}(x)\|$  holds for all  $x \in \mathbb{R}^n$ , see e.g. [62]. It follows that every  $p$ -norm on  $\mathbb{R}^n$ ,  $p \in [1, \infty]$ , is monotonic.

For a given norm  $\|\cdot\|$  we define the set

$$B_{[a,b]} := \{x \in \mathbb{R}^n : \|x\| \in [a, b]\}.$$

Consider a sequence  $\{y(l)\}_{l \in \mathbb{N}}$  with  $y(l) \in \mathbb{R}^m$ , (or for short  $y(\cdot) \subset \mathbb{R}^m$ ). Let  $\|\cdot\|$  be an arbitrary norm on  $\mathbb{R}^m$ , and define

$$\begin{aligned} |y|_{[0,k]} &:= \sup \{\|y(l)\| : l \in \{0, \dots, k\}\} && \in \mathbb{R}_+ \\ |y|_\infty &:= \sup \{\|y(l)\| : l \in \mathbb{N}\} && \in \mathbb{R}_+ \cup \{\infty\} \end{aligned}$$

If  $|y|_\infty < \infty$ , then the sequence  $y(\cdot)$  is called *bounded*.

### 1.3 Comparison functions

To state the stability results in this thesis we use the classes of *comparison functions*  $\mathcal{K}, \mathcal{K}_\infty, \mathcal{L}$ , and  $\mathcal{KL}$  as defined e.g. in [81, 82]. Since the introduction of input-to-state stability in [128], the usage of these comparison functions has become standard in control theory, especially concerning the stability analysis of nonlinear systems. Following the recommendable work [77], it was Wolfgang Hahn who termed those function classes by  $\mathcal{K}$  [55] and  $\mathcal{KL}$  [56]. It has been speculated that the letter  $\mathcal{K}$  is derived from Kamke, see [45, 77].

**Definition 1.3.** A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of *class*  $\mathcal{K}$  (or a  $\mathcal{K}$ -function, denoted by  $\alpha \in \mathcal{K}$ ) if it is strictly increasing, continuous, and satisfies  $\alpha(0) = 0$ . In particular, if  $\alpha \in \mathcal{K}$  is unbounded, it is said to be of *class*  $\mathcal{K}_\infty$  (or a  $\mathcal{K}_\infty$ -function). A function  $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of *class*  $\mathcal{L}$  (or an  $\mathcal{L}$ -function, denoted by  $\pi \in \mathcal{L}$ ), if it is continuous, strictly decreasing, and satisfies  $\lim_{s \rightarrow \infty} \pi(s) = 0$ .



**Definition 1.4.** A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of class  $\mathcal{KL}$  (or a  $\mathcal{KL}$ -function, denoted by  $\beta \in \mathcal{KL}$ ), if it is of class  $\mathcal{K}$  in the first argument and of class  $\mathcal{L}$  in the second argument.

The class  $\mathcal{K}_\infty$  is the set of all homeomorphisms of the interval  $[0, \infty)$ . This fact immediately implies the following proposition.

**Proposition 1.5.** *The pair  $(\mathcal{K}_\infty, \circ)$  is a non-commutative group.*

*Proof.* By the properties of any  $\mathcal{K}_\infty$ -functions  $\alpha_1, \alpha_2$  it is easy to see that  $\alpha_1 \circ \alpha_2 \in \mathcal{K}_\infty$ ,  $\alpha_1^{-1}$  exists and is of class  $\mathcal{K}_\infty$  and the identity map  $\text{id}$  is the identity element of  $(\mathcal{K}_\infty, \circ)$ , which shows that  $(\mathcal{K}_\infty, \circ)$  is a group. In particular, the group is non-commutative, i.e., there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  with  $\alpha_1 \circ \alpha_2 \neq \alpha_2 \circ \alpha_1$ , which can be seen by taking e.g.  $\alpha_1(s) = s^2$  and  $\alpha_2(s) = e^s - 1$  for  $s \geq 0$ .  $\square$

Due to the monotonicity property of  $\mathcal{K}_\infty$ -functions, it holds for all  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$  that

$$\alpha_1(\max\{\alpha_2, \alpha_3\}) = \max\{\alpha_1 \circ \alpha_2, \alpha_1 \circ \alpha_3\}.$$

In Section 4.2 we consider equalities of the form  $\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_k = \text{id}$ , where  $\alpha_1, \dots, \alpha_k \in \mathcal{K}_\infty$ ,  $k \in \mathbb{N}$ . A consequence of Proposition 1.5 for such equalities is that we can *permute the functions cyclically*, i.e., the following equivalence holds:

$$\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_k = \text{id} \iff \alpha_k \circ \alpha_1 \circ \dots \circ \alpha_{k-1} = \text{id}. \quad (1.2)$$

Furthermore, for  $\alpha \in \mathcal{K}_\infty$ ,  $j \in \mathbb{N}$ ,  $k \in \mathbb{N}$ ,  $k > 0$ , we denote by  $\gamma = \alpha^{j/k} \in \mathcal{K}_\infty$  a solution of the functional equation  $\gamma^k = \alpha^j$ . Its existence is shown in Proposition 4.16.

For two  $\mathcal{K}_\infty$ -functions  $\eta_1, \eta_2 \in \mathcal{K}_\infty$  with  $\eta_1 - \eta_2 \in \mathcal{K}_\infty$ , the inverse  $(\eta_1 - \eta_2)^{-1}$  is of the form  $\eta_1^{-1} \circ (\text{id} + \rho)$  with  $\rho \in \mathcal{K}_\infty$ , which follows by setting  $\rho := \eta_2 \circ (\eta_1 - \eta_2)^{-1} \in \mathcal{K}_\infty$ :

$$(\eta_1^{-1} \circ (\text{id} + \rho)) \circ (\eta_1 - \eta_2) = \eta_1^{-1} \circ (\eta_1 - \eta_2 + \eta_2 \circ (\eta_1 - \eta_2)^{-1} \circ (\eta_1 - \eta_2)) = \text{id}. \quad (1.3)$$

Here we have used Proposition 1.5 to conclude that the composition of  $\mathcal{K}_\infty$ -functions yields a  $\mathcal{K}_\infty$ -function. One particular case of this observation will be used in Chapter 4, and is thus stated explicitly.

**Lemma 1.6.** *Let  $\eta, \hat{\eta} \in \mathcal{K}_\infty$  such that  $\eta = \text{id} - \hat{\eta}$ , and  $\epsilon \in [0, 1]$ . Then we have*

(i)  $\text{id} - \epsilon \hat{\eta} \in \mathcal{K}_\infty$ ;

(ii) *there exists a function  $\rho \in \mathcal{K}_\infty$  such that  $(\text{id} + \rho) = (\text{id} - \epsilon \hat{\eta})^{-1}$ .*

*Proof.* The first statement follows directly from  $\text{id} - \epsilon\hat{\eta} = \eta + (1 - \epsilon)\hat{\eta} \in \mathcal{K}_\infty$ . The second statement follows from (1.3), where we set  $\eta_1 = \text{id}$ ,  $\eta_2 = \epsilon\hat{\eta}$ , and  $\rho = \epsilon\hat{\eta}(\text{id} - \epsilon\hat{\eta})^{-1}$ .  $\square$

A similar result to Lemma 1.6 is the following.

**Lemma 1.7.** *Let  $\eta \in \mathcal{K}_\infty$ . Then there exists a  $\hat{\eta} \in \mathcal{K}_\infty$  such that  $(\text{id} + \eta)^{-1} = \text{id} - \hat{\eta}$ . On the other hand, for given  $\hat{\eta} \in \mathcal{K}_\infty$  with  $(\text{id} - \hat{\eta}) \in \mathcal{K}_\infty$ , there exists a function  $\eta \in \mathcal{K}_\infty$  such that  $(\text{id} + \eta)^{-1} = \text{id} - \hat{\eta}$ .*

*Proof.* The first implication follows directly with  $\hat{\eta} = \eta \circ (\text{id} + \eta)^{-1}$ , see also [123, Lemma 2.4]. The other implication follows similarly, by setting  $\eta = \hat{\eta} \circ (\text{id} - \hat{\eta})^{-1}$ .  $\square$

## 1.4 Large-scale dynamical systems and stability properties

The abstract definition of a *dynamical system* usually consists of a structure containing a time domain, a state space, an input value space, and a state transition map, which has to satisfy some properties, see e.g. [130, Definition 2.1.2] or [60, Definition 2.1.1]. In this work we consider time-invariant discrete-time dynamical systems.

Let  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  be given, and define the *difference equation*

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N}. \quad (1.4)$$

Here  $u(k) \in \mathbb{R}^m$  denotes the input at time  $k \in \mathbb{N}$ . Note that an input is a function  $u : \mathbb{N} \rightarrow \mathbb{R}^m$ . By  $x(k, \xi, u(\cdot)) \in \mathbb{R}^n$  we denote the solution<sup>2</sup> of (1.4) at time  $k \in \mathbb{N}$ , starting in the initial state  $x(0) = \xi \in \mathbb{R}^n$  with input function  $u(\cdot) \subset \mathbb{R}^m$ .

*Remark 1.8.* The difference equation (1.4) implies a structure

$$\Sigma = (\mathbb{N}, \mathbb{R}^m, (\mathbb{R}^m)^\mathbb{N}, \mathbb{R}^n, \mathbb{R}^n, \varphi, \text{id}).$$

In the first argument,  $\mathbb{N}$  denotes the time domain. The input value space and the input function space are  $\mathbb{R}^m$  resp.

$$(\mathbb{R}^m)^\mathbb{N} := \{(u(0), u(1), u(2), \dots) : u(k) \in \mathbb{R}^m, k \in \mathbb{N}\}.$$

By  $\text{id}$  in the last argument, state and output are equal; in particular, from arguments 4 and 5 we see that both state space and output value space are  $\mathbb{R}^n$ . Finally,  $\varphi$  denotes the state transition map. Moreover, the axioms in [60, Definition 2.1.1]

<sup>2</sup>Usually, *trajectories* denote the evolution of a state of a dynamical system [60]. Another notion for trajectory that we do not use in this work is *motion* [141]. As the dynamical systems considered in this work are described by difference equations, *solutions* (to an initial value problem) of a system correspond to trajectories, and thus, both notions are used synonymously.

are satisfied. Thus, the structure  $\Sigma$  is a dynamical system in the sense of [60, Definition 2.1.1], see also [60, Example 2.1.23].

To be precise, the structure  $\Sigma$  is a *time-invariant* dynamical system, as the function  $G$  in (1.4) does not explicitly depend on the time  $k \in \mathbb{N}$ . Hence, we can always assume that the initial time is 0.  $\triangleleft$

As the difference equation (1.4) gives rise to a (time-invariant) dynamical system (Remark 1.8), we call the difference equation (1.4) a *discrete-time (dynamical) system*.

Similarly, for given  $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , the difference equation

$$x(k+1) = \tilde{G}(x(k)), \quad k \in \mathbb{N} \quad (1.5)$$

is called a discrete-time (dynamical) system, too. Clearly, (1.5) can be obtained from a discrete-time system of the form (1.4) by setting  $\tilde{G}(x) := G(x, 0)$ .

Next we discuss what we mean by *large-scale* dynamical systems. First let us mention that the term “large-scale” is often used in different settings. As already observed in e.g. [109], there is no precise definition of large-scale systems. We refer to Remark 1.9, where we recite the understanding of different authors about large-scale systems.

In this thesis we consider the following:

*A large-scale system is an interconnection of smaller subsystems.*

We do then call it the *overall* or *interconnected system*.

*Remark 1.9.* The authors in [109] consider a dynamical system to be *large* if “it possesses a certain degree of complexity in terms of structure and dimensionality”. Moreover, they divide problems concerned with large-scale systems into two broad areas: static problems (e.g. graph theoretic problems, routing problems) and dynamical problems, where the latter may in turn be separated into quantitative problems (e.g. numerical solution of equations describing large systems) and into qualitative problems (e.g. stability or instability in the sense of Lyapunov, boundedness of solutions, estimates of trajectory behavior and trajectory bounds, input-output properties of dynamical systems).

The author in [140] considers a large-scale system as “an interconnected system consisting of several subsystems interacting through various interconnection operators”. Similarly, the authors in [50] write that “in analyzing complex large-scale interconnected dynamical systems it is often desirable to treat the overall system as a collection of interacting subsystems.”

As described by [127] “advantage [of this decomposition] can be taken of the special structural features of a given system to devise feasible and efficient ‘piece-by-piece’ algorithms for solving large problems which were previously intractable or impractical to tackle with ‘one shot’ methods and techniques”. In other words, precisely those of [50, p.1], the advantage of this point of view is the following: “The behavior of the aggregate or composite (i.e., large-scale) system can then be predicted from the behaviors of the individual subsystems and their interconnections. The need for decentralized analysis and control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical system. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between systems sensors, actuators, and processors may render centralized control architectures impractical. Moreover, even when communication constraints do not exist, decentralized processing may be more economical.”

Examples of large-scale systems that are naturally arising can be found e.g. in the areas of economics (markets), ecology (swarms, multi-species communities), and engineering (power plants), which are treated in more detail in [127].  $\triangleleft$

To study stability properties of the discrete-time system (1.5), we call  $\bar{x} \in \mathbb{R}^n$  an *equilibrium point* of (1.5), if it satisfies  $\bar{x} = \tilde{G}(\bar{x})$ . Thus,  $\bar{x}$  is a *fixed point* of the function  $\tilde{G}$ . In addition, we can shift the equilibrium point  $\bar{x}$  to the origin as follows. Let  $\bar{x} \in \mathbb{R}^n$  be an equilibrium point of  $\tilde{G}$ . Consider the change of variables  $y = x - \bar{x}$ . If the evolution of  $x$  is described by (1.5), then the evolution of  $y$  can be described by the difference equation

$$y(k+1) = x(k+1) - \bar{x} = \tilde{G}(x(k)) - \bar{x} = \tilde{G}(y(k) + \bar{x}) - \bar{x} =: F(y(k))$$

for all  $k \in \mathbb{N}$ . Note that the origin  $\bar{y} = 0 \in \mathbb{R}^n$  is a fixed point of  $F$  since  $\bar{x}$  is a fixed point of  $\tilde{G}$ . Thus, we can always assume that the origin is an equilibrium point of the time-invariant discrete-time system (1.5).

Next, we define stability properties of the origin of the discrete-time system (1.5).

**Definition 1.10.** Consider the discrete-time system (1.5) and assume that the origin is an equilibrium point. Let  $\|\cdot\|$  be any arbitrary fixed norm on  $\mathbb{R}^n$ . Then we call the origin

- *stable*<sup>3</sup> if for any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| < \delta$  and all  $k \in \mathbb{N}$  we have

$$\|x(k, \xi)\| < \varepsilon;$$

- *unstable* if it is not stable;

---

<sup>3</sup>this property is also often called *Lyapunov stability*

- *attractive* if there exists an  $\delta > 0$  such that for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| < \delta$  we have

$$\lim_{k \rightarrow \infty} x(k, \xi) = 0;$$

- *globally attractive* if it is attractive for any  $\delta > 0$ ;
- *(globally) asymptotically stable* if it is stable and (globally) attractive.

We illustrate these stability concepts in Figure 1.1.

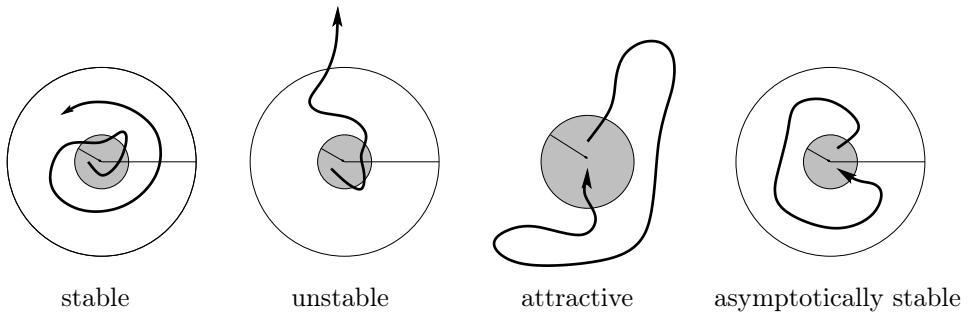


Figure 1.1: Visualization of the stability concepts of Definition 1.10.

In this thesis, we use an equivalent characterization of the stability concepts in terms of comparison functions as introduced in Section 1.3.

**Lemma 1.11.** *The origin of system (1.5) is*

- *stable if and only if there exists a  $\mathcal{K}$ -function  $\gamma$  and a constant  $c > 0$  such that for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| < c$  and all  $k \in \mathbb{N}$ ,*

$$\|x(k, \xi)\| \leq \gamma(\|\xi\|);$$

- *asymptotically stable if and only if there exists a  $\mathcal{KL}$ -function  $\beta$  and a constant  $c > 0$  such that for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| < c$  and all  $k \in \mathbb{N}$ ,*

$$\|x(k, \xi)\| \leq \beta(\|\xi\|, k); \tag{1.6}$$

- *globally asymptotically stable (GAS) if and only if (1.6) is satisfied for all initial states  $\xi \in \mathbb{R}^n$  and all  $k \in \mathbb{N}$ .*

This result is stated similarly in [82, Lemma 4.5] for time-varying continuous-time systems. The proof of Lemma 1.11 can be derived from the proof of [82, Lemma 4.5] by minor modifications and is therefore only sketched: Following the proof of [82,

Lemma 4.5], the idea for the stability part is that for given  $\varepsilon > 0$ , we can set  $\delta := \min\{c, \gamma^{-1}(\varepsilon)\}$  to satisfy the  $\varepsilon, \delta$ -criterion of Definition 1.10. Similarly, global asymptotic stability is obtained as  $x(k, \xi) \leq \beta(\|\xi\|, 0)$  implies stability of the origin, and

$$0 \leq \lim_{k \rightarrow \infty} \|x(k, \xi)\| \leq \lim_{k \rightarrow \infty} \beta(\|\xi\|, k) = 0$$

implies global attractivity of the origin, since  $\beta \in \mathcal{KL}$  is strictly decreasing to zero in the second argument.

For discrete-time systems with inputs of the form (1.4) we introduce the notion of *input-to-state stability* in Chapter 3, which is defined similarly to GAS of the origin in (1.6).

In contrast to continuous-time systems, the existence and uniqueness of solutions for discrete-time systems of the form (1.4) (or (1.5)) is guaranteed since  $G(x, u)$  is well-defined and unique for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ .

In order to check stability of the origin of the discrete-time system (1.5), the function  $\tilde{G}$  has to be continuous in zero, which follows directly from Definition 1.10. Thus, a common regularity condition is that the right-hand side function  $\tilde{G}$  in (1.5) (resp.  $G$  in (1.4)) is assumed to be continuous (see e.g. [71]). In this thesis, we also allow for discontinuities of the function  $\tilde{G}$  (resp.  $G$ ). We thus give the following definition, where the property defined therein will serve as a standing assumption in the remainder of this thesis.

**Definition 1.12.** A function  $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called *globally  $\mathcal{K}$ -bounded* if for some given norm  $\|\cdot\|$  there exists a class  $\mathcal{K}$ -function  $\omega$ , such that for all  $x \in \mathbb{R}^n$  we have

$$\|\tilde{G}(x)\| \leq \omega(\|x\|).$$

Firstly, note that the  $\mathcal{K}$ -function  $\omega$  depends on the norm  $\|\cdot\|$ . As all norms on  $\mathbb{R}^n$  are equivalent it is easy to see that the characterization of the global  $\mathcal{K}$ -boundedness property in Definition 1.12 is indeed independent of the choice of the norm  $\|\cdot\|$ . Secondly, global  $\mathcal{K}$ -boundedness does not require continuity of the map  $\tilde{G}(\cdot)$  (except at  $x = 0$ , which is a necessary condition for stability of the origin). On the other hand, any continuous map  $\tilde{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $\tilde{G}(0) = 0$  is  $\mathcal{K}$ -bounded. Furthermore, global  $\mathcal{K}$ -continuity implies that the origin is an equilibrium point of the discrete-time system (1.5). If the origin is GAS we can derive an obvious global  $\mathcal{K}$ -bound  $\omega$  from (1.6) as

$$\|\tilde{G}(\xi)\| = \|x(1, \xi)\| \leq \beta(\|\xi\|, 1) =: \omega(\|\xi\|).$$

Accordingly, for the discrete-time system (1.4) with inputs we define global  $\mathcal{K}$ -boundedness as follows.

**Definition 1.13.** The function  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  in (1.4) is called *globally  $\mathcal{K}$ -bounded* if for given norms  $\|\cdot\|_{\mathbb{R}^n}$  on  $\mathbb{R}^n$  and  $\|\cdot\|_{\mathbb{R}^m}$  on  $\mathbb{R}^m$  there exist  $\mathcal{K}$ -functions  $\omega_1$  and  $\omega_2$  that satisfy

$$\|G(x, u)\|_{\mathbb{R}^n} \leq \omega_1(\|x\|_{\mathbb{R}^n}) + \omega_2(\|u\|_{\mathbb{R}^m})$$

for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ .

Again, the property of global  $\mathcal{K}$ -boundedness of the function  $G$  is independent of the norms  $\|\cdot\|_{\mathbb{R}^n}$  and  $\|\cdot\|_{\mathbb{R}^m}$ ; only the  $\mathcal{K}$ -functions  $\omega_1$  and  $\omega_2$  depend on the choice of the norms. Moreover, global  $\mathcal{K}$ -boundedness of  $G$  immediately implies  $G(0, 0) = 0$ , and that  $G$  is continuous in  $(0, 0)$ .

We will see in Chapter 3 that any *input-to-state stable* discrete-time system of the form (1.4) is also globally  $\mathcal{K}$ -bounded. Thus, stability analysis of discrete-time systems under the assumption of global  $\mathcal{K}$ -boundedness is not restrictive while allowing for discontinuous dynamics.

## 1.5 Graphs

In this section we start by giving a formal definition of a directed graph, which is strongly related to the theory of nonnegative matrices, see Appendix A.1 and e.g. [8]. The correspondence derived can be extended to matrices of the form  $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  as in [123]. This is an essential idea in relating networks or interconnections of dynamical systems to directed graphs.

**Definition 1.14.** A *directed graph*  $G(\mathbf{V}, \mathbf{E})$  consists of a finite set of *vertices*  $\mathbf{V}$  and a set of *edges*  $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ . If  $G(\mathbf{V}, \mathbf{E})$  consists of  $N$  vertices, then we may identify  $\mathbf{V} = \{1, \dots, N\}$ . So if  $(i, j) \in \mathbf{E}$  then there is an edge from  $j$  to  $i$ .

We call the directed graph  $G(\mathbf{V}, \mathbf{E})$  *strongly connected* if for each pair  $(i, j) \in \mathbf{V} \times \mathbf{V}$  there exists a *path*

$$((i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k))$$

with  $i = i_0, j = i_k$  such that  $(i_{l-1}, i_l) \in \mathbf{E}$  for all  $i \in \{1, \dots, k\}$ .

In other words, a directed graph is strongly connected if every vertex can be reached from any other vertex along a path of (directed) edges.

To any directed graph  $G = G(\mathbf{V}, \mathbf{E})$  we can assign a matrix representing the graph.

**Definition 1.15.** The *adjacency matrix*  $A(G) = (a_{ij})_{i,j=1}^N \in \mathbb{R}_+^{N \times N}$  of a directed graph  $G$  is defined by  $a_{ij} = 1$  if  $(j, i) \in \mathbf{E}$  and  $a_{ij} = 0$  else.

A matrix  $A \in \mathbb{R}_+^{N \times N}$  is called *reducible* if there exists a permutation matrix  $P$  such that

$$A = P^T \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} P$$

for suitable, square matrices  $B$  and  $D$ . Else, we call  $A$  *irreducible*.

On the other hand, to any given nonnegative matrix  $A \in \mathbb{R}_+^{N \times N}$  we can associate a directed graph  $G(A)$  by setting  $\mathbf{V} := \{1, \dots, N\}$  and  $\mathbf{E} := \{(i, j) : a_{ji} > 0\}$ . The entries  $a_{ji}$  are called *weights* (of the edges), and the associated directed graph is called a *weighted directed graph*. Then the following relation holds.

**Theorem 1.16** ([8, Theorem 2.2.7]). *A matrix  $A \in \mathbb{R}_+^{N \times N}$  is irreducible if and only if  $G(A)$  is strongly connected.*

The significance of Theorem 1.16 lies in the fact that the strong connectedness of the graph can be ensured by a purely algebraic property.

Next, we consider matrices  $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ . Whereas nonnegative matrices consist only of positive or zero entries, the matrix  $\Gamma$  has functions as entries, which are either of class  $\mathcal{K}_\infty$  (in particular, positive definite) or the zero function. Thus, we can define an adjacency matrix  $A(\Gamma) = (a_{ij})_{i,j=1}^N$  by setting  $a_{ij} = 1$  if  $\gamma_{ij} \in \mathcal{K}_\infty$  and  $a_{ij} = 0$  if  $\gamma_{ij} \equiv 0$ . We call  $\Gamma$  irreducible if the matrix  $A(\Gamma)$  is.

In a directed graph corresponding to a matrix  $\Gamma = (\gamma_{ij})_{i,j=1}^N$ , paths from a vertex to itself are denoted as cycles:

**Definition 1.17.** A *k-cycle* in a matrix  $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  is a sequence of  $\mathcal{K}_\infty$ -functions

$$(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_{k-1} i_k})$$

of length  $k$ , i.e.,  $\gamma_{i_l i_{l+1}} \in \mathcal{K}_\infty$  for  $l \in \{0, \dots, k-1\}$ ,  $i_l \in \{1, \dots, N\}$  and  $i_0 = i_k$ . If  $i_l \neq i_j$  for all  $j \neq l$  other than  $i_0 = i_k$  then the  $k$ -cycle is called *minimal*.

If for each  $k \in \{1, \dots, N\}$  each  $k$ -cycle satisfies

$$\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \dots \circ \gamma_{i_{k-1} i_k} < \text{id}$$

for all  $i_0, \dots, i_k \in \{1, \dots, N\}$  with  $i_0 = i_k$  and  $k \leq N$ , then  $\Gamma$  is said to satisfy the *cycle condition*.

We call a function  $\gamma \in (\mathcal{K}_\infty \cup \{0\})$  *weakly contracting* if  $\gamma(t) < t$  for all  $t > 0$  or for short  $\gamma < \text{id}$ . Thus, the cycle condition says that any cycle in  $\Gamma$  is weakly contracting.

In Chapter 2 and Chapter 3 we study stability properties of networks (i.e., interconnections) of dynamical systems. The considered networks consist of  $N$  interconnected subsystems, which can be seen as a directed graph with vertex set



$\mathbf{V} = \{1, \dots, N\}$ , where each vertex corresponds to a subsystem. Moreover, the set of edges  $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$  is defined by the interconnection structure, i.e., if system  $j$  directly influences system  $i$ , then  $(i, j) \in \mathbf{E}$ .

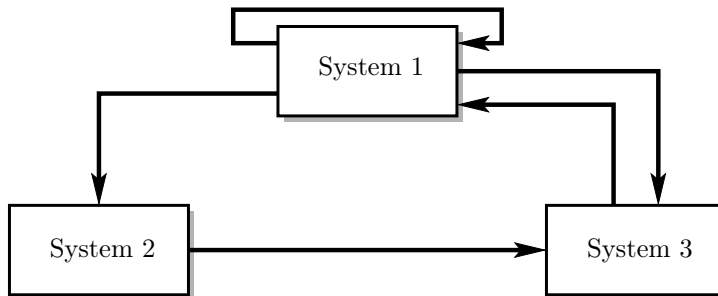
*Remark 1.18.* In small-gain theory, the classical approach is to assume that the interconnected systems have a disturbing influence on each other. In a qualitative form (e.g. (3.33)), this leads to a set of  $\mathcal{K}_\infty$ -functions  $\gamma_{ij}$  describing how much system  $i$  is affected by system  $j$ . Note that if system  $j$  is not affecting system  $i$ , then we set  $\gamma_{ij} = 0$ , where 0 denotes the zero function. In other words, there is no edge from  $j$  to  $i$  in the corresponding directed graph. Moreover, the  $\mathcal{K}_\infty$ -functions  $\gamma_{ij}$  are weights of the edges. Thus, the weighted directed graph is completely determined by the matrix  $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ .

From the viewpoint of interconnected systems, the  $\mathcal{K}_\infty$ -functions  $\gamma_{ij}$  correspond to *interconnection gains*. Thus, the matrix  $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  is usually called *gain matrix*.  $\triangleleft$

**Example 1.19.** Consider the interconnected discrete-time system given by

$$\begin{aligned} x_1(k+1) &= g_1(x_1(k), x_3(k)) \\ x_2(k+1) &= g_2(x_1(k)) \\ x_3(k+1) &= g_3(x_1(k), x_2(k)) \end{aligned} \quad k \in \mathbb{N}.$$

The right-hand side  $g_1$  of subsystem 1 depends on the states of system 1 and 3. Thus, in the corresponding interconnection graph there is an edge from system (vertex) 1 and 3 to system 1, but no edge from system 2 to system 1. The whole interconnection graph is depicted below.



We observe that the interconnection graph is strongly connected as any vertex can be reached by any other vertex.  $\triangleleft$

## 1.6 Small-gain conditions

In this section, we state so-called small-gain conditions that are used in the remainder of this work to impose stability criteria for the overall system, based on  $\mathcal{K}_\infty$ -functions derived from the subsystems. To formulate general small-gain conditions we use the following definition taken from [123].

**Definition 1.20.** A continuous function  $\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$  is called a *monotone aggregation function* if it satisfies

- (i) positive definiteness:  $\mu(s) \geq 0$  for all  $s \in \mathbb{R}_+^N$  and  $\mu(s) = 0$  iff  $s = 0$ ;
- (ii) increase:  $\mu(s_1) < \mu(s_2)$  if  $s_1 \leq s_2$ ,  $s_1 \neq s_2$ ;
- (iii) unboundedness:  $\mu(s) \rightarrow \infty$ , as  $\|s\| \rightarrow \infty$ ;

The space of monotone aggregation functions is denoted by  $\text{MAF}_N$ .

We observe that for any  $\mu \in \text{MAF}_N$ , and any  $i \in \{1, \dots, N\}$ , the function

$$v_i(r) := \mu(re_i) \tag{1.7}$$

is of class  $\mathcal{K}_\infty$ . Here,  $e_i$  denotes the  $i$ th unit vector. In this respect, the notion of monotone aggregation functions extends the notion of  $\mathcal{K}_\infty$ -functions.

The properties in Definition 1.20 can be extended to vectors in the sense that  $\mu = (\mu_1, \dots, \mu_N) \in \text{MAF}_N^N$ ,  $\mu_i \in \text{MAF}_N$ ,  $i \in \{1, \dots, N\}$ , defines a mapping  $\mu : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}^N$  by

$$A = (a_{ij})_{i,j=1}^N \mapsto \mu(A) := \begin{pmatrix} \mu_1(a_{11}, \dots, a_{1N}) \\ \vdots \\ \mu_N(a_{N1}, \dots, a_{NN}) \end{pmatrix}.$$

Generalizing this concept to matrices of the form  $\Gamma = (\gamma_{ij})_{i,j=1}^N \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ , we obtain the so-called *gain operator*  $\Gamma_\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  defined by

$$\Gamma_\mu(s) := (\mu \circ \Gamma)(s) := \begin{pmatrix} \mu_1(\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N)) \\ \vdots \\ \mu_N(\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N)) \end{pmatrix} \tag{1.8}$$

For the  $k$  times composition of this operator we write  $\Gamma_\mu^k$ . We call an operator of the form  $\Gamma_\mu$

- (i) *monotone* if  $\Gamma_\mu(s_1) \leq \Gamma_\mu(s_2)$  for all  $s_1, s_2 \in \mathbb{R}_+^N$  with  $s_1 \leq s_2$ ;
- (ii) *strictly increasing* if  $\Gamma_\mu(s_1) < \Gamma_\mu(s_2)$  for all  $s_1, s_2 \in \mathbb{R}_+^N$  with  $s_1 < s_2$ .

Note that if  $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  and  $\mu \in \text{MAF}_N^N$ , then  $\Gamma_\mu$  is monotone and satisfies  $\Gamma_\mu(0) = 0$ , see [122, Lemma 1.2.3].

For  $\delta_i \in \mathcal{K}_\infty$ ,  $D_i = (\text{id} + \delta_i)$ ,  $i \in \{1, \dots, N\}$ , we define the *diagonal operator*  $D : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  by

$$D(s) := (D_1([s]_1), \dots, D_N([s]_N)). \quad (1.9)$$

Now we are ready to state small-gain conditions that are significant for the remainder of this thesis.

**Definition 1.21.** The map  $\Gamma_\mu$  from (1.8) is said to satisfy the *small-gain condition* if

$$\Gamma_\mu(s) \not\geq s \quad \text{for all } s \in \mathbb{R}_+^N \setminus \{0\}. \quad (1.10)$$

The map  $\Gamma_\mu$  is said to satisfy the *strong small-gain condition* if there exists a diagonal operator  $D$  as in (1.9) such that

$$(D \circ \Gamma_\mu)(s) \not\geq s \quad \text{for all } s \in \mathbb{R}_+^N \setminus \{0\}. \quad (1.11)$$

The condition  $\Gamma_\mu(s) \not\geq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$ , or for short  $\Gamma_\mu \not\geq \text{id}$ , means that for any  $s > 0$  there exists at least one component  $i^* \in \{1, \dots, N\}$  such that  $[\Gamma_\mu(s)]_{i^*} < [s]_{i^*}$  holds. We can further assume that all  $\mathcal{K}_\infty$ -functions  $\delta_i$  of the diagonal operator  $D$  are identical by setting  $\delta(s) := \min_i \delta_i(s)$ . For short, we write  $D = \text{diag}(\text{id} + \delta)$ .

For any diagonal operator  $D = \text{diag}(\text{id} + \delta)$  with  $\delta \in \mathcal{K}_\infty$  there exist functions  $\delta_I, \delta_{II} \in \mathcal{K}_\infty$  such that for  $D_i := \text{diag}(\text{id} + \delta_i)$ ,  $i \in \{I, II\}$  it holds  $D = D_{II} \circ D_I$ , see [122, Lemma 1.1.4]. Moreover, we have the following equivalences of the strong small-gain condition for  $\Gamma_\mu$  from (1.8), which was proved in [122, Lemma 2.2.12]:

$$D \circ \Gamma_\mu \not\geq \text{id} \iff D_I \circ \Gamma_\mu \circ D_{II} \not\geq \text{id} \iff \Gamma_\mu \circ D \not\geq \text{id}. \quad (1.12)$$

For a given function  $\Gamma_\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ , the *set of decay*  $\Omega$  is defined by

$$\Omega := \{s \in \mathbb{R}_+^N : \Gamma_\mu(s) < s\}. \quad (1.13)$$

Points in  $\Omega$  are called *decay points*. The set  $\Omega$  is *radially unbounded* if for any  $x \in \mathbb{R}_+^N$  there exists a  $y \in \Omega$  such that  $x \leq y$ , see [123].

**Definition 1.22.** A continuous path  $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathcal{K}_\infty^N$  is called an  *$\Omega$ -path with respect to  $\Gamma_\mu : \mathbb{R}^N \rightarrow \mathbb{R}^N$*  if the following conditions are satisfied:

- (i) for each  $i$  the function  $\sigma_i^{-1}$  is locally Lipschitz continuous on  $(0, \infty)$ ;
- (ii) for every compact set  $K \subset (0, \infty)$  there are constants  $0 < c < C$  such that for all  $i \in \{1, \dots, N\}$  and all points of differentiability of  $\sigma_i^{-1}$  and we have

$$0 < c \leq (\sigma_i^{-1})'(r) < C \quad \text{for all } r \in K;$$

(iii)  $\sigma(r) \in \Omega(\Gamma_\mu)$  for all  $r > 0$ , i.e.,  $\Gamma_\mu(\sigma(r)) < \sigma(r)$  for all  $r > 0$ .

The following lemmas summarize some relations between the existence of  $\Omega$ -paths and the (strong) small-gain condition.

**Lemma 1.23.** *Let  $\Gamma_\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  be the monotone gain operator defined in (1.8) and  $D$  defined in (1.9). If there exists an  $\Omega$ -path  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  with respect to  $\Gamma_\mu$  (resp.  $D \circ \Gamma_\mu$ ) then  $\Gamma_\mu$  satisfies the (strong) small-gain condition (1.10) (resp. (1.11)).*

*Proof.* This follows from [123, Lemma 5.1] by noticing the following two facts: Firstly, the set  $\Omega$  is radially unbounded since  $\sigma_i \in \mathcal{K}_\infty$  for all  $i \in \{1, \dots, N\}$ . Secondly, there cannot exist a fixed point of  $\Gamma_\mu$  (resp.  $D \circ \Gamma_\mu$ ) despite the origin, as the  $\Omega$ -path is strictly decreasing.  $\square$

The converse implication in Lemma 1.23 is to the author's knowledge still not fully elaborated. Nevertheless, under additional, reasonable assumptions, the converse of Lemma 1.23 can be shown as illustrated next.

**Lemma 1.24.** *Let  $D$  be given by (1.9) and assume that the monotone gain operator  $\Gamma_\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  from (1.8) satisfies the strong small-gain condition (1.11). If  $\mu_i \in \text{MAF}_N$ ,  $i \in \{1, \dots, N\}$ , is sub-additive or  $\Gamma_\mu$  is irreducible, then there exists an  $\Omega$ -path  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  with respect to  $\hat{D} \circ \Gamma_\mu$ , where  $\hat{D} = \text{diag}(\text{id} + \hat{\delta})$ ,  $\hat{\delta} \in \mathcal{K}_\infty$ . In particular, we can choose  $\hat{D} = D$  if  $\Gamma$  is irreducible.*

*Proof.* If  $\mu_i$  is sub-additive then the result follows from [123, Theorem 5.10]. In particular, if  $\Gamma$  is irreducible, then the strong small-gain condition even implies the existence of an  $\Omega$ -path with respect to  $D \circ \Gamma_\mu$ , cf. [25, Theorem 5.2(ii)].  $\square$

*Remark 1.25.* In general, checking the (strong) small-gain condition (1.10) (resp. (1.11)) is nontrivial. One way is to make use of the previous lemmata, which show that if the underlying directed graph is strongly connected then the strong small-gain condition is equivalent to the existence of an  $\Omega$ -path  $\sigma$ . Hence, verification of the small-gain condition is performed (at least locally) by constructing an  $\Omega$ -path  $\sigma$ . The construction consists of two parts:

- (i) In a first step, a *decay point*  $w^*$  is computed, i.e., a point in the set of decay  $\Omega$  defined in (1.13), see [37, 125].
- (ii) In a second step, the (local)  $\Omega$ -path  $\sigma$  is constructed on  $(0, w^*]$  by piecewise linear interpolation of the sequence  $\{\Gamma_\mu^k(w^*)\}_{k \in \mathbb{N}}$ , see e.g. [123].

The same procedure applies to verify the strong small-gain condition, where the gain operator  $\Gamma_\mu$  is replaced by  $D \circ \Gamma_\mu$ . We emphasize that a direct consequence of (1.12)

is the following equivalence:

$$\begin{aligned}
 & \sigma_1 \in \mathcal{K}_\infty^N \quad \text{satisfies} \quad (D \circ \Gamma_\mu)(\sigma_1) < \sigma_1 \\
 \iff & \sigma_2 := D_{II}^{-1} \circ \sigma_1 \in \mathcal{K}_\infty^N \quad \text{satisfies} \quad (D_I \circ \Gamma_\mu \circ D_{II})(\sigma_2) < \sigma_2 \\
 \iff & \sigma_3 := D^{-1} \circ \sigma_1 \in \mathcal{K}_\infty^N \quad \text{satisfies} \quad (\Gamma_\mu \circ D)(\sigma_3) < \sigma_3.
 \end{aligned}$$

In some cases, there exist further equivalent characterizations of the small-gain condition that give the possibility of a simpler verification. These are the linear case (Lemma 1.27), the maximization case (Proposition 1.29), and the max linear case (Remark 1.31) that will be discussed in the remainder of this section.  $\triangleleft$

The interest in the gain operator of the form (1.8) lies in the fact that for large-scale interconnected systems of the form  $\dot{x}_i = f_i(x_1, \dots, x_N, u)$  ISS (input-to-state stability) conditions may be written in the form

$$\|x(t)\|_{\text{vec}} \leq \beta(\|x(0)\|_{\text{vec}}, t) + \Gamma_\mu(\|x\|_{[0,t]}) + \gamma(\|u\|_\infty),$$

see [24, Equation (3.19)], where the inequality is understood component-wise for  $\|x\|_{\text{vec}} = (\|x_1\|, \dots, \|x_N\|)$ . In particular, we consider inequalities of this type in Section 3.4.

Summation and maximization, as special cases of monotone aggregation, are in a way easier to treat. That is why in [122, 123] the author additionally requires sub-additivity of the monotone aggregation functions. This property is then used to show that upper bounds in an additive form can always be obtained. More precisely, for every  $\tilde{\mu} \in \text{MAF}_{N+1}$ ,  $\tilde{\beta} \in \mathcal{KL}$ , and  $\tilde{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ ,  $i \in \{1, \dots, N\}$ , there exist  $\mu \in \text{MAF}_{N-1}$ ,  $\beta \in \mathcal{KL}$ , and  $\gamma_i \in \mathcal{K}_\infty \cup \{0\}$ ,  $i \in \{1, \dots, N\}$ , such that for all  $r, t, \in \mathbb{R}_+$ ,  $u \in \mathbb{R}_+^N$  we have

$$\begin{aligned}
 & \tilde{\mu} \left( \tilde{\beta}(r, t), \tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_N([u]_N) \right) \\
 & \leq \beta(r, t) + \mu(\tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_{N-1}([u]_{N-1})) + \gamma_N([u]_N).
 \end{aligned} \tag{1.14}$$

This can be seen by setting  $\beta = \tilde{\mu}(\tilde{\beta}, 0, \dots, 0)$ ,  $\gamma_N := \tilde{\mu}(0, \dots, 0, \tilde{\gamma}_N)$ , and  $\mu([s]_1, \dots, [s]_{N-1}) = \tilde{\mu}(0, [s]_1, \dots, [s]_{N-1}, 0)$  (under a certain compatibility assumption [123, Assumption 2.3]).

We do now show that there also exist  $\bar{\mu} \in \text{MAF}_{N-1}$ ,  $\bar{\beta} \in \mathcal{KL}$ ,  $\bar{\gamma}_N \in \mathcal{K}_\infty \cup \{0\}$  such that the inequality

$$\begin{aligned}
 & \tilde{\mu} \left( \tilde{\beta}(r, t), \tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_N([u]_N) \right) \\
 & \leq \max \left\{ \bar{\beta}(r, t), \bar{\mu}(\tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_{N-1}([u]_{N-1})), \bar{\gamma}_N([u]_N) \right\}
 \end{aligned} \tag{1.15}$$

stays true without the assumption of sub-additivity. Note that the inequality (1.15) is stronger than inequality (1.14) as (1.14) is implied by (1.15).

**Proposition 1.26.** *For any  $\tilde{\mu} \in \text{MAF}_{N+1}$ ,  $\tilde{\beta} \in \mathcal{KL}$ , and  $\tilde{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ ,  $i \in \{1, \dots, N\}$  there exist  $\bar{\mu} \in \text{MAF}_{N-1}$ ,  $\bar{\beta} \in \mathcal{KL}$ ,  $\bar{\gamma}_N \in \mathcal{K}_\infty \cup \{0\}$  such that (1.15) holds.*

*Proof.* Define the function  $\tilde{\gamma} : \mathbb{R}_+^{N-1} \rightarrow \mathbb{R}_+$  by

$$\tilde{\gamma}(u) := \max_{i \in \{1, \dots, N-1\}} \tilde{\gamma}_i([u]_i)$$

and note (by considering the maximum of  $\tilde{\beta}$  and  $\tilde{\gamma}_i$ ) that the following inequality holds:

$$\begin{aligned} & \tilde{\mu} \left( \tilde{\beta}(r, t), \tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_N([u]_N) \right) \\ & \leq \max \left\{ \tilde{\mu} \left( \tilde{\beta}(r, t), \dots, \tilde{\beta}(r, t) \right), \tilde{\mu}(\tilde{\gamma}(u), \tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_{N-1}([u]_{N-1}), \tilde{\gamma}(u)), \right. \\ & \quad \left. \tilde{\mu}(\tilde{\gamma}_N([u]_N), \dots, \tilde{\gamma}_N([u]_N)) \right\} \\ & =: \max \left\{ \bar{\beta}(r, t), \bar{\mu}(\tilde{\gamma}_1([u]_1), \dots, \tilde{\gamma}_{N-1}([u]_{N-1})), \tilde{\gamma}_N([u]_N) \right\}, \end{aligned}$$

where

- (i)  $\bar{\beta}(r, t) := \tilde{\mu} \left( \tilde{\beta}(r, t), \dots, \tilde{\beta}(r, t) \right)$  is of class  $\mathcal{KL}$ ,
- (ii)  $\bar{\mu}([s]_1, \dots, [s]_{N-1}) := \tilde{\mu} \left( \left( \max_{i \in \{1, \dots, N-1\}} [s]_i \right), [s]_1, \dots, [s]_{N-1}, \left( \max_{i \in \{1, \dots, N-1\}} [s]_i \right) \right)$  is of class  $\text{MAF}_{N-1}$ , and
- (iii)  $\bar{\gamma}_N([s]_N) := \tilde{\mu}(\tilde{\gamma}_N([s]_N), \dots, \tilde{\gamma}_N([s]_N))$  is of class  $\mathcal{K}_\infty \cup \{0\}$ . □

In the following subsections we study two important cases of monotone aggregation functions, namely summation and maximization, in more detail.

### 1.6.1 Summation of linear gains

If we consider the monotone aggregation functions  $\mu_i(s) = \sum_{j=1}^N [s]_j$  for all  $s \in \mathbb{R}_+^N$  and  $i \in \{1, \dots, N\}$  then we obtain the corresponding gain operator  $\Gamma_\Sigma : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  as

$$\Gamma_\Sigma(s) := \begin{pmatrix} \gamma_{11}([s]_1) + \dots + \gamma_{1N}([s]_N) \\ \vdots \\ \gamma_{N1}([s]_1) + \dots + \gamma_{NN}([s]_N) \end{pmatrix}.$$

This case is called the *summation case* denoted by  $\mu = \Sigma$ . If the gains  $\gamma_{ij}$ ,  $i, j \in \{1, \dots, N\}$ , are linear functions, the gain operator is of the form  $\Gamma_\Sigma(s) = \Gamma s$  with  $\Gamma \in \mathbb{R}_+^{N \times N}$ . In particular,  $\Gamma_\Sigma$  is a linear map and we call it the *linear summation case*. In this case, we have the following equivalences of the small-gain condition (see [123, Lemma 1.1] and [24, Section 4.5]).

**Lemma 1.27.** *Let  $\Gamma \in \mathbb{R}_+^{N \times N}$ ,  $\mu = \Sigma$ , and  $\Gamma_\Sigma(s) = \Gamma s$ . Then the following are equivalent:*

- (i)  $\rho(\Gamma) < 1$ , where  $\rho(\Gamma)$  denotes the spectral radius;
- (ii)  $\Gamma s \not\geq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$ ;
- (iii)  $\Gamma^k \rightarrow 0$  for  $k \rightarrow \infty$ ;
- (iv) the system  $x(k+1) = \Gamma x(k)$  is globally asymptotically stable.

Indeed, the small-gain condition  $\Gamma_\mu \not\geq \text{id}$  in (1.10), originating from [24], stems from the linear case, and is in fact, under the assumption of irreducibility of  $\Gamma_\mu$ , equivalent to the equilibrium  $x^* = 0$  of the system  $x(k+1) = \Gamma_\mu(x(k))$  being GAS, see [24, Theorem 5.6].

*Remark 1.28* ([122]). In the linear case the small-gain condition and the strong small-gain condition are equivalent, which can be seen by setting  $D = \text{diag}((1+\epsilon) \text{id})$  with  $\epsilon > 0$  small enough.  $\triangleleft$

In the linear summation case an  $\Omega$ -path can be easily computed via the *Perron-Frobenius eigenvector*. We refer to the Appendix A.1, where this procedure is outlined.

### 1.6.2 Maximization of gains

If we consider the monotone aggregations functions  $\mu_i(s) = \max_{j \in \{1, \dots, N\}} [s]_j$  for all  $i \in \{1, \dots, N\}$  then we obtain the corresponding gain operator  $\Gamma_\oplus : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  as

$$\Gamma_\oplus(s) := \begin{pmatrix} \max\{\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N)\} \\ \vdots \\ \max\{\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N)\} \end{pmatrix}. \quad (1.16)$$

We call this case the *maximization case* and denote it by  $\mu = \oplus$ .

For the map  $\Gamma_\oplus$  defined in (1.16) we have the following equivalence, which gives the possibility to check the small-gain condition (1.10) (see [123, Theorem 6.4]).

**Proposition 1.29.** *The map  $\Gamma_\oplus : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  defined in (1.16) satisfies the small-gain condition (1.10) if and only if all cycles in the corresponding graph of  $\Gamma_\oplus$  are weakly contracting, i.e.,  $\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \dots \circ \gamma_{i_k i_0} < \text{id}$  for  $k \in \mathbb{N}$  and  $i_j \neq i_l$  for  $j \neq l$ .*

In the maximization case the converse implication of Lemma 1.23 holds true. For a proof see [25, Theorem 5.2(iii)].

**Lemma 1.30.** *Let  $\Gamma_\oplus : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  be the monotone gain operator from (1.16). If  $\Gamma_\oplus$  satisfies the small-gain condition (1.10) then there exists an  $\Omega$ -path with respect to  $\Gamma_\oplus$ .*

*Remark 1.31* ([123]). In the case where all  $\gamma_{ij}$  are linear,  $\Gamma_{\oplus}$  is a max linear operator (cf. [104]). The cycle condition is equivalent to the maximum cycle geometric mean  $\hat{\mu}(\Gamma)$  being less than one. The maximum cycle geometric mean is defined as the maximum of all  $k$ th roots of the  $k$ -cycles in  $\Gamma \in \mathbb{R}_+^{N \times N}$ ,  $k \leq N$ . Further, the maximum cycle geometric mean is a *max eigenvalue* of  $\Gamma$ , i.e., there exists a  $v \in \mathbb{R}_+^N$  such that

$$\Gamma \oplus v = \hat{\mu}(\Gamma)v \iff \max_{j \in \{1, \dots, N\}} \gamma_{ij}[v]_j = \hat{\mu}(\Gamma)[v]_i, \quad i \in \{1, \dots, N\}.$$

In particular,  $\hat{\mu}(\Gamma) \leq \rho(\Gamma)$  where  $\rho(\Gamma)$  denotes the spectral radius of  $\Gamma$ . ◁



# Stability analysis of large-scale discrete-time systems

In this chapter, we are interested in the stability analysis of discrete-time systems of the form

$$x(k+1) = G(x(k)), \quad k \in \mathbb{N},$$

where  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . In particular, we derive criteria to ensure *global asymptotic stability* (GAS) of the origin.

As outlined in the Introduction, GAS of the origin is equivalent to the existence of a Lyapunov function [105]. The existence of a Lyapunov function is ensured by *converse Lyapunov theorems*, where the proofs do usually construct a Lyapunov function by taking infinite series or the supremum over all solutions, see e.g. [71, 81, 116]. Thus, these converse Lyapunov functions are important from a theoretical point of view, but they do not lead to constructive methods. In general, Lyapunov functions for nonlinear systems are hard to find.

In the first part of this chapter, Section 2.2, an alternative approach to the construction of Lyapunov functions for discrete-time systems is proposed. The first ingredient of the proposed approach consists of a relaxation of the Lyapunov function concept, which was originally introduced in [1]: the Lyapunov function is allowed to decrease along the system solutions after a finite number of time steps, and not at every time step. This relaxation is thus termed *global finite-step Lyapunov function*. Firstly, we prove that the existence of a global finite-step Lyapunov function is sufficient to establish GAS of the system's origin. Secondly, a converse finite-step Lyapunov theorem is derived. This converse Lyapunov theorem is constructive as it yields an

explicit finite-step Lyapunov function. Here we use an additional assumption that is satisfied for a large class of discrete-time systems. In particular, this assumption is met if the origin is *globally exponentially stable*. Then, a way to construct a standard Lyapunov function based on the knowledge of a finite-step Lyapunov function and a corresponding natural number is given. The construction only depends on a finite sum and therefore can be implemented directly. Nevertheless, the Lyapunov function construction a priori requires the knowledge of a suitable natural number that determines the upper bound of summation. Several possibilities to systematically find such a suitable number are discussed for certain classes of systems.

In the next part of this chapter, Section 2.3, we focus on the stability analysis of *large-scale* discrete-time systems, i.e., we consider the system as an interconnection of several smaller discrete-time systems. It is generally difficult to prove global stability properties such as GAS for interconnected systems. One way to prove stability relies on the existence of Lyapunov functions for each subsystem and testing if a small-gain condition holds. However, as shown in the introduction (example in (3)), small-gain theorems come with certain conservatism. To reduce this conservatism in small-gain theory, we make use of the concept of a finite-step Lyapunov function as introduced in Section 2.1. To be more precise, we do not demand that each subsystem has to admit a Lyapunov function, and instead assume the existence of a Lyapunov-type function that decreases after a finite time. The proof of the resulting small-gain theorem is based on a construction of a finite-step Lyapunov function for the overall system from the Lyapunov-type inequalities. The distinctive feature of the proposed relaxation is the ability of the corresponding small-gain theorem to handle the case of 0-input unstable subsystems.

Eventually, as small-gain theorems are only sufficient criteria, we study the necessity of the hypothesis of the derived small-gain theorem. More specifically, we state a converse of our small-gain theorem under which a GAS system can be considered as the interconnection of subsystems that admit suitable Lyapunov-type functions and satisfy a classical small-gain condition. The converse is shown to hold under a reasonable assumption, which allows for a general class of discrete-time nonlinear systems, which includes the class of GES systems.

The last part of this chapter, Section 2.4, is to establish necessity of specific types of Lyapunov functions *via* the developed converse Lyapunov theorems from Section 2.2. Most remarkably, we show that the existence of conewise linear Lyapunov functions is sufficient and necessary for GES conewise linear systems. This has been one of the open problems in stability analysis of conewise linear systems [72]. As a by-product, a new method to construct polyhedral Lyapunov functions for linear systems is obtained, that is tractable even in state spaces of high dimension.

## 2.1 Problem statement

In this chapter we consider discrete-time systems of the form

$$x(k+1) = G(x(k)), \quad k \in \mathbb{N}, \quad (2.1)$$

where  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to satisfy the following standing assumption.

**Assumption 2.1.** The function  $G$  in (2.1) is globally  $\mathcal{K}$ -bounded.  $\triangleleft$

A definition of  $\mathcal{K}$ -boundedness is given in Definition 1.12. Recall from Section 1.4, that Assumption 2.1 is not restrictive, as it does not require continuity of the map  $G(\cdot)$  (except at  $x = 0$ , which is a necessary condition for (Lyapunov) stability of the origin). Moreover, the solution at time instance  $k \in \mathbb{N}$  starting in the initial state  $\xi \in \mathbb{R}^n$  is denoted by  $x(k, \xi) \in \mathbb{R}^n$ .

To study global asymptotic stability (GAS) of the origin of system (2.1), we recall the characterization of GAS from Lemma 1.11.

**Definition 2.2.** The origin of system (2.1) is called *globally asymptotically stable* if there exists a  $\mathcal{KL}$ -function  $\beta$  such that for all  $\xi \in \mathbb{R}^n$  and all  $k \in \mathbb{N}$  it holds

$$\|x(k, \xi)\| \leq \beta(\|\xi\|, k). \quad (2.2)$$

If the  $\mathcal{KL}$ -function in (2.2) can be chosen as

$$\beta(r, t) = C\mu^t r \quad (2.3)$$

with  $C \geq 1$  and  $\mu \in [0, 1)$ , then the origin of system (2.1) is called *globally exponentially stable* (GES).

*Remark 2.3.* (i) The GAS property in Definition 2.2 is sometimes called  $\mathcal{KL}$ -stability (e.g. [81]), and in [98, Proposition 2.5] it is shown that  $\mathcal{KL}$ -stability is equivalent to uniform global asymptotic stability (UGAS). Note that for time-invariant systems as considered in this thesis, every continuous GAS system is UGAS, see [71, Proposition 3.2].

(ii) The definition of GES is somehow misleading, since *global* only indicates that (2.2) with  $\beta$  as in (2.3) holds for all  $\xi \in \mathbb{R}^n$ . In particular, systems in which all solutions have an exponential rate of decay may fail to be GES. Since  $C \geq 1$  in (2.3) is chosen globally, it does not reflect the local behavior of a particular solution near 0. Note that this property is also often called *exponentially stable in the whole*, see e.g., [56, Sec. 2] and [101, Sec. 6.3].  $\triangleleft$

As conditions (2.2) and (2.3) imply  $\|G(\xi)\| = \|x(1, \xi)\| \leq C\mu^1 \|\xi\|$  for all  $\xi \in \mathbb{R}^n$ , we see that for any GES system the  $\mathcal{K}$ -bound  $\omega$  on  $G$  from (2.1) can always be chosen to be linear. Similarly, as already shown in Section 1.4, any GAS system is  $\mathcal{K}$ -bounded, and the  $\mathcal{K}$ -bound can be chosen as  $\omega(s) = \beta(s, 1)$ .

**Definition 2.4.** A function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a *global Lyapunov function* for system (2.1) if

- (i)  $W$  is *proper and positive definite*, i.e., there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that for all  $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq W(\xi) \leq \alpha_2(\|\xi\|),$$

- (ii) there exists a positive definite function  $\rho$  satisfying<sup>1</sup>  $\rho < \text{id}$  such that for all  $\xi \in \mathbb{R}^n$

$$W(x(1, \xi)) \leq \rho(W(\xi)).$$

*Remark 2.5.* In many prior works such as e.g., [71], the definition of a Lyapunov function requires the existence of a positive definite function  $\alpha_3$  such that

$$W(x(1, \xi)) - W(\xi) \leq -\alpha_3(\|\xi\|) \tag{2.4}$$

holds for all  $\xi \in \mathbb{R}^n$ . Let us briefly explain that a proper and positive definite function  $W$  satisfies inequality (2.4) if and only if it satisfies condition (ii) of Definition 2.4.

Assume that  $W$  satisfies an inequality of the form (2.4). By following similar steps as in [90, Theorem 2.3.5], we conclude that

$$\begin{aligned} 0 \leq W(x(1, \xi)) &\leq W(\xi) - \alpha_3(\|\xi\|) \\ &\leq (\text{id} - \alpha_3 \circ \alpha_2^{-1})(W(\xi)) = \rho(W(\xi)) \end{aligned}$$

holds with  $\rho := (\text{id} - \alpha_3 \circ \alpha_2^{-1}) \geq 0$ . We can without loss of generality assume that  $\rho$  is positive definite by possibly picking a larger  $\alpha_2$ . We further have  $0 \leq W(x(1, \xi)) \leq (\alpha_2 - \alpha_3)(\|\xi\|)$ , so  $\alpha_2 \geq \alpha_3$  and therefore  $\rho < \text{id}$ . Thus,  $W$  satisfies condition (ii) of Definition 2.4.

On the other hand, let  $W$  satisfy Definition 2.4. Then  $W$  also satisfies (2.4) with  $\alpha_3 := (\text{id} - \rho) \circ \alpha_1$  and  $\rho < \text{id}$  given from Definition 2.4.

Moreover, for the case  $\alpha_1(s) = as^\lambda, \alpha_2(s) = bs^\lambda, \alpha_3(s) = cs^\lambda$  for some  $a, b, c, \lambda > 0$  we have  $W(x(1, \xi)) \leq \rho W(\xi)$  with  $\rho := (1 - \frac{c}{b}) \in [0, 1)$ , see [90, Theorem 2.3.5].

Without loss of generality, we can assume that the positive definite function  $\rho$  in Definition 2.4 is of class  $\mathcal{K}_\infty$ . To see this, note that for any positive definite function  $\rho < \text{id}$  there exists a  $\mathcal{K}_\infty$ -function  $\tilde{\rho}$  with  $\rho < \tilde{\rho} < \text{id}$ .  $\triangleleft$

Next, the assumptions on global Lyapunov functions given in Definition 2.4 are relaxed as follows.

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<sup>1</sup>Recall that from the order relation introduced in Chapter 1 this notation means  $\rho(s) < s$  for all  $s > 0$ .

**Definition 2.6.** A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a *global finite-step Lyapunov function* for system (2.1) if

- (i)  $V$  is proper and positive definite, i.e., there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that for all  $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq W(\xi) \leq \alpha_2(\|\xi\|),$$

- (ii) there exists a finite  $M \in \mathbb{N}$  and a positive definite function  $\rho < \text{id}$  such that for all  $\xi \in \mathbb{R}^n$

$$V(x(M, \xi)) \leq \rho(V(\xi)).$$

It is worth pointing out that the concept of a global finite-step Lyapunov function was originally introduced in [1], which dealt with stability analysis of time-varying systems, although the term *finite-step* was not used therein. Clearly, any global Lyapunov function is a particular global finite-step Lyapunov function.

Observe that if  $V$  is a global finite-step Lyapunov function for system (2.1) then  $V$  is a global Lyapunov function for the system

$$\bar{x}(k+1) = G^M(\bar{x}(k)), \quad k \in \mathbb{N}, \quad \bar{x} \in \mathbb{R}^n. \quad (2.5)$$

Thus, global finite-step Lyapunov functions can be seen as global Lyapunov functions of the iterated system.

## 2.2 Stability analysis via finite-step Lyapunov functions

The aim of this section is to derive a constructive converse Lyapunov theorem for systems of the form (2.1). Firstly, we proceed by showing that any global finite-step Lyapunov function guarantees GAS of the origin. In the next step, we prove that the converse, i.e., the existence of a global finite-step Lyapunov function for a GAS system, also holds. Moreover, we can show that under appropriate conditions we can take norms as global finite-step Lyapunov functions. In particular, these conditions are satisfied for any GES system of the form (2.1). To obtain a converse Lyapunov theorem, an explicit construction of a global Lyapunov function from a global finite-step Lyapunov function is provided.

### 2.2.1 Finite-step Lyapunov functions

This section proceeds by showing that the existence of a global finite-step Lyapunov function is sufficient to conclude GAS of the origin. Recall that Assumption 2.1 is supposed to hold throughout this chapter, and thus continuity of  $G$  at the origin is implied.

**Theorem 2.7.** *The existence of a global finite-step Lyapunov function implies that the origin of system (2.1) is GAS.*

*Proof.* Assume that there exists a global finite-step Lyapunov function  $V$  as defined in Definition 2.6 with suitable  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $M \in \mathbb{N}$ ,  $M \geq 1$ , and positive definite  $\rho < \text{id}$ . First note that from the standing Assumption 2.1 we conclude that for any  $j \in \mathbb{N}$  we have

$$\|x(j, \xi)\| \leq \omega^j(\|\xi\|). \quad (2.6)$$

With this, for any  $k = lM + j$ ,  $l \in \mathbb{N}$ ,  $j \in \{0, \dots, M-1\}$  we have

$$\begin{aligned} \|x(k, \xi)\| &\leq \alpha_1^{-1}(V(x(k, \xi))) \\ &\leq \alpha_1^{-1}(V(x(lM, x(j, \xi)))) \\ &\leq \alpha_1^{-1} \circ \rho^l(V(x(j, \xi))) \\ &\leq \alpha_1^{-1} \circ \rho^l \circ \alpha_2(\|x(j, \xi)\|) \\ &\stackrel{(2.6)}{\leq} \alpha_1^{-1} \circ \rho^l \circ \alpha_2 \circ \omega^j(\|\xi\|) \\ &\leq \max_{i \in \{0, \dots, M-1\}} \alpha_1^{-1} \circ \rho^l \circ \alpha_2 \circ \omega^i(\|\xi\|) \\ &\leq \max_{i \in \{0, \dots, M-1\}} \alpha_1^{-1} \circ \rho^{\frac{k}{M}-1} \circ \alpha_2 \circ \omega^i(\|\xi\|) \\ &\leq \max_{i \in \{0, \dots, M-1\}} \alpha_1^{-1} \circ \bar{\rho}^{\frac{k}{M}-1} \circ \alpha_2 \circ \omega^i(\|\xi\|) \\ &=: \beta(\|\xi\|, k). \end{aligned}$$

Since  $\rho < \text{id}$  is positive definite there exists a  $\mathcal{K}$ -function  $\tilde{\rho}$  with  $\rho \leq \tilde{\rho} < \text{id}$ . Then it is easy to see that  $\beta$  is a  $\mathcal{KL}$ -function. By Definition 2.2 the origin of system (2.1) is GAS.  $\square$

*Remark 2.8.* An alternative proof of Theorem 2.7 can be obtained by recognizing the following equivalence for any  $M \in \mathbb{N}$

$$\text{the origin of (2.1) is GAS} \quad \iff \quad \text{the origin of (2.5) is GAS.}$$

Since any global finite-step Lyapunov function of system (2.1) is a global Lyapunov function of system (2.5), GAS of the origin of system (2.5) is implied, and with the above equivalence, also GAS of the origin of system (2.1).  $\triangleleft$

If the global finite-step Lyapunov function satisfies stronger conditions on the functions invoked then we can conclude GES of the origin.

**Corollary 2.9.** *Let the global  $\mathcal{K}$ -bound on  $G$  be  $\omega(s) = ws$  for all  $s \geq 0$  and some  $w > 0$ . Then the origin of system (2.1) is GES if there exists a global finite-step Lyapunov function  $V$  satisfying conditions (i) and (ii) of Definition 2.6 with*

$$\alpha_1(s) = a_1 s^\lambda, \quad \alpha_2(s) = a_2 s^\lambda, \quad \rho(s) = cs, \quad (2.7)$$

where  $0 < a_1 \leq a_2$ ,  $\lambda > 0$  and  $c \in [0, 1)$ .

*Proof.* The proof of Theorem 2.7 implies  $\|x(k, \xi)\| \leq C\mu^k\|\xi\|$  with  $C = \max_{i \in \{1, \dots, M-1\}} (\frac{a_2}{a_1 c})^{1/\lambda} \omega^i$  and  $\mu = c^{1/M\lambda} \in [0, 1)$ .  $\square$

Note that the assumption on  $\omega$  to be linear is necessary for the origin of the system being GES as shown after Remark 2.3.

Since any global finite-step Lyapunov function with  $M = 1$  is a global Lyapunov function, the “classical” Lyapunov theorem<sup>2</sup> can be obtained as a corollary of Theorem 2.7.

**Corollary 2.10.** *The existence of a global Lyapunov function  $W$  implies that the origin of system (2.1) is GAS. In particular, if  $W$  satisfies conditions (i) and (ii) of Definition 2.4 with  $\alpha_1, \alpha_2, \rho$  as in (2.7) then the origin of system (2.1) is GES.*

The following result states that if the map  $G$  is *contracting* (i.e.,  $G$  is globally  $\mathcal{K}$ -bounded with  $\omega < \text{id}$ ), then the origin of system (2.1) is GAS.

**Proposition 2.11.** *If the map  $G$  in (2.1) is globally  $\mathcal{K}$ -bounded with  $\mathcal{K}$ -function  $\omega < \text{id}$ , then system (2.1) is GAS.*

*Proof.* Take  $W(\xi) = \|\xi\|$  as a candidate Lyapunov function. Then clearly condition (i) of Definition 2.4 is satisfied with  $\alpha_1 = \alpha_2 = \text{id}$  and condition (ii) of Definition 2.4 is satisfied with  $\rho = \omega$  and  $\omega < \text{id}$ . So Corollary 2.10 applies, which concludes the proof.  $\square$

In this section we have shown that the existence of a global finite-step Lyapunov function is sufficient to conclude GAS of the origin. In the next section we proceed by showing that the existence of a global finite-step Lyapunov function is also necessary for systems with GAS origin.

### 2.2.2 A converse finite-step Lyapunov theorem

Since global Lyapunov functions are global finite-step Lyapunov functions, we can make use of the converse Lyapunov theorems [71, Theorem 1] (for continuous  $G$ ) resp. [116, Lemma 4] (for discontinuous  $G$ ) to state the following proposition.

**Proposition 2.12.** *If the origin of system (2.1) is GAS, then there exists a global finite-step Lyapunov function.*

Proposition 2.12 ensures the existence of a global finite-step Lyapunov function by using a standard converse Lyapunov theorem, which is obvious. Indeed, a converse result was not pursued in [1].

---

<sup>2</sup>also called *Lyapunov’s second or direct method*

Theorem 2.7 and Proposition 2.12 together show the equivalence between the existence of a global finite-step Lyapunov function and GAS of the origin. In Figure 2.1 we present a diagram that illustrates some equivalences of global finite-step Lyapunov functions.

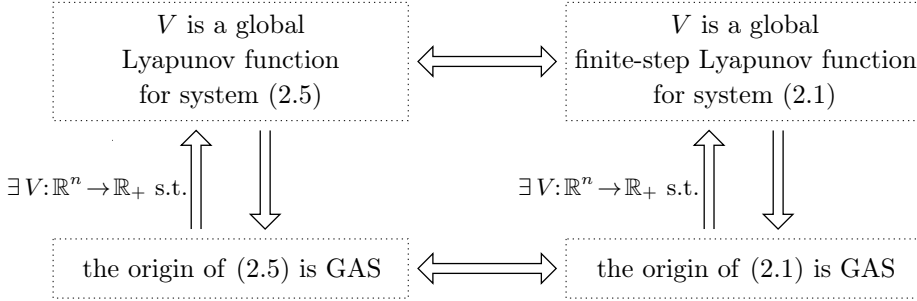


Figure 2.1: Some equivalences of global finite-step Lyapunov functions

Next, a constructive converse finite-step Lyapunov theorem is stated, which, in contrast to Proposition 2.12, does not rely on existing converse Lyapunov theorems, but on an appropriate assumption that is discussed in the following.

**Assumption 2.13.** There exists a  $\mathcal{KL}$ -function  $\beta$  satisfying (2.2) for system (2.1) and

$$\beta(r, M) < r \tag{2.8}$$

for some  $M \in \mathbb{N}$  and all  $r > 0$ . ◁

Under this assumption a global finite-step Lyapunov function can be given explicitly.

**Theorem 2.14.** *If Assumption 2.13 is satisfied, then for any function  $\eta \in \mathcal{K}_\infty$  and any norm  $\|\cdot\|$  the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$V(\xi) := \eta(\|\xi\|) \tag{2.9}$$

for all  $\xi \in \mathbb{R}^n$  is a global finite-step Lyapunov function for system (2.1).

*Proof.* Take any  $\eta \in \mathcal{K}_\infty$  and let  $V$  be defined as in (2.9). Then clearly,  $V$  is proper and positive definite. Let  $\beta \in \mathcal{KL}$  satisfy (2.2), and  $M \in \mathbb{N}$  satisfy (2.8). Then for any  $\xi \in \mathbb{R}^n$

$$\begin{aligned} V(x(M, \xi)) &= \eta(\|x(M, \xi)\|) \\ &\leq \eta \circ \beta(\|\xi\|, M) \\ &= \eta \circ \beta(\eta^{-1}(V(\xi)), M) =: \tilde{\rho}(V(\xi)), \end{aligned}$$



where  $\tilde{\rho}(\cdot) = \eta \circ \beta(\eta^{-1}(\cdot), M)$  satisfies  $\tilde{\rho} < \text{id}$  by (2.8). This shows condition (ii) of Definition 2.6. So  $V$  defined in (2.9) is a global finite-step Lyapunov function for system (2.1).  $\square$

For simplicity, we can set  $\eta = \text{id}$  in (2.9). On the other hand, as  $\eta \in \mathcal{K}_\infty$  can be chosen arbitrarily,  $\eta$  can be used as a scaling function to design the global finite-step Lyapunov function.

Let us briefly discuss Assumption 2.13. First of all, by Definition 2.6, the norm of any solution of a GES system is bounded by a  $\mathcal{KL}$ -function  $\beta(r, t) = C\mu^t r$  with  $C \geq 1, \mu \in [0, 1)$ , see (2.3). So we can find an  $M \in \mathbb{N}$  such that  $C\mu^M < 1$  by simply taking  $M \in \mathbb{N}$  with  $M > \log_\mu(1/C)$ , which immediately yields the following lemma.

**Lemma 2.15.** *If the origin of system (2.1) is GES, then Assumption 2.13 is satisfied.*

This now implies that for a GES system (2.1) any norm is a global finite-step Lyapunov function. Note that the norms in (2.2) and (2.9) coincide. In particular,  $M \in \mathbb{N}$  in (2.8) depends on the particular norm.

**Corollary 2.16.** *If the origin of system (2.1) is GES, then for any function  $\eta \in \mathcal{K}_\infty$  the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined in (2.9) is a global finite-step Lyapunov function for this system.*

*Proof.* Lemma 2.15 implies that Assumption 2.13 holds, so Theorem 2.14 applies and the result follows.  $\square$

The converse implication of Lemma 2.15, i.e., Assumption (2.13) implies that the origin of system (2.1) is GES, doesn't hold in general. This is shown in the following example.

**Example 2.17.** Consider the system

$$x(k+1) = G(x(k)) := \begin{cases} |x(k)| - x^2(k) & \text{if } |x(k)| \leq \frac{1}{2} \\ \frac{1}{2}|x(k)| & \text{if } |x(k)| > \frac{1}{2} \end{cases} \quad k \in \mathbb{N}. \quad (2.10)$$

The right-hand side function  $G : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$|G(\xi)| = \max\{|\xi| - |\xi|^2, \frac{1}{2}|\xi|\}, \quad \text{for all } \xi \in \mathbb{R}.$$

Hence,  $G$  is globally  $\mathcal{K}$ -bounded with  $\omega(s) = \max\{s - s^2, 0.5s\}$ . As  $\omega < \text{id}$ , we can apply Proposition 2.11 to conclude that the origin of system (2.10) is GAS. Moreover, for all  $k \in \mathbb{N}$  and all  $\xi \in \mathbb{R}$  we have

$$|x(k, \xi)| = |G^k(\xi)| = \omega^k(|\xi|) =: \beta(|\xi|, k),$$

where the function  $\beta$  is of class  $\mathcal{KL}$  as  $\omega \in \mathcal{K}_\infty$  satisfies  $\omega < \text{id}$ . Thus, Assumption 2.13 is satisfied for  $M = 1$ .

On the other hand, the origin of system (2.10) is not GES in the sense of Definition 2.2 as the decrease rate of any solution approaches 1, i.e.,

$$\lim_{k \rightarrow \infty} \frac{|x(k+1)|}{|x(k)|} = \lim_{k \rightarrow \infty} 1 - |x(k)| = 1.$$

◁

If we assume the origin of system (2.1) to be GAS only, then (2.8) does not have to hold globally, i.e., for all  $r > 0$ . But we can at least show that Assumption 2.13 is satisfied in a semi-global practical sense.

**Lemma 2.18.** *Let  $\beta \in \mathcal{KL}$ . Then for any  $0 < a < b < \infty$  there exists an  $M \in \mathbb{N}$ , such that (2.8) holds for all  $r \in [a, b]$ .*

*Proof.* Let  $\beta \in \mathcal{KL}$ . Using [139, Lemma 4.3] there exist  $\mathcal{K}_\infty$ -functions  $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$  such that  $\beta(r, t) \leq \sigma_1(\sigma_2(r)e^{-t})$  holds for all  $r, t \geq 0$ . Define for any  $r > 0$

$$M(r) := \min\{M \in \mathbb{N} : \sigma_1(\sigma_2(r)e^{-M}) < r\}.$$

Note that  $M(r)$  is well-defined for any  $r > 0$ . Let  $0 < a < b < \infty$  be given and define  $\bar{M} := \sup\{M(r) : r \in [a, b]\}$ . We will show that  $\bar{M} < \infty$ , which implies that for all  $r \in [a, b]$  it holds

$$\beta(r, \bar{M}) \leq \sigma_1(\sigma_2(r)e^{-\bar{M}}) \leq \sigma_1(\sigma_2(r)e^{-M(r)}) < r.$$

So assume to the contrary that  $\bar{M} = \infty$ . Then there exists a sequence  $\{r_l\}_{l \in \mathbb{N}} \in [a, b]$  such that  $\{M(r_l)\}_{l \in \mathbb{N}} \rightarrow \infty$  for  $l \rightarrow \infty$ . Since  $[a, b] \subset \mathbb{R}_+$  is compact, we can, without loss of generality, assume that the sequence  $\{r_l\}_{l \in \mathbb{N}}$  is convergent to a point  $r^* \in [a, b]$ , else take a convergent subsequence. Consequently, this means that in any open neighborhood  $U$  around  $r^*$  there exist infinitely many  $r_i \in U$  with  $M(r_i)$  pairwise distinct. On the other hand  $M(r^*)$  is well-defined, and, by continuity, there exists an open neighborhood  $\tilde{U}$  around  $r^*$  with  $M(\tilde{r}) \leq M(r^*)$  for all  $\tilde{r} \in \tilde{U}$ . But this contradicts the unboundedness of the sequence  $\{M(r_i)\}$ , where  $r_i \in U \subset \tilde{U}$ . So  $\bar{M} < \infty$ . □

This now implies that for any GAS system of the form (2.1), any scaled norm is a finite-step Lyapunov function for a set  $[a, b]$  as defined next.

**Definition 2.19.** Let  $0 < a < b < \infty$ . A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is an  $(a, b)$  *finite-step Lyapunov function* for system (2.1) if

- (i) there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that for all  $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|),$$

- (ii) there exists a finite  $M \in \mathbb{N}$  and a positive definite function  $\rho < \text{id}$  such that for all  $\xi \in V^{-1}([a, b]) := \{z \in \mathbb{R}^n : V(z) \in [a, b]\}$  we have

$$V(x(M, \xi)) \leq \rho(V(\xi)),$$

and for all  $\xi \in V^{-1}([0, a])$  we have

$$V(x(M, \xi)) \leq a.$$

Definition 2.19 implies that the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is decreasing at least any  $M$  steps towards the set  $[0, a]$  as long as  $\xi \in V^{-1}([a, b])$ . Finally, along solutions  $x(k, \xi)$  starting in  $\xi \in V^{-1}([0, a])$  we see that  $V(x(k, \xi))$  is within  $[0, a]$  at least for any  $k = lM$  with  $l \in \mathbb{N}$ . In this respect,  $(a, b)$  finite-step Lyapunov functions can be used to ensure practical asymptotic stability.

**Corollary 2.20.** *If the origin of system (2.1) is GAS, then for any function  $\eta \in \mathcal{K}_\infty$  and any  $0 < a < b < \infty$ , the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined in (2.9) is an  $(a, b)$  finite-step Lyapunov function for this system.*

*Proof.* Let  $\eta \in \mathcal{K}_\infty$  and  $0 < a < b < \infty$  be given, and define  $\hat{a} := \eta^{-1}(a)$  and  $\hat{b} := \eta^{-1}(b)$ . Then Lemma 2.18 implies the existence of an  $M \in \mathbb{N}$  such that (2.8) holds for all  $r \in [\hat{a}, \hat{b}]$ . Hence, the positive definite function  $\tilde{\rho}$  defined in the proof of Theorem 2.14 satisfies  $\tilde{\rho}(r) < r$  for all  $r \in [\eta(\hat{a}), \eta(\hat{b})]$ . Thus, for all  $\xi \in V^{-1}([\eta(\hat{a}), \eta(\hat{b})]) = V^{-1}([a, b])$  it holds

$$V(x(M, \xi)) \leq \tilde{\rho}(V(\xi)).$$

On the other hand, for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| \in [0, \hat{a}]$ , or, equivalently,  $\xi \in V^{-1}([0, a])$ , we have

$$V(x(M, \xi)) = \eta(\|x(M, \xi)\|) \leq \eta(\beta(\|\xi\|, M)) \leq \eta(\beta(\hat{a}, M)) = \eta(\hat{a}) = a.$$

Hence, the function  $V$  defined in (2.9) is an  $(a, b)$  finite-step Lyapunov function for system (2.1).  $\square$

We stress that the constant  $M \in \mathbb{N}$  chosen in Corollary 2.20 depends on the interval  $[a, b]$ . In general, a larger interval requires a larger  $M$ .

The meaning of Corollary 2.20 is that if we are not aware of a global Lyapunov function for system (2.1), we can nevertheless construct a finite-step Lyapunov function in (2.9) to ensure practical asymptotic stability of the origin. In this respect,  $M \in \mathbb{N}$  can be interpreted as a tuning parameter that regulates the size of the interval  $[a, b]$ .

### 2.2.3 Construction of a Lyapunov function from a finite-step Lyapunov function

In this section we construct a global Lyapunov function  $W$  for system (2.1) from the knowledge of a global finite-step Lyapunov function  $V$ . We like to emphasize that this construction does not depend on the particular structure of the global finite-step Lyapunov function in (2.9), but rather holds in general. The construction is as follows. For all  $\xi \in \mathbb{R}^n$  we define the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  by

$$W(\xi) := \sum_{j=0}^{M-1} V(x(j, \xi)). \quad (2.11)$$

The idea behind this construction is that the summands for  $W(\xi)$  are the same as for  $W(x(1, \xi))$  except that  $V(\xi)$  is replaced by  $V(x(M, \xi)) < V(\xi)$ . Hence,  $W$  is decreasing along trajectories of (2.1), which yields the following result.

**Theorem 2.21** (Construction of a global Lyapunov function I). *If  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a global finite-step Lyapunov function for system (2.1) with  $M \in \mathbb{N}$  satisfying condition (ii) of Definition 2.6, then  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined in (2.11) is a global Lyapunov function for system (2.1).*

*Proof.* For the global finite-step Lyapunov function  $V$  let  $\alpha_1, \alpha_2, \rho, M$  satisfy conditions (i) and (ii) of Definition 2.6. In the remainder of the proof, we will construct  $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\rho}$  such that they satisfy conditions (i) and (ii) of Definition 2.4 for the Lyapunov function candidate  $W$  defined in (2.11). Assume that  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  are given such that  $\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|)$ . Let  $\tilde{\alpha}_1 = \alpha_1$ , then by definition of  $W$  we have  $\tilde{\alpha}_1(\|\xi\|) \leq V(\xi) \leq W(\xi)$ . Since  $V$  is a global finite-step Lyapunov function we have for all  $k \in \mathbb{N}$

$$\begin{aligned} \|x(k, \xi)\| &\leq \alpha_1^{-1}(V(x(k, \xi))) \leq \max_{j \in \mathbb{N}} \alpha_1^{-1}(V(x(j, \xi))) \\ &\leq \max_{j \in \{0, \dots, M-1\}} \alpha_1^{-1}(V(x(j, \xi))) \\ &\leq \max_{j \in \{0, \dots, M-1\}} \alpha_1^{-1} \circ \alpha_2 \circ \omega^j(\|\xi\|) = \sigma(\|\xi\|), \end{aligned}$$

for  $\sigma := \max_{j \in \{0, \dots, M-1\}} \alpha_1^{-1} \circ \alpha_2 \circ \omega^j \in \mathcal{K}_\infty$ . Moreover,

$$\begin{aligned} W(\xi) &\leq M \cdot \max_{i \in \{0, \dots, M-1\}} V(x(i, \xi)) \\ &\leq M \cdot \max_{i \in \{0, \dots, M-1\}} \alpha_2(\|x(i, \xi)\|) \leq M \alpha_2 \circ \sigma(\|\xi\|). \end{aligned}$$

Define  $\tilde{\alpha}_2 = M \text{id} \circ \alpha_2 \circ \sigma$ . Hence, we have shown condition (i) of Definition 2.4,

$$\tilde{\alpha}_1(\|\xi\|) \leq W(\xi) \leq \tilde{\alpha}_2(\|\xi\|).$$

To show the decay of  $W$ , condition (ii) of Definition 2.4, note that for any  $\xi \in \mathbb{R}^n$

$$\begin{aligned}
 W(x(1, \xi)) &= \sum_{j=1}^M V(x(j, \xi)) \\
 &\leq \sum_{j=1}^{M-1} V(x(j, \xi)) + \rho(V(\xi)) \\
 &= W(\xi) - (\text{id} - \rho)(V(\xi)) \\
 &\leq W(\xi) - (\text{id} - \rho)(\alpha_1(\|\xi\|)) \\
 &\leq (\text{id} - (\text{id} - \rho) \circ \tilde{\alpha}_1 \circ \tilde{\alpha}_2^{-1})(W(\xi)) =: \tilde{\rho}(W(\xi)).
 \end{aligned}$$

As  $\tilde{\alpha}_1 \circ \tilde{\alpha}_2^{-1} \leq \text{id}$  it is easy to see that  $0 < \tilde{\rho} < \text{id}$ , and  $\tilde{\rho}$  is positive definite. This shows that  $W$  is a global Lyapunov function for system (2.1).  $\square$

In the next theorem, we present an alternative construction of a global Lyapunov function to the one given in (2.11). The construction is as follows. We define  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  by

$$W(\xi) := \max_{j \in \{0, \dots, M-1\}} \rho^{j/M}(V(x(M-1-j, \xi))). \quad (2.12)$$

Here the positive definite function  $\rho$  stems from the global finite-step Lyapunov function  $V$ . Note that if  $\rho \in \mathcal{K}_\infty$ , which we can assume without loss of generality, then  $\rho^{1/M}$  exists by Proposition 4.16. In particular, by  $\hat{\rho} := \rho^{1/M}$  we mean that  $\hat{\rho}$  satisfies  $\hat{\rho}^M = \rho$ , see Section 1.3. With this construction we obtain an analogue to Theorem 2.21.

**Theorem 2.22** (Construction of a global Lyapunov function II). *If  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a global finite-step Lyapunov function for system (2.1) with  $M \in \mathbb{N}$  satisfying condition (ii) of Definition 2.6, then  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined in (2.12) is a global Lyapunov function for system (2.1).*

*Proof.* The proof follows the lines of the proof of Theorem 2.21 and is therefore only sketched. It is not hard to see that by definition of  $W$  in (2.12) we have

$$\tilde{\alpha}_1(\xi) := \rho^{(M-1)/M} \circ \alpha_1(\xi) \leq \rho^{(M-1)/M}(V(\xi)) \stackrel{(2.12)}{\leq} W(\xi) \leq \tilde{\alpha}_2(\xi),$$

where  $\tilde{\alpha}_2 \in \mathcal{K}_\infty$  was defined in the proof of Theorem 2.21, and  $\alpha_1, \rho$  come from the global finite-step Lyapunov function  $V$ . To show the decay of  $W$ , condition (ii) of

Definition 2.4, note that for any  $\xi \in \mathbb{R}^n$ , we have

$$\begin{aligned}
 W(x(1, \xi)) &= \max_{j \in \{0, \dots, M-1\}} \rho^{j/M} (V(x(M-j, \xi))) \\
 &= \max \left\{ \max_{j \in \{1, \dots, M-1\}} \rho^{j/M} (V(x(M-j, \xi))), V(x(M, \xi)) \right\} \\
 &\leq \max \left\{ \max_{j \in \{1, \dots, M-1\}} \rho^{j/M} (V(x(M-j, \xi))), \rho(V(\xi)) \right\} \\
 &= \max_{j \in \{0, \dots, M-1\}} \rho^{(j+1)/M} (V(x(M-1-j, \xi))) \\
 &= \rho^{1/M} (W(\xi)).
 \end{aligned}$$

With  $\rho < \text{id}$  also  $\rho^{1/M} < \text{id}$  holds, which shows that  $W$  satisfies condition (ii) of Definition 2.4.  $\square$

#### 2.2.4 A converse Lyapunov theorem

In Section 2.2.2, we have presented converse finite-step Lyapunov theorems, which show that (scaled) norms are finite-step Lyapunov functions. Moreover, in Section 2.2.3, we have shown how to construct a Lyapunov function from a finite-step Lyapunov function. In this section we will bring those results together to deduce constructive converse Lyapunov theorems, which are corollaries of Theorem 2.14 and Theorem 2.21 resp. Theorem 2.22.

**Theorem 2.23** (Converse Lyapunov theorem I). *Let  $M \in \mathbb{N}$  satisfy Assumption 2.13. Then for any function  $\eta \in \mathcal{K}_\infty$  the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$W(\xi) := \sum_{j=0}^{M-1} \eta(\|x(j, \xi)\|) \quad (2.13)$$

for all  $\xi \in \mathbb{R}^n$  is a global Lyapunov function for system (2.1).

**Theorem 2.24** (Converse Lyapunov theorem II). *Let  $M \in \mathbb{N}$  satisfy Assumption 2.13. Then for any function  $\eta \in \mathcal{K}_\infty$  the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$W(\xi) := \max_{j \in \{0, \dots, M-1\}} \rho^{j/M} (\eta(\|x(M-1-j, \xi)\|)) \quad (2.14)$$

for all  $\xi \in \mathbb{R}^n$  is a global Lyapunov function for system (2.1).

*Remark 2.25.* The main difference of the construction of the Lyapunov functions in Theorem 2.23 and 2.24 in contrast to the constructions of Lyapunov functions in other converse Lyapunov theorems, is that in (2.13) we use a finite sum of solutions instead of an infinite series [71] (respectively, in (2.14) we use the maximum over a finite set instead of the supremum over all solutions and all times, [81, 116]).  $\triangleleft$

Since any GES system satisfies Assumption 2.13 (see Lemma 2.15), we obtain the following converse Lyapunov theorem for GES systems of the form (2.1).

**Corollary 2.26.** *If the origin of system (2.1) is GES then for any  $\eta \in \mathcal{K}_\infty$  there exists an  $M \in \mathbb{N}$  such that the function  $W$  defined in (2.13) resp. (2.14) is a global Lyapunov function for system (2.1).*

*Proof.* Since the origin of system (2.1) is GES, there exist  $C \geq 1$ , and  $\mu \in [0, 1)$  such that  $\|x(k, \xi)\| \leq C\mu^k \|\xi\|$ . Take  $M \in \mathbb{N}$  such that  $C\mu^M < 1$ , then Assumption 2.13 holds. Then by Theorem 2.23 resp. Theorem 2.24 the function  $W$  defined in (2.13) resp. (2.14) is a global Lyapunov function for system (2.1).  $\square$

Combining Corollary 2.20 and Theorem 2.21 resp. Theorem 2.22 we immediately obtain an  $(a, b)$  Lyapunov function (Definition 2.19 with  $M = 1$ ).

**Corollary 2.27.** *If the origin of system (2.1) is GAS, then for  $0 < a < b < \infty$  there exists an  $M \in \mathbb{N}$  such that the function  $W$  defined in (2.13) resp. (2.14) is an  $(a, b)$  Lyapunov function for system (2.1).*

For general systems of the form (2.1) the following procedure enables us to check asymptotic stability of the origin at least on a set  $[a, b]$ . Note that this procedure may fail if a suitable number  $M$  has to be chosen too large, or if the dynamics  $G$  are too complex.

*Procedure 2.28.* The stability analysis we proposed so far to construct a Lyapunov function (at least on a set  $[a, b]$ ) can be summarized as follows. Let a system of the form (2.1) be given.

[0] Set  $k = 1$ .

[1] Check  $\|G^k(\xi)\| < \|\xi\|$  for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| \in [a, b]$ , and  $\|G^k(\xi)\| \leq a$  for all  $\xi \in \mathbb{R}^n$  with  $\|\xi\| \in [0, a]$ . If these inequalities hold proceed with step [2]; else set  $k = k + 1$  and repeat.

[2] Define  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  by (2.13) or (2.14) with  $M = k$ .

If this procedure is successful then  $W$  is an  $(a, b)$  finite-step Lyapunov function for the overall system (2.1), and it particularly ensures practical asymptotic stability. If  $a = 0$  then we obtain a Lyapunov function and if, additionally,  $b = \infty$ , then the Lyapunov function is a global one.  $\triangleleft$

Computation of a suitable  $M \in \mathbb{N}$  can be done by iteratively checking the condition  $\|x(M, \xi)\| < \|\xi\|$  while increasing the value of  $M$ , which needs to be verified either globally or for a subset of  $\mathbb{R}^n$ . At least, if the origin is GES, there always exists an  $M$  large enough for which the condition holds globally. The difficulty of checking

this condition depends on the particular map  $G$ . Systematic methods for obtaining an  $M$  for linear and conewise linear maps are given in Section 2.4.

To conclude this section we apply our results, following Procedure 2.28, to an example.

### 2.2.5 Illustrative example

We consider the following discrete-time system

$$x(k+1) = \begin{pmatrix} [x(k)]_1 - 0.3[x(k)]_2 \\ 0.7[x(k)]_1 + 0.2 \frac{[x(k)]_2^2}{1+[x(k)]_2^2} \end{pmatrix}. \quad (2.15)$$

The aim is to prove that the origin of this system is GAS, and we want to use the results derived in this section to construct a global Lyapunov function.

Firstly, we show that the map

$$G(\xi) := \begin{pmatrix} [\xi]_1 - 0.3[\xi]_2 \\ 0.7[\xi]_1 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2} \end{pmatrix}$$

satisfies Assumption 2.1. To this end, observe that for all  $t \in \mathbb{R}$ , we have

$$\frac{t^2}{1+t^2} \leq \frac{|t|}{2}. \quad (2.16)$$

Thus, applying the infinity norm  $\|\cdot\|_\infty$ , and using the triangle inequality, we obtain

$$\begin{aligned} \|G(\xi)\|_\infty &= \max \left\{ |[ \xi ]_1 - 0.3[ \xi ]_2 |, \left| 0.7[ \xi ]_1 + 0.2 \frac{[ \xi ]_2^2}{1 + [ \xi ]_2^2} \right| \right\} \\ &\leq \max \{ |[ \xi ]_1 | + 0.3|[ \xi ]_2 |, 0.7|[ \xi ]_1 | + 0.1|[ \xi ]_2 | \} \leq 1.3\|\xi\|_\infty. \end{aligned}$$

Hence,  $G$  is globally  $\mathcal{K}$ -bounded with  $\omega(s) = 1.3s \in \mathcal{K}_\infty$  (with respect to the infinity norm  $\|\cdot\|_\infty$ ).

To show GAS of the origin of system (2.15) we follow the methodology given in Procedure 2.28. Iterating the dynamics map  $G$ , we see that for  $k = 3$  we obtain

$$G^3(\xi) = \begin{pmatrix} 0.58[\xi]_1 - 0.237[\xi]_2 - 0.06 \frac{[\xi]_2^2}{1+[\xi]_2^2} - 0.06 \frac{(0.7[\xi]_1 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+(0.7[\xi]_1 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2})^2} \\ \left( 0.7[\xi]_1 - 0.21[\xi]_2 + 0.2 \frac{(0.7[\xi]_1 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+(0.7[\xi]_1 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2})^2} \right)^2 \\ 0.553[\xi]_1 - 0.21[\xi]_2 - 0.042 \frac{[\xi]_2^2}{1+[\xi]_2^2} + 0.2 \frac{\left( 0.7[\xi]_1 - 0.21[\xi]_2 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2} \right)^2}{1 + \left( 0.7[\xi]_1 - 0.21[\xi]_2 + 0.2 \frac{[\xi]_2^2}{1+[\xi]_2^2} \right)^2} \end{pmatrix}.$$



Again, exploiting (2.16), and the fact that  $x(3, \xi) = G^3(\xi)$ , we obtain

$$\|x(3, \xi)\|_\infty \leq \max\{0.601|\xi_1| + 0.397|\xi_2|, 0.6335|\xi_1| + 0.253|\xi_2|\} \leq 0.998\|\xi\|_\infty.$$

We therefore conclude that  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by  $V(\xi) = \|\xi\|_\infty$  is a global finite-step Lyapunov function for system (2.15) with  $M = 3$  and  $\rho(s) = 0.998s$  yielding GAS of the origin by Theorem 2.7. Furthermore, as the  $\mathcal{K}$ -bound  $\omega$  and the function  $\rho < \text{id}$  are linear, and  $V$  satisfies condition (i) of Definition 2.6 with  $\alpha_1 = \alpha_2 = \text{id}$ , the origin of system (2.15) is GES by Corollary 2.9.

Finally, a global Lyapunov function can be constructed from the global finite-step Lyapunov function as in (2.13) (resp. (2.14)). In Figure 2.2, a contour plot of the global Lyapunov function constructed in (2.13) with  $\eta = \text{id}$ , i.e.,  $W(\xi) = \sum_{j=0}^2 \|x(j, \xi)\|_\infty$  is depicted. We furthermore include a plot of the trajectory starting in  $\xi = (3, -3)$ .

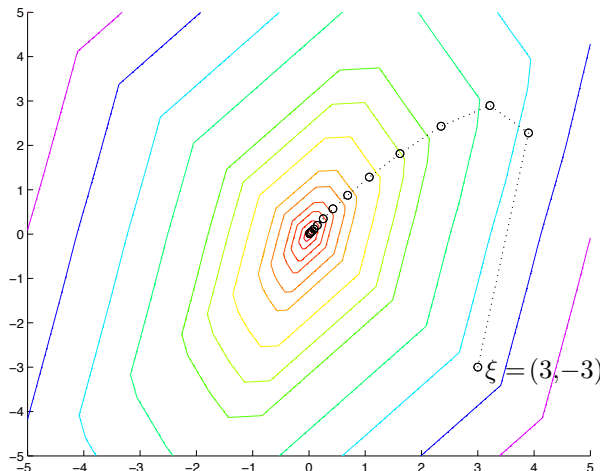


Figure 2.2: Contour plot of the Lyapunov function  $W(\xi) = \sum_{j=0}^2 \|x(j, \xi)\|_\infty$  for system (2.15), and the trajectory of system (2.15) starting in  $\xi = (3, -3)$ .

We note that as  $V(\xi) = \|\xi\|_\infty$  is a global finite-step Lyapunov function with  $M = 3$ , the function  $V$  is also a global finite-step Lyapunov function for any  $M = 3k$ , where  $k \in \mathbb{N}$ ,  $k \geq 1$ . Thus, the construction of the global Lyapunov function as in (2.13) (resp. (2.14)) only depends on finding a suitably large  $M \in \mathbb{N}$ . To end this section, we give a contour plot of a global Lyapunov function with  $M = 150$  in Figure 2.3, i.e.,  $\tilde{W}(\xi) := \sum_{j=0}^{149} \|x(j, \xi)\|_\infty$  for all  $\xi \in \mathbb{R}^2$ . Recall that usually proofs of converse

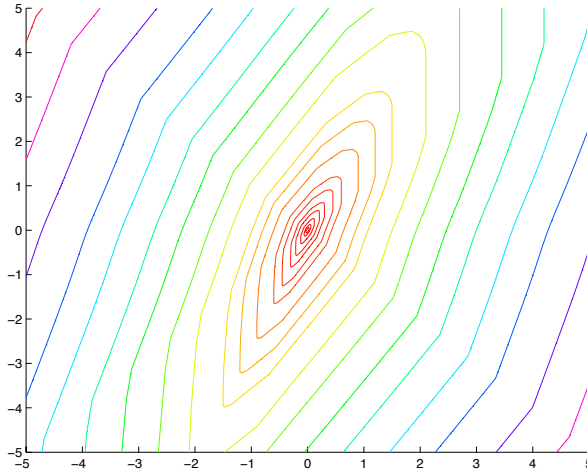


Figure 2.3: Contour plot of the Lyapunov function  $\tilde{W}(\xi) = \sum_{j=0}^{149} \|x(j, \xi)\|_\infty$  for system (2.15).

Lyapunov theorems use infinite series or suprema of solutions. Hence,  $W$  and  $\tilde{W}$  are approximations of the infinite series Lyapunov function construction, but these are already global Lyapunov functions.

### 2.3 Relaxed and non-conservative small-gain theorems

In Section 2.2, we have derived sufficient and necessary conditions in terms of existence of global (finite-step) Lyapunov functions to show GAS of a system's origin. For large-scale systems, this approach (following Procedure 2.28) might be computationally hard. In this section we present an alternative approach, the so-called *small-gain approach*, where we consider the system split into several smaller subsystems. GAS of the overall system's origin can then be deduced from properties of the subsystems.

We consider the interconnection of  $N \in \mathbb{N}$  discrete-time systems of the form

$$x_i(k+1) = g_i(x_1(k), \dots, x_N(k)) \in \mathbb{R}^{n_i}, \quad k \in \mathbb{N}, \quad (2.17)$$

with  $x_i(0) \in \mathbb{R}^{n_i}$  and  $g_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \rightarrow \mathbb{R}^{n_i}$  for  $i \in \{1, \dots, N\}$ . Let  $n = \sum_{i=1}^N n_i$ ,  $x = (x_1, \dots, x_N) \in \mathbb{R}^n$ , and  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $G = (g_1, \dots, g_N)$ . Then the overall system is of the form (2.1). Moreover, we assume that  $g_i(0) = 0$  for all  $i \in \{1, \dots, N\}$ . We call the states  $x_j$ ,  $j \neq i$ , the (*internal*) *inputs* for system  $i$ .

Throughout this section, we assume the standing Assumption 2.1 to hold, i.e., we require the map  $G$  to be globally  $\mathcal{K}$ -bounded. Observe that existing results on small-gain theory typically assume continuity of the map  $G$ . Moreover, at some places of this section, we consider the  $p$ -norm  $\|\cdot\|_p$  to simplify technicalities. Indeed, we could have used any monotonic norm.

### 2.3.1 Sufficient and necessary small-gain theorems

The basic idea of this section is to drop the assumption in (at least Lyapunov-based) small-gain theory that each subsystem has to admit a Lyapunov function. Instead, we require the existence of a Lyapunov-type function that decreases after a finite number of time steps. The first result, Theorem 2.29, states the sufficiency to conclude GAS of the origin. For this theorem, we recall the definition of the gain operator

$$\Gamma_{\oplus}(s) := \begin{pmatrix} \max\{\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N)\} \\ \vdots \\ \max\{\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N)\} \end{pmatrix}.$$

from (1.16).

**Theorem 2.29.** *Let (2.1) be the overall system of the interconnected systems (2.17). Assume that there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ , and  $\gamma_{ij} \in \mathcal{K}_{\infty} \cup \{0\}$ ,  $i, j \in \{1, \dots, N\}$ , defining the gain operator  $\Gamma_{\oplus}$  in (1.16), such that the following conditions hold:*

- (i) *For all  $i \in \{1, \dots, N\}$  there exist  $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_{\infty}$  such that for all  $\xi_i \in \mathbb{R}^{n_i}$  it holds*

$$\alpha_{1i}(\|\xi_i\|) \leq \mathcal{V}_i(\xi_i) \leq \alpha_{2i}(\|\xi_i\|). \quad (2.18)$$

- (ii) *For all  $i \in \{1, \dots, N\}$ , and all  $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n$  with  $\xi_i \in \mathbb{R}^{n_i}$  it holds*

$$\mathcal{V}_i(x_i(M, \xi)) \leq \max_{j \in \{1, \dots, N\}} \gamma_{ij}(\mathcal{V}_j(\xi_j)), \quad (2.19)$$

where  $x_i(M, \xi)$  denotes the solution of the  $i$ th subsystem (2.17).

- (iii) *The map  $\Gamma_{\oplus}$  from (1.16) induced by the functions  $\gamma_{ij}$  satisfies the small-gain condition  $\Gamma_{\oplus} \not\geq \text{id}$ .*

Then the origin of (2.1) is GAS.

In particular, there exists an  $\Omega$ -path  $\sigma \in \mathcal{K}_{\infty}^N$  such that the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by

$$W(\xi) := \sum_{j=0}^{M-1} \max_i \sigma_i^{-1}(\mathcal{V}_i(x_i(j, \xi))) \quad (2.20)$$

is a global Lyapunov function for the overall system (2.1).

Note that (2.19) can be written in compact form as

$$\begin{bmatrix} \mathcal{V}_1(x_1(M, \xi)) \\ \vdots \\ \mathcal{V}_N(x_N(M, \xi)) \end{bmatrix} \leq \Gamma_{\oplus} \left( \begin{bmatrix} \mathcal{V}_1(\xi_1) \\ \vdots \\ \mathcal{V}_N(\xi_N) \end{bmatrix} \right). \quad (2.21)$$

*Proof.* Firstly, condition (iii) and Lemma 1.30 imply the existence of an  $\Omega$ -path  $\sigma \in \mathcal{K}_{\infty}^N$  (Definition 1.22) with respect to  $\Gamma_{\oplus}$ , i.e., a function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  with  $\sigma_i \in \mathcal{K}_{\infty}$  for  $i \in \{1, \dots, N\}$  satisfying

$$\Gamma_{\oplus}(\sigma(r)) < \sigma(r) \quad \text{for all } r > 0, \quad (2.22)$$

or component-wise, for any  $i \in \{1, \dots, N\}$ ,  $\max_{j \in \{1, \dots, N\}} \gamma_{ij} \circ \sigma_j(r) < \sigma_i(r)$  for all  $r > 0$ . In the following let  $i, j, j' \in \{1, \dots, N\}$ . Define

$$V(\xi) := \max_i \sigma_i^{-1}(\mathcal{V}_i(\xi_i)). \quad (2.23)$$

We will show that  $V$  is a global finite-step Lyapunov function for the overall system (2.1). For this purpose note that condition (i) of Theorem 2.29 implies

$$V(\xi) \geq \max_i \sigma_i^{-1}(\alpha_{1i}(\|\xi_i\|)) \geq \alpha_1(\|\xi\|)$$

with  $\alpha_1 := \min_j \sigma_j^{-1} \circ \alpha_{1j} \circ \frac{1}{\kappa} \text{id} \in \mathcal{K}_{\infty}$ , where  $\kappa \geq 1$  comes from (1.1). On the other hand, we have

$$V(\xi) \leq \max_i \sigma_i^{-1}(\alpha_{2i}(\|\xi_i\|)) \leq \alpha_2(\|\xi\|)$$

with  $\alpha_2 := \max_i (\sigma_i^{-1} \circ \alpha_{2i}) \in \mathcal{K}_{\infty}$ , which shows that  $V$  is proper and positive definite. To show the decay of  $V$ , condition (ii) of Definition 2.6, note that (2.22) is equivalent to  $\max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j(r) < r$  for all  $r > 0$ . Define  $\rho := \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j$ , then  $\rho < \text{id}$  and we have

$$\begin{aligned} V(x(M, \xi)) &= \max_i \sigma_i^{-1}(\mathcal{V}_i(x_i(M, \xi))) \\ &\stackrel{(2.21)}{\leq} \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij}(\mathcal{V}_j(\xi_j)) = \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j \circ \sigma_j^{-1}(\mathcal{V}_j(\xi_j)) \\ &\leq \max_{i,j,j'} (\sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j) \circ (\sigma_{j'}^{-1}(\mathcal{V}_{j'}(\xi_{j'}))) = \rho(V(\xi)). \end{aligned}$$

This shows that  $V$  is a global finite-step Lyapunov function for system (2.1). So from Theorem 2.7 we conclude that the origin of system (2.1) is GAS. Furthermore, the function  $W$  defined in (2.23) is a global Lyapunov function, which follows from (2.11) and Theorem 2.21.  $\square$

Theorem 2.29 states sufficient conditions under which GAS of the overall system's origin is shown. In the following, we explain the meaning of the functions involved. In particular, we study the role of the number  $M \in \mathbb{N}$ .

Assume that we can choose  $M = 1$  in Theorem 2.29. Let  $i \in \{1, \dots, N\}$  and set  $\xi_j = 0$  for each  $j \in \{1, \dots, N\} \setminus \{i\}$ . Then condition (ii) implies

$$\mathcal{V}_i(x_i(1, (0, \dots, 0, \xi_i, 0, \dots, 0))) \leq \gamma_{ii}(\mathcal{V}_i(\xi_i)).$$

From condition (iii) and Proposition 1.29, we conclude that  $\gamma_{ii} < \text{id}$ . Moreover, by condition (i),  $\mathcal{V}_i$  is proper and positive definite. Hence,  $\mathcal{V}_i$  is a global Lyapunov function for the decoupled system

$$x_i(k+1) = g(0, \dots, 0, x_i(k), 0, \dots, 0). \quad (2.24)$$

This implies that the origin of the decoupled system (2.24) is GAS. Hence, each subsystem is 0-GAS<sup>3</sup>. In this case, Theorem 2.29 resembles results in [67,99]. Indeed, if  $M = 1$  then Theorem 2.29 is a small-gain theorem in the *classical* sense: The origin of each subsystem is at least 0-GAS and, by the small-gain condition, the (disturbing) influence of the subsystems on the interconnection structure is small enough. Hence, the origin of the overall system is GAS.

To better understand the distinguishing difference between the cases  $M = 1$  and  $M > 1$ , we consider the following example.

**Example 2.30.** Consider the 2-dimensional discrete-time system

$$\begin{aligned} x_1(k+1) &= x_1(k) - x_2(k) \\ x_2(k+1) &= \frac{2}{3}(x_1(k) - x_2(k)) \end{aligned}$$

with  $k \in \mathbb{N}$ ,  $x_1, x_2 \in \mathbb{R}$ . Clearly, the origin of the first subsystem is *not* 0-GAS, as  $x_1(k, \xi_1, 0) \equiv \xi_1$  for all  $k \in \mathbb{N}$ . Nevertheless, computing solutions for  $k = 2$  we obtain

$$x_1(2, \xi) = \frac{1}{3}(\xi_1 - \xi_2), \quad x_2(2, \xi) = \frac{2}{9}(\xi_1 - \xi_2)$$

for all  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ . Taking the norm, we can derive the estimates

$$|x_1(2, \xi)| \leq \max\left\{\frac{2}{3}|\xi_1|, \frac{2}{3}|\xi_2|\right\}, \quad |x_2(2, \xi)| \leq \max\left\{\frac{4}{9}|\xi_1|, \frac{4}{9}|\xi_2|\right\},$$

which implies condition (ii) of Theorem 2.29 with  $\mathcal{V}_i(\cdot) := |\cdot|$  and

$$\gamma_{11}(s) := \frac{2}{3}s, \quad \gamma_{12}(s) := \frac{2}{3}s, \quad \gamma_{21}(s) := \frac{4}{9}s, \quad \gamma_{22}(s) = \frac{4}{9}s.$$

By definition of  $\mathcal{V}_i$ , condition (i) of Theorem 2.29 holds. Moreover, by Proposition 1.29, also condition (iii) of Theorem 2.29 is satisfied. Hence, the origin of the 2-dimensional system is GAS.  $\triangleleft$

<sup>3</sup>I.e., the origin of each subsystem (2.24) is GAS, see also Definition 3.6.

Example 2.30 shows that Theorem 2.29 can be applied, although the origin of the first subsystem is not 0-GAS. In addition, as we will see in Section 2.3.2, the subsystems (2.17) may be 0-input unstable, i.e., the origin of subsystem (2.24) is unstable for some  $i \in \{1, \dots, N\}$ . This means that Theorem 2.29 is not a *classical* small-gain result, where the origin of each subsystem has to be at least 0-GAS. In any case, GAS of the overall system's origin can be guaranteed by Theorem 2.29. Thus, this theorem is a strict relaxation of previous small-gain theorems as the ones in [67, 99], which cannot handle the case of 0-input unstable subsystems (2.17).

To understand why subsystems do not have to be 0-GAS for  $M > 1$ , again, let  $i \in \{1, \dots, N\}$  and set  $\xi_j = 0$  for  $j \neq i$ . Then (2.19) implies

$$\mathcal{V}_i(x_i(M, (0, \dots, 0, \xi_i, 0, \dots, 0))) \leq \gamma_{ii}(\mathcal{V}_i(\xi_i)), \quad \text{for all } \xi_i \in \mathbb{R}^{n_i},$$

but, and this is important, it does *not* imply

$$\mathcal{V}_i(x_i(M, \xi_i, 0)) \leq \gamma_{ii}(\mathcal{V}_i(\xi_i)), \quad \text{for all } \xi_i \in \mathbb{R}^{n_i}.$$

Let us explain the difference of these two estimates. Firstly, the 0 in  $x_i(M, \xi_i, 0)$  means that all inputs  $x_j$ ,  $j \neq i$ , are set to zero (for all times  $k \in \mathbb{N}$ ). Hence,  $x_i(M, \xi_i, 0)$  denotes the solution of the decoupled system (2.24) at time  $M$  starting in the initial value  $\xi_i \in \mathbb{R}^{n_i}$ . On the other hand,  $x_i(M, (0, \dots, 0, \xi_i, 0, \dots, 0))$  denotes the solution of (2.17) at time  $M$  starting in the initial value  $\xi = (0, \dots, 0, \xi_i, 0, \dots, 0)$ . This means that the (internal) inputs  $x_j$ ,  $j \neq i$ , are zero at time  $k = 0$ , i.e.,  $x_j(0) = \xi_j = 0$  for all  $j \neq i$ , but they may be nonzero for times  $k \geq 1$ . The consequence is that the state of the  $i$ th subsystems can be fed back via the internal inputs  $x_j$ ,  $j \neq i$ . Hence, stabilizing feedback effects are implicitly taken into account by Theorem 2.29.

To make this observation more clear, consider Example 2.30 again. Obviously, the systems are in a feedback loop. If we consider the subsystems to be decoupled the dynamics are

$$x_1(k+1) = x_1(k), \quad x_2(k+1) = -\frac{2}{3}x_2(k), \quad \text{for all } k \in \mathbb{N}.$$

As observed in Example 2.30, the origin of the first subsystem is not 0-GAS as  $x_1(k, \xi_1, 0) \equiv \xi_1$  for all  $k \in \mathbb{N}$  and all  $\xi_1 \in \mathbb{R}$ . On the other hand, the solution of the first subsystem starting in the initial state  $(\xi_1, 0)$  at time  $k = 2$  is given by  $x_1(2, (\xi_1, 0)) = \frac{1}{3}\xi_1$ . This means that the second subsystem has a stabilizing effect on the first subsystem.

Summarizing, we can interpret the functions involved in Theorem 2.29 as follows:

- (i) The proper and positive definite functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $i \in \{1, \dots, N\}$  are used to characterize a decrease of the solutions of the subsystems (2.17). If

$M = 1$  then these functions are global Lyapunov functions for the decoupled systems (2.24). For  $M > 1$  the subsystems may be 0-input unstable. Thus, the functions  $\mathcal{V}_i$  are, in general, also not global finite-step Lyapunov function.

- (ii) The number  $M \geq 1$  denotes a time step after which a decrease of the function  $\mathcal{V}_i$  evaluated at the solution  $\xi_i(M, \xi)$  of the form (2.19) is guaranteed for each  $i \in \{1, \dots, N\}$ . In particular, the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined in (2.23) is shown to be a global finite-step Lyapunov function for the overall system, which decreases along solutions after  $M$  steps.
- (iii) The functions  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $i, j \in \{1, \dots, N\}$  in (2.19) describe, via a worst-case estimate, how the solution  $x_i(M, \xi)$  behaves depending on the size of  $\xi_j \in \mathbb{R}^{n_j}$ . From this point of view, we can consider  $\gamma_{ij}$  as the *gain* characterizing the (disturbing) effect of  $\xi_j$  on  $x_i(M, \xi)$ . In this spirit, we call Theorem 2.29 a *small-gain theorem*.

We emphasize that to check the conditions of Theorem 2.29, we have to compute the solutions  $x_i(j, \xi)$  for all  $i \in \{1, \dots, N\}$  and all  $j \in \{1, \dots, M\}$  in (2.19). As computing solutions of discrete-time systems corresponds to iterating the dynamics map  $G$  in (2.1), it is clear that deriving estimates of the form (2.19) might be challenging if  $G$  is complex or  $M$  is large. We refer to Remark 2.34, where we comment on finding a suitable  $M \in \mathbb{N}$ .

Next, we consider the case that the Lyapunov-type functions  $\mathcal{V}_i$  and the gains  $\gamma_{ij}$  in Theorem 2.29 are of a special form. Then GES of the origin of the overall system (2.1) can be derived.

**Theorem 2.31.** *The origin of system (2.1) is GES if there exists an  $M \in \mathbb{N}$ ,  $M \geq 1$ , functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $i \in \{1, \dots, N\}$ , and linear functions  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $i, j \in \{1, \dots, N\}$ , such that the following conditions hold.*

- (i) *There exist  $0 < c_1 \leq c_2$  and  $\lambda > 0$  such that (2.18) holds with  $\alpha_{1i}(s) = c_1 s^\lambda$  and  $\alpha_{2i}(s) = c_2 s^\lambda$ .*
- (ii) *For all  $\xi \in \mathbb{R}^n$ , (2.21) holds.*
- (iii) *The map  $\Gamma_\oplus$  from (1.16) satisfies the small-gain condition (1.10).*

*Proof.* We only give a sketch of the proof, which follows the proof of Theorem 2.29. Note that since the functions  $\gamma_{ij}$  are linear we can also choose a linear  $\Omega$ -path  $\sigma \in \mathcal{K}_\infty^N$ , see [37] or Appendix A.1. Then the constructed global finite-step Lyapunov function  $V$  in (2.23) satisfies Definition 2.6 for functions  $\alpha_1(s) = a_1 s^\lambda$ ,  $\alpha_2(s) = a_2 s^\lambda$  with  $\lambda > 0$  as above and suitable  $0 < a_1 \leq a_2$ . Furthermore, the function  $\rho < \text{id}$  can be chosen to be linear. Applying Corollary 2.9 proves GES of the origin of the overall system (2.1).  $\square$

An alternative proof of Theorem 2.31 can be found in [40].

Theorem 2.29 states sufficient conditions to prove GAS of the origin of system (2.1). Next, we study the necessity of Theorem 2.29. This means we consider the question whether the existence of particular functions  $\mathcal{V}_i$  and  $\gamma_{ij}$  satisfying the conditions of Theorem 2.29 can be ensured if the origin of the overall system (2.1) is GAS. This is achieved by taking  $\mathcal{V}_i$  as a norm. To simplify the presentation, we will use the  $p$ -norm  $\|\cdot\|_p$  in the remainder of this section. Note that we could have used any arbitrary norm on  $\mathbb{R}^n$ , but, in this case, the constant  $N^{1/p}$  in Assumption 2.32 and in the following has to be replaced by the constant  $\kappa \geq 1$  given in (1.1). A discussion of the following assumption can be found in the remainder of this section.

**Assumption 2.32.** Let system (2.1) be the interconnection of  $N$  subsystems given in (2.17). Moreover, system (2.1) admits a global Lyapunov function  $W$  that satisfies for some  $M \in \mathbb{N}$ ,  $M \geq 1$  and all  $s > 0$

$$\rho^M(s) < \alpha_1 \circ \left(\frac{1}{N^{1/p}} \text{id}\right) \circ \alpha_2^{-1}(s), \quad (2.25)$$

where  $\alpha_1, \alpha_2, \rho$  are related to the global Lyapunov function  $W$  as in Definition 2.6, and  $p \in [1, \infty]$  is arbitrary but fixed.  $\triangleleft$

Note that  $p \in [1, \infty]$  in Assumption 2.32 defines the norm  $\|\cdot\|_p$  that is used to define the functions  $\mathcal{V}_i$ ,  $i \in \{1, \dots, N\}$ , in the next theorem. In addition, recall that for  $p = \infty$ , by definition,  $\frac{1}{N^{1/\infty}} := 1$ .

Under Assumption 2.32 we can prove the converse of Theorem 2.29.

**Theorem 2.33.** *If system (2.1) satisfies Assumption 2.32, then there exist functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $i \in \{1, \dots, N\}$ , and  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $i, j \in \{1, \dots, N\}$ , such that the following holds:*

(i) *For all  $i \in \{1, \dots, N\}$  there exist  $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$  such that for all  $\xi_i \in \mathbb{R}^{n_i}$ , (2.18) holds.*

(ii) *For all  $\xi \in \mathbb{R}^n$  and each  $M \in \mathbb{N}$ ,  $M \geq 1$  satisfying (2.25) it holds (2.21).*

(iii) *The map  $\Gamma_\oplus$  from (1.16) satisfies the small-gain condition (1.10).*

*Proof.* By Assumption 2.32 system (2.1) admits a global Lyapunov function  $W$ . Thus, by Corollary 2.10 system (2.1) is GAS. From condition (ii) of Definition 2.6 we obtain by iteration

$$W(x(k, \xi)) \leq \rho^k(W(\xi)). \quad (2.26)$$

Take any  $\eta \in \mathcal{K}_\infty$  (for simplicity, take  $\eta = \text{id}$ ) and define, for  $i \in \{1, \dots, N\}$ ,

$$\mathcal{V}_i(\xi_i) := \eta(\|\xi_i\|_p). \quad (2.27)$$



Then condition (i) of Theorem 2.33 holds with  $\alpha_{1i} = \alpha_{2i} = \eta$  for all  $i \in \{1, \dots, N\}$ . Let  $M \in \mathbb{N}$  satisfy (2.25). Then

$$\begin{aligned}
 \mathcal{V}_i(x_i(M, \xi)) &= \eta(\|x_i(M, \xi)\|_p) \leq \eta(\|x(M, \xi)\|_p) \\
 &\leq \eta \circ \alpha_1^{-1}(W(x(M, \xi))) \\
 &\stackrel{(2.26)}{\leq} \eta \circ \alpha_1^{-1} \circ \rho^M(W(\xi)) \\
 &\leq \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2(\|\xi\|_p) \\
 &\stackrel{(2.28)}{\leq} \max_j \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2(N^{1/p} \|\xi_j\|_p) \\
 &\leq \max_j \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2 \circ (N^{1/p} \text{id}) \circ \eta^{-1}(\mathcal{V}_j(\xi_j)),
 \end{aligned}$$

where we used

$$\|\xi\|_p \leq \max_j N^{1/p} \|\xi_j\|_p, \quad (2.28)$$

which follows from (1.1). By (2.25) we obtain

$$\gamma := \eta \circ \underbrace{\alpha_1^{-1} \circ \rho^M \circ \alpha_2 \circ N^{1/p} \text{id}}_{< \text{id}} \circ \eta^{-1} < \text{id}.$$

Let  $\gamma_{ij} := \gamma$  for  $i, j \in \{1, \dots, N\}$  then  $\mathcal{V}_i(x_i(M, \xi)) \leq \max_j \gamma_{ij}(\mathcal{V}_j(\xi_j))$  holds for all  $i \in \{1, \dots, N\}$  showing condition (ii) of Theorem 2.33, and from  $\gamma_{ij} < \text{id}$  we conclude that condition (iii) of Theorem 2.33 holds. This concludes the proof.  $\square$

*Remark 2.34.* (i) The distinguishing feature of Assumption 2.32 in Theorem 2.33 is that the functions  $\mathcal{V}_i$  satisfying (2.18) and (2.21) can be chosen as (scaled) norms by (2.27).

(ii) The number  $M \in \mathbb{N}$  in Theorem 2.29 and Theorem 2.33 depends on the system dynamics (2.1) and, of course, on the functions  $\mathcal{V}_i$  and  $\gamma_{ij}$ . The goal of reducing the conservatism present in small-gain theorems is attained, however, for the price of finding a suitable  $M$ . Since verifying GAS of nonlinear systems is a difficult problem, also finding a suitable  $M$  is challenging. However, since the only constraint on  $M$  is that it is large enough, the developed results hold the promise of delivering applicable conditions. This is also demonstrated by the example provided in Section 2.3.2.

(iii) In addition, observe that when existing (Lyapunov-based) small-gain theorems can be applied<sup>4</sup>, the hypothesis of the relaxed small-gain theorem is verified with  $M = 1$ . Different from existing small-gain theorems, we propose to iterate the right-hand side  $G$  to find a suitable  $M$  such that the subsystems decrease in norm after  $M$  time instants. Thus, in general, the problem of finding a suitable  $M$  reduces

<sup>4</sup>This usually requires the construction of ISS Lyapunov functions for the subsystems to conclude GAS of the origin of the overall system, cf. [25].

to iterating the map  $G$ , which leads to systematic algorithms. Compare to finding a suitable ISS Lyapunov function, for which there is, in general, no systematic approach for nonlinear systems.  $\triangleleft$

If we combine Theorem 2.33 with Theorem 2.29, then Assumption 2.32 implies an explicit construction of a global Lyapunov function.

**Corollary 2.35.** *If system (2.1) satisfies Assumption 2.32, then for arbitrary  $\eta \in \mathcal{K}_\infty$ , and  $M \in \mathbb{N}$  satisfying (2.25), the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$W(\xi) := \sum_{j=0}^{M-1} \max_{i \in \{1, \dots, N\}} \eta(\|x_i(j, \xi)\|_p) \quad (2.29)$$

is a global Lyapunov function for system (2.1).

*Proof.* From the proof of Theorem 2.33 we see that  $\gamma_{ij} = \gamma < \text{id}$  for all  $i, j \in \{1, \dots, N\}$ . Thus  $\sigma \in \mathcal{K}_\infty^N$  with  $\sigma_i = \text{id}$  for all  $i \in \{1, \dots, N\}$  is an  $\Omega$ -path for  $\Gamma_\oplus$ . Then from (2.20) we see that (2.29) is a global Lyapunov function.  $\square$

An alternative Lyapunov function to the one given in (2.29) is obtained as

$$W(\xi) := \max_{i \in \{1, \dots, N\}} \max_{j \in \{0, \dots, M-1\}} \gamma^{j/M} (\eta(\|x_i(j, \xi)\|_p))$$

for all  $\xi \in \mathbb{R}^n$ , where  $\gamma < \text{id}$  is defined in the proof of Theorem 2.33. This follows as by definition of  $\rho$  in the proof of Theorem 2.29 we have  $\rho := \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j = \gamma$ , which implies the above alternative global Lyapunov function by (2.12).

We will see in Theorem 2.38 that Assumption 2.32 is satisfied if the underlying system is GES. On the other hand, Assumption 2.32 does not imply that system (2.1) is GES. This can be seen by system (2.10) in Example 2.17, which has been shown to be GAS, but not GES. In this example, the right-hand side function  $G$  is globally  $\mathcal{K}$ -bounded with  $\mathcal{K}$ -function  $\omega < \text{id}$ . Hence, as shown in the proof of Proposition 2.11, the function  $W : \mathbb{R} \rightarrow \mathbb{R}_+, \xi \mapsto \|\xi\|_p$  is a Lyapunov function with  $\alpha_1 = \alpha_2 = \text{id}$ ,  $\rho = \omega < \text{id}$ ,  $M = 1$ ,  $N = 1$ , satisfies (2.25), and, consequently, Assumption 2.32.

Unfortunately, a result connecting both Assumptions 2.13 and 2.32 by stating whether these assumptions imply each other or not, is still missing. But for  $N = 1$  both Assumptions 2.13 and 2.32 have the same implication as shown next.

**Theorem 2.36.** *For  $N=1$ , Assumptions 2.13 and 2.32 both imply that  $V(\xi) := \|\xi\|_p$  is a global finite-step Lyapunov function, and  $W(\xi) := \sum_{j=0}^{M-1} \|x(j, \xi)\|_p$  is a global Lyapunov function for system (2.1).*

*Proof.* Let Assumption 2.32 hold. Then Corollary 2.35 with  $N = 1$  and  $\eta = \text{id}$  implies that  $W(\xi) = \sum_{j=0}^{M-1} \|x(j, \xi)\|_p$  is a global Lyapunov function. In particular,

$V(\xi) = \|\xi\|_p$  is a global finite-step Lyapunov function for the overall system (2.1), which follows by

$$\|x(M, \xi)\|_p \leq \alpha_1^{-1}(W(x(M, \xi))) \leq \alpha_1^{-1} \circ \rho^M(W(\xi)) \leq \underbrace{\alpha_1^{-1} \circ \rho^M \circ \alpha_2}_{=:\tilde{\rho} < \text{id}}(\|\xi\|_p).$$

The same holds true under Assumption 2.13, i.e., by Theorem 2.14,  $V(\xi) = \|\xi\|_p$  is a global finite-step Lyapunov function with suitable  $M \in \mathbb{N}$ , and applying Theorem 2.23, again  $W(\xi) = \sum_{j=0}^{M-1} \|x(j, \xi)\|_p$  turns out to be a global Lyapunov function for system (2.1).  $\square$

Next, we provide sufficient conditions under which Assumption 2.32 holds. We start with the case that (2.25) is not globally satisfied.

**Theorem 2.37.** *Let  $W$  be a global Lyapunov function for system (2.1) with suitable  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and positive definite  $\rho$ . Then the following holds.*

- (i) *For any compact set  $[a, b] \subset (0, \infty)$  there exists an  $M \in \mathbb{N}$ ,  $M \geq 1$  such that (2.25) holds for any  $s \in [a, b]$ .*
- (ii) *If, additionally,  $\alpha_1, \alpha_2, \rho$  are continuously differentiable and satisfy  $\rho'(0) < 1$  and  $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$ , then for any compact set  $[0, b] \subset \mathbb{R}_+$  there exists an  $M \in \mathbb{N}$ ,  $M \geq 1$  such that (2.25) holds for any  $s \in [0, b]$ .*

*Proof.* (i) Let  $[a, b] \in (0, \infty)$  be given. Let  $M \in \mathbb{N}$ ,  $M \geq 1$ . Then (2.25) holds for any  $s \in [a, b]$  if and only if  $\beta(M, s) < s$  for any  $s \in [a, b]$ , where

$$\beta(k, r) := \alpha_2 \circ N^{1/p} \circ \alpha_1^{-1} \circ \rho^k(r), \quad k, r \in \mathbb{R}_+.$$

Since  $\beta \in \mathcal{KL}$ , the result follows from Lemma 2.18.

(ii) We will show that there exists an  $\varepsilon > 0$  sufficiently small, and an  $M \in \mathbb{N}$  such that for all  $s \in [0, \varepsilon)$  it holds  $\rho^M(s) < \eta(s)$ . From the first part it follows that there exists a finite  $\bar{M} \in \mathbb{N}$  such that (2.25) is satisfied for all  $s \in [\varepsilon, b]$ . The result follows by taking the maximum of  $M$  and  $\bar{M}$ .

Since  $\rho'(0) < 1$  there exists a  $c_1 < 1$  such that  $\rho(t) < c_1 t$  for all  $t \in [0, \varepsilon)$  and  $\varepsilon > 0$  sufficiently small, and since  $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$  there exists a  $c_2 > 0$  such that  $\alpha_1 \circ \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1}(t) > c_2 t$  for all  $t \in [0, \varepsilon)$ . Pick any  $M \in \mathbb{N}$  such that  $c_1^M < c_2$  then

$$\rho^M(t) < c_1^M t < c_2 t < \alpha_1 \circ \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1}(t)$$

for all  $t \in [0, \varepsilon)$ , which is (2.25) and concludes the proof.  $\square$

Note that it is not restrictive to assume  $\alpha_1, \alpha_2, \rho$  to be continuously differentiable on  $(0, \infty)$ , cf. [106].

The essence of Theorems 2.33 and 2.37-(i) is the following. Assume that system (2.1) admits a global Lyapunov function  $W$ . Then there exist an  $M \in \mathbb{N}$ , functions  $\mathcal{V}_i$  and  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$  satisfying the conditions (i) and (ii) of Theorem 2.33 for  $\xi_i \in \mathbb{R}^{n_i}$ , and condition (iii) of Theorem 2.33 for all  $s \in [a, b] \subset (0, \infty)$ . This may be used to show that the construction (2.27) and (2.20) can be used to obtain a Lyapunov function guaranteeing practical asymptotic stability of the origin of system (2.1). It is worth mentioning that the procedure only requires iterating the map  $G$  in (2.1) and searching for an  $M \in \mathbb{N}$  satisfying (2.25) on a particular set  $B_{[a,b]}$ .

Next, we briefly explain why the assumption on the derivatives in Theorem 2.37-(ii) is reasonable. Assume that system (2.1) admits a global Lyapunov function  $W$  for which the bounds in Theorem 2.37-(ii) are satisfied. If we fix  $\eta \in \mathcal{K}_\infty$ ,  $b > 0$ , and  $\mathcal{V}_i$  given by (2.27) then for any  $M \in \mathbb{N}$  large enough condition (iii) of Theorem 2.33 holds for all  $s \in [0, b]$ . Again, this can then be used to obtain a Lyapunov function that guarantees asymptotic stability of the origin of system (2.1). Note that the bounds on the derivatives are satisfied if the equilibrium point 0 is locally exponentially stable (i.e., there exists a local Lyapunov function with exponential bounds), see also Theorem 2.38-(i).

Now we state particular cases under which Assumption 2.32 holds globally.

**Theorem 2.38.** *Let  $W$  be a global Lyapunov function for system (2.1) with suitable  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and positive definite  $\rho$ . If one of the following conditions holds then Assumption 2.32 is globally satisfied.*

- (i) *For some  $a_1, a_2, \lambda > 0$ ,  $c \in [0, 1)$  we have  $\alpha_1(s) = a_1 s^\lambda$ ,  $\alpha_2(s) = a_2 s^\lambda$ , and  $\rho(s) = cs$ .*
- (ii) *We have  $\rho'(0) < 1$ ,  $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$  and  $\rho \in \mathcal{K} \setminus \mathcal{K}_\infty$ .*
- (iii) *We have  $\rho'(0) < 1$ ,  $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$ , as well as  $\liminf_{s \rightarrow \infty} (\alpha_1)'(s) \in (0, \infty)$ ,  $\liminf_{s \rightarrow \infty} (\alpha_2^{-1})'(s) \in (0, \infty)$  and  $\limsup_{s \rightarrow \infty} \rho'(s) \in (0, 1)$ .*

*Proof.* (i) From Remark 2.5-(ii) we obtain  $\rho := (1 - \frac{c}{b}) \in [0, 1)$ . Then (2.25) is equivalent to  $\rho^M < \frac{a}{N^{1/p}b}$ , and there always exists an  $M \in \mathbb{N}$  such that this condition holds.

(ii) Since  $\rho \in \mathcal{K} \setminus \mathcal{K}_\infty$  there exists a  $C > 0$  such that  $\rho(s) \leq C$  for all  $s \in \mathbb{R}_+$ . Let  $v := \alpha_1^{-1}(C) \in \mathbb{R}_+$ . From Theorem 2.37-(ii) there exists an  $M \in \mathbb{N}$  such that (2.25) holds for all  $s \in [0, \alpha_2(N^{1/p}v)]$ . Then for all  $s > \alpha_2(N^{1/p}v)$ ,

$$\begin{aligned} \rho^M(s) < \rho(s) \leq C &= \alpha_1 \left( \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1} \circ \alpha_2 \circ N^{1/p} \text{id} \right) (v) \\ &< \alpha_1 \circ \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1}(s). \end{aligned}$$

This shows that (2.25) holds for all  $s \in \mathbb{R}_+$ .

(iii) Define  $c_1 := \frac{1}{2} \liminf_{s \rightarrow \infty} (\alpha_1)'(s) > 0$ ,  $c_2 := \frac{1}{2} \liminf_{s \rightarrow \infty} (\alpha_2^{-1})'(s) > 0$  and  $c_3 := \frac{1}{2}(\limsup_{s \rightarrow \infty} \rho'(s) + 1) \in (0, 1)$ . Let  $T > 0$  be such that  $\alpha_1'(s) > c_1$  for all  $s > T$ . Then for all  $s > T$  we have

$$\alpha_1(s) = \underbrace{\int_0^T \alpha_1'(\tau) d\tau}_{=: \hat{K}_1} + \int_T^s \underbrace{\alpha_1'(\tau)}_{> c_1} d\tau > c_1 s + \underbrace{(\hat{K}_1 - c_1 T)}_{=: K_1}.$$

Similar observations for  $\alpha_2$  and  $\rho$  imply that there exists an  $\hat{s} > 0$  suitably large and constants  $K_1, K_2, K_3 \in \mathbb{R}$  such that for all  $s \geq \hat{s}$  we have

$$\alpha_1(s) > c_1 s + K_1 \quad (2.30)$$

$$\alpha_2^{-1}(s) > c_2 s + K_2 \quad (2.31)$$

$$\rho(s) < c_3 s + K_3. \quad (2.32)$$

Let  $s \geq \hat{s}_1$ , and  $\hat{s}_1$  suitably large, then equations (2.30) and (2.31) imply

$$\alpha_1 \circ \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1}(s) > \frac{c_1 c_2}{N^{1/p}} s + \left( \frac{c_1 K_2}{N^{1/p}} + K_1 \right). \quad (2.33)$$

Furthermore, from  $\rho < \text{id}$  and (2.32), we see that for all  $k \in \mathbb{N}$  and all  $s \geq \hat{s}_1$  we have

$$\rho^k(s) < c_3^k s + K_3 \sum_{j=0}^{k-1} c_3^j < c_3^k s + K_3 \sum_{j=0}^{\infty} c_3^j < c_3^k s + \frac{K_3}{1 - c_3} \quad (2.34)$$

by evaluating the geometric series. Let  $M_1 \in \mathbb{N}$  be such that  $c_3^{M_1} < \frac{c_1 c_2}{N^{1/p}}$  and define

$$\hat{s}_2 := \frac{K_3/(1 - c_3) - K_2 c_1/N^{1/p} - K_1}{c_1 c_2/N^{1/p} - c_3^{M_1}}.$$

Then this implies for all  $s > \hat{s}_2$

$$c_3^{M_1} s + \frac{K_3}{1 - c_3} < \frac{c_1 c_2}{N^{1/p}} s + \left( \frac{c_1 K_2}{N^{1/p}} + K_1 \right). \quad (2.35)$$

Altogether we conclude for all  $s > \hat{s} := \max\{\hat{s}_1, \hat{s}_2\}$

$$\begin{aligned} \rho^{M_1}(s) &\stackrel{(2.34)}{<} c_3^{M_1} s + \frac{K_3}{1 - c_3} \stackrel{(2.35)}{<} \frac{c_1 c_2}{N^{1/p}} s + \frac{c_1}{N^{1/p}} K_2 + K_1 \\ &\stackrel{(2.33)}{<} \alpha_1 \circ \frac{1}{N^{1/p}} \text{id} \circ \alpha_2^{-1}(s). \end{aligned} \quad (2.36)$$

From Theorem 2.37-(ii) we conclude that there exists an  $M_2 \in \mathbb{N}$  such that (2.25) holds for all  $s \in [0, \hat{s}]$ . Take  $M := \max\{M_1, M_2\}$ , then (2.25) holds on  $[0, \hat{s}]$ , since  $\rho^M \leq \rho^{M_2}$ . And (2.25) holds for all  $s \geq \hat{s}$  since  $\rho^M \leq \rho^{M_1}$  and (2.36).  $\square$

We note that the condition of Theorem 2.38-(i) implies that the origin of system (2.1) is GES, which follows from Theorem 2.31 with  $N = M = 1$ . The condition  $\rho \in \mathcal{K} \setminus \mathcal{K}_\infty$  of Theorem 2.38-(ii) indicates that there exists a compact set  $U$  containing the origin such that the system dynamic maps any point  $\xi \in \mathbb{R}^n$  in one step into  $U$ . This is the case if  $G$  in (2.1) is bounded. The conditions on the derivatives only require the study of the local behavior in 0. Furthermore, Theorem 2.38-(iii) considers the case where for large  $s > 0$ ,  $\alpha_1$  and  $\alpha_2$  are bounded from below and above by affine functions, and  $\rho$  is bounded from above by an affine function with slope less than one.

We highlight the following theorem, which combines the results of Theorems 2.31, 2.33 and 2.38-(i). This theorem shows that for GES systems the conditions imposed in Theorem 2.31 are sufficient and necessary (and hence, they are *non-conservative*).

**Theorem 2.39.** *The origin of the overall system (2.1) is GES if and only if there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ ,  $0 < c_1 \leq c_2$  and  $\lambda > 0$ , functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $i \in \{1, \dots, N\}$ , and linear functions  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $i, j \in \{1, \dots, N\}$ , such that the following conditions hold.*

- (i) Condition (2.18) holds with  $\alpha_{1i}(s) = c_1 s^\lambda$  and  $\alpha_{2i}(s) = c_2 s^\lambda$ .
- (ii) For all  $\xi \in \mathbb{R}^n$ , (2.21) holds.
- (iii) The map  $\Gamma_\oplus$  from (1.16) satisfies the small-gain condition (1.10).

Furthermore, there exists a global Lyapunov function for the overall system (2.1) of the form  $W(\xi) := \sum_{j=0}^{M-1} \max_i \varsigma_i \|x_i(j, \xi)\|_p$  with  $\varsigma_i > 0$ ,  $i \in \{1, \dots, N\}$ .

### 2.3.2 Illustrative example

In this section, we illustrate the results obtained in Section 2.3 by means of an example. In the following, we study the stability of an interconnected system, which is similar to the example in Section 2.2.5, with  $x_i \in \mathbb{R}$ ,  $i \in \{1, 2\}$ ,  $k \in \mathbb{N}$ :

$$\begin{aligned} x_1(k+1) &= 1.01x_1(k) - 0.3x_2(k) \\ x_2(k+1) &= x_1(k) + 0.3 \frac{x_2^2(k)}{1 + x_2^2(k)}. \end{aligned} \tag{2.37}$$

Firstly, the right-hand side map is globally  $\mathcal{K}$ -bounded, which follows by the same arguments as in Section 2.2.5. Furthermore, we observe that the origin of the first subsystem is 0-input unstable. So there does not exist a Lyapunov function for this subsystem. At this point, usually, existing (Lyapunov-based) small-gain theorems cannot be applied as they assume that the origin of each subsystem is (at least) GAS.

Suppose that Assumption 2.32 is satisfied. Following the proof of Theorem 2.33, by defining functions  $\mathcal{V}_i(\xi_i) := \eta(|\xi_i|)$  with  $\eta \in \mathcal{K}_\infty$ , we can find an  $M \in \mathbb{N}$ ,  $M \geq 1$  and functions  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$  such that the conditions (i)-(iii) of Theorem 2.33 are satisfied. Eventually, this leads to a global finite-step Lyapunov function for the overall system as constructed in (2.23), which implies GAS of the origin of system (2.37).

So let us start with  $\mathcal{V}_i(\xi_i) := |\xi_i|$ ,  $i = \{1, 2\}$ . Then we compute for all  $\xi \in \mathbb{R}^2$

$$\begin{aligned}\mathcal{V}_1(x_1(1, \xi)) &= |1.01\xi_1 - 0.3\xi_2| \leq \max\{2.02\mathcal{V}_1(\xi_1), 0.6\mathcal{V}_2(\xi_2)\}, \\ \mathcal{V}_2(x_2(1, \xi)) &= |\xi_1 + 0.3\frac{\xi_2^2}{1+\xi_2^2}| \leq \max\left\{2\mathcal{V}_1(\xi_1), 0.6\frac{\mathcal{V}_2^2(\xi_2)}{1+\mathcal{V}_2^2(\xi_2)}\right\}.\end{aligned}$$

From this estimate, we derive the gain  $\gamma_{11}(s) = 2.02s$ , which violates the small-gain condition (1.10). This is also clear as the origin of the first subsystem is 0-input unstable.

So we iterate the dynamics map, and see that for  $k = 3$ , we obtain

$$x(3, \xi) = \begin{pmatrix} 0.424301\xi_1 - 0.21603\xi_2 - 0.0909\frac{\xi_2^2}{1+\xi_2^2} - 0.09\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2} \\ 0.7201\xi_1 - 0.303\xi_2 - 0.09\frac{\xi_2^2}{1+\xi_2^2} + 0.3\frac{\left(1.01\xi_1 - 0.3\xi_2 + 0.3\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}\right)^2}{1+\left(1.01\xi_1 - 0.3\xi_2 + 0.3\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}\right)^2} \end{pmatrix}.$$

Using (2.16), we obtain the estimates

$$\begin{aligned}\mathcal{V}_1(x_1(3, \xi)) &\leq 0.424301|\xi_1| + 0.21603|\xi_2| + \frac{0.0909}{2}|\xi_2| + \frac{0.09}{2}(|\xi_1| + \frac{0.3}{2}|\xi_2|) \\ &= \max\{0.939\mathcal{V}_1(\xi_1), 0.537\mathcal{V}_2(\xi_2)\}, \\ \mathcal{V}_2(x_2(3, \xi)) &\leq 0.7201|\xi_1| + 0.303|\xi_2| + \frac{0.09}{2}|\xi_2| + \\ &\quad + \frac{0.3}{2}(1.01|\xi_1| + 0.3|\xi_2| + \frac{0.3}{2}(|\xi_1| + \frac{0.3}{2}|\xi_2|)) \\ &= \max\{1.789\mathcal{V}_1(\xi_1), 0.793\mathcal{V}_2(\xi_2)\}.\end{aligned}$$

From this we derive the linear functions

$$\gamma_{11}(s) = 0.939s, \quad \gamma_{12}(s) = 0.537s, \quad \gamma_{21}(s) = 1.789s, \quad \gamma_{22}(s) = 0.793s.$$

Since  $\gamma_{11} < \text{id}$ ,  $\gamma_{22} < \text{id}$  and  $\gamma_{12} \circ \gamma_{21} < \text{id}$ , we conclude from the cycle condition (Proposition 1.29) that the small-gain condition (1.10) is satisfied. Hence, Theorem 2.29 yields GAS of the origin of the overall system (2.37). Since the functions

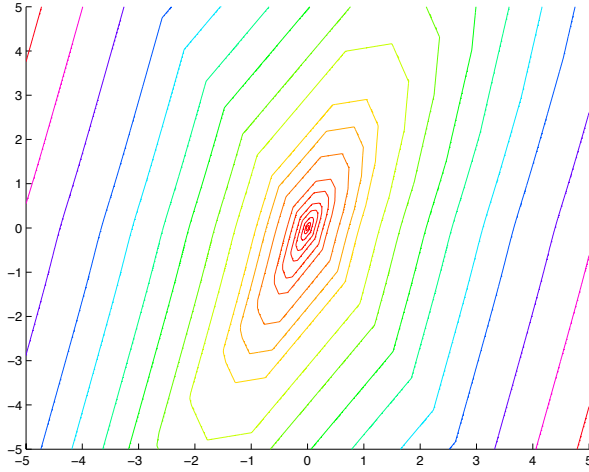


Figure 2.4: Contour plot of the Lyapunov function  $W(\xi) = \sum_{j=0}^2 \max \{2|x_1(j, \xi)|, \frac{25}{23}|x_2(j, \xi)|\}$  for the overall system (2.37) obtained via the small-gain approach (Theorem 2.33).

$\gamma_{ij}$  are linear as well as  $\alpha_1 = \alpha_2 = \text{id}$  satisfying (2.18), we can even apply Theorem 2.31 to conclude GES of the origin of the overall system (2.37).

Theorem 2.29 proves the GAS property of the interconnected system (2.1) by constructing a global finite-step Lyapunov function in (2.20). This construction is straightforward to implement, and is now executed for system (2.37). Firstly, we use the method proposed in [37] to compute the  $\Omega$ -path  $\sigma(r) := (0.5r, 0.92r)$ , which satisfies

$$\Gamma_{\oplus}(\sigma(r)) \approx \begin{pmatrix} 0.494r \\ 0.895r \end{pmatrix} < \begin{pmatrix} 0.5r \\ 0.92r \end{pmatrix} = \sigma(r)$$

for all  $r > 0$ . From the proof of Theorem 2.29 we can now conclude that  $V(\xi) := \max_i \sigma_i^{-1}(\mathcal{V}_i(\xi_i)) = \max\{2|\xi_1|, \frac{25}{23}|\xi_2|\}$  is a global finite-step Lyapunov function for the overall system (2.37). In addition, a global Lyapunov function  $W$  can be directly computed using (2.20) as

$$W(\xi) = \sum_{j=0}^2 \max \{2|x_1(j, \xi)|, \frac{25}{23}|x_2(j, \xi)|\}.$$

A contour plot of the global Lyapunov function  $W$  is shown in Figure 2.4.

*Remark 2.40.* Alternatively, we can use the approach presented in Section 2.2 to derive a global Lyapunov function as in Section 2.2.5. We omit the details, but



mention that for system (2.37) the function  $V(\xi) := \|\xi\|_2$  is a global finite-step Lyapunov function with  $M = 4$ . Hence, we obtain an alternative global Lyapunov function for this system as

$$\tilde{W}(\xi) = \sum_{k=0}^3 \|x(k, \xi)\|_2.$$

A contour plot of the global Lyapunov function  $\tilde{W}$  is shown in Figure 2.5.  $\triangleleft$

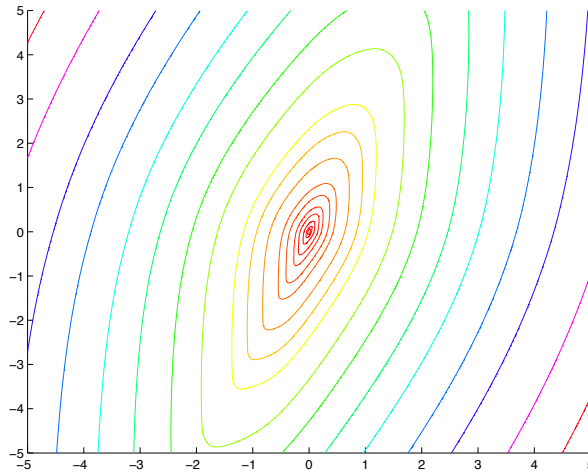


Figure 2.5: Contour plot of the Lyapunov function  $\tilde{W}(\xi) = \sum_{k=0}^3 \|x(k, \xi)\|_2$  for system (2.37) obtained via the converse Lyapunov function approach in Section 2.2 (Theorem 2.23).

Although the construction of the overall global Lyapunov function  $W$  via the small-gain approach in (2.20) requires the computation of an  $\Omega$ -path, we believe that for large-scale interconnections the small-gain approach is still more advisable than the converse Lyapunov function approach in Section 2.2 using the construction of a global Lyapunov function (2.13) (resp. (2.14)). The reason for this belief is that the choice of a suitable natural number  $M$  in (2.13) (resp. (2.14)) might be, in general, much higher than the choice for a suitable natural number  $M$  in (2.21), and thus for the construction of the Lyapunov function in (2.20). For instance, for system (2.37) the small-gain approach for constructing a global Lyapunov function yields  $M = 3$  as the smallest suitable natural number, whereas the direct construction of the global Lyapunov function yields  $M = 4$  as the smallest suitable natural number, as outlined in Remark 2.40.

## 2.4 Further applications

This section makes use of the developed converse Lyapunov theorems from Section 2.2 to obtain relevant implications for several classes of dynamical systems.

### 2.4.1 Continuous, polynomial and homogeneous dynamical systems

In [71] the authors prove a converse Lyapunov theorem under the assumption that the dynamics are continuous. In this case, the authors show that there also exists a continuous global Lyapunov function. Here we make use of the construction of the global Lyapunov function  $W$  in (2.13) in Theorem 2.23 (respectively (2.14) in Theorem 2.24) to show that the global Lyapunov function obtained is also continuous if the dynamics are continuous.

**Theorem 2.41.** *Let the origin of system (2.1) be GAS, and assume that Assumption 2.13 is satisfied. If  $G$  in (2.1) is continuous then there exists a continuous global Lyapunov function for system (2.1).*

*Proof.* Theorem 2.23 implies that

$$W(\xi) := \sum_{j=0}^{M-1} \eta(\|G^j(\xi)\|) \quad (2.38)$$

is a global Lyapunov function for system (2.1). Alternatively, Theorem 2.24 yields global Lyapunov function

$$W(\xi) := \max_{j \in \{0, \dots, M-1\}} \rho^{j/M} (\eta(\|G^{M-1-j}(\xi)\|))$$

as a global Lyapunov function for system (2.1). In both cases the composition of continuous functions  $(\eta, \rho, G)$  yields a continuous function, implying that  $W$  is a continuous global Lyapunov function.  $\square$

*Remark 2.42.* The converse Lyapunov theorem obtained in [71] does not only show that there exists a continuous global Lyapunov function, the authors also prove that there exists a smooth global Lyapunov function. Smoothness of the global Lyapunov function is achieved by using smoothing techniques, see e.g. [71], [81, Sec. 3]. However, for discrete-time systems, the existence of a smooth global Lyapunov function does not give any more insights of the system than a continuous global Lyapunov function. On the other hand, continuous global Lyapunov functions are important as even for discontinuous dynamics, a continuous global Lyapunov function already yields inherent robustness, see [94]. Additionally, if the system dynamics are discontinuous it is not possible to guarantee the existence of a continuous global Lyapunov function, see also [116].  $\triangleleft$

*Remark 2.43.* Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a polynomial function and suppose that Assumption 2.13 holds. Then there exists a polynomial global Lyapunov function for system (2.1). This follows by taking  $\eta(s) := s^2$  in the global Lyapunov function in (2.13), and noticing that the sum and the composition of polynomial functions yields a polynomial function.  $\triangleleft$

Next, we study another relevant type of map  $G$ , which was considered e.g. in [26, 27, 92, 121].

**Definition 2.44.** A function  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called *positively homogeneous of degree one* if for all  $\xi \in \mathbb{R}^n$  and all  $c > 0$  we have  $G(c\xi) = cG(\xi)$ .

**Theorem 2.45.** *Let  $G$  in (2.1) be positively homogeneous of degree one and let the origin of system (2.1) be GAS. Then there exists a global Lyapunov function for system (2.1) that is positively homogeneous of degree one.*

*If, in addition,  $G$  is continuous then there exists a continuous global Lyapunov function, which is positively homogeneous of degree one.*

*Proof.* If the map  $G$  is positively homogeneous of degree one and system (2.1) is GAS, it holds by Corollary V.3 in [92] that the origin of system (2.1) is GES, i.e., GAS is equivalent to GES for homogeneous dynamics. Let  $\eta \in \mathcal{K}_\infty$  be positively homogeneous of degree, then Corollary 2.26 yields a global Lyapunov function of the form (2.38). The proof is completed by observing that vector norms are positively homogeneous functions of degree one as well and that the composition of a finite number of such functions remains a positively homogeneous function of degree one. Hence, the function  $W$  defined in (2.38) is a global Lyapunov function, which is positively homogeneous of degree one. If, in addition,  $G$  is continuous then the constructed global Lyapunov function is also continuous by Theorem 2.41.  $\square$

In the next two sections we consider the cases of conewise linear and linear dynamical systems in more detail.

#### 2.4.2 Conewise linear dynamical systems

Now we focus on conewise linear systems, see e.g. the survey [135] or [9]. In [72] it was shown that conewise linear Lyapunov functions are sufficient for establishing GES for conewise linear systems and that such functions can be computed via linear programming. See also [95], which focuses on the discrete-time setting. The open question that remains to be answered is whether the existence of conewise linear Lyapunov functions is also necessary for GES conewise linear systems. In what follows we make use of the results of Section 2.2 to answer this question affirmatively, within the discrete-time setting.

To this end, a formal characterization of conewise linear dynamics is given. We need the following notion. A nonempty set  $C \subset \mathbb{R}^n$  is *convex* if for any two points  $\xi_1, \xi_2 \in C$  and  $\lambda \in [0, 1]$  we have  $\lambda\xi_1 + (1 - \lambda)\xi_2 \in C$ . The *dimension*  $\dim(C)$  of a convex set  $C$  is equal to the dimension of the smallest affine subspace  $U \subset \mathbb{R}^n$  containing  $C$ . We define the *relative interior* of a convex set  $C$  (denoted by  $\text{relint}(C)$ ) as its interior relative to the smallest affine subspace  $U \subset \mathbb{R}^n$  containing  $C$ . This is equivalent to the definition

$$\text{relint}(C) := \{\xi \in C : \forall \tilde{\xi} \in C \exists \lambda > 1 \text{ such that } \lambda\xi + (1 - \lambda)\tilde{\xi} \in C\}. \quad (2.39)$$

The *convex hull*  $\text{co}\{S\}$  of a set  $S \subset \mathbb{R}^n$  is the smallest convex set containing  $S$ , and  $\text{cl}\{S\}$  denotes the *closure* of  $S$ . A *ray* induced by a vector  $v \in \mathbb{R}^n$  is the set  $\langle v \rangle := \{cv : c \in \mathbb{R}_+\}$ .

In the following definition we define convex polyhedral cones. As this is the only type of cones considered in this thesis, we will for the sake of simplicity only speak of cones.

**Definition 2.46.** A nonempty set  $\mathcal{C} \subset \mathbb{R}^n$  is a (*convex polyhedral*) *cone* if  $\mathcal{C}$  is the convex hull of a finite number of rays, i.e.,  $\mathcal{C} := \text{co}\{\langle v_1 \rangle, \dots, \langle v_r \rangle\}$ . By  $\dim(\mathcal{C})$  we denote the number of linearly independent vectors  $v_1, \dots, v_r$ . If  $\mathcal{S}, \mathcal{C}$  are cones with  $\mathcal{S} \subset \mathcal{C}$  then  $\mathcal{S}$  is called a *subcone* of  $\mathcal{C}$ . If additionally  $\dim(\mathcal{S}) < \dim(\mathcal{C})$  then  $\mathcal{S}$  is a (*lower dimensional*) *subcone* of  $\mathcal{C}$ .

A finite set of cones  $\{\mathcal{C}_i \subset \mathbb{R}^n\}_{i \in \{1, \dots, l\}}$  defines an *l-conic partition* of  $\mathbb{R}^n$  if the following holds:

- (i)  $\bigcup_{i \in \{1, \dots, l\}} \text{relint}(\mathcal{C}_i) = \mathbb{R}^n$ ;
- (ii) for  $i \neq j$ ,  $i, j \in \{1, \dots, l\}$ , we have  $\text{relint}(\mathcal{C}_i) \cap \text{relint}(\mathcal{C}_j) = \emptyset$ .

Note that by definition of an *l-conic partition*, two cones can only intersect on the boundaries, and for any point  $\xi \in \mathbb{R}^n$  there exists a unique cone  $\mathcal{C}_i$  such that  $\xi$  is contained in the relative interior of  $\mathcal{C}_i$ . In particular, the cone  $\{0\}$  must be contained in the *l-conic partition*.

Next, consider the class of *conewise linear dynamics*, i.e.,

$$G(x) := A_i x \quad \text{if } x \in \text{relint}(\mathcal{C}_i); \quad i \in \{1, \dots, N\}, \quad (2.40)$$

where  $N \in \mathbb{N}$ ,  $A_i \in \mathbb{R}^{n \times n}$ ,  $i \in \{1, \dots, N\}$ , and the finite set of cones  $\{\mathcal{C}_i\}_{i \in \{1, \dots, N\}}$  defines an *N-conic partition* of  $\mathbb{R}^n$ . By the above considerations, the map  $G$  in (2.40) is well-defined. Observe that  $G$  satisfies the global  $\mathcal{K}$ -boundedness Assumption 2.1 with  $\omega(s) := \max_{i \in \{1, \dots, N\}} \|A_i\|s$ .

*Remark 2.47.* (i) Note that  $G$  in (2.40) is continuous if and only if for any  $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$  with  $i \neq j$ ,  $i, j \in \{1, \dots, N\}$ , it holds  $A_i \xi = A_j \xi$ , or, equivalently,  $\xi \in \ker(A_i - A_j)$ . Indeed, continuity of  $G$  is not required in our next result.

(ii) For any cone  $\mathcal{C}_i$ , the map  $G$  in (2.40) is only defined in the relative interior of  $\mathcal{C}_i$ , but can also be defined on the closed cone  $\mathcal{C}_i$ . In this case, well-posedness of  $G$ , i.e., that  $G$  is uniquely defined for any  $\xi \in \mathbb{R}^n$ , can only be guaranteed if there is a rule to decide which map is applied in points that lie on the boundary of several cones. In the case that  $G$  is continuous this is not an issue, see (i).  $\triangleleft$

Let  $x(\cdot, \xi)$  be the solution of the conewise linear system

$$x(k+1) = A_i x(k) \quad \text{if } x(k) \in \text{relint}(\mathcal{C}_i) \quad (2.41)$$

starting in  $x(0) = \xi \in \mathbb{R}^n$ . For any  $k \in \mathbb{N}$ , we associate the  $k$ -tuple  $(j_1, \dots, j_k)$  with  $j_i \in \{1, \dots, N\}$  that satisfies  $x(l, \xi) \in \text{relint}(\mathcal{C}_{j_{l+1}})$  for  $l \in \{0, \dots, k-1\}$ . Since  $G$  is well-defined, the associated  $k$ -tuple  $(j_1, \dots, j_k)$  is uniquely determined. Unifying these  $k$ -tuples for all  $\xi \in \mathbb{R}^n$  we obtain the set

$$\begin{aligned} \mathcal{I}_k := & \{(j_1, \dots, j_k) \in \{1, \dots, N\}^k : \\ & A_{j_k} \{ \dots \{ A_{j_2} \{ A_{j_1} \mathcal{C}_{j_1} \cap \mathcal{C}_{j_2} \} \cap \mathcal{C}_{j_3} \} \cap \dots \cap \mathcal{C}_{j_k} \} \neq \emptyset \}. \end{aligned}$$

Note that the set  $\mathcal{I}_k$  can be computed by basic operations (image under the linear mappings  $A_{j_i}$  from (2.41) and intersection) involving cones.

Furthermore, we define the set

$$\mathcal{A}_k := \left\{ \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] : (j_1, \dots, j_k) \in \mathcal{I}_k \right\},$$

where  $\left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] := A_{j_k} A_{j_{k-1}} \dots A_{j_1}$ , and for  $k = 0$  this product is defined as the identity matrix  $I$ .

The following theorem states that GES of the origin of a conewise linear system (2.41) is equivalent to the existence of a conewise linear Lyapunov function. Observe that conewise linear maps are positively homogeneous maps of degree one and, as such, GAS as defined in this work, or equivalently  $\mathcal{KL}$ -stability, is equivalent to GES by [92, Corollary V.3]. Thus, without loss of generality, we can state the following result in terms of GES.

**Theorem 2.48.** *The origin of the conewise linear system (2.41) is GAS, and thus also GES, if and only if it admits a global conewise linear Lyapunov function.*

*Proof.* In [95, Theorem 4.6] it is shown that the existence of a conewise linear Lyapunov function implies GAS of the origin of system (2.41), and hence even GES. So in this proof we consider the converse statement.

Let the origin of system (2.41) be GES. Then Corollary 2.16 implies that the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by  $V(\xi) := \|\xi\|_1$  is a global finite-step Lyapunov function for

system (2.41). Furthermore, let  $M \in \mathbb{N}$  satisfy condition (ii) of Definition 2.6. Hence, applying Corollary 2.26 we see that the function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by  $W(\xi) := \sum_{k=0}^{M-1} \|x(k, \xi)\|_1$  is a global Lyapunov function for system (2.41). We will now show that this global Lyapunov function is defined on a conic partition.

For any  $\xi \in \mathbb{R}^n$  let  $\iota := (j_1, \dots, j_M) \in \mathcal{I}_M$  be associated to the solution  $x(\cdot, \xi)$ , i.e.,

$$x(k, \xi) \in \text{relint}(\mathcal{C}_{j_{k+1}}) \quad \text{for } k \in \{0, \dots, M-1\}. \quad (2.42)$$

Note that the number  $\#\mathcal{I}_M$  of non-identical  $M$ -tuples in  $\mathcal{I}_M$  is at most  $N^M$ . Then for any  $\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$  we define the sets

$$\begin{aligned} \mathcal{M}_\iota &:= \{\xi \in \text{relint}(\mathcal{C}_{j_1}) : (2.42) \text{ holds}\} \\ &= \{\xi \in \text{relint}(\mathcal{C}_{j_1}) : \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \in \text{relint}(\mathcal{C}_{j_{k+1}}) \forall k \in \{1, \dots, M-1\}\}, \end{aligned} \quad (2.43)$$

and

$$\mathcal{D}_\iota := \text{cl}\{\mathcal{M}_\iota\}. \quad (2.44)$$

First observe, that by (2.43) and (2.44) an equivalent definition of  $\mathcal{D}_\iota$  is that  $\mathcal{D}_\iota$  is the largest subset  $\mathcal{S}$  of  $\mathcal{C}_{j_1}$  satisfying for all  $k \in \{1, \dots, M-1\}$

$$A_{j_k} \{ \dots \{ A_{j_2} \{ A_{j_1} \mathcal{S} \cap \mathcal{C}_{j_2} \} \cap \mathcal{C}_{j_3} \} \cap \dots \cap \mathcal{C}_{j_k} \} \subset \mathcal{C}_{j_{k+1}}. \quad (2.45)$$

We will now show that the sets  $\mathcal{D}_\iota$  are cones in the sense of Definition 2.46. So consider two cones  $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}$  with  $A_{j_1} \mathcal{C}_{j_1} \cap \mathcal{C}_{j_2} \neq \emptyset$ . If we can show that the largest subset  $\mathcal{S}_1 \subset \mathcal{C}_{j_1}$  with  $A_{j_1} \mathcal{S}_1 \subset \mathcal{C}_{j_2}$ , i.e.,  $A_{j_1} \mathcal{S}_1 = A_{j_1} \mathcal{C}_{j_1} \cap \mathcal{C}_{j_2}$ , is a cone, then the largest subset  $\mathcal{S}_2 \subset \mathcal{S}_1$  satisfying  $A_{j_2} A_{j_1} \mathcal{S}_2 = A_{j_2} A_{j_1} \mathcal{S}_1 \cap \mathcal{C}_{j_3}$  is a cone, and inductively, by (2.45),  $\mathcal{D}_\iota$  is a cone. To do this we first note that the intersection of two cones in the sense of Definition 2.46 is a cone. Furthermore, a cone under a linear map is a cone, since for  $A \in \mathbb{R}^{n \times n}$ , we have  $A \text{co}\{\langle v_1 \rangle, \dots, \langle v_r \rangle\} = \text{co}\{\langle Av_1 \rangle, \dots, \langle Av_r \rangle\}$ . On the other hand, the *pre-image* of a cone  $\mathcal{C}$  under a linear map  $A$  defined by  $\text{Inv}_A(\mathcal{C}) := \{\xi \in \mathbb{R}^n : A\xi \in \mathcal{C}\}$  is a cone [13, Proposition 5.1.8]. Hence,  $\mathcal{S}_1 = (\text{Inv}_{A_{j_1}}(A_{j_1} \mathcal{C}_{j_1}) \cap \mathcal{C}_{j_2}) \cap \mathcal{C}_{j_1}$  is a cone, and by the above argumentation, we conclude that  $\mathcal{D}_\iota$  is a cone.

Now we want to show that the finite set<sup>5</sup> of cones  $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$  forms a conic partition. Firstly, consider two cones  $\mathcal{D}_{\iota_1} \neq \mathcal{D}_{\iota_2}$ . We show that condition (ii) of Definition 2.46 holds. So assume to the contrary that  $\text{relint}(\mathcal{D}_{\iota_1}) \cap \text{relint}(\mathcal{D}_{\iota_2}) \neq \emptyset$ . Thus, there exists a  $\xi \in \text{relint}(\mathcal{D}_{\iota_1}) \cap \text{relint}(\mathcal{D}_{\iota_2})$ . In particular,  $\xi \neq 0$ . By (2.43) and (2.44), for any  $\iota \in \{j_1, \dots, j_M\}$ , we have  $\text{relint}(\mathcal{D}_\iota) = \mathcal{M}_\iota \subset \text{relint}(\mathcal{C}_{j_1})$ . Since the cones  $\{\mathcal{C}_i\}_{i \in \{1, \dots, N\}}$  form a conic partition, for any  $\xi \in \mathbb{R}^n$  the  $M$ -tuple  $\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$  for

---

<sup>5</sup>There exist at most  $N^M$  cones  $\mathcal{D}_\iota$ .

which (2.42) holds is uniquely determined. Thus, if  $\xi \in \text{relint}(\mathcal{D}_{\iota_1}) \cap \text{relint}(\mathcal{D}_{\iota_2})$ , then  $\iota_1 = \iota_2$ , showing condition (ii) of Definition 2.46.

Clearly,  $\bigcup_{\iota \in \mathcal{I}_M} \mathcal{D}_\iota = \mathbb{R}^n$ , since solutions  $x(k, \xi)$  are defined for any time  $k \in \mathbb{N}$  and any initial point  $\xi \in \mathbb{R}^n$ . To show condition (i) of Definition 2.46, we have to show that for all  $\xi \in \mathbb{R}^n$  there exists a cone  $\mathcal{D}_\iota$  such that  $\xi \in \text{relint}(\mathcal{D}_\iota)$ . So pick  $\xi \in \mathbb{R}^n$ , then, as  $\{\mathcal{C}_i\}_{i \in \{1, \dots, N\}}$  forms a conic partition, there exists a unique  $\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$  such that (2.42) holds. Assume to the contrary, that  $\xi \notin \text{relint}(\mathcal{D}_\iota)$ . Take any point  $\bar{\xi} \in \text{relint}(\mathcal{D}_\iota)$ . Then, by convexity of the cones  $\mathcal{C}_i$ , also  $x(k, \lambda\xi + (1 - \lambda)\bar{\xi}) \in \text{relint}(\mathcal{C}_{j_{k+1}})$  for all  $\lambda \in [0, 1]$  and all  $k \in \{0, \dots, M - 1\}$ . Thus, we can extend the line segment in  $\mathcal{D}_\iota$ , i.e., there exists a  $\varepsilon > 0$  such that  $x(k, \lambda\xi + (1 - \lambda)\bar{\xi}) \in \text{relint}(\mathcal{C}_{j_{k+1}})$  for all  $\lambda \in [0, 1 + \varepsilon]$  and all  $k \in \{0, \dots, M - 1\}$ . Then  $\lambda\xi + (1 - \lambda)\bar{\xi} \in \mathcal{D}_\iota$  for all  $\lambda \in [0, 1 + \varepsilon]$  implying, by (2.39),  $\xi \in \text{relint}(\mathcal{D}_\iota)$ . Therefore, the set  $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$  forms a conic partition of  $\mathbb{R}^n$ .

For any cone  $\mathcal{D}_\iota$  there exists a matrix  $P_\iota \in \mathbb{R}^{p \times n}$  with  $p \geq n$  such that

$$\sum_{k=0}^{M-1} \left\| \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_1 = \|P_\iota \xi\|_1 \quad (2.46)$$

for all  $\xi \in \text{relint}(\mathcal{D}_\iota)$ . This matrix  $P_\iota \in \mathbb{R}^{p \times n}$  can be chosen as

$$P_\iota = \left( \left[ \prod_{i=0}^{M-2} A_{j_{M-1-i}} \right]; \dots; A_{j_2} A_{j_1}; A_{j_1}; I \right),$$

where  $I$  denotes the identity matrix. In particular,  $p = Mn$ . Then the global Lyapunov function takes the explicit form

$$\begin{aligned} W(\xi) &= \sum_{k=0}^{M-1} \|x(k, \xi)\|_1 = \sum_{k=0}^{M-1} \left\| \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_1 \\ &= \|P_\iota \xi\|_1 \text{ if } \xi \in \text{relint}(\mathcal{D}_\iota). \end{aligned}$$

Since weighted 1-norms are conewise linear functions, see e.g. [84], and  $\bigcup_{\iota \in \mathcal{I}_M} \mathcal{D}_\iota$  defines a conic partition of  $\mathbb{R}^n$  we obtain that  $W$  is a conewise linear function, which concludes the proof.  $\square$

The proof of Theorem 2.48 essentially relies on refining the conic partition  $\{\mathcal{C}_i\}_{i \in \{1, \dots, N\}}$  to obtain the conic partition  $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$ . In the next example we will indicate how this refinement is obtained.

**Example 2.49.** Consider the vector space  $\mathbb{R}^3$ , and denote the  $i$ th unit vector in  $\mathbb{R}^3$

by  $e_i$ ,  $i \in \{1, 2, 3\}$ . Assume the partition of  $\mathbb{R}^3$  into the 8 orthants given by

$$\begin{aligned} \mathcal{C}_1 &= \text{co}\{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_2 &= \text{co}\{\langle e_1 \rangle, \langle e_2 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_3 &= \text{co}\{\langle e_1 \rangle, \langle -e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_4 &= \text{co}\{\langle e_1 \rangle, \langle -e_2 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_5 &= \text{co}\{\langle -e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_6 &= \text{co}\{\langle -e_1 \rangle, \langle e_2 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_7 &= \text{co}\{\langle -e_1 \rangle, \langle -e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_8 &= \text{co}\{\langle -e_1 \rangle, \langle -e_2 \rangle, \langle -e_3 \rangle\}. \end{aligned}$$

Note that this partition is not a conic partition in the sense of Definition 2.46, since  $\bigcup_{i \in \{1, \dots, 8\}} \text{relint } \mathcal{C}_i \neq \mathbb{R}^n$ . To achieve that all points in  $\mathbb{R}^n$  are contained in the interior of a cone, we have to add the 2-dimensional cones

$$\begin{aligned} \mathcal{C}_9 &= \text{co}\{\langle e_1 \rangle, \langle e_2 \rangle\}, & \mathcal{C}_{10} &= \text{co}\{\langle e_1 \rangle, \langle -e_2 \rangle\} \\ \mathcal{C}_{11} &= \text{co}\{\langle e_1 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_{12} &= \text{co}\{\langle e_1 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_{13} &= \text{co}\{\langle -e_1 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_{14} &= \text{co}\{\langle -e_1 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_{15} &= \text{co}\{\langle -e_1 \rangle, \langle -e_2 \rangle\}, & \mathcal{C}_{16} &= \text{co}\{\langle -e_1 \rangle, \langle e_2 \rangle\} \\ \mathcal{C}_{17} &= \text{co}\{\langle e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_{18} &= \text{co}\{\langle e_2 \rangle, \langle -e_3 \rangle\}, \\ \mathcal{C}_{19} &= \text{co}\{\langle -e_2 \rangle, \langle e_3 \rangle\}, & \mathcal{C}_{20} &= \text{co}\{\langle -e_2 \rangle, \langle -e_3 \rangle\}, \end{aligned}$$

the 1-dimensional cones

$$\begin{aligned} \mathcal{C}_{21} &= \langle e_1 \rangle, & \mathcal{C}_{22} &= \langle -e_1 \rangle, \\ \mathcal{C}_{23} &= \langle e_2 \rangle, & \mathcal{C}_{24} &= \langle -e_2 \rangle, \\ \mathcal{C}_{25} &= \langle e_3 \rangle, & \mathcal{C}_{26} &= \langle -e_3 \rangle, \end{aligned}$$

and the 0-dimensional cone

$$\mathcal{C}_{27} = \{0\}.$$

To see how the cones  $\mathcal{D}_l$  are generated consider the cone  $\mathcal{C}_9$  with corresponding linear map  $A_9 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ . Thus, for any point  $\xi \in \text{relint}(\mathcal{C}_9)$  we have  $G(\xi) = A_9 \xi$ . We see that for all  $\xi \in \text{relint}(\mathcal{C}_9)$  we have  $G(\xi) \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_9$ . In particular, we obtain the refinement of the cone  $\mathcal{C}_9$  into the cones

$$\begin{aligned} \mathcal{D}_{(9,1)} &= \text{co} \left\{ \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \langle e_2 \rangle \right\}, \\ \mathcal{D}_{(9,2)} &= \text{co} \left\{ \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \langle e_1 \rangle \right\}, \\ \mathcal{D}_{(9,9)} &= \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \end{aligned}$$

The partition of the cone  $\mathcal{C}_9$  into the cones  $\mathcal{D}_{(9,1)}$ ,  $\mathcal{D}_{(9,2)}$ , and  $\mathcal{D}_{(9,9)}$  is shown in Figure 2.6. We see that the number of cones  $\mathcal{D}_l$  generated may, in general, increase fast. ◁



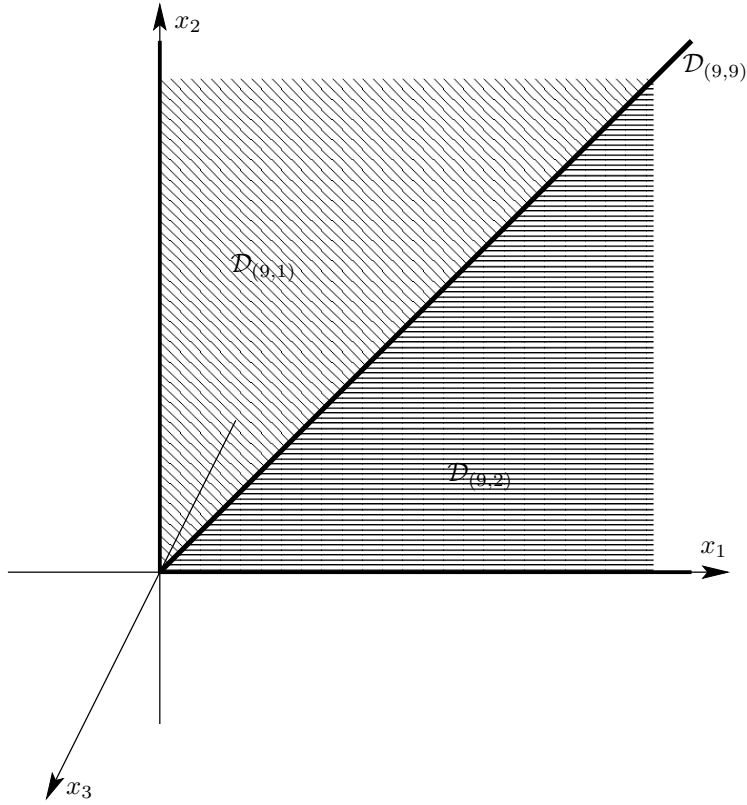


Figure 2.6: The partition of the cone  $\mathcal{C}_9$  into the cones  $\mathcal{D}_{(9,1)}$ ,  $\mathcal{D}_{(9,2)}$ , and  $\mathcal{D}_{(9,9)}$ .

If the dynamics of the conewise linear system (2.41) are continuous then Theorem 2.45 directly yields a continuous global Lyapunov function. In this case, the conewise linear dynamics (2.40) are well-defined on the intersection of two cones, see Remark 2.47. Thus the conewise linear system (2.41) can be written as

$$x(k+1) = A_i x(k) \quad \text{if } x(k) \in \mathcal{C}_i, \quad (2.47)$$

where

- (i)  $\bigcup_i \mathcal{C}_i = \mathbb{R}^n$ ;
- (ii) for  $i \neq j$  we have  $\text{relint}(\mathcal{C}_i) \cap \text{relint}(\mathcal{C}_j) = \emptyset$ ; and
- (iii) for  $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$  it holds  $A_i \xi = A_j \xi$ .

By continuity of the right-hand side of system (2.47), we do not need to worry about well-posedness at the points on the boundary of the cones  $\mathcal{C}_i$ . Hence, the proof of Theorem 2.48 can be simplified as outlined in Procedure 2.50.

*Procedure 2.50.* The following steps show in an algorithmic fashion how a continuous global conewise linear Lyapunov function can be obtained for a continuous conewise linear system (2.41). We emphasize that this method only illustrates how a continuous global conewise linear Lyapunov function can be constructed, whereas we do not treat the problem of an efficient implementation here.

- [1] Compute  $M \in \mathbb{N}$  satisfying condition (ii) of Definition 2.6 for  $V(\xi) := \|\xi\|_1$  as follows:
  - (i) Set  $k = 1$ ;
  - (ii) Compute  $\rho := \max_{(j_1, \dots, j_k) \in \mathcal{I}_k} \|\prod_{i=1}^k A_{j_i}\|_1$ ;
  - (iii) If  $\rho < 1$  set  $M = k$  and go to step [2]; else set  $k = k + 1$  and repeat with step (ii).

- [2] For any  $\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$  define the cones

$$\mathcal{D}_\iota := \left\{ \xi \in \mathcal{C}_{j_1} : \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \in \mathcal{C}_{j_{k+1}} \forall k \in \{1, \dots, M-1\} \right\}.$$

(Note that these cones are closed.)

- [3] Take those cones  $\mathcal{D}_{\iota_1}$  that are not contained in another cone  $\mathcal{D}_{\iota_2}$ , i.e.,

$$\mathcal{P}_M := \{ \mathcal{D}_{\iota_1} : \iota \in \mathcal{I}_M \text{ and } \forall \iota_2 \in \mathcal{I}_M \mathcal{D}_{\iota_1} \not\subset \mathcal{D}_{\iota_2} \}.$$

(Note that  $\bigcup_{\mathcal{D}_\iota \in \mathcal{P}_M} \mathcal{D}_\iota = \mathbb{R}^n$  and  $\text{reint}(\mathcal{D}_i) \cap \text{reint}(\mathcal{D}_j) = \emptyset$  if  $i \neq j$ .)

- [4] Define  $P_\iota$  as in (2.46).

- [5] Then

$$W(\xi) := \|P_\iota \xi\|_1 \quad \text{if } \xi \in \mathcal{D}_\iota$$

is a continuous global conewise linear Lyapunov function.

◁

*Remark 2.51.* For continuous conewise linear systems (2.41) an alternative approach to the proof of Theorem 2.48 is to approximate the Lyapunov function by a continuous conewise linear Lyapunov function. By the robustness of a continuous global Lyapunov function this also yields a conewise linear Lyapunov function. Note that by applying such an approximation, the information on the number of cones required for the conewise linear Lyapunov function is lost. ◁

The proof of Theorem 2.48 is constructive as it yields a global conewise linear Lyapunov function  $W$ . In this proof we use the *sum* formulation (2.13) of Corollary 2.26 for the 1-norm to construct the Lyapunov function  $W$ . Alternatively, we can use the *max* formulation (2.14) of Corollary 2.26 for the infinity norm as follows.

Since we have a conewise linear system, the positive definite function  $\rho$  in (2.14), corresponding to the global finite-step Lyapunov function  $V(\xi) = \|\xi\|_\infty$  in Definition 2.6, can be chosen as  $\rho(s) = cs$  with  $c \in [0, 1)$ . Using the same conic partition  $\mathbb{R}^n = \bigcup_{\iota \in \mathcal{I}_M} \mathcal{D}_\iota$  as in the proof of Theorem 2.48, we have for  $\xi \in \text{relint}(\mathcal{D}_\iota)$ ,

$$\begin{aligned} W(\xi) &= \max_{k \in \{0, \dots, M-1\}} \rho^{k/M} (\|x(k, \xi)\|_\infty) \\ &= \max_{k \in \{0, \dots, M-1\}} \rho^{k/M} \left( \left\| \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_\infty \right) \\ &= \max_{k \in \{0, \dots, M-1\}} \|c^{k/M} \left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi\|_\infty \\ &= \|P_\iota \xi\|_\infty \end{aligned}$$

with

$$P_\iota = \left( c^{\frac{M-1}{M}} \left[ \prod_{i=0}^{M-2} A_{j_{M-1-i}} \right]; \dots; c^{\frac{2}{M}} A_{j_2} A_{j_1}; c^{\frac{1}{M}} A_{j_1}; I \right),$$

where  $P_\iota \in \mathbb{R}^{nM \times n}$ . Hence,  $W$  is a global conewise linear Lyapunov function for system (2.41). This is the infinity norm analogue to the 1-norm construction in Theorem 2.48.

Besides establishing non-conservatism of conewise linear Lyapunov functions for stability analysis of conewise linear systems, Theorem 2.48 and the above paragraph for the infinity norm case, provide an explicit construction of such Lyapunov functions. The construction depends on finding an admissible value of the positive integer  $M$ , related to the finite-step Lyapunov condition, which hinges on computing the set  $\mathcal{A}_k$ .

**Example 2.52.** To illustrate the above results, consider the discontinuous dynamics (2.40) with  $N = 9$ , and

$$A_i = \begin{pmatrix} 0.197 & -0.241 \\ 1.845 & 1.703 \end{pmatrix} \quad \text{for } i \in \{1, 3, 5, 7, 9\},$$

and

$$A_i = \begin{pmatrix} -0.638 & -0.823 \\ 0.175 & 0.538 \end{pmatrix} \quad \text{for } i \in \{2, 4, 6, 8\}.$$

Note that  $A_1$  is unstable. The corresponding conic partition is defined by  $\{\mathcal{C}_i\}_{i \in \{1, \dots, 9\}}$ , where  $\mathcal{C}_i = \{x \in \mathbb{R}^2 : E_i x > 0\}$  for all  $i \in \{1, \dots, 4\}$  with

$$E_1 = -E_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad E_2 = -E_4 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

are the 2-dimensional cones. Furthermore, the 1-dimensional cones (rays) are

$$\begin{aligned} \mathcal{C}_5 &= \{x \in \mathbb{R}^2 : x_1 = x_2 \geq 0\}, \\ \mathcal{C}_6 &= \{x \in \mathbb{R}^2 : x_1 = x_2 \leq 0\}, \\ \mathcal{C}_7 &= \{x \in \mathbb{R}^2 : -x_1 = x_2 \geq 0\}, \\ \mathcal{C}_8 &= \{x \in \mathbb{R}^2 : -x_1 = x_2 \leq 0\}, \end{aligned}$$

and the 0-dimensional cone is  $\mathcal{C}_9 = \{0\}$ . To make use of the results developed in Section 2.2, we first indicate that the function  $V(x) = \|x\|_1$  is a global finite-step Lyapunov function with  $M = 18$ , which was established by computing  $\|\prod_{i=1}^k A_{j_i}\|_1$  for all  $(j_1, \dots, j_k) \in \mathcal{A}_k$  for  $k \in \{1, \dots, 18\}$ . Hence, the function

$$W(\xi) := \sum_{k=0}^{17} \|x(k, \xi)\|_1$$

is a global Lyapunov function for system (2.41).

In Figure 2.7 we show a contour plot of the constructed non-convex conewise linear Lyapunov function. Note that  $W$  is conewise linear with respect to a conic partition  $\mathcal{D}_\iota$ , which is finer than that given by  $\mathcal{C}_1, \dots, \mathcal{C}_9$ . This follows from the trajectory-wise definition of  $W$ . The effect can be seen in Figure 2.7, where discontinuities of  $W$  occur along rays. ◁

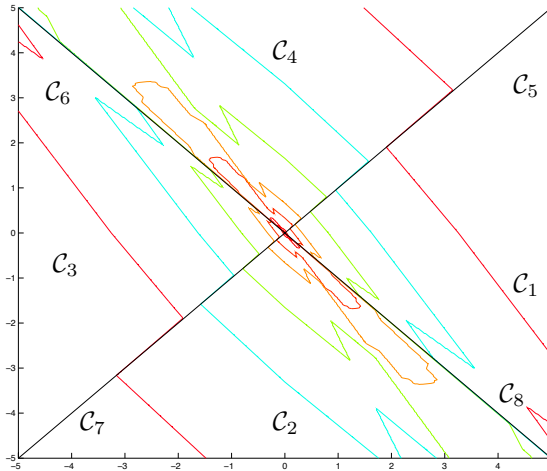


Figure 2.7: Contour plot of the Lyapunov function  $W(\xi) := \sum_{k=0}^{17} \|x(k, \xi)\|_1$  for Example 2.52.

**Example 2.53.** Consider Example 6.1 in [135]. In this example the author showed the global exponential stability of a conewise linear system, where two out of three of the linear dynamics<sup>6</sup> are unstable. The system is given by

$$x(k+1) = A_i x(k), \quad k \in \mathbb{N}, \quad x(k) \in \text{relint } \mathcal{C}_i$$

with

$$A_1 = \begin{bmatrix} -1.4078 & 0.1223 \\ 1.3846 & 0.4437 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2405 & 0.1223 \\ -0.3420 & 0.4437 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -0.5837 & -0.7019 \\ 0.5213 & 1.3070 \end{bmatrix},$$

and

$$\begin{aligned} \mathcal{C}_1 &= \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x \geq 0 \right\}, \\ \mathcal{C}_2 &= \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x \geq 0 \right\}, \\ \mathcal{C}_3 &= \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x \geq 0 \right\}. \end{aligned}$$

Accordingly we define

$$\mathcal{C}_4 := \mathcal{C}_1 \cap \mathcal{C}_2, \quad \mathcal{C}_5 := \mathcal{C}_2 \cap \mathcal{C}_3, \quad \mathcal{C}_6 := \mathcal{C}_1 \cap \mathcal{C}_3, \quad \mathcal{C}_7 = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 = \{0\},$$

and

$$A_4 = A_7 = A_1, \quad A_5 = A_2, \quad A_6 = A_3.$$

Similarly, as done in the previous example, one can compute  $M = 4$  for the infinity norm case, and  $M = 3$  for the 1-norm case, for which the corresponding finite-step Lyapunov function condition is met. Hence, the function  $V(\xi) := \|\xi\|_1$  is a global finite-step Lyapunov function by Theorem 2.14, and the function  $W(\xi) := \sum_{i=0}^2 \|x(k, \xi)\|_1$  is a global Lyapunov function for this system. A contour plot showing the sublevel sets of  $W$  is given in Figure 2.8.  $\triangleleft$

### 2.4.3 Linear dynamical systems

Interestingly, if the conewise linear dynamics reduce to standard linear dynamics, Theorem 2.48 implies that GES is equivalent to the existence of a global *polyhedral*<sup>7</sup> Lyapunov function of the form  $W(\xi) := \|P\xi\|_{1,\infty}$ . In this case, the Lyapunov weight matrix  $P \in \mathbb{R}^{p \times n}$  with  $p \geq n$  is not square in general, but of full column rank. Polyhedral Lyapunov functions [84, 102, 111] are in fact *convex* conewise linear functions,

<sup>6</sup>Please note the typo in [135]: The matrices  $A_2$  and  $A_3$  have to be interchanged.

<sup>7</sup>I.e., the sublevel sets of such a function are convex polyhedra with zero in their interior.

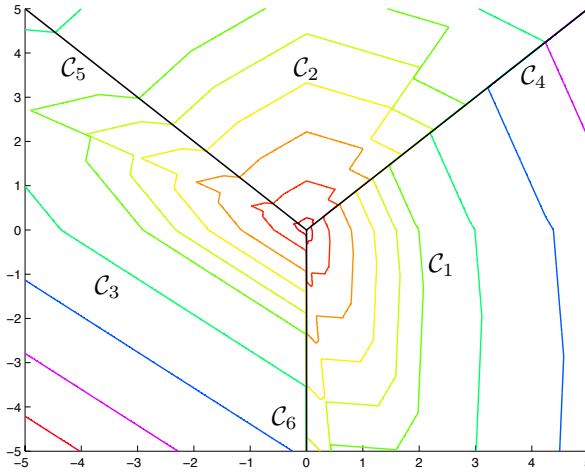


Figure 2.8: A contour plot of the Lyapunov function  $W(\xi) := \sum_{k=0}^2 \|x(k, \xi)\|_1$  in Example 2.53 .

which can be expressed as the maximum over a finite number of linear functions (see also [91] for further insights).

The above observation is stated formally next.

**Corollary 2.54.** *The origin of the linear system*

$$x(k+1) = Ax(k) \tag{2.48}$$

with  $k \in \mathbb{N}$ , and  $A \in \mathbb{R}^{n \times n}$  is GES if and only if there exists a global polyhedral Lyapunov function of the form  $W(\xi) = \|P\xi\|_1$  with  $P \in \mathbb{R}^{p \times n}$ ,  $p \geq n$ .

In particular, if  $M \in \mathbb{N}$  satisfies  $C\mu^M < 1$  with  $C \geq 1, \mu \in [0, 1)$  satisfying (2.3) and (2.2), then the number of rows of  $P$  can always be chosen as  $p = Mn$ .

We highlight that Corollary 2.54 explicitly gives an exact bound on the number of rows of  $P$ , which, by the best of the author’s knowledge, has not been solved elsewhere, see also Remark 2.56.

*Proof.* The sufficiency part was proven in [84]. So it remains to show that GES of the origin of the linear system (2.48) implies the existence of a matrix  $P \in \mathbb{R}^{p \times n}$  such that  $W(\xi) = \|P\xi\|_1$  is a global Lyapunov function. First note that by Corollary 2.26, GES implies the existence of an  $M \in \mathbb{N}$  such that the function  $W(\xi) := \sum_{k=0}^{M-1} \|x(k, \xi)\|_1 = \sum_{k=0}^{M-1} \|A^k \xi\|_1$  is a global Lyapunov function. In

particular,  $M$  satisfies  $C\mu^M < 1$ , where  $C \geq 1, \mu \in [0, 1)$  stem from (2.3) and (2.2). Exploiting the fact that

$$\sum_{k=0}^{M-1} \|A^k \xi\|_1 = \|P\xi\|_1$$

with

$$P := (A^{M-1}; A^{M-2}; \dots; A; I)$$

we obtain that  $W(\xi) = \|P\xi\|_1$  is a global Lyapunov function for the linear system (2.48). Note that  $P$  has full column rank because of the identity matrix in the last row block.  $\square$

*Remark 2.55.* The proof of Corollary 2.54 constructs a suitable global Lyapunov function  $W$  as a weighted 1-norm. Following the paragraph after Remark 2.51 it is easy to see that  $W(\xi) := \|P\xi\|_\infty$ , i.e., a weighted infinity norm, is a global polyhedral Lyapunov function as well. In this case, however, the matrix  $P$  is defined as

$$P := \left( (c^{1/M}A)^{M-1}; \dots; c^{2/M}A^2; c^{1/M}A; I \right),$$

where  $c \in (0, 1)$  and  $\rho(s) := cs$  satisfies condition (ii) of Definition 2.6 for the global finite-step Lyapunov function  $V(\xi) = \|\xi\|_\infty$ .  $\triangleleft$

*Remark 2.56.* In [111], [84] and, among several other works, [91], existence of a polyhedral Lyapunov function  $W(\xi) = \|P\xi\|_{1,\infty}$  with  $P \in \mathbb{R}^{p \times n}$  is established for GES linear systems. We stress that [111] treats the more general problem of difference inclusions. However, the proofs therein are rather complex and not constructive. In particular, no estimate of an upper bound on the number of rows  $p$  of the Lyapunov weight matrix  $P$  is given. This is in fact one of the non-trivial, open problems in the construction of polyhedral Lyapunov functions for linear systems, see e.g. [10, 11, 102]. In [10, 11] the problem is studied for continuous-time systems and lower bounds are given in terms of the geometry of the spectrum of  $A$ . Corollary 2.54 solves this problem by explicitly giving an admissible value of  $p$  for the 1-norm case, while an admissible value of  $p$  for the infinity norm case is given in Remark 2.55. In both cases  $p = Mn$ , where  $M$  is derived from the corresponding global finite-step Lyapunov function  $V(\xi) = \|\xi\|_{1,\infty}$ .  $\triangleleft$

Based on the above results and insights, we are in a position to provide a systematic procedure for constructing polyhedral Lyapunov functions for linear systems that is applicable in state spaces of high dimension. Note that this is attained without employing a (Jordan) decomposition of the  $A$  matrix or any further assumptions on the eigenvalues of  $A$ , as done in existing works on this topic, see e.g. [91] and the references therein. To this end, in view of the proof of Corollary 2.26, it is possible to obtain an admissible value for  $M$  analytically, for linear systems. The procedure is as follows.

*Procedure 2.57.* Consider the matrix  $A \in \mathbb{R}^{n \times n}$  from the linear system (2.48). Let  $Q \in \mathbb{R}^{n \times n}$  be any given symmetric and positive definite matrix, denoted by  $Q \succeq 0$ . We consider the linear matrix inequalities (LMIs)

$$\begin{aligned} \rho(A)^2 R - A^\top R A &\succeq Q, \\ c_2 I_n &\succeq R \succeq I_n, \end{aligned} \tag{2.49}$$

where  $c_2 \geq 1$  and  $\rho(A)$  denotes the spectral radius of  $A$ . There exists a matrix  $R$  that satisfies the LMIs (2.49) if and only if the origin of the linear system is GES, see e.g. [73, Corollary 3.2]. A matrix  $R$  that satisfies (2.49) can be found by solving the LMIs (2.49) while minimizing  $c_2$ . Thus, it follows from standard Lyapunov arguments, see e.g. [71], that the GES property  $\|x(k, \xi)\|_2 \leq C\mu^k \|\xi\|_2$  holds with  $\mu := \rho(A)$ ,  $C := \sqrt{c_2}$  for the Euclidean norm  $\|\cdot\|_2$ . Hence, it follows from the condition  $C\mu^M < 1$  that any  $M \in \mathbb{N}$  satisfying  $M > \log_\mu(\frac{1}{C})$  is admissible in the sense that it provides a valid global finite-step Lyapunov function.  $\triangleleft$

Note that for the 1-norm and the infinity norm case, we have to choose  $C := \sqrt{nc_2}$  in Procedure 2.57, which follows by the equivalences of norms in  $\mathbb{R}^n$ . The polyhedral Lyapunov function is then directly obtained from Corollary 2.54 or Remark 2.55, respectively.

An alternative to the LMI approach in Procedure 2.57 to computing a suitable  $M \in \mathbb{N}$  was obtained in [92].

*Procedure 2.58.* Take any norm  $\|\cdot\|$ . Then it holds  $\|x(k, \xi)\| = \|A^k \xi\| \leq \|A^k\| \|\xi\|$ . Define  $M := \min\{k \in \mathbb{N} : \|A^k\| < 1\}$ .  $\triangleleft$

As we can see in the following example, this second approach yields smaller values for  $M \in \mathbb{N}$  compared to the LMI approach.

**Example 2.59.** To illustrate the results for linear dynamics, consider system (2.48)

$$x(k+1) = Ax(k) := \begin{bmatrix} 1 & 0.4 \\ -0.2 & 0.9 \end{bmatrix} x(k), \quad k \in \mathbb{N}. \tag{2.50}$$

We construct global Lyapunov functions both for the 1-norm case and for the infinity norm case.

(i) In the 1-norm case we obtain  $M = 74$  by the LMI approach (2.49), whereas the second approach, Procedure 2.58, for computing  $M \in \mathbb{N}$  yields  $\|x(k, \xi)\|_1 \leq \|A^k\|_1 \|\xi\|_1$ , and  $\|A^{11}\|_1 < 1$ . Hence for  $M = 11$ , the function  $W_1(\xi) = \|P_1 \xi\|_1$  is a conewise linear Lyapunov function for system (2.50), where  $P_1 \in \mathbb{R}^{2 \times 2}$  can be



computed in a straightforward manner by Corollary 2.54 as

$$P_1 = \begin{bmatrix} -0.8193 & 0.3726 \\ -0.1863 & -0.9125 \\ -0.6764 & 0.7147 \\ -0.3573 & -0.8551 \\ -0.4753 & 1.0053 \\ -0.5027 & -0.7267 \\ -0.2314 & 1.2199 \\ -0.6099 & -0.5363 \\ 0.0365 & 1.3392 \\ -0.6696 & -0.2983 \\ 0.3068 & 1.3516 \\ -0.6758 & -0.0311 \\ 0.5576 & 1.2540 \\ -0.6270 & 0.2441 \\ 0.7680 & 1.0520 \\ -0.5260 & 0.5050 \\ 0.9200 & 0.7600 \\ -0.3800 & 0.7300 \\ 1.0000 & 0.4000 \\ -0.2000 & 0.9000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}.$$

In Figure 2.9 we provide a contour and surface plot of the Lyapunov function  $W_1$ .

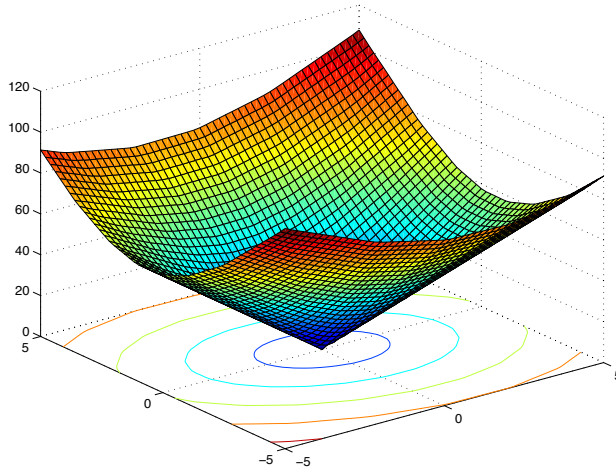


Figure 2.9: Surface plot of the Lyapunov function  $W_1$  of Example 2.59.

(ii) In the infinity norm case, again, we obtain  $M = 74$  by the LMI approach (2.49). Alternatively,  $\|A^{11}\|_\infty < 1$  implies that  $V(\xi) = \|\xi\|_\infty$  is a global finite-step Lyapunov function for the linear system (2.50) satisfying  $V(x(11, \xi)) \leq \|A^{11}\|_\infty V(\xi)$ . Hence for  $M = 11$ , the function  $W_\infty(\xi) = \|P_\infty \xi\|_\infty$  is a conewise linear Lyapunov function for system (2.50), where  $P_\infty \in \mathbb{R}^{22 \times 2}$  can be computed in a straightforward manner

by Remark 2.55 as

$$P_\infty = \begin{bmatrix} -0.7520 & 0.3420 \\ -0.1710 & -0.8375 \\ -0.6262 & 0.6616 \\ -0.3308 & -0.7916 \\ -0.4438 & 0.9387 \\ -0.4693 & -0.6785 \\ -0.2179 & 1.1488 \\ -0.5744 & -0.5051 \\ 0.0346 & 1.2720 \\ -0.6360 & -0.2834 \\ 0.2939 & 1.2949 \\ -0.6475 & -0.0298 \\ 0.5388 & 1.2117 \\ -0.6059 & 0.2359 \\ 0.7485 & 1.0253 \\ -0.5126 & 0.4922 \\ 0.9044 & 0.7471 \\ -0.3735 & 0.7176 \\ 0.9915 & 0.3966 \\ -0.1983 & 0.8923 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix},$$

where we took  $c = 0.91$ . ◁

**Example 2.60.** To illustrate the applicability of the developed methods in high dimension state spaces, consider the linear system (2.48) with

$$A = \begin{bmatrix} 0 & -0.3 & 0.1 & -0.1 & -0.3 & 0 & -0.1 & -0.1 & 0 & -0.4 \\ -0.4 & 0.4 & -0.3 & 0.1 & -0.2 & 0.1 & -0.4 & 0 & -0.2 & 0.4 \\ -0.4 & -0.1 & 0.1 & 0.2 & -0.3 & -0.3 & 0 & 0.2 & 0.3 & 0.3 \\ -0.4 & -0.3 & 0.2 & -0.1 & -0.1 & -0.1 & 0 & -0.1 & -0.2 & 0.4 \\ 0.3 & 0.2 & 0.4 & 0.2 & -0.2 & -0.4 & -0.3 & -0.4 & 0.1 & 0.2 \\ -0.4 & 0.4 & 0.2 & 0.3 & -0.2 & -0.1 & 0.4 & -0.2 & -0.3 & 0.4 \\ 0 & 0.2 & 0.2 & 0 & 0.2 & -0.3 & -0.3 & -0.4 & 0.2 & -0.1 \\ 0.4 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0.4 & 0.3 & 0.4 & 0 \\ 0.2 & 0 & -0.2 & -0.3 & -0.3 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0.1 & 0 & -0.1 & 0.4 & 0.3 & -0.2 & 0.3 & 0.4 & 0.3 \end{bmatrix}.$$

This systems is GES as the spectral radius of  $A$  is 0.9544, and hence, less than 1. So we can compute the value  $M \in \mathbb{N}$  by the LMI approach (2.49) as  $M = 39$ . Using the alternative approach, by computing the minimal  $k \in \mathbb{N}$  satisfying  $\|A^k\| < 1$ , Procedure 2.58, we obtain for the 1-norm case a value of  $M_1 = 17$ , and for the infinity-norm case a value of  $M_\infty = 20$ . Thus, by Corollary 2.54, we obtain the linear Lyapunov function  $W_1(\xi) = \|P_1\xi\|_1$ , where the matrix  $P_1 \in \mathbb{R}^{170 \times 10}$  is given by Corollary 2.54. Following Remark 2.55, we obtain the linear Lyapunov function  $W_\infty(\xi) = \|P_\infty\xi\|_\infty$  with matrix  $P_\infty \in \mathbb{R}^{200 \times 10}$ . ◁

## 2.5 Notes and references

The concept (but not the name) of a finite-step Lyapunov function was first introduced in [1] for time-varying continuous-time systems, where it was shown that its existence implies uniform asymptotic stability of the origin. The proof relies on the  $\varepsilon, \delta$ -criterion in contrast to our proof relying on the construction of a  $\mathcal{KL}$ -function  $\beta$ . We point out that for continuous-time systems, similar results to those presented in Section 2.2 may be derived. Nevertheless, this approach requires the computation of solutions of the systems in both continuous and discrete time. In discrete time, computing solutions reduces to iterating the dynamics map  $G$ . Thus, this approach

appears to be implementable in a wide range of applications. Otherwise, in continuous time solutions cannot be computed easily or exactly, in general. Recent works on finite-step Lyapunov functions, including the results presented in this chapter, are [31–35, 41, 92]. Note that e.g. in [31, 92] and several preprints the term “finite-time” Lyapunov function is used, but as this term has also been used in studying *finite-time stability*<sup>8</sup> we do now use the term *finite-step Lyapunov function*.

The existence of a Lyapunov function is guaranteed by converse Lyapunov theorems under the assumption that the origin of the system is GAS. Classic results in this direction are [5, 73, 107, 108] and also the seminal books [56, 142]. Extensions of this theory can be found e.g. in [4, 71, 81, 116, 139], and the references therein. For the discrete-time case, which is of interest in this work, the abstract construction of a global Lyapunov function for a converse theorem is performed by taking infinite series [71] or the supremum over all solutions and all times [81, 116]. For certain classes of dynamical systems, Lyapunov functions are guaranteed to exist in classes of functions that are computationally easy to describe; e.g. quadratic functions [73] for linear difference equations and polyhedral functions [4, 84, 111] for linear difference inclusions. But for most nonlinear systems we only know that  $C^\infty$  Lyapunov functions exist, which are a class of functions that is not computationally easy to describe.

Existing results of constructive converse Lyapunov theorems for general nonlinear systems are scarce and come with certain limitations, as discussed in the following. A relevant result for continuous-time systems can be found in [17], where the authors show the relation between control Lyapunov functions and solutions to generalized Zubov equations, i.e., a first order partial differential equation. A result relevant for *discrete-time* systems was given in [145], where it was shown that for a GES discrete-time system a Lyapunov function can be constructed by a finite sum of solutions. This was established under the assumption that the system dynamics are locally Lipschitz continuous. In [51, 52] (see also the monograph [53]) converse Lyapunov theorems for *continuous-time* systems are obtained *via* piecewise linear Lyapunov functions and linear programming. Some recent extensions to discrete-time systems are [42, 96], which we discuss in more detail next.

Firstly, in [96], the authors consider discrete-time systems with locally Lipschitz continuous dynamics. In this work, an alternative Lyapunov function construction is proposed using Yoshizawa functions, [142]. The idea is the following. Let system (2.1) satisfy an estimate of the form (2.2), and let, for given  $\mu \in (0, 1)$ , the  $\mathcal{K}_\infty$ -functions  $\alpha_1, \alpha_2$  satisfy

$$\alpha_1(\beta(s, k)) \leq \alpha_2(s)\mu^{2k}, \quad \forall s \geq 0, \quad \forall k \in \mathbb{N}. \quad (2.51)$$

---

<sup>8</sup>I.e., the equilibrium point is reached in a finite amount of time

Note that such  $\mathcal{K}_\infty$ -functions  $\alpha_1, \alpha_2$  always exist, see [96, Lemma 1]. If  $\alpha_1 \in \mathcal{K}_\infty$  is locally Lipschitz continuous then the function  $V : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by  $V(\xi) := \sup_{k \in \mathbb{N}} \alpha_1(\|x(k, \xi)\|) \mu^{-k}$ , called *discrete-time Yoshizawa function*, is a Lyapunov function for system (2.1). The domain of definition  $\mathcal{D}$  is assumed to be open, compact and includes the origin. Moreover, for each  $x \in \mathcal{D}$  there exists a positive integer  $K(x)$  such that

$$V(x) = \sup_{k \in \{0, \dots, K(x)\}} \alpha_1(\|x(k, \xi)\|) \mu^{-k}. \quad (2.52)$$

Although not explicitly stated in [96], we see that in the case of GES of the origin, i.e.,  $\beta(s, k) = C \tilde{\mu}^k s$  with  $C \geq 1$  and  $\tilde{\mu} \in (0, 1)$ , we can pick  $\alpha_1 = \text{id}$ ,  $\alpha_2 = C \text{id}$  and  $\mu = \sqrt{\tilde{\mu}}$  to satisfy (2.51). This can be used to show that we can pick  $K(x)$  the same for all  $x \in \mathcal{D}$ , which follows from the estimate [96, Equation (33)]. In this case, the Lyapunov function from (2.52) is similar to the one proposed in (2.14). However, the Yoshizawa construction requires the knowledge of the  $\mathcal{KL}$ -function  $\beta$ , which may be hard to characterize. Moreover, local Lipschitz continuity of a  $\mathcal{K}_\infty$ -function  $\alpha_1$  satisfying (2.51) has to be checked.

Secondly, in [42], the authors consider a linear programming problem, where the solution parametrizes a *continuous and piecewise affine* (CPA) Lyapunov function. The domain of the Lyapunov function is only limited by the size of the equilibrium's domain of attraction. Note that this approach requires the origin to be GES, and the dynamics to be  $\mathcal{C}^2$ , i.e., two times continuously differentiable.

Physical systems do often continuously depend on time. Standard examples are e.g. the pendulum, electrical circuits or neural networks, [82, 130]. The control of continuous-time systems is often performed digitally, see e.g. [103]. One way of applying digital control is the co-called *emulation approach*: A continuous-time controller is discretized (e.g. via *sample-and-hold*) and the discrete controller is implemented. Alternatively, one finds a discrete-time system for the continuous-time system, computes a discrete-time controller for the discrete-time system, which is then implemented at the continuous-time system. The discrete-time system is also called *sampled-data system*. Especially for linear systems, this second approach is often used, as an exact linear sampled-data system for the linear continuous-time system can be obtained from the variation of constants formula. For further reading on sampled-data control, we refer to [114] and the references therein. The disadvantage of the second approach is that for nonlinear systems an exact sampled-data system often cannot be found. On the other hand, if a sampled-data system of a continuous-time system is obtained, then the results developed in this chapter can be applied to the sampled-data system to check stability properties of the continuous-time system.

There exists a wide variety of small-gain theorems, and it seems that the first small-

gain result was obtained by Zames [143] in 1966. Since then, there has been an extensive study in that topic, and hence, small-gain theory is present in many monographs as e.g. [82, 109, 140]. More recent publications on small-gain results are [21, 23–25, 49, 68, 69, 75, 115, 124] for mainly continuous-time and hybrid systems, as well as [40, 65, 67, 89, 99] for discrete-time systems. Note that some of the above references consider *input-to-state stability*, which will be the subject of the next chapter. In particular, the construction of a global finite-step Lyapunov function within the small-gain results of Section 2.3 follows the construction procedure proposed in [25]. There the authors construct an overall ISS Lyapunov function using the ISS Lyapunov functions of the subsystems and an  $\Omega$ -path that is derived from the gains. It is worth mentioning that non-conservative small-gain results have been reported for interconnected systems in the frequency domain, mostly for specific settings related to robust analysis and synthesis, see [20, 83]. Moreover, the authors in [28, 64] show that classical small-gain theorems are, in general, not necessary, even if all subsystems' equilibria are GAS.

For the system classes considered in Section 2.4 there exists an extensive list of literature, so we only cite a selection, which are connected to the results presented in this chapter, and further refer to the references therein. For homogeneous systems these are [26, 27, 92, 121], for conewise linear systems these are [9, 72, 95, 135], and, in conclusion, for work on polyhedral Lyapunov functions we refer to [10, 11, 73, 84, 91, 92, 102, 111].



## Stability analysis of large-scale discrete-time systems with inputs

In Chapter 2 we have considered discrete-time systems of the form

$$x(k+1) = G(x(k)), \quad k \in \mathbb{N},$$

i.e., the updated state  $x(k+1)$  is defined by the map  $G$  and the previous state  $x(k)$ . Indeed, the focus of Chapter 2 has been to establish stability properties such as global asymptotic stability of the origin.

In this chapter, we consider discrete-time systems *with inputs* of the form

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N},$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . The input may be used as a control input, e.g. for feedback stabilization, but here, we treat the input as a *disturbance*. The stability analysis we establish in this chapter aims at providing conditions guaranteeing *input-to-state stability* (ISS) that has been introduced in [128]. The concept of ISS turned out to be fruitful for nonlinear control systems, not least because ISS can be characterized by Lyapunov functions, see [128, 132] for continuous-time systems and [70, 76] for discrete-time systems.

For discrete-time systems, ISS Lyapunov functions are assumed to decay at each time step (while neglecting the input). To relax this assumption we introduce the concept of dissipative finite-step ISS Lyapunov functions, where the function is assumed to decay after a *finite number of time steps* rather than at each time step. This extends the notion of a global finite-step Lyapunov function introduced in Definition 2.6 to

systems with inputs. Again, as in Chapter 2, we do not require continuity of the right-hand side  $G$ .

We provide, in a first step, an equivalent characterization of input-to-state stability in terms of the existence of a dissipative finite-step ISS Lyapunov function in Section 3.2. The sufficiency part follows the lines of [70, Lemma 3.5], which shows that the existence of a continuous (dissipative) ISS Lyapunov function implies ISS of the system. Necessity is shown using a converse ISS Lyapunov theorem [70, 88]. Moreover, for the case of exponentially input-to state stable (expISS) systems we can show that any norm is a dissipative finite-step ISS Lyapunov function.

As we have seen in Chapter 2, it may be difficult to show stability properties of large-scale nonlinear systems directly. Clearly, if there are additional inputs acting as disturbances, proving ISS of the system directly gets even more difficult. Thus, we consider the system as an interconnection of a number of smaller components. The classical ISS small-gain approach for studying input-to-state stability can be divided into two approaches:

- The classical *Lyapunov-based* small-gain approach considers the existence of ISS Lyapunov functions for each subsystem. Gains may then be derived from estimates of the ISS Lyapunov functions, see e.g. [23, 25, 65, 68, 74, 89, 99].
- In the classical *trajectory-based* small-gain approach, gains are derived from ISS estimates of the subsystems' trajectories, see e.g. [24, 67, 69].

In the second part of this chapter we present relaxed ISS small-gain theorems that have some advantages over classical ISS small-gain theorems such as:

- (i) we do not require continuity of the system dynamics;
- (ii) we do not require the subsystems to be ISS;
- (iii) we establish system classes for which these relaxed ISS small-gain conditions are also necessary.

On the other hand, we admit that these relaxed ISS small-gain theorems are, in general, technically challenging.

In Section 3.3 we state relaxed *Lyapunov-based* small-gain theorems that follow the same idea as proposed in Section 2.3: The requirement imposed is that Lyapunov-type functions for the subsystems have to decrease along solutions after a *finite number of time steps*. Indeed, each subsystem may be unstable when decoupled from the other subsystems. This is a crucial difference to classical ISS small-gain results, where it is implicitly assumed that the other subsystems act as perturbations. Here the subsystems may have a stabilizing effect on each other. This relaxation includes previous ISS small-gain theorems as special cases. We will show by means



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of an example that the relaxed ISS small-gain theorem applies to a larger class of interconnected systems. Furthermore, if the overall system is expISS, i.e., solutions of the unperturbed system are decaying exponentially, the relaxed ISS small-gain theorems are also necessary, i.e., they are non-conservative. The proofs of the ISS small-gain theorems presented give further insight in the system's behavior. For the sufficiency part, a dissipative finite-step ISS Lyapunov function is constructed from the Lyapunov-type functions and the gain functions involved. Moreover, for expISS systems suitable Lyapunov-type and gain functions are derived. This particularly implies a constructive methodology for applications.

The relaxed *trajectory-based* small-gain approach that we propose in Section 3.4 has the following idea. Usually, the trajectory-based small-gain approach considers the interconnection of systems that have the same stability properties, as e.g. ISS, global stability (GS) or the asymptotic gain property (AG), see [24, 69]. For instance, if all subsystems are ISS then ISS of the overall system is implied by a small-gain condition, where the gain operator is derived from trajectory estimates of the subsystems. In other cases, the interconnection might consist of systems with different stability properties. We mention the class of networked control systems, where information is sent via data channels, see [39]. There, the effects of the data channels might be modeled as error dynamics that are treated as subsystems. The error dynamics might not be ISS, but GS. Hence, we consider the case of the interconnection of ISS and GS subsystems. The small-gain theorem that we derive in Section 3.4 states sufficient conditions under which the whole interconnection is GS, while the interconnection of the ISS systems is ISS with respect to the GS systems and possibly additional external inputs.

The outline of this chapter is as follows. The problem statement including the definition of a dissipative finite-step ISS Lyapunov function is given in Section 3.1. Stability analysis via dissipative finite-step ISS Lyapunov functions is treated in Section 3.2, starting with some preliminary lemmas in Section 3.2.1. In particular, we state the sufficiency of the existence of dissipative finite-step ISS Lyapunov functions to conclude ISS in Section 3.2.2. Subsequently, in Section 3.2.3, we propose a particular converse dissipative finite-step ISS Lyapunov theorem that shows that for any expISS system any norm is a dissipative finite-step ISS Lyapunov function. The relaxed Lyapunov-based small-gain approach that we present in Section 3.3 is split as follows. Firstly, in Section 3.3.1, sufficient ISS small-gain theorems are presented that do not require each system to admit an ISS Lyapunov function. Secondly, in Section 3.3.2, we show the non-conservativeness of the relaxed ISS small-gain theorems for the class of expISS systems, by stating a converse of the presented ISS small-gain theorems. We conclude Section 3.3 by examining a nonlinear illustrative example in Section 3.3.3. Finally, we propose a relaxed trajectory-based small-gain approach in Section 3.4 and conclude with notes and references in Section 3.5.

### 3.1 Problem statement

We consider discrete-time systems of the form

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N}, \quad (3.1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Recall from Section 1.4 that  $u(k) \in \mathbb{R}^m$  denotes the input at time  $k \in \mathbb{N}$ , and  $x(k, \xi, u(\cdot)) \in \mathbb{R}^n$  denotes the state at time  $k \in \mathbb{N}$ , starting in the initial state  $x(0) = \xi \in \mathbb{R}^n$  with input  $u(\cdot) \subset \mathbb{R}^m$ .

Unless otherwise stated, we consider  $\|\cdot\|$  to be some arbitrary norm on  $\mathbb{R}^n$  resp.  $\mathbb{R}^m$ , and  $|\cdot|_\infty$  to be the supremum norm for input sequences  $u(\cdot) \subset \mathbb{R}^m$  as defined in Section 1.2. Moreover, we demand the map  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  to satisfy the following assumption throughout this chapter.

**Assumption 3.1.** The function  $G$  in (3.1) is *globally  $\mathcal{K}$ -bounded*, i.e., there exist  $\omega_1, \omega_2 \in \mathcal{K}$  such that

$$\|G(\xi, \nu)\| \leq \omega_1(\|\xi\|) + \omega_2(\|\nu\|) \quad (3.2)$$

for all  $\xi \in \mathbb{R}^n$  and  $\nu \in \mathbb{R}^m$ . ◁

We recall Definition 1.13, where global  $\mathcal{K}$ -boundedness for the function  $G$  is introduced. Note that in this chapter, the  $\mathcal{K}$ -functions  $\omega_1$  and  $\omega_2$  are always used to denote the  $\mathcal{K}$ -bounds in (3.2). As outlined in Section 1.4, Assumption 3.1 implies continuity of  $G$  in  $(0, 0)$  with  $G(0, 0) = 0$ , but it does not require the map  $G$  to be continuous elsewhere (as assumed e.g. in [70, 71, 99]) or (locally) Lipschitz (as assumed e.g. in [1, 2]). For further remarks on Assumption 3.1 see Remark 3.3 and Section 3.2.1.

The aim of this chapter is to check *input-to-state stability* of system (3.1), which is defined as follows.

**Definition 3.2.** We call system (3.1) *input-to-state stable* (ISS) (from  $u$  to  $x$ ) if there exists a  $\mathcal{KL}$ -function  $\beta$  and a  $\mathcal{K}$ -function  $\gamma$  such that for all initial states  $\xi \in \mathbb{R}^n$ , all bounded inputs  $u(\cdot) \subset \mathbb{R}^m$  and all  $k \in \mathbb{N}$  we have

$$\|x(k, \xi, u(\cdot))\| \leq \beta(\|\xi\|, k) + \gamma(|u|_\infty). \quad (3.3)$$

If the  $\mathcal{KL}$ -function in (3.3) can be chosen as

$$\beta(r, t) = C\mu^t r \quad (3.4)$$

with  $C \geq 1$  and  $\mu \in [0, 1)$ , then system (3.1) is called *exponentially input-to-state stable* (expISS).

An alternative definition of ISS replaces the sum in (3.3) by the maximum. Indeed, both definitions are equivalent, and the equivalence even holds for more general definitions of ISS using monotone aggregation functions, see Proposition 1.26.

*Remark 3.3.* Since we are interested in checking the ISS property of system (3.1), it is clear that the existence of functions  $\omega_1, \omega_2 \in \mathcal{K}$  satisfying the global  $\mathcal{K}$ -boundedness condition (3.2) is no restriction, since every ISS system necessarily satisfies (3.2). Particularly, by (3.3), we have for all  $\xi \in \mathbb{R}^n$  and  $\nu \in \mathbb{R}^m$

$$\|G(\xi, \nu)\| = \|x(1, \xi, \nu)\| \leq \beta(\|\xi\|, 1) + \gamma(\|\nu\|)$$

and we may choose  $\omega_1(\cdot) = \beta(\cdot, 1)$  and  $\omega_2(\cdot) = \gamma(\cdot)$  to obtain (3.2). Moreover, for expISS systems we can take  $\omega_1(s) = C\mu s$ , where  $C \geq 1$  and  $\mu \in [0, 1)$  stem from (3.4). In other words, any expISS system is globally  $\mathcal{K}$ -bounded with a linear function  $\omega_1 \in \mathcal{K}$ .  $\triangleleft$

The following lemma shows that by a suitable *change of coordinates*, i.e., a homeomorphism  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $T(0) = 0$  (see e.g. [78, 131]), we can always assume that the function  $\omega_1 \in \mathcal{K}$  in (3.2) is linear.

**Lemma 3.4.** *Consider system (3.1) and let Assumption 3.1 hold. Then there exists a change of coordinates  $T$  such that for  $z(k) := T(x(k))$  the induced system*

$$z(k+1) = \tilde{G}(z(k), u(k)), \quad \forall k \in \mathbb{N} \quad (3.5)$$

*satisfies (3.2) with linear  $\omega_1 \in \mathcal{K}_\infty$ .*

*Proof.* Consider a change of coordinates  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and define  $z(k) := T(x(k))$ , where  $x(k)$  comes from (3.1). Then  $z$  satisfies (3.5) with

$$\tilde{G}(z, u) := T(G(T^{-1}(z), u)).$$

Note that  $\tilde{G}(0, 0) = 0$  since  $T$  and its inverse fix the origin. Furthermore, let  $G$  satisfy the global  $\mathcal{K}$ -boundedness condition (3.2) with  $\omega_1, \omega_2 \in \mathcal{K}_\infty$ . Note that it is no restriction to assume  $\omega_1$  and  $\omega_2$  to be of class  $\mathcal{K}_\infty$  as any  $\mathcal{K}$ -function can be upper bounded by a  $\mathcal{K}_\infty$ -function. Without loss of generality, we assume that  $(2\omega_1 - \text{id}) \in \mathcal{K}_\infty$ , else increase  $\omega_1$ . Take any  $\lambda > 1$ . By [79, Lemma 19] there exists a  $\mathcal{K}_\infty$ -function  $\varphi$  satisfying

$$\varphi \circ 2\omega_1(s) = \lambda\varphi(s) \quad \forall s \geq 0. \quad (3.6)$$

Now we consider the particular change of coordinates defined by  $T(x) := \varphi(\|x\|) \frac{x}{\|x\|}$  for  $x \neq 0$ , and  $T(0) = 0$ . Clearly,  $T$  is continuous for  $x \neq 0$ . On the other hand,  $\|T(x)\| = \varphi(\|x\|)$ , so continuity of  $T$  in zero is implied by continuity of  $\varphi$  and  $\varphi(0) = 0$ . With  $z = T(x)$  a direct computation yields  $T^{-1}(z) := \varphi^{-1}(\|z\|) \frac{z}{\|z\|}$  for  $z \neq 0$  and  $T^{-1}(0) = 0$ . By the same arguments as above, also  $T^{-1}$  is continuous. Hence,  $T$  is a homeomorphism. Moreover, we obtain the following estimate

$$\begin{aligned} \|\tilde{G}(\tilde{\xi}, \tilde{\nu})\| &= \varphi \left( \left\| G \left( \frac{\varphi^{-1}(\|\tilde{\xi}\|)}{\|\tilde{\xi}\|} \tilde{\xi}, \tilde{\nu} \right) \right\| \right) \stackrel{(3.2)}{\leq} \varphi \left( \omega_1(\varphi^{-1}(\|\tilde{\xi}\|)) + \omega_2(\|\tilde{\nu}\|) \right) \\ &\leq \varphi \left( 2\omega_1(\varphi^{-1}(\|\tilde{\xi}\|)) \right) + \varphi \left( 2\omega_2(\|\tilde{\nu}\|) \right) \stackrel{(3.6)}{=} \lambda\|\tilde{\xi}\| + \varphi(2\omega_2(\|\tilde{\nu}\|)). \end{aligned}$$

So,  $\tilde{G}$  satisfies (3.2) with the linear function  $\omega_1(s) = \lambda s$ ,  $s \in \mathbb{R}_+$ , which concludes the proof.  $\square$

In Sections 3.3 and 3.4 we will derive small-gain theorems for an interconnection of subsystems that have weaker stability properties than ISS. As there exists an extensive amount of different stability notions, we only state those notions that will be used in the remainder of this work.

**Definition 3.5** ([70, Definition A.2]). We call system (3.1) *robustly stable* if there exists a  $\mathcal{K}_\infty$ -function  $\psi$  such that the origin of the system

$$x(k+1) = G(x(k), d(k)\psi(\|x(k)\|))$$

with  $d(k) \in [-1, 1]^m$  is GAS in the sense that

- (i) for every  $\varepsilon > 0$  there exists some  $\delta > 0$  such that  $\|x(k, \xi, d(\cdot))\| < \varepsilon$  for all  $k \geq 0$ , all  $d(\cdot) \subset [-1, 1]^m$  and all  $\|\xi\| < \delta$ ; and
- (ii)  $\lim_{k \rightarrow \infty} \|x(k, \xi, d(\cdot))\| = 0$  holds for all  $\xi \in \mathbb{R}^n$  and all  $d(\cdot) \subset [-1, 1]^m$ .

The notion of robust stability is somehow redundant as in [70] the authors show that for continuous dynamics, robust stability and ISS are equivalent. Nevertheless, we introduce the notion of robust stability as it will be needed in the proof of Lemma 3.33 later on. As we will see in this proof the equivalence of robust stability and ISS also holds true for discontinuous dynamics, see Remark 3.34.

**Definition 3.6.** We call system (3.1) *globally asymptotically stable with 0 input* (0-GAS) if there exists a  $\mathcal{KL}$ -function  $\beta$  such that for all initial states  $\xi \in \mathbb{R}^n$  and all  $k \in \mathbb{N}$  we have

$$\|x(k, \xi, 0)\| \leq \beta(\|\xi\|, k).$$

*Remark 3.7.* In [3] the author shows that in the discrete-time setting *integral input-to-state stability* (iISS) is equivalent to 0-GAS, at least for continuous dynamics. However, this equivalence does not hold for continuous-time systems.

Furthermore, results from this chapter can be used to derive results for systems without inputs by simply setting the external input  $u$  to zero. In this respect, several results of Chapter 2 can be derived immediately from the results in this chapter.  $\triangleleft$

**Definition 3.8.** We say that system (3.1) has the *global stability property* (GS) if there exist  $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$  such that for all initial states  $\xi \in \mathbb{R}^n$ , all inputs  $u(\cdot) \subset \mathbb{R}^m$  and all  $k \in \mathbb{N}$  we have

$$\|x(k, \xi, u(\cdot))\| \leq \sigma_1(\|\xi\|) + \sigma_2(\|u\|_\infty).$$

**Definition 3.9.** We say that system (3.1) has the *asymptotic gain property* (AG) if there exists  $\gamma \in \mathcal{K}_\infty$  such that for all initial states  $\xi \in \mathbb{R}^n$  and all inputs  $u(\cdot) \subset \mathbb{R}^m$  we have

$$\limsup_{k \rightarrow \infty} \|x(k, \xi, u(\cdot))\| \leq \gamma(\|u\|_\infty).$$

It is not difficult to see that ISS implies AG and GS. Moreover, if system (3.1) has continuous dynamics then also the converse implication holds, i.e., if system (3.1) is AG and GS then it is also ISS, see [70, Theorem 1] and [30, Theorem 2]. We emphasize that this converse implication also holds for discontinuous dynamics, which will be shown in Lemma 3.33.

To prove ISS of system (3.1) the concept of ISS Lyapunov functions is widely used, see Section 3.5. Note that the definition of an ISS Lyapunov function stated here does not require continuity of the ISS Lyapunov function.

**Definition 3.10.** A proper and positive function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called a *dissipative ISS Lyapunov function* for system (3.1) if there exist  $\sigma \in \mathcal{K}$  and a positive definite function  $\rho$  with  $(\text{id} - \rho) \in \mathcal{K}_\infty$  such that for any  $\xi \in \mathbb{R}^n$  and  $\nu \in \mathbb{R}^m$  we have

$$W(G(\xi, \nu)) \leq \rho(W(\xi)) + \sigma(\|\nu\|). \quad (3.7)$$

*Remark 3.11.* (i) In many prior works (e.g. [70,99]) the definition of an ISS Lyapunov function requires the existence of a  $\mathcal{K}_\infty$ -function  $\alpha_3$  and a  $\mathcal{K}$ -function  $\sigma$  such that

$$W(G(\xi, \nu)) - W(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\nu\|)$$

holds for all  $\xi \in \mathbb{R}^n, \nu \in \mathbb{R}^m$ . Note that this condition is equivalent to condition (3.7). The proof of this observation follows the same steps as in Remark 2.5, and is therefore omitted. However, a proof can be found in [38, Remark 3.7].

(ii) For systems with external inputs there are usually two forms of ISS Lyapunov functions. The first one is the dissipative form of Definition 3.10. The other type are frequently called implication-form ISS Lyapunov functions. These are proper and positive definite functions  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  satisfying

$$\|\xi\| \geq \chi(\|\nu\|) \quad \Rightarrow \quad W(G(\xi, \nu)) \leq \bar{\rho}(W(\xi)). \quad (3.8)$$

for all  $\xi \in \mathbb{R}^n, \nu \in \mathbb{R}^m$ , and some positive definite function  $\bar{\rho} < \text{id}$  and  $\chi \in \mathcal{K}$ . If the function  $G$  in (3.1) is *continuous* then conditions (3.7) and (3.8) are equivalent,

see [70, Remark 3.3] and [46, Proposition 3.3 and 3.6]. So the existence of a dissipative or implication-form ISS Lyapunov function implies ISS of the system if the dynamics are continuous, see [70, Lemma 3.5].

If  $G$  is *discontinuous* then the equivalence between the existence of dissipative and implication-form ISS Lyapunov functions is no longer satisfied. Indeed, any dissipative ISS Lyapunov function is an implication-form ISS Lyapunov function, but the converse does not hold in general, see [46]. In particular, for discontinuous dynamics, an implication-form ISS Lyapunov function is not sufficient to conclude ISS, see [89, Remark 2.1] and [46, Example 3.7].

(iii) To prove ISS of system (3.1), the authors in [46, Proposition 2.4] have shown that the assumption  $(\text{id} - \rho) \in \mathcal{K}_\infty$  in Definition 3.10 can be weakened to the condition  $(\text{id} - \rho) \in \mathcal{K}$  and  $\sup(\text{id} - \rho) > \sup \sigma$ . Moreover, for any ISS system (3.1) there exists a dissipative ISS Lyapunov function  $W$  with linear decrease function  $\rho$ , see [46, Theorem 2.6].  $\triangleleft$

We relax the condition (3.7) in Definition 3.10 by replacing the solution after one time step  $G(\xi, \nu) = x(1, \xi, \nu)$  by the solution after a finite number of time steps. This generalizes the definition of a global finite-step Lyapunov function as introduced in Definition 2.6 to systems (3.1) with inputs.

**Definition 3.12.** A proper and positive definite function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called a *dissipative finite-step ISS Lyapunov function* for system (3.1) if there exist an  $M \in \mathbb{N}$ ,  $\sigma \in \mathcal{K}$ , a positive definite function  $\rho$  with  $(\text{id} - \rho) \in \mathcal{K}_\infty$  such that for any  $\xi \in \mathbb{R}^n$  and  $u(\cdot) \subset \mathbb{R}^m$  we have

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(|u|_\infty). \quad (3.9)$$

We point out, however, that it is not sufficient to know a dissipative finite-step ISS Lyapunov function, but we also require to know the constant  $M$ , which may be hard to characterize. Certainly, the introduction of global finite-step Lyapunov functions in Chapter 2 has offered several useful implications. Thus, in the following two sections, we elaborate several similar results to those of Chapter 2 for systems with (external) inputs.

## 3.2 Stability analysis via dissipative finite-step ISS Lyapunov functions

In this section we prove that the existence of a dissipative finite-step ISS Lyapunov function as introduced in Definition 3.12 is sufficient and necessary to show ISS of system (3.1). For convenience, we split the section in three parts. In the first part,

Section 3.2.1, we state some preliminary lemmas that are needed in the remainder of this section. In Section 3.2.2 we state the sufficiency part of the above equivalence, namely that the existence of a dissipative finite-step ISS Lyapunov function implies ISS. Finally, in Section 3.2.3, we show the converse implication. In particular, we state a procedure, how a dissipative finite-step ISS Lyapunov function can be obtained for an expISS system.

### 3.2.1 Preliminary lemmata

In this section we state some lemmas that are needed in the subsequent section. The first lemma is a particular comparison lemma for finite-step dynamics.

**Lemma 3.13.** *Let  $M \in \mathbb{N} \setminus \{0\}$ ,  $L \in \mathbb{N} \cup \{\infty\}$ ,  $k_0 \in \{0, \dots, M-1\}$ , and  $y : \mathbb{N} \rightarrow \mathbb{R}_+$  be a function satisfying*

$$y((l+1)M + k_0) \leq \chi(y(lM + k_0)), \quad \forall l \in \{0, \dots, L\}, \quad (3.10)$$

where  $\chi \in \mathcal{K}_\infty$  satisfies  $\chi < \text{id}$ . Then there exists a  $\mathcal{KL}$ -function  $\beta_{k_0}$  such that the function  $y$  also satisfies

$$y(lM + k_0) \leq \beta_{k_0}(y(k_0), lM + k_0), \quad \forall l \in \{0, \dots, L\}.$$

In addition, if (3.10) is satisfied for all  $k_0 \in \{0, \dots, M-1\}$  then there exists a  $\mathcal{KL}$ -function  $\beta$  such that with  $y_M^{\max} := \max\{y(0), \dots, y(M-1)\}$  we have

$$y(k) \leq \beta(y_M^{\max}, k), \quad \forall k \in \{0, \dots, (L+1)M-1\}.$$

*Proof.* Let  $M \in \mathbb{N} \setminus \{0\}$ ,  $L \in \mathbb{N} \cup \{\infty\}$  and  $k_0 \in \{0, \dots, M-1\}$ . From (3.10) and the monotonicity property of  $\chi \in \mathcal{K}$ , we obtain

$$y((l+1)M + k_0) \leq \chi(y(lM + k_0)) \leq \dots \leq \chi^{l+1}(y(k_0))$$

for all  $l \in \{0, \dots, L\}$ . Note that since  $\chi < \text{id}$  we have  $\chi^l > \chi^{l+1}$ , and  $\chi^l(s) \rightarrow 0$  as  $l \rightarrow \infty$  for any  $s \in \mathbb{R}_+$ . Define  $t_{k_0, l} := lM + k_0$  and  $t_{k_0, l}^+ := (l+1)M + k_0$  for all  $l \in \mathbb{N}$ . Let  $\beta_{k_0} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be defined by

$$\beta_{k_0}(s, r) := \begin{cases} \frac{1}{M} \left( (t_{k_0, 0} - r) \chi^{-1}(s) + (r + M - k_0) \text{id}(s) \right) & r \in [0, t_{k_0, 0}], s \geq 0 \\ \frac{1}{M} \left( (t_{k_0, l}^+ - r) \chi^l(s) + (r - t_{k_0, l}) \chi^{l+1}(s) \right) & r \in [t_{k_0, l}, t_{k_0, l}^+], s \geq 0. \end{cases}$$

Note that this construction is similar to the one proposed in [71, Lemma 4.3]. Clearly,  $\beta_{k_0}$  is continuous and  $\beta_{k_0}(\cdot, r)$  is a  $\mathcal{K}$ -function for any fixed  $r \geq 0$ . On the other hand, for any fixed  $s \geq 0$ ,  $\beta_{k_0}(s, \cdot)$  is an  $\mathcal{L}$ -function, as it is linear affine on any interval  $[t_{k_0, l}, t_{k_0, l}^+]$  and strictly decreasing by

$$\beta_{k_0}(s, t_{k_0, l}) = \chi^l(s) > \chi^{l+1}(s) = \beta_{k_0}(s, t_{k_0, l}^+).$$

Hence,  $\beta_{k_0} \in \mathcal{KL}$ . Moreover, for all  $l \in \{0, \dots, L\}$  we have

$$y(lM + k_0) \leq \chi^l(y(k_0)) = \beta_{k_0}(y(k_0), lM + k_0),$$

which shows the first assertion of the lemma.

Now let (3.10) be satisfied for all  $k_0 \in \{0, \dots, M-1\}$ . Define

$$\beta(s, r) := \max_{k_0 \in \{0, \dots, M-1\}} \beta_{k_0}(s, r),$$

which is again a function of class  $\mathcal{KL}$ . For any  $k \in \{0, \dots, (L+1)M-1\}$  there exist unique  $l \in \{0, \dots, L\}$  and  $k_0 \in \{0, \dots, M-1\}$  such that  $k = lM + k_0$ , and we have

$$y(k) = y(lM + k_0) \leq \chi^l(y(k_0)) = \beta_{k_0}(y(k_0), lM + k_0) \leq \beta_{k_0}(y_M^{\max}, k) \leq \beta(y_M^{\max}, k)$$

with  $y_M^{\max} := \max\{y(0), \dots, y(M-1)\}$ . This concludes the proof.  $\square$

If the function  $\chi$  in Lemma 3.13 is linear, then the  $\mathcal{KL}$ -function  $\beta$  has a simpler form as we will see in the next lemma.

**Lemma 3.14.** *Let the assumptions of Lemma 3.13 be satisfied for all  $k_0 \in \{0, \dots, M-1\}$  with  $\chi(s) = \theta s$  and  $\theta \in (0, 1)$ . Let  $y_M^{\max} := \max\{y(0), \dots, y(M-1)\}$ , then for all  $k \in \{0, \dots, (L+1)M-1\}$  we have*

$$y(k) \leq \frac{y_M^{\max}}{\theta} \theta^{k/M}.$$

*Proof.* A direct computation yields that for any  $k_0 \in \{0, \dots, M-1\}$ , and any  $l \in \{0, \dots, L\}$  we have  $y(lM + k_0) \leq \chi(y((l-1)M + k_0)) \leq \chi^l(y(k_0)) = \theta^l y(k_0)$ . Hence, for any  $k = lM + k_0 \leq (L+1)M-1$ , with  $l \in \{0, \dots, L\}$  and  $k_0 \in \{0, \dots, M-1\}$  we have

$$y(k) \leq \max_{k_0 \in \{0, \dots, M-1\}} \{y(k_0)\theta^l\} \leq y_M^{\max} \theta^{k/M-1}.$$

This proves the lemma.  $\square$

As noted in Remark 3.3 the requirement on the existence of  $\mathcal{K}$ -functions  $\omega_1, \omega_2$  satisfying (3.2) in Assumption 3.1 is a necessary condition for system (3.1) to be ISS. The following lemma states that under the assumption of global  $\mathcal{K}$ -boundedness any trajectory of system (3.1) has a global  $\mathcal{K}$ -bound for any time step. This result is needed in Theorem 3.17 to show that the existence of a dissipative finite-step ISS Lyapunov function implies ISS of system (3.1).

**Lemma 3.15.** *Let system (3.1) satisfy Assumption 3.1. Then for any  $j \in \mathbb{N}$ , there exist  $\mathcal{K}$ -functions  $\vartheta_j, \zeta_j$  such that for all  $\xi \in \mathbb{R}^n$  and all bounded  $u(\cdot) \subset \mathbb{R}^m$  we have*

$$\|x(j, \xi, u(\cdot))\| \leq \vartheta_j(\|\xi\|) + \zeta_j(\|u\|_\infty). \quad (3.11)$$



Before we prove this lemma we note that, by definition, any trajectory of a GS system has a uniform global  $\mathcal{K}$ -bound, i.e., (3.11) is satisfied by taking  $\vartheta_j = \sigma_1$  and  $\zeta_j = \sigma_2$  for all  $j \in \mathbb{N}$ . On the other hand, if the system is not GS then we cannot find functions  $\sigma_1 \in \mathcal{K}_\infty$  resp.  $\sigma_2 \in \mathcal{K}_\infty$  upper bounding the functions  $\vartheta_j \in \mathcal{K}$ ,  $j \in \mathbb{N}$ , resp.  $\zeta_j \in \mathcal{K}$ ,  $j \in \mathbb{N}$ , in (3.11).

*Proof.* We prove the result by induction. Take any  $\xi \in \mathbb{R}^n$  and any input  $u(\cdot) \subset \mathbb{R}^m$ . For  $j = 0$  we have  $\|x(0, \xi, u(\cdot))\| = \|\xi\|$  satisfying (3.11) with  $\vartheta_0 = \text{id}$  and arbitrary  $\zeta_0 \in \mathcal{K}$ . For  $j = 1$  it follows by Assumption 3.1 that

$$\|x(1, \xi, u(\cdot))\| \leq \omega_1(\|\xi\|) + \omega_2(|u|_\infty),$$

where  $\omega_1, \omega_2 \in \mathcal{K}$  come from (3.2). So we can take  $\vartheta_1 := \omega_1$  and  $\zeta_1 := \omega_2$ .

Now assume, that there exist  $\vartheta_j, \zeta_j \in \mathcal{K}$  satisfying (3.11) for some  $j \in \mathbb{N}$ . Then

$$\begin{aligned} \|x(j+1, \xi, u(\cdot))\| &= \|G(x(j, \xi, u(\cdot)), u(j))\| \\ &\leq \omega_1(\|x(j, \xi, u(\cdot))\|) + \omega_2(|u|_\infty) \\ &\leq \omega_1(\vartheta_j(\|\xi\|) + \zeta_j(|u|_\infty)) + \omega_2(|u|_\infty) \\ &\leq \omega_1(2\vartheta_j(\|\xi\|)) + \omega_1(2\zeta_j(|u|_\infty)) + \omega_2(|u|_\infty) \\ &=: \vartheta_{j+1}(\|\xi\|) + \zeta_{j+1}(|u|_\infty). \end{aligned}$$

By induction, the assertion holds for all  $j \in \mathbb{N}$ . □

If the functions  $\omega_1, \omega_2$  in (3.2) are linear then the functions  $\vartheta_j, \zeta_j$  in Lemma 3.15 are also linear, and have an explicit construction in terms of  $\omega_1, \omega_2$ .

**Lemma 3.16.** *Let system (3.1) satisfy the global  $\mathcal{K}$ -boundedness condition of Assumption 3.1 with linear functions  $\omega_1(s) := w_1 s$  and  $\omega_2(s) := w_2 s$ , where  $w_1, w_2 > 0$ . Then (3.11) is satisfied with  $\vartheta_j(s) = w_1^j s$  and  $\zeta_j = w_2 \sum_{i=0}^{j-1} w_1^i s$ .*

*Proof.* Following the proof of Lemma 3.15, we inductively obtain for any  $j \in \mathbb{N}$ ,

$$\vartheta_{j+1}(s) = \omega_1(\vartheta_j(s)) = \omega_1^{j+1}(s) = w_1^{j+1} s,$$

and

$$\zeta_{j+1}(s) = \omega_1(\zeta_j(s)) + \omega_2(s) = w_2 \sum_{i=0}^j w_1^i s.$$

□

### 3.2.2 Dissipative finite-step ISS Lyapunov functions: sufficient criteria

We start this section by proving that the existence of a dissipative finite-step ISS Lyapunov function is sufficient to conclude ISS of system (3.1). As any dissipative

ISS Lyapunov function is a particular dissipative finite-step ISS Lyapunov function, this result includes [70, Lemma 3.5]. Furthermore, the class of dissipative ISS Lyapunov functions is a strict subset of the class of dissipative finite-step ISS Lyapunov functions. Hence, this result is more general than showing that the existence of a dissipative ISS Lyapunov function implies ISS of the underlying system.

**Theorem 3.17.** *If there exists a dissipative finite-step ISS Lyapunov function for system (3.1) then system (3.1) is ISS.*

The proof follows the lines of [70, Lemma 3.5], which establishes that the existence of a continuous dissipative ISS Lyapunov function implies ISS of the system. Note that in [70] the authors assume continuity, whereas in this chapter global  $\mathcal{K}$ -boundedness is considered.

*Proof.* Let  $V$  be a dissipative finite-step ISS Lyapunov function satisfying Definition 3.12 for system (3.1) with suitable  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $M \in \mathbb{N}$ ,  $\sigma \in \mathcal{K}$ , and a positive definite function  $\rho$  with  $(\text{id} - \rho) \in \mathcal{K}_\infty$ . Let  $\xi \in \mathbb{R}^n$  and fix any bounded input  $u(\cdot) \subset \mathbb{R}^m$ . We abbreviate the state  $x(k) := x(k, \xi, u(\cdot))$ . Let  $\nu \in \mathcal{K}_\infty$  be such that  $\text{id} - \nu \in \mathcal{K}_\infty$  and consider the set

$$\Delta := \{\xi \in \mathbb{R}^n : V(\xi) \leq \delta := (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(|u|_\infty)\}.$$

We will now show that for any  $k \in \mathbb{N}$  with  $x(k) \in \Delta$  we have  $x(k + lM) \in \Delta$  for all  $l \in \mathbb{N}$ . Using (3.9), a direct computation yields

$$\begin{aligned} V(x(k + M)) &\leq \rho(V(x(k))) + \sigma(|u|_\infty) \leq \rho(\delta) + \sigma(|u|_\infty) \\ &= -(\text{id} - \nu) \circ (\text{id} - \rho)(\delta) + \delta - \nu \circ (\text{id} - \rho)(\delta) + \sigma(|u|_\infty) \\ &= -(\text{id} - \nu) \circ (\text{id} - \rho)(\delta) + \delta \leq \delta. \end{aligned}$$

Hence,  $x(k + M) \in \Delta$  and by induction we get  $x(k + lM) \in \Delta$  for all  $l \in \mathbb{N}$ .

Let  $j_0 \in \mathbb{N} \cup \{\infty\}$  satisfy  $j_0 := \min\{k \in \mathbb{N} : x(k), \dots, x(k + M - 1) \in \Delta\}$ . By definition of  $j_0$  and by the above consideration, we see that  $x(k) \in \Delta$  for all  $k \geq j_0$ . Thus, we have

$$V(x(k)) \leq (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(|u|_\infty) =: \tilde{\gamma}(|u|_\infty). \quad (3.12)$$

For  $k < j_0$ , we have to consider two cases.

First, if  $x(k) \in \Delta$  then by definition of  $\Delta$  we have  $V(x(k)) \leq \tilde{\gamma}(|u|_\infty)$ . Secondly, if  $x(k) \notin \Delta$  let  $l \in \mathbb{N}$  and  $k_0 \in \{0, \dots, M - 1\}$  satisfy  $k = lM + k_0$ . Since  $x(k_0) \in \Delta$  implies  $x(lM + k_0) \in \Delta$ , we conclude  $x(k_0) \notin \Delta$ . Hence, by definition of  $\Delta$ ,  $V(x(k_0)) > (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(|u|_\infty)$ , or, equivalently,  $\sigma(|u|_\infty) < \nu \circ (\text{id} - \rho) \circ$

$V(x(k_0))$ , which implies

$$\begin{aligned} V(x(k_0 + M)) &\leq \rho(V(x(k_0))) + \sigma(|u|_\infty) \\ &< \rho(V(x(k_0))) + \nu \circ (\text{id} - \rho) \circ V(x(k_0)) \\ &= (\rho + \nu \circ (\text{id} - \rho)) \circ V(x(k_0)). \end{aligned}$$

Note that the function  $\chi := (\rho + \nu \circ (\text{id} - \rho))$  satisfies  $\chi = \text{id} - (\text{id} - \nu) \circ (\text{id} - \rho) < \text{id}$ . Let  $L := \sup\{l \in \mathbb{N} : V(x(lM + k_0)) \notin \Delta\}$ . Then we have for all  $l \in \{0, \dots, L\}$

$$V(x((l+1)M + k_0)) \leq \chi(V(x(lM + k_0))).$$

Note that the function  $\chi = \rho + \nu \circ (\text{id} - \rho)$  is continuous, positive definite and unbounded as  $\nu, (\text{id} - \rho) \in \mathcal{K}_\infty$ , and it satisfies  $\chi(0) = 0$ . Hence, we can without loss of generality assume that  $\chi \in \mathcal{K}_\infty$ , else pick  $\tilde{\chi} \in \mathcal{K}_\infty$  satisfying  $\chi \leq \tilde{\chi} < \text{id}$ . Applying Lemma 3.13 there exists a  $\mathcal{KL}$ -function  $\beta_{k_0}$  satisfying

$$V(x(lM + k_0)) \leq \beta_{k_0}(V(x(k_0)), lM + k_0)$$

for all  $l \in \{0, \dots, L\}$ . Moreover, for all  $l > L$ , we have  $V(x(lM + k_0)) \in \Delta$  implying  $V(x(lM + k_0)) \leq \tilde{\gamma}(|u|_\infty)$ . Thus, for all  $l \in \mathbb{N}$ , we have

$$V(x(lM + k_0)) \leq \max\{\beta_{k_0}(V(x(k_0)), lM + k_0), \tilde{\gamma}(|u|_\infty)\}. \quad (3.13)$$

It is important to note that both  $\tilde{\gamma}$  and  $\chi$  are independent on the choice of  $\xi \in \mathbb{R}^n$  and  $u(\cdot) \in l^\infty(\mathbb{R}^m)$ . In addition, by the proof of Lemma 3.13, also  $\beta_{k_0} \in \mathcal{KL}$  does not depend on  $\xi \in \mathbb{R}^n$  and  $u(\cdot) \in \mathbb{R}^m$ . Hence, (3.13) holds for all solutions  $x(k)$ .

Define the  $\mathcal{KL}$ -function

$$\tilde{\beta}(s, r) := \max_{k_0 \in \{0, \dots, M-1\}} \beta_{k_0}(s, r)$$

and  $V_M^{\max}(\xi, u(\cdot)) := \max_{j \in \{0, \dots, M-1\}} V(x(j, \xi, u(\cdot)))$ . Then for all  $k \in \mathbb{N}$ , all  $\xi \in \mathbb{R}^n$  and all  $u(\cdot) \in l^\infty(\mathbb{R}^m)$  we have

$$V(x(k)) \leq \max\left\{\tilde{\beta}(V_M^{\max}(\xi, u(\cdot)), k), \tilde{\gamma}(|u|_\infty)\right\}.$$

Consider  $\vartheta_j, \zeta_j \in \mathcal{K}$  from Lemma 3.15 and define  $\tilde{\vartheta} := \max_{j \in \{0, \dots, M-1\}} \alpha_2(2\vartheta_j)$  and  $\tilde{\zeta} := \max_{j \in \{0, \dots, M-1\}} \alpha_2(2\zeta_j)$ . Then for all  $\xi \in \mathbb{R}^n$  and  $u(\cdot) \in l^\infty(\mathbb{R}^m)$  we get

$$V_M^{\max}(\xi, u(\cdot)) \leq \max_{j \in \{0, \dots, M-1\}} \alpha_2(\|x(j)\|) \leq \tilde{\vartheta}(\|\xi\|) + \tilde{\zeta}(|u|_\infty).$$

So all in all we have for all  $k \in \mathbb{N}$ , all  $\xi \in \mathbb{R}^n$  and all  $u(\cdot) \in l^\infty$ ,

$$\begin{aligned} V(x(k)) &\leq \max\left\{\tilde{\beta}(\tilde{\vartheta}(\|\xi\|) + \tilde{\zeta}(|u|_\infty), k), \tilde{\gamma}(|u|_\infty)\right\} \\ &\leq \max\left\{\tilde{\beta}(2\tilde{\vartheta}(\|\xi\|), k) + \tilde{\beta}(2\tilde{\zeta}(|u|_\infty), 0), \tilde{\gamma}(|u|_\infty)\right\} \\ &\leq \tilde{\beta}(2\tilde{\vartheta}(\|\xi\|), k) + \left(\tilde{\beta}(2\tilde{\zeta}(|u|_\infty), 0) + \tilde{\gamma}(|u|_\infty)\right). \end{aligned}$$

Hence, we get (3.3) by defining  $\beta(s, r) := \alpha_1^{-1}(2\tilde{\beta}(2\tilde{\vartheta}(s), r))$  and  $\gamma(s) := \alpha_1^{-1}(2(\tilde{\beta}(2\tilde{\zeta}(|u|_\infty), 0) + \tilde{\gamma}(|u|_\infty)))$ . Note that for fixed  $r \geq 0$ ,  $\beta(\cdot, r)$  is a  $\mathcal{K}$ -function as the composition of  $\mathcal{K}$ -functions, and for fixed  $s > 0$ ,  $\beta(s, \cdot) \in \mathcal{L}$ , since the composition of  $\mathcal{K}$ - and  $\mathcal{L}$ -functions is of class  $\mathcal{L}$  (see [56, Section 24], [77, Section 2]), so really  $\beta \in \mathcal{KL}$ . Further note that the summation of class- $\mathcal{K}$  functions yields a class- $\mathcal{K}$  function, so  $\gamma \in \mathcal{K}$ .  $\square$

If we impose stronger conditions on the dissipative finite-step ISS Lyapunov function and on the dynamics, then we can conclude an exponential decay of the bound on the system's state.

**Theorem 3.18.** *Let system (3.1) be globally  $\mathcal{K}$ -bounded with linear  $\omega_1 \in \mathcal{K}_\infty$ . If there exists a dissipative finite-step ISS Lyapunov function  $V$  for system (3.1) satisfying for any  $\xi \in \mathbb{R}^n$  and any bounded  $u(\cdot) \subset \mathbb{R}^m$*

$$\begin{aligned} a\|\xi\|^\lambda &\leq V(\xi) \leq b\|\xi\|^\lambda, \\ V(x(M, \xi, u(\cdot))) &\leq cV(\xi) + d|u|_\infty \end{aligned}$$

with  $b \geq a > 0, c \in [0, 1)$  and  $d, \lambda > 0$ , then system (3.1) is expISS.

*Proof.* The proof follows the lines of the proof of Theorem 3.17. Hence, we will omit the detailed proof, and only give a sketch.

Consider the linear global  $\mathcal{K}$ -bound  $\omega_1 \in \mathcal{K}_\infty$ . We assume that  $\omega_2 \in \mathcal{K}$  is a linear function, too. This second assumption is only for simplifying the proof, but does not change the result. First note that in the proof of Theorem 3.17 we can choose  $\eta(s) = hs$  with  $h \in (0, 1)$  linear, and since  $\rho$  and  $\sigma$  are linear  $\mathcal{K}_\infty$ -functions, we obtain that

$$\tilde{\gamma}(s) := (\text{id} - \rho)^{-1} \circ \eta^{-1} \circ \sigma(s) = \frac{d}{h(1-c)}s$$

is a linear function. Furthermore, in the case that  $x(k) \notin \Delta$ , we see that

$$V(x(k+M, \xi, u(\cdot))) \leq (c + h(1-c))V(x(k, \xi, u(\cdot))).$$

Define  $\tilde{\mu} := (c + h(1-c)) < 1$ . In this case, using the comparison Lemma 3.14, we obtain the estimate

$$V(x(k, \xi, u(\cdot))) \leq \frac{(\tilde{\mu}^{1/M})^k}{\tilde{\mu}} V_M^{\max}(\xi, u(\cdot)),$$

where  $V_M^{\max}(\xi, u(\cdot)) := \max_{j \in \{0, \dots, M-1\}} V(x(j, \xi, u(\cdot)))$ . Let  $\omega_1(s) := w_1s$  and  $\omega_2(s) := w_2s$  for  $s \in \mathbb{R}_+$  and  $w_1, w_2 > 0$ . Using Lemma 3.16, the estimate (3.11) is

satisfied for  $\vartheta_j(s) = w_1^j s$  and  $\zeta_j(s) = w_2 \sum_{i=0}^{j-1} w_1^i s$ . Thus,

$$\begin{aligned} V(x(j, \xi, u(\cdot))) &\leq b(\|x(j, \xi, u(\cdot))\|)^\lambda \\ &\leq b\left(w_1^j \|\xi\| + w_2 \sum_{i=0}^{j-1} w_1^i |u|_\infty\right)^\lambda \\ &= b(\tilde{w}_1 \|\xi\| + \tilde{w}_2 |u|_\infty)^\lambda \end{aligned}$$

with  $\tilde{w}_1 := \max_{j \in \{0, \dots, M-1\}} w_1^j$ , and  $\tilde{w}_2 := \max_{j \in \{0, \dots, M-1\}} w_2 \sum_{i=0}^{j-1} w_1^i$ , and hence,

$$\begin{aligned} V(x(k, \xi, u(\cdot))) &\leq \max_{j \in \{0, \dots, M-1\}} \frac{\tilde{\mu}^{k/M}}{\tilde{\mu}} b\left(w_1^j \|\xi\| + w_2 \sum_{i=0}^{j-1} w_1^i |u|_\infty\right)^\lambda \\ &\leq \frac{b}{\tilde{\mu}} \tilde{\mu}^{k/M} (\tilde{w}_1 \|\xi\| + \tilde{w}_2 |u|_\infty)^\lambda. \end{aligned}$$

This implies that for all  $\xi \in \mathbb{R}^n$  and all bounded  $u(\cdot) \subset \mathbb{R}^m$  we have

$$\begin{aligned} \|x(k, \xi, u(\cdot))\| &\leq (a^{-1} V(x(k, \xi, u(\cdot))))^{1/\lambda} \\ &\leq \left(\frac{b}{a\tilde{\mu}} \tilde{\mu}^{k/M}\right)^{1/\lambda} (\tilde{w}_1 \|\xi\| + \tilde{w}_2 |u|_\infty) \\ &\leq \left(\frac{b\tilde{\omega}_1^{1/\lambda}}{a\tilde{\mu}}\right)^\lambda \mu^k \|\xi\| + \left(\frac{b\tilde{\omega}_2^{1/\lambda}}{a\tilde{\mu}}\right)^\lambda |u|_\infty, \end{aligned}$$

with  $\mu := \tilde{\mu}^{1/\lambda M} < 1$ . So, system (3.1) satisfies (3.3) with  $\beta$  as in (3.4), where  $C = \left(\frac{b\tilde{\omega}_1^{1/\lambda}}{a\tilde{\mu}}\right)^\lambda$  and  $\mu < 1$  as defined above. Hence, system (3.1) is expISS.  $\square$

### 3.2.3 Dissipative finite-step ISS Lyapunov functions: necessary criteria

While Theorem 3.17 shows the sufficiency of the existence of dissipative finite-step ISS Lyapunov functions to conclude ISS of system (3.1), we are now interested in the necessity. At this stage we can exploit the fact that any dissipative ISS Lyapunov function as introduced in Definition 3.10 is a particular dissipative finite-step ISS Lyapunov function satisfying Definition 3.12 with  $M = 1$ .

**Theorem 3.19.** *If system (3.1) is ISS then there exists a dissipative finite-step ISS Lyapunov function for system (3.1).*

*Proof.* If the right-hand side  $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  of system (3.1) is continuous, then [70, Theorem 1] implies the existence of a smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  satisfying  $\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|)$  and  $V(G(\xi, \mu)) - V(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\mu\|)$  for all  $\xi \in \mathbb{R}^n$ ,  $\mu \in \mathbb{R}^m$ , and suitable  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ ,  $\sigma \in \mathcal{K}$ . From Remark 3.11 and  $M = 1$  we conclude that  $V$  is a dissipative finite-step ISS Lyapunov function.

Similarly, if the right-hand side  $G$  is discontinuous, then the existence of a dissipative ISS Lyapunov function follows from [46, Lemma 2.3].  $\square$

Obviously, Theorem 3.19 makes use of the converse ISS Lyapunov theorem in [46, 70] to guarantee the existence of a dissipative (finite-step) ISS Lyapunov function. Converse Lyapunov theorems have, in general, the disadvantage that they are theoretical results and no (ISS) Lyapunov function can be explicitly constructed (see also the explanation in Chapter 2). Consequently, finding a suitable (finite-step) (ISS) Lyapunov function is a difficult task, in general.

However, for the case of expISS systems of the form (3.1), we can show that norms are dissipative finite-step ISS Lyapunov functions. The following result extends Corollary 2.16 to systems with inputs.

**Theorem 3.20.** *If system (3.1) is expISS then the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$V(\xi) := \|\xi\|, \quad \xi \in \mathbb{R}^n \quad (3.14)$$

*is a dissipative finite-step ISS Lyapunov function for system (3.1).*

*Proof.* System (3.1) is expISS if it satisfies (3.3) and (3.4) for constants  $C \geq 1$  and  $\mu \in [0, 1)$ . Take  $M \in \mathbb{N}$  such that  $C\mu^M < 1$ , and  $V$  as defined in (3.14). Clearly,  $V$  is proper and positive definite with  $\alpha_1 = \alpha_2 = \text{id} \in \mathcal{K}_\infty$ . In addition, for any  $\xi \in \mathbb{R}^n$ , we have

$$\begin{aligned} V(x(M, \xi, u(\cdot))) &= \|x(M, \xi, u(\cdot))\| \leq C\mu^M \|\xi\| + \gamma(|u|_\infty) \\ &= C\mu^M V(\xi) + \gamma(|u|_\infty) =: \rho(V(\xi)) + \sigma(|u|_\infty) \end{aligned}$$

where  $\rho(s) := C\mu^M s < s$  for all  $s > 0$ , since  $C\mu^M < 1$ . Note that  $(\text{id} - \rho)(s) = (1 - C\mu^M)s \in \mathcal{K}_\infty$  and  $\sigma := \gamma \in \mathcal{K}_\infty$ , which shows (3.9). So  $V$  defined in (3.14) is a dissipative finite-step ISS Lyapunov function for system (3.1).  $\square$

We emphasize that the hard task in Theorem 3.20 is finding a sufficiently large  $M \in \mathbb{N}$ . However, Theorem 3.20 suggests to take the norm as a candidate for a dissipative finite-step ISS Lyapunov function. Verification for this candidate function to be a dissipative finite-step ISS Lyapunov function can be done as outlined in the following procedure.

*Procedure 3.21.* Consider system (3.1) and assume that Assumption 3.1 holds.

[1] Set  $k = 1$ .

[2] Check

$$\|x(k, \xi, u(\cdot))\| \leq c\|\xi\| + \sigma(|u|_{[0, k]})$$

for all  $\xi \in \mathbb{R}^n$ ,  $u(\cdot) \subset \mathbb{R}^m$  with suitable  $c \in [0, 1)$  and  $\sigma \in \mathcal{K}_\infty$ . If the inequality holds set  $M = k$ ; else set  $k = k + 1$  and repeat.

If this procedure is successful, then  $V(\xi) := \|\xi\|$  is a dissipative finite-step ISS Lyapunov function for system (3.1). By Theorem 3.18 system (3.1) is expISS.  $\triangleleft$

Step [2] can be checked by noticing that the solution  $x(k, \xi, u(\cdot))$  for each  $k \in \mathbb{N}$  and all  $\xi \in \mathbb{R}^n$ ,  $u(\cdot) \subset \mathbb{R}^m$  can be expressed in a closed form by considering higher order iterates of  $G$  as defined by below. In this way, step [2] can be checked by using suitable norm estimates for  $\|G^k(\xi, u(1), \dots, u(k))\|$ . Compare to Section 3.3.3, where such estimates are obtained by means of an example.

To clarify the concept of dissipative finite-step ISS Lyapunov functions we now discuss the connection to higher order iterates of system (3.1).

Let  $G : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$  from (3.1) be given. Then, for any  $i \in \mathbb{N}$  with  $i \geq 1$ , we define the  $i$ th iterate of  $G$ , denoted by  $G^i : \mathbb{R}^n \times (\mathbb{R}^m)^i \mapsto \mathbb{R}^n$ , as follows:

$$\begin{aligned} \xi \in \mathbb{R}^n, w_1 = u_1 \in \mathbb{R}^m &\mapsto G^1(\xi, w_1) := G(\xi, u_1), \\ \xi \in \mathbb{R}^n, w_i := (u_1, \dots, u_i) \in (\mathbb{R}^m)^i &\mapsto G^i(\xi, w_i) := G(G^{i-1}(\xi, w_{i-1}), u_i) \end{aligned}$$

with  $i \in \mathbb{N}$ ,  $i \geq 2$ . Now fix any  $M \in \mathbb{N}$ , and consider the system

$$\bar{x}(k+1) = G^M(\bar{x}(k), w_M(k)) \quad (3.15)$$

with state  $\bar{x} \in \mathbb{R}^n$  and input function  $w_M(\cdot) = (u_1(\cdot), \dots, u_M(\cdot)) \subset (\mathbb{R}^m)^M$ . Firstly, for any  $k \in \mathbb{N}$  there exist unique  $l \in \mathbb{N}$  and  $i \in \{1, \dots, M\}$  such that  $k = lM + i$ . For  $u : \mathbb{N} \rightarrow \mathbb{R}^m$  define  $u_i$ ,  $i \in \{1, \dots, M\}$ , by

$$u_i(l) := u(lM + i), \quad l \in \mathbb{N}, i \in \{1, \dots, M\}$$

we call (3.15) the  $M$ -iteration corresponding to system (3.1). Note that  $|w_M|_\infty := \max\{|u_1|_\infty, \dots, |u_M|_\infty\} = |u|_\infty$ . It is not difficult to see that for all  $j \in \mathbb{N}$  and all  $\xi \in \mathbb{R}^n$  we have

$$x(jM, \xi, u(\cdot)) = \bar{x}(j, \xi, w_M(\cdot)). \quad (3.16)$$

Thus, if system (3.1) is ISS, i.e., it satisfies (3.3), then also the  $M$ -iteration (3.15) is ISS and satisfies

$$\begin{aligned} \|\bar{x}(j, \xi, w_M(\cdot))\| &\stackrel{(3.16)}{=} \|x(jM, \xi, u(\cdot))\| \\ &\leq \beta(\|\xi\|, jM) + \gamma(|u|_\infty) \\ &=: \bar{\beta}(\|\xi\|, j) + \gamma(|w_M|_\infty). \end{aligned}$$

Moreover, a dissipative finite-step ISS Lyapunov function for system (3.1) with suitable  $M \in \mathbb{N}$  is also a dissipative ISS Lyapunov function for the  $M$ -iteration (3.15).

Conversely, let system (3.15) be ISS then there exists a dissipative ISS Lyapunov function  $V$  for system (3.15) (see e.g. [70, Theorem 1] for continuous  $G^M$  or [46, Lemma 2.3] for discontinuous  $G^M$ ). From (3.16) we see that  $V$  is also a dissipative

finite-step ISS Lyapunov function for system (3.1), and by Theorem 3.17 we conclude that system (3.1) is ISS.

Summarizing, we obtain the following corollary.

**Corollary 3.22.** *System (3.1) is ISS if and only if the  $M$ th iterated system (3.15) is ISS. In particular, a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a dissipative Lyapunov function for system (3.15) if and only if it is a dissipative finite-step Lyapunov function for system (3.1).*

As finding ISS Lyapunov functions is a difficult task in general, we will see in the next section that finding a dissipative finite-step ISS Lyapunov function (or equivalently a dissipative Lyapunov function for a corresponding  $M$ -iteration) is sometimes easier. In particular, the next section is dedicated to deriving ISS small-gain theorems, where the proof relies heavily on constructing a dissipative finite-step ISS Lyapunov function for the overall system.

### 3.3 Relaxed small-gain theorems: a Lyapunov-based approach

In the remainder of this chapter we consider system (3.1) split into  $N$  subsystems of the form

$$x_i(k+1) = g_i(x_1(k), \dots, x_N(k), u(k)), \quad k \in \mathbb{N}, \quad (3.17)$$

with  $x_i(0) \in \mathbb{R}^{n_i}$  and  $g_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$  for  $i \in \{1, \dots, N\}$ . We further let  $n = \sum_{i=1}^N n_i$ ,  $x = (x_1, \dots, x_N) \in \mathbb{R}^n$ , then with  $G := (g_1, \dots, g_N)$  we call (3.1) the *overall system* of the subsystems (3.17).

The aim of this section is to derive an ISS small-gain theorem which relaxes the assumption of classical (Lyapunov-based) ISS small-gain theorems to admit ISS Lyapunov functions for each subsystem. We assume that each system has to admit a Lyapunov-type function that decreases after a finite number of time steps rather than at each time step. So the Lyapunov-type functions proposed here allow the subsystems to be 0-input unstable. The results in this section base on the small-gain theorems in Chapter 2 for systems without inputs, and the construction of ISS Lyapunov functions presented in [25].

The section is divided into two parts. In the first one, Section 3.3.1, we prove ISS of system (3.1) by constructing an overall dissipative finite-step ISS Lyapunov function. In Section 3.3.2, we show that the relaxed small-gain theorems derived are necessary, at least for expISS systems. We conclude in Section 3.3.3 with an example where the relaxed small-gain approach is applied.



## 3.3.1 Dissipative ISS small-gain theorems

In this section we derive small-gain theorems by constructing dissipative finite-step ISS Lyapunov functions for the overall system (3.1). We highlight that the proposed small-gain results do not require the subsystems (3.17) to be ISS. This fact is outlined in Remark 3.24.

We start with the case that the effect of the external input  $u$  can be captured via maximization.

**Theorem 3.23.** *Let (3.1) be given by the interconnection of the subsystems in (3.17). Assume that there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $\gamma_{iu} \in \mathcal{K} \cup \{0\}$ , and positive definite functions  $\delta_i$ , with  $d_i := (\text{id} + \delta_i) \in \mathcal{K}_\infty$ , for  $i, j \in \{1, \dots, N\}$  such that with  $\Gamma_\oplus$  defined in (1.16), and the diagonal operator  $D$  defined by  $D = \text{diag}(d_i)$  the following conditions hold.*

- (i) *For all  $i \in \{1, \dots, N\}$ , the functions  $\mathcal{V}_i$  are proper and positive definite.*
- (ii) *For all  $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n$  with  $\xi_i \in \mathbb{R}^{n_i}$ ,  $i \in \{1, \dots, N\}$ , and  $u(\cdot) \subset \mathbb{R}^m$  it holds that*

$$\begin{bmatrix} \mathcal{V}_1(x_1(M, \xi, u(\cdot))) \\ \vdots \\ \mathcal{V}_N(x_N(M, \xi, u(\cdot))) \end{bmatrix} \leq \max \left\{ \Gamma_\oplus \left( \begin{bmatrix} \mathcal{V}_1(\xi_1) \\ \vdots \\ \mathcal{V}_N(\xi_N) \end{bmatrix} \right), \begin{bmatrix} \gamma_{1u}(|u|_\infty) \\ \vdots \\ \gamma_{Nu}(|u|_\infty) \end{bmatrix} \right\}.$$

- (iii) *The small-gain condition<sup>1</sup>  $\Gamma_\oplus \circ D \not\leq \text{id}$  holds.*

*Then there exists an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  for  $\Gamma_\oplus \circ D$ . Moreover, if for all  $i \in \{1, \dots, N\}$  there exists a  $\mathcal{K}_\infty$ -function  $\hat{\alpha}_i$  satisfying*

$$\tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \tilde{\sigma}_i = \tilde{\sigma}_i^{-1} \circ (\text{id} + \delta_i)^{-1} \circ \tilde{\sigma}_i = \text{id} - \hat{\alpha}_i \quad (3.18)$$

*then the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by*

$$V(\xi) := \max_i (\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\mathcal{V}_i(\xi_i)). \quad (3.19)$$

*is a dissipative finite-step ISS Lyapunov function for system (3.1). In particular, system (3.1) is ISS.*

*Proof.* Assume that  $\mathcal{V}_i$  and  $\gamma_{ij}, \gamma_{iu}$  satisfy the hypothesis of the theorem. Denote  $\gamma_u(\cdot) := (\gamma_{1u}(\cdot), \dots, \gamma_{Nu}(\cdot))^\top$ . Then from condition (iii) and [25, Theorem 5.2-(iii)] it follows that there exists an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  such that

$$(\Gamma_\oplus \circ D)(\tilde{\sigma}(s)) < \tilde{\sigma}(s)$$

<sup>1</sup>Note that the strong small-gain condition in Definition 1.21 requires the functions  $\delta_i$  in the diagonal operator to be of class  $\mathcal{K}_\infty$ , whereas here we only require  $\delta_i$  to be positive definite.

holds for all  $s > 0$ . In particular,

$$\max_{i,j \in \{1, \dots, N\}} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ d_j \circ \tilde{\sigma}_j < \text{id}. \quad (3.20)$$

In the following let  $i, j \in \{1, \dots, N\}$ . Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be defined as in (3.19). The aim is to show that  $V$  is a dissipative finite-step ISS Lyapunov function for the overall system (3.1). Recall that condition (i) implies the existence of  $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$  such that for all  $\xi_i \in \mathbb{R}^{n_i}$  we have  $\alpha_{1i}(\|\xi_i\|) \leq \mathcal{V}_i(\xi_i) \leq \alpha_{2i}(\|\xi_i\|)$ . Thus,

$$V(\xi) \geq \max_i (\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\alpha_{1i}(\|\xi_i\|)) \geq \alpha_1(\|\xi\|)$$

with  $\alpha_1 := \min_j \tilde{\sigma}_j^{-1} \circ d_j^{-1} \circ \alpha_{1j} \circ \frac{1}{\kappa} \text{id} \in \mathcal{K}_\infty$ , where  $\kappa \geq 1$  comes from (1.1) in Chapter 1. On the other hand we have

$$V(\xi) \leq \max_i (\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\alpha_{2i}(\|\xi_i\|)) \leq \alpha_2(\|\xi\|)$$

with  $\alpha_2 := \max_i \tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \alpha_{2i} \in \mathcal{K}_\infty$ , which shows  $V$  defined in (3.19) is proper and positive definite. To show the decay of  $V$ , i.e., an inequality of the form (3.9), we define  $\sigma := \max_i \tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \gamma_{iu}$ , and obtain

$$\begin{aligned} V(x(M, \xi, u(\cdot))) &= \max_i (\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\mathcal{V}_i(x_i(M, \xi, u(\cdot)))) \\ &\stackrel{\text{cond. (ii)}}{\leq} \max_i (\tilde{\sigma}_i^{-1} \circ d_i^{-1}) \left( \max_j \left\{ \max_j \gamma_{ij}(\mathcal{V}_j(\xi_j)), \gamma_{iu}(|u|_\infty) \right\} \right) \\ &= \max \left\{ \max_{i,j} \tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \gamma_{ij}(\mathcal{V}_j(\xi_j)), \max_i \tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \gamma_{iu}(|u|_\infty) \right\} \\ &\leq \max \left\{ \max_{i,j} \underbrace{(\tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \tilde{\sigma}_i)}_{=\text{id} - \hat{\alpha}_i \text{ by (3.18)}} \circ \underbrace{(\tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ d_j \circ \tilde{\sigma}_j)}_{< \text{id by (3.20)}} \circ \underbrace{(\tilde{\sigma}_j^{-1} \circ d_j^{-1} \circ \mathcal{V}_j(\xi_j))}_{\leq V(\xi)}, \sigma(|u|_\infty) \right\} \\ &< \max \left\{ \max_i (\text{id} - \hat{\alpha}_i)(V(\xi)), \sigma(|u|_\infty) \right\}. \end{aligned}$$

Define  $\rho := \max_i (\text{id} - \hat{\alpha}_i)$ , then  $\rho \in \mathcal{K}_\infty$  by (3.18), and satisfies  $\text{id} - \rho = \min_i \hat{\alpha}_i \in \mathcal{K}_\infty$ . Noting that the maximum can be upper bounded by summation, this shows that  $V$  is a dissipative finite-step ISS Lyapunov function as defined in Definition 3.12. Then from Theorem 3.17 we conclude that system (3.1) is ISS.  $\square$

*Remark 3.24.* (i) To understand the assumptions imposed in Theorem 3.23 consider the case that  $M = 1$  and  $\delta_i \in \mathcal{K}_\infty$ ,  $i \in \{1, \dots, N\}$ . Firstly, by condition (ii), we have for any  $i \in \{1, \dots, N\}$

$$\mathcal{V}_i(x_i(1, \xi, u(\cdot))) \leq \max \left\{ \max_{j \in \{1, \dots, N\}} \gamma_{ij}(\mathcal{V}_j(\xi_j)), \gamma_{iu}(|u|_\infty) \right\}. \quad (3.21)$$

From the small-gain condition (iii) of Theorem 3.23 we conclude that  $\gamma_{ii} \circ (\text{id} + \delta_i) < \text{id}$  by considering the  $i$ th unit vector. Hence, since  $\delta_i \in \mathcal{K}_\infty$ ,  $\gamma_{ii} < (\text{id} + \delta_i)^{-1} = \text{id} - \hat{\delta}_i$

with  $\hat{\delta}_i \in \mathcal{K}_\infty$ , where the last equality follows from [123, Lemma 2.4] or Lemma 1.6. Thus, we can write (3.21) as

$$\mathcal{V}_i(x_i(1, \xi, u(\cdot))) \leq \underbrace{(\text{id} - \hat{\delta}_i)}_{=: \rho_i}(\mathcal{V}_i(\xi_i)) + \sum_{\substack{j=1 \\ j \neq i}}^N \gamma_{ij}(\mathcal{V}_j(\xi_j)) + \gamma_{iu}|u|_\infty.$$

Together with condition (i) this implies that the functions  $\mathcal{V}_i$  are dissipative ISS Lyapunov functions for the subsystems (3.17) with respect to both internal and external inputs.

Therefore, if  $M = 1$  and  $\delta_i \in \mathcal{K}_\infty$  then Theorem 3.23 is a dissipative small-gain theorem for discrete-time systems in the classical sense.

(ii) If  $M = 1$  and the functions  $\delta_i$  are only positive definite, then the functions  $\mathcal{V}_i$  are not necessarily dissipative ISS Lyapunov functions, as we cannot ensure that the decay of  $\mathcal{V}_i$  in terms of the function  $\rho_i$  satisfies  $\text{id} - \rho_i \in \mathcal{K}_\infty$ . In Theorem 3.23, even in the case  $M = 1$ , we do not necessarily assume that the subsystems are ISS.

(iii) Now consider the case  $M > 1$ . Typically in classical ISS small-gain theorems the systems are required to be ISS with respect to internal and external inputs, and the small-gain condition ensures that the (internal *and* external) inputs cannot destabilize the subsystem. In particular, the subsystems are 0-GAS.

However, in Theorem 3.23, the internal inputs  $x_j$  may have a stabilizing effect on system  $x_i$  in the first  $M$  time steps, whereas the external input  $u$  is considered as a disturbance. Thus, the subsystems do not have to be ISS, while the overall system is ISS. This observation is essential as it extends the classical idea of small-gain theory. In particular, the subsystem (3.17) can be 0-input unstable, see also Section 3.3.3, which is devoted to the discussion of an example. We further refer to the detailed explanation in Section 2.3.1, where we comment on how such a result can be embedded into the small-gain context.

(iv) In contrast to classical ISS small-gain theorems, condition (ii) of Theorem 3.23 requires the knowledge of solutions of the subsystems at time step  $M$  or at least estimates for these. Whereas computing solutions can be done by iterating the dynamics map  $G$ , computing gains satisfying condition (ii) might be challenging, in general.  $\triangleleft$

**Example 3.25.** In this example, we show that condition (3.18), which quantifies the robustness given by the scaling matrix  $D$ , is not trivially satisfied, even if  $\delta_i \in \mathcal{K}_\infty$ . To this end consider the functions

$$\tilde{\sigma}(s) := e^s - 1, \quad \tilde{\sigma}^{-1}(s) = \log(s + 1), \quad \hat{\delta}(s) = (s + 1) \left( 1 - \left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}} \right).$$

It is not hard to see that  $\tilde{\sigma}, \tilde{\sigma}^{-1} \in \mathcal{K}_\infty$ . Moreover, as we show in the Appendix A.3, also  $\hat{\delta} \in \mathcal{K}_\infty$  and  $(\text{id} - \hat{\delta}) \in \mathcal{K}_\infty$ . Similarly as in [123, Lemma 2.4], there exists<sup>2</sup> a function  $\delta \in \mathcal{K}_\infty$  such that  $(\text{id} + \delta)^{-1} = \text{id} - \hat{\delta}$ . Hence, we have for all  $s \in \mathbb{R}_+$ , see Appendix A.3,

$$\tilde{\sigma}^{-1} \circ (\text{id} + \delta)^{-1} \circ \tilde{\sigma}(s) = \tilde{\sigma}^{-1} \circ (\text{id} - \hat{\delta}) \circ \tilde{\sigma}(s) = s(1 - e^{-s}).$$

As  $\lim_{s \rightarrow \infty} s(1 - e^{-s}) - s = 0$ , there cannot exist a  $\mathcal{K}_\infty$ -function  $\hat{\alpha}$  satisfying (3.18).  $\triangleleft$

In condition (ii) of Theorem 3.23 the effect of internal and external inputs was captured via maximization. Next, we replace the maximum in condition (ii) of Theorem 3.23 by a sum. Note that in the case of summation, the small-gain condition invoked in Theorem 3.23 is not strong enough to ensure that  $V$  defined in (3.19) is a dissipative finite-step ISS Lyapunov function (see [25]), so we also have to change condition (iii) of Theorem 3.23. In particular, we assume that the functions  $\delta_i$  are of class  $\mathcal{K}_\infty$ , and not only positive definite. We recall from (1.12) that if the diagonal operator  $D = \text{diag}(\text{id} + \delta_i)$  is factorized into

$$D = D_2 \circ D_1, \quad D_j = \text{diag}(\text{id} + \delta_{ij}), \quad \delta_{ij} \in \mathcal{K}_\infty, \quad i \in \{1, \dots, N\}, \quad j \in \{1, 2\} \quad (3.22)$$

then  $D \circ \Gamma_\oplus \not\geq \text{id}$  is equivalent to  $D_1 \circ \Gamma_\oplus \circ D_2 \not\geq \text{id}$ .

**Theorem 3.26.** *Let (3.1) be given by the interconnection of the subsystems in (3.17). Let  $\delta_i, \delta_{i_1}, \delta_{i_2} \in \mathcal{K}_\infty$  for  $i \in \{1, \dots, N\}$  and  $D := \text{diag}(d_i) := \text{diag}(\text{id} + \delta_i)$  satisfy (3.22). Assume that there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ ,  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ , and  $\gamma_{iu} \in \mathcal{K} \cup \{0\}$  for  $i, j \in \{1, \dots, N\}$  such that with  $\Gamma_\oplus$  defined in (1.16) the following conditions hold.*

- (i) *For all  $i \in \{1, \dots, N\}$  the functions  $\mathcal{V}_i$  are proper and positive definite.*
- (ii) *For all  $\xi \in \mathbb{R}^n$  and  $u(\cdot) \subset \mathbb{R}^m$  it holds that*

$$\begin{bmatrix} \mathcal{V}_1(x_1(M, \xi, u(\cdot))) \\ \vdots \\ \mathcal{V}_N(x_N(M, \xi, u(\cdot))) \end{bmatrix} \leq \Gamma_\oplus \left( \begin{bmatrix} \mathcal{V}_1(\xi_1) \\ \vdots \\ \mathcal{V}_N(\xi_N) \end{bmatrix} \right) + \begin{bmatrix} \gamma_{1u}(|u|_\infty) \\ \vdots \\ \gamma_{Nu}(|u|_\infty) \end{bmatrix}.$$

- (iii) *The strong small-gain condition  $D \circ \Gamma_\oplus \not\geq \text{id}$  holds.*

*Then there exists an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  for  $D_1 \circ \Gamma_\oplus \circ D_2$ . Moreover, if for all  $i \in \{1, \dots, N\}$  there exists a  $\mathcal{K}_\infty$ -function  $\hat{\alpha}_i$  satisfying*

$$\tilde{\sigma}_i^{-1} \circ d_{i_2}^{-1} \circ \tilde{\sigma}_i = \text{id} - \hat{\alpha}_i \quad (3.23)$$

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<sup>2</sup>Note that [123, Lemma 2.4] argues that if  $\delta \in \mathcal{K}_\infty$  is given, there exists a suitable  $\hat{\delta} \in \mathcal{K}_\infty$  satisfying  $(\text{id} + \delta)^{-1} = \text{id} - \hat{\delta}$ . Conversely, for any  $\hat{\delta} \in \mathcal{K}_\infty$  with  $(\text{id} - \hat{\delta}) \in \mathcal{K}_\infty$  given there exists  $\delta \in \mathcal{K}_\infty$  satisfying  $(\text{id} + \delta)^{-1} = \text{id} - \hat{\delta}$ , follows by defining  $\delta := \hat{\delta} \circ (\text{id} - \hat{\delta})^{-1} \in \mathcal{K}_\infty$ .

then the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined by

$$V(\xi) := \max_i (\tilde{\sigma}_i^{-1} \circ d_{i2}^{-1})(\mathcal{V}_i(\xi_i)) \quad (3.24)$$

is a dissipative finite-step ISS Lyapunov function for system (3.1). In particular, system (3.1) is ISS.

*Proof.* Since the diagonal operator  $D$  is split as in (3.22), we can make use of the equivalences in (1.12), which show that  $D \circ \Gamma_{\oplus} \not\geq \text{id}$  if and only if  $D_1 \circ \Gamma_{\oplus} \circ D_2 \not\geq \text{id}$ . By [25, Theorem 5.2-(iii)] it follows that there exists an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_{\infty}^N$  for  $D_1 \circ \Gamma_{\oplus} \circ D_2$  satisfying

$$(D_1 \circ \Gamma_{\oplus} \circ D_2)(\tilde{\sigma}(r)) < \tilde{\sigma}(r) \quad \forall r > 0,$$

or, equivalently, for all  $i \in \{1, \dots, N\}$ ,

$$\max_{j \in \{1, \dots, N\}} d_{i1} \circ \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j(r) < \tilde{\sigma}_i(r) \quad \forall r > 0. \quad (3.25)$$

We show that this inequality implies the existence of a function  $\varphi \in \mathcal{K}_{\infty}$  such that for all  $r > 0$ , we have

$$\max_{i,j} \tilde{\sigma}_i^{-1} \circ (\gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j(r) + \gamma_{iu} \circ \varphi(r)) < r. \quad (3.26)$$

To do this, we assume, without loss of generality<sup>3</sup>, that  $\gamma_{iu} \in \mathcal{K}_{\infty}$ . Since  $d_{i1} = \text{id} + \delta_{i1}$  with  $\delta_{i1} \in \mathcal{K}_{\infty}$  for all  $i \in \{1, \dots, N\}$ , we can write (3.25) as

$$\max_{j \in \{1, \dots, N\}} \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j(r) + \max_{j \in \{1, \dots, N\}} \delta_{i1} \circ \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j(r) < \tilde{\sigma}_i(r). \quad (3.27)$$

Let  $i \in \{1, \dots, N\}$ . We consider two cases:

- (i) If  $\gamma_{ij} = 0$  for all  $j \in \{1, \dots, N\}$ , define

$$\varphi_i := \frac{1}{2} \gamma_{iu}^{-1} \circ \tilde{\sigma}_i \in \mathcal{K}_{\infty}.$$

- (ii) If  $\gamma_{ij} \in \mathcal{K}_{\infty}$  for at least one  $j \in \{1, \dots, N\}$ , define

$$\varphi_i := \max_{j \in \{1, \dots, N\}} \gamma_{iu}^{-1} \circ \delta_{i1} \circ \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j \in \mathcal{K}_{\infty}.$$

Note that we need  $\delta_{i1} \in \mathcal{K}_{\infty}$  for  $\varphi_i$  to be of class  $\mathcal{K}_{\infty}$ , as opposed to the proof of Theorem 3.23, where we only needed positive definiteness.

<sup>3</sup>If  $\gamma_{iu} \in \mathcal{K} \setminus \mathcal{K}_{\infty}$  take any  $\mathcal{K}_{\infty}$ -function upper bounding  $\gamma_{iu}$ . If  $\gamma_{iu} = 0$ , take e.g.  $\gamma_{iu} = \text{id}$ .

For both cases, the definition of  $\varphi_i \in \mathcal{K}_\infty$  together with (3.27) implies

$$\max_{j \in \{1, \dots, N\}} \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j(r) + \gamma_{iu} \circ \varphi_i(r) < \tilde{\sigma}_i(r).$$

for all  $r > 0$ . Then it is not hard to see that  $\varphi := \min_{i \in \{1, \dots, N\}} \varphi_i \in \mathcal{K}_\infty$  satisfies (3.26) for all  $r > 0$ .

In the following let  $i, j \in \{1, \dots, N\}$ . Consider the function  $V$  from (3.24). First note that  $V$  is proper and positive definite, which follows directly from the proof of Theorem 3.23.

To show the decay of  $V$ , i.e., an inequality of the form (3.9), we use (3.26), and obtain

$$\begin{aligned} V(x(M, \xi, u(\cdot))) &= \max_i (\tilde{\sigma}_i^{-1} \circ d_{i2}^{-1})(\mathcal{V}_i(x_i(M, \xi, u(\cdot)))) \\ &\stackrel{\text{cond. (ii)}}{\leq} \max_i (\tilde{\sigma}_i^{-1} \circ d_{i2}^{-1}) \left( \max_j \gamma_{ij}(\mathcal{V}_j(\xi_j)) + \gamma_{iu}(|u|_\infty) \right) \\ &\stackrel{(3.23)}{=} \max_{i,j} (\text{id} - \hat{\alpha}_i) \circ \tilde{\sigma}_i^{-1} (\gamma_{ij}(\mathcal{V}_j(\xi_j)) + \gamma_{iu}(|u|_\infty)) \\ &= \max_{i,j} (\text{id} - \hat{\alpha}_i) \circ \tilde{\sigma}_i^{-1} \left( \gamma_{ij} \circ d_{j2} \circ \tilde{\sigma}_j \circ \underbrace{(\tilde{\sigma}_j^{-1} \circ d_{j2}^{-1} \circ \mathcal{V}_j(\xi_{j'}))}_{\leq V(\xi)} + \gamma_{iu} \circ \varphi \circ \varphi^{-1}(|u|_\infty) \right) \\ &\stackrel{(3.26)}{<} \max_i (\text{id} - \hat{\alpha}_i) (\max\{V(\xi), \varphi^{-1}(|u|_\infty)\}) \\ &\leq \max_i (\text{id} - \hat{\alpha}_i)(V(\xi)) + \max_i (\text{id} - \hat{\alpha}_i)(\varphi^{-1}(|u|_\infty)). \end{aligned}$$

Define  $\rho := \max_i (\text{id} - \hat{\alpha}_i)$  and  $\sigma := \max_i (\text{id} - \hat{\alpha}_i) \circ \varphi^{-1}$ , then  $\text{id} - \rho = \min_i \hat{\alpha}_i \in \mathcal{K}_\infty$ . Hence, (3.9) is satisfied. Again, as in the proof of Theorem 3.23, this shows that  $V$  is a dissipative finite-step ISS Lyapunov function as defined in Definition 3.12. From Theorem 3.17 we conclude that system (3.1) is ISS.  $\square$

*Remark 3.27.* If Theorem 3.23 (resp. Theorem 3.26) is satisfied with  $M = 1$ , then the dissipative finite-step Lyapunov function  $V$  in (3.19) (resp. (3.24)) is a dissipative ISS Lyapunov function. In particular, we obtain the following special cases: If Theorem 3.26 is satisfied for  $M = 1$  then this gives a dissipative-form discrete-time version of [25, Corollary 5.6]. On the other hand, for  $M = 1$ , Theorem 3.23 includes the ISS variant of [67, Theorem 3] as a special case.  $\triangleleft$

*Remark 3.28.* Condition (3.23) in Theorem 3.23 crucially depends on the decomposition (3.22). In this remark we comment on the question whether or not condition 3.23 holds depending on the decomposition 3.22.

Let  $D_1, D_2$  as well as  $\hat{D}_1, \hat{D}_2$  be two compositions of  $D$  as in (3.22), i.e.,  $D_2 \circ D_1 = D = \hat{D}_2 \circ \hat{D}_1$ . A direct computation shows that if  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  is an  $\Omega$ -path for  $D_1 \circ \Gamma_\oplus \circ D_2$  then  $\hat{\sigma} := \hat{D}_1 \circ D_1^{-1} \circ \tilde{\sigma} \in \mathcal{K}_\infty^N$  is an  $\Omega$ -path for  $\hat{D}_1 \circ \Gamma_\oplus \circ \hat{D}_2$ :

$$\begin{aligned} (\hat{D}_1 \circ \Gamma_\oplus \circ \hat{D}_2)(\hat{\sigma}) &= (\hat{D}_1 \circ \Gamma_\oplus \circ D_2) \circ \underbrace{(D_1 \circ \hat{D}_1^{-1})(\hat{\sigma})}_{=\tilde{\sigma}} \\ &= (\hat{D}_1 \circ D_1^{-1}) \circ (D_1 \circ \Gamma_\oplus \circ D_2)(\tilde{\sigma}) \\ &< (\hat{D}_1 \circ D_1^{-1})(\tilde{\sigma}) = \hat{\sigma}. \end{aligned}$$

Moreover, if we assume that (3.23) holds, then we have

$$\begin{aligned} \hat{\sigma}_i^{-1} \circ \hat{d}_{i2}^{-1} \circ \hat{\sigma}_i &= (\tilde{\sigma}_i^{-1} \circ d_{i2}^{-1} \circ \tilde{\sigma}_i) \circ (\tilde{\sigma}_i^{-1} \circ \hat{d}_{i1} \circ d_{i1}^{-1} \circ \tilde{\sigma}_i) \\ &= (\text{id} - \hat{\alpha}_i) \circ (\tilde{\sigma}_i^{-1} \circ \hat{d}_{i1} \circ d_{i1}^{-1} \circ \tilde{\sigma}_i) \end{aligned}$$

with  $\hat{\alpha}_i \in \mathcal{K}_\infty$ ,  $i \in \{1, \dots, N\}$ . Unfortunately, from this equation we cannot conclude that a condition of the form (3.23) holds for the decomposition  $\hat{D}_1, \hat{D}_2$  and the  $\Omega$ -path  $\hat{\sigma}$  as defined above.

However, consider the special case of the decomposition (3.22) with  $D_2 = \text{diag}(\text{id})$  and  $D_1 = D = \text{diag}(d_i)$ . Let  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  be an  $\Omega$ -path for  $D_1 \circ \Gamma_\oplus \circ D_2 = D \circ \Gamma_\oplus$ . The aim is to find a decomposition  $\hat{D}_1, \hat{D}_2$  of  $D$  as in (3.22) such that (3.23) holds. Following the same steps as above, we see that  $\hat{\sigma} = \hat{D}_1 \circ D^{-1} \circ \tilde{\sigma} \in \mathcal{K}_\infty^N$  is an  $\Omega$ -path for  $\hat{D}_1 \circ \Gamma_\oplus \circ \hat{D}_2$ . Moreover, as  $D = \hat{D}_2 \circ \hat{D}_1$  implies  $d_i^{-1} = \hat{d}_{i1}^{-1} \circ \hat{d}_{i2}^{-1}$ , we see that

$$\begin{aligned} \hat{\sigma}_i^{-1} \circ \hat{d}_{i2}^{-1} \circ \hat{\sigma}_i &= (\tilde{\sigma}_i^{-1} \circ d_i \circ \hat{d}_{i1}^{-1}) \circ (\hat{d}_{i1} \circ d_i^{-1}) \circ (\hat{d}_{i1} \circ d_i^{-1} \circ \tilde{\sigma}_i) \\ &= (\tilde{\sigma}_i^{-1} \hat{d}_{i1} \circ d_i^{-1} \circ \tilde{\sigma}_i) \\ &= \tilde{\sigma}_i^{-1} \hat{d}_{i2}^{-1} \circ \tilde{\sigma}_i. \end{aligned}$$

Hence, a condition of the form (3.23) holds for the decomposition  $\hat{D}_1, \hat{D}_2$  with  $\Omega$ -path  $\hat{\sigma} \in \mathcal{K}_\infty^N$  if and only there exist  $\mathcal{K}_\infty$ -functions  $\hat{\alpha}_i$ ,  $i \in \{1, \dots, N\}$  satisfying

$$\tilde{\sigma}_i^{-1} \circ \hat{d}_{i2}^{-1} \circ \tilde{\sigma}_i = \text{id} - \hat{\alpha}_i. \quad (3.28)$$

Hence, to find a "good" decomposition  $\hat{D}_1, \hat{D}_2$  of  $D$ , i.e., a decomposition for which (3.23) holds, we can try to find  $\mathcal{K}_\infty$ -functions  $\hat{d}_{i2}^{-1} = (\text{id} + \hat{\delta}_{i2})^{-1}$ ,  $i \in \{1, \dots, N\}$  that satisfy

- (i) equation (3.28) for suitable  $\alpha_i \in \mathcal{K}_\infty$ ;
- (ii)  $\hat{d}_{i2} \circ \hat{d}_{i1} = d_i = \text{id} + \delta_i$  with suitable  $\hat{d}_{i1} = \text{id} + \hat{\delta}_{i1}$ .

However, an answer to the question whether there exists a decomposition satisfying these two conditions remains open.  $\triangleleft$

In Theorem 3.23 we introduced the diagonal operator  $D$ . In addition, (3.18) is assumed to hold. In the following corollary we impose further assumptions such that we do not need the diagonal operator  $D$  anymore. Under these stronger assumptions, system (3.1) is shown to be expISS.

**Corollary 3.29.** *Let (3.1) be given by the interconnection of the subsystems in (3.17). Assume that there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , linear functions  $\gamma_{ij} \in \mathcal{K}_\infty$ , and functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$  satisfying condition (i) of Theorem 3.23 with linear functions  $\alpha_{1i}, \alpha_{2i}$ . Let condition (ii) of Theorem 3.23 hold, and instead of condition (iii) of Theorem 3.23, let the small-gain condition (1.10) hold. Furthermore, assume that the  $\mathcal{K}$ -function  $\omega_1$  in Assumption 3.1 is linear. Then system (3.1) is expISS.*

*Proof.* We follow the proof of Theorem 3.23. By the small-gain condition (1.10) there exists an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_\infty^N$  satisfying  $\Gamma_\oplus(\tilde{\sigma})(r) < \tilde{\sigma}(r)$  for all  $r > 0$ , see [25]. Moreover, as the functions  $\gamma_{ij} \in \mathcal{K}_\infty$  are linear for all  $i, j \in \{1, \dots, N\}$ , we can also assume the  $\Omega$ -path functions  $\tilde{\sigma}_i \in \mathcal{K}_\infty$  to be linear, see [37]. Thus, the function

$$V(\xi) := \max_i \tilde{\sigma}_i^{-1}(\mathcal{V}_i(\xi_i)) \quad (3.29)$$

has linear bounds  $\alpha_1$  and  $\alpha_2$ . Furthermore, since  $\tilde{\sigma}_i$  and  $\gamma_{ij}$  are linear functions, we obtain (3.9) with linear function  $\rho := \max_{i,j} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j < \text{id}$ , and  $\sigma := \max_i \tilde{\sigma}_i^{-1} \circ \gamma_{iu}$ . Clearly,  $(\text{id} - \rho) \in \mathcal{K}_\infty$  by linearity of  $\rho$ . Thus,  $V$  is a dissipative finite-step ISS Lyapunov function for system (3.1). Since  $\omega_1$  in Assumption 3.1 is linear, we can apply Theorem 3.18 to show that system (3.1) is expISS.  $\square$

Note that the requirement that  $\omega_1$  is linear is necessary for the system to be expISS as outlined in Remark 3.3.

A similar reasoning applies in the case where the external input enters additively.

**Corollary 3.30.** *Let (3.1) be given by the interconnection of the subsystems in (3.17). Assume there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , linear functions  $\gamma_{ij} \in \mathcal{K}_\infty$ , and functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$  satisfying condition (i) of Theorem 3.26 with linear functions  $\alpha_{1i}, \alpha_{2i}$ . Let condition (ii) of Theorem 3.26 and the small-gain condition (1.10) hold. Furthermore, assume that  $\omega_1$  in Assumption 3.1 is linear. Then system (3.1) is expISS.*

*Proof.* We omit the details as the proof follows the lines of the proof of Theorem 3.26 combined with the argumentation of the proof of Corollary 3.29.

Firstly, the small-gain condition implies the existence of an  $\Omega$ -path  $\tilde{\sigma} \in \mathcal{K}_\infty^N$ , which is linear as the functions  $\gamma_{ij}$  are linear. In particular,  $\Gamma_\oplus(\tilde{\sigma})(r) < \sigma(r)$  for all  $r > 0$ . Next, note that the function  $V$  defined in (3.29) has linear bounds as shown in the proof of Corollary 3.29. It satisfies

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(|u|_\infty)$$



with  $\rho := \max_{i,j} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j$  and  $\sigma := \max_i \tilde{\sigma}_i^{-1} \circ \gamma_{iu}$ , which can be seen by a straightforward calculation, invoking condition (ii) of Theorem 3.26 and the linearity of the  $\Omega$ -path  $\tilde{\sigma}$ . Again as in the proof of Corollary 3.29,  $(\text{id} - \rho) \in \mathcal{K}_\infty$  by linearity of  $\rho$ . Thus,  $V$  defined in (3.29) is a dissipative finite-step ISS Lyapunov function for system (3.1). Since  $\omega_1$  in Assumption 3.1 is linear, we can apply Theorem 3.18, and the result follows.  $\square$

In this section we have presented sufficient criteria to conclude ISS, whereas in the next section we will study the necessity of these relaxed small-gain results.

### 3.3.2 Non-conservative expISS small-gain theorems

Classical small-gain theorems come with certain conservatism, as explained in the introduction. In the remainder of this section we show that the relaxation of classical small-gain theorems given in Theorems 3.23 and 3.26 is non-conservative at least for expISS systems, i.e., Corollary 3.29 and 3.30 provide conditions that are not only sufficient but also necessary.

**Theorem 3.31.** *Let system (3.1) be the overall system of the subsystems (3.17). Then system (3.1) is expISS if and only if*

- (i) *Assumption 3.1 holds with linear  $\omega_1$ , and*
- (ii) *there exist an  $M \in \mathbb{N}$ ,  $M \geq 1$ , linear functions  $\gamma_{ij} \in \mathcal{K}_\infty$ , proper and positive definite functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$  with linear bounds  $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$  such that the following holds:*
  - (a) *condition (ii) of Theorem 3.23 (and thus also condition (ii) of Theorem 3.26);*
  - (b) *the small-gain condition (1.10).*

*Proof.* Sufficiency is shown by Corollary 3.29 and Corollary 3.30, so we only have to prove necessity. Since system (3.1) is expISS, Assumption 3.1 holds with linear  $\omega_1$ , see Remark 3.3. Furthermore, the function  $V(\xi) := \|\xi\|$  for all  $\xi \in \mathbb{R}^n$  is a dissipative finite-step ISS Lyapunov function for system (3.1) by Theorem 3.20. Hence, there exist  $\tilde{M} \in \mathbb{N}$ ,  $\sigma \in \mathcal{K}$  and  $c < 1$  such that for all  $\xi \in \mathbb{R}^n$  and all bounded  $u(\cdot) \subset \mathbb{R}^m$  we have

$$\|x(\tilde{M}, \xi, u(\cdot))\| \leq c\|\xi\| + \sigma(\|u\|_\infty). \quad (3.30)$$

Define  $\mathcal{V}_i(\xi_i) := \|\xi_i\|$  for  $i \in \{1, \dots, N\}$ , where the norm for  $\xi_i \in \mathbb{R}^{n_i}$  is defined by (1.1). Then  $\mathcal{V}_i$  is proper and positive definite with  $\alpha_{1i} = \alpha_{2i} = \text{id}$  for all  $i \in \{1, \dots, N\}$ . Take  $\kappa \geq 1$  from (1.1), and define

$$l := \min\{\ell \in \mathbb{N} : c^\ell \kappa < \frac{1}{2}\},$$

which exists as  $c \in [0, 1)$ . Then we have

$$\begin{aligned}
 \mathcal{V}_i(x_i(l\tilde{M}, \xi, u(\cdot))) &= \|x_i(l\tilde{M}, \xi, u(\cdot))\| \leq \|x(l\tilde{M}, \xi, u(\cdot))\| \\
 &\stackrel{(3.30)}{\leq} c\|x((l-1)\tilde{M}, \xi, u(\cdot))\| + \sigma(|u|_\infty) \\
 &\stackrel{(3.30)}{\leq} c^l\|\xi\| + \sum_{j'=1}^l c^{j'-1}\sigma(|u|_\infty) \\
 &\stackrel{(1.1)}{\leq} \max_j c^l \kappa \|\xi_j\| + \sum_{j'=1}^l c^{j'-1}\sigma(|u|_\infty) \\
 &\leq \max \left\{ \max_j \gamma_{ij}(\xi_j), \gamma_{iu}(|u|_\infty) \right\} \\
 &\leq \max_j \gamma_{ij}(\xi_j) + \gamma_{iu}(|u|_\infty)
 \end{aligned}$$

with  $\gamma_{iu}(\cdot) := 2 \sum_{j=1}^l c^{j-1} \sigma(\cdot)$ , and  $\gamma_{ij} := 2c^l \kappa \text{id}$ . The last inequality shows condition (ii) of Theorem 3.26, while the second last inequality shows condition (ii) of Theorem 3.23 for  $M = l\tilde{M}$ . Finally, by definition of  $l \in \mathbb{N}$ , we have  $\gamma_{ij} < \text{id}$  for all  $i, j \in \{1, \dots, N\}$ . Hence, Proposition 1.29 implies the small-gain condition (1.10). This proves the theorem.  $\square$

The non-conservative expISS small-gain Theorem 3.31 is proved in a constructive way, i.e., it is shown that under the assumption that system (3.1) is expISS we can choose the Lyapunov-type functions  $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$  as norms, i.e.,  $\mathcal{V}_i(\cdot) = \|\cdot\|$ . Then there exist an  $M \in \mathbb{N}$  and linear gains  $\gamma_{ij} \in \mathcal{K}_\infty$  satisfying condition (ii) of Theorem 3.23, and thus also condition (ii) of Theorem 3.26, as well as the small-gain condition (1.10). This suggests the following procedure.

*Procedure 3.32.* Consider (3.1) as the overall system of the subsystems (3.17). Check that Assumption 3.1 is satisfied with a linear  $\omega_1$  (else the origin of system (3.1) cannot be expISS, see Remark 3.3). Define  $\mathcal{V}_i(\xi_i) := \|\xi_i\|$  for  $\xi_i \in \mathbb{R}^{n_i}$ , and set  $k = 1$ .

- [1] Compute  $\gamma_{iu} \in \mathcal{K}_\infty$  and linear functions  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$  satisfying

$$\mathcal{V}_i(x_i(k, \xi, u(\cdot))) = \|x_i(k, \xi, u(\cdot))\| \leq \max_{j \in \{1, \dots, N\}} \gamma_{ij} \|\xi_j\| + \gamma_{iu}(|u|_\infty).$$

- [2] Check the small-gain condition (1.10) with  $\Gamma_\oplus$  defined in (1.16). If (1.10) is violated set  $k = k + 1$  and repeat with step [1].

If this procedure is successful, then expISS of the overall system (3.1) is shown by Theorem 3.31. Moreover, a dissipative finite-step ISS Lyapunov function can be constructed via (3.29).  $\triangleleft$

Although Procedure 3.32 is straightforward, even for simple classes of systems, finding a suitable  $M \in \mathbb{N}$  may be computationally intractable, as already noted in Chapter 2. Nevertheless, a systematic way to find a suitable number  $M \in \mathbb{N}$  for certain classes of systems was discussed in Section 2.4 for systems without inputs. For systems with inputs, similar ideas might be applied.

In the next section we consider a nonlinear system and show how Procedure 3.32 can be applied.

### 3.3.3 Illustrative example

In the introduction of this thesis the conservatism of classical small-gain theorems was illustrated by a linear example without external inputs, which is GAS, but where the subsystems are not 0-GAS. In this section we consider a nonlinear example *with* external inputs and show how the relaxed small-gain theorems from Sections 3.3.1 and 3.3.2, and, in particular, Procedure 3.32, can be applied.

Consider the nonlinear system

$$x(k+1) = \begin{pmatrix} [x(k+1)]_1 \\ [x(k+1)]_2 \end{pmatrix} = \begin{pmatrix} [x(k)]_1 - 0.3[x(k)]_2 + u(k) \\ [x(k)]_1 + 0.3 \frac{[x(k)]_2^2}{1+[x(k)]_2^2} \end{pmatrix} \quad (3.31)$$

with  $x(\cdot) \in \mathbb{R}^2$ , and  $u(\cdot) \in \mathbb{R}$ . We show that this system is ISS by constructing a suitable dissipative finite-step ISS Lyapunov function following Procedure 3.32. We consider the system split into two subsystems. Note that the origin of the first subsystem is not 0-GAS<sup>4</sup>, hence not ISS. So we cannot find an ISS Lyapunov function for this subsystem. At this point, classical (Lyapunov-based) small-gain theorems would fail.

The converse small-gain results in Section 3.3.2 suggest to prove ISS by a search for suitable functions  $\mathcal{V}_i$  and  $\gamma_{ij} \in \mathcal{K}_\infty$  that satisfy the conditions of one of the small-gain theorems (eg. Theorem 3.23 or Corollary 3.29). Here, we follow Procedure 3.32. Firstly, the right-hand side function  $G$  of (3.31) is globally  $\mathcal{K}$ -bounded, since

$$\|G(\xi, \nu)\|_\infty \leq \max\{\|[\xi]_1\| + 0.3\|[\xi]_2\| + \|\nu\|, \|[\xi]_1\| + 0.3 \frac{[\xi]_2^2}{1+[\xi]_2^2}\} \leq 1.3\|\xi\|_\infty + \|\nu\|,$$

where we used (2.16).

---

<sup>4</sup>If the first subsystem is considered decoupled, i.e., if we set  $x_2 = u = 0$ , then this system is globally stable but not GAS. We could also make the first decoupled system unstable by letting  $x_1(k+1) = (1+\epsilon)x_1(k) - 0.3x_2(k) + u(k)$  and  $\epsilon > 0$  small enough, see also [35]. But here we let  $\epsilon = 0$  to simplify computations.

Let  $\mathcal{V}_i([\xi]_i) := |[\xi]_i|$ ,  $i \in \{1, 2\}$ . Then we compute for all  $\xi \in \mathbb{R}^2$ ,

$$\begin{aligned}\mathcal{V}_1(x_1(1, \xi, u(\cdot))) &= |[\xi]_1 - 0.3[\xi]_2 + u(0)| \leq \max\{2\mathcal{V}_1([\xi]_1), 0.6\mathcal{V}_2([\xi]_2)\} + |u|_\infty, \\ \mathcal{V}_2(x_2(1, \xi, u(\cdot))) &= |[\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2}| \leq \max\left\{2\mathcal{V}_1([\xi]_1), 0.6\frac{\mathcal{V}_2^2([\xi]_2)}{1+\mathcal{V}_2^2([\xi]_2)}\right\}.\end{aligned}$$

Since  $\gamma_{11}(s) = 2s$ , the small-gain condition is violated and we cannot conclude stability. Intuitively, this was expected from the above observation that the origin of the first subsystem is not ISS.

Computing solutions  $x(k, \xi, u(\cdot))$  with initial condition  $\xi \in \mathbb{R}^2$  and input  $u(\cdot) \subset \mathbb{R}$  we see that for  $k = 3$  we have

$$\begin{aligned}[x(3, \xi, u(\cdot))]_1 &= 0.4[\xi]_1 - 0.21[\xi]_2 - 0.09\frac{[\xi]_2^2}{1+[\xi]_2^2} + 0.7u(0) + u(1) + u(2) \\ &\quad - 0.09\frac{([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2} \\ [x(3, \xi, u(\cdot))]_2 &= 0.7[\xi]_1 - 0.3[\xi]_2 - 0.09\frac{[\xi]_2^2}{1+[\xi]_2^2} + u(0) + u(1) \\ &\quad + 0.3\frac{\left([\xi]_1 - 0.3[\xi]_2 + 0.3\frac{([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}\right)^2}{\left([\xi]_1 - 0.3[\xi]_2 + 0.3\frac{([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+([\xi]_1 + 0.3\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}\right)^2}.\end{aligned}$$

Using (2.16) again, we compute

$$\begin{aligned}\mathcal{V}_1([x(3, \xi, u(\cdot))]_1) &\leq 0.4|[\xi]_1| + 0.21|[\xi]_2| + 0.7|u(0)| + |u(1)| + |u(2)| \\ &\quad + \frac{0.09}{2}|[\xi]_2| + \frac{0.09}{2}(|[\xi]_1| + \frac{0.3}{2}|[\xi]_2|) \\ &= \max\{0.89\mathcal{V}_1([\xi]_1), 0.5235\mathcal{V}_2([\xi]_2)\} + 2.7|u|_\infty, \\ \mathcal{V}_2([x(3, \xi, u(\cdot))]_2) &\leq 0.7|[\xi]_1| + 0.3|[\xi]_2| + \frac{0.09}{2}|[\xi]_2| + |u(0)| + |u(1)| \\ &\quad + \frac{0.3}{2}(|[\xi]_1| + 0.3|[\xi]_2| + \frac{0.3}{2}(|[\xi]_1| + \frac{0.3}{2}|[\xi]_2|)) \\ &= \max\{1.745\mathcal{V}_1([\xi]_1), 0.78675\mathcal{V}_2([\xi]_2)\} + 2|u|_\infty.\end{aligned}$$

From this we derive the linear functions

$$\begin{aligned}\gamma_{11}(s) &= 0.89s, & \gamma_{12}(s) &= 0.5235s, & \gamma_{1u}(s) &= 2.7s, \\ \gamma_{21}(s) &= 1.745s, & \gamma_{22}(s) &= 0.78675s, & \gamma_{2u}(s) &= 2s,\end{aligned}$$

yielding the map  $\Gamma_\oplus : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$  from (1.16) as

$$\Gamma_\oplus((s_1, s_2)) = \begin{pmatrix} \max\{0.89s_1, 0.5235s_2\} \\ \max\{1.745s_1, 0.78675s_2\} \end{pmatrix}.$$

Since  $\gamma_{11} < \text{id}$ ,  $\gamma_{22} < \text{id}$  and  $\gamma_{12} \circ \gamma_{21} < \text{id}$ , we conclude from the cycle condition, Proposition 1.29, that the small-gain condition (1.10) is satisfied. Hence, from Corollary 3.30 we can now conclude that the origin of system (3.31) is expISS.

The small-gain results in Section 3.3.1, and in particular Corollary 3.30, prove the ISS property of the interconnected system (3.1) by constructing a dissipative finite-step ISS Lyapunov function. The following shows that this construction is straightforward to implement. To do this, we use the method proposed in [37] to compute an  $\Omega$ -path  $\tilde{\sigma}(r) := \begin{pmatrix} 0.5r \\ 0.9r \end{pmatrix}$  that satisfies

$$\Gamma_{\oplus}(\tilde{\sigma}(r)) = \begin{pmatrix} 0.47115r \\ 0.8725r \end{pmatrix} < \begin{pmatrix} 0.5r \\ 0.9r \end{pmatrix} = \tilde{\sigma}(r)$$

for all  $r > 0$ . From the proof of Corollary 3.30 we can now conclude that

$$V(\xi) := \max_i \tilde{\sigma}_i^{-1}(\mathcal{V}_i([\xi]_i)) = \max\{2|\xi|_1, \frac{10}{9}|\xi|_2\}$$

is a dissipative finite-step ISS Lyapunov function for the overall system (3.31). In particular, as shown in the proof of Corollary 3.30, we can compute

$$\rho(s) := \max_{i,j \in \{1,2\}} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j(s) = 0.9695s,$$

and,

$$\sigma(s) := \max_{i \in \{1,2\}} \tilde{\sigma}_i^{-1} \circ \gamma_{iu} = 5.4s$$

for which  $V$  satisfies  $V(x(3, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(|u|_{\infty})$  for all  $\xi \in \mathbb{R}^2$ .

In Figure 3.1 we plot the sublevel sets of the dissipative finite-step ISS Lyapunov function  $V$ . In addition, we plot the trajectory starting in the initial state  $\xi = (10, -15)$ , where we assume that the input  $u : \mathbb{N} \rightarrow \mathbb{R}$  is a disturbance that is uniformly distributed on  $[-0.2, 0.2]$ . We see that the trajectory converges to a neighborhood of the origin, and the norm of the trajectory decreases at least any 3 steps.

### 3.4 Relaxed small-gain theorems: a trajectory-based approach

In the last section we proposed relaxed *Lyapunov-based* small-gain theorems, where gains were derived from Lyapunov-type estimates such as e.g. condition (ii) of Theorem 3.23. In this section we propose small-gain theorems that are *trajectory-based*, i.e., gains are derived from estimates of system trajectories. In particular, the obtained small-gain theorem handles the interconnection of systems that are ISS, and systems that are globally stable (GS). Whereas in the last section the gain operator  $\Gamma_{\mu}$  defined in (1.8) was considered in the maximization case (i.e., in the

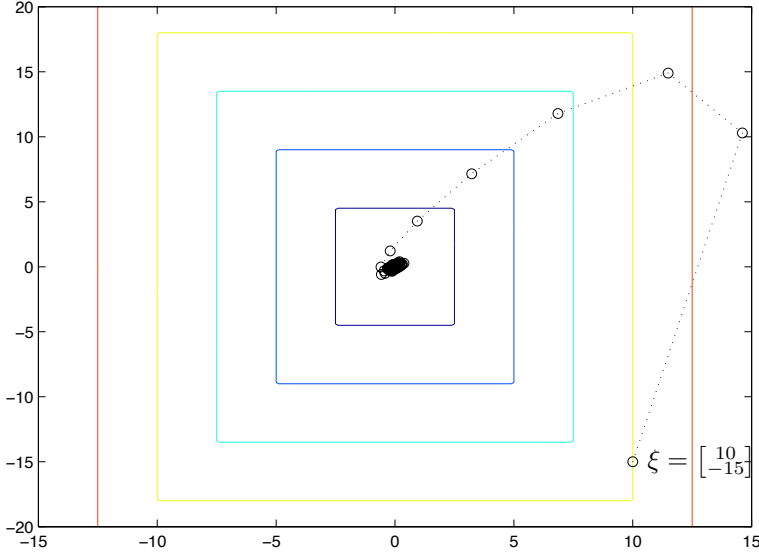


Figure 3.1: Sublevel sets of the dissipative finite-step ISS Lyapunov function  $V(\xi) = \max\{2\|\xi\|_1, \frac{10}{9}\|\xi\|_2\}$ , and a trajectory starting in  $\xi = (10, -15)$ , where  $u$  is uniformly distributed on  $[-0.2, 0.2]$ .

form  $\Gamma_{\oplus}$  defined in (1.16)), we do now consider the general case of interconnection using *monotone aggregation functions* as introduced in Definition 1.20.

To better distinguish between the subsystems we consider system (3.1) split into  $N + M$  subsystems of the form

$$x_i(k+1) = g_i(x_1(k), \dots, x_{N+M}(k), u(k)), \quad k \in \mathbb{N}, \quad (3.32)$$

with  $\xi_i := x_i(0) \in \mathbb{R}^{n_i}$  and  $g_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_{N+M}} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$  for  $i \in \{1, \dots, N+M\}$ . Let  $n = \sum_{i=1}^{N+M} n_i$ ,  $x = (x_1, \dots, x_{N+M}) \in \mathbb{R}^n$ , then with  $G = (g_1, \dots, g_{N+M})$  we call (3.1) the overall system of the subsystems (3.32). We collect the inputs (internal  $x_j$ ,  $j \neq i$ , and external  $u$ ) on system  $i$  in the input variable

$$w_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{N+M}, u).$$

As we do now have more than one input, we extend the notions of ISS, GS and AG given in Definitions 3.2, 3.8 and 3.9, respectively. We call the  $i$ th subsystem (3.32) *ISS* (from  $w_i$  to  $x_i$ ) if there exist  $\beta_i \in \mathcal{KL}$ ,  $\gamma_{ij} \in \mathcal{K}_{\infty} \cup \{0\}$  for  $j \in \{1, \dots, N+M\}$ ,

$\gamma_{iu} \in \mathcal{K}_\infty$ , and  $\mu_i \in \text{MAF}_{N+M}$  such that for all initial states  $\xi_i \in \mathbb{R}^{n_i}$  and all  $k \in \mathbb{N}$  the solution  $x_i(\cdot, \xi_i, w_i)$  of (3.32) satisfies an estimate of the form

$$\|x_i(k, \xi_i, w_i(\cdot))\| \leq \beta_i(\|\xi_i\|, k) + \mu_i(\gamma_{i1}(|x_1|_{[0,k]}), \dots, \gamma_{iN+M}(|x_{N+M}|_{[0,k]})) + \gamma_{iu}(|u|_\infty). \quad (3.33)$$

Note that instead of summation, we could also have used maximization or using a monotone aggregation function (see Proposition 1.26).

Further, we call the  $i$ th subsystem (3.32) *GS* if there exist  $\alpha_i \in \mathcal{K}_\infty$ ,  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $j \in \{1, \dots, N+M\}$ ,  $\gamma_{iu} \in \mathcal{K}_\infty$  and  $\mu_i \in \text{MAF}_{N+M}$  such that for all initial states  $\xi_i \in \mathbb{R}^{n_i}$  and all  $k \in \mathbb{N}$  the solution  $x_i(\cdot, \xi_i, w_i)$  of (3.32) satisfies an estimate of the form

$$\|x_i(k, \xi_i, w_i(\cdot))\| \leq \alpha_i(\|\xi_i\|) + \mu_i(\gamma_{i1}(|x_1|_{[0,k]}), \dots, \gamma_{iN+M}(|x_{N+M}|_{[0,k]})) + \gamma_{iu}(|u|_\infty). \quad (3.34)$$

We call the  $i$ th subsystem (3.32) *AG* if there exist  $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ ,  $j \in \{1, \dots, N+M\}$ ,  $\gamma_{iu} \in \mathcal{K}_\infty$  and  $\mu_i \in \text{MAF}_{N+M}$  such that for all initial states  $\xi_i \in \mathbb{R}^{n_i}$  and all  $k \in \mathbb{N}$  the solution  $x_i(\cdot, \xi_i, w_i)$  of (3.32) satisfies an estimate of the form

$$\limsup_{k \rightarrow \infty} \|x_i(k, \xi, w_i(\cdot))\| \leq \mu_i(\gamma_{i1}(|x_1|_{[0,k]}), \dots, \gamma_{iN+M}(|x_{N+M}|_{[0,k]})) + \gamma_{iu}(|u|_\infty). \quad (3.35)$$

Let the matrix  $\Gamma = (\gamma_{ij})_{i,j=1}^{N+M}$  and the map  $\Gamma_\mu : \mathbb{R}_+^{N+M} \rightarrow \mathbb{R}_+^{N+M}$  defined by

$$\Gamma_\mu(s) = \begin{pmatrix} \mu_1(\gamma_{11}([s]_1), \dots, \gamma_{1,N+M}([s]_{N+M})) \\ \vdots \\ \mu_{N+M}(\gamma_{N+M,1}([s]_1), \dots, \gamma_{N+M,N+M}([s]_{N+M})) \end{pmatrix} \quad (3.36)$$

be given from the interconnection structures (3.33), (3.34), or (3.35).

We start by restating small-gain theorems for interconnected discrete-time subsystems, where all subsystems are assumed to be GS, AG or ISS, respectively. These results were presented in [24, 122] for continuous-time systems. Note that in [24] the theorems are only proved for summation and maximization whereas in [122] the theorems are given in a more general form using monotone aggregation functions. The stated theorems also hold true for discrete-time systems in a suitable form, as already observed in [24, Section 4.4].

Nevertheless, in [24] the following results in the discrete-time framework are shown under the assumption that the dynamics of the subsystems (3.32) are continuous. To ensure that the results also hold true for discontinuous dynamics, we have to prove the following lemma first.

**Lemma 3.33.** *System (3.1) is ISS if and only if it is AG and GS.*

*Proof.* As mentioned before, this result has been proven for continuous dynamics in [70, Theorem 1] and in [30, Theorem 2]. Here, we prove the result also for the case that the dynamics are discontinuous. Firstly observe that ISS implies AG as the  $\mathcal{KL}$ -function  $\beta$  in (3.3) converges to zero as  $k$  goes to infinity. Moreover, ISS implies GS by defining  $\sigma_1(\cdot) := \beta(\cdot, 0)$  and  $\sigma_2 := \gamma$ . It remains to show that AG and GS together imply ISS of system (3.1). We prove this implication by showing that

- (1) AG + GS  $\Rightarrow$  (3.1) is robustly stable;
- (2) (3.1) is robustly stable  $\Rightarrow$  there exists a dissipative ISS Lyapunov function.

Then by Theorem 3.19 system (3.1) is ISS.

Now we prove the implications (1) and (2):

(1) For continuous dynamics this result is proven in [70, Lemma 3.11] by noticing that the GS property introduced in Definition 3.8 is qualitatively the same as the *uniformly bounded input bounded state* (UBIBS) property defined in [70]. If the dynamics are discontinuous, we can apply [70, Lemma 3.11] as well, as the proof does not utilize the continuity property.

(2) For continuous dynamics this implication is shown in [70, Lemma 3.10]. For the discontinuous case this implication is contained in the proof of [46, Theorem 2.6]. For the sake of completeness we give a sketch of the proof. First note that in [46] robust stability is defined as the difference inclusion

$$x(k+1) \subset G(x(k), \psi(\|x(k)\|)) \cdot [-1, 1]^m$$

being  $\mathcal{KL}$ -stable, which is qualitatively the same as Definition 3.5. The authors show that  $\mathcal{KL}$ -stability of the above difference inclusion (in our context, this is robust stability of system (3.1)) implies the existence of a Lyapunov function for the difference inclusion (see [46, Theorem 6.4]). Finally, as it is shown in the proof of [46, Theorem 2.6], the Lyapunov function for the difference inclusion is a dissipative ISS Lyapunov function for system (3.1).  $\square$

*Remark 3.34.* The proof of Lemma 3.33 immediately implies that system (3.1) is ISS if and only if it is robustly stable.  $\triangleleft$

The following two small-gain results are taken from [24, Section 4.4]. We note that continuity of the system dynamics is not needed in the particular proofs. Hence, continuity of the right-hand side is not required.

**Theorem 3.35** (small-gain global stability theorem). *Assume that each subsystem of (3.32) is GS, i.e., for all  $i \in \{1, \dots, N + M\}$  the solution  $x_i(\cdot, \xi_i, w_i(\cdot))$  of subsystem  $i$  satisfies a condition of the form (3.34). If  $\Gamma_\mu$  given in (3.36), which is*



derived from the interconnection structure in (3.34), satisfies the strong small-gain condition (1.11), then the overall system (3.1) is GS.

**Theorem 3.36** (small-gain theorem for asymptotic gains). *Assume that each subsystem of (3.32) is AG as given by (3.35). If  $\Gamma_\mu$  given in (3.36), which is derived from the interconnection structure in (3.35), satisfies the strong small-gain condition (1.11), then the overall system (3.1) is AG.*

For the next result we need the fact that AG and GS implies ISS, which has been shown in Lemma 3.33. Hence, the following small-gain theorem for ISS systems from [24, Proposition 4.8] also applies to discontinuous dynamics.

**Theorem 3.37** (general small-gain theorem). *Assume that each subsystem of (3.32) is ISS (from  $w_i$  to  $x_i$ ), i.e., for all  $i \in \{1, \dots, N + M\}$  the solution  $x_i(\cdot, \xi_i, w_i(\cdot))$  satisfies a condition of the form (3.33). Let  $\Gamma_\mu$  given in (3.36), which is derived from the interconnection structure in (3.33), satisfy the strong small-gain condition (1.11). Then the overall system (3.1) is ISS (from  $u$  to  $x$ ).*

Next, we consider the case that not all subsystems have the same stability properties. In particular, we assume that the first  $N$  subsystems (3.32) of (3.1) are ISS, and the remaining  $M$  subsystems are GS. We can thus view the overall system from two different viewpoints. The first one considers the overall system as the interconnection of all  $N + M$  subsystems, while the second one considers the overall systems as the interconnection of the first  $N$  ISS subsystems with the remaining subsystems as additional inputs. This point of view is indicated in Figure 3.2.

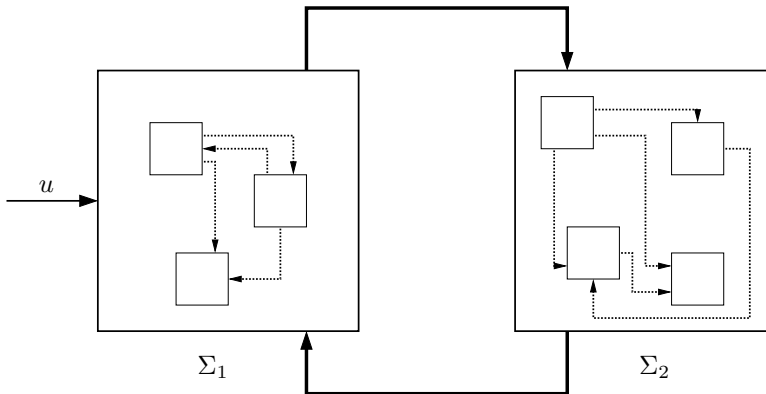


Figure 3.2: The interconnection of  $\Sigma_1$  (containing ISS subsystems) and  $\Sigma_2$  (containing GS subsystems)

Summarizing, we are interested in a result guaranteeing that the interconnected system

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N} \quad (3.1)$$

of all  $N+M$  subsystems given in (3.32) with  $x = (x_1, \dots, x_{N+M})$ ,  $G = (g_1, \dots, g_{N+M})$  is GS, and the interconnected system of the first  $N$  subsystems

$$\tilde{x}(k+1) = \tilde{G}(\tilde{x}(k), \tilde{u}(k)), \quad k \in \mathbb{N} \quad (3.37)$$

with  $\tilde{x} = (x_1, \dots, x_N)$ ,  $\tilde{G} = (g_1, \dots, g_N)$ , and input  $\tilde{u} = (x_{N+1}, \dots, x_{N+M}, u)$  is ISS (from  $\tilde{u}$  to  $\tilde{x}$ ).

This is stated in the next theorem and follows from the previous results in this section.

**Theorem 3.38.** *Assume that in (3.32) the subsystems  $i \in \{1, \dots, N\}$  are ISS and the subsystems  $i \in \{N+1, \dots, N+M\}$  are GS. If  $\Gamma_\mu$  given in (3.36) derived from (3.33) and (3.34) satisfies the strong small-gain condition (1.11) then the interconnected system (3.1) is GS. In addition, the interconnected system of the first  $N$  subsystems (3.37) is ISS from  $\tilde{u} = (x_{N+1}, \dots, x_{N+M}, u)$  to  $\tilde{x} = (x_1, \dots, x_N)$ .*

*Proof.* From (3.33) and (3.34) we see that the ISS subsystems  $i \in \{1, \dots, N\}$  are GS with  $\alpha_i(\|x_i^0\|) = \beta_i(\|x_i^0\|, 0)$ . So if  $\Gamma_\mu$  satisfies the strong small-gain condition (1.11) then it follows directly from Theorem 3.35 that the interconnected system (3.1) is GS.

Let us decompose the gain matrix  $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{(N+M) \times (N+M)}$  into

$$\Gamma = \left( \begin{array}{c|c} \Gamma_1 & \Gamma_2 \\ \hline \Gamma_3 & \Gamma_4 \end{array} \right).$$

where  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ ,  $\Gamma_2 \in (\mathcal{K}_\infty \cup \{0\})^{N \times M}$ ,  $\Gamma_3 \in (\mathcal{K}_\infty \cup \{0\})^{M \times N}$ , and  $\Gamma_4 \in (\mathcal{K}_\infty \cup \{0\})^{M \times M}$ .

Let  $D = \text{diag}(\text{id} + \delta)$  with  $\delta \in \mathcal{K}_\infty$  be a diagonal operator for which the strong small-gain condition (1.11) holds. Let  $\bar{D} : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  be defined by  $\bar{D} := \text{diag}(\text{id} + \delta)$ . Then we claim that  $(\bar{D} \circ \Gamma_{1,\mu})(s) \not\geq s$  for all  $s \neq 0$ , where  $\Gamma_{1,\mu}$  is defined by

$$\Gamma_{1,\mu}(s) := \left( \begin{array}{c} \mu_1(\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N), 0, \dots, 0) \\ \vdots \\ \mu_N(\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N), 0, \dots, 0) \end{array} \right).$$

Then we conclude from Theorem 3.37 that the system (3.37) is ISS from  $\tilde{u}$  to  $\tilde{x}$ . So assume to the contrary that there exists an  $\bar{s} \neq 0$  with  $(\bar{D} \circ \Gamma_{1,\mu})(\bar{s}) \geq \bar{s}$ . Set

$\tilde{s} = \begin{pmatrix} \bar{s} \\ 0 \end{pmatrix}$  then it follows

$$(D \circ \Gamma_\mu)(\tilde{s}) = \begin{pmatrix} (\bar{D} \circ \Gamma_{1,\mu})(\bar{s}) \\ * \end{pmatrix} \geq \begin{pmatrix} \bar{s} \\ 0 \end{pmatrix},$$

a contradiction to the strong small-gain condition (1.11).  $\square$

*Remark 3.39.* (i) As already mentioned above, Theorems 3.35, 3.36 and 3.37 have been derived for continuous-time systems in [24, 122]. On the other hand, also Theorem 3.38 can be stated for continuous-time systems. Moreover, we have derived an analogous result to Theorem 3.38 in a hybrid systems framework in [39].

(ii) Instead of considering the interconnection of ISS and GS subsystems we could consider the interconnection of ISS and AG subsystems. Indeed, the statement of Theorem 3.38 with GS replaced by AG remains valid as can be easily seen from the proof. A motivation for considering of ISS and GS subsystems can be found in Section 3.5.

(iii) In [86] resp. [23] (for hybrid resp. continuous-time systems) a small-gain theorem similar to Theorem 3.37 is given for interconnections of ISS systems with mixed ISS characterizations, i.e., some subsystems are stated in the summation case and others are stated in the maximization case. The condition that the interconnected system then is ISS is a small-gain condition considering that  $(D \circ \Gamma)(s) \not\geq s$  for all  $s \neq 0$ . Here  $D([\![s]\!]_1, \dots, [\![s]\!]_{N+M}) = (D_1([\![s]\!]_1), \dots, D_{N+M}([\![s]\!]_{N+M}))$  with  $D_i([\![s]\!]_i) = [s]_i$  if  $\mu_i = \oplus$  (the maximization case) and  $D_i([\![s]\!]_i) = (\text{id} + \delta)([s]_i)$  with  $\delta \in \mathcal{K}_\infty$  if  $\mu_i = \Sigma$ . The proof of Theorem 3.38 extends to this context in a straightforward manner, so we have the same result even for mixed ISS characterizations.  $\triangleleft$

Remark 3.39 in particular implies the following special case. First note that in the maximization case  $\mu = \oplus$  the gain operator  $\Gamma_\oplus : \mathbb{R}_+^{N+M} \rightarrow \mathbb{R}_+^{N+M}$  is of the form (1.16). From [86, Theorem 2.4.5] we know that if each subsystem of the system (3.32) is ISS and  $\Gamma_\oplus$  satisfies the small-gain condition (1.10) then the interconnected system (3.1) is ISS from  $u$  to  $x$ . So the same proof as in Theorem 3.38 now yields the following corollary, where we can even omit the diagonal operator.

**Corollary 3.40.** *Assume that for all  $i \in \{1, \dots, N\}$  the subsystems (3.32) satisfy the ISS estimate (3.33) with  $\mu_i = \oplus$ , and for all  $i \in \{N + 1, \dots, N + M\}$  the subsystems (3.32) satisfy the GS estimate (3.34) with  $\mu_i = \oplus$ . If  $\Gamma_\oplus$  satisfies the small-gain condition (1.10) then the interconnected system (3.1) is GS. In addition, the interconnected system of the first  $N$  subsystems (3.37) is ISS from  $\tilde{u}$  to  $\tilde{x}$ .*

### 3.5 Notes and references

The concept of input-to-state stability was introduced in [128] for continuous-time systems by E. Sontag. Since then, there has been an immense interest in this topic as it turned out that the concept of ISS is well-suited for nonlinear control systems. Two important research directions, where the concept of ISS emerges to be an elegant tool, are

- *(stability) analysis* of dynamical systems, where the input acts as a disturbance;
- *feedback (re-)design*<sup>5</sup> of dynamical systems, where the input acts as a feedback control.

For an overview on different aspects of ISS we refer to [131], which also includes a comprehensive list of references. An important observation has been derived in [132], where it has been shown that ISS fits well in the context of Lyapunov functions. In particular, the authors have shown that ISS is equivalent to the existence of an ISS Lyapunov function.

For discrete-time systems, the notion of ISS has been introduced in [70, 76]. In [70], the authors also prove that ISS is equivalent to the existence of an ISS Lyapunov function in the discrete-time framework. To be precise, the authors show that ISS implies the existence of a *dissipative* ISS Lyapunov function, and note that dissipative ISS Lyapunov functions are equivalent to *implication-form* ISS Lyapunov functions if the dynamics are continuous. As recently shown in [46], the equivalence between dissipative and implication-form ISS Lyapunov functions does not hold if the dynamics are discontinuous, see also Remark 3.11. Indeed, for discontinuous dynamics, the notion of an implication-form ISS Lyapunov function is not sufficient to conclude ISS of the system.

Recently, there has been extensive interest in the derivation of small-gain theorems for large-scale systems within the context of ISS systems. While stability conditions for large-scale systems have already been studied in the 1970s and early 1980s cf. [112, 127, 140] based on linear gains and Lyapunov techniques, nonlinear approaches are more recent. It has been recognized that input-to-state stability is well suited to the analysis of system interconnection. Early works on ISS small-gain theorems are [68, 69], where feedback interconnections of two systems were studied. For large-scale systems defined through the interconnection of a number of ISS subsystems there exist several small-gain type conditions guaranteeing the ISS property for the interconnected system, see e.g. [21, 23–25, 64, 75, 124] for continuous-time systems

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<sup>5</sup>Feedback redesign means that a feedback controller is optimized such that it does not only make the system GAS, but also robust to disturbances as noise, uncertainty or computation errors, see e.g. [131, Section 2.11 and 2.12].

and [22, 86, 115, 126] for hybrid<sup>6</sup> systems.

The first ISS small-gain theorems for discrete-time systems were presented in [70], which parallel the results of [69] and [68] for continuous-time systems. For interconnections consisting of more than two subsystems, small-gain theorems are presented in [67] and in [24], whereas in [67] ISS was defined in a maximum formulation and in [24] the results are given in a summation formulation. Further extensions to the formulation via maximization or summation are ISS formulations via *monotone aggregation functions*. In this formulation, the ISS small-gain results are shown to hold in a more general form, see [122]. In [25] the authors present an ISS small-gain theorem in a Lyapunov-based formulation that allows to construct an overall ISS Lyapunov function. This idea is picked up in [99], where the authors present a discrete-time version in a maximum formulation and construct an ISS Lyapunov function for the overall system. Other small-gain theorems for discrete-time systems can be found in [40, 48, 89], whereas these references do not require continuity of the dynamics. While in this chapter Lyapunov-based small-gain results have been obtained in a dissipative formulation, the authors in [48] consider an implication-form formulation. We claim that the results in [48] can be generalized by incorporating the finite-step idea.

There are several stability concepts related to the notions of global stability (GS), the asymptotic gain property (AG), robust stability and zero input global asymptotic stability (0-GAS) as introduced in Section 3.1. While the concept of integral input-to-state stability (iISS) is weaker than ISS for continuous-time systems, see [129], it is in fact stronger than the notion of 0-GAS. Nevertheless, for discrete-time systems, the author in [3] showed that iISS and 0-GAS are equivalent. Another concept for systems with outputs is the *input-to-output stability* (IOS), which is similarly defined as in (3.3) but for output trajectories and not for state trajectories. This concept generalizes the notion of ISS, which is the special case if the output is equal to the state. For further insight in IOS we refer to the [69, 134] for the continuous-time case and to [66] for the discrete-time case.

As already mentioned before, the results of Section 3.4 have also been derived for hybrid systems in [39]. The motivation for considering interconnections of ISS as well as GS systems came up in the context of *networked control systems*, see e.g. [18, 57, 136] and the references therein. Networked control systems consider the interconnection of systems *via* data channels that have communication imperfections as varying transmission delays, varying sampling/transmission intervals, packet loss, communication constraints and quantization effects. Much work done in the context of networked control systems aims at deriving bounds on the imperfections such that a desired property such as e.g. asymptotic stability is preserved. For example

<sup>6</sup>I.e., systems that have both continuous-time and discrete-time dynamics involved.

we mention the notions of MATI (maximum allowable transfer interval) and MAD (maximum allowable delay), which are defined in e.g. [57]. For instance, the authors in [12] derive ISS of the networked control system from a small-gain condition in terms of a MATI. In particular, as it is implied by [12, Equation (34)], the MATI is small if the corresponding gain is small.

The idea we propose in [39] is to consider the data channels as dynamical systems. Since it would seem unreasonable to demand of a data channel that it be ISS, e.g. if the signals are quantized with a fixed quantization region, we do only require the data channels be globally stable. Alternatively, as it is e.g. done in [12, 57], the imperfections can be modeled by error dynamics, which can be written as a dynamical system. Finally, as networked control systems over data channels do naturally lead to both continuous-time (e.g. the subsystems dynamics) and discrete-time (e.g. update rules after transmissions and logic variables) dynamics, the framework of hybrid systems as e.g. [14, 15, 43, 126] fits well in studying networked control systems.

## About the tightness of small-gain conditions

In Chapters 2 and 3 we have presented several small-gain results to study stability properties of interconnected discrete-time systems. These small-gain results rely heavily on a small-gain condition of the form  $\Gamma_\mu \not\geq \text{id}$  (resp.  $D \circ \Gamma_\mu \not\geq \text{id}$ ). Broadly speaking, the small-gain condition guarantees that the composition of the gains is small enough to avoid destabilizing effects of the interconnection. It is then clear that, in general, there is a tradeoff between the size of the gains, which are admissible at different places of the interconnection graph.

In the first part of this chapter we address the following design question: assume a large-scale system is given satisfying a small-gain condition, we wish to add further subsystems, and we are able to design the interconnection gains. Can we derive a uniform bounding gain  $\bar{\gamma}$  for the new gains depending on the given gains in order that a small-gain condition holds? Furthermore, we are interested in a *maximal gain*, i.e., a preferably large uniform bounding gain  $\bar{\gamma}$  for the new gains preserving a small-gain condition.

Considering several situations, we derive constructive methods for obtaining maximal gains  $\bar{\gamma}$ . In particular, we treat the linear summation case (defined in Section 1.6.1) and the maximization case (defined in Section 1.6.2) separately. For these cases, equivalent characterizations of the small-gain condition can be used to compute a maximal gain, as the “spectral radius less than one”-condition (Lemma 1.27) and the cycle condition (Proposition 1.29). For the general case, we use the characterization of strong small-gain conditions in terms of the existence of an  $\Omega$ -path (Lemma 1.23) to compute a set of uniform bounding gains  $\gamma \in \mathcal{K}_\infty$ , in order to

preserve a strong small-gain condition. For the general case, a complete answer to the question of computing maximal gains is open.

In the second part of this chapter, we treat a problem that comes up in deriving maximal gains in the maximization case. In this case, by the cycle condition (Proposition 1.29), the small-gain condition  $\Gamma_{\oplus} \not\geq \text{id}$  is equivalent to all cycles of the corresponding graph of  $\Gamma_{\oplus}$  being weakly contracting. To determine a qualitative uniform bounding gain for the new gains, we have to solve iterative functional  $\mathcal{K}_{\infty}$ -equations of the form

$$\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma = \text{id}, \quad (4.1)$$

where  $\alpha_1, \dots, \alpha_k \in \mathcal{K}_{\infty}$  are determined by the given gains and the interconnection structure. Note that (4.1) has, in general, non-unique solutions. Thus, we establish a subclass of  $\mathcal{K}_{\infty}$ , the class of *right-affine  $\mathcal{K}_{\infty}$ -functions*, in which a unique solution of the functional equation (4.1) exists. The proofs are constructive and can be implemented to numerically compute this unique solution.

The outline of this chapter is as follows. In Section 4.1 we start with a precise description of the problem of finding admissible uniformly bounding gains such that a (strong) small-gain condition is satisfied. For the maximization case and for the linear summation case we give construction methods to compute maximal gains (Sections 4.1.1 resp. 4.1.2). The general case is treated in Section 4.1.3. In Section 4.2 we consider iterative functional  $\mathcal{K}_{\infty}$ -equations of the form (4.1). As a motivation, some examples show the different behavior of solutions of functional equations of the form (4.1) in Section 4.2.1. We then restrict the functions  $\alpha_i$ ,  $i \in \{1, \dots, k\}$  to the class of right-affine  $\mathcal{K}_{\infty}$ -functions. In Sections 4.2.2 and 4.2.3 we show existence and uniqueness of solutions of the functional equation (4.1) within the class of right-affine  $\mathcal{K}_{\infty}$ -functions. To make the presentation more clear, we first consider the case  $k = 2$  in Section 4.2.2, and treat the general case  $k \geq 2$  afterwards in Section 4.2.3. In Section 4.3 we conclude the chapter with notes and references.

## 4.1 Gain construction methods

The (strong) small-gain condition  $\Gamma_{\mu} \not\geq \text{id}$  (resp.  $D \circ \Gamma_{\mu} \not\geq \text{id}$ ) is a sufficient condition to guarantee stability properties of interconnected systems (see Chapter 2 and 3, as well as the notes and references therein). In the context of ISS of two interconnected systems with suitable gains  $\gamma_{12}, \gamma_{21} \in \mathcal{K}_{\infty}$ , the small-gain condition (1.10) is satisfied if and only if

$$\gamma_{12} \circ \gamma_{21} < \text{id}. \quad (4.2)$$

It is obvious that a “smaller” function  $\gamma_{12} \in \mathcal{K}_{\infty}$  enlarges the set of gains  $\gamma_{21} \in \mathcal{K}_{\infty}$  satisfying (4.2). We are interested in quantifying the set of suitable functions  $\gamma_{21}$



that satisfies (4.2) for any fixed  $\gamma_{12} \in \mathcal{K}_\infty$ . For this particular setting it is clear that for fixed  $\gamma_{12} \in \mathcal{K}_\infty$ , we have

$$\mathcal{S} := \{\gamma_{21} \in \mathcal{K}_\infty : (4.2) \text{ holds}\} = \{\gamma_{21} \in \mathcal{K}_\infty : \gamma_{21} < \gamma_{12}^{-1}\}.$$

In this section we want to extend this problem of finding suitable gains such that the (strong) small-gain condition (1.10) (resp. (1.11)) holds. In particular, we raise the problem of determining a maximal gain, if we assume that all undetermined gains are the same.

So consider an interconnected system consisting of  $N + M$  subsystems. For the first  $N$  subsystems it is assumed that the gains are determined and we assume to know the gain matrices  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  and  $\Gamma_2 \in (\mathcal{K}_\infty \cup \{0\})^{N \times M}$ . For the remaining  $M$  subsystems only the interconnection structure is determined. For this purpose, consider a *weighted directed graph* of the subsystems, as the one given by Figure 2 in the introduction, described by a *weighted structure matrix*

$$\Gamma_{sub}^* := [\Gamma_3^* \ \Gamma_4^*] \in \mathbb{R}_+^{M \times (N+M)}, \quad \text{to obtain } \Gamma^* = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3^* & \Gamma_4^* \end{bmatrix}, \quad (4.3)$$

where  $[\Gamma_{sub}^*]_{ij} = 0$  if and only if system  $N + i$  does not depend on system  $j$ . If we choose a uniform gain  $\gamma$  in the structural entries, we obtain the gain operator  $\Gamma_\mu^\gamma$  of the form

$$\Gamma_\mu^\gamma := \mu \circ \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_{sub}^* \cdot \gamma \end{bmatrix} = \mu \circ \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3^* \cdot \gamma & \Gamma_4^* \cdot \gamma \end{bmatrix}. \quad (4.4)$$

**Definition 4.1.** Let  $\mu \in \text{MAF}_{N+M}^{N+M}$ ,  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$ ,  $\Gamma_2 \in (\mathcal{K}_\infty \cup \{0\})^{N \times M}$ , and consider the structure matrix  $\Gamma_{sub}^*$  from (4.3). A function  $\gamma \in \mathcal{K}_\infty$  is called (strongly) *admissible*, if  $\Gamma_\mu^\gamma$  defined in (4.4) satisfies a (strong) small-gain condition of the form (1.10) (resp. (1.11)).

Note that in (4.4) the entries are linear scalings of the form  $[\Gamma_{sub}^*]_{ij} \cdot \gamma$  corresponding to  $a_{(N+i)j}\gamma$  in Figure 2 in the introduction. Intuitively, if  $\Gamma$  satisfies a small-gain condition with  $\gamma = 0$  then “small”  $\gamma$  should be admissible. In general, it might be desirable to have large admissible  $\gamma$ , since this shows stability for a larger class of systems. The question is then how large admissible  $\gamma$  may be and how a maximal  $\gamma$  can be characterized.

We now state the problem in a more general form and do not assume that the entries placed in the structure matrix coincide. For this purpose let  $J \subset (\{1, \dots, M\} \times \{1, \dots, N + M\})$  be the set containing the non-zero entries of the structure entries of  $\Gamma_{sub}^*$  from (4.3), i.e.,

$$(i, j) \in J \iff [\Gamma_{sub}^*]_{ij} \neq 0.$$

It will be helpful to have precise notation linking the  $\mathcal{K}_\infty$ -functions  $([s]_1, \dots, [s]_{\#J}) =: s$  that are placed at the structure entries with the resulting gain matrix  $\Gamma^S$ .

Let  $\pi : \{1, \dots, \#J\} \rightarrow J$  be a bijective mapping. Define the map  $\Gamma_{sub}^S : \mathcal{K}_\infty^{\#J} \rightarrow (\mathcal{K}_\infty \cup \{0\})^{M \times (N+M)}$  by

$$s \mapsto [\Gamma_{sub}^S]_{ij} = \begin{cases} [\Gamma_{sub}^*]_{ij}[s]_{\pi^{-1}(i,j)} & \text{if } (i, j) \in J \\ 0 & \text{if } (i, j) \notin J. \end{cases}$$

We arrive at the bijective correspondence between

$$s \in \mathcal{K}_\infty^{\#J} \quad \text{and} \quad \Gamma^S := \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_{sub}^S & \end{pmatrix}, \quad (4.5)$$

and we denote  $\gamma_{ij} := [\Gamma^S]_{ij} \in \mathcal{K}_\infty \cup \{0\}$ . We further define the *potential decay sets*

$$\mathcal{S} := \{s \in \mathcal{K}_\infty^{\#J} : \Gamma_\mu^S \not\geq \text{id}\} \quad (4.6)$$

and

$$\mathcal{S}_{\text{strong}} := \{s \in \mathcal{K}_\infty^{\#J} : \exists D, D \circ \Gamma_\mu^S \not\geq \text{id}\} \quad (4.7)$$

where in the latter case  $D = \text{diag}(\text{id} + \delta)$ ,  $\delta \in \mathcal{K}_\infty$ , is the diagonal operator defined in (1.9). These potential decay sets can be viewed as the solution sets of the problem of finding  $\mathcal{K}_\infty$ -functions for the non-zero entries of (4.3) such that a (strong) small-gain condition (1.10) (resp. (1.11)) holds for  $\Gamma^S$ . Clearly, if some  $s \in \mathcal{S}$  exists, then the minimum over all component functions of  $s \in \mathcal{K}_\infty^{\#J}$  is an admissible function in the sense of Definition 4.1.

The questions from above are now whether the set  $\mathcal{S}$  (resp.  $\mathcal{S}_{\text{strong}}$ ) is nonempty. And if yes, how to characterize maximal  $\bar{\gamma} \in \mathcal{K}_\infty$  such that

$$\mathcal{S}_{\bar{\gamma}} := \{s \in \mathcal{K}_\infty^{\#J} : [s]_j(t) < \bar{\gamma}(t), j \in \{1, \dots, \#J\}, t > 0\} \subset \mathcal{S}, \quad (4.8)$$

(resp.  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}_{\text{strong}}$ ), provided such a  $\bar{\gamma}$  exists. This leads to the following definition.

**Definition 4.2.** Let  $\Gamma^*$  from (4.3) be given. A  $\mathcal{K}_\infty$ -function  $\bar{\gamma}$  is called *maximal* for the set  $\mathcal{S}$  (resp. for the set  $\mathcal{S}_{\text{strong}}$ ) if any  $\mathcal{K}_\infty$ -function  $\gamma < \bar{\gamma}$  is admissible (resp. strongly admissible), and any  $\mathcal{K}_\infty$ -function  $\gamma \geq \bar{\gamma}$  is not admissible (resp. not strongly admissible).

The remainder of this section is divided into three parts, where we use different equivalent characterizations of the (strong) small-gain condition. As we will see, the methods presented here for the case of interconnections modeled by general monotone aggregation functions, invoking the equivalent characterization of the existence of an  $\Omega$ -path, will in general not lead to a maximal gain  $\bar{\gamma}$  for which  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}_{\text{strong}}$  (Section 4.1.3). In particular, we have to require that the monotone aggregation function

is sub-additive to show that  $\mathcal{S}_{\text{strong}}$  is nonempty. But for the linear summation case (Section 4.1.2), and the maximization case (Section 4.1.1) other characterizations lead to a computable maximal  $\bar{\gamma}$ . In particular, we can use the “spectral radius less than one” (Lemma 1.27) and the “cycle condition” (Proposition 1.29) characterization here. Thus, the proof techniques involved are different.

#### 4.1.1 The maximization case

We now study the problem formulation (4.8) for the case of maximization,  $\mu = \oplus$ , see Section 1.6.2. In this case, the gain operator corresponding to  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  is

$$\Gamma_{1, \oplus}(s) := \begin{pmatrix} \max \{ \gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N) \} \\ \vdots \\ \max \{ \gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N) \} \end{pmatrix}. \quad (4.9)$$

Consider the gain matrix  $\Gamma^S$  from (4.5). Any  $k$ -cycle  $c^* = (\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_{k-1} i_k})$  of  $\Gamma^*$  that contains at least one structural entry, i.e.,  $(i_l - N, i_{l+1}) \in J$  for at least one  $l \in \{0, \dots, k-1\}$ , is called a *structural cycle*. By  $c^\gamma$  we denote the corresponding cycle in  $\Gamma^S$  with  $s = \gamma \cdot (1, \dots, 1)$ . Moreover,  $N^*$  is defined as the number of minimal<sup>1</sup> structural cycles of  $\Gamma^*$ . Let us further denote the composition by

$$\bigotimes c^\gamma = \gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \dots \circ \gamma_{i_{k-1} i_k}. \quad (4.10)$$

Note that the equation  $\bigotimes c^\gamma = \text{id}$  can also be written in the form

$$(\alpha_1 \circ \gamma) \circ \dots \circ (\alpha_l \circ \gamma) = \text{id}, \quad (4.11)$$

where the functions  $\alpha_k \in \mathcal{K}_\infty$ ,  $k \in \{1, \dots, l\}$ , subsume the known gains and the structural scalings. For instance, consider the cycle  $(\gamma_{i_0 i_1}, a_{i_1 i_2} \gamma, \gamma_{i_2 i_0})$ . Then  $\bigotimes c^\gamma = \alpha_1 \circ \gamma$  with  $\alpha_1 = \gamma_{i_2 i_0} \circ \gamma_{i_0 i_1} \circ a_{i_1 i_2} \text{id}$ , where we used (1.2).

**Theorem 4.3.** *Let  $\mu = \oplus$  and  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{N \times N}$  be given such that the small-gain condition  $\Gamma_{1, \oplus} \not\geq \text{id}$  is satisfied, with  $\Gamma_{1, \oplus}$  defined as in (4.9). Let  $\Gamma_2 \in (\mathcal{K}_\infty \cup \{0\})^{N \times M}$  and consider the structure matrix  $\Gamma_{\text{sub}}^*$  from (4.3). Assume that for any minimal structural cycle  $c_i^*$ ,  $i \in \{1, \dots, N^*\}$ , the functional equation  $\bigotimes c_i^{\gamma_i} = \text{id}$  exhibits a solution  $\gamma_i \in \mathcal{K}_\infty$ . If there exist more than one solution, fix one, and denote it by  $\gamma_i$ . Then, for*

$$\bar{\gamma}(t) := \min_{i \in \{1, \dots, N^*\}} \gamma_i(t), \quad t \geq 0 \quad (4.12)$$

we have  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}$ . Moreover, if there exists an index  $j$  and a  $T > 0$  such that  $\gamma_j \equiv \bar{\gamma}$  on  $[0, T)$ , then  $\bar{\gamma}$  is maximal.

<sup>1</sup>Recall the definition of a minimal cycle in Definition 1.17.

Note that as solutions of the functional equation (4.10) may not be unique, also maximal gains need not be unique, see also Remark 4.5.

*Proof.* We first show that every  $\mathcal{K}_\infty$ -function  $\gamma < \bar{\gamma}$  is admissible. From Proposition 1.29 we know that every  $k$ -cycle ( $k \leq N$ ) of  $\Gamma_1$  is weakly contracting since the small-gain condition  $\Gamma_{1,\oplus} \not\geq \text{id}$  is satisfied. Let  $s < \bar{\gamma} \cdot (1, \dots, 1)$ . By construction of  $\bar{\gamma}$ , every  $k$ -cycle in  $\Gamma^S$  is weakly contracting. From Proposition 1.29 it follows  $\Gamma_{\oplus}^S \not\geq \text{id}$  implying  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}$ .

For the proof of maximality, consider the structural cycle  $c^* := c_j^*$  satisfying  $\bigotimes c_j^{\gamma_j} = \text{id}$  and  $\gamma_j \equiv \bar{\gamma}$  on  $[0, T)$ . Denote  $\gamma := \gamma_j$  and consider the equation  $\bigotimes c^\gamma = \text{id}$  in the form of (4.11).

Let  $t_1 \leq T$  be such that  $(\alpha_1 \circ \bar{\gamma})(t_1) \leq T$ . Define  $t_k$  recursively for  $k \in \{2, \dots, l\}$  by

$$(\alpha_k \circ \bar{\gamma})(t_k) \leq t_{k-1}, \quad t_k \leq t_{k-1}.$$

Now consider  $\hat{\gamma} \geq \bar{\gamma}, \hat{\gamma} \neq \bar{\gamma}$ . For the cycle  $c$  in (4.11) we obtain, using the definition of  $t_k$  backwards recursively from  $l$  to 1,

$$\begin{aligned} \bigotimes c^{\hat{\gamma}}(t_l) &\geq (\alpha_1 \circ \hat{\gamma}) \circ \dots \circ (\alpha_{l-1} \circ \hat{\gamma}) \circ (\alpha_{l-2} \circ \hat{\gamma}) \circ \underbrace{(\alpha_l \circ \bar{\gamma})(t_l)}_{\leq t_{l-1} \leq T} \\ &\stackrel{t_l \leq T}{\geq} (\alpha_1 \circ \hat{\gamma}) \circ \dots \circ (\alpha_{l-2} \circ \hat{\gamma}) \circ \underbrace{(\alpha_{l-1} \circ \bar{\gamma}) \circ (\alpha_l \circ \gamma)(t_l)}_{\leq t_{l-2} \leq T} \\ &\stackrel{t_{l-1} \leq T}{\geq} (\alpha_1 \circ \hat{\gamma}) \circ \dots \circ (\alpha_{l-2} \circ \bar{\gamma}) \circ (\alpha_{l-1} \circ \gamma) \circ (\alpha_l \circ \gamma)(t_l) \\ &\quad \vdots \\ &\stackrel{t_1 \leq T}{\geq} \bigotimes c^\gamma(t_l) = t_l. \end{aligned}$$

Here we used the fact that  $\gamma \equiv \bar{\gamma}$  on  $[0, T)$  and  $t_k \leq T$  for all  $k \in \{1, \dots, l\}$ . In other words, we made use of the fact that, by monotonicity,  $\bigotimes c^{\hat{\gamma}}(t) \geq \bigotimes c^{\bar{\gamma}}(t)$  for all  $t \geq 0$ , and that  $t_l$  is chosen such that  $\bigotimes c^{\bar{\gamma}}(t) = \bigotimes c^\gamma(t) = t$  for all  $t \in [0, t_l]$ . So the cycle  $c^{\hat{\gamma}}$  is not weakly contracting, hence  $\mathcal{S}_{\hat{\gamma}} \not\subset \mathcal{S}$ .  $\square$

In some cases, a solution  $\gamma_i$  for the equation  $\bigotimes c_i^{\gamma_i} = \text{id}$  can be determined analytically as the next example shows.

**Example 4.4.** Consider the matrix

$$\Gamma^* = \begin{pmatrix} \gamma_{11} & 0 & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \text{id} & 0 & \frac{1}{3} \text{id}^3 \\ \text{id}^2 & \text{id}(1 - e^{-\text{id}}) & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.13)$$

and let  $\mu = \oplus$ . Clearly  $\Gamma_1 \in (\mathcal{K}_\infty \cup \{0\})^{2 \times 2}$  satisfies  $\Gamma_{1, \oplus} \not\geq \text{id}$  because the only  $k$ -cycles are  $(\gamma_{11})$  and  $(\gamma_{22})$ , and those are weakly contracting. In addition, in  $\Gamma^*$  we have another 1-cycle  $([\Gamma^*]_{33})$ , no 2-cycles, and one 3-cycle  $(\gamma_{21}, \gamma_{13}, [\Gamma^*]_{32})$ . So setting those two cycles equal to the identity map yields

$$\gamma = \text{id} \quad \text{and} \quad \left(\frac{1}{3}\gamma^3\right)^2 = \text{id}.$$

From the definition of  $\bar{\gamma}$  in (4.12) we get

$$\bar{\gamma}(t) = \min\{t, 3^{1/3}t^{1/6}\}.$$

So by Theorem 4.3,  $\Gamma_{\oplus}^S \not\geq \text{id}$  for any  $s < (\bar{\gamma}, \bar{\gamma})$ . ◁

*Remark 4.5.* In Section 4.2 we will consider the problem of solving iterative functional  $\mathcal{K}_\infty$ -equations of the form  $\bigotimes c^\gamma = \text{id}$  in more detail. In this remark we highlight some of the results that are useful for computing maximal gains.

(i) For a given structural cycle  $c^*$  a solution  $\gamma \in \mathcal{K}_\infty$  of the equation  $\bigotimes c^\gamma = \text{id}$  is not unique, in general, see Example 4.20. However, in Definition 4.18 we introduce the class of *right affine  $\mathcal{K}_\infty$ -functions*  $\mathcal{R}(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$  such that with  $\alpha_k \in \mathcal{R}(\mathcal{K}_\infty)$ , the functional equation  $\bigotimes c^\gamma = \text{id}$  has a unique solution within  $\mathcal{R}(\mathcal{K}_\infty)$ . Furthermore, this solution can be computed numerically, see Procedure 4.24.

(ii) If the solutions  $\gamma_i$  of the equations (4.11) are of class  $\mathcal{R}(\mathcal{K}_\infty)$ , then the final condition of Theorem 4.3 is automatically satisfied. Hence,  $\bar{\gamma}$  defined in (4.12) is maximal.

(iii) If  $\bigotimes c^\gamma = \text{id}$  cannot be solved analytically, i.e., if we cannot compute  $\bar{\gamma}$  in (4.12), we can at least construct an admissible  $\mathcal{K}_\infty$ -function  $\gamma$  as outlined in Remark 4.34 under the assumption that the  $\mathcal{K}_\infty$ -functions  $\alpha_k$  are differentiable in zero. ◁

#### 4.1.2 The linear summation case

Now we consider the linear summation case with  $\mu = \sum$  and given nonnegative matrices  $\Gamma_1 \in \mathbb{R}_+^{N \times N}$ ,  $\Gamma_2 \in \mathbb{R}_+^{N \times M}$ , where  $\Gamma_1$  has the property that the spectral radius satisfies  $\rho(\Gamma_1) < 1$ . Note that by Lemma 1.27,  $\rho(\Gamma_1) < 1$  is equivalent to the small-gain condition  $\Gamma_1 s \not\geq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$ . In addition, we consider the nonnegative structure matrices  $\Gamma_3 \in \mathbb{R}_+^{M \times N}$  and  $\Gamma_4 \in \mathbb{R}_+^{M \times M}$ . The problem of this section is to determine the largest scalar  $\bar{\gamma} > 0$  (possibly infinite) such that for all  $\gamma \in [0, \bar{\gamma})$  the small-gain condition

$$\Gamma^\gamma s \not\geq s \quad \text{for all } s \in \mathbb{R}_+^{N+M} \setminus \{0\} \tag{4.14}$$

is satisfied, where

$$\Gamma^\gamma := \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3\gamma & \Gamma_4\gamma \end{bmatrix}. \tag{4.15}$$

As  $\Gamma^\gamma \in \mathbb{R}_+^{(N+M) \times (N+M)}$  is a nonnegative matrix, we have the following implication

$$0 \leq \gamma_1 \leq \gamma_2 \quad \Rightarrow \quad \Gamma^0 \leq \Gamma^{\gamma_1} \leq \Gamma^{\gamma_2} \quad \Rightarrow \quad \rho(\Gamma_1) \leq \rho(\Gamma^{\gamma_1}) \leq \rho(\Gamma^{\gamma_2}), \quad (4.16)$$

see [8, Corollary 2.1.5] resp. Lemma A.3. Thus, a maximal<sup>2</sup> gain  $\bar{\gamma} \in \mathbb{R}_+ \cup \{\infty\}$  is characterized as follows.

**Lemma 4.6.** *Let  $\Gamma^\gamma$  be given by (4.15), and assume  $\rho(\Gamma_1) < 1$ . Let  $\bar{\gamma} \in \mathbb{R}_+ \cup \{\infty\}$  be defined as*

$$\bar{\gamma} := \inf\{\gamma > 0 : \rho(\Gamma^\gamma) = 1\}. \quad (4.17)$$

*Then  $\bar{\gamma}$  is maximal. In particular, for all  $\gamma \in [0, \bar{\gamma})$  the small-gain condition (4.14) holds, and (4.14) is violated for all  $\gamma \geq \bar{\gamma}$ .*

*Proof.* Consider  $\bar{\gamma}$  from (4.17). Firstly, note that as the eigenvalues depend continuously on the matrix entries the infimum in (4.17) is attained, and can be replaced by a minimum.

If  $\bar{\gamma} = \infty$  then for all  $\gamma \in \mathbb{R}_+$  we have  $\rho(\Gamma^\gamma) < 1$ , which is equivalent to the small-gain condition (4.14) by Lemma 1.27.

If  $\bar{\gamma} < \infty$  we distinguish the following two cases:

(i) Let  $\gamma \in [0, \bar{\gamma})$  then by (4.16), we have  $\rho(\Gamma^\gamma) \leq \rho(\Gamma^{\bar{\gamma}}) = 1$ . Moreover, by definition of  $\bar{\gamma}$  in (4.17),  $\rho(\Gamma^{\bar{\gamma}}) = 1$ . Hence,  $\rho(\Gamma^\gamma) < 1$ , and the small-gain condition (4.14) is satisfied by Lemma 1.27.

(ii) Let  $\gamma \geq \bar{\gamma}$  then by (4.16), we have  $\rho(\Gamma^\gamma) \geq \rho(\Gamma^{\bar{\gamma}}) = 1$ . Thus, the small-gain condition (4.14) is violated by Lemma 1.27.  $\square$

The characterization of the maximal gain  $\bar{\gamma}$  in (4.17) is in general inapplicable to compute  $\bar{\gamma}$  explicitly. Next, we focus on the computation of  $\bar{\gamma}$ . Here, we make use of tools from perturbation theory of linear systems, which are stated in the appendix section A.2.

Let us decompose  $\Gamma^\gamma$  into the following form

$$\Gamma^\gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} \Gamma_3 & \Gamma_4 \end{bmatrix} =: A + \gamma DE, \quad (4.18)$$

where  $A \in \mathbb{R}_+^{(N+M) \times (N+M)}$ ,  $D \in \mathbb{R}_+^{(N+M) \times M}$ ,  $E \in \mathbb{R}_+^{M \times (N+M)}$ . Then we define the *transfer function* of the decomposition (4.18) by

$$G(s) := E(sI - A)^{-1}D \quad \forall s \in \mathbb{C} \setminus \sigma(A). \quad (4.19)$$

---

<sup>2</sup>We allow  $\bar{\gamma} = \infty$  and call it a maximal gain, which means that any linear gain is admissible in the sense of Definition 4.2.

The maximal gain  $\bar{\gamma}$  in (4.17) can now be expressed in terms of the *stability radius*  $r_+(A; D, E)$  given in Definition A.4 as follows:

$$\begin{aligned}\bar{\gamma} &\stackrel{(4.17)}{=} \inf\{\gamma > 0 : \rho(\Gamma^\gamma) = 1\} \\ &= \inf\{\gamma > 0 : \rho(A + \gamma DE) \geq 1\} \\ &= r_+(A; D, E).\end{aligned}\tag{4.20}$$

As outlined in the appendix section A.2, stability radii for positive systems can be explicitly computed using the transfer function  $G$  from (4.19). Hence, the next theorem follows immediately from Theorem A.8.

**Theorem 4.7.** *Let  $\Gamma^\gamma$  be given in (4.15) and  $D, E$  be the decomposition matrices from (4.18). Let  $G$  be the transfer function defined in (4.19) and assume that  $\rho(\Gamma_1) < 1$ . Then the maximal gain  $\bar{\gamma}$  from (4.17) is given by*

$$\bar{\gamma} = (\rho(G(1)))^{-1}.$$

*Proof.* From Lemma 4.6 and (4.20), a maximal gain for  $\Gamma^\gamma$  is equal to the stability radius  $r_+(A; D, E)$  with respect to the decomposition (4.18). Since  $\rho(A) = \rho(\Gamma_1) < 1$  by assumption, we can apply Theorem A.8 to conclude

$$\bar{\gamma} \stackrel{(4.20)}{=} r_+(A; D, E) \stackrel{\text{Theorem A.8}}{=} (\rho(G(1)))^{-1},$$

where  $G(s)$  is the transfer function defined in (4.19). □

The importance of Theorem 4.7 is that in the linear summation case the (unique) maximal gain  $\bar{\gamma}$  of the  $(N + M) \times (N + M)$ -matrix  $\Gamma^\gamma$  in (4.15) can be easily obtained by computing the spectral radius of the nonnegative, lower-dimensional  $M \times M$ -matrix  $G(1)$ , see e.g. [110] for a reference that addresses the problem of computing the spectral radius of a nonnegative matrix. In particular, a straightforward calculation yields

$$G(1) = \Gamma_3(I - \Gamma_1)^{-1}\Gamma_2 + \Gamma_4.$$

In the next example we consider the question of how to choose the interconnection of several systems under some restrictions to achieve a maximal gain  $\bar{\gamma}$  for (4.8).

**Example 4.8.** In the following academic example we consider two interconnected linear systems with (Lyapunov-based or trajectory-based) gains  $\gamma_{12} = 0.9$  and  $\gamma_{21} = 0.8$ , so  $\Gamma_1 = \begin{pmatrix} 0 & 0.9 \\ 0.8 & 0 \end{pmatrix}$  has spectral radius less than one. Now we want to interconnect five additional systems to this interconnected system with the following restrictions:

- (i) the graph should be strongly connected;

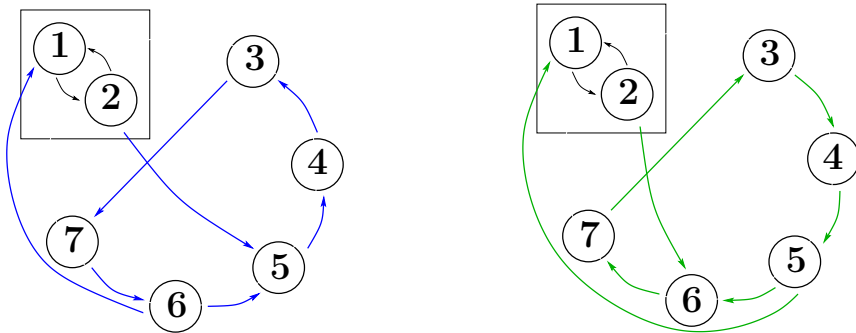
- (ii) the gains  $\gamma_{ii}$  satisfy  $\gamma_{ii} = 0$  for  $i \in \{3, \dots, 7\}$ ;
- (iii) we assume that there is no influence of the systems 3 and 7 on systems 1 and 2, and vice versa;
- (iv) we assume that there is no influence of system 3 on system 6, and there is no influence of system 5 on system 3;
- (v) we assume we use exactly seven additional interconnections.
- (vi) the weights of the gains correspond to a nearest neighbor order, the first nearest neighbors have a weight of 1, the second nearest neighbors have a weight of 2, and so on.

Summarizing, the (irreducible) gain matrix is of the form

$$\Gamma = \left( \begin{array}{cc|cccc} 0 & 0.9 & 0 & 2 & 3 & 2 & 0 \\ 0.8 & 0 & 0 & 2 & 3 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 3 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 & 2 & 1 & 0 \end{array} \right)$$

where exactly seven of the lower written entries  $(1, 2, 3)$  are non-zero and the other lower written entries are set to zero.

The problem in question is now to determine which interconnection, satisfying the above assumptions, leads to a largest gain  $\bar{\gamma} > 0$  such that  $\Gamma^\gamma$  satisfies the small-gain condition for all  $\gamma \in [0, \bar{\gamma})$ . Therefore, we compute  $\bar{\gamma}$  for all possible graphs satisfying the assumptions (i)-(vi). It turns out that the maximal  $\bar{\gamma}$  occurs in the cases with the following graphs





with gain matrices

$$\Gamma_{(1)} = \left( \begin{array}{cc|cccccc} 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) \quad \text{and} \quad \Gamma_{(2)} = \left( \begin{array}{cc|cccccc} 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{array} \right).$$

Here  $\bar{\gamma} = 0.47682$  satisfies  $\rho(\Gamma_{(i)}^\gamma) < 1, i \in \{1, 2\}$  for all  $\gamma \in [0, \bar{\gamma})$ . ◁

### 4.1.3 The general case

We now study the general nonlinear case. Firstly, note that in the maximization case (resp. in the linear summation case), we can determine a maximal gain  $\bar{\gamma} \in \mathcal{K}_\infty$  (resp.  $\bar{\gamma} \in \mathbb{R}_+$ ) satisfying  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}$  by using tools, which are equivalent to the small-gain condition (“cycle-condition” Proposition 1.29 resp. “spectral radius less than one”-condition Lemma 1.27). In the general nonlinear case, the operator  $D$  from (1.9) is required for the formulation of small-gain conditions; this diagonal operator will depend on the particular gain matrix. In our analysis, we thus rely on the concept of  $\Omega$ -paths introduced in Definition 1.22.

We start with a result on monotone aggregation functions that is used in the proof of the subsequent theorem.

**Lemma 4.9.** *Let  $\mu \in \text{MAF}_N$  be sub-additive, and  $\hat{\eta}, \eta \in \mathcal{K}_\infty$  satisfy  $\hat{\eta} = \text{id} - \eta$ . Assume that there exists an  $l \in \{1, \dots, N - 1\}$ , and functions  $\gamma_i \in \mathcal{K}_\infty$  for all  $i \in \{1, \dots, l\}$  satisfying*

$$\mu(\gamma_1(r), \dots, \gamma_l(r), 0, \dots, 0) < \eta(r) \quad \text{for all } r > 0. \quad (4.21)$$

Let  $\epsilon \in (0, 1)$ . Then there exists a  $\mathcal{K}_\infty$ -function  $\gamma_{l+1}$  such that we have

$$\mu(\gamma_1(r), \dots, \gamma_{l+1}(r), 0, \dots, 0) < (\text{id} - \epsilon\hat{\eta})(r) \quad \text{for all } r > 0.$$

*Proof.* Let  $\epsilon \in (0, 1)$ . Then we have

$$(\text{id} - \epsilon\hat{\eta}) = (\text{id} - \hat{\eta}) + (1 - \epsilon)\hat{\eta} = \eta + (1 - \epsilon)\hat{\eta}. \quad (4.22)$$

Consider the function  $v_{l+1}(r) := \mu(re_{l+1})$ , where  $e_{l+1}$  denotes the  $(l + 1)$ th unit vector. Note that  $v_{l+1} \in \mathcal{K}_\infty$  as observed in (1.7). Define

$$\gamma_{l+1}(r) := v_{l+1}^{-1} \circ \left( \frac{(1 - \epsilon)}{N} \hat{\eta} \right) (r) \quad \text{for all } r \in \mathbb{R}_+.$$

Clearly,  $\gamma_{l+1} \in \mathcal{K}_\infty$ , and it holds

$$\mu(0, \dots, 0, \gamma_{l+1}(r), 0, \dots, 0) < \frac{(1-\epsilon)}{N} \hat{\eta}(r) \quad \text{for all } r > 0. \quad (4.23)$$

By sub-additivity of  $\mu$ , we have for all  $r > 0$

$$\begin{aligned} \mu(\gamma_1(r), \dots, \gamma_{l+1}(r), 0, \dots, 0) &\leq \mu(\gamma_1(r), \dots, \gamma_l(r), 0, \dots, 0) \\ &\quad + \mu(0, \dots, 0, \gamma_{l+1}(r), 0, \dots, 0) \\ &\stackrel{(4.21), (4.23)}{<} \eta(r) + \frac{(1-\epsilon)}{N} \hat{\eta}(r) \\ &\stackrel{(4.22)}{<} (\text{id} - \epsilon \hat{\eta})(r), \end{aligned}$$

which concludes the proof.  $\square$

The following theorem shows in a constructive way that the set  $\mathcal{S}_{\text{strong}}$  defined in (4.7) is nonempty if the monotone aggregation function  $\mu$  is sub-additive.

**Theorem 4.10.** *Let  $\Gamma_1 \in (\{0\} \cup \mathcal{K}_\infty)^{N \times N}$ ,  $\Gamma_2 \in (\{0\} \cup \mathcal{K}_\infty)^{N \times M}$ , and the diagonal operator  $D = \text{diag}(\text{id} + \delta)$  with  $\delta \in \mathcal{K}_\infty$  be given. Assume that  $\mu \in \text{MAF}_{N+M}^{N+M}$  is sub-additive. Let  $\sigma^1 \in \mathcal{K}_\infty^N$  be an  $\Omega$ -path with respect to  $(D \circ \Gamma_{1,\mu})$ , where  $\Gamma_{1,\mu}$  is defined by*

$$\Gamma_{1,\mu}(s) := \begin{pmatrix} \mu_1(\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N), 0, \dots, 0) \\ \vdots \\ \mu_N(\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N), 0, \dots, 0) \end{pmatrix}.$$

*Then for any weighted structure matrix  $\Gamma_{\text{sub}}^* \in \mathbb{R}_+^{M \times (N+M)}$  from (4.3) there exists a strongly admissible  $\bar{\gamma} \in \mathcal{K}_\infty$ , thus  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}_{\text{strong}}$ . In particular,  $\mathcal{S}_{\text{strong}}$  is nonempty.*

*Proof.* To prove the statement we will construct an  $\Omega$ -path  $\sigma$  with respect to  $(\hat{D} \circ \Gamma_\mu)$ , with  $\hat{D} = \text{diag}(\text{id} + \hat{\delta})$ , and  $\hat{\delta}$  given below in (4.28). By assumption, there exists an  $\Omega$ -path  $\sigma^1 = (\sigma_1^1, \dots, \sigma_N^1) \in \mathcal{K}_\infty^N$  with respect to  $(D \circ \Gamma_{1,\mu})$ , i.e.,

$$(D \circ \Gamma_{1,\mu})(\sigma^1(r)) < \sigma^1(r) \quad \text{for all } r > 0. \quad (4.24)$$

Since  $D = \text{diag}(\text{id} + \delta)$ ,  $\delta \in \mathcal{K}_\infty$ , there exists, by Lemma 1.7,  $\mathcal{K}_\infty$ -functions  $\eta, \hat{\eta}$  with  $\eta = \text{id} - \hat{\eta}$  satisfying  $D^{-1} = \text{diag}(\eta)$ . So (4.24) is equivalent to

$$\Gamma_{1,\mu}(\sigma^1(r)) < D^{-1}(\sigma^1(r)) = \text{diag}(\eta) \circ \sigma^1(r) \quad \text{for all } r > 0$$

or row by row

$$\mu_i(\gamma_{i1}(\sigma_1^1(r)), \dots, \gamma_{iN}(\sigma_N^1(r)), 0, \dots, 0) < \eta(\sigma_i^1(r)) \quad (4.25)$$

for all  $r > 0$  and  $i \in \{1, \dots, N\}$ . Let  $\epsilon \in (0, 1)$ , then Lemma 1.6 implies that the function  $\text{id} - \epsilon\hat{\eta}$  is of class  $\mathcal{K}_\infty$ . By assumption,  $\mu$  is sub-additive. Applying Lemma 4.9 iteratively, we can determine functions  $\sigma_j^2 \in \mathcal{K}_\infty$ ,  $j \in \{1, \dots, M\}$  such that

$$\mu_i(\gamma_{i1}(\sigma_1^1(r)), \dots, \gamma_{iN}(\sigma_N^1(r)), \gamma_{i,N+1}(\sigma_1^2(r)), \dots, \gamma_{i,N+M}(\sigma_M^2(r))) < (\text{id} - \epsilon\hat{\eta}) \circ \sigma_i^1(r) \quad (4.26)$$

holds for all  $r > 0$ . This can be achieved by computing  $\sigma_j^2 \in \mathcal{K}_\infty$ ,  $j \in \{1, \dots, M\}$  for each  $i \in \{1, \dots, N\}$  and then taking the minimum for each  $j \in \{1, \dots, M\}$ .

Define  $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+^{N+M}$  by  $\sigma(r) := (\sigma^1(r), \sigma^2(r))$ , then (4.26) can equivalently be written as

$$(\mu \circ (\Gamma_1, \Gamma_2))(\sigma(r)) < \text{diag}(\text{id} - \epsilon\hat{\eta}) \circ \sigma^1(r), \quad (4.27)$$

where  $\mu = (\mu_1, \dots, \mu_N) \in \text{MAF}_{N+M}^N$ .

In the remainder of the proof, we show that we can choose the entries  $s \in \mathcal{K}_\infty^{\#J}$  of  $\Gamma^S$  (defined in (4.5)) in such a way, that  $\sigma$  is an  $\Omega$ -path with respect to  $(\hat{D} \circ \Gamma_\mu)$ ,  $\hat{D} = \text{diag}(\text{id} + \hat{\delta})$ , where

$$\hat{\delta} \in \mathcal{K}_\infty \text{ satisfies } (\text{id} + \hat{\delta}) = (\text{id} - \epsilon\hat{\eta})^{-1}, \quad (4.28)$$

which exists by Lemma 1.6. Since  $\sigma_i^1, \sigma_j^2 \in \mathcal{K}_\infty$  and  $\mu_i \in \text{MAF}_{N+M}$  is sub-additive, we can apply Lemma 4.9 to determine  $\mathcal{K}_\infty$ -functions  $\bar{\gamma}_i$ ,  $i \in \{1, \dots, M\}$ , such that for all  $r > 0$  we have

$$\mu_{i+N}(\bar{\gamma}_i(\sigma_{(i,j) \in J}(r))) < (\text{id} - \epsilon\hat{\eta}) \circ \sigma_{i+N}(r), \quad (4.29)$$

where we use the notation  $\mu_{i+N}(\bar{\gamma}_i(\sigma_{(i,j) \in J}(r))) := \mu_{i+N}(a_{i1}(r), \dots, a_{i,N+M}(r))$  with  $a_{ij}(r) = \begin{cases} [\Gamma_{sub}^*]_{ij} \bar{\gamma}_i(\sigma_j(r)) & \text{if } (i, j) \in J \\ 0 & \text{if } (i, j) \notin J \end{cases}$ . Define

$$\bar{\gamma}(r) := \min_{i \in \{1, \dots, M\}} \{\bar{\gamma}_i(r)\}. \quad (4.30)$$

With  $\Gamma_{sub}^{\bar{\gamma}} := \Gamma_{sub}^{\bar{\gamma}(1, \dots, 1)}$ , (4.29) is equivalent to

$$\left(\mu \circ \Gamma_{sub}^{\bar{\gamma}}\right)(\sigma(r)) < \text{diag}(\text{id} - \epsilon\hat{\eta}) \circ \sigma^2(r). \quad (4.31)$$

All in all we conclude from (4.27) and (4.31)

$$\Gamma_\mu^{\bar{\gamma}}(\sigma(r)) < \text{diag}(\text{id} - \epsilon\hat{\eta}) \circ \sigma(r) \quad \text{for all } r > 0$$

which is equivalent to  $(\hat{D} \circ \Gamma_\mu^{\bar{\gamma}})(\sigma(r)) < \sigma(r)$  with  $\hat{D} = \text{diag}(\text{id} + \hat{\delta})$  and  $\hat{\delta} \in \mathcal{K}_\infty$  given in (4.28). Thus, Lemma 1.23 shows that the strong small-gain condition  $(\hat{D} \circ \Gamma_\mu^{\bar{\gamma}}) \not\geq \text{id}$  is satisfied, which shows that  $\mathcal{S}_{\bar{\gamma}} \subset \mathcal{S}_{\text{strong}}$ .  $\square$

The proof that  $\mathcal{S}_{\text{strong}}$  is nonempty relies on the knowledge of an  $\Omega$ -path  $\sigma^1$  of  $\Gamma_{1,\mu}$ , which enables the construction of an  $\Omega$ -path  $\sigma$  for the overall system with  $\sigma = (\sigma^1, \sigma^2)$ . Such  $\Omega$ -paths can be computed efficiently using the algorithm in [37], see Remark 1.25. Clearly, since  $\Omega$ -paths are far away from being unique,  $\bar{\gamma}$  in (4.30) may vary within the different choices of  $\sigma^1$ . Moreover, since the operator  $\hat{D}$ , depending on the non-unique  $D$  and  $\varepsilon \in (0, 1)$ , influences  $\bar{\gamma}$ , the construction of  $\bar{\gamma}$  will in general be conservative. In addition, the method of proof of Theorem 4.10 relies on choosing an arbitrary path  $\sigma^2$ . Clearly, the derived strongly admissible gain  $\bar{\gamma}$  in (4.30) crucially depends on this choice of  $\sigma^2$ , but it is unclear how to choose an “optimal” path in terms of maximizing  $\bar{\gamma}$ .

In Theorem 4.10 we assume the existence of an  $\Omega$ -path  $\sigma^1$  with respect to  $(D \circ \Gamma_{1,\mu})$ . Note that from Lemma 1.24 it follows that if  $\Gamma^*$  defined in (4.3) is irreducible, then this assumption is necessary. Indeed, if there exists a strongly admissible gain  $\bar{\gamma} \in \mathcal{K}_\infty$ , then there exist a diagonal operator  $D$  and an  $\Omega$ -path  $\sigma = (\sigma^1, \sigma^2)$  with respect to  $(D \circ \Gamma_\mu^{\bar{\gamma}})$ . If  $\Gamma^*$  from (4.3) is reducible, then we can bring  $\Gamma^*$  into upper block triangular form, and apply Theorem 4.10 on each block separately.

The proof of Theorem 4.10 heavily relies on the fact that given an inequality of the form (4.25), we can determine functions  $\sigma_j^2 \in \mathcal{K}_\infty$ ,  $j \in \{1 \dots, M\}$ , such that an inequality of the form (4.26) holds. The following example shows that this step cannot, in general, be carried out without the assumption that  $\mu$  is sub-additive.

**Example 4.11.** Let  $\mu \in \text{MAF}_2$  be defined by

$$\mu(s_1, s_2) := 0.9s_1 + (s_1 + 1)s_2, \quad s_1, s_2 \in \mathbb{R}_+.$$

Indeed,  $\mu$  is a monotone aggregation function, which is not sub-additive. Let  $\gamma_1 := \text{id}$  then  $\mu(\gamma_1(r), 0) = 0.9r$  for all  $r \in \mathbb{R}_+$ . We claim that there does not exist a function  $\gamma_2 \in \mathcal{K}_\infty$  satisfying

$$\mu(\gamma_1(r), \gamma_2(r)) < r \quad \text{for all } r > 0. \quad (4.32)$$

To prove this claim, assume to the contrary that there exists a function  $\gamma_2 \in \mathcal{K}_\infty$  satisfying (4.32). Then, by definition of  $\mu$ , we have for all  $r > 0$

$$0.9r + (r + 1)\gamma_2(r) = \mu(\gamma_1(r), \gamma_2(r)) < r \quad \iff \quad \gamma_2(r) < 0.1 \frac{r}{r + 1} < 0.1.$$

Hence, any function  $\gamma_2$  satisfying (4.32) is bounded, which contradicts the unboundedness condition of a  $\mathcal{K}_\infty$ -function.  $\triangleleft$

We proceed with an example, where Theorem 4.10 is applied.

**Example 4.12.** We consider  $\Gamma_{(1)}^\gamma$  from Example 4.8. For this gain matrix we want to determine  $\bar{\gamma} > 0$  such that for  $\gamma \in [0, \bar{\gamma})$  the small-gain condition (4.14) is satisfied, by following the proof of Theorem 4.10. It is not hard to see that an  $\Omega$ -path  $\sigma^1$  for

the first two subsystems can be taken as  $\sigma^1 = (\text{id}, \text{id})$ . The only condition for the  $\Omega$ -path  $\sigma^2$  for the remaining five subsystems from (4.26) (with  $\varepsilon = 1$ , since we do not need the diagonal operator  $D$ ) is

$$0.8 \text{id} + 3\sigma_3^2 < \text{id} \quad \iff \quad \sigma_3^2 < \frac{2}{30} \text{id}.$$

So if we choose  $\sigma^2$  to be linear then any

$$\sigma^2(r) = vr, \quad v > 0, v \in \mathbb{R}_+^5 \text{ with } [v]_3 < \frac{2}{30} \tag{4.33}$$

will give an estimate for  $\bar{\gamma}$ . For instance, the choice  $\sigma^2 = \frac{1}{30}(1, \dots, 1)$  leads to (see (4.30))

$$\bar{\gamma} < 0.01634 \approx \min\{0.5, 1, 1, \frac{1/30}{1/30+2}, 1\}.$$

Compared to  $\bar{\gamma} = 0.47682$  obtained in Example 4.8, this shows the conservatism of the method of Theorem 4.10.

More suitable is optimizing  $\bar{\gamma}$  along all paths  $\sigma^2$  satisfying (4.33), i.e., solving the following system for  $\bar{\gamma} > 0$

$$\begin{aligned} 2\bar{\gamma}\sigma_5^2 &< \sigma_1^2, & \bar{\gamma}\sigma_1^2 &< \sigma_2^2, \\ \bar{\gamma}\sigma_2^2 &< \sigma_3^2 < \frac{2}{30} \text{id}, & 2\bar{\gamma} \text{id} + \bar{\gamma}\sigma_3^2 &< \sigma_4^2, \\ 2\bar{\gamma}\sigma_4^2 &< \sigma_5^2, \end{aligned}$$

which leads to  $\bar{\gamma}^5(2 \text{id} + \sigma_3^2) < \frac{1}{2}\sigma_3^2 < \frac{1}{30} \text{id}$  and to the less conservative estimate

$$\bar{\gamma} < 0.43804. \quad \triangleleft$$

The idea of using properties of the  $\Omega$ -paths rather than using explicit paths is also used in the next example.

**Example 4.13.** Recall Example 4.4 with  $\Gamma^*$  from (4.13). Firstly, note that  $\sigma^1(r) = (\sigma_1^1(r), \sigma_2^1(r))$  is an  $\Omega$ -path for  $\Gamma_{1, \oplus}$  if and only if

$$(\sigma_1^1(r))^2 < \sigma_2^1(r) \quad \text{for all } r > 0. \tag{4.34}$$

Since we are in the maximization case, we can omit the diagonal operator and (4.26) is satisfied if and only if

$$\sigma_1^2(r) < (3\sigma_1^1(r))^{1/3} \quad \text{for all } r > 0. \tag{4.35}$$

Then (4.29) is satisfied if and only if

$$\max\{\bar{\gamma}(\sigma_2^1(r)), \bar{\gamma}(\sigma_1^2(r))\} < \sigma_1^2(r)$$

holds for every  $r > 0$ . In particular,  $\bar{\gamma} < \text{id}$ . From (4.34) and (4.35) we conclude

$$\bar{\gamma}((\sigma_1^1(r))^2) < \bar{\gamma}(\sigma_2^1(r)) < \sigma_1^2(r) < (3\sigma_1^1(r))^{1/3} \text{ for all } r > 0 \iff \bar{\gamma}(\tau) < 3^{1/3}\tau^{1/6}.$$

Together we have  $\bar{\gamma}(\tau) < \max\{\tau, 3^{1/3}\tau^{1/6}\}$  for all  $\tau > 0$  which coincides with the maximal gain from (4.12). Note that fixing  $\sigma_1^1$ ,  $\sigma_2^1$  and  $\sigma_1^2$  would have led to a considerably more conservative estimate for  $\bar{\gamma}$ .  $\triangleleft$

*Remark 4.14.* Throughout this section we assume that  $\Gamma_2 \in (\mathcal{K}_\infty \cup \{0\})^{N \times M}$  is known. This assumption is further motivated in Chapters 2 and 3, where the entries of  $\Gamma_2$  describe how the first  $N$  subsystems are influenced by the last  $M$  subsystems. On the other hand, we could assume that the operator  $\Gamma^*(\gamma)$  in (4.4) is of the form

$$\Gamma^*(\gamma) := \begin{bmatrix} \Gamma_1 & \Gamma_2^* \cdot \gamma \\ \Gamma_3^* \cdot \gamma & \Gamma_4^* \cdot \gamma \end{bmatrix}.$$

By small modifications in the particular proofs, we obtain results similar to Theorem 4.3, Theorem 4.7 with  $A = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $D = I$  and  $E = \begin{bmatrix} 0 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}$ , as well as Theorem 4.10.  $\triangleleft$

*Remark 4.15.* To ease presentation, we assumed that the weights  $a_{(N+i)j} \in \mathbb{R}_+$ , collected in the matrix  $\Gamma_{sub}^*$  in (4.3), are nonnegative real numbers. These weights can also be chosen nonlinear, i.e., as  $\alpha_{(N+i)j} \in \mathcal{K}_\infty$ . The statement of the results of this section still holds true.  $\triangleleft$

## 4.2 Solutions of iterative functional $\mathcal{K}_\infty$ -equations

The purpose of this section is to study functional equations of the form

$$\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma = \text{id}, \tag{4.36}$$

with  $k \in \mathbb{N}$ , and  $\alpha_i \in \mathcal{K}_\infty$  for  $i \in \{1, \dots, k\}$ . Functional equations of the form (4.36) will also be referred to as *iterative functional  $\mathcal{K}_\infty$ -equations*, as the functions are applied iteratively, and we are looking for solutions  $\gamma$  of class  $\mathcal{K}_\infty$ . Of particular interest are existence and uniqueness as well as the numerical computation of solutions  $\gamma \in \mathcal{K}_\infty$  of the functional equation (4.36).

Functional equations of this class have been derived in Section 4.1 to determine the set of functions such that a corresponding cycle is weakly contracting. To be more precise, if  $\bar{\gamma} \in \mathcal{K}_\infty$  satisfies (4.36), then by the strict increase of  $\mathcal{K}_\infty$ -functions we have

$$\alpha_1 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma \quad \begin{cases} < \text{id} & \text{if } \gamma < \bar{\gamma} \\ \geq \text{id} & \text{if } \gamma \geq \bar{\gamma} \end{cases}.$$

From an abstract point of view, the results obtained in this section extend previous results on functional equations such as the special case from [87], which is given in the next proposition. Although functional equations have been extensively studied, the problem we pose appears to be solved only for the special case, where  $\alpha_1, \dots, \alpha_{k-1} = \text{id}$ .

**Proposition 4.16.** *Let  $\alpha \in \mathcal{K}_\infty$ . Then the iterative functional  $\mathcal{K}_\infty$ -equation*

$$\gamma^k = \alpha$$

*admits at least one solution  $\gamma \in \mathcal{K}_\infty$ , which is not unique, in general.*

*Proof.* The result follows directly from [87, Theorem 11.2.2] by noticing that  $\alpha \in \mathcal{K}_\infty$  is a strictly increasing, continuous self-mapping of  $[0, \infty) \subset \mathbb{R}$ .  $\square$

Proposition 4.16 can be used to ensure the existence of solutions of iterative functional  $\mathcal{K}_\infty$ -equations (4.36) of a special form.

**Corollary 4.17.** *Consider the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) with  $\alpha_i \in \mathcal{K}_\infty$  for all  $i \in \{1, \dots, k\}$ , and assume that at least  $k - 1$  of the functions  $\alpha_i \in \mathcal{K}_\infty$  are the same. Then there exists at least one solution  $\gamma \in \mathcal{K}_\infty$  of (4.36).*

*In particular, for  $k \in \{1, 2\}$  the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) always exhibits a solution.*

*Proof.* Let  $\alpha_i = \beta \in \mathcal{K}_\infty$  for all  $i \in \{1, \dots, k\}$  then clearly  $\gamma = \beta^{-1}$  solves (4.36). Now assume that one function  $\alpha_i$  is different from the others. Denote  $\beta := \alpha_j, j \neq i$ . Then defining  $\chi := \beta \circ \gamma$  and  $\alpha := \beta \circ \alpha_i^{-1}$ , and using the cyclic permutation property (1.2) leads to the iterative functional  $\mathcal{K}_\infty$ -equation  $\chi^k = \alpha$ , which has at least one solution by Proposition 4.16. Thus, a solution  $\gamma$  of (4.36) is given by  $\gamma = \beta^{-1} \circ \chi$ .

In particular, if  $k = 1$  then  $\gamma = \alpha_1^{-1}$  solves (4.36). If  $k = 2$ , then  $\gamma = \alpha_2^{-1} \circ \chi$  solves (4.36), where  $\chi \in \mathcal{K}_\infty$  is a solution of  $\chi^2 = \alpha_2 \circ \alpha_1^{-1}$ . This solution exists by Proposition 4.16.  $\square$

Proposition 4.16 ensures the existence of solutions for a special class of iterative functional  $\mathcal{K}_\infty$ -equations (4.36). Also for this special case, solutions may not be unique, and the proof of [87, Theorem 11.2.2] is not constructive. Hence, a solution cannot be computed, in general.

To apply the results in Section 4.1.1, in particular Theorem 4.3, we have to be able to compute a solution of (4.36). For this reason, we restrict the  $\mathcal{K}_\infty$ -functions  $\gamma, \alpha_i, i \in \{1, \dots, k\}$ , to the class of right affine  $\mathcal{K}_\infty$ -functions in which uniqueness of solutions will be shown in the remainder of this section.

**Definition 4.18.** A function  $\alpha \in \mathcal{K}_\infty$  is called *right affine*, resp. *left affine* in  $t \in [0, \infty)$ , if there exists an  $\varepsilon > 0$  such that  $\alpha$  is affine linear on  $[t, t + \varepsilon)$ , resp.  $(t - \varepsilon, t]$ . Then  $\alpha \in \mathcal{K}_\infty$  is called *right affine* (resp. *left affine*), if it is right affine (resp. left affine) in every  $t \in [0, \infty)$ .

The set of right affine resp. left affine functions is denoted by  $\mathcal{R}(\mathcal{K}_\infty)$  resp.  $\mathcal{L}(\mathcal{K}_\infty)$ . Further, we denote

$$\mathcal{I}(\mathcal{K}_\infty) := \{\alpha \in \mathcal{K}_\infty \mid \forall t \in (0, \infty) : \alpha \text{ is right or left affine in } t\}.$$

A *sampling point*  $t > 0$  for a function  $\alpha \in \mathcal{I}(\mathcal{K}_\infty)$  is defined by the property that  $\alpha$  is left and right affine in  $t$  and

$$\alpha'(t^-) := \lim_{\tau \nearrow t} \frac{\alpha(t) - \alpha(\tau)}{t - \tau} \neq \lim_{\tau \searrow t} \frac{\alpha(\tau) - \alpha(t)}{\tau - t} =: \alpha'(t^+),$$

i.e., its left and right hand derivatives differ. Loosely speaking,  $\alpha \in \mathcal{I}(\mathcal{K}_\infty)$  has a kink in a sampling point  $t$ . Let  $\mathcal{P}(\mathcal{K}_\infty) = \mathcal{R}(\mathcal{K}_\infty) \cap \mathcal{L}(\mathcal{K}_\infty)$ , then  $\mathcal{P}(\mathcal{K}_\infty)$  is the set of piecewise linear functions of class  $\mathcal{K}_\infty$ , i.e., the set of those functions in  $\mathcal{K}_\infty$  which locally have only finitely many *sampling points* and are affine linear between two successive sampling points. By definition, the subset  $\mathcal{F}(\mathcal{K}_\infty)$  of  $\mathcal{P}(\mathcal{K}_\infty)$  consists of those functions  $\alpha \in \mathcal{P}(\mathcal{K}_\infty)$  that have only finitely many sampling points.

If  $\alpha \in \mathcal{I}(\mathcal{K}_\infty)$  admits a strictly increasing or strictly decreasing convergent sequence  $\{t_k\}_{k \in \mathbb{N}}$  of sampling points, then the limit  $\lim_{k \rightarrow \infty} t_k =: t^*$  is called an *accumulation point* of  $\alpha$ . By definition, for  $\alpha \in \mathcal{R}(\mathcal{K}_\infty)$  there do not exist strictly decreasing sequences of sampling points. Furthermore, given  $\alpha \in \mathcal{R}(\mathcal{K}_\infty)$  and  $t \in [0, \infty)$ , there exists an  $\varepsilon > 0$  such that  $\alpha$  is affine linear on  $[t, t + \varepsilon)$ . Hence, there exists a  $t^* \in (t, \infty]$  such that  $\alpha$  has at most countably many sampling points in  $(t, t^*)$  and no other singularities in that interval. In particular,  $\alpha \in \mathcal{R}(\mathcal{K}_\infty)$  has at most countably many sampling points and countably many accumulation points.

The sets  $\mathcal{F}(\mathcal{K}_\infty)$ ,  $\mathcal{P}(\mathcal{K}_\infty)$ ,  $\mathcal{R}(\mathcal{K}_\infty)$  and  $\mathcal{I}(\mathcal{K}_\infty)$  satisfy the relation

$$\mathcal{F}(\mathcal{K}_\infty) \subset \mathcal{P}(\mathcal{K}_\infty) \subset \mathcal{R}(\mathcal{K}_\infty) \subset \mathcal{I}(\mathcal{K}_\infty).$$

Moreover, any of these sets, together with the composition  $\circ$ , is a group. We note that we do not consider the class  $\mathcal{L}(\mathcal{K}_\infty)$  in this work, but rely on the class  $\mathcal{R}(\mathcal{K}_\infty)$ . This will become clear in the proof of Lemma 4.22, where we exploit the knowledge of the right hand derivatives of the functions  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$  to construct a solution of (4.36).

#### 4.2.1 Motivating examples

We start with an example, where we study the behavior of solutions of the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) with  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{F}(\mathcal{K}_\infty)$ . We show that in this case we cannot conclude that a solution  $\gamma$  of (4.36) is of class  $\mathcal{P}(\mathcal{K}_\infty)$ . This example shows that the set  $\mathcal{P}(\mathcal{K}_\infty)$  is not appropriate for computing solutions of (4.36).



**Example 4.19.** Consider the following special case of (4.36),

$$\alpha \circ \gamma \circ \text{id} \circ \gamma = \text{id}, \tag{4.37}$$

where  $\alpha \in \mathcal{F}(\mathcal{K}_\infty)$ . We solve this equations for two different functions  $\alpha$ , which are given by

$$\alpha_1(t) = \begin{cases} 4t & t \in [0, 1] \\ \frac{1}{2}t + \frac{7}{2} & t \in [1, \infty) \end{cases}, \quad \text{and} \quad \alpha_2(t) = \begin{cases} \frac{1}{4}t & t \in [0, 4] \\ t - 3 & t \in [4, \infty) \end{cases}.$$

Using the procedure that will be given in Lemma 4.22, we can compute the unique solution of (4.37) in  $\mathcal{R}(\mathcal{K}_\infty)$  as

$$\gamma_1(t) = \begin{cases} \frac{1}{2}t & t \in [0, 4] \\ 4t - 14 & t \in [4, \frac{9}{2}] \\ \vdots & \\ \frac{1}{2}t + \frac{7 \cdot (2^n - 1)}{2^{n+1}} & t \in [\frac{2^{n-1} \cdot 14 - 5}{2^n}, \frac{2^{n-1} \cdot 14 - 3}{2^n}] \quad n \in \mathbb{N} \\ 4t - \frac{7 \cdot (2^n \cdot 3 - 1)}{2^n} & t \in [\frac{2^{n-1} \cdot 14 - 3}{2^n}, \frac{2^n \cdot 14 - 5}{2^{n+1}}] \quad n \in \mathbb{N} \\ \vdots & \\ \sqrt{2}t - \frac{7}{\sqrt{2}+1} & t \in [7, \infty) \end{cases}$$

and

$$\gamma_2(t) = \begin{cases} 2t & t \in [0, 2] \\ \frac{1}{2}t + 3 & t \in [2, 4] \\ \vdots & \\ 2t - 3n & t \in [3n + 1, 3n + 2] \quad n \in \mathbb{N} \\ \frac{1}{2}t + \frac{3}{2}n + 3 & t \in [3n + 2, 3n + 4] \quad n \in \mathbb{N} \\ \vdots & \end{cases}.$$

The functions  $\alpha_i$  and  $\gamma_i$  are shown in Figure 4.1. In both cases,  $\alpha_i \in \mathcal{F}(\mathcal{K}_\infty)$  for  $i \in \{1, 2\}$  with only one sampling point. However, the solutions  $\gamma_i$  have infinitely many sampling points that may accumulate (in the case  $i = 1$ ) or not (in the case  $i = 2$ ). ◁

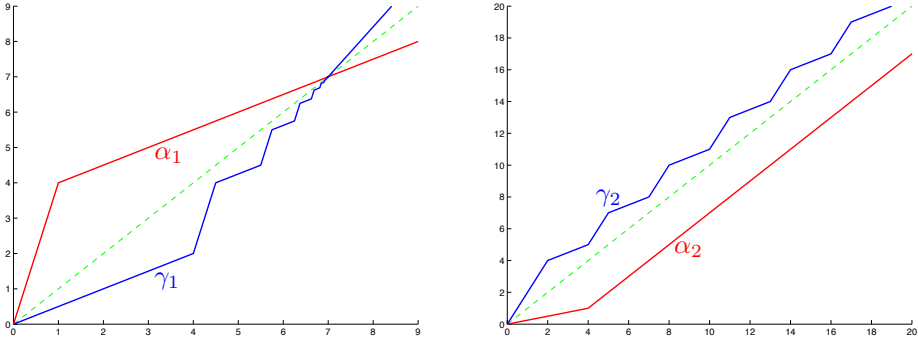


Figure 4.1:  $\gamma_i$  and  $\alpha_i$  satisfying  $\alpha_i \circ \gamma_i \circ \text{id} \circ \gamma_i = \text{id}$ .

The next example shows that even for linear  $\alpha_i \in \mathcal{F}(\mathcal{K}_\infty)$ ,  $i \in \{1, \dots, k\}$  we do not have unique solutions  $\gamma \in \mathcal{I}(\mathcal{K}_\infty)$  for the functional equation (4.36). Hence, to obtain unique solutions of (4.36), the set  $\mathcal{I}(\mathcal{K}_\infty)$  is too big to be a solution space.

**Example 4.20.** Consider the functional equation

$$\frac{1}{2} \text{id} \circ \gamma \circ 2 \text{id} \circ \gamma = \text{id}. \tag{4.38}$$

Clearly, a solution is given by  $\gamma_1 = \text{id}$ . Now consider the function

$$\gamma_2(t) := t + \begin{cases} \vdots \\ \hline -1/3t - 1/12 & t \in [1/4, 7/16] \\ t - 1/2 & t \in [7/16, 1/2] \\ \hline 1/2t - 1/4 & t \in [1/2, 3/4] \\ -1/2t + 1/2 & t \in [3/4, 1] \\ \hline -1/3t + 1/3 & t \in [1, 7/4] \\ t - 2 & t \in [7/4, 2] \\ \hline 1/2t - 1 & t \in [2, 7/2] \\ -1/2t + 5/2 & t \in [7/2, 4] \\ \hline \vdots \end{cases} \tag{4.39}$$

shown in Figure 4.2.

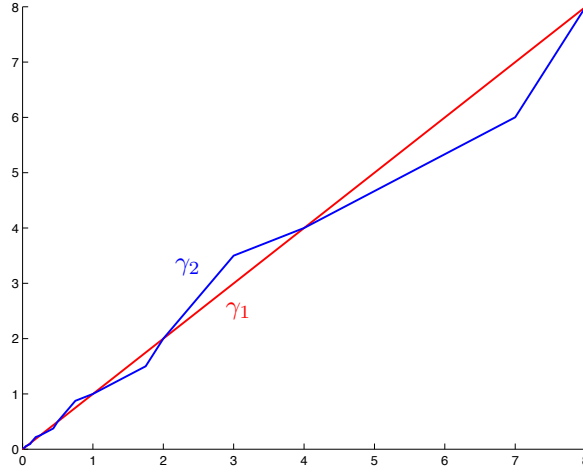


Figure 4.2: Solutions to functional equations of the form (4.36) are in general not unique.

This function is defined as follows. We start by defining  $\gamma_2$  on the interval  $[1/2, 1]$  as in (4.39), whence  $\gamma_2([1/2, 1]) = [1/2, 1]$ . We now construct  $\gamma_2$  such that (4.38) holds. By solving  $\gamma_2(2\gamma_2(t)) = 2t$  for  $t \in [1/2, 1]$ , the definition of  $\gamma_2$  on  $[1, 2]$  as in (4.39) follows. Continuing in this manner we can construct  $\gamma_2$  on all intervals  $[2^{k-2}, 2^{k-1}]$ ,  $k \in \mathbb{N}$  and solving this equation backwards,  $\gamma_2$  is obtained on  $[2^{-k-2}, 2^{-k-1}]$ ,  $k \in \{0, 1, \dots\}$ . By construction,  $\gamma_2 \in \mathcal{I}(\mathcal{K}_\infty)$  satisfies (4.38).  $\triangleleft$

#### 4.2.2 Solutions of iterative functional $\mathcal{K}_\infty$ -equations for $k = 2$

We start with the case  $k = 2$  to explain the behavior of solutions and to give clear ideas how to prove results in this context. From Proposition 4.16 we know that for  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) always admits a solution  $\gamma$ . But from Example 4.20 we know that even for linear functions  $\alpha_i$  we cannot expect a unique solution in  $\mathcal{K}_\infty$  of (4.36). In the next lemma we show that uniqueness of a solution of (4.36) is obtained for  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$  if only right affine solutions are considered.

**Lemma 4.21** (Unique representation). *Let  $k = 2$ ,  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $T > 0$ ,  $I = [0, T]$ . If for  $\gamma_1, \gamma_2 \in \mathcal{R}(\mathcal{K}_\infty)$  we have*

$$\forall t \in I : \alpha_1 \circ \gamma_i \circ \alpha_2 \circ \gamma_i(t) = t, \quad i \in \{1, 2\}, \tag{4.40}$$

then the restriction to  $\bar{I} := I \cup \alpha_2(\gamma_1(I)) \cup \alpha_2(\gamma_2(I))$  satisfies  $\gamma_{1|\bar{I}} \equiv \gamma_{2|\bar{I}}$ . In particular, there exists at most one right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  of (4.36).

*Proof.* Let  $k = 2$  and  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $i \in \{1, 2\}$ . Assume that there exist  $\gamma_1, \gamma_2 \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $\gamma_1 \neq \gamma_2$  solving (4.40). Taking the derivative of (4.40), we obtain for  $i \in \{1, 2\}$

$$(\alpha_1 \circ \gamma_i \circ \alpha_2 \circ \gamma_i)'(0^+) = \alpha_1'(0^+) \cdot \gamma_i'(0^+) \cdot \alpha_2'(0^+) \cdot \gamma_i'(0^+) = 1.$$

and so  $\gamma_1'(0^+) = \gamma_2'(0^+)$ . As  $\gamma_1, \gamma_2 \in \mathcal{R}(\mathcal{K}_\infty)$  this implies that there exists an interval  $[0, a]$  with  $\gamma_1(s) = \gamma_2(s)$  for all  $s \in [0, a]$ . Seeking a contradiction, assume  $a < T$  and without loss of generality  $\gamma_1(s) < \gamma_2(s)$  for all  $s \in (a, a + \varepsilon)$  for  $\varepsilon > 0$  small enough. Fix  $t \in (a, a + \varepsilon) \cap I$ . If  $\alpha_2(\gamma_2(t)) \leq t$ , then we obtain the contradiction

$$t = \alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_1(t) < \alpha_1 \circ \gamma_1 \circ \underbrace{\alpha_2 \circ \gamma_2(t)}_{\leq t} \leq \alpha_1 \circ \gamma_2 \circ \alpha_2 \circ \gamma_2(t) = t.$$

On the other hand, if  $\alpha_2(\gamma_2(t)) > t$ , then pick a  $\tilde{t} < t$  with  $\alpha_2(\gamma_2(\tilde{t})) = t$ , which is uniquely determined by monotonicity and continuity. Then as  $\tilde{t} \in I$

$$\tilde{t} = \alpha_1 \circ \gamma_2 \circ \alpha_2 \circ \gamma_2(\tilde{t}) > \alpha_1 \circ \gamma_1(t) = \alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_2(\tilde{t}) \geq \alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_1(\tilde{t}) = \tilde{t},$$

where we have used that  $\gamma_2 \geq \gamma_1$  on  $[0, a + \varepsilon)$  by construction. This shows  $\gamma_{1|I} \equiv \gamma_{2|I}$ , whence  $\alpha_2 \circ \gamma_1(I) = \alpha_2 \circ \gamma_2(I)$ . The claim for  $\bar{I}$  now follows from (4.40). The second statement is then immediate.  $\square$

The following lemma makes the existence result of Proposition 4.16 more precise. The proof is constructive and it ensures the existence of a solution of class  $\mathcal{R}(\mathcal{K}_\infty)$  if  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ . Furthermore, the proof yields an algorithm for computing solutions of (4.36) for  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ .

**Lemma 4.22** (Existence). *Let  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ . Then there exists a right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  of (4.36).*

*Proof.* The proof relies on an extension principle, which we describe next. Assume that  $\eta \in \mathcal{R}(\mathcal{K}_\infty)$  and the interval  $I = [0, T]$  are such that

$$\forall t \in I : \alpha_1 \circ \eta \circ \alpha_2 \circ \eta(t) = t, \tag{4.41}$$

and let  $I$  be the maximal closed interval on which (4.41) holds. This implies  $(\alpha_1 \circ \eta \circ \alpha_2 \circ \eta)'(T^+) \neq 1$  as otherwise the interval  $I$  in (4.41) can be extended to the right, contradicting the maximality of  $I$ . Define  $\tau := \max\{T, \alpha_2(\eta(T))\}$  and  $\tilde{\eta} \in \mathcal{R}(\mathcal{K}_\infty)$  by setting

$$\tilde{\eta}(t) := \begin{cases} \eta(t) & , \quad t \in [0, \tau] \\ m(t - \tau) + \eta(\tau) & , \quad t \in [\tau, \infty) \end{cases} \tag{4.42}$$

with

$$\begin{aligned} m &:= (\alpha'_1(\eta(T)^+) \cdot \alpha'_2(\eta(T)^+))^{-1/2}, & \text{if } T = \alpha_2(\eta(T)), \\ m &:= (\alpha'_1(\eta \circ \alpha_2 \circ \eta(T)^+) \cdot \alpha'_2(\eta(T)^+) \cdot \eta'(T^+))^{-1}, & \text{if } T < \alpha_2(\eta(T)), \\ m &:= (\alpha'_1(\eta \circ \alpha_2 \circ \eta(T)^+) \cdot \eta'(\alpha_2 \circ \eta(T)^+) \cdot \alpha'_2(\eta(T)^+))^{-1}, & \text{if } T > \alpha_2(\eta(T)). \end{aligned}$$

We claim that  $\tilde{\eta}$  satisfies (4.41) on a strictly larger interval than  $I$ . To this end note that in the case  $T = \alpha_2(\eta(T))$  we have

$$\begin{aligned} (\alpha_1 \circ \tilde{\eta} \circ \alpha_2 \circ \tilde{\eta})'(T^+) &= \alpha'_1(\tilde{\eta} \circ \alpha_2 \circ \tilde{\eta}(T)^+) \cdot \tilde{\eta}'(\alpha_2 \circ \tilde{\eta}(T)^+) \cdot \alpha'_2(\tilde{\eta}(T)^+) \cdot \tilde{\eta}'(T^+) \\ &= \alpha'_1(\tilde{\eta}(T)^+) \cdot \alpha'_2(\tilde{\eta}(T)^+) \cdot m^2 \\ &= \alpha'_1(\eta(T)^+) \cdot \alpha'_2(\eta(T)^+) \cdot m^2 = 1. \end{aligned}$$

Similar arguments apply in the cases  $T < \alpha_2(\eta(T)), T > \alpha_2(\eta(T))$  to show that  $(\alpha_1 \circ \tilde{\eta} \circ \alpha_2 \circ \tilde{\eta})'(T^+) = 1$ . Since  $\tilde{\eta}, \alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$  we conclude that there exists an  $\varepsilon > 0$  such that  $\alpha_1 \circ \tilde{\eta} \circ \alpha_2 \circ \tilde{\eta}(t) = t$  holds for all  $t \in [0, T + \varepsilon]$ . By continuity the equality holds on  $[0, T + \varepsilon]$ .

To show that there exists a function  $\eta \in \mathcal{R}(\mathcal{K}_\infty)$  and  $T > 0$  satisfying (4.41), define  $\gamma_0(t) := (\alpha'_1(0^+) \cdot \alpha'_2(0^+))^{-1/2}t$  for  $t \geq 0$ . Then  $(\alpha_1 \circ \gamma_0 \circ \alpha_2 \circ \gamma_0)'(0^+) = 1$ , and since  $\alpha_1, \alpha_2$  are right affine, there exists an  $\varepsilon > 0$  such that  $\alpha_1 \circ \gamma_0 \circ \alpha_2 \circ \gamma_0(t) = t$  for all  $t \in I_0 := [0, \varepsilon]$ .

Finally, we show the existence of a solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$ . We call  $\eta \in \mathcal{R}(\mathcal{K}_\infty)$  an extension of  $\gamma_0$  if the two functions coincide on  $I_0$ . Denote by  $I(\eta)$  the maximal closed interval containing 0 on which (4.41) holds for  $\eta$  and let  $\bar{I}(\eta) := I(\eta) \cup \alpha_2 \circ \eta(I(\eta))$ . By Lemma 4.21 two extensions  $\eta_1, \eta_2$  of  $\gamma_0$  coincide on  $\bar{I}(\eta_1) \cap \bar{I}(\eta_2)$ . Thus the definition

$$\gamma(t) = \{\eta(t) \mid \eta \text{ extends } \gamma_0, t \in \bar{I}(\eta)\} \quad (4.43)$$

is well defined on  $\bar{I}_{\max} := \cup \bar{I}(\eta)$ , where the union is over all extensions  $\eta$  of  $\gamma_0$ . If  $\bar{I}_{\max}$  and so  $I_{\max} = \cup I(\eta)$  is bounded, then the fact that extensions  $\eta$  solve (4.40) on  $I(\eta)$  implies that  $\gamma$  is strictly increasing and bounded on  $\bar{I}_{\max}$ . Thus  $\gamma$  can be continuously extended to  $\text{cl}\bar{I}_{\max}$  and then extended in an arbitrary manner to  $\infty$  so that the resulting function  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$ . Now  $\gamma$  extends  $\gamma_0$ ,  $I(\gamma) = \text{cl}I_{\max}$  and by the first step of the proof  $\gamma$  may be extended to a function  $\tilde{\gamma}$  which solves (4.40) on a larger interval than  $I_{\max}$ . This clearly contradicts the definition of  $I_{\max}$  and so (4.43) already defines a solution  $\gamma$  of (4.36).  $\square$

The main result of this section is now a corollary of the previous lemmata.

**Theorem 4.23** (Existence of a unique right affine solution). *Let  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ . Then there exists a unique right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  of (4.36).*

*Proof.* This follows directly from Lemma 4.21 and Lemma 4.22.  $\square$

We note that the method of proof of Lemma 4.22 lends itself to a constructive procedure.

*Procedure 4.24.* Define  $\gamma_0$  as in the proof of Lemma 4.22 and then iterate.

- [I] Compute  $t_{i+1} := \inf\{t > t_i \geq 0 : \alpha_1 \circ \gamma_i \circ \alpha_2 \circ \gamma_i(t) - t \neq 0\}$ .
- [II] If  $t_{i+1} = \infty$ , then (4.36) is satisfied for  $\gamma_i$  and we are done. Otherwise define the slope  $m$  as in the proof of Lemma 4.22 for  $T = t_{i+1}$ .
- [III] Define an extension  $\gamma_{i+1}$  as in (4.42) and continue with Step [II].  $\triangleleft$

This iteration will compute the solution up to an accumulation point. It is therefore of interest to have criteria for the existence of accumulation points, which can be used as initial values for computations beyond such points. This is discussed in the sequel.

In the remainder of this section we investigate when accumulation points occur. We start with the iterative functional  $\mathcal{K}_\infty$ -equation (4.36), where the functions  $\alpha_1, \alpha_2 \in \mathcal{P}(\mathcal{K}_\infty)$ , i.e., they are piecewise linear, but do not have accumulation points.

**Proposition 4.25.** *Let  $k = 2$ ,  $\alpha_1, \alpha_2 \in \mathcal{P}(\mathcal{K}_\infty)$ , and  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  be a solution of (4.36). If  $\gamma$  has an accumulation point  $t^* > 0$ , then*

$$t^* = \alpha_1(\gamma(t^*)) = \alpha_2(\gamma(t^*)).$$

*Proof.* An accumulation point  $t^*$  of  $\gamma$  is called *isolated*, if there exists an  $\varepsilon > 0$  such that  $(t^* - \varepsilon, t^* + \varepsilon)$  contains no other accumulation point. As  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  has at most countably many accumulation points, it follows that every accumulation point of  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  is either isolated or the limit of isolated accumulation points. Hence, it is sufficient to prove the claim for isolated accumulation points. The full statement then follows by continuity.

So assume to the contrary that  $t^* < \alpha_2(\gamma(t^*))$ . The case  $t^* > \alpha_2(\gamma(t^*))$  will be discussed in the remainder of the proof. Assume that  $t^*$  is the first accumulation point of  $\gamma$  which arises and let  $\{t_i\}_{i \in \mathbb{N}}$  be the (unique and strictly increasing) sequence of all sampling points less than  $t^*$  converging to  $t^*$ . By continuity there exists an index  $J \in \mathbb{N}$  such that for all  $i \geq J$

$$t_i < t^* < \alpha_2(\gamma(t_i)),$$

which implies  $\alpha_1(\gamma(t_i)) < \alpha_1(\gamma(t^*)) < t_i$ , in particular  $\alpha_1(\gamma(t^*)) < t^*$ . We can further assume, since  $\alpha_1, \alpha_2 \in \mathcal{P}(\mathcal{K}_\infty)$ , that for all  $i \geq J$ ,  $\gamma(t_i)$  is no sampling point of  $\alpha_2$  and  $\gamma \circ \alpha_2 \circ \gamma(t_i)$  is no sampling point of  $\alpha_1$ . If we denote by  $m_1$  the slope

of  $\alpha_1$  on  $[\gamma \circ \alpha_2 \circ \gamma(t_J), \gamma \circ \alpha_2 \circ \gamma(t^*)]$ , and by  $m_2$  the slope of  $\alpha_2$  on  $[\gamma(t_J), \gamma(t^*)]$ , then, by the update rules given in the proof of Lemma 4.22 we have

$$\gamma'(\alpha_2(\gamma(t_i))^+) = (m_1 \cdot m_2 \cdot \gamma'(t_i^+))^{-1}$$

for all  $i \geq J$ . So  $\gamma$  has a sampling point in  $\alpha_2(\gamma(t_i))$  only if  $\gamma$  has a sampling point in  $t_i$ . But then  $\gamma$  accumulates in  $t^*$  and in  $\alpha_2(\gamma(t^*))$ . Using the cyclic permutation (1.2) we see that  $\alpha_2 \circ \gamma \circ \alpha_1 \circ \gamma(t_i) = t_i$  holds for all  $t_i$ . Since the sequence  $\{t_i\}$  accumulates in  $t^*$ , and  $\alpha_1, \alpha_2 \in \mathcal{P}(\mathcal{K}_\infty)$  have no accumulation points, the same argumentation as above implies that  $\gamma$  accumulates in  $\alpha_1(\gamma(t^*))$ . The assumption  $t^* < \alpha_2(\gamma(t^*))$  implies

$$\alpha_1(\gamma(t^*)) < \alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma(t^*) = t^*,$$

which yields a contradiction to the assumption that  $t^*$  is the first accumulation point.

Now assume  $t^* > \alpha_2(\gamma(t^*))$ , or equivalently, by (4.36),  $t^* < \alpha_1(\gamma(t^*))$ . Note that (4.36) is satisfied if and only if  $\alpha_2 \circ \gamma \circ \alpha_1 \circ \gamma = \text{id}$ , by (1.2). Assume that  $t^*$  is the smallest accumulation point of  $\gamma$ . Then following the above argumentation, we conclude that  $\alpha_2(\gamma(t^*)) < t^*$  is a smaller accumulation point; a contradiction to the assumption that  $t^*$  is the smallest one.

Thus, we conclude

$$\alpha_2(\gamma(t^*)) = t^* = \alpha_1(\gamma(t^*)).$$

Now let  $t^*$  be an isolated accumulation point of  $\gamma$  and let  $t_-^*$  denote the previous accumulation point of  $t^*$  (i.e., there exists no other accumulation point of  $\gamma$  in  $(t_-^*, t^*)$ ), and let  $\{t_i\}_{i \in \mathbb{N}}$  be the (unique and strictly increasing) sequence of all sampling points  $t_i \in (t_-^*, t^*)$  that converges to  $t^*$ . We want to prove the statement by induction. So assume that the claim holds for all previous accumulation points  $\tilde{t}$ , i.e.,  $\alpha_1(\gamma(\tilde{t})) = \tilde{t} = \alpha_2(\gamma(\tilde{t}))$ . Then by the same arguments as before, under the assumption  $t^* < \alpha_2(\gamma(t^*))$  we conclude that  $\alpha_1(\gamma(t^*)) < t^*$  and  $\alpha_1(\gamma(t^*))$  is an accumulation point of  $\gamma$ . But then, by assumption,  $\alpha_1(\gamma(t^*))$  is one of the previous accumulation points and hence satisfies  $\alpha_1(\gamma(t^*)) = t^* = \alpha_2(\gamma(t^*))$ . This contradicts the assumption  $t^* < \alpha_2(\gamma(t^*))$  and concludes the proof.  $\square$

As we see from the proof of Proposition 4.25, an accumulation point  $t^* > 0$  implies  $t^* = \alpha_1(\gamma(t^*)) = \alpha_2(\gamma(t^*))$ . This observation can also be shown in a more general form.

**Proposition 4.26.** *Let  $k = 2$ ,  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\gamma \in \mathcal{K}_\infty$  be a solution of (4.36). If  $\alpha_1$  and  $\alpha_2$  intersect in  $\gamma(t^*)$ , then  $\alpha_1(\gamma(t^*)) = t^* = \alpha_2(\gamma(t^*))$ .*

*Proof.* Let  $y$  be an intersection point of  $\alpha_1, \alpha_2$ , i.e.,  $\alpha_1(y) = \alpha_2(y)$ . By strict increase and continuity of  $\gamma$  there exists a  $\tilde{t}$  such that  $\gamma(\tilde{t}) = y$ . We claim that  $\alpha_1(\gamma(\tilde{t})) =$

$\tilde{t} = \alpha_2(\gamma(\tilde{t}))$ . To show this assume that  $\alpha_1(\gamma(\tilde{t})) > \tilde{t}$ . Then we conclude

$$\tilde{t} = \alpha_1 \circ \gamma \circ (\alpha_2 \circ \gamma(\tilde{t})) = \alpha_1 \circ \gamma \circ (\alpha_1 \circ \gamma(\tilde{t})) > \alpha_1(\gamma(\tilde{t})) > \tilde{t},$$

a contradiction. The case  $\alpha_1(\gamma(\tilde{t})) < \tilde{t}$  follows by the same arguments, thus  $\alpha_1(\gamma(\tilde{t})) = \tilde{t}$ , and the results follows since  $\gamma(\tilde{t})$  satisfies  $\alpha_1(\gamma(\tilde{t})) = \alpha_2(\gamma(\tilde{t}))$ .  $\square$

If we do not assume  $\alpha_1, \alpha_2$  to be of class  $\mathcal{P}(\mathcal{K}_\infty)$  then Proposition 4.25 does not hold anymore. This is shown by the next example.

**Example 4.27.** Let us consider the equation (4.36) for  $k = 2$  with  $\alpha_1 = 3 \text{id} \in \mathcal{F}(\mathcal{K}_\infty)$  and  $\alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$  defined as follows. For any  $t \in [10(1 - \frac{1}{n}), 10(1 - \frac{1}{n+1}))$  with  $n \in \mathbb{N}$ , we define the right hand derivative

$$\alpha'_2(t^+) := \begin{cases} \frac{1}{3} & n \text{ is odd} \\ 3 & n \text{ is even} \end{cases}.$$

Then  $t^* = 10$  is an accumulation point of  $\alpha_2$ . For all  $t \geq 10$  we define  $\alpha'_2(t^+) := \frac{1}{2}$ . We note that  $\alpha_1$  and  $\alpha_2$  do not intersect, but the solution  $\gamma$ , obtained by applying Procedure 4.24, has accumulation points, see Figure 4.3.  $\triangleleft$

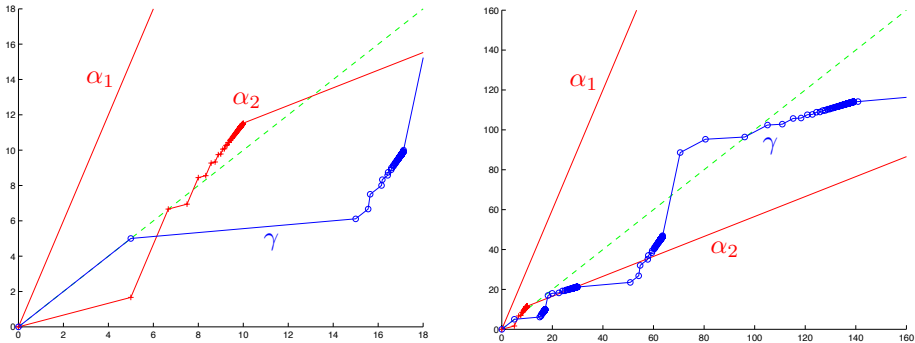


Figure 4.3: An accumulation point of  $\alpha$  can lead to infinitely many accumulation points of  $\gamma$ .

The phenomenon that occurred in Example 4.27 is characterized in the following proposition.

**Proposition 4.28.** *Let  $\gamma$  be a solution of (4.36) for  $k = 2$  and  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$ . If  $\gamma$  has an accumulation point  $t^* > 0$  and  $\alpha_1(\gamma(t^*)) \neq \alpha_2(\gamma(t^*))$ , then  $\alpha_1$  or  $\alpha_2$  has at least one accumulation point.*



*Proof.* Let without loss of generality  $t^* > 0$  be the first accumulation point of  $\gamma$  that arises. Else consider the reasoning from Proposition 4.25. By assumption  $\alpha_1(\gamma(t^*)) \neq \alpha_2(\gamma(t^*))$ . But this implies  $t^* \neq \alpha_2(\gamma(t^*))$  since  $\gamma$  solves (4.36). Since  $\gamma$  is unique by Lemma 4.21 it follows that  $\gamma$  satisfies the update rules (4.42) in the proof of Lemma 4.22. Let  $\{t_i\}_{i \in \mathbb{N}}$  be the strictly increasing sequence of sampling points of  $\gamma$  with limit  $t^*$ . Here we have to distinguish between the following two cases:

Assume  $t^* > \alpha_2(\gamma(t^*))$ . Then there exists an index  $J \in \mathbb{N}$  such that for all  $i \geq J$  it holds  $\alpha_2(\gamma(t_i)) < \alpha_2(\gamma(t^*)) < t_i < t^*$ . The update rule (4.42) for the case  $T > \alpha_2(\gamma(T))$  is applicable and we obtain

$$\gamma'(t_{i+1}^+) = (\alpha'_1(\gamma \circ \alpha_2 \circ \gamma(t_{i+1})^+) \cdot \gamma'(\alpha_2 \circ \gamma(t_{i+1})^+) \cdot \alpha'_2(\gamma(t_{i+1})^+))^{-1}.$$

Since  $t^*$  was the smallest accumulation point of  $\gamma$ ,  $\gamma'(\alpha_2(\gamma(t_{i+1})^+))$  is constant for  $i$  large enough. So  $\alpha_1$  accumulates in  $\gamma \circ \alpha_2 \circ \gamma(t^*)$  or  $\alpha_2$  accumulates in  $\gamma(t^*)$ .

Assume  $t^* < \alpha_2(\gamma(t^*))$ . Then there exists an index  $J \in \mathbb{N}$  such that for all  $i \geq J$  it holds  $t_i < t^* < \alpha_2(\gamma(t_i)) < \alpha_2(\gamma(t^*))$ . Let  $\tilde{t}_i$  be uniquely defined by the relation  $\alpha_2(\gamma(\tilde{t}_i)) = t_i < \alpha_2(\gamma(t_i))$ . By strict monotonicity  $\tilde{t}_i < t_j$  and in particular  $\tilde{t}_i < \alpha_2(\gamma(\tilde{t}_i))$ . Furthermore, by definition we have  $\tilde{t}_i \rightarrow \tilde{t}$  for  $i \rightarrow \infty$ , and hence  $\alpha_2 \circ \gamma(\tilde{t}) = t^*$  by continuity. But then from the update rule (4.42) for  $T < \alpha_2(\gamma(T))$  we conclude

$$\gamma'(t_{i+1}^+) = \gamma'(\alpha_2(\gamma(\tilde{t}_{i+1})^+)) = (\alpha'_1(\gamma \circ \alpha_2 \circ \gamma(\tilde{t}_{i+1})^+) \cdot \alpha'_2(\gamma(\tilde{t}_{i+1})^+) \cdot \gamma'(\tilde{t}_{i+1}^+))^{-1}.$$

From  $\alpha_2(\gamma(\tilde{t})) = t^* < \alpha_2(\gamma(t^*))$  we obtain  $\tilde{t} < t^*$ , and since  $t^*$  was the smallest accumulation point of  $\gamma$ ,  $\gamma'(\tilde{t}_{i+1}^+)$  is constant for  $i$  large enough. So  $\alpha_1$  accumulates in  $\gamma \circ \alpha_2 \circ \gamma(\tilde{t}) = \gamma(t^*)$  or  $\alpha_2$  accumulates in  $\gamma(\tilde{t})$ .  $\square$

If we restrict  $\alpha_1, \alpha_2$  to the class of  $\mathcal{P}(\mathcal{K}_\infty)$ , by Proposition 4.25, the existence of an accumulation point of  $\gamma$  implies that  $\alpha_1$  and  $\alpha_2$  intersect. From a numerical point of view this is important, since in a first step we can compute the intersection points  $y$  of  $\alpha_1$  and  $\alpha_2$ . From Proposition 4.25 and 4.26 we conclude that  $t^* := \alpha_1(y)$  is a possible accumulation point,  $\gamma(t^*) = y$  and since  $\alpha_2(\gamma(t^*)) = t^*$ , we have  $\gamma'((t^*)^+) = (\alpha'_1(\gamma(t^*)^+) \cdot \alpha'_2(\gamma(t^*)^+))^{-1/2}$ . So if  $t^*$  is an accumulation point we can (numerically) compute the solution on  $[0, t^*]$ , and then extend this solution as in (4.42) with  $\tau = t^*$  and repeat the Procedure 4.24.

In a similar way we can argue if  $\alpha_1, \alpha_2$  are of class  $\mathcal{R}(\mathcal{K}_\infty)$ . Again, intersection points of  $\alpha_1$  and  $\alpha_2$  lead to possible accumulation points, but also accumulation points of  $\alpha_1$  and  $\alpha_2$ . Note that there may exist infinitely many accumulation points of a solution  $\gamma$ . This can be seen e.g. in (4.42) case 2. If  $t^*$  is an accumulation point

of  $\gamma$ , but neither  $\gamma(t^*)$  is an accumulation point of  $\alpha_2$  nor is  $\gamma \circ \alpha_2 \circ \gamma(t^*)$  for  $\alpha_1$  then also  $\alpha_2(\gamma(t^*))$  is an accumulation point of  $\gamma$ .

We further note that neither the conditions in Proposition 4.25 nor in Proposition 4.28 are necessary. This is shown in the next example.

**Example 4.29.** Consider the functional equation (4.36) for  $k = 2$  with

$$\alpha_1(t) = \begin{cases} \frac{1}{3}t & t \in [0, 3] \\ 3t - 8 & t \in [3, \infty) \end{cases} \quad \text{and} \quad \alpha_2(t) = \begin{cases} 3t & t \in [0, 1] \\ \frac{1}{3}t + \frac{8}{3} & t \in [1, \infty) \end{cases} .$$

Then  $\alpha_1, \alpha_2 \in \mathcal{P}(\mathcal{K}_\infty)$  intersect, but  $\gamma = \text{id}$  has no accumulation point. In a similar way it is possible to construct  $\alpha_1, \alpha_2 \in \mathcal{R}(\mathcal{K}_\infty)$  that accumulate, but the solution  $\gamma = \text{id}$  of the corresponding functional equation (4.36) has no accumulation point. The functions  $\alpha_i$  and  $\gamma$  are shown in Figure 4.4.  $\triangleleft$

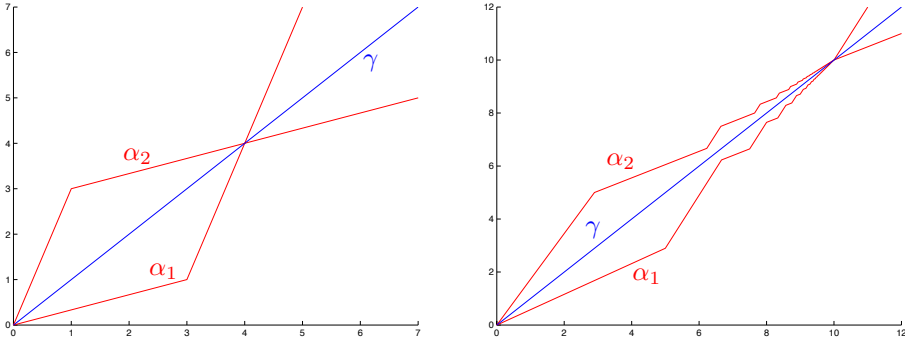


Figure 4.4: Neither intersection of  $\alpha_1$  and  $\alpha_2$  nor accumulation has to imply accumulation of  $\gamma$ .

#### 4.2.3 Solutions of iterative functional $\mathcal{K}_\infty$ -equations for $k \geq 2$

In this section we generalize the results of the previous section. To avoid confusingly long terms, we establish some notation first.

Let  $c = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_i \in \mathcal{K}_\infty$ ,  $i \in \{1, \dots, k\}$ , which we abbreviate by  $c \in \mathcal{K}_\infty^k$ . For  $\gamma \in \mathcal{K}_\infty$  we define

$$\bigotimes_k c^\gamma(t) := \alpha_1 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma(t). \tag{4.44}$$

With this notation the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) is equivalent to

$$\bigotimes_k c^\gamma = \text{id}. \tag{4.36'}$$

*Remark 4.30.* Although the notation  $\bigotimes_k c^\gamma$  in (4.44) is similar to the notation  $\bigotimes c^\gamma$  in (4.10) there is an important difference. In (4.44),  $c = (\alpha_1, \dots, \alpha_k)$  is simply a vector of  $k$   $\mathcal{K}_\infty$ -functions, which are in each case applied on  $\gamma \in \mathcal{K}_\infty$ , and then aggregated according to (4.44). Otherwise,  $c = (\gamma_{i_0 i_1}, \dots, \gamma_{i_{k-1} i_k})$  in (4.10) corresponds to a  $k$ -cycle in  $\Gamma^S$  with  $s = \gamma(1, \dots, 1)$ , hence the structural entries in the cycle are replaced by  $\gamma \in \mathcal{K}_\infty$ .

However,  $k \in \mathbb{N}$  in (4.44) is the number of functions  $\alpha_i$ , and thus,  $\gamma \in \mathcal{K}_\infty$  appears  $k$  times in (4.44). On the other hand,  $k \in \mathbb{N}$  in (4.10) is the length of the cycle  $c$ . Hence, the number of how often the function  $\gamma \in \mathcal{K}_\infty$  appears in (4.10) may be smaller than  $k \in \mathbb{N}$ .

Nevertheless, the iterative functional  $\mathcal{K}_\infty$ -equation  $\bigotimes c^\gamma = \text{id}$  with  $\bigotimes c^\gamma$  from (4.10) can always be written in the form (4.36') as outlined in (4.11).  $\triangleleft$

Given  $\alpha_1, \dots, \alpha_k \in \mathcal{K}_\infty$ , we define for any  $t \geq 0$

$$\begin{aligned} \eta^1(t) &:= t \\ \eta^2(t) &:= \alpha_k \circ \gamma(t) \\ \eta^3(t) &:= \alpha_{k-1} \circ \gamma \circ \alpha_k \circ \gamma(t) \\ &\vdots \\ \eta^k(t) &:= \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma(t). \end{aligned}$$

Then for each  $t \geq 0$  we define

$$\eta_{\max}(t) := \max_{j \in \{1, \dots, k\}} \eta^j(t) \tag{4.45}$$

and its underlying index set

$$J_{\max}(t) = \{j \in \{1, \dots, k\} : \eta^j(t) = \eta_{\max}(t)\}.$$

By construction,  $\eta_{\max}, \eta^j \in \mathcal{K}_\infty$  for all  $j \in \{1, \dots, k\}$ , since  $\gamma, \alpha_1, \dots, \alpha_k \in \mathcal{K}_\infty$ . By  $\eta_{\max, l}$  we denote the function in (4.45), which is defined for the equation  $\bigotimes c^{\gamma^l} = \text{id}$ . With this notation Lemma 4.21 may be generalized as follows.

**Lemma 4.31** (Unique representation). *Let  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$  for  $i \in \{1, \dots, k\}$ ,  $T > 0$ ,  $I = [0, T]$ . If for  $\gamma_1, \gamma_2 \in \mathcal{R}(\mathcal{K}_\infty)$  we have*

$$\forall t \in I : \bigotimes_k c^{\gamma^l}(t) = t, \quad l \in \{1, 2\}, \tag{4.46}$$

*then the restriction to  $\bar{I} := \eta_{\max, 1}(I) \cup \eta_{\max, 2}(I)$  satisfies  $\gamma_1|_{\bar{I}} \equiv \gamma_2|_{\bar{I}}$ . In particular, there exists at most one right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  of (4.36).*

*Proof.* Let  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$  for  $i \in \{1, \dots, k\}$ , or, in another notation,  $c = (\alpha_1, \dots, \alpha_k) \in \mathcal{R}(\mathcal{K}_\infty)^k$ . Assume that there exist two right affine functions  $\gamma_1$  and  $\gamma_2$  satisfying (4.46). In particular  $\gamma_1'(0^+) = \gamma_2'(0^+)$ , which follows by the same argument as in Lemma 4.21. So there exists an interval  $[0, a]$  with  $\gamma_1(t) = \gamma_2(t)$  for all  $t \in [0, a]$ , and seeking a contradiction assume that there exists an  $\varepsilon > 0$  small enough such that

$$\gamma_1(t) < \gamma_2(t) \quad \text{for all } t \in (a, a + \varepsilon). \quad (4.47)$$

Fix  $t \in (a, a + \varepsilon)$ . If  $\eta_{\max, i}(I) \subset [0, a]$ ,  $i \in \{1, 2\}$  there is nothing to show. Else, since  $\eta_{\max, 2} \in \mathcal{K}_\infty$ , there exists a  $\tilde{t} \leq t$  with  $\eta_{\max, 2}(\tilde{t}) = t$ . By choice of  $t$  we get

$$t = \eta_{\max, 2}(\tilde{t}) \geq \eta_{\max, 1}(\tilde{t}). \quad (4.48)$$

Let  $l := \max\{J_{\max, 2}(\tilde{t})\}$ , i.e.,  $\eta_2^l(\tilde{t}) = \eta_{\max, 2}(\tilde{t})$ . Since (4.36) is equivalent to (4.36'), we conclude

$$\begin{aligned} \tilde{t} &= \bigotimes_k c^{\gamma_2}(\tilde{t}) = (\alpha_1 \circ \gamma_2) \circ \dots \circ (\alpha_{k-l+1} \circ \gamma_2) \circ (\alpha_{k-l+2} \circ \gamma_2) \circ \dots \circ (\alpha_k \circ \gamma_2)(\tilde{t}) \\ &= (\alpha_1 \circ \gamma_2) \circ \dots \circ (\alpha_{k-l+1} \circ \gamma_2) \circ \eta_{\max, 2}(\tilde{t}) \\ &= (\alpha_1 \circ \gamma_2) \circ \dots \circ (\alpha_{k-l} \circ \gamma_2) \circ (\alpha_{k-l+1} \circ \gamma_2)(t) \\ &\stackrel{(4.47)}{>} (\alpha_1 \circ \gamma_2) \circ \dots \circ (\alpha_{k-l} \circ \gamma_2) \circ (\alpha_{k-l+1} \circ \gamma_1)(t) \\ &\stackrel{(4.48)}{\geq} (\alpha_1 \circ \gamma_2) \circ \dots \circ (\alpha_{k-l} \circ \gamma_2) \circ (\alpha_{k-l+1} \circ \gamma_1) \circ \eta_{\max, 1}(\tilde{t}) \\ &\geq \bigotimes_k c^{\gamma_1}(\tilde{t}) = \tilde{t}, \end{aligned}$$

where the last step follows since  $l$  is the maximal element in  $J_{\max}(\tilde{t})$ , which implies that  $(\alpha_i \circ \gamma_2)$ ,  $i \in \{1, \dots, k-l\}$  is evaluated in values less or equal to  $t \in [0, a + \varepsilon]$ . But in this case  $\gamma_2 \geq \gamma_1$  by (4.47) and since  $\gamma_1$  and  $\gamma_2$  are identical on  $[0, a]$ . But this is a contradiction to the assumption that  $\gamma_1$  and  $\gamma_2$  are distinct on  $[a, a + \varepsilon]$ . So  $\gamma_1$  and  $\gamma_2$  are identical on  $[0, t]$ ,  $t > a$ .

To show that  $\gamma_1$  and  $\gamma_2$  are identical on  $\bar{I}$  assume that  $\eta_{\max, i}(I) \not\subset [0, a]$  for at least one  $i \in \{1, 2\}$ . Then this implies the existence of a  $\tilde{t} \in I$  with  $\eta_{\max, 2}(\tilde{t}) = t > a$  satisfying (4.47) and (4.48). Again, this leads to a contradiction, showing that  $\eta_{\max, 2}(I) \subset [0, a]$ . Hence,  $\gamma_1|_{\bar{I}} \equiv \gamma_2|_{\bar{I}}$  for  $\bar{I} := \eta_{\max, 1}(I) \cup \eta_{\max, 2}(I)$ .  $\square$

A generalization of Lemma 4.22 is given now.

**Lemma 4.32** (Existence). *For the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) with  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $i \in \{1, \dots, k\}$ , there exists at least one right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$ .*

*Proof.* The proof follows the same steps as the proof of Lemma 4.22 by constructing a right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  for (4.36) with  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $i \in \{1, \dots, k\}$ .

Assume that  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$  and the interval  $I = [0, T]$  are such that

$$\forall t \in I : \bigotimes_k c^\gamma(t) = t, \quad (4.49)$$

and let  $I$  be the maximal closed interval on which (4.49) holds. This implies  $\bigotimes_k c^\gamma(T^+) \neq 1$  as otherwise (4.49) can be extended to the right.

To extend  $\gamma$  we define

$$m_\alpha(t) := \prod_{j=1}^k \alpha'_j (\gamma \circ \eta^{k+1-j}(t)^+)$$

as the product of slopes of the  $\alpha_i$  that occur by evaluating  $\bigotimes_k c^\gamma$  in  $t$ . And we define

$$m_\gamma^{\notin J}(t) := \prod_{j \notin J_{\max}(t)} \gamma'(\eta^j(t)^+)$$

as the product of slopes of  $\gamma$  that remain as they are. Then the update rule reads as follows. Set

$$\begin{aligned} \tau &:= \eta_{\max}(T) \\ m &:= (m_\alpha(T) \cdot m_\gamma^{\notin J}(T))^{-1/\#J_{\max}(T)} > 0, \end{aligned}$$

where  $\#J_{\max}(T)$  denotes the cardinality of  $J_{\max}(T)$ . Define  $\tilde{\gamma} \in \mathcal{R}(\mathcal{K}_\infty)$  by setting

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t) & , \quad t \in [0, \tau] \\ m(t - \tau) + \gamma(\tau) & , \quad t \in [\tau, \infty) \end{cases}.$$

Again, by construction,

$$\begin{aligned} \left( \bigotimes_k c^{\tilde{\gamma}} \right)'(T^+) &= \prod_{j=1}^k \left( \alpha'_j (\gamma \circ \eta^{k+1-j}(T)^+) \cdot \gamma'(\eta^{k+1-j}(T)^+) \right) \\ &= \prod_{j=1}^k \alpha'_j (\gamma \circ \eta^{k+1-j}(T)^+) \cdot \prod_{j \in J_{\max}} \gamma'(\eta^j(T)^+) \cdot \prod_{j \notin J_{\max}} \gamma'(\eta^j(T)^+) \\ &= m_\alpha(T) \cdot m^{\#J_{\max}} \cdot m_\gamma^{\notin J}(T) \\ &= 1, \end{aligned}$$

and since  $\gamma_i, \alpha_1, \dots, \alpha_k$  are right affine, this holds on an interval  $[0, T + \varepsilon)$  with  $\varepsilon > 0$  suitable, and the solution is extended.

To start set  $\gamma'_0(0^+) := \left( \prod_{i=1}^k \alpha'_i(0^+) \right)^{-1/k}$ . Then there exists an  $\varepsilon > 0$  such that (4.49) holds on  $[0, \varepsilon]$ .

The procedure described above extends to a solution  $\gamma$  of (4.36) by the same arguments as in Lemma 4.22.  $\square$

The main result of this section now follows directly.

**Theorem 4.33** (Existence of a unique right affine solution). *For the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) with  $\alpha_i \in \mathcal{R}(\mathcal{K}_\infty)$ ,  $i \in \{1, \dots, k\}$ , there exists exactly one right affine solution  $\gamma \in \mathcal{R}(\mathcal{K}_\infty)$ .*

*Proof.* This follows directly from Lemma 4.31 and Lemma 4.32. □

*Remark 4.34.* Consider the iterative functional  $\mathcal{K}_\infty$ -equation (4.36) with  $\alpha_1, \dots, \alpha_k \in \mathcal{K}_\infty$ , or equivalently,  $\bigotimes c_k^\gamma$  with  $c = (\alpha_1, \dots, \alpha_k)$ . We assume that  $\alpha'_i(0) < \infty$  for all  $i \in \{1, \dots, k\}$ . Then the functions  $\alpha_i$  can be approximated from above by a function  $\tilde{\alpha}_i \geq \alpha_i$  that is of class  $\mathcal{R}(\mathcal{K}_\infty)$ , for each  $i \in \{1, \dots, k\}$ . By Theorem 4.33, there exists a unique solution  $\tilde{\gamma} \in \mathcal{R}(\mathcal{K}_\infty)$  of the approximated equation  $\bigotimes \tilde{c}_k^\gamma = \text{id}$  with  $\tilde{c} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)$ . By monotonicity, we have  $\bigotimes c_k^\gamma \leq \bigotimes \tilde{c}_k^\gamma = \text{id}$ . This inequality is particularly useful for Theorem 4.3 as it implies that if we can approximate the entries of  $\Gamma_1$  and  $\Gamma_2$  from above by functions of class  $\mathcal{R}(\mathcal{K}_\infty) \cup \{0\}$ , then  $\bar{\gamma}$  defined in (4.12) is admissible in the sense of Definition 4.1. ◁

### 4.3 Notes and references

The topic of constructing gains that satisfy a small-gain condition was motivated by the small-gain results in Chapters 2 and 3. Therein, small-gain theorems for interconnected discrete-time systems have been derived that rely on a (strong) small-gain condition of the form (1.10) (resp. (1.11)). Moreover, there exist several small-gain results also for interconnected continuous-time and hybrid systems that use the *same* small-gain condition, see e.g. [22–25, 124].

In other publications, the small-gain conditions are *different* to the ones used herein. However, we list a few of those references to indicate how the results derived in this chapter can also be applied in these cases.

In [21] the authors present a small-gain condition of the form  $\Gamma \circ A^{-1}(s) \not\geq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$ . Here the function  $A^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  serves as a scaling operator for the gains  $\gamma_{ij}$  of the matrix  $\Gamma$ . Hence, by defining  $\tilde{\Gamma} := \Gamma \circ A^{-1}$ , and noticing that the weights in  $\Gamma_{sub}^*$  can also be chosen nonlinear (see Remark 4.15), the results of this chapter can be applied. In other publications as e.g. [67, 75, 99] the authors do not propose a gain operator of the form  $\Gamma_\mu$  as it is done here, but the small-gain condition is expressed in terms of a cycle condition. So it is straightforward to collect the gains into a matrix and write the cycle condition in the equivalent small-gain condition  $\Gamma_\oplus \not\geq \text{id}$ , see Proposition 1.29. Moreover, sometimes a small-gain condition of the form (1.10) is not explicitly stated, but does also apply. For instance, the small-gain condition in [40] is the same as (1.10), but simply written in a slightly different form.

Summarizing, the proposed gain construction methods are not restricted to interconnected systems as considered in Chapters 2 and 3. We emphasize that these methods may be useful in a variety of different setups.

The whole procedure of this chapter relies on the assumption that it is possible to determine the gain of an interconnection, thus to determine the gain matrices  $\Gamma_1$  and  $\Gamma_2$ . We do not treat this problem here, but note that results in this direction have been obtained in [45, 63, 144]. In the latter references [63, 144] tight integral ISS bounds for nonlinear systems are investigated.

In the maximization case (Section 4.1.1) we have seen that a maximal gain  $\bar{\gamma}$  can be obtained by solving an iterative functional  $\mathcal{K}_\infty$ -equation for each cycle (Theorem 4.3). Interestingly, as solutions of iterative functional  $\mathcal{K}_\infty$ -equation are, in general, not unique (Example 4.20), also maximal gains are not unique. Uniqueness is only obtained if the gains are required to be in a certain subclass of the class of  $\mathcal{K}_\infty$ -function. The subclass  $\mathcal{R}(\mathcal{K}_\infty)$  that we propose in Section 4.2 seems to be promising as solutions of iterative functional  $\mathcal{R}(\mathcal{K}_\infty)$ -equations exist, are unique within  $\mathcal{R}(\mathcal{K}_\infty)$ , and can be numerically computed as we have shown in Section 4.2. Moreover, if we consider the gains to be of class  $\mathcal{R}(\mathcal{K}_\infty)$  then there exists a unique maximal gain in  $\mathcal{R}(\mathcal{K}_\infty)$ , see Remark 4.5.

To compute maximal gains in the linear summation case (Section 4.1.2) we make use of the concept of stability radii. We refer to the textbook [60] for a comprehensive overview and further references. Of particular importance is [61], where the authors consider positive linear systems, i.e., systems of the form  $x(k+1) = Ax(k)$ ,  $k \in \mathbb{N}$  with nonnegative matrix  $A$ . In this case, the stability radius can be explicitly computed. As we have shown that the maximal gain in the linear summation case is equal to the stability radius, the results of [61] can be directly employed. We further refer to the appendix Section A.2, where we present definitions and results related to stability radii for positive linear systems in more detail.

Constructing admissible gains in the general case (Theorem 4.10) requires the existence of an  $\Omega$ -path. These paths can be computed with the recently developed algorithm [37] or, alternatively, [125]. As shown in the examples of Section 4.1.3 the obtained admissible gain is conservative. This conservatism comes from the fact that we first fix the path  $\sigma^1$  (which is not unique), and then fix the path  $\sigma^2$  (that we can choose). Not till then, we compute an admissible gain. So to reduce conservatism in the general case it would seem reasonable to find construction methods, which do not rely on  $\Omega$ -paths.

Iterative functional equations, as considered in Section 4.2, have been extensively studied in the second half of the 20th century and there exists an extensive literature; see for example the survey papers [6, 138] and the references therein. The history of

the subject goes back to work of Charles Babbage in 1815. The particular case of iterative functional  $\mathcal{K}_\infty$ -equations that we address has, to the best of the author's knowledge, not previously been studied in the literature.

Special cases of the iterative functional equation we proposed can be found in [85], where the author considers the problem of solving the functional equation  $x(x(t)) = f(t)$  with  $f$  piecewise linear from a numerical perspective. Solutions are computed by considering fixed points of  $f$ , which, in our case, correspond to intersection points of the functions  $\alpha_1, \alpha_2$ . Another special case is the functional equation  $x^N(t) = f(t)$ , for  $t \in \mathbb{R}$  and  $N \in \mathbb{N}$ , which is known as *finding iterative roots* (see [6,87]). From [87] it is known that iterative roots of class  $\mathcal{K}_\infty$  exist if the function  $f$  is of class  $\mathcal{K}_\infty$ ; such solutions are not unique, in general.



## Extensions and outlook

In this chapter, we give a variety of possible extensions of some results in this thesis. On the one hand, we state results that build upon the findings in this thesis, but have not been included for reasons of brevity. On the other hand, we describe open problems related to some of the results in this thesis that can be seen as a starting point for future research.

### 5.1 On computing the finite-step number $M \in \mathbb{N}$

In Chapter 2 we have studied the relation between global finite-step Lyapunov functions and global Lyapunov functions. A distinctive difference between global Lyapunov functions and global finite-step Lyapunov functions is that in the latter one we also require the knowledge of a finite-step number  $M \in \mathbb{N}$ , see Definition 2.6. In particular, we know from Corollary 2.16 that any norm is a global finite-step Lyapunov function for a system with globally exponentially stable origin. However, this result only shows existence of a suitably large finite-step number  $M$ . We note that the knowledge of  $M$  is particularly required for the construction of a global Lyapunov function in Theorem 2.21 and Theorem 2.22. Hence, deriving methods to compute the number  $M \in \mathbb{N}$  for different system classes is an important task.

In Section 2.4 we have studied several classes of dynamical systems by using the converse Lyapunov theorems from Section 2.2. In doing so, we have derived systematic ways to compute the finite-step number  $M$  for the classes of conewise linear and linear systems. In both cases (e.g. Procedure 2.50 for continuous conewise linear systems and Procedure 2.58 for linear systems), we have obtained conditions for computing  $M$  that invoke the matrices  $A_i$  resp.  $A$  of the dynamics. Hence, these

conditions on  $M$  are independent of the initial state, and thus easier to check. For general nonlinear systems we cannot expect to derive e.g. algebraic conditions for the number  $M$ . Nevertheless, we think that there are several interesting system classes for which procedures to compute a suitable finite-step number  $M$  can be derived. For instance, the class of discrete-time Lur'e systems [44, 100] seems to be feasible. Roughly speaking, a Lur'e system has both a linear and a nonlinear part. In the stability analysis of Lur'e systems as e.g. [44], the nonlinearity is assumed to be cone bounded. Thus, a similar reasoning as in Section 2.4.2 can be used to derive a method for computing  $M \in \mathbb{N}$  for discrete-time Lur'e systems with globally asymptotically stable origin.

## 5.2 Lyapunov functions for difference inclusions

In Section 2.4.2 we have studied conewise linear systems of the form (2.41) (discontinuous dynamics) and (2.47) (continuous dynamics). For discontinuous dynamics we cannot use the form (2.47) as we have to ensure well-posedness of the system. Well-posedness in this case means that the image of the right-hand side

$$G(\xi) := A_i \xi, \quad \xi \in \text{relint}(\mathcal{C}_i) \quad \text{resp.} \quad \xi \in \mathcal{C}_i$$

has to be uniquely determined for any  $\xi \in \mathbb{R}^n$ . For discontinuous dynamics  $G$  there exist points  $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$ ,  $i \neq j$  with  $A_i \xi \neq A_j \xi$ . To ensure well-posedness, we have extended the set of cones  $\mathcal{C}_i$  such that any point  $\xi \in \mathbb{R}^n$  lies in the relative interior of a unique cone  $\mathcal{C}_i$ , see (2.40).

Alternatively, we can allow the right-hand side  $G$  to be *set-valued*, i.e.,

$$G(\xi) := \{A_i \xi : \xi \in \mathcal{C}_i\},$$

where the set of cones  $\{\mathcal{C}_i\}_{i \in \{1, \dots, l\}}$  satisfies  $\bigcup_i \mathcal{C}_i = \mathbb{R}^n$ . Then for any point  $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$  we have  $\{A_i \xi, A_j \xi\} \subset G(\xi)$ . In this respect, we can regard a conewise linear system as a *difference inclusion*

$$x(k+1) \in G(x(k)), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \quad (5.1)$$

Difference inclusions have been studied e.g. in [27, 80, 81]. In [27] the authors consider difference inclusions under the assumption that the right-hand side  $G$  is *upper semicontinuous* [27, Definition 1], the image  $G(\xi)$  is nonempty and compact for any point  $\xi \in \mathbb{R}^n$ , and  $G$  is homogeneous of degree one. The main result in [27] is that  $\mathcal{KL}$ -stability (or, equivalently, GAS as defined in this thesis) of the difference inclusion (5.1) implies the existence of a homogeneous Lyapunov function. In [80, 81] the authors do impose the same assumptions as [27] except homogeneity and they derive a converse Lyapunov theorem for difference inclusions.

To define a Lyapunov function for the difference inclusion (5.1), let  $x(\cdot, \xi)$  denote a solution of the difference equation (5.1) from the initial value  $\xi \in \mathbb{R}^n$ . As the map  $G$  in (5.1) is set-valued there might exist more than one solution. So we denote the set of all solutions starting from  $\xi \in \mathbb{R}^n$  by  $\mathcal{S}(\xi)$ . A Lyapunov function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  for the difference inclusion (5.1) in its simplest<sup>1</sup> form has to satisfy the following two properties:

- (i)  $W$  is proper and positive definite;
- (ii) there exists  $\mu \in (0, 1)$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$\sup_{g \in G(\xi)} W(g) \leq \mu W(\xi).$$

With regard to the results obtained in Chapter 2, we propose the following definition.

**Definition 5.1.** A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a *global finite-step Lyapunov function* for the difference inclusion (5.1) if

- (i)  $V$  is proper and positive definite, i.e., it satisfies condition (i) of Definition 2.6;
- (ii) there exists a finite  $M \in \mathbb{N}$  and a positive definite function  $\rho < \text{id}$  such that for all  $\xi \in \mathbb{R}^n$  we have

$$\sup_{x(\cdot, \xi) \in \mathcal{S}(\xi)} V(x(M, \xi)) \leq \rho(V(\xi)).$$

If the right-hand side  $G$  is upper semicontinuous, and  $G(\xi)$  is nonempty and compact for any  $\xi \in \mathbb{R}^n$  then from [80, Theorem 10] we obtain that the existence of a global finite-step Lyapunov function is *equivalent* to the difference inclusion being  $\mathcal{KL}$ -stable. Moreover, several other results from Chapter 2 can be carried over to the case of difference inclusions such as e.g.

1. Theorem 2.14,
2. Corollary 2.16,
3. Theorem 2.21 with

$$W(\xi) := \sum_{j=0}^{M-1} \sup_{x(\cdot, \xi) \in \mathcal{S}(\xi)} V(x(j, \xi)),$$

---

<sup>1</sup>In [27, 81] the authors additionally consider Lyapunov functions with respect to one resp. two measures.

4. Theorem 2.22 with

$$W(\xi) := \max_{j \in \{0, \dots, M-1\}} \sup_{x(\cdot, \xi) \in \mathcal{S}(\xi)} \rho^{j/M}(V(x(j, \xi))).$$

Proofs are omitted as they follow the same lines as the proofs of the results for difference equations (discrete-time systems). Hence, studying  $\mathcal{KL}$ -stability of difference inclusions (5.1) is possible via the results of Section 2.2. Moreover, we claim that the small-gain results developed in Section 2.3 can be extended to derive stability criteria for interconnected difference inclusions.

### 5.3 Stabilization using the finite-step idea

We consider the following discrete-time system

$$x(k+1) = G(x(k), u(k)), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m.$$

While inputs  $u(\cdot) \subset \mathbb{R}^m$  have been treated as disturbances throughout this thesis, we do now consider it as a *control input*. To be precise, we want to compute a *stabilizing control law*  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  to ensure that the origin of the feedback system with  $u(k) = f(x(k))$ ,

$$x(k+1) = \hat{G}(x(k)) := G(x(k), f(x(k))), \quad k \in \mathbb{N}, \quad (5.2)$$

is GAS.

The idea we propose is to design a stabilization control law such that all solutions of the feedback system (5.2) satisfy an estimate of the form

$$\|x(M, \xi)\| < \rho(\|\xi\|),$$

where  $\xi \in \mathbb{R}^n$ ,  $M \in \mathbb{N}$  is fixed, and the function  $\rho < \text{id}$  is positive definite. Such a stabilizing control law can be derived, for instance, by solving an optimization problem (similarly as done e.g. in *model predictive control* (MPC) [16, 47, 90]) or by applying stabilization schemes as *event-triggered control* [119, 137].

### 5.4 Construction of dissipative ISS Lyapunov functions

In Chapter 2, particularly Theorem 2.21 and Theorem 2.22, we have established two constructions of a global Lyapunov function from a global finite-step Lyapunov function. For systems with inputs that act as disturbances, this construction cannot be applied to obtain a dissipative ISS Lyapunov function from a dissipative finite-step ISS Lyapunov function. For instance, the sum construction in (2.11) of a system  $x(k+1) = G(x(k))$  is of the form

$$W(\xi) := \sum_{j=0}^{M-1} V(x(j, \xi)),$$

where  $V$  is a global finite-step Lyapunov function with finite-step number  $M \in \mathbb{N}$ . If  $V$  is a dissipative finite-step ISS Lyapunov function for the system  $x(k+1) = G(x(k), u(k))$  satisfying

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(\|u\|_\infty),$$

with  $(\text{id} - \rho) \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$ , then the function

$$W(\xi, u(\cdot)) := \sum_{j=0}^{M-1} V(x(j, \xi, u(\cdot)))$$

depends explicitly on the input sequence  $u(\cdot) \subset \mathbb{R}^m$ . Hence,  $W$  cannot be a dissipative ISS Lyapunov function by definition. Alternatively, if we define

$$W(\xi) := \sum_{j=0}^{M-1} V(x(j, \xi, 0))$$

then additional assumptions are required to guarantee that at least<sup>2</sup> an inequality of the form

$$W(G(\xi, \nu)) - W(\xi) = \sum_{j=0}^{M-1} (V(x(j, G(\xi, \nu), 0)) - V(x(j, \xi, 0))) < \sigma(\|\nu\|)$$

for all  $\xi \in \mathbb{R}^n, \nu \in \mathbb{R}^m$  holds, where  $\sigma \in \mathcal{K}$ . Eventually, it is an interesting task to elaborate how a dissipative ISS Lyapunov function can be constructed from the knowledge of a dissipative finite-step ISS Lyapunov function.

## 5.5 On solving iterative functional $\mathcal{K}_\infty$ -equations

In Section 4.2, and also in [36], we have studied iterative functional  $\mathcal{K}_\infty$ -equations of the form

$$\alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma = \text{id} \tag{5.3}$$

with  $k \in \mathbb{N}$ , and  $\alpha_i \in \mathcal{K}_\infty$  for  $i \in \{1, \dots, k\}$ . Clearly, for  $k = 1$  the iterative functional  $\mathcal{K}_\infty$ -equation (5.3) has a unique solution, which follows from Proposition 1.5. From Corollary 4.17 we know that at least for  $k = 2$  a solution of (5.3) exists, which is in general not unique.

Whereas Section 4.2 focusses on establishing the class of right affine  $\mathcal{K}_\infty$ -functions  $\mathcal{R}(\mathcal{K}_\infty)$ , in which solutions exist and are unique within this class, the following questions remain open.

---

<sup>2</sup>Note that this inequality is necessary but not sufficient for  $W$  being a dissipative ISS Lyapunov function.

Firstly, for  $k > 2$  there exists, to the best of the author's knowledge, no result guaranteeing the existence of solutions of (5.3), except for the special case  $\gamma^k = \alpha$  proposed in Proposition 4.16. Here, we present an idea for proving existence, which, unfortunately, is not successful. Nevertheless, we present this idea as it gives more insight into the problem.

Consider the set

$$\mathcal{Z} = \{\gamma \in \mathcal{K}_\infty : \alpha_1 \circ \gamma \circ \alpha_2 \circ \gamma \circ \dots \circ \alpha_k \circ \gamma \leq \text{id}\}.$$

We observe the following.

- (i) The set  $\mathcal{Z}$  is partially ordered, which is implied by the partial order of  $\mathcal{K}_\infty$ .
- (ii) Any chain  $\{\gamma_i\}_{i \in \mathbb{N}} \subset \mathcal{Z}$ , i.e., a sequence  $\{\gamma_i\}_{i \in \mathbb{N}} \subset \mathcal{Z}$  with  $\gamma_i \leq \gamma_{i+1}$ ,  $\gamma_i \neq \gamma_{i+1}$  for all  $i \in \mathbb{N}$ , is upper bounded.

Assume that any chain  $\{\gamma_i\}_{i \in \mathbb{N}} \subset \mathcal{Z}$  has an upper bound in  $\mathcal{Z}$ . Then, by Zorn's lemma [146], the set  $\mathcal{Z}$  has at least one maximal element. Clearly, if  $\gamma \in \mathcal{K}_\infty$  satisfies (5.3) then  $\gamma$  is a maximal element of  $\mathcal{Z}$ . However, maximal elements in  $\mathcal{Z}$  do *not* have to satisfy (5.3) as we show in the next example.

**Example 5.2.** Let  $\epsilon \in (0, \frac{1}{3})$  and consider the  $\mathcal{F}(\mathcal{K}_\infty)$ -functions

$$\alpha_1(t) = \begin{cases} t & t \in [0, 1 - \epsilon) \\ \frac{5}{6}t + \frac{1-\epsilon}{6} & t \in [1 - \epsilon, 1 + \epsilon) \\ \frac{2}{3}t + \frac{1}{3} & t \in [1 + \epsilon, 2.5) \\ \frac{4}{9}t + \frac{8}{9} & t \in [2.5, \infty) \end{cases} \quad \text{and} \quad \alpha_2(t) = 4t \quad t \in [0, \infty).$$

Let

$$\gamma_1(t) := \begin{cases} \frac{1}{2}t & t \in [0, 2] \\ \frac{3}{4}t - \frac{1}{2} & t \in [2, \infty) \end{cases}.$$

Then we have

$$\alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_1(t) \begin{cases} = t & t \in [0, 1 - \epsilon] \\ < t & t \in (1 - \epsilon, 1 + \epsilon) \\ = t & t \in [1 + \epsilon, \infty) \end{cases}. \quad (5.4)$$

Clearly, for  $\epsilon = 0$  we would have  $\alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_1 = \text{id}$ .

Let us assume that there exists a  $\mathcal{K}_\infty$ -function  $\gamma_2$  satisfying  $\gamma_2 \in \mathcal{Z}$ ,  $\gamma_2 \geq \gamma_1$  and  $\gamma_2 \neq \gamma_1$ . By monotonicity of  $\mathcal{K}_\infty$ -functions and by (5.4) we conclude that  $\gamma_2(t) = \gamma_1(t)$  for all  $t \in [0, 1 - \epsilon] \cup [1 + \epsilon, \infty)$ . Further there exists a  $t^* \in (1 - \epsilon, 1 + \epsilon)$  with  $\gamma_2(t^*) > \gamma_1(t^*)$ . Define  $\tilde{t} := \gamma_1^{-1}(\alpha_2^{-1}(t^*)) = \frac{t^*}{2} \in (\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2})$ . Since  $\epsilon < \frac{1}{3}$  we have  $\tilde{t} \notin [1 - \epsilon, 1 + \epsilon]$ . This implies

$$\begin{aligned} \alpha_1 \circ \gamma_2 \circ \alpha_2 \circ \gamma_2(\tilde{t}) &= \alpha_1 \circ \gamma_2 \circ \alpha_2 \circ \gamma_1(\tilde{t}) = \alpha_1 \circ \gamma_2(t^*) \\ &> \alpha_1 \circ \gamma_1(t^*) = \alpha_1 \circ \gamma_1 \circ \alpha_2 \circ \gamma_1(\tilde{t}) = \tilde{t}, \end{aligned}$$

a contradiction to  $\gamma_2 \in \mathcal{Z}$ . So such an  $\gamma_2$  cannot exist. Hence,  $\gamma_1$  is a maximal element in  $\mathcal{Z}$ , but  $\gamma_1$  does not satisfy (5.3).  $\triangleleft$

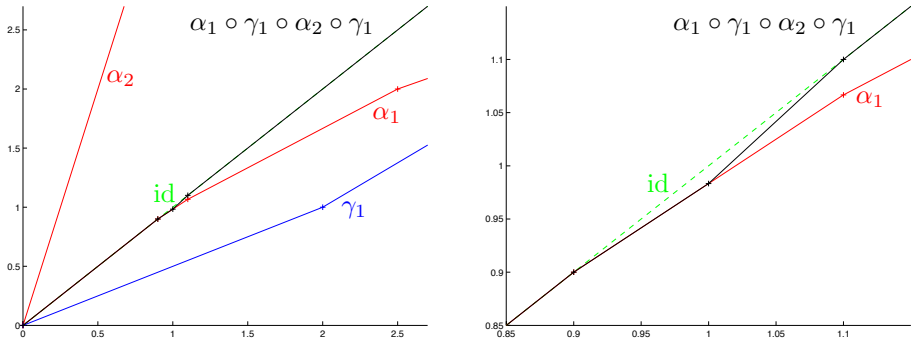


Figure 5.1: Maximal elements of  $\mathcal{Z}$  do not have to satisfy (5.3).

In this respect, solutions of the iterative functional  $\mathcal{K}_\infty$ -equation (5.3) cannot be equivalently described by maximal elements of  $\mathcal{Z}$ . Hence, the question for an existence result of solutions of (5.3) remains unsolved.

Secondly, as outlined in Remark 4.34, we can approximate a  $\mathcal{K}_\infty$ -function  $\alpha$  that satisfies  $\alpha'(0) < \infty$  from above by a  $\mathcal{R}(\mathcal{K}_\infty)$ -function  $\tilde{\alpha} \geq \alpha$ . This approximation is done in order to compute solutions of (5.3) that lead to an admissible (but not maximal) gain  $\tilde{\gamma}$  in Theorem 4.3.

On the other hand, if we are only interested in solutions of (5.3), we can approximate the  $\mathcal{K}_\infty$ -function  $\alpha_1, \dots, \alpha_k$  via piecewise linear  $\mathcal{K}_\infty$ -functions  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k$  by linear interpolation of the points  $\alpha_i(\Delta k)$  with  $k \in \mathbb{N}$ , where  $\Delta > 0$  is the sampling distance.

An interesting problem is then to derive an estimate of the deviation of the solution  $\gamma$  of (5.3) and a closest<sup>3</sup> solution  $\tilde{\gamma}$  of the approximated iterative functional  $\mathcal{R}(\mathcal{K}_\infty)$ -equation, i.e., for instance, an estimate of the form

$$\sup_{t \geq 0} \|\gamma(t) - \tilde{\gamma}(t)\| \leq \sup_{t \geq 0} \max_{i \in \{1, \dots, k\}} \phi(\|\alpha_i(t) - \tilde{\alpha}_i(t)\|)$$

with  $\phi \in \mathcal{K}_\infty$  to be found. Nevertheless, the following example indicates that such an accuracy estimate might be hard to obtain even if the domain of  $t$  is restricted to some finite interval  $[0, T]$ ,  $T > 0$ .

<sup>3</sup>Recall that solutions of (5.3) are not unique, in general.

**Example 5.3.** Let  $\alpha_1(t) = \max\{t, t^2\}$  and  $\alpha_2(t) = t(1 - e^{-t})$  for all  $t \geq 0$  and consider the iterative functional  $\mathcal{K}_\infty$ -equation (5.3). We approximate the  $\mathcal{K}_\infty$ -functions  $\alpha_1, \alpha_2$  by sampling and linear interpolation, where  $\Delta > 0$  is the constant sampling distance, i.e., we define the functions  $\tilde{\alpha}_i, i \in \{1, 2\}$  by

$$\tilde{\alpha}_i(t) := \alpha_i(k\Delta) + (\alpha_i((k+1)\Delta) - \alpha_i(k\Delta)) \left(\frac{t}{\Delta} - k\right), \quad t \in [k\Delta, (k+1)\Delta), \quad k \in \mathbb{N}.$$

Then we compute the solutions  $\gamma_\Delta \in \mathcal{R}(\mathcal{K}_\infty)$  of the sampled iterative functional  $\mathcal{R}(\mathcal{K}_\infty)$ -equation

$$\tilde{\alpha}_1 \circ \gamma \circ \tilde{\alpha}_2 \circ \gamma = \text{id}.$$

Since  $\alpha_1$  and  $\alpha_2$ , and also  $\tilde{\alpha}_1$  and  $\tilde{\alpha}_2$  do not intersect, the solutions  $\gamma_\Delta$  do not have accumulation points, see Proposition 4.25. The solutions are shown in Figure 5.2. Although the solutions  $\gamma_{0.001}$  (blue) and  $\gamma_{0.01}$  (green) are close near zero they deviate widely away from zero.  $\triangleleft$



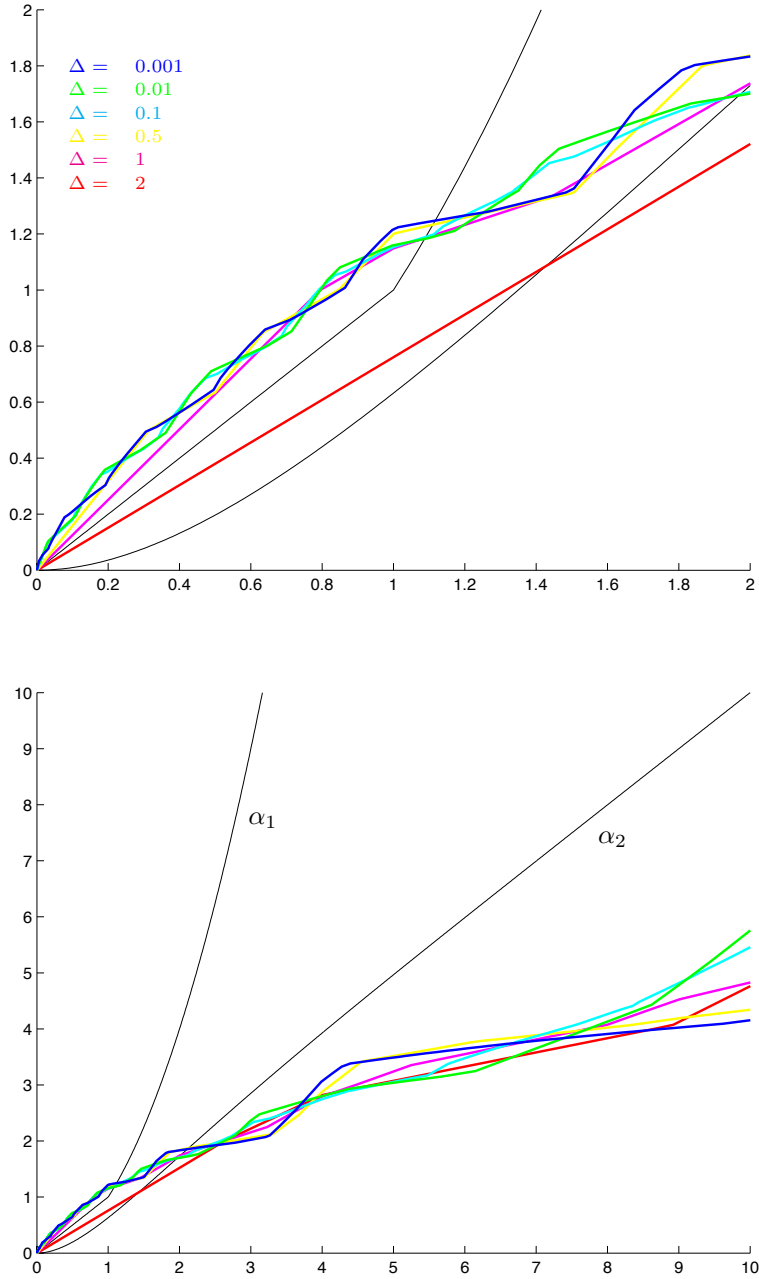


Figure 5.2: Solutions of the approximated iterative functional  $\mathcal{R}(\mathcal{K}_\infty)$ -equations  $\tilde{\alpha}_1 \circ \gamma \circ \tilde{\alpha}_2 \circ \gamma = \text{id}$  with different sampling distances  $\Delta > 0$ .



## Appendix

In this chapter, we recall some facts from the literature, which are on the one hand necessary for some results in this thesis, and, on the other hand, give further insight.

In the first section, we recall some results from the theory of *nonnegative matrices*. Indeed, if we are in the linear summation case (Section 1.6.1), i.e., the gain operator  $\Gamma_\mu$  in (1.8) consists of *linear* gains  $\gamma_{ij}$  that are aggregated via summation, then the gain operator is a linear map defined by a nonnegative matrix. Moreover, as the small-gain condition is equivalent to the spectral radius being less than one (Lemma 1.27), the theory of nonnegative matrices plays an important role.

In the second section, we study the concept of *stability radii* of positive linear systems. This concept describes bounds on perturbations such that a perturbed system is still GAS. The results recalled in this section are used in Section 4.1.2 to compute a maximal gain for the linear summation case.

In the third section, we recall l'Hôpital's rule, which is used to prove that the function  $\hat{\delta}$  from Example 3.25 is of class  $\mathcal{K}_\infty$ .

### A.1 Nonnegative matrices

In this thesis, we have stated several small-gain theorems, which heavily rely on the gain operator  $\Gamma_\mu : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  as defined in (1.8). We recall that in the linear summation case the gain operator  $\Gamma_\Sigma(s) = \Gamma s$  is a linear map with *nonnegative matrix*  $\Gamma \in \mathbb{R}_+^{N \times N}$ , see Section 1.6.1.

Moreover, we have seen that the small-gain condition  $\Gamma s \preceq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$  is

equivalent to the spectral radius  $\rho(\Gamma) < 1$ , and to the fact that the discrete-time linear system  $x(k+1) = \Gamma x(k)$ ,  $k \in \mathbb{N}$  is GAS, see Lemma 1.27.

In this section, we state further properties of nonnegative matrices related to the gain operator  $\Gamma_\Sigma$ , which are mainly taken from [8].

**Lemma A.1** ([8, Theorem 2.1.1]). *Let  $A \in \mathbb{R}_+^{N \times N}$ . Then the spectral radius  $\rho(A)$  is an eigenvalue of  $A$ , and there exists a nonnegative eigenvector  $v$  corresponding to  $\rho(A)$ .*

Perron [117] proved that for positive matrices, i.e.,  $[A]_{ij} > 0$  for all  $i, j \in \{1, \dots, N\}$ , there exists a positive eigenvector  $v$  corresponding to  $\rho(A)$ , and  $\rho(A)$  is a simple eigenvalue. Frobenius [29] gave the extension to nonnegative irreducible matrices. Thus, the following theorem is often called the (classical) *Perron-Frobenius theorem*.

**Theorem A.2** ([8, Theorem 2.1.4]). *If  $A \in \mathbb{R}_+^{N \times N}$  is irreducible, then the following holds:*

- (i)  $\rho(A)$  is a simple eigenvalue of  $A$ ;
- (ii) any eigenvalue of  $A$  of the same modulus is also simple;
- (iii)  $A$  has a positive eigenvector  $v$  corresponding to  $\rho(A)$  (called the Perron-Frobenius eigenvector);
- (iv) any nonnegative eigenvector of  $A$  is a multiple of  $v$ .

We recall that the irreducibility assumption on  $\Gamma$  is equivalent to the underlying interconnection graph of the subsystems being strongly connected, see Theorem 1.16 and Remark 1.18.

The small-gain theorems in Sections 2.3 and 3.3 prove stability properties of the overall system by constructing a (finite-step) (ISS) Lyapunov function for the overall system. This construction stems from [25], and hinges on an  $\Omega$ -path with respect to the gain operator  $\Gamma_\mu$ . In general, computing an  $\Omega$ -path is nontrivial, but can be obtained as outlined in Remark 1.25.

However, in the linear summation case an  $\Omega$ -path can be easily obtained if  $\Gamma \in \mathbb{R}_+^{N \times N}$  is irreducible. This can be seen as follows: By Lemma 1.27, the small-gain condition  $\Gamma s \not\geq s$  for all  $s \in \mathbb{R}_+^N \setminus \{0\}$  is equivalent to  $\rho(\Gamma) < 1$ . Since  $\Gamma$  is irreducible, there exists a positive Frobenius eigenvector  $v \in \mathbb{R}_+^N$  corresponding to  $\rho(\Gamma)$  satisfying  $\Gamma v = \rho(\Gamma)v < v$ . Thus,  $\sigma(r) := vr$  for all  $r \geq 0$  is a linear  $\Omega$ -path with respect to  $\Gamma$ , see Definition 1.22.

On the other hand, if  $\Gamma$  is reducible, then an  $\Omega$ -path can be obtained as in the proof of [123, Lemma 1.1]: Let  $E = (e_{ij})$ ,  $e_{ij} = 1$  for all  $i, j \in \{1, \dots, N\}$ . By continuity

of the spectrum there exists an  $\varepsilon > 0$  small enough such that  $\rho(\Gamma + \varepsilon E) < 1$ . We define  $\tilde{\Gamma} := \Gamma + \varepsilon E$  and observe that  $\tilde{\Gamma}$  is positive, hence irreducible. Applying the Perron-Frobenius theorem A.2 there exists a positive Frobenius eigenvector  $\tilde{v}$  with  $\Gamma \tilde{v} < \tilde{\Gamma} \tilde{v} = \rho(\tilde{\Gamma}) \tilde{v} < \tilde{v}$ . Therefore,  $\sigma(r) := \tilde{v} r$  is an  $\Omega$ -path with respect to  $\Gamma$ .

Related to this observation is the following monotonicity property of nonnegative matrices.

**Lemma A.3** ([8, Corollary 2.1.5]). *Let  $A, B \in \mathbb{R}_+^{N \times N}$ . If  $0 \leq A \leq B$ , then  $\rho(A) \leq \rho(B)$ . In particular, if  $0 \leq A < B$  and  $A + B$  is irreducible, then  $\rho(A) < \rho(B)$ .*

## A.2 Stability radii of positive linear discrete-time systems

Before restricting ourselves to *positive* linear discrete-time systems, we consider (complex) linear discrete-time system of the form

$$x(k+1) = Ax(k), \quad k \in \mathbb{N} \tag{A.1}$$

with  $A \in \mathbb{C}^{N \times N}$  and  $x \in \mathbb{C}^N$ . Assume that we have additional affine parameter perturbations of the dynamics  $A$  leading to a system of the form

$$x(k+1) = (A + D\Delta E)x(k), \quad k \in \mathbb{N}, \quad \Delta \in \mathcal{D}, \tag{A.2}$$

where  $D \in \mathbb{C}^{N \times L}$ ,  $E \in \mathbb{C}^{Q \times N}$  are given *structure matrices* and  $\mathcal{D} \subset \mathbb{C}^{L \times Q}$  is a given set of *perturbation matrices*, see [61]. We may also restrict the class of perturbation matrices to real and nonnegative matrices, and define

$$\mathcal{D}_{\mathbb{R}} = \mathcal{D} \cap \mathbb{R}^{L \times Q}, \quad \mathcal{D}_+ = \mathcal{D} \cap \mathbb{R}_+^{L \times Q}. \tag{A.3}$$

The problem under consideration is the following: Assume that the unperturbed system (A.1) is GAS. Can we impose an upper bound on the perturbation (in terms of a norm) such that the perturbed system (A.2) is still GAS? To answer this question affirmatively, we introduce the notation of stability radii, see [61, Definition 3.1].

**Definition A.4.** Let  $\mathcal{D} \subset \mathbb{C}^{L \times Q}$  and  $\|\cdot\|$  be a given norm on  $\text{span } \mathcal{D}$ . The *stability radius* with respect to perturbations of the form  $A + D\Delta E$  with  $\Delta \in \mathcal{D}$ , and  $D \in \mathbb{C}^{N \times L}$ ,  $E \in \mathbb{C}^{Q \times N}$ , is defined by

$$r_{\mathcal{D}} = r_{\mathcal{D}}(A; D, E) = \inf\{\|\Delta\| : \Delta \in \mathcal{D}, \rho(A + D\Delta E) \geq 1\}. \tag{A.4}$$

Note that the map  $\Delta \mapsto \rho(A + D\Delta E)$  is continuous. Hence, decreasing the set of perturbation matrices  $\mathcal{D}$  leads to an increase of the stability radius. In particular, we have (see [61, (Equation (19))])

$$0 < r_{\mathcal{D}}(A; D, E) \leq r_{\mathcal{D}_{\mathbb{R}}}(A; D, E) \leq r_{\mathcal{D}_+}(A; D, E).$$

*Remark A.5.* The concept and the computation of stability radii of linear system of the form (A.1) has been widely studied starting with [58]. We refer to [60] for a comprehensive list of references and further insights into this topic (especially [60, Chapter 5]).  $\triangleleft$

In general, the *complex* stability radius  $r_{\mathbb{C}}(A; D, E)$  and the *real* stability radius  $r_{\mathbb{R}}(A; D, E)$  are distinct. While real perturbations seem to be more natural in several applications, complex stability radii are easier to compute, see [59]. But if we restrict our attention to positive systems, both stability radii are equal and can be computed more easily, as outlined next.

We call a dynamical system *positive* if trajectories starting in the positive orthant remain in the positive orthant. Note that system (A.1) is positive if and only if the matrix  $A$  is nonnegative. Moreover, any positive time-invariant linear discrete-time systems can be written in the form (A.1) with nonnegative matrix  $A$ , see [61]. Considering the perturbed system (A.2), we assume the structure matrices to be nonnegative as well, i.e.,  $D \in \mathbb{R}_+^{N \times L}$  and  $E \in \mathbb{R}_+^{Q \times N}$ . In this case the complex, real and nonnegative stability radii are equal.

**Lemma A.6** ([61, Proposition 3.9]). *Suppose  $A \in \mathbb{R}_+^{N \times N}$  satisfies  $\rho(A) < 1$ ,  $D \in \mathbb{R}_+^{N \times L}$  and  $E \in \mathbb{R}_+^{Q \times N}$  are given nonnegative structure matrices, and  $\mathcal{D} \subset \mathbb{C}^{L \times Q}$  is a perturbation class endowed with an admissible<sup>1</sup> perturbation norm. Then with  $\mathcal{D}_{\mathbb{R}}$  and  $\mathcal{D}_+$  defined in (A.3), we have*

$$r_{\mathcal{D}}(A; D, E) = r_{\mathcal{D}_{\mathbb{R}}}(A; D, E) = r_{\mathcal{D}_+}(A; D, E).$$

To compute the spectral radius in Lemma A.6, we define the *transfer matrix* associated to the triplet  $(A, D, E)$  by

$$G(s) = E(sI - A)^{-1}D, \quad s \in \mathbb{C} \setminus \sigma(A). \tag{A.5}$$

Note that if the matrices  $A, D, E$  are nonnegative, then for all  $s > \rho(A)$  also  $G(s)$  is a nonnegative matrix and satisfies the following monotonicity property, see [61, Lemma 4.1],

$$\forall t > s > \rho(A) : \quad G(s) \geq G(t) \geq 0.$$

The transfer matrix can now be used to compute the (complex, real and nonnegative) stability radius as follows.

**Theorem A.7** ([61, Theorem 4.3]). *Suppose  $A \in \mathbb{R}_+^{N \times N}$  satisfies  $\rho(A) < 1$ ,  $D \in \mathbb{R}_+^{N \times L}$  and  $E \in \mathbb{R}_+^{Q \times N}$  are given nonnegative structure matrices,  $\mathbb{C}^L, \mathbb{C}^Q$  are provided*

---

<sup>1</sup>An admissible perturbation norm is an operator norm induced by monotonic norms, see [61, Definition 3.4] for a precise definition.

with monotonic norms and  $\mathcal{D} = \mathbb{C}^{L \times Q}$  is endowed with the induced operator norm. Then

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}_+}(A; D, E) = \|G(1)\|_{\mathcal{L}(\mathbb{C}^L, \mathbb{C}^Q)}^{-1},$$

where  $G(s)$  is the transfer matrix defined in (A.5),  $\|G(1)\|_{\mathcal{L}(\mathbb{C}^L, \mathbb{C}^Q)}$  is the operator norm of  $G(1) : \mathbb{C}^L \rightarrow \mathbb{C}^Q$  and, by definition,  $0^{-1} = \infty$ .

To conclude this section, the following theorem considers the special case, where  $\Delta = \delta I$ . Hence, the right-hand side of the perturbed linear system (A.2) has the form  $(A + D\Delta E) = (A + \delta DE)$ . This result is needed in Chapter 4 to derive a maximal gain in the linear summation case in Theorem 4.7. Here, the stability radius can be expressed by the spectral radius of the transfer matrix  $G(1)$ .

**Theorem A.8** ([61, Theorem 4.7]). *Suppose  $A \in \mathbb{R}_+^{N \times N}$  satisfies  $\rho(A) < 1$ ,  $D \in \mathbb{R}_+^{N \times L}$  and  $E \in \mathbb{R}_+^{L \times N}$  are given nonnegative structure matrices, and  $\Delta = \delta I$  with  $\delta \in \mathbb{C}$ . Then*

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}_+}(A; D, E) = (\rho(G(1)))^{-1},$$

where  $G(s)$  is the transfer matrix defined in (A.5).

### A.3 L'Hôpital's rule and completion of Example 3.25

Firstly, we recall l'Hôpital's rule to compute the limit of a quotient of two real functions. The result can be found in most of the textbooks on real analysis, e.g. in [7, Satz 4.2.4 and 4.2.5]. The rest of this section is dedicated to prove properties of the function  $\hat{\delta}$  from Example 3.25.

**Theorem A.9** (L'Hôpital's rule). *Let  $a \in \mathbb{R}$ , and  $f, g : [a, \infty) \rightarrow \mathbb{R}$  be differentiable, with  $g'(s) \neq 0$  for all  $s \in [a, \infty)$ . Assume that one of the following holds:*

- (i)  $\lim_{s \rightarrow \infty} f(s) = \lim_{s \rightarrow \infty} g(s) = 0$ ;
- (ii)  $\lim_{s \rightarrow \infty} f(s) = \pm\infty$ , and  $\lim_{s \rightarrow \infty} g(s) = \pm\infty$ .

If  $\lim_{s \rightarrow \infty} \frac{f'(s)}{g'(s)}$  exists then also  $\lim_{s \rightarrow \infty} \frac{f(s)}{g(s)}$  exists, and it holds

$$\lim_{s \rightarrow \infty} \frac{f(s)}{g(s)} = \lim_{s \rightarrow \infty} \frac{f'(s)}{g'(s)}. \tag{A.6}$$

Next, we make use of Theorem A.9 to show some properties of the function  $\hat{\delta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined in Example 3.25. The aim is to prove that  $\hat{\delta} \in \mathcal{K}_\infty$  and  $(\text{id} - \hat{\delta}) \in \mathcal{K}_\infty$ . In order to do this, we firstly establish properties of the function

$$h(s) := \left( \frac{1}{s+1} \right)^{\frac{1}{s+1}}, \quad s \in \mathbb{R}_+.$$

Note that the function  $\hat{\delta}$  from Example 3.25 can be written as

$$\hat{\delta}(s) = (s + 1)(1 - h(s)), \quad s \in \mathbb{R}_+. \quad (\text{A.7})$$

Clearly,  $h(0) = 1$  and for all  $s > 0$  we have  $h(s) \in (0, 1)$ . To compute the limit of  $h$  for  $s \rightarrow \infty$ , we make use of l'Hôpital's rules, Theorem A.9, and we obtain

$$\lim_{s \rightarrow \infty} h(s) = \lim_{s \rightarrow \infty} \left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}} = \lim_{s \rightarrow \infty} e^{\log\left(\frac{1}{s+1}\right) \cdot \left(\frac{1}{s+1}\right)} \stackrel{(\text{A.6})}{=} \lim_{s \rightarrow \infty} e^{-\frac{1}{s+1}} = 1.$$

Moreover, using the exponential representation of  $h$  again, the derivative can be computed as

$$h'(s) = -\frac{1}{(s + 1)^2} \left( 1 + \log \left( \frac{1}{s + 1} \right) \right) \left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}}, \quad s \in \mathbb{R}_+.$$

From (A.7) and the properties of  $h$ , we can now derive properties of  $\hat{\delta}$ .

- (i) As  $h$  is a continuous function on  $\mathbb{R}_+$ , also  $\hat{\delta}$  is continuous on  $\mathbb{R}_+$ . Moreover, from  $h(0) = 1$  and  $h(s) \in (0, 1)$  for all  $s > 0$  we conclude that  $\hat{\delta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is positive definite.
- (ii) The function  $\hat{\delta}$  is unbounded, since we have

$$\begin{aligned} \lim_{s \rightarrow \infty} \hat{\delta}(s) &= \lim_{s \rightarrow \infty} (s + 1)(1 - h(s)) \\ &= \lim_{s \rightarrow \infty} \frac{1 - h(s)}{\frac{1}{s+1}} \\ &\stackrel{(\text{A.6})}{=} \lim_{s \rightarrow \infty} \frac{-h'(s)}{-\frac{1}{(s+1)^2}} \\ &= \lim_{s \rightarrow \infty} - \left( 1 + \log \left( \frac{1}{s + 1} \right) \right) \underbrace{\left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}}}_{\rightarrow 1} = \infty. \end{aligned}$$

- (iii) To show that  $\hat{\delta}$  is strictly increasing, we compute the derivative

$$\begin{aligned} \hat{\delta}'(s) &= (1 - h(s)) - (s + 1)h'(s) \\ &= 1 - \left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}} + \frac{1}{s + 1} \left( 1 + \log \left( \frac{1}{s + 1} \right) \right) \left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}} \\ &= \underbrace{\frac{1}{s + 1}}_{>0} \underbrace{\left( (s + 1) \left( \frac{1}{s + 1} \right)^{-\frac{1}{s+1}} - s + \log \left( \frac{1}{s + 1} \right) \right)}_{=: f(s)} \underbrace{\left( \frac{1}{s + 1} \right)^{\frac{1}{s+1}}}_{>0}. \end{aligned}$$



Since  $\hat{\delta}$  is strictly increasing if  $\hat{\delta}'(s) > 0$  for all  $s \in \mathbb{R}_+$ , it remains to show that  $f(s) > 0$  for all  $s \in \mathbb{R}_+$ . By the coordinate transformation  $s \mapsto \frac{1}{t} - 1$ , we see that  $f(s) > 0$  for all  $s \in \mathbb{R}_+$  if and only if  $\hat{f}(t) > 0$  for all  $t \in (0, 1]$ , where the function  $\hat{f}$  is defined by

$$\hat{f}(t) := f\left(\frac{1}{t} - 1\right) = \frac{1}{t}t^{-t} - \frac{1}{t} + 1 + \log(t) \quad \text{for all } t \in (0, 1]. \quad (\text{A.8})$$

In particular, as we can see from Figure A.1, we have  $\hat{f}(t) \geq 1$  for all  $t \in (0, 1]$ . Equivalently,  $f(s) \geq 1 > 0$  for all  $s \in \mathbb{R}_+$ , which shows that  $\hat{\delta}'(s) > 0$  for all  $s \in \mathbb{R}_+$ . Thus,  $\hat{\delta}$  is strictly increasing.

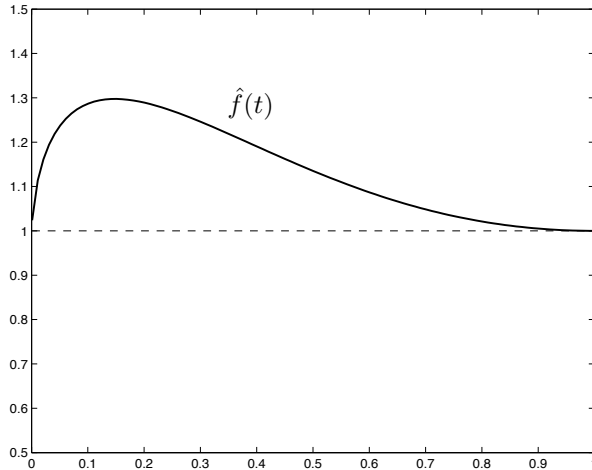


Figure A.1: The function  $\hat{f}$  defined in (A.8)

Summarizing, we have shown that the function  $\hat{\delta}$  satisfies  $\hat{\delta}(0) = 0$ , and it is continuous, unbounded and strictly increasing. Thus,  $\hat{\delta}$  is of class  $\mathcal{K}_\infty$ .

A similar reasoning can be used to show that also  $(\text{id} - \hat{\delta}) \in \mathcal{K}_\infty$ . Here, we prove this property in a different way. Let  $\tilde{\sigma}(s) = e^s - 1$  and  $\tilde{\sigma}^{-1}(s) = \log(s + 1)$  be given from Example 3.25. Define  $\nu(s) := s(1 - e^{-s})$  for all  $s \in \mathbb{R}_+$ . Note that  $\tilde{\sigma}, \tilde{\sigma}^{-1}, \nu \in \mathcal{K}_\infty$ .

Then we have

$$\begin{aligned}
 \tilde{\sigma} \circ \nu \circ \tilde{\sigma}^{-1}(s) &= \tilde{\sigma} \circ \left( \log(s+1)(1 - e^{-\log(s+1)}) \right) \\
 &= \tilde{\sigma} \circ \left( \log(s+1) \left( 1 - \frac{1}{s+1} \right) \right) \\
 &= e^{\log(s+1)\left(1 - \frac{1}{s+1}\right)} - 1 \\
 &= (s+1) \left( \frac{1}{s+1} \right)^{\frac{1}{s+1}} - 1 \\
 &= s - (s+1) \left( 1 - \left( \frac{1}{s+1} \right)^{\frac{1}{s+1}} \right) \\
 &= (\text{id} - \hat{\delta})(s).
 \end{aligned}$$

This equation shows that  $(\text{id} - \hat{\delta}) \in \mathcal{K}_\infty$ , since the composition of  $\mathcal{K}_\infty$ -functions is again of class  $\mathcal{K}_\infty$ , see Proposition 1.5.

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# List of symbols

## Notions

$(A_1; \dots; A_N)$	$:= (A_1^\top, \dots, A_N^\top)^\top$
$(v_1, \dots, v_N)$	$:= (v_1^\top, \dots, v_N^\top)^\top$
$[v]_i$	the $i$ th component of the vector $v$
$\alpha'(t^-), \alpha'(t^+)$	the left resp. right hand derivative
$\otimes c^\gamma$	as defined in (4.10)
$\otimes c_k^\gamma$	as defined in (4.44)
$\left[ \prod_{i=0}^{k-1} A_{j_{k-i}} \right]$	$:= A_{j_k} A_{j_{k-1}} \dots A_{j_1}$
$\rho(Q)$	the spectral radius of the matrix $Q$
$\sigma(Q)$	the spectrum of the matrix $Q$
$I$	the identity matrix
$v > w$	$\forall i \in \{1, \dots, N\} : [v]_i > [w]_i$
$v \geq w$	$\forall i \in \{1, \dots, N\} : [v]_i \geq [w]_i$
$v \not\geq w$	$\exists i \in \{1, \dots, N\} : [v]_i < [w]_i$
0-GAS	global asymptotic stability with zero input
AG	asymptotic gain property
expISS	exponential input-to-state stability
GAS	global asymptotic stability
GES	global exponential stability
GS	global stability property
ISS	input-to-state stability

## Functions

$0$	the zero function
$\text{diag}$	the diagonal operator defined in (1.9)
$\eta_{\max}$	as defined in (4.45)
$\Gamma_{\mu}$	the gain operator defined in (1.8)
$\Gamma_{\mu}^{\gamma}$	the gain operator defined in (4.4)
$\Gamma_{\oplus}$	the gain operator in the maximization case
$\Gamma_{\Sigma}$	the gain operator in the summation case
$\Gamma_{sub}^*$	the weighted structure matrix defined in (4.3)
$\text{id}$	the identity function
$\mathcal{K}$	the set of continuous, and strictly increasing functions $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\alpha(0) = 0$ , see Definition 1.3
$\mathcal{K}_{\infty}$	the set of unbounded $\mathcal{K}$ -functions, see Definition 1.3
$\mathcal{KL}$	the set of continuous functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which are of class $\mathcal{K}$ in the first argument, and of class $\mathcal{L}$ in the second argument, see Definition 1.4
$\mathcal{L}$	the set of continuous, and strictly decreasing functions $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\lim_{s \rightarrow \infty} \pi(s) = 0$ , see Definition 1.3
MAF	the set of monotone aggregation functions, see Definition 1.20
$\mathcal{I}(\mathcal{K}_{\infty})$	the set of piecewise linear $\mathcal{K}$ -functions
$\mathcal{F}(\mathcal{K}_{\infty})$	the set of piecewise linear $\mathcal{K}$ -functions with finitely many sampling points
$\mathcal{R}(\mathcal{K}_{\infty})$	the set of right-affine $\mathcal{K}$ -functions, see Definition 4.18
$\mathcal{V}_i$	functions $\mathcal{V}_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ are usually assumed to denote Lyapunov-type functions for the subsystems in Section 2.3 and Section 3.3
$V$	functions $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are usually assumed to denote global finite-step Lyapunov functions or dissipative finite-step ISS Lyapunov functions
$W$	functions $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are usually assumed to denote global Lyapunov functions or dissipative ISS Lyapunov functions
$\gamma \in \mathcal{K}_{\infty}$	usually denotes a gain
$\bar{\gamma} \in \mathcal{K}_{\infty}$	A maximal gain
$\sigma, \tilde{\sigma} \in \mathcal{K}_{\infty}^N$	usually denotes an $\Omega$ -path, see Definition 1.22

## Norms

$ \cdot $	the absolute value of a scalar
$\ \cdot\ _{\infty}$	the infinity norm on $\mathbb{R}^n$
$ \cdot _{[0,k]}$	the supremum norm for finite sequences
$\ \cdot\ _1$	the 1-norm on $\mathbb{R}^n$



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$\ \cdot\ _p$	the $p$ -norm on $\mathbb{R}^n$
$\ \cdot\ _2$	the Euclidean norm on $\mathbb{R}^n$
$ \cdot _\infty$	the supremum norm for sequences
$\ \cdot\ $	an arbitrary norm on $\mathbb{R}^n$ or an arbitrary operator norm on $\mathbb{R}^{l \times n}$

**Sets**

$\mathbb{C}$	the field of complex numbers
$\text{cl}\{S\}$	the closure of a set $S$
$\text{co}\{S\}$	the convex hull of a set $S$
$\mathcal{C}, \mathcal{D}$	A (convex polyhedral) cone
$\mathcal{S}, \mathcal{S}_{\text{strong}}$	the potential decay set defined in (4.6) and (4.7)
$\mathbb{N}$	the natural numbers including zero
$\Omega$	the set of decay defined in (1.13)
$\mathbb{R}$	the field of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{R}_+^n$	the cone of nonnegative real column vectors
$\mathbb{R}^n$	the vector space of real column vectors
$\text{relint}(S)$	the relative interior of a set $S$
$B_{[a,b]}$	$:= \{x \in \mathbb{R}^n : \ x\  \in [a, b]\}$



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