Discrete Moments of Zeta-Functions with respect to random and ergodic transformations

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Teerapat Srichan

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1. Gutachter: Prof. Dr. Jörn Steuding
2. Gutachter: Prof. Dr. Antanas Laurinčikas
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Notations

The following notations will be used throughout this thesis.

$\mathbb{N}$	is the set of all natural numbers: $\mathbb{N} = \{1, 2, ...\}$.
$\mathbb{N}_0$	is the set of all natural numbers and zero: $\mathbb{N}_0 = \{0, 1, 2, ...\}$.
$\mathbb{Z}$	is the set of all integer numbers.
$\mathbb{R}$	is the set of all real numbers.
$\mathbb{C}$	is the set of all complex numbers.
$\Re(s)$	is the real part of a complex number $s \in \mathbb{C}$.
$\Im(s)$	is the imaginary part of a complex number $s \in \mathbb{C}$.
$f(x) = O(g(x))$	means $\mid f(x) \mid \leq Cg(x)$ for $x \geq x_0$ and a certain $C > 0$. Here $f(x)$ is a complex function of the real variable x and $g(x)$ is positive function of x for $x \geq x_0$.
$f(x) \ll g(x)$	means the same as $f(x) = O(g(x))$.
$f(x) = o(g(x))$ as $x \to x_0$	means $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 0$ with x_0 possibly infinite.
$\mathrm{Res}_{s=s_0}\ F(s)$	is the residue of $F(s)$ at the point $s = s_0$.
$\Gamma(s)$	is the gamma-function defined by $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt,\ \Re(s) > 0$, otherwise by analytic continuation.
γ	is Euler's constant, defined by $\gamma = -\int_0^\infty e^{-x}\log x dx = 0.5772157...$.
$\rho = \beta + i\gamma$	denotes the non-trivial zeros of $\zeta(s)$; $\beta = \Re(\rho)$, $\gamma = \Im(\rho)$.
$N(T)$	denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the critical strip with $0 \leq \gamma < T$.
$N_0(T)$	denote the number of non-trivial zeros which lie on the critical line and have imaginary part $\gamma \in (0, T]$.
$N(\sigma, T)$	denote the number of non-trivial zeros with real part $\beta > \sigma$ and imaginary part $\gamma \in (0, T]$.
$N^*(T)$	denote the number of simple non-trivial zeros with imaginary part $\gamma \in (0, T]$.
$\exp(z)$	$= e^z$.
$e(z)$	$= e^{2\pi iz}$.
$\log x$	$= \mathrm{Log}_e x (= \ln x)$.
$\sum\limits_{n \leq x} f(n)$	denote a sum taken over all natural numbers n not exceeding x; the empty sum is defined to be zero.
$\sum\limits_{n \leq x}{}' f(n)$	denote the same as above, only $'$ denotes that when x is an integer one should take the last term in the sum as $\frac{f(x)}{2}$ and not as $f(x)$.
$\prod\limits_{j}$	denote a product taken over all possible values of the index j; the empty product is defined to be unity.
$\sum\limits_{d \mid n}$	denote a sum taken over all positive divisors of n.
$\Lambda(n)$	is the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^m$ and zero otherwise.
$\mu(n)$	is the Möbius function, defined as $\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$ (p_j's being different primes) and zero otherwise, and $\mu(1) = 1$.
$\pi(x)$	$= \sum\limits_{p \leq x} 1$, the number of primes not exceeding x.

$\mathrm{li}(x)$	$= \int_0^x \frac{dt}{\log t} = \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right).$
$d_k(n)$	is the number of ways n can be written as a product of $k \geq 2$ fixed factors; $d_2(n) = d(n)$ is the number of divisors of n
B_k	Bernoulli numbers
$\phi(n)$	is Euler's function defined as $\phi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$, where the product is over all prime divisors of n.
$\mathrm{meas}\{A\}$	is Lebesgue measure of the set A.

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Chapter 1

1 Introduction and statement of the main results

The Riemann zeta-function plays an important role in number theory by the relation between its zeros and the number of prime numbers less than a given magnitude. The question about the zeros of the Riemann zeta-function is one of the most famous open problems of mathematics.

Recently, the behaviour of the Riemann zeta-function has been studied in various directions, for example, the location of non-trivial zero of the zeta-functions, the order of growth of the zeta-function inside the critical strip, the mean-value behaviour of the zeta-functions, universality properties of the zeta-function, etc. These directions have been studied by analytic means. In this thesis, we shall study the zeta-function from a probabilistic point of view, namely in contexts of a random walk and an ergodic transformation.

In Chapter 1, we introduce the Riemann zeta-function and other zeta-functions, which shall appear in this thesis. Moreover, we provide the analytic tools for studying the behaviour of these zeta-functions.

In Chapter 2, we study the asymptotic behaviour of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$ by modelling the imaginary part t with a Cauchy random walk. We briefly discuss the technique of Lifshits and Weber [43] in the investigation of the almost sure asymptotic behaviour for the Riemann zeta-function $\zeta(s)$. Furthermore, we emulate their technique for the Hurwitz zeta-function $\zeta(s, a)$. Moreover, we use Atkinson's formula [2] in place of the technique of Lifshits and Weber in the case of Dirichlet L-functions $L(s, \chi)$ with a primitive character χ.

In Chapter 3, we study the behaviour of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$, when t is sampled by an ergodic transformation. Here the ergodic transformation $T : \mathbb{R} \to \mathbb{R}$ is given by

$$T0 := 0, \qquad Tx := \frac{1}{2}(x - \frac{1}{x}) \qquad \text{for } x \neq 0.$$

Its iterates $T^n x$ are defined by $T \circ T^{n-1}x$, for $n \geq 1$, and $T^0 x = x$. We discuss the distribution of values of the Riemann zeta-function $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to this ergodic transformation T following a work of J. Steuding in [60]. Moreover, we study the behaviour of the logarithmic derivative of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$, when t is sampled by an ergodic transformation. Here, we shall provide an equivalent formulation for the Riemann Hypothesis in terms of an ergodic transformation. We also study the behaviour of other zeta-functions in this sense.

In Chapter 4, we investigate the phenomenon of universality with respect to certain stochastic processes. Regarding the absolute value of an analytic function as analytic landscape over the complex plane, we discuss the question: how often does a random walk observe the phenomenon of universality? And: how soon does a random walk meet a given set?

1.1 The Riemann zeta-function

The Riemann zeta-function is a function of a complex variable s, for $\sigma := \Re(s) > 1$ given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

The Dirichlet series (1.1) is convergent for $\sigma > 1$, and uniformly in any finite region in which $\sigma \geq 1+\delta$, $\delta > 0$. In addition, for $\sigma > 1$, $\zeta(s)$ can be written as an infinite product over the prime numbers p:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}. \tag{1.2}$$

The infinite product (1.2) is known as Euler's product.

In most application of zeta-function theory information about $\zeta(s)$ for $\sigma \leq 1$ is of interest. B. Riemann [53] discovered that the function $\zeta(s)$ is regular for all values of s except a simple pole at $s = 1$ with residue 1. The Laurent expansion of $\zeta(s)$ in a neighborhood of its pole $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + ..., \tag{1.3}$$

where the coefficients γ_k in (1.3) are given by

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\sum_{m \leq N} \frac{1}{m} \log^k m - \frac{\log^{k+1} N}{k+1} \right),$$

and, in particular,

$$\gamma = \gamma_0 = \lim_{N\to\infty} \left(1 + \frac{1}{2} + ... + \frac{1}{N} - \log N \right) = 0.5772157...$$

is Euler's constant (see [27] p.4). B. Riemann also proved the functional equation for the Riemann zeta-function, which states that, for all complex s,

$$\pi^{\frac{-s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s). \tag{1.4}$$

The functional equation (1.4) shows $\zeta(-2n) = 0$ for $n = 1, 2, ...$, since the gamma-function has simple poles at the non-positive integers and for $s = 0$ the pole of $\zeta(1-s)$ cancels the pole of $\Gamma(\frac{s}{2})$. The zeros $s = -2n$ are called the "trivial zeros" of $\zeta(s)$. All other zeros lie inside the strip $0 < \sigma < 1$ are called "non-trivial zeros" of $\zeta(s)$. As already mentioned in the beginning, the location of the non-trivial zeros of $\zeta(s)$ is topic of the most famous conjectures of mathematics, namely the Riemann Hypothesis. This conjecture states that there are no zeros of $\zeta(s)$ to the right of the critical line $\sigma = \frac{1}{2}$.

In Riemann's nine pages memoir, he conjectured an asymptotic formula for the number $N(T)$ of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the critical strip with $0 \leq \gamma < T$, namely

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

which was proved by von Mangoldt in 1895/1905. For the ξ-function, defined by

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s),$$

Riemann conjectured the product representation

$$\xi(s) = e^{A+Bs} \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}},$$

where A, B are constants and the product is taken over all non-trivial zeros ρ of $\zeta(s)$. This conjecture was proved by Hadarmard [22] in 1893. In 1896, de la Vallée Poussin [66] proved that the zero-free region includes the vertical line $\Re(s) = 1$, i.e, $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$. Vinogradov [67] and Korobov [37] proved independently that $\zeta(s)$ has no zeros in the region

$$\Re(s) \geq 1 - \frac{c(\alpha)}{(\log \mid t \mid +1)^{\alpha}},$$

for any $\alpha > \frac{2}{3}$.

Hardy [23] showed that there are infinitely many zeros on the critical line. Selberg [54] proved that a positive proportion of all non-trivial of $\zeta(s)$ lie on the critical line: let $N_0(T)$ denote the number of non-trivial zeros which lie on the critical line and have imaginary part $\gamma \in (0, T]$, then

$$U := \liminf_{T \to \infty} \frac{N_0(T)}{N(T)} \geq C$$

with some constant $C > 0$. The lower bound U was improved by Levinson [42] who obtained that $U \geq 0.3437$. Conrey [19] obtained the better lower bound $U \geq 0.4088$. The best and most recent lower bound has been established by Bui, Conrey and Young [14], namely $U \geq 0.4105$.

Selberg [55] attempted to bound the number of possible zeros off the critical line. Let $N(\sigma, t)$ denote the number of non-trivial zeros with real part $\beta > \sigma$ and imaginary part $\gamma \in (0, T]$. Selberg showed that, for $\frac{1}{2} \leq \sigma \leq 1$,

$$N(\sigma, T) \ll T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T$$

uniformly in σ.

Moreover, there is a conjecture about the simplicity of the zeros of the Riemann zeta-function. Let $N^*(T)$ denote the number of simple non-trivial zeros with imaginary part $\gamma \in (0, T]$. Levinson [42] proved that

$$S := \liminf_{T \to \infty} \frac{N^*(T)}{N(T)} \geq \frac{1}{3}.$$

Unconditionally, Bui, Conrey and Young [14] proved that $S \geq 0.4058$. Recently, under the assumption of the Riemann Hypothesis, Bui and Heath-Brown [15] proved that $S \geq \frac{19}{27}$.

Let $(\gamma_n)_n$ denote the sequence of all positive imaginary parts of non-trivial zeros in ascending order. Littlewood [44] proved that the gap between consecutive ordinates γ_n, γ_{n+1} tends to zero, as $n \to \infty$. Littlewood obtained that, as $n \to \infty$,

$$\gamma_{n+1} - \gamma_n \ll \frac{1}{\log \log \log \gamma_n}.$$

Montgomery [46] investigated the behaviour of the pair correlation of ordinates γ, γ' of non-trivial zeros. Montgomery conjectured that, for any fixed $0 < \alpha < \beta$,

$$\lim_{T \to \infty} \frac{1}{N(T)} \sharp \{\gamma, \gamma' \in (0, T] : \alpha \leq \frac{(\gamma - \gamma') \log T}{2\pi} \leq \beta\} = \int_{\alpha}^{\beta} \left(1 - (\frac{\sin \pi u}{\pi u})^2\right) du.$$

This conjecture is called "Montgomery's pair correlation conjecture".

There is another famous conjecture, which is a consequence of the Riemann Hypothesis, namely the Lindelöf Hypothesis. The Lindelöf Hypothesis asserts that $\zeta(\frac{1}{2} + it) \ll \mid t \mid^{\epsilon}$ for any $\epsilon > 0$, as $| t |\to \infty$. Titchmarch [64] gave an equivalent form of the Lindelöf Hypothesis, namely that, for every positive ϵ and every $\sigma \geq \frac{1}{2}$,

$$\zeta(\sigma + it) \ll t^{\epsilon}.$$

Titchmarch showed further that the truth of the Riemann Hypothesis implies the Lindelöf Hypothesis and

$$\zeta(\frac{1}{2} + it) = O\left(\exp\left\{A\frac{\log t}{\log\log t}\right\}\right),$$

where A is a constant. Recently, Huxley [26] showed the best unconditional result in that direction, namely that, for every $\epsilon > 0$,

$$\zeta(\frac{1}{2} + it) \ll t^{\frac{32}{205}+\epsilon}.$$

In view of this it is natural to consider the order of $| \zeta(s) |$ with respect to the Lindelöf Hypothesis.

Lemma 1.1. *For $t \geq t_0 > 0$, uniformly in σ,*

$$\zeta(\sigma + it) \ll \begin{cases} 1 & \text{for } \sigma \geq 2 \\ \log t & \text{for } 1 \leq \sigma < 2 \\ t^{(1-\sigma)/2}\log t & \text{for } 0 \leq \sigma < 1 \\ t^{1/2-\sigma}\log t & \text{for } \sigma \leq 0, \end{cases}$$

and if $\mu(\sigma)$ is defined by

$$\mu(\sigma) = \lim_{t\to\infty}\sup\frac{\log | \zeta(\sigma + it) |}{\log t},$$

then $\mu(\sigma)$ is continuous, non-increasing and for $\sigma_1 \leq \sigma \leq \sigma_2$,

$$\mu(\sigma) \leq \mu(\sigma_1)\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} + \mu(\sigma_2)\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}$$

(see [27] Theorem 1.9 p.25).

In view of the function $\mu(\sigma)$, the Lindelöf Hypothesis is equivalent to $\mu(\frac{1}{2}) = 0$, respectively

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{for } \sigma \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Titchmarch gave also various equivalent formulations of the Lindelöf Hypothesis in terms of mean-values, namely

$$\frac{1}{T}\int_1^T | \zeta(\frac{1}{2} + it) |^{2k} \, dt = O(T^{\epsilon}), \qquad k = 1, 2, ... \tag{1.5}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T | \zeta(\sigma + it) |^{2k} \, dt = \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}, \qquad \sigma > \frac{1}{2}, \qquad k = 1, 2, ...$$

where $d_k(n)$ denotes the number of representations of integer n as a product of k factors. We collect the so far achieved relevant mean-value results in the following

Lemma 1.2. *For fixed $\sigma > 1$ and a fixed integer $k \geq 1$ we have*

$$\int_0^T \mid \zeta(\sigma+it) \mid^{2k} dt = T\sum_{n=1}^{\infty} d_k^2(n) n^{-2\sigma} + O(T^{2-\sigma+\epsilon}) + O(1).$$

For $\frac{1}{2} < \sigma < 1$ fixed

$$\int_0^T \mid \zeta(\sigma+it) \mid^2 dt = \zeta(2\sigma)T + O(T^{2-\sigma}\log T),$$
$$\int_0^T \mid \zeta(1+it) \mid^2 dt = \zeta(2)T + O(\log^2 T),$$
$$\int_0^T \mid \zeta(\frac{1}{2}+it) \mid^2 dt = T\log T + O(T\log^{\frac{1}{2}} T)$$

(see [24]).

In addition, Titchmarch proved that, for every positive integer $k > 2$,

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T \mid \zeta(\sigma+it) \mid^{2k} dt = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}}$$

if $\sigma > 1 - \frac{1}{k}$. In the study (1.5), Ingham (see [27] Theorem 5.1 p.129) estimated that

$$\int_1^T \mid \zeta(\frac{1}{2}+it) \mid^4 dt = \frac{1}{2\pi^2} T\log^4 T + O(T\log^3 T).$$

For no integer $k \geq 3$ up to now any mean-value estimate comparable to the equivalent formulation of the Lindelöf Hypothesis (1.5) has been proven.

In addition, the Lindelöf Hypothesis has a connection with the function $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT)$. It is known that $S(T) = O(\log T)$, while the truth of the Lindelöf Hypothesis would imply $S(T) = o(\log T)$ (see [64]). Studying the value-distribution of the zeta-function, Ghosh [21] showed that

$$\text{meas}\{T < t < T+H :\mid S(T) \mid < \sigma\sqrt{\log\log t}\} := \left(\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma} e^{-x^2} dx + o(1)\right) H,$$

is valid for $T^\alpha < H < T$ and any fixed $\alpha > \frac{1}{2}$. Assuming the Riemann Hypothesis, Ghosh proved that this result holds for any fixed $\alpha > 0$.

The most simple, however sometimes useful, approximation for $\zeta(s)$ is given in the following

Lemma 1.3. *For $0 < \sigma_0 \leq \sigma \leq 2$, $x \geq \frac{|t|}{\pi}$, $s = \sigma + it$,*

$$\zeta(s) = \sum_{n\leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}),$$

where the O-constant depends only on σ_0 (see [27] Theorem 1.8 p.21).

1.2 Other zeta-functions

In this thesis we shall also consider other zeta-functions.

The Dirichlet L-function is the most common zeta-function besides $\zeta(s)$; it is defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s} \qquad\qquad (\sigma>1),$$

where χ is a Dirichlet character mod q, $q\geq 1$. Since the coefficients are strongly multiplicative, there exists an Euler product representation. For $\sigma>1$, the Euler product of the Dirichlet L-function is written as

$$L(s,\chi)=\prod_{p}(1-\chi(p)p^{-s})^{-1},$$

and in the case of the principal character χ_0 mod q we have

$$L(s,\chi_0)=\zeta(s)\prod_{p|q}(1-p^{-s}).$$

Hence $L(s,\chi_0)$ has a first-order pole at $s=1$ like $\zeta(s)$, while $L(s,\chi)$ for $\chi\neq\chi_0$ is regular for $\sigma>0$ (since $\mid\sum_{n\leq x}\chi(n)\mid\leq q$ for any x). Moreover, $L(s,\chi)$ has an analytic continuation to the whole complex plane and $L(s,\chi)\neq 0$ for $\sigma\geq 1$.

If χ is a primitive, non-principal character mod q, let $\alpha=1$ if $\chi(-1)=-1$, and $\alpha=0$ if $\chi(-1)=1$, as well as

$$G(\chi,\eta)=\sum_{r \mod q}\chi(r)\eta^{\sigma}, \qquad \eta=e(\frac{1}{q})=e^{\frac{2\pi i}{q}},$$

$$E(\chi)=\begin{cases}G(\chi,\eta)q^{\frac{-1}{2}} & \text{if } \alpha=0\\ iG(\chi,\eta)q^{\frac{-1}{2}} & \text{if } \alpha=1.\end{cases}$$

Then the functional equation for the Dirichlet L-function is given by

$$\left(\frac{\pi}{q}\right)^{-(\frac{\alpha+s}{2})}\Gamma(\frac{\alpha+s}{2})L(s,\chi)=E(\chi)\left(\frac{\pi}{q}\right)^{-(\frac{\alpha+1-s}{2})}\Gamma(\frac{\alpha+1-s}{2})L(1-s,\bar{\chi}).$$

From this it follows that the trivial zeros of $L(s,\chi)$ are given by $s=0,-2,-4,..$ if $\chi(-1)=1$ and by $s=-1,-3,...$ if $\chi(-1)=-1$.

The analytic behaviour of $L(s,\chi)$ is related to the distribution of primes in arithmetic progressions. An important result is the Siegel-Walfisz theorem which states that, for $(l,q)=1$,

$$\pi(x;q,l)=\sum_{\substack{p\leq x\\ p\equiv l \mod q}}1=\frac{\operatorname{li}x}{\phi(q)}+O(x\exp(-C\sqrt{\log x})),$$

$(C>0)$, uniformly for $3\leq q\leq\log x$, $l\leq q$, where $A>0$ is any fixed constant and $\operatorname{li}x=\int_0^x\frac{dt}{\log t}$.

The Hurwitz zeta-function is a generalization of the Riemann zeta-function, which is defined by

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s} \qquad (\sigma>1, 0<x\leq 1).$$

If χ is a character mod q, then for $\sigma>1$,

$$L(s,\chi)=q^{-s}\sum_{a=1}^{q-1}\chi(a)\zeta(s,\frac{a}{q}).$$

Obviously $\zeta(s)=\zeta(s,1)$. The Hurwitz zeta-function has an analytic continuation to the whole complex plane with a simple pole at $s=1$ with residue 1. The Laurent series for the Hurwitz zeta-function in the neighborhood of $s=1$ is given by

$$\zeta(s,x)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\gamma_n(x)(s-1)^n \qquad (0<x\leq 1),$$

where the coefficients γ_n are given by

$$\gamma_n(x)=\frac{(-1)^n}{n!}\lim_{N\to\infty}\left(\sum_{m\leq N}\frac{1}{m+x}\log^n(m+x)-\frac{\log^{n+1}(N+x)}{n+1}\right).$$

Let $1\leq h\leq k$ be integers. For complex s, the functional equation for the Hurwitz zeta-function is given by

$$\zeta(1-s,\frac{h}{k})=\frac{2\Gamma(s)}{(2\pi k)^s}\sum_{r=1}^{k}\cos\left(\frac{\pi s}{2}-\frac{2\pi rh}{k}\right)\zeta(s,\frac{r}{k}).$$

The behaviour of the Hurwitz zeta-function differs in some aspects from $\zeta(s)$, for example; it does not have an Euler product except for $x=1,\frac{1}{2}$.

The logarithmic derivative of the Riemann zeta-function is obtained by differentiating the logarithm of $\zeta(s)$. Since for $\sigma>1$ the Euler product of $\zeta(s)$ is absolutely convergent and in view of

$$\log(1-z)^{-1}=\sum_{k=1}^{\infty}\frac{z^k}{k},$$

valid for $\mid z\mid<1$, we have

$$\log\zeta(s)=\sum_p\log(1-p^{-s})^{-1}=\sum_p\sum_{k=1}^{\infty}k^{-1}p^{-ks}.$$

By differentiation, we have, for $\sigma>1$,

$$\frac{\zeta'}{\zeta}(s)=-\sum_p\sum_{k=1}^{\infty}(\log p)p^{-ks}.$$

The logarithmic derivative of the Riemann zeta-function is for $\sigma>1$ written as

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^{\infty}\Lambda(n)n^{-s},$$

where $\Lambda(n)$ is the von Mangoldt Λ-function. The following lemmata provide useful properties:

Lemma 1.4. *For $s = \sigma + it$, $0 < \sigma < 1$,*

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \log 2\pi - \frac{\gamma}{2} - 1$$

(see [27] p.17).

Lemma 1.5. *For $s = \sigma + it$, $-1 \le \sigma \le 2$,*

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho, |t-\gamma|<1} \frac{1}{s-\rho} + O(\log t)$$

(see [27] p.26).

We note that, for $-1 \le \sigma \le 2$, $\frac{\zeta'}{\zeta}(\sigma+it) \ll \log^2 t$, $t \ge 2$, since $\mid \gamma - t \mid \ll \frac{1}{\log t}$ for all $\rho = \beta + i\gamma$ and the number of summands is here $\ll \log t$.

1.3 The value-distribution of the Riemann zeta-function

The distribution of values of the Riemann zeta-function is an important topic. Selberg (unpublished, the published proof is due to Joyner [31]) proved that the values of the Riemann zeta-function on the critical line are Gauss-normal distributed after a suitable normalization. Namely, let $\mathcal{R}$ be an arbitrary fixed rectangle in the complex plane whose sides are parallel to the real and the imaginary axes, then

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{t \in (0,t] : \frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log T}}\right\} = \frac{1}{2\pi}\int\int_{\mathcal{R}}\exp\left(-\frac{1}{2}(x^2+y^2)\right)dxdy.$$

Bohr and his collaborators [9], [10], [12] and [13] studied the value-distribution of the Riemann zeta-function to the right of the critical line. Bohr and Jessen [13] proved that for every $\sigma > 1$ the set $\{\log\zeta(\sigma+it) : t \in \mathbb{R}\}$ is dense in an area of the complex plane which is either simply connected and bounded by a convex curve or which is ring-shaped and bounded by two convex curve. For denseness results on the vertical line in $\frac{1}{2} < \sigma \le 1$, Bohr and Courant [9] proved that, for every $\sigma \in [\frac{1}{2}, 1]$, every $a \in \mathbb{C}$ and every $\epsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\{t \in [1,\infty) :\mid \zeta(\sigma+it) - a \mid< \epsilon\} > 0.$$

Laurinčikas [39] obtained probabilistic limit theorems for the values of the zeta-function on vertical lines and showed that, for every $\sigma > \frac{1}{2}$, there exists a Borel probability measure μ_σ such that, for every continuous and bounded function $f : \mathbb{C} \to \mathbb{C}$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\log\zeta(\sigma+it))dt = \int_{\mathbb{C}} f(z)d\mu(z).$$

If $\sigma \in (\frac{1}{2}, 1]$, then the support of μ_σ is the whole complex plane. Voronin [68] extended Bohr's denseness result to a multidimensional theorem, namely

$$\overline{\{\Delta_n\{\zeta(\sigma+it) : t \in [1,\infty)\}\}} = \mathbb{C}^{n+1},$$

for every $\sigma \in (\frac{1}{2}, 1]$ and every $n \in \mathbb{N}_0$, where

$$\Delta_n\{\zeta(\sigma+it)\} = \{\zeta(\sigma+it), \zeta'(\sigma+it), ..., \zeta^{(n)}(\sigma+it) : t \in [0, \infty)\}.$$

For denseness results on vertical lines in $\sigma < 1$, Garunkštis and Steuding [20] proved that for $\sigma \leq 0$,

$$\overline{\{\zeta(\sigma+it) : t \in [1, \infty)\}} \neq \mathbb{C},$$

and the same for $\sigma < \frac{1}{2}$ under assumption of the Riemann Hypothesis. It seems that proving results about denseness or non-denseness on vertical lines is in general a difficult problem.

Among the various topics of studying the value-distribution of the zeta-function inside the critical strip we shall focus on discrete moments in this thesis.

1.4 Motive

The main topics of this thesis are related to the following two papers;

1. " Sampling the Lindelöf Hypothesis with the Cauchy random walk" of Lifshits and Weber [43]

2. " Sampling the Lindelöf Hypothesis with an ergodic transformation" of Steuding [60].

Both of them study certain discrete moments of the zeta-function associated with random sequences and ergodic transformations, respectively.

Firstly, we discuss some results about discrete moments of the zeta-function. Kalpokas and Steuding [34] investigated the intersection of the curve $\mathbb{R} \ni t \mapsto \zeta(\frac{1}{2}+it)$ with the real axis. They showed that if the Riemann Hypothesis is true, the mean-value of those real value exists and it equals to 1. Namely, for any $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{\substack{0<t<T \\ \zeta(\frac{1}{2}+it) \in e^{i\phi}\mathbb{R}}} \zeta(\frac{1}{2}+it) = 2e^{i\phi}\cos\phi \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(T^{\frac{1}{2}+\epsilon});$$

note that the number of values $0 < t < T$ with $0 \neq \zeta(\frac{1}{2}+it) \in e^{i\phi}\mathbb{R}$ is asymptotically equal to $\frac{T}{2\pi}\log T$. Moreover, Kalpokas and Steuding obtained also a new discrete mean-square result for the zeta-function, namely, for any $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{\substack{0<t<T \\ \zeta(\frac{1}{2}+it) \in e^{i\phi}\mathbb{R}}} | \zeta(\frac{1}{2}+it) |^2 = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2 + (2\gamma + 2\cos 2\phi)\frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{T}{2\pi} + O(T^{\frac{1}{2}+\epsilon}),$$

where γ is the Euler constant. Continuing this work, Kalpokas, Korolev and Steuding [35] showed unconditionally that the zeta-function takes arbitrarily large positive and negative values on the critical line. They established a lower bound of the expected order for the discrete moment with arbitrary rational exponents and showed that, for any rational $k \geq 1$ and any $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{0<t_n(\phi)\leq T} | \zeta(\frac{1}{2}+it_n(\phi)) |^{2k} \ll T(\log T)^{k^2+1},$$

where $t_n(\phi)$ denotes the positive roots of the equation $e^{2i\phi} = \Delta(\frac{1}{2}+it)$, for $n \in \mathbb{N}$, in ascending order $(\Delta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right))$. Furthermore they derived an asymptotic formula for the third discrete moment, namely, for any $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{0<t_n(\phi)\leq T} (\zeta(\frac{1}{2}+it_n(\phi)))^3 = 2e^{3i\phi}\cos\phi\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right) + 2e^{3i\phi}\cos 3\phi\frac{T}{2\pi}\log\frac{T}{2\pi e} + O(T^{\frac{1}{2}+\epsilon}),$$

where P_3 is a computable polynomial of degree three.

Steuding and Wegert [61] investigated the values of the zeta-function on certain arithmetic progressions on vertical lines in the critical strip. They gave an interesting result about the mean-value depending on the difference of the arithmetic progression in a rather irregular way. Namely, for fixed $s \in \mathbb{C}\setminus\{1\}$ with $0 < \sigma = \Re(s) \leq 1$, $t = \Im(s) > 0$, let $d = \frac{2\pi}{\log l}$, where $l \geq 2$ is an integer; then, for $M \to +\infty$,

$$\frac{1}{M}\sum_{0\leq m<M}\zeta(s+imd) = \frac{1}{1-l^{-s}} + O(M^{-\sigma}\log M).$$

This shows that the zeta-function is small on average on such samples on vertical lines inside the critical strip. We shall observe a similar phenomenon in the study about the discrete moments of the zeta-function associated with random sequences or with an ergodic transformation.

For the discrete moment of the Riemann zeta-function associated with a random sequence Lifshits and Weber [43] studied the behaviour of the Riemann zeta-function $\zeta(\frac{1}{2}+it)$, when t is sampled by a Cauchy random walk. The Cauchy random walk S_n is defined by

$$S_n = X_1 + X_2 + ... + X_n, \qquad n = 1, 2, ...,$$

where $X_1, X_2, ...$ denotes an infinite sequence of independent Cauchy-distributed random variables (with characteristic function $\varphi(t) = e^{-|t|}$). The work of Lifshits and Weber shows the almost sure asymptotic behaviour of the system

$$\zeta(\frac{1}{2}+iS_n), \qquad n = 1, 2, ...$$

For this purpose they developed a complete second-order theory for this system and showed, by using an approximation formula of $\zeta(s)$, that it behaves almost surely like a system of non-correlated variables. For almost sure convergence, they proved that, for any real $b > 2$,

$$\sum_{k=1}^{n}\zeta(\frac{1}{2}+iS_k) \overset{\text{(a.s.)}}{=} n + O(n^{\frac{1}{2}}(\log n)^b).$$

In view of the almost sure convergence theorem of Lifshits and Weber, it follows that the expectation value of $\zeta(s)$ on a Cauchy random walk $s = \frac{1}{2}+iS_n$ is equal to one. This indicates that the values of the zeta-function are small on average. It is natural to ask what happens if we consider a Dirichlet L-function in place of the Riemann zeta-function? The answer of this question is part of this thesis.

Shirai [56] extended their work to a subclass of Lévy processes, and obtained the following almost sure convergence: let S_k be a symmetric α-stable process with $1 \leq \alpha \leq 2$. Then,

$$\sum_{k=1}^{n}\zeta(\frac{1}{2}+iS_k) \overset{\text{(a.s.)}}{=} n + O(n^{1-\frac{1}{2\alpha}}(\log n)^b),$$

for any $b > \frac{3}{2}$ if $1 < \alpha \le 2$; the same result holds for any $b > 2$ if $\alpha = 1$.

We note that the class of symmetric α-stable processes includes the Cauchy random walk in case of $\alpha = 1$ and the Brownian motion in case of $\alpha = 2$. By applying an extension of Rademacher-Menchoff type, Shirai showed that the expectation of

$$\frac{1}{n}\sum_{k=1}^{n}\zeta(\frac{1}{2}+iS_k)$$

equals one, is independent of α, and the only impact of α is visible in the remainder term. Moreover, Shirai listed some problems related to his topic; for example, he asked what can be said if Dirichlet L-function are considered in place of the Riemann zeta-function.

Sihun Jo and Minsuk Yang [30] investigated the second moment of the random sampling $\zeta(\frac{1}{2}+iX_t)$ of the Riemann zeta-function on the critical line, where X_t is a gamma process. They proved that if X_t is an increasing random sampling with gamma distribution, then for all sufficiently large t,

$$\mathbb{E}\left|\zeta(\frac{1}{2}+iX_t)\right|^2 = \log t + O(\sqrt{\log t}\log\log t).$$

They remarked that their probabilistic result is similar to the famous result obtained by Hardy and Littlewood [24] that, as $T \to \infty$,

$$\frac{1}{T}\int_0^T \left|\zeta(\frac{1}{2}+it)\right|^2 dt = \log T + O(1).$$

In fact, we can find this similarity in the work of Lifshits and Weber [43]. In case of the Cauchy random walk S_n, Lifshits and Weber proved that, for $n \ge 2$,

$$\mathbb{E}\left|\zeta(\frac{1}{2}+iS_n)\right|^2 = \log n + \gamma - 1 + +2\int_0^1 \phi(\alpha)d\alpha + 2\int_1^\infty \left(\phi(\alpha) - \frac{1}{2\alpha}\right)d\alpha,$$

where γ is the Euler constant and $\phi(\alpha)$ is defined by

$$\phi(\alpha) = \frac{\alpha e^\alpha - 2e^\alpha + \alpha + 2}{2\alpha^2(e^\alpha - 1)}.$$

In addition, the work of Sihun Jo and Minsuk Yang is interesting with respect to their method of proof. They begin by analytically extending the zeta-function to a suitable form and then investigate the moment of the sampling $\zeta(\frac{1}{2}+iX_t)$. In order to find the asymptotic formula for $\mathbb{E}\left|\zeta(\frac{1}{2}+iX_t)\right|^2$, they apply Fourier transformation and use the van der Corput method.

Another motive of this thesis follows from Steuding's article [60] "The distribution of the Riemann zeta-function $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to the ergodic transformation" (given by $Tx := \frac{1}{2}\left(x - \frac{1}{x}\right)$ for $x \neq 0$). Steuding showed that, for $\Re(s) > -\frac{1}{2}$, the mean-value of $\zeta(s+iT^n x)$ exists for almost all values $x \in \mathbb{R}$, as $n \to \infty$, and is independent of x. Moreover, he determined also the exact value of the mean-value of $\zeta(s+iT^n x)$; for example,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(2+iT^n x) = \zeta(3) = 1.20205...,$$

for almost all $x \in \mathbb{R}$. Moreover, the zeta-function is small on average on vertical lines inside the critical strip, since Steuding also obtained similarly that, for almost all $x \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(\frac{1}{2} + iT^n x) = \zeta(\frac{3}{2}) - \frac{8}{3} = -0.05429... .$$

Interestingly, in the situation of Lifshits and Weber [43], the expectation of $\zeta(\frac{1}{2} + iX)$ equals also $\zeta(\frac{3}{2}) - \frac{8}{3} = -0.05429...$, for X being a Cauchy distributed random variable. Moreover, Steuding applied the analyticity of $\frac{1}{\pi}\int_{\mathbb{R}} \zeta(s+i\tau)\frac{d\tau}{1+\tau^2}$ to show that

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} = \frac{\int_{\mathbb{R}} \zeta'(s+i\tau)\frac{d\tau}{1+\tau^2}}{\int_{\mathbb{R}} \zeta(s+i\tau)\frac{d\tau}{1+\tau^2}}$$

valid for $\Re(s) > 1$. It is a motive of this thesis to study the distributation of values of the logarithmic derivative of the Riemann zeta-function $\frac{\zeta'}{\zeta}(s)$ on vertical lines with respect to the ergodic transformation. In the study of the behaviour of the function $\frac{\zeta'}{\zeta}(s)$ via this approach, we shall prove an equivalent formulation of the Riemann Hypothesis in terms of the ergodic transformation. In addition, Steuding gave a similar equivalent formulation of the Riemann Hypothesis in terms of the ergodic transformation, i.e., for almost all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \log\left|\zeta(\frac{1}{2} + \frac{iT^n x}{2})\right| = \sum_{\Re(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|;$$

in particular, the Riemann hypothesis is true if, and only if, one and thus either side vanishes, the left-hand side for almost all real x. The vanishing aspect follows from a result [6] of Balazard, Saias and Yor, namely

$$\frac{1}{2\pi}\int_{\Re s=\frac{1}{2}} \frac{\log \mid \zeta(s) \mid}{\mid s \mid^2} \mid ds \mid = \sum_{\Re(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|.$$

In this thesis, we shall consider this theme with $\log\zeta(s)$ in place of the function $\log \mid \zeta(s) \mid$ inside the critical strip. In his work Steuding stated an equivalent formulation of the Lindelöf Hypothesis too, namely, the Lindelöf hypothesis is true if, and only if, for any $k \in \mathbb{N}$ and almost all $x \in \mathbb{R}$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \left|\zeta(\frac{1}{2} + iT^n x)\right|^{2k}$$

exists, which is also equivalent to the existence of the integrals

$$\int_{\mathbb{R}} \left|\zeta(\frac{1}{2} + it)\right|^{2k} \frac{dt}{1+t^2}.$$

Steuding gave the beautiful proof but did not provide an explicit formula for the latter integral. In this thesis some explicit formulas related to this and similar integrals are considered in case of $k = 1$. Finally, Steuding investigated also the behaviour of other functions than $\zeta(s)$ under the ergodic transformation, namely the Lerch zeta-function $L(\lambda, \alpha, s)$ with real parameters $\lambda, \alpha > 0$ and the Hurwitz zeta-function, for which an ergodic transformation of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has been considered.

Another topic of this thesis deals with some connections between universality theory and ergodic theory. The ergodic theorem plays a central role in probabilistic proofs of universality properties of $\zeta(s)$ due to Bagchi [3]; for details we refer to Steuding [59] and Laurinčikas [39]. Steuding [63] proved a new type of universality theorem for the Riemann zeta-function and other L-function by investigating the phenomenon of universality on orbits of certain ergodic transformations. Firstly, he introduced some notion to abbreviate the formulation of his results. Namely, for a domain $\mathcal{D}$, a family of analytic functions $\mathcal{L}_1, ..., \mathcal{L}_m$ is called a jointly universal family with respect to $\mathcal{D}$ if, for any collection of compact subsets $\mathcal{K}_1, ..., \mathcal{K}_m$ of $\mathcal{D}$ with connected complements, any family of continuous functions f_j defined on $\mathcal{K}$ which is analytic and non-vanishing in the interior of $\mathcal{K}$, any ergodic dynamical system $(\mathbb{R}, \mathcal{F}, \mathbb{P}, T)$, almost all real numbers x, and any poitive ϵ, there exists a positive integer n such that

$$\max_{1 \leq j \leq m} \max_{s \in \mathcal{K}_j} \mid \mathcal{L}_j(s + iT^n x) - f_j(s) \mid < \epsilon$$

holds. Here, the notion "for almost all" is an abbreviation for "all real numbers except a set of $\mathcal{P}$-measure zero". A family of analytic functions $\mathcal{L}_1, ..., \mathcal{L}_m$ is called a jointly universal family with respect to $\mathcal{D}$ if the assumption on the non-vanishing of the target functions f_j can be dropped. Then, he proved that, for $(\mathbb{R}, \mathcal{F}, \mathbb{P}, T)$ an ergodic system where $\mathbb{P}$ is a probability measure with a positive density function, a family of L-functions is jointly (strongly) universal with respect to some domain $\mathcal{D}$ if, and only if, it is jointly (strongly) ergodic universal with respect to $\mathcal{D}$; in this case,

$$\liminf_{N \to \infty} \frac{1}{N} \sharp \left\{ n \in \mathbb{N} \cap [1, N] : \max_{1 \leq j \leq m} \max_{s \in \mathcal{K}_j} \mid \mathcal{L}_j(s + iT^n x) - f_j(s) \mid < \epsilon \right\} > 0.$$

This investigation of Steuding shows that universality is a kind of ergodic phenomenon. In order to understand universality properties of zeta-functions from the viewpoint of dynamical systems, we shall investigate the phenomenon of universality with respect to certain stochastic processes in this thesis.

1.5 Statement of the main results

This thesis is devided into three parts. In the first part, we study the asymptotic behaviour of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$ by modelling the imaginary part t with a Cauchy random walk (see [58]). We emulate the technique of Lifshits and Weber [43] for the Hurwitz zeta-function $\zeta(s, a)$ and arrive at an analogous result in this case: *for any real* $b > 2$ *and* $0 < a \leq 1$,

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} a^{\frac{1}{2} + iS_k} \zeta(\frac{1}{2} + iS_k, a) - n}{n^{\frac{1}{2}} \log(1+n)^b} \qquad \xrightarrow{a.s.} \qquad 0,$$

and

$$\| \sup_{n \geq 1} \frac{| \sum_{k=1}^{n} a^{\frac{1}{2} + iS_k} \zeta(\frac{1}{2} + iS_k, a) - n|}{n^{\frac{1}{2}} \log(1+n)^b} \|_2 \qquad < \infty.$$

Moreover, we use Atkinson's formula [2] instead of the technique of Lifshits and Weber in case of Dirichlet L-functions $L(s, \chi)$ associated with a primitive character χ. To consider only primitive characters is sufficient in order to understand all Dirichlet L-functions. Here we obtain, *for any real* $b > 2$, $\sigma \geq \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} L(\sigma + iS_k, \chi) - n}{n^{\frac{1}{2}} \log(1+n)^b} \qquad \overset{a.s.}{\longrightarrow} \qquad 0,$$

and

$$\| \sup_{n\geq 1} \frac{|\sum_{k=1}^{n} L(\sigma + iS_k, \chi) - n|}{n^{\frac{1}{2}} \log(1+n)^b} \|_2 \qquad < \infty.$$

Both results indicate that, by the almost sure convergence theorem of Lifshits and Weber, the expectation value of $\zeta(s,a)$, $0 < a \leq 1$ and $L(s,\chi)$ with a primitive character χ on the Cauchy random walk also is to equal is to one. Heuristically, its always expectation one because of the constant term in the Dirichlet series expansion. This also shows that the values of these zeta-function are small on average.

In the second part, we study the behaviour of the logarithmic derivative zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$, when values $\frac{\zeta'}{\zeta}(\sigma + it)$ are sampled with t varying according to an ergodic transformation. Similar as in the work of Steuding we obtain: *let s be given with $\Re(s) > -\frac{1}{2}$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(s + iT^n x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s + i\tau) \frac{d\tau}{1+\tau^2} \qquad \textit{for almost all} \qquad x \in \mathbb{R}.$$

For $-\frac{1}{2} < \Re(s) < 1$, $\Re(s) \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(s + iT^n x) = \frac{\zeta'}{\zeta}(s+1) + \frac{2}{s(2-s)} - \sum_{\substack{\rho \\ \Re(\rho)=\Re(s)}} \frac{1}{1-(s-\rho)^2}$$
$$- \sum_{\substack{\rho \\ \Re(\rho)>\Re(s)}} \frac{2}{1-(s-\rho)^2},$$

where ρ denotes the non-trivial zeros of ζ.
For $\Re(s) > 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(s + iT^n x) = \frac{\zeta'}{\zeta}(s+1).$$

For the special case $s = 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(iT^n x) = \sum_{\rho} (\frac{1}{\rho} - \frac{1}{\rho+1}) + \log 2\pi - \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(\frac{3}{2}) + \gamma + 1\right).$$

For some real t,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(1 + i(t + T^n x)) = \frac{\zeta'}{\zeta}(2+it) + \frac{1}{1+t^2}.$$

Here, we find an equivalent formulation for the Riemann Hypothesis in terms of an ergodic transformation: *the Riemann Hypothesis is true if, and only if, for almost all* $x \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(\frac{1}{2} + iT^n x) = \frac{\zeta'}{\zeta}(\frac{3}{2}) + \frac{8}{3} - \sum_{\rho} \frac{1}{1-(\rho-\frac{1}{2})^2},$$

where the ρ *denotes the non-trivial zeros of* ζ. We also study the behaviour of the logarithmic of zeta-functions in this sense by using a lemma of Kai-Man Tsang [65] and obtain: *for* $\frac{1}{2} \leq \Re(s) \leq 2$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(s + iT^n x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log\zeta(2 + i(\Im(s)+u))}{1 + (i(\Re(s)-2)+u)^2} du$$
$$+ 2 \sum_{\substack{\rho \\ \Re(\rho)>\Re(s)}} \int_0^{\Re(\rho)-\Re(s)} \frac{d\alpha}{1+(\Im(\rho-s)-i\alpha)^2} - 2\int_{\min(1-\Re(s),0)}^{1-\Re(s)} \frac{d\alpha}{1+(\Im(s)+i\alpha)^2},$$

for almost all $x \in \mathbb{R}$. We also give an equivalent formulation for the Riemann Hypothesis in terms of ergodic transformation. *The Riemann Hypothesis is true if, and only if, for almost all* $x \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(\frac{1}{2} + iT^n x) = \log\zeta(\frac{3}{2}) - \log 3 = -0.138352....$$

In the third part of this thesis, we investigate the phenomenon of universality with respect to certain stochastic processes (see [57]). We shall prove: *assume that* Λ *is a lattice given by* (4.1) *and* $(s_n)_n$ *is a random walk on this lattice, defined by* (4.2). *Further suppose that* K *is a compact set with connected complement satisfying* (4.6), *and* g *is a non-vanishing continuous function on* K *which is analytic in the interior of* K. *Then, for any* $\epsilon > 0$, *almost surely*

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ n \leq N : \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon \right\} > 0.$$

Chapter 2

2 Sampling the Lindelöf Hypothesis with the Cauchy random walk

In this chapter, we study the asymptotic behaviour of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$, by modelling the imaginary part t with a Cauchy random walk. Let $X_1, X_2, ...$ denote an infinite sequence of independent Cauchy-distributed random variables (with characteristic function $\varphi(t) = e^{-|t|}$); then the imaginary part t is modelled by the sequence of partial sum $S_n = X_1 + ... + X_n$. This idea is due to Lifshits and Weber [43] in the investigation of the almost sure asymptotic behaviour of the system

$$\zeta(\frac{1}{2} + iS_n), \qquad n = 1, 2, ...$$

They proved that almost surely

$$\frac{1}{N} \sum_{1 \leq n \leq N} \zeta(\frac{1}{2} + iS_n) = 1 + o(N^{-\frac{1}{2}} (\log N)^b) \tag{2.1}$$

for any $b > 2$. Hence, Lifshits and Weber showed that the expectation value of $\zeta(s)$ on the Cauchy random walk $s = \frac{1}{2} + iS_n$ equals one, which implies that the values of the Riemann zeta-function are small on average.

In Section 2.1, we briefly discuss the technique of Lifshits and Weber in the investigation of the asymptotic behaviour for the Riemann zeta-function $\zeta(s)$.

In Section 2.2, we emulate the technique of Lifshits and Weber for the Hurwitz zeta-function $\zeta(s, a)$.

In Section 2.3, we use Atkinson's formula [2] and not the technique of Lifshits and Weber in order to derive corresponding results for Dirichlet L-functions $L(s, \chi)$ with a primitive character χ.

2.1 Sampling the Lindelöf Hypothesis for the Riemann zeta-function $\zeta(s)$

In this section, we discuss the technique of Lifshits and Weber.

The result (2.1) is based on a Proposition of Weber [69], namely

Proposition 2.1. *(Weber, 2006). Let $\{m_l, l \geq 1\}$ be a sequence of positive reals with partial sums $M_n = \sum_{l=1}^{n} m_l$ tending to infinity with n. Assume that*

$$\log \frac{M_n}{m_n} \sim \log M_n.$$

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave non-decreasing function. Then any sequence $\{\xi_l, l \geq 1\}$ of random variables satisfying the increment condition

$$\mathbb{E} \left| \sum_{l=i}^{j} \xi_l \right|^2 \leq \Phi\left(\sum_{l=1}^{j} m_l \right) \left(\sum_{l=i}^{j} m_l \right)$$

$(i \leq j)$ also satisfies for any $\tau > \frac{3}{2}$

$$\frac{\sum_{l=1}^{n}\xi_l}{\Phi(M_n)^{\frac{1}{2}}\log^{\tau}(1+M_n)M_n^{\frac{1}{2}}} \qquad \xrightarrow{a.s.} \qquad 0$$

and

$$\| \sup_{n\geq 1} \frac{|\sum_{l=1}^{n}\xi_l|}{\Phi(M_n)^{\frac{1}{2}}\log^{\tau}(1+M_n)M_n^{\frac{1}{2}}} \|_2 \qquad < \infty.$$

In order to investigate the almost-sure asymptotic behaviour of the system $\zeta(\frac{1}{2}+iS_n)$, $n = 1, 2, ...$, the increment condition in Proposition 2.1 is necessary. Lifshits and Weber replaced the sequence $\{\xi_n, n \geq 1\}$ by $\{W_n = \zeta(\frac{1}{2}+iS_n) - \mathbb{E}\zeta(\frac{1}{2}+iS_n), n \geq 1\}$ and developed a complete second-order theory of the system $\{W_n\}$. They obtained the following

Theorem 2.2. *(Lifshits and Weber, 2009). There exist a constant C, C_0 such that*

$$\mathbb{E}|W_n|^2 = \log n + C + o(1), \qquad n \to \infty,$$

and for $m > n+1$

$$|\mathbb{E}W_n\overline{W}_m| \leq C_0 \max\left(\frac{1}{2^{m-n}}, \frac{1}{n}\right).$$

The explicit value of C is

$$C = \gamma - 1 + 2\int_0^1 \phi(\alpha)d\alpha + 2\int_1^{\infty}\left(\phi(\alpha) - \frac{1}{2\alpha}\right)d\alpha,$$

where γ is the Euler constant and $\phi(\alpha)$ is defined by

$$\phi(\alpha) = \frac{\alpha e^{\alpha} - 2e^{\alpha} + \alpha + 2}{2\alpha^2(e^{\alpha}-1)}.$$

They applied this result to Proposition 2.1 with the choice $m_l \equiv 1$ and $\Phi(x) = \log(x+1)$ and obtained

Theorem 2.3. *(Lifshits and Weber, 2009). For any real $b > 2$*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\zeta(\frac{1}{2}+iS_k) - n}{n^{\frac{1}{2}}\log^{b} n} \qquad \xrightarrow{a.s.} \qquad 0,$$

and

$$\| \sup_{n\geq 1} \frac{|\sum_{k=1}^{n}\zeta(\frac{1}{2}+iS_k) - n|}{n^{\frac{1}{2}}\log^{b} n} \|_2 \qquad < \infty.$$

2.1.1 Sketch of the proof of Theorem 2.2

In order to estimate the covariance of the system $\{W_n = \zeta(\frac{1}{2} + iS_n) - \mathbb{E}\zeta(\frac{1}{2} + iS_n), n \geq 1\}$, Lifshits and Weber approximate $\zeta(\frac{1}{2} + iS_n)$ from the main terms of the approximation of $\zeta(s)$, (see Lemma 1.3). Therefore, the second-order theory of the system (W_n) follows from a study of the same kind concerning the auxiliary system

$$Z_n = \sum_{k\leq x} k^{-(\sigma+iS_n)} - \frac{x^{1-\sigma-iS_n}}{1-\sigma-iS_n}, \quad n = 1, 2, ..., \quad x > \frac{1}{2}.$$

Using the fourth moments estimate for ζ (see [27] Theorem 5.1 p.129),

$$\int_{-T}^{T} \mid \zeta(\frac{1}{2} + iu) \mid^4 du = O(T \log^4 T),$$

it follows that $Z_n(x)$ approximates the zeta-function well enough, that is for each positive integer n,

$$\lim_{x\to\infty} \mathbb{E} \mid Z_n(x) - \zeta(\frac{1}{2} + iS_n) \mid^2 = 0.$$

From this follow that

$$\mathbb{E}\zeta(\frac{1}{2} + iS_n) = \lim_{x\to\infty} \mathbb{E}Z_n(x)$$

and for any positive integer $m > n + 1$

$$\mathbb{E}\zeta(\frac{1}{2} + iS_n)\overline{\zeta(\frac{1}{2} + iS_m)} = \lim_{x\to\infty} \mathbb{E}Z_n(x)\overline{Z_m(x)}.$$

In order to calculate the first and second order moments of Z_n, and the correlation $\mathbb{E}Z_n(x)\overline{Z_m(x)}$, Lifshits and Weber defined $Z_n(x) = Z_n = Z_{n1} - Z_{n2}$ with

$$Z_{n1} = Z_{n1}(x) = \sum_{k\leq x} e^{-i(\log k)S_n} k^{-\sigma},$$

$$Z_{n2} = Z_{n2}(x) = \frac{x^{1-\sigma}}{(1-\sigma-iS_n)} e^{-i(\log x)S_n}.$$

Concerning the first moments, they obtained

Lemma 2.4. *For $x \to \infty$ we have*

$$\mathbb{E}Z_n \to \zeta(n+\sigma, a) - \frac{2n}{n^2 - (1-\sigma)^2},$$

for any integer n and $\sigma > 0$.

Hence, for $\sigma = \frac{1}{2}$,

$$\mathbb{E}\zeta(\frac{1}{2} + iS_n) = \lim_{x\to\infty} \mathbb{E}Z_n(x) = \zeta(n + \frac{1}{2}) - \frac{8n}{4n^2 - 1}. \tag{2.2}$$

For the second order moments Lifshits and Weber put

$$\mathbb{E}|Z_n|^2 = \mathbb{E}|Z_{n1}|^2 + \mathbb{E}|Z_{n2}|^2 - 2\Re\mathbb{E}Z_{n1}\bar{Z}_{n2}, \tag{2.3}$$

$$\mathbb{E}Z_n\bar{Z}_m = \mathbb{E}Z_{n1}\bar{Z}_{m1} - \mathbb{E}Z_{n1}\bar{Z}_{m2} - \mathbb{E}Z_{n2}\bar{Z}_{m1} + \mathbb{E}Z_{n2}\bar{Z}_{m2}, \tag{2.4}$$

and calculated explicit asymptotic formulas for these terms. They obtained the following

- **Exact formulae related to Z_{n2}**

Lemma 2.5. *For $m = n$ and for $m > n+1$ we have*

$$\mathbb{E} Z_{n2} \bar{Z}_{m2} = A + B x^{-n+1-\sigma} + C x^{-(m-n)+2(1-\sigma)},$$

where

$$A = \frac{4n(m-n)}{((m-n)^2 - 4(1-\sigma)^2)(n^2 - (1-\sigma)^2)},$$
$$B = \frac{2(m-n)}{(2n - m + (1-\sigma))(m+1-\sigma))(n-(1-\sigma))},$$
$$C = \frac{3n - m + 2(1-\sigma)}{(2(1-\sigma) - m + n)(2n - m + 1 - \sigma))(n + (1-\sigma))}.$$

If $m = n$,

$$\mathbb{E} Z_{n2} \overline{Z}_{n2} = \frac{x^{2(1-\sigma)}}{(1-\sigma)(n+1-\sigma)}. \tag{2.5}$$

For all $m \geq n+1$ we have

$$\mathbb{E} Z_{n1} \overline{Z}_{m2} = \sum_{k \leq x} \left\{ \frac{-2(m-n) k^{-n-\sigma}}{(m+1-\sigma)(2n-m+1-\sigma)} + \frac{2n k^{n-m+1-2\sigma}}{(m-1+\sigma)(2n-m+1-\sigma)} \right\}$$
$$- \sum_{k \leq x} \left\{ \frac{k^{n-\sigma} x^{-m+1-\sigma}}{(m-1+\sigma)} \right\},$$

and

$$\mathbb{E} Z_{m1} \overline{Z}_{n2} = \sum_{k \leq x} \left\{ \frac{2n k^{n-m+1-2\sigma}}{n^2 - (1-\sigma)^2} - \frac{k^{2n-m-\sigma} x^{-n+1-\sigma}}{(n-1+\sigma)} \right\}.$$

The behaviour of these expressions as $x \to \infty$ has been investigated by Lifshits and Weber only for $\sigma = \frac{1}{2}$.

- **Asymptotic formulae related to Z_{n2}**

Lemma 2.6. *Assume $\sigma = \frac{1}{2}$. For $m > n+1$*

$$\mathbb{E} Z_{n2} \overline{Z}_{m2} = \frac{4n(m-n)}{((m-n)^2 - 1)(n^2 - \frac{1}{4})} + o(1), \qquad x \to \infty, \tag{2.6}$$

while

$$\mathbb{E} Z_{n2} \overline{Z}_{n2} = \frac{2x}{n + \frac{1}{2}}. \tag{2.7}$$

For $m > n+1$ we obtain

$$\mathbb{E} Z_{n1} \overline{Z}_{m2} = \frac{-2(m-n)\zeta(n+\frac{1}{2})}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} + \frac{2n\zeta(m-n)}{(m-\frac{1}{2})(2n-m+\frac{1}{2})} + o(1), \qquad x \to \infty, \tag{2.8}$$

$$\mathbb{E} Z_{n1} \overline{Z}_{n2} = \frac{2x}{n+\frac{1}{2}} - \frac{1}{2n-1} + o(1), \qquad x \to \infty, \tag{2.9}$$

$$\mathbb{E} Z_{m1} \overline{Z}_{n2} = \frac{2n\zeta(m-n)}{n^2 - \frac{1}{4}} + o(1), \qquad x \to \infty. \tag{2.10}$$

For the double sum, Lifshits and Weber obtained

- **Asymptotic formulae related to** Z_{n1}

Lemma 2.7. *Let* $\sigma = \frac{1}{2}$*. For* $m > n+1$

$$\mathbb{E}Z_{n1}\overline{Z}_{m1} = \zeta(m-n+1) + \theta\left(\frac{1}{m-\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\zeta(m-n) + o(1), \qquad x \to \infty, \tag{2.11}$$

with $\theta = \theta(n,m) \in [0,1]$.

$$\mathbb{E}Z_{n1}\overline{Z}_{n1} = \frac{2x}{n+\frac{1}{2}} + K_n + o(1), \qquad x \to \infty \tag{2.12}$$

with

$$K_n = \log n + C + o(1), \qquad n \to \infty$$

and

$$C = \gamma - 1 + 2\left\{\int_0^1 \phi(\alpha)d\alpha + \int_1^\infty \left(\phi(\alpha) - \frac{e^\alpha - 1}{2\alpha(e^\alpha - 1)}\right)d\alpha\right\}. \tag{2.13}$$

In view of (2.2)

$$\begin{aligned}\mathbb{E}W_n\overline{W}_m &= \mathbb{E}\zeta_n\overline{\zeta_m} - \mathbb{E}\zeta_n\overline{\mathbb{E}\zeta_m} \\ &= \lim_{x\to\infty}\mathbb{E}Z_n(x)\overline{Z}_m(x) - \left(\zeta(n+\frac{1}{2}) - \frac{8n}{4n^2-1}\right)\left(\zeta(m+\frac{1}{2}) - \frac{8m}{4m^2-1}\right).\end{aligned}$$

The first claim of Theorem 2.2 follows from (2.12), (2.7), (2.9) and (2.3),

$$\begin{aligned}\mathbb{E}|Z_n(x)|^2 &= \frac{2x}{n+\frac{1}{2}} + K_n + \frac{2x}{n+\frac{1}{2}} - \frac{4x}{n+\frac{1}{2}} + \frac{2}{2n-1} + o(1) \\ &= K_n + \frac{2}{2n-1} + o(1), \qquad x \to \infty.\end{aligned}$$

Hence,

$$\mathbb{E}|W_n|^2 = K_n + \frac{2}{2n-1} - \left(\zeta(n+\frac{1}{2}) - \frac{8n}{4n^2-1}\right)^2.$$

The last claim of Theorem 2.2 follows from (2.4), (2.11), (2.8), (2.10) and (2.6) with a suitable approximation argument.

2.2 Sampling the Lindelöf Hypothesis for the Hurwitz zeta-function $\zeta(s,a)$

In the previous section we have considered the behaviour of $\zeta(s)$ with respect to a Cauchy random walk on critical line. In order to emulate the technique of Lifshits and Weber for the Hurwitz zeta-function $\zeta(s,a)$, where $0 < a \le 1$, we consider

$$W_n(a) := \zeta(\frac{1}{2} + iS_n, a) - \mathbb{E}\zeta(\frac{1}{2} + iS_n, a), \qquad n \ge 1.$$

We can state an analogous results for the Hurwitz zeta-function.

Theorem 2.8. *For $0 < a \le 1$ there exist a constant C, C_0 such that*

$$\mathbb{E}|W_n(a)|^2 = \log(n+a) + C + o(1), \qquad n \to \infty,$$

and for $m > n+1$

$$|\mathbb{E}W_n(a)\overline{W_m(a)}| \le C_0 \max\left(\frac{1}{(1+a)^{m-n}}, \frac{1}{n}\right).$$

The explicit value of C is

$$C = \gamma(a) - \frac{1}{a} + 2\int_0^1 \phi(\alpha)d\alpha + 2\int_1^\infty \left(\phi(\alpha) - \frac{1}{2\alpha}\right) d\alpha,$$

where $\gamma(a)$ is the generalized Euler constant defined by

$$\gamma(a) = \lim_{N\to\infty}\left(\sum_{m=0}^{N} \frac{1}{m+a} - \log(N+a)\right).$$

Applying Proposition 2.1 we obtain

Theorem 2.9. *For any real $b > 2$ and $0 < a \le 1$,*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a^{\frac{1}{2}+iS_k}\zeta(\frac{1}{2}+iS_k, a) - n}{n^{\frac{1}{2}}\log(1+n)^b} \qquad \xrightarrow{a.s.} \qquad 0,$$

and

$$\| \sup_{n\ge1} \frac{|\sum_{k=1}^n a^{\frac{1}{2}+iS_k}\zeta(\frac{1}{2}+iS_k, a) - n|}{n^{\frac{1}{2}}\log(1+n)^b} \|_2 \qquad < \infty.$$

Building on the technique of Lifshits and Weber, we need an approximation of the Hurwitz zeta-function to estimate the covariance of the system $(W_n(a))$. For this purpose we use

$$\zeta(s,a) = \sum_{m\le x}(m+a)^{-s} + \frac{(x+a)^{1-s}}{s-1} + O(x^{-\sigma}),$$

which holds uniformly for $\sigma \ge \sigma_0 > 0$, $2\pi \le |t| \le \pi x$ (see [40] Theorem 1.3, p.34). Then, the second-order theory of the system $(W_n(a))$ follows from a study of the same kind concerning the auxiliary system

$$Z_n(x,a) = \sum_{0\le k\le x}(k+a)^{-(\sigma+iS_n)} - \frac{(x+a)^{1-\sigma-iS_n}}{(1-\sigma-iS_n)}, \quad n = 1, 2, ..., \quad x > \frac{1}{2}.$$

Before we prove Theorem 2.8 and 2.9, we shall investigate the second-order theory of $(Z_n(x,a))$ in the next subsection and then we shall show that $(Z_n(x,a))$ approximates the Hurwitz zeta-function sufficiently well.

2.2.1 Second order theory of $(Z_n(x,a))$

We write $Z_n(x,a) = Z_{n1}(x,a) - Z_{n2}(x,a)$ with

$$Z_{n1}(x,a) = \sum_{0 \le k \le x} e^{-i(\log(k+a))S_n}(k+a)^{-\sigma},$$

and

$$Z_{n2}(x,a) = \frac{(x+a)^{1-\sigma}}{(1-\sigma-iS_n)} e^{-i(\log(x+a))S_n}.$$

In order to calculate the expectation, the following integral representation will be used repeatedly:

$$\frac{1}{1-s} = \int_0^1 u^{-s}du, \qquad \Re s < 1. \tag{2.14}$$

Concerning the first moments, we have for $x \ge 1$,

$$\begin{aligned}
\mathbb{E}Z_{n2}(x,a) &= \mathbb{E}\left\{\frac{(x+a)^{1-\sigma}}{(1-\sigma-iS_n)} e^{-i(\log(x+a))S_n}\right\} \\
&= (x+a)^{1-\sigma}\mathbb{E}\left\{e^{-i(\log(x+a))S_n}\int_0^1 e^{-(\log u)(\sigma+iS_n)}du\right\} \\
&= (x+a)^{1-\sigma}\int_0^1 u^{-\sigma}\mathbb{E}\left\{e^{-i(\log u(x+a))S_n}du\right\} \\
&= (x+a)^{1-\sigma}\int_0^1 u^{-\sigma}\left\{e^{-|\log u(x+a)|n}du\right\} \\
&= (x+a)^{1-\sigma}\left(\int_0^{\frac{1}{x+a}}\frac{(u(x+a))^n}{u^\sigma}du + \int_{\frac{1}{x+a}}^1 \frac{(u(x+a))^{-n}}{u^\sigma}du\right) \\
&= \left(\frac{2n}{n^2-(1-\sigma)^2} - \frac{(x+a)^{1-\sigma-n}}{n+\sigma-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}Z_{n1}(x,a) &= \mathbb{E}\left\{\sum_{0\le k\le x}(k+a)^{-(\sigma+iS_n)}\right\} = \sum_{0\le k\le x}\mathbb{E}\left\{(k+a)^{-(\sigma+iS_n)}\right\} \\
&= \sum_{0\le k\le x}(k+a)^{-\sigma}\mathbb{E}\left\{e^{-i(\log(k+a))S_n}\right\} = \sum_{0\le k\le x}(k+a)^{-\sigma-n}.
\end{aligned}$$

Therefore, for $x \to \infty$ we have

$$\begin{aligned}
\mathbb{E}Z_n(x,a) &= \mathbb{E}Z_{n1}(x,a) - \mathbb{E}Z_{n2}(x,a) \\
&= \sum_{0\le k\le x}(k+a)^{-\sigma-n} - \left(\frac{2n}{n^2-(1-\sigma)^2} - \frac{(x+a)^{1-\sigma-n}}{n+\sigma-1}\right) \\
&\to \zeta(n+\sigma,a) - \frac{2n}{n^2-(1-\sigma)^2},
\end{aligned} \tag{2.15}$$

for any integer n and $\sigma > 0$.

Next, we shall find asymptotic formulas for $\mathbb{E}|Z_n(x,a)|^2$ and $\mathbb{E}Z_n(x,a)\overline{Z_m(x,a)}$, where $m > n+1$.

Exact formulae related to $Z_{n2}(x,a)$
Using again (2.14), we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\frac{e^{i(\log(x+a))(S_m-S_n)}}{(1-\sigma+iS_m)(1-\sigma-iS_n)}\right\}\\
&=\int_0^1\int_0^1 (uv)^{-\sigma}\mathbb{E}\left\{e^{i(\log(x+a)+\log v)(S_m-S_n)+i(\log v-\log u)S_n}\right\}dudv\\
&=\int_0^1\int_0^1 (uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv.
\end{aligned}$$

Next, we split the square $[0,1]^2$ into four domains.
For the first domain, $u \le v$, $\frac{1}{x+a} \le v$, we have

$$\begin{aligned}
&\int_{\frac{1}{x+a}}^1\int_0^v (uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\int_{\frac{1}{x+a}}^1\int_0^v (uv)^{-\sigma}(x+a)^{-(m-n)}v^{-(m-n)}\left(\frac{u}{v}\right)^n dudv\\
&=\frac{(x+a)^{-2(1-\sigma)}-(x+a)^{-(m-n)}}{((m-n)-2(1-\sigma))(n+(1-\sigma))}.
\end{aligned}$$

Thus, for the first domain,

$$\begin{aligned}
&(x+a)^{2(1-\sigma)}\int_{\frac{1}{x+a}}^1\int_0^v (uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\frac{1-(x+a)^{-(m-n)+2(1-\sigma)}}{((m-n)-2(1-\sigma))(n+(1-\sigma))}.
\end{aligned}$$

For the second domain, $u \le v \le \frac{1}{x+a}$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{x+a}}\int_0^v (uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\int_0^{\frac{1}{x+a}}\int_0^v (uv)^{-\sigma}(x+a)^{m-n}v^{m-n}\left(\frac{u}{v}\right)^n dudv\\
&=\frac{(x+a)^{-2(1-\sigma)}}{(m-n+2(1-\sigma))(n+(1-\sigma))}.
\end{aligned}$$

Thus, for the second domain,

$$\begin{aligned}
&(x+a)^{2(1-\sigma)}\int_0^{\frac{1}{x+a}}\int_0^v (uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\frac{1}{(m-n+2(1-\sigma))(n+1-\sigma)}.
\end{aligned}$$

For the third domain, $u \geq v \geq \frac{1}{x+a}$, we have

$$\begin{aligned}
&\int_{\frac{1}{x+a}}^{1}\int_{v}^{1}(uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\int_{\frac{1}{x+a}}^{1}\int_{v}^{1}(uv)^{-\sigma}(x+a)^{-(m-n)}v^{-(m-n)}\left(\frac{v}{u}\right)^{n}dudv\\
&=\frac{(x+a)^{-(m-n)}}{n-(1-\sigma)}\left(\frac{(x+a)^{(m-n)-2(1-\sigma)}}{(m-n)-2(1-\sigma)}+\frac{(x+a)^{-(2n-m+1-\sigma)}}{2n-m+1-\sigma}\right)\\
&-\frac{(x+a)^{-(m-n)}}{n-(1-\sigma)}\left(\frac{1}{m-n-2(1-\sigma)}-\frac{1}{2n-m+1-\sigma}\right).
\end{aligned}$$

Thus, for the third domain,

$$\begin{aligned}
&(x+a)^{2(1-\sigma)}\int_{\frac{1}{x+a}}^{1}\int_{v}^{1}(uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\frac{1}{((m-n)-2(1-\sigma))(n-(1-\sigma))}-\frac{(x+a)^{-n+1-\sigma}}{(2n-m+1-\sigma)(n-(1-\sigma))}\\
&+\frac{(x+a)^{-(m-n)+2(1-\sigma)}}{(2n-m-n+1-\sigma)(m-n-2(1-\sigma))}.
\end{aligned}$$

For the fourth domain, $u \geq v$, $v \leq \frac{1}{x+a}$, we have

$$\begin{aligned}
&\int_{0}^{\frac{1}{x+a}}\int_{v}^{1}(uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\int_{0}^{\frac{1}{x+a}}\int_{v}^{1}(uv)^{-\sigma}(x+a)^{m-n}v^{m-n}\left(\frac{v}{u}\right)^{n}dudv\\
&=\frac{(x+a)^{m-n}}{n-(1-\sigma)}\left(\frac{(x+a)^{-(m-n)-2(1-\sigma)}}{(m-n)+2(1-\sigma)}-\frac{(x+a)^{-m-(1-\sigma)}}{m+1-\sigma}\right).
\end{aligned}$$

Thus, for the fourth domain,

$$\begin{aligned}
&(x+a)^{2(1-\sigma)}\int_{0}^{\frac{1}{x+a}}\int_{v}^{1}(uv)^{-\sigma}e^{-|\log(x+a)+\log v|(m-n)-|\log v-\log u|n}dudv\\
&=\frac{1}{(m-n+2(1-\sigma))(n-1-\sigma)}-\frac{(x+a)^{-n+1-\sigma}}{(m+1-\sigma)(n-1-\sigma)}.
\end{aligned}$$

By summing up the four domains, we arrive at,

Proposition 2.10. *For $m = n$ and for $m > n+1$ we have*

$$\mathbb{E}Z_{n2}(x,a)\overline{Z_{m2}(x,a)} = A + B(x+a)^{-n+1-\sigma} + C(x+a)^{-(m-n)+2(1-\sigma)}, \tag{2.16}$$

where

$$\begin{aligned}
A &= \frac{4n(m-n)}{((m-n)^2-4(1-\sigma)^2)(n^2-(1-\sigma)^2)},\\
B &= \frac{2(m-n)}{(2n-m+(1-\sigma))(m+1-\sigma))(n-(1-\sigma))},\\
C &= \frac{3n-m+2(1-\sigma)}{(2(1-\sigma)-m+n)(2n-m+1-\sigma))(n+(1-\sigma))}.
\end{aligned}$$

If $m = n$, we have

$$\mathbb{E}Z_{n2}(x,a)\overline{Z_{n2}(x,a)} = \frac{(x+a)^{2(1-\sigma)}}{(1-\sigma)(n+1-\sigma)}. \tag{2.17}$$

Now we want to derive an exact formula for $\mathbb{E}Z_{n1}\overline{Z}_{m2}$, for all $m \geq n$. By definition,

$$\begin{aligned}
\mathbb{E}Z_{n1}\overline{Z}_{m2} &= \mathbb{E}\left\{\sum_{0\leq k\leq x} e^{-i(\log(k+a))S_n}(k+a)^{-\sigma}\frac{(x+a)^{1-\sigma}e^{i(\log(x+a))S_m}}{(1-\sigma+iS_m)}\right\} \\
&= (x+a)^{1-\sigma}\sum_{0\leq k\leq x}(k+a)^{-\sigma}\int_0^1 v^{-\sigma}\mathbb{E}\left\{e^{-i(\log(k+a))S_n+i(\log(x+a)+\log v)S_m}\right\}dv \\
&= (x+a)^{1-\sigma}\sum_{0\leq k\leq x}(k+a)^{-\sigma}\int_0^1 v^{-\sigma}e^{-|\log v(x+a)|(m-n)-|\log\frac{v(x+a)}{k+a}|n}dv.
\end{aligned}$$

Here, we split the interval $[0,1]$ into three intervals. Firstly,

$$\int_0^{\frac{1}{x+a}} v^{-\sigma}e^{-|\log v(x+a)|(m-n)-|\log\frac{v(x+a)}{k+a}|n}dv = \frac{(x+a)^{\sigma-1}}{(m+1-\sigma)(k+a)^n}.$$

Secondly,

$$\int_{\frac{1}{x+a}}^{\frac{k+a}{x+a}} v^{-\sigma}e^{-|\log v(x+a)|(m-n)-|\log\frac{v(x+a)}{k+a}|n}dv = \frac{(x+a)^{\sigma-1}((k+a)^{n-m+1-\sigma}-(k+a)^{-n})}{(2n-m+1-\sigma)}.$$

Thirdly,

$$\int_{\frac{k+a}{x+a}}^1 v^{-\sigma}e^{-|\log v(x+a)|(m-n)-|\log\frac{v(x+a)}{k+a}|n}dv = \frac{(k+a)^{-m+n+1-\sigma}}{(x+a)^{1-\sigma}(m-1+\sigma)} - \frac{(k+a)^n}{(x+a)^m(m-1+\sigma)}.$$

By summing up the three results, multiplying each one by $(k+a)^{-\sigma}$, adding up over k, and multiplying each by $(x+a)^{1-\sigma}$, we get

$$\begin{aligned}
\mathbb{E}Z_{n1}(x,a)\overline{Z_{m2}(x,a)} = &\sum_{k\leq x}\left\{\frac{-2(m-n)(k+a)^{-n-\sigma}}{(m+1-\sigma)(2n-m+1-\sigma)} + \frac{2n(k+a)^{n-m+1-2\sigma}}{(m-1+\sigma)(2n-m+1-\sigma)}\right\} \\
&-\sum_{k\leq x}\left\{\frac{(k+a)^{n-\sigma}(x+a)^{-m+1-\sigma}}{(m-1+\sigma)}\right\}.
\end{aligned}$$

The calculation for $\mathbb{E}Z_{m1}(x,a)\overline{Z_{n2}(x,a)}$ is very similar. We have,

$$\begin{aligned}
\mathbb{E}Z_{m1}(x,a)\overline{Z_{n2}(x,a)} &= \mathbb{E}\left\{\sum_{0\leq k\leq x} e^{-i(\log(k+a))S_m}(k+a)^{-\sigma}\frac{(x+a)^{1-\sigma}e^{i(\log(x+a))S_n}}{1-\sigma+iS_n}\right\} \\
&= (x+a)^{1-\sigma}\sum_{0\leq k\leq x}(k+a)^{n-m-\sigma}\int_0^1 v^{-\sigma}e^{-|\log\frac{v(x+a)}{k+a}|n}dv.
\end{aligned}$$

We calculate this integral by splitting $[0,1]$ in two intervals. Firstly,

$$\int_0^{\frac{k+a}{x+a}} v^{-\sigma}e^{-|\log\frac{v(x+a)}{k+a}|n}dv = \frac{(k+a)^{1-\sigma}}{(n+1-\sigma)(x+a)^{1-\sigma}}.$$

Secondly,

$$\int_{\frac{k+a}{x+a}}^{1} v^{-\sigma} e^{-|\log \frac{v(x+a)}{k+a}| n} dv = \frac{(k+a)^{1-\sigma}}{(n-1+\sigma)(x+a)^{1-\sigma}} - \frac{(k+a)^{n}}{(x+a)^{n}(n-1+\sigma)}.$$

By summing up the two results, multiplying by $(k+a)^{n-m-\sigma}$, adding up over k, and multiplying by $(x+a)^{1-\sigma}$, we get

$$\mathbb{E} Z_{m1}(x,a)\overline{Z_{n2}(x,a)} = \sum_{k\le x}\left\{ \frac{2n(k+a)^{n-m+1-2\sigma}}{n^2-(1-\sigma)^2} - \frac{(k+a)^{2n-m-\sigma}(x+a)^{-n+1-\sigma}}{(n-1+\sigma)} \right\}.$$

We formulate again

Proposition 2.11. *For all $m \ge n+1$ we have*

(2.18)

$$\mathbb{E} Z_{n1}(x,a)\overline{Z_{m2}(x,a)} = \sum_{k\le x}\left\{ \frac{-2(m-n)(k+a)^{-n-\sigma}}{(m+1-\sigma)(2n-m+1-\sigma)} + \frac{2n(k+a)^{n-m+1-2\sigma}}{(m-1+\sigma)(2n-m+1-\sigma)} \right\} - \sum_{k\le x}\left\{ \frac{(k+a)^{n-\sigma}(x+a)^{-m+1-\sigma}}{(m-1+\sigma)} \right\},$$

and

$$\mathbb{E} Z_{m1}(x,a)\overline{Z_{n2}(x,a)} = \sum_{k\le x}\left\{ \frac{2n(k+a)^{n-m+1-2\sigma}}{n^2-(1-\sigma)^2} - \frac{(k+a)^{2n-m-\sigma}(x+a)^{-n+1-\sigma}}{(n-1+\sigma)} \right\}. \tag{2.19}$$

Asymptotic formulae related to $Z_{n2}(x,a)$

Here we only consider the case $\sigma = \frac{1}{2}$. In order to obtain asymptotic formulae related to $Z_{n2}(x,a)$, we take x to infinity in the results obtained in the previous step. We immediately deduce from (2.16)

$$\mathbb{E} Z_{n2}(x,a)\overline{Z_{m2}(x,a)} = \frac{4n(m-n)}{((m-n)^2-1)(n^2-\frac{1}{4})} + o(1), \qquad x\to\infty,$$

and from (2.17) we derive

$$\mathbb{E} Z_{n2}(x,a)\overline{Z_{n2}(x,a)} = \frac{2(x+a)}{n+\frac{1}{2}}.$$

Next, (2.18) implies that

$$\mathbb{E} Z_{n1}(x,a)\overline{Z_{m2}(x,a)} = \frac{-2(m-n)\zeta(n+\frac{1}{2},a)}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} + \frac{2n}{(m-\frac{1}{2})(2n-m+\frac{1}{2})}\sum_{k\le x}(k+a)^{n-m} - \frac{(x+a)^{-m+\frac{1}{2}}}{(m-\frac{1}{2})}\sum_{k\le x}(k+a)^{n-\frac{1}{2}} + o(1).$$

Note that for $m > n+1$ the second term converges and the third term is negligible, since

$$(x+a)^{-m+\frac{1}{2}}\sum_{k\le x}(k+a)^{n-\frac{1}{2}} \le (x+a)^{-m+\frac{1}{2}}(x)(x+a)^{n-\frac{1}{2}} = x(x+a)^{n-m} \le (x+a)^{n-m+1} = o(1).$$

Hence, for $m > n+1$ we obtain

$$\mathbb{E}Z_{n1}(x,a)\overline{Z_{m2}(x,a)} = \frac{-2(m-n)\zeta(n+\frac{1}{2},a)}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} + \frac{2n\zeta(m-n,a)}{(m-\frac{1}{2})(2n-m+\frac{1}{2})} + o(1).$$

When $m = n > 2$, we have, by the second-order Euler-Maclaurin formula,

$$\sum_{k\le x}(k+a)^{n-\frac{1}{2}} = \frac{(x+a)^{n+\frac{1}{2}}}{n+\frac{1}{2}} + \frac{(x+a)^{n-\frac{1}{2}}}{2} + o((x+a)^{n-\frac{1}{2}})$$

and obtain, for $x \to \infty$

$$\begin{aligned}\mathbb{E}Z_{n1}(x,a)\overline{Z_{n2}(x,a)} &= \frac{2nx}{n^2-\frac{1}{4}} - \frac{x+a}{n^2-\frac{1}{4}} - \frac{1}{2n-1} + o(1)\\ &= \frac{2x}{n+\frac{1}{2}} - \frac{1}{2n-1} - \frac{a}{(n^2-\frac{1}{4})} + o(1).\end{aligned}$$

Now we consider the last expectation (2.19), that is

$$\mathbb{E}Z_{m1}(x,a)\overline{Z_{n2}(x,a)} = \frac{2n}{n^2-\frac{1}{4}}\sum_{k\le x}(k+a)^{n-m} - \frac{(x+a)^{-n+\frac{1}{2}}}{(n-\frac{1}{2})}\sum_{k\le x}(k+a)^{2n-m-\frac{1}{2}}.$$

When $m > n+1$, the first term converges and the second term vanish, since with $x \to \infty$

$$\begin{aligned}(x+a)^{-n+\frac{1}{2}}\sum_{k\le x}(k+a)^{2n-m-\frac{1}{2}} &\le (x+a)^{-n+\frac{1}{2}}(x+a)(x+a)^{2n-m-\frac{1}{2}}\\ &= (x+a)^{n-m+1} = o(1).\end{aligned}$$

Thus we get, for $x \to \infty$,

$$\mathbb{E}Z_{m1}(x,a)\overline{Z_{n2}(x,a)} = \frac{2n\zeta(m-n,a)}{n^2-\frac{1}{4}} + o(1).$$

Proposition 2.12. *For all $m \ge n+1$ and $x \to \infty$ we have*

$$\mathbb{E}Z_{n2}(x,a)\overline{Z_{m2}(x,a)} = \frac{4n(m-n)}{((m-n)^2-1)(n^2-\frac{1}{4})} + o(1), \tag{2.20}$$

$$\mathbb{E}Z_{m1}(x,a)\overline{Z_{n2}(x,a)} = \frac{2n\zeta(m-n,a)}{n^2-\frac{1}{4}} + o(1), \tag{2.21}$$

$$\mathbb{E}Z_{n1}(x,a)\overline{Z_{m2}(x,a)} = \frac{-2(m-n)\zeta(n+\frac{1}{2},a)}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} + \frac{2n\zeta(m-n,a)}{(m-\frac{1}{2})(2n-m+\frac{1}{2})} + o(1). \tag{2.22}$$

For $n \ge 1$ and $x \to \infty$ we have

$$\mathbb{E}Z_{n2}(x,a)\overline{Z_{n2}(x,a)} = \frac{2(x+a)}{n+\frac{1}{2}}, \tag{2.23}$$

$$\mathbb{E}Z_{n1}(x,a)\overline{Z_{n2}(x,a)} = \frac{2x}{n+\frac{1}{2}} - \frac{1}{2n-1} - \frac{a}{(n^2-\frac{1}{4})} + o(1). \tag{2.24}$$

Asymptotic formulae related to $Z_{n1}(x,a)$

Let us fix $\sigma \in [\frac{1}{2}, 1]$ and m, n such that $m > n+1$. We have

$$\begin{aligned}
\mathbb{E} Z_{n1}(x,a)\overline{Z_{m1}(x,a)} &= \mathbb{E} \sum_{0 \le k,l \le x} (k+a)^{-(\sigma+iS_n)}(l+a)^{-(\sigma-iS_m)} \\
&= \sum_{0 \le k,l \le x} ((k+a)(l+a))^{-\sigma} \mathbb{E} e^{-i(\log(k+a))S_n} e^{i(\log(l+a))S_m} \\
&= \sum_{0 \le k,l \le x} ((k+a)(l+a))^{-\sigma} \mathbb{E} e^{i(\log(l+a)-\log(k+a))S_n + i(\log(l+a))(S_m - S_n)}.
\end{aligned}$$

We note that

$$\mathbb{E} e^{i(\log(l+a)-\log(k+a))S_n} = \begin{cases} (\frac{l+a}{k+a})^{-n} & \text{if} \quad l > k, \\ (\frac{l+a}{k+a})^{n} & \text{if} \quad l < k. \end{cases}$$

Thus we get

$$\begin{aligned}
\mathbb{E} Z_{n1}(x,a)\overline{Z_{m1}(x,a)} &= \sum_{0 \le k,l \le x} ((k+a)(l+a))^{-\sigma} \left(\frac{\min(k+a, l+a)}{\max(k+a, l+a)} \right)^n (l+a)^{-m+n} \\
&= S_1 + S_2 + S_0,
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{0 \le k \le x} (k+a)^{n-\sigma} \sum_{k+1 \le l \le x} (l+a)^{-m-\sigma}, \\
S_2 &= \sum_{0 \le l \le x} (l+a)^{2n-m-\sigma} \sum_{l+1 \le k \le x} (k+a)^{-n-\sigma} \\
&= \sum_{0 \le k \le x} (k+a)^{2n-m-\sigma} \sum_{k+1 \le l \le x} (l+a)^{-n-\sigma}, \\
S_0 &= \sum_{k \le x} (k+a)^{n-m-2\sigma}.
\end{aligned}$$

For $m > n+1$ and $\sigma = \frac{1}{2}$, we obviously have

$$S_0 = \zeta(m-n+1, a) + o(1), \qquad x \to \infty.$$

Next,

$$S_1 = \sum_{k=0}^{\infty} (k+a)^{n-\frac{1}{2}} \sum_{l=k+1}^{\infty} (l+a)^{-m-\frac{1}{2}} + o(1), \qquad x \to \infty.$$

In what follows and elsewhere $\theta_{k,m}$ denote constants in $[0,1]$, not necessarily the same at each appearance. Moreover, for $m > n+1$,

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+a)^{n-\frac{1}{2}} \sum_{l=k+1}^{\infty} (l+a)^{-m-\frac{1}{2}} &= \sum_{k=0}^{\infty} (k+a)^{n-\frac{1}{2}} \theta_{k,m} \int_k^{\infty} (u+a)^{-m-\frac{1}{2}} du \\
&= \sum_{k=0}^{\infty} (k+a)^{n-\frac{1}{2}} \theta_{k,m} \left[\frac{(u+a)^{-m+\frac{1}{2}}}{-m+\frac{1}{2}} \right]_k^{\infty} \\
&= \frac{\theta_{n,m}}{m-\frac{1}{2}} \sum_{k=0}^{\infty} (k+a)^{n-m} \\
&= \frac{\theta_{n,m}}{m-\frac{1}{2}} \zeta(m-n, a).
\end{aligned}$$

Exactly in the same way we obtain

$$S_2 = \sum_{k=0}^{\infty}(k+a)^{2n-m-\frac{1}{2}} \sum_{l=k+1}^{\infty}(l+a)^{-n-\frac{1}{2}} + o(1), \qquad x\to\infty,$$

and

$$\sum_{k=0}^{\infty}(k+a)^{2n-m-\frac{1}{2}} \sum_{l=k+1}^{\infty}(l+a)^{-n-\frac{1}{2}} = \frac{\acute{\theta}_{m,n}}{n-\frac{1}{2}}\zeta(m-n,a).$$

Thus, finally, we have

Proposition 2.13. *For $m > n+1$*

(2.25)

$$\mathbb{E}Z_{n1}(x,a)\overline{Z_{m1}(x,a)} = \zeta(m-n+1,a) + \theta\left(\frac{1}{m-\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\zeta(m-n,a) + o(1), \qquad x\to\infty,$$

with $\theta = \theta(n,m) \in [0,1]$.

Now we shall calculate $\mathbb{E}Z_{n1}(x,a)\overline{Z_{n1}(x,a)}$. The main term of Theorem 2.2 follows form results of this part. We already known that, for $\sigma = \frac{1}{2}$

$$\begin{aligned}\mathbb{E}Z_{n1}(x,a)\overline{Z_{n1}(x,a)} &= \sum_{0\le k,l\le x}((k+a)(l+a))^{-\frac{1}{2}}\left(\frac{\min(k+a,l+a)}{\max(k+a,l+a)}\right)^n \\ &= 2\sum_{0\le l\le x}(l+a)^{-\frac{1}{2}-n}\sum_{0\le k\le l}(k+a)^{-\frac{1}{2}+n} - \sum_{0\le l\le x}\frac{1}{l+a}.\end{aligned}$$

Applying the Euler-Maclaurin formula of the first order to the inner sum, we find

$$\begin{aligned}&2\sum_{0\le l\le x}(l+a)^{-\frac{1}{2}-n}\sum_{0\le k\le l}(k+a)^{-\frac{1}{2}+n} \\ &= 2\sum_{0\le l\le x}(l+a)^{-\frac{1}{2}-n}\left(\frac{a^{n-\frac{1}{2}}+(l+a)^{n-\frac{1}{2}}}{2} + \frac{(l+a)^{n+\frac{1}{2}}-a^{n+\frac{1}{2}}}{n+\frac{1}{2}} + \sum_{k\le l-1}A_k\right) \\ &= \frac{2(x+1)}{n+\frac{1}{2}} + \sum_{0\le l\le x}\frac{1}{l+a} + 2\left(\frac{a^{n-\frac{1}{2}}}{2} - \frac{a^{n+\frac{1}{2}}}{n+\frac{1}{2}}\right)\sum_{0\le l\le x}(l+a)^{-\frac{1}{2}-n} \\ &+ 2\sum_{0\le l\le x}(l+a)^{-\frac{1}{2}-n}\sum_{k\le l-1}A_k,\end{aligned}$$

where

$$A_k = (n-\frac{1}{2})\int_0^1(t-\frac{1}{2})(t+k+a)^{n-\frac{3}{2}}dt.$$

For $x\to\infty$ we get

$$\mathbb{E}Z_{n1}(x,a)\overline{Z_{n1}(x,a)} = \frac{2(x+1)}{n+\frac{1}{2}} + \frac{(2n-4a-1)a^{n-\frac{1}{2}}}{2n-1}\zeta(n+\frac{1}{2},a) + 2\sum_{k=0}^{\infty}A_k\sum_{l=k+1}^{\infty}(l+a)^{-\frac{1}{2}-n} + o(1).$$

Now, it remains to analyse the behaviour of the double sum. Here we shall use the same technique as Lifshits and Weber. We let

$$S = \sum_{k=1}^{\infty} A_k \sum_{l=k+1}^{\infty} (l+a)^{-\frac{1}{2}-n},$$

as $n \to \infty$. We denote

$$B_k = B_k(n) = \int_0^1 (t - \frac{1}{2})(t+k+a)^{n-\frac{3}{2}} dt,$$

$$D_k = D_k(n) = \sum_{l=k+1}^{\infty} (l+a)^{-\frac{1}{2}-n}, \qquad D'_k = D'_k(n) = \sum_{l=k+2}^{\infty} (l+a)^{-\frac{1}{2}-n}.$$

Then we have

$$\begin{aligned} S &= (n - \frac{1}{2}) \sum_{k=1}^{\infty} B_k D_k \\ &= (n - \frac{1}{2}) \left(\sum_{k=n}^{\infty} B_k D_k + \sum_{k=1}^{n-1} B_k \left(D'_k + (k+a+1)^{-n-\frac{1}{2}} \right) \right) \end{aligned}$$

We also aim to show that

$$\lim_{n\to\infty} (n - \frac{1}{2}) \sum_{k=n}^{\infty} B_k D_k = \int_0^1 \phi(\alpha) d\alpha, \tag{2.26}$$

$$\lim_{n\to\infty} (n - \frac{1}{2}) \sum_{k=1}^{n-1} B_k D'_k = \int_1^{\infty} \phi_1(\alpha) d\alpha, \tag{2.27}$$

$$\lim_{n\to\infty} \left((n - \frac{1}{2}) \sum_{k=1}^{n-1} B_k (k+1+a)^{-n-\frac{1}{2}} - \sum_{k=1}^{n-1} \frac{1}{2(k+1+a)} \right) = \int_1^{\infty} \phi_2(\alpha) d\alpha. \tag{2.28}$$

For (2.26), we have to show that

$$\liminf_{n\to\infty} n \sum_{k=n}^{\infty} B_k D_k \geq \int_0^1 \phi(\alpha) d\alpha \leq \limsup_{n\to\infty} n \sum_{k=n}^{\infty} B_k D_k. \tag{2.29}$$

Now we substitute $t = 1 - \frac{(k+a+1)v}{n}$ in B_k and get

$$B_k = \frac{(k+a+1)^{n-\frac{1}{2}}}{n} \int_0^{\beta_k} (\frac{1}{2} - \frac{v}{\beta_k})(1 - \frac{v}{n})^{n-\frac{3}{2}} dv,$$

where $\beta_k = \frac{n}{k+a+1}$. In view of

$$\lim_{n\to\infty} (1 - \frac{v}{n})^{n-\frac{3}{2}} = e^{-v},$$

we have

$$B_k \sim \frac{(k+a+1)^{n-\frac{1}{2}}}{n} \int_0^{\beta_k} e^{-v}(\frac{1}{2} - \frac{v}{\beta_k})dv.$$

By using the explicit formula

$$\int_0^{\beta} e^{-v}(\frac{1}{2} - \frac{v}{\beta})dv = \frac{e^{-\beta}+1}{2} + \frac{e^{-\beta}-1}{\beta},$$

we obtain

$$nB_k(k+a+1)^{-n+\frac{1}{2}} \sim \frac{1}{2}(1+e^{-\beta_k}) + \frac{e^{-\beta_k}-1}{\beta_k}.$$

Now we deal with D_k. We have

$$\begin{aligned} D_k\,(k+a+1)^{n+\frac{1}{2}} &= \sum_{h=1}^{\infty} \left(\frac{k+h+a}{k+1+a}\right)^{-n-\frac{1}{2}} = \sum_{h=0}^{\infty} \left(1 + \frac{h}{k+1+a}\right)^{-n-\frac{1}{2}} \\ &\sim \exp\left(-(\frac{h}{k+a+1})(n+\frac{1}{2})\right) \sim \left(1 - \exp\left(-\frac{n+\frac{1}{2}}{k+a+1}\right)\right)^{-1} \\ &\sim \left(1 - \exp\left(-\frac{n}{k+a+1}\right)\right)^{-1} \sim \left(1 - \exp\left(-\beta_k\right)\right)^{-1}. \end{aligned}$$

Thus we have, for $\beta_k = \frac{n}{k+a+1} \in (0,1]$

$$B_k D_k \sim \phi(\beta_k)\frac{1}{(k+a+1)^2}.$$

Since ϕ is uniformly continuous, we find

$$\begin{aligned} \int_{\frac{n}{k+a+1}}^{\frac{n}{k+a}} \phi(\alpha)d\alpha &\sim \phi(\frac{n}{k+a+1})\left(\frac{n}{k+a} - \frac{n}{k+a+1}\right) \\ &\sim \phi(\beta_k)\frac{n}{(k+a+1)^2}. \end{aligned}$$

For this, we obtain that

$$nB_kD_k \sim \int_{\frac{n}{k+a+1}}^{\frac{n}{k+a}} \phi(\alpha)d\alpha.$$

It remains to show (2.29). Exchanging k to $k+a$ makes no difference in our situation. For this aim, we find that for any (large) fixed $M>1$, uniformly in $k \in [n, Mn]$,

$$\liminf_{n\to\infty} n\sum_{k=n}^{\infty} B_kD_k \geq \lim_{n\to\infty} \sum_{k=n}^{Mn} \int_{\frac{n}{k+a+1}}^{\frac{n}{k+a}} \phi(\alpha)d\alpha = \int_{\frac{1}{M}}^{1} \phi(\alpha)d\alpha.$$

If M tend to infinity, we arrive at

$$\liminf_{n\to\infty} n\sum_{k=n}^{\infty} B_kD_k \geq \lim_{n\to\infty} \int_0^1 \phi(\alpha)d\alpha.$$

Similarly, we see that for any $M > 1$

$$\limsup_{n\to\infty} n \sum_{k=n}^{Mn} B_k D_k \le \lim_{n\to\infty} \int_{\frac{1}{M}}^{1} \phi(\alpha) d\alpha \le \lim_{n\to\infty} \int_0^1 \phi(\alpha) d\alpha.$$

Thus we only need to show that

$$\lim_{M\to\infty} \limsup_{n\to\infty} n \sum_{k>Mn} B_k D_k = 0. \tag{2.30}$$

We have already seen that

$$B_k = \frac{(k+a+1)^{n-\frac{1}{2}}}{n} \int_0^{\beta_k} (\frac{1}{2} - \frac{v}{\beta_k})(1-\frac{v}{n})^{n-\frac{3}{2}} dv.$$

In view of $\int_0^\beta \frac{1}{2} - \frac{v}{\beta} dv = 0$ and $|\frac{1}{2} - \frac{v}{\beta}| \le 1$, where $0 \le v \le \beta$, we have

$$|\int_0^\beta (\frac{1}{2} - \frac{v}{\beta})(1-\frac{v}{n})^{n-\frac{3}{2}} dv| \le \int_0^\beta |(1-\frac{v}{n})^{n-\frac{3}{2}} - 1| dv.$$

Since

$$|(1-\frac{v}{n})^{n-\frac{3}{2}} - 1| = \left(n - \frac{3}{2}\right) \int_{1-\frac{v}{n}}^1 y^{n-\frac{5}{2}} dy \le \left(n - \frac{3}{2}\right) \frac{v}{n} \le v,$$

we have

$$B_k \le \frac{(k+a+1)^{n-\frac{1}{2}}}{n} \beta_k^2.$$

To approximate D_k we split the sum in two parts

$$\begin{aligned} D_k\,(k+a+1)^{n+\frac{1}{2}} &= \sum_{h=0}^{\infty} \left(1 + \frac{h}{k+a+1}\right)^{-n-\frac{1}{2}} \\ &= \sum_{h=0}^{k+1} \left(1 + \frac{h}{k+a+1}\right)^{-n-\frac{1}{2}} + \sum_{h>k+1} \left(1 + \frac{h}{k+a+1}\right)^{-n-\frac{1}{2}}. \end{aligned}$$

By using $1 + s \ge e^{s\log 2}$, $0 \le s \le 1$, we have

$$\begin{aligned} \sum_{h=0}^{k+1} \left(1 + \frac{h}{k+a+1}\right)^{-n-\frac{1}{2}} &\le \sum_{h=0}^{\infty} \exp\left(-\left(n+\frac{1}{2}\right)\frac{h}{k+a+1}\log 2\right) \\ &= \left(1 - \exp\left(-\left(n+\frac{1}{2}\right)\frac{1}{k+a+1}\log 2\right)\right)^{-1} \\ &\le (1 - \exp(-4\beta_k))^{-1} \le C(\beta_k)^{-1}, \end{aligned}$$

for all $0 \le \beta_k \le 1$.
Secondly,

$$\begin{aligned} \sum_{h>k+1} \left(1 + \frac{h}{k+a+1}\right)^{-n-\frac{1}{2}} \frac{k+a+1}{k+a+1} &\le (k+a+1) \int_1^\infty (1+x)^{-n-\frac{1}{2}} dx \\ &= \frac{k+a+1}{n-\frac{1}{2}} 2^{-n-\frac{1}{2}} \\ &\le 2^{\frac{3}{2}} \frac{k+a+1}{n} \le C\beta_k^{-1}, \end{aligned}$$

Thus, we obtain that

$$D_k(k+a+1)^{n+\frac{1}{2}} \le C\beta_k^{-1}.$$

It follows that

$$\begin{aligned} nB_kD_k \le (k+a+1)^{n-\frac{1}{2}}\beta_k^2(k+a+1)^{-n-\frac{1}{2}}C\beta_k^{-1} &= C(k+a+1)^{-1}\beta_k \\ &= C\frac{n}{(k+a+1)^2}, \end{aligned}$$

whence

$$n\sum_{k>Mn} B_kD_k \le Cn\sum_{k>Mn}\frac{1}{(k+a+1)^2} \le \frac{Cn}{Mn} \le \frac{C}{M},$$

and (2.30) follows. The proof of (2.26) is complete.

By the same method of proof we also obtain (2.27). The only different point is that for any (large) fixed $M>1$, uniformly in $k \in [\frac{n}{M}, n]$, we have

$$nB_kD_k' \sim \int_{\frac{n}{k+a+1}}^{\frac{n}{k+a}} \phi_1(\alpha)d\alpha.$$

By continuity of ϕ_1, we have

$$\int_{\frac{n}{k+a+1}}^{\frac{n}{k+a}} \phi_1(\alpha)d\alpha \sim \phi_1(\frac{n}{k+a+1})\left(\frac{n}{k+a}-\frac{n}{k+a+1}\right) \sim \phi_1(\beta_k)\left(\frac{n}{k+a+1}\right)^2,$$

where $\beta_k = \frac{n}{k+a+1} \in [1, M]$.

Finally, we shall prove (2.28). We have to investigate the limiting behaviour of the sum

$$\begin{aligned} \sum_{k=1}^{n-1}(k+1+a)^{-n-\frac{1}{2}}B_k &= \sum_{k=1}^{n-1}(k+1+a)^{-n-\frac{1}{2}}\int_0^1(t-\frac{1}{2})(t+k+a)^{n-\frac{3}{2}}dt \\ &= \sum_{k=1}^{n-1}\int_0^1\frac{(t+k+a)^{n-\frac{3}{2}}(t-\frac{1}{2})}{(k+1+a)^{n-\frac{3}{2}}(k+a+1)^2}dt. \end{aligned}$$

Changing again the variable $t = 1-\frac{k+a+1}{n}v$, we have

$$\begin{aligned} &\sum_{k=1}^{n-1}\int_{\frac{n}{k+a+1}}^{0}\frac{(1-\frac{k+a+1}{n}v+k+a)^{n-\frac{3}{2}}(1-\frac{k+a+1}{n}v-\frac{1}{2})}{(k+1+a)^{n-\frac{3}{2}}(k+a+1)^2}d(1-\frac{k+a+1}{n}v) \\ &= \sum_{k=1}^{n-1}\frac{k+a+1}{n}\int_0^{\frac{n}{k+a+1}}\frac{\left(1-\frac{v}{n}\right)^{n-\frac{3}{2}}\left(\frac{1}{2}-\frac{(k+a+1)v}{n}\right)}{(k+a+1)^2}dv \\ &= \sum_{k=1}^{n-1}\frac{1}{2n(k+a+1)}\int_0^{\frac{n}{k+a+1}}\left(1-\frac{v}{n}\right)^{n-\frac{3}{2}}dv - \sum_{k=1}^{n-1}\frac{1}{n^2}\int_0^{\frac{n}{k+a+1}}\left(1-\frac{v}{n}\right)^{n-\frac{3}{2}}vdv. \end{aligned} \tag{2.31}$$

We write the last sum of (2.31) as one integral:

$$\frac{1}{n}\sum_{k=1}^{n-1}\int_0^{\frac{n}{k+a+1}}\left(1-\frac{v}{n}\right)^{n-\frac{3}{2}}vdv = \frac{1}{n}\int_0^{\infty}\left(1-\frac{v}{n}\right)^{n-\frac{3}{2}}\sharp\left\{k : k+a+1 \le n, k+a+1 \le \frac{n}{v}\right\}vdv.$$

next we split the integral over the domains $[0,1]$ and $(1,\infty)$, and obtain

$$\int_0^1 \left(1-\frac{v}{n}\right)^{n-\frac{3}{2}} \frac{\sharp\{k : k+a+1 \le n\}}{n} v dv + \int_1^\infty \left(1-\frac{v}{n}\right)^{n-\frac{3}{2}} \frac{\sharp\left\{k : k+a+1 \le \frac{n}{v}\right\}}{n} v dv.$$

In view of the theorem of dominated convergence, the first integral converges to $\int_0^1 e^{-v} v dv$ and the second to $\int_1^\infty e^{-v} dv$. Thus, we have

$$\lim_{n\to\infty} \frac{n-\frac{1}{2}}{n^2} \sum_{k=1}^{n-1} \int_0^{\frac{n}{k+a+1}} \left(1-\frac{v}{n}\right)^{n-\frac{3}{2}} v dv = \int_0^1 e^{-v} v dv + \int_1^\infty e^{-v} dv = 1-e^{-1}.$$

Now we consider the first term of (2.31). After we multiply by $n-\frac{1}{2}$ we get

$$\begin{aligned}
&(n-\frac{1}{2}) \sum_{k=1}^{n-1} \frac{1}{2n(k+a+1)} \int_0^{\frac{n}{k+a+1}} \left(1-\frac{v}{n}\right)^{n-\frac{3}{2}} dv \\
&= \frac{n-\frac{1}{2}}{n} \left(\sum_{k=1}^{n-1} \frac{X_k}{2(k+a+1)} - \sum_{k=1}^{n-1} \frac{Y_k}{2(k+a+1)} + \sum_{k=1}^{n-1} \frac{Z_k}{2(k+a+1)} \right),
\end{aligned}$$

where

$$X_k = \int_0^\infty e^{-v} dv = 1, \qquad Y_k = \int_{\frac{n}{k+a+1}}^\infty e^{-v} dv = e^{\frac{-n}{k+a+1}}, \qquad Z_k = \int_0^{\frac{n}{k+a+1}} \left((1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}\right) dv.$$

We find that the first sum is

$$\sum_{k=1}^{n-1} \frac{1}{2(k+a+1)} + O\left(\frac{\log(n+a)}{n}\right) = \sum_{k=1}^{n-1} \frac{1}{2(k+a+1)} + o(1).$$

For the second term, we have to show that

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{Y_k}{2(k+a+1)} = \frac{1}{2} \int_1^\infty e^{-\alpha} \frac{1}{\alpha} d\alpha.$$

For this purpose, we consider the following subdivision $t_1 = \frac{n}{2+a}, ..., t_k = \frac{n}{k+1+a}, ..., t_{n-1} = \frac{n}{n+a} \le 1$. We have

$$t_{k-1} - t_k = \frac{n}{k+a} - \frac{n}{k+a+1} = \frac{n}{(k+a+1)(k+a)}.$$

We fix a large integer M and write

$$\begin{aligned}
\frac{1}{2} \int_1^{t_M} e^{-\alpha} \frac{1}{\alpha} d\alpha \le \sum_{k=M+1}^{n-1} \int_{t_k}^{t_{k-1}} e^{-\alpha} \frac{1}{2\alpha} d\alpha &\le \sum_{k=M+1}^{n-1} e^{-t_k} \frac{1}{2t_k} (t_k - t_{k-1}) \\
&= \sum_{k=M+1}^{n-1} e^{-t_k} \frac{1}{2t_k} \frac{n}{(k+a+1)(k+a)} \\
&= \sum_{k=M+1}^{n-1} e^{\frac{-n}{k+a+1}} \frac{k+a+1}{n} \frac{n}{2(k+a+1)^2} \frac{k+a+1}{k+a} \\
&\le \frac{M+a+1}{M+a} \sum_{k=M+1}^{n-1} e^{\frac{-n}{k+a+1}} \frac{k+a+1}{n} \frac{n}{2(k+a+1)^2}.
\end{aligned}$$

Since $t_M = \frac{n}{M+a+1} \to \infty$, as n tends to infinity, M fixed, we see that

$$\liminf_{n\to\infty} \sum_{k=1}^{n-1} \frac{Y_k}{2(k+a+1)} \geq \frac{1}{2}\frac{M+a}{M+a+1}\int_1^\infty e^{-\alpha}\frac{1}{\alpha}d\alpha.$$

With M tending to infinity, we establish that

$$\liminf_{n\to\infty} \sum_{k=1}^{n-1} \frac{Y_k}{2(k+a+1)} \geq \frac{1}{2}\int_1^\infty e^{-\alpha}\frac{1}{\alpha}d\alpha.$$

The upper bound is obtained similary and thus we arrive at

$$\sum_{k=1}^{n-1} \frac{Y_k}{2(k+a+1)} \to \frac{1}{2}\int_1^\infty e^{-\alpha}\frac{1}{\alpha}d\alpha,$$

as $n \to \infty$. Now we want to show that

$$\sum_{k=1}^{n-1} \frac{Z_k}{(k+a+1)} = \sum_{k=1}^{n-1} \frac{1}{(k+a+1)} \int_0^{\frac{n}{k+a+1}} \left((1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}\right) dv \longrightarrow 0,$$

as $n \to \infty$. We have

$$\begin{aligned}
&|\sum_{k=1}^{n-1} \frac{1}{(k+a+1)} \int_0^{\frac{n}{k+a+1}} \left((1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}\right) dv| \\
&\leq \int_0^{\frac{n}{k+a+1}} |(1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}|dv \left(\sum_{k=1}^{n-1} \frac{1}{(k+a+1)}\right) \\
&\leq (\log(n+a)) \int_0^{\frac{n}{2}} |(1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}|dv.
\end{aligned}$$

In the proof of Lifshits and Weber, the latter integral is close to zero when n tends to infinity. We split the integration domain $[0, \frac{n}{2}]$ in $[0, n^{\frac{1}{4}}]$ and $(n^{\frac{1}{4}}, \frac{n}{2}]$, $n \geq 3$. For the second domain, we use the elementary estimate

$$(1-\frac{v}{n})^{n-\frac{3}{2}} \leq \exp(-\frac{v(n-\frac{3}{2})}{n}) \leq e^{\frac{-v}{2}}, \qquad n \geq 3.$$

We thus get the estimate

$$(\log n)(\frac{n}{2})e^{\frac{-n^{\frac{1}{4}}}{2}}.$$

For the first domain, we have

$$|e^{-v} - (1-\frac{v}{n})^{n-\frac{3}{2}}| = |e^{-v} - (1-\frac{v}{n})^n(1-\frac{v}{n})^{-\frac{3}{2}}| = |e^{-v} - (1-\frac{v}{n})^n h|,$$

while

$$\begin{aligned}
\max\left\{e^{-v} - (1-\frac{v}{n})^n h, (1-\frac{v}{n})^n h - e^{-v}\right\} &\leq \max\left\{e^{-v} - (1-\frac{v}{n})^n, (h-1)e^{-v}\right\} \\
&\leq e^{-v} - (1-\frac{v}{n})^n + (h-1)e^{-v}.
\end{aligned}$$

We use the following estimate

$$e^{-v} - (1-\frac{v}{n})^n \leq \frac{(v)^2}{2n}, \qquad v \leq n,$$

and also

$$h = (1-\frac{v}{n})^{\frac{-3}{2}} \leq (1-\frac{n^{\frac{1}{4}}}{2})^{\frac{-3}{2}}.$$

It follows that

$$|\sum_{k=1}^{n-1} \frac{1}{(k+a+1)} \int_0^{\frac{n}{k+a+1}} \left((1-\frac{v}{n})^{n-\frac{3}{2}} - e^{-v}\right) dv| \leq (\log n)\left(\frac{n}{2}e^{\frac{-n^{\frac{1}{4}}}{2}} + \frac{n^{\frac{3}{4}}}{6n} + (1-\frac{n^{\frac{1}{4}}}{2})^{\frac{-3}{2}} - 1\right).$$

Now letting n tends to infinity, we obtain our aim. By collecting the three terms we have

$$(n-\frac{1}{2})\left[\sum_{k=1}^{n-1}(k+a+1)^{-n-\frac{1}{2}}B_k - \sum_{k=1}^{n-1}\frac{1}{2(k+a+1)}\right] \longrightarrow e^{-1} - 1 - \frac{1}{2}\int_1^{\infty} e^{-\alpha}\frac{1}{\alpha}d\alpha,$$

as $n \to \infty$. Summing up we obtain the following

Proposition 2.14.

$$\mathbb{E}Z_{n1}\overline{Z}_{n1} = \frac{2(x+1)}{n+\frac{1}{2}} + K_n + o(1), \qquad x \to \infty \tag{2.32}$$

with

$$K_n = \log(n+a) + C + o(1), \qquad n \to \infty$$

and

$$C = \gamma(a) - \frac{1}{a} + 2\left\{\int_0^1 \phi(\alpha)d\alpha + \int_1^{\infty}\left(\phi(\alpha) - \frac{e^{\alpha}-1}{2\alpha(e^{\alpha}-1)}\right)d\alpha\right\}.$$

2.2.2 Good approximation of $\zeta(\frac{1}{2}+iS_n, a)$

In this subsection, we shall show that $Z_n(x,a)$ provides a sufficiently good approximation to $\zeta(\frac{1}{2}+iS_n,a)$ in the following sense.

Proposition 2.15. *For each positive integer n,*

$$\lim_{x\to\infty} \mathbb{E}|Z_n(x,a) - \zeta(\frac{1}{2}+iS_n,a)|^2 = 0. \tag{2.33}$$

Consequently,

$$\mathbb{E}\zeta(\frac{1}{2}+iS_n,a)\overline{\zeta(\frac{1}{2}+iS_m,a)} = \lim_{x\to\infty}\mathbb{E}Z_n(x,a)\overline{Z_m(x,a)} \tag{2.34}$$

and

$$\mathbb{E}\zeta(\frac{1}{2}+iS_n,a) = \lim_{x\to\infty}\mathbb{E}Z_n(x,a). \tag{2.35}$$

Now let $p_n(u) = \frac{n}{\pi(n^2+u^2)}$ denote the distribution density of S_n.

Lemma 2.16. *(Lifshits and Weber [43], 2009) Let $\alpha \in \mathbb{R}$ and $x \geq 1$. Then,*

$$\left| \int_{|u|\geq x} e^{i\alpha u} p_n(u) du \right| \leq \frac{C(n)}{|\alpha| x^2},$$

where the constant $C(n)$ depends on n only.

Lemma 2.17. *For any fixed n, we have*

$$\lim_{x\to\infty} \int_{|u|\geq x} \left| \sum_{m\leq x} (m+a)^{-(\frac{1}{2}+iu)} \right|^2 p_n(u) du = 0.$$

We prove Lemma 2.17 in the same way as Lifshits and Weber [43].
Proof. We have

$$\begin{aligned} &\int_{|u|\geq x} \left| \sum_{m\leq x} (m+a)^{-(\frac{1}{2}+iu)} \right|^2 p_n(u) du \\ &= \sum_{m_1\leq x} \sum_{m_2\leq x} \frac{1}{(m_1+a)^{\frac{1}{2}} (m_2+a)^{\frac{1}{2}}} \int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u) du. \end{aligned}$$

We consider two cases: let $\beta = \frac{1}{2}$.
The first case: if $|m_2 - m_1| < (m_1+a)^\beta$. Then

$$\left| \int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u) du \right| \leq \int_{|u|\geq x} p_n(u) du \leq \int_{|u|\geq x} \frac{C(n)}{u^2} du \leq \frac{C(n)}{x}.$$

Therefore,

$$\begin{aligned} &\sum_{\substack{m_1,m_2\leq x \\ |m_2-m_1|<(m_1+a)^\beta}} \frac{1}{(m_1+a)^{\frac{1}{2}} (m_2+a)^{\frac{1}{2}}} \left| \int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u) du \right| \\ &\leq \frac{C(n)}{x} \sum_{m_1\leq x} \sum_{\substack{m_2\leq x \\ |m_2-m_1|<(m_1+a)^\beta}} \frac{1}{(m_1+a)^{\frac{1}{2}} (m_2+a)^{\frac{1}{2}}} \leq \frac{C(n)}{x} \sum_{m_1\leq x} \frac{2(m_1+a)^\beta}{m_1^{\frac{1}{2}} (m_1 - (m_1+a)^\beta)^{\frac{1}{2}}} \\ &\leq C\frac{C(n)}{x} \sum_{m_1\leq x} (m_1+a)^{\beta-1} \leq C\frac{C(n)}{x} (x+a)^\beta. \end{aligned}$$

Thus

$$\lim_{x\to\infty} \sum_{\substack{m_1,m_2\leq x \\ |m_2-m_1|<(m_1+a)^\beta}} \frac{1}{(m_1+a)^{\frac{1}{2}} (m_2+a)^{\frac{1}{2}}} \left| \int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u) du \right| = 0.$$

For the second case, we assume $|m_2 - m_1| \geq (m_1+a)^\beta$. If $m_2 - m_1 \geq (m_1+a)^\beta$, then by assigning $\psi := \log(\frac{m_2+a}{m_1+a})$ we get

$$|\psi| \geq \log\left(\frac{m_1 + a + (m_1+a)^\beta}{m_1+a}\right) = \log\left(1 + (m_1+a)^{\beta-1}\right) \geq C(m_1+a)^{\beta-1};$$

if $m_1 - m_2 \geq (m_1+a)^\beta$, then

$$m_1 - m_2 \geq (m_1+a)(m_1+a)^{\beta-1} \geq (m_2+a)(m_1+a)^{\beta-1}.$$

And consequently

$$|\psi| = \log\left(\frac{m_1+a}{m_2+a}\right) \geq \log\left(1+(m_1+a)^{\beta-1}\right) \geq C(m_1+a)^{\beta-1}.$$

In view of Lemma 2.16, we have

$$\begin{aligned}\left|\int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u)du\right| &= \left|\int_{|u|\geq x} e^{iu\psi} p_n(u)du\right| \\ &\leq \frac{C(n)}{|\psi|x^2} \leq \frac{C(n)}{(m_1+a)_1^{\beta-1}x^2} = \frac{C(n)}{x^2}(m_1+a)_1^{1-\beta},\end{aligned}$$

and

$$\begin{aligned}&\sum_{\substack{m_1,m_2\leq x\\ |m_2-m_1|\geq(m_1+a)^\beta}} \frac{1}{(m_1+a)^{\frac{1}{2}}(m_2+a)^{\frac{1}{2}}} \left|\int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u)du\right| \\ &\leq \frac{C(n)}{x^2}\sum_{m_1,m_2\leq x}\frac{(m_1+a)_1^{1-\beta}}{(m_1+a)^{\frac{1}{2}}(m_2+a)^{\frac{1}{2}}} \\ &\leq \frac{C(n)}{x^2}\sum_{m_1,m_2\leq x}(m_1+a)^{-\beta+\frac{1}{2}}(m_2+a)^{-\frac{1}{2}} \\ &\leq \frac{C(n)}{x^2}\left(\sum_{m_1\leq x}(m_1+a)^{-\beta+\frac{1}{2}}\right)\left(\sum_{m_2\leq x}(m_2+a)^{-\frac{1}{2}}\right) \\ &\leq \frac{C(n)}{x^2}x^{-\beta+\frac{3}{2}}x^{\frac{1}{2}} = C(n)x^{-\beta}.\end{aligned}$$

Thus

$$\lim_{x\to\infty}\sum_{\substack{m_1,m_2\leq x\\ |m_2-m_1|\geq(m_1+a)^\beta}} \frac{1}{(m_1+a)^{\frac{1}{2}}(m_2+a)^{\frac{1}{2}}} \left|\int_{|u|\geq x} e^{iu\log\left(\frac{m_2+a}{m_1+a}\right)} p_n(u)du\right| = 0.$$

Lemma 2.18. *For any fixed n, we have*

$$\lim_{x\to\infty}\int_{|u|\geq x}\left|\frac{(x+a)^{\frac{1}{2}-iu}}{\frac{1}{2}-iu}\right|^2 p_n(u)du = 0.$$

Proof.

$$\begin{aligned}\int_{|u|\geq x}\left|\frac{(x+a)^{\frac{1}{2}-iu}}{\frac{1}{2}-iu}\right|^2 p_n(u)du &\leq (x+a)\int_{|u|\geq x}\frac{1}{|u|^2}p_n(u)du \leq C(n)(x+a)\int_{|u|\geq x}\frac{1}{|u|^4}du \\ &\leq C(n)x^{-2}.\end{aligned}$$

Thus, this integral tends to zero, as x tends to infinity.

Now we shall prove Proposition 2.15 in the same way as Lifshits and Weber. The only differences are to replace the Riemann zeta-function by the Hurwitz zeta-function and the use of the fourth moment estimate for the Hurwitz zeta-function.

Proof of Proposition 2.15 Let

$$Z_u(x,a) = \sum_{0\le k\le x} e^{-i(\log(k+a))u}(k+a)^{-\frac{1}{2}} - \frac{(x+a)^{\frac{1}{2}}}{(\frac{1}{2}-iu)} e^{-i(\log(x+a))u}$$

We have

$$\begin{aligned}\mathbb{E}|Z_n(x,a)-\zeta(\frac{1}{2}+iS_n,a)|^2 &= \int_{-\infty}^{\infty} |Z_u(x,a)-\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du \\ &\le \int_{|u|\le x} |Z_u(x,a)-\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du + 2\int_{|u|>x} |Z_u(x,a)|^2 p_n(u)du \\ &+ 2\int_{|u|>x} |\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du.\end{aligned}$$

For the first integral, we have by the approximation of the Hurwitz zeta-function,

$$\int_{|u|\le x} |Z_n(x,a)-\zeta(\frac{1}{2}+iS_n)|^2 p_n(u)du \le \max_{|u|\le x} |Z_u(x,a)-\zeta(\frac{1}{2}+iu)|^2 \le \frac{C}{|x|}.$$

Thus,

$$\lim_{x\to\infty} \int_{|u|\le x} |Z_u(x,a)-\zeta(\frac{1}{2}+iu)|^2 p_n(u)du = 0.$$

The second integral, we observe

$$\begin{aligned}&\int_{|u|>x} |Z_u(x,a)|^2 p_n(u)du \\ &\le 2\int_{|u|>x} \left|\frac{1}{(k+a)^{\frac{1}{2}+iu}}\right|^2 p_n(u)du + 2\int_{|u|>x} \left|\frac{(x+a)^{1-(\frac{1}{2}+iu)}}{1-(\frac{1}{2}+iu)}\right|^2 p_n(u)du,\end{aligned}$$

and this tends to zero, as a consequence of Lemmas 2.17 and 2.18. For the third integral, we use that (see in [32])

$$\int_{|u|\le T} |\zeta(\frac{1}{2}+iu,a)|^2 du \le CT(\log\frac{T}{2\pi})^4.$$

We have

$$\begin{aligned}
\int_{|u|>x} |\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du &\le \sum_{m:2^m\ge x} \int_{|u|\in[2^{m-1},2^m]} |\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du \\
&\le \sum_{m:2^m\ge x} \left(\max_{|u|\ge 2^{m-1}} p_n(u)\right) \int_{|u|\in[2^{m-1},2^m]} |\zeta(\frac{1}{2}+iu,a)|^2 du \\
&\le \sum_{m:2^m\ge x} \frac{C(n)}{2^{2m}} \left(\int_{|u|\le 2^m} |\zeta(\frac{1}{2}+iu,a)|^4 du\right)^{\frac{1}{2}} 2^{\frac{m}{2}} \\
&\le \sum_{m\ge \frac{\log x}{\log 2}} \frac{C(n)}{2^{2m}} \left(2^m[m\log 2]^4\right)^{\frac{1}{2}} 2^{\frac{m}{2}} \\
&\le C\cdot C(n) \sum_{m\ge \frac{\log x}{\log 2}} \frac{m^2}{2^m}.
\end{aligned}$$

Thus,

$$\lim_{x\to\infty} \int_{|u|>x} |\zeta(\frac{1}{2}+iu,a)|^2 p_n(u)du \le C\cdot C(n) \lim_{x\to\infty} \sum_{m\ge \frac{\log x}{\log 2}} \frac{m^2}{2^m} = 0,$$

and the proof is complete.

2.2.3 Proof of Theorem 2.8

In view of (2.32), (2.23) and (2.24), we have

$$\begin{aligned}
\mathbb{E}|Z_n(x,a)|^2 &= \mathbb{E}|Z_{n1}(x,a)|^2 + E|Z_{n2}(x,a)|^2 - 2\Re\mathbb{E}Z_{n1}\overline{Z_{n2}(x,a)} \\
&= \frac{2(x+1)}{n+\frac{1}{2}} + K_n + \frac{2(x+a)}{n+\frac{1}{2}} - 2\left\{\frac{2x}{n+\frac{1}{2}} - \frac{1}{2n-1} - \frac{a}{n^2-\frac{1}{4}}\right\} + o(1) \\
&= K_n + \frac{8an+8a+12n-2}{4n^2-1} + o(1).
\end{aligned}$$

Hence,

$$\mathbb{E}|\zeta_n|^2 = \lim_{x\to\infty} \mathbb{E}|Z_n(x,a)|^2 = K_n + \frac{8an+8a+12n-2}{4n^2-1} < \infty.$$

Since

$$\mathbb{E}W_n(a)\overline{W_m(a)} = \mathbb{E}\zeta(\frac{1}{2}+iS_n,a)\overline{\zeta(\frac{1}{2}+iS_m,a)} - \mathbb{E}\zeta(\frac{1}{2}+iS_n,a)\overline{\mathbb{E}\zeta(\frac{1}{2}+iS_m,a)},$$

and by Proposition 2.15 and (2.15), we obtain

$$\mathbb{E}W_n(a)\overline{W_m(a)} = \lim_{x\to\infty} \mathbb{E}Z_n(x,a)\overline{Z_m(x,a)} - \left(\zeta(n+\frac{1}{2},a) - \frac{8n}{4n^2-1}\right)\left(\zeta(m+\frac{1}{2},a) - \frac{8m}{4m^2-1}\right).$$

In particular

$$\mathbb{E}|W_n(a)|^2 = \left(K_n + \frac{8an+8a+12n-2}{4n^2-1}\right) - \left(\zeta(n+\frac{1}{2},a) - \frac{8n}{4n^2-1}\right)^2$$

and the first claim of Theorem 2.8 follows. For the second assertion, we apply

$$\mathbb{E}Z_n\overline{Z}_m = \mathbb{E}Z_{n1}\overline{Z}_{m1} - \mathbb{E}Z_{n1}\overline{Z}_{m2} - \mathbb{E}Z_{n2}\overline{Z}_{m1} + \mathbb{E}Z_{n2}\overline{Z}_{m2}$$

with (2.25), (2.22), (2.21) and (2.20). Thus we have

$$\begin{aligned}
&\mathbb{E}Z_n(x,a)\overline{Z_m(x,a)} - \mathbb{E}Z_n(x,a)\overline{\mathbb{E}Z_m(x,a)}\\
&\leq \left|\zeta(m-n+1,a) - \left(\zeta(n+\frac{1}{2},a) - \frac{8n}{4n^2-1}\right)\left(\zeta(m+\frac{1}{2},a) - \frac{8m}{4m^2-1}\right)\right|\\
&+ \left(\frac{1}{m-\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\zeta(m-n,a) + \left|\frac{2(m-n)\zeta(n+\frac{1}{2},a)}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} - \frac{2n\zeta(m-n,a)}{(m-\frac{1}{2})(2n-m+\frac{1}{2})}\right|\\
&+ \left|\frac{2n\zeta(m-n,\frac{a}{q})}{n^2-\frac{1}{4}} - \frac{4n(m-n)}{((m-n)^2-1)(n^2-\frac{1}{4})}\right| + o(1).
\end{aligned}$$

In view of Proposition 2.15, we obtain for any fixed pair of integers n, m with $m > n+1$ that

$$\begin{aligned}
&|\mathbb{E}W_n(a)\overline{W_m(a)}|\\
&\leq \left|\zeta(m-n+1,a) - \left(\zeta(n+\frac{1}{2},a) - \frac{8n}{4n^2-1}\right)\left(\zeta(m+\frac{1}{2},a) - \frac{8m}{4m^2-1}\right)\right|\\
&+ \left(\frac{1}{m-\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right)\zeta(m-n,a)\\
&+ \left|\frac{2(m-n)\zeta(n+\frac{1}{2},a)}{(m+\frac{1}{2})(2n-m+\frac{1}{2})} - \frac{2n\zeta(m-n,a)}{(m-\frac{1}{2})(2n-m+\frac{1}{2})}\right|\\
&+ \left|\frac{2n\zeta(m-n,a)}{n^2-\frac{1}{4}} - \frac{4n(m-n)}{((m-n)^2-1)(n^2-\frac{1}{4})}\right|.
\end{aligned}$$

However,

$$\begin{aligned}
&\zeta(m-n+1,a) - \left(\zeta(n+\frac{1}{2},a) - \frac{8n}{4n^2-1}\right)\left(\zeta(m+\frac{1}{2},a) - \frac{8m}{4m^2-1}\right)\\
&= \zeta(m-n+1,a) - \left(\zeta(n+\frac{1}{2},a)\right)\left(\zeta(m+\frac{1}{2},a)\right)\\
&+ \left(\frac{8m}{4m^2-1}\right)\zeta(n+\frac{1}{2},a) + \left(\frac{8n}{4n^2-1}\right)\zeta(m+\frac{1}{2},a) - \frac{64mn}{(4n^2-1)(4m^2-1)}.
\end{aligned}$$

Now

$$\begin{aligned}
&\zeta(m-n+1,a) - \zeta(n+\frac{1}{2},a)\zeta(m+\frac{1}{2},a)\\
&= \sum_{k=0}^{\infty}\frac{1}{(k+a)^{m-n+1}} - \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(k+a)^{\frac{1}{2}+n}(l+a)^{\frac{1}{2}+m}}\\
&= \sum_{k=1}^{\infty}\frac{1}{(k+a)^{m-n+1}} - \frac{1}{a^{\frac{1}{2}+m}}\sum_{k=1}^{\infty}\frac{1}{(k+a)^{\frac{1}{2}+n}} + \frac{1}{a^{\frac{1}{2}+n}}\sum_{l=1}^{\infty}\frac{1}{(l+a)^{\frac{1}{2}+m}}\\
&+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(k+a)^{\frac{1}{2}+n}(l+a)^{\frac{1}{2}+m}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\zeta(m-n+1,a)-\left(\zeta(n+\frac{1}{2},a)\right)\left(\zeta(m+\frac{1}{2},a)\right)\right| \\
&\leq C\max\left(\frac{1}{(1+a)^{m-n+1}},\frac{1}{(1+a)^{n+\frac{1}{2}}},\frac{1}{(1+a)^{m+\frac{1}{2}}}\right) \\
&\leq C\max\left(\frac{1}{(1+a)^{m-n}},\frac{1}{(1+a)^{n}}\right).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\sup_{m>n+1}\frac{m}{4m^2-1} &= O(\frac{1}{n}), \\
\frac{n}{4n^2-1} &= O(\frac{1}{n}),
\end{aligned}$$

so that

$$\sup_{m>n+1}\left|\frac{8m\zeta(n+\frac{1}{2},a)}{4m^2-1}+\frac{8n\zeta(m+\frac{1}{2},a)}{4n^2-1}-\frac{64mn}{(4n^2-1)(4m^2-1)}\right| = O(\frac{1}{n}).$$

For the other terms, we have, uniformly in m with $m>n+1$,

$$\begin{aligned}
\left(\frac{1}{m-\frac{1}{2}}+\frac{1}{n-\frac{1}{2}}\right)\zeta(m-n,a) &= O(\frac{1}{n}) \\
\frac{2n\zeta(m-n,a)}{n^2-\frac{1}{4}} &= O(\frac{1}{n}) \\
\frac{4n(m-n)}{((m-n)^2-1)(n^2-\frac{1}{4})} &= O(\frac{1}{n}).
\end{aligned}$$

The last term can be treated as in the proof of Lifshits and Weber, namely by

$$\left|\frac{2(m-n)}{(n+\frac{1}{2})(2n-m+\frac{1}{2})}(\zeta(m-n,a)-\zeta(n+\frac{1}{2},a))\right| \leq C\max\left(\frac{1}{(1+a)^{m-n}},\frac{1}{(1+a)^{n}}\right).$$

Therefore, for $m>n+1$

$$|\mathbb{E}W_n(a)\overline{W_m(a)}| \leq C\max\left(\frac{1}{(1+a)^{m-n}},\frac{1}{n}\right),$$

as claimed in Theorem 2.8.

2.2.4 The proof of Theorem 2.9

In this subsection, we shall prove Theorem 2.9. As we have seen in Section 2.1 the increment condition in Proposition 2.1 is necessary. Here we shall consider the increments

$$\mathbb{E}\left|\sum_{\substack{i\leq n\leq j\\ n \text{ even}}}W_n(a)\right|^2, \qquad \mathbb{E}\left|\sum_{\substack{i\leq n\leq j\\ n \text{ odd}}}W_n(a)\right|^2.$$

Using Theorem 2.8, we have

$$\mathbb{E}\left|\sum_{\substack{i\le n\le j\\ n \text{ even}}} W_n\right|^2 = \sum_{\substack{i\le n\le j\\ n \text{ even}}} \mathbb{E}\,|W_n|^2 + \sum_{\substack{i\le n\le j\\ n \text{ even}}}\sum_{\substack{i\le m\le j\\ m \text{ even}}} |\mathbb{E}W_n\overline{W_m}|$$
$$\le C\sum_{\substack{i\le n\le j\\ n \text{ even}}} \log(n+a) + C\sum_{\substack{i\le n<m\le j\\ n,m \text{ even}}} \max\left(\frac{1}{(1+a)^{m-n}}, \frac{1}{n}\right).$$

However,

$$\sum_{\substack{i\le n\le j\\ n \text{ even}}}\sum_{\substack{i\le m\le j\\ m \text{ even}}} \frac{1}{n} \le \left(\sum_{n\le j}\frac{1}{n}\right)\left(\sum_{i\le m\le j} 1\right) \le C(\log j)(j-i)$$

and

$$\sum_{\substack{i\le n\le j\\ n \text{ even}}}\sum_{\substack{i\le m\le j\\ m \text{ even}}} \frac{1}{(1+a)^{m-n}} \le \left(\sum_{i\le n\le j} 1\right)\left(\sum_{m>n}\frac{1}{(1+a)^{m-n}}\right)$$
$$\le (j-i)\left(\sum_{h\ge 1}(1+a)^{-h}\right) \le C(j-i).$$

Therefore

$$\mathbb{E}\left|\sum_{\substack{i\le n\le j\\ n \text{ even}}} W_n(a)\right|^2 \le C(\log(j+a))(j-i).$$

Similarly as for the odd part, we find that there exists a constant c such that, for any $j>i$,

$$\mathbb{E}\left|\sum_{i\le n\le j} W_n(a)\right|^2 \le C(\log(j+a))(j-i).$$

Applying this result to $W_n(a)$ with the choice $m_l \equiv 1$ and $\Phi(x) = \log(n+1)$, we obtain the assertion of Theorem 2.9.

2.3 Sampling the Lindelöf Hypothesis for Dirichlet L-functions

In this chapter, we investigate the almost-sure asymptotic behaviour of the system

$$L_n(\sigma,\chi) := L(\sigma + iS_n, \chi), \qquad n = 1, 2, ...$$

for $\sigma \ge \frac{1}{2}$, where χ is a primitive character modulo q. For any positive integer n, let

$$WL_n(\sigma,\chi) := L(\sigma + iS_n,\chi) - \mathbb{E}L(\sigma + iS_n,\chi) = L_n(\sigma,\chi) - \mathbb{E}L_n(\sigma,\chi).$$

In our situation, we obtain

Theorem 2.19. *For any real $b > 2$, $\sigma \geq \frac{1}{2}$,*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} L(\sigma + iS_k, \chi) - n}{n^{\frac{1}{2}} \log(1+n)^b} \qquad \overset{a.s.}{\longrightarrow} \qquad 0,$$

and

$$\| \sup_{n\geq 1} \frac{|\sum_{k=1}^{n} L(\sigma + iS_k, \chi) - n|}{n^{\frac{1}{2}} \log(1+n)^b} \|_2 \qquad < \infty.$$

We notice that this is pretty similar to Theorem 2.3 and Theorem 2.9 from the previous subsection. However, here we shall use an alternative proof in the case of Dirichlet L-functions associated with a primitive character χ. Instead of working with an approximation by a finite sum we shall incorporate the Dirichlet L-function directly by using Atkinson's formula. In our situation we need to show that, for $\sigma > 0$ and a primitive character χ modulo q,

$$\mathbb{E}\left\{\sum_{k=1}^{\infty} \chi(k) k^{-(\sigma+iS_n)}\right\} = \sum_{k=1}^{\infty} \mathbb{E}\left\{\chi(k) k^{-(\sigma+iS_n)}\right\} \tag{2.36}$$

and

$$\mathbb{E}\left\{L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)}\right\} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{E}\left\{\chi(k)\overline{\chi(l)} k^{-\sigma-iS_n} l^{-\sigma+iS_m}\right\}. \tag{2.37}$$

Notice that interchanging summation and expectation as in (2.36) and (2.37) is not possible for the Riemann zeta-function when $0 < \sigma < 1$ (the case considered by Lifshits and Weber). We shall show the proof of (2.36) and (2.37) as part of the proof of Lemma 2.20 in the next section. Here, Atkinson's formula is used to calculate the correlation $\mathbb{E}L(\sigma + iS_n, \chi)\overline{L(\sigma + iS_m, \chi)}$ whenever $m > n+1$. The idea of Atkinson's formula is to consider the product $\zeta(s_1)\zeta(s_2)$ and divide it according to

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1),$$

where

$$\zeta_2(s_1, s_2) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} m_1^{-s_1}(m_1 + m_2)^{-s_2}.$$

Hence we can avoid the lengthy proof using an approximation of the Dirichlet L-function as in Section 2 of [43].

2.3.1 The proof of Theorem 2.19

In this section, we prove Theorem 2.19. We develop a complete second-order theory of the system $\{WL_n(\sigma,\chi), n \geq 1\}$. For this aim we first show

Lemma 2.20. *For $\sigma > \frac{1}{2}$, there exists a constant C_1 such that*

$$\mathbb{E}\sum_{\chi} |WL_n(\sigma,\chi)|^2 = \frac{2\phi^2(q)}{q^{2\sigma}}\zeta(2\sigma - 1)\Gamma(2\sigma - 1)\frac{\Gamma(n+1-\sigma)}{\Gamma(n+\sigma)}$$
$$+ \phi(q)L(2\sigma, \chi_0) + C_1 + o(1),$$

where

$$C_1 = \lim_{n\to\infty} \left(\frac{4\phi(q)}{q^{2\sigma-1}} \sum_{e|q} \frac{\mu(e)}{e} \sum_{h=1}^{\infty} h^{1-2\sigma} \sum_{u=1}^{\infty} \int_0^\infty \frac{y^{n-\sigma}}{(1+y)^{n+\sigma}} \cos(\frac{2\pi huqy}{e}) dy \right).$$

For $\sigma = \frac{1}{2}$, there exists a constant C_2 such that

$$\mathbb{E} \sum_{\chi} |WL_n(\frac{1}{2},\chi)|^2 = \frac{\phi^2(q)}{q} \log(n+\frac{1}{2}) + C_2 + o(1),$$

where

$$C_2 = \lim_{n\to\infty} \left(4\phi(q) \sum_{e|q} \frac{\mu(e)}{e} \sum_{v=1}^{\infty} d(v) \int_0^\infty \frac{y^{n-\frac{1}{2}}}{(1+y)^{n+\frac{1}{2}}} \cos(\frac{2\pi qyv}{e}) dy \right)$$
$$+ \gamma + \gamma_q - \log 2\pi + \log q,$$

with Euler's constant γ and $\gamma_q = \gamma + \sum_{p|q} \frac{\log p}{p-1}$, where the summation is over all prime divisors p of q.
For $m > n+1$, $\sigma \in [\frac{1}{2}, 1)$, there exists a constant C_3 (dependending on q) such that

$$|\mathbb{E} \sum_{\chi} WL_n(\sigma,\chi)\overline{WL_m(\sigma,\chi)}| \le C_3 \max\left\{ \frac{1}{2^{m-n}}, \frac{1}{n} \right\}.$$

Proof of Lemma 2.20 In order to investigate the covariance structure, we study the behavior of the moments of first and second order of $L_n(\sigma,\chi)$, and the correlation $\mathbb{E}\sum_\chi L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)}$, from which the second order distances $\mathbb{E}\sum_\chi |L_n(\sigma,\chi) - L_m(\sigma,\chi)|^2$, $m > n$, can be derived easily. The first moments are given by

$$\mathbb{E}L_n(\sigma,\chi) = \mathbb{E}L(\sigma + iS_n, \chi) = \frac{n}{\pi} \int_{\mathbb{R}} L(\sigma + i\tau, \chi) \frac{d\tau}{n^2 + \tau^2}.$$

The integrand on the right-hand side is a regular function of τ except at $\tau = \pm ni$ in the τ-plane, since for primitive χ the Dirichlet L-function is a regular function for $\sigma > 0$. In order to calculate the expectation of $L(\sigma + iS_n, \chi)$ we apply the calculus of residues. For a sufficiently large parameter $R > 1 + \sigma$, we denote the counterclockwise oriented semicircle of radius R centered at the origin located in the lower half of the τ-plane by Γ_R. Then

$$\int_{-R}^{R} L(\sigma + i\tau, \chi) \frac{d\tau}{n^2+\tau^2} = \int_{\Gamma_R} L(\sigma + i\tau, \chi) \frac{d\tau}{n^2+\tau^2} - 2\pi i \mathrm{Res}_{\tau=-ni} \frac{L(\sigma+i\tau,\chi)}{n^2+\tau^2}.$$

By the functional equation for the Dirichlet L-function and Stirling's formula, we have, for $\frac{1}{2} \le \alpha \le \frac{3}{4}$,

$$L(1-\alpha+it,\chi) \ll_q |L(\alpha+it,\chi)|(1+|t|)^{\alpha-\frac{1}{2}}$$

(see [52]). From this, we deduce

$$\int_{\Gamma_R} L(\sigma+i\tau,\chi)\frac{d\tau}{n^2+\tau^2} \ll_q \frac{R}{n^2+R^2} \max_{\tau\in\Gamma_R} |L(\sigma+i\tau,\chi)| \ll_q R^{-\frac{1}{2}}.$$

The integral tends to zero as $R \to \infty$. Next we compute the residue

$$\mathrm{Res}_{\tau=-ni}\frac{L(\sigma+i\tau,\chi)}{n^2+\tau^2} = \lim_{\tau\to -ni}(\tau+ni)\frac{L(\sigma+i\tau,\chi)}{n^2+\tau^2} = \frac{L(n+\sigma,\chi)}{-2ni}.$$

Therefore, we have

$$\mathbb{E}L(\sigma+iS_n,\chi) = L(n+\sigma,\chi) \tag{2.38}$$

for any integer n and $\sigma \geq \frac{1}{2}$. Moreover, (2.38) follows from

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{E}\left\{\chi(k)k^{-(\sigma+iS_n)}\right\} &= \sum_{k=1}^{\infty}\chi(k)k^{-\sigma}\mathbb{E}\left\{\exp\left(-i(\log k)S_n\right)\right\} = \sum_{k=1}^{\infty}\chi(k)k^{-\sigma-n}\\ &= L(n+\sigma,\chi) = \mathbb{E}\left\{\sum_{k=1}^{\infty}\chi(k)k^{-(\sigma+iS_n)}\right\}.\end{aligned}$$

The follwoing calculations yield an asymptotic formula for $\mathbb{E}\sum_{\chi}L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)}$ whenever $m > n+1$. Let us fix $\sigma \in [\frac{1}{2},1)$ and $m > n+1$. Here, since we aim at proving (2.37) by use of the method of Lifshits and Weber, we need an approximation of the Dirichlet L-function (see [52]). For $0<\sigma<1$ and $t>0$, let $x > C\frac{qt}{2\pi}$, where C is a positive constant; then

$$L(s,\chi) = \sum_{n\leq x}\chi(n)n^{-s} + O(q^{\frac{1}{2}}x^{-\sigma}(\log(q+2))).$$

We can consider the second-order theory of the system $WL_n(\sigma,\chi)$ from a study of the same kind concerning the system

$$Z_n(\sigma,\chi) = \sum_{n\leq x}\chi(n)n^{-\sigma-iS_n}, \qquad n=1,2,..,x>0.$$

Since, for $\sigma>0$, $\mathbb{E}|Z_n(\sigma,\chi)-L(\sigma+iS_n,\chi)|^2 = O(qx^{-2\sigma}(\log^2(q+2))) \to 0$, when $x\to\infty$, we can easily show that $Z_n(\sigma,\chi)$ approximates the Dirichlet L-function sufficiently well. It follows that

$$\begin{aligned}\mathbb{E}L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)} &= \lim_{x\to\infty}\mathbb{E}Z_n(\sigma,\chi)\overline{Z_m(\sigma,\chi)}\\ &= \lim_{x\to\infty}\mathbb{E}\sum_{k\leq x}\chi(k)k^{-\sigma-iS_n}\sum_{l\leq x}\overline{\chi(l)}l^{-\sigma+iS_m}\\ &= \lim_{x\to\infty}\sum_{k\leq x}\sum_{l\leq x}\mathbb{E}\chi(k)\overline{\chi(l)}k^{-\sigma-iS_n}l^{-\sigma+iS_m}\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mathbb{E}\chi(k)\overline{\chi(l)}k^{-\sigma-iS_n}l^{-\sigma+iS_m}.\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
\mathbb{E}\sum_{\chi \mod q} L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)} &= \mathbb{E}\left\{\sum_{\chi \mod q}\sum_{k,l=1}^{\infty}\chi(k)k^{-(\sigma+iS_n)}\overline{\chi(l)}l^{-(\sigma-iS_m)}\right\}\\
&= \sum_{\chi \mod q}\sum_{k,l=1}^{\infty}\chi(k)\overline{\chi(l)}(kl)^{-\sigma}\mathbb{E}\left\{\exp(-i(\log k)S_n)\exp(i(\log l)S_m)\right\}\\
&= \sum_{\chi \mod q}\sum_{k,l=1}^{\infty}\chi(k)\overline{\chi(l)}(kl)^{-\sigma}\mathbb{E}\left\{\exp(i(\log l-\log k)S_n+i(\log l)(S_m-S_n))\right\}.
\end{aligned}$$

In order to evaluate this expression further we consider the value of $\mathbb{E}\exp\left(i(\log l-\log k)S_n\right)$.
If $l>k$, then $\mathbb{E}\exp\left(i(\log l-\log k)S_n\right)=(\frac{l}{k})^{-n}$.
If $l<k$, then $\mathbb{E}\exp\left(i(\log l-\log k)S_n\right)=(\frac{l}{k})^{n}$.
Thus we get

$$\mathbb{E}\sum_{\chi \mod q} L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)} = \sum_{\chi \mod q}\sum_{k,l=1}^{\infty}\chi(k)\overline{\chi(l)}(kl)^{-\sigma}\left(\frac{\min(k,l)}{\max(k,l)}\right)^n l^{-m+n}.$$

As in [45], the double series is studied according to whether $k=l$, $k>l$ or $k<l$. Besides we shall also use the orthogonality of the Dirichlet characters involved. Hence, this sum is

$$\phi(q)\left\{L(m-n+2\sigma,\chi_0)+\sum_{\substack{r=1\\(r,q)=1}}^{\infty}\sum_{h=1}^{\infty}\frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}}+\sum_{\substack{r=1\\(r,q)=1}}^{\infty}\sum_{h=1}^{\infty}\frac{r^{2n-m-\sigma}}{(r+qh)^{n+\sigma}}\right\}+o(1), \tag{2.39}$$

where χ_0 is the principal character modulo q. For $\sigma\geq\frac{1}{2}$, $m>n+1$, (2.39) holds by analytic continuation. In view of the convergence of the double series, for $\sigma\geq\frac{1}{2}$, $m>n+1$, we have

$$\sum_{\substack{r=1\\(r,q)=1}}^{\infty}\sum_{h=1}^{\infty}\frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}}=\sum_{e|q}\mu(e)\sum_{h=1}^{\infty}\sum_{r=1}^{\infty}\frac{(er)^{n-\sigma}}{(er+qh)^{m+\sigma}}.$$

By Poisson's summation formula,

$$\begin{aligned}
&\sum_{r=1}^{\infty}\frac{(er)^{n-\sigma}}{(er+qh)^{m+\sigma}}\\
&=\int_0^{\infty}\frac{(ex)^{n-\sigma}}{(ex+qh)^{m+\sigma}}dx+2\sum_{u=1}^{\infty}\int_0^{\infty}\frac{(ex)^{n-\sigma}}{(ex+qh)^{m+\sigma}}\cos(2\pi ux)dx\\
&=\frac{(qh)^{1-2\sigma+n-m}}{e}\left(\int_0^{\infty}\frac{y^{n-\sigma}}{(1+y)^{m+\sigma}}dy+2\sum_{u=1}^{\infty}\int_0^{\infty}\frac{y^{n-\sigma}}{(1+y)^{m+\sigma}}\cos(\frac{2\pi uqsy}{t})dy\right).
\end{aligned}$$

Now we sum with respect to h and e and use the identities

$$\int_0^{\infty}\frac{y^{n-\sigma}}{(1+y)^{m+\sigma}}dy=\Gamma(m-n+2\sigma-1)\frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)},\qquad \sum_{e|q}\mu(e)e^{-1}=\phi(q)q^{-1}.$$

This gives

$$\sum_{\substack{r=1\\(r,q)=1}}^{\infty}\sum_{h=1}^{\infty}\frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}}=\frac{\phi(q)}{q^{m-n+2\sigma}}\zeta(m-n+2\sigma-1)\Gamma(m-n+2\sigma-1)\frac{\Gamma(n+1-\sigma)}{\Gamma(m+\sigma)}$$
$$+g_q(m,n),$$

where

$$g_q(m,n)=\frac{2}{q^{m-n+2\sigma-1}}\sum_{e|q}\frac{\mu(e)}{e}\sum_{h=1}^{\infty}h^{-m+n-2\sigma+1}\sum_{u=1}^{\infty}\int_0^{\infty}\frac{y^{n-\sigma}}{(1+y)^{m+\sigma}}\cos(\frac{2\pi huqy}{e})dy. \tag{2.40}$$

Exactly in the same way we obtain for the last sum in (2.39)

$$\sum_{\substack{r=1\\(r,q)=1}}^{\infty}\sum_{h=1}^{\infty}\frac{r^{2n-m-\sigma}}{(r+qh)^{n+\sigma}}$$
$$=\frac{\phi(q)}{q^{m-n+2\sigma}}\zeta(m-n+2\sigma-1)\Gamma(m-n+2\sigma-1)\frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)}+f_q(m,n),$$

where

$$f_q(m,n)=\frac{2}{q^{m-n+2\sigma-1}}\sum_{e|q}\frac{\mu(e)}{e}\sum_{h=1}^{\infty}h^{-m+n-2\sigma+1}\sum_{u=1}^{\infty}\int_0^{\infty}\frac{y^{2n-m-\sigma}}{(1+y)^{n+\sigma}}\cos(\frac{2\pi huqy}{e})dy.$$

Thus,

(2.41)

$$\mathbb{E}_{\chi}\sum_{\text{mod }q}L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)}$$
$$=\frac{\phi^2(q)}{q^{m-n+2\sigma}}\zeta(m-n+2\sigma-1)\Gamma(m-n+2\sigma-1)\left(\frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)}+\frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)}\right)$$
$$+\phi(q)L(m-n+2\sigma,\chi_0)+\phi(q)(g_q(m,n)+f_q(m,n)).$$

Now we return to (2.40) and consider the convergence of its right-hand side. We have, for $\sigma\ge\frac{1}{2}$ and $k\ge 1$,

$$2\int_0^{\infty}\frac{y^{n-\sigma}}{(1+y)^{m+\sigma}}\cos(2\pi ky)dy$$
$$=k^{\sigma-n-1}\int_0^{\infty}\frac{y^{n-\sigma}}{(1+\frac{y}{k})^{m+\sigma}}\left(e^{2\pi iy}+e^{-2\pi iy}\right)dy$$
$$=k^{\sigma-n-1}\int_0^{i\infty}\frac{y^{n-\sigma}}{(1+\frac{y}{k})^{m+\sigma}}e^{2\pi iy}dy+k^{\sigma-n-1}\int_0^{-i\infty}\frac{y^{n-\sigma}}{(1+\frac{y}{k})^{m+\sigma}}e^{-2\pi iy}dy$$
$$\ll\left|\frac{k^{\sigma-n-1}}{\sigma-n-1}\right|$$

uniformly for $m>n+1$. It follows that the double series (2.40) is absolutely convergent for $\sigma\ge\frac{1}{2}$ and $m>n+1$, by comparison with

$$\sum_{h=1}^{\infty}|h^{-m-\sigma}|\sum_{u=1}^{\infty}|u^{\sigma-n-1}|=o(1)\qquad\text{as}\qquad m,n\to\infty.$$

Next we shall find an asymptotic formula for $\mathbb{E}\sum_\chi |L_n(\sigma,\chi)|^2$. For this purpose we put $m = n$ in equation (2.41); then we have, for $\sigma \in (\frac{1}{2}, 1)$,

$$\mathbb{E}\sum_{\chi \bmod q} |L_n(\sigma,\chi)|^2 = \frac{2\phi^2(q)}{q^{2\sigma}}\zeta(2\sigma-1)\Gamma(2\sigma-1)\frac{\Gamma(n+1-\sigma)}{\Gamma(n+\sigma)} + \phi(q)L(2\sigma,\chi_0) + g_q(n,n),$$

where

$$g_q(n,n) = \frac{4\phi(q)}{q^{2\sigma-1}}\sum_{e|q}\frac{\mu(e)}{e}\sum_{h=1}^{\infty}h^{1-2\sigma}\sum_{u=1}^{\infty}\int_0^\infty \frac{y^{n-\sigma}}{(1+y)^{n+\sigma}}\cos(\frac{2\pi huqy}{e})dy.$$

In view of (2.41), the case $\sigma = \frac{1}{2}$ is exceptional. Here we use the fact that $g_q(n,n)$ and $f_q(n,n)$ are continuous in n and write $2\sigma - 1 = \delta$, $|\delta| < \frac{1}{2}$, with the aim of letting $\delta \to 0$. Then the first two terms on the right-hand side give us

$$\frac{2\phi^2(q)}{q^{1+\delta}}\zeta(\delta)\Gamma(\delta)\frac{\Gamma(1-\sigma+n)}{\Gamma(1-\sigma+n+\delta)} + \phi(q)L(1+\delta,\chi_0).$$

Using Taylor's formula for the gamma-function terms, the functional equation for $\zeta(s)$ and the Laurent expansion of $\zeta(s)$ at the pole at $s = 1$

$$\zeta(s) = (s-1) + \gamma + O(|s-1)|,$$

and writing

$$\gamma_q = \gamma + \sum_{p|q}\frac{\log p}{p-1},$$

where γ is Euler's constant, we obtain

$$\begin{aligned}
&\phi(q)L(1+\delta,\chi_0) + \frac{\phi^2(q)}{q}\zeta(1-\delta)\left(\frac{q}{2\pi}\right)^{-\delta}\sec(\frac{\pi\delta}{2})\frac{\Gamma(1-\sigma+n)}{\Gamma(1-\sigma+n+\delta)} \\
&= \frac{\phi^2(q)}{q}\left\{\frac{1}{\delta} + \gamma_q - \left(\frac{1}{\delta} - \gamma\right)\left(1 - \delta\log\frac{q}{2\pi}\right)\left(1 - \delta\frac{\Gamma'}{\Gamma}(1-\sigma+n)\right)\right\} + O(|\delta|) \\
&= \frac{\phi^2(q)}{q}\left\{\frac{\Gamma'}{\Gamma}(1-\sigma+n) + \gamma + \gamma_q + \log\frac{q}{2\pi}\right\} + O(|\delta|).
\end{aligned}$$

Hence, making $\delta \to 0$ and setting $\sigma = \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{E}\sum_{\chi \bmod q} |L_n(\frac{1}{2},\chi)|^2 &= \frac{\phi^2(q)}{q}\left\{\frac{\Gamma'}{\Gamma}(n+\frac{1}{2}) + \gamma + \gamma_q + \log\frac{q}{2\pi}\right\} + G_q(n) + o(1) \\
&= \frac{\phi^2(q)}{q}\left\{\log(n+\frac{1}{2}) - \frac{1}{2n-1} + \gamma + \gamma_q + \log\frac{q}{2\pi}\right\} \\
&\quad + G_q(n) + o(\frac{1}{n^2}),
\end{aligned} \tag{2.42}$$

where

$$G_q(n) = 4\phi(q)\sum_{e|q}\frac{\mu(e)}{e}\sum_{v=1}^{\infty}d(v)\int_0^\infty \frac{y^{n-\frac{1}{2}}}{(1+y)^{n+\frac{1}{2}}}\cos(\frac{2\pi qyv}{e})dy.$$

In the next step, we estimate the covariance of the system $\{WL_n(\sigma,\chi), n \geq 1\}$. Recall that

$$WL_n(\sigma,\chi) := L(\sigma+iS_n,\chi) - \mathbb{E}L(\sigma+iS_n,\chi).$$

Since

$$\mathbb{E}\sum_{\chi} WL_n(\sigma,\chi)\overline{WL(\sigma,\chi)_m} = \mathbb{E}\sum_{\chi} L_n(\sigma,\chi)\overline{L_m(\sigma,\chi)} - \sum_{\chi} \mathbb{E}L_n(\sigma,\chi)\overline{\mathbb{E}L_m(\sigma,\chi)},$$

we obtain from (2.41) and (2.39), for $\sigma \in [\frac{1}{2}, 1)$,

(2.43)

$$\begin{aligned}
&\mathbb{E}\sum_{\chi} WL_n(\sigma,\chi)\overline{WL(\sigma,\chi)_m} \\
&= \frac{\phi^2(q)}{q^{m-n+2\sigma}}\zeta(m-n+2\sigma-1)\Gamma(m-n+2\sigma-1)\left(\frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)} + \frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)}\right) \\
&+ \phi(q)L(m-n+2\sigma,\chi_0) + \phi(q)(g_q(m,n)+f_q(m,n)) - \sum_{\chi}\left(L(n+\sigma,\chi)\overline{L(m+\sigma,\chi)}\right).
\end{aligned}$$

In view of (2.43), for $\sigma \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
\mathbb{E}\sum_{\chi} |WL_n(\sigma,\chi)|^2 &= \frac{2\phi^2(q)}{q^{2\sigma}}\zeta(2\sigma-1)\Gamma(2\sigma-1)\frac{\Gamma(n+1-\sigma)}{\Gamma(n+\sigma)} \\
&+ \phi(q)L(2\sigma,\chi_0) + g_q(n,n) - \sum_{\chi}|L(n+\sigma,\chi)|^2.
\end{aligned}$$

For $\sigma = \frac{1}{2}$, we get similarly

$$\begin{aligned}
\text{(2.44)} \qquad \mathbb{E}\sum_{\chi} |WL_n(\tfrac{1}{2},\chi)|^2 &= \frac{\phi^2(q)}{q}\left\{\log(n+\frac{1}{2}) - \frac{1}{2n-1} + \gamma + \gamma_q + \log\frac{q}{2\pi}\right\} \\
&+ G_q(n) - \sum_{\chi}\left|L(n+\frac{1}{2},\chi)\right|^2.
\end{aligned}$$

Now we estimate (2.43) for $\sigma \geq \frac{1}{2}$, $m > n+1$, by

$$\begin{aligned}
&|\mathbb{E}\sum_{\chi} WL_n(\sigma,\chi)\overline{WL(\sigma,\chi)_m}| \\
&\leq \frac{\phi^2(q)}{q^{m-n+2\sigma}}\zeta(m-n+2\sigma-1)\Gamma(m-n+2\sigma-1)\left(\frac{\Gamma(m-\sigma+1)}{\Gamma(m+\sigma)} + \frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)}\right) \\
&+ |\phi(q)L(m-n+2\sigma,\chi_0) - \sum_{\chi}\left(L(n+\sigma,\chi)\overline{L(m+\sigma,\chi)}\right)| + |\phi(q)(g_q(m,n)+f_q(m,n))|.
\end{aligned}$$

We observe

$$
\begin{aligned}
&\phi(q)L(m-n+2\sigma,\chi_0)-\sum_{\chi}\Big(L(n+\sigma,\chi)\overline{L(m+\sigma,\chi)}\Big)\\
&=\phi(q)\sum_{k=1}^{\infty}\chi_0(k)k^{-m+n-2\sigma}-\phi(q)\sum_{k=1}^{\infty}\chi_0(k)k^{-m-\sigma}-\phi(q)\sum_{l=1}^{\infty}\chi_0(l)l^{-n-\sigma}\\
&-\sum_{\chi}\sum_{k=2}^{\infty}\overline{\chi(k)}k^{-m-\sigma}\sum_{l=2}^{\infty}\chi(l)l^{-n-\sigma}.
\end{aligned}
$$

It follows that

$$
|\phi(q)L(m-n+2\sigma,\chi_0)-\sum_{\chi}\Big(L(n+\sigma,\chi)\overline{L(m+\sigma,\chi)}\Big)|\leq C_q\left(\frac{1}{2^{m-n}},\frac{1}{2^n}\right).
$$

Regarding the other terms we find in view of the absolutely convergence of the double series (2.40), for $m>n+1$,

$$
|\phi(q)(g_q(m,n)+f_q(m,n))|=O(\frac{1}{q^{m-n}}).
$$

Finally considering the last term, we have

$$
\frac{\Gamma(m-n+2\sigma-1)\Gamma(n+1-\sigma)}{\Gamma(m+\sigma)}=O(\frac{1}{n}),
$$

hence

$$
\frac{\phi^2(q)\zeta(m-n+2\sigma-1)}{q^{m-n+2\sigma-1}}\frac{\Gamma(m-n+2\sigma-1)\Gamma(n+1-\sigma)}{\Gamma(m+\sigma)}=O_q(\frac{1}{n}).
$$

Thus, for $q\geq 2$, there exists a constant C (depending only on q) such that

$$
|\mathbb{E}\sum_{\chi}WL_n(\sigma,\chi)\overline{WL_m(\sigma,\chi)}|\leq C\max(\frac{1}{2^{m-n}},\frac{1}{n}).
$$

Here and in the sequel C denotes a positive constant, not necessarily the same at each appearance.

Proof Theorem 2.19 Now we consider the asymptotic behaviour along the Cauchy random walk. The essential step consists of controlling the increments

$$
\mathbb{E}\sum_{\chi}\left|\sum_{\substack{i\leq n\leq j\\ n\text{ even}}}WL_n(\sigma,\chi)\right|^2,\qquad \mathbb{E}\sum_{\chi}\left|\sum_{\substack{i\leq n\leq j\\ n\text{ odd}}}WL_n(\sigma,\chi)\right|^2.
$$

Since the two increments can be treated in exactly the same way, we consider only the first one. We use Lemma 2.20. For $\sigma \geq \frac{1}{2}$, we have

$$\begin{aligned}\mathbb{E}\sum_{\chi}\left|\sum_{\substack{i\leq n\leq j\\ n \text{ even}}} WL_n(\sigma,\chi)\right|^2 &= \sum_{\substack{i\leq n\leq j\\ n \text{ even}}} \mathbb{E}\sum_{\chi} |WL_n(\sigma,\chi)|^2 \\ &+ \sum_{\substack{i\leq n\leq j\\ n \text{ even}}}\sum_{\substack{i\leq m\leq j\\ m \text{ even}}} |\mathbb{E}\sum_{\chi} WL_n(\sigma,\chi)\overline{WL_m(\sigma,\chi)}| \\ &\leq C_q \sum_{\substack{i\leq n\leq j\\ n \text{ even}}} \log n + C_q \sum_{\substack{i\leq n<m\leq j\\ n,m \text{ even}}} \frac{1}{2^{m-n}}.\end{aligned}$$

However,

$$\begin{aligned}\sum_{\substack{i\leq n\leq j\\ n \text{ even}}}\sum_{\substack{i\leq m\leq j\\ m \text{ even}}} \frac{1}{2^{m-n}} &\leq \left(\sum_{i\leq n\leq j} 1\right)\left(\sum_{m>n} \frac{1}{2^{m-n}}\right) \\ &\leq (j-i)\left(\sum_{h\geq 1} 2^{-h}\right) \leq C(j-i),\end{aligned}$$

with some positive constant C. Therefore,

$$\mathbb{E}\sum_{\chi}\left|\sum_{\substack{i\leq n\leq j\\ n \text{ even}}} WL_n(\sigma,\chi)\right|^2 \leq C(\log j)(j-i).$$

And similarly for the odd part, we find that there exists a constant C such that, for any $j > i$,

$$\mathbb{E}\sum_{\chi}\left|\sum_{i\leq n\leq j} WL_n(\sigma,\chi)\right|^2 \leq C(\log j)(j-i).$$

Now the conclusion of Theorem 2.19 is easily obtained from Proposition 2.1. We apply this result to $WL_n(\sigma,\chi)$ with the choice $m_l \equiv 1$ and $\Phi(x) = \log(n+1)$ and obtain the assertion of Theorem 2.19.

Remark 1. If we put $q = 1$ in Lemma 2.20, for $\sigma = \frac{1}{2}$ some of our results are contained in Theorem 2.2, however, our constants take another form.

Remark 2. For $\sigma \in [\frac{1}{2}, 1)$, we can deduce

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} \mathbb{E}|\zeta(\sigma+iS_n,\frac{a}{q})|^2 = \frac{q^{2\sigma}}{\phi(q)} \sum_{\chi \mod q} \mathbb{E}|L(\sigma+iS_n,\chi)|^2.$$

Thus,

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} \mathbb{E}|\zeta(\sigma+iS_n,\frac{a}{q}) - \mathbb{E}\zeta(\sigma+iS_n,\frac{a}{q})|^2 \leq \frac{1}{q}\log(n+1-\sigma), \qquad n\to\infty.$$

Chapter 3

3 Sampling the Riemann Hypothesis with an ergodic transformation

In this chapter, we study the behaviour of the logarithmic derivative of the Riemann zeta-function on vertical lines $\sigma + it$, $t \in \mathbb{R}$, where the values $\frac{\zeta'}{\zeta}(\sigma + it)$ are sampled with t varying according to an ergodic transformation. Here, our ergodic transformation $T : \mathbb{R} \to \mathbb{R}$ is given by

$$Tx := \begin{cases} \frac{1}{2}(x - \frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{3.1}$$

Its iterates $T^n x$ are defined by $T \circ T^{n-1}$, for $n \geq 1$ and $T^0 x = x$.

In Section 3.1, we discuss the distribution of value of the Riemann zeta-function $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to the ergodic transformation T following a work of Steuding in [60].

In Section 3.2, we study the behaviour of the logarithmic derivative of zeta-functions on vertical lines $\sigma + it$, $t \in \mathbb{R}$, with respect to the ergodic transformation T. Here, we shall also give an equivalent formulation for the Riemann Hypothesis in terms of our ergodic transformation.

In Section 3.3, we also study the behaviour of the logarithm of the Riemann zeta-function in this sense by using a lemma of Kai-Man Tsang [65]. Here, we shall also give another equivalent formulation for the Riemann Hypothesis in terms of ergodic transformation.

In Section 3.4, we study the behaviour of an arithmetical function $\alpha(s + i\mathbb{R})x^{i\mathbb{R}}$ with respect to our ergodic transformation.

In Section 3.5, we study the behaviour of the moments of zeta-function. In particular, we deal with a problem concerning the explicit evaluation of the integral in Theorem 3.2.

Throughout this chapter, ρ denote non-trivial zeros of ζ.

3.1 Sampling the Lindelöf Hypothesis with an ergodic transformation

In this section, we discuss the investigation the distribution of value of the Riemann zeta-function $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to the ergodic transformation T from above due to a work of Steuding in [60].

Recently, Steuding showed that, for $\Re(s) > \frac{1}{2}$, the mean value of $\zeta(s + iT^n x)$ exists for almost all values $x \in \mathbb{R}$, as $n \to \infty$, and is independent of x. Moreover Steuding also determined its values.

Theorem 3.1. *(Steuding, 2012) Let s be given with $\Re(s) > -\frac{1}{2}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leq n < N} \zeta(s + iT^n x) = \frac{1}{\pi} \int_{\mathbb{R}} \zeta(s + i\tau) \frac{d\tau}{1 + \tau^2} \qquad \text{for almost all} \qquad x \in \mathbb{R}.$$

For $\Re(s) < 1$, $\Re(s) \neq 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leq n < N} \zeta(s + iT^n x) = \zeta(s + 1) - \frac{2}{s(2 - s)}.$$

For $\Re(s) > 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(s + iT^n x) = \zeta(s+1).$$

For the special case $s = 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(iT^n x) = \gamma - \frac{1}{2},$$

where γ *denote the Euler constant and for some real* t,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(1 + i(t + T^n x)) = \zeta(2 + it) - \frac{1}{1+t^2}.$$

From Theorem 3.1, the mean $\frac{1}{N}\sum_{0\le n<N} \zeta(s + iT^n x)$ provide ergodic samples for testing the Lindelöf Hypothesis and their almost sure convergence indicates that most of values of the zeta function are not too big. The most interesting case is $s = \frac{1}{2}$,for which, for almost all x,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \zeta(\frac{1}{2} + iT^n x) = \zeta(\frac{3}{2}) - \frac{8}{3} = -0.05429.... \tag{3.2}$$

For illustration, Steuding gave numerical results for

$$c_k = \frac{1}{10^k} \sum_{0\le n<10^k} \zeta(\frac{1}{2} + iT^n 42);$$

he computed

$$\begin{aligned} c_4 &= -0.04092... + i0.00288...,\\ c_5 &= -0.05357... + i0.00022...,\\ c_6 &= -0.05362... - i0.00043.... \end{aligned}$$

Moreover, Steuding showed an equivalent formulation of the Lindelöf Hypothesis in terms of the ergodic transformation T.

Theorem 3.2. *(Steuding, 2012)*
The Lindelöf Hypothesis is true if, and only if, for any $k \in \mathbb{N}$ *and almost all* $x \in \mathbb{R}$, *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} |\zeta(\frac{1}{2} + iT^n x)|^{2k} \tag{3.3}$$

exist, which is also equivalent to the existence of the integrals

$$\int_{\mathbb{R}} |\zeta(\frac{1}{2} + it)|^{2k} \frac{dt}{1+t^2}. \tag{3.4}$$

However, the investigation concerning the Cesàro means (3.3) and the explicit evaluation of (3.4) are an interesting object.

By elementary means on the approximation of $\zeta(s)$, Lifshits and Weber [43] showed that the result (2.2) $\mathbb{E}(\frac{1}{2} + iS_1) = \zeta(\frac{3}{2}) - \frac{8}{3}$, which yield a result of Steuding in (3.2).

3.1.1 Sketch of the proof of Theorem 3.1

The proof of Theorem 3.1 consists of two parts. In the first part, the pointwise ergodic theorem of Birkhoff is applied in order to show that for $\Re(s) > -\frac{1}{2}$, the mean value of $\zeta(s+iT^n x)$ exists for almost all values $x \in \mathbb{R}$, as $n \to \infty$, and is independent of x. In the second part, the residue theorem is applied to determine the explicit evaluation of the integrals.

- Applying the pointwise ergodic theorem of Birkhoff [8]

We call again the pointwise ergodic theorem of Birkhoff. Given a measure preserving transformation T on a measurable space (X, μ) and an integrable function f, the limit of the Cesàro means

$$\frac{1}{N}\sum_{0\le n<N} f(T^n x)$$

exists as $N \to \infty$ for almost all $x \in X$; if the measure space is finite and T ergodic, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N} f(T^n x) = \frac{1}{\mu(X)}\int_X f d\mu. \tag{3.5}$$

We have already known that our transformation T is ergodic, which satisfies this theorem. Hence, we have to only show that the function $\tau \to \frac{\zeta(s+i\tau)}{1+\tau^2}$ is Lebesque integrable on $\mathbb{R}$ for fixed $\Re(s) > -\frac{1}{2}$. For this we can check by Lemma 1.1 and the functional equation for ζ.

- Applying the residue theorem

In order to apply residue theorem, we first consider the three poles of the function $\tau \to \frac{\zeta(s+i\tau)}{1+\tau^2}$ at $\tau = \pm i$ and $\tau = i(s-1)$. Now we use the residue theorem inside the semicircle of radius R centered at the origin located in the lower half of the τ-plane, where $R > 1+|s|$. Thus we have

$$\frac{1}{2\pi i}\int_{\mathbb{R}}\zeta(s+i\tau)\frac{d\tau}{1+\tau^2} = \frac{1}{2\pi i}\int_{I_R}\zeta(s+i\tau)\frac{d\tau}{1+\tau^2} - \sum(s), \tag{3.6}$$

where $\sum(s)$ is the sum of residue inside $[-R, R]$ and I_R is the counterclockwise oriented semicircle. We use Lemma1.1 once more to show that the integral on the left-hand side of (3.6) tends to zero, when $R \to \infty$. Finally, we distinguish several cases according to be the location of the poles and conclude with the calculation of their residues.

Remark 1. Montgomery and Vaughan proposed the following claim as an exercise (see [47] p.338). Suppose throughout that $0 < \delta \le \frac{1}{2}$. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. If $\sigma_0 > \max(\delta, \sigma_c)$, then

$$\sum_{n\le x} a_n((\frac{x}{n})^{\delta} - (\frac{n}{x})^{\delta}) = \frac{\delta}{i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\alpha(w)\frac{x^w}{w^2-\delta^2}dw. \tag{3.7}$$

By taking $\alpha(w) = \zeta(\frac{1}{2}+it+w)$, we have

$$\begin{aligned}\zeta(\frac{1}{2}+it+\delta) = x^{-\delta}\sum_{n\le x} n^{\frac{-1}{2}-it}((\frac{x}{n})^{\delta} - (\frac{n}{x})^{\delta}) + \frac{\delta x^{-\delta}}{\pi}\int_{-\infty}^{\infty}\zeta(\frac{1}{2}+it+iu)x^{iu}\frac{du}{u^2+\delta^2} \\ - \frac{2\delta x^{\frac{1}{2}-\delta-it}}{(\frac{1}{2}-it)^2-\delta^2}.\end{aligned} \tag{3.8}$$

We replace $x=1$ and $\delta=1$ in (3.8), which is allowed in this case, we obtain Theorem 3.1 in the case $s=\frac{1}{2}+it$.

Remark 2. If we set $\delta=\frac{1}{2}$, then we have

$$\sum_{n\le x} a_n \frac{x^2-n^2}{\sqrt{nx}} = \frac{1}{i\pi}\int_{2\sigma_0-i\infty}^{2\sigma_0+i\infty} \alpha(w/2)\frac{x^{w/2}}{w^2-1}dw.$$

Considering the residue arising from the pole of $\alpha(\frac{w}{2})$ and at $w=1$, the integral

$$\frac{x^{\frac{-1}{2}}}{\pi}\int_{-\infty}^{\infty}\alpha(i\tau/2)\frac{x^{i\tau/2}}{\tau^2+1}d\tau$$

appears. Applying the pointwise ergodic theorem, we may be obtain that

$$\sum_{n\le x} a_n \frac{x^2-n^2}{\sqrt{nx}} = \lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\alpha(\frac{iT^n y}{2})x^{\frac{iT^n y}{2}} + \text{Term of residues},$$

for almost all $y\in\mathbb{R}$. We will give more details in Section 3.4.

3.2 Sampling the Riemann Hypothesis for the logarithmic derivative of the Riemann zeta-function with an ergodic transformation

In order to study the Riemann Hypothesis, we shall study the distribution of values of the logarithmic derivative of the Riemann zeta-function $\frac{\zeta'}{\zeta}(s)$ on vertical lines with respect to the ergodic transformation T as in the work of Steuding. We obtain the following

Theorem 3.3. *Let s be given with $\Re(s)>-\frac{1}{2}$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(s+iT^n x) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} \qquad \text{for almost all} \qquad x\in\mathbb{R}. \tag{3.9}$$

For $-\frac{1}{2}<\Re(s)<1$, $\Re(s)\neq 0$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(s+iT^n x) = \frac{\zeta'}{\zeta}(s+1)+\frac{2}{s(2-s)} - \sum_{\substack{\rho\\ \Re(\rho)=\Re(s)}}\frac{1}{1-(s-\rho)^2} - \sum_{\substack{\rho\\ \Re(\rho)>\Re(s)}}\frac{2}{1-(s-\rho)^2}. \tag{3.10}$$

For $\Re(s)>1$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(s+iT^n x) = \frac{\zeta'}{\zeta}(s+1). \tag{3.11}$$

For the special case $s=0$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(iT^n x) = \sum_{\rho}(\frac{1}{\rho}-\frac{1}{\rho+1})+\log 2\pi-\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(\frac{3}{2})+\gamma+1\right). \tag{3.12}$$

For some real t,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(1+i(t+T^n x))=\frac{\zeta'}{\zeta}(2+it)+\frac{1}{1+t^2}. \tag{3.13}$$

Remark 1. We can see that our results are also independent on x. Now we check (3.10) for some $s\in(-\frac{1}{2},1)$. We test the left hand side of (3.10) by setting

$$L_s(k)=\frac{1}{10^k}\sum_{0\le n<10^k}\frac{\zeta'}{\zeta}(s+iT^n x) \qquad \text{and} \qquad l(s)=\frac{\zeta'}{\zeta}(s+1)+\frac{2}{s(2-s)}.$$

With the initial value $x=1.16$ we find

	$L_s(3)$	$L_s(4)$	$L_s(5)$	$l(s)$	$l(s)-\Re(L_s(5))$
$s=0.30$	1.0793 + i 0.0054	1.0230 - i 0.0050	1.0191 - i 0.0010	1.11347	0.09442
$s=0.40$	1.1055 + i 0.0132	1.0466 - i 0.0054	1.0407 - i 0.0009	1.13463	0.09389
$s=0.45$	1.1237 + i 0.0206	1.0629 - i 0.0059	1.0545 - i 0.0003	1.14727	0.09276
$s=0.50$	1.1518 + i 0.0271	1.1211 + i 0.0424	1.1182 - i 0.0356	1.16143	0.04323
$s=0.55$	1.1813 + i 0.0221	1.1658 - i 0.0059	1.1742 - i 0.0003	1.17725	0.00305
$s=0.60$	1.2042 + i 0.0164	1.1873 - i 0.0055	1.1930 - i 0.0010	1.19489	0.00189
$s=0.70$	1.2483 + i 0.0132	1.2324 - i 0.0059	1.2354 - i 0.0010	1.23632	0.00092

Now we consider (3.10) in case of $s=0.55$; we have, for almost all $x\in\mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(0.55+iT^n x)=1.17725-\sum_{\substack{\rho\\ \Re(\rho)=0.55}}\frac{1}{1-(0.55-\rho)^2}-\sum_{\substack{\rho\\ \Re(\rho)>0.55}}\frac{2}{1-(0.55-\rho)^2}. \tag{3.14}$$

From the above table, we find that $L_{0.55}(k)$ tends to $l(0.55)$, as $k\to\infty$. This indicates that the sums on the right-hand side of (3.14) which taken all non-trivial zero of ζ are zero. Thus, there should be no non-trivial zero ρ of ζ with $\Re(\rho)\ge 0.55$.

In case of $s=0.45$, we have, for almost all $x\in\mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(0.45+iT^n x)=1.14727-\sum_{\substack{\rho\\ \Re(\rho)=0.45}}\frac{1}{1-(0.45-\rho)^2}-\sum_{\substack{\rho\\ \Re(\rho)>0.45}}\frac{2}{1-(0.45-\rho)^2}. \tag{3.15}$$

From the table, we find that the value of $l(0.45)-L_{0.45}(k)$ does not tend to zero, as $k\to\infty$. This indicates that the sums on the right-hand side of (3.15) do not vanish. Thus, the real part of the non-trivial zero of ζ should be ≥ 0.45, resp. that the real part of all non-trivial zeros are in $[0.45, 0.55)$.

Now we consider (3.10) in a special case $s=\frac{1}{2}$, we have, for almost all $x\in\mathbb{R}$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(\frac{1}{2}+iT^n x)&=\frac{\zeta'}{\zeta}(\frac{3}{2})+\frac{8}{3}-\sum_{\substack{\rho\\ \Re(\rho)=\frac{1}{2}}}\frac{1}{1-(\frac{1}{2}-\rho)^2}-\sum_{\substack{\rho\\ \Re(\rho)>\frac{1}{2}}}\frac{2}{1-(\frac{1}{2}-\rho)^2}\\ &=1.16143...-\sum_{\substack{\rho\\ \Re(\rho)=\frac{1}{2}}}\frac{1}{1-(\frac{1}{2}-\rho)^2}-\sum_{\substack{\rho\\ \Re(\rho)>\frac{1}{2}}}\frac{2}{1-(\frac{1}{2}-\rho)^2}.\end{aligned} \tag{3.16}$$

We consider the value of $l(s) - \Re(L_s(5))$ in the above table. We find that, for $s = 0.30$, 0.40 and 0.45, these values are nearly $2(l(\frac{1}{2}) - \Re(L_{\frac{1}{2}}(5)))$. This indicates that the last sum of (3.16) should vanish. That means there should be no non-trivial zero of ζ with real part $> \frac{1}{2}$. Moreover, the values of $l(s) - \Re(L_s(5))$ in case of $s = 0.30$, 0.40 and 0.45, tell us that the sum which taken over ρ with $\Re(\rho) = \Re(s)$ should be also zero, since these value are not different from each other. Therefore, it should be true that the real part of all non-trivial zeros of ζ is $\frac{1}{2}$. There is an interesting link to a recent result of Büthe, Franke, Jost & Kleinjung [16]

Lemma 3.4. *(J.Büthe, Franke, Jost & Kleinjung, 2013) We have*

$$\sum_{\rho} \frac{1}{1-(\rho-\frac{1}{2})^2} \leq 0.05, \tag{3.17}$$

where the sums are taken over all zeros of ζ in the critical strip, counted according to their multiplicity.

We note that the value of $l(\frac{1}{2}) - \Re(L_{\frac{1}{2}}(5))$ in the above table satisfies (3.17). This also indicates that there should be no non-trivial zero of ζ with real part $> \frac{1}{2}$. Moreover, Montgomery and Vaughan [47] gave a result about the summation in (3.17), namely

Lemma 3.5. *(see. [47] p. 434) For $s = \sigma + it$, $\sigma > 1$,*

$$\sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma-\frac{1}{2})^2 + (t - \Im(\rho))^2} = \Re(\frac{\zeta'}{\zeta}(s)) + \frac{1}{2}\Re(\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1)) - \frac{1}{2}\log\pi + \frac{\sigma-1}{(\sigma-1)^2+t^2}. \tag{3.18}$$

Here, we put $s = \frac{3}{2}$ in (3.18), then we have

$$\sum_{\rho} \frac{1}{1+(\Im(\rho))^2} = \frac{\zeta'}{\zeta}(\frac{3}{2}) + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{7}{4}) - \frac{1}{2}\log\pi + 2 = 0.0461388.... \tag{3.19}$$

We note that this sum in (3.19) is nearly the value of $l(\frac{1}{2}) - \Re(L_{\frac{1}{2}}(5))$ in the above table.

As a consequence of (3.16) we find an equivalent formulation of the Riemann Hypothesis.

Theorem 3.6. *The Riemann Hypothesis is true if, and only if, for almost all $x \in \mathbb{R}$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \frac{\zeta'}{\zeta}(\frac{1}{2} + iT^n x) = \frac{\zeta'}{\zeta}(\frac{3}{2}) + \frac{8}{3} - \sum_{\rho} \frac{1}{1-(\rho-\frac{1}{2})^2}, \tag{3.20}$$

where ρ denotes the non-trivial zeros of ζ.

3.2.1 Proof of Theorem 3.3

In order to apply the pointwise ergodic theorem we have to check that the function $\tau \to \frac{\zeta'}{\zeta}(s + i\tau)\frac{1}{1+\tau^2}$ is Lebesgue integrable on $\mathbb{R}$ for fixed $\Re(s) > -\frac{1}{2}$. For this, we need an approximation of the logarithmic derivative of the Riemann zeta-function. In view of Lemma 1.5, we have $\frac{\zeta'}{\zeta}(s+i\tau) \ll \log^2 \tau$, hence the function $\tau \to \frac{\zeta'}{\zeta}(s+i\tau)\frac{1}{1+\tau^2}$ is also Lebesque integrable on $\mathbb{R}$. Moreover, if $\zeta(s)$ has a zero of multiplicity m at $1 + it_1$, then $\frac{\zeta'}{\zeta}(s) \sim \frac{m}{s-(1+it_1)}$, when s is near $1+it_1$. Therefore, for s near $1+it_1$, the function $\tau \to \frac{\zeta'}{\zeta}(s+i\tau)\frac{1}{1+\tau^2}$ is also Lebesgue integrable on $\mathbb{R}$. Since, for s near $1+it_1$,

$$\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} \ll 1 + m\int_{t_1-\epsilon}^{t_1+\epsilon} \frac{d\tau}{(\tau-t_1)(1+\tau^2)} < \infty.$$

Now it remains to calculate the integral by the residue theorem. In view of an approximation of the logarithmic derivative of the Riemann zeta-function $\frac{\zeta'}{\zeta}(s)$, we have, for almost all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \sum_{0\le n<N} \frac{\zeta'}{\zeta}(s+iT^n x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} \qquad \text{for} \qquad \Re(s) > -\frac{1}{2}.$$

Here we apply calculus of residues. This integrand is a regular function of τ apart from the pole at $\tau = \pm i$, $\tau = i(s-1)$, the simple pole $\tau = i(s+2m)$, $m = 1, 2, ...$ and $\tau = i(s-\rho)$, where ρ denotes a non-trivial zero of ζ. We shall distinguish several cases according to the location of $i(s-1)$.

In the first case $i(s-1)$ lies in the lower half of the τ-plane, that means $-\frac{1}{2} < \Re(s) < 1, s \neq 0$. Moreover, we suppose that $i(s-1) \neq -i$, resp. $\Re(s) \neq 0$. Then the integrand has following distinct simple poles; $\tau = -i$, $\tau = i(s-1)$, and $\tau = i(s-\rho)$, where ρ is a non-trivial zero of ζ such that $\Re(\rho) > \Re(s)$. Moreover, there are simple poles $\tau = i(s-\rho)$, where ρ is a non-trivial zero of ζ such that $\Re(\rho) = \Re(s)$, which are on the real axis of the τ-plane. For a sufficiently large parameter $R > 1 + |s|$ denote by I_R the counterclockwise oriented semicircle of radius R centered at $\tau = -\Im(s)$ and located in the lower half of the τ-plane. For $\Re(\rho) = \Re(s)$, we denote I_ϵ the counterclockwise oriented semicircles of radius ϵ centered at $\tau = \Im(\rho) - \Im(s)$ and located in the lower half of the τ-plane. Then

$$\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} = \int_{I_R} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} - \sum_{\substack{\rho \\ \Re(\rho)=\Re(s)}} \int_{I_\epsilon} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} - 2\pi i \sum(s),$$

where $\sum(s)$ is the sum of residue inside $([-R-\Im(s), R-\Im(s)] \setminus I_\epsilon)$ and I_R is the counterclockwise oriented semicircle.

Now we compute the residues.

$$\mathrm{Res}_{\tau=-i} \frac{\frac{\zeta'}{\zeta}(s+i\tau)}{1+\tau^2} = \lim_{\tau\to -i}(\tau+i)\frac{\frac{\zeta'}{\zeta}(s+i\tau)}{1+\tau^2} = \frac{i}{2}\frac{\zeta'}{\zeta}(s+1).$$

In view of Lemma 1.4,

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1) + \sum_\rho(\frac{1}{s-\rho} + \frac{1}{\rho}) + \log 2\pi - \frac{\gamma}{2} - 1, \tag{3.21}$$

we have

$$\begin{aligned}
&\mathrm{Res}_{\tau=i(s-1)} \frac{\frac{\zeta'}{\zeta}(s+i\tau)}{1+\tau^2} \\
&= \frac{1}{1+(i(s-1))^2}\lim_{\tau\to i}(\tau - i(s-1))(\frac{1}{1-(s+i\tau)} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+i\tau}{2}+1\right) \\
&+ \sum_\rho(\frac{1}{(s+i\tau)-\rho} + \frac{1}{\rho}) + \log 2\pi - \frac{\gamma}{2} - 1) \\
&= \frac{i}{s(2-s)}.
\end{aligned}$$

Similary, for a non-trivial zero ρ of ζ, with $\Re(\rho) > \Re(s)$, we have

$$\mathrm{Res}_{\tau=i(s-\rho)} \frac{\frac{\zeta'}{\zeta}(s+i\tau)}{1+\tau^2} = \frac{-i}{1-(s-\rho)^2}.$$

In order to evaluate the integral over I_ϵ, we set $\Im(s) = t$, for some real t and use the parameterization $\tau = \epsilon\exp(i\varphi) - t + \Im(\rho_1)$ and (3.21) again, and find

$$\begin{aligned}
&\int_{I_\epsilon} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} \\
&= \int_\pi^{2\pi} \frac{\zeta'}{\zeta}(\rho_1 + i\epsilon\exp(i\varphi))\frac{i\epsilon\exp(i\varphi)d\varphi}{1+(\epsilon\exp(i\varphi)-t+i\Im(\rho_1))^2} \\
&= \int_\pi^{2\pi} \left\{ \frac{1}{1-\rho_1-i\epsilon\exp(i\varphi)} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(1+\frac{\rho_1+i\epsilon\exp(i\varphi)}{2}) + \sum_\rho (\frac{1}{\rho_1-\rho+i\epsilon\exp(i\varphi)} + \frac{1}{\rho}) + O(1) \right\} \times \\
&\times \frac{i\epsilon\exp(i\varphi)d\varphi}{1+(t-\Im(\rho_1))^2+O(\epsilon)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \int_{I_\epsilon} \frac{\zeta'}{\zeta}(\frac{1}{2}+i(t+\tau))\frac{d\tau}{1+\tau^2} \\
&= \int_\pi^{2\pi} \lim_{\epsilon\to 0^+} \left\{ \frac{d\varphi}{1+(t-\Im(\rho_1))^2+O(\epsilon)} + O(\epsilon) \right\} = \frac{\pi}{1+(t-\Im(\rho_1))^2} \\
&= \frac{\pi}{1-(s-\rho_1)^2}.
\end{aligned}$$

Now only the computation of the integral term on I_R remains. In view of Lemma 1.5, we have

$$\int_{I_R} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} \ll \frac{R}{1+R^2}\max_{\tau\in I_R}|\frac{\zeta'}{\zeta}(s+i\tau)| \ll \frac{R}{1+R^2}\log^2 R.$$

As $R\to\infty$, this integral tends to zero. Then, for $-\frac{1}{2} < \Re(s) < 1$, $\Re(s)\neq 0$

$$\begin{aligned}
\text{(3.22)}\qquad \frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} = \frac{\zeta'}{\zeta}(s+1) + \frac{2}{s(2-s)} - \sum_{\substack{\rho \\ \Re(\rho)=\Re(s)}} \frac{1}{1-(s-\rho)^2} \\
- \sum_{\substack{\rho \\ \Re(\rho)>\Re(s)}} \frac{2}{1-(s-\rho)^2}.
\end{aligned}$$

In view of (3.22), for almost all $x\in\mathbb{R}$, we obtain (3.10).

Now we suppose that $i(s-1)$ lies in the upper half of the τ-plane, that means $\Re(s) > 1$. Then the integrand has only one pole in the lower half of the plane because $0 < \Re(\rho) < 1$; resp. $\Re(s) - \Re(\rho) > 0$ and $\Re(s) + 2m$ is always positive. With the same reasoning as above we have, for $\Re(s) > 1$,

$$\text{(3.23)}\qquad \frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{d\tau}{1+\tau^2} = \frac{\zeta'}{\zeta}(s+1).$$

In view of (3.23), for almost all $x\in\mathbb{R}$, we obtain (3.11).

Now we assume that the integrand has a double pole in the lower half plane, that is $-i = i(s-1)$, resp. $s = 0$ and also simple poles at $\tau = -i\rho$, where ρ are the non-trivial zeros of ζ. In order to compute the residue we use again (3.21) and also the Laurent expantion of $\frac{1}{\tau - i}$ near this pole. This gives, as $\tau \to -i$,

$$
\begin{aligned}
&\frac{\zeta'}{\zeta}(i\tau)\frac{1}{1+\tau^2} \\
&= \left(\frac{i}{\tau+i} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{i\tau}{2}+1\right) + \sum_\rho\left(\frac{1}{i\tau-\rho}+\frac{1}{\rho}\right) + \log 2\pi - \frac{\gamma}{2} - 1\right)\left(\frac{i}{2(\tau+i)} + \frac{1}{4} + O(|\tau+i|)\right).
\end{aligned}
$$

Hence,

$$
\mathrm{Res}_{\tau=-i}\frac{\frac{\zeta'}{\zeta}(i\tau)}{1+\tau^2} = -\frac{i}{4}\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) + \frac{i}{2}\sum_\rho\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right) + \frac{i}{2}\log 2\pi - \frac{i\gamma}{4} - \frac{i}{4}.
$$

For the simple pole at $\tau = -i\rho$, we have

$$
\mathrm{Res}_{\tau=-i\rho}\frac{\frac{\zeta'}{\zeta}(i\tau)}{1+\tau^2} = \frac{i}{1-\rho^2}.
$$

Then, for $s = 0$,

$$
\frac{1}{\pi}\int_{\mathbb{R}}\frac{\zeta'}{\zeta}(i\tau)\frac{d\tau}{1+\tau^2} = \sum_\rho\left(\frac{1}{\rho} - \frac{1}{\rho+1}\right) + \log 2\pi - \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) + \gamma + 1\right). \tag{3.24}
$$

In view of (3.24), for almost all $x \in \mathbb{R}$, we obtain (3.12).

The last case is that $i(s-1)$ lies on the real axis; that is $s = 1 + it$ for some real number t. Here we denote by γ_ϵ and γ_R the counterclockwise oriented semicircles of radius ϵ and R, respectively, both centered at $\tau = -t$ and located in the lower half of the τ-plane. Then, for sufficiently large R,

$$
\begin{aligned}
&\left(\int_{-t-R}^{-t-\epsilon} + \int_{-t+\epsilon}^{-t+R}\right)\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2} \\
&= \left(\int_{\gamma_R} - \int_{\gamma_\epsilon}\right)\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2} - 2\pi i \mathrm{Res}_{\tau=-i}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{1}{1+\tau^2}.
\end{aligned}
$$

In view of Lemma 1.5, we see that

$$
\lim_{R\to\infty}\int_{\gamma_R}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2} = 0.
$$

In order to evalute the integral over γ_ϵ we use the parameterization $\tau = \epsilon\exp(i\varphi) - t$ and (3.21) again, and find

$$
\begin{aligned}
&\int_{\gamma_\epsilon}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2} \\
&= \int_\pi^{2\pi}\frac{\zeta'}{\zeta}(1+i\epsilon\exp(i\varphi))\frac{i\epsilon\exp(i\varphi)d\varphi}{1+(\epsilon\exp(i\varphi)-t)^2} \\
&= \int_\pi^{2\pi}\left\{\frac{-1}{i\epsilon\exp(i\varphi)} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{i\epsilon\exp(i\varphi)+3}{2}\right) + \sum_\rho\left(\frac{1}{1+i\epsilon\exp(i\varphi)-\rho}+\frac{1}{\rho}\right) + \log 2\pi - \frac{\gamma}{2} - 1\right\} \\
&\times \frac{i\epsilon\exp(i\varphi)d\varphi}{1+t^2+O(\epsilon)}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{\gamma_\epsilon}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2}\\
&=\int_\pi^{2\pi}\lim_{\epsilon\to 0^+}\left\{\frac{-d\varphi}{1+t^2+O(\epsilon)}+O(\epsilon)\right\}=-\frac{\pi}{1+t^2}.
\end{aligned}$$

Inserting this and

$$\mathrm{Res}_{\tau=-i}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{1}{1+\tau^2}=\frac{i}{2}\frac{\zeta'}{\zeta}(2+it),$$

we have, for some real t,

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{\zeta'}{\zeta}(1+i(t+\tau))\frac{d\tau}{1+\tau^2}=\frac{\zeta'}{\zeta}(2+it)+\frac{1}{1+t^2}. \tag{3.25}$$

In view of (3.25), for almost all $x\in\mathbb{R}$, we obtain (3.13). The proof of the theorem is complete.

3.2.2 Proof of Theorem 3.6

Assume the Riemann Hypothesis, then it is clearly that the last term in (3.16) vanishes. Now we assume the equality (3.20) holds for almost all real $x\in\mathbb{R}$. If the Riemann Hypothesis is not true, then there is a non-trivial zero of $\zeta(s)$, $\rho=\beta+i\gamma$ with $\beta>\frac{1}{2}$ and also its conjugate is a non-trivial zero. Now we consider, for $z\in\mathbb{C}\setminus\{\pm 1\}$,

$$\frac{1}{1-\bar{z}^2}+\frac{1}{1-z^2}=\frac{1-\Re(z)}{1-2\Re(z)+\mid z\mid^2}+\frac{1+\Re(z)}{1+2\Re(z)+\mid z\mid^2}.$$

We put $z=\frac{1}{2}-\beta-i\gamma$, we have $0<\Re(z)<\frac{1}{2}$, then $1\pm\Re(z)>0$ and $1\pm 2\Re(z)+\mid z\mid^2>0$. Thus

$$\frac{1}{1-(\frac{1}{2}-\beta-i\gamma)^2}+\frac{1}{1-(\frac{1}{2}-\beta+i\gamma)^2}>0,$$

This leads to the last term in (3.16) is positive and contradiction for almost all real $x\in\mathbb{R}$. This proves the theorem.

Remark 2. We could determine the appearing integral by the explicit formula of the Riemann zeta-function which connects a sum over the zero of $\zeta(s)$ with a sum over prime numbers (see in [28] Theorem 5.12 p.109).

Lemma 3.7. *Let $g\in C_c^\infty(\mathbb{R})$ and let*

$$h(r)=\int_{-\infty}^{\infty}g(u)e^{iur}du.$$

Put $\Gamma_R(s)=\pi^{-s/2}\Gamma(\frac{s}{2})$, the local factor at the infinite place for the Euler product of $\zeta(s)$. Then

$$\begin{aligned}
\sum_\rho h(\Im(\rho))&=h(\frac{i}{2})+h(\frac{-i}{2})+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\left\{\frac{\Gamma_R'}{\Gamma_R}(\frac{1}{2}+ir)+\frac{\Gamma_R'}{\Gamma_R}(\frac{1}{2}-ir)\right\}dr\\
&-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}(g(\log n)+g(-\log n)).
\end{aligned} \tag{3.26}$$

In view of

$$-\frac{\zeta'}{\zeta}(s)-\frac{\zeta'}{\zeta}(1-s)=\frac{\Gamma_R'}{\Gamma_R}(s)+\frac{\Gamma_R'}{\Gamma_R}(1-s).$$

and we apply Lemma 3.6 with $h(r)=\frac{1}{1+r^2}$ then we have

$$\sum_{\rho}\frac{1}{1+(\Im(\rho))^2}=\frac{8}{3}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\zeta'}{\zeta}(\frac{1}{2}+ir)+\frac{\zeta'}{\zeta}(\frac{1}{2}-ir)\right\}\frac{dr}{1+r^2}+\frac{\zeta'}{\zeta}(\frac{3}{2}). \tag{3.27}$$

This give us (3.20).

Remark 3. In view of Remark 2, we can extend our problem to the logarithmic derivative of an Dirichlet L-function. In oder to calculate the exactly value of the appearing integral, we can apply the corresponding explicit formula that can be found in [28]. Hughes and Rudnick [25] provide an explicit formula for $L(s,\chi)$, namely

$$\sum_{\rho}h(\Im(\rho_\chi))=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\left\{\log q+G_\chi(r)\right\}dr-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)(\chi(n)+\overline{\chi(n)}), \tag{3.28}$$

where

$$G_\chi(r)=\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir+a(\chi))+\frac{\Gamma'}{\Gamma}(\frac{1}{2}-ir+a(\chi))-\frac{1}{2}\log\pi,$$

$a(\chi)=0$ if χ is even and $a(\chi)=1$ if χ is odd. We could also apply (3.28) with $h(r)=\frac{1}{1+r^2}$.

3.3 Sampling Riemann Hypothesis for the logarithm of zeta-functions

In this section, we study the behaviour of other zeta-functions related to the function $\frac{\zeta'}{\zeta}(s)$ in the previous section. Here, we consider the function $\log\zeta(s)$ and apply a lemma of Kai-Man Tsang [65]:

Lemma 3.8. *(Kai-Man Tsang, 1986) Suppose $\Re(s)\in[\frac{1}{2},2]$. Let $V(x+iy)$ be an analytic function in the horizontal strip: $\Re(s)-2\le y\le 0$ satisfying the growth condition*

$$\sup_{\sigma-2\le y\le 0}\mid V(x+iy)\mid\ll\frac{1}{\mid x\mid\log^2\mid x\mid}. \tag{3.29}$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty}\log\zeta(s+iu)V(u)du&=\int_{-\infty}^{\infty}\log\zeta(2+i(\Im(s)+u))V(i(\Re(s)-2)+u)du\\&+2\pi\sum_{\substack{\rho\\ \Re(\rho)>\Re(s)}}\int_0^{\Re(\rho)-\Re(s)}V(\Im(\rho-s)-i\alpha)d\alpha-2\pi\int_{\min(1-\Re(s),0)}^{1-\Re(s)}V(-\Im(s)-i\alpha)d\alpha.\end{aligned} \tag{3.30}$$

Here, we take $V(x+iy)=\frac{1}{1+(x+iy)^2}$. Clearly, the function $V(x+iy)$ satisfies the growth condition (3.29). In view of (3.30), we have

$$\begin{aligned}\int_{-\infty}^{\infty}\log\zeta(s+iu)\frac{du}{1+u^2}&=\int_{-\infty}^{\infty}\frac{\log\zeta(2+i(\Im(s)+u))}{1+(i(\Re(s)-2)+u)^2}du\\&+2\pi\sum_{\substack{\rho\\ \Re(\rho)>\Re(s)}}\int_0^{\Re(\rho)-\Re(s)}\frac{d\alpha}{1+(\Im(\rho-s)-i\alpha)^2}-2\pi\int_{\min(1-\Re(s),0)}^{1-\Re(s)}\frac{d\alpha}{1+(\Im(s)+i\alpha)^2}.\end{aligned}$$

We apply the pointwise ergodic theorem and obtain

Theorem 3.9. *For* $\frac{1}{2} \leq \Re(s) \leq 2$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(s+iT^n x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log\zeta(2+i(\Im(s)+u))}{1+(i(\Re(s)-2)+u)^2}du \tag{3.31}$$

$$+2\sum_{\substack{\rho \\ \Re(\rho)>\Re(s)}} \int_0^{\Re(\rho)-\Re(s)} \frac{d\alpha}{1+(\Im(\rho-s)-i\alpha)^2} - 2\int_{\min(1-\Re(s),0)}^{1-\Re(s)} \frac{d\alpha}{1+(\Im(s)+i\alpha)^2},$$

for almost all $x \in \mathbb{R}$.

Now we discuss Theorem 3.9 and Theorem 3.3 for different values of s.

- $\Re(s) > 1$

In this case, the last two terms vanish. Thus we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(s+iT^n x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\zeta(2+i(t+u))\frac{du}{1+(i(\sigma-2)+u)^2}.$$

By Cauchy's theorem, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(s+iT^n x) = \log\zeta(s+1), \qquad \text{for almost all} \qquad x\in\mathbb{R}. \tag{3.32}$$

If we differentiate the logarithm of both sides on $\Re(s)$, we obtain (3.11) again.

- $\frac{1}{2} < \Re(s) < 1$

By Cauchy's theorem again, we have

(3.33)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(s+iT^n x) = \log(s+1) + 2\sum_{\substack{\rho \\ \Re(\rho)>\Re(s)}} \int_0^{\Re(\rho)-\Re(s)} \frac{d\alpha}{1+(\Im(\rho-s)-i\alpha)^2}$$

$$-2\int_0^{1-\Re(s)} \frac{d\alpha}{1+(\Im(s)+i\alpha)^2},$$

for almost all $x \in \mathbb{R}$. If we differentiate the functions on both sides on $\Re(s)$, we obtain (3.10).

- The special case $s = \frac{1}{2}$

In this case, we find that

(3.34)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \log\zeta(\frac{1}{2}+iT^n x) = \log\zeta(\frac{3}{2}) - \log 3 + 2\sum_{\substack{\rho \\ \Re(\rho)>\frac{1}{2}}} \int_0^{\Re(\rho)-\frac{1}{2}} \frac{d\alpha}{1+(\Im(\rho)-i\alpha)^2}$$

$$= -0.138352... + 2\sum_{\substack{\rho \\ \Re(\rho)>\frac{1}{2}}} \int_0^{\Re(\rho)-\frac{1}{2}} \frac{d\alpha}{1+(\Im(\rho)-i\alpha)^2}.$$

We let

$$J_s(k) = \frac{1}{10^k} \sum_{0 \le n < 10^k} \log \zeta(s + iT^n x), \qquad j(s) = \log \frac{\zeta(s+1)\Re(s)}{2 - \Re(s)}.$$

With the initial value $x = 1.18$ we find

	$J_s(4)$	$J_s(5)$	$j(s)$	$\mid j(s) - \Re(J_s(5)) \mid$
$s = 0.30$	-0.349 + i 0.006	-0.347 + i 0.001	-0.3655	0.0186
$s = 0.40$	-0.246 + i 0.008	-0.244 + i 0.001	-0.2531	0.0093
$s = 0.45$	-0.194 + i 0.008	-0.191 + i 0.001	-0.1961	0.0047
$s = 0.488888$	-0.153 + i 0.008	-0.150 + i 0.001	-0.1512	0.001
$s = 0.50$	-0.140 + i 0.008	-0.138 + i 0.001	-0.1384	4.06×10^{-5}
$s = 0.55$	-0.082 + i 0.008	-0.080 + i 0.001	-0.0798925	2.11×10^{-6}
$s = 0.60$	-0.023 + i 0.007	-0.021 + i 0.001	-0.0205968	5.13×10^{-5}
$s = 0.70$	0.099 + i 0.007	0.101 + i 0.001	0.10089	1.5×10^{-5}

The results of the above table indicate that the second sum in (3.31) should vanish for $\Re(s) \ge 2$. Therefore, it should be true that there are no non-trivial zeros ρ of ζ with $\Re(\rho) > \frac{1}{2}$.

Theorem 3.10. *The Riemann Hypothesis is true if, and only if, for almost all $x \in \mathbb{R}$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \le n < N} \log \zeta(\frac{1}{2} + iT^n x) = \log \zeta(\frac{3}{2}) - \log 3 = -0.138352.... \tag{3.35}$$

Proof. Assume the Riemann Hypothesis, then the last sum in (3.34) vanishes. Now we assume the equality (3.35) holds for almost all real $x \in \mathbb{R}$. If the Riemann Hypothesis is not true, then there is a non-trivial zero of $\zeta(s)$, $\rho = \beta + i\gamma$ with $\Re(\rho) > \frac{1}{2}$ and also its conjugate is a non-trivial zero. Thus we consider, for $\alpha \in (0, \Re(\rho) - \frac{1}{2})$,

$$\frac{1}{1 + (\gamma + i\alpha)^2} + \frac{1}{1 + (\gamma - i\alpha)^2} = \frac{2 + 2\gamma^2 - 2\alpha^2}{1 + 2\gamma^2 - 2\alpha^2 + (\alpha^2 + \gamma^2)^2} > 0.$$

This leads to the last term in (3.34) is positive and contradiction for almost all real $x \in \mathbb{R}$. This proves the theorem.

Remark 1. There is an interesting link to a recent result of Lahoucine and Zine El Abidine [38]. Namely, let σ_0 be any fixed number in $\mathbb{R}$ and $a > 0$ then the integral

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\log \mid \zeta(\sigma_0 + it) \mid}{a^2 + t^2} dt$$

exists and

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\log \mid \zeta(\sigma_0 + it) \mid}{a^2 + t^2} dt = \log \left| \frac{a + \sigma_0 - 1}{a + \mid 1 - \sigma_0 \mid} \zeta(a + \sigma_0) \right| + \sum_{\Re(\rho) > \sigma_0} \log \left| \frac{a - \sigma_0 + \rho}{a + \sigma_0 - \rho} \right|.$$

Remark 2. Steuding [60] gave an equivalent formulation of the Riemann Hypothesis in terms of the ergodic transformation under investigation.

Theorem 3.11. *(Steuding, 2012) For almost all $x \in \mathbb{R}$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \log \mid \zeta(\frac{1}{2} + \frac{1}{2} i T^n x) \mid = \sum_{\Re(\rho)>\frac{1}{2}} \log \mid \frac{\rho}{1-\rho} \mid .$$

In particular, the Riemann Hypothesis is true if, and only if, one and thus either side vanishes, the left-hand side for almost all real x.

Theorem 3.10 follows with the help of a result in [6] of Balazard, Saias and Yor, that is

$$\frac{1}{2\pi} \int_{\Re s=\frac{1}{2}} \frac{\log \mid \zeta(s) \mid}{\mid s \mid^2} \mid ds \mid = \sum_{\Re(\rho)>\frac{1}{2}} \log \mid \frac{\rho}{1-\rho} \mid .$$

Notice that $\frac{\log|\zeta(s)|}{|s|^2}$ is integrable on $s = \frac{1}{2} + i\mathbb{R}$.

Remark 3. In [7], Balazard and Saias asked the following yet unsolved questions.

- Assume the Riemann Hypothesis. Is it true that

$$\lim_{n\to\infty} \inf_{(a_1,\ldots,a_n)\in\mathbb{C}^n} \int_{\Re s=\frac{1}{2}} \mid 1 - \zeta(s) \sum_{k=1}^{n} a_k k^{-s} \mid^2 \frac{\mid ds \mid}{\mid s \mid^2} = 0 \qquad ?$$

- Is it true that

$$\lim_{n\to\infty} \int_{\Re s=\frac{1}{2}} \mid 1 - \zeta(s) \sum_{k=1}^{n} \mu(k) k^{-s} \mid^2 \frac{\mid ds \mid}{\mid s \mid^2} = 0 \qquad ?$$

- Assume the Riemann Hypothesis. Is it true that

$$\lim_{n\to\infty} \int_{\Re s=\frac{1}{2}} \mid 1 - \zeta(s) \sum_{k=1}^{n} \mu(k)(1 - \frac{\log k}{\log n}) k^{-s} \mid^2 \frac{\mid ds \mid}{\mid s \mid^2} = 0 \qquad ?$$

We shall study these questions by the pointwise ergodic theorem in a similar way as in Theorem 3.10.

3.4 Sampling Riemann Hypothesis on the arithmetical function

In this section, we study some details with respect to Remark 2 in Section 3.1. In fact, we shall study the behaviour of an arithmetical function $\alpha(s+i\mathbb{R})x^{i\mathbb{R}}$ with respect to our ergodic transformation. However, we start with an illustration by computing

$$T(k) = \frac{1}{10^k} \sum_{0\le n<10^k} 2^{iT^n y} \zeta(\frac{1}{2} + iT^n y).$$

We obtain

k	4	5	6
y=0.0007	-0.0471557 + 0.0028i	-0.0477072 - 0.00104i	-0.0465638 - 0.000235i
y=3.8	-0.0465345 - 0.00289i	-0.0455023 - 0.0005598i	-0.0463815 - 0.0005666i

If we set $x = 2$ in (3.8), we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2^{iu}\zeta(\frac{1}{2}+iu)}{u^2+1} du = 2\zeta(\frac{3}{2}) - \frac{3}{2} - \frac{8\sqrt{2}}{3} = -0.0464855....$$

We note that these computations show a slow tendency towards -0.0464855....

3.4.1 Summation formulae and the ergodic transformations

Now we shall derive the summation formula for certain arithmetical functions in terms of our ergodic transformation. Let

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a Dirichlet series with abscissa of convergence σ_c. The idea follows from the proof of Perron's formula with a weight function. For our purpose we take a weight function $w(x)$, and define the related summatory weight function

$$A_w(x) = \sum_{n=1}^{\infty} a_n w(\frac{n}{x}).$$

Let $K(s)$ denote the Mellin transform of $w(x)$,

$$K(s) = \int_0^{\infty} w(x) x^{s-1} dx.$$

Then we expect that

$$\alpha(s)K(s) = \int_0^{\infty} A_w(x) x^{-s-1} dx$$

holds for $\sigma > \sigma_c$ and

$$A_w(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)K(s)x^s ds,$$

for $\sigma_0 > \max(0, \sigma_c)$.

Now we apply this setting to our situation. We start with a kernel $K(s) = \frac{1}{s^2-1}$. Its inverse Mellin transform is for $\Re(s) > 1$ given by

$$w(x) = \begin{cases} \frac{1}{2}(\frac{1}{x} - x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < \infty. \end{cases}$$

Thus, we obtain

Lemma 3.12. *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. For $\sigma_0 > 1$, we have*

$$\sum_{n \leq x} a_n (\frac{x}{n} - \frac{n}{x}) = \frac{1}{i\pi} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s^2 - 1} ds. \tag{3.36}$$

Example 1. Now we return to the question from Remark 2 in Section 3.1. We consider $\alpha(s) = \zeta(u+s)$. For $0 < \Re(u) < 1$, we obtain

Theorem 3.13. *For almost all $y \in \mathbb{R}$, $0 < \Re(u) < 1$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leq n < N} \zeta(u + iT^n y) x^{iT^n y} = x\zeta(u+1) - \sum_{m \leq x} \frac{1}{m^u} \left(\frac{x}{m} - \frac{m}{x}\right) - \frac{2}{u(2-u)} x^{1-u}.$$

Proof. We study the behaviour of function $\zeta(u+i\mathbb{R})x^{i\mathbb{R}}$ with respect to our ergodic transformation, that is

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(u+iT^n y)x^{iT^n y}.$$

For $0<\Re(u)<1$, the function $\zeta(u+i\tau)\frac{x^{i\tau}}{1+\tau^2}$ is Lebesgue integrable. Moreover, by the ergodic pointwise theorem, we have , for almost all $y\in\mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(u+iT^n y)x^{iT^n y}=\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta(u+i\tau)\frac{x^{i\tau}}{1+\tau^2}d\tau. \tag{3.37}$$

Now we apply Lemma 3.12 in order to calculate the explicit value of the integral in (3.37). We take $\alpha(s)=\zeta(u+s)$, and consider the residue arising from the poles at $s=1-u$ and at $s=1$. Thus, the proof is complete.

Example 2. (The divisor function) We consider $\alpha(s)=\zeta^2(s+u)$, for $0<\Re(u)<1$. We obtain

Theorem 3.14. *For almost all $y\in\mathbb{R}$, for $0<\Re(u)<1$,*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta^2(u+iT^n y)x^{iT^n y}&=\zeta^2(u+1)x-\sum_{m\le x}\frac{d(m)}{m^u}\left(\frac{x}{m}-\frac{m}{x}\right)-\frac{2x^{1-u}\log x}{u(2-u)}\\&\quad-\frac{4(1-u)x^{1-u}}{(u(2-u))^2}-\frac{4\gamma x^{1-u}}{u(2-u)}.\end{aligned}$$

Proof. As in the proof of Theorem 3.13, by the ergodic pointwise theorem, we have, for almost all $y\in\mathbb{R}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta^2(u+iT^n y)x^{iT^n y}=\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta^2(u+i\tau)\frac{x^{i\tau}}{1+\tau^2}d\tau. \tag{3.38}$$

Again, for $0<\Re(u)<1$, the function $\zeta^2(u+i\tau)\frac{x^{i\tau}}{1+\tau^2}$ is Lebesgue integrable. Now, we apply Lemma 3.12 with $\alpha(s)=\zeta^2(u+s)$, then we have, for $\sigma_0>1$,

$$\sum_{m\le x}\frac{d(m)}{m^u}\left(\frac{x}{m}-\frac{m}{x}\right)=\frac{1}{i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\zeta^2(u+s)\frac{x^s}{s^2-1}ds. \tag{3.39}$$

By Cauchy's theorem, we can shift the path of integration such that

$$\frac{1}{i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\zeta^2(s+u)\frac{x^s}{s^2-1}ds=-\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta^2(u+i\tau)\frac{x^{i\tau}}{\tau^2+1}d\tau+2\delta\Sigma(s), \tag{3.40}$$

where $\Sigma(s)$ denotes the sum of residues from the function $\zeta^2(u+s)\frac{x^s}{s^2-1}$. It has one simple pole at $s=1$ with residue

$$\text{Res}_{s=1}\zeta^2(u+s)\frac{x^s}{s^2-1} = \zeta^2(u+1)\frac{x}{2}, \tag{3.41}$$

and a double pole at $s=1-u$ with residue

$$\text{Res}_{s=1-u}\zeta^2(u+s)\frac{x^s}{s^2-1} = -\frac{x^{1-u}\log x}{u(2-u)} - \frac{2(1-u)x^{1-u}}{u^2(2-u)^2} - \frac{2\gamma x^{1-u}}{u(2-u)}. \tag{3.42}$$

We insert (3.40)-(3.42) in (3.39), thus, for $0 < \Re(u) < 1$,

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta^2(u+i\tau)\frac{x^{i\tau}}{\tau^2+1}d\tau &= \zeta^2(u+1)x - \sum_{m\le x}\frac{d(m)}{m^u}\left(\frac{x}{m}-\frac{m}{x}\right) \\ &\quad - \frac{2x^{1-u}\log x}{u(2-s)} - \frac{4(1-u)x^{1-u}}{u^2(2-u)^2} - \frac{4\gamma x^{1-u}}{u(2-u)}.\end{aligned} \tag{3.43}$$

In view of (3.38) this finishes the proof of theorem.

We illustrate the results with another computation. Letting

$$A(k) = \frac{1}{10^k}\sum_{0\le n<10^k} 2^{iT^n y}\zeta^2(\frac{1}{2}+iT^n y),$$

then we obtain

k	4	5	6
y=0.31	0.203747 - 0.0249392i	0.222069 + 0.0036938i	0.216707 - 0.001151i
y=29.765	0.196296 + 0.0029673i	0.222079 + 0.0037476i	0.217298 + 0.00089883i

If we set $x=2$ in Theorem 3.14, then we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2^{i\tau}\zeta^2(\frac{1}{2}+i\tau)}{\tau^2+1}d\tau &= 2\zeta^2(\frac{3}{2}) - \frac{3}{2} - \frac{8\sqrt{2}\log 2}{3} - \frac{32\sqrt{2}}{9} - \frac{16\gamma\sqrt{2}}{3} \\ &= 0.214966....\end{aligned}$$

For a special case $u=\frac{1}{2}$, we have

Corollary 3.15. *For almost all $y \in \mathbb{R}$, there is a constant C such that*

$$\sum_{m\le x}\frac{d(m)}{\sqrt{m}}\left(\frac{x}{m}-\frac{m}{x}\right) = \zeta^2(\frac{3}{2})x - \frac{8}{3}\sqrt{x}\log x - \frac{32}{9}\sqrt{x} - \frac{16\gamma}{3}\sqrt{x} + C,$$

where $C = \lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta^2(\frac{1}{2}+iT^n y)x^{iT^n y}$.

Remark 1. The square of the Riemann zeta-function $\zeta^2(s)$ plays an important role in determining the asymptotic behaviour of the sum $D(x) = \sum_{n\le x} d(n)$ as $x\to\infty$. The link relies on the following formula:

$$D(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta^2(\tau)x^\tau}{\tau}d\tau, \qquad c>1. \tag{3.44}$$

In view of (3.44) for $c = 1+\epsilon$, $\epsilon > 0$ we have

$$D(x) = x\log x + (2\gamma - 1)x + \Delta(x). \tag{3.45}$$

By partial summation we have, for $\Re(s) > 1$,

(3.46)
$$\zeta^2(s) = \sum_{n\le x} d(n)n^{-s} + \frac{(\log x + 2\gamma)x^{1-s}}{s-1} + \frac{x^{1-s}}{(s-1)^2} + O(x^{\epsilon+\frac{1}{3}-\sigma}) + s\int_x^\infty \tau^{-s-1}\Delta(\tau)d\tau.$$

This integral is absolutely convergent for $\Re(s) > \frac{1}{3}$. We may be connect Corollary 3.15 to study more details about $\Delta(x)$.

Example 3. (The two-dimensional divisor function) The two-dimensional divisor problems may be considered in just the same way. For $1 \le a \le b \in \mathbb{N}$, we denote by $d(a,b;k)$ the number of representations of k as $k = n_1^a n_2^b$, where n_1, n_2 are natural numbers. The Dirichlet series of the arithmetical function $d(a,b;n)$ is for $\Re(s) > \frac{1}{a}$ represented by

$$F(s) = \sum_{n=1}^\infty d(a,b;n)n^{-s} = \zeta(as)\zeta(bs). \tag{3.47}$$

Let

$$D(a,b;x) = \sum_{1\le k\le x} d(a,b;k) = \zeta(\frac{b}{a})x^{\frac{1}{a}} + \zeta(\frac{a}{b})x^{\frac{1}{b}} + \Delta(a,b;x). \tag{3.48}$$

Applying Lemma 3.12 again and the pointwise ergodic theorem, we obtain

Theorem 3.16. *For almost all $y \in \mathbb{R}$, for $0 \le \Re(s) < \frac{1}{b}$,*

(3.49)
$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(a(s+iT^ny))\zeta(b(s+iT^ny))x^{iT^ny}$$
$$= \zeta(as+a)\zeta(bs+b)x + \frac{2}{a}\frac{\zeta(b/a)x^{\frac{1}{a}-s}}{(\frac{1}{a}-s)^2-1} + \frac{2}{b}\frac{\zeta(a/b)x^{\frac{1}{b}-s}}{(\frac{1}{b}-s)^2-1} - \sum_{m\le x}\frac{d(a,b;m)}{m^s}\left(\frac{x}{m}-\frac{m}{x}\right).$$

We shall an illustrative computation by defining

$$B(k) = \frac{1}{10^k}\sum_{0\le n<10^k} 2^{iT^ny}\zeta(\frac{1}{4}+iT^ny)\zeta(\frac{3}{4}+3iT^ny).$$

Then

k	4	5	6
y= 0.0891	0.0881092 - 0.000673i	0.0802388 - 0.002664i	0.0771856 - 0.00113381i
y=-5.765	0.0947122 - 0.00576725i	0.0843676 - 0.000260385i	0.0771863 + 0.00028565i

In view of (3.49), for $x = 2$, $a = 1$, $b = 3$, and $s = \frac{1}{4}$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(\frac{1}{4}+iT^ny)\zeta(\frac{3}{4}+3iT^ny)2^{iT^ny}=2\zeta(\frac{5}{4})\zeta(\frac{15}{4})-\frac{32}{7}\zeta(3)2^{\frac{3}{4}}-\frac{288}{429}\zeta(\frac{1}{3})2^{\frac{1}{12}}-\frac{3}{2}$$
$$=0.0763561....$$

There is a connection with the error term in the divisor problem for $\Delta(a,b;x)$. If we set $s=0$ in (3.49) we have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(aiT^ny)\zeta(biT^ny)x^{iT^ny}&=\zeta(a)\zeta(b)x+\frac{1}{1-a}\zeta(\frac{b}{a})x^{\frac{1}{a}}+\frac{1}{1-b}\zeta(\frac{a}{b})x^{\frac{1}{b}}\\&-\frac{1}{1+a}\zeta(\frac{b}{a})x^{\frac{1}{a}}-\frac{1}{1+b}\zeta(\frac{a}{b})x^{\frac{1}{b}}-\sum_{m\le x}d(a,b;m)\left(\frac{x}{m}-\frac{m}{x}\right)\qquad\text{for almost all } y\in\mathbb{R}.\end{aligned}\tag{3.50}$$

The following result is due to Ivić

Lemma 3.17. *(see [27] Lemma 14.1 p.399) Let* $1\le a<b<1$. *We have*

$$\sum_{m\le x}d(a,b;m)\frac{x}{m}=\frac{1}{1-a}\zeta(\frac{b}{a})x^{\frac{1}{a}}+\frac{1}{1-b}\zeta(\frac{a}{b})x^{\frac{1}{b}}+\zeta(a)\zeta(b)x+\Delta(a,b;x)+O(1).\tag{3.51}$$

In view of Lemma 3.17, we obtain, for almost all $y\in\mathbb{R}$,

(3.52)
$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(aiT^ny)\zeta(biT^ny)x^{iT^ny}&=\sum_{m\le x}d(a,b;m)\frac{m}{x}-\frac{1}{1+a}\zeta(\frac{b}{a})x^{\frac{1}{a}}-\frac{1}{1+b}\zeta(\frac{a}{b})x^{\frac{1}{b}}\\&-\Delta(a,b;x)+O(1).\end{aligned}$$

Example 4. (The pair correlation function) Montgomery [46] introduced the pair correlation function

$$F(x,T)=\sum_{0<\gamma,\gamma'\le T}x^{i(\gamma-\gamma')}w(\gamma-\gamma')$$

for any real x and $T\ge 2$, where $w(u)$ is a suitable weight function. The sum is a double sum over the imaginary parts of the non-trivial zeros of $\zeta(s)$. Montgomery proved an asymptotic formula for $F(x,T)$ as $T\to\infty$.

Theorem 3.18 (Montgomery)**.** *For* $1\le x\le T$ *and* $T\ge 2$

$$F(x,T)\sim\frac{T}{2\pi}\log x+\frac{T}{2\pi x^2}\log^2 T.$$

In proving Theorem 3.18, Montgomery defined a function $L(x,t)$ which is a special sum over the zeros of $\zeta(s)$ localized near t, namely

$$L(x,t)=2\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2}.\tag{3.53}$$

Here, by using Lemma 3.12, we study (3.53) in terms of our ergodic transformation. We apply Lemma 3.12 , with $\alpha(s)=\frac{\zeta'}{\zeta}(\frac{1}{2}+it+s)$. As the proof of Theorem 3.3, we obtain

Theorem 3.19. *For almost all $y \in \mathbb{R}$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\frac{\zeta'}{\zeta}(\frac{1}{2}+it+iT^ny)x^{iT^ny} = \frac{\zeta'}{\zeta}(\frac{3}{2}+it)x+\frac{2x^{\frac{1}{2}-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)} \tag{3.54}$$
$$-\sum_{\substack{\rho\\ \Re(\rho)=\frac{1}{2}}}\frac{x^{i(\Im(\rho)-t)}}{1+(t-\Im(\rho))^2}-\sum_{\substack{\rho\\ \Re(\rho)>\frac{1}{2}}}\frac{2x^{i(\Im(\rho)-t)}}{1+(t-\Im(\rho))^2}-\sum_{m\le x}\Lambda(m)m^{\frac{-1}{2}-it}(\frac{x}{m}-\frac{m}{x}).$$

There is a related result due to T.H. Chan [17], namely

Lemma 3.20. *(T. H. Chan, 2004) For $1 \le x \le T$ and $T \ge 2$*

$$2\sum_{\gamma}\frac{x^{i(\gamma-t)}}{1+(t-\gamma)^2} = -\frac{1}{x}\sum_{n\le x}\Lambda(n)n^{\frac{1}{2}-it}-x\sum_{n\le x}\Lambda(n)n^{-\frac{3}{2}-it}+\frac{2x^{\frac{1}{2}-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}$$
$$+\frac{\log T}{x}+\frac{1}{x}\left(\frac{\zeta'}{\zeta}(\frac{3}{2}-it)-\log 2\pi\right)+O(\frac{1}{xT}).$$

3.5 Sampling Lindelöf Hypothesis for moments of the zeta-function

In this section we shall deal with a problem concerning the explicit evaluation of the integral in Theorem 3.2.

3.5.1 Sampling Lindelöf Hypothesis for an approximation of the Riemann zeta-function

We shall use the approximation of the Riemann zeta-function to calculate the explicit value of the integral in Theorem 3.2. This idea follows from the method of Lifshits and Weber [43] in Section 2.1. In view of this we consider

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\sum_{k\le x}\frac{1}{k^{\frac{1}{2}+iT^ny}}-\frac{x^{\frac{1}{2}-iT^ny}}{\frac{1}{2}-iT^ny}|^2. \tag{3.55}$$

Firstly, we set

$$a_n(x)=a_n=\sum_{k\le x}\frac{1}{k^{\frac{1}{2}+iT^ny}},\qquad b_n(x)=b_n=\frac{x^{\frac{1}{2}-iT^ny}}{\frac{1}{2}-iT^ny}.$$

In view of

$$|\sum_{k\le x}\frac{1}{k^{\frac{1}{2}+iT^ny}}-\frac{x^{\frac{1}{2}-iT^ny}}{\frac{1}{2}-iT^ny}|^2=|a_n|^2+|b_n|^2-2\Re(a_n\bar{b}_n),$$

we write (3.55) as $A_1+A_2-2\Re(A_3)$, where

$$A_1=\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|a_n|^2,$$
$$A_2=\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|b_n|^2,$$
$$A_3=\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}a_n\bar{b}_n.$$

Using the pointwise ergodic theorem, for almost all $y \in \mathbb{R}$, we have

$$A_1 = \frac{1}{\pi}\int_{\mathbb{R}} |\sum_{k\le x} \frac{1}{k^{\frac12+i\tau}}|^2 \frac{d\tau}{1+\tau^2},$$

$$A_2 = \frac{1}{\pi}\int_{\mathbb{R}} |\frac{x^{\frac12-i\tau}}{\frac12-i\tau}|^2 \frac{d\tau}{1+\tau^2},$$

$$A_3 = \frac{1}{\pi}\int_{\mathbb{R}} \sum_{k\le x} \frac{1}{k^{\frac12+i\tau}} \frac{x^{\frac12+i\tau}}{\frac12+i\tau}\frac{d\tau}{1+\tau^2}.$$

We have

$$\begin{aligned} A_1 &= \frac{1}{\pi}\int_{\mathbb{R}} |\sum_{k\le x} \frac{1}{k^{\frac12+i\tau}}|^2 \frac{d\tau}{1+\tau^2} = \frac{1}{\pi}\int_{\mathbb{R}} \sum_{k,l\le x} \frac{1}{k^{\frac12+i\tau} l^{\frac12-i\tau}} \frac{d\tau}{1+\tau^2} \\ &= \sum_{k,l\le x} (kl)^{\frac{-1}{2}} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i(\log l-\log k)\tau}\frac{d\tau}{1+\tau^2} = \sum_{k,l\le x}(kl)^{\frac{-1}{2}}\frac{\min(k,l)}{\max(k,l)} \\ &= 2\sum_{l\le x} l^{\frac{-3}{2}} \sum_{k\le l} k^{\frac12} - \sum_{l\le x}\frac{1}{l}. \end{aligned}$$

Euler's summation formula gives

$$\sum_{k=1}^{l} \sqrt{k} = \frac{2}{3}l^{\frac32} + \frac12 l^{\frac12} - \frac16 + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{\frac12}{2k-1}(l^{\frac32-2k}-1) + \theta(m)\frac{B_{2m+2}}{2m+2}\binom{\frac12}{2m-1}(l^{\frac12-2m}-1), \tag{3.56}$$

where $\theta(m) \in [0,1]$. Summing up we arrive at

$$\begin{aligned} A_1 &= \frac{4x}{3} - \frac13\sum_{l\le x} l^{\frac{-3}{2}} + \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac12}{2k-1}\left(\sum_{l\le x} l^{-2k} - \sum_{l\le x} l^{\frac{-3}{2}}\right) \\ &+ \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac12}{2m-1}\left(\sum_{l\le x} l^{-2k-1} - \sum_{l\le x} l^{\frac{-3}{2}}\right) \\ &= \frac{4x}{3} + \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac12}{2k-1}\sum_{l\le x} l^{-2k} + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac12}{2m-1}\sum_{l\le x} l^{-2k-1} \\ &- \left(\frac13 + \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac12}{2k-1} + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac12}{2m-1}\right)\sum_{l\le x} l^{\frac{-3}{2}}. \end{aligned}$$

Next, we compute A_2 by

$$\begin{aligned} A_2 &= \frac{1}{\pi}\int_{\mathbb{R}} |\frac{x^{\frac12-i\tau}}{\frac12-i\tau}|^2\frac{d\tau}{1+\tau^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^{\frac12-i\tau}}{\frac12-i\tau}\frac{x^{\frac12+i\tau}}{\frac12+i\tau}\frac{d\tau}{1+\tau^2} \\ &= \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{d\tau}{(\frac14+\tau^2)(1+\tau^2)} \\ &= \frac{4x}{3}. \end{aligned}$$

Finally, we compute A_3 as

$$\begin{aligned}A_3 &= \frac{1}{\pi}\int_{\mathbb{R}} \sum_{k\le x} \frac{1}{k^{\frac{1}{2}+i\tau}} \frac{x^{\frac{1}{2}+i\tau}}{\frac{1}{2}+i\tau}\frac{d\tau}{1+\tau^2}\\ &= \sum_{k\le x}\left(\frac{8}{3} - \frac{2}{x^{\frac{1}{2}}}k^{\frac{1}{2}}\right)\end{aligned}$$

In view of (3.56) we have

$$\begin{aligned}A_3 = &\frac{8x}{3} - \frac{2}{x^{\frac{1}{2}}}\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{6} + \sum_{k=1}^{m}\frac{B_{2k}}{2k}\binom{\frac{1}{2}}{2k-1}(x^{\frac{3}{2}-2k}-1)\right)\\ &- \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}(x^{-2m} - x^{\frac{-1}{2}})\\ = &\frac{4x}{3} - 1 + \frac{1}{3x^{\frac{1}{2}}} - \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1}(x^{1-2k} - x^{\frac{-1}{2}})\\ &- \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}(x^{-2m} - x^{\frac{-1}{2}}).\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\sum_{k\le x}\frac{1}{k^{\frac{1}{2}+iT^n y}} - \frac{x^{\frac{1}{2}-iT^n y}}{\frac{1}{2}-iT^n y}|^2 \\ &= \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1}\sum_{l\le x} l^{-2k} + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}\sum_{l\le x} l^{-2k-1}\\ &- \left(\frac{1}{3} + \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1} + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}\right)\sum_{l\le x} l^{\frac{-3}{2}} + 2 - \frac{2}{3x^{\frac{1}{2}}}\\ &+ 2\sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1}(x^{1-2k} - x^{\frac{-1}{2}}) + 2\theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}(x^{-2m} - x^{\frac{-1}{2}}).\end{aligned} \tag{3.57}$$

In view of (3.2) in [43] in the case $m = n = 1$, we have

$$\int_{-\infty}^{\infty}|\zeta(\frac{1}{2}+i\tau)|^2\frac{d\tau}{1+\tau^2} = \lim_{x\to\infty}\int_{-\infty}^{\infty}|\sum_{k\le x}\frac{1}{k^{\frac{1}{2}+i\tau}} - \frac{x^{\frac{1}{2}-i\tau}}{\frac{1}{2}-i\tau}|^2\frac{d\tau}{1+\tau^2}. \tag{3.58}$$

From (3.57), (3.58) and the pointwise ergodic theorem, we obtain

Theorem 3.21. *For almost all* $y \in \mathbb{R}$

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^n y)|^2\\ &= \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1}\zeta(2k) + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}\zeta(2k+1)\\ &- \left(\frac{1}{3} + \sum_{k=1}^{m}\frac{B_{2k}}{k}\binom{\frac{1}{2}}{2k-1} + \theta(m)\frac{B_{2m+2}}{m+1}\binom{\frac{1}{2}}{2m-1}\right)\zeta(\frac{3}{2}) + 2,\end{aligned}$$

where $\theta(m) \in (0, 1)$.

Remark 1. We shall provide computations of our result.
For $m = 1$, we have

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2\\&=2-\frac{1}{3}\zeta(\frac{3}{2})-B_2\frac{1}{2}\left(\zeta(\frac{3}{2})-\zeta(2)\right)-\theta(1)\frac{B_4}{2}\frac{1}{2}\left(\zeta(\frac{3}{2})-\zeta(3)\right)\\&=2-0.870792-0.0806201+\theta(1)\frac{1.41032}{120}.\end{aligned}$$

Thus, for $m = 1$, we have

$$1.04859<\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2<1.06034$$

For $m = 2$, we have

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2\\&=2-\frac{1}{3}\zeta(\frac{3}{2})-\frac{1}{12}\left(\zeta(\frac{3}{2})-\zeta(2)\right)+\frac{1}{960}\left(\zeta(\frac{3}{2})-\zeta(4)\right)-\theta(2)\frac{1}{2016}\left(\zeta(\frac{3}{2})-\zeta(5)\right)\\&=2-0.870792-0.0806201+0.0015938-\theta(2)\frac{1.57545}{2016}.\end{aligned}$$

Thus, for $m = 2$, we have

$$1.0494<\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2<1.05018.$$

For $m = 3$, we have

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2\\&=2-\frac{1}{3}\zeta(\frac{3}{2})-\frac{1}{12}\left(\zeta(\frac{3}{2})-\zeta(2)\right)+\frac{1}{960}\left(\zeta(\frac{3}{2})-\zeta(4)\right)\\&-\frac{1}{4608}\left(\zeta(\frac{3}{2})-\zeta(6)\right)+\theta(3)\frac{7}{30720}\left(\zeta(\frac{3}{2})-\zeta(7)\right)\\&=1.05018-0.000346144+\theta(3)\frac{7}{30720}\left(\zeta(\frac{3}{2})-\zeta(7)\right).\end{aligned}$$

Thus, for $m = 3$, we have

$$1.04983<\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^ny)|^2<1.0502.$$

We put again

$$T(k)=\frac{1}{10^k}\sum_{0\le n<10^k}\left|\zeta(\frac{1}{2}+iT^ny)\right|^2,$$

k	1	2	3	4	5	6
y=1.1	0.982324	1.03874	1.01059	1.02478	1.04848	1.05059
y=100.98	0.84383	0.910918	1.0323	1.03455	1.04655	1.05023

3.5.2 Sampling Lindelöf Hypothesis for moments of the Riemann zeta-function by spectral decomposition

Here, we give an explicit expression for the integral in Theorem 3.2. This method is based on Motohashi's exact formula [48]. It state that

Lemma 3.22. *Let a function $g(r)$ be real-valued for $r \in \mathbb{R}$ and that there exists a large constant $A > 0$ such that $g(r)$ is regular and $g(r) \ll (\mid r \mid +1)^{-A}$, for $\mid \Im(r) \mid \leq A$. Then, we have*

$$\int_{-\infty}^{\infty} \mid \zeta(\frac{1}{2}+it) \mid^2 g(t)dt = \int_{-\infty}^{\infty} (\Re(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+it) + 2\gamma - \log(2\pi)))g(t)dt + 2\pi\Re(g(\frac{i}{2})) + 4\sum_{n=1}^{\infty} d(n) \int_0^{\infty} (y^2+y)^{\frac{-1}{2}} g_c(\log(1+\frac{1}{y}))\cos(2\pi ny)dy,$$

where

$$g_c(x) = \int_{-\infty}^{\infty} g(t)\cos(xt)dt.$$

We apply Lemma 3.22 with $g(t) = 1/(1+t^2)$ and obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mid \zeta(\frac{1}{2}+it) \mid^2 \frac{dt}{1+t^2} = \frac{\Gamma'}{\Gamma}(\frac{3}{2}) + 2\gamma - \log(2\pi) + \frac{8}{3} + 4\sum_{n=1}^{\infty} d(n) \int_0^{\infty} \frac{y^{\frac{1}{2}}}{(y+1)^{\frac{3}{2}}} \cos(2\pi ny)dy.$$

We note that $\frac{\Gamma'}{\Gamma}(\frac{3}{2}) + 2\gamma - \log(2\pi) + \frac{8}{3} = 2.01971....$ This coincides almost with our results in section 3.5.1. Thus, we can rewrite Theorem 3.21

Theorem 3.23. *For almost all $y \in \mathbb{R}$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} |\zeta(\frac{1}{2}+iT^n y)|^2 = \frac{\Gamma'}{\Gamma}(\frac{3}{2}) + 2\gamma - \log(2\pi) + \frac{8}{3} + 4\sum_{n=1}^{\infty} d(n) \int_0^{\infty} \frac{y^{\frac{1}{2}}}{(y+1)^{\frac{3}{2}}} \cos(2\pi ny)dy. \tag{3.59}$$

Moreover, Motohashi considered the general case of $s = \sigma + it$.

Lemma 3.24. *For any $0 < \sigma < 1$,*

$$\int_{-\infty}^{\infty} \mid \zeta(\sigma+it) \mid^2 g(t)dt$$
$$= \zeta(2\sigma)g^*(0) + 2\zeta(2\sigma-1)\Re(\tilde{g}(2\sigma-1,\sigma)) - 4\pi\zeta(2\sigma-1)\Re(g((\sigma-1)i))$$
$$+ 4(2\pi)^{2\sigma-2}\Im\left(\sum_{n=1}^{\infty} d_{2\sigma-1}(n) \int_{2-i\infty}^{2+i\infty} (2\pi n)^{-w} \sin\left(\frac{(2\sigma-w)\pi}{2}\right)\Gamma(w+1-2\sigma)\tilde{g}(w,\sigma)dw\right),$$

where g^* is the Fourier transform $g^*(u) = \int_{-\infty}^{\infty} g(t)e^{iut}dt$, $\tilde{g}(s,\lambda)$ is the Mellin transform $\tilde{g}(s,\lambda) = \int_0^{\infty} y^{s-1}(1+y)^{-\lambda} g^*(\log(1+y))dy$ and $d_{2\sigma-1}(n) = \sum_{d|n} d^{2\sigma-1}$.

For our situation, we put

$$g(t) = \frac{1}{1+t^2} \qquad \text{with} \qquad g^*(u) = \pi e^{-|u|},$$

and

$$\tilde{g}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta+1-\alpha)}{\Gamma(\beta+1)}\pi.$$

From this and the pointwise ergodic transformation we deduce

Theorem 3.25. *For almost all $y \in \mathbb{R}$ and $0 < \sigma < 1$, $\sigma \neq \frac{1}{2}$*

(3.60)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \le n < N} |\zeta(\sigma + iT^n y)|^2 = \zeta(2\sigma) + 2\zeta(2\sigma-1)\frac{\Gamma(2\sigma-1)\Gamma(2-\sigma)}{\Gamma(\sigma+1)} - \frac{4\zeta(2\sigma-1)}{\sigma(2-\sigma)}$$
$$+ 4(2\pi)^{2\sigma-2}\Im\left(\sum_{n=1}^{\infty} d_{2\sigma-1}(n) \int_{2-i\infty}^{2+i\infty} (2\pi n)^{-w} \sin\left(\frac{(2\sigma-w)\pi}{2}\right)\Gamma(w+1-2\sigma)\frac{\Gamma(w)\Gamma(\sigma-w+1)}{\Gamma(\sigma+1)}dw\right).$$

Remark 1. We can extend this idea to other zeta-functins; for example, the investigation of the values of the Dirichlet L- function, namely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \le n < N} |L(\frac{1}{2} + iT^n x, \chi)|^2, \qquad \text{for almost} \qquad x \in \mathbb{R}, \tag{3.61}$$

where χ is supposed to be a primitive Dirichlet character mod q. In [48], Motohashi investigated the square mean of Dirichlet L-function:

$$Q(u, v; \chi) = \frac{1}{\phi(q)} \sum_{\chi \mod q} L(u, \chi)L(v, \bar{\chi}). \tag{3.62}$$

We could approach our problem from the investigation (3.62) of Motohashi. One point of view, we can also investigate further is the behaviour of the Hurwitz zeta-function.

Remark 2. The interesting term $\Re(\frac{\Gamma'}{\Gamma}(\frac{1}{2} + it))$ in Lemma 3.22 appears in many place of analytic number theory. Katsurada and Matsumoto [33] showed that

$$\int_1^2 |\ \zeta(\frac{1}{2} + it, \alpha)\ |^2\ d\alpha = \gamma - \log 2\pi + \Re(\frac{\Gamma'}{\Gamma}(\frac{1}{2} + it)) - 2\Re(\sum_{n=0}^{\infty} \frac{\zeta(\frac{1}{2} + n + it)}{\frac{1}{2} + n + it}). \tag{3.63}$$

Steuding studied also a behaviour of this by an ergodic transformation of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given a real number θ, the circle rotation R_θ is defined by $R_\theta : \mathbb{T} \to \mathbb{T}$, $R_\theta x = x + \theta$ mod 1.

Theorem 3.26. *(Steuding, 2012) Let θ be irrational. For almost all $x \in [0, 1)$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \le n < N} |\zeta_1(\frac{1}{2} + it, R_\theta^n x)|^2 = \int_1^2 |\ \zeta(\frac{1}{2} + it, \alpha)\ |^2\ d\alpha, \tag{3.64}$$

where $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$.

If we divide both sides of (3.61) by $(1+t^2)\pi$ and integrate with respect to the variable t in $\mathbb{R}$, then

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{1}^{2} \mid \zeta(\frac{1}{2}+it,\alpha) \mid^2 d\alpha\frac{dt}{1+t^2} &= \gamma-\log 2\pi+\frac{1}{\pi}\int_{-\infty}^{\infty}\Re(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+it))\frac{dt}{1+t^2}\\ &\quad-\frac{2}{\pi}\int_{-\infty}^{\infty}\Re(\sum_{n=0}^{\infty}\frac{\zeta(\frac{1}{2}+n+it)}{\frac{1}{2}+n+it})\frac{dt}{1+t^2}\\ &=\gamma-\log 2\pi+\frac{\Gamma'}{\Gamma}(\frac{3}{2})+\frac{16}{3}-4\sum_{n=0}^{\infty}\frac{\zeta(\frac{3}{2}+n)}{2n+3}.\end{aligned}$$

We now introduce our ergodic transformation and the pointwise ergodic theorem. We obtain, for almost $x,y\in\mathbb{R}$,

$$\lim_{N,M\to\infty}\frac{1}{MN}\sum_{0\le m<M}\sum_{0\le n<N}|\zeta_1(\frac{1}{2}+iT^m y,R_\theta^n x)|^2=\gamma-\log 2\pi+\frac{\Gamma'}{\Gamma}(\frac{3}{2})+\frac{16}{3}-4\sum_{n=0}^{\infty}\frac{\zeta(\frac{3}{2}+n)}{2n+3}.$$

In view of Theorem 3.23, we have

Theorem 3.27. *Let θ be irrational. For almost $(x,y)\in[0,1)\times\mathbb{R}$*

$$\begin{aligned}&\lim_{N,M\to\infty}\frac{1}{MN}\sum_{0\le m<M}\sum_{0\le n<N}|\zeta_1(\frac{1}{2}+iT^m y,R_\theta^n x)|^2-\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\frac{1}{2}+iT^n y)|^2\\ &=\frac{8}{3}-\gamma-4\sum_{n=1}^{\infty}d(n)\int_0^\infty\frac{u^{\frac{1}{2}}}{(u+1)^{\frac{3}{2}}}\cos(2\pi nu)du-4\sum_{n=0}^{\infty}\frac{\zeta(\frac{3}{2}+n)}{2n+3}.\end{aligned}$$

Remark 3. In order to investigate the distribution of values of the Riemann zeta-function with respect to the real part of s with an ergodic transformation on $[0,1)$, we may also think about the Gauss transformation given by

$$T0:=0,\qquad Tx:=\frac{1}{x}\mod 1,\qquad \text{for } 0<x<1.$$

The invariant probability density of T is then given by

$$\mu[a,b]=\frac{1}{\log 2}\int_a^b\frac{dx}{1+x}.$$

In view of the pointwise ergodic theorem we note

Theorem 3.28. *For almost all $x\in[0,1)$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}\zeta(T^n x+it)=\frac{1}{\log 2}\int_0^1\zeta(\tau+it)\frac{d\tau}{1+\tau}.$$

Remark 4. In [48], Motohashi give a useful result of the k-th moment of the Riemann zeta-function by

$$M_k(\zeta;g)=\int_{-\infty}^{\infty}\mid\zeta(\frac{1}{2}+it)\mid^k g(t)dt,$$

where k is an arbitrary fixed number, and the weight function g is assumed to be even, entire, real on $\mathbb{R}$, and of fast decay in any fixed horizontal strip.

Chapter 4

4 Does a random walker meet universality ?

In this chapter, we investigate the phenomenon of universality with respect to certain stochastic processes.

In Section 4.1, we explain the notion of universality. Roughly speaking, Voronin's universality theorem implies that any finite analytic landscape can be found-up to an arbitrarily small error in the analytic landscape of the Riemann zeta-function.

In Section 4.2, we interpret the absolute value of an analytic function as analytic landscape over the complex plane, extend our results to other random walks and consider how soon a random walk will meet a given set?

4.1 Voronin's universality theorem

The universality property asserts that any analytic function can be approximated uniformly on compact subsets by translations of $\zeta(s)$. Voronin's universality theorem was refined by Reich [51] and Bagchi [3]. The strongest version of Voronin's theorem has the form:

Theorem 4.1. *(Voronin's universality theorem) Suppose that $\mathcal{K}$ is a compact subset of the strip $\frac{1}{2} < \Re(s) < 1$ with connected complement, and let g be a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \mid \zeta(s+i\tau) - g(s) \mid < \epsilon \right\} > 0.$$

Bagchi considered the Riemann Hypothesis in terms of universality. In sense of Voronin's universality theorem, the Riemann zeta-function can approximate itself, if and only if the Riemann hypothesis is true. That is, the Riemann hypothesis is true, if and only if, for any compact subset $\mathcal{K}$ of $\frac{1}{2} < \sigma < 1$ with connected complement and any $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \max_{s\in\mathcal{K}} \mid \zeta(s+i\tau) - \zeta(s) \mid < \epsilon \right\} > 0.$$

Antanas Laurinčikas'probabilistic approach to universality and his monograph [39] in particular have pushed research in this field strongly forward; a probabilistic proof of the universality property for the Riemann zeta-function or some of its relatives heavily depends on properties of weakly convergent probability measures in appropriate function spaces in combination with non-trivial results from arithmetic.

Universality is a phenomenon which is restricted to the right open half $\mathcal{D}$ of the critical strip, i.e., $\mathcal{D} := \{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$. As mentioned in the introduction, Garunkštis and Steuding [20] showed that the set of vectors $\overline{\{\zeta(\sigma+it) : t \in [1,\infty)\}} \neq \mathbb{C}$ for $\sigma \leq 0$. Moreover, Andersson [1] proved that universality cannot take place on the line $1 + i\mathbb{R}$. Neither in the half-plane of absolute convergence $\Re(s) > 1$ of the Dirichlet series expansion of $\zeta(s)$ nor to the left of the critical line universality is possible due to growth estimates.

A discrete version of Voronin's universality theorem was proved by Reich [51]:

Theorem 4.2. *Suppose that $\mathcal{K}$ is a compact subset of the strip $\frac{1}{2} < \Re(s) < 1$ with connected complement, and let G be a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the* *interior of $\mathcal{K}$. Then, for any positive real numbers Δ and $\epsilon > 0$,*

$$\liminf_{M\to\infty} \frac{1}{M}\sharp\left\{m \le M : \max_{s\in\mathcal{K}} \mid \zeta(s+i\Delta m) - G(s) \mid < \epsilon\right\} > 0.$$

4.2 A random walk

As in Section 2.1, we let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent, identically distributed discrete random variables. For each positive integer n, we call

$$s_n = X_1 + X_2 + ... + X_n$$

a random walk. In order to investigate the phenomenon of universality with respect to certain stochastic processes, we need a notion of a random walk in the complex plane.

4.2.1 A random walk in the complex plane

Denoting by $s_0 \in \mathbb{C}$ the starting position of our random walk, it moves on the affine lattice

$$\Lambda := s_0 + \lambda\mathbb{Z}[i], \tag{4.1}$$

where $\lambda > 0$ is real, $i = \sqrt{-1}$ is the imaginary unit in the upper half-plane, and $\mathbb{Z}[i]$ is the ring of Gaussian integers $a + bi$ with $a, b \in \mathbb{Z}$. We assume that at each time unit the random walker steps one space unit further with equal probability $\frac{1}{4}$ to either possible direction on Λ, and we denote its random position by the sequence of s_n defined by

$$s_n = s_{n-1} + \lambda\delta_n \qquad \text{with} \qquad \delta_n \in \{\pm 1, \pm i\}, \qquad n \in \mathbb{N}, \tag{4.2}$$

and probability $\mathbb{P}(+1)$=$\mathbb{P}(-1)$=$\mathbb{P}(+i)$=$\mathbb{P}(-i) = \frac{1}{4}$. Hence, $(s_n)_{n\in\mathbb{N}_0}$ is a random walk on the affine lattice Λ. We shall answer the interesting question: for any given $\epsilon > 0$, does there exist n such that

$$\max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon?, \tag{4.3}$$

where K is a compact set with connected complement, and g is an arbitrary continuous function on K which is analytic in the interior of K. In order to tackle this problem, we may think about applying Voronin's universality theorem. Of course, the best we can expect is a positive answer almost surely. The assumption in Theorem 4.1 that the target function g is non-vanishing follows from a simple application of Rouché's theorem. Hence, in view of the location of zeta zeros, as predicted by the Riemann Hypothesis, we shall suppose that g has no zero inside K.

The obvious approach towards answering the question concerning (4.3) is to find an instance of the random walk in the associated set of universality

$$\mathcal{U} := \left\{z \in \mathbb{C} : \max_{s\in K} \mid \zeta(s+z) - g(s) \mid < \epsilon\right\}, \tag{4.4}$$

and

$$K + \mathcal{U} \subset \mathcal{D}. \tag{4.5}$$

Here $K+\mathcal{U}$ is the union of points $s+z$ with $s \in K$ and $z \in \mathcal{U}$. In particular, the latter condition implies that the horizontal expansion of K has to be less than $\frac{1}{2}$ as necessary condition for K.

The set of shifts s_n under consideration is discrete. Accordingly we need a discrete version of Voronin's universality theorem in the form of Theorem 4.2, that is Reich's discrete universality theorem [51].

We aim to find some $n \in \mathbb{N}_0$ for which $s_n = z \in \mathcal{U}$. In view of Reich's Theorem 4.2 we may hope to find $s_n = i\Delta m$ with some appropriate m. However, this is only possible if, and only if, $\Lambda = s_0 + i\lambda\mathbb{Z}[i]$ has a non-empty intersection with $i\Delta\mathbb{Z}$; notice that this intersection may consist only of a few points which might lead to have no instances of universality at all. Nevertheless, if λ is sufficiently small, we may apply Reich's theorem with $\Delta = \lambda$ and $\mathcal{K} := K + \kappa := \{s + \kappa : s \in K\}$ where κ is any of those lattice points in Λ for which the translate $K + \kappa$ lies to the right of the critical line. In view of (4.5), we thus need

$$K + \kappa \subset \mathcal{D}. \tag{4.6}$$

If (4.5) is satisfied, then (4.6) is possible for sufficiently small λ depending on K. Under this assumption, setting $G(s+\kappa) = g(s)$ for $s \in K$, we have

$$\max_{s\in\mathcal{K}} \mid \zeta(s+i\Delta m) - G(s) \mid = \max_{s\in K} \mid \zeta(s+\kappa+i\lambda m) - G(s+\kappa) \mid = \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid$$

provided $s_n = i\lambda m + \kappa$ for some m. In view of Theorem 4.2 this quantity can be made smaller than ϵ if the random walk s_n will intersect with the set $i\lambda\mathcal{M} + \kappa \subset \Lambda$, where

$$\mathcal{M} := \left\{ m \in \mathbb{N} : \max_{s\in\mathcal{K}} \mid \zeta(s+i\lambda m) - G(s) \mid < \epsilon \right\}$$

is non-empty. It was Pólya [49], [50] who showed that a symmetric random walk in one or in two dimensions is recurrent, i.e., with probability one our one-dimensional random walk with starting point at s_0 will return to s_0; here the attribute symmetric refers to equidistribution of the probability measure (which holds true in our case). We may also express this recurrence by

$$\liminf_{n\to\infty} \mid s_n - s_0 \mid = 0 \qquad \text{almost surely.}$$

Moreover, by translation, given any lattice point $z = i\lambda m + \kappa \in \Lambda$ with $m \in \mathcal{M}$, almost every realization of $(s_n)_n$ will reach z an infinity of times. Hence, it follows from Theorem 4.2 that (4.3) holds almost surely infinitely often. We shall show that this even happens in a rather regular way. For this purpose denote by η_1 the minimum of all positive integers n for which s_n will hit some point $z_1 = i\lambda m + \kappa \in \Lambda$ such that $m \in \mathcal{M}$. We may interpret this quantity η_1 as the first hitting time of the random walk under consideration to an admissible lattice point, and η_1 is indeed a stopping time in the sense of stochastic processes. We may consider the symmetric random walk $(s_n)_{n>\eta_1}$ starting at $s_{\eta_1} = z_1$ and define subsequent hitting times η_l as the minima of those n with $n > \eta_{l-1}$ such that s_n will hit some point $z_l = i\lambda m + \kappa \in \Lambda$ with $m \in \mathcal{M}$. Clearly, the subsequent hitting times η_l exist almost surely and all the visits to such points z_l lead to realizations of (4.3). It is well-known that the hitting times η_l are integer valued, independent and identically distributed random variables with finite first moment $\mathbb{E} \mid \eta_1 \mid$ and, in our case, positive mean $\mu = \mathbb{E}\eta_1 > 0$ (see [36]). Hence, it follows from the strong law of large numbers that

$$\lim_{L\to\infty} \frac{1}{L} \sum_{l\leq L} \eta_l = \mu.$$

In view of the divergence of the η_l to infinity, for any N, there exists almost surely some M for which $\eta_M \leq N \leq \eta_{M+1}$ and

$$\sharp\left\{n \leq N : \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon\right\} = M \sim M+1 \sim \frac{1}{\mu}\sum_{l\leq M+1}\eta_l > \frac{1}{\mu}\eta_{M+1}.$$

This implies

$$\frac{1}{N}\sharp\left\{n \leq N : \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon\right\} > \left(\frac{1}{\mu} - \epsilon\right)\frac{\eta_{M+1}}{N} > \frac{1}{\mu} - \epsilon$$

for some small ϵ which trends to zero as $N \to$. Thus we have proved

Theorem 4.3. *Assume that Λ is a lattice given by* (4.1) *and $(s_n)_n$ is a random walk on this lattice, defined by* (4.2). *Further suppose that K is a compact set with connected complement satisfying* (4.6), *and g is a non-vanishing continuous function on K which is analytic in the interior of K. Then, for any $\epsilon > 0$, almost surely*

$$\liminf_{N\to\infty} \frac{1}{N}\sharp\left\{n \leq N : \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon\right\} > 0.$$

Notice that for the proof of the almost sure frequency of (4.3) we have not made use of the positive lower density estimate in Reich's Theorem 4.2. Applying Theorem 4.3 with $g = \zeta$, we immediately obtain that if $\zeta(s) \neq 0$ for $s \in K$, then, for any $\epsilon > 0$, almost surely

$$\liminf_{N\to\infty} \frac{1}{N}\sharp\left\{n \leq N : \max_{s\in K} \mid \zeta(s+s_n) - \zeta(s) \mid < \epsilon\right\} > 0,$$

which actually is stronger than (4.3). On the contrary, if $\zeta(s') = 0$ for some $s' \in K$, then (4.3) is satisfied if $s_n = 0$, respectively, if $0 \in \Lambda$. Hence, we may not expect an equivalent of the Riemann Hypothesis in terms of self-approximation with respect to random walks.

Remark 1. We have mentioned the recurrence of a symmetric random walk in one or in two dimensions. We could interpret the phenomenon of universality with respect to a simple random walk in one dimension. In general, a simple random walk s_n in dimension $d \in \mathbb{N}$ is defined by $s_0 = x \in \mathbb{Z}^d$ and for $n \in \mathbb{N}$, by $s_n = x + X_1 + X_2 + ... + X_n$, where $\{X_k\}_{k=1}^{\infty}$ is a sequence of independent, identically distributed discrete random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$\mathbb{P}(X_k = \pm e_i) = \frac{1}{2d},$$

$i = 1, ..., d$ and $\{e_i\}_{i\in\{1,...,d\}}$ is the standard orthonormal basis of $\mathbb{R}^d$.

In order to consider the same question as (4.3), we think of our simple random walk in one dimension moving on the vertical $L := i\mathbb{R}$ (up or down). As in the proof of Theorem 4.3 and by use of Pólya's result on recurrence we find

Theorem 4.4. *Assume that $(s_n)_n$ is a simple random walk on the vertical line $L := i\lambda\mathbb{Z}$, where $\lambda > 0$. Further suppose that K is a compact set with connected complement satisfying*

$L + K \subset \mathcal{D}$, and g is a non-vanishing continuous function on K which is analytic in the interior of K. Then, for any $\epsilon > 0$, almost surely

$$\liminf_{N\to\infty} \frac{1}{N}\sharp \left\{ n \le N : \max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon \right\} > 0.$$

Sketch of the proof. Our aim is to find an instance of the random walk in the associated set of universality $\mathcal{U}$ in (4.4) and the condition (4.5). Then we have to find some $n \in \mathbb{N}_0$ for which $s_n = z \in \mathcal{U}$. In view of Theorem 4.2, we may hope that $s_n = i\Delta m$ with some appropriate m if and only if $L \cap i\Delta\mathbb{Z} \neq \emptyset$. For λ is sufficiently small, we apply Theorem 4.2 with $\Delta = \lambda$ and $\mathcal{K} := K + \iota := \{s + \iota : s \in K\}$ where ι is any of those lattice points in L, where λ for which the translation $K + \iota$ lies to the right of the critical line. With the same reasoning as for (4.5), we need to make sure that

$$K + \iota \subset \mathcal{D}. \tag{4.7}$$

If $K + \mathcal{U} \subset \mathcal{D}$ is satisfied, then $K + \iota \subset \mathcal{D}$ is possible for sufficiently small λ depending on K. Under this assumption, if we set $G(s + \iota) = g(s)$ for $s \in K$, then we have

$$\max_{s\in K} \mid \zeta(s+s_n) - g(s) \mid < \epsilon$$

provided $s_n = i\lambda m + \iota$ for some m. In view of the recurrence property proved by Pólya, we can show that this quantity can be made smaller than ϵ.

Remark 2. The related question how soon a random walk $(s_n)_n$ will meet the set $i\lambda\mathcal{M}+\kappa$ such that (4.3) holds is linked to the problem of hitting times for domains in the set of reachablility of a stochastic process. Firstly, we discuss this question in the one-dimensional case and refer to a result about the expression for the probability $P(s_n = x)$ that the simple random walk s_n started at the origin is at a given location $x \in \mathbb{Z}$ at time $n \in \mathbb{Z}$. In fact, Lawler and Limic gave a result about $P(s_n = x)$ in [41]:

Theorem 4.5. *(Lawler and Limic, 2010) For a simple random walk in $\mathbb{Z}$, if $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ with $\mid x \mid \le n$ and $x + n$ even,*

$$P(s_n = x) = \sqrt{\frac{2}{n\pi}} e^{-x^2/2n} \exp\left(O(\frac{1}{n} + \frac{x^4}{n^3}) \right).$$

In particular, if $\mid x \mid \le n^{\frac{3}{4}}$, then

$$P(s_n = x) = \sqrt{\frac{2}{n\pi}} e^{-x^2/2n} \exp\left(O(1 + \frac{1}{n} + \frac{x^4}{n^3}) \right).$$

Appendix

The analytic tools

Here we list some analytic tools for the study of the behaviour of zeta-functions. The reader is referred to the monograph by Ivić [27].

Summation techniques

A simple summation technique is the partial summation.

The partial summation *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of complex numbers and* $f(t)$ *is a continuously differentiable functin on* $[1, x]$. *Set*

$$A(t) = \sum_{n \le t} a_n.$$

Then

$$\sum_{n \le x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

In addition, there are several useful techniques.

The Euler-Maclaurin summation formula *Let* k *be a nonnegative integer and* f *be* $(k+1)$ *times differentiable on* $[a, b]$ *with* $a, b \in \mathbb{Z}$. *Then*

$$\sum_{a < n \le b} f(n) = \int_a^b f(t) dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!} \left(f^{(r)}(b) - f^{(r)}(a) \right) B_{r+1} + \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(t) f^{k+1}(t) dt,$$

where $B_r(x)$ *is the* r*-th Bernoulli function.*

The Poisson summation formula *Let* $f(x)$ *be a function of a real variable with bounded first derivative on* $[a, b]$ *with* $a, b \in \mathbb{Z}$. *Then*

$$\sideset{}{'}\sum_{a \le n \le b} f(n) = \int_a^b f(t) dt + 2 \sum_{n=1}^{\infty} \int_a^b (t) f(x) \cos 2\pi n x dx.$$

Here $\sideset{}{'}\sum$ *means that* $\frac{1}{2} f(a)$ *and* $\frac{1}{2} f(b)$ *are to be taken instead of* $f(a)$ *and* $f(b)$ *respectively.*

Atkinson's formula

Usually, Atkinson's formula is used to study the mean-square of the zeta-function. In this thesis, we apply an idea from the proof of Atkinson's formula to derive a product representation of the zeta-function, that is **Atkinson's dissection**. For $\Re(u)$, $\Re(v) > 1$, we have

$$\zeta(u)\zeta(v) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-u} n^{-v} = \zeta(u+v) + f(u, v) + f(v, u),$$

where

$$f(u, v) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} r^{-u} (r+s)^{-v}.$$

Using the Poisson summation formula, we have, *for* $0 < \Re(u) < 1$, $0 < \Re(v) < 1$, $u + v \neq 1$,

$$\zeta(u)\zeta(v) = \zeta(u+v) + \zeta(u+v-1)\Gamma(u+v-1)\left(\frac{\Gamma(1-u)}{\Gamma(v)} - \frac{\Gamma(1-v)}{\Gamma(u)}\right) + g(u,v) + g(v,u),$$

where

$$g(u,v) = 2\sum_{n=1}^{\infty} \sigma_{1-u-v}(n) \int_0^{\infty} y^{-u}(1-y)^{-v}\cos(2\pi n y)dy;$$

here, $\sigma_k = \sum_{d|n} d^k$ *is the sum of the k-th powers of divisors of* n*.* Furthermore, for studying the behaviour of zeta-function on the critical line we notice, *for* $0 < \Re(u) < 1$,

$$\zeta(u)\zeta(1-u) = \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(1-u) - \frac{\Gamma'}{\Gamma}(u)\right) + 2\gamma - \log 2\pi + g(u, 1-u) + g(1-u, u).$$

The residue theorem

Let $\mathbb{C}^1(D)$ be the set of differentiable functions $f : D \to \mathbb{C}$. There is a useful criterion of being analytic, which is known as **the Cauchy-Riemann equation**. Namely, if f is an analytic function in D, then $\frac{\partial f}{\partial \bar{z}} = 0$ for all $z \in D$. Let $\mathbb{C}^w(D)$ be the set of function in $\mathbb{C}^1(D)$ which satisfy the Cauchy-Riemann equation for all $z \in D$.

The residue theorem

Let $f \in \mathbb{C}^w(D/\{z_i\}_{i=1}^n)$ *be a function,* D *be an open set containing* $\{z_i\}_{i=1}^n$ *with the boundary* $\partial D = \gamma$,

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_{i=1}^{n} Res(f, z_i).$$

Probability Theory

Here, we provide some concepts of probability Theory. We refer to the book of Jacod and Protter [29].

As the usual probabilistic notion, let Ω be an abstract space. Let 2^{Ω} denote all subsets of Ω and let $\mathcal{A}$ be a subset of 2^{Ω} .

$\mathcal{A}$ is an **algebra** if it satisfies the following properties:

1. $\emptyset \in \mathcal{A}$ and $\Omega \in \mathcal{A}$;

2. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, where A^c is the complement of A;

3. $\mathcal{A}$ is closed under finite unions and finite intersections.

$\mathcal{A}$ is a **σ-algebra** if it satisfies the properties (1),(2) and $\mathcal{A}$ is closed under under countable unions and finite intersections.

If $\mathcal{C} \subset 2^{\Omega}$, the σ-algebra generated by $\mathcal{C}$, and wriiten $\sigma(\mathcal{C})$, is the smallest σ-algebra containing $\mathcal{C}$.

A probability measure defined on a σ-algebra $\mathcal{A}$ of Ω is a function $\mathbb{P} : \mathcal{A} \to [0,1]$ that satisfies:

1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$

2. For every countable sequence $(A_n)_{n \geq 1}$ of elements of $\mathcal{A}$, pairwise disjoint, one has

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Assume that $\Omega = \mathbb{R}$ and $\mathcal{A}$ be the borel σ-algebra of $\mathbb{R}$

The distribution function induced by a probability $\mathbb{P}$ on $(\mathbb{R}, \mathcal{A})$ is the function

$$F(x) = \mathbb{P}((-\infty, x]),$$

for $x \in \mathbb{R}$.
If f is positive and $\int_{-\infty}^{\infty} f(x)dx = 1$, the function $F(x) = \int_{-\infty}^{x} f(y)dy$ is a distribution function of a probability on $\mathbb{R}$ and the function f is called **the density function**. There are the important distribution function, for example,

1. **The Gamma distribution with parameters** α**,** β is defined by

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0, \end{cases}$$

for $0 < \alpha < \infty$ and $0 < \beta < \infty$.

2. **The Normal distribution with parameters** (μ, σ^2) is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

if $-\infty < x < \infty$. It is also known as **the Gaussian distribution**.

3. **The Cauchy distribution with parameters** α**,** β is defined by, for $0 < \alpha < \infty$ and $0 < \beta < \infty$,

$$f(x) = \frac{1}{\beta x} \frac{1}{1 + (x - \alpha)^2/\beta^2},$$

if $-\infty < x < \infty$.

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces. A function $X : E \to F$ is a called **measurable** if $X^{-1}(\Lambda) \in \mathcal{E}$, for all $\Lambda \in \mathcal{F}$. When $(E, \mathcal{E}) = (\Omega, \mathcal{A})$, a measurable function X is called **a random variable**. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A random variable X is called **simple** if it takes on only a finite number of values and hence can be written in the from

$$X = \sum_{i=1}^{n} a_i 1_{A_i},$$

where $a_i \in \mathbb{R}$, $A_i \in \mathcal{A}$, $1 \leq i \leq n$ and the function $1_A(x)$ is **the indicator function**, which is defined by

$$1_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

The expectation of a simple random variable is defined by

$$\mathbb{E}\{X\} = \sum_{i=1}^{n} a_i \mathbb{P}(A_i).$$

(It is also written $\mathbb{E}\{X\} = \int X(\omega)\mathbb{P}(d\omega)$). Let $\mathcal{L}^1$ denote the set of all integrable (finite expectation) random variables. There are the important properties of the expectation in following:

1. If $X = Y$ almost surely (a.s), then $\mathbb{E}\{X\} = \mathbb{E}\{Y\}$,

   (The statement "$X = Y$ almost surely" means $\mathbb{P}(\{\omega : X(\omega) = Y(\omega)\}) = 1$).

2. (Monotone convergence theorem): If the sequence of random variables X_n are positive and increasing a.s. to X, then $\lim_{n\to\infty} \mathbb{E}\{X_n\} = \mathbb{E}\{X\}$.

3. (Fatou's lemma): If the sequence of random variables X_n satisfy $X_n \geq Y$ a.s. ($Y \in \mathcal{L}^1$), all n, we have $\mathbb{E}\{\liminf_{n\to\infty} X_n\} \leq \liminf_{n\to\infty} \mathbb{E}\{X_n\}$.

4. (Lebesque's dominated convergence theorem): If the sequence of random variables X_n converge a.s. to X and if $| X_n | \leq Y$ a.s. $\in \mathcal{L}^1$, all n, then $X_n \in \mathcal{L}^1$, $X \in \mathcal{L}^1$, and $\mathbb{E}\{X_n\} \to \mathbb{E}\{X\}$.

5. If the sequence of random variables X_n are all positive, then

$$\mathbb{E}\left\{\sum_{n=1}^{\infty} X_n\right\} = \sum_{n=1}^{\infty} \mathbb{E}\{X_n\}.$$

6. If $\sum_{n=1}^{\infty} \mathbb{E}\{| X_n |\} < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges a.s. and the sum of this series is integrable and moreover the interchange of the expectationand summation also holds.

The Variance of X, written $\sigma^2(X)$, is

$$\mathrm{Var}(X) = \sigma^2(X) \equiv \mathbb{E}\{(X - \mathbb{E}\{X\})^2\}.$$

Let X be Cauchy random variable with identity function

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Note that the mean of a Cauchy random variable X does not exist. Since,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \frac{1}{\pi}\left\{\int_{-\infty}^{0} \frac{x}{1+x^2}dx + \int_{0}^{\infty} \frac{x}{1+x^2}dx\right\}.$$

Note that, $\frac{x}{1+x^2} \geq 0$ for all $x \geq 0$ and $\frac{x}{1+x^2} \geq \frac{1}{2x}$ for all $x > 1$, then we have

$$\int_0^{\infty} \frac{x}{1+x^2}dx \geq \int_0^1 0dx + \int_1^{\infty} \frac{1}{2x}dx = \infty.$$

And also $\int_{-\infty}^{0} \frac{x}{1+x^2}dx = -\infty$, because $\frac{x}{1+x^2}$ is an odd function. Thus, the improper integral over $(-\infty, \infty)$ cannot be defined.

Convergence of random variables

A sequence of random variables $(X_n)_{n\geq 1}$ **converges almost surely** to a random variable X if

$$\mathbb{P}\{\omega : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\} = 0.$$

Usually, we abbreviate almost sure convergence by

$$\lim_{n\to\infty} X_n \overset{\text{(a.s.)}}{=} X.$$

A sequence of random variables $(X_n)_{n\geq 1}$ **converges in** L^p to X $(1 \leq p < \infty)$ if $\mid X_n \mid, \mid X \mid$ are in L^p and:

$$\lim_{n\to\infty} \mathbb{E}\{\mid X_n - X \mid^p\} = 0,$$

and we write

$$X_n \overset{L^p}{\to} X.$$

A sequence of random variables $(X_n)_{n\geq 1}$ **converges in probability** to X if for any $\epsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}(\mid X_n - X \mid > \epsilon) = 0,$$

and we write

$$X_n \overset{\mathbb{P}}{\to} X.$$

There are some useful properties in following: Let f be a continuous function.

1. If $\lim_{n\to\infty} X_n \overset{\text{(a.s.)}}{=} X$, then $\lim_{n\to\infty} f(X_n) \overset{\text{(a.s.)}}{=} f(X)$.

2. If $X_n \overset{\mathbb{P}}{\to} X$, then $f(X_n) \overset{\mathbb{P}}{=} f(X)$.

Weak convergence

Let μ_n and μ be probability measures on $\mathbb{R}$. The sequence μ_n **converges weakly** to μ if $\int f(x)\mu_n(dx)$ converges to $\int f(x)\mu(dx)$ for each f which is real-valued, continuous and bounded on $\mathbb{R}$.

Let $(X_n)_{n\geq 1}$, X be $\mathbb{R}$-valued random variables. We say X_n **converges in distribution** to X if the distribution measures $\mathbb{P}^{X_n}$ converge weakly to $\mathbb{P}^X$ and write $X_n \overset{\mathcal{D}}{\to} X$.

There is an useful property in following: Let $(X_n)_{n\geq 1}$, X be $\mathbb{R}$-valued random variables. Then $X_n \overset{\mathcal{D}}{\to} X$ if and only if

$$\lim_{n\to\infty} \mathbb{E}\{f(X_n)\} = \mathbb{E}\{f(X)\},$$

for all continuous, bounded function f on $\mathbb{R}$.

The basic ergodic theory

We refer to the monograph of Geon Ho Choe [18] and Steuding [62].

The measure preserving transformation

Let (X_1, μ_1) and (X_2, μ_2) be measure spaces. A mapping $T : X_1 \to X_2$ is **measurable** if $T^{-1}(E)$ is measurable for every measurable subset $E \subset X_2$. The mapping T is **measure preserving** if $\mu_1(T^{-1}E) = \mu_2(E)$ for every measurable subset $E \subset X_2$. The mapping T is a **transformation** if $X_1 = X_2$ anf $\mu_1 = \mu_2$. The measure μ is T**-invariant** (or invariant under

T) if a measurable transformation $T : X \to X$ preserves the measure μ. The transformation T is an **invertible measure preserving transformation** if T is invertible and if both T and T^{-1} are measurable and measure preserving. Let $(X, \mathcal{A}, \mu)$ be a measure space. The μ-invariant T is **ergodic** if $E \in \mathcal{A}$ satisfies $T^{-1}E = E$ if, and only if, $\mu(E) = 0$ or 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. A transformation $T : X \to X$ preserves μ if, and only if, for any Lebesque integrable function f we have*

$$\int_X f(x)d\mu = \int_X f(T(x))d\mu$$

.

We give the information about our transformation. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}\left(x - \frac{1}{x}\right) & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Note that, the inverse image of an interval (α, β), written $T^{-1}(\alpha, \beta)$ is

$$(\alpha - \sqrt{\alpha^2 + 1}, \beta - \sqrt{\beta^2 + 1}) \cup (\alpha + \sqrt{\alpha^2 + 1}, \beta + \sqrt{\beta^2 + 1}),$$

hence, T is measurable. Using the substitution $\tau = Tx$, we have $d\tau = \frac{1}{2}(1 + \frac{1}{x^2})$ and

$$\int_{-\infty}^{\infty} f(Tx)\frac{dx}{1 + x^2} = \int_{-\infty}^{\infty} f(\tau)\frac{d\tau}{1 + \tau^2}.$$

Thus, the transformation T is measure preserving and has a finite invariant density function

$$\rho(x) = \frac{1}{1 + x^2}.$$

The Birkhoff Ergodic Theorem

Let (X, μ) be a probability space. If T is μ-invariant and f is integrable, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = f^*(x)$$

for some Lebesque integrable function f^* with $f^*(Tx) = f^*(x)$ for almost every x. Furthermore, if T is ergodic, then f^* is constant and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f d\mu$$

for almost every x.

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